Statistical programming

Overview of probability. Simulation

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Today’s session

- Overview of probability
  - Probability
  - Random variables
  - The Central limit theorem
  - Simulation

A brief review of probability

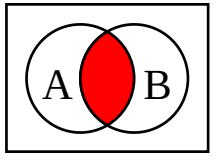
Event

- An *event* is a set of possible results in a particular experiment.
- The event containing all possible events/results is called the *sample space*.
- For completeness, the *impossible event* is defined and it is denoted as $\emptyset$.
- Example:
  - “Face 6” is a possible event when rolling a six-sided dice.
- Events are nicely represented with Euler diagrams:

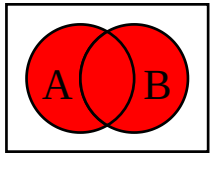
Basic event operations

Given two events A and B.

- $A \cap B$. Intersection of events A and B. Event defined as A and B occur.



- $A \cup B$. Union of events A and B. Event defined as either A or B occur.



- A^c. Complement of an event A. Event defined as A does not occur.

Probability (of an event)

- The *probability of an event* measures "how likely" the event is or "how big" with respect the sample space the event is.
- The probability of an event A is denoted as $\mathrm{P}(A)$.
- Probabilities are quantities between 0 and 1, i.e. for any event A, $0 \leq \mathrm{P}(A) \leq 1$.
- $\mathrm{P}(\emptyset) = 0$ and $\mathrm{P}(\Omega) = 1$, where Ω denotes the sample space.

What are probabilities?

- **Frequentist**. Probabilities are long run relative frequencies of events.
- **Bayesian**. Probabilities are used to quantify our uncertainty about events.
- Example:
  - "Face 6" event has probability $1/6$ when rolling a six-sided dice.

Conditional probability

The conditional probability of one event A given another event B is defined as

$$\mathrm{P}(A|B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}.$$

- $\mathrm{P}(A|B)$ measures how likely is to happen A once we know B has happened.
- We can thing conditioning as if we were reducing our sample space with events where B occurs.
- Example: “Face 6” event has probability $1/3$ once we are told the result is even.

Bayes’ rule

For two events A and B. Then,

$$\begin{aligned} P(A|B) &= \frac{\mathrm{P}(A)\mathrm{P}(B|A)}{\mathrm{P}(B)} \\ &= \frac{\mathrm{P}(A)\mathrm{P}(B|A)}{\mathrm{P}(A)\mathrm{P}(B|A)+\mathrm{P}(A^c)\mathrm{P}(B|A^c)}. \end{aligned}$$

Simulating events with R (1)

- We can randomize a set of possible events with function `sample()`.

```
cards = c('2', '3', '4', '5', '6', '7', '8', '9', '10', 'J', 'Q', 'K', 'A')
sample(cards)
#>  [1] "Q"  "10" "J"  "8"  "9"  "7"  "A"  "4"  "3"  "K"  "6"  "5"  "2"
```

- We can pick a certain number of events:

```
sample(cards, 5)
#> [1] "A" "9" "8" "7" "Q"
```

- By default, `sample()` picks elements without replacement. To force replacement:

```
sample(cards, 5, replace = TRUE)
#> [1] "Q" "7" "3" "A" "A"
```

Simulating events with R (2)

- We can assign a probability to each event:

```
breast_cancer = c('yes', 'no')
sample(breast_cancer, 200, replace = TRUE, prob = c(0.004, 0.996))
#>   [1] "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"
#>  [13] "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"
#>  [25] "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"
#>  [37] "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"
#>  [49] "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"
#>  [61] "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"
#>  [73] "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"
#>  [85] "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"
#>  [97] "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"
#> [109] "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"
#> [121] "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"
#> [133] "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"
#> [145] "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"
#> [157] "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"
#> [169] "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "yes" "no"  "no"  "no"
#> [181] "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"
#> [193] "no"  "no"  "no"  "no"  "no"  "no"  "no"  "no"
```

Activity: Medical diagnosis

Suppose you are a women in your 40s, and you decide to have a medical test for breast cancer called a mammogram.

- If the test is positive, what is the probability you have cancer?

Information:

- The probability of having breast cancer at 40s is 0.004.
- The test has a **sensitivity** of 80%. In other words, if you have breast cancer, the test will be positive with probability 0.8.
- The test has a **specificity** 90%. In other words, if you don't have breast cancer, the test will be negative with probability 0.9.

Activity: Medical diagnosis

```r
library(tidyverse)
N = 100000
test_result_sampling = function(breast_cancer){
  if(breast_cancer == 'yes'){
    sample(c('+', '-'), 1, prob = c(0.8, 0.2))
  }else{
    sample(c('+', '-'), 1, prob = c(0.1, 0.9))
  }
}
women40s = tibble(
  breast_cancer = sample(c('yes', 'no'), N, replace = TRUE, prob = c(0.004, 0.996)),
  test_result = map_chr(breast_cancer, ~test_result_sampling(.x))
)
```

- What is the proportion of women with breast cancer in the population having a positive test?

Activity: Medical diagnosis

Use probability theory to calculate the exact probability of a women having breast cancer once we know she got a positive test.

Hint. Consider the following events:

- A = “To have cancer”, A^c = “Not to have cancer” and
- B = “Test is positive”.

Independence between events

- Two events A and B are said to be independent, denoted $A \perp B$, if

$$\mathrm{P}(A) = \mathrm{P}(A|B).$$

- A practical equivalent definition is A and B are independent if

$$\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B).$$

Conditional independence

- Two events A and B are said to be conditionally independent given C, denoted $A \perp B|C$, if

$$\mathrm{P}(A|C) = \mathrm{P}(A|B \cap C),$$

or

$$\mathrm{P}(A \cap B|C) = \mathrm{P}(A|C)\mathrm{P}(B|C).$$

Activity

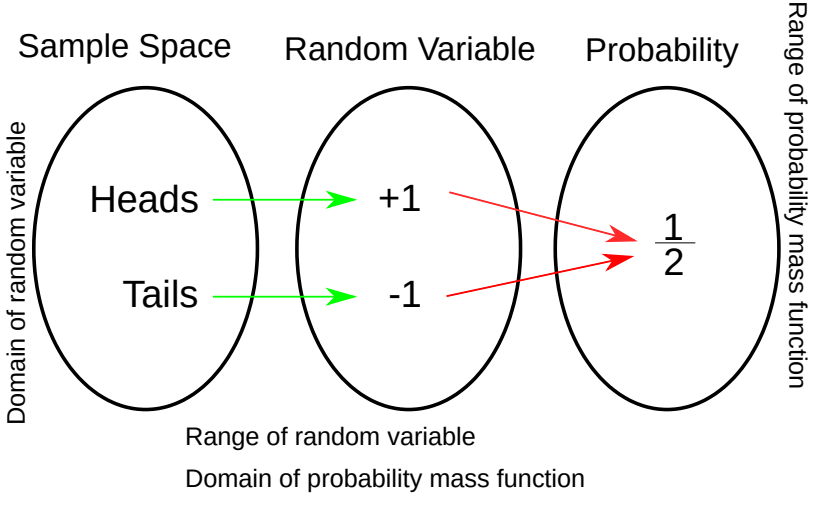
- Check if event **A = “face is even”** and event **B = “the face is smaller or equal than 4”** are independent when rolling a 6-faced dice.
- Check if event **A = “face is 6 in the blue dice”** and **B = “face is 6 in the red dice”** are independent when rolling a blue and red 6-faced dice.
- Check if event **A = “face is 6 in the blue dice”** and **B = “face is 6 in the red dice”** are conditionally independent when rolling a blue and red 6-faced dice when we know about event **C = “the sum is even”**.

Activity: Monty Hall problem

> Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?

Random variables

- A random variable (r.v.) is a function from a sample space to a measurable space, like the real numbers.
- We can thing of a random variable as a variable taking values randomly.
- If values are discrete, the function assigning a probability to its values is called *probability mass function* (pmf).

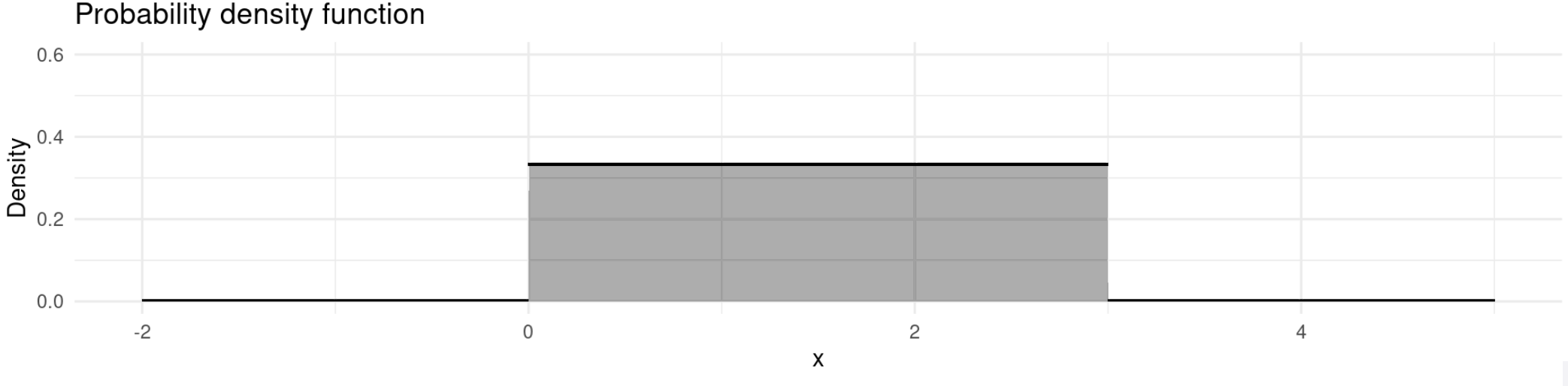


Discrete random variables

- Discrete r.v. take values in a countable set of elements. For example a finite set of numbers or the integers.
- Discrete r.v. can be characterised with giving $\mathrm{P}(X = x)$ for each possible value x. Function $p(x) = \mathrm{P}(X = x)$ is called the *probability mass function* (pmf).
- **Example:** Discrete r.v. X is uniquely determined by
  - $X \in \{1, 2, 3\}$ and
  - the pmf: $p(1) = 0.25, p(2) = 0.5$ and $p(3) = 0.25$.

Continuous random variables

- Continuous random variables take values in the real line.
- Possible events for continuous r.v. are combinations of subsets the real line.
- Continuous r.v. are characterised by providing $\mathrm{P}(X \in E)$ for any event E.
- The *probability density function* (pdf) of a coninuous r.v. is a function $f(x)$ such that ((X E) = _{x E} f(x) dx.)
- **Example:** Continuous r.v. X is uniquely determined with the pdf $f(x)$ defined as $f(x) = \frac{1}{3}$ if $x \in (0, 3)$ and $f(x) = 0$ otherwise.



The expected value or mean of a r.v.

- The *expected value* of a r.v. X, denoted $\mathbb{E}[X]$, is the mean value we expect after an infinite number of runs.
  - For a discrete r.v. X, the expected value is

$$\mathbb{E}[X] = \sum_x x\ p(x) \qquad [\sum_x x\ \mathrm{P}(X = x)]$$

  - For a continuous r.v. X, the expected value is

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x\, f(x) dx$$

- Example: the expected value of the r.v. with the value of rolling a 6-faced dice is

$$1\frac{1}{6} + 2\frac{1}{6} + 3\frac{1}{6} + 4\frac{1}{6} + 5\frac{1}{6} + 6\frac{1}{6} = 3.5$$

More expected values

- The variance of a r.v., denoted $\text{var}[X]$, is defined as

$$\text{var}[X] = \mathbb{E}[\ (X - \mathbb{E}[X])^2\ ].$$

- The covariance between two r.v.'s X and Y, denoted $\text{cov}[X, Y]$, is defined as

$$\text{cov}[X, Y] = \mathbb{E}[\ (X - \mathbb{E}[X])(Y - \mathbb{E}[Y])\ ].$$

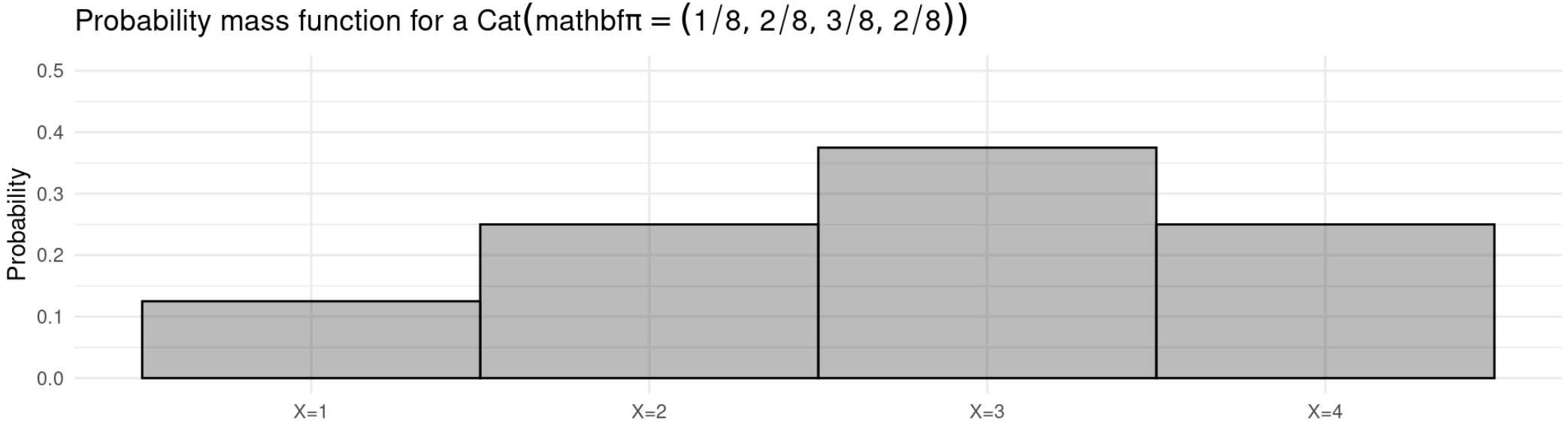
- See visualisation at https://seeing-theory.brown.edu/basic-probability/index.html
- For any function $g(x)$, we can defined the expected value of $g(X)$ as
  - $\mathbb{E}[g(X)] = \sum_x g(x)\ p(x)$ when X is discrete or
  - $\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x)\ f(x)dx$ when continuous.

Some common probability distributions

Categorical distribution

The categorical distribution is the distribution obtained after associating natural number 1 to k to a set of events with a certain probability $\boldsymbol{\pi} = (\pi_1, \dots, \pi_k)$.

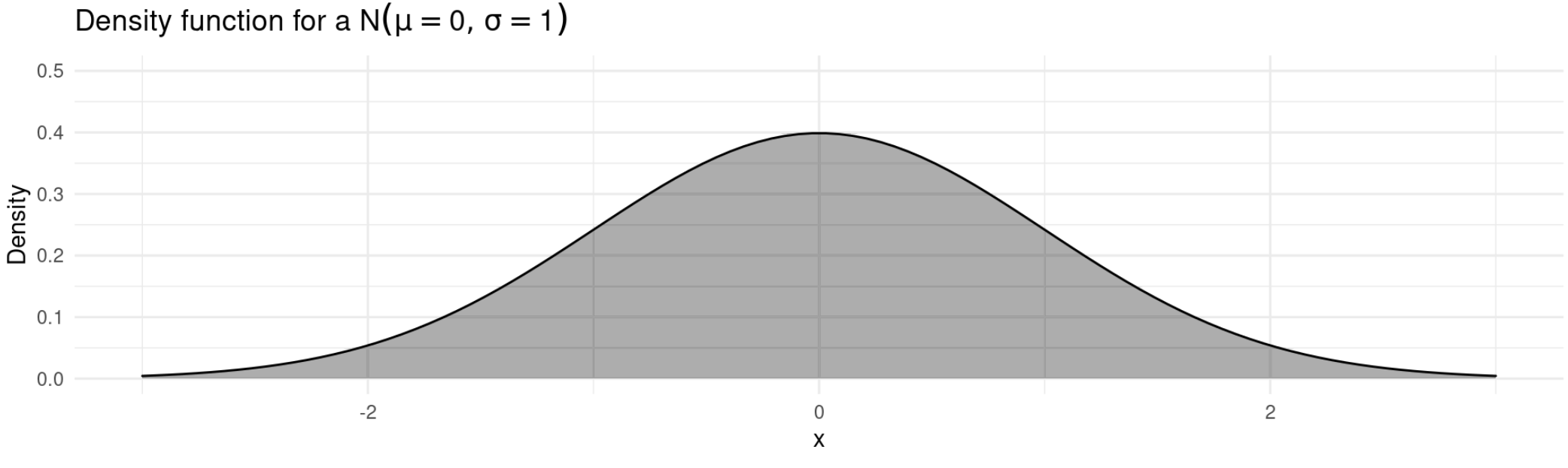
- For a categorical r.v. X, denoted $X \sim Cat(\boldsymbol{\pi} = (\pi_1, \dots, \pi_k))$,
  - $\mu = \mathbb{E}[X] = \sum_{i \leq k} i \ \pi_k$ and
  - $\text{var}[X] = \sum_{i \leq k} (i - \mu)^2 \ \pi_k$.



Gaussian (normal) distribution

The gaussian (normal) distribution is the distribution that reduces the probability exponentially (with velocity $1/\sigma^2$) for events far from a certain center (μ).

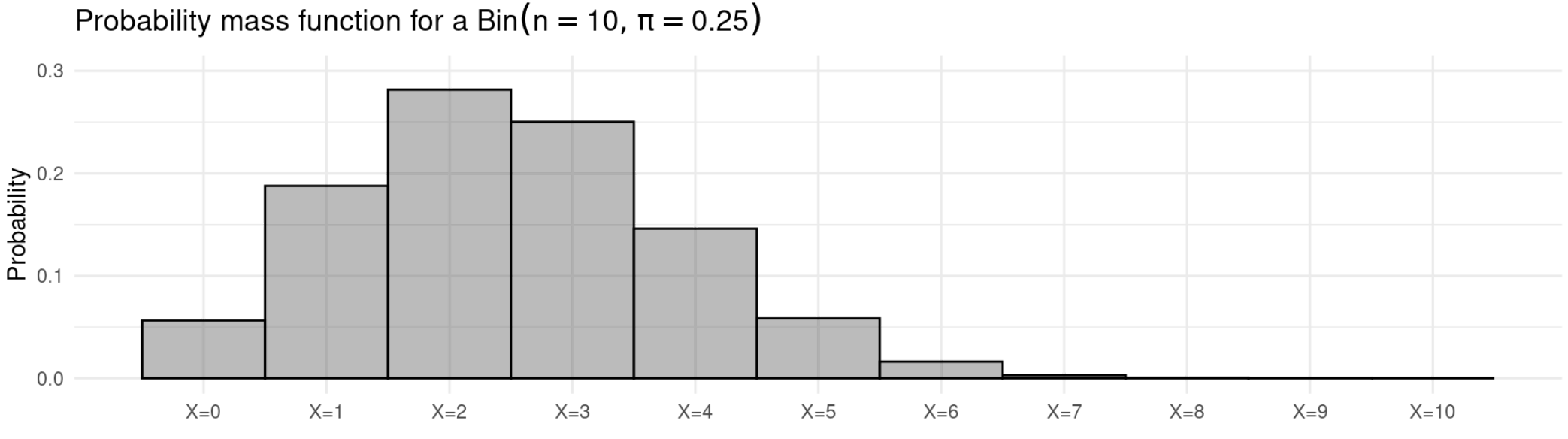
- For a gaussian r.v. $X, X \sim N(\mu, \sigma)$,
  - $\mathbb{E}[X] = \mu$ and
  - $\text{var}[X] = \sigma^2$.



Binomial distribution

The binomial distribution counts the number of success after repeating a certain experiments n times when the probability of success is π.

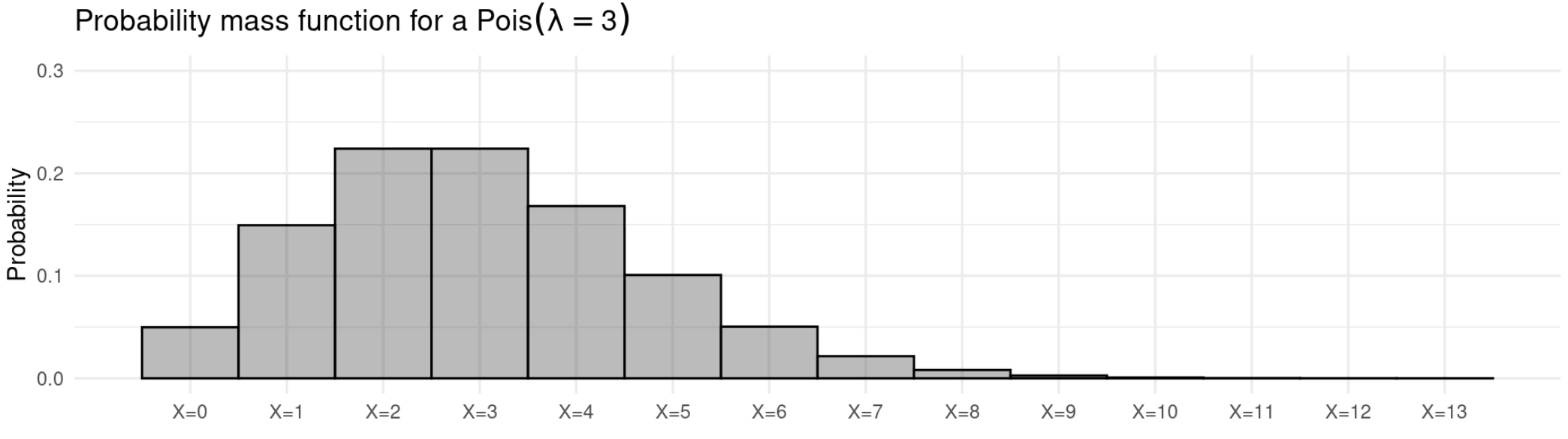
- For a binomial r.v. $X, X \sim Bin(n, \pi)$,
  - $\mathbb{E}[X] = n\pi$ and
  - $\text{var}[X] = n\pi(1 - \pi)$.



Poisson distribution

The Poisson distribution counts the number of success detected in an interval of time or space when the expected number of successes is λ.

- For a Poisson r.v. $X, X \sim Poiss(\lambda)$,
  - $\mathbb{E}[X] = \lambda$ and
  - $\text{var}[X] = \lambda$.



Generating r.v. with R

- Categorical r.v.

```
sample(1:4, 6, prob = c(1/8,2/8,3/8,2/8), replace=TRUE)
#> [1] 2 4 4 4 2 2
```

- Gaussian r.v.

```
rnorm(6, mean = 0, sd = 1)
#> [1]  1.6741280 -1.5617505  1.0604928  1.3124341  1.5795088 -0.5374763
```

- Binomial r.v.

```
rbinom(6, size = 10, prob = 0.25)
#> [1] 5 3 3 1 3 2
```

- Poisson r.v.

```
rpois(6, lambda = 3)
#> [1] 8 2 2 2 2 5
```

Activity

Thing about possible variables that can be modelled with one of the four seen distributions:

- Categorical distribution
- Gaussian distribution
- Binomial distribution
- Poisson distribution

Think about possible parameters to use.

Joint probability distribution

The *joint probability distribution* is a multivariate model for random variables. The joint probability distribution of r.v. $X_1, \ldots, X_k$ is completely determined by providing

- the probability of all possible events for discrete r.v.'s

$$p(x_1, \ldots, x_k) = \mathrm{P}(X_1 = x_1 \cap \cdots \cap X_k = x_k),$$

- or the multivariate density for continous r.v's

$$f(x_1, \ldots, x_k).$$

Transformations of random variables

- If we transform a r.v. with a function $g(x)$ we get another r.v.
- Linear transformations. If we linearly transform a r.v. $\boldsymbol{X} = (X_1, \ldots, X_k)$ with $f(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}$ then
  - $\mathbb{E}[f(\boldsymbol{X})] = \boldsymbol{A}\boldsymbol{\mu} + \boldsymbol{b}$ where $\boldsymbol{\mu} = \mathbb{E}[\boldsymbol{X}]$, and
  - $\text{cov}[f(\boldsymbol{X})] = \boldsymbol{A}\boldsymbol{\Sigma}\boldsymbol{A}^T$ where $\boldsymbol{\Sigma} = \text{cov}[\boldsymbol{X}]$.
- For non-linear transformations we can simulate the r.v. to obtain these expected values.

Example: The log-normal distribution is defined as the distribution obtained after exponentiating a gaussian r.v., $f(x) = e^x$, $Y = e^X$, where $X \sim N(\mu, \sigma)$. Check using simulating gaussian r.v. that

- $\mathbb{E}[Y] = e^{\mu+\sigma^2/2}$ and
- $\text{var}[Y] = (e^{\sigma^2} - 1)e^{2\mu+\sigma^2}$.

The central limit theorem

If $X_1, \ldots, X_n$ are r.v. equally distributed with expected value $\mathbb{E}[X]$ and variance $\mathrm{var}[X]$. Then, when n is big, we have

$$\bar{X} = \frac{X_1 + \cdots + X_n}{n} \sim N(\mathbb{E}[X], \sqrt{\mathrm{var}[X]/n})$$

Next plot shows the distribution of 1000 realisations of a r.v. obtained as the average of 30 binomial distributions with parameters $n = 10$ and $\pi = 0.25$. Blue line represents the gaussian distribution with $\mu = n\pi$ and $\sigma^2 = \sqrt{n\pi(1-\pi)/30}$

The central limit theorem

If $X_1, \ldots, X_n$ are r.v. equally distributed with expected value $\mathbb{E}[X]$ and variance $\text{var}[X]$. Then, when n is big, we have

$$\frac{\bar{X} - \mathbb{E}[X]}{\sqrt{\text{var}[X]/n}} \sim N(0, 1)$$

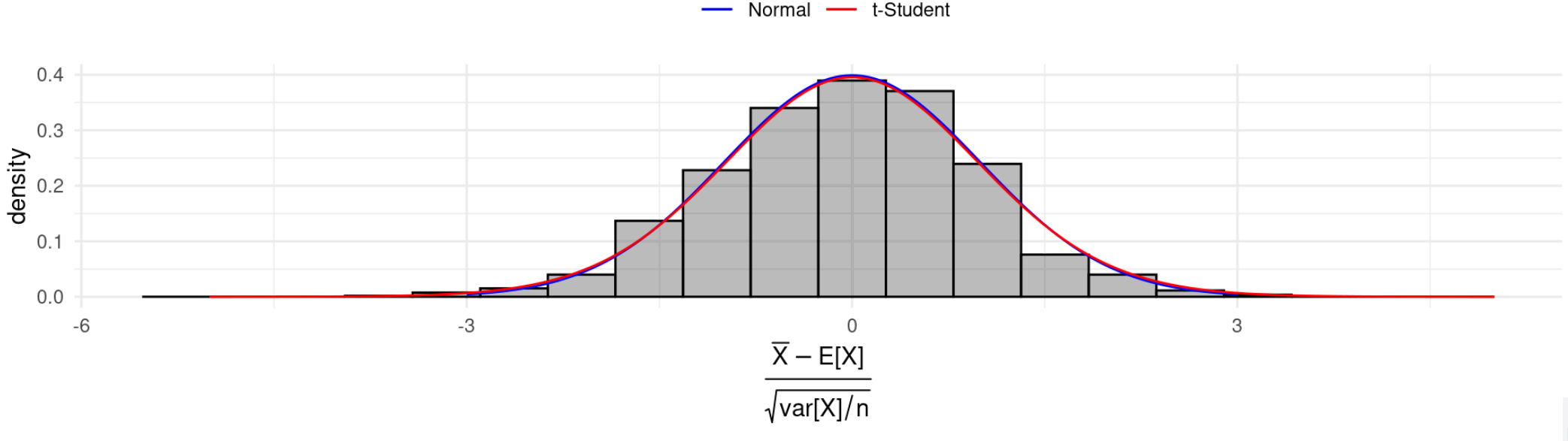
Next plot shows the distribution of 1000 standardised realisations of a r.v. obtained as the average of 30 binomial distributions with parameters $n = 10$ and $\pi = 0.25$. Blue line represents the gaussian distribution with $\mu = 0$ and $\sigma^2 = 1$

Sampling distribution

If $x_1, \ldots, x_n$ is a sample i.i.d with expected value $\mathbb{E}[X]$. Then, when n is big, we have

$$\frac{\bar{x} - \mathbb{E}[X]}{\sqrt{s_x/n}} \sim t_{n-1}$$

Next plot shows the distribution of 1000 standardised realisations of a r.v. obtained as the average of 30 binomial distributions with parameters $n = 10$ and $\pi = 0.25$. Normal and Student distributionss are compared.



Final activity

- Fix some r.v. Y after some transformation $f(x)$.
  - For example by transforming certain binomial distribution with function $f(x) = \sin(x)$.
- Take a sample of size 50 and calculate the mean of this sample. We we call it: x_1.
- Repeat the process 1000 times obtaining a sample of means: $X_m = \{x_1, \dots, x_{1000}\}$.
- Visualise the resulting distribution of sample X_m.
- Think about how we can use x_i to obtain information of $\mathbb{E}[Y]$?

That’s all for today

Next week session

- Overview of probability and statistics
  - Chi-squared test: goodness of fit, independence
  - Student’s t-test: one sample, paired samples, independent two-sample
  - One-way ANOVA test
  - Normality test
  - Homocedasticity test