Gröbner Bases of some Undirected Graphs

Max Comstock

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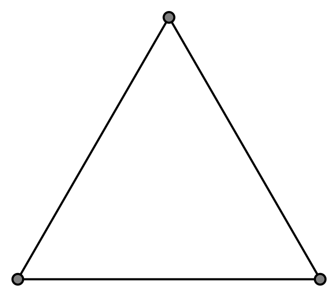
1 Three Vertices with One Hamiltonian Cycle

We will begin by examining a few simple graphs with a single Hamiltonian cycle and varying numbers of vertices. In the case of undirected graphs, we should find that the Gröbner basis follow a pattern, although it is more complicated than the pattern for directed graphs with a single Hamiltonian cycle. Our first graph has three vertices:



To create a system of polynomial equations for this graph, we treat it as a directed graph with two arcs for each edge of the original graph, one going in each direction. This leads to a more complicated polynomial equation and therefore a more complicated Gröbner basis than the simple directed case. For this graph, the system of equations is

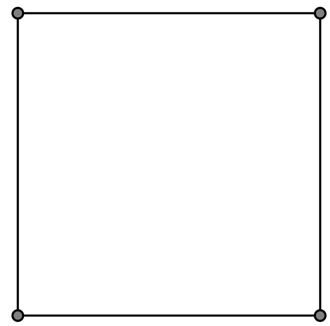
$$\begin{aligned} z^2 + z + 1 &= 0 \\ x_i^3 - 1 &= 0 \quad 1 \leq i \leq 3 \\ (zx_1 - x_2)(zx_1 - x_3) &= 0 \\ (zx_2 - x_3)(zx_2 - x_1) &= 0 \\ (zx_3 - x_1)(zx_3 - x_2) &= 0 \end{aligned}$$

By taking the reduced Gröbner basis of the ideal generated by these polynomials, we find

$$\{x_3^3 - 1, x_2^2 + x_2x_3 + x_3^2, x_1 + x_2 + x_3\}$$

2 Four Vertices with One Hamiltonian Cycle

Next, we examine the case with four vertices:



In this case, the graph corresponds to the system of equations

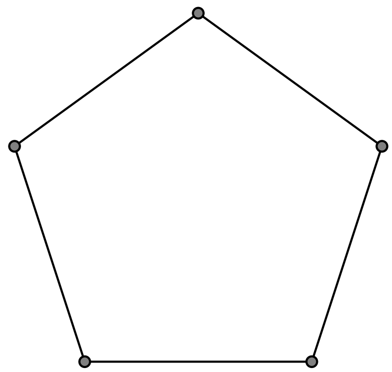
$$\begin{aligned} z^2 + 1 &= 0 \\ x_i^4 - 1 &= 0 \quad 1 \le i \le 4 \\ (zx_1 - x_2)(zx_1 - x_4) &= 0 \\ (zx_2 - x_3)(zx_2 - x_1) &= 0 \\ (zx_3 - x_4)(zx_3 - x_2) &= 0 \\ (zx_4 - x_1)(zx_4 - x_3) &= 0 \end{aligned}$$

The reduced Gröbner basis of the ideal generated by these polynomials is

$$\{x_4^4 - 1, x_3^2 + x_4^2, x_2 + x_4, x_1 + x_3\}$$

3 Five Vertices with One Hamiltonian Cycle

We will now examine a similar graph with five vertices:



We find that the following system of equations represents the Hamiltonian cycles of the graph:

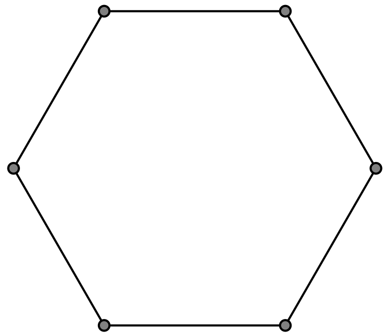
$$
\begin{aligned}
z^4 + z^3 + z^2 + z + 1 &= 0 \\
x_i^5 - 1 &= 0 \quad 1 \leq i \leq 5 \\
(zx_1 - x_2)(zx_1 - x_5) &= 0 \\
(zx_2 - x_3)(zx_2 - x_1) &= 0 \\
(zx_3 - x_4)(zx_3 - x_2) &= 0 \\
(zx_4 - x_5)(zx_4 - x_3) &= 0 \\
(zx_5 - x_1)(zx_5 - x_4) &= 0
\end{aligned}
$$

The Gröbner basis of the ideal generated by these polynomials is

$$
\begin{aligned}
&\{x_5^5 - 1, x_4^2 + x_4x_5z^3 + x_4x_5z^2 + x_4x_5 + x_5^2, x_3 + x_4z^3 + x_4z^2 + x_4 + x_5, \\
&x_2 - x_4z^3 - x_4z^2 - x_4 - x_5z^3 - x_5z^2 - x_5, x_1 + x_4 + x_5z^3 + x_5z^2 + x_5\}
\end{aligned}
$$

4 Six Vertices with One Hamiltonian Cycle

Finally, we extend our previous results to the case with six vertices:



In this case, the system of polynomial equations is

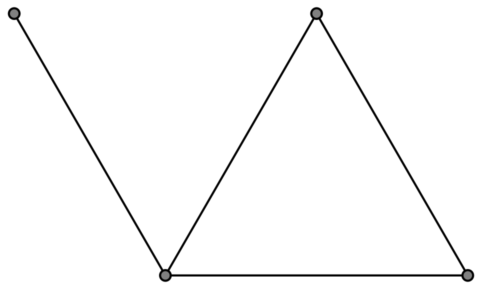
$$\begin{aligned} z^2 - z + 1 &= 0 \\ x_i^6 - 1 &= 0 \quad 1 \leq i \leq 6 \\ (zx_1 - x_2)(zx_1 - x_6) &= 0 \\ (zx_2 - x_3)(zx_2 - x_1) &= 0 \\ (zx_3 - x_4)(zx_3 - x_2) &= 0 \\ (zx_4 - x_5)(zx_4 - x_3) &= 0 \\ (zx_5 - x_6)(zx_5 - x_4) &= 0 \\ (zx_6 - x_1)(zx_6 - x_5) &= 0 \end{aligned}$$

The Gröbner basis of the ideal generated by these polynomials is

$$\{x_6^6 - 1, x_5^2 - x_5x_6 + x_6^2, x_4 - x_5 + x_6, x_3 + x_6, x_2 + x_5, x_1 + x_5 - x_6\}$$

5 Four Vertices with no Hamiltonian Cycles

We will now move on to a graph with no Hamiltonian cycles. The solution to the polynomial system in this case should have a very different form.



The system of equations representing the Hamiltonian cycles in this graph is

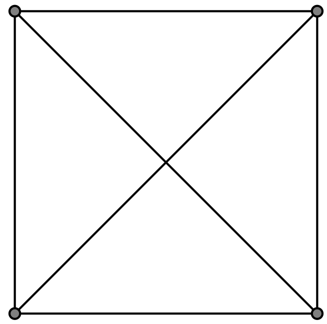
$$\begin{aligned}
z^2 + 1 &= 0 \\
x_i^4 - 1 &= 0 \quad 1 \le i \le 4 \\
zx_1 - x_2 &= 0 \\
(zx_2 - x_3)(zx_2 - x_4)(zx_2 - x_1) &= 0 \\
(zx_3 - x_4)(zx_3 - x_2) &= 0 \\
(zx_4 - x_2)(zx_4 - x_3) &= 0
\end{aligned}$$

The Gröbner basis of the ideal generated by these polynomials is

$$\{1\}$$

6 Complete Graph with Four Vertices

We will now examine the complete four-vertex graph.



In this case, the graph corresponds to the system of equations

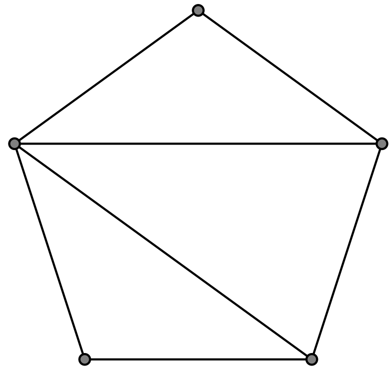
$$\begin{aligned}
z^2 + 1 &= 0 \\
x_i^4 - 1 &= 0 \quad 1 \leq i \leq 4 \\
(zx_1 - x_2)(zx_1 - x_4)(zx_1 - x_3) &= 0 \\
(zx_2 - x_3)(zx_2 - x_1)(zx_2 - x_4) &= 0 \\
(zx_3 - x_4)(zx_3 - x_2)(zx_3 - x_1) &= 0 \\
(zx_4 - x_1)(zx_4 - x_3)(zx_4 - x_2) &= 0
\end{aligned}$$

The Gröbner basis of the ideal generated by these polynomials is

$$\{x_4^4 - 1, x_3^3 + x_3^2x_4 + x_3x_4^2 + x_4^3, x_2^2 + x_2x_3 + x_2x_4 + x_3^2 + x_3x_4 + x_4^2, x_1 + x_2 + x_3 + x_4\}$$

7 Five Vertices with One Hamiltonian Cycle and Additional Edges

This graph is similar to the previous graph with five vertices, and also has only one cycle. However, it has additional vertices that should not affect the solutions.



The system of polynomial equations for this graph is

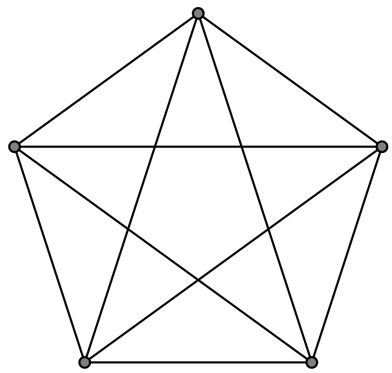
$$
\begin{aligned}
z^4 + z^3 + z^2 + z + 1 &= 0 \\
x_i^5 - 1 &= 0 \quad 1 \leq i \leq 5 \\
(zx_1 - x_2)(zx_1 - x_3)(zx_1 - x_4)(zx_1 - x_5) &= 0 \\
(zx_2 - x_3)(zx_2 - x_1) &= 0 \\
(zx_3 - x_4)(zx_3 - x_2)(zx_3 - x_1) &= 0 \\
(zx_4 - x_5)(zx_4 - x_3)(zx_4 - x_1) &= 0 \\
(zx_5 - x_1)(zx_5 - x_4) &= 0
\end{aligned}
$$

The Gröbner basis of the ideal generated by these polynomials is

$$
\begin{aligned}
\{&x_5^5 - 1, x_4^2 + x_4x_5z^3 + x_4x_5z^2 + x_4x_5 + x_5^2, x_3 + x_4z^3 + x_4z^2 + x_4 + x_5, \\
&x_2 - x_4z^3 - x_4z^2 - x_4 - x_5z^3 - x_5z^2 - x_5, x_1 + x_4 + x_5z3 + x_5z2 + x_5\}
\end{aligned}
$$

8 Five Vertices with Multiple Hamiltionian Cycles

Now we will examine some graphs with multiple Hamiltonian cycles. We begin with a graph with five vertices:



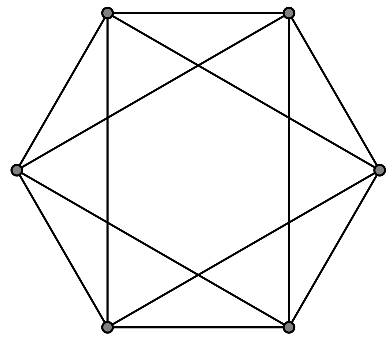
The system of equations for this graph is

$$
\begin{aligned}
z^4 + z^3 + z^2 + z + 1 &= 0 \\
x_i^5 - 1 &= 0 \quad 1 \leq i \leq 5 \\
(zx_1 - x_2)(zx_1 - x_3)(zx_1 - x_4)(zx_1 - x_5) &= 0 \\
(zx_2 - x_1)(zx_2 - x_3)(zx_2 - x_4)(zx_2 - x_5) &= 0 \\
(zx_3 - x_1)(zx_3 - x_2)(zx_3 - x_4)(zx_3 - x_5) &= 0 \\
(zx_4 - x_1)(zx_4 - x_2)(zx_4 - x_3)(zx_4 - x_5) &= 0 \\
(zx_5 - x_1)(zx_5 - x_2)(zx_5 - x_3)(zx_5 - x_4) &= 0
\end{aligned}
$$

The Gröbner basis of the ideal generated by these polynomials is

9 Six Vertices with Multiple Hamiltonian Cycles (4-Regular)

We will now examine a graph with multiple Hamiltonian cycles that has six vertices:

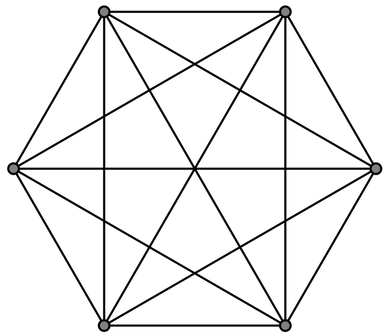


This graph gives us the system of equations:

$$
\begin{aligned}
z^2 - z + 1 &= 0 \\
x_i^6 - 1 &= 0 \quad 1 \le i \le 6 \\
(zx_1 - x_2)(zx_1 - x_3)(zx_1 - x_5)(zx_1 - x_6) &= 0 \\
(zx_2 - x_1)(zx_2 - x_3)(zx_2 - x_4)(zx_2 - x_6) &= 0 \\
(zx_3 - x_1)(zx_3 - x_2)(zx_3 - x_4)(zx_3 - x_5) &= 0 \\
(zx_4 - x_2)(zx_4 - x_3)(zx_4 - x_5)(zx_4 - x_6) &= 0 \\
(zx_5 - x_1)(zx_5 - x_3)(zx_5 - x_4)(zx_5 - x_6) &= 0 \\
(zx_6 - x_1)(zx_6 - x_2)(zx_6 - x_4)(zx_6 - x_5) &= 0
\end{aligned}
$$

10 Six Vertices with Multiple Hamiltonian Cycles (5-Regular)

Finally, we will examine a graph similar to the previous graph, but with more edges so it is a complete graph.



This graph gives us the system of equations:

$$\begin{aligned}
z^2 - z + 1 &= 0 \\
x_i^6 - 1 &= 0 \quad 1 \le i \le 6 \\
(zx_1 - x_2)(zx_1 - x_3)(zx_1 - x_4)(zx_1 - x_5)(zx_1 - x_6) &= 0 \\
(zx_2 - x_1)(zx_2 - x_3)(zx_2 - x_4)(zx_2 - x_5)(zx_2 - x_6) &= 0 \\
(zx_3 - x_1)(zx_3 - x_2)(zx_3 - x_4)(zx_3 - x_5)(zx_3 - x_6) &= 0 \\
(zx_4 - x_1)(zx_4 - x_2)(zx_4 - x_3)(zx_4 - x_5)(zx_4 - x_6) &= 0 \\
(zx_5 - x_1)(zx_5 - x_2)(zx_5 - x_3)(zx_5 - x_4)(zx_5 - x_6) &= 0 \\
(zx_6 - x_1)(zx_6 - x_2)(zx_6 - x_3)(zx_6 - x_4)(zx_6 - x_5) &= 0
\end{aligned}$$