# 1 Graph Model of Set Theory

- Directed graphs: $G = \langle V, A \rangle$
- A graph is **well-founded** if it has no looping paths and no infinite descending paths
- A graph is **extensional** if for any $v_0, v_1$ such that $v_0$ has the same incoming arrows as $v_1$, $v_0 = v_1$
- Two graphs are isomorphic if there is a function (*isomorphism*) $\sigma$ between them such that:
  - $\sigma$ is a bijection (surjection + injection)
  - $vA_0u \leftrightarrow \sigma(v)A_1\sigma(u)$
- An automorphism is an isomorphism between some graph and itself (the identity is a trivial one)
- $G$ is a subgraph of $G'$ if $V \subseteq V'$ and $v_0Av_1 \leftrightarrow v_0A'v_1$ for all $v_0, v_1 \in V$
- $G$ is *maximal* in some property $\Phi$ if $G$ possesses $\Phi$ and there exists no graph $G'$ such that:
  - $G'$ possesses $\Phi$; and
  - $G$ is a proper subgraph of $G'$
- Let $G$ be a *maximal* well-founded graph with no non-trivial automorphisms
- Equivalently, $G$ is a maximal well-founded graph which is extensional
- $G$ is then an *intended model* of Set Theory

# 2 First Order Logic

- Logical symbols: $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\exists$, $\forall$, (, )
- Non-logical symbols: constant symbols $(a, b, c)$, relation symbols $(P, Q, R)$, function symbols $(f, g, h)$
- Language: $\mathcal{L} = \{a, b, c, \ldots, P, Q, R, \ldots, f, g, h, \ldots\}$
- Individual variables will be denoted $v_1, v_2, \ldots$
- Metavariables / arbitrary variables will be denoted $x, y, z, \ldots$
- The set of terms, $Term$ is defined recursively:
  - If $t$ is a constant symbol or individual variable, $t$ is a term
  - If $t_1, \ldots, t_n$ are terms and $f$ is a function symbol with arity $n$, $f(t_1, \ldots, t_n)$ is a term
  - Nothing else is a term
- A string $\varphi$ is an *atom* if $\varphi = Rt_1, \ldots, t_n$ where $t_1, \ldots, t_n$ are terms and $R$ is a relation with arity $n$
- If $t$ is a term with no variables occurring in it, $t$ is a *closed term*
- The set of Well Formed Formulae $WFF$ is defined recursively, with atoms as the base case
  - If $\varphi$ consists of a combination of logical operators over well formed formulae, $\varphi \in WFF$
- $x$ is *free* in $\varphi \in WFF$ if:
  - If $\varphi = Rt_1, \ldots, t_n$ and $x = t_i$ for some $i$
  - If $\varphi$ consists of logical operators over well formed formula, where $x$ is free in at least one
  - If $\varphi$ is a quantification over a formula where $x$ is free, and $x$ is not the bound variable
- If $\varphi$ has no free variables, $\varphi$ is a sentence, $\varphi \in Sent$

# 3 Model Theory

- A model $\mathcal{M}$ requires:
  - A language $\mathcal{L}$
  - A domain $M$ of objects
  - An interpretation:
    * Constant symbols $c$ are interpreted by some object from the domain: $c^{\mathcal{M}} \in M$
    * Relation symbols $R$ (with arity $n$) are interpreted by a set of tuples of objects within the domain; so $R^{\mathcal{M}} \subseteq \{\langle m_1, \ldots, m_n \rangle \mid m_1, \ldots, m_n \in M\}$
    * Function symbols $f$ with arity $n$ are interpreted by functions taking some $m_1, \ldots, m_n \in M$, and returning some $m \in M$
- A model of $\mathcal{L} = \{a, b, c, \ldots, P, Q, R, \ldots, f, g, h, \ldots\}$ would then be:

$$\mathcal{M} = \left\langle M, a^{\mathcal{M}}, b^{\mathcal{M}}, c^{\mathcal{M}}, \ldots, P^{\mathcal{M}}, Q^{\mathcal{M}}, R^{\mathcal{M}}, \ldots, f^{\mathcal{M}}, g^{\mathcal{M}}, h^{\mathcal{M}}, \ldots \right\rangle$$

- If $t$ is a closed term of $\mathcal{L}$ and $\mathcal{M}$ is a model of $\mathcal{L}$, then the *denotation* $t^{\mathcal{M}}$ is:
  - If $t$ is a constant symbol $c$, $t^{\mathcal{M}} = c^{\mathcal{M}}$
  - If $t = f(t_1, \ldots, t_n)$, $t^{\mathcal{M}} = f^{\mathcal{M}}(t_1^{\mathcal{M}}, \ldots, t_n^{\mathcal{M}})$
- A sentence $\varphi$ in some language $\mathcal{L}$ may then be *true* in $\mathcal{M}$ ($\mathcal{M} \models \varphi$)
- $\varphi$ is *satisfiable* if there exists some model $\mathcal{M}$ in the language $\mathcal{L}$ of $\varphi$ such that $\mathcal{M} \models \varphi$
- $\varphi$ is *valid* ($\models \varphi$) if for every model $\mathcal{M}$ in the language $\mathcal{L}$ of $\varphi$, $\mathcal{M} \models \varphi$
- Given $\Gamma \subseteq Sent_{\mathcal{L}}$ and $\varphi \in Sent_{\mathcal{L}}$, we say $\varphi$ *is a consequence of* $\Gamma$ ($\Gamma \models \varphi$) if for every model $\mathcal{M}$ of $\mathcal{L}$, if $\mathcal{M} \models \gamma$ for all $\gamma \in \Gamma$, then $\mathcal{M} \models \varphi$
- $\varphi$ is *derivable* from $\Gamma$ ($\Gamma \vdash \varphi$) if:
  - (Ax) $\varphi$ is an axiom of first order logic
  - (Ass) $\varphi \in \Gamma$
  - (MP) $\Gamma \vdash \psi \to \varphi$ and $\Gamma \vdash \psi$
  - (UG) If $\Gamma \vdash \varphi$, then $\Gamma \vdash \forall y \varphi(y \mapsto x)$ when:
    * $x$ is not free in any formula in $\Gamma$;
    * $y = x$; or
    * $y$ is not free in $\varphi$
- A model can describe a structure *externally* (as in the graph model), or *internally* using sentences true in the intended model
- A *theory* $\Gamma$ is a set of sentences closed under consequence (i.e. $\Gamma \vdash \varphi \to \varphi \in \Gamma$)
  - It is *consistent* if there is no sentence $\varphi$ such that $\Gamma \vdash \varphi \wedge \neg\varphi$
  - It is *complete* if for all $\varphi$ we have either $\Gamma \vdash \varphi$ or $\Gamma \vdash \neg\varphi$
  - It is *categorical* if there is exactly one model $\mathcal{M}$ such that $\mathcal{M} \models \Gamma$
- A theory is **algebraic** if it has multiple intended models, and **non-algebraic** if it has one unique intended model

# 4 The Axioms

## 4.1 Terms & Definitions

- A *term* is well-defined in Set Theory if it *exists* and is *unique*
- Term: $x$ is a *subset* of $y$ ($x \subseteq y$) if $\forall z(z \in x \to z \in y)$

## 4.2 Extensionality

$$\forall x \forall y(\forall z(z \in x \leftrightarrow z \in y) \to x = y)$$

- If sets $x$ and $y$ have exactly the same members, then they are the same set

## 4.3 Foundation

$$\forall x(\exists y \in x \to \exists z(z \in x \wedge \forall w(w \in x \to w \notin z)))$$

- If $x$ is non-empty, then it has an $\in$-minimal member (there is some $z \in x$ such that every member of $x$ is not a member of $z$)

## 4.4 Pairing

$$\forall x \forall y \exists z(x \in z \wedge y \in z)$$

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## 4.5 Union

$$\forall x \exists z \forall y \forall w(y \in w \wedge w \in x \to y \in z)$$

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## 4.6 Powerset

$$\forall x \exists y \forall z(z \subseteq x \to z \in y)$$

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## 4.7 Separation

$$\forall x_0 \ldots \forall x_n \forall w \exists y \forall z(z \in y \leftrightarrow z \in w \wedge \varphi(z, x_0 \ldots, x_n))$$

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## 4.8 Replacement

$$aa$$

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### 4.9 Infinity

$$\exists x(\exists yy \in x \land \forall z(z \in x \to \{z\} \in x))$$

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### 4.10 Choice

$$aa$$

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## 5 Models, Structures & Sequences

## 6 The Ordinals

## 7 Transfinite Induction and Recursion

## 8 The Cardinals

## 9 Infinite Cardinals & The Axiom of Choice