Gröbner Bases of some Undirected Graphs Using an Alternate Cycle Encoding

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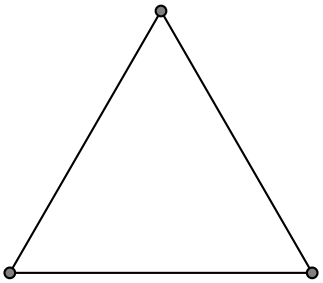
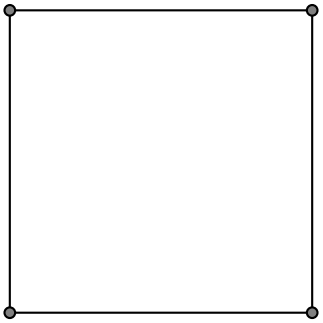
1 Three Vertices with One Hamiltonian Cycle



$$y_1 + y_2 + y_3 - 3 = 0$$
$$y_1(y_1 - 1) = 0$$
$$y_2(y_2 - 1) = 0$$
$$y_3(y_3 - 1) = 0$$
$$(x_1 - 1)(x_1 - 2)(x_1 - 3) = 0$$
$$(x_2 - 1)(x_2 - 2)(x_2 - 3) = 0$$
$$(x_3 - 1)(x_3 - 2)(x_3 - 3) = 0$$
$$y_1(x_1 - y_2x_2 + y_2)(x_1 - y_2x_2 - y_2(3-1))(x_1 - y_3x_3 + y_3)(x_1 - y_3x_3 - y_3(3-1)) = 0$$
$$y_2(x_2 - y_1x_1 + y_1)(x_2 - y_1x_1 - y_1(3-1))(x_2 - y_3x_3 + y_3)(x_2 - y_3x_3 - y_3(3-1)) = 0$$
$$y_3(x_3 - y_1x_1 + y_1)(x_3 - y_1x_1 - y_1(3-1))(x_3 - y_2x_2 + y_2)(x_3 - y_2x_2 - y_2(3-1)) = 0$$

$$\{x_3^3 - 6x_3^2 + 11x_3 - 6, x_2^2 + x_2x_3 - 6x_2 + x_3^2 - 6x_3 + 11, x_1 + x_2 + x_3 - 6, y_3 - 1, y_2 - 1, y_1 - 1\}$$

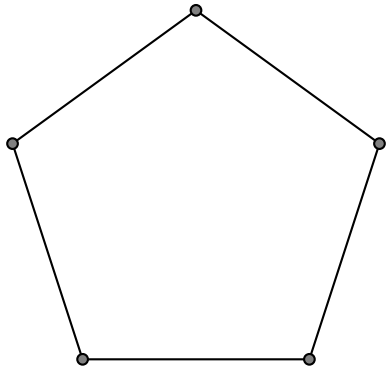
2 Four Vertices with One Hamiltonian Cycle



$$
\begin{aligned}
y_1 + y_2 + y_3 + y_4 - 4 &= 0 \\
y_1(y_1 - 1) &= 0 \\
y_2(y_2 - 1) &= 0 \\
y_3(y_3 - 1) &= 0 \\
y_4(y_4 - 1) &= 0 \\
(x_1 - 1)(x_1 - 2)(x_1 - 3)(x_1 - 4) &= 0 \\
(x_2 - 1)(x_2 - 2)(x_2 - 3)(x_3 - 4) &= 0 \\
(x_3 - 1)(x_3 - 2)(x_3 - 3)(x_3 - 4) &= 0 \\
(x_4 - 1)(x_4 - 2)(x_4 - 3)(x_4 - 4) &= 0 \\
y_1(x_1 - y_2x_2 + y_2)(x_1 - y_2x_2 - 3y_2)(x_1 - y_4x_4 + y_4)(x_1 - y_4x_4 - 3y_4) &= 0 \\
y_2(x_2 - y_1x_1 + y_1)(x_2 - y_1x_1 - 3y_1)(x_2 - y_3x_3 + y_3)(x_2 - y_3x_3 - 3y_3) &= 0 \\
y_3(x_3 - y_2x_2 + y_2)(x_3 - y_2x_2 - 3y_2)(x_3 - y_4x_4 + y_4)(x_3 - y_4x_4 - 3y_4) &= 0 \\
y_4(x_4 - y_1x_1 + y_1)(x_4 - y_1x_1 - 3y_1)(x_4 - y_3x_3 + y_3)(x_4 - y_3x_3 - 3y_3) &= 0
\end{aligned}
$$

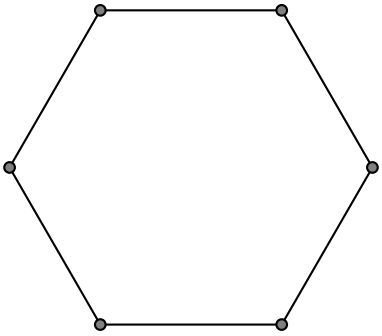
$$
\begin{aligned}
\{&x_4^4 - 10x_4^3 + 35x_4^2 - 50x_4 + 24, 3x_3^2 + 4x_3x_4^3 - 30x_3x_4^2 + 68x_3x_4 - 60x_3 - 10x_4^3 + 75x_4^2 - 170x_4 + 129, \\
&3x_2 - 4x_4^3 + 30x_4^2 - 65x_4 + 30, 3x_1 + 3x_3 + 4x_4^3 - 30x_4^2 + 68x_4 - 60, y_4 - 1, y_3 - 1, y_2 - 1, y_1 - 1\}
\end{aligned}
$$

3 Five Vertices with One Hamiltonian Cycle



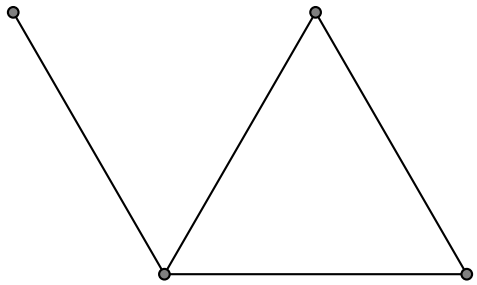
$$y_1 + y_2 + y_3 + y_4 + y_5 - 5 = 0$$
$$y_1(y_1 - 1) = 0$$
$$y_2(y_2 - 1) = 0$$
$$y_3(y_3 - 1) = 0$$
$$y_4(y_4 - 1) = 0$$
$$y_5(y_5 - 1) = 0$$
$$(x_1 - 1)(x_1 - 2)(x_1 - 3)(x_1 - 4)(x_1 - 5) = 0$$
$$(x_2 - 1)(x_2 - 2)(x_2 - 3)(x_3 - 4)(x_2 - 5) = 0$$
$$(x_3 - 1)(x_3 - 2)(x_3 - 3)(x_3 - 4)(x_3 - 5) = 0$$
$$(x_4 - 1)(x_4 - 2)(x_4 - 3)(x_4 - 4)(x_4 - 5) = 0$$
$$(x_5 - 1)(x_5 - 2)(x_5 - 3)(x_5 - 4)(x_5 - 5) = 0$$
$$y_1(x_1 - y_2x_2 + y_2)(x_1 - y_2x_2 - 4y_2)(x_1 - y_5x_5 + y_5)(x_1 - y_5x_5 - 4y_5) = 0$$
$$y_2(x_2 - y_1x_1 + y_1)(x_2 - y_1x_1 - 4y_1)(x_2 - y_3x_3 + y_3)(x_2 - y_3x_3 - 4y_3) = 0$$
$$y_3(x_3 - y_2x_2 + y_2)(x_3 - y_2x_2 - 4y_2)(x_3 - y_4x_4 + y_4)(x_3 - y_4x_4 - 4y_4) = 0$$
$$y_4(x_4 - y_3x_3 + y_3)(x_4 - y_3x_3 - 4y_3)(x_4 - y_5x_5 + y_5)(x_4 - y_5x_5 - 4y_5) = 0$$
$$y_5(x_5 - y_1x_1 + y_1)(x_5 - y_1x_1 - 4y_1)(x_5 - y_4x_4 + y_4)(x_5 - y_4x_4 - 4y_4) = 0$$

4 Six Vertices with One Hamiltonian Cycle



$$y_1 + y_2 + y_3 + y_4 + y_5 + y_6 - 6 = 0$$
$$y_1(y_1 - 1) = 0$$
$$y_2(y_2 - 1) = 0$$
$$y_3(y_3 - 1) = 0$$
$$y_4(y_4 - 1) = 0$$
$$y_5(y_5 - 1) = 0$$
$$y_6(y_6 - 1) = 0$$
$$(x_1 - 1)(x_1 - 2)(x_1 - 3)(x_1 - 4)(x_1 - 5)(x_1 - 6) = 0$$
$$(x_2 - 1)(x_2 - 2)(x_2 - 3)(x_3 - 4)(x_2 - 5)(x_2 - 6) = 0$$
$$(x_3 - 1)(x_3 - 2)(x_3 - 3)(x_3 - 4)(x_3 - 5)(x_3 - 6) = 0$$
$$(x_4 - 1)(x_4 - 2)(x_4 - 3)(x_4 - 4)(x_4 - 5)(x_4 - 6) = 0$$
$$(x_5 - 1)(x_5 - 2)(x_5 - 3)(x_5 - 4)(x_5 - 5)(x_5 - 6) = 0$$
$$(x_6 - 1)(x_6 - 2)(x_6 - 3)(x_6 - 4)(x_6 - 5)(x_6 - 6) = 0$$
$$y_1(x_1 - y_2x_2 + y_2)(x_1 - y_2x_2 - 5y_2)(x_1 - y_6x_6 + y_6)(x_1 - y_6x_6 - 5y_6) = 0$$
$$y_2(x_2 - y_1x_1 + y_1)(x_2 - y_1x_1 - 5y_1)(x_2 - y_3x_3 + y_3)(x_2 - y_3x_3 - 5y_3) = 0$$
$$y_3(x_3 - y_2x_2 + y_2)(x_3 - y_2x_2 - 5y_2)(x_3 - y_4x_4 + y_4)(x_3 - y_4x_4 - 5y_4) = 0$$
$$y_4(x_4 - y_3x_3 + y_3)(x_4 - y_3x_3 - 5y_3)(x_4 - y_5x_5 + y_5)(x_4 - y_5x_5 - 5y_5) = 0$$
$$y_5(x_5 - y_4x_4 + y_4)(x_5 - y_4x_4 - 5y_4)(x_5 - y_6x_6 + y_6)(x_5 - y_6x_6 - 5y_6) = 0$$
$$y_6(x_6 - y_1x_1 + y_1)(x_6 - y_1x_1 - 5y_1)(x_6 - y_5x_5 + y_5)(x_6 - y_5x_5 - 5y_5) = 0$$

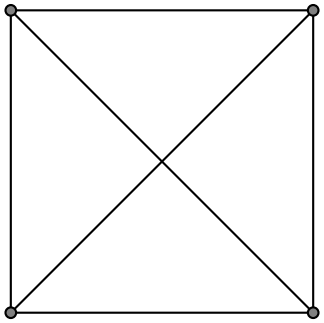
5 Four Vertices with no Hamiltonian Cycles



$$y_1 + y_2 + y_3 + y_4 - 4 = 0$$
$$y_1(y_1 - 1) = 0$$
$$y_2(y_2 - 1) = 0$$
$$y_3(y_3 - 1) = 0$$
$$y_4(y_4 - 1) = 0$$
$$(x_1 - 1)(x_1 - 2)(x_1 - 3)(x_1 - 4) = 0$$
$$(x_2 - 1)(x_2 - 2)(x_2 - 3)(x_3 - 4) = 0$$
$$(x_3 - 1)(x_3 - 2)(x_3 - 3)(x_3 - 4) = 0$$
$$(x_4 - 1)(x_4 - 2)(x_4 - 3)(x_4 - 4) = 0$$
$$y_1(x_1 - y_2x_2 + y_2)(x_1 - y_2x_2 - 3y_2) = 0$$
$$y_2(x_2 - y_1x_1 + y_1)(x_2 - y_1x_1 - 3y_1)(x_2 - y_3x_3 + y_3)(x_2 - y_3x_3 - 3y_3) = 0$$
$$y_3(x_3 - y_2x_2 + y_2)(x_3 - y_2x_2 - 3y_2)(x_3 - y_4x_4 + y_4)(x_3 - y_4x_4 - 3y_4) = 0$$
$$y_4(x_4 - y_2x_2 + y_1)(x_4 - y_2x_2 - 3y_2)(x_4 - y_3x_3 + y_3)(x_4 - y_3x_3 - 3y_3) = 0$$

$\{1\}$

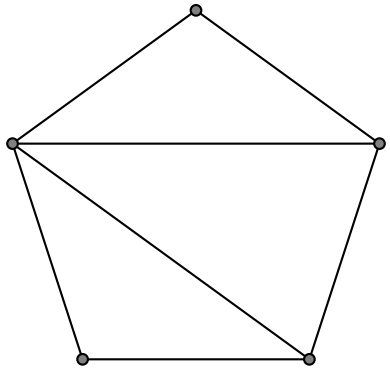
6 Complete Graph with Four Vertices



$$
\begin{aligned}
y_1 + y_2 + y_3 + y_4 - 4 &= 0 \\
y_1(y_1 - 1) &= 0 \\
y_2(y_2 - 1) &= 0 \\
y_3(y_3 - 1) &= 0 \\
y_4(y_4 - 1) &= 0 \\
(x_1 - 1)(x_1 - 2)(x_1 - 3)(x_1 - 4) &= 0 \\
(x_2 - 1)(x_2 - 2)(x_2 - 3)(x_3 - 4) &= 0 \\
(x_3 - 1)(x_3 - 2)(x_3 - 3)(x_3 - 4) &= 0 \\
(x_4 - 1)(x_4 - 2)(x_4 - 3)(x_4 - 4) &= 0 \\
y_1(x_1 - y_2x_2 + y_2)(x_1 - y_2x_2 - 3y_2) & \\
(x_1 - y_3x_3 + y_3)(x_1 - y_3x_3 - 3y_3)(x_1 - y_4x_4 + y_4)(x_1 - y_4x_4 - 3y_4) &= 0 \\
y_2(x_2 - y_1x_1 + y_1)(x_2 - y_1x_1 - 3y_1) & \\
(x_2 - y_3x_3 + y_3)(x_2 - y_3x_3 - 3y_3)(x_2 - y_4x_4 + y_4)(x_2 - y_4x_4 - 3y_4) &= 0 \\
y_3(x_3 - y_1x_1 + y_1)(x_3 - y_1x_1 - 3y_1) & \\
(x_3 - y_2x_2 + y_2)(x_3 - y_2x_2 - 3y_2)(x_3 - y_4x_4 + y_4)(x_3 - y_4x_4 - 3y_4) &= 0 \\
y_4(x_4 - y_1x_1 + y_1)(x_4 - y_1x_1 - 3y_1) & \\
(x_4 - y_2x_2 + y_2)(x_4 - y_2x_2 - 3y_2)(x_4 - y_3x_3 + y_3)(x_4 - y_3x_3 - 3y_3) &= 0
\end{aligned}
$$

$$
\begin{aligned}
&\{x_4^4 - 10x_4^3 + 35x_4^2 - 50x_4 + 24, x_3^3 + x_3^2x_4 - 10x_3^2 + x_3x_4^2 - 10x_3x_4 + 35x_3 + x_4^3 - 10x_4^2 + 35x_4 - 50, \\
&x_2^2 + x_2x_3 + x_2x_4 - 10x_2 + x_3^2 + x_3x_4 - 10x_3 + x_4^2 - 10x_4 + 35, x_1 + x_2 + x_3 + x_4 - 10, \\
&y_4 - 1, y_3 - 1, y_2 - 1, y_1 - 1\}
\end{aligned}
$$

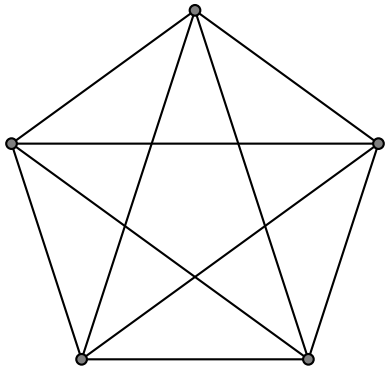
7 Five Vertices with One Hamiltonian Cycle and Additional Edges



$$y_1 + y_2 + y_3$$

$$(x_1 - 1)(x_1 - 2)(x_1 - 3)(x_1$$
$$(x_2 - 1)(x_2 - 2)(x_2 - 3)(x_2$$
$$(x_3 - 1)(x_3 - 2)(x_3 - 3)(x_3$$
$$(x_4 - 1)(x_4 - 2)(x_4 - 3)(x_4$$
$$(x_5 - 1)(x_5 - 2)(x_5 - 3)(x_5$$
$$y_1(x_1 - y_2x_2 + y_2)(x_1 - y_2x_2 - 4y_2)(x_1 - y_5x_5 + y_5)(x_1 -$$
$$y_2(x_2 - y_1x_1 + y_1)(x_2 - y_1x_1 - 4y_1)(x_2 - y_3x_3 + y_3)(x_2 - y_3x_3 - 4y_3)(x_2 - y_5x_5 + y_5)(x_2 -$$
$$y_3(x_3 - y_2x_2 + y_2)(x_3 - y_2x_2 - 4y_2)(x_3 - y_4x_4 + y_4)(x_3 - y_4x_4 - 4y_4)(x_3 - y_5x_5 + y_5)(x_3 -$$
$$y_4(x_4 - y_3x_3 + y_3)(x_4 - y_3x_3 - 4y_3)(x_4 - y_5x_5 + y_5)(x_4 -$$
$$y_5(x_5 - y_1x_1 + y_1)(x_5 - y_1x_1 - 4y_1)(x_5 - y_2x_2 + y_2)(x_5 - y_2x_2 - 4y_2)(x_5 - y_3x_3 + y_3)(x_5 - y_3x_3 - 4y_3)(x_5 - y_4x_4 + y_4)(x_5 -$$

8 Five Vertices with Multiple Hamiltionian Cycles

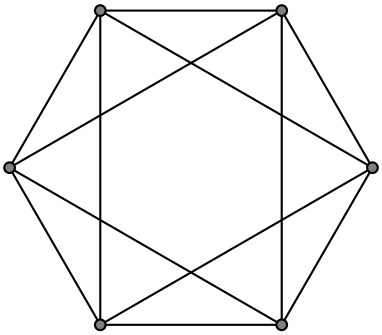


$$y_1 + y_2 + y_3$$

$$(x_1 - 1)(x_1 - 2)(x_1 - 3)(x_1$$
$$(x_2 - 1)(x_2 - 2)(x_2 - 3)(x_2$$
$$(x_3 - 1)(x_3 - 2)(x_3 - 3)(x_3$$
$$(x_4 - 1)(x_4 - 2)(x_4 - 3)(x_4$$
$$(x_5 - 1)(x_5 - 2)(x_5 - 3)(x_5$$

$$y_1(x_1 - y_2x_2 + y_2)(x_1 - y_2x_2 - 4y_2)(x_1 - y_3x_3 + y_3)(x_1 - y_3x_3 - 4y_3)(x_1 - y_4x_4 + y_4)(x_1 - y_4x_4 - 4y_4)(x_1 - y_5x_5 + y_5)(x_1$$
$$y_2(x_2 - y_1x_1 + y_1)(x_2 - y_1x_1 - 4y_1)(x_2 - y_3x_3 + y_3)(x_2 - y_3x_3 - 4y_3)(x_2 - y_4x_4 + y_4)(x_2 - y_4x_4 - 4y_4)(x_2 - y_5x_5 + y_5)(x_2$$
$$y_3(x_3 - y_1x_1 + y_1)(x_3 - y_1x_1 - 4y_1)(x_3 - y_2x_2 + y_2)(x_3 - y_2x_2 - 4y_2)(x_3 - y_4x_4 + y_4)(x_3 - y_4x_4 - 4y_4)(x_3 - y_5x_5 + y_5)(x_3$$
$$y_4(x_4 - y_1x_1 + y_1)(x_4 - y_1x_1 - 4y_1)(x_4 - y_2x_2 + y_2)(x_4 - y_2x_2 - 4y_2)(x_4 - y_3x_3 + y_3)(x_4 - y_3x_3 - 4y_3)(x_4 - y_5x_5 + y_5)(x_4$$
$$y_5(x_5 - y_1x_1 + y_1)(x_5 - y_1x_1 - 4y_1)(x_5 - y_2x_2 + y_2)(x_5 - y_2x_2 - 4y_2)(x_5 - y_3x_3 + y_3)(x_5 - y_3x_3 - 4y_3)(x_5 - y_4x_4 + y_4)(x_5$$

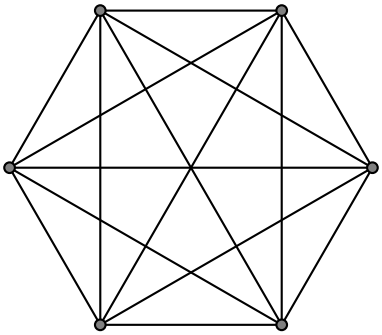
9 Six Vertices with Multiple Hamiltonian Cycles (4-Regular)



$$y_1 + y_2 + y_3 + y_4$$

$$(x_1 - 1)(x_1 - 2)(x_1 - 3)(x_1 - 4)(x_1$$
$$(x_2 - 1)(x_2 - 2)(x_2 - 3)(x_3 - 4)(x_2$$
$$(x_3 - 1)(x_3 - 2)(x_3 - 3)(x_3 - 4)(x_3$$
$$(x_4 - 1)(x_4 - 2)(x_4 - 3)(x_4 - 4)(x_4$$
$$(x_5 - 1)(x_5 - 2)(x_5 - 3)(x_5 - 4)(x_5$$
$$(x_6 - 1)(x_6 - 2)(x_6 - 3)(x_6 - 4)(x_6$$
$$y_1(x_1 - y_2x_2 + y_2)(x_1 - y_2x_2 - 5y_2)(x_1 - y_3x_3 + y_3)(x_1 - y_3x_3 - 5y_3)(x_1 - y_5x_5 + y_5)(x_1 - y_5x_5 - 5y_5)(x_1 - y_6x_6 + y_6)(x_1 -$$
$$y_2(x_2 - y_1x_1 + y_1)(x_2 - y_1x_1 - 5y_1)(x_2 - y_3x_3 + y_3)(x_2 - y_3x_3 - 5y_3)(x_2 - y_4x_4 + y_4)(x_2 - y_4x_4 - 5y_4)(x_2 - y_6x_6 + y_6)(x_2 -$$
$$y_3(x_3 - y_1x_1 + y_1)(x_3 - y_1x_1 - 5y_1)(x_3 - y_2x_2 + y_2)(x_3 - y_2x_2 - 5y_2)(x_3 - y_4x_4 + y_4)(x_3 - y_4x_4 - 5y_4)(x_3 - y_5x_5 + y_5)(x_3 -$$
$$y_4(x_4 - y_2x_2 + y_2)(x_4 - y_2x_2 - 5y_2)(x_4 - y_3x_3 + y_3)(x_4 - y_3x_3 - 5y_3)(x_4 - y_5x_5 + y_5)(x_4 - y_5x_5 - 5y_5)(x_4 - y_6x_6 + y_6)(x_4 -$$
$$y_5(x_5 - y_1x_1 + y_1)(x_5 - y_1x_1 - 5y_1)(x_5 - y_3x_3 + y_3)(x_5 - y_3x_3 - 5y_3)(x_5 - y_4x_4 + y_4)(x_5 - y_4x_4 - 5y_4)(x_5 - y_6x_6 + y_6)(x_5 -$$
$$y_6(x_6 - y_1x_1 + y_1)(x_6 - y_1x_1 - 5y_1)(x_6 - y_2x_2 + y_2)(x_6 - y_2x_2 - 5y_2)(x_6 - y_4x_4 + y_4)(x_6 - y_4x_4 - 5y_4)(x_6 - y_5x_5 + y_5)(x_6 -$$

10 Six Vertices with Multiple Hamiltonian Cycles (5-Regular)



$$y_1(x_1 - y_2x_2 + y_2)(x_1 - y_2x_2 - 5y_2)(x_1 - y_3x_3 + y_3)(x_1 - y_3x_3 - 5y_3)(x_1 - y_4x_4 + y_4)(x_1 - y_4x_4 - 5y_4)(x_1 - y_5x_5 + y_5)(x_1 -$$
$$y_2(x_2 - y_1x_1 + y_1)(x_2 - y_1x_1 - 5y_1)(x_2 - y_3x_3 + y_3)(x_2 - y_3x_3 - 5y_3)(x_2 - y_4x_4 + y_4)(x_2 - y_4x_4 - 5y_4)(x_2 - y_5x_5 + y_5)(x_2 -$$
$$y_3(x_3 - y_1x_1 + y_1)(x_3 - y_1x_1 - 5y_1)(x_3 - y_2x_2 + y_2)(x_3 - y_2x_2 - 5y_2)(x_3 - y_4x_4 + y_4)(x_3 - y_4x_4 - 5y_4)(x_3 - y_5x_5 + y_5)(x_3 -$$
$$y_4(x_4 - y_1x_1 + y_1)(x_4 - y_1x_1 - 5y_1)(x_4 - y_2x_2 + y_2)(x_4 - y_2x_2 - 5y_2)(x_4 - y_3x_3 + y_3)(x_4 - y_3x_3 - 5y_3)(x_4 - y_5x_5 + y_5)(x_4 -$$
$$y_5(x_5 - y_1x_1 + y_1)(x_5 - y_1x_1 - 5y_1)(x_5 - y_2x_2 + y_2)(x_5 - y_2x_2 - 5y_2)(x_5 - y_3x_3 + y_3)(x_5 - y_3x_3 - 5y_3)(x_5 - y_4x_4 + y_4)(x_5 -$$
$$y_6(x_6 - y_1x_1 + y_1)(x_6 - y_1x_1 - 5y_1)(x_6 - y_2x_2 + y_2)(x_6 - y_2x_2 - 5y_2)(x_6 - y_3x_3 + y_3)(x_6 - y_3x_3 - 5y_3)(x_6 - y_4x_4 + y_4)(x_6 -$$