Week 7

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Recall: Encoding graphs as polynomials

Proposition

Let $G = (V, A)$ be a simple directed graph on vertices $V = \{1, \ldots, n\}$. Assume that the characteristic of $\mathbb{K}$ is relatively prime to n and that $z \in \mathbb{K}$ is a primitive n-th root of unity. Consider the following system in $\mathbb{K}[x_1, \ldots, x_n]$:

$$H_G = \{x_i^n - 1 = 0, \prod_{j \in \delta^+(i)} (zx_i - x_j) = 0 \,:\, i \in V\}$$

Here, $\delta^+(i)$ denotes those vertices j which are connected to i by the arc going from i to j in G. The system H has a solution over $\mathbb{K}$ if and only if G has a Hamiltonian cycle.

Source: "Recognizing Graph Theoretic Properties with Polynomial Ideals" by J.A. De Loera, C. Hillar, P.N. Malkin, and M. Omar.

Recall: Graphs with one Hamiltonian cycle

Definition

Let z be a fixed primitive k-th root of unity. If C is a directed cycle of length k in a directed graph, with vertex set $\{v_1, \ldots, v_k\}$, the cycle encoding of C is the following set of k polynomials:

$$g_i = \begin{cases} x_{v_{k-i}} - z^{k-i} x_{v_k} & i = 1, \ldots, k-1 \\ x_{v_k}^k - 1 & i = k \end{cases}.$$

Note: define $H_{G,C} = \langle g_1, \ldots, g_i \rangle$. The g_i's form a reduced Gröbner basis (which must be unique) for $H_{G,C}$.

Goal for this week

My current goal is to prove the following conjecture:

Conjecture

For a graph G with n vertices and one or more Hamiltonian cycles, $x_n^n - 1$ is in the reduced Gröbner basis of H_G.

From the previous slide, this is true for graphs with a single Hamiltonian cycle.

Extending to Graphs with multiple Hamiltonian cycles

We will use the following theorem to relate the encoding of a single cycle to multiple cycles:

Theorem

Let G be a connected directed graph with n vertices. Then,

$$H_G = \bigcap_{C} H_{G,C},$$

where C ranges over all Hamiltonian cycles of the graph G.

More work to do

While this theorem reveals a potential strategy for proving the conjecture, computing the Gröbner basis of intersecting ideals is still very complicated.

Can we reduce the problem further?

Elimination Theory

One way to reduce the scope of this problem is to examine an *elimination ideal*, defined as follows:

Definition

Given $I = \langle f_1, \dots, f_s \rangle \subseteq k[x_1, \dots, x_n]$ *the* ℓ*-th* ***elimination ideal*** I_ℓ *is the ideal of* $k[x_{\ell+1}, \dots, x_n]$ *defined by*

$$I_\ell = I \cap k[x_{\ell+1}, \dots, x_n].$$

We would like to show that $H_G \cap k[x_n] = \langle x_n^n - 1 \rangle$ for any graph G with one or more Hamiltonian cycles.
Can Gröbner bases help us do this?

Elimination Theory

Fortunately, they can. We can use the Elimination Theorem:

Theorem

Let $I \subseteq k[x_1, \ldots, x_n]$ be an ideal and let G be a Gröbner basis of I with respect to lex order where $x_1 > x_2 > \cdots > x_n$. Then, for every $0 < \ell < n$, the set

$$G_\ell = G \cap k[x_{\ell+1}, \ldots, x_n]$$

is a Gröbner basis of the ℓ-th elimination ideal I_ℓ.

Source: Cox, Little, O'Shea

Example with two Hamiltonian cycles

Let G be a graph with n vertices and two Hamiltonian cycles, represented by the cycle ideals H_{G,C_1} and H_{G,C_2}. Then we find

$$\begin{aligned} H_G \cap k[x_n] &= (H_{G,C_1} \cap H_{G,C_2}) \cap k[x_n] \\ &= (H_{G,C_1} \cap k[x_n]) \cap (H_{G,C_2} \cap k[x_n]) \\ &= \langle x_n^n - 1 \rangle \cap \langle x_n^n - 1 \rangle \\ &= \langle x_n^n - 1 \rangle. \end{aligned}$$

Therefore, the polynomial $x_n^n - 1$ is the sole generator of the polynomials in H_G that contain x_n alone.