Conjectures

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Lemma 1. *For a graph G with n vertices and one or more Hamiltonian cycles, we find $x_n^n - 1$ is in the reduced Gröbner basis of H_G. Furthermore, this is the only polynomial in the reduced Gröbner basis that contains x_n and no other variables.*

Proof. We will use induction on the number of Hamiltonian cycles of the graph. For the base case, consider a graph with a single Hamiltonian cycle. Let G_1 be such a graph with vertex set $\{v_1, \ldots, v_n\}$, and ω be an n-th root of unity. Lemma 3.3 and Theorem 3.9 in De Loera, Hillar, Malkin, Omar tells us that the graph ideal of G_1 has a reduced Gröbner basis of the form

$$H_{G_1} = \langle g_1, \ldots, g_n \rangle,$$

where

$$g_i = \begin{cases} x_{v_{n-i}} - \omega^{n-i} x_{v_n} & i = 1, \ldots, n-1 \\ x_{v_n}^n - 1 & i = n \end{cases}$$

Next, consider the graph G_i with i Hamiltonian cycles. From Theorem 3.9, we know

$$H_{G_i} = H_{G_i,C_1} \cap \cdots \cap H_{G_i,C_i}.$$

Suppose that our hypothesis holds for $H_{G_i,C_1} \cap \cdots \cap H_{G_i,C_{i-1}}$. Then, by the Elimination Theorem, $(H_{G_i,C_1} \cap \cdots \cap H_{G_i,C_{i-1}}) \cap \mathbb{K}[x_n] = \langle x_n^n - 1 \rangle$. Then we find

$$\begin{aligned} H_{G_i} \cap \mathbb{K}[x_n] &= (H_{G_i,C_1} \cap \cdots \cap H_{G_i,C_i}) \cap \mathbb{K}[x_n] \\ &= ((H_{G_i,C_1} \cap \cdots \cap H_{G_i,C_{i-1}}) \cap \mathbb{K}[x_n]) \cap (H_{G_i,C_i} \cap \mathbb{K}[x_n]) \\ &= \langle x_n^n - 1 \rangle \cap \langle x_n^n - 1 \rangle \\ &= \langle x_n^n - 1 \rangle. \end{aligned}$$

Thus, by the Elimination Theorem, we find that $x_n^n - 1$ is in the reduced Gröbner basis of H_{G_i} and is the sole generator of its polynomials in $\mathbb{K}[x_n]$.

Lemma 2. *Let G be a graph with n vertices and one or more Hamiltonian cycles, and k be a natural number (not including zero). Then we find, for each i such that $1 \leq i \leq n$, that $\mathrm{LT}(g) = x_i^k$ for some g in the reduced Gröbner basis of H_G.*

Proof. We know from the construction of H_G that there is a polynomial with the leading term x_i^n for all $1 \leq i \leq n$. From the algorithm for finding a reduced Gröbner basis, the only way to eliminate one of these terms is by adding polynomial f where $\mathrm{LT}(f)$ divides x_i^n. But then $\mathrm{LT}(f) = x_i^k$.

Conjecture 1. *Let G be a graph with n vertices, and $k \geq 2$. There exists g in the reduced Gröbner basis of H_G such that $\mathrm{LT}(g) = x_i^k$ for some i such that $1 \leq i < n$ if and only if G has more than one Hamiltonian cycle.*