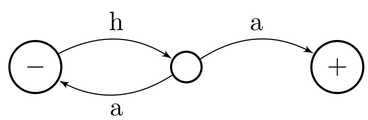
1 Finite State Automata

1.1 Alphabets & Strings

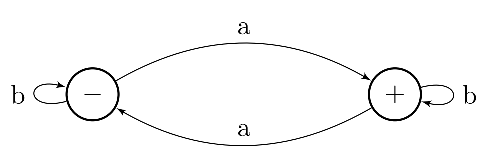
- Let A be a set; then A^n is the set of all finite sequences $a_1 \dots a_n$ with $a_i \in A$, $1 \leq i \leq m$
  - Elements of A are *letters* or *symbols*
  - Elements of A^n are *words* or *strings* over A of length m
- ε is the special *empty string*, the only string of length 0
- $A^+ = \bigcup_{m \geq 1} A^m$ – the set of non-empty strings over A of any length
- $A^* = A^+ \cup \varepsilon = \bigcup_{m \geq 0} A^m$ – the set of (possibly empty) strings over A of any length
- If $\alpha = a_1 \dots a_m$, $\beta = b_1 \dots b_m \in A^*$, then define $\alpha\beta$ to be $a_1 \dots a_m b_1 \dots b_m \in A^{m+n}$. This gives binary 'product' or *concatenation* on A^*
- For $\alpha \in A^+$, define $\alpha^n, n \in \mathbb{N}$ by $\alpha^0 = \varepsilon$, and $\alpha^{n+1} = \alpha^n \alpha$
- A *language* with alphabet A is a subset of A^*

1.2 Definition of an FSA

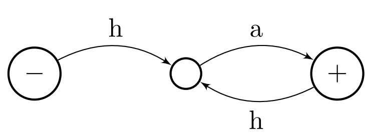
- A Finite State Automaton (FSA) is a tuple $M = (Q, F, A, \tau, q_0)$
  - Q is a finite set of states
  - $F \subseteq Q$ is the set of final states
  - A is the alphabet
  - $\tau \subseteq Q \times A \times Q$ is the set of transitions
  - $q_0 \in Q$ is the initial state
- The transition diagram of an FSA is a directed graph with:
  - Vertex set Q
  - An edge for each transition; $(q, a, q') \in \tau$ corresponds to an edge from q to q' with label a
  - Initial state q_0 labelled with $-$
  - Final states labelled with $+$
  - Example: a *non-deterministic* 'haha machine', with $A = \{h, a\}$



- A *computation* of M is a sequence $q_0, a_1, q_1, a_2, \ldots, a_n, q_n$ with $n \geq 0$ where $(q_i, a_{i+1}, q_{i+1}) \in \tau$ for $0 \leq i \leq n-1$
  - The *label* on the computation is $a_1 \ldots a_m$
  - The computation is *successful* if $q_n \in F$
  - A string $a_1 \ldots a_n$ is *accepted* by M if there is a successful computation with label $a_1 \ldots a_n$, and it is *rejected* otherwise
- The language recognised by M is $\mathcal{L}(M) = \{w \in A^* \mid w \text{ is accepted by } M\}$
- There is a one-to-one correspondence between computations of M and paths in the graph from q_0
- Example: $A = \{a, b\}$ of an FSA accepting only words with an odd number of 'a's



- An FSA is deterministic (a DFA) if for all $q \in Q, a \in A$ there is exactly one $q' \in Q$ such that $(q, a, q') \in \tau$
- Example: DFA for the 'haha machine'



- Note this machine lacks a transition for a when in the initial state – though technically required for a DFA, it is easily fixed by adding an 'error state' to catch what would otherwise be missing transitions

1.3 Deterministic FSAs

- For a DFA M, define the transition function $\delta : Q \times A \to Q$ by $q' = \delta(a, q)$, where q' is the unique element such that $(q, a, q') \in \tau$
- If $\mathcal{L}$ is a language with alphabet A, then the following are equivalent:
  1. $\mathcal{L}$ is recognised by an FSA
  2. $\mathcal{L}$ is recognised by a DFA
- Given a non-deterministic FSA $M = (Q, F, A, \tau, q_0)$, an equivalent DFA $M' = (Q', F', A, \tau', q_0')$ may be generated by the *powerset method*:
  - $Q' = \mathcal{P}(Q) \setminus \emptyset$ (i.e. the set of all subsets of Q that aren't empty)
  - $F' = \{X \in Q' \,|\, q \in X \text{ for some } q \in F\}$
  - For $X \in Q', a \in A$, define $\delta(X, a) := \{q \in Q \,|\, (x, a, q) \in \tau \text{ for some } x \in X\}$
  - $\tau' = \{(X, a, \delta(X, a)) \,|\, X \in Q', a \in A\}$
  - $q_0' = \{q_0\}$
- Proof: show that $\mathcal{L}(M) = \mathcal{L}(M')$
  - $\mathcal{L}(M) \subseteq Lang(M')$:
    * Given $w \in \mathcal{L}(M)$, $q_0 a_1 \dots a_n q_n$ is a successful computation of M
    * Then define $q_i' = \delta(q_{i-1}', a_i)$ for $1 \leq i \leq n$
    * $q_0', a_1, q_1' \dots a_n, q_n'$ will be a successful computation of M'
    * Therefore $w \in \mathcal{L}(M')$
  - $\mathcal{L}(M') \subseteq Lang(M)$:
    * Let $w = a_1 \dots a_n \in L(M')$, and $q_0', a_1, q_1' \dots a_n, q_n'$ be a successful computation of M
    * Each q_i' cannot be the empty set
    * By definition of τ', $\exists q_1 \in q_1'$ s.t. $(q_0, a_1, q_1) \in \tau$
    * Then we can find $q_i \in q_i'$ s.t. $(q_{i-1}, a_i, q_i) \in \tau$ for $1 \leq i \leq n$
    * For q_n we further require $q_n \in F$
    * Therefore, $q_0, a_1, q_1, a_2, \dots a_n, q_n$ is a successful computation
    * Therefore $w \in \mathcal{L}(M)$

1.4 The Pumping Lemma

- The Pumping Lemma says that for any $\mathcal{L}$ recognised by an FSA M, there is a certain word length beyond which all words can be split into sections as xyz, where xy^nz is also in the language
- Formally there is an integer $p > 0$ s.t. any word $w \in L$ with $|w| \geq p$ is of the form $w = xyz$, where $|y| > 0$, $|xy| \leq p$ and $xy^iz \in \mathcal{L}$ for $i \geq 0$
- Proof:
  - Let p be the number of states in M, and suppose $w = a_1 \dots a_n \in \mathcal{L}$, where $n \geq p$
  - A successful computation $q_0, a_1, \dots, q_n$ has to pass through a certain state at least twice (by the pigeonhole principle)
  - Therefore, $\exists r < s$ s.t. $q_r = q_s$; choose minimal such s
  - Now put $x = a_1 \dots a_r$, $y = a_{r+1} \dots a_s$ (note $|y| > 0$), and $z = a_{s+1} \dots a_n$
  - By minimality of s, $q_0, \dots q_{s-1}$ are distinct, and $|xy| = s \leq p$
  - Then, note that $q_r, a_{r+1}, \dots, q_s$ is a loop, which may be validly repeated $i \geq 0$ times
  - Therefore, $xy^iz \in \mathcal{L}$
- Corollary: here exist languages which are not computable by an FSA
- Example: there is no FSA which can recognise $\mathcal{L} = \{a^nb^n \mid n \in \mathbb{N}\}$
- Proof:
  - Assume for a contradiction there exists an FSA M which can recognise $\mathcal{L}$
  - Let p be the number from the pumping lemma, and choose $n \geq p$ and consider $w = a^nb^n$
  - By the pumping lemma, $\exists x, y, z$ s.t. $a^nb^n = xyz$, with $|y| \geq 1$ and $|xy| \leq p \leq n$
  - Then y is written entirely in terms of the letter a, and $|y| \geq 1$
  - By the pumping lemma, $xy^iz \in \mathcal{L}$ for all i
  - So choose $i = 0$, then some $w = a^kb^n \in \mathcal{L}$ s.t. $k < n$, which is a contradiction

2 Turing Machines

2.1 Definition

- A Turing machine is a tuple $T = (Q, F, A, I, \tau, q_0)$
  - Q is a finite set of states
  - $F \subseteq Q$ is the set of final states
  - A is a finite set, the tape alphabet, with a distinguished blank symbol $B \in A$
  - I is a subset of $A \setminus \{B\}$, the input alphabet
  - $\tau \subseteq Q \times A \times Q \times A \times \{L, R\}$ is the set of transitions
  - $q_0 \in Q$ is the initial state
- As in an FSA, non-determinism is allowed
- The tape is infinite in both directions, but only ever contains a finite number of non-blank symbols
- A *tape description* for T is a triple (a, α, β) with $a \in A$, and $\alpha : \mathbb{N} \to A$ and $\beta : \mathbb{N} \to A$ being functions with $a(n) = B$ and $B(n) = B$ for all but finitely many $n \in \mathbb{N}$
  - So the tape looks like: $\dots BBB\beta(l)\beta(l-1)\dots\beta(0)\underline{a}\alpha(0)\alpha(1)\dots\alpha(r)BBB\dots$, with $l, r \in \mathbb{N}$
- A *configuration* of T is a tuple (q, a, α, β) where $q \in Q$ and (a, α, β) is a tape description
- If $c = (q, a, \alpha, \beta)$ is a configuration, a configuration c' is obtained (reachable) from c by a single move if one of the following holds:
  - $(q, a, q', a', L) \in \tau$ and $c' = (q', \beta(0), \alpha', \beta')$ where: $\alpha'(0) = a', \alpha'(n) = \alpha(n-1), n > 0$ and $\beta'(n) = \beta(n+1), n \geq 0$, or
  - $(q, a, q', a', R) \in \tau$ and $c' = (q', \alpha(0), \alpha', \beta')$ where: $\alpha'(n) = \alpha(n+1), n \geq 0$ and $\beta'(0) = a', \beta'(n) = \beta(n-1), n > 0$
- A *computation* of T is a finite sequence of configurations $c_1, \dots, c_n = c'$ where $n \geq 1$ and c_{i+1} is obtained from c_i by a single move, for $1 \leq i \leq n-1$
- A configuration is *terminal* if no configuration is reachable from it
- A computation halts if c' is terminal (i.e. there is no configuration reachable from c')
- We may write $c \underset{T}{\rightarrow} c'$ if there is a computation starting at c and ending at c'

2.2 Turing Machine as Language Recogniser

- For $w = a_1 \dots a_n \in A^*$, let $c_w = (a_0, \underline{a_1} \dots a_n)$ (recall $\underline{a_1} \dots a_n$ is a tape description (a, α, β))
- If $w = \varepsilon$, we put $c_w = (q_0, \underline{B})$
- The TM T *accepts* if $c_w \underset{T}{\rightarrow} c'$ for some $c' = (q, a, \alpha, \beta)$ with $q \in F$
- The language recognised by T is $\mathcal{L}(T) = \{w \in I^* \mid w \text{ is accepted by } T\}$
- Note that $\mathcal{L}(T)$ is a language over I rather than over A
- T is deterministic if for every $(q, a) \in Q \times A$ there is *at most one* element of τ starting with (q, a)
- Then, there is at most one config c' obtained from c by a single move; set $\delta(c) = c'$
- $\delta : C \to C$ is then a partial function

2.3 Numerical Turing Machines

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3 Partial Recursive Functions

4 First Order Logic