Week 5 Presentation

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Recall: Encoding graphs as polynomials

Proposition

Let $G = (V, A)$ be a simple directed graph on vertices $V = \{1, \ldots, n\}$. Assume that the characteristic of $\mathbb{K}$ is relatively prime to n and that $z \in \mathbb{K}$ is a primitive n-th root of unity. Consider the following system in $\mathbb{K}[x_1, \ldots, x_n]$:

$$H_G = \{x_i^n - 1 = 0, \prod_{j \in \delta^+(i)} (zx_i - x_j) = 0 \,:\, i \in V\}$$

Here, $\delta^+(i)$ denotes those vertices j which are connected to i by the arc going from i to j in G. The system H has a solution over $\mathbb{K}$ if and only if G has a Hamiltonian cycle.

Source: "Recognizing Graph Theoretic Properties with Polynomial Ideals" by J.A. De Loera, C. Hillar, P.N. Malkin, and M. Omar.

Recall: Graphs with one Hamiltonian cycle

Definition

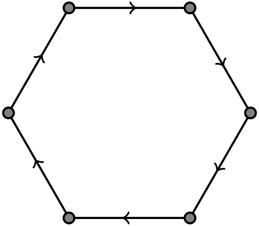
Let z be a fixed primitive k-th root of unity. If C is a directed cycle of length k in a directed graph, with vertex set $\{v_1, \ldots, v_k\}$, the cycle encoding of C is the following set of k polynomials:

$$g_i = \begin{cases} x_{v_{k-i}} - z^{k-i} x_{v_k} & i = 1, \ldots, k-1 \\ x_{v_k}^k - 1 & i = k \end{cases}.$$

Note: define $H_{G,C} = \langle g_1, ldots, g_i \rangle$. The g_i's form a reduced Gröbner basis (which must be unique) for H_G, C.

Example

The graph

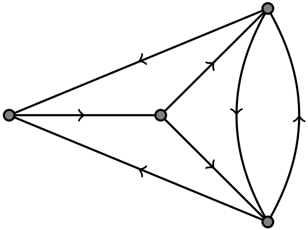


gives us the following Gröbner basis for H_G:

$$\{x_6^6 - 1, x_5 - x_6 z^5, x_4 - x_6 z^4, x_3 - x_6 z^3, x_2 - x_6 z^2, x_1 - x_6 z\}.$$

Example with two Hamiltonian cycles

The graph



has two Hamiltonian cycles. The Gröbner basis for H_G is

$$\{x_4^4 - 1, x_3^2 + x_4^2, 2x_2 + (z+1)x_3 + (z+1)x_4,$$
$$2x_1 + (-z+1)x_3 + (-z+1)x_4\}.$$

Conjectures

Conjecture

For a graph G with n vertices and one or more Hamiltonian cycles, we find $x_n^n - 1 \in H_G$.

Conjecture

Let G be a graph with n vertices and one or more Hamiltonian cycles, and k be a natural number (not including zero). Then we find, for each i such that $1 \leq i \leq n$, that $\mathrm{LT}(g) = x_i^k$ for some $g \in H_G$.

Conjecture

Let G be a graph with n vertices, and $k \geq 2$. There exists $g \in H_G$ such that $\mathrm{LT}(g) = x_i^k$ for some i such that $1 \leq i < n$ if and only if G has more than one Hamiltonian cycle.