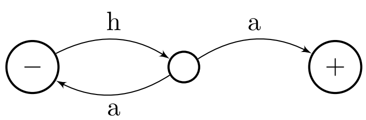
1 Finite State Automata

1.1 Alphabets & Strings

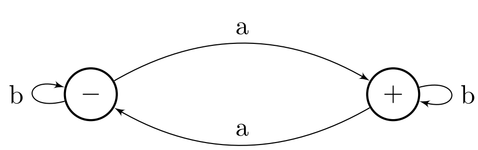
- Let A be a set; then A^n is the set of all finite sequences $a_1 \dots a_n$ with $a_i \in A$, $1 \leq i \leq m$
  - Elements of A are *letters* or *symbols*
  - Elements of A^n are *words* or *strings* over A of length m
- ε is the special *empty string*, the only string of length 0
- $A^+ = \bigcup_{m \geq 1} A^m$ – the set of non-empty strings over A of any length
- $A^* = A^+ \cup \varepsilon = \bigcup_{m \geq 0} A^m$ – the set of (possibly empty) strings over A of any length
- If $\alpha = a_1 \dots a_m$, $\beta = b_1 \dots b_m \in A^*$, then define $\alpha\beta$ to be $a_1 \dots a_m b_1 \dots b_m \in A^{m+n}$. This gives binary 'product' or *concatenation* on A^*
- For $\alpha \in A^+$, define $\alpha^n, n \in \mathbb{N}$ by $\alpha^0 = \varepsilon$, and $\alpha^{n+1} = \alpha^n \alpha$
- A *language* with alphabet A is a subset of A^*

1.2 Definition of an FSA

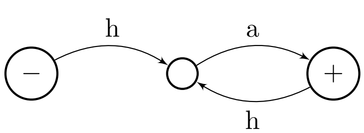
- A Finite State Automaton (FSA) is a tuple $M = (Q, F, A, \tau, q_0)$
  - Q is a finite set of states
  - $F \subseteq Q$ is the set of final states
  - A is the alphabet
  - $\tau \subseteq Q \times A \times Q$ is the set of transitions
  - $q_0 \in Q$ is the initial state
- The transition diagram of an FSA is a directed graph with:
  - Vertex set Q
  - An edge for each transition; $(q, a, q') \in \tau$ corresponds to an edge from q to q' with label a
  - Initial state q_0 labelled with $-$
  - Final states labelled with $+$
  - Example: a *non-deterministic* 'haha machine', with $A = \{h, a\}$



- A *computation* of M is a sequence $q_0, a_1, q_1, a_2, \ldots, a_n, q_n$ with $n \geq 0$ where $(q_i, a_{i+1}, q_{i+1}) \in \tau$ for $0 \leq i \leq n - 1$
  - The *label* on the computation is $a_1 \ldots a_m$
  - The computation is *successful* if $q_n \in F$
  - A string $a_1 \ldots a_n$ is *accepted* by M if there is a successful computation with label $a_1 \ldots a_n$, and it is *rejected* otherwise
- The language recognised by M is $\mathcal{L}(M) = \{w \in A^* \mid w \text{ is accepted by } M\}$
- There is a one-to-one correspondence between computations of M and paths in the graph from q_0
- Example: $A = \{a, b\}$ of an FSA accepting only words with an odd number of 'a's



- An FSA is deterministic (a DFA) if for all $q \in Q, a \in A$ there is exactly one $q' \in Q$ such that $(q, a, q') \in \tau$
- Example: DFA for the 'haha machine'



- Note this machine lacks a transition for a when in the initial state – though technically required for a DFA, it is easily fixed by adding an 'error state' to catch what would otherwise be missing transitions

1.3 Deterministic FSAs

- For a DFA M, define the transition function $\delta : Q \times A \to Q$ by $q' = \delta(a, q)$, where q' is the unique element such that $(q, a, q') \in \tau$
- If $\mathcal{L}$ is a language with alphabet A, then the following are equivalent:
  1. $\mathcal{L}$ is recognised by an FSA
  2. $\mathcal{L}$ is recognised by a DFA
- Given a non-deterministic FSA $M = (Q, F, A, \tau, q_0)$, an equivalent DFA $M' = (Q', F', A, \tau', q_0')$ may be generated by the *powerset method*:
  - $Q' = \mathcal{P}(Q) \setminus \emptyset$ (i.e. the set of all subsets of Q that aren't empty)
  - $F' = \{X \in Q' \,|\, q \in X \text{ for some } q \in F\}$
  - For $X \in Q', a \in A$, define $\delta(X, a) := \{q \in Q \,|\, (x, a, q) \in \tau \text{ for some } x \in X\}$
  - $\tau' = \{(X, a, \delta(X, a)) \,|\, X \in Q', a \in A\}$
  - $q_0' = \{q_0\}$
- Proof: show that $\mathcal{L}(M) = \mathcal{L}(M')$
  - $\mathcal{L}(M) \subseteq Lang(M')$:
    - Given $w \in \mathcal{L}(M)$, $q_0 a_1 \dots a_n q_n$ is a successful computation of M
    - Then define $q_i' = \delta(q_{i-1}', a_i)$ for $1 \leq i \leq n$
    - $q_0', a_1, q_1' \dots a_n, q_n'$ will be a successful computation of M'
    - Therefore $w \in \mathcal{L}(M')$
  - $\mathcal{L}(M') \subseteq Lang(M)$:
    - Let $w = a_1 \dots a_n \in L(M')$, and $q_0', a_1, q_1' \dots a_n, q_n'$ be a successful computation of M
    - Each q_i' cannot be the empty set
    - By definition of τ', $\exists q_1 \in q_1'$ s.t. $(q_0, a_1, q_1) \in \tau$
    - Then we can find $q_i \in q_i'$ s.t. $(q_{i-1}, a_i, q_i) \in \tau$ for $1 \leq i \leq n$
    - For q_n we further require $q_n \in F$
    - Therefore, $q_0, a_1, q_1, a_2, \dots a_n, q_n$ is a successful computation
    - Therefore $w \in \mathcal{L}(M)$

1.4 The Pumping Lemma

- The Pumping Lemma says that for any $\mathcal{L}$ recognised by an FSA M, there is a certain word length beyond which all words can be split into sections as xyz, where xy^nz is also in the language
- Formally there is an integer $p > 0$ s.t. any word $w \in L$ with $|w| \geq p$ is of the form $w = xyz$, where $|y| > 0$, $|xy| \leq p$ and $xy^iz \in \mathcal{L}$ for $i \geq 0$
- Proof:
  - Let p be the number of states in M, and suppose $w = a_1 \dots a_n \in \mathcal{L}$, where $n \geq p$
  - A successful computation $q_0, a_1, \dots, q_n$ has to pass through a certain state at least twice (by the pigeonhole principle)
  - Therefore, $\exists r < s$ s.t. $q_r = q_s$; choose minimal such s
  - Now put $x = a_1 \dots a_r$, $y = a_{r+1} \dots a_s$ (note $|y| > 0$), and $z = a_{s+1} \dots a_n$
  - By minimality of s, $q_0, \dots q_{s-1}$ are distinct, and $|xy| = s \leq p$
  - Then, note that $q_r, a_{r+1}, \dots, q_s$ is a loop, which may be validly repeated $i \geq 0$ times
  - Therefore, $xy^iz \in \mathcal{L}$
- Corollary: here exist languages which are not computable by an FSA
- Example: there is no FSA which can recognise $\mathcal{L} = \{a^nb^n \mid n \in \mathbb{N}\}$
- Proof:
  - Assume for a contradiction there exists an FSA M which can recognise $\mathcal{L}$
  - Let p be the number from the pumping lemma, and choose $n \geq p$ and consider $w = a^nb^n$
  - By the pumping lemma, $\exists x, y, z$ s.t. $a^nb^n = xyz$, with $|y| \geq 1$ and $|xy| \leq p \leq n$
  - Then y is written entirely in terms of the letter a, and $|y| \geq 1$
  - By the pumping lemma, $xy^iz \in \mathcal{L}$ for all i
  - So choose $i = 0$, then some $w = a^kb^n \in \mathcal{L}$ s.t. $k < n$, which is a contradiction

2 Turing Machines

2.1 Definition

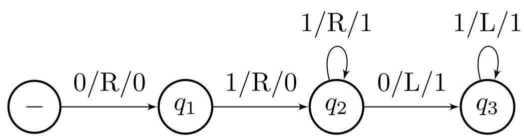
- A Turing machine is a tuple $T = (Q, F, A, I, \tau, q_0)$
  - Q is a finite set of states
  - $F \subseteq Q$ is the set of final states
  - A is a finite set, the tape alphabet, with a distinguished blank symbol $B \in A$
  - I is a subset of $A \setminus \{B\}$, the input alphabet
  - $\tau \subseteq Q \times A \times Q \times A \times \{L, R\}$ is the set of transitions
  - $q_0 \in Q$ is the initial state
- As in an FSA, non-determinism is allowed
- The tape is infinite in both directions, but only ever contains a finite number of non-blank symbols
- A *tape description* for T is a triple (a, α, β) with $a \in A$, and $\alpha : \mathbb{N} \to A$ and $\beta : \mathbb{N} \to A$ being functions with $a(n) = B$ and $B(n) = B$ for all but finitely many $n \in \mathbb{N}$
  - So the tape looks like: $\dots BBB\beta(l)\beta(l-1)\dots\beta(0)\underline{a}\alpha(0)\alpha(1)\dots\alpha(r)BBB\dots$, with $l, r \in \mathbb{N}$
- A *configuration* of T is a tuple (q, a, α, β) where $q \in Q$ and (a, α, β) is a tape description
- If $c = (q, a, \alpha, \beta)$ is a configuration, a configuration c' is obtained (reachable) from c by a single move if one of the following holds:
  - $(q, a, q', a', L) \in \tau$ and $c' = (q', \beta(0), \alpha', \beta')$ where: $\alpha'(0) = a', \alpha'(n) = \alpha(n-1), n > 0$ and $\beta'(n) = \beta(n+1), n \geq 0$, or
  - $(q, a, q', a', R) \in \tau$ and $c' = (q', \alpha(0), \alpha', \beta')$ where: $\alpha'(n) = \alpha(n+1), n \geq 0$ and $\beta'(0) = a', \beta'(n) = \beta(n-1), n > 0$
- A *computation* of T is a finite sequence of configurations $c_1, \dots, c_n = c'$ where $n \geq 1$ and c_{i+1} is obtained from c_i by a single move, for $1 \leq i \leq n-1$
- A configuration is *terminal* if no configuration is reachable from it
- A computation halts if c' is terminal (i.e. there is no configuration reachable from c')
- We may write $c \underset{T}{\to} c'$ if there is a computation starting at c and ending at c'

2.2 Turing Machine as Language Recogniser

- For $w = a_1 \dots a_n \in A^*$, let $c_w = (a_0, \underline{a_1} \dots a_n)$ (recall $\underline{a_1} \dots a_n$ is a tape description (a, α, β))
- If $w = \varepsilon$, we put $c_w = (q_0, \underline{B})$
- The TM T *accepts* if $c_w \underset{T}{\to} c'$ for some $c' = (q, a, \alpha, \beta)$ with $q \in F$
- The language recognised by T is $\mathcal{L}(T) = \{w \in I^* \mid w \text{ is accepted by } T\}$
- Note that $\mathcal{L}(T)$ is a language over I rather than over A
- T is deterministic if for every $(q, a) \in Q \times A$ there is *at most one* element of τ starting with (q, a)
- Then, there is at most one config c' obtained from c by a single move; set $\delta(c) = c'$
- $\delta : C \to C$ is then a partial function

2.3 Numerical Turing Machines: TMs as Function Calculators

- We want to use TMs to describe a partial function $f : \mathbb{N}^n \to \mathbb{N}$
- A *numerical TM* is a deterministic TM $T = (Q, F, A, I, \tau, q_0)$ with:
  - $F = I = \emptyset$
  - $A = \{0, 1\}$, with 0 as the blank symbol
- In a numerical TM, the final states F and input alphabets I are not relevant
- For $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{N}^n$, define the tape description $Tape(\mathbf{x}) = \underline{0}1^{x_1}01^{x_2}0\ldots01^{x_n}$
- Define the partial function $\varphi_{T,n} : \mathbb{N}^n \to \mathbb{N}$ as follows:
  - Let $\mathbf{x} \in \mathbb{N}^n$ be given
  - The initial config of T is $(q_0, Tape(\mathbf{x}))$
  - If T halts with tape $\underline{0}1^y = Tape(y)$ for some $y \in \mathbb{N}$, then $\varphi_{T,n}(\mathbf{x}) = y$
  - Otherwise, $\varphi_{T,n}$ is undefined
- If $f : \mathbb{N}^n \to \mathbb{N} = \varphi_{T,n}$ for some numerical TM T, then f is *TM computable*
- Note that when considering TMs as language recognisers, halting is regarded as an error – but for a numerical TM, it is fine *so long as* it ends with a configuration of the form $(q, \underline{0}1^y)$ with $y \in \mathbb{N}$
- Example: an addition function $S : \mathbb{N}^2 \to \mathbb{N}$



- Ultimate theorem: All TM computable functions are partial recursive, and conversely all partial recursive functions are TM computable

3 Partial Recursive Functions

3.1 Partial Functions, Definition by Composition & Primitive Recursion

- Classes of functions:
  - Let P be the set of partial functions, $P = \{f \mid f \text{ is a partial function } \mathbb{N}^n \to \mathbb{N} \text{ for some } n > 0\}$
  - Let T be the set of total functions, $T = \{f \in P \mid f \text{ is total}\}$
  - A *class* of functions means a subset of P, and a class of total functions means a subset of T
  - Goal: build a class of functions which we might call 'computable'
- Let $g : \mathbb{N}^r \to \mathbb{N}, h_1 \dots h_r : \mathbb{N}^n \to \mathbb{N}$ be partial functions.

  Then the partial function $f : \mathbb{N}^n \to \mathbb{N}$ obtained from $g, h_1, \dots, h_r$ by composition is defined by:

  $$f(\mathbf{x}) = g(h_1(\mathbf{x}), \dots, h_r(\mathbf{x}))$$

  - We write $f = g \circ (h_1, \dots, h_r)$
- Let $g : \mathbb{N}^n \to \mathbb{N}, h : \mathbb{N}^{n+1} \to \mathbb{N}$ be partial functions.

  Then the partial function $f : \mathbb{N}^{n+1} \to \mathbb{N}$ obtained from g and h by primitive recursion is defined by:

  $$\begin{aligned} &f(\mathbf{x}, 0) = g(\mathbf{x}) \\ &f(\mathbf{x}, y+1) = h(\mathbf{x}, y, f(\mathbf{x}, y)) \end{aligned}$$

  - For a given $\mathbf{x}$, $f(\mathbf{x}, y)$ is defined for no y, for all y, or for $0 \le y \le r$ for some $r \in \mathbb{N}$
  - Where the 'counter' parameter is placed does not matter - it could equally be at the start

3.2 Primitive Recursive Functions

- We define the *initial functions* to be the following functions:
  - The zero function $z : \mathbb{N} \to \mathbb{N}$, such that $z(x) = 0$ for all $x \in \mathbb{N}$
  - The successor function $\sigma : \mathbb{N} \to \mathbb{N}$, such that $\sigma(x) = x + 1$ for all $x \in \mathbb{N}$
  - The projection functions $\pi_{i,n} : \mathbb{N}^n \to \mathbb{N}$, where for $n \ge 1$ and $1 \le i \le n$, $\pi_{i,n}(x_1, \dots, x_n) = x_i$
- A class $\mathcal{C}$ of total functions is *primitively recursively closed* if:
  - $\mathcal{C}$ contains all the initial functions
  - $\mathcal{C}$ is closed under composition
  - $\mathcal{C}$ is closed under primitive recursion
- The smallest primitively recursively closed class (i.e. the intersection of all prim. rec. closed classes) is called *the class of primitive recursive functions*
- Example: addition function $S : \mathbb{N}^2 \to \mathbb{N}$, such that $S(x, y) = x + y$

  $$\begin{aligned} S(x, 0) &= g(x), g = \pi_{1,1} \\ S(x, y+1) &= S(x, y) + 1 \\ &= \sigma(S(x, y)) \\ &= h(x, y, S(x, y)), h = \sigma \circ \pi_{3,3} \end{aligned}$$

- Useful tips for showing a function is in a primitively recursively closed class $\mathcal{C}$:
  - Given $f : \mathbb{N}^n \to \mathbb{N}$ is in $\mathcal{C}$

    If $g : \mathbb{N}^m \to \mathbb{N}$ is defined by $g(x_1, \ldots, x_m) = f(y_1, \ldots, y_n)$ where each y_i is either a constant or x_j for some j, then $g \in \mathcal{C}$ – lets you manipulate arity
  - To show a unary function $f : \mathbb{N} \to \mathbb{N}$ is in $\mathcal{C}$ by primitive recursion, define $f' : \mathbb{N}^2 \to \mathbb{N}$ such that $f'(x, y) = f'(y)$; then, if f' can be shown to be in $\mathcal{C}$, f will be also
  - Let $a \in \mathbb{N}$ and $h : \mathbb{N} \to \mathbb{N}$ be in $\mathcal{C}$

    Then, for $f : \mathbb{N} \to \mathbb{N}$, if $f(0) = a$ and $f(y + 1) = h(f(y))$, $f \in \mathcal{C}$
- A *primitive recursive definition* of $f : \mathbb{N}^n \to \mathbb{N}$ is a finite sequence of functions $f_0, f_1, \ldots, f_k = f$, where for each i:
  - f_i is initial; or
  - f_i is obtained from composition of some functions f_j, $j < i$; or
  - f_i is obtained by primitive recursion from two of f_j, $j < i$
- Example: addition function S can be defined by $\pi_{1,1}, \pi_{3,3}, \sigma, \sigma \circ \pi_{3,3}$
- The class $\mathcal{C}_1$ of primitive recursive functions is the same as the class $\mathcal{C}_2$ of functions that have a primitive recursive definition (seems trivial, but isn't!)

  Prove by showing $\mathcal{C}_1 \subseteq \mathcal{C}_2$ (i.e. $\mathcal{C}_2$ is prim. rec. closed) and that $\mathcal{C}_2 \subseteq \mathcal{C}_1$ (i.e. $\mathcal{C}_2$ is contained in any prim. rec. closed class)
- Let $\mathcal{C}$ be a prim. rec. closed class, and let $g : \mathbb{N}^{n+1} \to \mathbb{N}$ be in $\mathcal{C}$; then the functions $f_1 : \mathbb{N}^{n+1} \to \mathbb{N}$ and $f_2 : \mathbb{N}^{n+1} \to \mathbb{N}$ defined by:

  $f_1(\mathbf{x}, y) = \sum_{t=0}^{y} g(\mathbf{x}, t)$

  $f_2(\mathbf{x}, y) = \prod_{t=0}^{y} g(\mathbf{x}, t)$

  are also in $\mathcal{C}$
- Useful prim. rec. functions:
  - Proper subtraction $x \dot{-} y = max\,\{x - y, 0\}$
  - Sign $sg(x) = \begin{cases} 0 \text{ if } x = 0 \\ 1 \text{ if } x \geq 0 \end{cases}$

3.3 Predicates

- A predicate $P(x_1, \dots, x_n)$ of n variables is a statement concerning $x_i \in \mathbb{N}$ which is either true or false
- We can identify P with the set $A_P = \{\mathbf{x} \in \mathbb{N}^n \mid P(\mathbf{x}) \text{ is true}\}$

  E.g. $P(x, y)$ means "x divides y", so $A_P = \{(1,6), (2,6), (3,6), (6,6), (1,3) \dots\}$
- The *characteristic function* of a set $\chi_A : \mathbb{N}^n \to \{0, 1\}$ of $A \subseteq \mathbb{N}^n$ is defined by:

$$\chi_A(\mathbf{x}) = \begin{cases} 1 \text{ if } \mathbf{x} \in A \\ 0 \text{ if } \mathbf{x} \notin A \end{cases}$$

- For a predicate P, we define χ_P to be χ_{A_P}
- Let $\mathcal{C}$ be a prim. rec. closed class; then a subset $A \subseteq \mathbb{N}^n$ is in $\mathcal{C}$ if $\chi_A \in \mathcal{C}$

  So a predicate P of n variables is in $\mathcal{C}$ if $\chi_P \in \mathcal{C}$
- If $A, B \subseteq \mathbb{N}^n$ are in $\mathcal{C}$, then $A \cup B$, $A \cap B$ and $\mathbb{N}^n \setminus A$ are in C

  So if P, Q are predicates of n variables in $\mathcal{C}$, $P \vee Q$, $P \wedge Q$ and $\neg P$ are in $\mathcal{C}$

  Proof: $\chi_{A \cup B}(x) = sg(\chi_A(x) + \chi_B(x))$, $\quad \chi_{A \cap B} = \chi_A(x) \cdot \chi_B(x)$, $\quad \chi_{\mathbb{N}^n \setminus A}(x) = 1 \dot{-} \chi_A(x)$
- The predicates $x = y$, $x \neq y$, $x \leq y$, $x < y$, $x \geq y$, $x > y$ are prim. rec.

  Proof: Note that $\chi_{\neq}(x, y) = sg(|1 - 3|)$ and $\chi_{\geq}(x, y) = sg(x \dot{-} y)$
- Bounded quantifiers:

  Assume P is a pred. of $n + 1$ variables in $\mathcal{C}$; then Q, R of $n + 1$ variables defined below are in $\mathcal{C}$:

  $Q(x_1, \dots x_n, z)$ is true if and only if $\exists_{y \leq z}(P(x_1, \dots, x_n, y)$ is true)

  $R(x_1, \dots x_n, z)$ is true if and only if $\forall_{y \leq z}(P(x_1, \dots, x_n, y)$ is true)

  Proof: $\chi_Q(\mathbf{x}, z) = sg(\sum_{y=0}^{z} \chi_P(\mathbf{x}, y))$, and $\chi_R(\mathbf{x}, z) = \prod_{y=0}^{z} \chi_P(\mathbf{x}, y)$

3.4 More Primitive Recursive Functions

3.4.1 Bounded Minimisation

Let P be a pred. of $n + 1$ variables. Define $f : \mathbb{N}^{n+1} \to \mathbb{N}$ by:

$$f(\mathbf{x}, z) = \begin{cases} \text{the least } y \leq z \text{ s.t. } P(\mathbf{x}, y) \text{ is true} \\ z + 1 \text{ if no such } y \text{ exists} \end{cases}$$

Then, $f(\mathbf{x}, z) = \mu\ y \leq z\ P(\mathbf{x}, y)$, called *bounded minimisation.* We have that if $P \in \mathcal{C}$ (a prim. rec. closed class), then f is in $\mathcal{C}$.

Proof. Define $g(\mathbf{x}, t) = \prod_{y=0}^{t} sg(1 \dot{-} \chi_P(\mathbf{x}, y) \text{ is true})$. Note that $g(\mathbf{x}, t) = \begin{cases} 0 \text{ if } \exists_{y \leq t} P(x, y) \text{ is true} \\ 1 \text{ if } \forall_{y \leq t} P(x, y) \text{ is false} \end{cases}$

Let $y \leq z$ be the least s.t. $P(\mathbf{x}, y)$ is true.

Then the values of g look like:

t	0	1	...	$y-1$	y	$y+1$	...	z
$g(\mathbf{x}, t)$	1	1	...	1	0	0	...	0

Let $f(\mathbf{x}, z) = \sum_{t=0}^{z} g(\mathbf{x}, t)$, then we will have f as required for bounded minimsation. If there is no such y, then by the definition of g we would have $f(\mathbf{x}, z) = z + 1$

□

3.4.2 Definition By Cases

Let $f_1, \ldots, f_k : \mathbb{N}^n \to \mathbb{N}$ be in prim. rec. closed $\mathcal{C}$ and let $P_1, \ldots, P_k$ be predicates in $\mathcal{C}$ of n variables. Suppose that for each $\mathbf{x} \in \mathbb{N}^n$ *exactly* one of $P_1(\mathbf{x}), \ldots, P_k(\mathbf{x})$ is true. Define $f : \mathbb{N}^n \to \mathbb{N}$ by:

$$f(\mathbf{x}) = f_i(\mathbf{x}) \text{ if } P_i(\mathbf{x}) \text{ is true}$$

Then f is in $\mathcal{C}$.

Proof. $f(\mathbf{x}) = f_1(\mathbf{x}) \cdot \chi_{P_1}(\mathbf{x}) + \cdots + f_k(\mathbf{x}) \cdot \chi_{P_k}(\mathbf{x})$

□

3.4.3 Iteration

Let X be a set, with a partial function $f : X \to X$. The *iterate* of f is the partial function $F : X \times \mathbb{N} \to X$ defined by:

$$\begin{aligned} F(x, 0) &= x \\ F(x, n+1) &= f(F(x, n)) \end{aligned}$$

We have a notion of a function $f : \mathbb{N}^n \to \mathbb{N}$ being in a class $\mathcal{C}$. This can be extended to functions $f : \mathbb{N}^n \to \mathbb{N}^k$ by saying that f is in $\mathcal{C}$ if $\pi_{i,k} \circ f$ is in $\mathcal{C}$ for each $1 \leq i \leq k$.

A class $\mathcal{C}$ is closed under iteration if, whenever $f : \mathbb{N}^n \to \mathbb{N}^n$ is in $\mathcal{C}$, then its iterate $F : \mathbb{N}^{n+1} \to \mathbb{N}^n$ is in $\mathcal{C}$.

Let $\mathcal{C}$ be a prim. rec. closed class. Then if $f : \mathbb{N}^n \to \mathbb{N}^n$ is in $\mathcal{C}$, its iterate $F : \mathbb{N}^{n+1} \to \mathbb{N}^n$ is also in $\mathcal{C}$. So any prim. rec. closed class is closed under iteration.

Proof. This shows only the $n = 1$ case.

Define $f' : \mathbb{N}^3 \to \mathbb{N}$ by $f'(x, y, z) = f(z)$, which is in $\mathcal{C}$.

Then the iterate of f is defined by:

$$\begin{aligned} F(z, 0) &= z \\ F(z, y+1) &= f(F(z, y)) = f'(x, y, F(x, y)) \end{aligned}$$

This is defined by primitive recursion, so F is in $\mathcal{C}$.

□

3.5 Recursive and Partial Recursive Functions

3.5.1 Minimisation

Let $f : \mathbb{N}^{n+1} \to \mathbb{N}$ be a partial function. The function obtained from f by *minimisation* is the partial function $g : \mathbb{N}^n \to \mathbb{N}$ defined by

$$g(\mathbf{x}) = \begin{cases} r & \text{if } f(\mathbf{x}, r) = 0 \text{ and for } s < r, f(\mathbf{x}, s) \text{ is defined and not } 0 \\ \textit{undefined} & \text{otherwise} \end{cases}$$

We write $g(\mathbf{x}) = \mu y(f(\mathbf{x}, y) = 0)$. It is also called the μ-operator or unbounded search operator. The function g may be partial, even if f is total, and vice versa.

Note that it is not *quite* accurate to say $g(x) = \mu y(f(\mathbf{x}, y) = 0)$ is the least y s.t. $f(x, y) = 0$; if there is some least y s.t. $f(x, y) = 0$, but $f(x, s)$ is undefined for some $s < y$, then $g(x)$ is undefined.

3.5.2 The Class of Recursive Functions

- A total function $f(\mathbf{x}, y)$ is *regular* if for any $\mathbf{x} \in \mathbb{N}^n$, there exists $y \in \mathbb{N}$ such that $f(\mathbf{x}, y) = 0$
- The regular functions are exactly those to which we can apply minimisation and end up with a total function
- The function g is obtained from f by *regular minimisation* if $g(\mathbf{x}) = \mu y(f(\mathbf{x}, y) = 0)$ where f is regular
- The *class of recursive functions* is the smallest class $\mathcal{C}$ of total functions which is primitively recursively closed and is closed under regular minimisation
- Note that there are recursive functions which are *not* primitive recursive
- Example: the two-argument Ackermann function defined by:

$$A(m, n) = \begin{cases} n + 1 & \text{if } m = 0 \\ A(m-1, 1) & \text{if } m > 0 \text{ and } n = 0 \\ A(m-1, A(m, n-1)) & \text{if } m > 0 \text{ and } n > 0 \end{cases}$$

- The *class of partial recursive functions* is the smallest class of partial functions which contains the initial functions, and is closed under composition, primitive recursion and minimisation
  - Note that this is *not* a primitively recursively closed class – that term only applies to a class of total functions

4 Equivalence of Partial Recursive and TM Computable Functions

A key theorem is that all partial recursive functions are Turing Machine computable, and vice versa.

4.1 TM Computable Functions Are Partial Recursive

Recall that a partial function $f : \mathbb{N}^n \to \mathbb{N}$ is TM computable if $f = \varphi_{T,n}$ for some numerical TM T.

Let $T = (Q, F, A, I, \tau, q_0)$ be a numerical Turing machine (i.e. deterministic, $F = I = \emptyset$, $A = \{0, 1\}$).

Recall that $\varphi_{T,n}(\mathbf{x}) = \begin{cases} y & \text{if the computation starting with } (q_0, \underline{0}1^{x_1} \dots 01^{x_n}) \text{ halts with } (q, \underline{0}1^y) \\ \textit{undefined} & \text{otherwise} \end{cases}$

It is convenient to modify T slightly. Add two new states p and h, and the transitions:

- (q, a, p, a, L) for all $(q, a) \in Q \times A$ s.t. no element in τ starts with (q, a)
- (p, a, h, a, R) for all $a \in A$ (i.e. for $a = 0$ and $a = 1$)
- (h, a, p, a, L) for all $a \in A$

Call the new machine T', so $Q' = Q \cup \{p, h\}$, with C' being the set of configurations.

Then T' is still deterministic, and transitions have the form:

$$(q, a, N(q, a), R(q, a), D(q, a)) \in Q' \times A \times Q' \times A \times \{L, R\}$$

where N, R, D are functions on $Q' \times A$.

Then, we number the states such that $Q = \{0, 1, \dots, r - 1\}$, where $h = 0$ and $p = 1$. We encode $L = 0$ and $R = 1$.

Now, $Q' \times A$ is a finite subset of $\mathbb{N}^2$; put $N(x, y) = R(x, y) = D(x, y) = 0$ for $(x, y) \in \mathbb{N}^2 \setminus (Q' \times A)$. Then, N, R, D are primitive recursive functions $\mathbb{N}^2 \to \mathbb{N}$.

Define $Code : C' \to \mathbb{N}$ by $Code(q, a, \alpha, \beta) = 2^q 3^a 5^{\sigma(\alpha)} 7^{\sigma(\beta)}$, where σ encodes a function in the binary representation of an integer:

$$\sigma(f) = f(0) + 2 \cdot f(1) + 2^2 \cdot f(2) + \dots$$

Then $Code$ is an injective (one-to-one) function.

There is a primitive recursive function $Next : \mathbb{N} \to \mathbb{N}$ s.t. $Next(Code(c)) = Code(\delta(c))$, for $c \in C'$ where δ is the transition function of T'.

Proof. Let $c = (q, a, \alpha, \beta)$; let $x \in \mathbb{N} = Code(c) = 2^q 3^a 5^{\sigma(\alpha)} 7^{\sigma(\beta)}$.

Then, we express $Next(x) = Code(\delta(c))$ in terms of x.

First, note that $q = \log_2 x$ and $a = \log_3 x$ (here log simply retrieves the exponents, it is not the normal logarithm function from calculus/analysis).

That $Next$ is primitive recursive is then proved in two cases:

Proof when $D(q, a) = 0$: We have that $\delta(c) = (q', a', \alpha', \beta')$, where $q' = N(q, a)$ and $a' = \beta(0)$.

$Next(x) = Code(\delta(c)) = 2^{N(q,a)} 3^{\beta(0)} 5^{\sigma(\alpha')} 7^{\sigma(\beta')}$

$N(q, a) = N(\log_2 x, \log_3 x)$; the log and N functions are primitive recursive.

$\beta(0) = rem(2, log_7(x))$ where rem is the remainder function (which is prim. rec.).

$\sigma(\alpha') = R(q, a) + 2\alpha(0) + 2^2\alpha(1) + \dots = R(\log_2 x, \log_3 x) + 2 \log_5 x$, where R is prim. rec.

$\sigma(\beta') = \beta(1) + 2\beta(2) + 2^2\beta(3) + \cdots = quo(2, \sigma(\beta)) = quo(2, \log_7 x)$, where *quo* is the quotient / 'integer division' function (which is prim. rec.).

□

Proof when $D(q, a) = 1$*:* TBA □

□

5 First Order Logic