# Graph Hamiltonicity and Gröbner Bases

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# Gröbner bases

## Definition

*The **initial term** of a polynomial $f \in R$ with respect to $\prec$, denoted $in_{\prec}(f)$, is the largest monomial in $f$ with respect to $\prec$.*

## Definition

*Given an ideal $I$ of $R$ and a term order $\prec$, a finite subset $\mathcal{G}$ of $I$ is a **Gröbner basis** with respect to $\prec$ if the ideal*

$$in_{\prec}(I) = \langle in_{\prec}(f) : f \in I \rangle,$$

*is generated by the initial terms of $\mathcal{G}$.*

# A way to algebraically encode Hamiltonian cycles

## Proposition

*Let* $G = (V, A)$ *be a simple directed graph on vertices* $V = \{1, \ldots, n\}$*. Assume that the characteristic of* $\mathbb{K}$ *is relatively prime to* $n$ *and that* $z \in \mathbb{K}$ *is a primitive* $n$*-th root of unity. Consider the following system in* $\mathbb{K}[x_1, \ldots, x_n]$*:*

$$H_G = \{x_i^n - 1 = 0, \prod_{j \in \delta^+(i)} (zx_i - x_j) = 0 \ : \ i \in V\}$$

*Here,* $\delta^+(i)$ *denotes those vertices* $j$ *which are connected to* $i$ *by the arc going from* $i$ *to* $j$ *in* $G$*. The system* $H$ *has a solution over* $\mathbb{K}$ *if and only if* $G$ *has a Hamiltonian cycle.*

Source: "Recognizing Graph Theoretic Properties with Polynomial Ideals" by J.A. De Loera, C. Hillar, P.N. Malkin, and M. Omar.

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# Simple case: directed graphs

## Definition

*Let $z$ be a fixed primitive $k$-th root of unity. If $C$ is a directed cycle of length $k$ in a directed graph, with vertex set $\{v_1, \ldots, v_k\}$, the **cycle encoding** of $C$ is the following set of $k$ polynomials:*

$$g_i = \begin{cases} x_{v_{k-i}} - z^{k-i} x_{v_k} & i = 1, \ldots, k-1 \\ x_{v_k}^k - 1 & i = k \end{cases}.$$

Note: define $H_{G,C} = \langle g_1, \ldots, g_i \rangle$. The $g_i$'s form a reduced Gröbner basis (which must be unique) for $H_{G,C}$.

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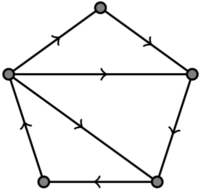
# Simple example



Let $z$ be a primitive $5^{\text{th}}$ root of unity. We are looking for solutions to the system $H$:

$$\begin{aligned}
x_i^5 - 1 &= 0 \quad 1 \le i \le 5 \\
(zx_1 - x_2)(zx_1 - x_3)(zx_1 - x_4) &= 0 \\
zx_2 - x_3 &= 0 \\
zx_3 - x_4 &= 0 \\
zx_4 - x_5 &= 0
\end{aligned}$$

# Simple example (continued)

Process for a single Hamiltonian cycle:

- Let $H_G$ be the ideal generated by the polynomials in the system $H$.
- If we find a Gröbner basis for $H_G$ with respect to the ordering $x_5 < x_4 < x_3 < x_2 < x_1$, we find it is a generating set for $H_{G,C}$.

In our example, the reduced Gröbner basis is

$$\{x_5^5 - 1, x_4 - x_5 z^4, x_3 - x_5 z^3, x_2 - x_5 z^2, x_1 - x_5 z\}.$$

# How to represent a primitive $n$th root of unity

**Definition**

*An $n$th root of unity is **primitive** if it is not a $k$th root of unity for some smaller $k$:*

$$z^k \neq 1 \quad (k = 1, 2, \ldots, n-1).$$

# Cyclotomic poynomial $\Phi_n$

## Definition

*The n**th cyclotomic polynomial** is the unique irreducible polynomial with integer coefficients whose roots are the nth primitive roots of unity.*

Examples:

$$\Phi_3(z) = z^2 + z + 1$$
$$\Phi_4(z) = z^2 + 1$$
$$\Phi_5(z) = z^4 + z^3 + z^2 + z + 1$$
$$\Phi_6(z) = z^2 - z + 1$$

# Using $\Phi_n$ to create a field with $n$th roots of unity

We will begin with the ring Our goal is to create the ring

$$\mathbb{K}[x_1, \ldots, x_n]$$

where $\mathbb{K}$ is a field containing the primitive $n$th roots of unity. We can create such a field as follows:

$$\mathbb{K} = \frac{\mathbb{Q}[z]}{\Phi_n(z)}.$$

Using the relationship

$$\left(\frac{\mathbb{Q}[z]}{\Phi_n(z)}\right)[x_1, \ldots, x_n] \cong \frac{\mathbb{Q}[x_1, \ldots, x_n, z]}{\Phi_n},$$

We can choose

$$\mathbb{K}[x_1, \ldots, x_n] = \frac{\mathbb{Q}[x_1, \ldots, x_n, z]}{\Phi_n(z)}.$$

# Implementation using Macaulay2

```
R = QQ[x_1, x_2, x_3, x_4, x_5, x_6, z];
CYCLOTOMIC_POLY = z^2 - z + 1;
R = R/CYCLOTOMIC_POLY;
IDEAL_GEN = {x_1^6 - 1, x_2^6 - 1, x_3^6 - 1,
         x_4^6 - 1, x_5^6 - 1, x_6^6 - 1,
 z*x_1 - x_2,
 z*x_2 - x_3,
 z*x_3 - x_4,
 z*x_4 - x_5,
 z*x_5 - x_6,
 z*x_6 - x_1};
idealOfGraph = ideal IDEAL_GEN;
graphBasis = flatten entries gens gb idealOfGraph;
print toString graphBasis;
```

# Using these results to prove a conjecture

We want to prove the following conjecture:

**Conjecture**

*For a graph $G$ with $n$ vertices and one or more Hamiltonian cycles, $x_n^n - 1$ is in the reduced Gröbner basis of $H_G$.*

This is true for graphs with a single cycle, from a previous theorem.

# Extending to Graphs with multiple Hamiltonian cycles

We will use the following theorem to relate the encoding of a single cycle to multiple cycles:

**Theorem**

*Let $G$ be a connected directed graph with $n$ vertices. Then,*

$$H_G = \bigcap_{C} H_{G,C},$$

*where $C$ ranges over all Hamiltonian cycles of the graph $G$.*

# More work to do

While this theorem reveals a potential strategy for proving the conjecture, computing the Gröbner basis of intersecting ideals is still very complicated.

Can we reduce the problem further?

# Elimination Theory

One way to reduce the scope of this problem is to examine an *elimination ideal*, defined as follows:

## Definition

*Given* $I = \langle f_1, \ldots, f_s \rangle \subseteq k[x_1, \ldots, x_n]$ *the* $\ell$*-th* ***elimination ideal*** $I_\ell$ *is the ideal of* $k[x_{\ell+1}, \ldots, x_n]$ *defined by*

$$I_\ell = I \cap k[x_{\ell+1}, \ldots, x_n].$$

We would like to show that $H_G \cap k[x_n] = \langle x_n^n - 1 \rangle$ for any graph $G$ with one or more Hamiltonian cycles.
Can Gröbner bases help us do this?

# Elimination Theory

Fortunately, they can. We can use the Elimination Theorem:

**Theorem**

*Let $I \subseteq k[x_1, \ldots, x_n]$ be an ideal and let $G$ be a Gröbner basis of $I$ with respect to lex order where $x_1 > x_2 > \cdots > x_n$. Then, for every $0 < \ell < n$, the set*

$$G_\ell = G \cap k[x_{\ell+1}, \ldots, x_n]$$

*is a Gröbner basis of the $\ell$-th elimination ideal $I_\ell$.*

Source: Cox, Little, O'Shea

# Example with two Hamiltonian cycles

Let $G$ be a graph with $n$ vertices and two Hamiltonian cycles, represented by the cycle ideals $H_{G,C_1}$ and $H_{G,C_2}$. Then we find

$$\begin{aligned} H_G \cap k[x_n] &= (H_{G,C_1} \cap H_{G,C_2}) \cap k[x_n] \\ &= (H_{G,C_1} \cap k[x_n]) \cap (H_{G,C_2} \cap k[x_n]) \\ &= \langle x_n^n - 1 \rangle \cap \langle x_n^n - 1 \rangle \\ &= \langle x_n^n - 1 \rangle. \end{aligned}$$

Therefore, the polynomial $x_n^n - 1$ is the sole generator of the polynomials in $H_G$ that contain $x_n$ alone.

# Future work

A number of conjectures from this summer still remain. For instance, we present the following conjecture:

## Conjecture

*Let $G$ be a graph with $n$ vertices, and $k \geq 2$. There exists $g$ in the reduced Gröbner basis of $H_G$ such that $\mathrm{LT}(g) = x_i^k$ for some $i$ such that $1 \leq i < n$ if and only if $G$ has more than one Hamiltonian cycle.*

# Potential applications

Hopefully, by proving facts about the structure of graph ideals, we can learn more about graphs themselves. One such application would be to prove or disprove Sheehan's Conjecture:

## Conjecture

*Every 4-regular graph has at least two Hamiltonian cycles.*