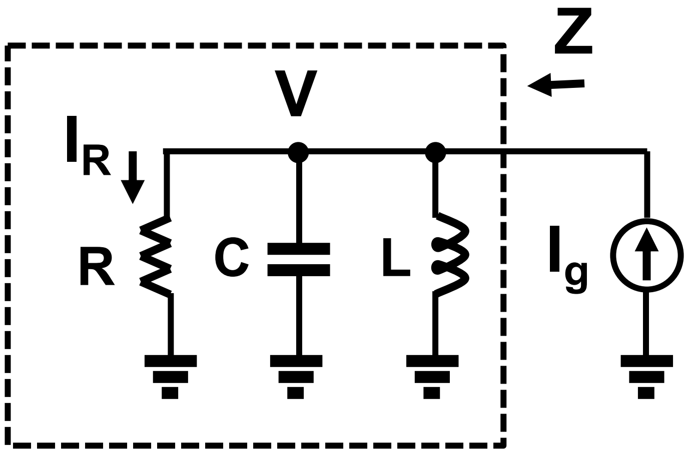
Passive Networks: Resonant Circuits and Transformers

Prof. Salvatore Levantino

Politecnico di Milano

Resonant Circuits

Resonant Circuits



Impedance in Laplace Transform:

$$Z(s) = \frac{V}{I_g} = R \cdot \frac{I_R}{I_g} = R \cdot H(s)$$

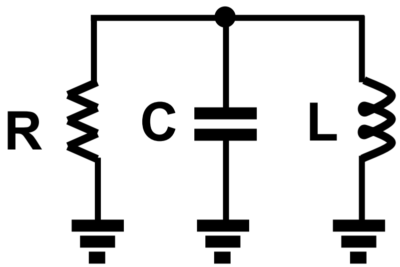
Definition

$$Q = \omega_0 RC$$

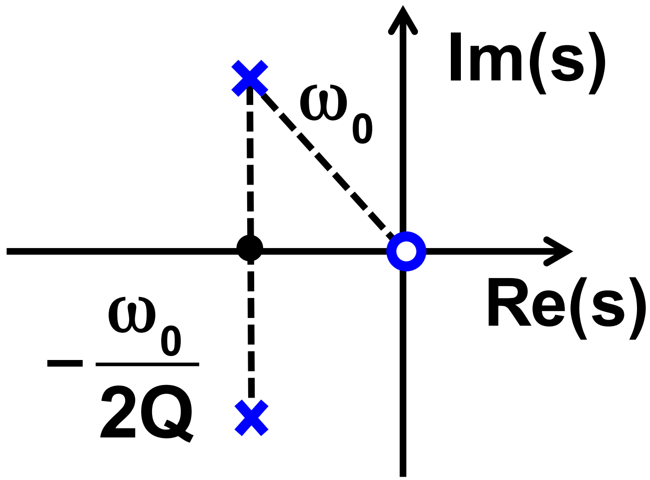
$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$H(s) = \frac{\frac{s}{RC}}{s^2 + \frac{s}{RC} + \frac{1}{LC}} = \frac{s\frac{\omega_0}{Q}}{s^2 + s\frac{\omega_0}{Q} + \omega_0^2}$$

Resonant Circuits: Complex Singularities



$$H(s)=\frac{\frac{s}{RC}}{s^2+\frac{s}{RC}+\frac{1}{LC}}=\frac{s\frac{\omega_0}{Q}}{s^2+s\frac{\omega_0}{Q}+\omega_0^2}$$

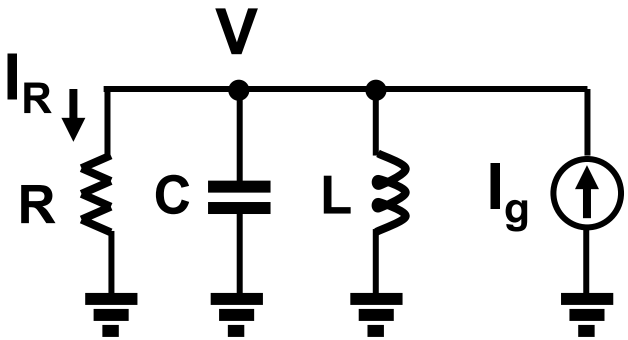


Damping Factor

$$\Rightarrow \zeta=\frac{-\mathrm{Re}\left(\omega_p\right)}{\left|\omega_p\right|}=\frac{1}{2Q}$$

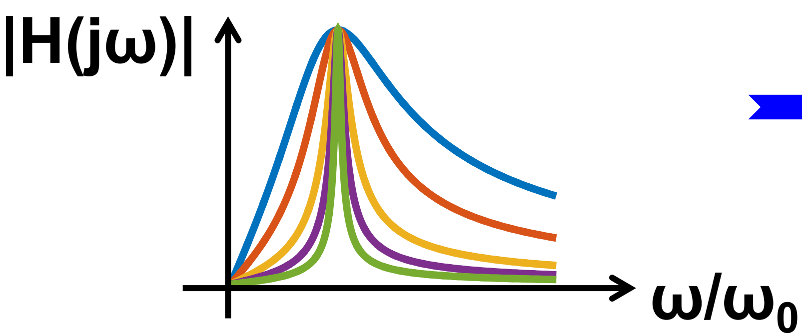
- **Damping factor is inversely proportional to Q**

Resonant Circuits: Network Functions



$$Z(j\omega) = \frac{V}{I_g} = R \cdot \frac{I_R}{I_g} = R \cdot H(j\omega)$$

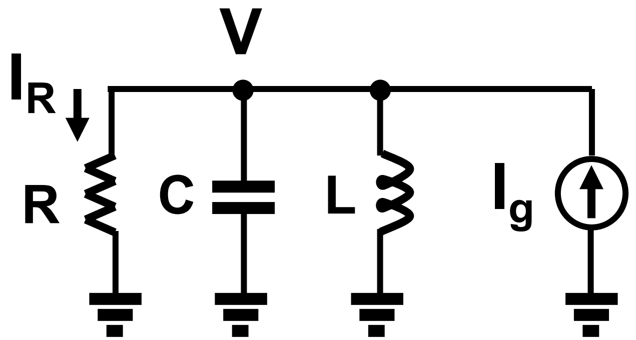
$$H(j\omega) = \frac{1}{1 + jR\left(\omega C - \frac{1}{\omega L}\right)} = \frac{1}{1 + jQ\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)}$$



➡

- **Band-pass frequency response dependent on Q**

Resonant Circuits: -3dB Bandwidth



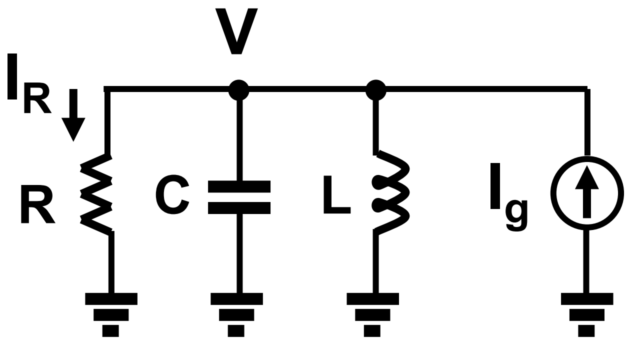
$$\left|H(j\omega)\right|^2 = \frac{1}{1+Q^2\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)^2} = \frac{1}{2}$$

$$\Rightarrow \quad \frac{\omega}{\omega_0}-\frac{\omega_0}{\omega} = \pm\frac{1}{Q} \quad \Rightarrow \quad \frac{\omega_{1,2}}{\omega_0} = \mp\frac{1}{2Q}+\sqrt{\frac{1}{4Q^2}+1}$$

$$\Rightarrow \quad \frac{\Delta\omega}{\omega_0} = \frac{\omega_2-\omega_1}{\omega_0} = \frac{1}{Q}$$

- **Q factor is the ratio of the center frequency over the -3dB BW of the network function**

Resonant Circuits: Energy Relationship

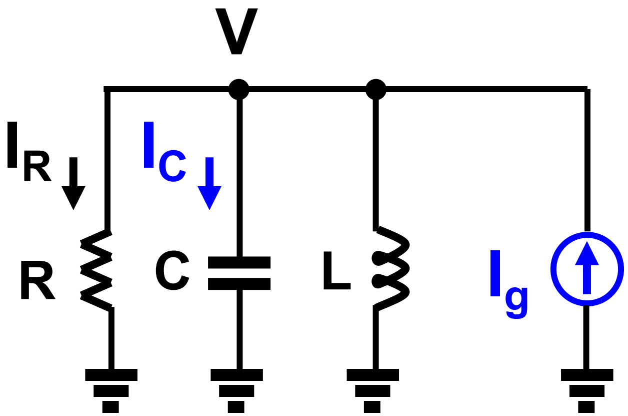


$$V(t) = \mathrm{Re}\left\{\underset{\text{Phasor}}{\overline{V}} \cdot e^{j\omega_0 t}\right\}$$

$$Q = \omega_0 RC = \omega_0 \frac{\frac{1}{2} C \left|\overline{V}\right|^2}{\frac{1}{2} \frac{\left|\overline{V}\right|^2}{R}} = \omega_0 \frac{E_{stored}}{P_{diss.}} = 2\pi \cdot \frac{E_{stored}}{E_{diss.per\,cycle}}$$

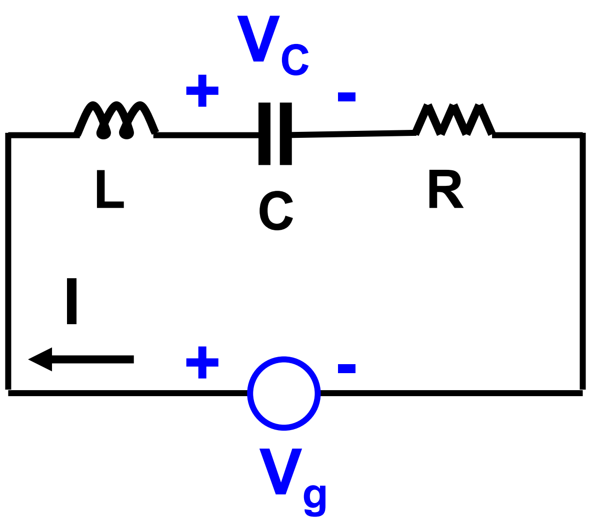
- **Q factor is proportional to the ratio of the energy stored over the energy dissipated in one oscillation cycle**

Current/Voltage Amplification at Resonance



Current Amplification at Resonance

$$\left|\overline{I_C}\right| = \omega_0 C \cdot \left|\overline{V}\right| = \omega_0 C \cdot \left|\overline{I_g}\right| R = Q \cdot \left|\overline{I_g}\right|$$

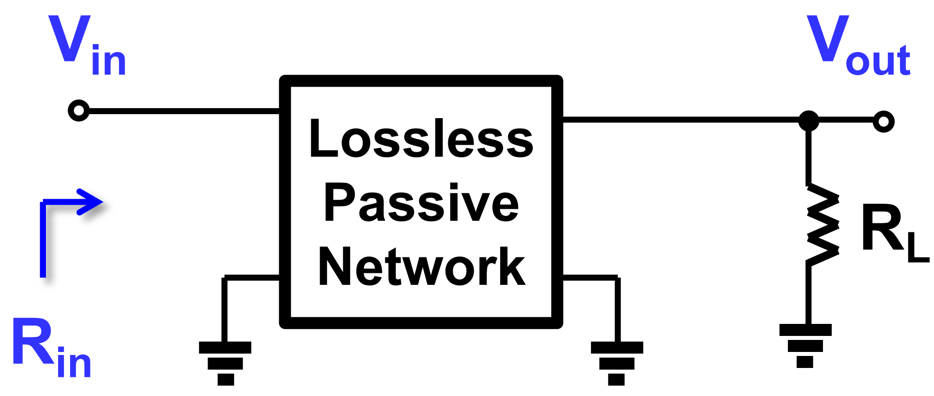


Voltage Amplification at Resonance

$$\left|\overline{V_C}\right| = \frac{\left|\overline{I}\right|}{\omega_0 C} = \frac{\left|\overline{V_g}\right|}{\omega_0 RC} = Q \cdot \left|\overline{V_g}\right|$$

Impedance Transformation

Impedance Transformations: General Result



$$G = \frac{V_{out}}{V_{in}}$$

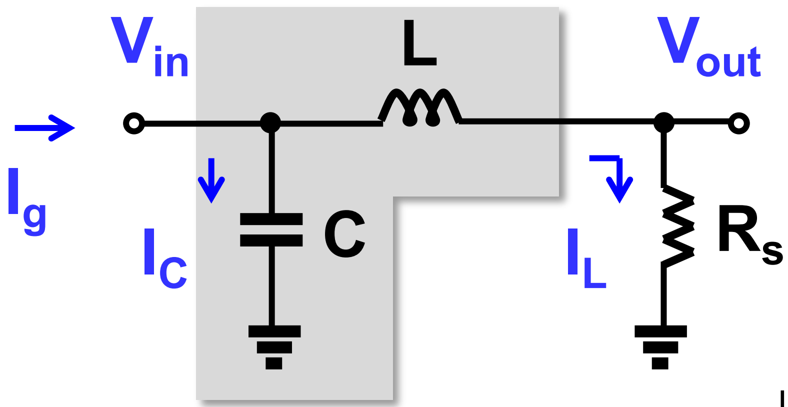
Voltage gain

$$\frac{1}{2}\frac{V_{in}^2}{R_{in}} = \frac{1}{2}\frac{V_{out}^2}{R_L} \quad \Rightarrow \quad R_{in} = R_L \cdot \frac{V_{in}^2}{V_{out}^2} = \frac{R_L}{G^2}$$

Impedance transformation

- *Upward* transformation if $G < 1$
- *Downward* transformation if $G > 1$

L-match Networks (Small Losses)



For small losses, <u>at resonance</u>:

$$\left|\overline{I_C}\right| = Q_L \cdot \left|\overline{I_g}\right| = \left|\overline{I_L}\right|$$

$$\left|\overline{V_{out}}\right| = \left|\overline{I_L}\right| R_s = \left|\overline{V_{in}}\right| \omega_0 C R_s = \left|\overline{V_{in}}\right| / Q_L$$

Voltage attenuation

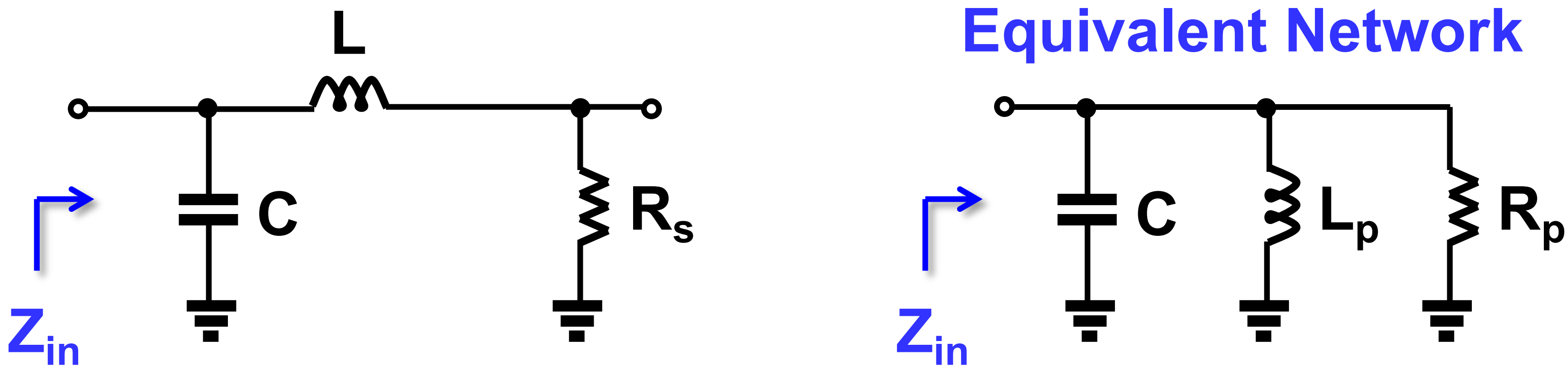
↓

$$\left|Z_{in}\right| = \frac{\left|\overline{V_{in}}\right|}{\left|\overline{I_g}\right|} \approx \frac{Q_L \left|\overline{V_{out}}\right|}{\left|\overline{I_L}\right| / Q_L} = R_s \cdot Q_L^2$$

Upward impedance transformation

$$\boxed{Q_L = \frac{\omega_0 L}{R_s} >> 1}$$

L-match Networks (General Case)



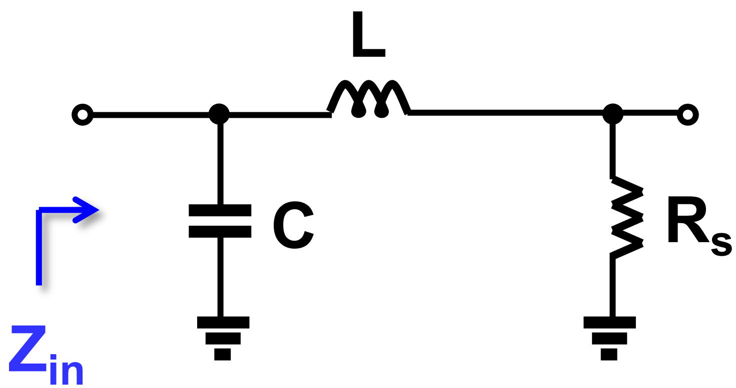
$$\frac{j\omega R_p L_p}{R_p + j\omega L_p} = R_s + j\omega L$$

$$\boxed{Q_L = \frac{\omega L}{R_s}}$$

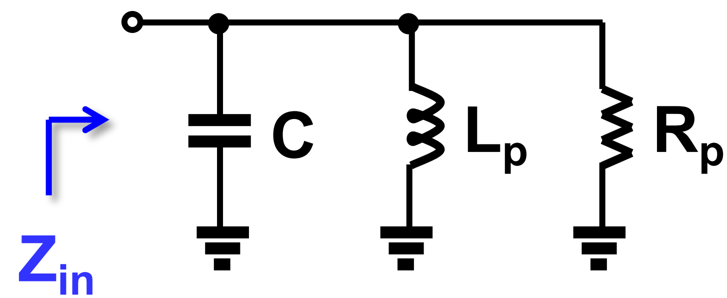
$$\downarrow$$

$$\frac{j\omega R_p L_p}{R_p + j\omega L_p} = R_s\left(1 + jQ_L\right) \rightarrow \begin{cases} \omega L_p R_p = \omega L_p R_s + R_s R_p Q_L \\ 0 = R_s R_p - R_s Q_L \omega L_p \end{cases}$$

L-match Networks (Continued)



Equivalent Network

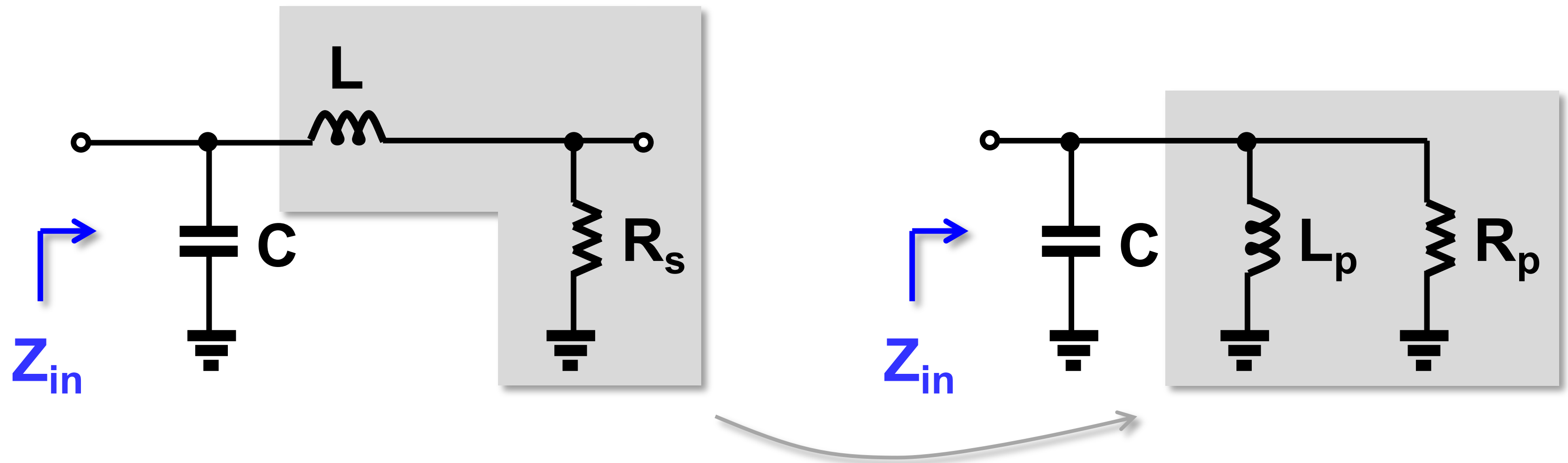


$$\begin{cases} \omega L_p R_p = \omega L_p R_s + R_s R_p Q_L \\ 0 = R_s R_p - R_s Q_L \omega L_p \end{cases} \rightarrow \begin{cases} \dfrac{R_s}{Q_L} + R_s Q_L = \dfrac{R_p}{Q_L} \\ L_p = \dfrac{R_p}{\omega Q_L} \end{cases}$$

$$\rightarrow \begin{cases} R_p = R_s \cdot \left(1 + Q_L^2\right) \\ L_p = L \cdot \dfrac{1 + Q_L^2}{Q_L^2} \end{cases}$$

Note that R_p and L_p in general depends on frequency

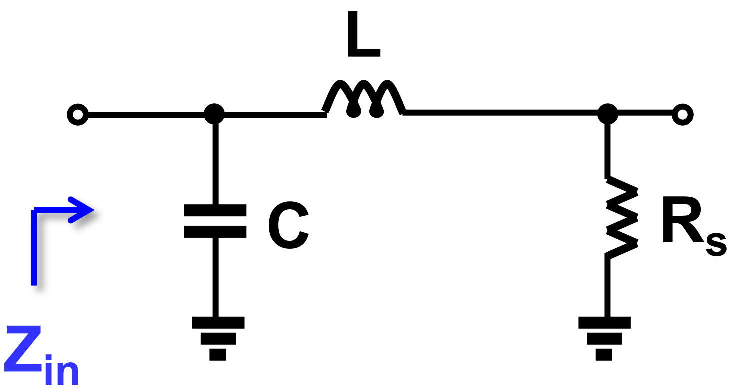
L-match Networks: Series/Parallel Transform

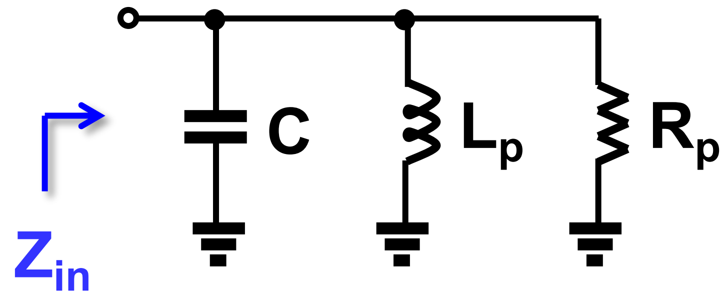


Series-Parallel Transformation

$$\begin{cases} R_p = R_s \cdot \left(1 + Q_L^2\right) \\ L_p = L \cdot \dfrac{1 + Q_L^2}{Q_L^2} \end{cases}$$

L-match Network as Impedance Transformer





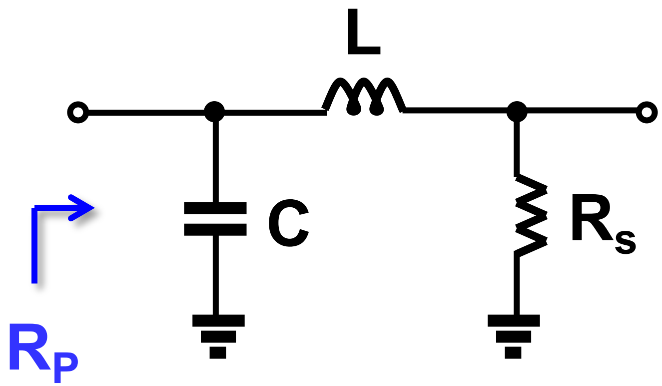
At $\omega_0 = \dfrac{1}{\sqrt{L_p C}}$ $\rightarrow$ $Z_{in}(j\omega_0) = R_p$

$\downarrow$

$$Z_{in}(j\omega_0) = R_s \cdot \left(1 + Q_L^2\right)$$

Upward impedance transformation

L-match Network: Practical Design Rules



Given frequency and transformation ratio:

ω_0, R_P/R_S

1. $$R_P = R_s \cdot \left(1+Q_L^2\right) \rightarrow Q_L = \sqrt{\frac{R_P}{R_S}-1}$$

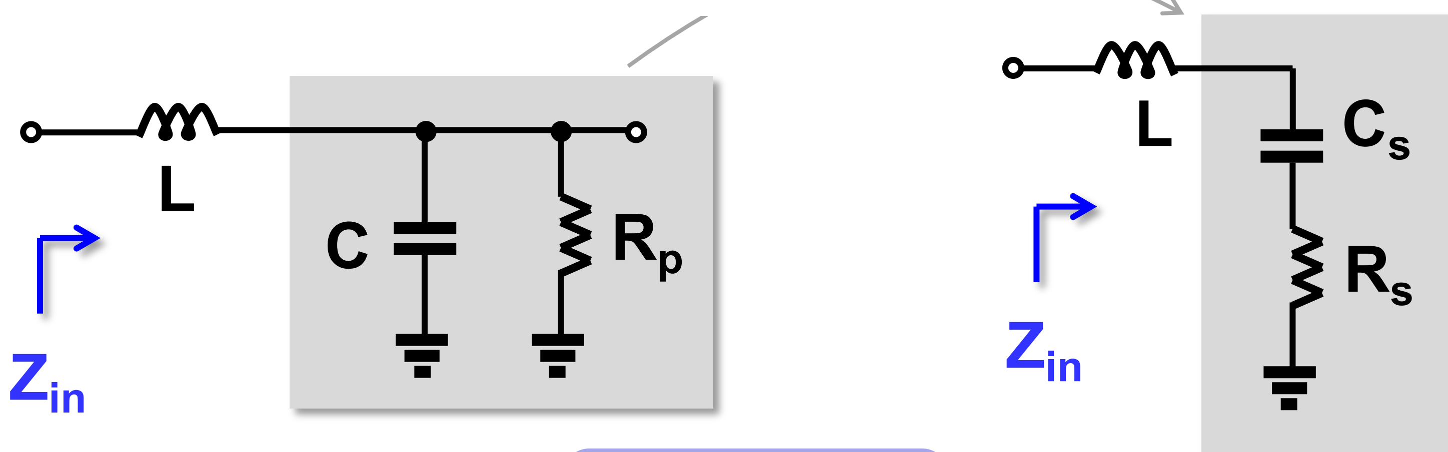
2. $$Q_L = \frac{\omega_0 L}{R_s} \rightarrow L = \frac{Q_L R_S}{\omega_0}$$

3. $$\omega_0 = \frac{1}{\sqrt{L_p C}},\quad L_p = L\cdot\frac{1+Q_L^2}{Q_L^2} \rightarrow C = \frac{Q_L^2}{\omega_0^2 L\left(1+Q_L^2\right)}$$

Large transformation
↓
Narrowband network

Downward L-match network

Parallel-to-Series Transformation



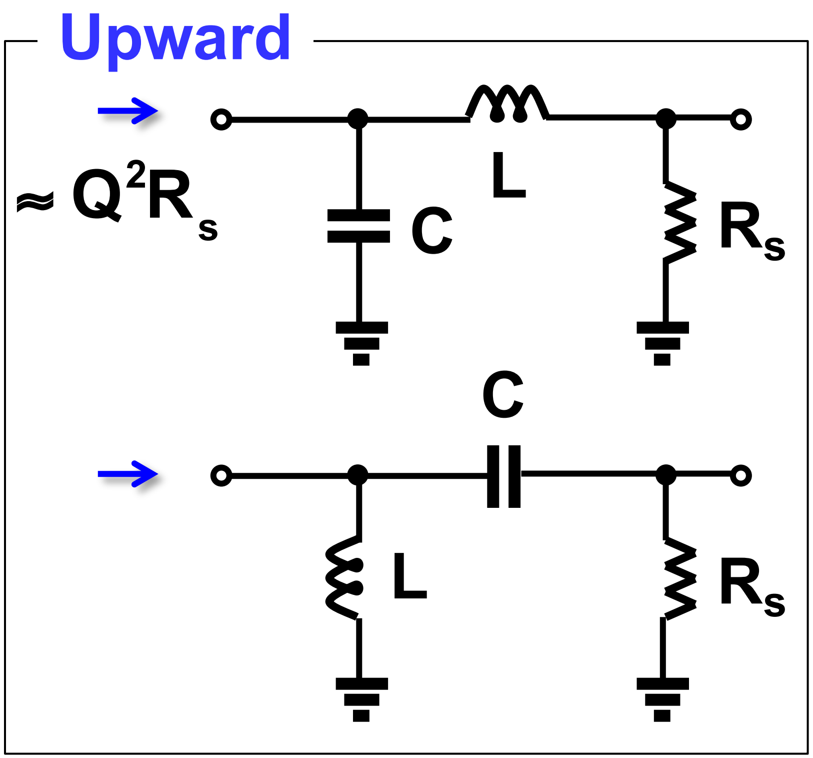
$$\boxed{Q_C = \omega C R_p}$$

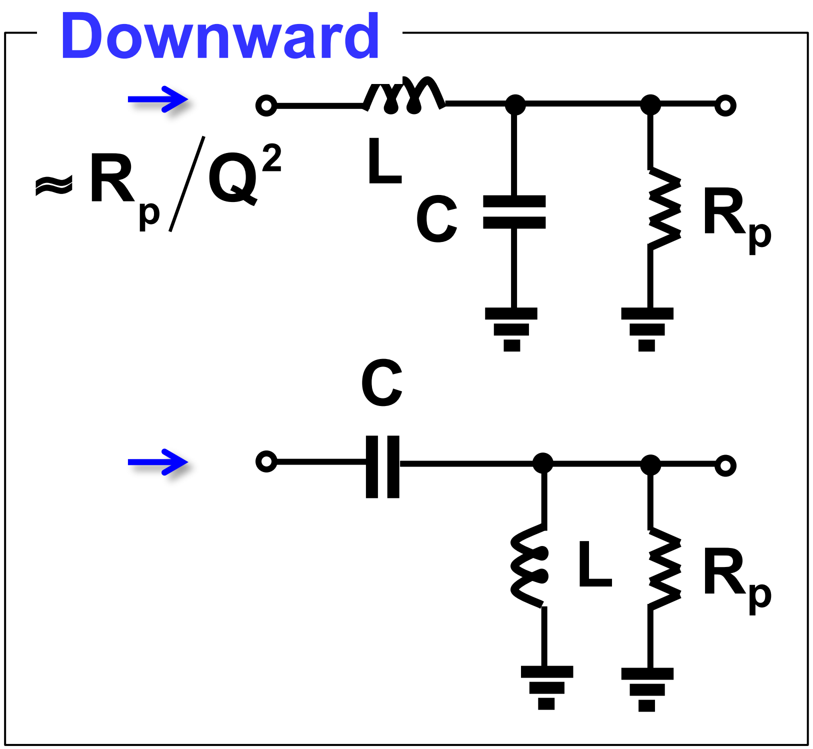
$$\begin{cases} R_s = \dfrac{R_p}{\left(1+Q_C^2\right)} \\ C_s = C \cdot \dfrac{1+Q_C^2}{Q_C^2} \end{cases} \rightarrow \quad \omega_0 = \frac{1}{\sqrt{LC_s}}$$

$$Z_{in}\left(j\omega_0\right) = R_p \Big/ \left(1+Q_C^2\right)$$

Downward transformation

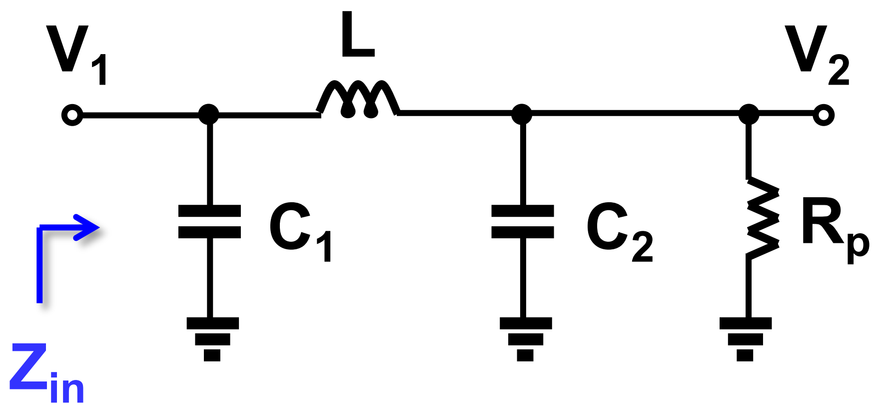
Other L-match networks





- DC blocking
- Absorption of stray capacitances
- Frequency response

Π-match networks: Low Losses



Colpitts Network

Physical Interpretation

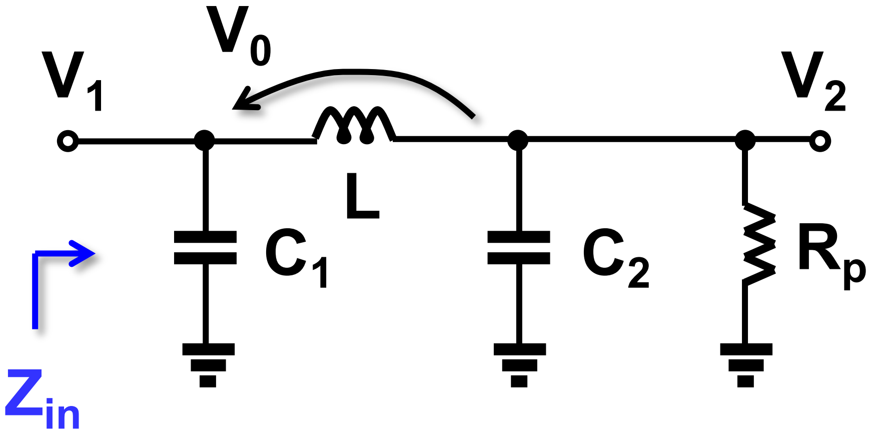
$$sC_1V_1 \cong -sC_2V_2 \quad \Rightarrow \quad \frac{V_1}{V_2} \cong -\frac{C_2}{C_1}$$

Low-losses

Up/Downward transformation

$$\frac{1}{2}\frac{V_1^2}{R_{in}} = \frac{1}{2}\frac{V_2^2}{R_p} \quad \Rightarrow \quad R_{in} = R_p \cdot \frac{V_1^2}{V_2^2} \cong R_p \cdot \left(\frac{C_2}{C_1}\right)^2$$

Π-match networks: Quality factor

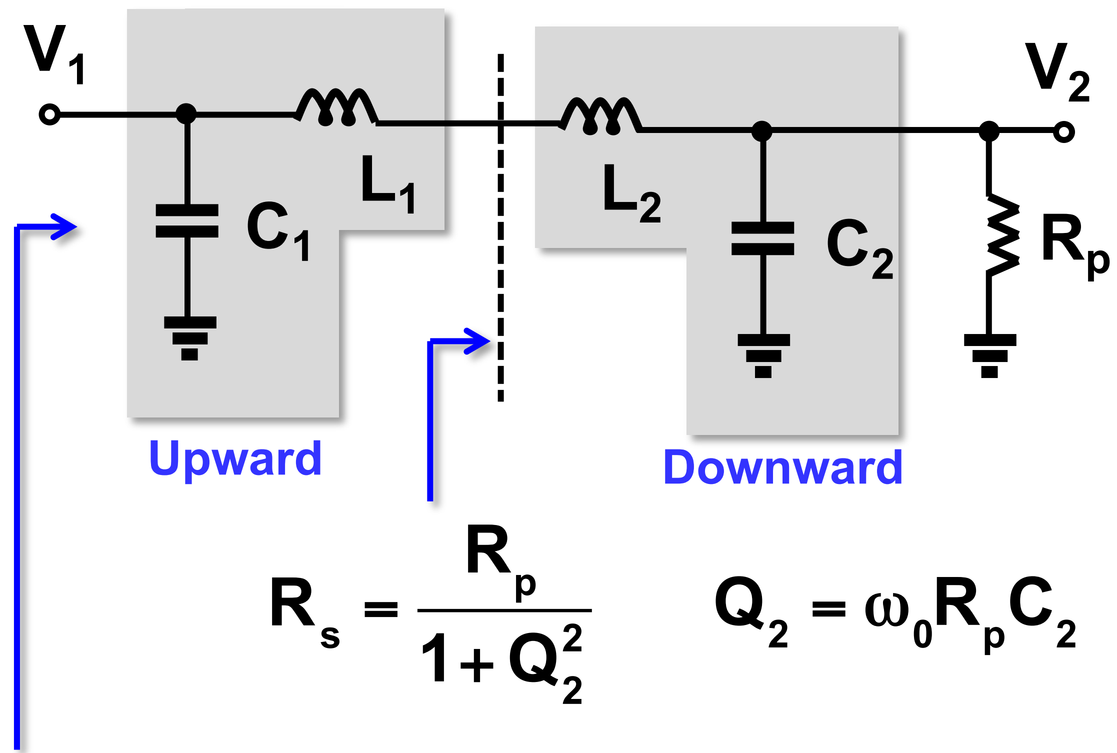


Colpitts Network

$$Q \cong \omega_0 \frac{\frac{1}{2}\frac{C_1 C_2}{C_1+C_2}\left|\overline{V_0}\right|^2}{\frac{1}{2}\left(\frac{C_1}{C_1+C_2}\right)^2 \frac{\left|\overline{V_0}\right|^2}{R_p}} = \omega_0 R_p C_2 \cdot \left(1+\frac{C_2}{C_1}\right)$$

Q factor of Π-match larger than L-match at same transformation ratio

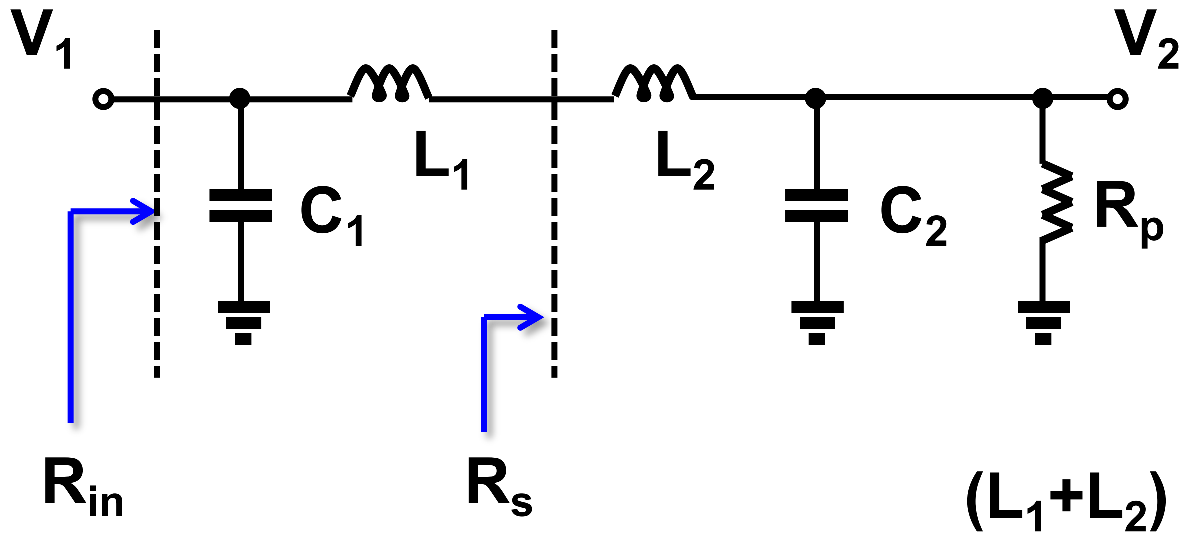
Π-match Network (General Case)



$$R_s = \frac{R_p}{1+Q_2^2} \qquad Q_2 = \omega_0 R_p C_2$$

$$R_{in} = R_s\left(1+Q_1^2\right) = R_p \frac{1+Q_1^2}{1+Q_2^2} \qquad Q_1 = \omega_0 L_1 / R_s$$

Π-match Network: Design Rules (I)



Given frequency, transformation ratio, total Q factor

ω_0, R_{in}/R_p, Q

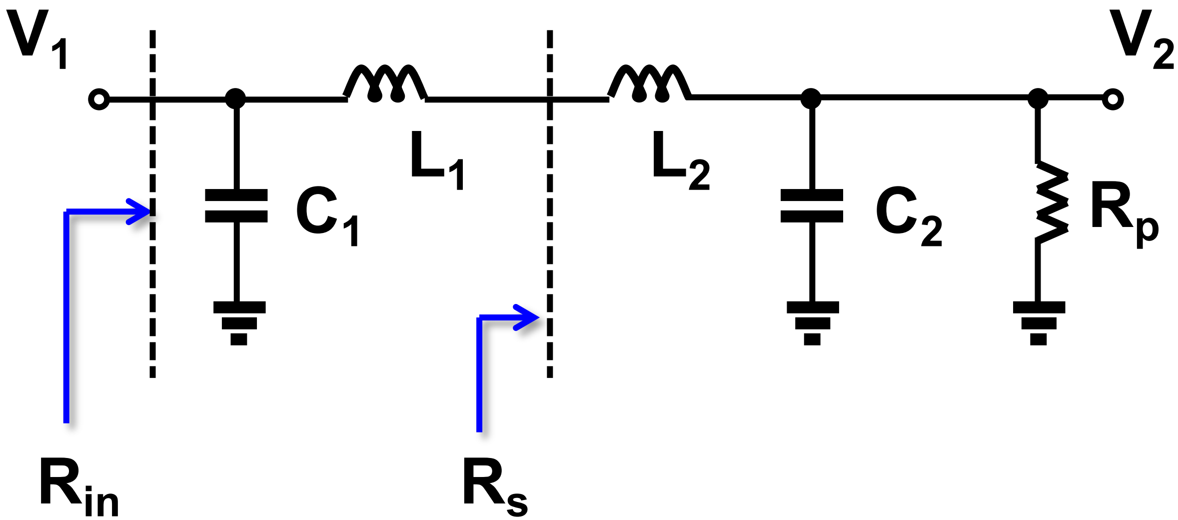
$$Q = \frac{\omega_0 (L_1 + L_2)}{R_s} =$$

2. ↗ (L_1+L_2)

1. ↗ R_s

$$= Q_1 + Q_2 = \sqrt{\frac{R_{in}}{R_s} - 1} + \sqrt{\frac{R_p}{R_s} - 1}$$

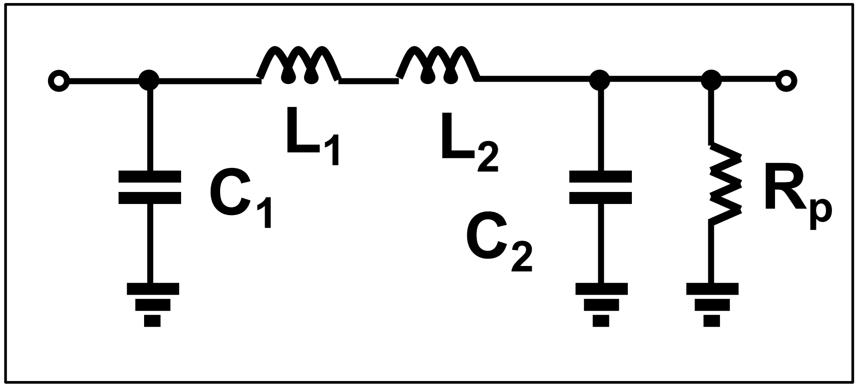
Π-match Network: Design Rules (II)

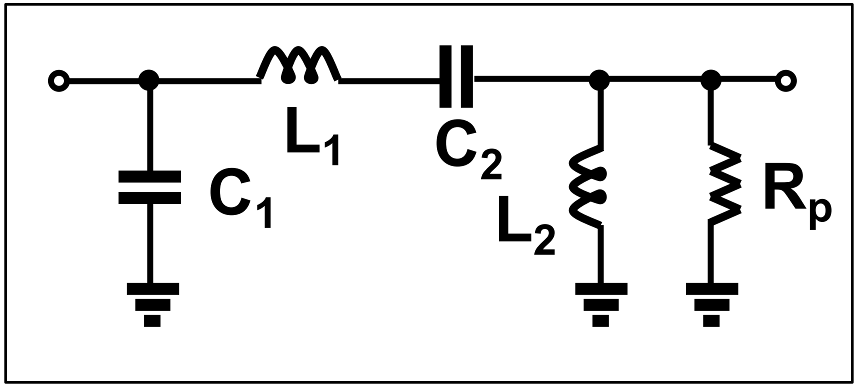


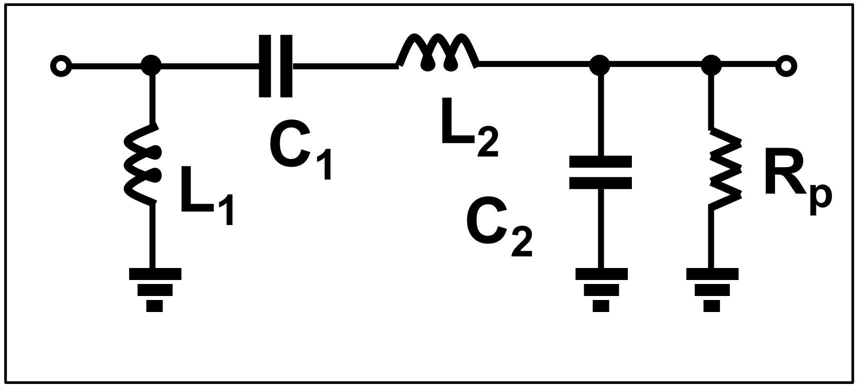
$$Q_2 = \omega_0 R_p C_2 \xrightarrow{3.} C_2 \qquad Q_1 = \omega_0 L_1 / R_s \xrightarrow{4.} L_1$$

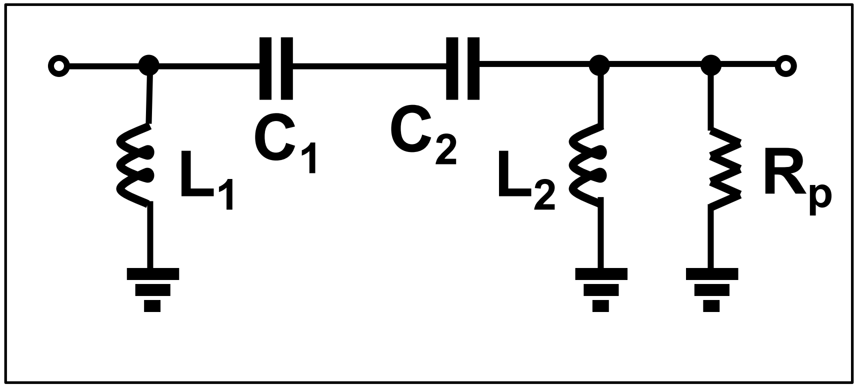
$$\omega_0 = \frac{1}{\sqrt{L_2 C_2 \frac{1+Q_2^2}{Q_2^2}}} = \frac{1}{\sqrt{C_1 L_1 \frac{1+Q_1^2}{Q_1^2}}} \xrightarrow{5.} C_1, L_2$$

Other Π-match Networks

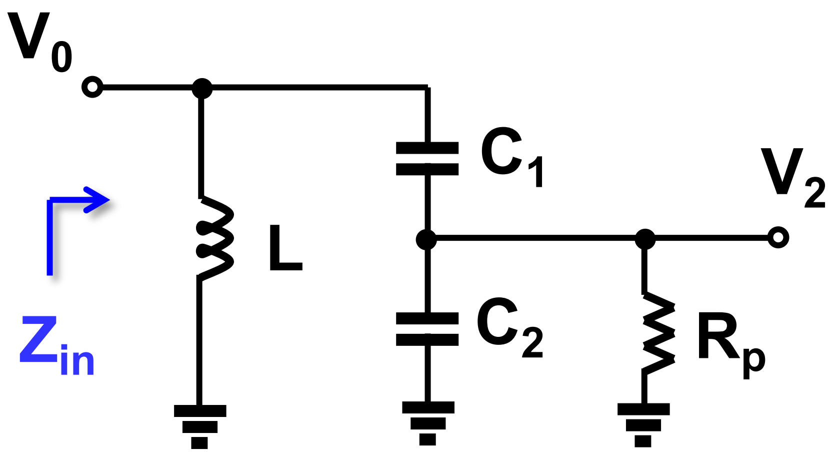








Resonator with Tapped Capacitor (Low Losses)



Colpitts Network

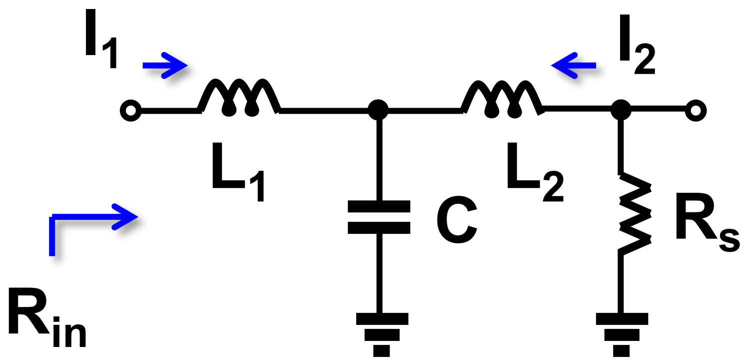
$$\frac{V_2}{V_0} \cong \frac{C_1}{C_1 + C_2}$$

Low-losses

Upward transformation

$$\frac{1}{2}\frac{V_0^2}{R_{in}} = \frac{1}{2}\frac{V_2^2}{R_p} \quad \Rightarrow \quad R_{in} = R_p \cdot \frac{V_0^2}{V_2^2} \cong R_p \cdot \left(1 + \frac{C_2}{C_1}\right)^2$$

T-match Network (Low Losses)

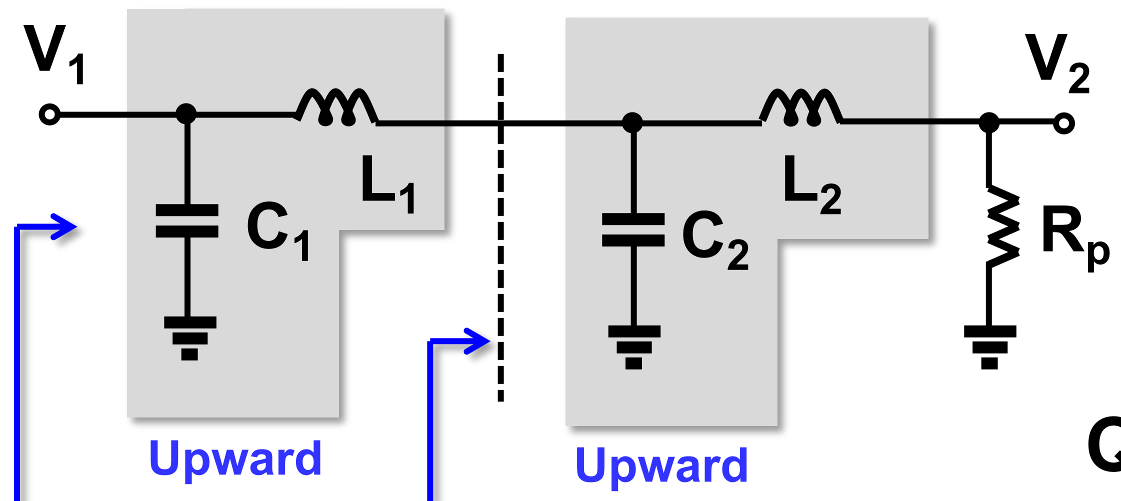


$$sL_1I_1 \cong sL_2I_2 \quad \Rightarrow \quad \frac{I_1}{I_2} \cong \frac{L_2}{L_1}$$

$$\frac{1}{2}I_1^2R_{in} = \frac{1}{2}I_2^2R_s \quad \Rightarrow \quad R_{in} = R_s \cdot \left(\frac{I_2}{I_1}\right)^2 \cong R_s \cdot \left(\frac{L_1}{L_2}\right)^2$$

$$Q \cong \omega_0 \frac{\frac{1}{2}L_1\left|\overline{I_1}\right|^2 + \frac{1}{2}L_2\left|\overline{I_2}\right|^2}{\frac{1}{2}\left|\overline{I_2}\right|^2 R_s} = \frac{\omega_0 L_2}{R_s} \cdot \left(1 + \frac{L_2}{L_1}\right)$$

Cascaded L-Match Networks (General)



$$Q_2 = \frac{\omega_0 L_2}{R_p}$$

$$R_s = R_p\left(1+Q_2^2\right)$$

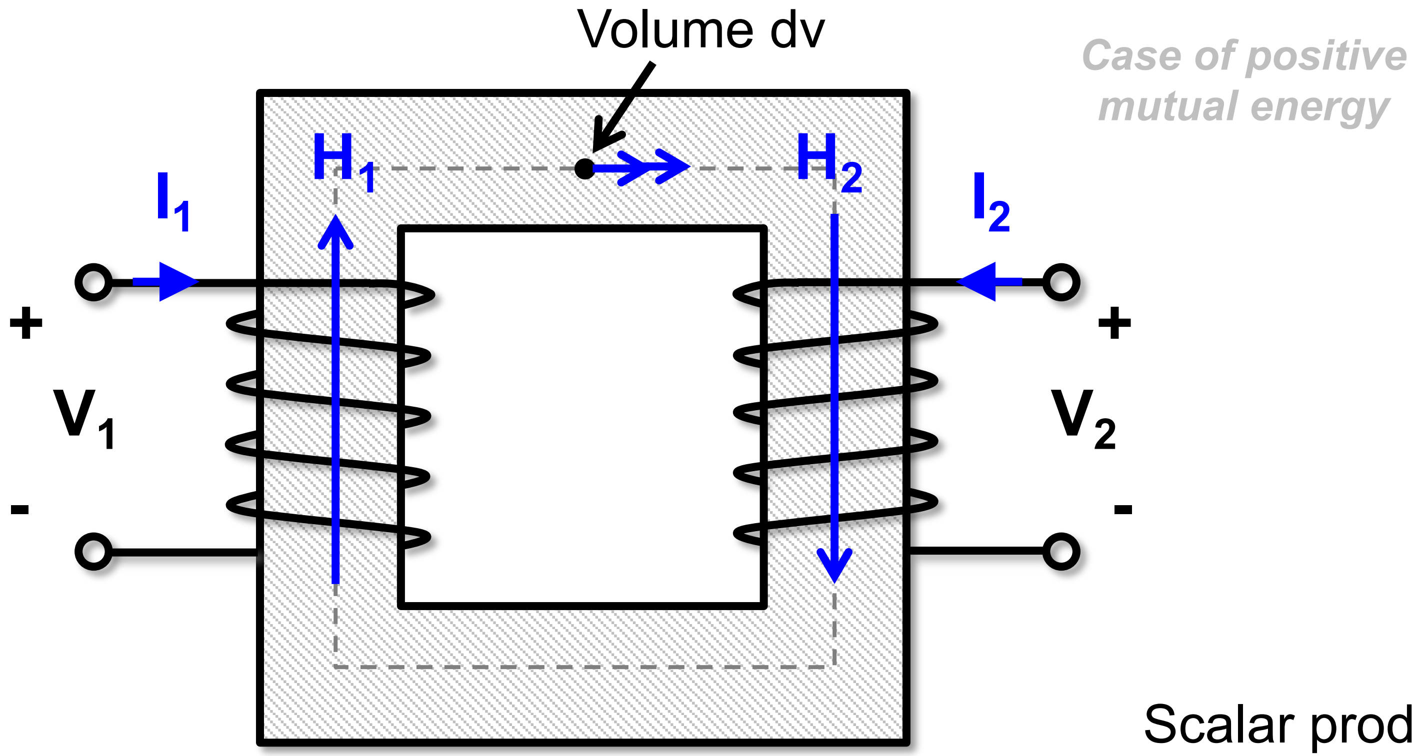
$$Q_1 = \frac{\omega_0 L_1}{R_s}$$

$$R_{in} = R_s\left(1+Q_1^2\right) = R_p\left(1+Q_1^2\right)\left(1+Q_2^2\right)$$

Q factor of cascaded L-match lower than L-match at same transformation ratio

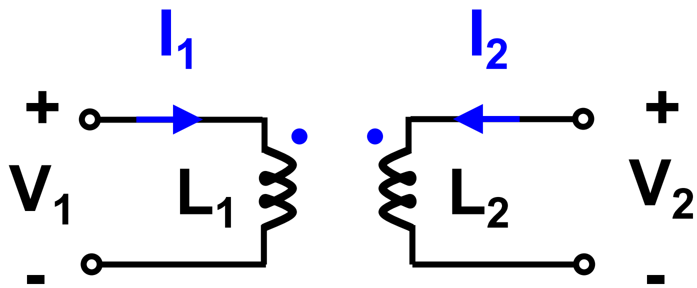
Transformers

Transformers: Physical View



$$\text{Energy} = \frac{\mu}{2}\left|\vec{H}_1 + \vec{H}_2\right|^2 dv = \underbrace{\frac{\mu}{2}\left|\vec{H}_1\right|^2 dv}_{\text{Energy of coil 1}} + \underbrace{\frac{\mu}{2}\left|\vec{H}_2\right|^2 dv}_{\text{Energy of coil 2}} + \underbrace{\mu \vec{H}_1 \cdot \vec{H}_2 dv}_{\text{Mutual Energy}}$$

Transformers: Circuit View



Case of positive mutual energy

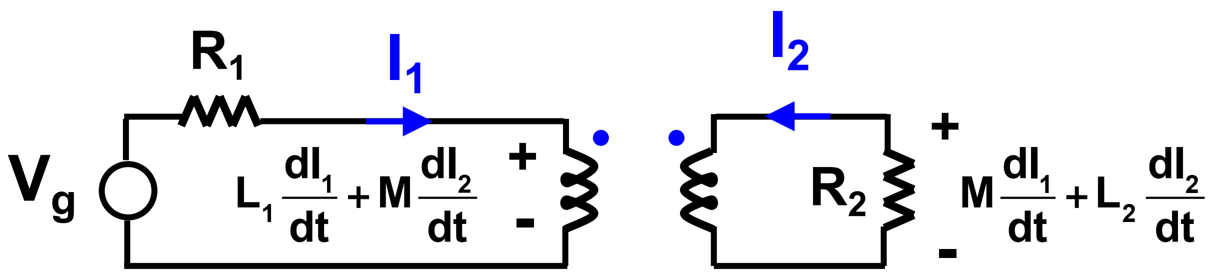
$$\begin{cases} \phi_1 = L_1 I_1 + M I_2 \\ \phi_2 = M I_1 + L_2 I_2 \end{cases} \qquad \begin{cases} V_1 = \dot{\phi}_1 \\ V_2 = \dot{\phi}_2 \end{cases}$$

Passive sign convention

$$\text{Energy} = \int_0^t \left(V_1 I_1 + V_2 I_2\right) dt' = \underbrace{\frac{1}{2} L_1 I_1^2(t)}_{\text{Energy of coil 1}} + \underbrace{\frac{1}{2} L_2 I_2^2(t)}_{\text{Energy of coil 2}} + \underbrace{M \cdot I_1(t) I_2(t)}_{\text{Mutual Energy}}$$

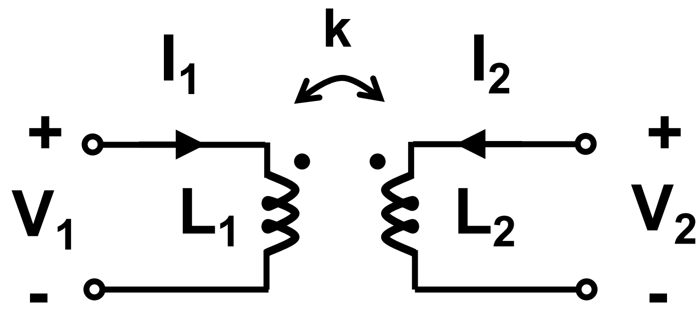
- **Sign of mutual energy depends on the sign of M and on the sign of $I_1 I_2$**
- **Dot convention: M is positive if both currents enter or leave the dotted terminal**

Transformers: More on dot convention



- If a current **enters** the dotted terminal of a coil, the reference polarity of the voltage induced in the other coil is **positive** at its dotted terminal.
- If a current **leaves** the dotted terminal of a coil, the reference polarity of the voltage induced in the other coil is **negative** at its dotted terminal.

Coupling Coefficient



$$\begin{cases}\phi_1 = L_1 I_1 + M I_2 \\ \phi_2 = M I_1 + L_2 I_2\end{cases}$$

Mutual Inductance

Self-Inductance

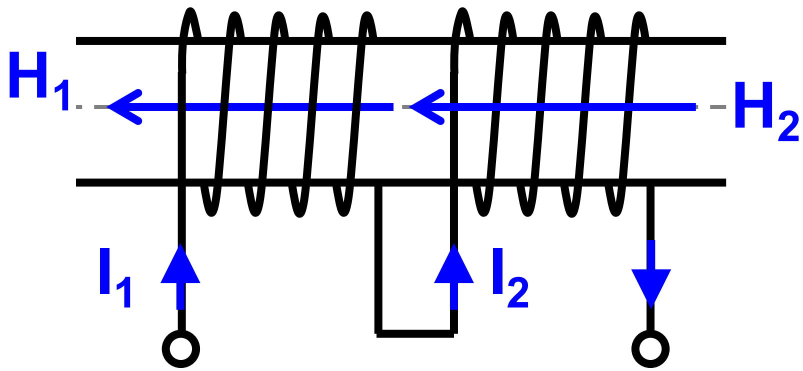
- Definition of coupling coefficient

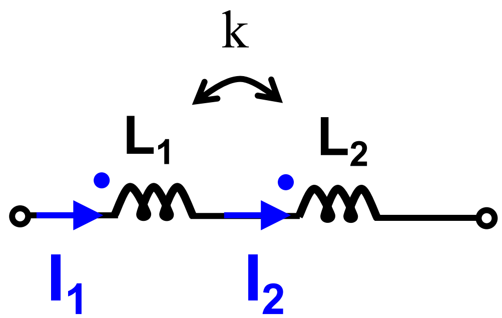
$$k = \frac{|M|}{\sqrt{L_1 L_2}}$$

- Conservation of energy implies that

$$0 \leq k \leq 1$$

Example: Series of Two Coupled Inductors (I)





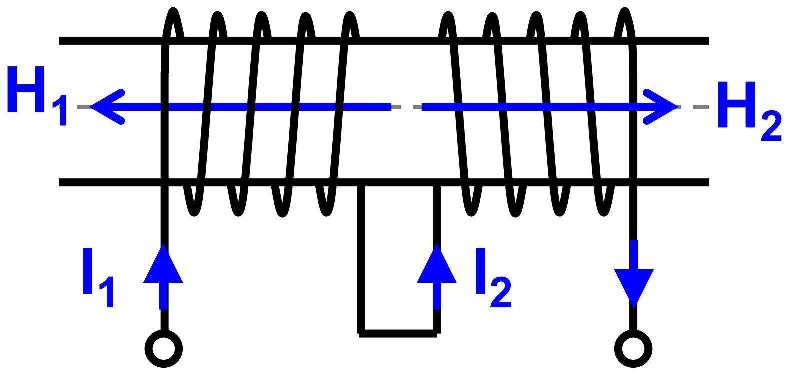
Case of positive mutual energy (M is positive)

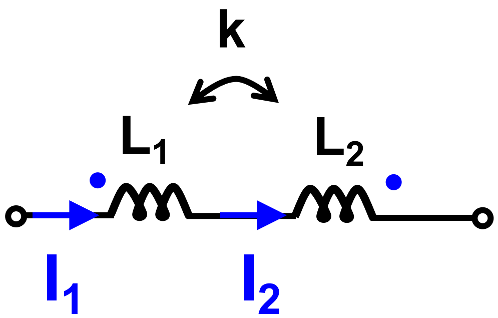
$$\phi = \phi_1 + \phi_2 = L_1 I_1 + M I_2 + M I_1 + L_2 I_2 = \underbrace{\left(L_1 + L_2 + 2M\right)}_{\text{Total Inductance } L_{tot}} I$$

- **If $L_1 = L_2 = L$:**

$$L_{tot} = 2\left(1 + k\right)L$$

Example: Series of Two Coupled Inductors (II)





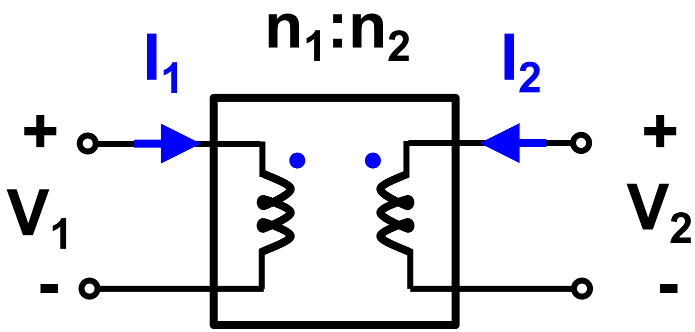
Case of negative mutual energy (M is negative)

$$\phi = \phi_1 + \phi_2 = L_1 I_1 - |M| I_2 - |M| I_1 + L_2 I_2 = \underbrace{\left(L_1 + L_2 - 2|M|\right)}_{\text{Total Inductance } L_{tot}} I$$

- **If $L_1 = L_2 = L$:**

$$L_{tot} = 2(1-k)L$$

Ideal Transformer



(*) Magnetomotive Force (m.m.f.)

$$\mathcal{F} = \Phi \cdot \mathcal{R}_m = \frac{\Phi}{\Lambda}$$

Reluctance (H⁻¹) ***Permeance (H)***

i. No flux dispersion

(k = 1)

$$\begin{cases} \phi_1 = n_1 \phi \\ \phi_2 = n_2 \phi \end{cases}$$

$$\Downarrow$$

$$\frac{V_1}{V_2} = \frac{n_1}{n_2}$$

ii. Infinite self-inductance

$(L_1, L_2 \rightarrow \infty)$

$$\text{m.m.f.} = n_1 I_1 + n_2 I_2 = \frac{\Phi}{\Lambda} = 0$$

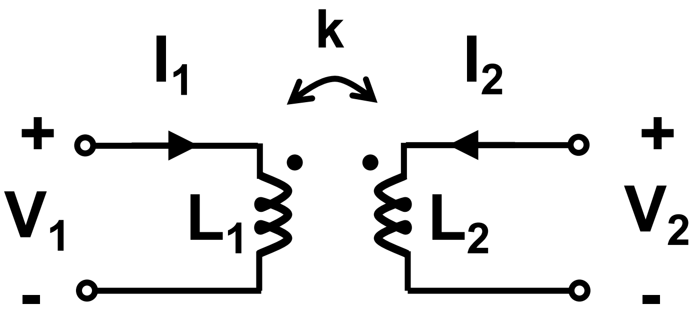
Ampere's Law ***Hopkinson's Law (*)***

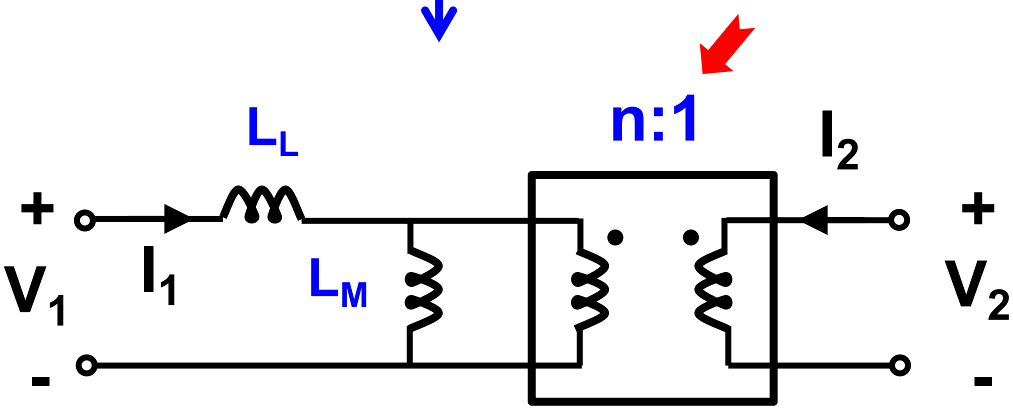
$$\Downarrow$$

$$\frac{I_1}{I_2} = -\frac{n_2}{n_1}$$

Lossless

Equivalent Model of Coupled Inductors (I)





Equivalent Model

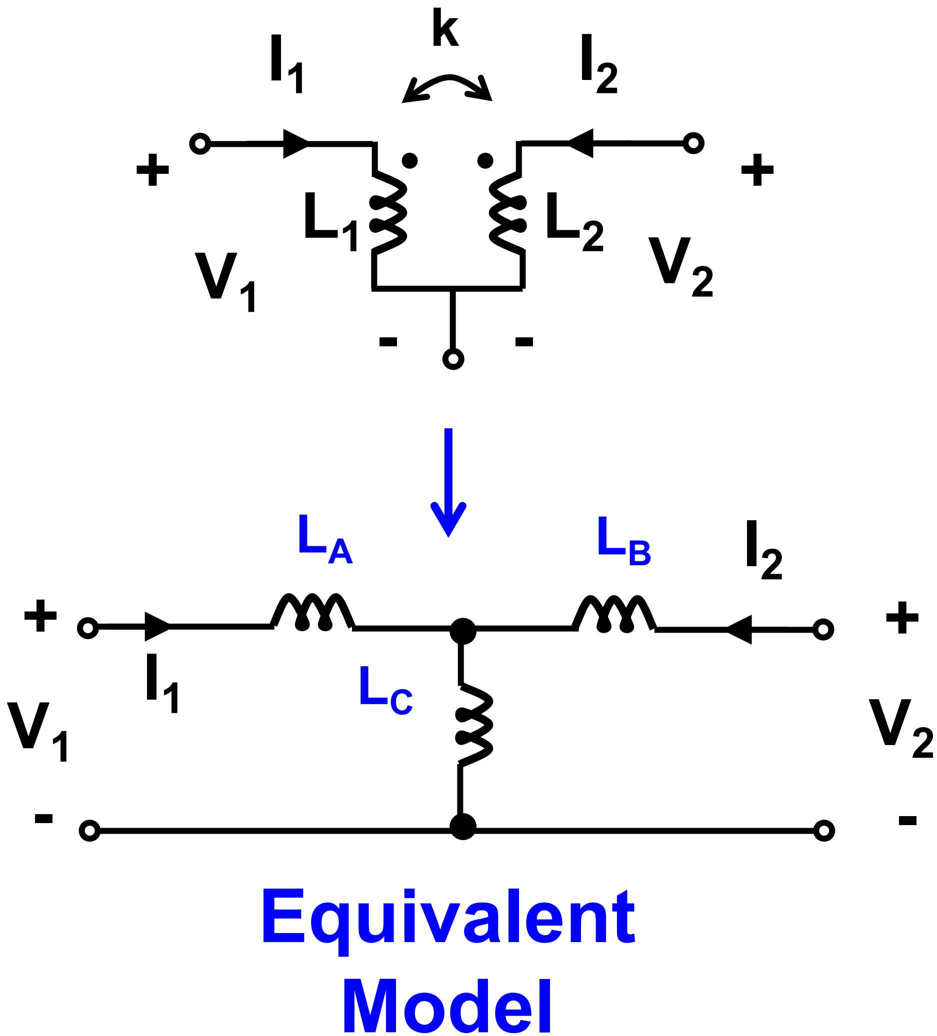
Leakage Inductance

Magnetizing Inductance

$$\begin{cases} L_L = \left(1-k^2\right)L_1 \\ L_M = k^2 L_1 \\ n = k\sqrt{L_1/L_2} \end{cases}$$

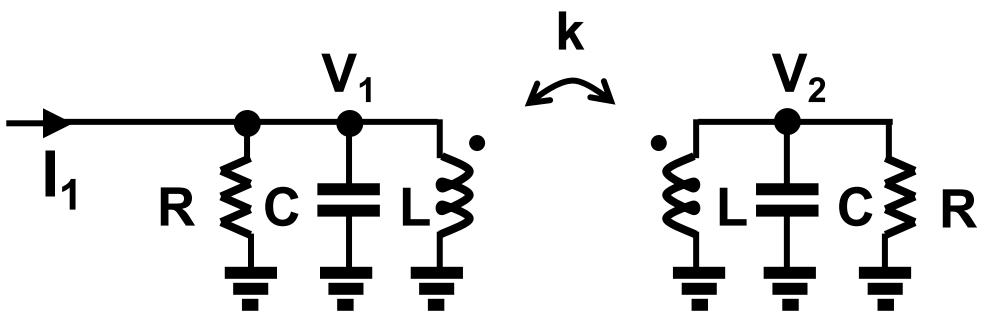
Turns Ratio

Equivalent Model of Coupled Inductors (II)



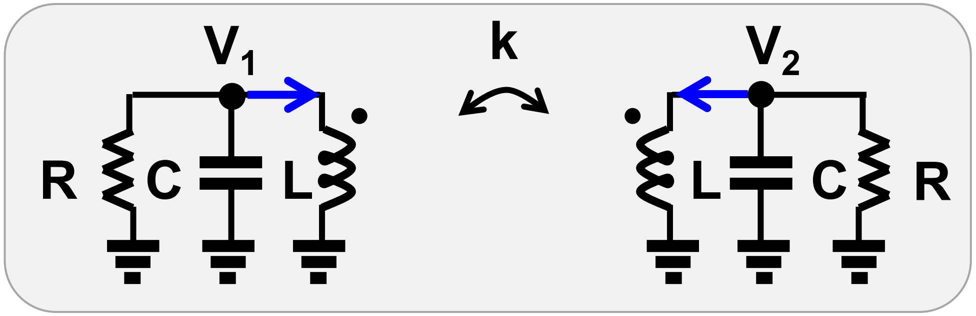
$$
\begin{cases}
L_A = L_1 - M \\
L_B = L_2 - M \\
L_C = M
\end{cases}
$$

Example: Coupled Resonators

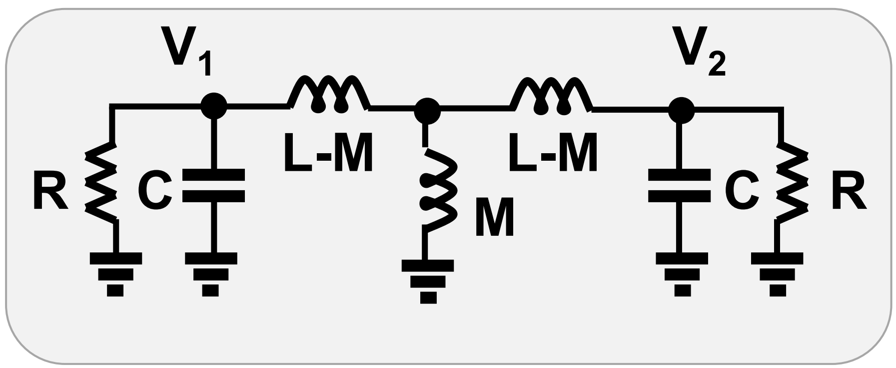


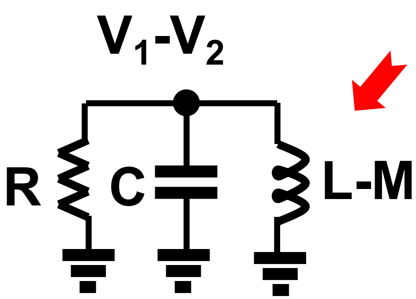
- **Calculate the network resonant frequency**
- **Calculate the frequency response of V_2/I_1**

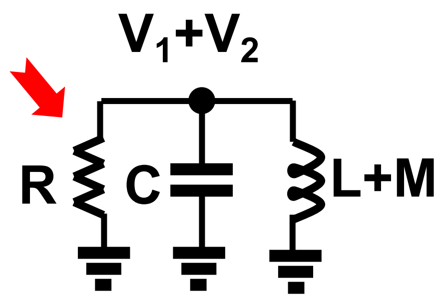
Example: Coupled Resonators (Solution)



M = kL



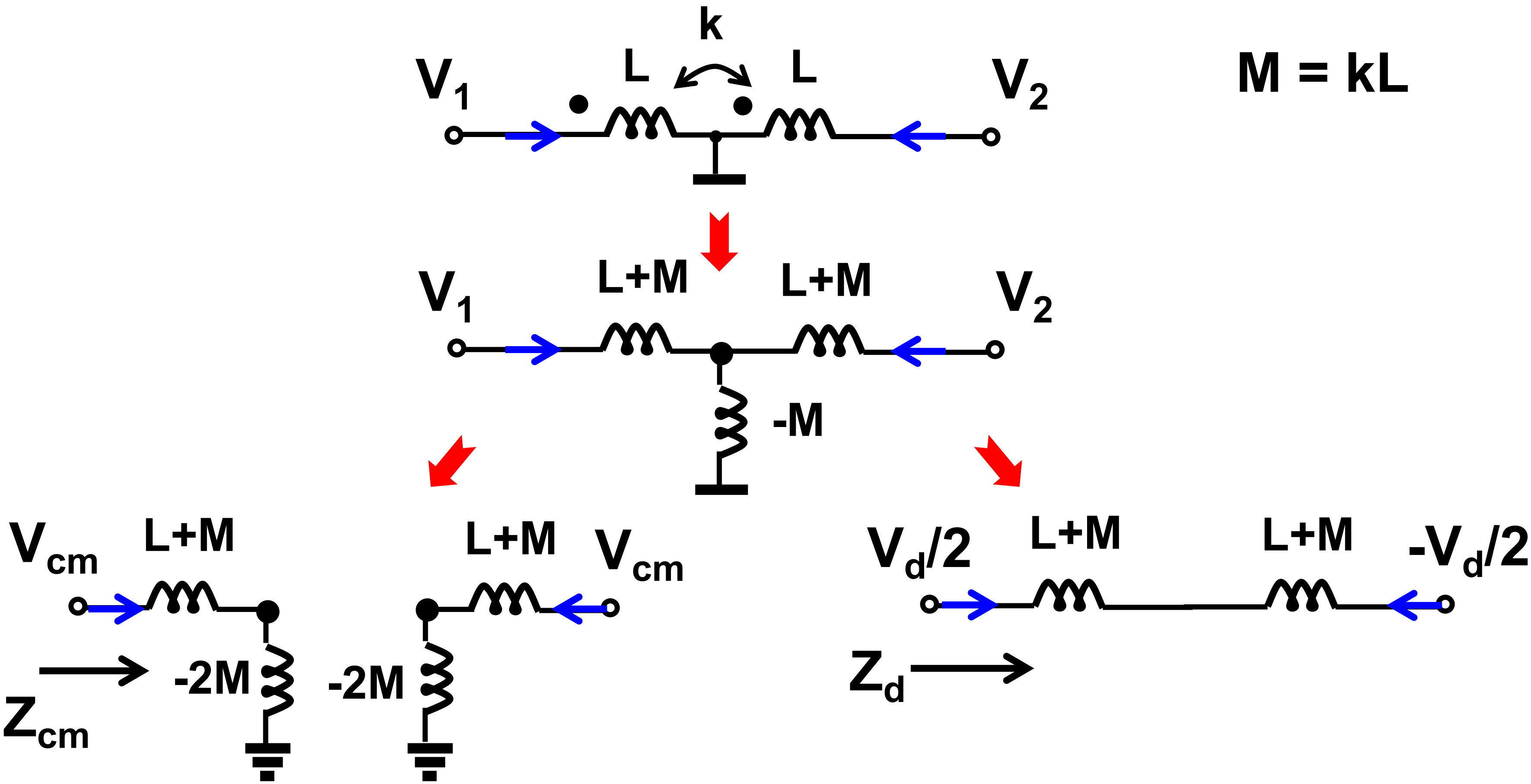




$$\omega_{diff.} = \frac{1}{\sqrt{(L-M)C}}$$

$$\omega_{com.} = \frac{1}{\sqrt{(L+M)C}}$$

Example: Common-Mode Killer



If k = 1, M = L: $Z_{cm} = 0$, $Z_d = j\omega 4L$

The circuit "kills" the common-mode voltage