Set

DEFINITION

A ***set*** is a (possibly infinite) collection of items and is notated with curly braces (for example, $\{1,2,3\}$ is the set containing the numbers 1, 2, and 3). We call the items in a set ***elements***.

If X is a set and a is an element of X, we may write $a \in X$, which is read "a is an element of X."

If X is a set, a ***subset*** Y of X (written $Y \subseteq X$) is a set such that every element of Y is an element of X. Two sets are called ***equal*** if they are subsets of each other (i.e., $X = Y$ if $X \subseteq Y$ and $Y \subseteq X$).

We can define a subset using ***set-builder notation***. That is, if X is a set, we can define the subset

$$Y = \{a \in X \ : \ \text{some rule involving } a\},$$

which is read "Y is the set of a in X **such that** some rule involving a is true." If X is intuitive, we may omit it and simply write $Y = \{a : \text{some rule involving } a\}$. You may equivalently use "|" instead of ":", writing $Y = \{a \mid \text{some rule involving } a\}$.

DEFINITION

Some common sets are

$$\mathbb{N} = \{\text{natural numbers}\} = \{\text{non-negative whole numbers}\}.$$

$$\mathbb{Z} = \{\text{integers}\} = \{\text{whole numbers, including negatives}\}.$$

$$\mathbb{R} = \{\text{real numbers}\}.$$

$$\mathbb{R}^n = \{\text{vectors in } n\text{-dimensional Euclidean space}\}.$$

1 Which of the following statements are true?

1.1 (a) $3 \in \{1, 2, 3\}$.

(b) $1.5 \in \{1, 2, 3\}$.

(c) $4 \in \{1, 2, 3\}$.

(d) “b”$\in \{x : x \text{ is an English letter}\}$.

(e) “ò”$\in \{x : x \text{ is an English letter}\}$.

(f) $\{1, 2\} \subseteq \{1, 2, 3\}$.

(g) For some $a \in \{1, 2, 3\}$, $a \geq 3$.

(h) For any $a \in \{1, 2, 3\}$, $a \geq 3$.

(i) $1 \subseteq \{1, 2, 3\}$.

(j) $\{1, 2, 3\} = \{x \in \mathbb{R} : 1 \leq x \leq 3\}$.

(k) $\{1, 2, 3\} = \{x \in \mathbb{Z} : 1 \leq x \leq 3\}$.

2 Write the following in set-builder notation

2.1 The subset $A \subseteq \mathbb{R}$ of real numbers larger than $\sqrt{2}$.

2.2 The subset $B \subseteq \mathbb{R}^2$ of vectors whose first coordinate is twice the second.

DEFINITION

Unions & Intersections

Two common set operations are ***unions*** and ***intersections***. Let X and Y be sets.

$$\text{(union)} \quad X \cup Y = \{a : a \in X \text{ or } a \in Y\}.$$

$$\text{(intersection)} \quad X \cap Y = \{a : a \in X \text{ and } a \in Y\}.$$

3 Let $X = \{1, 2, 3\}$ and $Y = \{2, 3, 4, 5\}$ and $Z = \{4, 5, 6\}$. Compute

3.1 $X \cup Y$

3.2 $X \cap Y$

3.3 $X \cup Y \cup Z$

3.4 $X \cap Y \cap Z$

4 Draw the following subsets of $\mathbb{R}^2$.

4.1 $V = \left\{ \vec{x} \in \mathbb{R}^2 \;:\; \vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix} \text{ for some } t \in \mathbb{R} \right\}.$

4.2 $H = \left\{ \vec{x} \in \mathbb{R}^2 \;:\; \vec{x} = \begin{bmatrix} t \\ 0 \end{bmatrix} \text{ for some } t \in \mathbb{R} \right\}.$

4.3 $D = \left\{ \vec{x} \in \mathbb{R}^2 \;:\; \vec{x} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ for some } t \in \mathbb{R} \right\}.$

4.4 $N = \left\{ \vec{x} \in \mathbb{R}^2 \;:\; \vec{x} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ for all } t \in \mathbb{R} \right\}.$

4.5 $V \cup H$.

4.6 $V \cap H$.

4.7 Does $V \cup H = \mathbb{R}^2$?

DEFINITION

Linear Combination

A ***linear combination*** of the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ is a vector

$$\vec{w} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_n \vec{v}_n.$$

The scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ are called the ***coefficients*** of the linear combination.

5 Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and $\vec{w} = 2\vec{v}_1 + \vec{v}_2$.

5.1 Write $\vec{w}$ as a column vector. When $\vec{w}$ is written as a linear combination of $\vec{v}_1$ and $\vec{v}_2$, what are the coefficients of $\vec{v}_1$ and $\vec{v}_2$?

5.2 Is $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

5.3 Is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

5.4 Is $\begin{bmatrix} 4 \\ 0 \end{bmatrix}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

5.5 Can you find a vector in $\mathbb{R}^2$ that isn't a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

5.6 Can you find a vector in $\mathbb{R}^2$ that isn't a linear combination of $\vec{v}_1$?

6 Recall the *Magic Carpet Ride* task where the hover board could travel in the direction $\vec{h} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and the magic carpet could move in the direction $\vec{m} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

6.1 Rephrase the sentence *"Gauss can be reached using just the magic carpet and the hover board"* using formal mathematical language.

6.2 Rephrase the sentence *"There is nowhere Gauss can hide where he is inaccessible by magic carpet and hover board"* using formal mathematical language.

6.3 Rephrase the sentence "$\mathbb{R}^2$ *is the set of all linear combinations of* $\vec{h}$ *and* $\vec{m}$" using formal mathematical language.

Non-negative & Convex Linear Combinations

DEFINITION

The linear combination $\vec{w} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n$ is called a ***non-negative*** linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ if $\alpha_1, \alpha_2, \ldots, \alpha_n \geq 0$.

If $\alpha_1, \alpha_2, \ldots, \alpha_n \geq 0$ and $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$, then $\vec{w}$ is called a ***convex*** linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$.

7 Let

$$\vec{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \vec{c} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \vec{d} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \qquad \vec{e} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}.$$

7.1 Out of $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$, and $\vec{e}$, which vectors are

(a) linear combinations of $\vec{a}$ and $\vec{b}$?

(b) non-negative linear combinations of $\vec{a}$ and $\vec{b}$?

(c) convex linear combinations of $\vec{a}$ and $\vec{b}$?

7.2 If possible, find two vectors $\vec{u}$ and $\vec{v}$ so that

(a) $\vec{a}$ and $\vec{c}$ are non-negative linear combinations of $\vec{u}$ and $\vec{v}$ but $\vec{b}$ is not.

(b) $\vec{a}$ and $\vec{e}$ are non-negative linear combinations of $\vec{u}$ and $\vec{v}$.

(c) $\vec{a}$ and $\vec{b}$ are non-negative linear combinations of $\vec{u}$ and $\vec{v}$ but $\vec{d}$ is not.

(d) $\vec{a}$, $\vec{c}$, and $\vec{d}$ are convex linear combinations of $\vec{u}$ and $\vec{v}$.

Otherwise, explain why it's not possible.

8 Let A be the set of points $(x, y) \in \mathbb{R}^2$ such that $y = 2x + 1$.

8.1 Describe A using set-builder notation.

8.2 Draw A as a subset of $\mathbb{R}^2$.

8.3 Add the vectors $\vec{a} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\vec{d} = \vec{b} - \vec{a}$ to your drawing.

8.4 For which $t \in \mathbb{R}$ is it true that $\vec{a} + t\vec{d} \in A$? Explain using your picture.

DEFINITION

Vector Form of a Line

A line ℓ is written in ***vector form*** if it is expressed as

$$\vec{x} = t\vec{d} + \vec{p}$$

for some vector $\vec{d}$ and point $\vec{p}$. That is, $\ell = \{\vec{x} \;:\; \vec{x} = t\vec{d} + \vec{p} \text{ for some } t \in \mathbb{R}\}$. The vector $\vec{d}$ is called a ***direction vector*** for ℓ.

9 Let $\ell \subseteq \mathbb{R}^2$ be the line with equation $2x + y = 3$, and let $L \subseteq \mathbb{R}^3$ be the line with equations $2x + y = 3$ and $z = y$.

9.1 Write ℓ in vector form. Is vector form of ℓ unique?

9.2 Write L in vector form.

9.3 Find another vector form for L where both "$\vec{d}$" and "$\vec{p}$" are different from before.

10 Let A, B, and C be given in vector form by

$$\overbrace{\vec{x} = t\begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix}}^{A} \quad \overbrace{\vec{x} = t\begin{bmatrix}-1\\1\\1\end{bmatrix} + \begin{bmatrix}-1\\1\\2\end{bmatrix}}^{B} \quad \overbrace{\vec{x} = t\begin{bmatrix}2\\-1\\1\end{bmatrix} + \begin{bmatrix}1\\1\\1\end{bmatrix}}^{C}.$$

10.1 Do the lines A and B intersect? Justify your conclusion.

10.2 Do the lines A and C intersect? Justify your conclusion.

10.3 Let $\vec{p} \neq \vec{q}$ and suppose X has vector form $\vec{x} = t\vec{d} + \vec{p}$ and Y has vector form $\vec{x} = t\vec{d} + \vec{q}$. Is it possible that X and Y intersect?

DEFINITION

Vector Form of a Plane

A plane $\mathcal{P}$ is written in ***vector form*** if it is expressed as

$$\vec{x} = t\vec{d}_1 + s\vec{d}_2 + \vec{p}$$

for some vectors $\vec{d}_1$ and $\vec{d}_2$ and point $\vec{p}$. That is, $\mathcal{P} = \{\vec{x} : \vec{x} = t\vec{d}_1 + s\vec{d}_2 + \vec{p}$ for some $t, s \in \mathbb{R}\}$. The vectors $\vec{d}_1$ and $\vec{d}_2$ are called ***direction vectors*** for $\mathcal{P}$.

11 Recall the intersecting lines A and B given in vector form by

$$\overbrace{\vec{x} = t\begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix}}^{A} \qquad \overbrace{\vec{x} = t\begin{bmatrix}-1\\1\\1\end{bmatrix} + \begin{bmatrix}-1\\1\\2\end{bmatrix}}^{B}.$$

Let $\mathcal{P}$ the plane that contains the lines A and B.

11.1 Find two direction vectors in $\mathcal{P}$.

11.2 Write $\mathcal{P}$ in vector form.

11.3 Describe how vector form of a plane relates to linear combinations.

11.4 Write $\mathcal{P}$ in vector form using different direction vectors and a different point.

12 Let $\mathcal{Q} \subseteq \mathbb{R}^3$ be a plane with equation $x + y + z = 1$.

12.1 Find three points in $\mathcal{Q}$.

12.2 Find two direction vectors for $\mathcal{Q}$.

12.3 Write $\mathcal{Q}$ in vector form.

Span

DEFINITION

The ***span*** of a set of vectors V is the set of all linear combinations of vectors in V. In set notation: $\operatorname{span} V$ is defined to be the set of vectors:

$$\{\vec{v} : \vec{v} = \alpha_1\vec{v}_1 + \cdots + \alpha_n\vec{v}_n \text{ for some } \vec{v}_1, \ldots, \vec{v}_n \in V \text{ and scalars } \alpha_1, \ldots, \alpha_n\}$$

13 Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

13.1 Draw $\operatorname{span}\{\vec{v}_1\}$.

13.2 Draw $\operatorname{span}\{\vec{v}_2\}$.

13.3 Describe $\operatorname{span}\{\vec{v}_1, \vec{v}_2\}$.

13.4 Describe $\operatorname{span}\{\vec{v}_1, \vec{v}_3\}$.

13.5 Describe $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

14 Let $\ell_1 \subseteq \mathbb{R}^2$ be the line with equation $x - y = 0$ and $\ell_2 \subseteq \mathbb{R}^2$ the line with equation $x - y = 4$.

14.1 If possible, describe ℓ_1 as a span. Otherwise explain why it's not possible.

14.2 If possible, describe ℓ_2 as a span. Otherwise explain why it's not possible.

14.3 Does the expression $\mathrm{span}(\ell_1)$ make sense? If so, what is it? How about $\mathrm{span}(\ell_2)$?

DEFINITION

Set Addition

If A and B are sets of vectors, then the ***set sum*** of A and B, denoted $A+B$, is

$$A+B=\{\vec{x} : \vec{x}=\vec{a}+\vec{b} \text{ for some } \vec{a}\in A \text{ and } \vec{b}\in B\}.$$

15 Let $A = \left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$, $B = \left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right\}$, and $\ell = \operatorname{span}\left\{\begin{bmatrix}1\\-1\end{bmatrix}\right\}$.

15.1 Draw A, B, and $A+B$ in the same picture.

15.2 Is $A+B$ the same as $B+A$?

15.3 Draw $\ell + A$.

15.4 Consider the line ℓ_2 given in vector form by $\vec{x} = t\begin{bmatrix}1\\-1\end{bmatrix} + \begin{bmatrix}1\\2\end{bmatrix}$. Can ℓ_2 be described using only a span? What about using a span and set addition?

Linearly Dependent & Independent (Geometric)

DEFINITION

We say the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are ***linearly dependent*** if for at least one i,

$$\vec{v}_i \in \operatorname{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_n\}.$$

Otherwise, they are called ***linearly independent***.

16 Let $\vec{u}=\begin{bmatrix}1\\0\\0\end{bmatrix}$, $\vec{v}=\begin{bmatrix}0\\1\\0\end{bmatrix}$, and $\vec{w}=\begin{bmatrix}1\\1\\0\end{bmatrix}$.

16.1 Describe $\operatorname{span}\{\vec{u},\vec{v},\vec{w}\}$.

16.2 Is $\{\vec{u},\vec{v},\vec{w}\}$ linearly independent? Why or why not?

Let $X=\{\vec{u},\vec{v},\vec{w}\}$.

16.3 Give a subset $Y \subseteq X$ so that $\operatorname{span} Y = \operatorname{span} X$ and Y is linearly independent.

16.4 Give a subset $Z \subseteq X$ so that $\operatorname{span} Z = \operatorname{span} X$ and Z is linearly independent and $Z \neq Y$.

DEF

Trivial Linear Combination

We say a linear combination $a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n$ is ***trivial*** if $a_1 = a_2 = \cdots = a_n = 0$.

17 Recall $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

17.1 Consider the linearly dependent set $\{\vec{u}, \vec{v}, \vec{w}\}$ (where $\vec{u}, \vec{v}, \vec{w}$ are defined as above). Can you write $\vec{0}$ as a non-trivial linear combination of vectors in this set?

17.2 Consider the linearly independent set $\{\vec{u}, \vec{v}\}$. Can you write $\vec{0}$ as a non-trivial linear combination of vectors in this set?

We now have an equivalent definition of linear dependence.

Linearly Dependent & Independent (Algebraic)

DEF

The vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are ***linearly dependent*** if there is a non-trivial linear combination of $\vec{v}_1, \ldots, \vec{v}_n$ that equals the zero vector.

18.1 Explain how this new definition implies the old one.

18.2 Explain how the old definition implies this new one.

Since we have old def $\implies$ new def, and new def $\implies$ old def ($\implies$ should be read aloud as 'implies'), the two definitions are *equivalent* (which we write as new def $\iff$ old def).

19 Suppose for some unknown $\vec{u}, \vec{v}, \vec{w}$, and $\vec{a}$,

$$\vec{a} = 3\vec{u} + 2\vec{v} + \vec{w} \qquad \text{and} \qquad \vec{a} = 2\vec{u} + \vec{v} - \vec{w}.$$

19.1 Could the set $\{\vec{u}, \vec{v}, \vec{w}\}$ be linearly independent?

Suppose that

$$\vec{a} = \vec{u} + 6\vec{r} - \vec{s}$$

is the *only* way to write $\vec{a}$ using $\vec{u}, \vec{r}, \vec{s}$.

19.2 Is $\{\vec{u}, \vec{r}, \vec{s}\}$ linearly independent?

19.3 Is $\{\vec{u}, \vec{r}\}$ linearly independent?

19.4 Is $\{\vec{u}, \vec{v}, \vec{w}, \vec{r}\}$ linearly independent?

Norm

DEFINITION

The ***norm*** of a vector $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ is the length/magnitude of $\vec{v}$.

It is written $\|\vec{v}\|$ and can be computed from the Pythagorean formula

$$\|\vec{v}\| = \sqrt{v_1^2 + \cdots + v_n^2}.$$

Dot Product

DEFINITION

If $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ are two vectors in n-dimensional space, then the ***dot product*** of $\vec{a}$ an $\vec{b}$ is

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

Equivalently, the dot product is defined by the geometric formula

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta$$

where θ is the angle between $\vec{a}$ and $\vec{b}$.

20

Let $\vec{a} = \begin{bmatrix}1\\1\end{bmatrix}$, $\vec{b} = \begin{bmatrix}3\\1\end{bmatrix}$, and $\vec{u} = \begin{bmatrix}1\\2\\1\end{bmatrix}$.

20.1 (a) Draw a picture of $\vec{a}$ and $\vec{b}$.

(b) Compute $\vec{a} \cdot \vec{b}$.

(c) Find $\|\vec{a}\|$ and $\|\vec{b}\|$ and use your knowledge of the multiple ways to compute the dot product to find θ, the angle between $\vec{a}$ and $\vec{b}$. Label θ on your picture.

20.2 Draw the graph of cos and identify which angles make cos negative, zero, or positive.

20.3 Draw a new picture of $\vec{a}$ and $\vec{b}$ and on that picture draw

(a) a vector $\vec{c}$ where $\vec{c} \cdot \vec{a}$ is negative.

(b) a vector $\vec{d}$ where $\vec{d} \cdot \vec{a} = 0$ and $\vec{d} \cdot \vec{b} < 0$.

(c) a vector $\vec{e}$ where $\vec{e} \cdot \vec{a} = 0$ and $\vec{e} \cdot \vec{b} > 0$.

(d) Could you find a vector $\vec{f}$ where $\vec{f} \cdot \vec{a} = 0$ and $\vec{f} \cdot \vec{b} = 0$? Explain why or why not.

20.4 Recall the vector $\vec{u}$ whose coordinates are given at the beginning of this problem.

(a) Write down a vector $\vec{v}$ so that the angle between $\vec{u}$ and $\vec{v}$ is $\pi/2$. (Hint, how does this relate to the dot product?)

(b) Write down another vector $\vec{w}$ (in a different direction from $\vec{v}$) so that the angle between $\vec{w}$ and $\vec{u}$ is $\pi/2$.

(c) Can you write down other vectors different than both $\vec{v}$ and $\vec{w}$ that still form an angle of $\pi/2$ with $\vec{u}$? How many such vectors are there?

THEOREM

For a vector $\vec{v} \in \mathbb{R}^n$, the formula

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$$

always holds.

DEF

Distance

The ***distance*** between two vectors $\vec{u}$ and $\vec{v}$ is $\|\vec{u}-\vec{v}\|$.

DEF

Unit Vector

A vector $\vec{v}$ is called a ***unit vector*** if $\|\vec{v}\| = 1$.

21 Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$.

21.1 Find the distance between $\vec{u}$ and $\vec{v}$.

21.2 Find a unit vector in the direction of $\vec{u}$.

21.3 Does there exists a *unit vector* $\vec{x}$ that is distance 1 from $\vec{u}$?

21.4 Suppose $\vec{y}$ is a unit vector and the distance between $\vec{y}$ and $\vec{u}$ is 2. What is the angle between $\vec{y}$ and $\vec{u}$?

Orthogonal

DEF

Two vectors $\vec{u}$ and $\vec{v}$ are ***orthogonal*** to each other if $\vec{u} \cdot \vec{v} = 0$. The word orthogonal is synonymous with the word perpendicular.

22.1 Find two vectors orthogonal to $\vec{a} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$. Can you find two such vectors that are not parallel?

22.2 Find two vectors orthogonal to $\vec{b} = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$. Can you find two such vectors that are not parallel?

22.3 Suppose $\vec{x}$ and $\vec{y}$ are orthogonal to each other and $\|\vec{x}\| = 5$ and $\|\vec{y}\| = 3$. What is the distance between $\vec{x}$ and $\vec{y}$?

23

23.1 Draw $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and *all* vectors orthogonal to it. Call this set A.

23.2 If $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\vec{x}$ is orthogonal to $\vec{u}$, what is $\vec{x} \cdot \vec{u}$?

23.3 Expand the dot product $\vec{u} \cdot \vec{x}$ to get an equation for A.

23.4 If possible, express A as a span.

DEF

Normal Vector

A ***normal vector*** to a line (or plane or hyperplane) is a non-zero vector that is orthogonal to all direction vectors for the line (or plane or hyperplane).

24 Let $\vec{d} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{p} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and define the lines

$$\ell_1 = \operatorname{span}\{\vec{d}\} \qquad \text{and} \qquad \ell_2 = \operatorname{span}\{\vec{d}\} + \{\vec{p}\}.$$

24.1 Find a vector $\vec{n}$ that is a normal vector for both ℓ_1 and ℓ_2.

24.2 Let $\vec{v} \in \ell_1$ and $\vec{u} \in \ell_2$. What is $\vec{n} \cdot \vec{v}$? What about $\vec{n} \cdot \vec{u}$?

24.3 A line is expressed in *normal form* if it is represented by an equation of the form $\vec{n} \cdot (\vec{x} - \vec{q}) = 0$ for some $\vec{n}$ and $\vec{q}$. Express ℓ_1 and ℓ_2 in normal form.

25

Let $\vec{n} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

25.1 Use set-builder notation to write down the set, X, of all vectors orthogonal to $\vec{n}$. Describe this set geometrically.

25.2 Describe X using an equation.

25.3 Describe X as a span.

Projection

DEF

Let X be a set. The ***projection*** of the vector $\vec{v}$ onto X, written $\operatorname{proj}_X \vec{v}$, is the closest point in X to $\vec{v}$.

26 Let $\vec{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $\ell = \mathrm{span}\{\vec{a}\}$.

26.1 Draw $\vec{a}$, $\vec{b}$, and $\vec{v}$ in the same picture.

26.2 Find $\mathrm{proj}_{\{\vec{b}\}} \vec{v}$, $\mathrm{proj}_{\{\vec{a},\vec{b}\}} \vec{v}$.

26.3 Find $\mathrm{proj}_{\ell} \vec{v}$. (Recall that a quadratic $at^2 + bt + c$ has a minimum at $t = -\frac{b}{2a}$).

26.4 Is $\vec{v} - \mathrm{proj}_{\ell} \vec{v}$ a normal vector for ℓ? Why or why not?

27 Let K be the line given in vector form by $\vec{x} = t\begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}1\\0\end{bmatrix}$ and let $\vec{c} = \begin{bmatrix}1\\3\end{bmatrix}$.

27.1 Make a sketch with $\vec{c}$, K, and $\operatorname{proj}_K \vec{c}$ (you don't need to compute $\operatorname{proj}_K \vec{c}$ exactly).

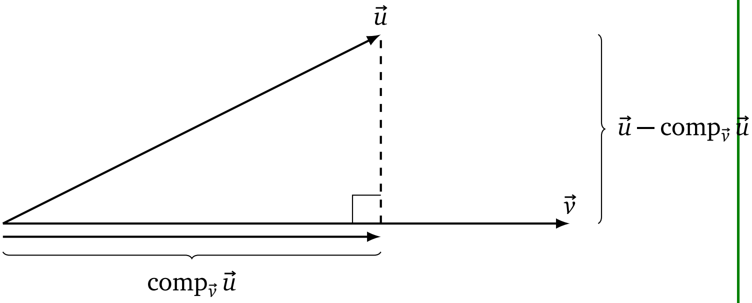
27.2 What should $(\vec{c} - \operatorname{proj}_K \vec{c}) \cdot \begin{bmatrix}1\\2\end{bmatrix}$ be? Explain.

27.3 Use your formula from the previous part to find $\operatorname{proj}_K \vec{c}$ *without* computing any distances.

Component

DEFINITION

Let $\vec{u}$ and $\vec{v} \neq \vec{0}$ be vectors. The ***component of $\vec{u}$ in the $\vec{v}$ direction***, written $\text{comp}_{\vec{v}}\,\vec{u}$, is the vector in the direction of $\vec{v}$ so that $\vec{u} - \text{comp}_{\vec{v}}\,\vec{u}$ is orthogonal to $\vec{v}$.



28 Let $\vec{a}, \vec{b} \in \mathbb{R}^3$ be unknown vectors.

28.1 List two conditions that $\operatorname{comp}_{\vec{b}} \vec{a}$ must satisfy.

28.2 Find a formula for $\operatorname{comp}_{\vec{b}} \vec{a}$.

29 Let $\vec{d} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

29.1 Draw $\vec{d}$, $\vec{u}$, $\mathrm{span}\{\vec{d}\}$, and $\mathrm{proj}_{\mathrm{span}\{\vec{d}\}} \vec{u}$ in the same picture.

29.2 How do $\mathrm{proj}_{\mathrm{span}\{\vec{d}\}} \vec{u}$ and $\mathrm{comp}_{\vec{d}} \vec{u}$ relate?

29.3 Compute $\mathrm{proj}_{\mathrm{span}\{\vec{d}\}} \vec{u}$ and $\mathrm{comp}_{\vec{d}} \vec{u}$.

29.4 Compute $\mathrm{comp}_{-\vec{d}} \vec{u}$. Is this the same as or different from $\mathrm{comp}_{\vec{d}} \vec{u}$? Explain.

DEFINITION

Subspace

A ***subspace*** $V \subseteq \mathbb{R}^n$ is a non-empty subset such that

(i) $\vec{u}, \vec{v} \in V$ implies $\vec{u} + \vec{v} \in V$.

(ii) $\vec{u} \in V$ implies $k\vec{u} \in V$ for all scalars k.

Subspaces give a mathematically precise definition of a "flat space through the origin."

30 For each set, draw it and explain whether or not it is a subspace of $\mathbb{R}^2$.

30.1 $A = \left\{ \vec{x} \in \mathbb{R}^2 \,:\, \vec{x} = \begin{bmatrix} a \\ 0 \end{bmatrix} \text{ for some } a \in \mathbb{Z} \right\}$.

30.2 $B = \left\{ \vec{x} \in \mathbb{R}^2 \,:\, \vec{x} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$.

30.3 $C = \left\{ \vec{x} \in \mathbb{R}^2 \,:\, \vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix} \text{ for some } t \in \mathbb{R} \right\}$.

30.4 $D = \left\{ \vec{x} \in \mathbb{R}^2 \,:\, \vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ for some } t \in \mathbb{R} \right\}$.

30.5 $E = \left\{ \vec{x} \in \mathbb{R}^2 \,:\, \vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix} \text{ or } \vec{x} = \begin{bmatrix} t \\ 0 \end{bmatrix} \text{ for some } t \in \mathbb{R} \right\}$.

30.6 $F = \left\{ \vec{x} \in \mathbb{R}^2 \,:\, \vec{x} = t \begin{bmatrix} 3 \\ 1 \end{bmatrix} \text{ for some } t \in \mathbb{R} \right\}$.

30.7 $G = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

30.8 $H = \operatorname{span}\{\vec{u}, \vec{v}\}$ for some unknown vectors $\vec{u}, \vec{v} \in \mathbb{R}^2$.

DEF

Basis

A ***basis*** for a subspace V is a linearly independent set of vectors, $\mathcal{B}$, so that $\operatorname{span}\mathcal{B} = V$.

DEF

Dimension

The ***dimension*** of a subspace V is the number of elements in a basis for V.

31 Let $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, and $V = \operatorname{span}\{\vec{u}, \vec{v}, \vec{w}\}$.

31.1 Describe V.

31.2 Is $\{\vec{u}, \vec{v}, \vec{w}\}$ a basis for V? Why or why not?

31.3 Give a basis for V.

31.4 Give another basis for V.

31.5 Is $\operatorname{span}\{\vec{u}, \vec{v}\}$ a basis for V? Why or why not?

31.6 What is the dimension of V?

32 Let $\vec{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, $\vec{c} = \begin{bmatrix} 7 \\ 8 \\ 8 \end{bmatrix}$ (notice these vectors are linearly independent) and let $P = \operatorname{span}\{\vec{a}, \vec{b}\}$ and $Q = \operatorname{span}\{\vec{b}, \vec{c}\}$.

32.1 Give a basis for and the dimension of P.

32.2 Give a basis for and the dimension of Q.

32.3 Is $P \cap Q$ a subspace? If so, give a basis for it and its dimension.

32.4 Is $P \cup Q$ a subspace? If so, give a basis for it and its dimension.

33 Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 3 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$.

33.1 Compute the product $A\vec{x}$.

33.2 Write down a system of equations that corresponds to the matrix equation $A\vec{x} = \vec{b}$.

33.3 Let $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ be a solution to $A\vec{x} = \vec{b}$. Explain what x_0 and y_0 mean in terms of *linear combinations* (hint: think about the columns of A).

33.4 Let $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ be a solution to $A\vec{x} = \vec{b}$. Explain what x_0 and y_0 mean in terms of *intersecting lines* (hint: think about systems of equations).

34 Let $\vec{u}=\begin{bmatrix}1\\2\\3\end{bmatrix}$, $\vec{v}=\begin{bmatrix}4\\5\\6\end{bmatrix}$, $\vec{w}=\begin{bmatrix}7\\8\\9\end{bmatrix}$.

34.1 How could you determine if $\{\vec{u},\vec{v},\vec{w}\}$ was a linearly independent set?

34.2 Can your method be rephrased in terms of a matrix equation? Explain.

35 Consider the system represented by

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{b}.$$

35.1 If $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, is the set of solutions to this system a point, line, plane, or other?

35.2 If $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, is the set of solutions to this system a point, line, plane, or other?

36 Let $\vec{d}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\vec{d}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$. Let $\mathcal{P}$ be the plane given in vector form by $\vec{x} = t\vec{d}_1 + s\vec{d}_2$. Further, suppose M is a matrix so that $M\vec{r} \in \mathcal{P}$ for any $\vec{r} \in \mathbb{R}^2$.

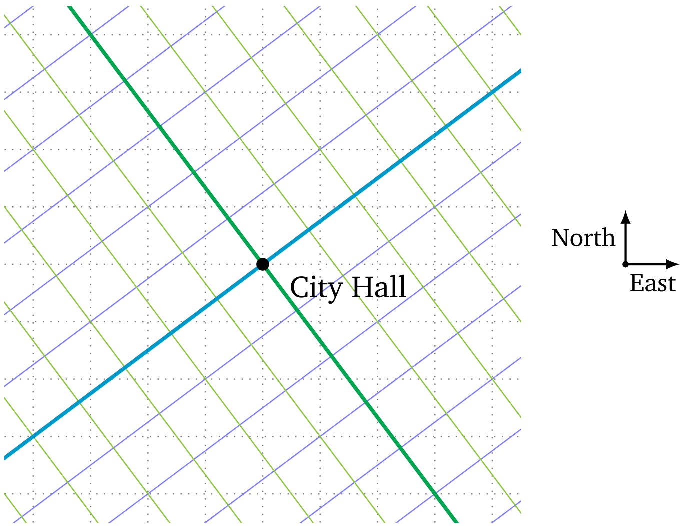
36.1 How many rows does M have?

36.2 Find such an M.

36.3 Find necessary and sufficient conditions (phrased as equations) for $\vec{n}$ to be a normal vector for $\mathcal{P}$.

36.4 Find a matrix K so that non-zero solutions to $K\vec{x} = \vec{0}$ are normal vectors for $\mathcal{P}$. How do K and M relate?

37 The fictional town of Oronto is not aligned with the usual compass directions. The streets are laid out as follows:



Instead, every street is parallel to the vector $\vec{d}_1 = \frac{1}{5}\begin{bmatrix} 4 \text{ east} \\ 3 \text{ north} \end{bmatrix}$ or $\vec{d}_2 = \frac{1}{5}\begin{bmatrix} -3 \text{ east} \\ 4 \text{ north} \end{bmatrix}$. The center of town is City Hall at $\vec{0} = \begin{bmatrix} 0 \text{ east} \\ 0 \text{ north} \end{bmatrix}$.

Locations in Oronto are typically specified in *street coordinates*. That is, as a pair (a, b) where a is how far you walk along streets in the $\vec{d}_1$ direction and b is how far you walk in the $\vec{d}_2$ direction, provided you start at city hall.

37.1 The points $A = (2, 1)$ and $B = (3, -1)$ are given in street coordinates. Find their east-north coordinates.

37.2 The points $X = (4, 3)$ and $Y = (1, 7)$ are given in east-north coordinates. Find their street coordinates.

37.3 Define $\vec{e}_1 = \begin{bmatrix} 1 \text{ east} \\ 0 \text{ north} \end{bmatrix}$ and $\vec{e}_2 = \begin{bmatrix} 0 \text{ east} \\ 1 \text{ north} \end{bmatrix}$. Does $\operatorname{span}\{\vec{e}_1, \vec{e}_2\} = \operatorname{span}\{\vec{d}_1, \vec{d}_2\}$?

37.4 Notice that $Y = 5\vec{d}_1 + 5\vec{d}_2 = \vec{e}_1 + 7\vec{e}_2$. Is the point Y better represented by the pair $(5, 5)$ or by the pair $(1, 7)$? Explain.

Representation in a Basis

DEFINITION

Let $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ be a basis for a subspace V and let $\vec{v} \in V$. The ***representation of $\vec{v}$ in the $\mathcal{B}$ basis***, notated $[\vec{v}]_{\mathcal{B}}$, is the column matrix

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}.$$

where $\alpha_1, \ldots, \alpha_n$ uniquely satisfy $\vec{v} = \alpha_1\vec{b}_1 + \cdots + \alpha_n\vec{b}_n$.

Conversely,

$$\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}_{\mathcal{B}} = \alpha_1\vec{b}_1 + \cdots + \alpha_n\vec{b}_n$$

is notation for the linear combination of $\vec{b}_1, \ldots, \vec{b}_n$ with coefficients $\alpha_1, \ldots, \alpha_n$.

38 Let $\mathcal{E} = \{\vec{e}_1, \vec{e}_2\}$ be the standard basis for $\mathbb{R}^2$ and let $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$ where $\vec{c}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}_{\mathcal{E}}$ and $\vec{c}_2 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}_{\mathcal{E}}$ be another basis for $\mathbb{R}^2$.

38.1 Express $\vec{c}_1$ and $\vec{c}_2$ as a linear combination of $\vec{e}_1$ and $\vec{e}_2$.

38.2 Express $\vec{e}_1$ and $\vec{e}_2$ as a linear combination of $\vec{c}_1$ and $\vec{c}_2$.

38.3 Let $\vec{v} = 2\vec{e}_1 + 2\vec{e}_2$. Find $[\vec{v}]_{\mathcal{E}}$ and $[\vec{v}]_{\mathcal{C}}$.

38.4 Can you find a matrix X so that $X[\vec{w}]_{\mathcal{C}} = [\vec{w}]_{\mathcal{E}}$ for any $\vec{w}$?

38.5 Can you find a matrix Y so that $Y[\vec{w}]_{\mathcal{E}} = [\vec{w}]_{\mathcal{C}}$ for any $\vec{w}$?

38.6 What is YX?

DEFINITION

Orientation of a Basis

The ordered basis $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ is ***right-handed*** or ***positively oriented*** if it can be continuously transformed to the standard basis (with $\vec{b}_i \mapsto \vec{e}_i$) while remaining linearly independent throughout the transformation. Otherwise, $\mathcal{B}$ is called ***left-handed*** or ***negatively oriented***.

39 Let $\{\vec{e}_1, \vec{e}_2\}$ be the standard basis for $\mathbb{R}^2$ and let $\vec{u}_\theta$ be a unit vector. Let θ be the angle between $\vec{u}_\theta$ and $\vec{e}_1$ measured counter-clockwise starting at $\vec{e}_1$.

39.1 For which θ is $\{\vec{e}_1, \vec{u}_\theta\}$ a linearly independent set?

39.2 For which θ can $\{\vec{e}_1, \vec{u}_\theta\}$ be continuously transformed into $\{\vec{e}_1, \vec{e}_2\}$ and remain linearly independent the whole time?

39.3 For which θ is $\{\vec{e}_1, \vec{u}_\theta\}$ right-handed? Left-handed?

39.4 For which θ is $\{\vec{u}_\theta, \vec{e}_1\}$ (in that order) right-handed? Left-handed?

39.5 Is $\{2\vec{e}_1, 3\vec{e}_2\}$ right-handed or left-handed? What about $\{2\vec{e}_1, -3\vec{e}_2\}$?

40 $\mathcal{R} : \mathbb{R}^2 \to \mathbb{R}^2$ is the transformation that rotates vectors counter-clockwise by 90°.

40.1 Compute $\mathcal{R}\begin{bmatrix}1\\0\end{bmatrix}$ and $\mathcal{R}\begin{bmatrix}0\\1\end{bmatrix}$.

40.2 Compute $\mathcal{R}\begin{bmatrix}1\\1\end{bmatrix}$. How does this relate to $\mathcal{R}\begin{bmatrix}1\\0\end{bmatrix}$ and $\mathcal{R}\begin{bmatrix}0\\1\end{bmatrix}$?

40.3 What is $\mathcal{R}\left(a\begin{bmatrix}1\\0\end{bmatrix}+b\begin{bmatrix}0\\1\end{bmatrix}\right)$?

40.4 Write down a matrix R so that $R\vec{v}$ is $\vec{v}$ rotated counter-clockwise by 90°.

DEFINITION

Linear Transformation

Let V and W be subspaces. A function $T : V \to W$ is called a ***linear transformation*** if

$$T(\vec{u}+\vec{v}) = T\vec{u} + T\vec{v} \qquad \text{and} \qquad T(\alpha\vec{v}) = \alpha T\vec{v}$$

for all vectors $\vec{u}, \vec{v} \in V$ and all scalars α.

41 Classify the following as linear transformations or not.

41.1 (a) $\mathcal{R}$ from before (rotation counter-clockwise by 90°).

(b) $W : \mathbb{R}^2 \to \mathbb{R}^2$ where $W\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x^2 \\ y \end{bmatrix}$.

(c) $T : \mathbb{R}^2 \to \mathbb{R}^2$ where $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+2 \\ y \end{bmatrix}$.

(d) $\mathcal{P} : \mathbb{R}^2 \to \mathbb{R}^2$ where $\mathcal{P}\begin{bmatrix} x \\ y \end{bmatrix} = \operatorname{comp}_{\vec{u}}\begin{bmatrix} x \\ y \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

DEFINITION

Image of a Set

Let $L : V \to W$ be a transformation and let $X \subseteq V$ be a set. The ***image of the set X under L***, denoted $L(X)$, is the set

$$L(X) = \{\vec{x} \in W \ : \ \vec{x} = L(\vec{y}) \text{ for some } \vec{y} \in X\}.$$

42 Let $S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : 0 \leq x, y \leq 1 \right\} \subseteq \mathbb{R}^2$ be the filled-in unit square and let $C = \{\vec{0}, \vec{e}_1, \vec{e}_2, \vec{e}_1 + \vec{e}_2\} \subseteq \mathbb{R}^2$ be the corners of the unit square.

42.1 Find $\mathcal{R}(C)$, $W(C)$, and $T(C)$ (where $\mathcal{R}$, W, and T are from the previous question).

42.2 Draw $\mathcal{R}(S)$, $T(S)$, and $\mathcal{P}(S)$ (where $\mathcal{R}$, T, and $\mathcal{P}$ are from the previous question).

42.3 Let $\ell = \{\text{all convex combinations of } \vec{a} \text{ and } \vec{b}\}$ be a line segment with endpoints $\vec{a}$ and $\vec{b}$ and let A be a linear transformation. Must $A(\ell)$ be a line segment? What are its endpoints?

42.4 Explain how images of sets relate to the *Italicizing N* task.

43 Define $\mathcal{P}$ to be projection onto $\operatorname{span}\{\vec{u}\}$ where $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, and let $\mathcal{R}$ be rotation counter-clockwise by $90°$.

43.1 Find a matrix P so that $P\vec{x} = \mathcal{P}(\vec{x})$ for all $\vec{x} \in \mathbb{R}^2$.

43.2 Find a matrix R so that $R\vec{x} = \mathcal{R}(\vec{x})$ for all $\vec{x} \in \mathbb{R}^2$.

43.3 Write down matrices A and B for $\mathcal{P} \circ \mathcal{R}$ and $\mathcal{R} \circ \mathcal{P}$.

43.4 How do the matrices A and B relate to the matrices P and R?

DEFINITION

Range

The ***range*** (or ***image***) of a linear transformation $T : V \to W$ is the set of vectors that T can output. That is,

$$\operatorname{range}(T) = \{\vec{y} \in W \;:\; \vec{y} = T\vec{x} \text{ for some } \vec{x} \in V\}.$$

DEFINITION

Null Space

The ***null space*** (or ***kernel***) of a linear transformation $T : V \to W$ is the set of vectors that get mapped to zero under T. That is,

$$\operatorname{null}(T) = \{\vec{x} \in V \ : \ T\vec{x} = \vec{0}\}.$$

44 Let $\mathcal{P} : \mathbb{R}^2 \to \mathbb{R}^2$ be projection onto $\operatorname{span}\{\vec{u}\}$ where $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ (like before).

44.1 What is the range of $\mathcal{P}$?

44.2 What is the null space of $\mathcal{P}$?

45 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be an arbitrary linear transformation.

45.1 Show that the null space of T is a subspace.

45.2 Show that the range of T is a subspace.

DEFINITION

Induced Transformation

Let M be an $n \times m$ matrix. We say M ***induces*** a linear transformation $T_M : \mathbb{R}^m \to \mathbb{R}^n$ defined by

$$[T_M \vec{v}]_{\mathcal{E}'} = M[\vec{v}]_{\mathcal{E}},$$

where $\mathcal{E}$ is the standard basis for $\mathbb{R}^m$ and $\mathcal{E}'$ is the standard basis for $\mathbb{R}^n$.

46 Let M be a 2×2 matrix and let $\vec{v} \in \mathbb{R}^2$. Further, let T_M be the transformation induced by M.

46.1 What is the difference between "$M\vec{v}$" and "$M[\vec{v}]_{\mathcal{E}}$"?

46.2 What is $[T_M \vec{e}_1]_{\mathcal{E}}$?

46.3 Can you relate the columns of M to the range of T_M?

DEFINITION

Fundamental Subspaces

Associated with any matrix M are three fundamental subspaces: the ***row space*** of M, denoted $\text{row}(M)$, is the span of the rows of M; the ***column space*** of M, denoted $\text{col}(M)$, is the span of the columns of M; and the ***null space*** of M, denoted $\text{null}(M)$, is the set of solutions to $M\vec{x} = \vec{0}$.

47 Consider $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

47.1 Describe the row space of A.

47.2 Describe the column space of A.

47.3 Is the row space of A the same as the column space of A?

47.4 Describe the set of all vectors perpendicular to the rows of A.

47.5 Describe the null space of A.

47.6 Describe the range and null space of T_A, the transformation induced by A.

48

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \qquad C = \mathrm{rref}(B) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

48.1 How does the row space of B relate to the row space of C?

48.2 How does the null space of B relate to the null space of C?

48.3 Compute the null space of B.

49

$$P = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} \qquad Q = \operatorname{rref}(P) = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

49.1 How does the column space of P relate to the column space of Q?

49.2 Describe the column space of P and the column space of Q.

DEFINITION

Rank

For a linear transformation $T : V \to W$, the ***rank*** of T, denoted $\operatorname{rank}(T)$, is the dimension of the range of T.

For an $n \times m$ matrix M, the ***rank*** of M, denoted $\operatorname{rank}(M)$, is the number of pivots in $\operatorname{rref}(M)$.

50 Let $\mathcal{P}$ be projection onto span$\{\vec{u}\}$ where $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, and let $\mathcal{R}$ be rotation counter-clockwise by 90°.

50.1 Describe range($\mathcal{P}$) and range($\mathcal{R}$).

50.2 What is the rank of $\mathcal{P}$ and the rank of $\mathcal{R}$?

50.3 Let P and R be the matrices corresponding to $\mathcal{P}$ and $\mathcal{R}$. What is the rank of P and the rank of R?

50.4 Make a conjecture about how the rank of a transformation and the rank of its corresponding matrix relate. Can you justify your claim?

51 Determine the rank of

51.1 (a) $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

52 Consider the homogeneous system

$$\begin{array}{llll} x & +2y & +z & = 0 \\ x & +2y & +3z & = 0 \\ -x & -2y & +z & = 0 \end{array} \tag{1}$$

and the non-augmented matrix of coefficients $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \\ -1 & -2 & 1 \end{bmatrix}$.

52.1 What is rank(A)?

52.2 Give the general solution to system (1).

52.3 Are the column vectors of A linearly independent?

52.4 Give a non-homogeneous system with the same coefficients as (1) that has

(a) infinitely many solutions

(b) no solutions.

53

53.1 The rank of a 3×4 matrix A is 3. Are the column vectors of A linearly independent?

53.2 The rank of a 4×3 matrix B is 3. Are the column vectors of B linearly independent?

THEOREM

Rank-nullity Theorem

The ***nullity*** of a matrix is the dimension of the null space.

The rank-nullity theorem for a matrix A states

$$\text{rank}(A) + \text{nullity}(A) = \#\text{ of columns in } A.$$

54.1 Is there a version of the rank-nullity theorem that applies to linear transformations instead of matrices? If so, state it.

55 The vectors $\vec{u}, \vec{v} \in \mathbb{R}^9$ are linearly independent and $\vec{w} = 2\vec{u} - \vec{v}$. Define $A = [\vec{u}|\vec{v}|\vec{w}]$.

55.1 What is the rank and nullity of A^T?

55.2 What is the rank and nullity of A?

56.1 Apply the row operation $R_3 \mapsto R_3 + 2R_1$ to the 3×3 identity matrix and call the result E_1.

56.2 Apply the row operation $R_3 \mapsto R_3 - 2R_1$ to the 3×3 identity matrix and call the result E_2.

DEF An ***elementary matrix*** is the identity matrix with a single row operation applied.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

56.3 Compute E_1A and E_2A. How do the resulting matrices relate to row operations?

56.4 Without computing, what should the result of applying the row operation $R_3 \mapsto R_3 - 2R_1$ to E_1 be? Compute and verify.

56.5 Without computing, what should E_2E_1 be? What about E_1E_2? Now compute and verify.

DEF

The ***inverse*** of a matrix A is a matrix B such that $AB = I$ and $BA = I$. In this case, B is called the inverse of A and is notated by A^{-1}.

57 Consider the matrices

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ -3 & -6 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \qquad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

57.1 Which pairs of matrices above are inverses of each other?

$$B = \begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix}$$

58.1 Use two row operations to reduce B to $I_{2\times 2}$ and write an elementary matrix E_1 corresponding to the first operation and E_2 corresponding to the second.

58.2 What is $E_2 E_1 B$?

58.3 Find B^{-1}.

58.4 Can you outline a procedure for finding the inverse of a matrix using elementary matrices?

59

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad C = [A|\vec{b}] \qquad A^{-1} = \begin{bmatrix} 9 & -3 \\ -5 & \\ -2 & 1 \end{bmatrix}$$

59.1 What is $A^{-1}A$?

59.2 What is rref(A)?

59.3 What is rref(C)? (Hint, there is no need to actually do row reduction!)

59.4 Solve the system $A\vec{x} = \vec{b}$.

60.1 For two square matrices X, Y, should $(XY)^{-1} = X^{-1}Y^{-1}$?

60.2 If M is a matrix corresponding to a non-invertible linear transformation T, could M be invertible?

61 Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ where $\vec{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{b}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and let $X = [\vec{b}_1|\vec{b}_2]$ be the matrix whose columns are $\vec{b}_1$ and $\vec{b}_2$.

61.1 Compute $[\vec{e}_1]_{\mathcal{B}}$ and $[\vec{e}_2]_{\mathcal{B}}$.

61.2 Compute $X[\vec{e}_1]_{\mathcal{B}}$ and $X[\vec{e}_2]_{\mathcal{B}}$. What do you notice?

61.3 Find the matrix X^{-1}. How does X^{-1} relate to change of basis?

62 Let $\mathcal{E} = \{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ be the standard basis for $\mathbb{R}^n$. Given a basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ for $\mathbb{R}^n$, the matrix $X = [\vec{b}_1|\vec{b}_2|\cdots|\vec{b}_n]$ converts vectors from the $\mathcal{B}$ basis into the standard basis. In other words,

$$X[\vec{v}]_{\mathcal{B}} = [\vec{v}]_{\mathcal{E}}.$$

62.1 Should X^{-1} exist? Explain.

62.2 Consider the equation

$$X^{-1}[\vec{v}]_{?} = [\vec{v}]_{?}.$$

Can you fill in the "?" symbols so that the equation makes sense?

62.3 What is $[\vec{b}_1]_{\mathcal{B}}$? How about $[\vec{b}_2]_{\mathcal{B}}$? Can you generalize to $[\vec{b}_i]_{\mathcal{B}}$?

63 Let $\vec{c}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}_{\mathcal{E}}$, $\vec{c}_2 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}_{\mathcal{E}}$, $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$, and $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$. Note that $A^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$ and that A changes vectors from the $\mathcal{C}$ basis to the standard basis and A^{-1} changes vectors from the standard basis to the $\mathcal{C}$ basis.

63.1 Compute $[\vec{c}_1]_{\mathcal{C}}$ and $[\vec{c}_2]_{\mathcal{C}}$.

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that stretches in the $\vec{c}_1$ direction by a factor of 2 and doesn't stretch in the $\vec{c}_2$ direction at all.

63.2 Compute $T\begin{bmatrix} 2 \\ 1 \end{bmatrix}_{\mathcal{E}}$ and $T\begin{bmatrix} 5 \\ 3 \end{bmatrix}_{\mathcal{E}}$.

63.3 Compute $[T\vec{c}_1]_{\mathcal{C}}$ and $[T\vec{c}_2]_{\mathcal{C}}$.

63.4 Compute the result of $T\begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{\mathcal{C}}$ and express the result in the $\mathcal{C}$ basis (i.e., as a vector of the form $\begin{bmatrix} ? \\ ? \end{bmatrix}_{\mathcal{C}}$).

63.5 Find $[T]_{\mathcal{C}}$, the matrix for T in the $\mathcal{C}$ basis.

63.6 Find $[T]_{\mathcal{E}}$, the matrix for T in the standard basis.

DEFINITION

Similar Matrices

A matrices A and B are called ***similar matrices***, denoted $A \sim B$, if A and B represent the same linear transformation but in possibly different bases. Equivalently, $A \sim B$ if there is an invertible matrix X so that

$$A = XBX^{-1}.$$

DEFINITION

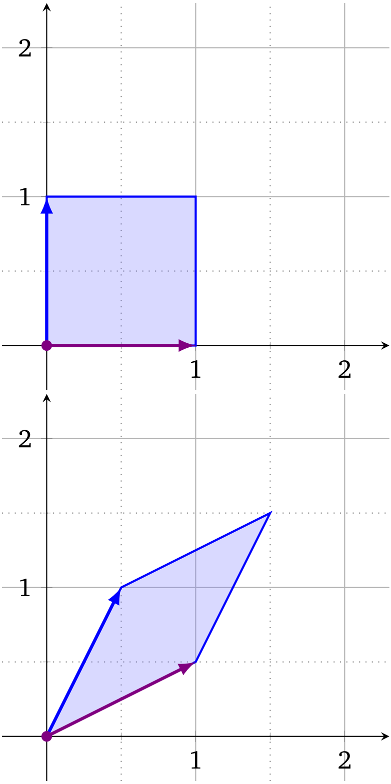
Unit n-cube

The ***unit n-cube*** is the n-dimensional cube with sides given by the standard basis vectors and lower-left corner located at the origin. That is

$$C_n = \left\{ \vec{x} \in \mathbb{R}^n : \vec{x} = \sum_{i=1}^{n} \alpha_i \vec{e}_i \text{ for some } \alpha_1, \ldots, \alpha_n \in [0,1] \right\} = [0,1]^n.$$

The sides of the unit n-cube are always length 1 and its volume is always 1.

64 The picture shows what the linear transformation T does to the unit square (i.e., the unit 2-cube).



64.1 What is $T\begin{bmatrix}1\\0\end{bmatrix}$, $T\begin{bmatrix}0\\1\end{bmatrix}$, $T\begin{bmatrix}1\\1\end{bmatrix}$?

64.2 Write down a matrix for T.

64.3 What is the volume of the image of the unit square (i.e., the volume of $T(C_2)$)? You may use trigonometry.

DEFINITION

Determinant

The ***determinant*** of a linear transformation $X : \mathbb{R}^n \to \mathbb{R}^n$ is the oriented volume of the image of the unit n-cube. The determinant of a square matrix is the determinant of its induced transformation.

65 We know the following about the transformation A:

$$A\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}2\\0\end{bmatrix} \qquad \text{and} \qquad A\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}.$$

65.1 Draw C_2 and $A(C_2)$, the image of the unit square under A.

65.2 Compute the area of $A(C_2)$.

65.3 Compute $\det(A)$.

66 Suppose R is a rotation counter-clockwise by 30°.

66.1 Draw C_2 and $R(C_2)$.

66.2 Compute the area of $R(C_2)$.

66.3 Compute $\det(R)$.

67 We know the following about the transformation F:

$$F\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix} \qquad \text{and} \qquad F\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}.$$

67.1 What is $\det(F)$?

Volume Theorem I

THM

For a square matrix M, $\det(M)$ is the oriented volume of the parallelepiped (n-dimensional parallelogram) given by the column vectors of M.

Volume Theorem II

THM

For a square matrix M, $\det(M)$ is the oriented volume of the parallelepiped (n-dimensional parallelogram) given by the row vectors of M.

68.1 Explain Volume Theorem I using the definition of determinant.

68.2 Based on Volume Theorems I and II, how should $\det(M)$ and $\det(M^T)$ relate for a square matrix M?

69 Let $D = \{\vec{x} : \|\vec{x}\| \leq 1\}$ be the unit disk. You know the following about the linear transformations M, T, and S.

M is defined by $\vec{x} \mapsto 2\vec{x}$; T has determinant 2; and S has determinant 3.

69.1 Find the oriented volumes of $M(C_2)$, $T(C_2)$, and $S(C_2)$.

69.2 How does the volume of $S(C_2 + \{\vec{e}_1\})$ compare to the volume of $S(C_2)$?

69.3 What is the oriented volume of $S(D)$?

69.4 What is the oriented volume of $T \circ M(C_2)$? What is $\det(T \circ M)$?

- E_f is $I_{3\times 3}$ with the first two rows swapped.
- E_m is $I_{3\times 3}$ with the third row multiplied by 6.
- E_a is $I_{3\times 3}$ with $R_1 \mapsto R_1 + 2R_2$ applied.

70.1 What is $\det(E_f)$?

70.2 What is $\det(E_m)$?

70.3 What is $\det(E_a)$?

70.4 What is $\det(E_f E_m)$?

70.5 What is $\det(4I_{3\times 3})$?

70.6 What is $\det(W)$ where $W = E_f E_a E_f E_m E_m$?

71 $$U = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 3 & -2 & 4 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

71.1 What is $\det(U)$?

71.2 V is a square matrix and $\mathrm{rref}(V)$ has a row of zeros. What is $\det(V)$?

71.3 P is projection onto $\mathrm{span}\left\{\begin{bmatrix} -1 \\ -1 \end{bmatrix}\right\}$. What is $\det(P)$?

72 Suppose you know $\det(X) = 4$.

72.1 What is $\det(X^{-1})$?

72.2 Derive a relationship between $\det(Y)$ and $\det(Y^{-1})$ for an arbitrary matrix Y.

72.3 Suppose Y is not invertible. What is $\det(Y)$?

Eigenvector

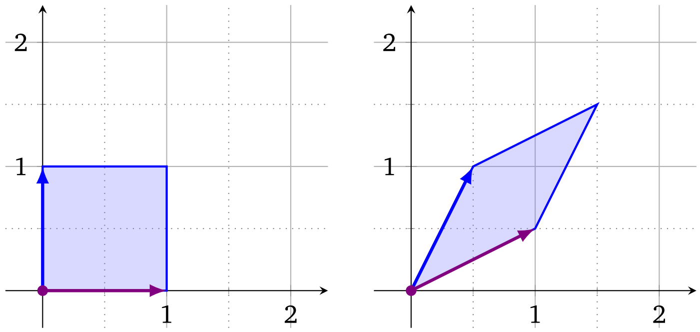
DEFINITION

Let X be a linear transformation. An ***eigenvector*** for X is a non-zero vector that doesn't change directions when X is applied. That is, $\vec{v} \neq \vec{0}$ is an eigenvector for X if

$$X\vec{v} = \lambda\vec{v}$$

for some scalar λ. We call λ the ***eigenvalue*** of X corresponding to the eigenvector $\vec{v}$.

73 The picture shows what the linear transformation T does to the unit square (i.e., the unit 2-cube).



73.1 Give an eigenvector for T. What is the eigenvalue?

73.2 Can you give another?

74 For some matrix A,

$$A\begin{bmatrix}3\\3\\1\end{bmatrix}=\begin{bmatrix}2\\2\\2/3\end{bmatrix}\qquad\text{and}\qquad B=A-\tfrac{2}{3}I.$$

74.1 Give an eigenvector and a corresponding eigenvalue for A.

74.2 What is $B\begin{bmatrix}3\\3\\1\end{bmatrix}$?

74.3 What is the dimension of $\mathrm{null}(B)$?

74.4 What is $\det(B)$?

75 Let $C = \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}$ and $E_\lambda = C - \lambda I$.

75.1 For what values of λ does E_λ have a non-trivial null space?

75.2 What are the eigenvalues of C?

75.3 Find the eigenvectors of C.

DEF

Characteristic Polynomial

For a matrix A, the ***characteristic polynomial*** of A is

$$\operatorname{char}(A) = \det(A - \lambda I).$$

76 Let $D = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$.

76.1 Compute $\mathrm{char}(D)$.

76.2 Find the eigenvalues of D.

77 Suppose $\text{char}(E) = -\lambda(\lambda-2)(\lambda+3)$ for some unknown 3×3 matrix E.

77.1 What are the eigenvalues of E?

77.2 Is E invertible?

77.3 What can you say about $\text{nullity}(E)$, $\text{nullity}(E-3I)$, $\text{nullity}(E+3I)$?

78 Consider

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

and notice that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are eigenvectors for A. Let T_A be the transformation induced by A.

78.1 Find the eigenvalues of A.

78.2 Find the characteristic polynomial of A.

78.3 Compute $A\vec{w}$ where $w = 2\vec{v}_1 - \vec{v}_2$.

78.4 Compute $T_A\vec{u}$ where $\vec{u} = a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3$ for unknown scalar coefficients a, b, c.

Notice that $\mathcal{V} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for $\mathbb{R}^3$.

78.5 If $[\vec{x}]_{\mathcal{V}} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$ is $\vec{x}$ written in the $\mathcal{V}$ basis, compute $T_A\vec{x}$ in the $\mathcal{V}$ basis.

79 The transformation P^{-1} takes vectors in the standard basis and outputs vectors in their $\mathcal{V}$-basis representation. Here, A, T_A, and $\mathcal{V}$ come from Problem .

79.1 Describe in words what P does.

79.2 Describe how you can use P and P^{-1} to easily compute $T_A\vec{y}$ for any $\vec{y} \in \mathbb{R}^3$.

79.3 Can you find a matrix D so that

$$PDP^{-1} = A?$$

79.4 $[\vec{x}]_{\mathcal{V}} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$. Compute $A^{100}\vec{x}$.

DEF

Diagonalizable

A matrix is ***diagonalizable*** if it is similar to a diagonal matrix.

80 For an $n \times n$ matrix T, suppose its eigenvectors $\{\vec{v}_1, \dots \vec{v}_n\}$ form a basis for $\mathbb{R}^n$. Let $\lambda_1, \dots, \lambda_n$ be the corresponding eigenvalues.

80.1 Is T diagonalizable (i.e., similar to a diagonal matrix)? If so, explain how to obtain its diagonalized form.

80.2 What if one of the eigenvalues of T is zero? Is T diagonalizable?

80.3 What if the eigenvectors of T did not form a basis for $\mathbb{R}^n$. Would T be diagonalizable?

Eigenspace

DEFINITION

Let A be a matrix with eigenvalues $\{\lambda_1, \ldots, \lambda_m\}$. The ***eigenspace*** of A corresponding to the eigenvalue λ_i is the null space of $A - \lambda_i I$. That is, it is the space spanned by all eigenvectors that have the eigenvalue λ_i.

The ***geometric multiplicity*** of an eigenvalue λ_i is the dimension of the eigenspace corresponding to λ_i. The ***algebraic multiplicity*** of λ_i is the number of times λ_i occurs as a root of the characteristic polynomial of A (i.e., the number of times $x - \lambda_i$ occurs as a factor).

81 Let $F = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ and $G = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$.

81.1 Is F diagonalizable? Why or why not?

81.2 Is G diagonalizable? Why or why not?

81.3 What is the geometric and algebraic multiplicity of each eigenvalue of F? What about the multiplicities for each eigenvalue of G?

81.4 Suppose A is a matrix where the geometric multiplicity of one of its eigenvalues is smaller than the algebraic multiplicity of the same eigenvalue. Is A diagonalizable? What if all the geometric and algebraic multiplicities match?