Problems on S-Polynomials and Gröbner Bases

Max Comstock

June 12, 2014

Problem

Consider the polynomial ring $R = \mathbb{C}[x_1, x_2, \ldots, x_n]$. Let $\prec$ be a monomial order in which $x_n \prec x_{n-1} \prec \cdots \prec x_2 \prec x_1$. Let $I \subseteq R$ be the ideal given by

$$I = (x_1^3 - 1, x_2^3 - 1, \ldots, x_n^3 - 1).$$

Determine, with proof, whether or not the set $\{x_1^3 - 1, x_2^3 - 1, \ldots, x_n^3 - 1\}$ is a Gröbner basis for I. (Hint: Compute the S-polynomials for every pair of polynomials in the set.)

Solution

Let $G = \{x_1^3 - 1, x_2^3 - 1, \ldots, x_n^3 - 1\}$. Using Buchberger's Criterion, we can show that, for all pairs $i \neq j$, the remainder on division of $S(g_i, g_j)$ by G is zero. Choose arbitrary i and j such that $i \neq j$, and $g_i, g_j \in G$. Suppose, without loss of generality, that $i < j$. Then we find

$$S(g_i, g_j) = \frac{x^\gamma}{x_i^3} g_i - \frac{x^\gamma}{x_j^3} g_j.$$

Note that $x^\gamma = x_i^3 x_j^3$. This means that the S-polynomial is

$$\begin{aligned} S(g_i, g_j) &= \frac{x_i^3 x_j^3}{x_i^3} g_i - \frac{x_i^3 x_j^3}{x_j^3} g_j \\ &= x_j^3 g_i - x_i^3 g_j \\ &= x_j^3 (x_i^3 - 1) - x_i^3 (x_j^3 - 1) \\ &= x_i^3 x_j^3 - x_j^3 - (x_i^3 x_j^3 - x_i^3) \\ &= x_i^3 - x_j^3. \end{aligned}$$

From the division algoritm, we find

$$S(g_i, g_j) = x_i^3 - x_j^3 = (x_i^3 - 1) - (x_j^3 - 1) = g_i - g_j.$$

Similarly, for $S(g_j, g_i)$, we find

$$S(g_j, g_i) = x_j^3 - x_i^3 = (x_j^3 - 1) - (x_i^3 - 1) = g_j - g_i.$$

Thus, the remainder on the S-polynomial on division by G is zero. Hence G is a Gröbner basis for I.