Summer 2014

Graph Hamiltonicity and Gröbner Bases

Max Comstock

Introduction

The purpose of this project is to examine an algebraic method used to find Hamiltonian cycles in graphs, by creating a system of polynomial equations that has solutions corresponding to cycles in the graph. The ideal generated from these polynomials is guaranteed to have certain properties depending on the number of Hamiltonian cycles in the graph. The results include a summary of existing theorems about these ideals, along with a few discoveries about the form of their reduced Gröbner bases.

Gröbner Bases

A central object in our study is Gröbner bases. Given an ideal I of R and a term order $\prec$, a finite subset $\mathcal{G}$ of I is a *Gröbner basis* with respect to $\prec$ if the ideal

$$in_{\prec}(I) = \langle in_{\prec}(f) : f \in I \rangle,$$

is generated by the initial terms of $\mathcal{G}$.

The ideal $in_{\prec}(I)$ is called the *initial ideal* of I with respect to $\prec$. The Gröbner basis $\mathcal{G}$ is *minimal* if no leading term of $f \in \mathcal{G}$ divides any other leading term of polynomials in $\mathcal{G}$.

The Gröbner basis $\mathcal{G}$ is *reduced* if no leading term of $f \in \mathcal{G}$ divides any monomial in any other polynomials in $\mathcal{G}$. If $I \neq \{0\}$ is a polynomial ideal, then I has a unique reduced Gröbner basis for any given monomial ordering.

From (1).

Graph Ideals

Let $G = (V, A)$ be a simple directed graph on vertices $V = \{1, \ldots, n\}$. Assume that the characteristic of $\mathbb{K}$ is relatively prime to n and that $\omega \in \mathbb{K}$ is a primitive n-th root of unity. Consider the following system in $\mathbb{K}[x_1, \ldots, x_n]$:

$$H_G = \{x_i^n - 1 = 0, \prod_{j \in \delta^+(i)} (\omega x_i - x_j) = 0 \,:\, i \in V\}.$$

Here, $\delta^+(i)$ denotes those vertices j which are connected to i by an arc going from i to j in G. The system H has a solution over $\overline{\mathbb{K}}$ if and only if G has a Hamiltonian cycle.

From (3).

Procedure

To detect whether a graph has any Hamiltonian cycles, we can use the following procedure given a graph G:

- Calculate the graph ideal H_G for G.
- Use software to find a reduced Gröbner basis for H_G.
- If the Gröbner basis is not $\{1\}$, the graph has a Hamiltonian cycle.
- We want to learn as much as we can about the structure of this Gröbner basis and how it relates to the structure of the graph.

For Further Information

- E-mail address: `mcomstock@g.hmc.edu`.
- More links to come.

Unique Hamiltonicity

We know that a directed graph has a unique Hamiltonian cycle if a Gröbner basis of H_G has the form $H_G = \langle g_1, \ldots, g_k \rangle$, where

$$g_i = \begin{cases} x_{v_{k-i}} - \omega^{k-1} x_{v_k} & i = 1, \ldots, k-1 \\ x_{v_k}^k - 1 & i = k. \end{cases}$$

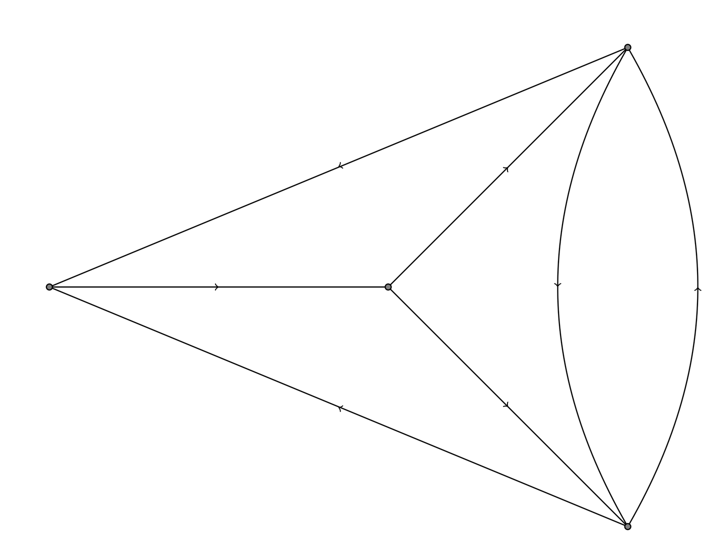
Proof can be found in (3).

New Discoveries

- Let G be a graph with n vertices and one or more Hamiltonian cycles, and k be a natural number (not including zero). Then we find, for each i such that $1 \leq i \leq n$, that $\mathrm{LT}(g) = x_i^k$ for some g in the reduced Gröbner basis of H_G.
- For a graph G with n vertices and one or more Hamiltonian cycles, we find $x_n^n - 1$ is in the reduced Gröbner basis of H_G. Furthermore, this is the only polynomial in the reduced Gröbner basis that contains x_n and no other variables.

Sample Calculation

Graph:



Reduced Gröbner basis: $\{x_4^4 - 1, x_3^2 + x_4^2, 2x_2 + (z+1)x_3 + (z+1)x_4, 2x_1 + (-z+1)x_3 + (-z+1)x_4\}$.

References

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Acknowledgments

I would like to thank Professors Omar and Ruiz for all of their guidance, as well as the members of their research teams for their support and feedback.

Advisor: Professor Omar