Week 2 Presentation

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Sheehan's Conjecture (1975)

Conjecture

Every 4-regular graph has at least two Hamiltonian cycles.

A way to algebraically encode Hamiltonian cycles

Proposition

Let $G = (V, A)$ *be a simple directed graph on vertices* $V = \{1, \dots, n\}$*. Assume that the characteristic of* $\mathbb{K}$ *is relatively prime to* n *and that* $z \in \mathbb{K}$ *is a primitive* n*-th root of unity. Consider the following system in* $\mathbb{K}[x_1, \dots, x_n]$*:*

$$H_G = \{x_i^n - 1 = 0, \prod_{j \in \delta^+(i)} (zx_i - x_j) = 0 \ : \ i \in V\}$$

Here, $\delta^+(i)$ *denotes those vertices* j *which are connected to* i *by the arc going from* i *to* j *in* G*. The system* H *has a solution over* $\mathbb{K}$ *if and only if* G *has a Hamiltonian cycle.*

Source: "Recognizing Graph Theoretic Properties with Polynomial Ideals" by J.A. De Loera, C. Hillar, P.N. Malkin, and M. Omar.

Simple case: directed graphs

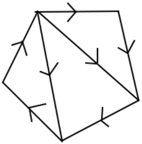
Definition

Let z be a fixed primitive k-th root of unity. If C is a directed cycle of length k in a directed graph, with vertex set $\{v_1, \ldots, v_k\}$, the cycle encoding of C is the following set of k polynomials:

$$g_i = \begin{cases} x_{v_{k-i}} - z^{k-i} x_{v_k} & i = 1, \ldots, k-1 \\ x_{v_k}^k - 1 & i = k \end{cases}.$$

Note: define $H_{G,C} = \langle g_1, \ldots, g_i \rangle$. The g_i's form a reduced Gröbner basis (which must be unique) for $H_{G,C}$.

Simple example



Let z be a primitive 5^{th} root of unity. We are looking for solutions to the system H:

$$\begin{aligned}
x_i^5 - 1 &= 0 \quad 1 \leq i \leq 5 \\
(zx_1 - x_2)(zx_1 - x_3)(zx_1 - x_4) &= 0 \\
zx_2 - x_3 &= 0 \\
zx_3 - x_4 &= 0 \\
zx_4 - x_5 &= 0 \\
zx_5 - x_1 &= 0
\end{aligned}$$

Simple example (continued)

Process for a single Hamiltonian cycle: Let H_G be the ideal generated by the polynomials in the system H. If we find a Gröbner basis for H_G with respect to the ordering $x_5 < x_4 < x_3 < x_2 < x_1$, we find it is a generating set for $H_{G,C}$. In our example, the reduced Gröbner basis is

$$\{x_5^5 - 1, x_4 - x_5 z^4, x_3 - x_5 z^3, x_2 - x_5 z^2, x_1 - x_5 z\}.$$

Harder for multiple Hamiltonian cycles...

When there are multiple Hamiltionian cycles, it turns out that

$$H_G = \bigcap_C H_{G,C}.$$

This makes it difficult to tell exactly how many Hamiltonian cycles you have, but we can easily see when there are two or more.