1 Graph Model of Set Theory

- Directed graphs: $G = \langle V, A \rangle$
- A graph is **well-founded** if it has no looping paths and no infinite descending paths
- A graph is **extensional** if for any v_0, v_1 such that v_0 has the same incoming arrows as v_1, $v_0 = v_1$
- Two graphs are isomorphic if there is a function (*isomorphism*) σ between them such that:
  - σ is a bijection (surjection + injection)
  - $vA_0u \leftrightarrow \sigma(v)A_1\sigma(u)$
- An automorphism is an isomorphism between some graph and itself (the identity is a trivial one)
- G is a subgraph of G' if $V \subseteq V'$ and $v_0Av_1 \leftrightarrow v_0A'v_1$ for all $v_0, v_1 \in V$
- G is *maximal* in some property Φ if G possesses Φ and there exists no graph G' such that:
  - G' possesses Φ; and
  - G is a proper subgraph of G'
- Let G be a *maximal* well-founded graph with no non-trivial automorphisms
- Equivalently, G is a maximal well-founded graph which is extensional
- G is then an *intended model* of Set Theory

2 First Order Logic

- Logical symbols: $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\exists$, $\forall$, (,)
- Non-logical symbols: constant symbols (a, b, c), relation symbols (P, Q, R), function symbols (f, g, h)
- Language: $\mathcal{L} = \{a, b, c, \ldots, P, Q, R, \ldots, f, g, h, \ldots\}$
- Individual variables will be denoted $v_1, v_2, \ldots$
- Metavariables / arbitrary variables will be denoted $x, y, z, \ldots$
- The set of terms, $Term$ is defined recursively:
  - If t is a constant symbol or individual variable, t is a term
  - If $t_1, \ldots, t_n$ are terms and f is a function symbol with arity n, $f(t_1, \ldots, t_n)$ is a term
  - Nothing else is a term
- A string φ is an *atom* if $\varphi = Rt_1, \ldots, t_n$ where $t_1, \ldots, t_n$ are terms and R is a relation with arity n
- If t is a term with no variables occurring in it, t is a *closed term*
- The set of Well Formed Formulae WFF is defined recursively, with atoms as the base case
  - If φ consists of a combination of logical operators over well formed formulae, $\varphi \in WFF$
- x is *free* in $\varphi \in WFF$ if:
  - If $\varphi = Rt_1, \ldots, t_n$ and $x = t_i$ for some i
  - If φ consists of logical operators over well formed formula, where x is free in at least one
  - If φ is a quantification over a formula where x is free, and x is not the bound variable
- If φ has no free variables, φ is a sentence, $\varphi \in Sent$

3 Model Theory

- A model $\mathcal{M}$ requires:
  - A language $\mathcal{L}$
  - A domain M of objects
  - An interpretation:
    * Constant symbols c are interpreted by some object from the domain: $c^{\mathcal{M}} \in M$
    * Relation symbols R (with arity n) are interpreted by a set of tuples of objects within the domain; so $R^{\mathcal{M}} \subseteq \{\langle m_1, \ldots, m_n \rangle \mid m_1, \ldots, m_n \in M\}$
    * Function symbols f with arity n are interpreted by functions taking some $m_1, \ldots, m_n \in M$, and returning some $m \in M$
- A model of $\mathcal{L} = \{a, b, c, \ldots, P, Q, R, \ldots, f, g, h, \ldots\}$ would then be:

$$\mathcal{M} = \left\langle M, a^{\mathcal{M}}, b^{\mathcal{M}}, c^{\mathcal{M}}, \ldots, P^{\mathcal{M}}, Q^{\mathcal{M}}, R^{\mathcal{M}}, \ldots, f^{\mathcal{M}}, g^{\mathcal{M}}, h^{\mathcal{M}}, \ldots \right\rangle$$

- If t is a closed term of $\mathcal{L}$ and $\mathcal{M}$ is a model of $\mathcal{L}$, then the *denotation* $t^{\mathcal{M}}$ is:
  - If t is a constant symbol c, $t^{\mathcal{M}} = c^{\mathcal{M}}$
  - If $t = f(t_1, \ldots, t_n)$, $t^{\mathcal{M}} = f^{\mathcal{M}}(t_1^{\mathcal{M}}, \ldots, t_n^{\mathcal{M}})$
- A sentence φ in some language $\mathcal{L}$ may then be *true* in $\mathcal{M}$ ($\mathcal{M} \models \varphi$)
- φ is *satisfiable* if there exists some model $\mathcal{M}$ in the language $\mathcal{L}$ of φ such that $\mathcal{M} \models \varphi$
- φ is *valid* ($\models \varphi$) if for every model $\mathcal{M}$ in the language $\mathcal{L}$ of φ, $\mathcal{M} \models \varphi$
- Given $\Gamma \subseteq Sent_{\mathcal{L}}$ and $\varphi \in Sent_{\mathcal{L}}$, we say φ *is a consequence of* Γ ($\Gamma \models \varphi$) if for every model $\mathcal{M}$ of $\mathcal{L}$, if $\mathcal{M} \models \gamma$ for all $\gamma \in \Gamma$, then $\mathcal{M} \models \varphi$
- φ is *derivable* from Γ ($\Gamma \vdash \varphi$) if:
  - (Ax) φ is an axiom of first order logic
  - (Ass) $\varphi \in \Gamma$
  - (MP) $\Gamma \vdash \psi \to \varphi$ and $\Gamma \vdash \psi$
  - (UG) If $\Gamma \vdash \varphi$, then $\Gamma \vdash \forall y \varphi(y \mapsto x)$ when:
    * x is not free in any formula in Γ;
    * $y = x$; or
    * y is not free in φ
- A model can describe a structure *externally* (as in the graph model), or *internally* using sentences true in the intended model
- A *theory* Γ is a set of sentences closed under consequence (i.e. $\Gamma \vdash \varphi \to \varphi \in \Gamma$)
  - It is *consistent* if there is no sentence φ such that $\Gamma \vdash \varphi \land \neg\varphi$
  - It is *complete* if for all φ we have either $\Gamma \vdash \varphi$ or $\Gamma \vdash \neg\varphi$
  - It is *categorical* if there is exactly one model $\mathcal{M}$ such that $\mathcal{M} \models \Gamma$
- A theory is **algebraic** if it has multiple intended models, and **non-algebraic** if it has one unique intended model

4 The Axioms

4.1 Terms & Definitions

- A *term* is well-defined in Set Theory if it *exists* and is *unique*
- Term: x is a *subset* of y ($x \subseteq y$) if $\forall z(z \in x \to z \in y)$

4.2 Extensionality

$$\forall x \forall y(\forall z(z \in x \leftrightarrow z \in y) \to x = y)$$

- If sets x and y have exactly the same members, then they are the same set

4.3 Foundation

$$\forall x(\exists y \in x \to \exists z(z \in x \wedge \forall w(w \in x \to w \notin z)))$$

- If x is non-empty, then it has an $\in$-minimal member (there is some $z \in x$ such that every member of x is not a member of z)

4.4 Separation

$$\forall x_0 \ldots \forall x_n \forall w \exists y \forall z(z \in y \leftrightarrow z \in w \wedge \varphi(z, x_0 \ldots, x_n))$$

- Given any set w, there is a set y consisting of exactly the elements z from w such that $\varphi(z)$. The φs are *separated out* from w to get y
- Why do we need a w? We would like to do naïve comprehension: $\exists y \forall x(x \in y \leftrightarrow \varphi(x))$
  - *(Russell) Any theory including naïve comprehension is inconsistent*
  - This also means there is no universal set, i.e. a set x such that $\forall y(y \in x \leftrightarrow y = y)$

4.5 Pairing

$$\forall x \forall y \exists z(x \in z \wedge y \in z)$$

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4.6 Union

$$\forall x \exists z \forall y \forall w(y \in w \wedge w \in x \to y \in z)$$

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4.7 Powerset

$$\forall x \exists y \forall z(z \subseteq x \to z \in y)$$

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4.8 Replacement

$$aa$$

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4.9 Infinity

$$\exists x(\exists y y \in x \land \forall z(z \in x \to \{z\} \in x))$$

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4.10 Choice

$$aa$$

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5 Models, Structures & Sequences

6 The Ordinals

7 Transfinite Induction and Recursion

8 The Cardinals

9 Infinite Cardinals & The Axiom of Choice