Week 3 Presentation

Max Comstock

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How to represent a primitive nth root of unity

Definition

An nth root of unity is **primitive** *if it is not a kth root of unity for some smaller k:*

$$z^k \neq 1 \quad (k = 1, 2, \ldots, n-1).$$

Cyclotomic poynomial Φ_n

Definition

*The n**th cyclotomic polynomial** is the unique irreducible polynomial with integer coefficients whose roots are the nth primitive roots of unity.*

Examples:

$$\begin{aligned}
\Phi_3(z) &= z^2 + z + 1 \\
\Phi_4(z) &= z^2 + 1 \\
\Phi_5(z) &= z^4 + z^3 + z^2 + z + 1 \\
\Phi_6(z) &= z^2 - z + 1
\end{aligned}$$

Using Φ_n to create a field with nth roots of unity

We will begin with the ring Our goal is to create the ring

$$\mathbb{K}[x_1, \ldots, x_n]$$

where $\mathbb{K}$ is a field containing the primitive nth roots of unity. We can create such a field as follows:

$$\mathbb{K} = \frac{\mathbb{Q}[z]}{\Phi_n(z)}.$$

Using the relationship

$$\left(\frac{\mathbb{Q}[z]}{\Phi_n(z)}\right)[x_1, \ldots, x_n] \cong \frac{\mathbb{Q}[x_1, \ldots, x_n, z]}{\Phi_n},$$

We can choose

$$\mathbb{K}[x_1, \ldots, x_n] = \frac{\mathbb{Q}[x_1, \ldots, x_n, z]}{\Phi_n(z)}.$$

Implementation using Macaulay2

```
R = QQ[x_1, x_2, x_3, x_4, x_5, x_6, z];
CYCLOTOMIC_POLY = z^2 - z + 1;
R = R/CYCLOTOMIC_POLY;
IDEAL_GEN = {x_1^6 - 1, x_2^6 - 1, x_3^6 - 1,
         x_4^6 - 1, x_5^6 - 1, x_6^6 - 1,
 z*x_1 - x_2,
 z*x_2 - x_3,
 z*x_3 - x_4,
 z*x_4 - x_5,
 z*x_5 - x_6,
 z*x_6 - x_1};
idealOfGraph = ideal IDEAL_GEN;
graphBasis = flatten entries gens gb idealOfGraph;
print toString graphBasis;
```