Pure Mathematics

Chapter 1

Foundations

1.1 Discrete Maths

1.1.1 The Axiom of Choice

1.1.1.1 Statement of the Problem - Intuitive

The problem that the Axiom of Choice addresses was famously described by Bertrand Russell using the example of choosing from an infinite number of pairs of shoes and an infinite number of pairs of socks. From pairs of shoes it is easy to see how you can have a strategy for making a choice but for pairs of identical socks it is not at all obvious how to achieve the same thing.

In fact, a set like a pair of socks cannot exist because the definition of a set of objects requires that the objects are distinguishable. As a result, just looking at Bertrand Russell's example, it is tempting to think that the problem wouldn't arise. However, we can instead think of an infinite collection of non-empty sets containing *distinguishable* objects: In this case we *still* cannot develop a choice function because there is no single strategy that can be applied to make choices from all the sets. This is a more important point than it may at first seem because the problem of choosing a strategy for each of the infinite sets is a problem of making an arbitrary choice for an infinite number of sets - i.e. the very same problem that the Axiom of Choice is attempting to address!

1.1.1.2 Statement of the Problem - Formal

If a set B is non-empty then this can be expressed in first-order predicate logic as the truth of

$$\phi(B) := (\exists x)(x \in B).$$

Then, through the use of *existential instantiation* we can posit an element, say $c \in B$.

Definition 1. In formal logic, **Existential Instantiation** is the inference,

$$\exists x \,.\, P(x) \implies P(c)$$

where c is a new constant symbol. The symbol c must not have been previously used in the proof nor must it appear in the final conclusion.

So existential instantiation gives us a way, in formal logic, of "selecting" an arbitrary element from a single set without the need for a strategy for making a choice. Furthermore, we can concatenate atomic predicate clauses together to make an arbitrarily-long (but finite) compound predicate statement. For example,

$$\phi(B_1) \wedge \phi(B_2) \wedge \cdots \wedge \phi(B_n)$$

which enables us to say,

$$\exists(x_1, x_2, \ldots, x_n) \,.\, (x_1 \in B_1) \wedge (x_2 \in B_2) \wedge \cdots \wedge (x_n \in B_n)$$

and so instantiate an n-tuple $(c_1, c_2, \ldots, c_n)$ with an element c_i from each set B_i. In this way, formal mathematical logic allows us to make a "selection" from an arbitrary *finite* number of non-empty sets.

However, the first-order logic that is the standard for the logical formalization of mathematics is not able to model infinite domains as the Compactness Theorem implies that first-order logic cannot uniquely determine infinite sets. We cannot, therefore, use the same logical approach to formalize a choice from an infinite number of sets. For this reason the capability needs to be introduced as an axiom.

1.1.1.3 The Axiom of Choice

Definition 2. Let A be a non-empty set of non-empty sets. A **choice function** on A is a function,

$$f : A \longmapsto \bigcup_{a \in A} \text{ s.t. } \forall a \in A,\ f(a) \in a.$$

Axiom 1.1.1. ***(The Axiom of Choice)*** *If A is a non-empty set of non-empty sets then there exists a choice function for A.*

1.1.1.4 The Well-Ordering Theorem (a.k.a Zermelo's Theorem)

Definition 3. A set is **well-ordered** by a strict total order if every non-empty subset has a minimal element under the ordering.

Axiom 1.1.2. ***(Well-Ordering Theorem)*** *Every set can be well-ordered.*

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*The **Well-Ordering Principle** is usually taken to be the proposition that the positive integers are well-ordered (which may be axiomatic or proven by induction depending on the method of constructing the natural numbers) but is sometimes used synonymously with the Well-Ordering Theorem.*

Although, for historic reasons, this is known as a *theorem*, it has been found to be unprovable from the axioms of mathematics and must be accepted as an axiom itself.

It has also been found to be equivalent to the Axiom of Choice. That's to say, every set can be well-ordered if every collection of sets has a choice function and every collection of sets has a choice function if their union is a well-ordered set.

A point of interest when proving the Axiom of Choice using the Well-Ordering Theorem: We assert the well-ordering on the union of the collection of sets rather than asserting a (potentially different) well-ordering on each of the sets - which would equally suffice as a choice function. The reason for this is that if we attempt to assert an individual well-ordering on each of the sets then we are again falling into the problem of existential instantiation over an infinite structure - which, itself, requires the Axiom of Choice. This is the same point as was mentioned in the problem statement.

For uncountable sets the well-ordering may be inexpressible. Specfically, in the case of the reals $\mathbb{R}$, it has been proven that any well-ordering of $\mathbb{R}$ must be inexpressible.

1.1.1.5 Zorn's Lemma

To understand Zorn's Lemma some concepts relating to partial orders are needed.

Partial Orders

Definition 4. Let $(P, \leq)$ be a partial order and let $A \subseteq P$. An element $p \in P$ is an **upper bound** for A if $a \leq p$ for all $a \in A$.

Definition 5. Let $(P, \leq)$ be a partial order. An element $m \in P$ is a **maximal element** if there is no $p \neq m \in P$ such that $m \leq p$.

Partial orders implicitly define totally ordered subsets, or *chains*, within which there may be a maximal element. So, there can be multiple maximal elements for each chain in the poset.

Example of a Partial Order

(1) Let $(\mathbb{N} \setminus \{1\}, \preceq)$ be the relation "divides by". More precisely, for $m, n \in \mathbb{N}$

$$n \preceq m \iff m|n.$$

Then, under this order, every prime number is a maximal element. If we were to modify the order slightly to include the element 1, the partial order $(\mathbb{N}, \preceq)$ has a single, global maximal element - the number 1.

Axiom 1.1.3. ***(Zorn's Lemma)*** *Let $(P, \leq)$ be a non-empty partial order such that every totally ordered subset has an upper bound. Then P has a maximal element.*

Proposition 1.1.1. *Zorn's Lemma implies the Axiom of Choice.*

Proof. Let A be a non-empty set of non-empty sets. Define a *partial choice function* over A to be a function that defines a choice from some of the sets in A but not others. Define an *extends* relation between functions such that if f and g are functions then g *extends* f if and only if $dom(f) \subseteq dom(g)$ and $g(x) = f(x)$ for all $x \in dom(f)$.
Now if we postulate the existence of a collection C of all the partial choice functions over A then the *extends* relation between the members of C is a partial order. Denote this partial order $(A, \leq)$. For any chain in C we can take the union of all the domains of the functions in the chain — which will actually be the domain of the final function in the chain, say f — and form a function g such that $g(x) = f(x)$ for all $x \in dom(f)$. The function f is an upper bound on the chain.
Therefore we can apply Zorn's lemma to assert the existence of a maximal element h. The function h, being maximal, cannot be extended by any function in C and must, therefore, have a domain equal to the entire set A. It follows then that h is a choice function over all the set A and satisfies the Axiom of Choice. □

Theorem 1.1.1. *Every vector space has a basis*

Proof. Let V be an arbitrary vector space and let S be the set of all linearly independent subsets of V. The inclusion relation $\subseteq$ is a partial order over the members of S. For every chain in this partial order $s_1 \subseteq s_2 \subseteq \cdots$ if we take the union of the sets in the chain $U \coloneqq s_1 \cup s_2 \cup \cdots$, then U is an upper bound of the chain and, since U is the union of sets of linearly independent vectors in S, $U \in S$ also. Therefore, Zorn's Lemma tells us that S has a maximal element.
Let B be a maximal element of S. By membership of S, B is a linearly independent set of vectors in V. Since B is a maximal element in S it follows that $\forall s \in S$, $s \subseteq B$. Suppose that there is some vector v in V such that the set $B \cup \{\vec{\boldsymbol{v}}\}$ is linearly independent. Then the set $B \cup \{\vec{\boldsymbol{v}}\}$ is a linearly independent set of vectors in V and so is a member of S but is not a subset of B. This contradicts the maximality of B. We can therefore conclude that no such v exists.
Therefore, there exists a set B of linearly independent vectors in V that spans

the space V and is the largest spanning set of linearly independent vectors that can be found in V. It is, therefore, a basis of V. □

> *Note that when looking for an upper bound to the chain in the set S we don't just take the last element of the chain because the chain can be infinite.*

references: http://www.math.toronto.edu/ivan/mat327/docs/notes/11-choice.pdf

For further study: https://www.mn.uio.no/math/tjenester/kunnskap/kompendier/acwozl.pdf.

1.1.2 Logic

1.1.2.1 Basic Identities

Negation of Universals and Existentials

$$\neg(\forall n \,.\, P(n)) \equiv \exists n \,.\, \neg P(n)$$
$$\neg(\exists n \,.\, P(n)) \equiv \forall n \,.\, \neg P(n)$$

De Morgan's Laws

$$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$$
$$\neg(P \vee Q) \equiv \neg P \wedge \neg Q$$

Implication

$$P \implies Q \equiv \neg Q \vee P$$
$$P \implies Q \equiv \neg Q \implies \neg P$$

Note the (informally agreed) operator precedence rules for logical operators: Stanford.

1.1.3 Functions

Definition 6. **(Function)** A *function* or *mapping* associates every element in its *domain* set with a single element in its *codomain* set.

Informally, a function sets up a relation between sets that is either many-to-1 or 1-to-1 but cannot be 1-to-many or many-to-many.

Notation. If f is a function with domain X and codomain Y, then we write:

$$f : X \longmapsto Y.$$

Definition 7. **(Function Image)** Let $f : X \longmapsto Y$. Then the set $I \subseteq Y$ such that,

$$I = \{\, y \in Y \mid \exists x \in X \,.\, f(x) = y \,\}$$

is known as the *image* of Y.

Notation. There are various notations for the image of a function, depending on context, but a common notation when dealing with general functions is

$$f(X) = \{\, y \in Y \mid \exists x \in X \,.\, f(x) = y \,\}.$$

This notation allows for a natural notation for the image of subsets of the domain: If $Z \subset X$ then the image of Z under f can be written $f(Z)$.

Definition 8. **(Identity Function)** The *identity function* associates every element in its *domain* set with the same element. As a result its codomain is necessarily equal to its domain.
The identity function from the set X to itself is often denoted id_X.

Definition 9. **(Function Inverse)** If f is a function $f : X \longmapsto Y$ then g is an inverse function of f iff g is a function $Y \longmapsto X$ such that

$$fg = id_Y \quad \text{and} \quad gf = id_X$$

which is to say,

$$(fg)(y) = y \quad \text{and} \quad (gf)(x) = x.$$

Definition 10. **(Function Left/Right Inverse)** TODO: needs review! If f is a function $f : X \longmapsto Y$ and $I \subseteq Y$ is the image of f, then g is a *left inverse* function of f iff g is a function $I \longmapsto X$ such that

$$fg = id_I \quad \text{and} \quad gf = id_X$$

which is to say, for $i \in I$ and for any $x \in X$,

$$(fg)(i) = i \quad \text{and} \quad (gf)(x) = x.$$

Similarly, h is a *right inverse* function of f iff h is a function $I \longmapsto X$ such that

$$fg = id_Y \quad \text{and} \quad gf = id_I$$

which is to say, for $i \in I$ and for any $y \in Y$,

$$(fg)(y) = y \quad \text{and} \quad (gf)(i) = i.$$

Definition 11. **(Function Pre-Image)** Let $f : X \longmapsto Y$ and let $Z \subset Y$. Then the set $P \subseteq X$ such that,

$$P = \{\, x \in X \mid \exists z \in Z \,.\, f(x) = z \,\}$$

is known as the *pre-image* or *inverse image* of Z under f.

Notation. The most common notation for the pre-image of a set Z is $f^{-1}(Z)$ but care must be taken not to confuse this notation with an inverse function; the pre-image is a set and, in fact, the function f may be uninvertible.

Definition 12. **(Surjection, Injection, Bijection)** A function $f : X \longmapsto Y$ is *surjective* iff

$$\forall y \in Y \;.\; \exists x \in X \;.\; f(x) = y;$$

injective iff

$$\forall x_1, x_2 \in X \;.\; f(x_1) = f(x_2) \implies x_1 = x_2;$$

and *bijective* iff both injective and surjective.

Notation. The set $S \subset \mathbb{N}$ defined as

$$S = \{\, n \in \mathbb{N} \mid n < m + 1 \,\}$$

has a standard notation $\mathbb{N}_m$.

Contrast this with the notation $\mathbb{F}^m$ (2.4.2) which refers to the set of all functions $\mathbb{N}_m \longmapsto \mathbb{F}$.

Definition 13. **(Function Composition)** If f and g are two functions,

$$f : X \longmapsto Y \quad \text{and} \quad g : Z \longmapsto X$$

then the composition of f and g, denoted $f \circ g$ is the function defined as,

$$f \circ g : Z \longmapsto Y \text{ s.t. } (f \circ g)(z) = f(g(z)).$$

Function composition is associative:

$$f \circ g \circ h = (f \circ g) \circ h = f \circ (g \circ h).$$

Notation. The composition of functions f and g is also often denoted fg although this can cause confusion with the pointwise product.

Proposition 1.1.2. *A bijection is a 1-to-1 mapping.*

Proof. If a function $f : X \longmapsto Y$ is bijective then it is injective and surjective. Surjectivity of the function means that every $y \in Y$ is mapped to by at least one $x \in X$ and injectivity means that every $y \in Y$ is mapped to be at most one $x \in X$. □

Proposition 1.1.3. *The restriction of an injection to a subset of the domain is an injection.*

Proof. Let $f : X \longmapsto Y$ be an injection so that, for any $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$ then $x_1 = x_2$. If we define the restriction

$$g : Z \subset X \longmapsto Y \text{ s.t. } g(z) = f(z),$$

then we clearly also have, for any $z_1, z_2 \in Z$,

$$g(z_1) = g(z_2) \iff f(z_1) = f(z_2) \iff z_1 = z_2.$$

□

The same is not, in general, true of surjections.

Proposition 1.1.4. *Composition preserves injectivity and surjectivity.*

Proof. Let

$$f : X \longmapsto Y \quad \text{and} \quad g : Y \longmapsto Z.$$

Then the composition $h = gf$ is a function $h : X \longmapsto Z$.

Assume f and g are both injections. Then, for any $x_1, x_2 \in X$,

$$\begin{aligned} & h(x_1) = h(x_2) \\ \iff \quad & g(f(x_1)) = g(f(x_2)) && \text{by defn. of composition} \\ \iff \quad & f(x_1) = f(x_2) && \text{by injectivity of } g \\ \iff \quad & x_1 = x_2. && \text{by injectivity of } f \end{aligned}$$

Therefore, the composition $h = gf$ is an injection.

Assume f and g are both surjections. Then, by surjectivity of g, for any $z \in Z$, there is some $y \in Y$ such that $g(y) = z$. Furthermore, by surjectivity of f, for any $y \in Y$, there is some $x \in X$ such that $f(x) = y$. Therefore,

$$\forall z \in Z \,.\, \exists x \in X \,.\, g(f(x)) = z.$$

Therefore, the composition $h = gf$ is a surjection. □

Corollary 1.1.1. *A composition of bijections is a bijection.*

Corollary 1.1.2. *The set of bijective functions forms a group (see: 2.1.1) with function composition.*

Proof.

- Function composition is associative by definition.
- The composition of bijections is a bijection (by Corollary 1.1.1) so function composition closes over the set of bijective functions.
- Bijective functions have inverses under function composition (by Proposition 1.1.7).

□

Proposition 1.1.5. *A function has a left inverse iff the function is injective.*

TODO: needs review!

Proof. Let $f : X \longmapsto Y$ and let $I \subseteq Y$ be the image of f.

Assume f is an injection. Then for any $x_1, x_2 \in X$, if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$ so for each $i \in I$ there exists a unique $x \in X$ such that $f(x) = i$. Therefore we can define a function,

$$g : I \longmapsto X \text{ s.t. } g(i) = x$$

and we have, for all $i \in I$,

$$f(g(i)) = i \implies fg = id_I$$

and also, for all $x \in X$,

$$g(f(x)) = x \implies gf = id_X.$$

Conversely, assume f has a left inverse $g : I \longmapsto X$. Then we have, for all $x \in X$,

$$g(f(x)) = x \implies gf = id_X.$$

Suppose, for contradiction, that f is not injective. Then there exist some $x_1, x_2 \in X$ with $x_1 \neq x_2$ but $f(x_1) = f(x_2)$. But this, together with $gf = id_X$ means that $x_1 = g(f(x_1)) = g(f(x_2)) = x_2$ which is a contradiction. □

Proposition 1.1.6. *A function has a right inverse iff the function is surjective.*

Proof. TODO: needs review! □

Proposition 1.1.7. *A function has an inverse iff the function is bijective.*

Proof. Let $f : X \longmapsto Y$.

Suppose f is a bijection. Then, by Proposition 1.1.2, f maps one and only one x to one and only one y. Therefore, we can define

$$g : Y \longmapsto X \text{ s.t. } g(y) = x \quad \text{where} \quad f(x) = y$$

and then we have

$$\forall x \in X \,.\, g(f(x)) = x \quad \text{and} \quad \forall y \in Y \,.\, f(g(y)) = y$$

which is to say that g is an inverse of f.

Suppose f has an inverse g. Then, by definition of the inverse (9), we have

$$gf = id_X \iff \forall x \in X \,.\, g(f(x)) = x \tag{1}$$

and also

$$\forall y \in Y \,.\, fg = id_Y \iff f(g(y)) = y. \tag{2}$$

Using (1) we can reason, for $x_1, x_2 \in X$,

$$\begin{aligned} & f(x_1) = f(x_2) \\ \iff \quad & g(f(x_1)) = g(f(x_2)) \quad \text{by defn. of function 6} \\ \iff \quad & x_1 = x_2 \qquad\qquad\qquad\qquad \because gf = id_X \end{aligned}$$

which is to say that f is injective.
Using (2) we can reason, for all $y \in Y$,

$$\exists g(y) \in X \,.\, f(g(y)) = y \implies \exists x \in X \,.\, f(x) = y$$

which is to say that f is surjective.
So f is a bijection. □

Proposition 1.1.8. *The inverse of a function (if it exists) is unique.*

Proof. Let $f : X \longmapsto Y$ and assume that g and h are inverses of f.

The proposition can be proven in a number of equivalent ways. One way is to use only the associativity of function composition to reason,

$$g(y) = (hf)(g(y)) = ((hf)g))(y) = (h(fg))(y) = h((fg)(y)) = h(y).$$

We also can use just the definition of a function to reason,

$$\forall x \in X \,.\, g(f(x)) = x = h(f(x)) \implies \forall y \in Y \,.\, g(y) = h(y).$$

Alternatively, we can use the group properties of the group formed by function composition over the set of invertible functions to reason,

$$gf = id_X = hf \iff gfg = hfg \iff g \circ id_Y = h \circ id_Y \iff g = h.$$

□

Compare the proof of uniqueness of inverses in Group Theory 2.1.1.1.

Proposition 1.1.9. *The inverse of a function is a bijection.*

Proof. Using the definition of function inverses (9) it's clear to see that if g is an inverse of f then f is also an inverse of g. Therefore, by Proposition 1.1.7, g is bijective. □

1.1.3.1 Cardinality

Definition 14. **(Set Cardinality)** If there exists a bijection between $\mathbb{N}_m$ and a set X then we say that X has *cardinality* equal to m, denoted $|X| = m$.

Theorem 1.1.2. ***(Pigeonhole Principle)*** *Let m be a natural number. Then, for all $n \in \mathbb{N}$, if there exists an injection $\mathbb{N}_n \longmapsto \mathbb{N}_m$ then $n \leq m$.*

Proof. We will prove this by induction on the cardinality of the domain of the injection. The base case of $n = 1$ holds trivially as $m \in \mathbb{N}$ and so, by Proposition 1.2.3, $1 = n \leq m$.

For the induction step we assume that the proposition holds for some $n = k > 1$ so that, if there is an injection $\mathbb{N}_k \longmapsto \mathbb{N}_m$, then $k \leq m$.

Now we consider the case $n = k + 1$: Assume we have an injection $f : \mathbb{N}_{k+1} \longmapsto \mathbb{N}_m$. Since $k + 1 > k > 1$, there are at least two distinct elements in $\mathbb{N}_{k+1}$ — which is to say $\mathbb{N}_2 \subseteq \mathbb{N}_{k+1}$. Since f is an injection, we have $f(1) \neq f(2)$ and so there are at least two distinct elements in $\mathbb{N}_m$ and we deduce that $m > 1$.

Since $m > 1$, let $s + 1 = m$. Either there is some $a \in \mathbb{N}_k$ such that $f(a) = m = s + 1$ or there is not.

- If there is no $a \in \mathbb{N}_k$ such that $f(a) = m = s + 1$ then the restriction of f to $\mathbb{N}_k$, by Proposition 1.1.3, is an injection and, by the induction hypothesis we can deduce that

$$k \leq s \iff k + 1 \leq s + 1 = m.$$

- Conversely, if there is some $a \in \mathbb{N}_k$ such that $f(a) = m = s + 1$ then, because f is an injection and so we must have $f(k + 1) \neq f(a)$, we can deduce that

$$f(k + 1) < m = s + 1 \iff f(k + 1) \leq s.$$

So we can define an injection from $\mathbb{N}_k$ to $\mathbb{N}_s$ using $f(k+1)$ as follows,

$$g : \mathbb{N}_k \longmapsto \mathbb{N}_s \text{ s.t. } g(x) = \begin{cases} f(k+1) & x = a \\ f(x) & x \neq a. \end{cases}$$

The function g is guaranteed to be injective because f is injective and all we've done is change the mapping of $a \in \mathbb{N}_k$ to return the element mapped to by $k+1 \notin \mathbb{N}_k$. Since g is an injection, we can now employ the induction hypothesis to deduce that

$$k \leq s \iff k+1 \leq s+1 = m. \qquad \square$$

Proposition 1.1.10. *If a function $f : X \longmapsto Y$ is injective then $|X| \leq |Y|$.*

Proof. Let $|X| = m$ and the $|Y| = n$. Then, by the definition of cardinality (14), there exist bijections

$$g_1 : \mathbb{N}_m \longmapsto X \quad \text{and} \quad g_2 : \mathbb{N}_n \longmapsto Y.$$

Since bijections are also injections, and inverses are also bijections (Proposition 1.1.9), g_1 and ${g_2}^{-1}$ are injections. Therefore, by Proposition 1.1.4, the composition $h = {g_2}^{-1} f g_1$ is an injection. Since h is an injection $\mathbb{N}_m \longmapsto \mathbb{N}_n$, by Theorem 1.1.2, we have $m \leq n$ and so, by the definition of cardinality, $|X| \leq |Y|$. $\square$

Proposition 1.1.11. *If a function $f : X \longmapsto Y$ is surjective then $|Y| \leq |X|$.*

Proof. Let $f : X \longmapsto Y$ be a surjection so that, by definition, for every $y \in Y$, there exists at least one $x \in X$ such that $f(x) = y$. Then $|Y| \leq |X|$.

More formally: The surjectivity of f means that every $y \in Y$ has a non-empty pre-image

$$I(y) = \{\, x \in X \mid f(x) = y \,\}.$$

So we can define a function

$$g : Y \longmapsto X \text{ s.t. } g(y) = x$$

for $x \in I(y)$ (we assume that some selection rule is available). Since g maps each y to an element in its pre-image and these pre-images, by definition, must be disjoint, g is an injection. Therefore, by Proposition 1.1.10, we have $|Y| \leq |X|$. $\square$

Corollary 1.1.3. *If a function $f : X \longmapsto Y$ is bijective then $|Y| = |X|$.*

Proof. By definition of bijection, f is an injection. Also, by Proposition 1.1.9, f^{-1} is a bijection and therefore also an injection. So we have an injection in both directions

$$f : X \longmapsto Y \quad \text{and} \quad f^{-1} : Y \longmapsto X.$$

Therefore, by Proposition 1.1.10,

$$|X| \leq |Y| \ \wedge \ |Y| \leq |X| \implies |X| = |Y| \,.$$

□

Corollary 1.1.4. *The cardinality of the image of a function is less than or equal to that of the domain of the function. That's to say, if I is the image of the function,*

$$f : X \longmapsto Y$$

then

$$|I| \leq |X| \,.$$

Proof. By the definition of the image of a function, the function is a surjection onto the image. Therefore, by Proposition 1.1.11, we have $|I| \leq |X|$. □

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(2) *In any room full of people, there will always be at least two people with the same number of friends in the room.*

Say there are n people in the room. Then, if there is a person in the room that is friends with all the other people in the room, then that person has $n - 1$ friends in the room. This is the maximum number of friends anyone can have in the room, and it also means that there is no-one with 0 friends in the room. Conversely, if there is someone with 0 friends in the room then there can't be anyone with $n - 1$ friends in the room.
So, if we set up a function $f : P \longmapsto \mathbb{N} \cup \{0\}$ from the set of people

P to the number of friends they have in the room, then the image of f can contain 0 or $n-1$ but not both. So, the image of f,

$$f(P) \subseteq \mathbb{N}_{n-2} \cup \{0\} \quad \text{or} \quad f(P) \subseteq \mathbb{N}_{n-1}.$$

Either way, the maximum cardinality of the image of f is $n-1$ which is obviously less than n, the number of people. Therefore, there cannot be any injection from the set of people to the set of numbers of friends and there must be two people with the same number of friends.

(3) *If there are 5 points in the plane $\mathbb{R}^2$ with integer co-ordinates then there must be at least one pair, say (x_1, y_1) and (x_2, y_2), whose midpoint, given by*

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right),$$

has integer co-ordinates.

A midpoint co-ordinate — say the x co-ordinate with formula $\frac{x_1+x_2}{2}$ — will be an integer when the sum $x_1 + x_2$ is even. By Proposition 1.2.4, this will happen when both x_1 and x_2 are even or both are odd.
For each of the 5 points the x and y co-ordinates may be odd or even in 4 different combinations: (odd, odd), (odd, even), (even, even), (even, odd). Since there are 5 points and only 4 possible combinations of odd and even, there cannot be an injection from points to odd/even combinations and, therefore, there must be, at least, one pair of points with the same combination. The midpoint of such a pair of points has integer co-ordinates.

(4) *Let $(a_1, a_2, \ldots, a_n)$ be a list of integers for $n > 1$. Then there exists a non-empty sublist whose sum is divisible by n.*

Let s_j be the partial sum $\sum_{i=1}^{j} a_i$ for $1 \leq j \leq n$ and let $m_j = s_j \bmod n$. Since there are only n distinct values in modulo-n, the values m_j must either:

- include all the modulo-n values including 0, in which case the sublist $(a_1, a_2, \ldots, a_j)$ for the j such that $m_j = 0$ has a sum that divides by n;
- or else there are $1 \leq i < j \leq n$ such that $m_j = m_i$ and then the sublist $(a_{i+1}, a_{i+2}, \ldots, a_j)$ has a sum that divides by n.

TODO: complete this

1.1.3.2 Infinite Sets

Definition 15. **(Infinite Set)** A set X is described as *infinite* if there is no $n \in \mathbb{N}$ such that $|X| = n$.

Proposition 1.1.12. *The set of natural numbers* $\mathbb{N}$ *is infinite.*

Proof. Suppose for contradiction that there is some $n \in \mathbb{N}$ such that $|\mathbb{N}| = n$. Then there exists a bijection,

$$f : \mathbb{N}_n \longmapsto \mathbb{N}.$$

Consider the value,

$$m = \sum_{i \in \mathbb{N}_n} f(i) = f(1) + f(2) + \cdots + f(n).$$

By closure of addition of naturals we have $m \in \mathbb{N}$ but for the naturals we also have,

$$\forall n \in \mathbb{N} \,.\, 1 \leq n \implies \forall i \in \mathbb{N}_n \,.\, f(i) < m.$$

Therefore, there is no $i \in \mathbb{N}_n$ such that $f(i) = m$ which contradicts the surjectivity of f and therefore also the bijectivity of f. □

Note a certain similarity here between this proof and Euler's proof of the infinitude of the primes.

1.1.4 Modular Arithmetic

1.1.4.1 Modular Arithmetic

Definition 16. **(Congruence Modulo m)** Two integers a and b are said to be *congruent modulo* m — denoted $a \equiv b \pmod{m}$ — iff integer division of a and b produces the same remainder. That's to say, if there exist $p, m \in \mathbb{Z}$ such that,

$$a = pm + r \quad \text{and} \quad b = qm + r$$

with $0 \leq r < m$.
In such a case, we have

$$b - a = qm - pm = m(q - p)$$

which leads to another description of congruence modulo m: a and b are *congruent modulo* m iff

$$m \mid (b - a).$$

Notation. The notation $a \equiv b \pmod{n}$ means that a is congruent to b in modulo-n. The notation $a \bmod n$ refers to the modulo operation which is the remainder between 0 and $n - 1$ of the integer division of a by n (some computer implementations return a negative remainder if the dividend or divisor is negative (see: blog post about this issue)).

The relationship between the notations is that: if the modulo operation is assumed to return the remainder b such that $0 \leq b < n$, then

$$a \equiv b \pmod{n} \iff a \bmod n = b.$$

Proposition 1.1.13. *Let $m \in \mathbb{N}$ and $a, b, c, d \in \mathbb{Z}$ with*

$$a \equiv b \pmod{m} \quad \textit{and} \quad c \equiv d \pmod{m}.$$

Then,

(i) $a + c \equiv b + d \pmod{m}$

(ii) $a - c \equiv b - d \pmod{m}$

(iii) $ac \equiv bd \pmod{m}$

(iv) $\forall z \in \mathbb{Z} \,.\, za \equiv zb \pmod{m}$

(v) $\forall n \in \mathbb{N} \,.\, a^n \equiv b^n \pmod{m}$

Proof.
Let $a = n_a m + r_1, b = n_b m + r_1, c = n_c m + r_2, d = n_d m + r_2$.

(i) $a + c \equiv b + d \pmod{m}$

$$a + c = n_a m + r_1 + n_c m + r_2 = (n_a + n_c)m + (r_1 + r_2)$$

(ii) $a - c \equiv b - d \pmod{m}$

$$a - c = n_a m + r_1 - n_c m - r_2 = (n_a - n_c)m + (r_1 - r_2)$$

(iii) $ac \equiv bd \pmod{m}$

$$ac = (n_a m + r_1)(n_c m + r_2) = (n_a n_c + r_1 n_c + r_2 n_a)m + r_1 r_2.$$

(iv) $\forall z \in \mathbb{Z} \,.\, za \equiv zb \pmod{m}$

$$za = z(n_a m + r_1) = (zn_a)m + zr_1$$

$$zb = z(n_b m + r_1) = (zn_b)m + zr_1$$

(v) $\forall n \in \mathbb{N} \,.\, a^n \equiv b^n \pmod{m}$

$$a^n = (n_a m + r_1)^n = p_0 m^n + p_1 m^{n-1} + \cdots + p_{n-1} m + {r_1}^n$$

$$b^n = (n_b m + r_1)^n = q_0 m^n + q_1 m^{n-1} + \cdots + q_{n-1} m + {r_1}^n$$

□

Proposition 1.1.14. *A number $x \in \mathbb{Z}_m$ has a multiplicative inverse if and only if $gcd(x, m) = 1$.*

Proof. Assume x^{-1} is a multiplicative inverse for $x \in \mathbb{Z}_m$. Then,

$$x^{-1}x = 1 \iff x^{-1}x \equiv 1 \pmod{m} \iff x^{-1}x = am + 1, \quad a \in \mathbb{Z}.$$

This means that we must have $1 = am + bx$ for some $a, b \in \mathbb{Z}$. Now if we have $d = gcd(x, m)$ then by **??** we must have $d \mid 1$. Therefore $d = 1$.

Clearly, also, if we have $gcd(x, m) = 1$ then we also have $1 = am + bx$ for some $a, b \in \mathbb{Z}$ and by following the previous logic in reverse we obtain that $b = x^{-1}$ is the multiplicative inverse of $x \in \mathbb{Z}_m$. □

---

(5) There is a well-known test for divisibility of an integer by 9: If the digits sum to a value that is divisible by 9, then the number divides by 9. So, 18 divides by 9 because $1 + 8 = 9$. This can be explained with modulo-9 arithmetic.

An integer in standard decimal format has the form,

$$n = d_0 + d_1 10^1 + d_2 10^2 + \cdots + d_k 10^k.$$

Since $10 \equiv 1 \pmod{9}$, we can apply (v) of Proposition 1.1.13 to deduce that, for all $i \in \mathbb{N}$,

$$10^i \equiv 1 \pmod{9}$$

and then applying (i) of Proposition 1.1.13, we have

$$d_0 + d_1 10^1 + d_2 10^2 + \cdots + d_k 10^k \equiv d_0 + d_1 + d_2 + \cdots + d_k \pmod{9}.$$

So, if the sum of the digits of n divides by 9 then n divides by 9.

(6) Suppose we need to determine if there exist any integers a and b that satisfy

$$7a^2 - 15b^2 = 1.$$

Since 15 is divisible by 5, in modulo-5 arithmetic subtracting $15b^2$ is an identity operation. That's to say,

$$7a^2 - 15b^2 = 7a^2 \pmod{5}.$$

So the equation becomes,

$$7a^2 \equiv 1 \pmod 5.$$

Now modulo-5 consists of $\{0, 1, 2, 3, 4\}$ so, applying (5) Proposition 1.1.13, we have

$$a^2 \pmod 5 \in \{0, 1, 4, 9 \equiv 4, 16 \equiv 1\} = \{0, 1, 4\}$$

and so, applying (iv) of Proposition 1.1.13, we have

$$7a^2 \pmod 5 \in \{0, 7 \equiv 2, 28 \equiv 3\} = \{0, 2, 3\}.$$

Since there are no possibilities that are congruent 1 in modulo-5, the equation cannot be satisfied.

Note that we could have chosen to work in modulo-7 but then we would have needed to consider a greater number of possiblities because modulo-7 has 8 distinct values as opposed to the 6 in modulo-5.

1.2 Numbers

1.2.1 Natural Numbers

1.2.1.1 Pre-requisites

Certain assumptions are required before even the definition of natural numbers. Most of these correspond to Euclid's "Common Notions" (Cornell Uni - Euclid's Definitions, Postulates and Common Notions). In particular, these give general notions of equivalence and substitutability of equal objects,

$$a = b \implies c + a = c + b.$$

Also required are some fundamental axioms of logic often referred to as the Laws of Thought - Wikipedia.

1.2.1.2 Peano Axioms

Axiom 1.2.1. *Closure under addition:*
For all $a, b \in \mathbb{N}$ *we have* $a + b \in \mathbb{N}$.

Axiom 1.2.2. *Closure under multiplication:*
For all $a, b \in \mathbb{N}$ *we have* $a \times b \in \mathbb{N}$.

Axiom 1.2.3. *Commutative Law for addition:*
For all $a, b \in \mathbb{N}$ *we have* $a + b = b + a$.

Axiom 1.2.4. *Associative Law for addition:*
For all $a, b, c \in \mathbb{N}$ *we have* $(a + b) + c = a + (b + c)$.

Axiom 1.2.5. *Commutative Law for multiplication:*
For all $a, b \in \mathbb{N}$ *we have* $a \times b = b \times a$.

Axiom 1.2.6. *Associative Law for multiplication:*
For all $a, b, c \in \mathbb{N}$ *we have* $(a \times b) \times c = a \times (b \times c)$.

Axiom 1.2.7. *Multiplicative Identity:*
There is a special element of $\mathbb{N}$*, denoted by 1, which has the property that for all* $n \in \mathbb{N},\ n \times 1 = n$*.*

Axiom 1.2.8. *Additive cancellation:*
For all $a, b, c \in \mathbb{N}$ *if* $a + c = b + c$ *then* $a = b$*.*

Axiom 1.2.9. *Multiplicative cancellation:*
For all $a, b, c \in \mathbb{N}$ *if* $a \times c = b \times c$ *then* $a = b$*.*

Axiom 1.2.10. *Distributive Law:*
For all $a, b, c \in \mathbb{N},\ a \times (b + c) = (a \times b) + (b \times c)$*.*

Axiom 1.2.11. *Definition of "less than":*
For all $a, b \in \mathbb{N},\ a < b$ *if and only if there is some* $c \in \mathbb{N}$ *s.t.* $a + c = b$*.*

Axiom 1.2.12. *Trichotomous property:*
For all $a, b \in \mathbb{N}$ *exactly one of the following is true:* $a = b,\ a < b,\ b < a$*.*

Not formally one of Peano's axioms but also required is the following axiom:

Axiom 1.2.13. *Well-Ordering Principle: see 1.1.1.4:*
Every non-empty subset of $\mathbb{N}$ *has a least element.*

Notation. We also write ab for $a \times b$.

Proposition 1.2.1. *If* $a, b \in \mathbb{N}$ *satisfy* $a \times b = a$*, then* $b = 1$*.*

Proof.

$$
\begin{aligned}
& a \times b = a = a \times 1 && \text{by Multiplicative Identity axiom} \\
\iff\ & b \times a = 1 \times a && \text{by Commutative Law for multiplication} \\
\iff\ & b = 1 && \text{by Multiplicative cancellation}
\end{aligned}
$$

□

Proposition 1.2.2. *If* $a, b, c \in \mathbb{N}$ *and* $a < b$ *then* $a \times c < b \times c$*.*

Proof.

$$
\begin{aligned}
& a < b \implies a + d = b \text{ for some } d \in \mathbb{N} && \text{by Definition of "less than"} \\
\therefore \quad & b \times c = (a + d) \times c = (a \times c) + (d \times c) && \text{by Distributive Law} \\
\therefore \quad & a \times c < (a \times c) + (d \times c) = b \times c && \text{by defn. "less than" and closure}
\end{aligned}
$$

□

Proposition 1.2.3. 1 *is the least element of* $\mathbb{N}$.

Proof. Assume m is the least element of $\mathbb{N}$. Then, also $m < 1$. So, by Proposition 1.2.2,

$$m < 1 \implies m \times m < 1 \times m = m$$

But, closure of multiplication and $m \times m < m$ together contradict the assumption that m is the least element of $\mathbb{N}$.
Therefore m cannot be less than 1. Since we know that $1 \in \mathbb{N}$ and that the minimum element of $\mathbb{N}$, m, cannot be less than 1, it follows that 1 must be the minimum element of $\mathbb{N}$ and $m = 1$. □

1.2.1.3 Odd and Even Numbers

Definition 17. **(Even number)** An *even* number, $n \in \mathbb{Z}$, is one that satisfies,

$$\exists\, m \in \mathbb{Z} \;\cdot\; n = 2m.$$

Definition 18. **(Odd number)** An *odd* number, $n \in \mathbb{Z}$, is one that satisfies,

$$\exists\, m \in \mathbb{Z} \;\cdot\; n = 2m + 1$$

Proposition 1.2.4. ***(Laws of addition of odd and even numbers)***

(i) The sum of even numbers is even;

(ii) the sum of odd numbers is even;

(iii) the sum of an odd number and an even number is odd.

Proof.

(i) Let $a = 2m$ and $b = 2n$. Then

$$a + b = 2m + 2n = 2(m + n).$$

(ii) Let $a = 2m + 1$ and $b = 2n + 1$. Then

$$a + b = 2m + 1 + 2n + 1 = 2(m + n + 1).$$

(iii) Let $a = 2m$ and $b = 2n + 1$. Then

$$a + b = 2m + 2n + 1 = 2(m + n) + 1.$$

□

1.2.1.4 Induction

Theorem 1.2.1. ***(Induction Principle)*** *Let $f : \mathbb{N} \longmapsto X$ be a bijective function from the naturals to some set of objects X and let $P : X \longmapsto \mathbb{B}$ be a predicate. Then, for $N < k \in \mathbb{N}$,*

$$P(f(N)) \wedge [\, P(f(k) \implies P(f(k+1)\,] \implies \forall n > N \in \mathbb{N} \,.\, P(f(n)).$$

Proof. Assume for contradiction that

$$P(f(N)) \wedge [\, P(f(k) \implies P(f(k+1)\,] \qquad (*)$$

but there exists some $n > N \in \mathbb{N}$ such that $\neg P(f(n))$. Then, the set

$$S = \{\, n \in \mathbb{N} \mid n > N \wedge \neg P(f(n)) \,\}$$

is non-empty. By the Well-Ordering Principle (1.1.1.4), there is a least element of S. Let a be the least element of S.

Since, by Proposition 1.2.3, 1 is the least element of $\mathbb{N}$, either $N = 1$ or $1 < N$. In either case, $1 < a$. Therefore, by the definition of $<$, $\exists b \in \mathbb{N}$ such that $1 + b = a$. By commutativity of addition therefore, $b + 1 = a$ and, again invoking the definition of $<$, we have $b < a$. Since $b < a$, $b \notin S$ and we have $P(f(b))$. But, by (*), we have

$$P(f(b)) \implies P(f(b+1)) = P(f(a)).$$

But this implies that $a \notin S$ which contradicts the definition of a. □

Corollary 1.2.1. ***(Strong Induction Principle)*** *Let $f : \mathbb{N} \longmapsto X$ be a bijective function from the naturals to some set of objects X and let $P : X \longmapsto \mathbb{B}$ be a predicate. Then, for $N < k \in \mathbb{N}$,*

$$[\forall k \leq N \,.\, P(f(k))] \;\wedge\; [\forall k \leq N \,.\, P(f(k) \implies P(f(k+1)]$$
$$\implies \forall n > N \in \mathbb{N} \,.\, P(f(n)).$$

1.2.2 Integers

1.2.2.1 Construction of Integers from Peano Numbers

Definition 19. **(Integers)** An integer n is defined as an equivalence class of a relation on the set of ordered pairs of naturals,

$$R \subset (\mathbb{N} \times \mathbb{N}) \times (\mathbb{N} \times \mathbb{N})$$

defined as, for $a, b, c, d \in \mathbb{N}$,

$$(a, b)\ R\ (c, d) \iff a + d = c + b.$$

Using $[x]$ to denote the equivalence class of x, addition and multiplication are defined as follows:

- $n_1 + n_2 = [(a, b)] + [(c, d)] = [(a + c, b + d)]$
- $n_1 \times n_2 = [(a, b)] \times [(c, d)] = [(ac + bd, ad + bc)]$

Ordering is defined as

$$[(a, b)] < [(c, d)] \iff a + d < b + c.$$

Notation. We assign the standard notations to the integers as defined above (where $[x]$ denotes the equivalence class of x):

- $[(1, 1)] = 0$
- $[(n + 1, 1)] = n \quad \text{and} \quad [(1, n + 1)] = -n$

Definition 20. **(Positivity and Negativity)** The most common convention is that 0 is neither positive nor negative and so, the positive integers $\mathbb{Z}^+$ do not include 0 (likewise the negative integers $\mathbb{Z}^-$). To include 0 we must refer to the non-negative integers $\mathbb{Z}_0^+$ or $\mathbb{N}_0$.

Axiom 1.2.14. ***(Well-Ordering Principle for Integers)*** *Any non-empty set of integers that has a lower bound, has a least member.*

1.2.2.2 Divisibility and Primality

Definition 21. **(Integer Divisibility)** For integers a and b, we say that a *divides* b — denoted $a \mid b$ — iff

$$\exists z \in \mathbb{Z} \,.\, b = za.$$

The integer 0 has the following properties w.r.t. integer divisibility,

$$\forall z \in \mathbb{Z} \,.\, z \mid 0 \quad \text{and} \quad 0 \mid x \implies x = 0.$$

Note that Integer Divisibiity "$\mid$" is a boolean-valued operator: it doesn't provide the result of the division. This is especially relevant given the convention that $0 \mid 0$ *as division by 0, in fact, is undefined.*

Definition 22. **(Euclidean Division)** Given two integers a and b, with $b \neq 0$, if we find two integers q and r such that

$$a = bq + r, \qquad 0 \leq r < |b|$$

where $|b|$ denotes the absolute value of b, then this process is referred to as *Euclidean Division* or *Integer Division*.

In the above: a is called the *dividend*, b is called the *divisor*, q is called the *quotient* and r is called the *remainder*.

Definition 23. **(Divisor)** The *divisors* or *factors* of an integer are the integers (not including itself but including 1) that evenly divide it — which is to say, the divisors for which Euclidean (Integer) Division produces a remainder of 0 or alternatively, the set of divisors of an integer a is

$$D_a = \{\, m \in \mathbb{Z} \mid m \mid a \,\}.$$

Sometimes the term *divisor* is used to refer to the positive such integers only but sometimes these are, more specifically, referred to as *proper divisors*. The use of the term *divisor* in the context of integer division is yet another usage of the term.

Theorem 1.2.2. ***(Division Theorem a.k.a. Remainder Theorem.)*** *Given two integers a and b, with $b \neq 0$, there exist unique integers q and r such that*

$$a = bq + r, \qquad 0 \leq r < |b|$$

where $|b|$ denotes the absolute value of b.

Proof. Consider first the case $b < 0$. Setting $b' = -b$ and $q' = -q$, the equation $a = bq + r$ may be rewritten as $a = b'q' + r$ and the inequality $0 \leq r < |b|$ may be rewritten as $0 \leq r < |b'|$. This reduces the existence for the case $b < 0$ to that of the case $b > 0$.
Similarly, if $a < 0$ and $b > 0$, setting $a' = -a$, $q' = -q - 1$, and $r' = b - r$, the equation $a = bq + r$ may be rewritten as $a' = bq' + r'$, and the inequality $0 \leq r < |b|$ may be rewritten as $0 \leq r' < |b|$. Thus the proof of the existence is reduced to the case $a \geq 0$ and $b > 0$ — which will be considered in the remainder of the proof.

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and let $Q = \{\, m \in \mathbb{N}_0 \mid bm \leq a \,\}$. Then Q is non-empty because $0 \in Q$ and Q is finite because

$$bm \leq a \ \wedge \ b > 0 \implies m \leq bm \leq a$$

so Q is an upper-bounded set of positive integers and, by 1.2.14 therefore, has a maximum element. Let q be the maximum element and let

$$r = a - bq > 0.$$

Since q is the maximum element of Q,

$$b(q+1) > a \iff bq + b > a \iff b > a - bq = r.$$

Therefore $a = bq + r$ and $0 \leq r < b$.

It remains to be shown that the integers q and r are unique. Suppose that q' and r' are also integers satisfying

$$a = bq' + r' = bq + r.$$

Suppose w.l.o.g. that $q \geq q'$. Then,

$$0 \leq r = a - bq \leq a - bq' = r' < b$$

which implies that

$$\begin{aligned} & 0 \leq r' - r < b \\ \iff \quad & 0 \leq (a - bq') - (a - bq) < b \\ \iff \quad & 0 \leq bq - bq' < b \\ \iff \quad & 0 \leq b(q - q') < b \\ \iff \quad & 0 \leq q - q' < 1. \end{aligned}$$

But q and q' are both positive integers so, by closure of addition of integers, $q - q'$ is also an integer. Therefore

$$0 \leq q - q' < 1 \implies q - q' = 0 \iff q = q' \iff r = r'.$$

□

Greatest Common Divisor (a.k.a. Highest Common Factor)

Definition 24. **(Common Divisor (gcd))** A *common divisor* of two integers a and b is an integer $c \in \mathbb{Z}$ such that

$$c \in D_a \cap D_b = \{\, z \in \mathbb{Z} \mid (z \mid a) \wedge (z \mid b) \,\}.$$

A common divisor of a and b divides any integer linear combination of a and b: If z is a common divisor of a and b then there exist $p, q \in \mathbb{Z}$ such that

$$ma + nb = mzp + nzq = z(mp + nq).$$

Definition 25. **(Greatest Common Divisor (gcd))** The *greatest common divisor* of two integers a and b — at least one of which is non-zero — is the greatest divisor of both a and b.
That's to say, if d is the greatest common divisor of a and b then,

$$d = \max D_a \cap D_b = \max\{\, z \in \mathbb{Z} \mid (z \mid a) \ \wedge \ (z \mid b) \,\}$$
$$= \max\{\, z \in \mathbb{Z} \mid \exists m, n \in \mathbb{Z} \,.\, a = mz \ \wedge \ b = nz \,\}.$$

Definition 26. **(Coprime Numbers)** Two integers are said to be *coprime* if their gcd is 1.

Proposition 1.2.5. *The greatest common divisor always exists.*

Proof. Let a and b be integers with at least one of them non-zero and let

$$D_a \cap D_b = \{\, z \in \mathbb{Z} \mid \exists m, n \in \mathbb{Z} \,.\, a = mz \ \wedge \ b = nz \,\}.$$

$D_a \cap D_b$ is non-empty because 1 is always a member. Furthermore, each $z \in D_a \cap D_b$ must be such that $z \leq \min\{|a|, |b|\}$. Therefore, $D_a \cap D_b$ is a non-empty set of upper-bounded integers and so, by Axiom 1.2.14, has a unique maximum — which is the gcd. □

Proposition 1.2.6. *The greatest common divisor is always positive.*

Proof. Let a and b be integers with at least one of them non-zero and let d be the greatest common divisor of a and b so that,

$$d = \max D_a \cap D_b = \max\{\, z \in \mathbb{Z} \mid \exists m, n \in \mathbb{Z} \,.\, a = mz \ \wedge \ b = nz \,\}.$$

Firstly, notice that $0 \notin D_a \cap D_b$ because at least one of a and b is non-zero. So $d \neq 0$.

Suppose $d < 0$. Then, since d is the maximum of $D_a \cap D_b$, for each $z \in D_a \cap D_b$ we have $z < 0$ and also, for some $m, n \in \mathbb{Z}$,

$$a = mz \quad \text{and} \quad b = nz.$$

But then we also have,

$$a = (-m)(-z) \quad \text{and} \quad b = (-n)(-z)$$

which implies that, for $p = -m$, $q = -n \in \mathbb{Z}$,

$$a = p(-z) \ \wedge \ b = q(-z).$$

Therefore $-z \in D_a \cap D_b$. But

$$z < 0 \implies -z > 0 > d.$$

This contradicts the definition of d as the maximum of $D_a \cap D_b$.

Therefore $d > 0$. □

Proposition 1.2.7. *If d is the gcd of a and b such that, for some $m, n \in \mathbb{Z}$,*

$$a = md \ \wedge \ b = nd,$$

then m and n are coprime.

Proof. Let e be the gcd of m and n. By Proposition 1.2.6, we have $e \in \mathbb{Z}^+$. Suppose, for contradiction, that m and n are not coprime so that $e \neq 1$. Then, because $e \in \mathbb{Z}^+$, we must have $e > 1$. Since e is a common divisor of m and n, there exist some $p, q \in \mathbb{Z}$ such that $m = pe$ and $n = qe$. Then we also have,

$$a = md = ped \ \wedge \ b = nd = qed$$

meaning that ed is a common divisor of a and b. Since $e > 1$ it follows that $ed > d$ which contradicts the assumption that d is the greatest common divisor of a and b.
Therefore, m and n are coprime. □

Corollary 1.2.2. *If d is the gcd of a and b and $e \in D_a \cap D_b$ — that is e is a common divisor of a and b — then $e \mid d$.*

Proof. By Proposition 1.2.7, we have coprime $m, n \in \mathbb{Z}$ such that

$$a = md \ \wedge \ b = nd$$

and by $e \in D_a \cap D_b$ we have

$$e \mid a = md \quad \text{and} \quad e \mid b = nd.$$

Since m and n are coprime, their only common divisors are 1 and -1. So the above implies that either:

- $e \in \{-1, 1\}$: in which case $e \mid d$ as it divides every integer;
- $e \notin \{-1, 1\}$: in which case we must have $e \mid d$.

□

Proposition 1.2.8. *If euclidean division (22) of a by b yields*

$$a = bq + r$$

then

$$\gcd a, b = \gcd b, r.$$

Proof. Since $r = a - bq$, any divisor of r must also divide a. Therefore, the set of divisors of r and b is the same set as the set of divisors of a and b. As a result, they have the same maximum — which is the gcd. □

> *Note that the result is not true of* $\gcd a, r$ *because r is not the remainder after dividing by a.*

Euclid's Algorithm for the gcd

Proposition 1.2.8 leads to the following algorithm for calculating the gcd of a and b.

Algorithm 1 gcd a,b

Input: Two integers a and b, not both zero.
Output: The greatest common divisor of a and b.

if $b = 0$ **then**
  return a.
else
  $r \leftarrow$ remainder after integer division (22) of a by b.
  return gcd b, r
end if

Analysis of the gcd algorithm:

Denote the arguments a and b on the i-th iteration of the algorithm as a_i and b_i.

On each iteration, $a_i = b_{i-1}$ and the argument b_i becomes the remainder after integer division of the argument a_{i-1} by b_{i-1} of the previous iteration. So, for each iteration i, we have

$$a_i = b_{i-1} \quad \text{and} \quad 0 \leq b_i < |b_{i-1}| \,.$$

So the arguments b_i are becoming non-negative and decreasing in absolute value and the arguments a_i are taking on the previous iteration's value of b_i. Therefore both arguments are decreasing towards 0 but b_i leads the way each time. Since this is integer arithmetic, eventually b_i will become 0 while a_i is not yet 0, and so the recursion will terminate.
Let the arguments to the 1st iteration be $a_1 = a$, $b_1 = b$. Then the 2nd iteration of the algorithm executes with $a_2 = b_1$, $b_2 = a_1 - b_1 q_1$ and the 3rd iteration, $a_3 = b_2$, $b_3 = a_2 - b_2 q_2$ and so on. If the algorithm terminates on the n-th iteration, then the final state is $a_n = b_{n-1}$, $b_n = 0$. The result returned is

$$\begin{aligned} d &= a_n = b_{n-1} \\ &= a_{n-2} - b_{n-2} q_{n-2} \\ &= a_{n-2} - (a_{n-3} - b_{n-3} q_{n-3}) q_{n-2} \\ &= b_{n-3} - (a_{n-3} - b_{n-3} q_{n-3}) q_{n-2} \\ &= (1 + q_{n-3} q_{n-2}) b_{n-3} - a_{n-3} \\ &\vdots \end{aligned}$$

we can continue unpacking the result until we get to an expression in terms of a_1 and b_1.

Example:

$$\begin{aligned}\gcd 8, 14 &= \gcd 14, 8 \\ &= \gcd 8, 6 \\ &= \gcd 6, 2 \\ &= \gcd 2, 0 \\ &= 2.\end{aligned}$$

Then we can unpack the result as follows,

$$\begin{aligned}2 &= 8 - (1)6 \\ &= 8 - (1)(14 - (1)8) \\ &= (2)8 - (1)14.\end{aligned}$$

Proposition 1.2.9. *Let a and b be integers, not both zero and let $d = \gcd a, b$. Then*

$$\exists m, n \in \mathbb{Z} \;.\; d = ma + nb.$$

Proof. The proof is the analysis of the gcd algorithm above. □

Proposition 1.2.10. *Let a and b be integers with $d = \gcd a, b$. Then, for any $c \in \mathbb{Z}$,*

$$(c \mid a) \;\wedge\; (c \mid b) \implies (c \mid d).$$

Proof. By Proposition 1.2.9,

$$\exists m, n \in \mathbb{Z} \;.\; d = ma + nb$$

and also

$$c \mid a \implies c \mid ma \quad \text{and} \quad c \mid b \implies c \mid nb$$

and therefore

$$c \mid (ma + nb) = d.$$

□

Proposition 1.2.11. *Let a and b be coprime integers. Then,*

$$(a \mid r) \ \wedge \ (b \mid r) \implies (ab \mid r).$$

Proof. By the definition of coprime numbers (26), we have $\gcd a, b = 1$. Then, by Proposition 1.2.9, we also have some $m, n \in \mathbb{Z}$ such that,

$$1 = ma + nb.$$

Furthermore,

$$a \mid r \implies \exists p \in \mathbb{Z} \,.\, r = pa \quad \text{and} \quad b \mid r \implies \exists q \in \mathbb{Z} \,.\, r = qb.$$

Putting these together we have,

$$r = r \times 1 = r(ma + nb) = rma + rnb = (qb)ma + (pa)nb = ab(qm + pn).$$

So $ab \mid r$. □

Note that this is not generally true. Take the example of 2 and 4:

$$2 \mid 4 \quad \textit{and} \quad 4 \mid 4 \quad \textit{but} \quad (2 \times 4) \nmid 4.$$

Lowest Common Multiple

The lowest common multiple of two numbers is formed by the multiplication of all the prime factors that occur in the two numbers where repititions of prime factors are important. That's to say, the lowest common multiple of 4 and 8 is not 2 (which is the highest common factor/greatest common divisor) but 8 because in 8, the factor 2 occurs three times (as 2^3) and it occurs twice in 4,

$$lcm(4, 8) = lcm(2 \times 2, 2 \times 2 \times 2) = 2 \times 2 \times 2.$$

The general formula for the lowest common multiple may be expressed in terms of the gcd as follows

$$d = gcd(a, b) \implies lcm(a, b) = d \times (a/d) \times (b/d).$$

Prime Numbers

The concept of primality has been extended to negative numbers and complex numbers (see this exchange for a discussion: math.stackexchange) but the standard definition is limited to the context of non-negative integers. That is the sole context of the following discussion of primality.

Definition 27. **(Prime and Composite Numbers)** A *prime* number has precisely two divisors: 1 and itself. A *composite* number has more than two divisors. The special cases 0 and 1 are neither prime nor composite.

A positive integer $a > 1$ is *prime* iff there *does not* exist $m, n \in \mathbb{Z}^+$

$$\frac{a}{m} = n \ \wedge \ m \notin \{1, a\}$$

and is *composite* if there *does* exist some such $m, n \in \mathbb{Z}^+$.

The reasons for 0 and 1 not being categorized either as prime or as composite are described in this conversation: math.stackexchange.

> *"Any number either is prime or is measured by some prime number."*
> *Euclid, Elements Book VII, Proposition 32*

Proposition 1.2.12. *Every positive integer either is prime or is divided by some prime.*

Proof. A positive integer $z > 1 \in \mathbb{Z}^+$ is either prime or is composite. If it is composite then there exists $m, n \in \mathbb{Z}$ such that,

$$\frac{z}{m} = n.$$

But then, n is also a positive integer greater than 1 that either is prime or is composite and so we can repeat the process until we necessarily reach a

number that is not composite and is greater than 1. Such a number is, by definition, prime. □

Proposition 1.2.13. ***(Euclid's Lemma)*** *If p is a prime number then, for integers a and b,*

$$(p \mid ab) \implies (p \mid a) \vee (p \mid b).$$

Proof. If $p \nmid a$ then, since p is prime, p and a are coprime. Then, by Proposition 1.2.11,

$$(p \mid ab) \wedge (a \mid ab) \implies (pa \mid ab).$$

But $pa \mid ab$ clearly implies that $p \mid b$. □

Theorem 1.2.3. ***(Fundamental Theorem of Arithmetic)*** *Every integer greater than 1 can be expressed as a product of primes that is unique upto ordering.*

Proof of existence

Proof. It must be shown that every integer greater than 1 is either prime or a product of primes. First, 2 is prime. Then, by strong induction, assume this is true for all numbers greater than 1 and less than n. If n is prime, there is nothing more to prove. Otherwise, there are integers a, b where $n = ab$, and $1 < a \leq b < n$. By the induction hypothesis, $a = p_1 p_2 ... p_j$ and $b = q_1 q_2 ... q_k$ are products of primes. But then $n = ab = p_1 p_2 ... p_j q_1 q_2 ... q_k$ is a product of primes. □

Proof of uniqueness

Proof. Suppose, to the contrary, that there is an integer that has two distinct prime factorizations. Let n be the least such integer and write $n = p_1 p_2 ... p_j = q_1 q_2 ... q_k$, where each p_i and q_i is prime. (Note that j and k are both at least 2.) We see that p_1 divides $q_1 q_2 ... q_k$, so p_1 divides some q_i by Euclid's lemma. Without loss of generality, say that p_1 divides q_1. Since p_1 and q_1 are both prime, it follows that $p_1 = q_1$. Returning to our factorizations of n, we may cancel these two terms to conclude that $p_2 ... p_j = q_2 ... q_k$. We now have two distinct prime factorizations of some integer strictly smaller than n, which contradicts the minimality of n. □

1.2.2.3 Some Proofs on the Integers

Proposition 1.2.14. *For any integer m, $\sqrt{m}$ is rational iff m is a square, i.e. $m = a^2$ for some integer a.*

To begin with we show the easier direction of implication: $(m = a^2) \implies (\sqrt{m}$ is rational$)$.

Proof. Assume $m, a, b \in \mathbb{Z}$.

$$\begin{aligned} & m = a^2 \\ \iff & \sqrt{m} = |a| \\ & = a/b \text{ for } b = 1 \text{ or } -1. \end{aligned}$$

□

Now the other (harder) direction, $(\sqrt{m}$ is rational$) \implies (m = a^2)$.

Proof. Assume $m, a, b \in \mathbb{Z}$. $(\sqrt{m}$ is rational$)$ can be formalized as:

$$\exists\, m, a, b \in \mathbb{Z} \cdot (\sqrt{m} = \frac{a}{b}) \ \wedge \ (a \text{ and } b \text{ are coprime})$$

$$\begin{aligned} \sqrt{m} &= \frac{a}{b} \\ \implies m &= \frac{a^2}{b^2} \\ \iff mb^2 &= a^2 \end{aligned}$$

But a and b are coprime so they don't share any prime factors. This means that a^2 and b^2 also don't share any prime factors. So, if $|b| > 1$, the prime factorization of mb^2 is necessarily different from that of a^2 meaning that $mb^2 \neq a^2$ contradicting the hypothesis of coprimality. On the other hand, if $|b| = 1$, then b has no prime factors (its prime factorization is empty) and so mb^2 has the same prime factorization as m which may be equal to that of a^2 in the case that $m = a^2$. □

Proposition 1.2.15. *For all nonnegative integers $a > b$ the difference of squares $a^2 - b^2$ does not give a remainder of 2 when divided by 4.*

Beginner's attempt - try proof by contradiction:

$$\begin{aligned} a^2 - b^2 &= 4n + 2 \\ 2k &= 4n + 2 && \text{by } a^2 - b^2 \text{ even} \\ k &= 2n + 1 \implies && k \text{ is some odd number.} \end{aligned}$$

So, proof by contradiction is our first instinct but doesn't seem to get us anywhere. Instead, proceed by cases:

Case a, b are even:

$$\begin{aligned} &\exists\, k, l \in \mathbb{Z} \;\cdot\; a = 2k, b = 2l \\ &\implies a^2 - b^2 = 4k^2 - 4l^2 \\ &\qquad\quad = 4\left(k^2 - l^2\right) \\ &\qquad\quad = 4m \;\text{ where }\; m \in \mathbb{Z} \end{aligned}$$

So 4 divides $a^2 - b^2$ with 0 remainder.

Case a, b are odd:

$$\begin{aligned} &\exists\, k, l \in \mathbb{Z} \;\cdot\; a = 2k + 1, b = 2l + 1 \\ &\implies a^2 - b^2 = \left(4k^2 + 4k + 1\right) - \left(4l^2 + 4l + 1\right) \\ &\qquad\quad = 4\left(k^2 + k - l^2 - l\right) \\ &\qquad\quad = 4m \;\text{ where }\; m \in \mathbb{Z} \end{aligned}$$

So, again, 4 divides $a^2 - b^2$ with 0 remainder.

Case a even, b odd:

$$\begin{aligned} &\exists\, k, l \in \mathbb{Z} \;\cdot\; a = 2k, b = 2l + 1 \\ &\implies a^2 - b^2 = 4k^2 - \left(4l^2 + 4l + 1\right) \end{aligned}$$

$$= 4\left(k^2 - l^2 - l\right) - 1$$
$$= 4m + 3 \quad \text{where} \quad m = k^2 - l^2 - l - 1 \in \mathbb{Z}$$

So, here, 4 divides $a^2 - b^2$ with 3 remainder. So the proposition is proven as we have proven all the possible cases.
TODO: There is also another approach given in the Cambridge University Discrete Mathematics lecture notes

1.2.3 Absolute Value

Definition 28. The **absolute value** function is defined,

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

Proposition 1.2.16. $|a|\,|b| = |ab|$.

Proof. By the definition of absolute value,

$$|ab| = \begin{cases} ab & ab \geq 0 \\ -ab & ab < 0. \end{cases}$$

Extending the definition to the product of absolute values,

$$|a|\,|b| = \begin{cases} ab & a, b \geq 0 \\ -ab & a < 0, b \geq 0 \\ -ab & a \geq 0, b < 0 \\ ab & a, b < 0. \end{cases}$$

We can see that these are equivalent because,

$$ab \text{ is } \begin{cases} \geq 0 & a, b \geq 0 \text{ or } a, b < 0 \\ < 0 & a < 0, b \geq 0 \text{ or } a \geq 0, b < 0. \end{cases}$$

□

1.2.3.1 The Triangle Inequality

$$|x| \geq x, |y| \geq y \implies |x| + |y| \geq x + y$$

$$|x+y| = \begin{cases} |x|+|y| & x,y \geq 0 \\ |-|x|+|y|| & x<0, y \geq 0 \\ ||x|-|y|| & x \geq 0, y<0 \\ |-(|x|+|y|)| & x,y<0 \end{cases} \iff \begin{cases} ||x|+|y|| & x,y \geq 0 \text{ or } x,y<0) \\ ||x|-|y|| & x<0, y \geq 0 \text{ or } x \geq 0, y<0 \end{cases}$$

Clearly, $||x|+|y|| \geq ||x|-|y||$ so that,

$$|x+y| \leq ||x|+|y|| = |x|+|y|$$

and this is known as the "triangle inequality".

Proposition 1.2.17. $|x-y| \leq |x-z|+|y-z|$

Proof.

$$|x-y| = |(x-z)+(z-y)| \leq |x-z|+|z-y| = |x-z|+|y-z|$$

□

Proposition 1.2.18. $|x-y| \geq ||x|-|y||$

Proof. Need to show $-|x-y| \leq |x|-|y| \leq |x-y|$. So, prove as two separate inequalities:

$$\begin{aligned} & |y| = |x+(y-x)| \leq |x|+|y-x| \\ \iff \qquad & -|y-x| = -|x-y| \leq |x|-|y| \end{aligned}$$

$$\begin{aligned} & |x| = |(x-y)+y| \leq |x-y|+|y| \\ \iff \qquad & |x|-|y| \leq |x-y| \end{aligned}$$

□

1.2.4 Rational Numbers

TODO: construction of the rationals from the integers.

1.2.5 Real Numbers

Question: Should the reals be considered a superclass of the naturals or the other way around? Answer: We would have to use the approach that is used in computer programming languages. That's to say, reals are a wider type (so similar to a base class) and operations are, effectively defined over the wider type. So, if a natural number is combined under some operation with a real number then the result is a real number. TODO: some words about constructing the reals from the rationals.

1.2.6 Complex Numbers

Definition 29. Complex numbers are the members of the set

$$\mathbb{C} = \{\, a + bi \mid a, b \in \mathbb{R} \,\}$$

and i is the imaginary number such that $i^2 = -1$.

In some contexts (e.g. physics), j is sometimes used as the imaginary number.

If $z = a + bi$ is a complex number then $\mathrm{Re}(z) = a$ and $\mathrm{Im}(z) = b$.

1.2.6.1 Roots of Quadratics

Proposition 1.2.19. *For any numbers a and b,*

$$(a^2 + b^2) - 2ab = (a - b)^2.$$

Proof.

$$(a - b)^2 = a^2 - 2ab + b^2 \iff (a^2 + b^2) - 2ab = (a - b)^2.$$

□

Corollary 1.2.3. *Let $z \in \mathbb{C}, a, b \in \mathbb{R}$ and*

$$p(z) = (z + a)(z + b) = z^2 + (a + b)z + ab$$

be a complex polynomial with real coefficients. Then the discriminant of $p(z)$ is

$$(a + b)^2 - 4ab = a^2 + b^2 + 2ab - 4ab = (a^2 + b^2) - 2ab = (a - b)^2.$$

So the roots of $p(z)$ are:

- *real-valued and distinct if:*

$$(a - b)^2 > 0;$$

- *real-valued and equal if:*
$$(a-b)^2 = 0;$$
- *complex-valued and conjugates if:*
$$(a-b)^2 < 0.$$

1.2.6.2 The Modulus

Definition 30. The **modulus** of a complex number, $z = a + bi$, is the quantity defined as,
$$|z| = \sqrt{a^2 + b^2}.$$

Proposition 1.2.20. *The modulus of a complex number is greater than or equal to the real part or the imaginary part. That's to say, for any $z \in \mathbb{C}$,*
$$|z| \geq \mathrm{Re}(z) \wedge |z| \geq \mathrm{Im}(z).$$

Proof. From the definition, if $z = a + bi$ then $\mathrm{Re}(z) = a$ and $\mathrm{Im}(z) = b$ and the modulus,
$$|z| = \sqrt{a^2 + b^2} = \sqrt{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2} \geq \mathrm{Re}(z), \mathrm{Im}(z).$$
□

Proposition 1.2.21. *Properties of the modulus of complex numbers (z, z_1, z_2 refer to complex numbers):*

(i) Real-valued: $\forall z \,.\, |z| \in \mathbb{R}$.

(ii) Positive definiteness (wikipedia):
$$|0| = 0 \quad \textit{and} \quad \forall z \neq 0 \,.\, |z| > 0.$$

(iii) Homomorphism w.r.t. scalar multiplication: $|z_1 z_2| = |z_1|\,|z_2|$.

(iv) Triangle Inequality: $|z_1 + z_2| \leq |z_1| + |z_2|$.

Proof.
Using the definition of a complex number 1.2.6:

(i) For $z = a + ib$ we have

$$a, b \in \mathbb{R} \implies \sqrt{a^2 + b^2} \in \mathbb{R}.$$

(ii) For $z = a + ib$ we have $a, b \in \mathbb{R}$ so we can use the properties of the real numbers to deduce,

$$a = b = 0 \implies a^2 + b^2 = 0 \implies |z| = \sqrt{a^2 + b^2} = 0.$$

$$a, b \neq 0 \implies a^2 + b^2 > 0 \implies |z| = \sqrt{a^2 + b^2} > 0.$$

(iii) Firstly observe that, for $z_1 = a_1 + b_1 i$, $z_2 = a_2 + b_2 i$ we have,

$$\begin{aligned} z_1 z_2 &= (a_1 + b_1 i)(a_2 + b_2 i) \\ &= a_1 a_2 + (a_1 b_2 + a_2 b_1)i - b_1 b_2 \\ &= (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)i. \end{aligned}$$

Then,

$$\begin{aligned} |z_1 z_2| &= [(a_1 a_2 - b_1 b_2)^2 + (a_1 b_2 + a_2 b_1)^2]^{\frac{1}{2}} \\ &= [{a_1}^2 {a_2}^2 - 2a_1 a_2 b_1 b_2 + {b_1}^2 {b_2}^2 + {a_1}^2 {b_2}^2 + 2a_1 a_2 b_1 b_2 + {a_2}^2 {b_1}^2]^{\frac{1}{2}} \\ &= [{a_1}^2 {a_2}^2 + {b_1}^2 {b_2}^2 + {a_1}^2 {b_2}^2 + {a_2}^2 {b_1}^2]^{\frac{1}{2}} \\ &= [({a_1}^2 + {b_1}^2)({a_2}^2 + {b_2}^2)]^{\frac{1}{2}} \\ &= [({a_1}^2 + {b_1}^2)]^{\frac{1}{2}} \, [({a_2}^2 + {b_2}^2)]^{\frac{1}{2}} \\ &= |z_1| \, |z_2| \, . \end{aligned}$$

(iv) Let $z_1 = a_1 + b_1 i$, $z_2 = a_2 + b_2 i$. Then,

$$\begin{aligned} |z_1 + z_2|^2 &= |(a_1 + a_2) + (b_1 + b_2)i|^2 \\ &= (a_1 + a_2)^2 + (b_1 + b_2)^2 \\ &= {a_1}^2 + {a_2}^2 + 2a_1 a_2 + {b_1}^2 + {b_2}^2 + 2b_1 b_2 \\ &= ({a_1}^2 + {a_2}^2 + {b_1}^2 + {b_2}^2) + 2(a_1 a_2 + b_1 b_2), \end{aligned}$$

$$
\begin{aligned}
(|z_1| + |z_2|)^2 &= (\sqrt{{a_1}^2 + {b_1}^2} + \sqrt{{a_2}^2 + {b_2}^2})^2 \\
&= {a_1}^2 + {b_1}^2 + 2\sqrt{({a_1}^2 + {b_1}^2)({a_2}^2 + {b_2}^2)} + {a_2}^2 + {b_2}^2 \\
&= ({a_1}^2 + {a_2}^2 + {b_1}^2 + {b_2}^2) + 2\sqrt{({a_1}^2 + {b_1}^2)({a_2}^2 + {b_2}^2)}.
\end{aligned}
$$

So

$$
\begin{aligned}
& |z_1 + z_2|^2 \leq (|z_1| + |z_2|)^2 \\
\iff \quad & (a_1 a_2 + b_1 b_2) \leq \sqrt{({a_1}^2 + {b_1}^2)({a_2}^2 + {b_2}^2)} \\
\iff \quad & (a_1 a_2 + b_1 b_2)^2 \leq ({a_1}^2 + {b_1}^2)({a_2}^2 + {b_2}^2) \\
\iff \quad & (a_1 a_2)^2 + (b_1 b_2)^2 + 2a_1 a_2 b_1 b_2 \leq \\
& \qquad\qquad (a_1 a_2)^2 + (a_1 b_2)^2 + (b_1 a_2)^2 + (b_1 b_2)^2 \\
\iff \quad & 2a_1 a_2 b_1 b_2 \leq (a_1 b_2)^2 + (b_1 a_2)^2 \\
\iff \quad & 2(a_1 b_2)(b_1 a_2) \leq (a_1 b_2)^2 + (b_1 a_2)^2
\end{aligned}
$$

By Proposition 1.2.19, for any numbers $p = a_1 b_2, q = b_1 a_2$,

$$2pq \leq p^2 + q^2.$$

Equality is attained when

$$a_1 b_2 = b_1 a_2 \iff a_1 + b_1 = \alpha(a_2 + b_2)$$

for $\alpha = \frac{a_1}{a_2} = \frac{b_1}{b_2}$. *This is actually an instance of Cauchy-Schwarz (Theorem 2.6.2).*

□

Corollary 1.2.4. *If* $\mathbb{C}$ *is considered to be a 1-dimensional vector space over itself, then the modulus function is a vector norm (definition: 2.6.1.3, properties: Proposition 2.6.12) in this space.*

1.2.6.3 The Exponential Form

Definition 31. The **exponential form** of a complex number $z = a + ib$ is defined as

$$z = re^{i\theta} \quad \text{for} \quad r, \theta \in \mathbb{R}$$

where r is the modulus $|z|$ and θ is an angle in radians. This implies that

$$re^{i\alpha} = re^{i(\alpha+2n\pi)} \quad \text{for} \quad n \in \mathbb{Z}.$$

The angle $\theta = \alpha + 2n\pi \in (-\pi, \pi]$ is known as the **principal argument** and is denoted **arg(z)**.

TODO: De Moivres' Formula

$$e^{(a+bi)t} = e^a(\cos t + i \sin t)^b = e^a(\cos bt + i \sin bt)$$

where we have used

$$\begin{aligned}(\cos t + i \sin t)^2 &= \cos^2 t + i \sin 2t - \sin^2 t \\ &= \cos 2t + i \sin 2t\end{aligned}$$

$$\begin{aligned}(\cos 2t + i \sin 2t)^2 &= \cos^2 2t + 2i \sin 2t \cos 2t - \sin^2 2t \\ &= \cos 4t + i \sin 4t.\end{aligned}$$

(more difficult to prove for non-even integer powers)

1.2.6.4 The Complex Conjugate

Definition 32. **(Complex Conjugate)** For a complex number $z = a + ib$, the **conjugate** of z, denoted $\overline{z}$, is defined as,

$$\overline{z} = a - ib.$$

Proposition 1.2.22. *Properties of the complex conjugate:*

(i) $z = \overline{z} \iff \mathrm{Im}(z) = 0$

(ii) $z = re^{i\theta} \iff \overline{z} = re^{-i\theta}$

(iii) $\overline{z+w} = \overline{z} + \overline{w}$

(iv) $\overline{zw} = \overline{z}\,\overline{w}$

(v) $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$

(vi) $z\overline{z} = |z|^2$

Proof.

(i)

$$\begin{aligned} & z = \overline{z} \\ \iff \quad & a + ib = a - ib \\ \iff \quad & ib = -ib \\ \iff \quad & b = -b \implies b = 0. \end{aligned}$$

(ii) Since cosine is an even function and sine is an odd function, we have

$$\begin{aligned} & z = re^{i\theta} = r(\cos\theta + i\sin\theta) \\ \iff \quad & \overline{z} = r(\cos\theta - i\sin\theta) \qquad \text{using defn. of conjugate} \\ \iff \quad & \overline{z} = r(\cos(-\theta) + i\sin(-\theta)) \\ \iff \quad & \overline{z} = re^{i(-\theta)} = re^{-i\theta}. \end{aligned}$$

(iii)

$$\begin{aligned} \overline{z+w} &= \overline{(a+bi)+(c+di)} \\ &= \overline{(a+c)+i(b+d)} \\ &= (a+c)-(b+d)i \\ &= (a-bi)+(c-di) \\ &= \overline{z} + \overline{w}. \end{aligned}$$

(iv)

$$\begin{aligned}\overline{zw} &= \overline{(a+bi)(c+di)} \\ &= \overline{(ac-bd)+(ad+bc)i} \\ &= (ac-bd)-(ad+bc)i \\ &= (a-bi)(c-di) \\ &= \overline{z}\,\overline{w}.\end{aligned}$$

(v)

$$\begin{aligned}\overline{\left(\frac{z}{w}\right)} &= \overline{\left(\frac{z\,\overline{w}}{w\,\overline{w}}\right)} \\ &= \overline{\left(\frac{z\,\overline{w}}{|w|}\right)} = \frac{\overline{z}\,w}{|w|} \\ &= \frac{\overline{z}\,w}{w\,\overline{w}} = \frac{\overline{z}}{\overline{w}}\end{aligned}$$

(vi)

$$\begin{aligned}z\overline{z} &= (a+bi)(a-bi) \\ &= a^2 - b^2 i^2 \\ &= a^2 + b^2 = |z|^2\end{aligned}$$

and also

$$\begin{aligned}z\overline{z} &= re^{i\theta}\cdot re^{-i\theta} \\ &= r^2 = |z|^2.\end{aligned}$$

□

Chapter 2

Algebra

2.1 Group Theory

2.1.1 Groups

Definition 33. A binary operation is a function,

$$f : G \times G \longmapsto G$$

which - by the definition of a function - maps a unique tuple from $G \times G$ to a unique value in the codomain G.

Definition 34. Let G be a set and $*$ a binary operation on G and denote this $(G, *)$. Then $(G, *)$ is a **group** if:

closure $\forall x, y \in G, x * y \in G$;

associativity $\forall x, y, z \in G, (x * y) * z = x * (y * z)$;

identity $\exists e \in G$ s.t. $\forall x \in G, e * x = x * e = x$;

inverse $\forall x \in G, \exists x^{-1} \in G$ s.t. $x * x^{-1} = x^{-1} * x = e$.

These are known as the **group axioms**.

Definition 35. The group is an **Abelian group** if it has the additional property:

commutativity $\forall x, y \in G, x * y = y * x \in G$.

Notation. from here on we will use juxtaposition notation for the group operation (so $xy = x * y$) and (usually) 1 for the identity element instead of e. This is known as *multiplicative notation*.

Theorem 2.1.1. *Suppose an associative law of composition is given on a set S. Then there is a unique way to define a product of n elements $a_1, \ldots, a_n$ for any $n \in \mathbb{N}$.*

Proof. Denote the product of n elements as $[a_1 \ldots a_n]$. We show that a product can be defined with the following properties:

(i) $[a_1] = a_1$;

(ii) $[a_1 a_2] = a_1 * a_2$ is defined by the law of composition;

(iii) for any integer i such that $1 \leq i \leq n$, $[a_1 \ldots a_n] = [a_1 \ldots a_i][a_{i+1} \ldots a_n]$.

Following a proof by induction, firstly note that the product is defined for $n \leq 2$ by (i) and (ii) and that (ii) also satisfies the requirement (iii). Then, assume that the product is defined for $n \leq 2$ and that this product is the unique product satisfying (iii).
Then the induction step is to show that,

$$[a_1 \ldots a_n] = [a_1 \ldots a_{n-1}][a_n]$$

TODO: complete this from Artin[56] □

2.1.1.1 Corollaries of the group axioms

The group operation is defined to map a unique tuple in $G \times G$ to a unique value in G so that if we have $x, y \in G$ then $f((x, y)) = f(x, y) = xy \in G$ and for $a, b, c \in G$,

$$a = b \iff (c, a) = (c, b) \implies f((c, a)) = f((c, b)) \iff ca = cb$$

$$\therefore a = b \implies ca = cb$$

Then, using all the group axioms - associativity, inverse and identity,

$$ca = cb \implies c^{-1}(ca) = c^{-1}(cb) \iff (c^{-1}c)a = (c^{-1}c)b \iff 1a = 1b \iff a = b$$

Therefore we have the principle of cancellation,

$$ca = cb \implies a = b$$

Note that, since we have used the axioms of inverse and identity and the definitions of these require these elements to exhibit these properties from both the left and the right, the principle of cancellation can also be shown from both the left and the right. So, also,

$$ac = bc \implies a = b$$

There are (at least) two approaches to finding the other consequences of the group axioms.

First approach. We begin by noticing that the law of cancellation implies that,

unique identity and inverses $\forall a, x, b \in G, ax = b$ has a unique solution because,

$$ax = ax' \iff x = x'$$

That unique solution is $a^{-1}b$. If $b = a$ we have $ax = a$ and x, by identity axiom, is an identity element. Since, the solution to this equation - x - is unique, it follows that there is a unique value that is the identity element. Then, if we let b be this unique identity element we have $ax = 1$ and the unique solution, x, is the inverse of a, i.e. a^{-1}. Therefore, the inverses of group elements are also unique.

Second approach. This approach begins by showing the uniqueness of the identity element solely using the defintion of the identity. Here, for clarity, we revert to using e to denote the identity element.

unique identity Assume there are two identity elements, e, e'. Then, by the definition of the identity $ee' = e'e = e = e'$ so that there is a single value that has the property of the identity element.

Then, using the definition of the inverse we have,

unique inverses Assume there are two distinct inverses of an element a: a^{-1} and a'. Then,

$$\begin{aligned} & aa^{-1} = 1 = aa' && \text{defn. of inverse, uniqueness of identity} \\ \iff \quad & a^{-1} = a' && \text{law of cancellation} \end{aligned}$$

Some Examples of Groups

- $(\mathbb{R} \setminus \{0\}, \times)$ is a group whereas $(\mathbb{R}, \times)$ is not a group because 0 has no multiplicative inverse.
- $(\mathbb{R}, +)$ is a group.
- The set of $n \times n$ invertible matrices is called the General Linear group and denoted GL_n
- Let G denote the set of matrices

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \;\middle|\; a, b \in \mathbb{Z}_7,\ a \neq 0 \right\}.$$

  Then G is a group with respect to matrix multiplication (where all additions and multiplications are carried out in $\mathbb{Z}_7$). This is because closure can be shown by,

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a' & b' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} aa' & ab' + b \\ 0 & 1 \end{pmatrix} \in G$$

  where the result is in G because $aa' \neq 0$. Next we need to show that we have inverses. So we need to show existence in G of matrices such that,

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a' & b' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \iff \begin{pmatrix} aa' & ab' + b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

  which implies that $aa' = 1$ and $ab' + b = 0$. Because we are in $\mathbb{Z}_7$ every non-zero element has a multiplicative inverse so we have,

$$a' = a^{-1} \quad \text{and} \quad b' = -a^{-1}b$$

so that,

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} a^{-1} & -a^{-1}b \\ 0 & 1 \end{pmatrix}.$$

2.1.1.2 Permutations and Symmetric Groups

Definition 36. A **permutation** is a bijection from a set to itself. Since permutations are bijective, they are invertible and since they are functions, function composition defines an associative law of composition over them. As a result, they form a group.

Definition 37. The **symmetric group** defined over a set is the group whose elements are the permutations of the objects of the set and whose law of composition is the composition of functions. The name probably comes from the study of symmetries of geometric objects that were eventually realised to be equivalent to permutations of the vertices.

Definition 38. A **generating set of a group** is a subset such that every element of the group can be expressed as a combination (under the group operation) of finitely many elements of the subset and their inverses.

Notation. The symmetric group over the integers from 1 to n is denoted S_n. The symmetric group over a set G may be denoted $Sym(G)$.

S_2 The symmetric group S_2 consists of the two elements i and τ which are, respectively, the identity map and the transposition which interchanges 1 and 2. The group composition law is described by the fact that the identity map is the identity of the composition and by the relation $\tau\tau = \tau^2 = i$. Which

results in the multiplication table:

$$\begin{aligned} i \cdot i &= i \\ i \cdot \tau &= \tau \\ \tau \cdot i &= \tau \\ \tau \cdot \tau &= i \end{aligned}$$

Note that the law of composition is commutative.

$\boldsymbol{S_3}$ The symmetric group S_3 contains 3! elements. It is the smallest group whose law of composition is not commutative. It can be described using any two permutations of $\{1, 2, 3\}$. For example, if we take,

$$x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \; y = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then the permutations are,

$$\{1, x, x^2, y, xy, x^2y\} = \{\, x^i y^j \mid 0 \leq i \leq 2, \; 0 \leq j \leq 1 \,\}$$

These are the elements of the group. The composition law over these elements is the function composition of these permutation functions and its multiplication table is characterized by the rules:

$$x^3 = 1, \; y^2 = 1, \; yx = x^2y$$

These are derived directly from the permutations themselves. Note that this composition law is not associative as $yx \neq xy$.
Any product of the elements x, y and of their inverses can be brought into the form $x^i y^j$ with i, j taking the ranges given above by repeated application of the above rules. To do so, we move all occurrences of y to the right side using the last relation and bring the exponents into the indicated ranges using the first two relations:

$$\begin{aligned} x^{-1}y^3x^2y = x^2yx^2y = x^2(yx)xy = x^2(x^2y)xy = x^4(yx)y \\ = x^4(x^2y)y = x^6y^2 = (x^3)^2y^2 = 1 \cdot 1 = 1 \end{aligned}$$

Rules like these that determine a complete multiplication table are called *defining relations* for the group.

2.1.2 Subgroups

Definition 39. A subset H of a group G is called a **subgroup** if it has the following properties:

- **Closure:** If $a \in H$ and $b \in H$ then $ab \in H$.
- **Identity:** $1 \in H$.
- **Inverses:** If $a \in H$ then $a^{-1} \in H$.

These conditions show that the subset H is a group with respect to the induced law of composition created by applying the law of composition of G on the members of H. Note that the associative property is not mentioned because the associativity of the composition of members of G automatically carries over to H.

Notation. If H and G are groups then we may write $H \leq G$ to indicate that H is a subgroup of G.

Note that an alternative, more compact, formulation of the definition of a subgroup is as follows.

Let G be a group and $\emptyset \neq H \subseteq G$. Then H is a subgroup if

$$\boldsymbol{x, y \in H \implies x^{-1}y \in H}.$$

This is because,

$$[(x, y \in H \implies xy \in H) \wedge (x \in H \implies x^{-1} \in H)] \iff (x, y \in H \implies x^{-1}y \in H).$$

The implication,

$$[(x, y \in H \implies xy \in H) \wedge (x \in H \implies x^{-1} \in H)] \implies (x, y \in H \implies x^{-1}y \in H)$$

is obvious. In the other direction,

$$(x, y \in H \implies x^{-1}y \in H) \implies [(x, y \in H \implies xy \in H) \wedge (x \in H \implies x^{-1} \in H)]$$

is because, if we set $x = y$,

$$x^{-1}x = e \in H \implies x^{-1}e = x^{-1} \in H$$

and then, for $x \neq y$,

$$x^{-1}, y \in H \implies xy \in H.$$

Every group, at a minimum, has two trivial subgroups: the maximal subgroup - the group itself; and the minimal subgroup - the set containing just the identity. A subgroup that is neither of these is known as a *proper subgroup*.

Proposition 2.1.1. *Suppose H and K are subgroups of G such that neither $H \subseteq K$ nor $K \subseteq H$. Then $H \cup K$ is not a subgroup of G.*

*Note that it might be possible, for this proof, to just show that we have no reason to think that $H \cup K$ is a subgroup (i.e. the definitions don't require it) so, in general, it is not. But, actually, we can show something much stronger: that $H \cup K$ **cannot** be a subgroup. If we can easily show something stronger then in most cases it's going to add clarity.*

Proof. Since neither $H \subseteq K$ nor $K \subseteq H$ we can conclude that $H \setminus K$ and $K \setminus H$ are both non-empty. So, select an element from each,

$$h \in H \setminus K,\ k \in K \setminus H.$$

Then we have $h, k \in H \cup K$ and if $H \cup K$ were a group then the closure property of the group would require that

$$hk \in H \cup K.$$

Assume $hk \in H \cup K$. Then, $hk \in H$ or $hk \in K$. If $hk \in H$ then the group properties of H require that

$$h^{-1}hk = k \in H$$

which contradicts the selection of k. We have a similar situation if $hk \in K$. Therefore, $hk \notin H \cup K$. □

2.1.2.1 Additive Groups of Integers

Important examples are the subgroups of the additive group of integers $\mathbb{Z}^+$. Denote the subset of $\mathbb{Z}^+$ consisting of all multiples of a given integer b by $b\mathbb{Z}$ such that,

$$b\mathbb{Z} = \{\, n \in \mathbb{Z} \mid n = bk,\ k \in \mathbb{Z} \,\}$$

Proposition 2.1.2. *For any integer b, the subset $b\mathbb{Z}$ is a subgroup of $\mathbb{Z}^+$ and every subgroup of $\mathbb{Z}^+$ is of the form $b\mathbb{Z}$ for some integer b.*

Proof. $b\mathbb{Z}$ is a subgroup of $\mathbb{Z}^+$ because,

- $b(0) = 0 \in b\mathbb{Z}$;
- If $a_1, a_2 \in b\mathbb{Z}$ then $a_1 = bk_1, a_2 = bk_2$ for $k_1, k_2 \in \mathbb{Z}$ and so $a_1 + a_2 = bk_1 + bk_2 = b(k_1 + k_2) \in b\mathbb{Z}$
- For any $a = bk \in b\mathbb{Z}$, $-a = b(-k) \in b\mathbb{Z}$

Now we need to prove that any subgroup of $\mathbb{Z}^+$ is $b\mathbb{Z}$ for some b. Let H be an arbitrary subgroup of $\mathbb{Z}^+$. Then by subgroup properties,

- $0 \in H$;
- If $a_1, a_2 \in H$ then $a_1 + a_2 \in H$
- For any $a \in H$, $-a \in H$

We proceed to show that there is always some integer b such that $H = b\mathbb{Z}$. Firstly, if H is the minimal subgroup $\{0\}$ then H trivially conforms to $b\mathbb{Z}$ with $b = 0$.
Otherwise, $\exists a \in H$ s.t. $a \neq 0$ then also $\exists -a \in H$ s.t. $-a \neq 0$. One of these must be a positive non-zero integer so there is at least one such member of H. We take b to be the smallest positive non-zero integer in H. Then,

$b\mathbb{Z} \in H$

- $b \in H$ (by selection) so by subgroup properties $b+b \in H$ and $(b+b)+b \in H$ and $b + \cdots + b \in H$
- By subgroup properties $b \in H \implies -b \in H$

So, $\{\, bk \in \mathbb{Z} \mid k \in \mathbb{Z} \,\}$ is in H.

***H* ∈ *b*ℤ** Take any $n \in H$. Using division with remainder and dividing by b we get,

$$n = bq + r \quad q \in \mathbb{Z},\ 0 \leq r < b$$

But, since $b\mathbb{Z} \in H$ this means that $bq \in H$ and so $-bq \in H$. Therefore $n - bq = r \in H$. But $0 \leq r < b$ and, by assumption, b is the smallest positive *non-zero* integer in H and so, $r = 0$. So, every $n \in H$ divides by b. □

2.1.2.2 Greatest Common Divisor

If we extend this to groups which are generated by two integers a, b, then we have a subgroup of $\mathbb{Z}^+$,

$$a\mathbb{Z} + b\mathbb{Z} = \{\, n \in \mathbb{Z} \mid n = ar + bs \quad r, s \in \mathbb{Z} \,\}$$

This is known as the subgroup *generated* by a, b because it is the smallest subgroup which contains a and b. Proposition 2.1.2 tells us that it has the form $d\mathbb{Z}$ for some integer d.

Corollary 2.1.1. *If d is the positive integer which generates the subgroup $a\mathbb{Z} + b\mathbb{Z}$ then d is the greatest common divisor of a and b and so,*

- *d can be written in the form $d = ar + bs$ for some integers r and s.*
- *d divides a and b.*
- *If an integer e divides a and b, it also divides d.*

Proof. The first property follows directly from the definition of the subgroup. The second property is a result of the fact that a, b are in the subgroup $a\mathbb{Z} + b\mathbb{Z}$ so that $d\mathbb{Z} = a$ and $d\mathbb{Z} = b$. The third property is evident because $d = ar + bs = ek_1r + ek_2s = e(k_1r + k_2s)$. □

2.1.2.3 Deductions about Subgroups

Proposition 2.1.3. *Suppose $G = \{g_1, g_2, \ldots, g_n\}$ is a finite group of order n and that $x \in G$. Then $\{xg_1, xg_2, \ldots, xg_n\} = G$.*

Proof. Let $X = \{xg_1, xg_2, \ldots, xg_n\}$. By closure in G, every element of X must be in G and by the inverses property in G every element of X is distinct. So, there are n distinct elements of X, each of which are members of G. Since the order of G is n we can conclude that $X = G$. □

Proposition 2.1.4. *Suppose that G is a finite group and that H is a non-empty subset of G such that $x, y \in H \implies xy \in H$. Then H is a subgroup.*

Proof. H is non-empty so it contains at least one element, say x. H is closed under the group operation so it must also contain $x^2, x^3, \ldots$. But G is finite so the order of x in G must be finite also and so $\exists n \in \mathbb{N}$ s.t. $x^n = e$. But also $x^n = e \iff x^{n-1} = x^{-1}$. Therefore, for every element in H, the inverse of the element is also in H. □

Proposition 2.1.5. *Suppose that p is a prime number and we have integers $1 \leq x, g, h < p$ such that $xg \equiv xh \pmod{p}$. Then $g = h$ and $x \in \mathbb{Z}_p^*$ has an inverse.*

Proof. If $xg \equiv xh \pmod{p}$ then the difference between xg and xh is a multiple of p. But Euclid's Lemma (https://en.wikipedia.org/wiki/Euclid's_lemma) tells us that, because p is prime,

$$p \mid (xg - xh) = x(g - h) \implies (p \mid x) \wedge (p \mid (g - h)).$$

But since we have $x, (g - h) < p$ it is impossible for p to divide either of them unless they are 0. Only $g - h$ can be 0. Therefore,

$$g - h = 0 \iff g = h.$$

So, if we define a function $f : \mathbb{Z}_p^* \longmapsto \mathbb{Z}_p^*$ such that $f(a) = xa$ for some fixed $x \in \mathbb{Z}_p^*$ then f is injective because

$$f(a) = f(b) \iff xa \equiv xb \pmod{p} \iff a \equiv b \pmod{p}.$$

This means that f maps the $p - 1$ different values of $\mathbb{Z}_p^*$ to $p - 1$ different values in $\mathbb{Z}_p^*$. Therefore f is a bijection and there exists an inverse function f^{-1} such that $f^{-1}(xa) = a$. □

2.1.2.4 Examples of Subgroups

(7) $(\mathbb{Z}_p^*, \otimes)$ for prime p is a subgroup of the integers. This can be seen as closure of modular multiplication is clear and the existence of inverses has been shown in Proposition 2.1.5.
This is not the case however, for non-prime p. For example $(\mathbb{Z}_6^*, \otimes)$ is not a subgroup as it does not have inverses. We can see this by looking at the values generated by selecting a non-identity element and multiplying it by all the elements in $\mathbb{Z}_6^*$:

$$2 \otimes 1 = \mathbf{2},\ 2 \otimes 2 = \mathbf{4},\ 2 \otimes 3 = \mathbf{0},\ 2 \otimes 4 = \mathbf{2},\ 2 \otimes 5 = \mathbf{4}.$$

As can be seen, it doesn't generate all the values of $\mathbb{Z}_6^*$ but repeats a subset of them. Compare with the same for $\mathbb{Z}_7^*$:

$$2 \otimes 1 = \mathbf{2},\ 2 \otimes 2 = \mathbf{4},\ 2 \otimes 3 = \mathbf{6},\ 2 \otimes 4 = \mathbf{1},\ 2 \otimes 5 = \mathbf{3},\ 2 \otimes 6 = \mathbf{5}.$$

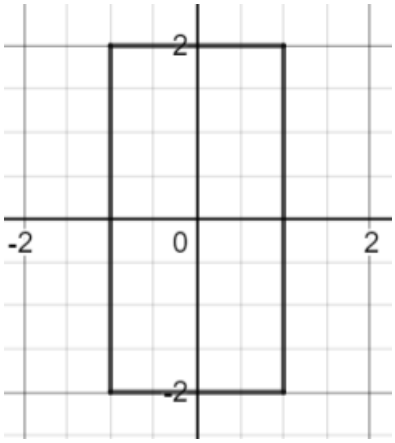
In the case of prime p all the values are generated so that multplication by other elements is a bijective function with a corresponding inverse.

(8) Let R be a non-square rectangle in $\mathbb{R}^2$ with corners having coordinates $(-1,-1), (-1,2), (1,2), (1,-1)$. Then there are four symmetries i, a, b, c of R, as follows:

- i is the identity
- a is reflection in the x-axis
- b is reflection in the y-axis
- c is a rotation of π radians around the origin.

These symmetry operations form a group whose group table is as follows.

	i	a	b	c
i	i	a	b	c
a	a	i	c	b
b	b	c	i	a
c	c	b	a	i



2.1.2.5 Cyclic Subgroups

Definition 40. If we take a single member of a group (along with its inverse and the identity), the subgroup generated by that element takes the form (using multiplicative notation),

$$H = \{x^{-(n-1)}, \cdots, x^{-2}, x^{-1}, 1, x, x^2, \cdots, x^{n-1}\}$$

where, either, $x^n = 1$ so that there are n distinct values in the group, or else n is infinite and the values never repeat. This is known as a **cyclic group** and also as the **subgroup generated by x** and is denoted by $\langle x \rangle$.

The cyclic subgroup, $\langle x \rangle$, generated by x is the smallest subgroup of G containing x in the sense that, if $H \leq G$ and $x \in H$ then $\langle x \rangle \subseteq H$.

Proposition 2.1.6. *Every cyclic group is Abelian.*

Proof. In a cyclic group every element has the form x^i for $i \in \mathbb{Z}$. So we have,

$$x^m x^n = x^{m+n} = x^n x^m$$

for all elements x^i in the group. □

Proposition 2.1.7. *The set S of integers n such that $x^n = 1$ is a subgroup of $\mathbb{Z}^+$.*

Proof. If $x^m = 1$ and $x^n = 1$, then $x^{m+n} = x^m x^n = 1$ also so we have closure of addition. Since $x^0 = 1$, 0 is in the subgroup so we have an identity. Finally, for some n in the subgroup, $x^n = 1 \iff x^{-n} = x^n x^{-n} = x^0 = 1$ so n being in the subgroup implies that $-n$ is also in the subgroup and we have inverses. □

Corollary 2.1.2. *It follows from S being a subgroup of $\mathbb{Z}^+$ and from Proposition 2.1.2 that S has the form $m\mathbb{Z}$ where m is the smallest positive integer such that $x^m = 1$. Therefore, in H, the m elements $1, x, x^2, \cdots, x^{m-1}$ are all different and any element in H will simplify to one of them: for $n \in S$, $n = mq + r$ such that $x^n = (x^m)^q x^r = 1^q x^r = x^r$.*

2.1.2.6 Order

Definition 41. The **order** of a group G is the number of distinct elements it contains. It is typically denoted $|G|$.
An element of a group is said to have **order** m (possibly infinity) if the cyclic subgroup it generates has order m. This means that m is the smallest positive integer with the property $x^m = 1$ or, if the order is infinite, that, $x^m \neq 1$ for all $m \neq 0$.

Theorem 2.1.2. *An element and its inverse have the same order.*

Proof. Firstly we need to consider the case that an element x has infinite order. In this case, $\nexists m \in \mathbb{N}$ s.t. $x^m = e$. Now suppose that $\exists n \in \mathbb{N}$ s.t. $(x^{-1})^n = e$. Then we have,

$$(x^{-1})^n = (x^n)^{-1} = e \iff e = x^n$$

which contradicts the hypothesis that x has infinite order. Therefore x^{-1} has infinite order also. Clearly also this argument can be used in reverse to show the reverse implication also holds.

Now consider the case that x has finite order. Let $x \in G$ be an arbitrary member of an arbitrary group such that $x^m = e$. Then $x^{-1} = x^{m-1}$ and if we consider powers $i \in \mathbb{N}$ of the inverse $(x^{-1})^i = (x^{m-1})^i$ then the order is the lowest value of $i(m-1)$ such that $x^{i(m-1)} = e$. But we know that the lowest

power of x equal to e is m so we're looking for the lowest multiple of m that has the form $i(m-1)$. So we require,

$$m \mid i(m-1) = im - i \iff (m \mid im) \wedge (m \mid i)$$

which clearly requires that $m \mid i$. Also, clearly, the lowest such i is $i = m$.

Another way to show this is to say that if $x^m = e$ then,

$$(x^{-1})^m = (x^m)^{-1} = e$$

so that the order of x^{-1} is less than or equal to m. Conversely, if x^{-1} has order n then,

$$x^n = ((x^{-1})^{-1})^n = ((x^{-1})^n)^{-1} = e^{-1} = e$$

so that the order of x is less than or equal to n. Thus we have,

$$m \leq n,\ n \leq m \implies m = n.$$

□

Theorem 2.1.3. *An element has order 2 iff it is equal to its inverse.*

Proof. Let $x \in G$ be an arbitrary member of an arbitrary group such that $x^2 = e$. Then by Theorem 2.1.2 we have,

$$e = x^2 = (x^{-1})^2 = x^{-2} \iff x = x^{-1}.$$

Also,

$$x = x^{-1} \iff x^2 = e.$$

□

Theorem 2.1.4. *A group of finite order cannot have any element of infinite order.*

Proof. If G is a group and $x \in G$ has infinite order then,

$$\begin{aligned}
& x^m = x^n \\
\iff \quad & x^{m-n} = 1 = x^{n-m} \\
\iff \quad & x^{|m-n|} = 1 \\
\therefore \quad & |m-n| = 0. && \text{because order of x is infinite}
\end{aligned}$$

So, there are no two distinct powers of x that produce the same object so that $\langle x \rangle \leq G$, the cyclic group generated by x, is infinite. Since $\langle x \rangle \subseteq G$ this requires that G also be infinite. □

Theorem 2.1.5. *If a group element x has finite order m then:*

1. *Let $n \in \mathbb{Z}$. If $n = km + r$ where $k, r \in \mathbb{Z}$ and $0 \leq r \leq m-1$, then $x^n = x^r$.*
2. *For $n \in \mathbb{N}$, $x^n = 1 \iff m|n$.*
3. *$1, x, x^2, \ldots, x^{m-1}$ is a complete, repetition-free, list of elements of $\langle x \rangle$.*
4. *The subgroup $\langle x \rangle$ generated by x has cardinality m.*

Theorem 2.1.6. *In an Abelian group the set of all elements of finite order forms a subgroup.*

Proof. Let S be the set of all elements of finite order,

$$S = \{\, g \in G \mid \exists m \in \mathbb{N} \,.\, g^m = e \,\}.$$

- Firstly, S is non-empty because it contains the identity.
- Secondly, if we have $a, b \in S$ then $a^i = b^j = e$ for some $i, j \in \mathbb{N}$. Then, if we let $m = i \times j$,

  $$(ab)^m = a^m b^m = (a^i)^j (b^j)^i = e$$

  where $(ab)^m = a^m b^m$ is valid *only* because the group is Abelian.
- Lastly, S contains all inverses because,

  $$a^i = e \iff a^{i-1} = a^{-1} \iff (a^{-1})^i = (a^i)^{i-1} = e.$$

  Therefore $a^{-1} \in S$.

□

Theorem 2.1.7. *If every non-identity element of a group has order 2 then the group is Abelian.*

Proof. Let $x, y \in G$ be two arbitrary elements of order 2 of an arbitrary group. Then by Theorem 2.1.2 we have $x = x^{-1},\ y = y^{-1},\ xy = xy^{-1}$ and so,

$$xy = (xy)^{-1} = y^{-1}x^{-1} = yx.$$

Note that $(xy)^{-1} = y^{-1}x^{-1}$ relies only on the associativity of the group operation and is therefore valid for all groups.
We can also show it this way,

$$(xy)^2 = e \iff xyxy = e \iff yxy = xe = x \iff yx = xy.$$

□

Theorem 2.1.8. *If a finite group has even-numbered order then it must have at least one element of order 2.*

Proof. By the group properties we know that the group contains the identity element – which has order 1 – and, for every non-identity element, the group also contains its inverse. Also, since the group is finite, every element must have finite order. Now, if every non-identity element is distinct from its inverse then the order of the group will be odd (because of the identity and then every other element is paired with its inverse). For the group's order to be even we must have at least one non-identity element that is not distinct from its inverse which, by Theorem 2.1.3, is equivalent to having order 2. □

Corollary 2.1.3. *If a finite group has even-numbered order then it must have an odd number of elements of order 2.*

Proof. Let G be a finite group with even-numbered order and M be the number of elements that are distinct from their inverse and N be the number of elements that are not distinct from their inverse (these correspond to elements with order greater than 2 and elements with order 2 respectively). Then the order of G can be expressed as,

$$|G| = 1 + 2M + N.$$

Therefore, $|G|$ is even if N is an odd natural number. □

Theorem 2.1.9. *In an infinite cyclic group all elements have infinite order.*

Proof. Let $G = \langle x \rangle$ be an infinite cyclic group. Suppose there is some non-identity element of G, x^n with finite order m. Then,

$$(x^n)^m = e \iff x^{nm} = e$$

which contradicts the hypothesis that $\langle x \rangle$ is infinite. □

Note that the elements of an infinite cyclic group having infinite order does not mean that they generate the group. For example in an infinite cyclic group $\langle x \rangle$, the element x^2 generates the cyclic subgroup,

$$\ldots x^{-4}, x^{-2}, e, x^2, x^4, \ldots$$

which is infinite but clearly doesn't generate the whole group $\langle x \rangle$.

Theorem 2.1.10. *An infinite cyclic group has 2 generators.*

Proof. Let $G = \langle x \rangle$ be an infinite cyclic group and suppose there is some non-identity element of G, x^n that generates the group. To show this we only need to show that x^n can generate x because, since x is a member of the group, it is obviously necessary to generate it but, also, if we generate x then we can generate all the other members of the group since they are powers of x.
So let there be an integer a such that $(x^n)^a = x^{an} = x \iff x^n = x^{1/a}$. The cyclic group $\langle x \rangle$ only contains integer powers of x so it therefore follows that $|a| = 1$ which implies that $n = 1$ or -1 and $x^n = x$ or x^{-1}.

We could also say that,

$$x^{an} = x \iff x^{an-1} = e$$

but this implies that the order of x is finite and so contradicts the hypothesis that x generates an infinite cyclic group.

□

Theorem 2.1.11. *Let $G = \langle x \rangle$ be a finite cyclic group of order n. If r is a positive integer then $G = \langle x^r \rangle$ if and only if the greatest common divisor of n and r is 1.*

Proof. Members of $\langle x^r \rangle$ have the form $(x^r)^a$ for some $a \in \mathbb{Z}$. For integers b, i,

$$(x^r)^a = x^{ar} = x^{bn+i} = x^{bn}x^i = ex^i = x^i$$

so that the generated elements are x^i where $i = ar - bn$ is the remainder when dividing ar by n. If $d = gcd(n, r)$ then $d \mid i$ and the generated elements are powers of x that are multiples of d. Therefore, to generate every power of x it is necessary to have $d = 1$. Conversely, we can see – by the same argument in reverse – that it is sufficient if $d = 1$ to generate all the powers of x.

Alternatively, we can say if $d > 1$ then n/d is a positive integer less than n and r/d is a positive integer so,

$$(x^r)^{n/d} = (x^n)^{r/d} = e^{r/d} = e$$

which shows that the order of x^r is less than or equal to n/d which is less than n. Therefore the order of the cyclic group it generates is less than n and so it cannot be equal to G.
Conversely, if $d = 1$ then n and r are coprime and so we have,

$$(x^r)^m = e \iff x^{rm} = e \implies n \mid rm \implies n \mid m \implies m \geq n.$$

This says that the order of x^r in G is greater than or equal to n, the order of G. Well, clearly it cannot be greater than the order of G so it follows therefore, that the order of x^n is n. Since $|\langle x^r \rangle| = |G|$ we can conclude that $\langle x^r \rangle = G$. □

Theorem 2.1.12. *A group G is such that G contains at least 2 elements and the only subgroups of G are $\{e\}$ and G itself. Then G is a finite cyclic group of prime order.*

Proof. G contains at least 2 elements so there is at least one non-identity element x. The only subgroups of G are the whole group and $\{e\}$ but $\langle x \rangle$ cannot equal $\{e\}$ so it must equal G. Therefore G is the cyclic group generated by x.
But if G were the infinite cyclic group generated by x then only x and x^{-1} would generate the group and all other non-identity elements – say x^n for $n > 1 \in \mathbb{N}$ – would generate subgroups $\langle x^n \rangle \neq G$. Therefore, G cannot be infinite and is therefore finite.
Now, we have a finite cyclic group where every non-identity element generates

the group. Let $|G| = n$. Then, for every m s.t. $0 < m < n$, $\langle x^m \rangle = G$ and, by Theorem 2.1.11, m and n are coprime. Therefore, n is prime.
Here we could also use a proof by contradiction: Assume n is not prime and it has factors $r, s > 1 \in \mathbb{N}$. Then,

$$(x^r)^s = x^{rs} = x^n = e$$

so that the order of x^r in G is less than or equal to s which is less than n (because it is a factor of n). It follows then that $\langle x^r \rangle \neq G$, contradicting the definition of G. Therefore n is prime. □

Proposition 2.1.8. *Suppose the elements x, y in a group G have orders m, n respectively and that the $gcd(m, n) = 1$. Then $\langle x \rangle \cap \langle y \rangle = \{e\}$ and, if x and y commute, then the order of xy in G is mn.*

Proof. One way to approach this is to say that for $z \in \langle x \rangle \cap \langle y \rangle$ we have some $0 \leq i < m, 0 \leq j < n$ such that,

$$z = x^i = y^j \iff x^{im} = e = y^{jm},\ x^{in} = y^{jn} = e$$

so that $x^{in} = y^{jm} = e \iff (m \mid in \quad \text{and} \quad n \mid jm)$. Note that,

$$(m \mid in \quad \text{and} \quad n \mid jm) \iff m, n \mid in + jm.$$

Now, applying the fact that $gcd(m, n) = 1$ we see that both m and n must divide 1. But both m and n are orders of elements and so, by definition, greater than 1. The only other alternative is that both $i, j = 0$ which results in $z = x^0 = y^0 = e$.

We could also have said,

$$x^{im} = e = x^{in} \iff x^{im-in} = e \iff m \mid im - in.$$

In this case we can apply the fact that $gcd(m, n) = 1$ to the statement that $m \mid i(m - n)$ to deduce that: either $m \mid (m - n) \iff m \mid 1$ which is impossible because m must be greater than 1; or $m \mid i$ which is also impossible because $i < m$. So, again, we are only left with the alternative that $i = 0$ which results in $z = x^0 = y^0 = e$.

Next we prove the order of $xy \in G$ and we begin by noting that, if x and y commute, then $(xy)^r = x^r y^r$. So if we assume that xy has order r then we must have $m, n \mid r$ and the lowest such r is the order. Well, the lowest common multiple of m, n is defined according to the gcd as described in the Number Theory treatment of Modular Arithmetic (1.2.2.2) as,

$$d = gcd(m, n) \implies lcd(m, n) = d \cdot (m/d) \cdot (n/d).$$

Clearly then, if $d = gcd(m, n) = 1$, then the lowest common multiple is mn and so the order of $xy \in G$ is mn.

Or to describe it a different way: $(xy)^{mn} = x^{mn}y^{mn} = ee = e$ so that the order of xy,

$$|xy| \leq mn.$$

Conversely any r such that $(xy)^r = e$ must have $m, n \mid r$ and the lowest common multiple of m and n is mn so

$$r \geq mn.$$

Therefore, $|xy| = mn$. □

2.1.2.7 Examples of Cyclic Subgroups

(9) **Cyclic group with order 3**

$$G = \{1, x, x^2\}$$

where $x^3 = 1$ is a cyclic group of order 3 generated by the element x. Note that, since this is a group, it must also contain the inverses, x^{-1}, x^{-2} but $x^3 = 1$ so $x^{-1} = x^2$ and $x^{-2} = x$.

(10) **Symmetries of an equilateral triangle**

Consider an equilateral triangle with vertices labeled A, B, C:

$$\begin{array}{c} A \\ B \;\; C. \end{array}$$

Every permutation of the vertices is a transformation that produces an object that occupies the same space as the original, i.e. a *symmetry*. If we take one of them, say, the clockwise rotation one place that results in,

$$\begin{array}{cc} & B \\ C & A \end{array}$$

and we name this r, then clearly – since there are 3 vertices – performing this same rotation 3 times leaves us back where we started. So, using function composition as the law of composition and multiplicative notation, $r^3 = i$ where i is the identity transformation. Also the inverse of r is r^2. So, we have a group consisting of $\{i, r, r^2\}$ and function composition. Notice the resemblance of this group to the previous group $\{1, x, x^2\}$; this group is *isomorphic* to the cyclic group of order 3.

(11) **Group $(\mathbb{Z}_5^*, \otimes)$**

Consider the element 2 modulo 5. Using multiplicative notation we have,

$$2^2 = 4, 2^3 = 8 = 3, 2^4 = 16 = 1, 2^5 = 32 = 2.$$

So $2^1 = 2^5 \iff 1 = 2^4$ meaning that the element 2 has order 4 in the group and we see, as expected that the group it generates, $\langle 2 \rangle$ has 4 members. In this case, the members are all the members of the group – that's to say, *the element 2 generates the whole group*. If we consider the element 4 we have,

$$4^2 = 16 = 1, 4^3 = 64 = 4.$$

So this element oscillates between 1 and 4 and so, the cyclic subgroup that it generates $\langle 4 \rangle$ has order 2.
Since the group $(\mathbb{Z}_5^*, \otimes) = \langle 2 \rangle$ it can also be described as a cyclic group. This will be the case for any such group modulo a prime number – i.e. $(\mathbb{Z}_p^*, \otimes)$ where p is prime.

(12) **Cyclic group with infinite order**

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

under matrix multiplication (which is commutative in this case), generates a cyclic group of infinite order because

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}.$$

(13) **Cyclic groups in a non-Abelian group**

Consider the following two elements in $GL(2, \mathbb{R})$,

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.$$

Both of these elements have finite order as,

$$(A^2)^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$(B^2)B = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

But their product AB does not have finite order.

$$AB = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad (AB)^n = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & -n \\ 0 & 1 \end{pmatrix}$$

for any $n \in \mathbb{N}$.

(14) **The Klein Four Group,** V is the simplest group that is not cyclic (it cannot be generated by a single element). It appears in many forms but, as an example, it can be realized as the group consisting of the four matrices,

$$\begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}$$

Any two non-identity elements generate V.

2.1.3 Isomorphisms

Definition 42. An **isomorphism** is a bijection between two groups that preserves the structure of the groups by being compatible with the law of composition of both groups. More formally, two groups are **isomorphic** if there exists a bijection $\phi : G \longmapsto G'$ such that,

$$\phi(ab) = \phi(a)\phi(b) \text{ for all } a, b \in G$$

where ab represents composition according to the law of composition of G and $\phi(a)\phi(b)$ represents composition according to the law of composition of G'.

*An **isomorphism** is a **bijection** between two **groups**. That's to say, it is already assumed in the definition of an isomorphism that the codomain G' is a group.*

Proposition 2.1.9. *As a consequence of this sole property that, across the bijection, the respective laws of composition are preserved, all other properties of the groups are also preserved.*

Proof. Let e be the identity in G and $e' = \phi(e) \in G'$, and $1'$ be the identity element in G' then,

- Since G' is a group, it has the inverses property that every element has an inverse so,

$$\begin{aligned} & e' = \phi(e) = \phi(ee) = \phi(e)\phi(e) = e'e' && \text{using preservation of law of composition} \\ \iff \quad & (e')^{-1}e' = ((e')^{-1}e')e' && \text{using the inverses property of } G' \\ \iff \quad & 1' = e' \end{aligned}$$

  which implies that e' is the identity in G' so that ϕ maps the identity in G to the identity in G'.

- We can use the fact just shown that $\phi(e) = e' = 1'$ to show,

$$1' = e' = \phi(e) = \phi(aa^{-1}) = \phi(a)\phi(a^{-1}) \quad \text{using preservation of law of composition}$$

$$\iff \qquad \phi(a)^{-1}1' = \phi(a)^{-1}\phi(a)\phi(a^{-1}) \qquad \text{using the inverses property of } G'$$
$$\iff \qquad \phi(a)^{-1} = \phi(a^{-1})$$

which shows that ϕ maps $a^{-1} \in G$ to $\phi(a)^{-1} \in G'$.

□

For example, if $e \in G$ is the identity of G mapped to an element $e' = \phi(e) \in G'$, then for any $a \in G$ mapped to $a' = \phi(a) \in G'$,

$$a' = \phi(a) = \phi(ea) = \phi(e)\phi(a) = e'a'$$

And $a' = e'a' = a'e'$ means that e' is the identity in G'. Furthermore, the order of elements in G and G' will also be the same as,

$$a^n = e \iff e' = \phi(e) = \phi(a^n) = \phi(a)^n = (a')^n$$

Since two isomorphic groups have the same properties, it is often convenient to identify them with each other when speaking informally. For example, the symmetric group S_n of permutations of $\{1, \cdots, n\}$ is isomorphic to the group of permutation matrices, a subgroup of $GL_n(\mathbb{R})$ and we often blur the distinction between these two groups.

Notation. Sometimes when two groups are isomorphic this is indicated using the notation,

$$G \approx G'$$

2.1.3.1 Examples

- Let $C = \{\cdots, a^{-2}, a^{-1}, 1, a, a^2, \cdots\}$ be an infinite cyclic group. Then the map,

  $$\phi : \mathbb{Z}^+ \longmapsto C \text{ s.t. } \phi(n) = a^n$$

  is an isomorphism where the preservation of the respective laws of composition can be seen as,

  $$\phi(m+n) = a^{m+n} = a^m a^n = \phi(m)\phi(n)$$

  and also $n + (-n) = 0$ and,

  $$\phi(-n) = a^{-n} = (a^n)^{-1}.$$

- Let G be the set of real matrices of the form,

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$$

This is a subgroup of $GL_2(\mathbb{R})$ and so, its law of composition is the same as that of $GL_2(\mathbb{R})$, i.e. matrix multiplication.

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x+y \\ 0 & 1 \end{bmatrix}$$

So, G is isomorphic to $\mathbb{R}^+$, the additive group of reals.

Definition 43. The groups isomorphic to a given group G form what is called the **isomorphism class** of G. Groups are often classified into isomorphism classes, for example, there is one isomorphism class of groups of order 3 and there are two classes of groups of order 4 and five classes of 12.

Proposition 2.1.10. *There is only one isomorphism class for each order of cyclic group.*

Proof. Any two cyclic groups of the same order are isomorphic because, if

$$G = \{1, x, x^2, \cdots, x^{n-1}\},\ G' = \{1, y, y^2, \cdots, y^{n-1}\}$$

are two cyclic groups of order n then the map $\phi(x^i) = y^i$ is an isomorphism. □

Proposition 2.1.11. ***Cayley's Theorem*** *states that every group is isomorphic to a group of permutations of the same underlying set, or in other words, to a subgroup of the symmetric group acting on the group.*

Proof. Let G be a group and $x \in G$ and define $f_x : G \longmapsto G$ as $f_x(g) = xg$. Then f_x is a bijection because it has an inverse ${f_x}^{-1}(g) = x^{-1}g = f_{x^{-1}}(g)$. Therefore f_x is a permutation.

Now we define a map, from G to the symmetric group of permutations of G, that maps each element x in G to the permutation defined by f_x. Let $\phi : G \longmapsto Sym(G)$ be defined as $\phi(x) = f_x$ then,

- ϕ is homomorphic because

$$(f_x \circ f_{x'})(g) = f_x(f_{x'}(g)) = x(x'g) = xx'g = f_{xx'}(g)$$

and so,

$$\phi(xx') = f_{xx'} = f_x \circ f_{x'} = \phi(x) \circ \phi(x').$$

- ϕ is injective because if $x, x' \in G$ such that $x \neq x'$ then $f_x(e) = x \neq x' = f_{x'}(e)$ is sufficient to show that $f_x \neq f_{x'}$. Or alternatively, the kernel of ϕ comprises the elements $k \in G$ such that $f_k(g) = kg = g \iff k = e$ so that the kernel is the trivial subgroup $\{e\}$.

Since ϕ is homomorphic, its image $im\,\phi$ is a subgroup of $Sym(G)$ and since ϕ is injective, it is in bijective correspondence with its image $im\,\phi \leq Sym(G)$. Therefore G is isomorphic to a subgroup of $Sym(G)$. □

2.1.3.2 Automorphisms

Definition 44. The domain and codomain of an isomorphism can be the same set of objects so that $\phi : G \longmapsto G$. This is known as an **automorphism**.

Example Let $G = \{1, x, x^2\}$ be a cyclic group of order 3 so that $x^3 = 1$. The transposition which interchanges x and x^2 is an automorphism of G,

$$\begin{array}{ccc} 1 & \longmapsto & 1 \\ x & \longmapsto & x^2 \\ x^2 & \longmapsto & x \end{array}$$

	1	x	x^2
1	1	x	x^2
x	x	x^2	1
x^2	x^2	1	x

$\longmapsto$

	1	x^2	x
1	1	x^2	x
x^2	x^2	x	1
x	x	1	x^2

This is because the group is cyclic and x and x^2 have the same order ($x^3 = 1$ and also $(x^2)^3 = x^6 = (x^3)^2 = 1^2 = 1$). So the law of composition is preserved.

2.1.3.3 Conjugation

The most important example of automorphism is conjugation.

Definition 45. **Conjugation** by $b \in G$ is the map from G to itself defined by,

$$\phi(a) = bab^{-1}$$

with the result that,

$$ba = \phi(a)b$$

so that we can think of conjugation of a by b as the way that we need to change a if we want to move the multiplication by b to the other side.

This is an automorphism (known as an *inner automorphism*) because it

- is compatible with law of composition,

$$\phi(xy) = bxyb^{-1} = bxb^{-1}byb^{-1} = \phi(x)\phi(y).$$

- has an inverse so it is bijective,

$$(\phi^{-1} \circ \phi)(a) = \phi^{-1}(\phi(a)) = b^{-1}(bab^{-1})b = (b^{-1}b)a(b^{-1}b) = a.$$

  Note that this is different from the inverse element of a corresponding under the mapping ϕ,

$$\phi(a)\phi(a^{-1}) = bab^{-1}ba^{-1}b^{-1} = ba(1)a^{-1}b^{-1} = b(1)b^{-1} = 1.$$

A couple more important properties of the conjugate are as follows.

(i) In an abelian group where the composition law is commutative, conjugation becomes the identity map.

$$ba = ab \iff bab^{-1} = a \iff \phi(a) = a.$$

(ii) The inverse of the conjugate $bab^{-1} = b^{-1}ab$.

2.1.4 Homomorphisms

Definition 46. A **homomorphism** is a mapping (not necessarily bijective) between two groups, $\phi : G \longmapsto G'$, such that,

$$\phi(ab) = \phi(a)\phi(b) \text{ for all } a, b \in G$$

where ab represents composition according to the law of composition of G and $\phi(a)\phi(b)$ represents composition according to the law of composition of G'.

So, the difference between a *homomorphism* and a *isomorphism* is that the latter is bijective whereas the former is not. As a result, a *homomorphism* may be one-way only.

*A **homomorphism** is a **mapping** between two **groups**. That's to say, it is already assumed in the definition of a homomorphism that the codomain G' is a group.*

Examples of homomorphisms

(15) Let $C = \{a^{n-1}, \cdots, a^{-2}, a^{-1}, 1, a, a^2, \cdots, a^{n-1}\}$ be a finite cyclic group. Then the map,

$$\phi : \mathbb{Z}^+ \longmapsto C \text{ s.t. } \phi(n) = a^n$$

is a homomorphism. Note that if C were an infinite cyclic group then this would be an isomorphism.

(16) the sign of a permutation $sign : S_n \longmapsto \pm 1$

(17) the determinant function $det : GL_n(\mathbb{R}) \longmapsto \mathbb{R}^\times$

(18) an arguably trivial example is called the *inclusion* map $i : H \longmapsto G$ of a subgroup H into a group G, defined by $i(x) = x$. It functions as the identity for elements in the subgroup H but, since it is not surjective, there is no inverse mapping.

2.1.4.1 Image of a homomorphism

Since a homomorphism is not bijective it has an image different to the codomain group,

$$im\ \phi = \{\, x \in G' \mid \exists a \in G \text{ s.t. } \phi(a) = x \,\}$$

The image of a homomorphism is a subgroup of the codomain group G' because the homomorphism preserves the group structure as described in Proposition 2.1.9.

Notation. The image of the mapping ϕ with domain G is sometimes denoted $\phi(G)$.

2.1.4.2 Kernel of a homomorphism

Definition 47. The **kernel** of a homomorphism is the set of elements in the domain that are mapped to the identity,

$$ker\ \phi = \{\, a \in G \mid \phi(a) = 1' \,\}$$

Proposition 2.1.12. *The kernel of a homomorphism is a subgroup of the domain group G.*

Proof. If $a, b \in ker\ \phi$ then,

- closure: $\phi(ab) = \phi(a)\phi(b) = 1' \cdot 1' = 1'$ which shows that

$$a, b \in ker\ \phi \implies ab \in ker\ \phi.$$

- identity: By Proposition 2.1.9, $1' = e' = \phi(e)$ and so $e \in ker\ \phi$.
- inverses: Since $a \in ker\ \phi$, then

$$\begin{aligned} & 1' = e' = \phi(e) = \phi(aa^{-1}) = \phi(a)\phi(a^{-1}) = 1'\phi(a^{-1}) \\ \iff \quad & 1' = \phi(a^{-1}) \end{aligned}$$

so that $a \in ker\ \phi \iff a^{-1} \in ker\ \phi$.

□

Proposition 2.1.13. *If $\phi : G \longmapsto G'$ is a group homomorphism with kernel N then, for $a, b \in G$,*

$$\phi(a) = \phi(b) \iff \exists n \in N, \text{ s.t. } b = an$$

or, equivalently, $a^{-1}b \in N$.

Proof.

$$\begin{aligned}
& b = an \\
\implies \quad & \phi(b) = \phi(an) \\
\implies \quad & \phi(b) = \phi(a)\phi(n) && \text{by homomorphism property} \\
\implies \quad & \phi(b) = \phi(a)1' && \text{n is in the kernel} \\
\implies \quad & \phi(b) = \phi(a)
\end{aligned}$$

$$\begin{aligned}
& \phi(b) = \phi(a) \\
\implies \quad & \phi(a)^{-1}\phi(b) = 1' && \text{codomain is a group so has inverses} \\
\implies \quad & \phi(a^{-1})\phi(b) = 1' && \text{by Proposition 2.1.9} \\
\implies \quad & \phi(a^{-1}b) = 1' && \text{by homomorphism property} \\
\implies \quad & a^{-1}b = n \in N \\
\implies \quad & b = an
\end{aligned}$$

□

Theorem 2.1.13. *A homomorphism is injective iff its kernel is the trivial subgroup $\{e\}$.*

When asked to prove the proposition that a homomorphism is injective iff its kernel is the trivial subgroup $\{e\}$, it's tempting to begin proving each direction of the bidirectional implication with a proof by contradiction (e.g. "Assuming there is a non-identity element in the kernel...") but the direct positive proof can be made very quick and simple with the above Proposition 2.1.13.

Proof. Let $\phi : G \longmapsto G'$ be a homomorphism.

Assume that ϕ is injective and $k \in ker\,\phi$. Remembering that we always at least have $e_G \in ker\,\phi$,

$$\phi(k) = e_{G'} = \phi(e_G) \iff k = e_G$$

where the last implication is by the injectivity of ϕ.

Now assume that $ker\,\phi = \{e_G\}$, $a, b \in G$. Then,

$$\begin{aligned}
& \phi(a) = \phi(b) && \\
\iff \quad & \phi(a)\phi(b)^{-1} = e_{G'} && \\
\iff \quad & \phi(a)\phi(b^{-1}) = \phi(ab^{-1}) = e_{G'} && \text{using homomorphism properties} \\
\iff \quad & ab^{-1} \in ker\,\phi && \\
\iff \quad & ab^{-1} = e_G && \text{by assumption } ker\,\phi = \{e_G\} \\
\iff \quad & a = b. && \square
\end{aligned}$$

Corollary 2.1.4. *A homomorphism is an isomorhpism if its kernel contains only the identity and its image is the whole of the codomain (i.e. it's surjective).*

2.1.4.3 Examples of Kernels of Homomorphisms

(19) The determinant function 17, $det : GL_n(\mathbb{R}) \longmapsto \mathbb{R}^\times$, has a kernel,

$$\{\,\text{real } n \times n \text{ matrices } A \mid det\,A = 1\,\},$$

which is a subgroup of $GL_n(\mathbb{R})$ known as the *special linear group* $SL_n(\mathbb{R})$.

(20) The sign of a permutation 16 has a kernel that is the set of *even* permutations,

$$A_n = \{\text{even permutations}\},$$

which is a subgroup of the symmetric group S_n and is known as the *alternating group*, A_n.

(21) The map from the additive group of integers to a finite cyclic group 15,

$$\phi : \mathbb{Z}^+ \longmapsto C \text{ s.t. } \phi(n) = a^n$$

has the kernel,

$$ker\ \phi = \{\, n \in \mathbb{Z}^+ \mid a^n = 1 \,\}$$

which has been proven to be a subgroup in Proposition 2.1.7.

2.1.5 Equivalence Relations and Partitions

Notation. In the following treatment of equivalence relations we will use the notation $a \sim b$ to denote the equivalence of a and b; $\overline{a}$ to indicate the equivalence class of a; and $\overline{S}$ to indicate the partition of S comprised of equivalence classes such as the class $\overline{a} = \overline{b}$ which includes both a and b.

Any map of sets $\phi : S \longmapsto T$ defines an equivalence relation on the domain S such that $a \sim b$ iff $\phi(a) = \phi(b)$. We will refer to this as the *equivalence relation determined by the map.* The corresponding partition is made up of the sets of elements in the domain S that are mapped to the same element in the codomain T.

Definition 48. Let $\phi : S \longmapsto T$ be a map, then the **inverse image** of an element $t \in T$ is defined as,

$$\phi^{-1}(t) = \{\, s \in S \mid \phi(s) = t \,\}$$

and can also be applied to a set $U \in T$ as,

$$\phi^{-1}(U) = \{\, s \in S \mid \phi(s) \in U \,\}$$

Note that in this notation, ϕ^{-1} **does not indicate an inverse function** as the inverse of the function may not exist but the inverse image is nevertheless defined.
The inverse images - the sets $\phi^{-1}(t)$ for all $t \in T$ - may also be called the **fibres** of the map ϕ.

Clearly, the non-empty fibres of the map ϕ form a partition of S. We can express this partition of S as a bijection, that we shall call $\overline{\phi}$, between the fibres of ϕ in S and the element of the image of S to which their members are mapped,

$$\overline{\phi} : \overline{S} \longmapsto im\ \phi$$

so that,

$$\overline{\phi}(\overline{s}) = \phi(s).$$

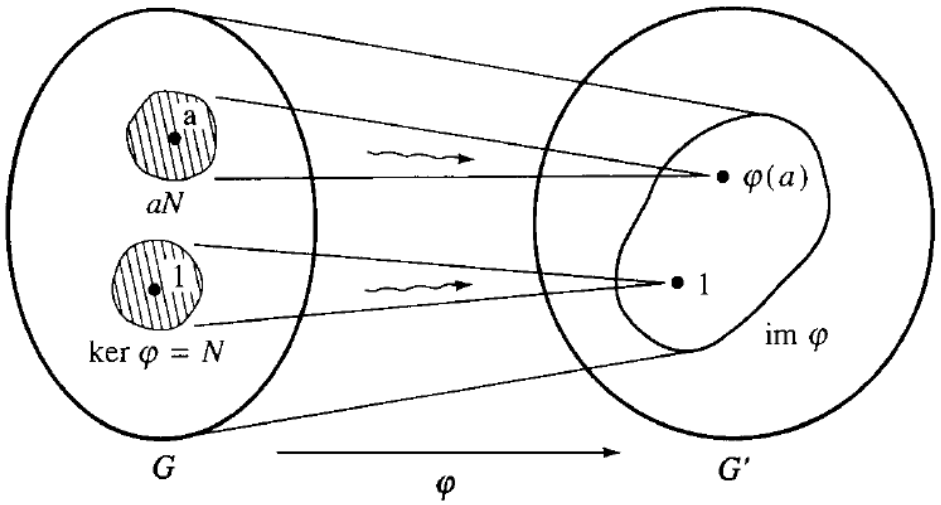


Figure 2.1: A schematic diagram of a group homomorphism

2.1.5.1 Congruence

Since a homomorphism maps the identity to the identity and inverses to inverses (Proposition 2.1.9), we can deduce that the inverse image of the identity in G' is going to contain at least the identity of G and that the inverse image of an element $(a')^{-1} \in G'$ will contain at least the element $a^{-1} \in G$. So, in terms of equivalence classes we can say that, for a homomorphism ϕ,

$$1 = e \in \phi^{-1}(1') \implies \overline{\phi}(\overline{1}) = 1'$$

$$a^{-1} \in \phi^{-1}((a')^{-1}) \implies \overline{\phi}(\overline{a^{-1}}) = (a')^{-1}$$

Definition 49. The equivalence relation determined by a homomorphism is known as **congruence** and is commonly denoted using $\equiv$ instead of $\sim$. For a homomorphism ϕ,

$$a \equiv b \iff \phi(a) = \phi(b).$$

Since ϕ is a homomorphism we also have,

$$a \equiv b \iff \phi(ac) = \phi(bc),\ \phi(a^{-1}) = \phi(b^{-1}).$$

More generally, a **congruence relation** is an equivalence relation on an algebraic structure (such as a group, ring, or vector space) that is compatible with the structure in the sense that algebraic operations done with equivalent elements will yield equivalent elements.

2.1.5.2 Congruence Examples

(22) The modulus function of complex numbers forms a homomorphism from the multiplicative group of complex numbers to the multiplicative group of reals,

$$\phi : \mathbb{C}^\times \longmapsto \mathbb{R}^\times \text{ s.t. } \phi(a) = |a|$$

and the induced equivalence relation is $a \equiv b \iff |a| = |b|$. The fibres of this map are the concentric circles about 0. They are in bijective correspondence with elements of $im\ \phi$, the set of positive reals.

2.1.6 Cosets

The set of elements of the form an - described in Proposition 2.1.13 - is denoted by aN and is called a *coset* of N in G.

Definition 50. A coset can be defined for any subgroup H of a group G. A **left coset** is a subset of the form,

$$aH = \{\, ah \mid h \in H \,\}.$$

Cosets are not, in general, subgroups. This can be easily seen as the left coset aH does not contain the identity as, although H contains the identity, aH contains $a1 = a$.

Note that the arbitrary subgroup H could also be thought of as a coset $1H = H$ and also that the left cosets aH are equivalence classes for the congruence relation,

$$a \equiv b \iff b = ah,\ h \in H.$$

This is a congruence because, for some arbitrary $c \in G$,

$$1 \equiv c \iff \exists h \in H \text{ s.t. } c = 1h = h.$$

That's to say, the elements that are congruent to the identity are precisely the members of the subgroup H so that it plays a similar role to the kernel N in Proposition 2.1.13. Furthermore, since the congruence relation is an equivalence relation it forms a partition of the domain G.

Proposition 2.1.14. *For a group G with a subgroup H and $x \in G$, the coset xH is equal to H iff $x \in H$.*

Proof. Assume $x \in H$. Then $\forall xh \in xH\ .\ xh \in H$. Therefore $xH \subseteq H$. Conversely, $x^{-1} \in H$ so,

$$\forall h \in H\ .\ x^{-1}h \in H \implies x(x^{-1}h) = h \in xH.$$

Therefore $H \subseteq xH$ and so $H = xH$.

Now assume that $H = xH$. Since $e \in H$ then $xe = x \in xH = H$ and so $x \in H$. □

Proposition 2.1.15. *The left cosets of a subgroup partition the group.*

Proof. The left cosets are equivalence classes and, as a result, they partition the group. □

2.1.6.1 Examples of cosets

(23) The coset of an element with the kernel N,

$$aN = \{\, g \in G \mid g = an,\ n \in N \,\}$$

is the set of all elements that are *congruent* to a. The *congruence classes* are precisely the cosets aN for each $a \in G$. They are also the nonempty *fibres* of the homomorphic map.

(24) 2.1.1.2 Continuing the example of the symmetric group S_3 represented as

$$G = \{1, x, x^2, y, xy, x^2y\}$$

with group multiplication rules,

$$x^3 = 1, y^2 = 1, yx = x^2y.$$

The element xy has order 2 so it generates a cyclic subgroup $H = \{1, xy\}$ of order 2. The left cosets of H in G are the three sets,

$$\{1, xy\} = 1H = xyH,\ \{x, x^2y\} = xH = x^2yH,\ \{x^2, y\} = x^2H = yH.$$

Note that they do partition the group G. Also, notice that the cosets aH for $a \in H$ produce the subgroup H itself as should be expected as the group properties of the subgroup dictate that all products of its elements are already present in the subgroup. For this reason, the cosets aH that are distinct from H are those such that $a \notin H$.

(25) Let $G = (\mathbb{R}^3, +)$ be the group of 3d vectors with vector addition and $\vec{\boldsymbol{w}} \in G$. Then if

$$H = \{\, \vec{\boldsymbol{x}} \in G \mid \vec{\boldsymbol{w}}^T \vec{\boldsymbol{x}} = \vec{\mathbf{0}} \,\}$$

then H is a subgroup, $H \leq G$. H is a vector space representing a plane through the origin in $\mathbb{R}^3$ and it's cosets are

$$\vec{\boldsymbol{v}} + H = \{\, \vec{\boldsymbol{v}} + \vec{\boldsymbol{h}} \mid \vec{\boldsymbol{v}} \in G, \vec{\boldsymbol{h}} \in H \,\}$$

which are the affine spaces representing the translated planes, parallel to H, but not passing through the origin. Once again we see that the cosets partition the space even if there may be an infinite number of them.

2.1.6.2 The index of a subgroup

Definition 51. The **index** of a subgroup is the number of left cosets it forms in the parent group.

Notation. The **index** of a subgroup H in G is denoted by $[G : H]$.

In the example (24) the index of H is 3. Note that if G were to contain infinitely many elements then the index of a subgroup may also be infinite.

Proposition 2.1.16. *Each coset aH has the same number of elements as H.*

Proof. As usual, equal cardinality is demonstrated by showing the existence of a bijection. It is clear that there is a bijective map between the subgroup H and any coset aH because the map $H \longmapsto aH$ is,

- injective because $ah = ah' \implies h = h'$ because by group properties a has an inverse in G;
- surjective because every $c \in aH$ has the form ah and is therefore mapped to by some $h \in H$.

□

2.1.6.3 Lagrange's Theorem

Since the left cosets of H in G form a partition of G and their order is the same as that of H we see that the order of G is the order of H multiplied by its index in G. This results in a formula known as the *Counting Formula* as follows,

$$|G| = |H| \cdot [G : H].$$

If G is of infinite order and H is finite, then the index of H in G will be infinite.

Theorem 2.1.14. ***Lagrange's Theorem:*** *Let G be a finite group, and let H be a subgroup of G. The order of H divides the order of G.*

Corollary 2.1.5. *Let G be a finite group, and let a be an element of G. Then the order of a divides the order of G. That's to say, the order of the cyclic group generated by a, $|\langle a \rangle|$, divides $|G|$.*

Corollary 2.1.6. *If G is a group of order n, then $g^n = e$ for every element g of G.*

Proof. This is clearly a consequence of the previous corollary. If we let the order of g be m, then by the previous corollary,

$$m \mid n \iff n = km \text{ for } k \in \mathbb{N} \iff g^n = g^{km} = (g^m)^k = e^k = e. \qquad \square$$

Corollary 2.1.7. *Suppose that a group G has p elements and that p is a prime integer. Let $a \in G$ be any element, not the identity. Then G is the cyclic group $\{1, a, \ldots, a^{p-1}\}$ generated by a.*

Proof. Since $a \neq 1$ by selection, it has order greater than 1. Since its order must divide the order of G, which is prime, its order is equal to the order of G, p. So, the order of the nonidentity element a is the same as the order of G and so it generates the whole group. $\square$

Corollary 2.1.8. *All groups with some prime order, p, are in the same isomorphism class.*

Proof. Any group with prime order p is the cyclic group of order p and by Proposition 2.1.10 there is only a single isomorphism class for each cyclic group of a given order. □

Proposition 2.1.17. *Suppose the elements x, y in a group G have orders m, n respectively and that the $gcd(m, n) = 1$. Then $\langle x \rangle \cap \langle y \rangle = \{e\}$.*

> *Here we will prove, using Lagrange's Theorem, something that we previously proved here (Proposition 2.1.8) using modular arithmetic. Notice how the proofs are similar but the proof with Lagrange's Theorem allows us to remain within Group Theory.*

Proof. Firstly, note that the intersection of the two cyclic groups,

$$H = \langle x \rangle \cap \langle y \rangle$$

is a subgroup both of the parent group G **and** of $\langle x \rangle$ and $\langle y \rangle$. So Lagrange's Theorem tells us that its order must divide into the order of the parent group **and** the orders of the cyclic groups of x and y. Therefore, we have,

$$|H| \mid m \quad \text{and} \quad |H| \mid n.$$

Now, applying the fact that the $gcd(m, n) = 1$ we see that $|H| \mid 1$ and therefore $|H| = 1$. Furthermore, any group of order 1 must be the minimal group $\{e\}$. □

Proposition 2.1.18. *Suppose that H is a subgroup of G and $x \in G$. Then there exists some $k \in \mathbb{N}$, $1 \leq k \leq [G : H]$ s.t. $x^k \in H$.*

Proof. Let $n = [G : H]$ be the index of H in G. Then there are precisely n cosets of H in G. But $x \in G \implies x^m \in G$ for any $m \in \mathbb{N}$ (we don't need to consider the negative powers of x because they are inverses of positive powers and are similar for these purposes) and so we have cosets of the form $x^m H$ for each $m \in \mathbb{N}$. Therefore, amongst the $n + 1$ cosets generated by, $x^i H$ for $i \in \{0, 1, \ldots, n\}$ we must have at least one repetition of the same coset.
So, for some fixed x^i, x^j with $0 \leq i, j \leq n$ and $i \neq j$, we have,

$$x^i H = x^j H$$

$$\begin{aligned}
&\iff && \forall h \in H \,.\, x^i h \in x^j H \\
&\iff && \forall h \in H \,.\, \exists h' \in H \,.\, x^i h = x^j h' \\
&\iff && \forall h \in H \,.\, \exists h' \in H \,.\, x^{i-j} = h^{-1} h' \in H
\end{aligned}$$

Since necessarily we have $1 \leq i - j \leq n$ we let $k = i - j$ and then $x^k \in H$ as required. □

2.1.6.4 Example applications of Lagrange Theorem

(26) **Fermat's Little Theorem**: *If p is a prime number then*

$$a^p \equiv a \bmod p \text{ for all } a \in \mathbb{Z}.$$

> *We need to be a little careful here. We might assume – given that we are multiplying the integer a in modulo p that the group we want to use is $(\mathbb{Z}_p, \otimes)$. However, this is not a group! The reason is that $\mathbb{Z}_p$ contains 0 which has no inverse under the proposed law of composition, multiplication.*
> *If, however, we take $\mathbb{Z}_p^*$ where the $*$ means $\mathbb{Z}/\{0\}$ then we have a set of $p-1$ distinct elements. Over this set we can form the multiplicative group $G = (\mathbb{Z}_p^*, \otimes)$ because the primality of p means that every element has a multiplicative inverse.*
> *Note that this is **not** a group of prime order. The primality of p is essential to make sure that every element has a multiplicative inverse but, since we also have to eliminate 0 for the same reason, the order is $p-1$ which is not necessarily prime.*

Proof. Take the set $\mathbb{Z}_p^*$ under multiplication and some arbitrary $a \in \mathbb{Z}$.

(i) Primality of p means that it is possible to find $1 = na + mp$ for $m, n \in \mathbb{Z}$ (see Corollary 2.1.1). This implies that there exists a multiplicative inverse of every non-zero element in modulo p. Specifically, n is the inverse of a because $na = (-m)p + 1 \iff na \bmod p \equiv 1$.

(ii) Existence of the multiplicative inverses implies that we have a group $G = (\mathbb{Z}_p^*, \otimes)$.

(iii) G being a group implies that, for any element $a \in G$, by Corollary 2.1.6 we have $a^{p-1} = 1$.

(iv) In G, $a^{p-1} = 1 \iff a^p = a$ which translates to $a^p \equiv a \bmod p$.

□

2.1.6.5 Lagrange's Theorem and Homomorphisms

The Counting Formula can also be applied when a homomorphism is given. Let $\phi : G \longmapsto G'$ be a homomorphism. As we saw in coset example 23, the left cosets of $ker\ \phi$ are the fibres of the map ϕ. They are in bijective correspondence with the elements of the image. Therefore,

$$[G : ker\ \phi] = |im\ \phi| .$$

Which implies that,

Corollary 2.1.9. *If $\phi : G \longmapsto G'$ is a homomorphism of finite groups then,*

$$|G| = |ker\ \phi| \cdot |im\ \phi| .$$

As a result, $|ker\ \phi|$ divides $|G|$, and $|im\ \phi|$ divides both $|G|$ and $|G'|$.

2.1.6.6 Restriction of a Homomorphism to a Subgroup

A useful way of understanding the structure of a complicated group is to understand its subgroups and then derive an understanding of the parent group from knowledge about the subgroups it contains. This frequently involves the application of Lagrange's Theorem. *Restriction of a Homomorphism to a subgroup* refers to studying the behaviour of a homomorphism on subgroups of the parent group.

Suppose that $\phi : G \longmapsto G'$ is a homomorphism and that H is a subgroup of G. Then we may *restrict ϕ to H* to obtain a homomorphism whose domain is a subset of the original,

$$\phi|_H : H \longmapsto G'.$$

This *restriction* is a homomorphism because ϕ is a homomorphism and the restriction domain is a group. Clearly, the kernel of the restricted homomorphism is the intersection of the domain H with $ker\ \phi$.

2.1.6.7 Examples of using Lagrange's Theorem with a homomorphism restricted to a subgroup

(27) Referring again to the sign of a permutation (16) $S_n \longmapsto \{-1, 1\}$: the order of the codomain of this homomorphism is clearly 2. Suppose we form the restriction of this homomorphism to a subgroup H of S_n. Then, denoting the image by $\phi|_H(H)$, by Corollary 2.1.9 we have that $|\phi|_H(H)|$ divides both 2 and $|H|$.
So, if the subgroup H has odd order then $|\phi|_H(H)| = 1$ and – since $\phi|_H(H)$ must be a group because the group structure is preserved across the homomorphism – $\phi|_H(H) = \{1\}$. This means that H is in the kernel of the sign map and that the subgroup of permutations in S_n represented by H consists of only even permutations.
Therefore, every permutation whose order in S_n is odd is an even permutation (since the cyclic group that it generates has odd order). However, we can not make any conclusions about permutations of even order; they may be even or odd permutations.

2.1.6.8 Right Cosets

Right cosets also exist and are defined as,

$$Ha = \{\, g \in G \mid g = ha,\ h \in H \,\}$$

and these are equivalence classes for the *right congruence* relation,

$$a \equiv b \iff b = ha,\ h \in H.$$

Right cosets are not necessarily the same as left cosets. For instance, continuing the example in 24, the right cosets of the subgroup $\{1, xy\}$ of S_3 are,

$$\{1, xy\} = H1 = Hxy,\ \{x, y\} = Hx = H,\ \{x^2, x^2y\} = Hx^2 = Hx^2y.$$

Note that this generates a different partition of G then was generated by the left cosets.

2.1.7 Normal Subgroups and Centers

2.1.7.1 Normal Subgroups

Definition 52. A subgroup N of a group G is called a **normal subgroup** if it has the property that,

$$\forall\, a \in N, b \in G,\ bab^{-1} \in N$$

which is to say, that the conjugate by any element of G of any element in N is also in N.

Proposition 2.1.19. *A subset H of a group G is normal if and only if every left coset is also a right coset. If H is normal then,*

$$\forall a \in G,\ aH = Ha.$$

Proof. Suppose that H is normal. For any $h \in H$ and any $a \in G$,

$$ah = (aha^{-1})a.$$

Since H is normal, the conjugate by h of a is also in H, that's to say, $aha^{-1} \in H$ which implies that $(aha^{-1})a \in Ha$. Therefore, any arbitrary member of aH is also a member of Ha. Clearly, the same proof also works in the other direction so that any member of Ha is also a member of aH and the two cosets are equal. So, we have shown that (H is normal) $\implies$ (left and right cosets of H are equal).
Now we need to show that (left and right cosets of H are equal) $\implies$ (H is normal). Firstly, clearly the above logic doesn't apply if H is not normal; there will be at least one element whose conjugate is not in H so $aH \neq Ha$. However, it could still be the case that each left coset is also a right coset if, for every a in G, there is some b in G such that $aH = Hb$. However, this is not possible because aH and Ha both contain a which means that in a given partition of G they must be the same partition. So $aH \neq Ha$ implies that the partitions are different; Ha creates different equivalence classes. Therefore (left and right cosets of H are equal) $\implies$ (H is normal). □

This is really the point of normal subgroups: That their cosets contain multiplication from both sides. As a result, when the cosets themselves are used as members of groups (see: Quotient Groups 2.1.8.6), we can define a group composition operation between them such that $aHbH = abH$ and the operation is well defined (i.e. equal arguments give equal results) because,

$$aHbH = abHH = abH \quad and \quad aH = a'H \implies abH = aHb = a'Hb = a'bH$$

where $aH = Ha$ because H is normal. .

2.1.7.2 Examples of Normal Subgroups

(28) The kernel of a homomorphism is a normal subgroup because,

$$\begin{aligned}
& a \in ker\ \phi \iff \phi(a) = 1 \\
\implies \quad & \phi(bab^{-1}) = \phi(b) \cdot 1 \cdot \phi(b^{-1}) \\
\iff \quad & \phi(bab^{-1}) = \phi(b)\phi(b)^{-1} && \text{using Proposition 2.1.9} \\
\iff \quad & \phi(bab^{-1}) = 1.
\end{aligned}$$

For example,

a. $SL_n(\mathbb{R})$ 19 is a normal subgroup of $GL_n(\mathbb{R})$ *even though it is not Abelian*, which can be seen as for $M \in GL_n(\mathbb{R}), A, B \in SL_n(\mathbb{R})$,

$$AB \neq BA \quad \text{and} \quad det\, A = 1,\ det\, M^{-1} = 1/(det\, M)$$

so that,

$$det\, M^{-1}AM = (det\, M^{-1}) \cdot 1 \cdot (det\, M) = (det\, M)/(det\, M) = 1.$$

b. A_n 20 is a normal subgroup of the symmetric group S_n.

c. In fact, any subgroup of $GL_n(\mathbb{F})$ with some fixed determinant $d \in \mathbb{R}$ will be a normal subgroup.

$$N = \{\, A \in GL_n(\mathbb{F}) \mid det\, A = d \,\}$$

For $A \in N, M \in GL_n(\mathbb{F})$. $det(MAM^{-1}) = d$ by the same logic as in example a. These conjugate matrices are known as *similar* matrices.

(29) Any subgroup of an abelian group is normal because when the composition law is commutative, as was mentioned in the section on conjugation,

$$ba = ab \iff bab^{-1} = abb^{-1} = a$$

so that conjugation becomes the identity map and so, trivially, all conjugates of elements in a subgroup are also in the subgroup.

Subgroups of non-abelian groups, however, need not be normal. For example,

(30) Group T of invertible upper triangular matrices is not a normal subgroup of $GL_2(\mathbb{R})$. To show this note,

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, BAB^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

where $A \in T, B \in GL_2(\mathbb{R})$ but $BAB^{-1} \notin T$.

Proposition 2.1.20. *Let $\phi : G \longmapsto G'$ be a homomorphism and let H' be a subgroup of G'. Denote the inverse image $\phi^{-1}(H') = \{ x \in G \mid \phi(x) \in H' \}$ by $\tilde{H}$. Then,*

(i) $\tilde{H}$ is a subgroup of G.

(ii) If H' is a normal subgroup of G' then $\tilde{H}$ is a normal subgroup of G.

(iii) $\tilde{H}$ contains ker ϕ.

(iv) The restriction of phi to $\tilde{H}$ defines a homomorphism $\tilde{H} \longmapsto H'$ whose kernel is ker ϕ.

Proof. Proofs are as follows:

(i) $\tilde{H}$ is a subgroup of G because ϕ is a homomorphism and its image, H', is a group (which is required for a homomorphism).

(ii) If H' is a normal subgroup of G' then $\tilde{H}$ is a normal subgroup of G because for every element in $\tilde{H}$ the mapped element is in H'. Then, since H' is normal, the conjugates of the mapped element are also in H' which means that their inverse images are in $\tilde{H}$. Since the map is homomorphic, the inverse images of the conjugates in G' are the respective conjugates in G.

(iii) $\tilde{H}$ contains $ker\ \phi$ because it contains every element in G that maps to an element in H' and, since H' is a group, it includes the identity of G'. Therefore $\tilde{H}$ contains every element that maps to the identity of G' which is $ker\ \phi$.

(iv) The restriction of phi to $\tilde{H}$ is clearly a homomorphism and, since it contains $ker\ \phi$, its kernel is equal to the kernel of ϕ.

□

2.1.7.3 The Center of a Group

Definition 53. The **center** of a group G is the set of elements that commute with every element of G,

$$Z = \{\, z \in G \mid zx = xz,\ \forall x \in G \,\}.$$

We can also define,

$$C(x) = \{\, g \in G \mid gx = xg \,\}$$

as the set of elements in G that commute with a single fixed element x.

Notation. The **center** of a group G may be denoted by Z or by $Z(G)$.

The center of a group, Z, is a subgroup of G. This can be easily seen as, first of all, Z is non-empty because the identity is in the center of any group. Then, also, the center Z is closed under the group operation,

$$\forall a, b \in Z, x \in G\ .\ (ab)x = axb = x(ab)$$

and it contains the inverses,

$$\forall a \in Z, x \in G\ .\ ax = xa \iff a^{-1}ax = x = a^{-1}xa \iff xa^{-1} = a^{-1}x.$$

2.1.7.4 Examples of group centers

(31) Let $G = GL(2, \mathbb{R})$ be the group of invertible 2x2 matrices with real coefficients and take two elements in G,

$$M = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Then we can identify the center of M by observing that an arbitrary matrix in $C(M)$ satisfies,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2a & b \\ 2c & d \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2a & 2b \\ c & d \end{pmatrix}$$

which gives $b = 2b,\ c = 2c$ implying that b and c are 0. So,

$$C(M) = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \,\middle|\, a, d \in \mathbb{R} \setminus \{0\} \right\}.$$

While matrices in the center of N satisfy,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & a+b \\ c & c+d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ c & d \end{pmatrix}$$

which gives $a = a + c,\ c + d = d,\ a + b = b + d$ implying that $c = 0$ and $a = d$. Therefore,

$$C(N) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \,\middle|\, a, b \in \mathbb{R},\ a \neq 0 \right\}.$$

> *Note that in both cases some coefficients were required to be non-zero because to be members of the general linear group they must be invertible and so their determinant must be non-zero.*

(32) The center of the general linear group $GL_n(\mathbb{R})$ is the group of *scalar matrices* of the form cI for $c \in \mathbb{R}$, i.e. matrices of the form,

$$\begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$$

in $GL_2(\mathbb{R})$. Note that, for *diagonal* matrices whose elements on the main diagonal are all non-zero but not-necessarily equal as in *scalar* matrices, multiplication is commutative with other diagonal matrices but not generally so with other matrices in the general linear group.

2.1.8 Products Groups and Quotient Groups

Definition 54. If we take the cartesian product of two sets then:

- if the two sets are the underlying sets of two distinct groups then we have no way to combine them (as there is no common group operation) but we can take the pairing and define a component-wise multiplication over the pairs where each component is multiplied using that group's composition operation. In this way we create a new group over the pairs.
- if the two sets are subsets of a common group then there is a common group operation between them and so we can multiply them using this group operation. The result is another subset of the common group (not necessarily a subgroup).

Both of these may at times be referred to as **Product Groups** but the first one is more specifically referred to as a **Direct Product** and the second one may be referred to as a **Product Set**.

2.1.8.1 Direct Products

Definition 55. Let G, G' be two groups. The **direct product** is the set $G \times G'$ with component-wise multiplication using the group composition operation for the group corresponding to the component. Its order is the product of the orders of G and G'.

Notation. The **direct product** of the two groups G, G' may be denoted by $G \times G'$ or GG'. In the case of Abelian groups the direct product may be referred to as the **direct sum** and denoted $G \oplus G'$.

So, if $a, b \in G$ and $a', b' \in G'$ then

- $(a, a'), (a, b'), (b, a'), (b, b') \in G \times G'$
- $(a, a')(b, b') = (ab, a'b')$
- the identity is $(1, 1)$ and $(a, a')^{-1} = (a^{-1}, a'^{-1})$.

Definition 56. The **projections** of a direct product $G \times G'$ are the maps p, p' such that,

$$p(x, x') = x, \quad p'(x, x') = x'.$$

Proposition 2.1.21. *The **mapping property of direct products**: Let H be any group. The homomorphisms $\Phi : H \longmapsto G \times G'$ are in bijective correspondence to pairs (ϕ, ϕ') of homomorphisms*

$$\phi : H \longmapsto G, \qquad \phi' : H \longmapsto G'.$$

The kernel of Φ is the intersection $(ker\, \phi) \cap (ker\, \phi')$.

Proof. Given a pair of homomorphisms (ϕ, ϕ') we can define $\Phi(x) = (\phi(x), \phi'(x))$. Then this is homomorphic because,

$$\Phi(xy) = (\phi(xy), \phi'(xy)) = (\phi(x), \phi'(x))(\phi(y), \phi'(y)) = \Phi(x)\Phi(y).$$

Conversely, given such a Φ we can recover the pair of homomorphisms with the group projections as such (outer parentheses omitted for clarity),

$$\phi(x), \phi'(x) = p(\Phi(x)), p'(\Phi(x)).$$

Since the correspondence is invertible, it is a bijection.

Clearly, also,

$$\Phi(x) = (\phi(x), \phi'(x)) = (1, 1) \iff (\phi(x) = 1) \wedge (\phi'(x) = 1)$$

so that $ker\, \Phi = (ker\, \phi) \cap (ker\, \phi')$. □

Proposition 2.1.22. *Let r, s be coprime integers. A cyclic group of order rs is isomorphic to the product of a cyclic group of order r and a cyclic group of order s.*

Proof. Let $C = \{1, x, x^2, \ldots, x^{rs-1}\}$, $C_1 = \{1, y, y^2, \ldots, y^{r-1}\}$, $C_2 = \{1, z, z^2, \ldots, z^{s-1}\}$ and define the map $\phi : C \longmapsto C_1 \times C_2$ as,

$$\phi(x^i) = (y^i, z^i).$$

Then ϕ is homomorphic because it is comprised of two homomorphisms (by the mapping proper Proposition 2.1.21),

$$\phi_1(x^i) = y^i \quad \text{and} \quad \phi_2(x^i) = z^i.$$

And ϕ is injective because,

$$\phi(x^i) = (1,1) \iff (y^i = 1) \wedge (z^i = 1) \iff (r \mid i) \wedge (s \mid i)$$

but r and s are coprime so this requires that $i = rs$ which is also the order of $x \in C$. So we have,

$$\phi(x^i) = (1,1) \iff x^i = x^{rs} = 1.$$

Therefore $ker\,\phi = \{1\}$ and, by Theorem 2.1.13, ϕ is injective.

Since ϕ is injective, its image has the same order as that of the domain C so we have,

$$|im\,\phi| = |C| = rs = |C \times C|$$

and ϕ is therefore surjective.

Therefore ϕ is a bijection and isomorphic. □

*Note that this is **only** the case for cyclic groups whose order is the product of two coprime numbers. For example, a cyclic group of order 4 is not isomorphic to a product of two cyclic groups of order 2 as every element in a product group $C_2 \times C_2$ has order 1 or 2. Whereas a cyclic group of order 4 has two elements of order 4 (the generating element and its inverse).*

Let $C_4 = \{1, x, x^2, x^3\}$, $C_2 = \{1, y\}$ and define the map $\phi : C_4 \longmapsto C_2 \times C_2$ as $\phi(x^i) = (y^i, y^i)$. Then,

$$\phi(x^i) = (1,1) \iff y^i = 1 \iff 2 \mid i$$

so that we have $ker\,\phi = \{1, x^2\}$ and so ϕ is not injective.

2.1.8.2 Product Sets

Definition 57. Let A and B be subsets of the group G and denote the **product set** of A and B by

$$AB = \{\, x \in G \mid x = ab \text{ for some } a \in A \quad \text{and} \quad b \in B \,\}.$$

Note that this notation is the same as one of the alternatives for the notation of the direct product so we need to be clear which is intended when we see this notation.

2.1.8.3 Relationship Between the Types of Product Groups

Proposition 2.1.23. *Let H and K be subgroups of G.*

(i) If $H \cap K = \{1\}$, the product map $p : H \times K \longmapsto G$ defined by $p(h,k) = hk$ is injective. Its image is the subset HK.

(ii) If either H or K is a normal subgroup of G, then the product sets HK and KH are equal and are subgroups of G.

(iii) If H and K are normal, $H \cap K = \{1\}$, and $HK = G$, then G is isomorphic to the direct product $H \times K$.

Proof. Proofs of each property are as follows.

(i) If we assume that $H \cap K = \{1\}$ then, for $h, h' \in H$, $k, k' \in K$,

$$p(h,k) = p(h',k') \iff hk = h'k' \iff (h')^{-1}h = k'k^{-1}$$

so that $(h')^{-1}h = k'k^{-1} \in H \cap K = \{1\}$ therefore,

$$(h')^{-1}h = 1 \iff h = h', \qquad k'k^{-1} = 1 \iff k' = k.$$

Therefore p is injective.

(ii) Assume w.l.o.g. that K is a normal subgroup. Then for all $k \in K, g \in G$, $g^{-1}kg \in K$ and, in particular, for $h \in H, h^{-1}kh \in K$. Therefore,

$$hk \in HK \implies h(h^{-1}kh) = kh \in HK$$

and conversely, using the fact that, $h^{-1} \in H \implies hkh^{-1} \in K$,

$$kh \in KH \implies (hkh^{-1})h = hk \in KH.$$

Therefore $HK = KH$ and this implies that HK is a subgroup because, for $h, h' \in H, k, k' \in K$,

- HK is closed because

  $$kh' \in KH = HK \implies kh' = h''k'' \in HK$$

  for some $h'' \in H$, $k'' \in K$, and so,

  $$hk, h'k' \in HK \implies (hk)(h'k') = h(kh')k' = h(h''k'')k' = (hh'')(k''k') \in HK.$$

- HK has inverses because for $hk \in HK$,

  $$h^{-1}k^{-1} \in HK \implies k^{-1}h^{-1} \in HK.$$

(iii) If $H \cap K = \{1\}$ then the product map p is injective and if $HK = G$ then $im\, p = G$ so p is surjective also and, therefore, is a bijection between $H \times K$ and G.

To show that p is a homomorphism between the direct product and the product set HK we need to show that,

$$p((h,k)(h',k')) = p((hh',kk')) = hh'kk' = hkh'k' = p((h,k))p((h',k'))$$

which will be true if $h'k = kh'$ which, in turn, will be the case if products in HK are commutative.
Now we have $H \cap K = \{1\}$ and so,

$$hk = kh \iff k^{-1}hk = h \iff k^{-1}hkh^{-1} = 1$$

implies that $H \cap K = \{k^{-1}hkh^{-1}\}$. So, if we can show that $k^{-1}hkh^{-1}$ is in both H and K then we have a homomorphism.
Since, in this case, both H and K are normal we have,

$$h, h^{-1} \in H,\ k \in K \implies k^{-1}hk \in H,\ hkh^{-1} \in K$$

which, by the group closure of H and K gives,

$$(k^{-1}hk)h \in H \quad \text{and} \quad k^{-1}(hkh^{-1}) \in K.$$

Therefore, if both H and K are normal subgroups and $H \cap K = \{1\}$, then $hk = kh$ for all $h \in H$, $k \in K$. This, in turn, means that the product map p is a homormorphism between the direct product $H \times K$ and the product set HK. Since p is also bijective, it is an isomorphism.

□

It is important to note that the product map of two subgroups $H \times K \longmapsto HK = G$ will not be a group homomorphism unless the two subgroups commute with each other.

2.1.8.4 Examples of Product Groups

(33) There is a group with subgroups of orders $1 \ldots 12$. It is a direct product of cyclic groups of orders

$$2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 7 \times 11 = |G| = 27,720$$

so it looks like,

$$G = C_2 \times C_2 \times C_2 \times C_3 \times C_3 \times C_5 \times C_7 \times C_{11}.$$

2.1.8.5 Products of Cosets

It is possible to define a law of composition on the cosets of normal subgroups. This is because,

$$aH = Ha \implies aHbH = abHH = abH$$

so that we may define a law of composition such that $aH * bH = abH$ which closes over the set of cosets of H. The identity element of this composition is $eH = H$ and the inverse of the element aH is $a^{-1}H$.

*Note that this **only** applies to normal subgroups. The reason is that if H is not normal then there exists $h \in H$ and $a \in G$ such that $aha^{-1} \notin H$ which means that $S = aHa^{-1}H$ is not in any coset.*
This last claim can be proven if we observe – remembering that cosets partition the group – that S contains $a1a^{-1}1 = 1$ which means that it has to be in H (TODO: in the kernel?). However, S also contains $aha^{-1}1 = aha^{-1} \notin H$ so, since these are equivalence classes, S cannot be in H.

2.1.8.6 Quotient Groups

Definition 58. Suppose N is a normal subgroup of a group G. Then the quotient group G/N is the set of cosets of N in G with the coset product. Its order is the index of N in G, $[G : N]$.

Notation. Sometimes – when it is not necessary to specify the subgroup against which the cosets are being formed – the set of cosets in G is denoted $\overline{G}$ and a member coset aH is denoted $\overline{a} \in \overline{G}$.

Theorem 2.1.15. *Every normal subgroup of a group G is the kernel of a homomorphism.*

Proof. For any normal subgroup $N \leq G$, if we define the map,

$$\pi : G \longmapsto G/N$$

then π is homomorphic because $\pi(ab) = abN = aNbN = \pi(a)\pi(b)$.
Now, Proposition 2.1.14 tells us that

$$\pi(x) = 1N = N \iff x \in N$$

which implies that $ker\,\pi = N$. (We could also observe that the cosets are equivalence classes and the kernel is the equivalence class containing the identity. The coset that contains the identity is the original subgroup $N = 1N$.) □

Theorem 2.1.16. ***First Isomorphism Theorem****: Let $\phi : G \longmapsto G'$ be a surjective group homomorphism, and let $N = ker\,\phi$. Then G/N is isomorphic to G' by the map $\overline{\phi}$ which sends the coset $\overline{a} = aN$ to $\phi(a)$,*

$$\overline{\phi}(\overline{a}) = \phi(a).$$

Proof. The non-empty fibres of ϕ are the cosets aN as seen in the example 23. So, G/N can be thought of either as the cosets of the kernel of ϕ or as the non-empty fibres of ϕ. Then, $\overline{\phi}$ bijectively maps the cosets in G/N with the elements of the $im\,\phi$ and, because ϕ is surjective, we have $im\,\phi = G'$ so we have a bijection $\overline{\phi} : G/N \longmapsto G'$.
Also, the map $\overline{\phi}$ is homomorphic because coset multiplication is consistent with multiplication in the group,

$$\overline{\phi}(\overline{ab}) = \phi(ab) = \phi(a)\phi(b) = \overline{\phi}(\overline{a})\overline{\phi}(\overline{b}).$$

□

2.1.8.7 Examples of Quotient Groups

(34) Let $G = (\mathbb{Z}, +)$ and $H = \{\, 4n \mid n \in \mathbb{Z} \,\}$. Then the cosets of H are $\{\, z + 4n \mid z \in \mathbb{Z} \,\}$ and the product of two cosets,

$$(z_1 + H) + (z_2 + H) = (z_1 + z_2 + H).$$

(35) Let $G = (\mathbb{R}, +)$ and $H = \{\, 2n\pi \mid n \in \mathbb{Z} \,\}$. Then the cosets are the possible angles. This is an example of an infinite quotient group. The affine spaces in example 25 are another example of an infinite quotient group.

(36) In example 22 we saw that the modulus of complex numbers is a homomorphism from complex numbers to the reals. So, its kernel is the unit circle – the set of complex numbers of modulus 1. The cosets of the unit circle are the concentric circles,

$$C_r = \{\, z \mid |z| = r \,\}.$$

Applying the product of cosets gives us $C_rC_s = C_{rs}$ which works out because,

$$|(a + bi)(c + di)| = |(ac - bd) + (ad + bc)i|$$

$$
\begin{aligned}
&\iff && |(a+bi)(c+di)| = \sqrt{(ac-bd)^2+(ad+bc)^2} \\
&\iff && |(a+bi)(c+di)| = \sqrt{(a^2c^2+b^2d^2-2abcd)+(a^2d^2+b^2c^2+2abcd)} \\
&\iff && |(a+bi)(c+di)| = \sqrt{(a^2+b^2)(c^2+d^2)} \\
&\iff && |(a+bi)(c+di)| = \sqrt{(a^2+b^2)}\sqrt{(c^2+d^2)} \\
&\iff && |(a+bi)(c+di)| = |(a+bi)|\,|(c+di)|\,.
\end{aligned}
$$

TODO: Notes on Quotient Groups: Artin[81]

2.1.8.8 Modular Arithmetic

TODO: Describe modular arithmetic in terms of cosets: Artin[79] TODO: include in examples Abstract Maths ex. 14.9

2.2 Fields

2.2.1 Infinite Fields

Definition 59. A **field** F is a set together with two laws of composition, addition and multiplication, satisfying the following axioms:

(i) Addition makes F into an abelian group F^+ (or $(F, +)$). Its identity element is denoted 0.

(ii) Multiplication is associative and commutative and makes $(F \setminus \{0\}, \times)$ into a group. Its identity element is denoted 1.

(iii) Distributive law: For all $a, b, c \in F$, $(a + b)c = ac + bc$.

The distributive law establishes a relationship between the two laws of composition such that, for $a \in \mathbb{F}$,

$$a + a = 1a + 1a = (1 + 1)a.$$

In so doing, it establishes a relationship between the two groups: the additive group and the multiplicative group.

Theorem 2.2.1. ***(Difference of Cubes)*** *For $a, b \in \mathbb{F}$,*

$$a^3 - b^3 = (a - b)(a^2 + b^2 + ab).$$

Proof.

$$\begin{aligned}
&(a - b)(a^2 + b^2 + ab) \\
&= (a^3 + ab^2 + a^2b) - (ba^2 + b^3 + bab) && \text{by distributive law} \\
&= (a^3 + ab^2 + a^2b) - (a^2b + b^3 + ab^2) && \text{by commutativity of multiplication} \\
&= a^3 + ab^2 + a^2b + (-a^2b) + (-b^3) + (-ab^2) && \text{by distributive law} \\
&= (a^3 + (-b^3)) + (ab^2 + (-ab^2)) + (a^2b + (-a^2b)) && \text{by associativity of } + \\
&= (a^3 + (-b^3)) + 0 + 0 = a^3 - b^3. && \text{by additive inverse}
\end{aligned}$$

□

Note that the above proof does not require the multiplicative inverse property of fields and so a field is sufficient but not necessary. For this reason, the above theorem holds also in the integers $\mathbb{Z}$ although it wouldn't be defined in the naturals $\mathbb{N}$ as, e.g. $(-b^3)$ would be undefined.

2.2.1.1 Real-Number Exponentiation

Positive exponents

Let $a \in \mathbb{F}$. Then

$$\begin{aligned} 2a &= a + a, \\ 3a &= a + a + a, \\ a^2 = aa &= \underbrace{a + a + \cdots + a}_{a \text{ times}}, \\ a^3 = aaa &= \underbrace{a + a + \cdots + a}_{a \text{ times}} \underbrace{a + a + \cdots + a}_{a \text{ times}}. \end{aligned}$$

So, exponentiation by a positive integer can be defined in any field using the properties of the additive group.

Let $a \in \mathbb{Q}$. Then

$$\frac{7}{2}a = \underbrace{a + a + a}_{\lfloor \frac{7}{2} \rfloor = 3 \text{ times}} + \frac{1}{2}a,$$

$$q = a^{\frac{1}{2}} \in \mathbb{Q},$$

$$q^2 = qq = \underbrace{q + q + \cdots + q}_{q \text{ times}},$$

$$\exists m, n \in \mathbb{N} \text{ s.t. } q = \frac{m}{n} \implies$$

$$qq = \frac{m}{n}\frac{m}{n} = \underbrace{\frac{m}{n} + \frac{m}{n} + \cdots + \frac{m}{n}}_{\lfloor \frac{m}{n} \rfloor \text{ times}} + \left(\frac{m \bmod n}{n}\right)\frac{m}{n}.$$

So, exponentiation by a rational number can be defined in $\mathbb{Q}$.

Therefore, since the reals can be expressed as summations of series of rationals, exponentiation by any real number can be defined in $\mathbb{R}$.

2.2.1.2 Subfields

Definition 60. A field F is a **subfield of** $\mathbb{C}$ if the following properties hold:

- If $a, b \in F$, then $a + b \in F$.
- If $a \in F$, then $-a \in F$.
- If $a, b \in F$, then $ab \in F$.
- If $a \in F$ and $a \neq 0$, then $a^{-1} \in F$.
- $1 \in F$.

Note that using the first, second and last of these axioms we can deduce that $1 - 1 = 0$ *is an element of* F*.*

Also notice that addition on the field makes $(F, +)$ *into an abelian group and multiplication makes* $(F \setminus \{0\}, \times)$ *into an abelian group also. Conversely, any subset of* F *for which this is also true is a* ***subfield****.*

Definition 61. **(Ordered Field)** A field $(\mathbb{F}, +, \times)$ together with a total order $\leq$ on $\mathbb{F}$ is an **ordered field** if the order satisfies the following properties. $\forall a, b, c \in \mathbb{F}$,

(i) $a \leq b \implies a + c \leq b + c$,

(ii) $(0 \leq a) \wedge (0 \leq b) \implies 0 \leq a \times b$.

2.2.2 The Complex Field

Proposition 2.2.1. *For every* $\alpha \in \mathbb{C}$*, there exists a unique* $\beta \in \mathbb{C}$ *such that* $\alpha + \beta = 0$*.*

Proof. By contradiction: Say there are two such elements, β, γ such that,

$$\begin{aligned} \alpha + \beta = 0 = \alpha + \gamma \\ (\alpha + \beta) + \beta = (\alpha + \beta) + \gamma \\ 0 + \beta = \beta = 0 + \gamma = \gamma \end{aligned}$$

□

Proposition 2.2.2. *For every* $\alpha \in \mathbb{C}$ *with* $\alpha \neq 0$*, there exists a unique* $\beta \in \mathbb{C}$ *such that* $\alpha\beta = 1$*.*

Proof. By contradiction: Say there are two such elements, β, γ then,

$$\alpha\beta = 1 = \alpha\gamma$$

$$\beta = \frac{1}{\alpha} = \gamma \qquad \square$$

Proposition 2.2.3. *The complex field $\mathbb{C}$ is not an ordered field.*

Proof. By definition (2.2.1.2) an ordered field must satisfy the properties: $\forall a, b, c \in \mathbb{F}$,

(i) $a \leq b \implies a + c \leq b + c$,

(ii) $(0 \leq a) \wedge (0 \leq b) \implies 0 \leq a \times b$.

The second of these implies that if $0 \leq i$ then,

$$\begin{aligned} & 0 \leq i \times i = i^2 \\ \iff \quad & 0 \leq -1. \end{aligned}$$

Conversely, if $0 \leq -i$ then,

$$\begin{aligned} & 0 \leq -i \times -i = i^2 \\ \iff \quad & 0 \leq -1. \end{aligned}$$

Since these are the only two possibilities for an ordering of 0 and i, any ordering of the two elements results in $0 \leq -1$. This results in a contradiction because,

$$\begin{aligned} & 0 \leq -1 \times -1 = 1 \\ \iff \quad & 0 \leq 1 \end{aligned}$$

and we cannot have both $0 \leq 1$ and $0 \leq -1$. $\square$

2.2.2.1 The Fundamental Theorem of Algebra

Theorem 2.2.2 (Fundamental Theorem of Algebra)**.** *Every non-constant single-variable polynomial with complex coefficients has at least one complex root.*

Proof. wikipedia $\square$

Corollary 2.2.1. *Every non-constant degree-n single-variable polynomial with complex coefficients has exactly n complex roots when counted with the multiplicity.*

Proof. This is equivalent to the Fundamental Theorem of Algebra because if we have a degree-n polynomial with complex coefficients,

$$p_n(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

then the theorem tells us that this polynomial has at least one complex root. Let r_n be a root of $p_n(z)$. Then,

$$p_n(z) = (z - r_n) p_{n-1}(z).$$

But since $p_{n-1}(z)$ is a degree-$n-1$ polynomial with complex coefficients, the theorem also assures us that it too has at least one complex root, say r_{n-1} so that

$$p_n(z) = (z - r_n)(z - r_{n-1}) p_{n-2}(z).$$

Clearly, we can proceed in this way until we obtain

$$p_n(z) = a(z - r_n)(z - r_{n-1}) \cdots (z - r_1)$$

for some constant a. Since the roots r_i may not be distinct, this shows that $p_n(z)$ has n roots when counted with the multiplicity. □

2.2.2.2 Examples

(37) Find all the roots of $x^3 = 1$ for $x \in \mathbb{C}$.
Since $x^3 - 1 = (x-1)(x^2 + x + 1)$, we have (via zero-factor theorem) possible roots from,

$$x - 1 = 0 \iff x = 1$$

$$x^2 + x + 1 = 0 \implies x = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3}\, i}{2}$$

More generally,

$$(a+bi)+(a-bi)=2a$$

and since also,

$$\left[\frac{-1+\sqrt{3}\,i}{2}\right]^2=\frac{-1-\sqrt{3}\,i}{2}$$

as well as the reverse,

$$\left[\frac{-1-\sqrt{3}\,i}{2}\right]^2=\frac{-1+\sqrt{3}\,i}{2}$$

this means that if $x=\frac{-1\pm\sqrt{3}\,i}{2}$ then x^2+x is of the form $(a+bi)+(a-bi)=2a$ and so we have that $x^2+x=-1 \iff x^2+x+1=0$.

In addition,

$$(a+bi)(a-bi)=a^2+b^2$$

which means that if $x=\frac{-1\pm\sqrt{3}\,i}{2}$ then $x^3=x^2x$ is of the form $(a+bi)(a-bi)=a^2+b^2$ so we have that $x^3=\frac{-1}{2}^2+\frac{\sqrt{3}}{2}^2=\frac{1}{4}+\frac{3}{4}=1$.

So we see that - allowing for complex x - the cubic polynomial x^3-1 has 3 roots as we should expect from the Fundamental Theorem of Algebra.

(38) Consider the roots of the complex polynomial $x^3=-1$.

If we use the exponential form of the complex number x then we have,

$$(re^{i\theta})^3=r^3e^{3i\theta}=-1.$$

So, if we let $r=1$, we are looking for an angle θ (in radians) such that,

$$\cos(3\theta)+i\sin(3\theta)=-1.$$

Since, for $z=-1=-1+0i$, the imaginary part is 0, we know that $i\sin(3\theta)=0$ so,

$$\sin(3\theta)=0 \implies 3\theta=n\pi \quad \text{for} \quad n\in\mathbb{Z}.$$

Any $n \in \mathbb{Z}$ will generate an angle that satisfies the equation but, from the definition of the complex exponential (ref: 1.2.6.3), we know that the complex numbers represented by exponentials in this way, cycle every 2π-length interval of θ. Since we are only interested in finding all the unique solutions (i.e. the unique complex numbers that are roots of the polynomial), we can restrict the search to a single 2π-length interval of θ. Common practices are to use either $[0, 2\pi)$ or $(-\pi, \pi]$.

Using the interval $\theta \in [0, 2\pi)$, the valid values of the parameter θ are

$$\theta \in \left\{0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}\right\}.$$

But we also need that $\cos(3\theta) = -1$ so,

$$\cos(3\theta) = -1 \implies 3\theta = \pi + 2n\pi \quad \text{for} \quad n \in \mathbb{Z}$$

which gives the following valid values for θ,

$$\theta \in \left\{\frac{\pi}{3}, \pi, \frac{5\pi}{3}\right\}.$$

The solutions are generated from the intersection of the two sets of valid values for θ. So, the solutions are

$$x \in \{e^{\frac{i\pi}{3}}, e^{i\pi}, e^{\frac{5i\pi}{3}}\}.$$

A quicker way to have generated the solutions in this case would be to use Euler's Identity (wikipedia) to say,

$$x^3 = -1 \implies x^3 = e^{i\pi} \cdot e^{2ni\pi} \quad \text{for} \quad n \in \mathbb{Z}$$

which implies that

$$x = e^{\frac{i\pi}{3}} \cdot e^{\frac{2ni\pi}{3}} = e^{\frac{i\pi(1+2n)}{3}} \quad \text{for} \quad n \in \mathbb{Z}$$

where, if we define a homomorphism between multiplicative groups from angles θ to the unit circle of complex numbers of modulus 1,

$$\phi(\theta) = \cos\theta + i\sin\theta$$

then the kernel of ϕ is

$$2n\pi \quad \text{for} \quad n \in \mathbb{Z}.$$

(39) Find the roots of the complex polynomial $z^6 = -1$.

$$\begin{aligned}
& x^6 = -1 \\
\iff \quad & x^6 = e^{i\pi} && \text{Euler's equation} \\
\iff \quad & x = e^{\frac{i\pi}{6}} \cdot e^{\frac{2n\pi}{6}} \quad \text{for} \quad n \in \mathbb{Z} \\
\iff \quad & x \in \{e^{\frac{i\pi}{6}}, e^{\frac{3i\pi}{6}}, e^{\frac{5i\pi}{6}}, e^{\frac{7i\pi}{6}}, e^{\frac{9i\pi}{6}}, e^{\frac{11i\pi}{6}}\} \\
\iff \quad & x \in \{e^{\frac{i\pi}{6}}, e^{\frac{i\pi}{2}}, e^{\frac{5i\pi}{6}}, e^{\frac{7i\pi}{6}}, e^{\frac{3i\pi}{2}}, e^{\frac{11i\pi}{6}}\}.
\end{aligned}$$

Now, observing that

$$e^{\frac{11i\pi}{6}} = e^{\frac{-i\pi}{6}} \quad \text{and} \quad e^{\frac{3i\pi}{2}} = e^{\frac{-i\pi}{2}} \quad \text{and} \quad e^{\frac{7i\pi}{6}} = e^{\frac{-5i\pi}{6}},$$

we see that, by the properties of the complex conjugate (Proposition 1.2.22), these pairs are conjugates. So, if we define

$$z_1 = e^{\frac{i\pi}{6}}, \ z_2 = e^{\frac{i\pi}{2}}, \ z_3 = e^{\frac{5i\pi}{6}},$$

then we have factorized the polynomial into linear factors,

$$\begin{aligned}
& x^6 + 1 = 0 \\
\iff \quad & (x - z_1)(x - \overline{z_1})(x - z_2)(x - \overline{z_2})(x - z_3)(x - \overline{z_3}) = 0.
\end{aligned}$$

2.2.3 Finite Fields

TODO: Finite Fields (Artin[98])

2.3 Matrices

2.3.1 Basic Operations on Matrices

Definition 62. Matrix **equality** is defined component-wise so that if $A = B$ then A and B must have the same dimension as well as equal values in each component.

Definition 63. An **identity** element e is defined as $ea = ae = a$.

The definition of an identity element above is in any context (not just for matrices). For matrices this has certain consequences.

Proposition 2.3.1. *Identity matrices must be square*

Proof. For a matrix A and an identity matrix I, $AI = IA = A$ which means that AI, IA and A must all have the same dimensions. If A is of dimension $m \times n$ then I must have dimension $n \times m$ but then AI has dimension $m \times m$ while IA has dimension $n \times n$. We conclude that $m = n$ and both matrices are square. □

***If** A, B, C **are matrices s.t.** $AB = AC$**, can we, in general, conclude that** $B = C$**?**

The answer is no, as the following example shows:

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & -1 \\ 3 & 5 \end{pmatrix}, \qquad C = \begin{pmatrix} 8 & 0 \\ -4 & 4 \end{pmatrix}$$

$$A = B = \begin{pmatrix} 0 & 0 \\ 4 & 4 \end{pmatrix}$$

This is because multiplication by A has no inverse (i.e. it's not a bijection and A^{-1} does not exist) as we can see by the fact that $|A| = 0$.

If A, B, C ***are matrices s.t.*** $A+5B = A+5C$***, can we, in general, conclude that*** $B = C$***?***

The answer is yes because the matrix addition and scalar multiplication always have inverses. The inverse of $+A$ is $-A$ and the inverse of scalar multiplication by 5 is scalar multiplication by $\frac{1}{5}$. So we can say,

$$\begin{aligned} & A + 5B = A + 5C \\ \iff \quad & A + 5B - A = A + 5C - A \\ \iff \quad & 5B = 5C \\ \iff \quad & \left(\frac{1}{5}\right) 5B = \left(\frac{1}{5}\right) 5C \\ \iff \quad & B = C \end{aligned}$$

2.3.1.1 Matrix multiplication

Multiplication of matrices proceeds as a collection of dot-products of individual vectors. As a result, its properties are largely dependent on the properties of the dot-product (see: 2.4.22).

Matrix multiplication treats the two operand matrices as collections of vectors with the first matrix having the vectors as rows and the second having the vectors as columns.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

This difference in orientation of the vectors in the two operands results in the multiplication not being commutative - the order matters. So, the first property of the dot-product is not preserved but the others are preserved (albeit with a slight modification for the last one).

$$\begin{aligned} \alpha \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} &= \begin{bmatrix} \alpha(ae + bg) & \alpha(af + bh) \\ \alpha(ce + dg) & \alpha(cf + dh) \end{bmatrix} \\ &= \alpha \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix} \end{aligned}$$

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix}\right)\begin{bmatrix} i & j \\ k & l \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}\begin{bmatrix} i & j \\ k & l \end{bmatrix}$$
$$= \begin{bmatrix} i(a+e)+k(b+f) & j(a+e)+l(b+f) \\ i(c+g)+k(d+h) & j(c+g)+l(d+h) \end{bmatrix}$$
$$= \begin{bmatrix} ia+kb & ja+lb \\ ic+kd & jc+ld \end{bmatrix} + \begin{bmatrix} ie+kf & je+lf \\ ig+kh & jg+lh \end{bmatrix}$$
$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} i & j \\ k & l \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix}\begin{bmatrix} i & j \\ k & l \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{bmatrix}$$

So, to summarize:

If A, B, C are matrices and α is a scalar then,

- $\alpha AB = (\alpha A)B = A(\alpha B) = \alpha(AB)$
- $(A+B)C = C(A+B) = AC + BC$
- AA^T is a symmetric matrix with positive values along the diagonal

2.3.1.2 The Zero Matrix

Definition 64. The **zero matrix** is the matrix additive identity and multiplicative unit. Therefore, if 0_n here denotes the $n \times n$ zero matrix then, for any square $n \times n$ matrix M,

$$M + 0_n = M = 0_n + M \quad \text{and} \quad M0_n = 0_n = 0_n M.$$

Obviously, for non-square matrices, the multiplication will not be commutative and so only the left- or right-multiplication will be applicable.

Theorem 2.3.1. *Any matrix A such that for all vectors $\vec{\boldsymbol{v}}$ in the space $A\vec{\boldsymbol{v}} = \vec{\mathbf{0}}$, is the zero matrix.*

Proof. Let A be such a matrix so that for all $\vec{\boldsymbol{v}}$ in the vector space V over which the matrix is defined,

$$A\vec{\boldsymbol{v}} = \vec{\mathbf{0}}.$$

Then, for any other arbitrary matrix M defined over the same vector space, and having the same dimensions as A,

$$(M + A)\vec{\boldsymbol{v}} = M\vec{\boldsymbol{v}} + A\vec{\boldsymbol{v}} = M\vec{\boldsymbol{v}} = A\vec{\boldsymbol{v}} + M\vec{\boldsymbol{v}} = (A + M)\vec{\boldsymbol{v}}$$

which implies that $M + A = M = A + M$.

In addition, if M has appropriate dimensions for left-multiplication of A then,

$$(MA)\vec{\boldsymbol{v}} = M(A\vec{\boldsymbol{v}}) = M\vec{\mathbf{0}} = \vec{\mathbf{0}}$$

and if M has appropriate dimensions for right-multiplication of A then,

$$(AM)\vec{\boldsymbol{v}} = A(M\vec{\boldsymbol{v}}) = A\vec{\boldsymbol{w}} = \vec{\mathbf{0}}.$$

It therefore follows that,

$$MA = 0 \quad \text{and} \quad AM = 0$$

where 0 denotes the zero matrix of appropriate dimensions. □

2.3.1.3 Block Matrix multiplication

In the description of matrix multiplication as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

$a, b, \ldots, h$ could also be blocks of matrices. In which case, the resulting calculations — $ae + bg$ etc. — refer to matrix multiplication ae where a and e are treated as matrices and must have compatible dimensions. This can be seen as follows.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & \cdots \\ a_{21}b_{11} + a_{22}b_{21} & \cdots \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & \cdots \\ \vdots & \end{bmatrix}$$
$$= \begin{bmatrix} (a_{11}b_{11} + a_{12}b_{21}) + a_{13}b_{31} & \cdots \\ \vdots & \end{bmatrix}$$
$$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} + \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} \begin{bmatrix} b_{31} & b_{32} \end{bmatrix}.$$

So we have, for example, working in blocks,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} AE + BG & AF + BH \\ CE + DG & CF + DH \end{bmatrix},$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \implies AF = -BH,\ CE = -DG,\ AE+BG = CF+DH = I,$$

$$\begin{bmatrix} A \\ B \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} = \begin{bmatrix} I & E \\ F & G \end{bmatrix} \implies AC = I \iff C = A^{-1}.$$

Block multiplication preserves the block dimensions

For example, let

$$A \in \mathbb{R}^{2\times 2},\ B \in \mathbb{R}^{2\times 1},\ C \in \mathbb{R}^{1\times 2},\ D \in \mathbb{R}.$$

Then,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A^2 + BC & AB + BD \\ CA + DC & CB + D^2 \end{bmatrix}$$

where

$$A^2 + BC \in \mathbb{R}^{2\times 2}$$
$$AB + BD \in \mathbb{R}^{2\times 1}$$
$$CA + DC \in \mathbb{R}^{1\times 2}$$
$$CB + D^2 \in \mathbb{R}.$$

2.3.1.4 Matrix transpose

Proposition 2.3.2. $(AB)^T = B^T A^T$

Proof. Denote the ith row of the matrix A as $A[i :]$ and the jth column of the matrix B as $B[: j]$ and a matrix whose components at (i, j) are the dot-products of the ith row of the matrix A with the jth column of the matrix B as $(\langle A[i :], B[: j]\rangle)$. Then,

$$(AB)^T = (\langle A[i :], B[: j]\rangle)^T = (\langle A[j :], B[: i]\rangle)$$
$$B^T A^T = (\langle B^T[i :], A^T[: j]\rangle) = (\langle B[: i], A[j :]\rangle)$$

So, by commutativity of dot-product, $(AB)^T = B^T A^T$. □

Proposition 2.3.3. $(A^T)^{-1} = (A^{-1})^T$

Proof.

$$\begin{aligned} & I = AA^{-1} = (AA^{-1})^T = (A^{-1})^T A^T \\ \iff \quad & I(A^T)^{-1} = (A^{-1})^T A^T (A^T)^{-1} \\ \iff \quad & (A^T)^{-1} = (A^{-1})^T. \end{aligned}$$

□

Proposition 2.3.4. $(A + B)^T = A^T + B^T$

Proof. In $A + B$ the (i, j) element is $A_{ij} + B_{ij}$ so the (i, j) element of $(A + B)^T$ is $A_{ji} + B_{ji}$ which clearly is also the (i, j) element of $A^T + B^T$. □

Proposition 2.3.5. *If two $m \times n$ matrices A and B are defined over a vector space V then,*

$$\left[\forall \vec{\boldsymbol{x}}, \vec{\boldsymbol{y}} \in V \ . \ \vec{\boldsymbol{x}}^T A \vec{\boldsymbol{y}} = \vec{\boldsymbol{x}}^T B \vec{\boldsymbol{y}} \right] \implies A = B.$$

Proof. Firstly, observe that, for any arbitrary matrix M,

$$\vec{\boldsymbol{e_i}}^T M \vec{\boldsymbol{e_j}} = m_{ij}$$

where m_{ij} denotes the i,j-th element of the matrix M. So, if we let $\vec{\boldsymbol{x}}, \vec{\boldsymbol{y}}$ range over $\vec{\boldsymbol{e_i}}, \vec{\boldsymbol{e_j}}$ for $1 \leq i \leq m$, $1 \leq j \leq n$, then, for A and B satisfying the condition, we must have

$$\forall i, j \; . \; \vec{\boldsymbol{e_i}}^T A \vec{\boldsymbol{e_j}} = \vec{\boldsymbol{e_i}}^T B \vec{\boldsymbol{e_j}}$$

which implies that

$$\forall i, j \; . \; a_{ij} = b_{ij}$$

where a_{ij}, b_{ij} denote the i,j-th element of the matrices A, B respectively.

Since the matrices A and B are equal in all elements, $A = B$. □

2.3.1.5 Matrix inverse

Definition 65. Inverse property is: If there exists a matrix B such that $AB = BA = I$ then B is the **inverse** of A and A is the inverse of B.

This definition is inherently bound up with the definition of the identity ($\exists$ a matrix I s.t. $AI = IA = A$) and both define the identity and inverse elements as commutatively producing their result under matrix multiplication. Since matrix multiplication is not, in general, commutative there is no guarantee that if $AB = I$ then $BA = I$. An example of this failing is,

$$A = \begin{bmatrix} 1 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 \end{bmatrix} = I_1, BA = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \neq I_2$$

But we could have guessed this because Proposition 2.3.1 showed that identity matrices must be square and its product with a matrix must be defined from both the left and the right, i.e. $IA = AI = A$ meaning that the matrix A must have the same dimensions as I. So, for non-square matrices, no identity can exist. If there is no identity, then the inverse is not defined either.

Proposition 2.3.6. *If the inverses of the matrices A and B both exist then so does the inverse of the product AB and it is equal to $B^{-1}A^{-1}$.*

Proof.

$$\begin{aligned} & (AB)(AB)^{-1} = I \\ \iff \quad & (A^{-1}A)B(AB)^{-1} = A^{-1}I \\ \iff \quad & (B^{-1}B)(AB)^{-1} = B^{-1}A^{-1} \\ \iff \quad & (AB)^{-1} = B^{-1}A^{-1} \end{aligned}$$

and since B^{-1} and A^{-1} both exist then their product exists. Furthermore, this holds for a product of any finite sequence of invertible matrices $A_1A_2\cdots A_n$ which can easily be shown by induction on the associative product. □

2.3.1.6 Conjugate Matrix

Definition 66. The conjugate matrix $\overline{A}$ is formed by taking the complex conjugate of every element in A. That's to say, if the i,j-th elements of A are a_{ij} then the corresponding elements of $\overline{A}$ are $\overline{a_{ij}}$.

Proposition 2.3.7. $\overline{A\vec{\boldsymbol{v}}} = \overline{A}\,\overline{\vec{\boldsymbol{v}}}$

Proof. Let the i,j-th elements of A be denoted a_{ij} and the i-th element of a vector $\vec{\boldsymbol{v}}$ be denoted v_i. Then, using the properties of the complex conjugate (1.2.22),

$$A\vec{\boldsymbol{v}} = \begin{bmatrix} v_1a_{11} + \cdots + v_na_{1n} \\ \vdots \\ v_1a_{n1} + \cdots + v_na_{nn} \end{bmatrix}$$

$$\implies \quad \overline{A\vec{\boldsymbol{v}}} = \begin{bmatrix} \overline{v_1a_{11} + \cdots + v_na_{1n}} \\ \vdots \\ \overline{v_1a_{n1} + \cdots + v_na_{nn}} \end{bmatrix} = \begin{bmatrix} \overline{v_1a_{11}} + \cdots + \overline{v_na_{1n}} \\ \vdots \\ \overline{v_1a_{n1}} + \cdots + \overline{v_na_{nn}} \end{bmatrix} = \overline{A}\,\overline{\vec{\boldsymbol{v}}}. \quad \square$$

2.3.1.7 Hermitian Conjugate (a.k.a. Conjugate Transpose, a.k.a Adjoint)

Definition 67. If A is a matrix in $\mathbb{C}^{n\times n}$ then the **hermitian conjugate** of A is defined as

$$A^* = \overline{A}^T.$$

The term "adjoint" is also often used to refer to the hermitian conjugate but this name is also used for the transpose of the cofactor matrix (see: wikipedia).

Notation. The **hermitian conjugate** of A is commonly denoted A^* or A^H.

Since the two operations to form the hermitian conjugate — conjugation and transposition — commute with each other, the conjugation doesn't affect the usual properties of the matrix transpose. So we have:

- $(A^*)^* = A$ *(compare* $(A^T)^T = A$*)*
- $(A+B)^* = A^* + B^*$ *(compare Proposition 2.3.4)*
- $(AB)^* = B^*A^*$ *(compare Proposition 2.3.2)*

Proposition 2.3.8. *For a matrix A over a field $\mathbb{F}$ which may be the reals or the complex field, for a scalar $k \in \mathbb{F}$,*

$$(kA)^* = \overline{k}A^*.$$

Proof. Using the fact that a scalar is invariant under transposition and the properties of the complex conjugate 1.2.22,

$$(kA)^* = \overline{(kA)^T} = \overline{kA^T} = \overline{k}\overline{A^T}.$$

□

2.3.2 Basic properties of Matrices

Definition 68. If a_{ij} is an entry of a matrix in the ith row and jth column then the **main diagonal** of the matrix is the collection of entries a_{ij} with $i = j$.

The main diagonal is most often spoken of with respect to square matrices but the definition does not require that the matrix be square (wikipedia).

2.3.2.1 Trace

Definition 69. The **trace** of a matrix is the sum of the diagonal entries.

2.3.2.2 Symmetric Matrices

Definition 70. A **symmetric** matrix is a matrix that is invariant under transposition.

This obviously requires that the matrix be square and that the upper off-diagonal elements mirror the lower off-diagonal elements.

Proposition 2.3.9. *If A and B are symmetric matrices then $(BA)^T = AB$.*

Proof. Proposition 2.3.2 gives us the result that,

$$(BA)^T = A^T B^T$$

and the definition of symmetric matrices tells us that for symmetric A and B,

$$A^T = A \quad \text{and} \quad B^T = B.$$

Therefore $(BA)^T = AB$. □

Proposition 2.3.10. *A is a symmetric matrix in $\mathbb{R}^{n\times n}$ iff, for any $\vec{\boldsymbol{x}}, \vec{\boldsymbol{y}} \in \mathbb{R}^n$,*

$$A\vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{y}} = \vec{\boldsymbol{x}} \cdot A\vec{\boldsymbol{y}}.$$

Proof. Assume A is a symmetric matrix in $\mathbb{R}^{n\times n}$. Then, using $\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}} = \vec{\boldsymbol{v}}^T \vec{\boldsymbol{w}}$,

$$A\vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{y}} = (A\vec{\boldsymbol{x}})^T \vec{\boldsymbol{y}} = \vec{\boldsymbol{x}}^T A^T \vec{\boldsymbol{y}} = \vec{\boldsymbol{x}}^T A\vec{\boldsymbol{y}} = \vec{\boldsymbol{x}} \cdot A\vec{\boldsymbol{y}}.$$

Conversely, if we assume that A is a (not-necessarily symmetric) matrix in $\mathbb{R}^{n\times n}$ and $A\vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{y}} = \vec{\boldsymbol{x}} \cdot A\vec{\boldsymbol{y}}$ then, if we use a_j to denote the j-th column of the matrix A and a_{ij} to denote the i,j-th element of A,

$$\begin{aligned} & A\vec{\boldsymbol{e_j}} \cdot \vec{\boldsymbol{e_i}} = \vec{\boldsymbol{e_j}} \cdot A\vec{\boldsymbol{e_i}} \\ \iff \quad & a_j \cdot \vec{\boldsymbol{e_i}} = \vec{\boldsymbol{e_j}} \cdot a_i \\ \iff \quad & a_{ij} = a_{ji}. \end{aligned}$$

□

2.3.2.3 Upper Triangular Matrices

An **upper triangular matrix** is also known as a **row echelon matrix**.

2.3.2.4 Reduced Row Echelon Form Matrices

Definition 71. A **reduced row echelon form matrix** is an upper triangular matrix with the added conditions that the pivot values are all 1 and the other components in the same column as a pivot are all 0.

Reading off the nullspace of a RREF matrix

Let

$$A = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_2 & F \\ 0 & 0 \end{bmatrix}.$$

Then the nullspace of A is, for $t \in \mathbb{R}$

$$t \begin{bmatrix} -F \\ I_1 \end{bmatrix} = t \begin{bmatrix} -a \\ -b \\ 1 \end{bmatrix}.$$

So the general formula is

$$\begin{bmatrix} -F \\ I_k \end{bmatrix}$$

where k is the dimension of the nullspace (kernel) of A.
If the free variables are not next to each other then the blocks are broken up. For example for

$$A = \begin{bmatrix} 1 & a & 0 & c \\ 0 & b & 1 & d \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

the nullspace of A is

$$\begin{bmatrix} -a & -c \\ 1 & 0 \\ -b & -d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

2.3.2.5 Diagonal Matrices

Definition 72. A **diagonal** matrix is a matrix whose entries outside the main diagonal are all zero.

- *The definition of a diagonal matrix does **not** require that all the diagonal entries are nonzero so, for example, the zero matrix is a diagonal matrix.*
- *Diagonal matrices are **usually understood to be** a subset of symmetric matrices but this is not strictly necessary as the definition of the main diagonal permits an interpretation that includes rectangular matrices (see wikipedia).*

Proposition 2.3.11. *If $A = a_{ij}$, $B = b_{ij}$ are square diagonal matrices then multiplying them results in a square diagonal matrix $C = c_{ij}$ whose diagonal entries $c_{11}, \ldots, c_{nn}$ are the products of the corresponding diagonal entries in A and B. That's to say*

$$c_{11}, \ldots, c_{nn} = a_{11}b_{11}, \ldots, a_{nn}b_{nn}.$$

Proof. If $A = a_{ij}$, $B = b_{ij}$ are square diagonal matrices then,

$$AB = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & 0 & \cdots & 0 \\ 0 & b_{22} & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & b_{nn} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & 0 & \cdots & 0 \\ 0 & a_{22}b_{22} & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & a_{nn}b_{nn} \end{bmatrix}.$$

□

Corollary 2.3.1. *Multiplication of square diagonal matrices is commutative.*

Proof. The multiplication described in Proposition 2.3.11 is commutative because the diagonal entries of the two matrices are the same whichever way around the multiplication is performed and the multiplication of the individual entries itself is commutative. □

Corollary 2.3.2. *Square diagonal matrices with only nonzero diagonal entries are invertible.*

Proof. A result of the multiplication described in Proposition 2.3.11 is that, if A is a square diagonal matrix with nonzero diagonal entries, then we can obtain the identity matrix by multiplication with a matrix B whose diagonal elements are the reciprocal of the corresponding entries in A. So, if the jjth entry in A is a_{jj}, then the corresponding entry in B is $1/a_{jj}$. For example,

$$AB = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} 1/a_{11} & 0 & \cdots & 0 \\ 0 & 1/a_{22} & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 1/a_{nn} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Furthermore, by 2.3.1, this multiplication is commutative so we have,

$$AB = BA = I.$$

Therefore $B = A^{-1}$. Furthermore, this matrix always exists as long as the entries of the matrix are drawn from a field (so that they have multiplicative inverses) and the diagonal entries are nonzero. □

Proposition 2.3.12. *Square diagonal matrices with only nonzero diagonal entries form an abelian group under multiplication.*

Proof. Let S be the set of $n \times n$ diagonal matrices with nonzero diagonal entries. Then,

- $I_n \in S$ so S is nonempty.
- By Proposition 2.3.11, if A, B are members of S, then AB is also a square diagonal matrix with nonzero diagonal entries and so is also a member of S.
- By 2.3.2 members of S are invertible and are members of S.
- By 2.3.1 multiplication of members of S is commutative.

Therefore, $S \leq GL_n(\mathbb{F})$. □

2.3.2.6 Self-adjoint

Definition 73. A *self-adjoint operator* on a finite-dimensional complex vector space V with inner product $\langle \cdot, \cdot \rangle$ is a linear map A such that,

$$\langle A\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}} \rangle = \langle \vec{\boldsymbol{v}}, A\vec{\boldsymbol{w}} \rangle.$$

2.3.2.7 Hermitian Matrices

Definition 74. A matrix A is a **hermitian matrix** iff it is equal to its hermitian conjugate, i.e.

$$A = A^*.$$

Proposition 2.3.13. *All real symmetric matrices are hermitian.*

Proof. Let A be a real symmetric matrix. Then,

$$\overline{A} = A \quad \text{and} \quad A = A^T.$$

Therefore,

$$A^* = \overline{A}^T = A. \qquad \square$$

Proposition 2.3.14. *If V is a finite-dimensional complex vector space with an orthonormal basis defined and A is a matrix w.r.t. the orthonormal basis then, A is self-adjoint iff A is hermitian. That's to say,*

$$\langle A\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}} \rangle = \langle \vec{\boldsymbol{v}}, A\vec{\boldsymbol{w}} \rangle \iff A = A^*.$$

Proof.

$$\langle A\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}} \rangle == \vec{\boldsymbol{w}}^* A^* \vec{\boldsymbol{v}} =$$

$$\begin{aligned} & \langle A\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}} \rangle = \langle \vec{\boldsymbol{v}}, A\vec{\boldsymbol{w}} \rangle && \text{by self-adjoint property} \\ \iff \quad & \vec{\boldsymbol{w}}^* A\vec{\boldsymbol{v}} = (A\vec{\boldsymbol{w}})^* \vec{\boldsymbol{v}} && \text{by standard complex inner product} \\ \iff \quad & \vec{\boldsymbol{w}}^* A\vec{\boldsymbol{v}} = \vec{\boldsymbol{w}}^* A^* \vec{\boldsymbol{v}}. \end{aligned}$$

Then, by Proposition 2.3.5, this implies that $A = A^*$ which is the hermitian property. Clearly, the same reasoning can also be applied in reverse. $\square$

(40) An example of a hermitian matrix is

$$\begin{bmatrix} 1 & 1+2i & 4-i \\ 1-2i & -3 & i \\ 4+i & -i & 2 \end{bmatrix}.$$

The diagonal entries must always be real because they need to be equal to their conjugates while the off diagonal entries must be the conjugates of their corresponding entries across the diagonal (i.e. $a_{ij} = \overline{a_{ji}}$).

2.3.2.8 Unitary Matrices

Definition 75. **(Unitary Matrix)** A matrix $A \in \mathbb{C}^{n\times n}$ is said to be *unitary* iff

$$A^*A = AA^* = I.$$

Proposition 2.3.15. *A matrix A is unitary iff:*

(i) A^* is also unitary.

(ii) If another matrix B is also unitary then the product AB is unitary.

Proof.

(i) If we let $C = A^*$ then $A = C^*$ and $CC^* = I$ which means that $C = A^*$ is unitary. Clearly this logic also works in reverse so that A is unitary $\iff$ A^* is unitary

$$A \text{ is unitary } \iff A^* \text{ is unitary.}$$

(ii) If we assume that both A and B are unitary then, using the properties of the hermitian conjugate (2.3.1.7) in addition to the unitary property,

$$(AB)(AB)^* = (AB)(B^*A^*) = A(BB^*)A^* = AA^* = I$$

which shows that their product AB is unitary.
Conversely, if we assume only that B is unitary then, if the product AB is unitary, this implies that

$$(AB)(AB)^* = A(BB^*)A^* = AA^* = I$$

and so, as a result, A is also shown to be unitary. (TODO: does this prove the inverse exists on both sides?)

□

Proposition 2.3.16. *A real unitary matrix is orthogonal.*

Proof. If A is real then $\overline{A} = A \implies A^* = \overline{A}^T = A^T$. Therefore,

$$A^* = A \iff A^T = A.$$

That's to say, for a real matrix, the unitary property is equivalent to the orthogonal property. □

Theorem 2.3.2. ***(Schur Decomposition)*** *Let $A \in \mathbb{C}^{n\times n}$ be an arbitrary square matrix. Then A is unitarily similar to an upper triangular matrix. That's to say, there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that*

$$T = U^*AU$$

is upper triangular.

Proof. Let $\vec{\boldsymbol{v}}_1$ be an eigenvector of A (which we know exists by Proposition 2.5.15) corresponding to the eigenvalue λ_1 and normalise it to the unit vector $\vec{\boldsymbol{u}}_1$. If we extend $\vec{\boldsymbol{u}}_1$ to a basis of the space $\vec{\boldsymbol{u}}_1, \vec{\boldsymbol{v}}_2, \ldots, \vec{\boldsymbol{v}}_n$ then, by Gram-Schmidt (2.6.2.4), we can obtain an orthonormal basis of the space $B = \{\vec{\boldsymbol{u}}_1, \vec{\boldsymbol{u}}_2, \ldots, \vec{\boldsymbol{u}}_n\}$. Let U_0 be the matrix whose columns are the elements of B. By Proposition 2.6.17 then, U_0 is unitary.

Furthermore, if the matrix A is expressed w.r.t. the basis B,

$$U_0{}^*AU_0 = \begin{bmatrix} \lambda_1 & * \\ 0 & A_1 \end{bmatrix}$$

then the block A_1 is an $(n-1)\times(n-1)$ matrix. We can then recurse on the matrix A_1 by applying the same technique to obtain a unitary $(n-1)\times(n-1)$ matrix U_1 such that,

$$U_1{}^* A_1 U_1 = \begin{bmatrix} \lambda_2 & * \\ 0 & A_2 \end{bmatrix}.$$

If we let

$$U = U_0 \begin{bmatrix} 1 & 0 \\ 0 & U_1 \end{bmatrix}$$

then the matrix U is unitary because,

$$\begin{bmatrix} 1 & 0 \\ 0 & U_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & U_1 \end{bmatrix}^* = \begin{bmatrix} 1 & 0 \\ 0 & U_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & U_1{}^* \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & U_1{}^* U_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so that U is a product of unitary matrices which, by Proposition 2.3.15, is unitary. If we apply the matrix U to the original matrix A,

$$U^* A U = \begin{bmatrix} \lambda_1 & * & * \\ 0 & \lambda_2 & * \\ 0 & 0 & A_2 \end{bmatrix}.$$

We can proceed in this way until we obtain,

$$U^* A U = \begin{bmatrix} \lambda_1 & * & * & * \\ 0 & \lambda_2 & * & * \\ 0 & 0 & \ddots & * \\ 0 & 0 & 0 & \lambda_n \end{bmatrix}$$

which is an upper-triangular matrix. □

Compare this proof with Proposition 2.5.28.

2.3.2.9 Normal Matrices

Definition 76. **(Normal Matrix)** A square matrix A is called *normal* iff

$$AA^* = A^*A.$$

That's to say, the matrix A commutes with its hermitian conjugate (2.3.1.7).

Compare with normal subgroups in Group Theory 2.1.7.1.

Proposition 2.3.17. *The following classes of matrix are all normal:*

(i) Hermitian matrices (2.3.2.7)

(ii) Unitary matrices (2.3.2.8)

(iii) Diagonal matrices (2.3.2.5)

Proof. (i) $AA^* = AA = A^*A$.

(ii) $AA^* = I = A^*A$.

(iii) If a matrix D is diagonal then we can define it as

$$D = \text{diag}(a_1, a_2, \ldots, a_n)$$

where the scalars $a_i \quad \text{for} \quad 1 \leq i \leq n$ are the entries on the main diagonal and, by definition, all the other entries are 0. Then, since diagonal matrices are symmetric and so invariant to taking the transpose,

$$D^* = \overline{D}^T = \text{diag}(\overline{a_1}, \overline{a_2}, \ldots, \overline{a_n}).$$

Since both D and D^* are diagonal we can use 2.3.1 to deduce that,

$$DD^* = D^*D.$$

□

Proposition 2.3.18. *A normal upper-triangular matrix is diagonal.*

Proof. Let A be a normal upper-triangular matrix and let a_{ij} denote the i,j-th entry of A. Using the normal property of A we have,

$$\begin{aligned} & AA^* = A^*A \\ \iff \quad & \vec{\boldsymbol{e_i}}^T AA^* \vec{\boldsymbol{e_j}} = \vec{\boldsymbol{e_i}}^T A^* A \vec{\boldsymbol{e_j}} \\ \iff \quad & (A^*\vec{\boldsymbol{e_i}})^*(A^*\vec{\boldsymbol{e_j}}) = (A\vec{\boldsymbol{e_i}})^*(A\vec{\boldsymbol{e_j}}). \end{aligned}$$

If we let $j = i$ then this last result gives us

$$\begin{aligned} & (A^*\vec{\boldsymbol{e_i}})^*(A^*\vec{\boldsymbol{e_i}}) = (A\vec{\boldsymbol{e_i}})^*(A\vec{\boldsymbol{e_i}}) \\ \iff \quad & \|A^*\vec{\boldsymbol{e_i}}\|^2 = \|A\vec{\boldsymbol{e_i}}\|^2 \\ \iff \quad & \|A^*\vec{\boldsymbol{e_i}}\| = \|A\vec{\boldsymbol{e_i}}\| \\ \iff \quad & \left\|A^T\vec{\boldsymbol{e_i}}\right\| = \|A\vec{\boldsymbol{e_i}}\|. \qquad \because \|\overline{z}\| = \|z\| \end{aligned}$$

In other words, the norm of the i-th row is equal to the norm of the i-th column. If we now look at the upper-triangular property we see that:

- The first column of A, by the upper-triangular property, must consist solely of the a_{11} entry. Since the norm of the first row is equal to the norm of the first column, the first row must also consist solely of this element a_{11}.
- Then the second column of A, by the upper-triangular property, must have only zeroes after the element in the second row a_{22}. But we have also just shown that the first row is all zeroes after the first element so we know also that $a_{21} = 0$. Therefore, the second column exists solely of the element a_{22} and by a similar reasoning to the above, the norm implies that the second row must also consist solely of the entry a_{22}.
- Continuing reasoning in this way for the remaining columns of A we show that A is diagonal.

□

(41) The matrix

$$\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$$

is unitary and normal (because it's diagonal) but not hermitian. On the other hand, the matrix

$$\begin{bmatrix} 2i & 0 \\ 0 & i \end{bmatrix}$$

is — being diagonal — normal but is not unitary or hermitian.

2.3.2.10 Similarity and Equivalence of Matrices

Definition 77. **(Matrix Similarity)** If A and B are two *square* $n \times n$ matrices then they are said to be *similar* iff there exists an $n \times n$ invertible matrix P such that,

$$AP = PB \iff A = PBP^{-1}.$$

Similar matrices represent the same linear operator under different bases, with P being the change of basis matrix. In $GL_n\mathbb{F}$ similar matrices are the conjugates (see *conjugation in Group Theory*: 2.1.3.3) but in subgroups of $GL_n(\mathbb{F})$ the definition of conjugacy may be more restrictive.

Definition 78. **(Matrix Equivalence)** If A and B are two *rectangular* $m \times n$ matrices then they are said to be *equivalent* iff there exists an $n \times n$ invertible matrix P and an $m \times m$ invertible matrix Q such that,

$$AP = QB \iff A = QBP^{-1}.$$

Equivalent matrices represent the same linear transformation under two different choices of a pair of bases for the domain and codomain, with P and Q being the change of basis matrices.

Proposition 2.3.19. *Let A be the matrix of a linear operator T with respect to the basis B of dimension n. The matrices A' which represent T with respect to other bases are those of the form,*

$$A' = PAP^{-1}$$

where $P \in GL_n(\mathbb{F})$.

Proof. A linear operator is a specialization of a linear transformation where the domain and codomain are the same set so we can use the definition (2.5.2.2) of the matrix of a linear transformation w.r.t. to two different bases and simply set the two bases to the same set B. This produces,

$$T(\vec{\boldsymbol{x}}) = A\vec{\boldsymbol{x}}_B = \vec{\boldsymbol{b}}_B$$

so that,

$$[B]A = [T(\vec{\boldsymbol{b}}_1) \cdots T(\vec{\boldsymbol{b}}_n)] \iff A = [B]^{-1}[T(\vec{\boldsymbol{b}}_1) \cdots T(\vec{\boldsymbol{b}}_n)].$$

Now let there be another basis B' related to B by,

$$[B] = [B']P \iff [B]P^{-1} = [B'],$$

$$P\vec{\boldsymbol{x}}_B = \vec{\boldsymbol{x}}_{B'}.$$

To define a matrix A' that performs the same linear transformation as A w.r.t. to the basis B' we need,

$$T(\vec{\boldsymbol{x}}) = A'\vec{\boldsymbol{x}}_{B'} = \vec{\boldsymbol{b}}_{B'}$$

and

$$[B']A' = [T(\vec{\boldsymbol{b}}'_1) \cdots T(\vec{\boldsymbol{b}}'_n)] \iff A' = [B']^{-1}[T(\vec{\boldsymbol{b}}'_1) \cdots T(\vec{\boldsymbol{b}}'_n)].$$

If we have the vectors of B' encoded in B-coordinates then we can use the transformed version of the basis B to produce the transformed version of the basis B',

$$[T(\vec{\boldsymbol{b}}'_1) \cdots T(\vec{\boldsymbol{b}}'_n)] = [T(\vec{\boldsymbol{b}}_1) \cdots T(\vec{\boldsymbol{b}}_n)][B]^{-1}[B'] = [T(\vec{\boldsymbol{b}}_1) \cdots T(\vec{\boldsymbol{b}}_n)]P^{-1}.$$

If we now note that,

$$A = [B]^{-1}[T(\vec{\boldsymbol{b}}_1) \cdots T(\vec{\boldsymbol{b}}_n)] \quad \text{and} \quad A' = [B']^{-1}[T(\vec{\boldsymbol{b}}'_1) \cdots T(\vec{\boldsymbol{b}}'_n)]$$

then,

$$\begin{aligned} & A' = [B']^{-1}[T(\vec{\boldsymbol{b}}_1)\cdots T(\vec{\boldsymbol{b}}_n)]P^{-1} \\ \iff \quad & A' = [B']^{-1}[B]AP^{-1} \\ \iff \quad & A' = PAP^{-1}. \end{aligned}$$

□

Intuition of Similar Matrices

$$A' = PAP^{-1} \iff A'P = PA$$

P is the change of basis matrix such that $\vec{\boldsymbol{x}}_{B'} = P\vec{\boldsymbol{x}}_B$. So P is $B_{B'}$ the basis vectors of B w.r.t. to B'. Another way of looking at it: $P = [B']^{-1}[B]$ so it decodes coordinates in B-coords to standard coordinates and then encodes them into B'-coords. Meanwhile A transforms the basis vectors of B and then encodes the result in B-coords so the schematic of A is $A = [B]^{-1}[T(B)]$. So

$$PA = P[B]^{-1}[T(B)] = [B']^{-1}[B][B]^{-1}[T(B)] = [B']^{-1}[T(B)].$$

By similar reasoning the schematic of A' is $A' = [B']^{-1}[T(B')]$. But also we have,

$$A' = PAP^{-1} = ([B']^{-1}[B])([B]^{-1}[T(B)])([B]^{-1}[B']) = [B']^{-1}[T(B)][B]^{-1}[B']$$

so that,

$$\begin{aligned} & [B']^{-1}[T(B')] = [B']^{-1}[T(B)][B]^{-1}[B'] \\ \iff \quad & [T(B')] = [T(B)][B]^{-1}[B']. \end{aligned}$$

What this last result is saying is that if we take the basis vectors of B' and encode them in B-coordinates and then apply the result to the transformed basis of B then the result is the transformed basis of B' defined in standard coordinates.

So, we have,

$$A'P = PA = [B']^{-1}[T(B)]$$
$$P^{-1}A' = AP^{-1} = [B]^{-1}[T(B)][B]^{-1}[B'] = [B]^{-1}[T(B')].$$

Theorem 2.3.3. *If A is similar to a matrix $\tilde{A} = P^{-1}AP$ then*

$$A^n = (P\tilde{A}P^{-1})^n = P\tilde{A}^nP^{-1}.$$

Proof. Remembering that matrix multiplication is not, in general, commutative,

$$\begin{aligned} A^n &= (P\tilde{A}P^{-1})(P\tilde{A}P^{-1})\cdots(P\tilde{A}P^{-1}) \\ &= P\tilde{A}(P^{-1})(P)\tilde{A}(P^{-1}P)\cdots(P^{-1}P)\tilde{A}P^{-1} \\ &= P\tilde{A}^nP^{-1}. \end{aligned}$$

□

Proposition 2.3.20. *Similarity of matrices is an equivalence relation.*

Proof. For any $M, N \in GL_n(\mathbb{F})$, similarity is an equivalence relation because of the following properties.

Reflexivity:

$$\begin{aligned} N &= I^{-1}NI \\ \therefore N &\sim N \end{aligned}$$

Symmetry:

$$\begin{aligned} & N = P^{-1}MP \\ \iff \quad & NP^{-1} = P^{-1}M(PP^{-1}) \\ \iff \quad & NP^{-1} = P^{-1}M \\ \iff \quad & PNP^{-1} = (PP^{-1})M \\ \iff \quad & PNP^{-1} = M \\ \iff \quad & R^{-1}NR = M, \;\; R \in X \\ \therefore \quad & N \sim M \iff M \sim N \end{aligned}$$

Transitivity:

$$\begin{aligned} & N = P^{-1}MP, \;\; M = Q^{-1}AQ \\ \implies \quad & N = P^{-1}(Q^{-1}AQ)P \end{aligned}$$

$$
\begin{aligned}
&\iff && N = (P^{-1}Q^{-1})A(QP) \\
&\iff && N = R^{-1}AR, \;\; R \in X \\
&\therefore && (N \sim M) \wedge (M \sim Q) \iff (N \sim Q)
\end{aligned}
$$

□

Proposition 2.3.21. *Similar matrices have the same determinant.*

Proof. Let $A' = PAP^{-1}$ where P is a change of basis matrix. Then,

$$
\begin{aligned}
\det A' &= \det PAP^{-1} \\
&= (detP) \cdot (\det A) \cdot (\det P^{-1}) \\
&= (\det P) \cdot (\det A) \cdot (1/\det P) \\
&= \det A.
\end{aligned}
$$

□

2.3.3 Matrices as linear transformations

2.3.3.1 Multiplying a vector by a matrix on the left: $A\vec{\boldsymbol{x}} = \vec{\boldsymbol{y}}$

Left multiplication of a matrix A of dimension $m \times n$ on a column vector $\vec{\boldsymbol{x}}$ of dimension $n \times 1$ transforms it to a column vector $\vec{\boldsymbol{y}}$ of dimension $m \times 1$.

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$

This can be thought of a function from the space of n-dimensional vectors from which $\vec{\boldsymbol{x}}$ is drawn to the space of m-dimensional vectors in which $\vec{\boldsymbol{y}}$ resides. So, for real-valued vectors, the function would be a function $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ such that,

$$f(x_1, \cdots, x_n) = \vec{\boldsymbol{y}} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$

Or else, this could be thought of as m n-ary functions of the form $f : \mathbb{R}^n \mapsto \mathbb{R}$,

$$\begin{aligned} f_1(x_1, \cdots, x_n) &= a_{11}x_1 + \cdots + a_{1n}x_n = y_1 \\ &\vdots \\ f_m(x_1, \cdots, x_n) &= a_{m1}x_1 + \cdots + a_{mn}x_n = y_m \end{aligned}$$

In this case, each row of the matrix is a real-valued function in n variables. Each of these functions is *homogenous linear* (a function of the form $a_1x_1 + \cdots + a_kx_k + c$ for scalars $a_1, \cdots, a_k, c$ and $c = 0$) and so the system of functions is called a *linear transformation*.

Example of $A\vec{\boldsymbol{x}} = \vec{\boldsymbol{y}}$

(42) Consider the following system of equations (represented by an augmented matrix) for some constants a, b,

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 3 & -1 & -1 & 0 \\ 1 & 1 & a & b \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 0 & 1 & \frac{-7}{2} & -6 \\ 0 & 0 & a+5 & b+8 \end{array}\right]$$

where the second matrix represents a system of linear equations with the same solutions as the first. The second form tells us that:

- The system has **precisely one solution** if $a \neq -5$, so that we have an upper triangular matrix.
- The system has **infinitely many solutions** if $a = -5$ and $b = -8$, so that the bottom row is all zeroes and there is a free variable.
- The system has **no solutions** if $a = -5$ but $b \neq -8$, so that the bottom row of the matrix is zeroes but the corresponding component in the augmented column is non-zero; meaning that the system is inconsistent.

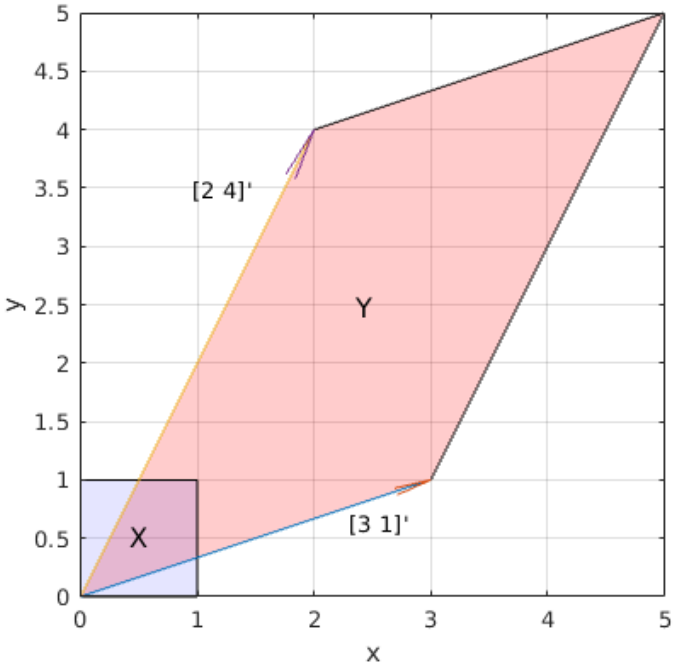
2.3.3.2 Multiplying a matrix of vectors by a matrix on the left: $AX = Y$

Looking at the matrix as a linear transformation from one co-ordinate space to another, consider $AX = Y$ where X is a matrix - which may be considered a collection of vectors - transformed by the matrix A into the matrix - or collection of vectors - Y.

We transform the unit square in the source space, X in $\mathbb{R}^2$, using the 2D transformation matrix A, into its image in the destination space, Y in $\mathbb{R}^2$.

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}, \; X = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$AX = Y = \begin{bmatrix} 0 & 3 & 5 & 2 \\ 0 & 1 & 5 & 4 \end{bmatrix}$$



2.3.3.3 Change of Co-ordinates

If a matrix A has columns comprised of the axes of a co-ordinate system, then $\vec{\boldsymbol{x}}$ defined against the standard basis is, in the other system, $A^{-1}\vec{\boldsymbol{x}}$. This is because the axes in A are defined relative to the standard basis axes. Therefore, if $\vec{\boldsymbol{x_A}}$ is a vector defined against the axes in A, then the same vector against the standard basis axes $\vec{\boldsymbol{x}}$ would be $A\vec{\boldsymbol{x_A}} = \vec{\boldsymbol{x}}$ and so $\vec{\boldsymbol{x_A}} = A^{-1}\vec{\boldsymbol{x}}$.

2.3.3.4 Types of Transformations

There are 3 basic types of transformation:

- **Rigid body** - preserves distances and angles.

  Examples: translation and rotation.

- **Conformal** - preserves angles.

  Examples: translation, rotation and uniform scaling.

- **Affine** - preserves parallelism.

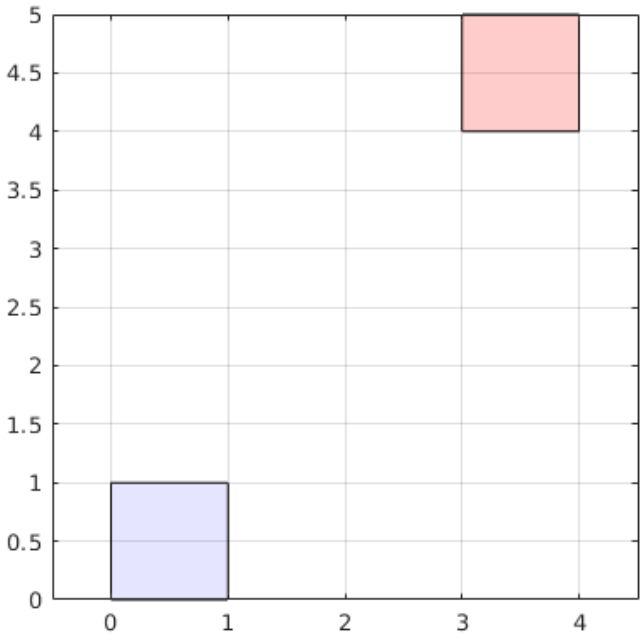
  Examples: translation, rotation, uniform and non-uniform scaling, shearing and reflection.

2.3.3.5 Rigid Body

Translation So as to perform the translation as multiplication by a transformation matrix we take the approach of homogeneous coordinates (see:https://en.wikipedia.org/wiki/Homogeneous_coordinates) so we form matrix with the identity in the first two columns and then a third column with the translation vector. Then, we add a row of ones to the vectors we will translate and the output vectors also have a 1 in the third row that is ignored.

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}, \; X = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

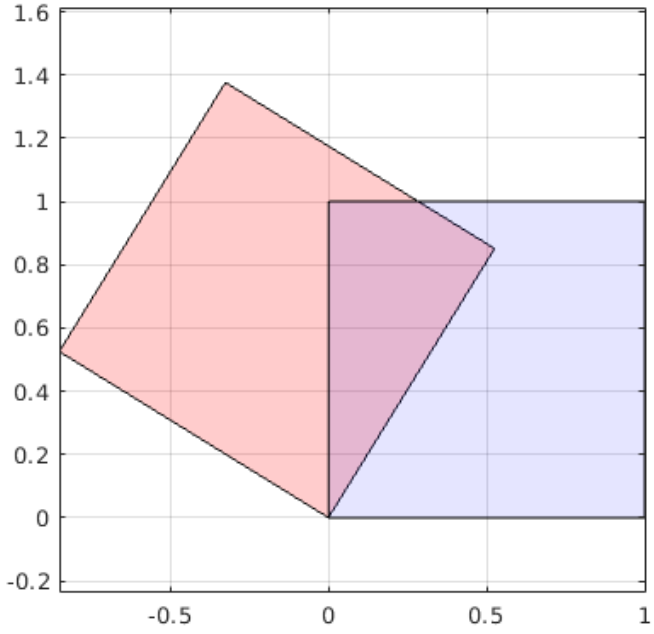
$$AX = Y = \begin{bmatrix} 3 & 4 & 4 & 3 \\ 4 & 4 & 5 & 5 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$



Rotation

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},\ X = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$AX = Y = \begin{bmatrix} 0 & 0.5253 & -0.3256 & -0.8509 \\ 0 & 0.8509 & 1.3762 & 0.5253 \end{bmatrix}$$

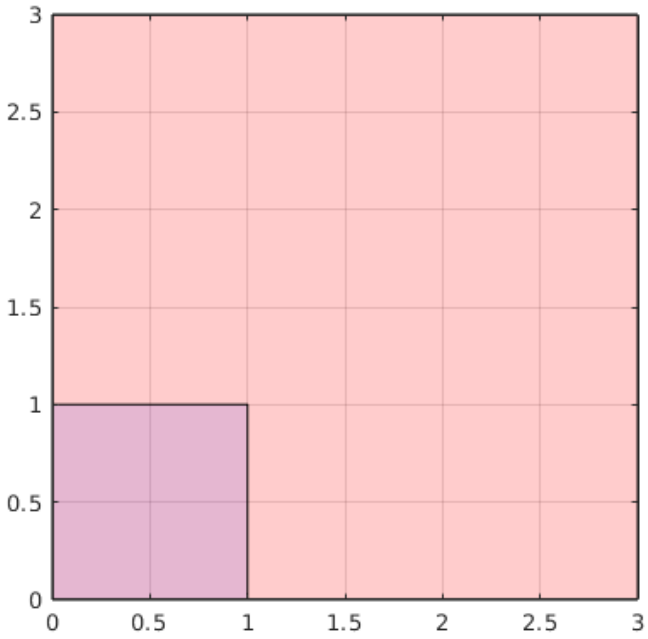


2.3.3.6 Conformal

Uniform Scaling is scaling by an equal amount in each dimension.

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, \; X = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$AX = Y = \begin{bmatrix} 0 & 3 & 3 & 0 \\ 0 & 0 & 3 & 3 \end{bmatrix}$$

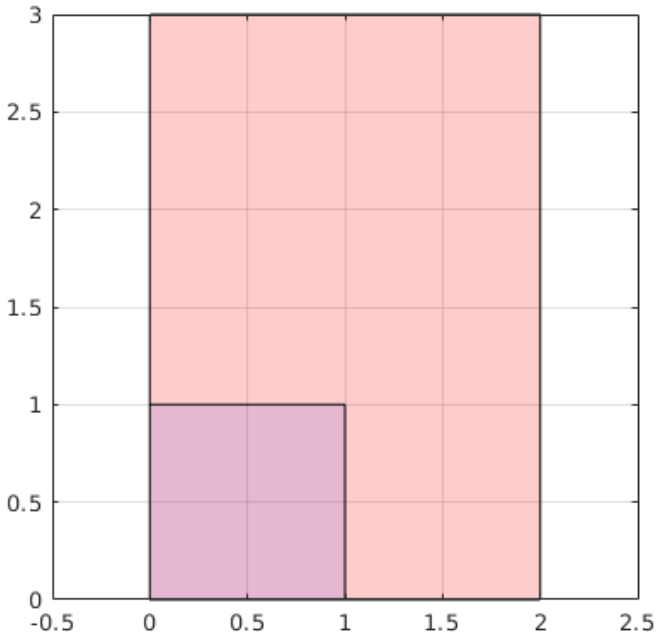


2.3.3.7 Affine

Non-uniform Scaling is scaling by different amounts in the different dimensions. (The example shown here preserves the angles but for other shapes, a triangle for example, the angles would not be preserved.)

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \; X = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

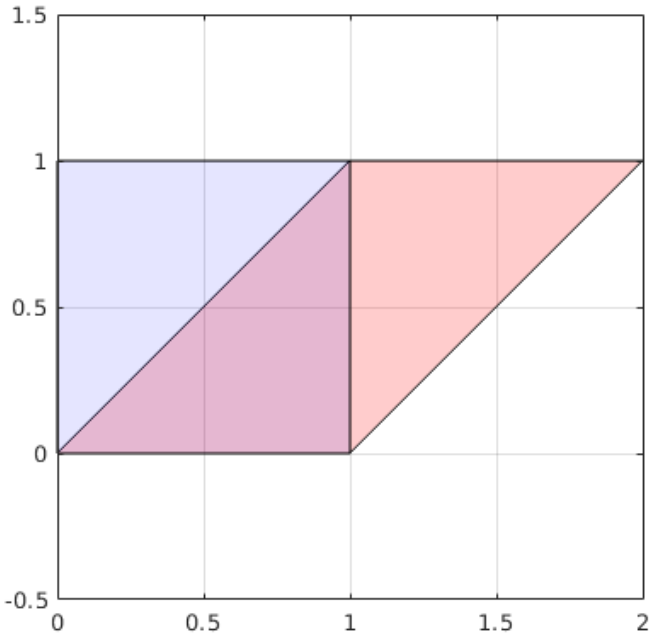
$$AX = Y = \begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 3 & 3 \end{bmatrix}$$



Shearing

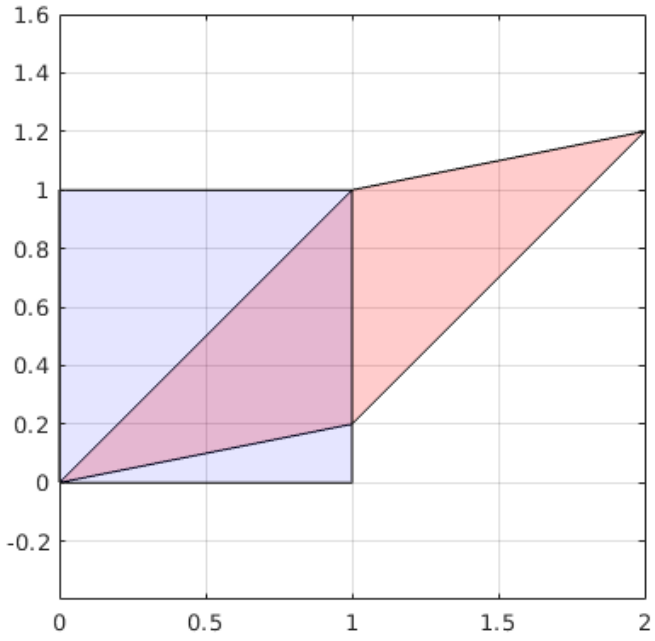
$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},\ X = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$AX = Y = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$



$$A = \begin{bmatrix} 1 & 1 \\ 0.2 & 1 \end{bmatrix},\ X = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

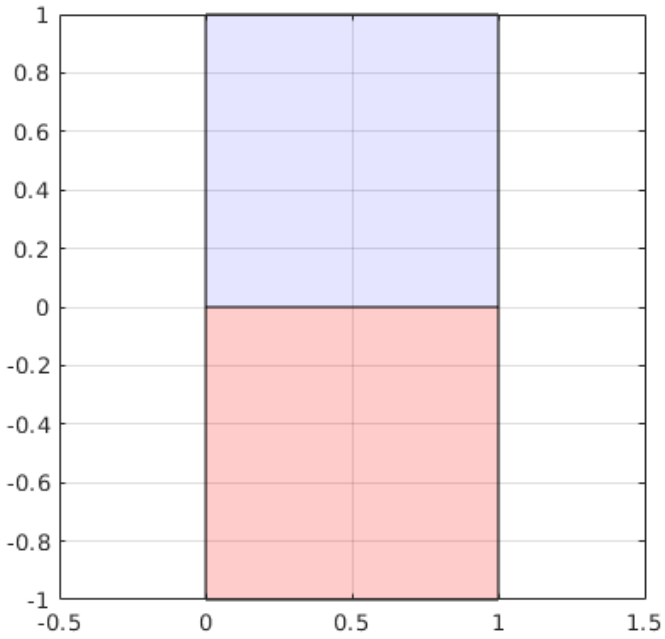
$$AX = Y = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 0.2 & 1.2 & 1 \end{bmatrix}$$



Reflection

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ X = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$AX = Y = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 \end{bmatrix}$$


1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-0.5
0
0.5
1
1.5

2.3.4 Elementary Matrices and Row Operations

Notation. The **matrix units** - matrices with a single non-zero component whose value is 1 are traditionally named $\boldsymbol{e_{ij}}$ where i, j is the matrix co-ordinate of the 1.

An arbitrary matrix $A = (a_{ij})$ may be expressed as a sum of such unit matrices as $A = a_{11}e_{11} + \cdots + a_{nn}e_{nn}$.

$$e_{ij} = \begin{bmatrix} \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots \\ \cdots & 1 & \cdots \\ \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots \end{bmatrix}$$

So matrix units can be used to analyse matrix addition but to analyse matrix multiplication some square matrices called **elementary matrices** are more useful.

Multiplying a matrix from the left (so doing row operations), there are 3 types of elementary matrix:
Adding rows: $\boldsymbol{I + ae_{ij}}$ ***for*** $\boldsymbol{i \neq j}$

$$\begin{bmatrix} 1 & & & \\ & \cdot & a & \\ & & \cdot & \\ & & & 1 \end{bmatrix}$$

This adds a times some row to another row.
Swapping rows: $\boldsymbol{I + e_{ij} + e_{ji} - e_{ii} - e_{jj}}$ ***for*** $\boldsymbol{i \neq j}$

$$\begin{bmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{bmatrix}$$

This swaps the rows i and j.

Scalar-multiplying a row: $\boldsymbol{I + (c-1)e_{ii}}$ ***for*** $\boldsymbol{c \neq 0}$

$$\begin{bmatrix} 1 & & & \\ & \cdot & & \\ & & c & \\ & & & 1 \end{bmatrix}$$

This multiplies row i by c.

Proposition 2.3.22. *Elementary matrices are invertible and their inverses are also elementary matrices.*

Proof. Proceed by cases on the 3 elementary types of elementary matrices.

Case $\boldsymbol{I + ae_{ij}}$ If R_i is row i and R_j is row j, then this matrix performs $R_i + aR_j$. Clearly this can be "undone" by performing $R_i - aR_j$. So the matrix, $I - ae_{ij}$ is the inverse and clearly this is also an elementary matrix of the same type.

Case $\boldsymbol{I - e_{ii} - ejj + e_{ij} + e_{ji}}$ This matrix swaps 2 rows in a permutation that is its own inverse.

Case $\boldsymbol{I + (c-1)e_{ii}}$ This matrix performs cR_i and so it is "undone" by performing $c^{-1}R_i$ (which for a real-valued matrix would be $\left(\frac{1}{c}\right)R_i$) and this inverse matrix is also an elementary matrix of the same type. □

Proposition 2.3.23. *Suppose $AX = B$ and a series of elementary row operations on $[A \mid B]$ produces $[A' \mid B']$, then the solutions of $A'X = B'$ are the same as those of $AX = B$.*

Proof. First note that the series of elementary row operations is described as multiplication on the left by a series of elementary matrices say, $E_1, E_2, \cdots, E_n$ so that,

$$[A' \mid B'] = [(E_n \cdots E_2 E_1)A \mid (E_n \cdots E_2 E_1)B]$$

Now, let $(E_n \cdots E_2 E_1) = E$ and notice that, since each of the individual E_i is invertible the product of them is also invertible by Proposition 2.3.6 so,

$$A'X = B' \iff EAX = EB$$

and the existence of the inverse E^{-1} means that the law of cancellation is in effect so,

$$EAX = EB \iff AX = B$$
$$\therefore A'X = B' \iff AX = B. \qquad \square$$

Note that this is why Gaussian Elimination can employ row reduction to produce a linear system that is simpler to solve: elementary row operations on the matrix form linear combinations of the equations in a system of linear equations. This means that any vector that was a solution to the original system of equations (i.e. was a member of the intersection of the linear vector spaces represented by the equations) will also be a solution to the new system. Gaussian Elimination proceeds in this way until — if we obtain the row reduced echelon form — it has produced a system in which the linear vector spaces represented by the rows of the matrix are parallel to the co-ordinate system so that we can simply read off the co-ordinates of the solution (if it is a single point).

Proposition 2.3.24. *Let A be a square matrix. The following conditions are equivalent:*

- *A can be reduced to the identity by a sequence of elementary row operations.*
- *A is a product of elementary matrices.*
- *A is invertible.*
- *The system of homogeneous equations $AX = 0$ has only the trivial solution $X = 0$.*

Proof. If A can be reduced to the identity by a sequence of elementary row operations then,

$$(E_n \cdots E_2 E_1)A = I$$

and by Proposition 2.3.22 and Proposition 2.3.6 the matrix $(E_n \cdots E_2 E_1)$ is invertible so,

$$A = (E_n \cdots E_2 E_1)^{-1} I = (E_n \cdots E_2 E_1)^{-1} = {E_1}^{-1} {E_2}^{-1} \cdots {E_n}^{-1}$$

and, also by Proposition 2.3.22, A is a product of elementary matrices and is invertible.
Furthermore, if $AX = 0$ then $X = A^{-1}0 = 0$ - i.e. the only solution to $AX = 0$ is $X = 0$. □

2.3.4.1 The Reduced Row Echelon Form

The **reduced row echelon form** of a matrix is described in 2.3.2.4. It is obtained for a given matrix by a sequence of elementary row operations on the matrix according to the procedure known as Gaussian Elimination.

Example of Gaussian Elimination

(43) Let

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 2 & -4 \\ 3 & 2 & -1 \end{bmatrix}.$$

Gaussian Elimination on A produces the reduced row echelon form (rref) matrix

$$A' = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}.$$

The rref matrix A' tells us that the third column of A is obtainable as a linear combination of the first two columns.

If we break up the matrix A into the blocks suggested by the rref

$$A = \begin{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} & \begin{bmatrix} 1 \\ -4 \end{bmatrix} \\ \begin{bmatrix} 3 & 2 \end{bmatrix} & \begin{bmatrix} -1 \end{bmatrix} \end{bmatrix}$$

we see that

$$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}^{-1} = \frac{1}{6} \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}$$

and

$$\frac{1}{6}\begin{bmatrix}2 & -1\\ 0 & 3\end{bmatrix}\begin{bmatrix}1\\ -4\end{bmatrix} = \begin{bmatrix}1\\ -2\end{bmatrix}$$

so that

$$\begin{bmatrix}\begin{bmatrix}3 & 1\\ 0 & 2\end{bmatrix}^{-1} & \begin{bmatrix}0\\ 0\end{bmatrix}\\ \begin{bmatrix}0 & 0\end{bmatrix} & \begin{bmatrix}0\end{bmatrix}\end{bmatrix}\begin{bmatrix}\begin{bmatrix}3 & 1\\ 0 & 2\end{bmatrix} & \begin{bmatrix}1\\ -4\end{bmatrix}\\ \begin{bmatrix}3 & 2\end{bmatrix} & \begin{bmatrix}-1\end{bmatrix}\end{bmatrix} = \begin{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} & \begin{bmatrix}1\\ -2\end{bmatrix}\\ \begin{bmatrix}0 & 0\end{bmatrix} & \begin{bmatrix}0\end{bmatrix}\end{bmatrix}.$$

Note that the matrix multiplication by blocks works out:

$$\begin{bmatrix}\begin{bmatrix}3 & 1\\ 0 & 2\end{bmatrix}^{-1} & \begin{bmatrix}0\\ 0\end{bmatrix}\\ \begin{bmatrix}0 & 0\end{bmatrix} & \begin{bmatrix}0\end{bmatrix}\end{bmatrix}\begin{bmatrix}\begin{bmatrix}3 & 1\\ 0 & 2\end{bmatrix} & \begin{bmatrix}1\\ -4\end{bmatrix}\\ \begin{bmatrix}3 & 2\end{bmatrix} & \begin{bmatrix}-1\end{bmatrix}\end{bmatrix} = \begin{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} & \begin{bmatrix}1\\ -2\end{bmatrix}+\begin{bmatrix}0\\ 0\end{bmatrix}\\ \begin{bmatrix}0 & 0\end{bmatrix}+\begin{bmatrix}0 & 0\end{bmatrix} & \begin{bmatrix}0\end{bmatrix}+\begin{bmatrix}0\end{bmatrix}\end{bmatrix}.$$

From this it's clear that the Gaussian Elimination algorithm amounts to left multiplication by the inverse of the invertible part of the matrix (in this case R).

Another way of looking at it arises from observing that

$$AA' = \begin{bmatrix}3 & 1 & 1\\ 0 & 2 & -4\\ 3 & 2 & -1\end{bmatrix}\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & -2\\ 0 & 0 & 0\end{bmatrix} = \begin{bmatrix}3 & 1 & 1\\ 0 & 2 & -4\\ 3 & 2 & -1\end{bmatrix} = A.$$

So, the rref matrix of A is also a right identity of A. The final column of the rref matrix representing the free variable, is an alternative to the standard basis vector $\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}$ when taking linear combinations of the columns of A.

Using the rref to solve $A\vec{x} = \vec{b}$

If we are attempting to solve an equation of the form $A\vec{x} = \vec{b}$ then forming the augmented matrix and performing Gaussian Elimination left multiplies

the vector $\vec{\boldsymbol{b}}$ by the inverse of the invertible part of the matrix to obtain an element in the reverse image of $\vec{\boldsymbol{b}}$.

For example, let $\vec{\boldsymbol{b}} = \langle 3, 4, 5 \rangle$ then, forming the augmented matrix $[A \,|\, \vec{\boldsymbol{b}}]$ and performing row reduction to obtain the rref,

$$\left[\begin{array}{ccc|c} 3 & 1 & 1 & 3 \\ 0 & 2 & -4 & 4 \\ 3 & 2 & -1 & 5 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & \frac{1}{3} \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

from which we infer that

$$\vec{\boldsymbol{x}} = \begin{bmatrix} \frac{1}{3} \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

for $t \in \mathbb{F}$.

Note that:

- *If $\vec{\boldsymbol{b}}$ were not in the span of the first two columns of A then elimination would show that the rows of A were inconsistent. In this case, for $\vec{\boldsymbol{b}}$ to be in the span of A it had to be a point in the plane $x + \frac{y}{2} - z = 0$.*
- *If we were to swap the columns of A around we would find the same kernel but the particular solution would be different because it would be expressed w.r.t to different vectors.*

Nature of the Solution

The solution found by this method is, naturally, a coset of the kernel. In the language of linear algebra it is usually described as a particular solution + a general solution. Geometrically, in this case, the general solution (the kernel/nullspace) is a line and the particular solution is a point on the line. But, more interestingly, the particular solution found by this method is the solution that effectively has the kernel zero-ed out. That's to say,

$$\vec{\boldsymbol{x}} = \begin{bmatrix} \frac{1}{3} \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} \\ 2 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}.$$

For example, if we define $t' = t - 1$ then,

$$\vec{\boldsymbol{x}} = \begin{bmatrix} \frac{-2}{3} \\ 4 \\ 1 \end{bmatrix} + t' \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

is also, equally, a solution. But the solution found by Gaussian Elimination will always consist of a vector (the particular solution) with zeroes in the components corresponding to any free-variable columns in the matrix.

2.3.5 Determinants

1×1

The determinant of a 1×1 matrix is just its unique component entry,

$$det \begin{bmatrix} a \end{bmatrix} = a$$

2×2

The determinant of a 2×2 matrix is given by the formula,

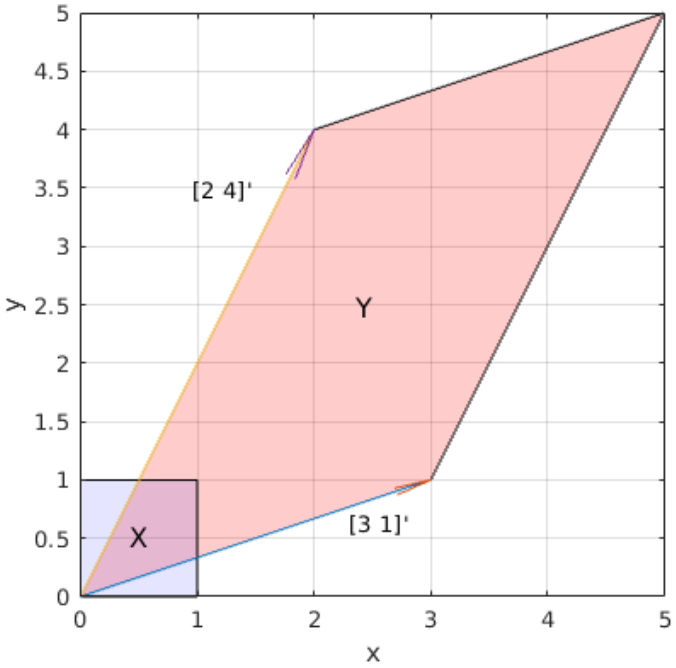
$$det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

Returning to our example of a 2d operator:

We transform the unit square in the source space, X in $\mathbb{R}^2$, using the 2D transformation matrix A, into its image in the destination space, Y in $\mathbb{R}^2$.

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}, \; X = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$AX = Y = \begin{bmatrix} 0 & 3 & 5 & 2 \\ 0 & 1 & 5 & 4 \end{bmatrix}$$



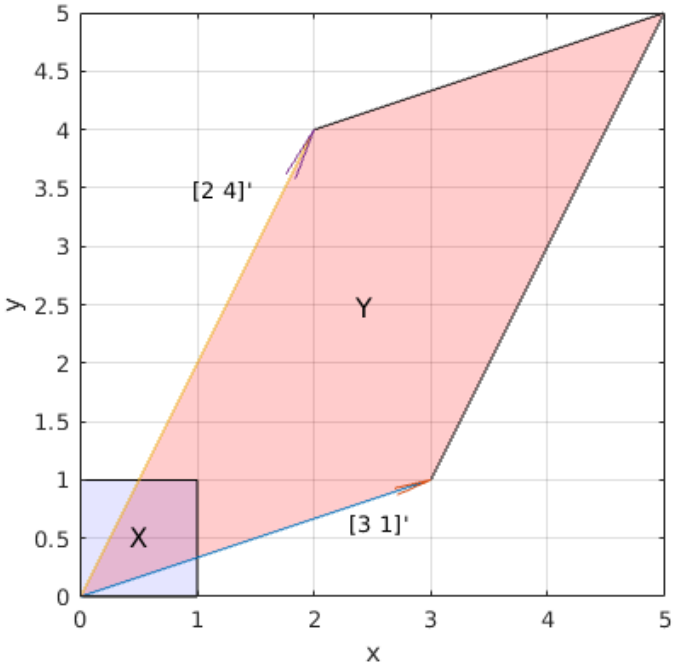
We see that $det\ A = 10$ and the parallelogram, Y, that is the image of the unit square, X, under the transformation represented by A has area,

$$\text{area } = b \cdot h = |\langle 3, 1\rangle| \cdot |\langle 2-3, 4-1\rangle| = \sqrt{10} \cdot \sqrt{10} = 10$$

And the determinant would be 0 in the case that the columns were proportional (representing co-linear vectors) and the determinant would be negative if the orientation of the output vectors were reversed w.r.t. the input vectors.
So, if we swap either the columns or the rows of the transformation matrix, A, the determinant comes out -10.
Swapping the columns:

$$A_c = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix},\ X = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$AX = Y = \begin{bmatrix} 0 & 2 & 5 & 3 \\ 0 & 4 & 5 & 1 \end{bmatrix}$$

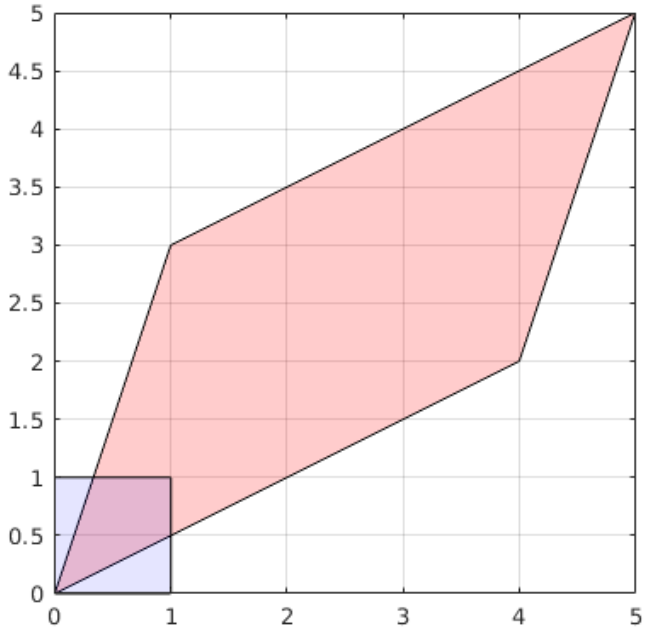


Note that the result looks exactly the same - it's just that now the x-vector $\langle 1, 0\rangle$, produces $\langle 4, 2\rangle$ and the y-vector, $\langle 0, 1\rangle$ produces $\langle 3, 1\rangle$.

Swapping the rows:

$$A_r = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}, \; X = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$AX = Y = \begin{bmatrix} 0 & 1 & 5 & 4 \\ 0 & 3 & 5 & 2 \end{bmatrix}$$



Note that if we swap **both** the columns and the rows then we get back to a transformation with determinant 10.

$$A_{rc} = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix},\ X = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$AX = Y = \begin{bmatrix} 0 & 4 & 5 & 1 \\ 0 & 2 & 5 & 3 \end{bmatrix}$$

which produces the same parallelogram as the previous one but with columns reversed.

Summary So we find that,

$$\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}, \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} \text{ have determinant } > 0$$

$$\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} \text{ have determinant } < 0$$

If the product of the components on the diagonal of the matrix is greater than the components on the bottom-left to top-right diagonal then the determinant is > 0, if the reverse is true then the determinant is < 0, and if they are equal then the determinant $= 0$.
Note that, for the determinant to be 0 in our example, we need something like $detA = (4 \times 3) - (4 \times 3)$ which, due to the commutativity of multiplication can be achieved by both,

$$\begin{bmatrix} 4 & 3 \\ 4 & 3 \end{bmatrix}, \begin{bmatrix} 4 & 4 \\ 3 & 3 \end{bmatrix}$$

but in both cases the columns are proportional and therefore co-linear.
$\boldsymbol{n \times n}$

The determinant of an $n \times n$ matrix is defined recursively as:

- if $n = 1$ then $det\, A = a_{11}$, i.e. the determinant is equal to the sole component.
- else if $n > 1$ then, defining A_{ij} as the matrix formed by leaving out the ith row and the jth column,

$$det\, A = a_{11} det\, A_{11} - a_{12} det\, A_{12} + \cdots \pm a_{1n} det\, A_{1n}$$

In the $n = 2$ case, each $det\, A_{ij}$ has $n = 1$ and so is simply equal to the sole component that is neither on the same row or column as the component a_{ij} that is multiplying it. This feature of the determinant calculation continues recursively for higher dimension matrices so that the calculation is always comprised of terms that are a product of components on each of the different columns and rows. In fact, it comprises the products of all such possible combinations of components.
For example, when $n = 2$ the only combinations are,

$$\{a_{11}, a_{22}\} \text{ and } \{a_{12}, a_{21}\}$$

so there are only 2 terms in the determinant calculation.
When $n = 3$ the possible combinations are,

$$\{a_{11}, a_{22}, a_{33}\},\ \{a_{11}, a_{32}, a_{23}\},$$

$$\{a_{12}, a_{21}, a_{33}\},\ \{a_{12}, a_{31}, a_{23}\},$$
$$\{a_{13}, a_{21}, a_{32}\},\ \{a_{13}, a_{31}, a_{22}\}$$

so there are 6 terms in the determinant calculation. Notice that each term is generated by a different permutation of the columns while holding the rows fixed in ascending order and that the sign of each term is governed by how many permutations the permutation of columns is away from ascending order, $1, 2, \cdots, n$.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$
$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

2.3.5.1 Consequences

From this feature of the calculation we can see a number of the important properties of the determinant.

Proposition 2.3.25. *$det\, I = 1$*

Proof. Whatever the dimension of the identity matrix there will be only one combination of rows and columns that is the diagonal along which the 1s of the identity matrix reside. So, there will be a single term of the determinant calculation that is a product of 1s and all other terms will contain at 0s. In addition, the term that is along the diagonal has a positive sign in the determinant calculation. Therefore the result is 1. □

Proposition 2.3.26. *$det\, A$ is linear in the rows of the matrix*

Proof. If p and q are row vectors and we have matrices A_p, A_q, A_{pq} in which are present, respectively, the row vector p, the one q, and the row $p + q$, then linearity implies that $det\, A_{pq} = det\, A_p + det\, A_q$. This can be seen since every term of the determinant calculation of A_{pq} will contain one of the components in the row $p + q$. So each term of the calculation will take the form,

$$(p + q)a_{ij} \cdots a_{mn} = p(a_{ij} \cdots a_{mn}) + q(a_{ij} \cdots a_{mn})$$

The other implication of linearity is that - if a row is multiplied by a scalar, c, to produce A_c then $det\, A_c = c\, det\, A$. Using a similar reasoning to the

previous argument we have each term of the determinant taking the form,

$$c\, a_{ij} \cdots a_{mn}$$

which obviously results in the determinant being multiplied by c. □

Proposition 2.3.27. *If two columns are exchanged in the matrix then the determinant is multiplied by* -1

Proof. If columns p and q are exchanged then the components of p and q appear in terms with signs reversed. Since the components of p and q appear in every term of the determinant calculation, every term has the sign reversed. So the determinant is multiplied by -1. □

Proposition 2.3.28. *det* $A = 0$ *if there are two identical columns in the matrix*

Proof. If column p is identical to column q then we can swap columns p and q and we will have the same matrix so the determinant must also remain the same. But Proposition 2.3.27 proved that swapping two columns causes the determinant to be multiplied by -1. So, if A_{pq} is the matrix A after swapping the columns,

$$det\, A_{pq} = det\, A = -det\, A \iff det\, A = 0$$

□

Proposition 2.3.29. *Adding a multiple of one column to another leaves the determinant unchanged*

Proof. By combining Proposition 2.3.26 and Proposition 2.3.28 we find that if the columns of A are,

$$\vec{\boldsymbol{x_1}}, \vec{\boldsymbol{x_2}}, \cdots, \vec{\boldsymbol{x_p}}, \vec{\boldsymbol{x_q}}, \cdots, \vec{\boldsymbol{x_n}}$$

and the columns of A_c are,

$$\vec{\boldsymbol{x_1}}, \vec{\boldsymbol{x_2}}, \cdots, \vec{\boldsymbol{x_p}} + c\vec{\boldsymbol{x_q}}, \vec{\boldsymbol{x_q}}, \cdots, \vec{\boldsymbol{x_n}}$$

then,

$$\begin{aligned} det\, A_c &= det\,(\vec{\boldsymbol{x_1}}, \vec{\boldsymbol{x_2}}, \cdots, \vec{\boldsymbol{x_p}}, \vec{\boldsymbol{x_q}}, \cdots, \vec{\boldsymbol{x_n}}) + c \cdot det\,(\vec{\boldsymbol{x_1}}, \vec{\boldsymbol{x_2}}, \cdots, \vec{\boldsymbol{x_q}}, \vec{\boldsymbol{x_q}}, \cdots, \vec{\boldsymbol{x_n}}) \\ &= det\,(\vec{\boldsymbol{x_1}}, \vec{\boldsymbol{x_2}}, \cdots, \vec{\boldsymbol{x_p}}, \vec{\boldsymbol{x_q}}, \cdots, \vec{\boldsymbol{x_n}}) + c \cdot 0 \\ &= det\, A \end{aligned}$$

□

2.3.5.2 Better formulation (from Rudin's Principles of Mathematical Analysis)

Let $a(i,j)$ be the component in the ith row and jth column of the matrix A and,

$$sign(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases}$$

$$s(j_1, \cdots, j_n) = \prod_{p<q} sign(j_q - j_p)$$

Then the determinant,

$$det\, A = \sum s(j_1, \cdots, j_n) a(1, j_1) \cdots a(n, j_n)$$

defined over all n-tuples of n distinct values, $j_1, \cdots, j_n$ with $1 \leq j_r \leq n$ (i.e, permutations of $[1, n] \subset \mathbb{N}$) with each term being produced by a different permutation.

From this we can see that,

- **The determinant of the identity matrix is 1**
  Every term of the determinant will contain at least one 0 apart from the term that traverses the main diagonal, which is all 1s. We can see that there is only one such term because the main diagonal has $i = j$ and so there is only one such $j_1, \cdots, j_n$ that satisfies this.

- **The determinant is linear in the rows or columns of the matrix, holding the others constant**
  If a column, j_r, is multiplied by a scalar α and another column, j_k, is added to it, then the resulting determinant takes the form,

$$\begin{aligned} & det\, A = \sum_i s(j_1, \cdots, j_n) a(i, j_1) \cdots (\alpha a(i, j_r) + a(i, j_k)) \cdots a(i, j_n) \\ \iff & det\, A = \alpha a(i, j_r) \sum_i s(j_1, \cdots, j_n) a(i, j_1) \cdots a(i, j_n) + \\ & \qquad a(i, j_k) \sum_i s(j_1, \cdots, j_n) a(i, j_1) \cdots a(i, j_n) \end{aligned}$$

- **If two columns are exchanged then the determinant is multiplied by -1**

If columns p and q are exchanged then this is equivalent to swapping j_p and j_q in the n-tuple so that $s(j_1, j_2, \cdots, j_n)$ changes sign and so the determinant is multiplied by -1.

- **If two columns are equal then the determinant will be 0**
  If two columns are the same then this is equivalent to a repetition of a value in the tuple $j_1, \cdots, j_n$ and so,

$$\exists\, p, q \text{ s.t. } sign(j_q - j_p) = 0 \implies s(j_1, \cdots, j_n) = 0$$

  which results in every term of the determinant being 0.
  This can also be proven by using the previous property that tells us that the determinant is multiplied by -1 when we exchange the identical columns but - since the columns are identical - the resultant matrix is the same - which means that the determinant remains unchanged. Therefore, the determinant must be 0.

Proposition 2.3.30. *If A and B are $n \times n$ matrices, then*

$$det\, BA = det\, A\, det\, B$$

Proof. Let the columns of A be the vectors, $\vec{\boldsymbol{x_1}}, \vec{\boldsymbol{x_2}}, \cdots, \vec{\boldsymbol{x_n}}$ so that for each column j,

$$\vec{\boldsymbol{x_j}} = \sum_i a(i,j)\vec{\boldsymbol{e_i}}$$

and define,

$$\Delta_B(\vec{\boldsymbol{x_1}}, \vec{\boldsymbol{x_2}}, \cdots, \vec{\boldsymbol{x_n}}) = \Delta_B(A) = det\, BA$$

so that,

$$det\,(B\vec{\boldsymbol{x_1}}, B\vec{\boldsymbol{x_2}}, \cdots, B\vec{\boldsymbol{x_n}}) = \Delta_B(\vec{\boldsymbol{x_1}}, \vec{\boldsymbol{x_2}}, \cdots, \vec{\boldsymbol{x_n}})$$

Since $B\vec{\boldsymbol{x_j}}$ is linear in $\vec{\boldsymbol{x_j}}$, $\Delta_B(\vec{\boldsymbol{x_1}}, \vec{\boldsymbol{x_2}}, \cdots, \vec{\boldsymbol{x_n}})$ is linear in each $\vec{\boldsymbol{x_j}}$ and so,

$$\begin{aligned}
\Delta_B(\vec{\boldsymbol{x_1}}, \vec{\boldsymbol{x_2}}, \cdots, \vec{\boldsymbol{x_n}}) &= \Delta_B(\sum_i a(i,1)\vec{\boldsymbol{e_i}}, \vec{\boldsymbol{x_2}}, \cdots, \vec{\boldsymbol{x_n}}) \\
&= \sum_i a(i,1)\Delta_B(\vec{\boldsymbol{e_i}}, \vec{\boldsymbol{x_2}}, \cdots, \vec{\boldsymbol{x_n}}) \\
&= \sum_{i_1} a(i_1,1) \sum_{i_2} a(i_2,2) \cdots \sum_{i_n} a(i_n,n)\, \Delta_B(\vec{\boldsymbol{e_{i_1}}}, \vec{\boldsymbol{e_{i_2}}}, \cdots, \vec{\boldsymbol{e_{i_n}}}) \\
&= \sum a(i_1,1)a(i_2,2)\cdots a(i_n,n)\, \Delta_B(\vec{\boldsymbol{e_{i_1}}}, \vec{\boldsymbol{e_{i_2}}}, \cdots, \vec{\boldsymbol{e_{i_n}}})
\end{aligned}$$

the sum being extended over all n-tuples, $(i_1, \cdots, i_n)$ such that $1 \leq i_j \leq n$. Also, by referring again to the properties of the determinant we see that,

$$\Delta_B(\vec{\boldsymbol{e}_{i_1}}, \vec{\boldsymbol{e}_{i_2}}, \cdots, \vec{\boldsymbol{e}_{i_n}}) = t(i_1, i_2, \cdots, i_n)\Delta_B(\vec{\boldsymbol{e}_1}, \vec{\boldsymbol{e}_2}, \cdots, \vec{\boldsymbol{e}_n})$$

where $t(i_1, i_2, \cdots, i_n) = 1, 0, -1$ similar to the function s previously. So, we end up with,

$$det\, BA = \sum a(i_1, 1)a(i_2, 2) \cdots a(i_n, n)t(i_1, i_2, \cdots, i_n)\, det\, B = det\, A\, det\, B$$

□

Proposition 2.3.31. *A linear operator A on $\mathbb{R}^n$ is invertible if and only if $det\, A \neq 0$*

Proof. If A is invertible then, $AA^{-1} = I$ and, using Proposition 2.3.25 and Proposition 2.3.30, we have,

$$det\, AA^{-1} = det\, A \cdot det\, A^{-1} = 1$$

so $det\, A$ cannot be 0.

Furthermore, if the columns of A are not independent then there is some linear combination of the columns that produces $\vec{\mathbf{0}}$. Since, by Proposition 2.3.29 we know that adding multiples of columns to other columns leaves the determinant unchanged, this means that the determinant is equal to the determinant of a matrix with $\vec{\mathbf{0}}$ as a column. Such a matrix has determinant 0, so the determinant of A is also 0. □

Corollary 2.3.3. *For invertible matrices,*

$$det\, A^{-1} = \frac{1}{det\, A}$$

Corollary 2.3.4. *The determinant is the only function that has the described properties.*

Proof. Every matrix, A, can be transformed by multiplication by elementary matrices to a row-reduced form, R, which is either the identity matrix - in the case that A is invertible - or a matrix with the last row zeroes - in the case where A is not invertible. So, the determinant of the row-reduced matrix, R, is either 1 or 0. Meanwhile, the determinants of the elementary matrices are:

- Add multiple of row to another row - determinant is 1 because this operation maintains the determinant of the identity.
- Swap two rows - determinant is -1 - determinant is -1 because this operation multiplies the determinant of the identity by -1.
- Multiply a row by some scalar c - determinant is c because this operation multiplies the determinant of the identity by c.

So, we have,

$$R = E_1E_2\cdots E_nA \implies det\, R = det\, E_1E_2\cdots E_n \cdot det\, A$$

where $det\, E_1E_2\cdots E_n$ is a known, non-zero quantity - say d_e. Since the determinant of R is either 0 or 1 this leaves the determinant of A being either 0 or $\frac{1}{d_e}$.

So, the value of the determinant of an arbitrary matrix, A, is wholly determined by the properties described. □

Proposition 2.3.32. *The determinant of any square matrix is equal to that of its transpose. That's to say, for an arbitrary square matrix A,*

$$det\, A^T = det\, A.$$

Proof. The determinant formula from 2.3.5.2 gives us:

$$det\, A = \sum s(j_1, \cdots, j_n)a(1, j_1)\cdots a(n, j_n)$$

where $a(1, j)$ is the (i, j)th element of the matrix A and the summation is over all n-tuples of n distinct values, $j_1, \cdots, j_n$ with $1 \leq j_r \leq n$. So we can also deduce that,

$$det\, A^T = \sum s(j_1, \cdots, j_n)a(j_1, 1)\cdots a(j_n, n).$$

Clearly for the identity permutation $j_1, \ldots, j_n = 1, \ldots, n$ the term of the summation generated is the same: in both cases it is the product of the elements along the main diagonal $a(1, 1)\cdots a(n, n)$. On the other hand, if we take a minimal permutation, for example, where $j_1 = 2$, $j_2 = 1$ so that the

first two elements are swapped, then we see that the term of the summation generated in the determinant of A is

$$-a(1,2)a(2,1)a(3,j_3)\cdots a(n,j_n)$$

while the term of the summation generated in the determinant of A^T is

$$-a(2,1)a(1,2)a(3,j_3)\cdots a(n,j_n)$$

so that the same permuatation in the n-tuple $j_1,\cdots,j_n$ produces the exact same terms of the summation in the determinant of A and of A^T.
It's easy to see in fact, that this will happen with any permutation as the permutation of the $j_1,\cdots,j_n$, in the determinant of A, permutes the column indices while selecting in order from the rows whereas, in the determinant of A^T, it permutes the row indices while selecting from the columns in order. So, the net result is the same in both cases. Since each permutation of the n-tuple produces equal terms of the summation, the complete summation produced by all the permutations will be the same.
Therefore $det\, A^T = det\, A$. □

2.3.5.3 Principal Minors

Definition 79. If A is an $n \times n$ matrix then the *principal minors* of A are the n determinants of the sub-matrices A_k where A_k is the matrix formed from the first k rows and columns of A.

That's to say, if the i,j-th element of A is denoted a_{ij} then A_k is the matrix comprised of the entries

$$\{\, a_{ij} \mid 1 \leq i,j \leq k \,\}.$$

In this case, the eigenvalue t could be any real number and so the principal minors method has failed to determine its sign.

2.3.6 Permutation Matrices

Definition 80. A permutation p is a bijective map from a set S to itself. If a matrix P is the matrix associated with a permutation p then:

- the jth column of the matrix is the basis vector $e_{p(j)}$,
- P is a sum of the matrix units, $P = e_{p(1)1} + e_{p(2)2} + \cdots + e_{p(n)n} = \sum_j e_{p(j)j}$.

Proposition 2.3.33. *If P, Q are permutation matrices associated with the permutations p, q then the matrix that corresponds to the permutation $p \circ q$ is PQ*

Proof. $pq(i) = p(q(i))$ and $PQX = P(QX)$ □

Proposition 2.3.34. *A permutation matrix P is invertible and its inverse is the transpose, $P^{-1} = P^T$*

Proof. A left-multiplying permutation matrix for a permutation, p, maps each row from the input matrix using a column j in the permutation matrix, to the output row, $p(j)$. Since the permutation, by definition, is bijective, we know that this mapping is one-to-one and invertible. If we transpose the matrix P to P^T, swapping rows and columns in the permutation matrix, then the new matrix, P^T maps input rows $p(j)$ into output rows j which is clearly the inverse permutation. □

Since a permutation matrix is a the result of permuting the rows of the identity matrix, clearly, its determinant is ± 1. A permutation is referred to as *odd* or *even* depending on whether its determinant is -1 or 1 respectively. Its determinant is called the *sign of the permutation*,

$$sign\, p = det\, p = \pm 1$$

The determinant of an arbitrary $n \times n$ matrix can be described as,

$$det\, A = \sum_p \left[det \sum_j a_{p(j)j} e_{p(j)j} \right]$$

$$
\begin{aligned}
&= \sum_p \left[(a_{p(1)1} \cdots a_{p(n)n}) \cdot \ det \sum_j e_{p(j)j} \right] \\
&= \sum_p \Big[(a_{p(1)1} \cdots a_{p(n)n}) \cdot \ det\, P \Big] \\
&= \sum_p \Big[(sign\, p)(a_{p(1)1} \cdots a_{p(n)n}) \Big]
\end{aligned}
$$

This is the same formula as earlier and is known as the *complete expansion* of the determinant.

2.3.7 Cramer's Rule

Expansion by minors on the jth column:

$$det\, A = (-1)^{j+1} a_{1j}\, det\, A_{1j} + (-1)^{j+2} a_{2j}\, det\, A_{2j} + \cdots + (-1)^{j+n} a_{nj}\, det\, A_{nj}$$

Expansion by minors on the ith row:

$$det\, A = (-1)^{i+1} a_{i1}\, det\, A_{i1} + (-1)^{i+2} a_{i2}\, det\, A_{i2} + \cdots + (-1)^{i+n} a_{in}\, det\, A_{in}$$

Definition 81. If we form a matrix with elements $\alpha_{ij} = (-1)^{i+j}\, det\, A_{ij}$ and then transpose it we get the **adjoint matrix**.

Notation. The adjoint of A is denoted $adj\, A$.

Following we use $[x]$ to denote a matrix as distinguished from a scalar.

Let $d = det\, A$. Then, if we multiply the adjoint matrix of $[A]$ by $[A]$ we get,

$$[adj\, A][A] = \begin{bmatrix} d & & & & & \\ & d & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & d \end{bmatrix}$$

The off-diagonal elements come out zero because they involve a determinant calculation that involves the same row (or column) repeated and so those determinants are zero.

Theorem 2.3.4.

$$[adj\, A][A] = (det\, A)[I] = [A][adj\, A]$$

Corollary 2.3.5.

$$\frac{1}{det\, A}[adj\, A][A] = [I] \iff [A^{-1}] = \frac{1}{det\, A}[adj\, A]$$

This formulation of the inverse of a matrix can be used to write the solution to a system of linear equations (reverting to the normal notation) $AX = B$ as, multiplying on the left by A^{-1},

$$X = A^{-1}B = \frac{1}{det\, A}(adj\, A)B$$

so that X is a vector whose components, x_j, are expressed as,

$$\begin{aligned} x_j &= \frac{1}{det\, A}(b_1\alpha_{1j} + \cdots + b_n\alpha_{nj}) \\ &= \frac{1}{det\, A}(b_1(-1)^{1+j}\, det\, A_{1j} + \cdots + b_n(-1)^{n+j}\, det\, A_{nj}) \end{aligned}$$

which is the expansion by minors of A on the jth column but with the components a_{ij} of A replaced with the components of the vector B, divided by the determinant of A.

2.3.8 Linear Algebra of Polynomials

2.3.8.1 Uniqueness of Polynomials

Proposition 2.3.35. *Three points uniquely identify a quadratic polynomial.*

Proof. Assume that there are two distinct quadratic polynomials $p(x)$, $q(x)$ that share 3 points. That is,

$$p(x_1) = q(x_1),\ p(x_2) = q(x_2),\ p(x_3) = q(x_3).$$

Then we can define another quadratic polynomial $r(x) = p(x) - q(x)$ with the property that,

$$r(x_1) = p(x_1) - q(x_1) = 0 = r(x_2) = r(x_3).$$

In other words, $r(x)$ has 3 roots: x_1, x_2, x_3. But $r(x)$ is a quadratic polynomial and has a maximum of 2 roots. Therefore there is non such (nonzero) polynomial. □

2.3.8.2 Uniqueness of Coefficients of Polynomials

TODO: is this the best place for this?

Theorem 2.3.5. *If a polynomial is identically zero (i.e. the zero function), then all coefficients are 0.*

Proof. An obvious way to prove this is to begin by assuming that we have a degree-m polynomial,

$$p(x) = a_0 + a_1 x + \cdots + a_m x^m$$

where any coefficients a_i with $i < m$ may be zero. Then we can reason that if $p(x)$ is zero for all $x \in \mathbb{R}$ then its derivative $p'(x)$ must also be identically zero. In this way we can extend the implication to lower and lower degree polynomials by differentiation until $p^{(m)}$ is a degree zero polynomial,

$$p^{(m)}(x) = m!\, a_m = 0 \implies a_m = 0.$$

But $a_m = 0$ contradicts the hypothesis that $p(x)$ is a degree-m polynomial.

Another (better?) proof is given in Linear Algebra Done Right as follows.

Let

$$x = \frac{|a_0| + |a_1| + \cdots + |a_{m-1}|}{|a_m|} + 1$$

so that $x \geq 1$ and so $|a_0| \leq |a_0|\, x$. Then, using the triangle inequality,

$$|a_0 + a_1 x| \leq |a_0| + |a_1 x| \leq |a_0 x| + |a_1 x| = (|a_0| + |a_1|)x.$$

Using induction with an inductive step,

$$\begin{aligned}
& \left|a_0 + a_1 x + \cdots + a_{n-1}x^{n-1}\right| \leq (|a_0| + \cdots + |a_{n-1}|)x^{n-1} \\
\implies \qquad & |a_0 + a_1 x + \cdots + a_n x^n| \leq \left|a_0 + a_1 x + \cdots + a_{n-1}x^{n-1}\right| + |a_n x^n| \\
& \qquad\qquad \leq (|a_0| + \cdots + |a_{n-1}|)x^{n-1} + |a_n|\, x^n \\
& \qquad\qquad \leq (|a_0| + \cdots + |a_{n-1}| + |a_n|)x^n
\end{aligned}$$

we can deduce that

$$\left|a_0 + a_1 x + \cdots + a_{m-1}x^{m-1}\right| \leq (|a_0| + \cdots + |a_{m-1}|)x^{m-1}.$$

Now,

$$\begin{aligned}
|a_m x^m| = |a_m|\, x^m &= (|a_m|\, x)x^{m-1} \\
&= \left[|a_m| \left(\frac{|a_0| + |a_1| + \cdots + |a_{m-1}|}{|a_m|} + 1\right)\right] x^{m-1} \\
&= (|a_0| + |a_1| + \cdots + |a_{m-1}| + |a_m|)\, x^{m-1}.
\end{aligned}$$

Therefore, combining with the previous result, we have,

$$\left|a_0 + a_1 x + \cdots + a_{m-1}x^{m-1}\right| \leq (|a_0| + \cdots + |a_{m-1}|)x^{m-1} < |a_m x^m|\,.$$

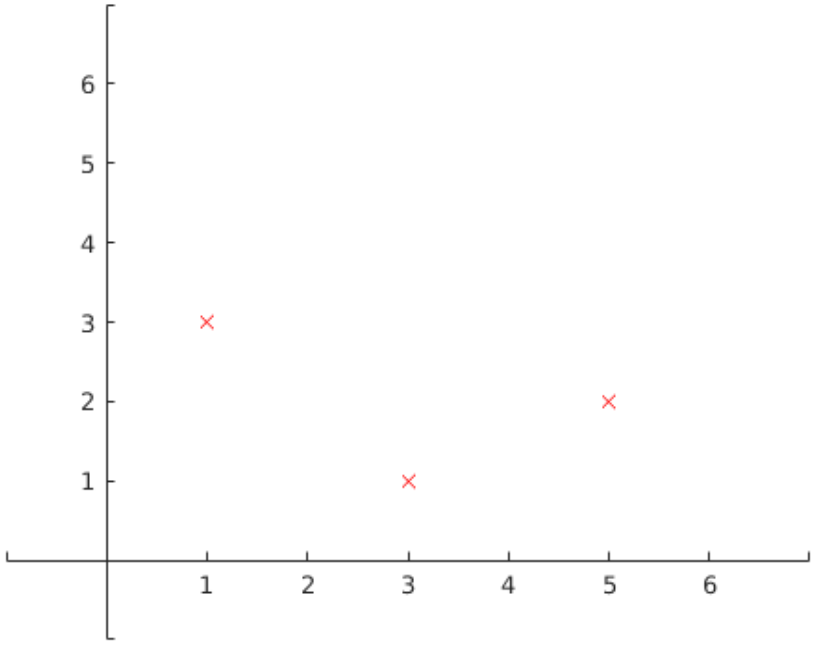
We can now reason that,

$$\begin{aligned}
& \left|a_0 + a_1 x + \cdots + a_{m-1}x^{m-1}\right| < |a_m x^m| \\
\implies \qquad & a_0 + a_1 x + \cdots + a_{m-1}x^{m-1} \neq -a_m x^m \\
\iff \qquad & a_0 + a_1 x + \cdots + a_{m-1}x^{m-1} + a_m x^m \neq 0.
\end{aligned}$$

□

2.3.8.3 Lagrange Polynomials

If we look for quadratic polynomials, $p(x)$, that pass throught the 3 points (1, 3), (3, 1) and (5, 2):



Then the first has roots at $x = 1, 3$ and passes through the point (5, 2). So, we have:

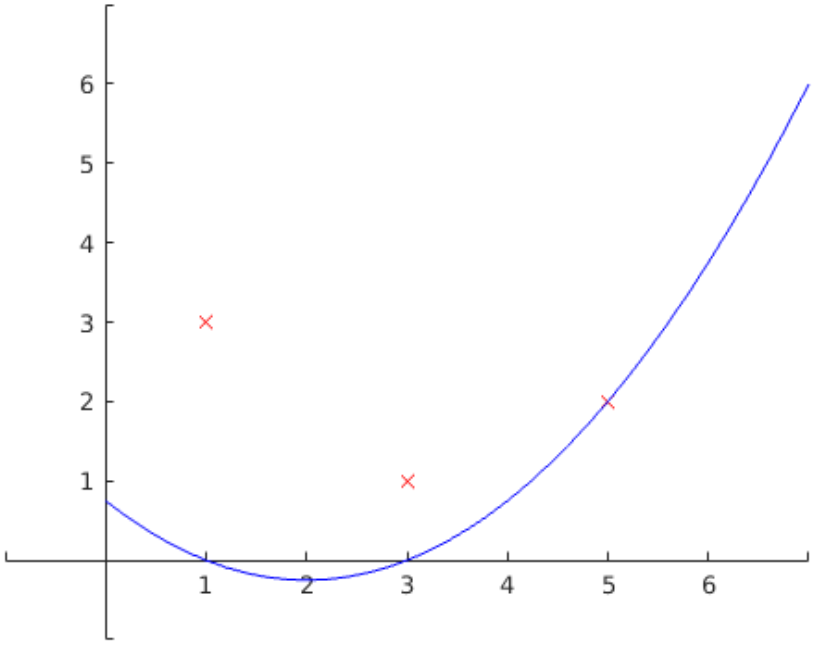
$$p(1) = p(3) = 0, p(5) = 2$$

meaning that $(x - 1)$ and $(x - 3)$ are factors. Therefore,

$$\begin{aligned} p(x) &= \alpha(x-1)(x-3) \\ &= \alpha(x^2 - 4x + 3) \end{aligned}$$

$$\begin{array}{rcl} & p(5) & = 2 \\ \implies & \alpha(5^2 - 4(5) + 3) & = 2 \\ \iff & 8\alpha & = 2 \\ \iff & \alpha & = \frac{1}{4} \end{array}$$

$$\therefore p(x) = \frac{1}{4}(x^2 - 4x + 3)$$

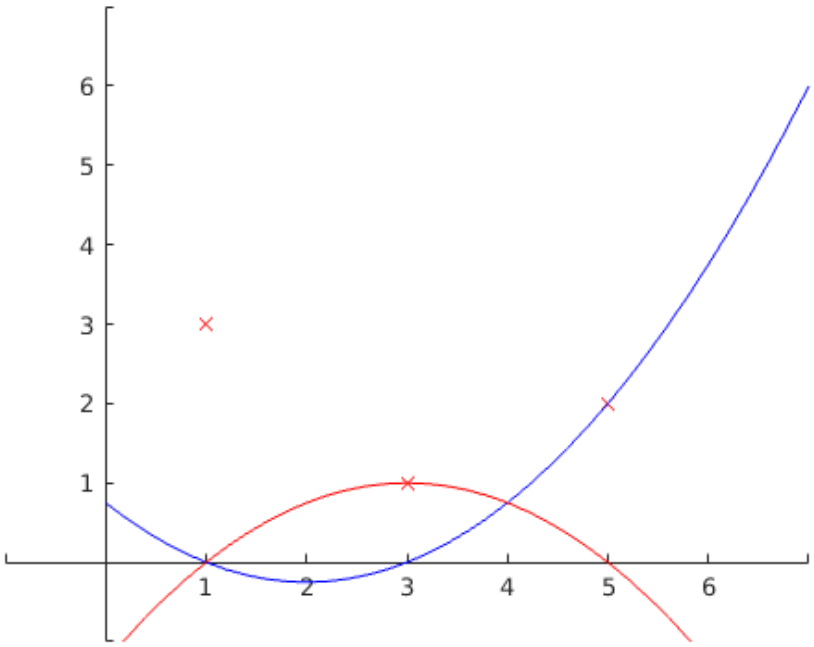


The second has roots at $x = 1, 5$ and passes throught the point (3, 1):

$$\begin{array}{rl} p(x) & = \alpha(x - 1)(x - 5) \\ & = \alpha(x^2 - 6x + 5) \end{array}$$

$$\begin{array}{rcl} & p(3) & = 1 \\ \implies & \alpha(3^2 - 6(3) + 5) & = 1 \\ \iff & -4\alpha & = 1 \\ \iff & \alpha & = -\frac{1}{4} \end{array}$$

$$\therefore p(x) = -\frac{1}{4}(x^2 - 6x + 5)$$

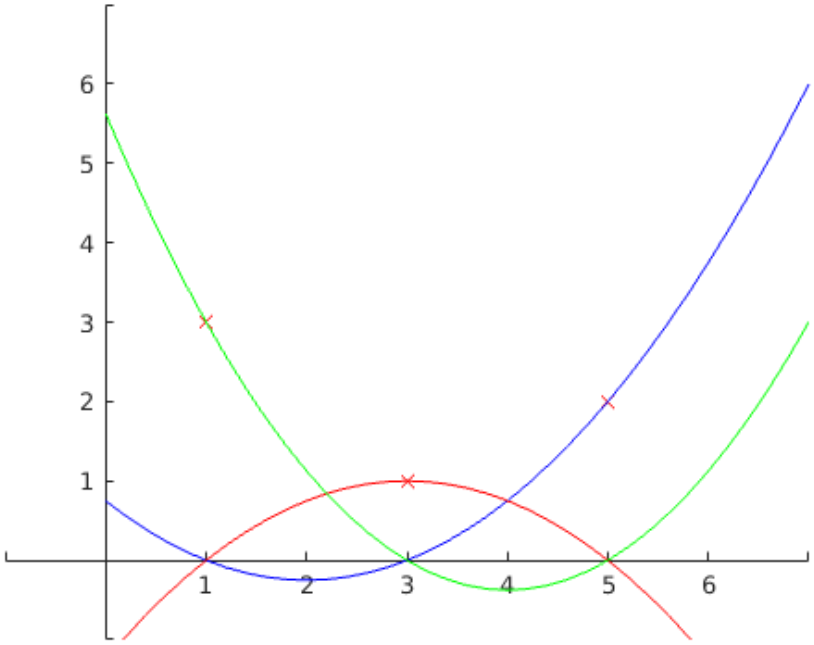


The third has roots at $x = 3, 5$ and passes through the point (1, 3):

$$\begin{array}{rl} p(x) & = \alpha(x - 3)(x - 5) \\ & = \alpha(x^2 - 8x + 15) \end{array}$$

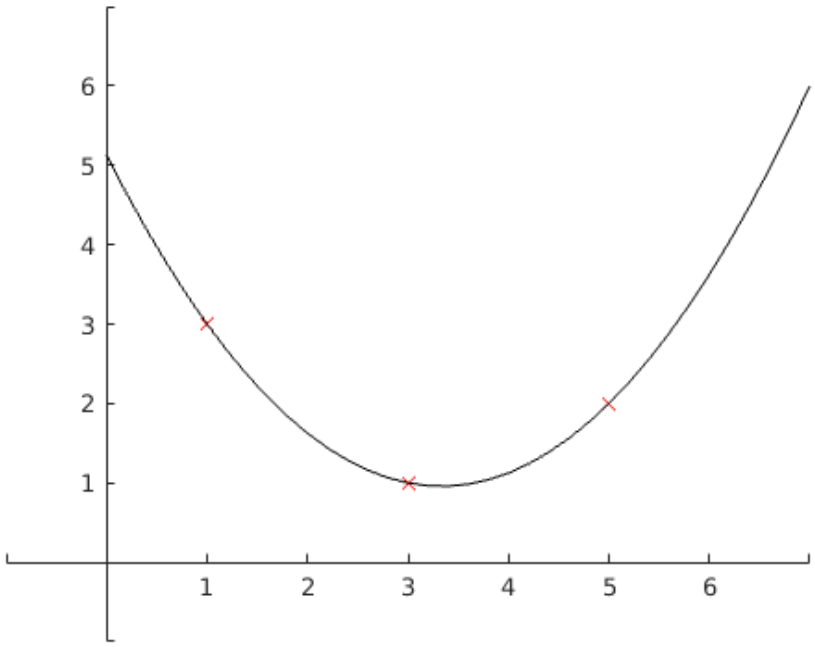
$$\begin{array}{lcl} & p(1) & = 3 \\ \implies & \alpha(1^2 - 8(1) + 15) & = 3 \\ \iff & 8\alpha & = 3 \\ \iff & \alpha & = \frac{3}{8} \end{array}$$

$$\therefore p(x) = \frac{3}{8}(x^2 - 8x + 15)$$



Adding them together we get,

$$\begin{aligned} & \tfrac{1}{4}(x^2 - 4x + 3) - \tfrac{1}{4}(x^2 - 6x + 5) + \tfrac{3}{8}(x^2 - 8x + 15) \\ = & (\tfrac{1}{4} - \tfrac{1}{4} + \tfrac{3}{8})x^2 + (-1 + \tfrac{3}{2} - 3)x + (\tfrac{3}{4} - \tfrac{5}{4} + \tfrac{45}{8}) \\ = & \ \tfrac{3}{8}x^2 - \tfrac{5}{2}x + \tfrac{41}{8} \end{aligned}$$



2.3.8.4 Matrix approach

We are looking for the unique quadratic polynomial $p(x) = \alpha_1 x^2 + \alpha_2 x + \alpha_3$ that satisfies

$$p(1) = 3,\ p(3) = 1,\ p(5) = 2$$

so this gives us the simultaneous equations:

$$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 &= 3 \\ 9\alpha_1 + 3\alpha_2 + \alpha_3 &= 1 \\ 25\alpha_1 + 5\alpha_2 + \alpha_3 &= 2 \end{aligned}$$

which can be expressed as a matrix equation,

$$\begin{bmatrix} 1 & 1 & 1 \\ 9 & 3 & 1 \\ 25 & 5 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}.$$

Now,

$$\begin{bmatrix} 1 & 1 & 1 \\ 9 & 3 & 1 \\ 25 & 5 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1/8 & -1/4 & 1/8 \\ -1 & 3/2 & -1/2 \\ 15/8 & -5/4 & 3/8 \end{bmatrix}$$

so we have,

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 1/8 & -1/4 & 1/8 \\ -1 & 3/2 & -1/2 \\ 15/8 & -5/4 & 3/8 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3/8 - 1/4 + 1/4 \\ -3 + 3/2 - 1 \\ 45/8 - 5/4 + 3/4 \end{bmatrix} = \begin{bmatrix} 3/8 \\ -5/2 \\ 41/8 \end{bmatrix}.$$

Therefore $p(x) = (3/8)x^2 + (-5/2)x + (41/8)$.

2.3.8.5 Unified view

Consider each of the component lagrange polynomials:

- $p_1(x) = \frac{1}{4}(x^2 - 4x + 3) = (1/4)x^2 - x + (3/4)$

$$\vec{\boldsymbol{p}}_1 = \begin{bmatrix} 1/4 \\ -1 \\ 3/4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 & 1 \\ 9 & 3 & 1 \\ 25 & 5 & 1 \end{bmatrix} \vec{\boldsymbol{p}}_1 = \begin{bmatrix} 1/4 - 1 + 3/4 \\ 9/4 - 3 + 3/4 \\ 25/4 - 5 + 3/4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}.$$

- $p_2(x) = -\frac{1}{4}(x^2 - 6x + 5)$

$$\vec{\boldsymbol{p}}_2 = \begin{bmatrix} -1/4 \\ 3/2 \\ -5/4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 & 1 \\ 9 & 3 & 1 \\ 25 & 5 & 1 \end{bmatrix} \vec{\boldsymbol{p}}_2 = \begin{bmatrix} -1/4 + 3/2 - 5/4 \\ -9/4 + 9/2 - 5/4 \\ -25/4 + 15/2 - 5/4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

- $p_3(x) = \frac{3}{8}(x^2 - 8x + 15)$

$$\vec{\boldsymbol{p}}_3 = \begin{bmatrix} 3/8 \\ -3 \\ 45/8 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 & 1 \\ 9 & 3 & 1 \\ 25 & 5 & 1 \end{bmatrix} \vec{\boldsymbol{p}}_3 = \begin{bmatrix} 3/8 - 3 + 45/8 \\ 27/8 - 9 + 45/8 \\ 75/8 - 15 + 45/8 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}.$$

So,

$$\begin{bmatrix} 1 & 1 & 1 \\ 9 & 3 & 1 \\ 25 & 5 & 1 \end{bmatrix} (\vec{\boldsymbol{p}}_1 + \vec{\boldsymbol{p}}_2 + \vec{\boldsymbol{p}}_3) = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}.$$

2.4 Vector Spaces

2.4.1 Definition of a Vector Space

Let F denote a field which is a subfield of $\mathbb{C}$ and V denote a vector space over F.

Definition 82. **Addition, Scalar Multiplication**

- An **addition** on a set V is a function that assigns an element $u+v \in V$ to each pair of elements $u, v \in V$.
- A **scalar multiplication** on a set V is a function that assigns an element $\lambda v \in V$ to each $\lambda \in F$ and each $v \in V$.

Note that both functions are closed over V.

A vector space is a set V along with an addition on V and a scalar multiplication on V such that the following properties hold:

commutativity $\vec{\boldsymbol{u}} + \vec{\boldsymbol{v}} = \vec{\boldsymbol{v}} + \vec{\boldsymbol{u}}$ for all $\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}} \in V$;

associativity $(\vec{\boldsymbol{u}} + \vec{\boldsymbol{v}}) + \vec{\boldsymbol{w}} = \vec{\boldsymbol{u}} + (\vec{\boldsymbol{v}} + \vec{\boldsymbol{w}})$ and $(ab)\vec{\boldsymbol{v}} = a(b\vec{\boldsymbol{v}})$ for all $\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}, \vec{\boldsymbol{w}} \in V$ and all $a, b \in F$;

additive identity there exists an element $\vec{\mathbf{0}} \in V$ such that $\vec{\boldsymbol{v}} + \vec{\mathbf{0}} = \vec{\boldsymbol{v}}$ for all $\vec{\boldsymbol{v}} \in V$;

additive inverse for every $\vec{\boldsymbol{v}} \in V$ there exists $\vec{\boldsymbol{w}} \in V$ such that $\vec{\boldsymbol{v}} + \vec{\boldsymbol{w}} = \vec{\mathbf{0}}$;

multiplicative identity $1\vec{\boldsymbol{v}} = \vec{\boldsymbol{v}}$ for all $\vec{\boldsymbol{v}} \in V$;

distributive properties $a(\vec{\boldsymbol{u}} + \vec{\boldsymbol{v}}) = a\vec{\boldsymbol{u}} + a\vec{\boldsymbol{v}}$ and $(a+b)\vec{\boldsymbol{u}} = a\vec{\boldsymbol{u}} + b\vec{\boldsymbol{u}}$ for all $a, b \in F$ and $\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}} \in V$;

Proposition 2.4.1. *A vector space contains a unique additive identity element.*

Proof. If $\vec{\mathbf{0}}'$ is also an additive identity then by the additive identity property,

$$\vec{\mathbf{0}} + \vec{\mathbf{0}}' = \vec{\mathbf{0}}$$

but since $\vec{\mathbf{0}}$ is also an additive identity,

$$\vec{\mathbf{0}}' + \vec{\mathbf{0}} = \vec{\mathbf{0}}'$$

Then, by the commutativity of vector addition,

$$\vec{\mathbf{0}} = \vec{\mathbf{0}} + \vec{\mathbf{0}}' = \vec{\mathbf{0}}' + \vec{\mathbf{0}} = \vec{\mathbf{0}}' \qquad \square$$

Proposition 2.4.2. *A vector space contains a unique additive inverse for each element.*

Proof. If $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{w}}$ are both additive inverses of $\vec{\boldsymbol{u}}$ then, by the additive inverse property we have,

$$\vec{\boldsymbol{u}} + \vec{\boldsymbol{v}} = \vec{\mathbf{0}} \text{ and also } \vec{\boldsymbol{u}} + \vec{\boldsymbol{w}} = \vec{\mathbf{0}}$$

using the uniqueness of the additive identity,

$$\vec{\boldsymbol{u}} + \vec{\boldsymbol{v}} = \vec{\mathbf{0}} = \vec{\boldsymbol{u}} + \vec{\boldsymbol{w}}$$

Then, if we add one of the additive inverses of $\vec{\boldsymbol{u}}$ to both sides,

$$\vec{\boldsymbol{u}} + \vec{\boldsymbol{v}} + \vec{\boldsymbol{v}} = \vec{\boldsymbol{u}} + \vec{\boldsymbol{w}} + \vec{\boldsymbol{v}}$$

and use the associativity of vector addition,

$$\begin{aligned} (\vec{\boldsymbol{u}} + \vec{\boldsymbol{v}}) + \vec{\boldsymbol{v}} &= (\vec{\boldsymbol{u}} + \vec{\boldsymbol{v}}) + \vec{\boldsymbol{w}} \\ \vec{\mathbf{0}} + \vec{\boldsymbol{v}} &= \vec{\mathbf{0}} + \vec{\boldsymbol{w}} \\ \vec{\boldsymbol{v}} &= \vec{\boldsymbol{w}} \end{aligned}$$

$\square$

Vector Subtraction

Definition 83. Because additive inverses are unique we can use the notation $-\vec{\boldsymbol{v}}$ to denote the additive inverse of $\vec{\boldsymbol{v}}$. Then we define $\vec{\boldsymbol{w}} - \vec{\boldsymbol{v}}$ to mean $\vec{\boldsymbol{w}} + -\vec{\boldsymbol{v}}$.

$$\vec{\boldsymbol{u}} - \vec{\boldsymbol{v}} \coloneqq \vec{\boldsymbol{u}} + -\vec{\boldsymbol{v}}$$

Proposition 2.4.3. $0\vec{\boldsymbol{v}} = \vec{\mathbf{0}}$ *for every* $\vec{\boldsymbol{v}} \in V$.

Note that this proposition is asserting something about scalar multiplication and the additive identity of V. The only part of the definition of a vector space that connects scalar multiplication and vector addition is the distributive property. Therefore the distributive property must be used in this proof.

Proof. Firstly take,

$$\vec{\boldsymbol{v}} + 0\vec{\boldsymbol{v}} = 0\vec{\boldsymbol{v}} + 1\vec{\boldsymbol{v}}$$

and then use the properties of the underlying field to say

$$(0+1)\vec{\boldsymbol{v}} = 1\vec{\boldsymbol{v}} = \vec{\boldsymbol{v}}$$

Now we have shown that,

$$\vec{\boldsymbol{v}} + 0\vec{\boldsymbol{v}} = \vec{\boldsymbol{v}}$$

which, by the definition and uniqueness of the additive identity, shows that $0\vec{\boldsymbol{v}} = \vec{\mathbf{0}}$. But if we want to continue algebraically we can now add the additive inverse to both sides,

$$(\vec{\boldsymbol{v}} + -\vec{\boldsymbol{v}}) + 0\vec{\boldsymbol{v}} = (\vec{\boldsymbol{v}} + -\vec{\boldsymbol{v}})$$

$$\vec{\mathbf{0}} + 0\vec{\boldsymbol{v}} = 0\vec{\boldsymbol{v}} = \vec{\mathbf{0}}$$ □

Another, simpler proof exists.

Proof. Using the underlying field properties and the distributivity of scalar vector multiplication,

$$0\vec{\boldsymbol{v}} = (0+0)\vec{\boldsymbol{v}} = 0\vec{\boldsymbol{v}} + 0\vec{\boldsymbol{v}}$$

and then adding the additive inverse to both sides,

$$(0\vec{\boldsymbol{v}} + -(0\vec{\boldsymbol{v}})) = (0\vec{\boldsymbol{v}} + -(0\vec{\boldsymbol{v}})) + 0\vec{\boldsymbol{v}}$$
$$\vec{\mathbf{0}} = \vec{\mathbf{0}} + 0\vec{\boldsymbol{v}} = 0\vec{\boldsymbol{v}}$$ □

Proposition 2.4.4. $a\vec{\mathbf{0}} = \vec{\mathbf{0}}$ *for every* $a \in F$.

Proof. Using the distributivity of scalar multiplication of vectors and the additive identity,

$$a\vec{\mathbf{0}} = a(\vec{\mathbf{0}} + \vec{\mathbf{0}}) = a\vec{\mathbf{0}} + a\vec{\mathbf{0}}$$

Then, adding the additive inverse to both sides,

$$\begin{aligned}(a\vec{\mathbf{0}} + -(a\vec{\mathbf{0}})) &= a\vec{\mathbf{0}} + (a\vec{\mathbf{0}} + -(a\vec{\mathbf{0}})) \\ \vec{\mathbf{0}} &= a\vec{\mathbf{0}} + \vec{\mathbf{0}} = a\vec{\mathbf{0}}\end{aligned}$$

□

Proposition 2.4.5. $(-1)\vec{\boldsymbol{v}} = -\vec{\boldsymbol{v}}$ *for every* $\vec{\boldsymbol{v}} \in V$.

Proof. Using the distributivity of scalar multiplication of vectors and the underlying field properties we have,

$$(-1)\vec{\boldsymbol{v}} + \vec{\boldsymbol{v}} = (-1)\vec{\boldsymbol{v}} + 1\vec{\boldsymbol{v}} = (-1+1)\vec{\boldsymbol{v}} = 0\vec{\boldsymbol{v}} = \vec{\mathbf{0}}$$

Now we could add the additive inverse to both sides to show that,

$$\begin{aligned}(-1)\vec{\boldsymbol{v}} + (\vec{\boldsymbol{v}} + -\vec{\boldsymbol{v}}) &= \vec{\mathbf{0}} + -\vec{\boldsymbol{v}} \\ (-1)\vec{\boldsymbol{v}} + \vec{\mathbf{0}} &= \vec{\mathbf{0}} + -\vec{\boldsymbol{v}} \\ (-1)\vec{\boldsymbol{v}} &= \vec{\boldsymbol{v}}\end{aligned}$$

□

But we already have,

$$(-1)\vec{\boldsymbol{v}} + \vec{\boldsymbol{v}} = \vec{\mathbf{0}}$$

and this, by the definition of the additive inverse, proves that $(-1)\vec{\boldsymbol{v}}$ is an additive inverse of $\vec{\boldsymbol{v}}$. Since we have previously proven the uniqueness of the additive inverse in Proposition 2.4.2 we can conclude, in fact, that $(-1)\vec{\boldsymbol{v}} = -\vec{\boldsymbol{v}}$ the unique additive inverse of v.

2.4.1.1 Vectors as magnitude and direction

A vector is often described as an object with magnitude and direction. So how does this relate to the definition of vector spaces?

Magnitude

If we take the distributive property of scalar multiplication of vectors over addition and the multiplicative identity we get:

$$\vec{\boldsymbol{v}} + \vec{\boldsymbol{v}} = 1\vec{\boldsymbol{v}} + 1\vec{\boldsymbol{v}} = (1+1)\vec{\boldsymbol{v}} = 2\vec{\boldsymbol{v}}.$$

This is how magnitudes behave: if we sum a magnitude with itself we get 2 times the magnitude. If we compare this with sets (not in a measure space), for example, where addition of sets is set union and, if A is a set then,

$$A + A = A.$$

That's to say, sets (without a measure defined on them) do not behave as magnitudes; if we add a set to itself we get the original set. As a result, it would be difficult or inappropriate to model sets as vectors.

> *Sets have their own sort of algebra: see wikipedia.*

Direction

If we take Proposition 2.4.5 we see that

$$(-1)\vec{\boldsymbol{v}} = -v \quad \text{and} \quad (-1)\vec{\boldsymbol{v}} + \vec{\boldsymbol{v}} = \vec{\mathbf{0}}.$$

That's to say, a vector's magnitude is cancelled by being added to another vector with the same magnitude multiplied by -1. This is how the directionality comes into the definition. This abstract vector space definition only defines that a vector and its additive inverse have opposing direction; a relationship of direction between vectors that are not additive inverses is not defined. If an inner product is defined over the vector space then this provides a concept of angle between vectors.

> *A coordinate space is not necessarily required to define an inner product. For example, the expectation of the product of two random variables is an inner product: wikipedia.*

Minimal Representation

Since a magnitude and a direction are required to define vectors, the minimal numerical representation of a vector is a signed number. We can see this in

basic physics where, if a system is defined in a single spatial dimension, forces are described as a signed real number with the sign indicating the direction.

2.4.1.2 Vector Spaces as Groups

Vector addition on a vector space V is an associative, commutative law of composition on the set of vectors so that $\boldsymbol{(V, +)}$ is an abelian group. Notice also, however, that scalar multiplication defines a law of composition between vectors in V and scalars in the field F. This is known as an **external law of composition** on the vector space. This is an important part of the definition of vectors as, it defines a relationship between the additive group of vectors V^+ and the field F in much the same way that the distributive law does for the additive and multiplicative groups inside fields (compare 2.2). Specifically the relationship takes the form,

$$(1 + 1)\vec{\boldsymbol{v}} = \vec{\boldsymbol{v}} + \vec{\boldsymbol{v}} = 2\vec{\boldsymbol{v}}.$$

It is typically the case that when modelling some system with vectors, the system contains more structure than is represented by a vector space. Furthermore, if we view a vector space as an abelian group then the group only describes a part of the vector space structure. These abstractions — vector space and group — allow us to see what is generalizable about a system and what is specific to the system in question.

2.4.1.3 Isomorphisms between vector spaces

Definition 84. **(Isomorphism of Vector Spaces)** An isomorphism between vector spaces $\phi : V \longmapsto V'$ – where V and V' are defined over the same field F – is defined as a bijective map compatible with the vector laws of composition.

That's to say, for all $\vec{\boldsymbol{v}}, \vec{\boldsymbol{v'}} \in V$, $c \in F$,

$$\phi(\vec{\boldsymbol{v}} + \vec{\boldsymbol{v'}}) = \phi(\vec{\boldsymbol{v}}) + \phi(\vec{\boldsymbol{v'}}) \quad \text{and} \quad \phi(c\vec{\boldsymbol{v}}) = c\phi(\vec{\boldsymbol{v}}).$$

In many contexts isomorphic vector spaces can be regarded as different representations of the same thing and used interchangeably. However, in some contexts the different identities of isomorphic vector spaces are relevant. So, it is important that we distinguish carefully between "equal" vector spaces and merely isomorphic vector spaces.

For example, any finite vector spaces of the same dimension and defined on the same field are isomorphic so two distinct lines in the cartesian planes represent two different but isomorphic 1-dimensional vector spaces in $\mathbb{R}^2$*. Clearly there are many situations where regarding these as equal would be nonsense however, in a good number of situations we are only interested in the fact that they are 1-dimensional vector subspaces of* $\mathbb{R}^2$ *and their difference can be ignored.*

Examples

(44) The space $\mathbb{F}^n$ of n-dimensional row vectors is isomorphic to the space of n-dimensional column vectors.

(45) If we treat the set of complex numbers $\mathbb{C}$ as a vector space *over the reals* then the map $\phi : \mathbb{R}^2 \longmapsto \mathbb{C}$ defined as,

$$\phi(a, b) = a + bi$$

is an isomorphism. If we, however, treat $\mathbb{C}$ as a complex vector space (a vector space defined over the field of complex numbers $\mathbb{C}$) then it cannot be isomorphic to the real vector space $\mathbb{R}^2$ because scalar multiplication by a complex scalar is undefined in $\mathbb{R}^2$.

$\mathbb{R}^2$ and $\mathbb{C}$ *are* isomorphic in the category of sets though; this means that there is a bijection between the sets and doesn't take into account any structure on the sets.

2.4.2 The notation F^S

Notation. If S is a set then $\boldsymbol{F^S}$ denotes the set of functions $S \mapsto F$.

Addition is defined as, for $f, g, (f+g) \in F^S$,

$$(f+g)(x) = f(x) + g(x)$$

for all $x \in S$.

Scalar multiplication is defined as, for $\lambda \in F, \lambda f \in F^S$,

$$(\lambda f)(x) = \lambda f(x)$$

for all $x \in S$.

Example: If S is the interval $[0,1]$ and $F = \mathbb{R}$ then $\mathbb{R}^{[0,1]}$ is the set of real-valued functions on the interval $[0,1]$. $\mathbb{R}^{[0,1]}$ is a vector space with additive identity $0 : [0,1] \mapsto \mathbb{R}$ defined as $0(x) = 0$ and the additive inverse of some function $f \in \mathbb{R}^{[0,1]}$ is the function defined as $(-f)(x) = -f(x)$.
Any *non-empty* set S in conjunction with a subset of $\mathbb{C}$ would similarly produce a vector space. In fact, the vector space F^n can be thought of as the space of functions from the set $\{1, 2, 3, \ldots, n\}$ to F. For example, vectors in 3-dimensional space can be viewed as:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \equiv f : \{1,2,3\} \mapsto \mathbb{R} \text{ with } f(t) = \begin{cases} x & t = 1 \\ y & t = 2 \\ z & t = 3 \end{cases}$$

2.4.3 Polynomials as a vector space

A very important example involves treating a polynomial as a vector. A function $p : F \mapsto F$ is called a polynomial with coefficients in F if there exist $a_0, \ldots, a_m \in F$ such that,

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$$

for all $z \in F$.
Then we can define a vector space, $P(F)$, to be the set of all polynomials with coefficients in F.

Addition on $P(F)$ is defined as,

$$(p+q)(z) = p(z) + q(z) \qquad \text{for } p, q \in P(F), z \in F$$

whose associativity is clear from the definition and the commutativity can be shown by,

$$\begin{aligned}((p+q)+r)(z) &= (p+q)(z) + r(z) \\ &= p(z) + q(z) + r(z) \\ &= p(z) + (q+r)(z) \\ &= (p+(q+r))(z)\end{aligned}$$

Scalar multiplication on $P(F)$ is defined as,

$$(ap)(z) = ap(z) \qquad \text{for } p \in P(F), a, z \in F$$

whose associativity can be shown by substituting (ab) for a in the definition,

$$[(ab)p](z) = (ab)p(z)$$

Then, by the associativity of the multiplication of the elements of the field F we have,

$$(ab)p(z) = a[b(p(z)]$$

then we use the definition in reverse,

$$a[b(p(z)]) = a[(bp)(z)] = [a(bp)](z)$$

(compare with $(ab)\vec{\boldsymbol{v}} = a(b\vec{\boldsymbol{v}})$)

modeling Concretely, each $p(z) \in P(F)$ is a vector that could be modeled, say, as

$$\vec{\boldsymbol{p}} = \{\, (a_0, a_1, \ldots, a_m) \mid p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m \in P(F) \,\}$$

2.4.4 Subspaces of vector spaces

Definition 85. A set U is a subspace of V if it is a subset of V and if the same addition and multiplication over U forms a vector space.

Considering the required properties of a vector space, we can see that commutativity and associativity of the addition; associativity of the scalar multiplication; and distributivity of the scalar multiplication over the addition; will all be satisfied as we have the same addition and multiplication over a subset of the elements in V. That's to say, the vector space properties ensure that these properties hold $\forall \vec{\boldsymbol{v}} \in V$ and we have $\forall \vec{\boldsymbol{u}} \in U, \vec{\boldsymbol{u}} \in V$.
Furthermore, the multiplicative identity also holds $\forall \vec{\boldsymbol{v}} \in V$ so will also hold for every element of U.

So what remains to be proven to satisfy the requirements of a subspace?

- Existence of the additive identity
- Existence of an additive inverse for every element of U
- Closure of the addition and scalar multiplication over U

Note, however that - having proved in Proposition 2.4.5 that multiplication by -1 gives the additive inverse - closure of the scalar multiplication over U also implies the presence in U of the additive inverse of every element of U. So, actually, what we need to prove for U to be a subspace is only,

- $\vec{\mathbf{0}} \in U$
- Closure of the addition and scalar multiplication over U

2.4.4.1 Examples of Subspaces

(46) An example of a subspace of the polynomials, $P(F)$ is,

$$\{\, p \in P(F) \mid p(3) = 0 \,\}$$

Members of this subspace include:

- $p(z) = 3 - z$
- $p(z) = 9 - z^2$
- $p(z) = 3 - z + 3z^2 - z^3$
- $p(z) = 12z - 4z^2$
- ...etc.

To verify this we need to show that addition and multiplication are closed over this set and that $\vec{\mathbf{0}}$ is a member of the set. It's easy to see that $\vec{\mathbf{0}}$ is a member of the set as,

$$p(3) = 0 + 0(3) + 0(3)^2 + \cdots + 0(3)^m = 0$$

as required. Scalar multiplication is closed as,

$$ap(3) = a(0) = 0$$

whereas addition can be shown to be closed as,

$$(q + r)(3) = q(3) + r(3) = 0 + 0 = 0$$

Note that for values of $z \neq 3$, the closure of these functions is the same as for the general case of $P(F)$.

2.4.4.2 Sums and Direct Sums

Definition 86. If $U_1, \ldots, U_m$ are subspaces of V then their sum is defined as

$$U_1 + \cdots + U_m = \{ \vec{\boldsymbol{u_1}} + \cdots + \vec{\boldsymbol{u_m}} \mid \vec{\boldsymbol{u_1}} \in U_1, \ldots, \vec{\boldsymbol{u_m}} \in U_m \}.$$

The sum of the subspaces of V is also a subspace of V because,

- Closure of addition

$$\begin{aligned} &(\vec{\boldsymbol{u_1}} + \vec{\boldsymbol{u_2}} + \cdots + \vec{\boldsymbol{u_m}}) + (\vec{\boldsymbol{u_1'}} + \vec{\boldsymbol{u_2'}} + \cdots + \vec{\boldsymbol{u_m'}}) \\ &= (\vec{\boldsymbol{u_1}} + \vec{\boldsymbol{u_1'}}) + (\vec{\boldsymbol{u_2}} + \vec{\boldsymbol{u_2'}}) + \cdots + (\vec{\boldsymbol{u_m}} + \vec{\boldsymbol{u_m'}}) \\ &= \vec{\boldsymbol{v_1}} + \vec{\boldsymbol{v_2}} + \cdots + \vec{\boldsymbol{v_m}} \qquad \text{where } \vec{\boldsymbol{v_1}} \in U_1, \vec{\boldsymbol{v_2}} \in U_2, \ldots, \vec{\boldsymbol{v_m}} \in U_m \end{aligned}$$

- Closure of scalar multiplication

$$\begin{aligned} & a(\vec{\boldsymbol{u_1}} + \vec{\boldsymbol{u_2}} + \cdots + \vec{\boldsymbol{u_m}}) && \text{where } a \in F \\ =\, & a\vec{\boldsymbol{u_1}} + a\vec{\boldsymbol{u_2}} + \cdots + a\vec{\boldsymbol{u_m}} \\ =\, & \vec{\boldsymbol{v_1}} + \vec{\boldsymbol{v_2}} + \cdots + \vec{\boldsymbol{v_m}} && \text{where } \vec{\boldsymbol{v_1}} \in U_1, \vec{\boldsymbol{v_2}} \in U_2, \ldots, \vec{\boldsymbol{v_m}} \in U_m \end{aligned}$$

- Existence of $\vec{\mathbf{0}}$

$$\begin{aligned} & U_1, U_2, \ldots, U_m \text{ are subspaces} \\ \implies & \vec{\mathbf{0}} \in U_1, \vec{\mathbf{0}} \in U_2, \ldots, \vec{\mathbf{0}} \in U_m \\ \implies & \vec{\mathbf{0}} + \vec{\mathbf{0}} + \cdots + \vec{\mathbf{0}} \in U_1 + U_2 + \cdots + U_m \end{aligned}$$

Note though, that this may not be the only way of producing $\vec{\mathbf{0}}$ from the sum of vectors of these subspaces. That's to say, there could be some $(\vec{\boldsymbol{u_1}} + \vec{\boldsymbol{u_2}} + \cdots + \vec{\boldsymbol{u_m}}) = \vec{\mathbf{0}}$ and this is a key difference from direct sums.

Proposition 2.4.6. *$U_1 + U_2 + \cdots + U_m$ is the smallest subspace of V containing $U_1, U_2, \ldots, U_m$.*

Proof. $U_1 + U_2 + \cdots + U_m$ is a subspace of V that contains $U_1, U_2, \ldots, U_m$ because we can obtain U_i by setting all the u_j for $j \neq i$ to $\vec{\mathbf{0}}$.

If a subspace of V contains $U_1, U_2, \ldots, U_m$ then, by the closure of addition, it must also contain $U_1 + U_2 + \cdots + U_m$.

Therefore the smallest subspace of V that contains $U_1, U_2, \ldots, U_m$ is $U_1 + U_2 + \cdots + U_m$. $\square$

Definition 87. If $U_1, \ldots, U_m$ are subspaces of V then their **direct sum** is defined as,

$$U_1 \oplus \cdots \oplus U_m = \{\, \vec{\boldsymbol{u_1}} + \cdots + \vec{\boldsymbol{u_m}} \mid \vec{\boldsymbol{u_1}} \in U_1, \ldots, \vec{\boldsymbol{u_m}} \in U_m \,\}$$

such that,

$$\vec{\boldsymbol{u_1}} + \cdots + \vec{\boldsymbol{u_m}} = \vec{\mathbf{0}} \implies \vec{\boldsymbol{u_1}} = \vec{\mathbf{0}}, \ldots, \vec{\boldsymbol{u_m}} = \vec{\mathbf{0}}.$$

In this case, the subspaces $U_1, \ldots, U_m$ are said to be **independent**.

That the unique way of obtaining $\vec{\mathbf{0}}$ is for all of the vectors from each of the subspaces to be $\vec{\mathbf{0}}$ is equivalent to there only being a single unique way of obtaining each resultant vector from an addition of the vectors from the individual subspaces. This can be seen as,

$$\begin{aligned}\vec{\boldsymbol{u_1}} + \vec{\boldsymbol{u_2}} + \cdots + \vec{\boldsymbol{u_m}} &= \vec{\boldsymbol{u'_1}} + \vec{\boldsymbol{u'_2}} + \cdots + \vec{\boldsymbol{u'_m}} \\ (\vec{\boldsymbol{u_1}} + \vec{\boldsymbol{u_2}} + \cdots + \vec{\boldsymbol{u_m}}) - (\vec{\boldsymbol{u'_1}} + \vec{\boldsymbol{u'_2}} + \cdots + \vec{\boldsymbol{u'_m}}) &= \vec{\mathbf{0}} \\ (\vec{\boldsymbol{u_1}} - \vec{\boldsymbol{u'_1}}) + (\vec{\boldsymbol{u_2}} - \vec{\boldsymbol{u'_2}}) + \cdots + (\vec{\boldsymbol{u_m}} - \vec{\boldsymbol{u'_m}}) &= \vec{\mathbf{0}}\end{aligned}$$

Therefore, since vector spaces always contain $\vec{\mathbf{0}}$ and so we will always have the representation,

$$\vec{\mathbf{0}} + \vec{\mathbf{0}} + \cdots + \vec{\mathbf{0}} = \vec{\mathbf{0}}$$

if this is the unique representation of $\vec{\mathbf{0}}$ then it follows that,

$$\begin{aligned}(\vec{\boldsymbol{u_1}} - \vec{\boldsymbol{u'_1}}) = \vec{\mathbf{0}}, (\vec{\boldsymbol{u_2}} - \vec{\boldsymbol{u'_2}}) = \vec{\mathbf{0}}, \ldots, (\vec{\boldsymbol{u_m}} - \vec{\boldsymbol{u'_m}}) &= \vec{\mathbf{0}} \\ \implies \vec{\boldsymbol{u_1}} = \vec{\boldsymbol{u'_1}}, \vec{\boldsymbol{u_2}} = \vec{\boldsymbol{u'_2}}, \ldots, \vec{\boldsymbol{u_m}} &= \vec{\boldsymbol{u'_m}}\end{aligned}$$

which means that these are the same representation. And this clearly holds in reverse also as, if there is a single way of representing each resultant vector then there must be a single way of representing $\vec{\mathbf{0}}$ and due to the definition of a vector space we must always have the representation of all $\vec{\mathbf{0}}$. Therefore, this is the only representation of $\vec{\mathbf{0}}$.

Note that this is a condition on the contents of the subspaces and not on the way that the addition is performed. So, the difference between vector space sum $(U_1 + U_2)$ and vector space direct sum $(U_1 \oplus U_2)$ is not in the operator itself but in the operands they operate over.

For two subspaces, say, U_1, U_2 this condition on the subspaces reduces to the requirement that $U_1 \cap U_2 = \{\vec{\mathbf{0}}\}$ which can be seen as,

$$\begin{aligned}\vec{\boldsymbol{u_1}} + \vec{\boldsymbol{u_2}} &= \vec{\mathbf{0}} \\ \vec{\boldsymbol{u_1}} + -\vec{\boldsymbol{u}}_{\boldsymbol{1}} + \vec{\boldsymbol{u_2}} &= \vec{\mathbf{0}} + -\vec{\boldsymbol{u}}_{\boldsymbol{1}} \\ \vec{\boldsymbol{u_2}} &= -\vec{\boldsymbol{u}}_{\boldsymbol{1}} \\ \implies -\vec{\boldsymbol{u}}_{\boldsymbol{1}} \in U_2 \implies \vec{\boldsymbol{u_1}} &\in U_2\end{aligned}$$

So, for two subspaces, obtaining $\vec{\mathbf{0}}$ as the sum of vectors from the subspaces implies a vector in common between them. So, for $\vec{\mathbf{0}} + \vec{\mathbf{0}}$ to be the only way of obtaining $\vec{\mathbf{0}}$ implies that $\vec{\mathbf{0}}$ is the only vector in common.

However, for more than two subspaces, say U_1, U_2, U_3, the situation is different as we could have,

$$\begin{aligned} & \vec{\boldsymbol{u_1}} + \vec{\boldsymbol{u_2}} + \vec{\boldsymbol{u_3}} = \vec{\mathbf{0}} \\ \iff & \vec{\boldsymbol{u_1}} + -\vec{\boldsymbol{u_1}} + \vec{\boldsymbol{u_2}} + -\vec{\boldsymbol{u_2}} + \vec{\boldsymbol{u_3}} = \vec{\mathbf{0}} + -\vec{\boldsymbol{u_1}} + -\vec{\boldsymbol{u_2}} \\ & \iff \vec{\boldsymbol{u_3}} = -\vec{\boldsymbol{u_1}} + -\vec{\boldsymbol{u_2}} \end{aligned}$$

which does not imply any vectors held in common.

2.4.5 Vector Space Problems

Prove that $-(-\vec{v}) = \vec{v}$ for every $\vec{v} \in V$

$$\begin{aligned} -(-\vec{\boldsymbol{v}}) &= -[(-1)\vec{\boldsymbol{v}}] && \text{using Proposition 2.4.5} \\ &= (-1)[(-1)\vec{\boldsymbol{v}}] && \text{using Proposition 2.4.5 again} \\ &= [(-1)(-1)]\vec{\boldsymbol{v}} && \text{using associativity of scalar multiplication} \\ &= \vec{\boldsymbol{v}} && \text{using field properties} \end{aligned}$$

Or, a quicker way is,

$$\begin{aligned} -\vec{\boldsymbol{v}} + -(-\vec{\boldsymbol{v}}) &= \vec{\mathbf{0}} && \text{using additive identity of } -\vec{\boldsymbol{v}} \\ (-\vec{\boldsymbol{v}} + \vec{\boldsymbol{v}}) + -(-\vec{\boldsymbol{v}}) &= \vec{\mathbf{0}} + \vec{\boldsymbol{v}} && \text{adding } \vec{\boldsymbol{v}} \text{ to both sides} \\ -(-\vec{\boldsymbol{v}}) &= \vec{\boldsymbol{v}} \end{aligned}$$

Prove that if $a \in F, \vec{v} \in V$, and $a\vec{v} = \vec{0}$, then $a = 0$ or $\vec{v} = \vec{0}$. We follow a proof by cases.

Case $a \neq 0$:

$$\begin{aligned} a\vec{\boldsymbol{v}} = \vec{\mathbf{0}}, a \neq 0 \implies & a^{-1}a\vec{\boldsymbol{v}} = a^{-1}\vec{\mathbf{0}} && \text{using field properties} \\ \iff & 1\vec{\boldsymbol{v}} = b\vec{\mathbf{0}} && \text{where } b = a^{-1} \in F \\ \iff & \vec{\boldsymbol{v}} = \vec{\mathbf{0}} && \text{using Proposition 2.4.4 and multiplicative identity} \end{aligned}$$

Case $\vec{v} \neq \vec{0}$:

$$
\begin{aligned}
a\vec{v} = \vec{0}, \vec{v} \neq \vec{0} &\implies a\vec{v} = a\vec{v} + -a\vec{v} \\
&\iff a\vec{v} = (a + -a)\vec{v} = 0\vec{v} && \text{using field properties} \\
\text{Wrong!}\ a\vec{v} = \vec{0} &\implies a\vec{v} = a\vec{v} + -a\vec{v} \\
&\text{without need for } \vec{v} \neq \vec{0}
\end{aligned}
$$

This indicates that you are proving something that doesn't need proving. In actual fact,

Case $a = 0$: Actually, in this case there is nothing to be proven as we know from Proposition 2.4.3 that $0\vec{v} = \vec{0}$. So we have collectively exhaustive cases by looking at $a = 0$ and $a \neq 0$ and we only need to show that $a \neq 0 \implies \vec{v} = \vec{0}$ which we have already done.

Case $\vec{v} \neq \vec{0}$:

$$
\begin{aligned}
& a\vec{v} = \vec{0}, \vec{v} \neq \vec{0} \implies a\vec{v} = \vec{0} = \vec{v} + -\vec{v} \\
\iff\ & a\vec{v} + (-\vec{v}) = -\vec{v} && \text{applying additive inverse} \\
\iff\ & a\vec{v} + (-1)\vec{v} = -\vec{v} \\
\iff\ & (a + (-1))\vec{v} = (-1)\vec{v} && \text{using distributive law} \\
\iff\ & a - 1 = -1 && \text{using injectivity of scalar multiplication} \\
\iff\ & a = 0. && \text{using field properties}
\end{aligned}
$$

Give an example of a nonempty subset U of $\mathbb{R}^2$ such that U is closed under scalar multiplication but U is not a subspace of $\mathbb{R}^2$. For all $\lambda \in \mathbb{R}$ the set $\{\, \lambda\vec{v} \mid \vec{v} \in \{(1,1)(-1,1)\}\,\}$ is closed under scalar multiplication but not addition.

Is $\mathbb{R}^2$ a subspace of the complex vector space $\mathbb{C}^2$? The definition of a subspace of $\mathbb{C}^2$ is a set of vectors which is a subset of those in $\mathbb{C}^2$ and that forms a vector space under the same addition and scalar multiplication of $\mathbb{C}^2$. The scalar multiplication of the vector space $\mathbb{C}^2$ is multiplication by scalars

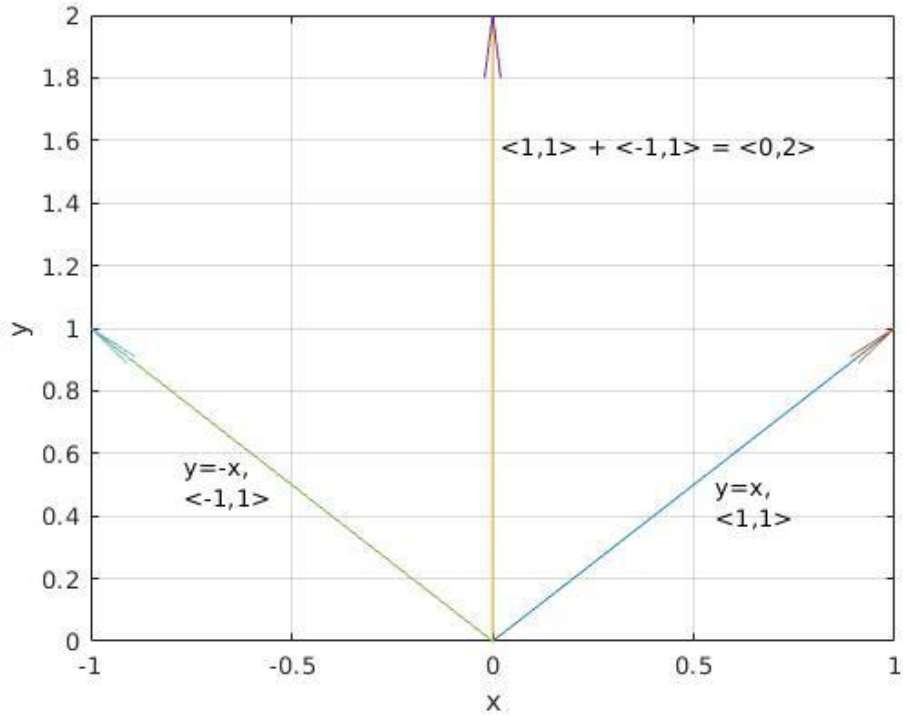


Figure 2.2: The blue arrows are vectors whose scalar multiples will all be in the same line as the blue arrows but the red arrow shows what happens if we add them; the result lies outside of both lines.

$\lambda \in \mathbb{C}$.
For a vector, $\vec{\boldsymbol{v}} \in \mathbb{R}^2$, scaling it by a complex number, $\lambda\vec{\boldsymbol{v}}$ will result in a vector that is not necessarily in $\mathbb{R}^2$.

Is $\{\,(a,b,c) \in \mathbb{C}^3 \mid a^3 = b^3\,\}$ a subspace of $\mathbb{C}^3$? For $x \in \mathbb{C}$, x^3 has roots, $1, \frac{-1+\sqrt{3}i}{2}, \frac{-1-\sqrt{3}i}{2}$ so we don't have $a = b$ as we would if we were ranging over the reals.
Concretely, we can have, $(1, \frac{-1+\sqrt{3}i}{2}, 0)$ and $(1, \frac{-1-\sqrt{3}i}{2}, 0)$ but,

$$(1, \frac{-1+\sqrt{3}i}{2}, 0) + (1, \frac{-1-\sqrt{3}i}{2}, 0) = (2, -1, 0)$$

where $(2, -1, 0) \notin \{\,(a,b,c) \in \mathbb{C}^3 \mid a^3 = b^3\,\}$ meaning that addition over this set is not closed. Therefore, this is not a subspace.

Give an example of a non-empty subset U of $\mathbb{R}^2$ such that U is closed under addition and under taking additive inverses (meaning $-\vec{\boldsymbol{u}} \in U$ whenever $\vec{\boldsymbol{u}} \in U$), but U is not a subspace of $\mathbb{R}^2$. First thought might be $\mathbb{R}^2 - \{\vec{\mathbf{0}}\}$ but this is Wrong!. If the subset is closed under addition and under taking additive inverses then it means that $\vec{\boldsymbol{u}} + -\vec{\boldsymbol{u}} = \vec{\mathbf{0}} \in U$ and so the set $\mathbb{R}^2 - \{\vec{\mathbf{0}}\}$ is not closed under addition and taking additive inverses.
The set $\{\,(x, y) \in \mathbb{R}^2 \mid x, y \in \mathbb{Z}\,\}$ however, is closed under addition because integer addition is closed and under taking additive inverses but scalar multiplication where the scalars range over the reals, will produce non-integer values for x and y.

Is the set of periodic functions over the reals a subspace of $\mathbb{R}^R$? vector problems, periodic functions The definition of two periodic functions over the reals is

$$\exists p > 0 \in \mathbb{R} \cdot f(x) = f(x + p)$$
$$\exists q > 0 \in \mathbb{R} \cdot g(x) = g(x + q)$$

Then for their sum to be periodic we need,

$$\exists \alpha, \beta \in \mathbb{Z}, m \in \mathbb{R} \cdot (m = \alpha p) \wedge (m = \beta q)$$
$$\iff \frac{q}{p} = \frac{\alpha}{\beta} \in \mathbb{Q}$$
$$\therefore (f + g)(x) = (f + g)(x + m) = f(x + m) + g(x + m)$$
$$\iff \frac{q}{p} \in \mathbb{Q}.$$

Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained within the other. Let A, B be subspaces of V and $\vec{\boldsymbol{a}} \in A$, $\vec{\boldsymbol{b}} \in B$ and,

$$C = A \cup B = \{\,\vec{\boldsymbol{c}} \mid \vec{\boldsymbol{c}} \in A \ \vee \ \vec{\boldsymbol{c}} \in B\,\}.$$

Since $\vec{\boldsymbol{a}}, \vec{\boldsymbol{b}} \in C$ we have (C subspace of V) $\iff \forall \alpha, \beta \in F \cdot (\alpha\vec{\boldsymbol{a}} + \beta\vec{\boldsymbol{b}}) \in C$. Then,

$$\vec{\boldsymbol{b}} \in A \implies \forall \alpha, \beta \in F \cdot (\alpha\vec{\boldsymbol{a}} + \beta\vec{\boldsymbol{b}}) \in A \text{ (by subspace properties)}$$

$$\implies (\alpha\vec{\boldsymbol{a}} + \beta\vec{\boldsymbol{b}}) \in C.$$

A similar argument holds for $\vec{\boldsymbol{a}} \in B$. Conversely,

$$\begin{aligned}\forall\alpha, \beta \in F \cdot (\alpha\vec{\boldsymbol{a}} + \beta\vec{\boldsymbol{b}}) \in C &\implies ((\alpha\vec{\boldsymbol{a}} + \beta\vec{\boldsymbol{b}}) \in A) \vee ((\alpha\vec{\boldsymbol{a}} + \beta\vec{\boldsymbol{b}}) \in B) \\ &\implies ((\alpha\vec{\boldsymbol{a}} - \alpha\vec{\boldsymbol{a}} + \beta\vec{\boldsymbol{b}}) = \beta\vec{\boldsymbol{b}} \in A) \vee ((\alpha\vec{\boldsymbol{a}} + \beta\vec{\boldsymbol{b}} - \beta\vec{\boldsymbol{b}}) = \alpha\vec{\boldsymbol{a}} \in B) \\ &\implies (\vec{\boldsymbol{b}} \in A) \vee (\vec{\boldsymbol{a}} \in B)\end{aligned}$$

$$\begin{aligned}\therefore (\text{C subspace of V}) &\iff \forall\alpha, \beta \in F \cdot (\alpha\vec{\boldsymbol{a}} + \beta\vec{\boldsymbol{b}}) \in C \\ &\iff (\vec{\boldsymbol{b}} \in A) \vee (\vec{\boldsymbol{a}} \in B) \\ &\equiv (B \subseteq A) \vee (A \subseteq B).\end{aligned}$$

Can a vector space over an infinite field be a finite union of proper subspaces? Assume that our vector space V is a finite union of proper subspaces, hence

$$V = \bigcup_{i=1}^{n} U_i.$$

Now, pick a non-zero vector $\vec{\boldsymbol{x}} \in U_1$, and pick another vector $\vec{\boldsymbol{y}} \in V \setminus U_1$.

There are infinitely many vectors $\vec{\boldsymbol{x}}+k\vec{\boldsymbol{y}}$, where $k \in K^*$ (K is our infinite field). Note that $\vec{\boldsymbol{x}}+k\vec{\boldsymbol{y}}$ is not in U_1, hence must be contained in some U_j where $j \neq 1$.

Then since $k \in K^*$, we can have $\vec{\boldsymbol{x}} + k_1\vec{\boldsymbol{y}}$, $\vec{\boldsymbol{x}} + k_2\vec{\boldsymbol{y}} \in U_j$, which implies that it also contains $\vec{\boldsymbol{y}}$ and hence also $\vec{\boldsymbol{x}}$, hence $U_1 \subset U_j$.

Explanation: There are infinitely many vectors $\vec{\boldsymbol{x}} + k\vec{\boldsymbol{y}}$ and only finitely many U_i so they cannot all be in different U_i so we have,

$$\begin{aligned}&\exists\, k_1,\, k_2 \in K^* \cdot \vec{\boldsymbol{x}} + k_1\vec{\boldsymbol{y}},\, \vec{\boldsymbol{x}} + k_2\vec{\boldsymbol{y}} \in U_j \\ \implies &(\vec{\boldsymbol{x}} + k_1\vec{\boldsymbol{y}}) - (\vec{\boldsymbol{x}} + k_2\vec{\boldsymbol{y}}) = (k_1 - k_2)\vec{\boldsymbol{y}} \in U_j \\ &\implies \vec{\boldsymbol{y}} \in U_j \implies \vec{\boldsymbol{x}} \in U_j\end{aligned}$$

Hence

$$V = \bigcup_{i=2}^{n} U_i.$$

Evidently, this can be continued, hence a contradiction arises.

Prove or give a counterexample: if U_1, U_2, W are subspaces of V such that $V = U_1 \oplus W$ and $V = U_2 \oplus W$ then $U_1 = U_2$. Counter example: $V = \mathbb{F}^2$, $U_1 = \{\, (x, 0) \in \mathbb{F}^2 \mid x \in F \,\}$, $U_2 = \{\, (0, x) \in \mathbb{F}^2 \mid x \in F \,\}$, $W = \{\, (x, x) \in \mathbb{F}^2 \mid x \in F \,\}$.

Let U_e denote the set of real-valued even functions on $\mathbb{R}$ and let U_o denote the set of real-valued odd functions on $\mathbb{R}$. Show that $\mathbb{R}^R = U_e \oplus U_o$. Every function $f \in \mathbb{R}^R$ can be expressed as the sum of an even function and an odd function as,

$$f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} = g(x) + h(x)$$

where $g(x) \in U_e$ and $h(x) \in U_o$. So, $U_e + U_o$ spans $\mathbb{R}^R$.
Furthermore,

$$\begin{aligned} f(x) \in (U_e \cap U_o) &\implies (f(-x) = f(x)) \wedge (f(-x) = -f(x)) \\ &\implies f(x) = -f(x) \\ &\implies f(x) = 0 \end{aligned}$$

Since $f(x) = 0$ is the additive identity of this space, this shows that the intersection is $\vec{\mathbf{0}}$. So, $\mathbb{R}^R = U_e \oplus U_o$.

2.4.6 Finite Sets of Vectors

2.4.6.1 Span and Linear Independence

Definition 88. The **span** of a nonempty finite set of vectors S – written *span* S – is defined as the set of **finite** linear combinations of elements of S,

$$\{\, \alpha_1\vec{\boldsymbol{v_1}} + \alpha_2\vec{\boldsymbol{v_2}} + \cdots + \alpha_k\vec{\boldsymbol{v_n}} \mid \vec{\boldsymbol{v_i}} \in S,\, \alpha_j \in F \,\}.$$

The span of an empty set of vectors is defined to be $\{\vec{\mathbf{0}}\}$.

The span of a set S is also sometimes known as the **subspace generated by** S.

Definition 89. A **linear relation** among a nonempty finite set of vectors S is any relation of a **finite** number of elements of S of the form,

$$c_1\vec{\boldsymbol{v_1}} + \cdots + c_n\vec{\boldsymbol{v_n}} = \vec{\mathbf{0}}$$

where $c_i \in F$.
A **linearly independent** set of vectors is a nonempty set among which there is no linear relation except the trivial relation where all $c_1, \ldots, c_n = 0$.

The empty set is defined to be linearly independent.

Note that if we talk about vector spaces being linearly independent this means that their sum is a direct sum. That's to say, if spaces $U_1, \ldots, U_k$ are linearly independent and $\vec{\boldsymbol{u}}_i \in U_i$ for $1 \leq i \leq k$, then

$$\vec{\boldsymbol{u}}_i + \cdots + \vec{\boldsymbol{u}}_k = \vec{\mathbf{0}} \iff \vec{\boldsymbol{u}}_1 = \cdots = \vec{\boldsymbol{u}}_k = \vec{\mathbf{0}}.$$

Proposition 2.4.7. *If a set of vectors contains the zero vector $\vec{\mathbf{0}}$ then it cannot be linearly independent.*

Proof. Assume an arbitrary set of vectors $\vec{\boldsymbol{v_1}}, \ldots, \vec{\boldsymbol{v_n}}$ and assume it contains some $\vec{\boldsymbol{v_i}} = \vec{\mathbf{0}}$. Then we have the linear relation $c_i\vec{\boldsymbol{v_i}} = \vec{\mathbf{0}}$ with some $c_i \neq 0$. □

Corollary 2.4.1. *If a set of vectors contains any repitition then it cannot be linearly independent.*

Proof. If a set of vectors contains the same vector twice then subtracting one from the other is a non-trivial linear combination of the set of vectors equalling $\vec{\mathbf{0}}$. □

Corollary 2.4.2. *A set of vectors is linearly independent iff, for any vector that may be expressed as a linear combination of vectors in the set, the expression is unique.*

Proof. If the expression is not unique then the two different representations may be subtracted one from the other to produce a non-trivial linear combination resulting in $\vec{\mathbf{0}}$ which contradicts the hypothesis that they are linearly independent. Therefore linearly independent implies unique representation.

Conversely, if the set of vectors is not linearly independent then there exists some non-trivial linear combination that results in $\vec{\mathbf{0}}$. This, in turn, implies that there exists some linear combination of a subset of the vectors that is equal to a linear combination of the remaining vectors. Since these two linear combinations are equal, they represent two different expressions of the same resultant vector. Therefore unique representation implies linearly independent. □

Proposition 2.4.8. *The span of a list of vectors is the smallest subspace containing those vectors.*

Note that a vector space over $\mathbb{R}$ or $\mathbb{C}$ is an uncountable set as - while the dimensions of the vector space may be finite - closure under scalar multiplication means that the vectors in the space are continuously valued as the field providing the scalars is continuously valued.
This means that the notion of the *smallest* subspace cannot refer to the cardinality of the set and must refer to ordering based on subset. So, the

smallest subspace containing a list of vectors is a subspace that contains the list of vectors and, of which, there is no proper subset which also contains the list of vectors.

Proof.

$$\begin{aligned} \text{Let } S &:= \mathit{span}(\vec{\boldsymbol{v_1}}, \vec{\boldsymbol{v_2}}, \ldots, \vec{\boldsymbol{v_k}}) \\ &:= \{\, \alpha_1\vec{\boldsymbol{v_1}} + \alpha_2\vec{\boldsymbol{v_2}} + \cdots + \alpha_k\vec{\boldsymbol{v_k}} \mid \alpha_1, \alpha_2, \ldots, \alpha_k \in F \,\} \\ \text{and let } V &:= \text{ the smallest vector space containing } \vec{\boldsymbol{v_1}}, \vec{\boldsymbol{v_2}}, \ldots, \vec{\boldsymbol{v_k}}. \end{aligned}$$

then S contains every linear combination of $\vec{\boldsymbol{v_1}}, \vec{\boldsymbol{v_2}}, \ldots, \vec{\boldsymbol{v_k}}$ and nothing else and so is a vector space containing $\vec{\boldsymbol{v_1}}, \vec{\boldsymbol{v_2}}, \ldots, \vec{\boldsymbol{v_k}}$,

$$V \subseteq S$$

Additionally, any vector space containing the vectors $\vec{\boldsymbol{v_1}}, \vec{\boldsymbol{v_2}}, \ldots, \vec{\boldsymbol{v_k}}$ must contain all their linear combinations, $\mathit{span}(\vec{\boldsymbol{v_1}}, \vec{\boldsymbol{v_2}}, \ldots, \vec{\boldsymbol{v_k}})$,

$$S \subseteq V$$

Therefore there is no proper subset of $\mathit{span}(\vec{\boldsymbol{v_1}}, \vec{\boldsymbol{v_2}}, \ldots, \vec{\boldsymbol{v_k}})$ that is also a vector space containing $\vec{\boldsymbol{v_1}}, \vec{\boldsymbol{v_2}}, \ldots, \vec{\boldsymbol{v_k}}$, and so $\mathit{span}(\vec{\boldsymbol{v_1}}, \vec{\boldsymbol{v_2}}, \ldots, \vec{\boldsymbol{v_k}})$ is the smallest vector space containing $\vec{\boldsymbol{v_1}}, \vec{\boldsymbol{v_2}}, \ldots, \vec{\boldsymbol{v_k}}$,

$$(V \subseteq S) \wedge (S \subseteq V) \iff V = S \qquad \square$$

Proposition 2.4.9. *Let L be a linearly independent set of vectors in V and $\vec{\boldsymbol{v}} \in V$. If we add $\vec{\boldsymbol{v}}$ to the set L then the resultant set L' is linearly independent iff $\vec{\boldsymbol{v}} \notin \mathit{span}\, L$.*

Proof. Clearly if $\vec{\boldsymbol{v}} \in \mathit{span}\, L$ then the resultant set is linearly dependent.
If $v \notin \mathit{span}\, L$ however, then if we attempt to form a linear relation of the vectors in L' then we find,

$$c_1\vec{\boldsymbol{v_1}} + \cdots + c_n\vec{\boldsymbol{v_n}} + b\vec{\boldsymbol{v}} = \vec{\boldsymbol{0}}$$

implies that $b \neq 0$ because that would leave a linear relation between the vectors of L which is not possible because L is linearly independent. Therefore,

$$\vec{\boldsymbol{v}} = -(c_1/b)\vec{\boldsymbol{v_1}} + \cdots + -(c_n/b)\vec{\boldsymbol{v_n}}$$

which contradicts the assumption that $v \notin \mathit{span}\, L$. $\square$

Proposition 2.4.10. *If we add a vector $\vec{\boldsymbol{v}} \in V$ to a set of vectors L in V to make a new set L', then $span\, L = span\, L'$ iff $\vec{\boldsymbol{v}} \in span\, L$.*

Proof. Clearly if $\vec{\boldsymbol{v}} \in span\, L$ then adding it to the set L doesn't change its span, so $span\, L = span\, L'$.
Conversely, by construction of L' we have $\vec{\boldsymbol{v}} \in L' \implies \vec{\boldsymbol{v}} \in span\, L'$ so if $span\, L = span\, L'$ then we also have $\vec{\boldsymbol{v}} \in span\, L$. □

Proposition 2.4.11. *Length of every linearly independent list in a space is less than or equal to the length of a spanning list in the same space.*

Proof. Let $U = \vec{\boldsymbol{u_1}}, \vec{\boldsymbol{u_2}}, \ldots, \vec{\boldsymbol{u_m}}$ be a linearly independent list of vectors in V and $W = \vec{\boldsymbol{w_1}}, \vec{\boldsymbol{w_2}}, \ldots, \vec{\boldsymbol{w_n}}$ be a spanning list of vectors in V.
If we take $\vec{\boldsymbol{u_1}}$ from U and add it to W then - since the other vectors in W are a spanning list - W must be linearly dependent. That's to say,

$$\begin{aligned}
\exists\, \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R} \cdot \alpha_1 \vec{\boldsymbol{w_1}} + \cdots + \alpha_n \vec{\boldsymbol{w_n}} &= \vec{\boldsymbol{u_1}} \\
&\iff \alpha_1 \vec{\boldsymbol{w_1}} + \cdots + \alpha_n \vec{\boldsymbol{w_n}} - \vec{\boldsymbol{u_1}} = -\alpha_i \vec{\boldsymbol{w_i}} \\
&\iff \frac{-\alpha_1}{\alpha_i} \vec{\boldsymbol{w_1}} + \cdots + \frac{-\alpha_n}{\alpha_i} \vec{\boldsymbol{w_n}} + \frac{1}{\alpha_i} \vec{\boldsymbol{u_1}} = \vec{\boldsymbol{w_i}}
\end{aligned}$$

So, $\vec{\boldsymbol{w_i}}$ is in the span of $\vec{\boldsymbol{u_1}}, \vec{\boldsymbol{w_2}}, \ldots, \vec{\boldsymbol{w_n}}$ and we can drop $\vec{\boldsymbol{w_i}}$ from the list, W, and it will still span the vector space.
We can keep doing this with the remaining vectors in U - each time the vector to be removed will be some $\vec{\boldsymbol{w_i}}$ because all the $\vec{\boldsymbol{u_i}}$ are linearly independent - and all the while W remains a spanning list. We continue until we have replaced (potentially) all n vectors in W, which would happen if $m > n$. At this point we would have the spanning list $W = \vec{\boldsymbol{u_1}}, \vec{\boldsymbol{u_2}}, \ldots, \vec{\boldsymbol{u_n}}$ and $(m - n)$ remaining vectors in U.
Now, since W spans the space, the $(m - n)$ vectors that remain in U will be in the span of W. But, all the vectors that originally came from U were linearly independent, so it is impossible for any vectors in U to be in the span of W (which now comprises only vectors that originally came from U).
We therefore conclude that there can be no remaining vectors in U and, consequently that m cannot be greater than n, i.e. $m \leq n$. □

Summary of Span and Linear Independence

- *The span of a set of vectors changes when adding a vector not in the existing span.*
- *A linearly independent set of vectors continues to be linearly independent when adding a vector not in the existing span.*

The fundamental point of linear independence and span is that if the span of two sets of vectors is not completely disjoint (ignoring the zero vector) then the sets are not linearly independent. Any nonzero vector in common between the two spans implies a linear relation between the sets,

$$\begin{aligned} & \vec{u_1} + ... + \vec{u_n} = \vec{v} = \vec{w_1} + ... + \vec{w_n} \\ \iff \quad & \vec{0} = (\vec{w_1} + ... + \vec{w_n}) - (\vec{u_1} + ... + \vec{u_n}). \end{aligned}$$

2.4.6.2 Bases

Definition 90. A **basis** for a vector space V is a set of vectors that is both linearly independent and spans the space V. The empty set is therefore a basis for the zero vector space $\{\vec{0}\}$.
Since a basis of V spans the space, any vector in V may be expressed as as a linear combination of the vectors in the basis set and, since the basis set is linearly independent, this expression is unique.

Compare with the ***generating set of a group*** *(2.1.1.2).*

Proposition 2.4.12. *A set of vectors $B = \{\vec{v_1}, \ldots, \vec{v_n}\}$ in V is a basis iff every $\vec{w} \in V$ can be written in a single unique way as a linear combination of vectors in B.*

Proof. If B is a basis of V then, by definition, B spans V and so every $\vec{\boldsymbol{w}} \in V$ can be written as a linear combination of vectors in B. Furthermore, also by the definition of a basis, the vectors in B are linearly independent so, by corollary 2.4.2 the linear combination is unique.

Conversely if every $\vec{\boldsymbol{w}} \in V$ can be written in a single unique way as a linear combination of vectors in B then B both spans the space and is linearly independent by corollary 2.4.2. □

2.4.6.3 Examples of Bases

(47) If we take an arbitrary finite set of vectors $S = \vec{\boldsymbol{s_1}}, \ldots, \vec{\boldsymbol{s_n}}$ then the space $V = V(S)$ of linear combinations of elements of S is the set of all expressions of the form,

$$a_1\vec{\boldsymbol{s_1}}, \ldots, a_n\vec{\boldsymbol{s_n}}, \qquad a_i \in \mathbb{F}.$$

In this space addition and multiplication are carried out assuming no relations among the elements of S so that,

$$(a_1\vec{\boldsymbol{s_1}}+\cdots+a_n\vec{\boldsymbol{s_n}})+(b_1\vec{\boldsymbol{s_1}}+\cdots+b_n\vec{\boldsymbol{s_n}}) = (a_1+b_1)\vec{\boldsymbol{s_1}}+\cdots+(a_n+b_n)\vec{\boldsymbol{s_1}}$$

and

$$c(a_1\vec{\boldsymbol{s_1}} + \cdots + a_n\vec{\boldsymbol{s_n}}) = ca_1\vec{\boldsymbol{s_1}} + \cdots + ca_n\vec{\boldsymbol{s_n}}.$$

Then the mapping $\phi : \mathbb{F}^n \longmapsto V(S)$ defined as,

$$\phi(a_1, \ldots, a_n) = a_1\vec{\boldsymbol{s_1}}, \ldots, a_n\vec{\boldsymbol{s_n}}$$

is an isomorphism.

> *Note that if the assumption of no relation between the elements of S is not valid then ϕ may fail to be isomorphic.*

$V(S)$ is often referred to as *the space with basis S* or *the space of formal linear combinations of S*. If S is an infinite set then $V(S)$ is defined to be the set of all *finite* linear combinations of the elements of S.

This crops up frequently in applications, when taking weightings of different features for example; this isomorphism allows us to treat them as vectors in $\mathbb{F}^n$.

2.4.6.4 Finite-Dimensional Vector Spaces

Definition 91. A vector space is called **finite-dimensional** if there is some finite set of vectors that spans the space.

Proposition 2.4.13. *Any finite set which spans a finite-dimensional space contains a basis for the space.*

Proof. Let S be a spanning set of vectors in the space V.

If S is not linearly independent then there is some $\vec{\boldsymbol{v}} \in S$ such that $\vec{\boldsymbol{v}} \in span\,(S \setminus \{\vec{\boldsymbol{v}}\})$. So we can remove $\vec{\boldsymbol{v}}$ from the set and S still spans the space. We may continue doing this until S is linearly independent, at which point we have found the minimal subset of S that spans the space. This remaining subset is a basis of the space. □

Corollary 2.4.3. *Any finite-dimensional vector space has a basis.*

Proof. This follows from the previous proposition and the definition of a finite-dimensional vector space. □

Proposition 2.4.14. *Any set (including infinite sets) that spans a finite-dimensional vector space, contains a finite subset which spans the space.*

Proof. Let V be a finite-dimensional vector space and S be a spanning set of V. By the definition of V as finite-dimensional there exists some finite set that spans V. Let W be a finite set of vectors that spans V. Then, since S also spans V, every vector in W may be expressed as a finite linear combination of vectors in S. The set of all the members of S that participate in the linear combinations required to produce the set W is a finite subset of S that spans V. □

Proposition 2.4.15. *Let V be a finite-dimensional vector space. Any linearly independent set $L \subseteq V$ can be extended by adding vectors to obtain a basis of V.*

Proof. If L spans V then it is already a basis.
Assume L does not span V. Then there exists some $\vec{\boldsymbol{v}} \in V$ such that $\vec{\boldsymbol{v}} \notin \mathit{span}\, L$. If we add $\vec{\boldsymbol{v}}$ to L then the resulting set, say L', continues to be linearly independent and may or may not span V. If it does then L' is a basis. If not then we can continue to repeat the same process until it does span the space at which point we have a basis. □

Proposition 2.4.16. *For finite subsets of a vector space, any linearly independent set has cardinality less than or equal to that of any spanning set in the same space.*

> *Proposition 2.4.15 only tells us that, for any linearly independent set of vectors, there exists some basis whose cardinality is greater than or equal to that of the original set. What we want to prove here is the condition on the cardinality exists between any linearly independent set and spanning set in the same space.*

Proof. We will show two different ways of proving this: one using an algorithm on lists and the other using simultaneous equations.

(i) **proof using lists**

Let V be a finite-dimensional vector space, S a spanning list $\vec{\boldsymbol{s_1}}, \ldots, \vec{\boldsymbol{s_m}}$ and L a linearly independent list $\vec{\boldsymbol{l_1}}, \ldots, \vec{\boldsymbol{l_n}}$ in V, and assume that $m < n$.
Now, since S is a spanning list, every element of L is in its span and so if we remove the first element $\vec{\boldsymbol{l_1}}$ from L and add it to S then S will definitely contain a linear relation. If we then remove some element $\vec{\boldsymbol{s_i}}$ from the original spanning list that participates in a linear relation (i.e. $\vec{\boldsymbol{s_i}} \in \mathit{span}\, S$) then we have a modified list S, of the same length as the original, but with an element replaced by $\vec{\boldsymbol{l_1}}$ and this list continues to span the space.
We can repeat this task m times until all elements $\vec{\boldsymbol{s_i}}$ from the original spanning list have been removed and we have a spanning list $S = \vec{\boldsymbol{l_1}}, \ldots, \vec{\boldsymbol{l_m}}$ and the remaining $n - m$ elements are still in L. But the original $L = \vec{\boldsymbol{l_1}}, \ldots, \vec{\boldsymbol{l_n}}$ was linearly independent and no linear relation exists between them so it is impossible that the first m elements span the

space because this would mean that they would participate in a linear relation with the remaining $n - m$ elements.
So we have obtained a contradiction and we therefore conclude that $m \geq n \quad \square$.

(ii) **proof using simultaneous equations**

Let V be a vector space, S a finite spanning set $\vec{\boldsymbol{s_1}}, \ldots, \vec{\boldsymbol{s_m}}$ and L a finite linearly independent set $\vec{\boldsymbol{l_1}}, \ldots, \vec{\boldsymbol{l_n}}$ in V, and assume that $m < n$. Since S spans the space, every $\vec{\boldsymbol{l_j}} \in L$ can be expressed as a linear combination of vectors $\vec{\boldsymbol{s_i}} \in S$ of the form,

$$\vec{\boldsymbol{l_j}} = a_{1j}\vec{\boldsymbol{s_1}} + \cdots + a_{mj}\vec{\boldsymbol{s_m}} = \sum_{i=1}^{m} a_{ij}\vec{\boldsymbol{s_i}}$$

for $1 \leq j \leq n$. A linear relation on the vectors of L would look like,

$$\begin{aligned} & c_1\vec{\boldsymbol{l_1}} + \cdots + c_n\vec{\boldsymbol{l_n}} = \vec{\boldsymbol{0}} \\ \iff \qquad & c_1 \sum_{i=1}^{m} a_{i1}\vec{\boldsymbol{s_i}} + \cdots + c_n \sum_{i=1}^{m} a_{in}\vec{\boldsymbol{s_i}} = \vec{\boldsymbol{0}} \end{aligned}$$

which expands into m simultaneous equations as follows.

$$\begin{gathered} c_1 a_{11}\vec{\boldsymbol{s_1}} + \cdots + c_n a_{1n}\vec{\boldsymbol{s_1}} = 0 \\ \vdots \\ c_1 a_{m1}\vec{\boldsymbol{s_m}} + \cdots + c_n a_{mn}\vec{\boldsymbol{s_m}} = 0 \end{gathered}$$

As we can see, each of the m equations has a has a factor $\vec{\boldsymbol{s_i}}$ in each term and so this may be factored out to give the following system.

$$\begin{gathered} \vec{\boldsymbol{s_1}}(c_1 a_{11} + \cdots + c_n a_{1n}) = 0 \\ \vdots \\ \vec{\boldsymbol{s_m}}(c_1 a_{m1} + \cdots + c_n a_{mn}) = 0 \end{gathered}$$

So now we have m equations such that the i-th equation will hold if either $\vec{\boldsymbol{s_i}} = \vec{\boldsymbol{0}}$ or $c_1 a_{i1} + \cdots + c_n a_{in} = 0$.

Assume that for all $\vec{s_i} \in S,\ \vec{s_i} \neq \vec{0}$. Then we end up with the following system.

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ . & & \\ . & & \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} c_1 \\ . \\ . \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ . \\ . \\ 0 \end{bmatrix}$$

It can be shown using matrix row reduction that a system such has this has non-trivial solutions if $m < n$. Therefore if $m < n$, there is a linear relation between the vectors of L contradicting its construction as linearly independent. Therefore $m \geq n$.

However, there may be some $\vec{s_i} \in S$ s.t. $\vec{s_i} = \vec{0}$ but only one because S is a set not a list. If that is all there is (i.e. $S = \{\vec{0}\}$) then the space $V = \{\vec{0}\}$ and the only linearly independent set of vectors in the zero space is the empty set. In this case $n = 0$ and so we cannot have $m < n$. Therefore $m \geq n$ also.
On the other hand, if there are non-zero elements in S then we can remove the equation i such that $\vec{s_i} = \vec{0}$ so that we will have $m - 1$ simultaneous equations in our system. But now the linear dependence of L follows if $(m-1) < n \iff m < (n+1)$ and clearly $m < n \implies m < n + 1$ so once again $m < n$ implies that L is not linearly independent.

Proposition 2.4.17. *Any two bases of the same finite-dimensional vector space have the same number of elements. In other words: For a given vector space, the cardinality of bases is fixed.*

Proof. Let L, L' be finite subsets of the finite-dimensional vector space V such that both are bases. Then both L and L' are linearly independent and span the space. Therefore we have,

$$|L| \leq \left|L'\right| \quad \text{and} \quad |L| \geq \left|L'\right|$$

which implies that $|L| = |L'|$. □

2.4.6.5 Dimension of Finite-Dimensional Vector Spaces

Definition 92. The **dimension** of a finite-dimensional vector space v is the number of vectors in a basis. The dimension will be denoted by $dim\, V$.

Theorem 2.4.1. *The dimension of a finite-dimensional vector space is an upper bound on the cardinality of a linearly independent set of vectors in the space and a lower bound on the cardinality of a spanning set of vectors in the same space.*

Proof. This theorem follows from Proposition 2.4.16. □

Theorem 2.4.2. *Any linearly independent set of vectors in a finite-dimensional vector space V of cardinality $dim\, V$ is a basis.*

Proof. Let $L \subset V$ be linearly independent set of vectors in V with $|L| = dim\, V$. By Proposition 2.4.15, any linearly independent set in V may be extended with zero or more vectors to obtain a basis ov V. But any basis of V has dimension $dim\, V = |L|$. Therefore we extend L with zero vectors to obtain a basis. □

Proposition 2.4.18. *If $W \subseteq V$ is a subspace of a finite-dimensional vector space then,*

$$dim\, W \leq dim\, V \quad and \quad (dim\, W = dim\, V) \iff (W = V).$$

Proof. Firstly note that W must also be finite-dimensional because by definition there is a finite set of vectors that spans V and since $W \subseteq V$, the same set must also span W.
Every basis of W is also a linearly independent set in V and so, by Proposition 2.4.16, it cannot have cardinality greater than any basis of V. Any basis in V has cardinality $dim\, V$ so the cardinality of any basis of W must be less than or equal to $dim\, V$. Therefore $dim\, W \leq dim\, V$.

If $dim\, W = dim\, V$ on the other hand, every basis of W has the same cardinality as every basis of V. Since every basis of W is also a linearly independent set

in V with cardinality equal to $dim\, V$, by Theorem 2.4.2 it is a basis of V. By similar logic in reverse, any basis of V is also a basis of W.
Let B be such a basis. Then,

$$\vec{\boldsymbol{v}} \in W \iff \vec{\boldsymbol{v}} \in span\, B \iff \vec{\boldsymbol{v}} \in V.$$

Therefore $W = V$. □

Theorem 2.4.3. *Let W_1, W_2 be subspaces of a finite-dimensional vector space. Then,*

$$dim(W_1 + W_2) = dim\, W_1 + dim\, W_2 - dim(W_1 \cap W_2).$$

Proof. It is easy to show that the intersection of two subspaces is a subspace. So define a basis of $W_1 \cap W_2$ as $B = \{\vec{\boldsymbol{u_1}}, \ldots, \vec{\boldsymbol{u_r}}\}$ where $r = dim(W_1 \cap W_2)$. Then B is a linearly independent set in W_1 and can be extended to a basis of W_1,

$$B_{W_1} = \{\vec{\boldsymbol{u_1}}, \ldots, \vec{\boldsymbol{u_r}}, \vec{\boldsymbol{v_1}}, \ldots, \vec{\boldsymbol{v_{m-r}}}\}$$

where $m = dim\, W_1$.
By the same reasoning we can extend B to a basis of W_2,

$$B_{W_2} = \{\vec{\boldsymbol{u_1}}, \ldots, \vec{\boldsymbol{u_r}}, \vec{\boldsymbol{w_1}}, \ldots, \vec{\boldsymbol{w_{n-r}}}\}$$

where $n = dim\, W_2$.
Now if we can show that,

$$B' = B_{W_1} \cup B_{W_2} = \{\vec{\boldsymbol{u_1}}, \ldots, \vec{\boldsymbol{u_r}}, \vec{\boldsymbol{v_1}}, \ldots, \vec{\boldsymbol{v_{m-r}}}, \vec{\boldsymbol{w_1}}, \ldots, \vec{\boldsymbol{w_{n-r}}}\}$$

is a basis of $W_1 + W_2$ then the proof will follow easily.
Clearly, B' spans $W_1 + W_2$ as B' contains a basis of both W_1 and W_2 and so is able to express any sum of two vectors chosen from the two spaces.
Linear Independence is a little more complicated however. Consider that, since B_{W_1} is linearly independent, there can be no linear relation between the vectors $\vec{\boldsymbol{u_i}}$ and $\vec{\boldsymbol{v_i}}$ and similarly there can be no such relation between the vectors $\vec{\boldsymbol{u_i}}$ and $\vec{\boldsymbol{w_i}}$. Therefore, if there were to be a linear relation among the vectors it would have to involve the vectors $\vec{\boldsymbol{v_i}}$ with the vectors from B_{W_2} or the vectors $\vec{\boldsymbol{w_i}}$ with the vectors from B_{W_1}. If we model the linear relation as a relation between vectors that are linear combinations of the vectors $\vec{\boldsymbol{u_i}}, \vec{\boldsymbol{v_i}}$ and $\vec{\boldsymbol{w_i}}$ we get,

$$\vec{\boldsymbol{u}} + \vec{\boldsymbol{v}} + \vec{\boldsymbol{w}} = \vec{\boldsymbol{0}} \iff \vec{\boldsymbol{v}} = -\vec{\boldsymbol{u}} - \vec{\boldsymbol{w}} \in W_2$$

therefore $\vec{\boldsymbol{v}} \in W_1 \cap W_2$ and so $\vec{\boldsymbol{v}}$ can be expressed as a linear combination of the vectors in B. But this implies a linear relation among the vectors of B_{W_1} which are linearly independent by construction. Therefore

$$\vec{\boldsymbol{v}} = \vec{\mathbf{0}} \implies -\vec{\boldsymbol{u}} - \vec{\boldsymbol{w}} = \vec{\mathbf{0}} \iff -\vec{\boldsymbol{u}} = \vec{\boldsymbol{w}}$$

which implies that there is a linear relation among the vectors of B_{W_2} which are also linearly independent by construction and so $\vec{\boldsymbol{u}} = \vec{\boldsymbol{w}} = \vec{\mathbf{0}}$. □

Corollary 2.4.4. *Let $W_1, \ldots, W_n$ be a set of subspaces of a finite-dimensional vector space. Then,*

$$dim(W_1 + \cdots + W_n) \leq dim\, W_1 + \cdots + dim\, W_n$$

with the equality case iff the spaces $W_1, \ldots, W_2$ are independent.

Proof. This follows by induction on Theorem 2.4.3.
If we take the case of $n = 2$ as our base case then,

$$dim(W_1 + W_2) = dim\, W_1 + dim\, W_2 - dim(W_1 \cap W_2) \leq dim\, W_1 + dim\, W_2.$$

Furthermore, if W_1, W_2 are independent then $W_1 \cap W_2 = \{\vec{\mathbf{0}}\}$ so that,

$$dim(W_1 + W_2) = dim\, W_1 + dim\, W_2 - 0 = dim\, W_1 + dim\, W_2.$$

For the induction step: Let $W' = W_1 + \cdots + W_{n-1}$ and take as the induction hypothesis that,

$$dim(W') \leq dim\, W_1 + \cdots + dim\, W_{n-1}$$

with equality being when the spaces are independent. Then observe that, by the associativity of addition of vectors and hence of vector spaces,

$$\begin{aligned} dim(W_1 + \cdots + W_n) &= dim((W_1 + \cdots + W_{n-1}) + W_n) \\ &= dim\, W' + dim\, W_n - dim(W' \cap W_n) && \text{by Theorem 2.4.3} \\ &\leq dim\, W' + dim\, W_n \\ &\leq dim\, W_1 + \cdots + dim\, W_{n-1} + dim\, W_n && \text{by induction hypothesis.} \end{aligned}$$

Furthermore, if $dim\, W' = dim\, W_1 + \cdots + dim\, W_{n-1}$ and if the spaces W' and W_n are independent then the intersection,

$$W' \cap W_n = \{\vec{\mathbf{0}}\}$$

which gives,

$$\begin{aligned} dim(W_1 + \cdots + W_n) &= dim((W_1 + \cdots + W_{n-1}) + W_n) \\ &= dim\, W' + dim\, W_n - 0 && \text{by Theorem 2.4.3} \\ &= dim\, W_1 + \cdots + dim\, W_{n-1} + dim\, W_n && \text{by induction hypothesis.} \end{aligned}$$

This shows that if the spaces are independent then

$$dim(W_1 + \cdots + W_n) = dim\, W_1 + \cdots + dim\, W_n$$

but it is also easy to reverse the logic and show that if the above equality holds then, if we add a space to the sum of other spaces, then the intersection of the added space with the sum of the other spaces must have dimension 0 and therefore must be $\{\vec{\mathbf{0}}\}$. In this way we can also prove the converse implication that the spaces must be independent. □

Proposition 2.4.19. *Two finite-dimensional vector spaces may be isomorphic only if they have the same dimension.*

Proof. We want to prove that if $\phi : W \longmapsto V$ is an isomorphism of vector spaces then it follows that $dim\, W = dim\, V$.
Assume for contradiction that ϕ is indeed an isomorphism between the vector spaces W and V but,

$$dim\, W = m > dim\, V = n.$$

Then any basis of $\vec{\boldsymbol{w_1}}, \ldots, \vec{\boldsymbol{w_m}} \in W$ is a linearly independent set of vectors in W with the property, for $c_1, \ldots, c_m \in \mathbb{F}$,

$$\begin{aligned} & c_1\vec{\boldsymbol{w_1}} + \cdots + c_m\vec{\boldsymbol{w_m}} = \vec{\mathbf{0}} \iff c_1, \ldots, c_m = 0 \\ \iff \quad & \phi(c_1\vec{\boldsymbol{w_1}} + \cdots + c_m\vec{\boldsymbol{w_m}}) = \phi(\vec{\mathbf{0}}) = \vec{\mathbf{0}} \iff c_1, \ldots, c_m = 0 \\ \iff \quad & c_1\phi(\vec{\boldsymbol{w_1}}) + \cdots + c_m\phi(\vec{\boldsymbol{w_m}}) = \vec{\mathbf{0}} \iff c_1, \ldots, c_m = 0. \end{aligned}$$

But we have $\phi(\vec{\boldsymbol{w_1}}), \ldots, \phi(\vec{\boldsymbol{w_m}}) \in V$ so that the result obtained implies that $\phi(\vec{\boldsymbol{w_1}}), \ldots, \phi(\vec{\boldsymbol{w_m}})$ is a linearly independent set of vectors in V of cardinality $m > n = dim\, V$. By Theorem 2.4.1 this cannot be and we have obtained a contradiction. □

TODO: example application of calculating the order of $GL_2(\mathbb{F})$ when $\mathbb{F} = \mathbb{F}_p$ is a prime field. Artin[114]

2.4.6.6 Direct Sums in Finite-Dimensional Vector Spaces

Proposition 2.4.20. *Let $W_1, \ldots, W_n$ be subspaces of a finite-dimensional vector space V, and let B_i be a basis for W_i. Then, the union*

$$\bigcup_{i=1}^{n} B_i$$

is a disjoint union and forms a basis of V iff

$$V = W_1 \otimes \cdots \otimes W_n.$$

Proof. Let $B = \bigcup_i B_i$ be a basis of V. Then B is a linearly independent set and so,

$$\forall \vec{\boldsymbol{b}} \in B \,.\, \vec{\boldsymbol{b}} \notin span(B \setminus \{\vec{\boldsymbol{b}}\})$$

which also means that,

$$\forall B_i \subset B \,.\, span\, B_i \cap span(B \setminus B_i) = \vec{\mathbf{0}}.$$

This implies that the sets B_i are independent spaces. Since their union is a basis of V then together they span V. Since the span of each B_i is W_i then, $V = W_1 \oplus \cdots \oplus W_n$.
Conversely, if $V = W_1 \oplus \cdots \oplus W_n$ then each subspace W_i is independent so that, for every $\vec{\boldsymbol{b}} \in B_i$, $\vec{\boldsymbol{b}} \notin span(B \setminus B_i)$. This means that if we begin with B_1 and add to it the elements of B_2 then the resulting set remains linearly independent and we can continue this until we have $B = \bigcup_i B_i$ as a linearly independent set. Then B contains all the basis vectors of every W_i and therefore, by $V = W_1 \oplus \cdots \oplus W_n$, B spans the space V. It is therefore a basis of V. □

Proposition 2.4.21. *Let W be a subspace of a finite-dimensional vector space V. Then there is another subspace W' such that $W \oplus W' = V$.*

Proof. Any basis of W is a linearly independent set in V and can be extended to a basis of V by adding a set of linearly independent vectors S. Then S is a basis of a subspace W' such that any vector in V is in $W + W'$ and since their bases are linearly independent this sum of spaces is a direct sum $W \oplus W'$. Therefore, $W \oplus W' = V$. □

Notes about Proposition 2.4.21:

- *It is not necessary that there be only one such subspace W'. In fact, often there are an infinite number of such subspaces. All such spaces are isomorphic.*
- *The proposition that all subspaces have a complement space can also be proven for infinite-dimensional spaces if we use the Axiom of Choice (1.1.1.3).*

2.4.6.7 Vector Complement Spaces and Quotient Spaces

Definition 93. **(Complement Space)** A subspace W' of a finite-dimensional vector space V such that $W \oplus W' = V$ is called a *complement space*. While there may be many such spaces W', they are equivalent upto isomorphism.

Definition 94. The subspace W' of V such that $W \oplus W' = V$ is called the **quotient space of V by W** and is denoted V/W. Wrong!TODO: copy correct definition from wikipedia

Further information: wikipedia.

(48) Consider the vector space $\mathbb{R}^2$ and let W be the subspace created by the linear combinations of the vector $(1,1)^T$ — i.e. the line $y = x$ shown in red in the diagrams below.

Figure 2.3 shows two of the cosets of W in $\mathbb{R}^2$ as green and blue lines. The cosets such as these are the elements of the quotient space and they are the infinite number of lines parallel to $y = x$. They are parallel because they are affine spaces whose associated vector space is the linear combinations of $(1,1)^T$ — these linear combinations form the kernel of the quotient map and the elements of the quotient space are the additive cosets with the kernel of the quotient map.

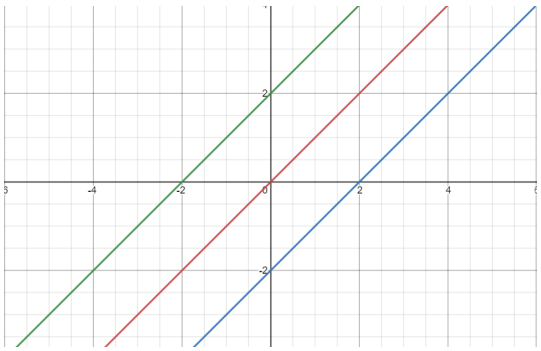


Figure 2.3: The green and blue lines are both cosets of W in $\mathbb{R}^2$ and elements in the quotient space.

Figure 2.4 shows two of the complement spaces as green and blue lines. Such complement spaces are the subspaces W' of $\mathbb{R}^2$ such that $W \oplus W' = \mathbb{R}^2$. Any non-parallel line is such a subspace. Each point on such a line represents a vector that is an element of the complement space and is also a point in a line parallel to $y = x$. In this way, any one of the possible complement spaces W' is isomorphic to the quotient space.

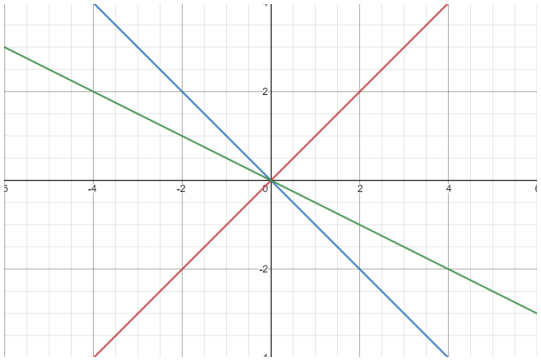


Figure 2.4: The green and blue lines are both complement spaces — subspaces W' such that $W \oplus W' = \mathbb{R}^2$.

2.4.7 Infinite Sets of Vectors

The definition of a vector space defines what it means to add two vectors and so, by extension, arbitrarily large *finite* sums of vectors but not what it means to add an infinite number of vectors. Therefore, we define the span and linear independence of infinite sets of vectors as conditions over finite subsets of the vectors.

2.4.7.1 Span and Linear Independence of Infinite Sets of Vectors

Definition 95. The **span of an infinite set of vectors** S is defined to be the set of **finite** linear combinations of its elements,

$$\{ c_1\vec{\boldsymbol{v_1}} + \cdots + c_r\vec{\boldsymbol{v_r}} \mid \vec{\boldsymbol{v_i}} \in S,\, c_i \in \mathbb{F} \}$$

where r is finite but may be arbitrarily large.

Definition 96. An **infinite set of vectors** is defined to be **linearly independent** if there is no linear relation among a **finite** subset of them,

$$c_1\vec{\boldsymbol{v_1}} + \cdots + c_n\vec{\boldsymbol{v_n}} = \vec{\mathbf{0}}$$

where $c_i \in F$.

These definitions of span and linear independence are compatible with the corresponding definitions for finite sets of vectors.

2.4.7.2 Infinite-Dimensional Vector Spaces

Definition 97. A vector space is called **infinite-dimensional** if there is no finite set of vectors that spans the space.

2.4.7.3 Examples of Infinite-Dimensional Vector Spaces

(49) The space $\mathbb{R}^\infty$ of infinite real vectors $(a) = (a_1, a_2, a_3, \dots)$ can also be thought of as the space of sequences $\{a_n\}$ of real numbers. It has many important subspaces:

a. Convergent sequences: $C = \{\, (a) \in \mathbb{R}^\infty \mid \lim_{n\to\infty} a_n \text{ exists} \,\}$.
b. Bounded sequences: $l^\infty = \{\, (a) \in \mathbb{R}^\infty \mid \{a_n\} \text{ is bounded} \,\}$.
c. Absolutely convergent series: $l^1 = \{\, (a) \in \mathbb{R}^\infty \mid \sum_1^\infty |a_n| < \infty \,\}$.
d. Sequences with finitely many nonzero terms:

$$Z = \{\, (a) \in \mathbb{R}^\infty \mid a_n = 0 \text{ for all but finitely many } n \,\}.$$

2.4.7.4 Bases of Infinite-Dimensional Vector Spaces

Definition 98. As with finite sets, a **basis of an infinite-dimensional vector space** is a linearly independent set which spans the space.

(50) Let $S = (e_1, e_2, \dots)$ be the infinite set of standard basis vectors in $\mathbb{R}^\infty$. S does not span $\mathbb{R}^\infty$ because the vector $\vec{\boldsymbol{w}} = (1, 1, 1, \dots)$ is not a *finite* linear combination of the elements of S. It *is*, however, a basis of the vector space Z in 49d.

It can be shown, using the Axiom of Choice, that every vector space has a basis (Theorem 1.1.1) but a basis of $\mathbb{R}^\infty$ will be uncountably infinite and so will not be expressable as $(\vec{\boldsymbol{v_1}}, \vec{\boldsymbol{v_2}}, \dots)$.

From the University of Michigan Maths Dept. Linear Algebra Supplement on Infinite-Dimensional Vector Spaces:

Let $\mathbb{R}$ be the set of real numbers considered as a vector space over the field $\mathbb{Q}$ of rational numbers. What could possibly be a basis? The elements $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}, \dots$ can be shown to be linearly independent, but they certainly

don't span $\mathbb{R}$, as we also need elements like $\pi, \pi^2, \pi^3, \ldots$, which also form a linearly independent set. In fact, because $\mathbb{Q}$ is countable, one can show that the subspace of $\mathbb{R}$ generated by any countable subset of $\mathbb{R}$ must be countable. Because $\mathbb{R}$ itself is uncountable, no countable set can be a basis for $\mathbb{R}$ over $\mathbb{Q}$. This means that any basis for $\mathbb{R}$ over $\mathbb{Q}$, if one exists, is going to be difficult to describe.

If we were to look for a basis of the functions over the reals we could consider the indicator functions for all $r \in \mathbb{R}$,

$$i_r(x) \coloneqq \begin{cases} 1 & x = r \\ 0 & x \neq r \end{cases}.$$

This set of functions is clearly linearly independent and spans the set of functions $\mathbb{R} \longmapsto \mathbb{R}$. Since there is one such indicator function for each real number and the real numbers are uncountable, this set of indicator functions is also uncountable.

Theorem 2.4.4. *A linear operator over an infinite-dimensional vector space does not conform to the Dimension Formula for finite-dimensional vector spaces.*

Proof. The shift operator described in a note to Proposition 2.5.11) is an example of a linear transformation over an infinite-dimensional vector space that doesn't conform to the dimension formula. As noted there, the reason is that infinite sets may have proper subsets of equal cardinality. So we can have an image that is clearly "smaller" than the space but nevertheless has the same dimensionality and so the map has a trivial kernel anyway.
Conversely the differentiation operator has a non-trivial kernel (the set of constant polynomials) but when differentiating an infinite power series the result is still an infinite power series. □

There is much more to this topic. For more details see the University of Michigan Maths Dept. Linear Algebra Supplement on Infinite-Dimensional Vector Spaces.

2.4.8 Coordinate Vector Spaces

Definition 99. A **coordinate vector** is a representation of a vector as an ordered list of numbers that describes the vector in terms of a particular ordered basis.

Definition 100. Let $B = \{\vec{\boldsymbol{v}}_1, \ldots, \vec{\boldsymbol{v}}_n\}$ be a basis of a finite-dimensional vector space V. Then for every $\vec{\boldsymbol{v}} \in V$ it is possible to express $\vec{\boldsymbol{v}}$ as a linear combination of the vectors in B in the form,

$$\vec{\boldsymbol{v}} = c_1\vec{\boldsymbol{v}}_1 + \cdots + c_n\vec{\boldsymbol{v}}_n.$$

Then, the **coordinate vector of $\vec{\boldsymbol{v}}$ with respect to the basis** B is,

$$\vec{\boldsymbol{v}}_B = \langle c_1, \ldots, c_n \rangle.$$

*The definition of a basis 2.4.6.2 requires that it is a set of **linearly independent** vectors. It is worth noting what results if we attempt to use a linearly dependent set of vectors as a basis because in real-world situations this often occurs (for example in data analysis) when the full relationship between objects may not be known.*
Imagine a "basis" $B = \{x, 3x\}$. Co-ordinate vectors defined against this basis are not unique —

$$(3)x + (1)3x = (6)x + (0)3x = \cdots$$

In fact, from the basis and the co-ordinate vectors, we appear to working in a 2-dimensional space but in fact, because there is a linear relation between the two "basis" vectors we are actually working in a 1-dimensional space. As a result, algebraic analysis will be missing one relation. For example, take the vectors against "basis" B,

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \end{bmatrix} \right\}.$$

It appears that there is a single linear relation between these vectors given by

the nullspace of their matrix as follows:

$$\begin{bmatrix}1 & 1 & 1\\ 0 & 3 & 5\end{bmatrix} \rightsquigarrow \begin{bmatrix}1 & 0 & -2/3\\ 0 & 1 & 5/3\end{bmatrix} \implies \textit{nullspace} = t\begin{bmatrix}2/3\\ -5/3\\ 1\end{bmatrix} \qquad \textit{for } t \in \mathbb{R}.$$

Whereas in actual fact all vectors in this space are colinear, being of the form αx for some $\alpha \in \mathbb{R}$. If we apply the co-ordinate vectors to their "basis" we see that,

$$\begin{bmatrix}1\\ 0\end{bmatrix} = (1)x = \boldsymbol{x}, \begin{bmatrix}1\\ 3\end{bmatrix} = (1)x + (3)3x = \mathbf{10}\boldsymbol{x}, \begin{bmatrix}1\\ 5\end{bmatrix} = (1)x + (5)3x = \mathbf{16}\boldsymbol{x}.$$

Then we can see that the true extent of the relationship between the vectors is given by,

$$\begin{bmatrix}1 & 10 & 16\end{bmatrix} \rightsquigarrow s\begin{bmatrix}-10\\ 1\\ 0\end{bmatrix} + t\begin{bmatrix}-16\\ 0\\ 1\end{bmatrix}$$

for $s, t \in \mathbb{R}$.
As we can see, the true relation (with respect to x) between the vectors is 2-dimensional. The relation found when using the "basis" B is a 1-dimensional subspace within this space corresponding to the values $s = -5/3,\ t = 1$.

Definition 101. The **dot product** of two coordinate vectors in an n-dimensional coordinate space is defined as,

$$\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}} = v_1 w_1 + \cdots + v_n w_n = \vec{\boldsymbol{v}}^T \vec{\boldsymbol{w}}.$$

Notation. The **dot product** may also be denoted $\langle \vec{\boldsymbol{v}}, \vec{\boldsymbol{w}} \rangle$ but this notation may also refer to a literal coordinate vector, i.e.

$$\begin{bmatrix}3\\ 7\end{bmatrix} = \langle 3, 7 \rangle = (3, 7)^T.$$

Proposition 2.4.22. *The properties of the dot product are:*

(i) $\langle \vec{\boldsymbol{x}}, \vec{\boldsymbol{y}} \rangle = \langle \vec{\boldsymbol{y}}, \vec{\boldsymbol{x}} \rangle$

(ii) $\alpha\langle \vec{\boldsymbol{x}}, \vec{\boldsymbol{y}} \rangle = \langle \alpha\vec{\boldsymbol{x}}, \vec{\boldsymbol{y}} \rangle = \langle \vec{\boldsymbol{x}}, \alpha\vec{\boldsymbol{y}} \rangle$

(iii) $\langle \vec{\boldsymbol{x}} + \vec{\boldsymbol{y}}, \vec{\boldsymbol{z}} \rangle = \langle \vec{\boldsymbol{x}}, \vec{\boldsymbol{z}} \rangle + \langle \vec{\boldsymbol{y}}, \vec{\boldsymbol{z}} \rangle$

(iv) $\langle \vec{\boldsymbol{x}}, \vec{\boldsymbol{x}} \rangle \geq 0$ *and* $\langle \vec{\boldsymbol{x}}, \vec{\boldsymbol{x}} \rangle = 0 \iff \vec{\boldsymbol{x}} = 0$

Proof. Proofs of these properties follow from the definition of the dot product.

(i) $\langle \vec{\boldsymbol{x}}, \vec{\boldsymbol{y}} \rangle = \langle \vec{\boldsymbol{y}}, \vec{\boldsymbol{x}} \rangle$ follows from the commutativity of multiplication in a field,

$$x_i y_i = y_i x_i.$$

(ii) $\alpha\langle \vec{\boldsymbol{x}}, \vec{\boldsymbol{y}} \rangle = \langle \alpha\vec{\boldsymbol{x}}, \vec{\boldsymbol{y}} \rangle = \langle \vec{\boldsymbol{x}}, \alpha\vec{\boldsymbol{y}} \rangle$ follows from the associativity of multiplication in a field,

$$\alpha x_i y_i = x_i \alpha y_i.$$

(iii) $\langle \vec{\boldsymbol{x}}+\vec{\boldsymbol{y}}, \vec{\boldsymbol{z}} \rangle = \langle \vec{\boldsymbol{x}}, \vec{\boldsymbol{z}} \rangle + \langle \vec{\boldsymbol{y}}, \vec{\boldsymbol{z}} \rangle$ follows from the distributivity of multiplication over addition in a field,

$$(x_i + y_i) z_i = x_i z_i + y_i z_i.$$

(iv) $\langle \vec{\boldsymbol{x}}, \vec{\boldsymbol{x}} \rangle \geq 0$ and $\langle \vec{\boldsymbol{x}}, \vec{\boldsymbol{x}} \rangle = 0 \iff \vec{\boldsymbol{x}} = 0$ follows from its form as a sum of squares,

$$x_1^2 + x_2^2 + \cdots + x_n^2.$$

□

2.4.8.1 Examples of coordinate vectors

(51) Let $\vec{\boldsymbol{x}} = \langle 3, 3 \rangle \in \mathbb{R}^2$ and B be the set $\{\langle 2, 0 \rangle, \langle 1, 3 \rangle\}$ so that B is a non-standard basis of the space $\mathbb{R}^2$. Then we can also define $\vec{\boldsymbol{x}}$ with respect to the basis B as follows.

$$\vec{\boldsymbol{x}} = 1 \cdot \begin{bmatrix} 2 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Therefore, the **coordinate vector of $\vec{\boldsymbol{x}}$ with respect to the basis** B is defined as,

$$\vec{\boldsymbol{x}}_B = \langle 1, 1 \rangle.$$

2.4.8.2 Bases as Matrices

Notation. If $B = \{\vec{\boldsymbol{v}}_1, \ldots, \vec{\boldsymbol{v}}_n\}$ is a set of vectors then, when it seems necessary to clearly differentiate, the matrix whose columns are the elements of B will be denoted $[B]$. However, for convenience, when the mathematical context makes clear whether we are referring to a set of vectors or a matrix we will use B to indicate either the set or the matrix.

Proposition 2.4.23. *If $\vec{\boldsymbol{x}}_B$ is a coordinate vector of $\vec{\boldsymbol{x}}$ w.r.t. the basis B then left multiplication by the matrix $[B]$ whose columns are the elements of B converts $\vec{\boldsymbol{x}}_B$ to its standard coordinate form $\vec{\boldsymbol{x}}$,*

$$\vec{\boldsymbol{x}} = [B]\vec{\boldsymbol{x}}_B.$$

Proof. Let $B = \{\vec{\boldsymbol{v}}_1, \ldots, \vec{\boldsymbol{v}}_n\}$ be a basis and the basis matrix

$$[B] = \begin{bmatrix} \vec{\boldsymbol{v}}_1 & \ldots & \vec{\boldsymbol{v}}_n \end{bmatrix}$$

and $\vec{\boldsymbol{x}} = \langle x_1, \ldots, x_n \rangle$ is a vector in standard coordinates. Then if we calculate $\vec{\boldsymbol{x}}$ using the basis B,

$$\vec{\boldsymbol{x}} = x_{B1}\vec{\boldsymbol{v}}_1 + \cdots + x_{Bn}\vec{\boldsymbol{v}}_n$$

we get a coordinate vector w.r.t. B,

$$\vec{\boldsymbol{x}}_B = \langle x_{B1}, \ldots, x_{Bn} \rangle.$$

So, clearly, to recover the standard coordinate vector we need to apply $\vec{\boldsymbol{x}}_B$ to the basis against which it was defined,

$$[B]\vec{\boldsymbol{x}}_B = \begin{bmatrix} \vec{\boldsymbol{v}}_1 & \ldots & \vec{\boldsymbol{v}}_n \end{bmatrix} \begin{bmatrix} x_{B1} \\ \vdots \\ x_{Bn} \end{bmatrix} = x_{B1}\vec{\boldsymbol{v}}_1 + \cdots + x_{Bn}\vec{\boldsymbol{v}}_n = \vec{\boldsymbol{x}}. \qquad \square$$

Corollary 2.4.5. *If $\vec{\boldsymbol{x}}$ is a coordinate vector w.r.t. the standard basis and B is an alternative basis then left multiplication by the matrix $[B]^{-1}$ converts $\vec{\boldsymbol{x}}$ to $\vec{\boldsymbol{x}}_B$ its form w.r.t. the basis B.*

Proof.

$$[B]\vec{\boldsymbol{x}}_B = \vec{\boldsymbol{x}} \iff \vec{\boldsymbol{x}}_B = [B]^{-1}\vec{\boldsymbol{x}}.$$

$\square$

One way to think of the action of the basis matrix is that it is used to encode/decode coordinates into/from its basis coordinates.
For example, if $\vec{\boldsymbol{x}}_B$ is a coordinate vector w.r.t. the basis B, then left-multiplying it by the basis matrix $[B]$,

$$\vec{\boldsymbol{x}} = [B]\vec{\boldsymbol{x}}_B$$

decodes the coordinate vector into standard coordinates. Conversely we can use the inverse of the basis matrix,

$$\vec{\boldsymbol{x}}_B = [B]^{-1}\vec{\boldsymbol{x}}$$

to encode a standard coordinate vector into B-coordinates.

2.4.8.3 Relationship with Abstract Vector Spaces

Proposition 2.4.24. *Every vector space V of dimension n is isomorphic to the space $\mathbb{F}^n$ of column vectors.*

Proof. Let $\phi : \mathbb{F}^n \longmapsto V$ be defined as $\phi(\vec{\boldsymbol{x}}) = B\vec{\boldsymbol{x}}$ where B is a matrix whose columns are a basis of V. The map ϕ is surjective because the columns of B span the space V and it is injective because they are linearly independent. So ϕ is a bijection.
The structure of the vector space is preserved because,

$$\phi(\vec{\boldsymbol{x_1}} + \vec{\boldsymbol{x_2}}) = B(\vec{\boldsymbol{x_1}} + \vec{\boldsymbol{x_2}}) = B\vec{\boldsymbol{x_1}} + B\vec{\boldsymbol{x_2}} = \phi(\vec{\boldsymbol{x_1}}) + \phi(\vec{\boldsymbol{x_2}})$$

and

$$\phi(c\vec{\boldsymbol{x}}) = B(c\vec{\boldsymbol{x}}) = cB\vec{\boldsymbol{x}} = c\phi(\vec{\boldsymbol{x}}).$$

□

*Note that, by Proposition 2.4.19, $\mathbb{F}^n$ is **not** isomorphic to $\mathbb{F}^m$ for $m \neq n$. Every finite-dimensional vector space V is isomorphic to $\mathbb{F}^n$, for some uniquely determined integer n.*
So, the finite-dimensional vector spaces are completely classified by Proposition 2.4.24 and any problem on finite-dimensional vector spaces may

be reduced to a problem on column vectors and matrices.

It is a result of Proposition 2.4.24 that we can use coordinate spaces as vector spaces.

2.4.8.4 Using Coordinate Spaces to analyse Vectors

Span If we have a set of n vectors in $\mathbb{F}^m$ then we can determine if a vector $\vec{\boldsymbol{b}}$ is in the span of the set of vectors by solving the system,

$$A\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}}$$

where $\vec{\boldsymbol{x}} \in \mathbb{F}^n$ and A is an $m \times n$ matrix whose columns are the set of vectors. If there is some $\vec{\boldsymbol{x}}$ that satisfies the equation then $\vec{\boldsymbol{b}}$ is in the span.

Linear Independence If we have a set of n vectors in $\mathbb{F}^m$ then we can determine linear independence by solving a system of homogeneous linear equations,

$$A\vec{\boldsymbol{x}} = \vec{\mathbf{0}}$$

where $\vec{\boldsymbol{x}} \in \mathbb{F}^n$ and A is an $m \times n$ matrix whose columns are the set of vectors. If there is a non-trivial solution — a non-zero $\vec{\boldsymbol{x}}$ for which $A\vec{\boldsymbol{x}} = \vec{\mathbf{0}}$ — then the set of vectors is not linearly independent.

2.4.8.5 Change of Basis

Definition 102. If we represent an abstract vector $\vec{\boldsymbol{v}}$ as a coordinate vector with respect to two different bases $B = \{\vec{\boldsymbol{b}}_1, \ldots, \vec{\boldsymbol{b}}_n\}$ and $B' = \{\vec{\boldsymbol{b}}'_1, \ldots, \vec{\boldsymbol{b}}'_n\}$ then we have,

$$\vec{\boldsymbol{v}} = a_1\vec{\boldsymbol{b}}_1 + \cdots + a_n\vec{\boldsymbol{b}}_n = a'_1\vec{\boldsymbol{b}}'_1 + \cdots + a'_n\vec{\boldsymbol{b}}'_n.$$

Using the notation $\vec{\boldsymbol{x}}_B = \langle a_1, \ldots, a_n\rangle$, $\vec{\boldsymbol{x}}_{B'} = \langle a'_1, \ldots, a'_n\rangle$ and letting B, B' from here on refer to the matrices whose columns are the elements of B, B', we can rewrite the previous equation with matrices as,

$$\vec{\boldsymbol{v}} = B\vec{\boldsymbol{x}}_B = B'\vec{\boldsymbol{x}}_{B'}.$$

Then the **change of basis** from B to B' is the mapping,

$$\vec{\boldsymbol{x}}_{B'} = (B')^{-1}B\vec{\boldsymbol{x}}_B = P\vec{\boldsymbol{x}}_B.$$

So we have,

$$(B')^{-1}B = P \iff B = B'P$$

which shows that P is the mapping between the two bases. This is known as the **matrix of change of basis**.

Note that another way to think about this is to say that $\vec{\boldsymbol{x}}$ is the coordinate vector of $\vec{\boldsymbol{v}}$ with respect to the standard basis and,

$$\vec{\boldsymbol{x}} = B\vec{\boldsymbol{x}}_B = B'\vec{\boldsymbol{x}}_{B'}$$

so we always use the standard basis as a reference.

Proposition 2.4.25. *If $B \in \mathbb{F}^{n\times n}$ is a basis of a finite vector space V then, for $P \in GL_n(\mathbb{F})$,*

$$BP^{-1} = B'$$

is another basis of V.

Proof. As a member of $GL_n(\mathbb{F})$, P is invertible and so is a bijective mapping. It follows then that each of the basis vectors forming the columns of B are in the span of the columns of B' and so the columns of B' must also span the space V. Furthermore, since we must also have $B' \in \mathbb{F}^{n\times n}$, there are n columns in B' and so they are a spanning set with cardinality equal to the set of columns of B which is a basis of V. Therefore, by Theorem 2.4.2, the columns of B' are a basis of V. □

2.4.8.6 Matrix of Change of Basis

Definition 103. Let B, B' be two different bases of the same space. Then, the matrix P such that,

$$B = B'P \quad \text{and} \quad \vec{\boldsymbol{x}}_{B'} = P\vec{\boldsymbol{x}}_B$$

is known as the **matrix of change of basis** between B and B'.

The matrix of change of basis P contains the vectors of one basis defined w.r.t. to the other basis. So,

$$B = B'P$$

tells us that if we apply the coordinate vectors in P to the basis B' we get the basis vectors of B.
Conversely, if we have a coordinate vector $\vec{\boldsymbol{x}}_B$ defined w.r.t. to B and we treat P as though it were a basis and apply the coordinate vector $\vec{\boldsymbol{x}}_B$ to it,

$$\vec{\boldsymbol{x}}_{B'} = P\vec{\boldsymbol{x}}_B$$

we get the same vector defined w.r.t. to the basis B'.

2.4.8.7 Euclidean Coordinate Spaces

Definition 104. The **Euclidean coordinate spaces** are the real coordinate spaces $\mathbb{R}^n$ equipped with the **standard norm**,

$$\|\vec{\boldsymbol{v}}\| = \sqrt{v_1^2 + \cdots + v_n^2}$$

which is considered to be the **length** of the vector in the space $\mathbb{R}^n$.

- *The Euclidean norm (or standard norm) is the square root of the dot product of the vector with itself,*

$$\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{v}} = v_1^2 + \cdots + v_n^2 = \|\vec{\boldsymbol{v}}\|^2 .$$

- *In a Euclidean coordinate space the metric (wikipedia:metric), or distance function, is the length of the vector between the two points considered to be position vectors in the corresponding Euclidean vector space. So, the distance d between the point whose position vector is $\vec{\boldsymbol{v}}$ and the point whose position vector is $\vec{\boldsymbol{w}}$ is,*

$$d = \|\vec{\boldsymbol{v}} - \vec{\boldsymbol{w}}\| = \|\vec{\boldsymbol{w}} - \vec{\boldsymbol{v}}\| .$$

Proposition 2.4.26. *The dot product of two distinct vectors in a Euclidean space is related to their lengths by the formula,*

$$\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}} = \|\vec{\boldsymbol{v}}\| \|\vec{\boldsymbol{w}}\| \cos\theta$$

where θ is the angle between the vectors.

Proof. The properties of the dot product (Proposition 2.4.22) tell us that,

$$\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}} = \vec{\boldsymbol{w}} \cdot \vec{\boldsymbol{v}} \quad \text{and} \quad (\vec{\boldsymbol{v}} - \vec{\boldsymbol{w}}) \cdot \vec{\boldsymbol{x}} = \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{x}} - \vec{\boldsymbol{w}} \cdot \vec{\boldsymbol{x}}$$

so if we let $\vec{\boldsymbol{x}} = (\vec{\boldsymbol{v}} - \vec{\boldsymbol{w}})$ also we obtain,

$$(\vec{\boldsymbol{v}} - \vec{\boldsymbol{w}}) \cdot (\vec{\boldsymbol{v}} - \vec{\boldsymbol{w}}) = \vec{\boldsymbol{v}} \cdot (\vec{\boldsymbol{v}} - \vec{\boldsymbol{w}}) - \vec{\boldsymbol{w}} \cdot (\vec{\boldsymbol{v}} - \vec{\boldsymbol{w}})$$

$$= \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{v}} - \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}} - \vec{\boldsymbol{w}} \cdot \vec{\boldsymbol{v}} + \vec{\boldsymbol{w}} \cdot \vec{\boldsymbol{w}}$$
$$= \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{v}} + \vec{\boldsymbol{w}} \cdot \vec{\boldsymbol{w}} - 2\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}}.$$

So this gives us a formula for the square of the length of the displacement vector between $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{w}}$ or, in other words, the square of the distance between $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{w}}$. Now, if we imagine $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{w}}$ and the vector $\vec{\boldsymbol{v}} - \vec{\boldsymbol{w}}$ forming a triangle we can also refer to the law of cosines which tells us that,

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

where θ is the angle subtended by a and b and c is the side opposite the angle. Letting,

$$a = \|\vec{\boldsymbol{v}}\|\,,\; b = \|\vec{\boldsymbol{w}}\|\,,\; c = \|\vec{\boldsymbol{v}} - \vec{\boldsymbol{w}}\|$$

we obtain,

$$\|\vec{\boldsymbol{v}} - \vec{\boldsymbol{w}}\|^2 = \|\vec{\boldsymbol{v}}\|^2 + \|\vec{\boldsymbol{w}}\|^2 - 2\|\vec{\boldsymbol{v}}\|\|\vec{\boldsymbol{w}}\|\cos\theta.$$

But the Euclidean norm is the square root of the dot product so also,

$$\|\vec{\boldsymbol{v}} - \vec{\boldsymbol{w}}\|^2 = (\vec{\boldsymbol{v}} - \vec{\boldsymbol{w}}) \cdot (\vec{\boldsymbol{v}} - \vec{\boldsymbol{w}})$$

therefore

$$\begin{aligned}
& \|\vec{\boldsymbol{v}}\|^2 + \|\vec{\boldsymbol{w}}\|^2 - 2\|\vec{\boldsymbol{v}}\|\|\vec{\boldsymbol{w}}\|\cos\theta = \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{v}} + \vec{\boldsymbol{w}} \cdot \vec{\boldsymbol{w}} - 2\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}} \\
\iff \quad & \|\vec{\boldsymbol{v}}\|^2 + \|\vec{\boldsymbol{w}}\|^2 - 2\|\vec{\boldsymbol{v}}\|\|\vec{\boldsymbol{w}}\|\cos\theta = \|\vec{\boldsymbol{v}}\|^2 + \|\vec{\boldsymbol{w}}\|^2 - 2\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}} \\
\iff \quad & -2\|\vec{\boldsymbol{v}}\|\|\vec{\boldsymbol{w}}\|\cos\theta = -2\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}} \\
\iff \quad & \|\vec{\boldsymbol{v}}\|\|\vec{\boldsymbol{w}}\|\cos\theta = \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}}. \qquad \square
\end{aligned}$$

This relationship is the foundation of analytic geometry.

Orthogonality in Euclidean Spaces

Definition 105. Two nonzero vectors $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{w}}$ are said to be **orthogonal** if their dot product is zero, i.e.

$$\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}} = 0.$$

Theorem 2.4.5. *Two nonzero vectors $\vec{v}$ and $\vec{w}$ are perpendicular to each other if and only if their dot product is zero.*

Proof. If the dot product $\vec{v} \cdot \vec{w} = 0$ then, by Proposition 2.4.26,

$$\vec{v} \cdot \vec{w} = \|\vec{v}\|\|\vec{w}\| \cos\theta = 0$$

where θ is the angle between $\vec{v}$ and $\vec{w}$. Since $\vec{v}$ and $\vec{w}$ are both nonzero, the product of their lengths $\|\vec{v}\|\|\vec{w}\| > 0$. Then we must have $\cos\theta = 0 \iff \theta \in \{\pi/2, 3\pi/2\}$. So, $\vec{v} \cdot \vec{w} = 0$ implies that the vectors are perpendicular.
Conversely, we can use the same logic in reverse to show that, for perpendicular vectors, $\cos\theta = 0$ which means that the dot product will be 0. □

Corollary 2.4.6. *Geometric orthogonality — that's to say perpendicularity — is equivalent to the dot product definition of orthogonality.*

2.4.8.8 Complex Coordinate Spaces

Definition 106. A **Complex vector space** is a space of vectors defined over the field $\mathbb{C}$ and, as such, the vector coordinates and the scalars are complex numbers. Similarly to the Euclidean spaces, a standard norm is defined,

$$\|\vec{\boldsymbol{v}}\| = \sqrt{v_1\overline{v_1} + \cdots + v_n\overline{v_n}} = \sqrt{|v_1|^2 + \cdots + |v_n|^2}$$

which is considered to be the **length** of the vector in the space $\mathbb{C}^n$.

Proposition 2.4.27. *The additive group in vector space $\mathbb{C}^n$ is isomorphic to the additive group in Euclidean space $\mathbb{R}^{2n}$.*

Proof. If we take the standard basis of $\mathbb{C}^2$,

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

then an arbitrary vector in $\mathbb{C}^2$ is expressed as,

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} a+bi \\ c+di \end{bmatrix}.$$

We can also express this as,

$$a \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

The mapping between these two representations is an isomorphism of the additive groups $(\mathbb{C}^2, +)$ and $(\mathbb{R}^4, +)$. Clearly the same approach can be followed for any greater dimension $n > 2$. □

This isomorphism of additive groups is not compatible with complex scalar multiplication so the vector spaces cannot be said to be an isomorphism of vector spaces (ref: 2.4.1.3). To see this take the example of the vector $(i, 1)^T \in \mathbb{C}^2$ *and the scalar multiplication,*

$$i \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ i \end{bmatrix}.$$

Clearly there is no real scalar α *such that,*

$$\alpha \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

However, in a situation where scalar multiplication can be restricted to only real-valued scalars it may be possible to consider the spaces as isomorphic.

Proposition 2.4.28. *For the conjugate pair* $\vec{\boldsymbol{v}}, \overline{\vec{\boldsymbol{v}}} \in \mathbb{C}^n$*,*

$$\text{Lin}\left\{\vec{\boldsymbol{v}},\, \overline{\vec{\boldsymbol{v}}}\right\} = \text{Lin}\left\{\text{Re}\,\vec{\boldsymbol{v}},\, \text{Im}\,\vec{\boldsymbol{v}}\right\}$$

where $\text{Re}\,\vec{\boldsymbol{v}}, \text{Im}\,\vec{\boldsymbol{v}}$ *are vectors with real-valued components in* $\mathbb{C}^n$*.*

Proof.
Let $\vec{\boldsymbol{v}} = \begin{bmatrix} a+bi \\ c+di \end{bmatrix}$ and so $\overline{\vec{\boldsymbol{v}}} = \begin{bmatrix} a-bi \\ c-di \end{bmatrix}$. These vectors can be expressed as

$$\begin{bmatrix} a \\ c \end{bmatrix} + i \begin{bmatrix} b \\ d \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a \\ c \end{bmatrix} - i \begin{bmatrix} b \\ d \end{bmatrix}.$$

Therefore we have,

$$\vec{\boldsymbol{v}}, \overline{\vec{\boldsymbol{v}}} \in \text{Lin}\left\{ \begin{bmatrix} a \\ c \end{bmatrix}, \begin{bmatrix} b \\ d \end{bmatrix} \right\} = \text{Lin}\left\{\text{Re}\,\vec{\boldsymbol{v}},\, \text{Im}\,\vec{\boldsymbol{v}}\right\}$$

and so,

$$\text{Lin}\left\{\vec{\boldsymbol{v}},\, \overline{\vec{\boldsymbol{v}}}\right\} = \text{Lin}\left\{\text{Re}\,\vec{\boldsymbol{v}},\, \text{Im}\,\vec{\boldsymbol{v}}\right\}$$

as required. □

Proposition 2.4.29. *Let $A \in \mathbb{C}^{n\times n}$ and $\vec{\boldsymbol{v}} \in \mathbb{C}^n$ be any vector in the complex space with $\vec{\boldsymbol{u}} = \operatorname{Re}\vec{\boldsymbol{v}}$. Then,*

$$A\vec{\boldsymbol{v}} + A\overline{\vec{\boldsymbol{v}}} = 2A\vec{\boldsymbol{u}}.$$

Proof. Let $A \in \mathbb{C}^{n\times n}$ be decomposed into

$$A = \operatorname{Re} A + i \operatorname{Im} A = B + iC$$

and a vector $\vec{\boldsymbol{v}} \in \mathbb{C}^n$ be decomposed into

$$\vec{\boldsymbol{v}} = \operatorname{Re}\vec{\boldsymbol{v}} + i \operatorname{Im}\vec{\boldsymbol{v}} = \vec{\boldsymbol{u}} + i\vec{\boldsymbol{w}}.$$

Then

$$\begin{aligned} A\vec{\boldsymbol{v}} &= (B + iC)(\vec{\boldsymbol{u}} + i\vec{\boldsymbol{w}}) \\ &= B\vec{\boldsymbol{u}} + iB\vec{\boldsymbol{w}} + iC\vec{\boldsymbol{u}} + i^2C\vec{\boldsymbol{w}} \\ &= (B\vec{\boldsymbol{u}} - C\vec{\boldsymbol{w}}) + i(C\vec{\boldsymbol{u}} + B\vec{\boldsymbol{w}}), \end{aligned}$$

$$\begin{aligned} A\overline{\vec{\boldsymbol{v}}} &= (B + iC)(\vec{\boldsymbol{u}} - i\vec{\boldsymbol{w}}) \\ &= B\vec{\boldsymbol{u}} - iB\vec{\boldsymbol{w}} + iC\vec{\boldsymbol{u}} - i^2C\vec{\boldsymbol{w}} \\ &= (B\vec{\boldsymbol{u}} + C\vec{\boldsymbol{w}}) + i(C\vec{\boldsymbol{u}} - B\vec{\boldsymbol{w}}). \end{aligned}$$

So

$$A\vec{\boldsymbol{v}} + A\overline{\vec{\boldsymbol{v}}} = 2B\vec{\boldsymbol{u}} + 2iC\vec{\boldsymbol{u}} = 2(B + iC)\vec{\boldsymbol{u}} = 2A\vec{\boldsymbol{u}}. \qquad \square$$

2.5 Linear Transformations

2.5.1 Basic Properties of Linear Transformations

The analogue for vector spaces of a homomorphism of groups is a map,

$$T : V \longmapsto W$$

from one vector space over a field $\mathbb{F}$ to another, which is compatible with addition and scalar multiplication:

$$T(\vec{\boldsymbol{v_1}} + \vec{\boldsymbol{v_2}}) = T(\vec{\boldsymbol{v_1}}) + T(\vec{\boldsymbol{v_2}}) \quad \text{and} \quad T(c\vec{\boldsymbol{v_1}}) = cT(\vec{\boldsymbol{v_1}}),$$

for all $\vec{\boldsymbol{v_1}}, \vec{\boldsymbol{v_2}} \in V$, $c \in \mathbb{F}$.

Note that another way of describing this is that ***linear combinations are preserved across linear transformations****. That's to say, if*

$$\vec{\boldsymbol{u}} = \alpha_1\vec{\boldsymbol{v}}_1 + \cdots + \alpha_n\vec{\boldsymbol{v}}_n \quad \textit{and} \quad \vec{\boldsymbol{w}} = \alpha_1 f(\vec{\boldsymbol{v}}_1) + \cdots + \alpha_n f(\vec{\boldsymbol{v}}_n)$$

then, if f is linear we also have,

$$f(\vec{\boldsymbol{u}}) = \vec{\boldsymbol{w}}.$$

Definition 107. A homomorphism between two vector spaces that is also compatible with scalar multiplication is called a **linear transformation** or **linear map or mapping**. That's to say, if T is a linear map over a vector space defined over a field $\mathbb{F}$; each $\vec{\boldsymbol{v}}_i$ is drawn from the vector space; and each α_i is drawn from $\mathbb{F}$ then,

$$T\left(\sum_i \alpha_i\vec{\boldsymbol{v}}_i\right) = \sum_i \alpha_i T(\vec{\boldsymbol{v}}_i).$$

The compatibility with addition of vectors implies that a linear transformation is a homomorphism between additive groups of vectors.

Linear transformations ***preserve*** *linear combinations in their arguments but this must be distinguished carefully from* ***being equal*** *to linear combinations of their arguments. A typical linear transformation is* ***not*** *expressible as a linear combination of it's sole argument and — in fact — the image of a vector under a linear map only fails to be linearly independent to the original vector in the case that the original vector is an eigenvector $A\vec{\boldsymbol{v}} = c\vec{\boldsymbol{v}}$.*
It is, however, worth noting that a linear map between finite vector spaces may be thought of as the application of a coordinate vector to a different basis than that against which it was originally defined. Therefore, any linear combination of objects may be thought of as a linear map between the space of coefficients and the space of objects. For example, the linear combination,

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$$

may be thought of as a linear map from the space of coefficients α_i to the space of objects x_i,

$$T\left(\begin{bmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_n\end{bmatrix}\right) = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n.$$

Corollary 2.5.1. *As with all homomorphisms, linear maps always map the identity to the identity. For linear maps this means mapping the zero vector to the zero vector.*

Proposition 2.5.1. *A linear map is homogeneous of degree 1.*

Proof. $L(\alpha\vec{\boldsymbol{v}}) = \alpha L(\vec{\boldsymbol{v}})$ for all $\alpha \in \mathbb{F},\ \vec{\boldsymbol{v}} \in V$. □

Proposition 2.5.2. *Linear Dependence is always preserved across any linear transformation.*

Proof. Let $\alpha_1\vec{\boldsymbol{v}}_1 + \alpha_2\vec{\boldsymbol{v}}_2 + \cdots + \alpha_n\vec{\boldsymbol{v}}_n = \vec{\mathbf{0}}$ be a linear relation between the vectors $\{\vec{\boldsymbol{v}}_1, \ldots, \vec{\boldsymbol{v}}_n\}$. If L is a linear map then,

$$\begin{aligned} & L(\alpha_1\vec{\boldsymbol{v}}_1 + \alpha_2\vec{\boldsymbol{v}}_2 + \cdots + \alpha_n\vec{\boldsymbol{v}}_n) = L(\vec{\mathbf{0}}) \\ \iff \qquad & \alpha_1 L(\vec{\boldsymbol{v}}_1) + \alpha_2 L(\vec{\boldsymbol{v}}_2) + \cdots + \alpha_n L(\vec{\boldsymbol{v}}_n) = \vec{\mathbf{0}}. \end{aligned}$$

□

2.5.1.1 The Kernel and the Image of a Linear Transformation

Definition 108. Let $T : V \longmapsto W$ be any linear transformation. Then the **kernel (or nullspace)** of T is defined as,

$$ker\, T = \{\, \vec{\boldsymbol{v}} \mid T(\vec{\boldsymbol{v}}) = \vec{\mathbf{0}} \,\}$$

and the **image** of T as,

$$im\, T = \{\, \vec{\boldsymbol{w}} \in W \mid \exists \vec{\boldsymbol{v}} \in V \,.\, \vec{\boldsymbol{w}} = T(\vec{\boldsymbol{v}}) \,\}.$$

Proposition 2.5.3. *The kernel of $T : V \longmapsto W$ is a subspace of V and the image is a subspace of W.*

Proof. T is a homomorphism between additive groups of vectors and so the proof that the kernel and image are subspaces is the same as for general homomorphisms (see 2.1.4.1). □

Proposition 2.5.4. *The fibres of a linear transformation are the additive cosets of the kernel.*

Proof. Let $T : V \longmapsto W$ be any linear transformation with kernel $K = ker\, T$. Then, for any fixed $\vec{\boldsymbol{v}} \in V$,

$$\forall \vec{\boldsymbol{k}} \in K \,.\, T(\vec{\boldsymbol{v}} + \vec{\boldsymbol{k}}) = T(\vec{\boldsymbol{v}}) + T(\vec{\boldsymbol{k}}) = T(\vec{\boldsymbol{v}}) + \vec{\mathbf{0}} = T(\vec{\boldsymbol{v}}).$$

So every element in the additive coset $\vec{\boldsymbol{v}} + K$ maps to the same value $T(\vec{\boldsymbol{v}})$ in W. Therefore we have,

$$im\, T = \{\, \vec{\boldsymbol{w}} \in W \mid \exists (\vec{\boldsymbol{v}} + K) \subseteq V \,.\, \{\vec{\boldsymbol{w}}\} = T(\vec{\boldsymbol{v}} + K) \,\}.$$

We could also express this using the inverse image as in 2.1.5 but the notation would be easily confused for the inverse transformation. □

Corollary 2.5.2. *Let $T : V \longmapsto W$ be any linear transformation with kernel $K = ker\, T$ and let there be $\vec{\boldsymbol{v}} \in V$, $\vec{\boldsymbol{w}} \in W$ such that*

$$T(\vec{\boldsymbol{v}}) = \vec{\boldsymbol{w}}.$$

Then,

$$T(\vec{\boldsymbol{x}}) = \vec{\boldsymbol{w}} \iff \vec{\boldsymbol{x}} \in (\vec{\boldsymbol{v}} + K).$$

Corollary 2.5.3. *Let $T : V \longmapsto W$ be any linear transformation with kernel $K = \ker T$. T is injective iff the kernel is trivial $K = \{\vec{\mathbf{0}}\}$.*

Proposition 2.5.5. *Linear Independence is preserved across a linear transformation iff the transformation is injective.*

Proof. Let $T : V \longmapsto W$ be any linear transformation with kernel $K = ker\, T$. If the kernel is nontrivial then there exists a nonempty basis of the kernel $B_K = \{\vec{\boldsymbol{k_1}}, \ldots, \vec{\boldsymbol{k_n}}\}$. Being a basis B_K is linearly independent so that,

$$\alpha_1 \vec{\boldsymbol{k_1}} + \cdots + \alpha_n \vec{\boldsymbol{k_n}} = \vec{\mathbf{0}} \iff \alpha_1, \ldots, \alpha_n = 0.$$

However, $T(B_K)$ the image of B_K under T is

$$\{\vec{\mathbf{0}}, \ldots, \vec{\mathbf{0}}\}$$

which, by Proposition 2.4.7, is obviously not linearly independent.

We can also see it more directly from the definition of a linear relation.

$$\begin{aligned} & T(\alpha_1 \vec{\boldsymbol{k_1}} + \cdots + \alpha_n \vec{\boldsymbol{k_n}}) = \vec{\mathbf{0}} \\ \iff \quad & \alpha_1 T(\vec{\boldsymbol{k_1}}) + \cdots + \alpha_n T(\vec{\boldsymbol{k_n}}) = \vec{\mathbf{0}} \\ \iff \quad & \alpha_1 \vec{\mathbf{0}} + \cdots + \alpha_n \vec{\mathbf{0}} = \vec{\mathbf{0}} \end{aligned}$$

So clearly,

$$\alpha_1 T(\vec{\boldsymbol{k_1}}) + \cdots + \alpha_n T(\vec{\boldsymbol{k_n}}) = \vec{\mathbf{0}} \implies \alpha_1, \ldots, \alpha_n = 0$$

which proves that if T is not injective then it does not preserve linear independence.

Conversely, if T does not preserve linear independence then, if $U = \{\vec{\boldsymbol{u}}_1, \ldots, \vec{\boldsymbol{u}}_n\} \subset V$ is a linearly independent set in V, there is a nontrivial linear relation between the vectors in $T(U)$ the image of U under T. That's to say,

$$\alpha_1 T(\vec{\boldsymbol{u}}_1) + \cdots + \alpha_n T(\vec{\boldsymbol{u}}_n) = \vec{\mathbf{0}} \quad \text{where } \prod_{i=1}^{n} \alpha_i \neq 0.$$

But this means that,

$$\begin{aligned} & \alpha_1 T(\vec{\boldsymbol{u}}_1) + \cdots + \alpha_n T(\vec{\boldsymbol{u}}_n) = \vec{\mathbf{0}} \\ \iff \quad & T(\alpha_1 \vec{\boldsymbol{u}}_1 + \cdots + \alpha_n \vec{\boldsymbol{u}}_n) = \vec{\mathbf{0}} \\ \iff \quad & \alpha_1 \vec{\boldsymbol{u}}_1 + \cdots + \alpha_n \vec{\boldsymbol{u}}_n \in ker\, T. \end{aligned}$$

Since U is linearly independent there is no nontrivial linear relation between its elements and since $\prod_{i=1}^{n} \alpha_i \neq 0$ we can conclude that,

$$\alpha_1 \vec{\boldsymbol{u}}_1 + \cdots + \alpha_n \vec{\boldsymbol{u}}_n \neq \vec{\mathbf{0}} \implies ker\, T \neq \{\vec{\mathbf{0}}\}$$

and therefore this proves that if T does not preserve linear independence then it is not injective. □

*Note that **linear dependence**, however, is preserved across any linear map (Proposition 2.5.2).*

2.5.1.2 Examples of Linear Transformations

(52) As previously seen in section 2.3.3, matrix multiplication on the left is a linear transformation. Let A be an $m \times n$ matrix with entries in $\mathbb{F}$ and consider A as an operator on column vectors $A : \mathbb{F}^n \longmapsto \mathbb{F}^m$. The kernel of A is the set of vectors that are solutions to $A\vec{\boldsymbol{x}} = \vec{\mathbf{0}}$ while

the image (or range) is the set of vectors $\vec{\boldsymbol{b}}$ such that $A\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}}$ has a solution.
The solutions $\{\, \vec{\boldsymbol{x}} \in \mathbb{F}^n \mid A\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}} \,\}$ for some fixed $\vec{\boldsymbol{b}} \in \mathbb{F}^m$ are the additive coset $\vec{\boldsymbol{v}} + K$ where K is the kernel of A and $\vec{\boldsymbol{v}} \in \mathbb{F}^n$ is such that $A\vec{\boldsymbol{v}} = \vec{\boldsymbol{b}}$. Compare with 23.

(53) Also previously seen in 2.4.3 is that polynomials can be modeled as vectors. Let P_n be the vector space of real polynomials of degree $\leq n$. Then the derivative is a linear transformation $P_n \longmapsto P_{n-1}$. The kernel of the derivative is the set of degree 0 polynomials (i.e. constant functions) and the additive cosets of the kernel are $f(x) + c$, for $f(x) \in P_n$ and constant c.

2.5.1.3 The Dimension of a Linear Transformation

Definition 109. The dimension of the image is called the **rank** while the dimension of the kernel is known as the **nullity**.

Dimension and rank also exist for Groups (see wikipedia) where it refers to the minimal generating set for the group.

Theorem 2.5.1. ***(The Dimension Formula.)*** *Let $T : V \longmapsto W$ be a linear transformation, and assume that V is finite dimensional. Then,*

$$\dim V = \dim(\ker T) + \dim(\operatorname{im} T) = \operatorname{rank} + \operatorname{nullity}\,.$$

Proof. Let $\{\vec{\boldsymbol{k_1}}, \ldots, \vec{\boldsymbol{k_m}}\}$ be a basis of $\ker T$. Then, by Proposition 2.4.15, it may be extended to a basis of V,

$$B = \{\vec{\boldsymbol{k_1}}, \ldots, \vec{\boldsymbol{k_m}}, \vec{\boldsymbol{u_1}}, \ldots, \vec{\boldsymbol{u_n}}\}.$$

So, for any $\vec{\boldsymbol{v}} \in V$, $\vec{\boldsymbol{v}}$ may be expressed as a linear combination of the vectors in B. Therefore, for any $\vec{\boldsymbol{w}} \in \operatorname{im} T$,

$$\vec{\boldsymbol{w}} = T(\alpha_1\vec{\boldsymbol{k_1}} + \cdots + \alpha_m\vec{\boldsymbol{k_m}} + \beta_1\vec{\boldsymbol{u_1}} + \cdots + \beta_n\vec{\boldsymbol{u_n}})$$

$$
\begin{aligned}
\iff \quad & = T(\alpha_1 \vec{\boldsymbol{k_1}}) + \cdots + T(\alpha_m \vec{\boldsymbol{k_m}}) + T(\beta_1 \vec{\boldsymbol{u_1}}) + \cdots + T(\beta_n \vec{\boldsymbol{u_n}}) \\
\iff \quad & = \vec{\mathbf{0}} + T(\beta_1 \vec{\boldsymbol{u_1}}) + \cdots + T(\beta_n \vec{\boldsymbol{u_n}}) \\
\iff \quad & = \beta_1 T(\vec{\boldsymbol{u_1}}) + \cdots + \beta_n T(\vec{\boldsymbol{u_n}})
\end{aligned}
$$

This shows that $B' = \{T(\vec{\boldsymbol{u_1}}), \ldots, T(\vec{\boldsymbol{u_n}})\}$ spans $im\, T$. Furthermore, if there were a linear relation between the elements of B' then,

$$
\begin{aligned}
& \beta_1 T(\vec{\boldsymbol{u_1}}) + \cdots + \beta_n T(\vec{\boldsymbol{u_n}}) = \vec{\mathbf{0}} \\
\iff \quad & T(\beta_1 \vec{\boldsymbol{u_1}} + \cdots + \beta_n \vec{\boldsymbol{u_n}}) = \vec{\mathbf{0}} \\
\iff \quad & \beta_1 \vec{\boldsymbol{u_1}} + \cdots + \beta_n \vec{\boldsymbol{u_n}} \in ker\, T \\
\iff \quad & \beta_1 \vec{\boldsymbol{u_1}} + \cdots + \beta_n \vec{\boldsymbol{u_n}} = \alpha_1 \vec{\boldsymbol{k_1}} + \cdots + \alpha_m \vec{\boldsymbol{k_m}}
\end{aligned}
$$

where this last result implies a linear relation between the vectors of B. Since B is a basis this linear relation can only be the trivial relation and so $\beta_1, \ldots, \beta_n = 0$ and B' is linearly independent also. □

Notes about the Dimension Formula:

- *The Dimension Formula does not imply that the range of a linear operator and its kernel partition the space (as in a direct sum). The kernel may be in the range. For example, the operator*

  $$T(a, b) = (0, a)$$

  has equal range and nullspace as

  $$R(T) = N(T) = \{\, (0, y) \mid y \in \mathbb{F} \,\}.$$

  In this case $T^2 = T_0$, the zero operator.

- *This formula bears a resemblance to Lagrange's Theorem applied to homomorphisms of finite groups (2.1.9),*

  $$|G| = |ker\, \phi| \cdot |im\, \phi|\,.$$

  The difference, however, is that the Dimension Formula of Linear Transformations is dealing with the generators of a group while

Lagrange's Theorem is dealing with the orders of the groups. The orders of the groups are the number of elements in the group that are generated by the generators of the group. In the case of a real vector space, the vectors generated by the basis vectors are uncountably infinite due to scalar multiplication by real numbers and so cardinality doesn't apply in the same way.

- *This formula* ***only applies to finite-dimensional vector spaces****. This should be clear as we simply cannot do this kind of arithmetic with ∞. For example, if the rank is infinite then the dimension of the kernel would be $\infty - \infty = ?$.*

Theorem 2.5.2. *If $T : V \longmapsto W$ is a linear transformation over a finite-dimensional vector space V, then the quotient space of V by the kernel of T is a space that is in bijective correspondence to the range of T.*

Proof. Let $B_K = \{\vec{\boldsymbol{k}}_1, \ldots, \vec{\boldsymbol{k}}_k\}$ be a basis of the kernel of T. By Proposition 2.4.15, it may be extended to a basis of V,

$$B = \{\vec{\boldsymbol{k}}_1, \ldots, \vec{\boldsymbol{k}}_k, \vec{\boldsymbol{b}}_1, \ldots, \vec{\boldsymbol{b}}_{n-k}\}.$$

Then, for any $\vec{\boldsymbol{v}} \in V$,

$$\begin{aligned} T\vec{\boldsymbol{v}} &= T(\alpha_1\vec{\boldsymbol{k}}_1 + \cdots + \alpha_k\vec{\boldsymbol{k}}_k + \alpha_{k+1}\vec{\boldsymbol{b}}_1 + \cdots + \alpha_n\vec{\boldsymbol{b}}_{n-k}) \\ &= T(\alpha_1\vec{\boldsymbol{k}}_1 + \cdots + \alpha_k\vec{\boldsymbol{k}}_k) + T(\alpha_{k+1}\vec{\boldsymbol{b}}_1 + \cdots + \alpha_n\vec{\boldsymbol{b}}_{n-k}) \\ &= \vec{\boldsymbol{0}} + T(\alpha_{k+1}\vec{\boldsymbol{b}}_1 + \cdots + \alpha_n\vec{\boldsymbol{b}}_{n-k}) \\ &= \alpha_{k+1}T(\vec{\boldsymbol{b}}_1) + \cdots + \alpha_n T(\vec{\boldsymbol{b}}_{n-k}) \end{aligned}$$

Therefore $B_R = \{T(\vec{\boldsymbol{b}}_1), \ldots, T(\vec{\boldsymbol{b}}_{n-k})\}$ spans the range of T. A linear relation between the elements of B_R would imply,

$$\begin{aligned} & \alpha_1 T(\vec{\boldsymbol{b}}_1) + \cdots + \alpha_{n-k}T(\vec{\boldsymbol{b}}_{n-k}) = \vec{\boldsymbol{0}} \\ \iff \quad & T(\alpha_1\vec{\boldsymbol{b}}_1 + \cdots + \alpha_{n-k}\vec{\boldsymbol{b}}_{n-k}) = \vec{\boldsymbol{0}}, \end{aligned}$$

which means that the vector,

$$\alpha_1\vec{\boldsymbol{b}}_1 + \cdots + \alpha_{n-k}\vec{\boldsymbol{b}}_{n-k} \in \ker T$$

which is impossible by construction of the basis B as it would imply a linear relation with the elements of B_K.
Therefore, B_R is linearly independent and a basis of the range (i.e. the image) of T. Furthermore, $B_{K'} = \{\vec{\boldsymbol{b}}_1, \ldots, \vec{\boldsymbol{b}}_{n-k}\}$ is a basis of the quotient space of V by the kernel of T and the map

$$\phi : \operatorname{span} B_{K'} \longmapsto \operatorname{span} B_R = R(T) \;\text{ s.t. }\; \phi(\vec{\boldsymbol{v}}) = T\vec{\boldsymbol{v}}$$

is a bijection because:

- For any $\vec{\boldsymbol{v}} \in R(T)$, there exists some $\beta_1, \ldots, \beta_{n-k}$ such that,

  $$\vec{\boldsymbol{v}} = \beta_1 T(\vec{\boldsymbol{b}}_1) + \cdots + \beta_{n-k} T(\vec{\boldsymbol{b}}_{n-k})) = T(\beta_1 \vec{\boldsymbol{b}}_1 + \cdots + \beta_{n-k}\vec{\boldsymbol{b}}_{n-k})$$

  and, clearly, $\beta_1 \vec{\boldsymbol{b}}_1 + \cdots + \beta_{n-k}\vec{\boldsymbol{b}}_{n-k} \in \operatorname{span} B_{K'}$. Therefore ϕ is surjective.

- For any $\vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2 \in V$ such that $\phi(\vec{\boldsymbol{v}}_1) = \phi(\vec{\boldsymbol{v}}_2)$ we have,

  $$\begin{aligned} & T\vec{\boldsymbol{v}}_1 = T\vec{\boldsymbol{v}}_2 \\ \iff \quad & T(\alpha_1 \vec{\boldsymbol{b}}_1 + \cdots + \alpha_{n-k}\vec{\boldsymbol{b}}_{n-k}) = T(\beta_1 \vec{\boldsymbol{b}}_1 + \cdots + \beta_{n-k}\vec{\boldsymbol{b}}_{n-k}) \\ \iff \quad & \alpha_1 T(\vec{\boldsymbol{b}}_1) + \cdots + \alpha_{n-k} T(\vec{\boldsymbol{b}}_{n-k}) = \beta_1 T(\vec{\boldsymbol{b}}_1) + \cdots + \beta_{n-k} T(\vec{\boldsymbol{b}}_{n-k}) \end{aligned}$$

  which, by the linear independence of B_R implies that $\alpha_i = \beta_i,\ 1 \leq i \leq n-k$. This in turn implies that,

  $$\begin{aligned} & \alpha_1 \vec{\boldsymbol{b}}_1 + \cdots + \alpha_{n-k}\vec{\boldsymbol{b}}_{n-k} = \beta_1 \vec{\boldsymbol{b}}_1 + \cdots + \beta_{n-k}\vec{\boldsymbol{b}}_{n-k} \\ \iff \quad & \vec{\boldsymbol{v}}_1 = \vec{\boldsymbol{v}}_2. \end{aligned}$$

  Therefore ϕ is injective. □

Note that this is the linear algebra version of Theorem 2.1.16 and also a combination of Proposition 2.5.4 and the definition of the vector quotient space 2.4.6.7.

2.5.1.4 The Algebra of Linear Transformations

Notation. If T_1 and T_2 are being used to denote linear maps then T_1T_2 will denote their function composition and any other common operations performed on them (e.g. addition, subtraction, scalar multiplication or division, etc.) will denote a pointwise function definition. That's to say, for example,

$$(T_1 + T_2)\vec{\boldsymbol{v}} = T_1\vec{\boldsymbol{v}} + T_2\vec{\boldsymbol{v}}.$$

Proposition 2.5.6. *Any linear combination of linear maps is a linear map.*

Proof. Let $M = \sum_i \alpha_i T_i$ be a map formed as an arbitrary linear combination of linear maps T_i. Then, following the pointwise definition,

$$\begin{aligned}
M(\beta_1\vec{\boldsymbol{v}}_1 + \beta_2\vec{\boldsymbol{v}}_2) &= \sum_i \alpha_i T_i(\beta_1\vec{\boldsymbol{v}}_1 + \beta_2\vec{\boldsymbol{v}}2) && \text{pointwise defn.} \\
&= \sum_i \alpha_i T_i \beta_1\vec{\boldsymbol{v}}_1 + \alpha_i T_i \beta_2\vec{\boldsymbol{v}}2 && \text{linearity of } T_i \\
&= \sum_i \alpha_i T_i \beta_1\vec{\boldsymbol{v}}_1 + \sum_i \alpha_i T_i \beta_2\vec{\boldsymbol{v}}2 && \text{'+' associative, commutative} \\
&= \beta_1 \sum_i \alpha_i T_i\vec{\boldsymbol{v}}_1 + \beta_2 \sum_i \alpha_i T_i\vec{\boldsymbol{v}}2 && \text{linearity of } T_i \\
&= \beta_1 M(\vec{\boldsymbol{v}}_1) + \beta_2 M(\vec{\boldsymbol{v}}2) && \text{pointwise defn..}
\end{aligned}$$

In fact, we can extend this to any arbitrary linear combination of vectors,

$$
\begin{aligned}
M\left(\sum_j \beta_j \vec{\boldsymbol{v}}_j\right) &= \sum_i \alpha_i T_i \left(\sum_j \beta_j \vec{\boldsymbol{v}}_j\right) && \text{pointwise defn.} \\
&= \sum_i \sum_j \alpha_i T_i \beta_j \vec{\boldsymbol{v}}_j && \text{linearity of } T_i \\
&= \sum_j \sum_i \alpha_i T_i \beta_j \vec{\boldsymbol{v}}_j && \text{'+' associative, commutative} \\
&= \sum_j \beta_j \sum_i \alpha_i T_i \vec{\boldsymbol{v}}_j && \text{linearity of } T_i \\
&= \sum_j \beta_j M(\vec{\boldsymbol{v}}_j) && \text{pointwise defn.}.
\end{aligned}
$$

□

2.5.2 Linear Transformations as Matrices

Proposition 2.5.7. *Left multiplication by a $m \times n$ matrix is a linear transformation $\mathbb{F}^n \longmapsto \mathbb{F}^m$.*

Proof. Let $T : \mathbb{F}^n \longmapsto \mathbb{F}^m$ be a linear transformation. Then T is a map from n-vectors to m-vectors that is compatible with the vector space operations. Let A be an $m \times n$ matrix then $A\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}}$ where $\vec{\boldsymbol{x}} \in \mathbb{F}^n$ and $\vec{\boldsymbol{b}} \in \mathbb{F}^m$ showing that $T(\vec{\boldsymbol{x}}) = A\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}}$ is a map of the form $\mathbb{F}^n \longmapsto \mathbb{F}^m$. Furthermore,

$$T(\vec{\boldsymbol{x_1}} + \vec{\boldsymbol{x_2}}) = A(\vec{\boldsymbol{x_1}} + \vec{\boldsymbol{x_2}}) = A\vec{\boldsymbol{x_1}} + A\vec{\boldsymbol{x_2}} = T(\vec{\boldsymbol{x_1}}) + T(\vec{\boldsymbol{x_2}})$$

and

$$T(c\vec{\boldsymbol{x}}) = A(c\vec{\boldsymbol{x}}) = cA\vec{\boldsymbol{x}} = cT(\vec{\boldsymbol{x}})$$

which shows that left multiplication preserves vector addition and scalar multiplication so that $T(\vec{\boldsymbol{x}}) = A\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}}$ is a linear map as required. □

Theorem 2.5.3. *Every linear transformation $\mathbb{F}^n \longmapsto \mathbb{F}^m$ is left multiplication by a particular $m \times n$ matrix.*

Proof. For any $\vec{\boldsymbol{x}} = \langle x_1, \ldots, x_n \rangle \in \mathbb{F}^n$ we can write it as,

$$x_1\vec{\boldsymbol{e_1}} + \cdots + x_n\vec{\boldsymbol{e_n}}.$$

Therefore if $T : \mathbb{F}^n \longmapsto \mathbb{F}^m$ then,

$$T(\vec{\boldsymbol{x}}) = T(x_1\vec{\boldsymbol{e_1}} + \cdots + x_n\vec{\boldsymbol{e_n}}) = T(\vec{\boldsymbol{e_1}})x_1 + \cdots + T(\vec{\boldsymbol{e_n}})x_n \in \mathbb{F}^m$$

and so letting $A \in \mathbb{F}^{m \times n}$ be,

$$A = \begin{bmatrix} T(\vec{\boldsymbol{e_1}}) & \cdots & T(\vec{\boldsymbol{e_n}}) \end{bmatrix}$$

we have,

$$T(\vec{\boldsymbol{x}}) = A\vec{\boldsymbol{x}}.$$ □

Corollary 2.5.4. *Any linear transformation between spaces isomorphic to $\mathbb{F}^n$ and $\mathbb{F}^m$ (refer to Proposition 2.4.24 and 47) is left multiplication by a particular $m \times n$ matrix.*

This is why linear transformations from a space to itself can be wholly characterized by what they do to the axes and also why every such linear transformation can be considered a change of basis and vice-versa.
Conceptually, a linear transformation changes the coordinates of a selection of transformed vectors. The confusion comes about because we implement the matrix of the linear transformation A by transforming the basis against which the coordinates are applied,

$$A = \begin{bmatrix} T(\vec{\boldsymbol{e_1}}) & \cdots & T(\vec{\boldsymbol{e_n}}) \end{bmatrix}.$$

Whereas a change of basis transforms the basis against which coordinates are applied ***and then updates the coordinates to balance out the change****. This can be seen if we deconstruct the change of basis formula:*

$$B\vec{\boldsymbol{x}}_B = B'\vec{\boldsymbol{x}}_{B'} \iff \vec{\boldsymbol{x}}_{B'} = (B')^{-1}B\vec{\boldsymbol{x}}_B$$

$$\vec{\boldsymbol{x}}_{B'} = P\vec{\boldsymbol{x}}_B = (B')^{-1}B\vec{\boldsymbol{x}}_B$$

This can be thought of as first obtaining the coordinates against the standard basis $B\vec{\boldsymbol{x}}_B$ and then applying the inverse of the target basis so as to obtain the equivalent coordinates against the target basis. But, note, we could also consider the whole thing as a linear transformation represented by P.

The biggest difference, however, is that a linear transformation can also be between different spaces — say from n-dimensional space to m-dimensional space — in which case it cannot be thought of as a change of basis as a vector $\vec{\boldsymbol{v}} \in \mathbb{F}^n$ cannot be equivalently expressed using a basis of $\mathbb{F}^m$ because the two spaces are not isomorphic.

2.5.2.1 Examples of Linear Transformations as Matrices

(54) Let $T : \mathbb{R}^2 \longmapsto \mathbb{R}^2$ be a linear transformation such that,

$$T(\vec{\boldsymbol{e_1}}) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad T(\vec{\boldsymbol{e_2}}) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

The transformation T has been completely described in this way because, for any $\vec{\boldsymbol{x}} = \langle x_1, x_2 \rangle \in \mathbb{R}^2$ we have,

$$T(\vec{\boldsymbol{x}}) = T(x_1\vec{\boldsymbol{e_1}} + x_2\vec{\boldsymbol{e_2}})$$

$$\iff \qquad T(\vec{\boldsymbol{x}}) = x_1 T(\vec{\boldsymbol{e_1}}) + x_2 T(\vec{\boldsymbol{e_2}})$$
$$\iff \qquad T(\vec{\boldsymbol{x}}) = x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ 2x_1 \end{bmatrix}.$$

So, T is also left multiplication by the matrix,

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}.$$

(55) Consider a linear map $T : \mathbb{R}^n \longmapsto \mathbb{R}^m$ and the matrix representing it $A \in \mathbb{R}^{mn}$. Suppose the equation $A\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}}$ has the known solution

$$\vec{\boldsymbol{x}} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \qquad s, t \in \mathbb{R}.$$

What can be said about the linear transformation T from looking at the solution $\vec{\boldsymbol{x}}$?

- The dimension of the domain, $n = 5$, is clear since the solution $\vec{\boldsymbol{x}}$ is a vector in the domain space.
- The nullity - dimension of the kernel - is 2, since there are two free variables s, t. The basis of the 2-dimensional nullspace is the two vectors being multiplied by these variables.
- Using the dimension formula (Theorem 2.5.1) we can deduce that the rank of T is n - nullity. So the rank is $5 - 2 = 3$. This is also supported by the fact that the particular solution has 3 non-zero components.

What can **not** be said?

- The dimension of the codomain m cannot be derived from looking at the solution $\vec{\boldsymbol{x}}$. The dimension of the image of T is given by its range because the kernel maps to the origin in the codomain and so the image of the kernel has dimension zero. Since we are not told whether or not the linear map is surjective we cannot know the dimension of the codomain space - only the image of T in the codomain space.

(56) The minimal linear transformation is multiplication by a minimal matrix — a one-by-one matrix, i.e. a scalar. So

$$T(\vec{\boldsymbol{v}}) = A\vec{\boldsymbol{v}} = a\vec{\boldsymbol{v}}.$$

Since the real number line, for example, may be considered a one-dimensional vector space, multiplication of two real numbers, for example, may be considered as a linear transformation.

2.5.2.2 Linear Transformations and Change of Basis

It is often possible to achieve powerful simplifications of problems by selecting appropriate bases. In this section we will look at linear transformations represented by matrices between arbitrary bases of spaces. So here we are looking at the relationship between linear transformations and change of basis.

Definition 110. If the matrix A of a linear transformation $T : V \longmapsto W$ is defined as, for $\vec{\boldsymbol{x}} \in V$,

$$T(\vec{\boldsymbol{x}}) = A\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}} \in W$$

then **the matrix of T with respect to the bases $B \subset V$ and $B' \subset W$** is defined as the matrix A that satisfies,

$$A\vec{\boldsymbol{x}}_B = \vec{\boldsymbol{b}}_{B'} \in W$$

and also

$$T(\vec{\boldsymbol{x}}) = [B']A\vec{\boldsymbol{x}}_B = \vec{\boldsymbol{b}}$$

2.5.2.3 Intuition of the Matrix of T with respect to Bases

Let $T : V \longmapsto W$ be a linear transformation and let $B_V = \{\vec{\boldsymbol{v_1}}, \ldots, \vec{\boldsymbol{v_n}}\}$ be a basis of V and $B_W = \{\vec{\boldsymbol{w_1}}, \ldots, \vec{\boldsymbol{w_m}}\}$ be a basis of W. Then $T(\vec{\boldsymbol{v_i}}) \in W$ and so there is some $m \times n$ matrix $A = (a_{ij})$ such that,

$$\begin{bmatrix} \vec{\boldsymbol{w_1}} & \cdots & \vec{\boldsymbol{w_m}} \end{bmatrix} A = \begin{bmatrix} T(\vec{\boldsymbol{v_1}}) & \cdots & T(\vec{\boldsymbol{v_n}}) \end{bmatrix}.$$

where,

$$T(\vec{\boldsymbol{v_j}}) = \begin{bmatrix} \vec{\boldsymbol{w_1}} & \cdots & \vec{\boldsymbol{w_m}} \end{bmatrix} \begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix} = a_{1j}\vec{\boldsymbol{w_1}} + \cdots + a_{mj}\vec{\boldsymbol{w_m}} = \sum_i a_{ij}\vec{\boldsymbol{w_i}}$$

so that the jth column of the matrix A is the coordinate vector of $T(\vec{\boldsymbol{v_j}})$ with respect to the basis B_W.

Substituting $B_W = \{\vec{\boldsymbol{w_1}}, \ldots, \vec{\boldsymbol{w_m}}\}$ we can obtain an expression for A,

$$\begin{aligned} & \begin{bmatrix} \vec{\boldsymbol{w_1}} & \cdots & \vec{\boldsymbol{w_m}} \end{bmatrix} A = \begin{bmatrix} T(\vec{\boldsymbol{v_1}}) & \cdots & T(\vec{\boldsymbol{v_n}}) \end{bmatrix} \\ \iff \quad & [B_W]A = \begin{bmatrix} T(\vec{\boldsymbol{v_1}}) & \cdots & T(\vec{\boldsymbol{v_n}}) \end{bmatrix} \\ \iff \quad & A = [B_W]^{-1} \begin{bmatrix} T(\vec{\boldsymbol{v_1}}) & \cdots & T(\vec{\boldsymbol{v_n}}) \end{bmatrix}. \end{aligned}$$

The matrix A is referred to as the **matrix of T with respect to the bases B_V and B_W** and conforms to,

$$A\vec{\boldsymbol{x}}_{B_V} = \vec{\boldsymbol{b}}_{B_W}.$$

This can be seen as,

$$\begin{aligned} & A\vec{\boldsymbol{x}}_{B_V} = [B_W]^{-1} \begin{bmatrix} T(\vec{\boldsymbol{v_1}}) & \cdots & T(\vec{\boldsymbol{v_n}}) \end{bmatrix} \vec{\boldsymbol{x}}_{B_V} \\ \iff \quad & A\vec{\boldsymbol{x}}_{B_V} = [B_W]^{-1}\vec{\boldsymbol{b}}_{B_V} \\ \iff \quad & A\vec{\boldsymbol{x}}_{B_V} = \vec{\boldsymbol{b}}_{B_W} \end{aligned}$$

so that $\vec{\boldsymbol{x}}_{B_V}$ — the coordinate vector with respect to B_V — is first transformed by applying the coordinates to the transformed version of the basis B_V and then these coordinates are converted to B_W coordinates by left multiplication by $[B_W]^{-1}$.

If we chose different bases for the spaces we would get a different matrix. If the bases are the standard bases then the matrix is the standard matrix for the transformation.

2.5.2.4 Examples of Linear Transform Matrices w.r.t. Bases

(57) Let $T : \mathbb{R}^2 \longmapsto \mathbb{R}^2$ be a linear transform defined (against the standard basis) by,

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}2\\3\end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}3\\2\end{bmatrix}.$$

Then, if we define the matrix of T with respect to the standard basis only then we have,

$$A = \begin{bmatrix} T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) & T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) \end{bmatrix} = \begin{bmatrix}2 & 3\\3 & 2\end{bmatrix}$$

and, if we define a vector $\vec{\boldsymbol{x}} = \langle 1, 1\rangle$ against the standard basis we can see that,

$$T(\vec{\boldsymbol{x}}) = 1 \cdot T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + 1 \cdot T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = A\vec{\boldsymbol{x}} = \begin{bmatrix}2 & 3\\3 & 2\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}5\\5\end{bmatrix}.$$

If we now define B, an alternative basis of $\mathbb{R}^2$, as

$$B = \left\{\begin{bmatrix}3\\0\end{bmatrix}, \begin{bmatrix}0\\2\end{bmatrix}\right\}$$

then we have,

$$[B] = \begin{bmatrix}3 & 0\\0 & 2\end{bmatrix} \quad \text{and} \quad [B]^{-1} = \frac{1}{6}\begin{bmatrix}2 & 0\\0 & 3\end{bmatrix}$$

and the linear transform matrix **of coordinate vectors w.r.t. the basis** B is defined as,

$$A = \begin{bmatrix} T\left(\begin{bmatrix}3\\0\end{bmatrix}\right) & T\left(\begin{bmatrix}0\\2\end{bmatrix}\right) \end{bmatrix} = \begin{bmatrix}6 & 6\\9 & 4\end{bmatrix}.$$

Now *this* matrix A expects coordinate vectors w.r.t. B and so if we convert $\vec{\boldsymbol{x}}$ to basis B as follows,

$$\vec{\boldsymbol{x}}_B = [B]^{-1}\vec{\boldsymbol{x}} = \frac{1}{6}\begin{bmatrix}2 & 0\\ 0 & 3\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1/3\\1/2\end{bmatrix}$$

then we find that,

$$T(\vec{\boldsymbol{x}}) = A\vec{\boldsymbol{x}}_B = \begin{bmatrix}6 & 6\\ 9 & 4\end{bmatrix}\begin{bmatrix}1/3\\1/2\end{bmatrix} = \begin{bmatrix}6/3+6/2\\9/3+4/2\end{bmatrix} = \begin{bmatrix}5\\5\end{bmatrix}.$$

If we were to define another basis of $\mathbb{R}^2$ called B' and construct the matrix of T with respect to B and B' then the matrix A would become,

$$A = [B']^{-1}\begin{bmatrix}T\left(\begin{bmatrix}3\\0\end{bmatrix}\right) & T\left(\begin{bmatrix}0\\2\end{bmatrix}\right)\end{bmatrix}$$

and to get the result in standard coordinates we would need to apply the result to the basis vectors of B',

$$T(\vec{\boldsymbol{x}}) = [B']A\vec{\boldsymbol{x}}_B.$$

In the case where we want the result to be in the same basis as the argument vector $\vec{\boldsymbol{x}}_B$ we still need to modify the matrix A. In this case A becomes,

$$A = [B]^{-1}\begin{bmatrix}T\left(\begin{bmatrix}3\\0\end{bmatrix}\right) & T\left(\begin{bmatrix}0\\2\end{bmatrix}\right)\end{bmatrix}$$

This A still expects $\vec{\boldsymbol{x}}$ to be in B coordinates but outputs a result defined in B coordinates rather than standard coordinates.

$$A\vec{\boldsymbol{x}}_B = \frac{1}{6}\begin{bmatrix}2 & 0\\ 0 & 3\end{bmatrix}\begin{bmatrix}6 & 6\\ 9 & 4\end{bmatrix}\vec{\boldsymbol{x}}_B = \begin{bmatrix}2 & 2\\ 9/2 & 2\end{bmatrix}\begin{bmatrix}1/3\\1/2\end{bmatrix} = \begin{bmatrix}5/3\\5/2\end{bmatrix}.$$

So, to get the result in standard coordinates we need to apply the result to the basis vectors of B',

$$T(\vec{\boldsymbol{x}}) = [B](A\vec{\boldsymbol{x}}_B) = \begin{bmatrix}3 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}5/3\\5/2\end{bmatrix} = \begin{bmatrix}5\\5\end{bmatrix}.$$

Proposition 2.5.8. *Let A be the matrix of a linear transformation $T : V \longmapsto W$ with respect to the bases B_V, B_W of dimension n and m respectively. The matrices A' which represent T with respect to other bases are those of the form,*

$$A' = QAP^{-1}$$

where $Q \in GL_m(\mathbb{F}), P \in GL_n(\mathbb{F})$.

Proof. Let $B'_V = \{\vec{\boldsymbol{v}}'_1, \ldots, \vec{\boldsymbol{v}}'_n\}, B'_W = \{\vec{\boldsymbol{w}}'_1, \ldots, \vec{\boldsymbol{w}}'_n\}$ be alternative bases with respect to which we want to find A', the matrix of T. Also let,

$$B_V = B'_V P \quad \text{and} \quad B_W = B'_W Q.$$

Then for $\vec{\boldsymbol{v}}'_i \in B'_V$, $T(\vec{\boldsymbol{v}}') \in \mathit{span}\, B'_W$ so there exists a matrix A' such that, using the notation $T(B_V)$ to indicate the image of the set B_V under T and $[B_V]$ for the matrix whose columns are the elements of B_V,

$$\begin{aligned}
& \begin{bmatrix} \vec{\boldsymbol{w}}'_1 & \cdots & \vec{\boldsymbol{w}}'_m \end{bmatrix} A' = \begin{bmatrix} T(\vec{\boldsymbol{v}}'_1) & \cdots & T(\vec{\boldsymbol{v}}'_n) \end{bmatrix} \\
\iff \quad & [B'_W]A' = [T(B'_V)] \\
\iff \quad & A' = [B'_W]^{-1}[T(B'_V)] \\
\iff \quad & A' = [B'_W]^{-1}[T(B_V P^{-1})] \\
\iff \quad & A' = Q[B_W]^{-1}[T(B_V)]P^{-1} \qquad \text{P is coefficient matrix} \\
\iff \quad & A' = QAP^{-1}. \qquad \square
\end{aligned}$$

2.5.2.5 Simplification of the Matrix of a Transformation

Proposition 2.5.9. *Let $T : V \longmapsto W$ be a linear transformation of rank r. Bases B_V, B_W may be chosen so that the matrix of T takes the form,*

$$A = \begin{bmatrix} I_r & \vdots \\ \cdots & 0 \end{bmatrix}.$$

Proof. Let $U = \{\vec{\boldsymbol{u}}_1, \ldots, \vec{\boldsymbol{u}}_k\}$ be a basis of the kernel of T where $k = dim(ker\, T)$. Then U may be extended to a basis of V (Proposition 2.4.15),

$$B_V = \{\vec{\boldsymbol{v}}_1, \ldots, \vec{\boldsymbol{v}}_r, \vec{\boldsymbol{u}}_1, \ldots, \vec{\boldsymbol{u}}_k\}.$$

Then, let $T(\vec{\boldsymbol{v}}_i) = \vec{\boldsymbol{w}}_i$ so that,

$$[T(B_V)] = \begin{bmatrix} \vec{\boldsymbol{w}}_1 & \cdots & \vec{\boldsymbol{w}}_r & \vec{\mathbf{0}} & \cdots & \vec{\mathbf{0}} \end{bmatrix}.$$

As shown in Theorem 2.5.1, $\{\vec{\boldsymbol{w}}_1, \ldots, \vec{\boldsymbol{w}}_r\}$ is a basis of the image of T and can also be extended to a basis of W,

$$B_W = \{\vec{\boldsymbol{w}}_1, \ldots, \vec{\boldsymbol{w}}_r, \vec{\boldsymbol{x}}_1, \ldots, \vec{\boldsymbol{x}}_{m-r}\}.$$

So, the matrix A of T with respect to the bases B_V, B_W satisfies,

$$\begin{aligned} & & A &= [B_W]^{-1}[T(B_V)] \\ \iff & & [B_W]A &= [T(B_V)] \\ \iff & \begin{bmatrix} \vec{\boldsymbol{w}}_1 & \ldots & \vec{\boldsymbol{w}}_r & \vec{\boldsymbol{x}}_1 & \ldots & \vec{\boldsymbol{x}}_{m-r} \end{bmatrix} A &= \begin{bmatrix} \vec{\boldsymbol{w}}_1 & \ldots & \vec{\boldsymbol{w}}_r & \vec{\mathbf{0}} & \ldots & \vec{\mathbf{0}} \end{bmatrix}. \end{aligned}$$

If we look at the components of the matrices we see,

$$\begin{bmatrix} w_{11}\ldots & w_{1r} & x_{11}\ldots & x_{1(m-r)} \\ \vdots & & & \\ w_{r1}\ldots & w_{rr} & x_{r1}\ldots & x_{r(m-r)} \\ \vdots & & & \\ w_{m1}\ldots & w_{mr} & x_{m1}\ldots & x_{m(m-r)} \end{bmatrix} \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \\ a_{r1} & \ldots & a_{rn} \\ \vdots & & \\ a_{m1} & \ldots & a_{mn} \end{bmatrix} = \begin{bmatrix} w_{11}\ldots & w_{1r} & 0\ldots & 0 \\ \vdots & & & \\ w_{r1}\ldots & w_{rr} & 0\ldots & 0 \\ \vdots & & & \\ w_{m1}\ldots & w_{mr} & 0\ldots & 0 \end{bmatrix}$$

which makes it clear that A has the form,

$$A = \begin{bmatrix} 1 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & & & \\ 0 & \ldots & 1 & 0 & \ldots & 0 \\ \vdots & & & & & \\ 0 & \ldots & 0 & 0 & \ldots & 0 \end{bmatrix}$$

as required. □

Corollary 2.5.5. *By Proposition 2.5.7, left multiplication by any matrix is a linear transformation and so is equivalent to left multiplication by a matrix of the form*

$$\begin{bmatrix} I_r & \vdots \\ \cdots & 0 \end{bmatrix}$$

but with reference to different coordinate systems.

2.5.2.6 Example of Simplification by Selecting Bases

(58) Continuing the example 43 of Gaussian Elimination for row reducing a matrix we have a matrix

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 2 & -4 \\ 3 & 2 & -1 \end{bmatrix}$$

whose rref form is

$$A' = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}.$$

The rref tells us that the first two columns of A are a basis of the image and the kernel is

$$c \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \qquad c \in \mathbb{R}.$$

We can use this information to form a matrix A_{PQ} — which is the matrix A expressed with respect to the bases P and Q — such that A_{PQ} is maximally simplified. Following the procedure used in the proof of Proposition 2.5.9, we begin by finding a basis of the domain space by extending a basis of the kernel.

Since the kernel is one-dimensional and the domain is three-dimensional, we can extend the basis of the kernel given above to a basis of the domain by adding two of the three standard basis vectors. So, the chosen basis of the domain is

$$P = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

Again following the procedure from the same proof, we next form a basis of the codomain space that extends a basis of the image which, in this case, is the first two columns of the matrix A. So, we can

choose the basis,

$$Q = \left\{ \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

We could have chosen anything for the final column just so long as it is linearly independent of the first two columns.

Then,

$$\begin{aligned} A_{PQ} &= Q^{-1}AP \\ &= \begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & 0 \\ 3 & 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 1 & 1 \\ 0 & 2 & -4 \\ 3 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \frac{1}{6} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3 & 0 \\ -6 & -3 & 6 \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ 0 & 2 & -4 \\ 3 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \frac{1}{6} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3 & 0 \\ -6 & -3 & 6 \end{bmatrix} \begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & 0 \\ 3 & 2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

2.5.3 Linear Operators and Eigenvectors

Definition 111. A **linear operator** is a linear transformation from a space to itself. That's to say, the domain and codomain of the transformation are the same space and considered with respect to the same basis.

Definition 112. A linear operator is called **singular** if it does not have an inverse and **nonsingular** if it has an inverse.

Definition 113. If T is a linear operator on a vector space (finite or infinite) over a field $\mathbb{F}$ and $p(x)$ is a polynomial over $\mathbb{F}$,

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n$$

then we can define the **polynomial of the linear operator** T as

$$p(T) = a_0 I + a_1 T + \cdots + a_n T^n.$$

Since any power of the operator T^n is also a linear operator and the polynomial is a linear combination of these, by Proposition 2.5.6, **the polynomial $p(T)$ is also a linear operator**.

Similarly for a square matrix A,

$$p(A) = a_0 I + a_1 A + \cdots + a_n A^n.$$

To show the validity of polynomials of linear operators:

Let T be an arbitrary linear map over a vector space V defined over a field $\mathbb{F}$. Define $T + a$ as the linear map $T + aI$ and similarly $T - b = T - bI$. Lastly, define $(T + a)(T - b)$ as the composition of the linear maps,

$$(T + aI) \circ (T - bI).$$

Then,

$$\begin{aligned}(T+a)(T-b) &= T(T-b)+a(T-b) && \textit{pointwise definition}\\ &= TT-Tb+aT-ab && \textit{linearity}\\ &= T^2+(a-b)T-ab.\end{aligned}$$

In fact, polynomials over a field such as the reals can be viewed as linear combinations of a linear map – the simplest linear map: scalar multiplication.

If we regard the real variable x as the function "multiply by x for any x in the field" $f_1(t)=xt$, then $x+a$ is the linear combination of this function with the function "multiply by the constant a" (defined pointwise) which results in: $f_2(t)=(x+a)t=xt+at$. Then

$$\begin{aligned}(x+a)^2 &= (x+a)(x+a)\\ &= (x+a)f_2(t)\\ &= (x+a)(xt+at)\\ &= x^2t+axt+axt+a^2t\\ &= x^2t+2axt+a^2t\\ &= (x^2+2ax+a^2)t.\end{aligned}$$

A full treatment of this topic requires Modules (wikipedia).

Proposition 2.5.10. *A linear operator $T:V\longmapsto V$ is bijective iff it has a trivial kernel.*

Proof. By Theorem 2.1.13 of homomorphisms, if T has a trivial kernel then it is injective which implies that $|im\,T|=|V|$ so that it is also surjective (because the domain is equal to the codomain). Therefore, it is bijective. Conversely, if it is bijective then it is injective and so it has a trivial kernel. □

Corollary 2.5.6. *A linear operator $T:V\longmapsto V$ is isomorphic iff it has a trivial kernel.*

Corollary 2.5.7. *A linear operator is nonsingular iff it has a trivial kernel.*

Note that these are not true of linear transformations in general because, for a transformation between different vector spaces of different dimensions, injectivity does not imply bijectivity and invertibility.

Proposition 2.5.11. *The following conditions on a linear operator $T : V \longmapsto V$ on a finite-dimensional vector space are equivalent:*

(i) $\ker T > 0$.

(ii) $\operatorname{im} T < V$.

(iii) If A is the matrix of the operator with respect to an arbitrary basis, then $\det A = 0$.

(iv) T is singular

Proof. The first two of these properties follow from the dimension formula for finite-dimensional vector spaces. The third follows since an operator with a non-trivial kernel is noninvertible (Proposition 2.5.10) and so its matrix will also be noninvertible; noninvertible matrices have determinant 0. The last property follows from the definition of singular and the fact that T is noninvertible. □

Again it's worth noting the contrast with transformations in general: for a transformation whose domain vector space is lower dimension than its codomain, the transformation may be injective but not surjective (kernel is trivial but $\operatorname{im} T < V$), and conversely, if the domain is higher dimension than the codomain then the transformation may be surjective while not injective ($\ker T > 0$ but image covers codomain). Furthermore, the third condition relating to the determinant, only applies to operators because the determinant is a property of square matrices.

Note that the first two properties do not hold for infinite-dimensional vector spaces. For example, let $V = \mathbb{R}^\infty$ be the space of sequences $a_1, a_2, \ldots$. Then the "shift operator" is defined as,

$$T(a_1, a_2, \ldots) = (0, a_1, a_2, \ldots).$$

This is a linear operator because,

$$\begin{aligned}\alpha T(a_1, a_2, \dots) + \beta T(b_1, b_2, \dots) &= \alpha(0, a_1, a_2, \dots) + \beta(0, b_1, b_2, \dots) \\ &= (0, \alpha a_1, \alpha a_2, \dots) + (0, \beta b_1, \beta b_2, \dots) \\ &= (0, \alpha a_1 + \beta b_1, \alpha a_2 + \beta b_2, \dots) \\ &= T(\alpha a_1 + \beta b_1, \alpha a_2 + \beta b_2, \dots).\end{aligned}$$

However, while clearly for this operator we have $\operatorname{im} T < V$, nevertheless the kernel of this operator is the trivial kernel $\{\vec{\mathbf{0}}\}$. This just shows further that the dimension formula (Theorem 2.5.1) for finite-dimensional vector spaces does not apply for infinite-dimensional spaces.
The explanation of this is that infinite sets can have proper subsets that have equal cardinality. So we can have an image whose dimensions are a proper subset of the dimensions of the space but the image, nevertheless, has the same dimensionality as the space. This can only happen with infinite-dimensional spaces.

2.5.3.1 Invariant Subspaces

Definition 114. Let $T : V \longmapsto V$ be a linear operator on a vector space. A subspace W of V is called an **invariant subspace** or a **T-invariant subspace** if it is carried to itself by the operator,

$$TW \subseteq W.$$

In other words, W is T-invariant if $T(\vec{\boldsymbol{w}}) \in W$ for all $\vec{\boldsymbol{w}} \in W$. In this case, T may be referred to as defining an operator on W that is the **restriction of T to W**.

Notation. Denote the restriction of T to W by T_w.

Proposition 2.5.12. *If T is a linear operator over a vector space V with range $R(T)$ and kernel/nullspace $N(T)$ then the following subspaces of V*

are all T-invariant:

(i) V

(ii) $\{\vec{\mathbf{0}}\}$

(iii) $R(T)$

(iv) $N(T)$

Proof.

(i) V
Since T is a linear operator, $T : V \longmapsto V$, we have, for all $\vec{\boldsymbol{v}} \in V$, $T\vec{\boldsymbol{v}} \in V$.

(ii) $\{\vec{\mathbf{0}}\}$
Any linear transformation, by homogeneity (Corollary 2.5.1), maps the origin to itself.

(iii) $R(T)$
By the definition of the range of T we must have for all $\vec{\boldsymbol{v}} \in V$, $T\vec{\boldsymbol{v}} \in R(T)$ and since T is a linear operator, $T : V \longmapsto V$,

$$R(T) \subseteq V \implies \forall \vec{\boldsymbol{r}} \in R(T),\ T\vec{\boldsymbol{r}} \in R(T).$$

(iv) $N(T)$
By the definition of the nullspace/kernel of T we have: for all $\vec{\boldsymbol{v}} \in N(T)$, $T\vec{\boldsymbol{v}} = \vec{\mathbf{0}}$. Since the origin is a member of all vector spaces, this implies that also for all $\vec{\boldsymbol{v}} \in N(T)$, $T\vec{\boldsymbol{v}} \in N(T)$.

□

Proposition 2.5.13. *Let T be an operator over a vector space V and W a subspace of V such that $V = R(T) \oplus W$ where $R(T)$ denotes the range of T. If W is T-invariant then $W \subseteq N(T)$ where $N(T)$ denotes the nullspace of T. Furthermore, if V is finite-dimensional then $W = N(T)$.*

Proof. Let $U = R(T)$ so that $V = U \oplus W$ and for all $\vec{\boldsymbol{v}} \in V$, there exists $\vec{\boldsymbol{u}} \in U, \vec{\boldsymbol{w}} \in W$ such that

$$\vec{\boldsymbol{v}} = \vec{\boldsymbol{u}} + \vec{\boldsymbol{w}} \quad \text{and} \quad \vec{\boldsymbol{u}} + \vec{\boldsymbol{w}} = \vec{\mathbf{0}} \implies \vec{\boldsymbol{u}} = \vec{\boldsymbol{w}} = \vec{\mathbf{0}}.$$

Since W is T-invariant, for any $\vec{\boldsymbol{w}} \in W$

$$T\vec{\boldsymbol{w}} \in W.$$

But $U = R(T)$ is the range of T, so it must also be that

$$T\vec{\boldsymbol{w}} \in U.$$

Therefore, there exists some $\vec{\boldsymbol{w}}_1 \in W$, $\vec{\boldsymbol{u}}_1 \in U$ such that

$$T\vec{\boldsymbol{w}} = \vec{\boldsymbol{w}}_1 = \vec{\boldsymbol{u}}_1.$$

Since,

$$\vec{\boldsymbol{w}}_1 = \vec{\boldsymbol{u}}_1 \iff \vec{\boldsymbol{u}}_1 - \vec{\boldsymbol{w}}_1 = \vec{\boldsymbol{0}} \implies \vec{\boldsymbol{w}}_1 = \vec{\boldsymbol{u}}_1 = \vec{\boldsymbol{0}}$$

we obtain the result that, for all $\vec{\boldsymbol{w}} \in W$,

$$T\vec{\boldsymbol{w}} = \vec{\boldsymbol{0}}.$$

This implies that $W \subseteq N(T)$.

Furthermore, if V is finite-dimensional, the dimension formula for linear operators over finite-dimensional vector spaces (Theorem 2.5.1) tells us that,

$$\dim R(T) + \dim N(T) = \dim V \iff \dim N(T) = \dim V - \dim R(T).$$

So we have,

$$\dim N(T) = \dim V - \dim U$$

but also

$$V = U \oplus W \implies \dim V = \dim U + \dim W \iff \dim V - \dim U = \dim W$$

which gives us

$$\dim W = \dim N(T).$$

We can therefore deduce,

$$[\, W \subseteq N(T) \,] \wedge [\, \dim W = \dim N(T) \,] \implies W = N(T).$$

□

Examples of Invariant Subspaces

(59) Let $T : \mathbb{R}^3 \longmapsto \mathbb{R}^3$ be defined by,

$$T(a, b, c) = (a + b,\, b + c,\, 0).$$

Then the xy-plane $\{\, (x, y, 0) \mid x, y \in \mathbb{R} \,\}$ and the x-axis $\{\, (x, 0, 0) \mid x \in \mathbb{R} \,\}$ are T-invariant subspaces of $\mathbb{R}^3$.

Invariant Subspaces as Matrix Blocks

Let $W \subseteq V$ be a T-invariant subspace of V with basis $B_W = \vec{\boldsymbol{w}}_1, \ldots, \vec{\boldsymbol{w}}_k$. Then, by Proposition 2.4.15, B_W can be extended to a basis of V,

$$B_V = \{\vec{\boldsymbol{w}}_1, \ldots, \vec{\boldsymbol{w}}_k, \vec{\boldsymbol{v}}_1, \ldots, \vec{\boldsymbol{v}}_{n-k}\}.$$

If we look at the matrix A of T with respect to this basis B_V we find a characteristic pattern.

$$A = [B_V]^{-1}[T(B_V)] = [B_V]^{-1}[T(\vec{\boldsymbol{w}}_1) \cdots T(\vec{\boldsymbol{w}}_k)\; T(\vec{\boldsymbol{v}}_1) \cdots T(\vec{\boldsymbol{v}}_{n-k})]$$

with $T(\vec{\boldsymbol{w}}_1), \ldots, T(\vec{\boldsymbol{w}}_k) \in W$ so that,

$$T(\vec{\boldsymbol{w}}_i) = \alpha_1 \vec{\boldsymbol{w}}_1 + \cdots + \alpha_k \vec{\boldsymbol{w}}_k.$$

This means that when we express $T(\vec{\boldsymbol{w}}_i)$ with respect to the basis B_V the coordinate vectors will take the form,

$$(\alpha_1, \ldots, \alpha_k, 0, \ldots, 0)^T.$$

As a result, the matrix A will take the form,

$$A = \begin{bmatrix} C & D \\ 0 & E \end{bmatrix}$$

where C is a $k \times k$ matrix that represents the restriction of T to W.

A description of a matrix of the form of the description of A here is known as a ***block decomposition****.*

On the other hand, if $V = W_1 \oplus W_2$ where *both* W_1 *and* W_2 are T-invariant subspaces then A takes the form,

$$A = \begin{bmatrix} C & 0 \\ 0 & E \end{bmatrix}$$

where, as before, C is a $k \times k$ matrix that represents the restriction of T to W_1 but, this time, also E is a $(n-k) \times (n-k)$ matrix that represents the restriction of T to W_2.

Matrices with the form of the matrix,

$$\begin{bmatrix} C & 0 \\ 0 & E \end{bmatrix}$$

are known as ***block diagonal matrices*** *or* ***diagonal block matrices****.*

Cyclic Subspaces

Definition 115. Let T be a linear operator on a vector space V and let $\vec{\boldsymbol{v}} \neq \vec{\mathbf{0}} \in V$. Then the subspace,

$$W = \text{span}\{\vec{\boldsymbol{v}},\, T\vec{\boldsymbol{v}},\, T^2\,\vec{\boldsymbol{v}}, \dots\}$$

is called the **T-cyclic subspace of V generated by $\vec{\boldsymbol{v}}$**.
The T-cyclic subspace of V generated by any element of V, is T-invariant by construction.

refer: cyclic subgroups in Group Theory 2.1.2.5

Proposition 2.5.14. *The T-cyclic subspace of V generated by $\vec{\boldsymbol{v}} \in V$ is the smallest T-invariant subspace of V containing $\vec{\boldsymbol{v}}$.*

Proof. Let W be the T-cyclic subspace of V generated by $\vec{\boldsymbol{v}} \in V$.
Clearly, from the definition, W contains $\vec{\boldsymbol{v}}$ so W is a subspace containing $\vec{\boldsymbol{v}}$.
Furthermore, if W is to be T-invariant then $T\vec{\boldsymbol{v}}$ must also be in W. But then for W to be T-invariant we also need that

$$T(T\vec{\boldsymbol{v}}) = T^2\,\vec{\boldsymbol{v}}$$

is in W too. In fact, we need the closure of the composition of T applied to $\vec{\boldsymbol{v}}$.
Therefore, W is the smallest T-cyclic subspace of V containing $\vec{\boldsymbol{v}}$. □

Examples

(60) Let $T : \mathbb{R}^3 \longmapsto \mathbb{R}^3$ be defined by,

$$T(a, b, c) = (-b + c,\ a + c,\ 3c).$$

To obtain the T-cyclic subspace generated by $\vec{\boldsymbol{e_1}} = (1, 0, 0)$ we compose T repeatedly:

$$\begin{aligned} T\vec{\boldsymbol{e_1}} &= (0, 1, 0) = \vec{\boldsymbol{e_2}} \\ T^2\,\vec{\boldsymbol{e_1}} &= T(0, 1, 0) = (-1, 0, 0) = -\vec{\boldsymbol{e_1}} \\ T^3\,\vec{\boldsymbol{e_1}} &= T(-1, 0, 0) = (0, -1, 0) = -\vec{\boldsymbol{e_2}} \\ T^4\,\vec{\boldsymbol{e_1}} &= T(0, -1, 0) = (1, 0, 0) = \vec{\boldsymbol{e_1}} \end{aligned}$$

which produces the cycle of elements:

$$\vec{\boldsymbol{e_1}} \to \vec{\boldsymbol{e_2}} \to -\vec{\boldsymbol{e_1}} \to -\vec{\boldsymbol{e_2}} \to \vec{\boldsymbol{e_1}}.$$

Therefore the subspace generated is

$$\text{span}\{\vec{\boldsymbol{e_1}}, \vec{\boldsymbol{e_2}}\} = \{\ (x, y, 0)\ \mid\ x, y \in \mathbb{R}\ \}.$$

(61) Let T be the linear operator on $P(\mathbb{R})$, the polynomials with real coefficients, defined by differentiation,

$$T(p(x)) = \mathrm{D}_x\, p.$$

Then the T-cyclic subspace generated by x^2 is $\text{span}\{x^2, 2x, 2\}$.

2.5.3.2 Eigenvectors

Definition 116. An **eigenvector** of a linear operator T is a nonzero vector $\vec{\boldsymbol{v}}$ with the property under T that,

$$T(\vec{\boldsymbol{v}}) = c\vec{\boldsymbol{v}}$$

for some constant $c \in \mathbb{F}$. The constant c is called an **eigenvalue**.

Eigenvectors may also be referred to as **characteristic vectors** and eigenvalues as **characteristic values**.

The **eigenspace** of an eigenvalue is the subspace formed by the eigenvectors associated with the eigenvalue and the zero vector.

Note:

- *To have eigenvectors, T must be an operator as the image of the eigenvector $\vec{\boldsymbol{v}} \in V$ under T,*

  $$T(\vec{\boldsymbol{v}}) = c\vec{\boldsymbol{v}} \in V$$

  is, by definition, in the same space as the eigenvector itself (if there were a change of basis then we would not consider it to be the same vector). In fact, this is the key of the relationship between eigenvectors and invariant subspaces: The space spanned by eigenvectors of T is T-invariant. The bases of T-invariant subspaces, however, need not be eigenvectors. For example, a rotation of a plane in a 3d space has the plane as a T-invariant subspace but the only eigenvector is the axis of rotation, perpendicular to the plane.

- *An eigenvector may not be $\vec{\mathbf{0}}$ but an eigenvalue may be 0. As a consequence of this, every nonzero vector in the kernel is an eigenvector (with eigenvalue 0). As a consequence of this, if W is a 2-dimensional T-invariant subspace, for example, and its image under T is one-dimensional (a line inside the plane of W) then vectors in W that are not in the line of the image will be in the kernel of T and so also eigenvectors but with the eigenvalue 0. It is also worth*

noting that they are still considered to be parallel to their image under T because, by convention, the zero vector $\vec{\mathbf{0}}$ is considered to be parallel to all vectors.

- *If we speak of an eigenvector of a square matrix then we are referring to an eigenvector of **left multiplication** by the matrix, i.e. a nonzero vector $\vec{\boldsymbol{x}}$ such that,*

$$A\vec{\boldsymbol{x}} = c\vec{\boldsymbol{x}}.$$

Clearly if $\vec{\boldsymbol{x}}_B$ is the coordinate vector of $\vec{\boldsymbol{v}} \in V$ with respect to B — a basis of V — and A is the matrix of T with respect to the basis B, then

$$A\vec{\boldsymbol{x}}_B = c\vec{\boldsymbol{x}}_B \iff T(\vec{\boldsymbol{v}}) = c\vec{\boldsymbol{v}}.$$

As a result, all similar matrices have the same eigenvalues.

Proposition 2.5.15. *Let T be a linear operator on a finite-dimensional vector space V.*

(i) If V has dimension n, then T has at most n eigenvalues.

(ii) If $\mathbb{F}$ is the field of complex numbers and $V \neq 0$, then T has at least one eigenvalue, and hence it has an eigenvector.

Proof.

(i) For any field $\mathbb{F}$, a polynomial of degree n can have at most n different roots (see Artin[373]). Since T is defined over a vector space of dimension n, the degree of the characteristic polynomial of T is n. Then, by Theorem 2.5.5 we can have a maximum of n eigenvalues.

(ii) Every polynomial of positive degree with complex coefficients has at least one complex root. This fact is called the Fundamental Theorem of Algebra (wikipedia).

□

Theorem 2.5.4. *The eigenspace of an eigenvalue is a vector subspace.*

Proof. Let S be the set of all eigenvectors of a linear operator T corresponding to a particular eigenvalue c. Then the eigenspace is defined as,

$$E = S \cup \{\vec{\mathbf{0}}\}.$$

Then E is a subspace because it contains the zero vector and

$$\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}} \in E \implies T(\alpha\vec{\boldsymbol{v}}+\beta\vec{\boldsymbol{w}}) = \alpha(c\vec{\boldsymbol{v}})+\beta(c\vec{\boldsymbol{w}}) = c(\alpha\vec{\boldsymbol{v}}+\beta\vec{\boldsymbol{w}}) \implies \alpha\vec{\boldsymbol{v}}+\beta\vec{\boldsymbol{w}} \in E.$$

□

Corollary 2.5.8. *Any linear combination of eigenvectors with eigenvalue c is also an eigenvector with eigenvalue c.*

Proposition 2.5.16. *Eigenvectors corresponding to different eigenvalues are linearly independent. That's to say: Let $\vec{\boldsymbol{v}}_1, \ldots, \vec{\boldsymbol{v}}_r \in V$ be eigenvectors for a linear operator T, with distinct eigenvalues $c_1, \ldots, c_r$. Then the set $\{\vec{\boldsymbol{v}}_1, \ldots, \vec{\boldsymbol{v}}_r\}$ is linearly independent.*

Proof. Assume for contradiction that there exists a linear relation between the set of eigenvectors,

$$\alpha_1\vec{\boldsymbol{v}}_1 + \cdots + \alpha_r\vec{\boldsymbol{v}}_r = \vec{\mathbf{0}}.$$

Linearity of T gives us,

$$T(\alpha_1\vec{\boldsymbol{v}}_1 + \cdots + \alpha_r\vec{\boldsymbol{v}}_r) = \alpha_1 T(\vec{\boldsymbol{v}}_1) + \cdots + \alpha_r T(\vec{\boldsymbol{v}}_r) = T(\vec{\mathbf{0}}) = \vec{\mathbf{0}}$$

while the eigenvector property gives us,

$$\alpha_1 T(\vec{\boldsymbol{v}}_1) + \cdots + \alpha_r T(\vec{\boldsymbol{v}}_r) = \alpha_1 c_1\vec{\boldsymbol{v}}_1 + \cdots + \alpha_r c_r\vec{\boldsymbol{v}}_r$$

so we have the simultaneous equations,

$$\begin{aligned} \alpha_1\vec{\boldsymbol{v}}_1 + \cdots + \alpha_r\vec{\boldsymbol{v}}_r &= \vec{\mathbf{0}} \\ \alpha_1 c_1\vec{\boldsymbol{v}}_1 + \cdots + \alpha_r c_r\vec{\boldsymbol{v}}_r &= \vec{\mathbf{0}}. \end{aligned}$$

If we multiply the first equation by c_r and subtract the second equation from it we get,

$$\alpha_1(c_r - c_1)\vec{\boldsymbol{v}}_1 + \cdots \alpha_{r-1}(c_r - c_{r-1})\vec{\boldsymbol{v}}_{r-1} = \vec{\mathbf{0}}.$$

Since all the eigenvalues are distinct, for $i \neq j,\ c_i - c_j \neq 0$ and the eigenvectors $\vec{\boldsymbol{v}}_i$, by definition, are nonzero. So, this equation implies that either $\alpha_1, \ldots, \alpha_{r-1} = 0$ or there is a linear relation between the vectors $\vec{\boldsymbol{v}}_1, \ldots, \vec{\boldsymbol{v}}_{r-1}$.
This dependence of the properties of the r-length list on the properties of the $(r-1)$-length list signals that we can set up a proof by induction using the hypothesis that a k-length list is linearly independent.
If we use $k = 2$ as the base case, set up the linear relation and use the eigenvector property as before then this results in,

$$\alpha_1(c_2 - c_1)\vec{\boldsymbol{v}}_1 = \vec{\boldsymbol{0}}.$$

As before, both $c_2 - c_1$ and $\vec{\boldsymbol{v}}_1$ are nonzero so this implies that $\alpha_1 = 0$. This, in turn, implies that $\alpha_2 = 0$ also and so, the list of length $k = 2$ is linearly independent.
Then the induction step is to assume that the list of length $k = r - 1$ is linearly independent and show that this implies that the list of length $k = r$ is linearly independent. We have already shown that if we set up a linear relation on a list of eigenvectors of length $k = r$ then the eigenvector property implies that,

$$\alpha_1(c_r - c_1)\vec{\boldsymbol{v}}_1 + \cdots \alpha_{r-1}(c_r - c_{r-1})\vec{\boldsymbol{v}}_{r-1} = \vec{\boldsymbol{0}}.$$

Now we can use the induction hypothesis to assert that $\vec{\boldsymbol{v}}_1, \ldots, \vec{\boldsymbol{v}}_{r-1}$ are linearly independent implying that $\alpha_1, \ldots, \alpha_{r-1} = 0$. This, in turn, implies that $\alpha_r = 0$ meaning that $\vec{\boldsymbol{v}}_1, \ldots, \vec{\boldsymbol{v}}_r$ is linearly independent. □

Proposition 2.5.17. *If we consider an $n \times n$ matrix A with real entries as a matrix in $\mathbb{C}^{n\times n}$ and it has a complex eigenvalue λ with a corresponding eigenvector $\vec{\boldsymbol{v}}$, then the complex conjugate $\overline{\lambda}$ is also an eigenvalue and has a corresponding eigenvector $\overline{\vec{\boldsymbol{v}}}$, the complex conjugate of $\vec{\boldsymbol{v}}$.*

Proof.

$$\begin{aligned} & A\vec{\boldsymbol{v}} = \lambda\vec{\boldsymbol{v}} \\ \iff \quad & \overline{A\vec{\boldsymbol{v}}} = \overline{\lambda\vec{\boldsymbol{v}}} \\ \iff \quad & \overline{A}\,\overline{\vec{\boldsymbol{v}}} = \overline{\lambda}\,\overline{\vec{\boldsymbol{v}}} \quad \text{by 2.3.7.} \end{aligned}$$

But A is a matrix with only real entries and so $\overline{A} = A$. So the previous result implies that

$$A\overline{\vec{\boldsymbol{v}}} = \overline{\lambda}\,\overline{\vec{\boldsymbol{v}}}$$

which means that $\overline{\lambda}$ is an eigenvalue of A with a corresponding eigenvector $\overline{\vec{\boldsymbol{v}}}$. □

Proposition 2.5.18. *Let $A \in \mathbb{C}^{n\times n}$ be a matrix in complex space but that has only real entries and let A have a real eigenvalue λ. Then we can find a basis of the eigenspace of λ containing only real vectors.*

Proof. Let $A \in \mathbb{C}^{n\times n}$ have only real entries and a real eigenvalue λ. Suppose $\vec{\boldsymbol{v}}$ is a complex eigenvector of A corresponding to λ. Then,

$$\begin{aligned} & A\vec{\boldsymbol{v}} = \lambda\vec{\boldsymbol{v}} \\ \iff \quad & \overline{A\vec{\boldsymbol{v}}} = \overline{\lambda\vec{\boldsymbol{v}}} \\ \iff \quad & A\overline{\vec{\boldsymbol{v}}} = \lambda\overline{\vec{\boldsymbol{v}}}. \quad \text{A and λ are real} \end{aligned}$$

So $\overline{\vec{\boldsymbol{v}}}$ is also an eigenvector of λ. Then, by Proposition 2.4.28 the two real vectors $\operatorname{Re}\vec{\boldsymbol{v}}$ and $\operatorname{Im}\vec{\boldsymbol{v}}$ span the same space as $\vec{\boldsymbol{v}}$ and $\overline{\vec{\boldsymbol{v}}}$ and so can also be chosen as eigenvectors for λ. □

> *It may also be possible to prove this using the definition of the eigenspace as the nullspace of the matrix $(A - \lambda I)$ but the above proof is constructive.*

Proposition 2.5.19. *If two matrices A and B have the same eigenvalues (with the same multiplicity) and corresponding eigenvectors and one of the matrices is diagonalisable, then $A = B$.*

Proof. w.l.o.g. assume that A is diagonalisable. Since B has the same eigenvectors as A, it must have the same dimension and since it has the same eigenvalues with the same multiplicity as A, it must also be diagonalisable.

By diagonalisability,

$$D_1 = {P_1}^{-1} A P_1 \quad \text{and} \quad D_2 = {P_2}^{-1} B P_2.$$

Since B has the same eigenvalues as A, the diagonal entries must be the same upto order (TODO: add reference). We can choose them to be in the same order and then we have,

$$D_1 = D_2.$$

Each of these eigenvalues in the diagonal entries has an associated eigenvector in the matrices P_1 and P_2 and, since these eigenvectors are the same in both A and B, and we have chosen the order of the eigenvalues in the diagonalisation to be the same, then the order of the eigenvectors in P_1 and P_2 also must be the same. So we have,

$$P_1 = P_2.$$

Therefore,

$$A = P_1 D_1 {P_1}^{-1} = P_2 D_2 {P_2}^{-1} = B.$$

□

Examples of Eigenvectors and Eigenvalues

(62) If we take the matrix,

$$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$$

we can determine its eigenvectors by finding the solutions to the equation,

$$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

This gives us two simultaneous equations in three unknowns: the two vector dimensions x_1, x_2 and the eigenvalue c,

$$\begin{aligned} 3x_1 + x_2 &= cx_1 \\ 2x_2 &= cx_2 \end{aligned}$$

which imply that,

$$6x_1 = 2cx_1 - cx_2 \iff (6/c - 2)x_1 = -x_2 \iff x_2 = (2 - 6/c)x_1.$$

So, if $x_1 = 1$ then $x_2 = 2 - (6/c)$ and we have the follwing eigenvector/eigenvalue pairs.

$$c = 3 : \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; c = 1 : \begin{bmatrix} 1 \\ -4 \end{bmatrix} \text{ Wrong!}$$

Why does this not work? There is an additional constraint not expressed in the linear system here: eigenvectors are nonzero by definition. This means that we have the additional constraint,

$$x_1 x_2 \neq 0$$

which cannot be expressed in a linear system of equations. When $c \notin \{2, 3\}$ both x_1 and x_2 are zero and the vector is not an eigenvector. So, how can we systematically restrict the values of c to only those that produce valid eigenvectors? If we follow the alternative logic:

$$\begin{aligned}
& \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
\iff \quad & \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - c \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{\mathbf{0}} \\
\iff \quad & \left(\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} I - cI \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{\mathbf{0}} \\
\iff \quad & \begin{bmatrix} 3-c & 1 \\ 0 & 2-c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{\mathbf{0}}
\end{aligned}$$

Now if we let

$$A = \begin{bmatrix} 3-c & 1 \\ 0 & 2-c \end{bmatrix}$$

then the nullspace of A is the space of vectors $(x_1, x_2)^T$ that satisfy this equation. If A is invertible and nonsingular then this space is the trivial space $\{\vec{\mathbf{0}}\}$ but, if A is singular however, then there exist nonzero vectors that satisfy this equation. Therefore, it is precisely the nonzero vectors that are in the nullspace of this matrix A when it is singular that are the eigenvectors we are looking for. So, if we first determine the values of c for which A is singular, we can then determine the eigenvectors. Since, by Proposition 2.5.11, A is singular if and only if $det\, A = 0$, we are looking for *precisely the values of c which make $det\, A = 0$.*

(63) The minimal linear transformation seen in 56 — which is just scalar multiplication — has every vector as its eigenvectors because

$$T(\vec{\boldsymbol{v}}) = A\vec{\boldsymbol{v}} = a\vec{\boldsymbol{v}}$$

for all $\vec{\boldsymbol{v}} \in V$. So every vector in V is an eigenvector with eigenvalue a.

2.5.3.3 Matrices of Eigenvectors

If $\vec{\boldsymbol{v}}_1 \in V$ is a eigenvector of a linear transformation T and we extend the set $\{\vec{\boldsymbol{v}}_1\}$ (by Proposition 2.4.15) to a basis of V, say $\{\vec{\boldsymbol{v}}_1, \ldots, \vec{\boldsymbol{v}}_n\}$, then the matrix of T will have the block form,

$$\begin{bmatrix} c & B \\ 0 & D \end{bmatrix} = \begin{bmatrix} c & \ldots & \ldots \\ 0 & \ldots & \ldots \\ \vdots & \ldots & \ldots \\ 0 & \ldots & \ldots \end{bmatrix}$$

where c is the eigenvalue of $\vec{\boldsymbol{v}}_1$. This is the same block decomposition as that shown for T-invariant spaces in 2.5.3.1 with the case of a 1-dimensional invariant subspace.

Proposition 2.5.20. *If A is the matrix of a linear operator T with respect to a basis B then the matrix A is diagonal iff every basis vector in B is an eigenvector of T.*

Proof. The defining property of the matrix A is that the j-th column is the coordinates of the image of the j-th basis vector in B under T,

$$A(:j) = T(\vec{\boldsymbol{v}}_j) = a_{1j}\vec{\boldsymbol{v}}_1 + \cdots + a_{nj}\vec{\boldsymbol{v}}_n.$$

For an eigenvector $\vec{\boldsymbol{v}}_j$, $T(\vec{\boldsymbol{v}}_j) = c\vec{\boldsymbol{v}}_j = a_{jj}\vec{\boldsymbol{v}}_j$ so that $a_{jj} = c$ the eigenvalue and for all a_{ij} such that $i \neq j$, $a_{ij} = 0$. □

Corollary 2.5.9. *The matrix of a linear operator T over a vector space V is similar to a diagonal matrix iff there exists some basis of V solely comprised of eigenvectors of T.*

Proposition 2.5.21. *Similar matrices have the same eigenvalues.*

Proof. Similar matrices represent the same transformation with respect to different bases and for a matrix A representing a transformation T with respect to an arbitrary basis B,

$$T(\vec{\boldsymbol{v}}) = c\vec{\boldsymbol{v}} \iff A\vec{\boldsymbol{v}}_B = c\vec{\boldsymbol{v}}_B.$$

That's to say, the eigenvalues are not dependent on the basis with respect to which a coordinate vector is defined. □

2.5.4 Diagonalisation

Definition 117. **(Diagonalisation)** The process of determining a diagonal matrix that is similar to a given matrix of a linear operator is known as **diagonalisation**.

Definition 118. **(Eigenbasis)** If a linear operator T is defined over a vector space V, an *eigenbasis* of T is a basis of V consisting solely of eigenvectors of T.

If A is the matrix of T w.r.t. the standard basis and P is the matrix whose columns are the elements of the eigenbasis of T then, by Proposition 2.5.20,

$$P^{-1}AP = D$$

is diagonal.

2.5.4.1 Existence of Eigenvectors

- Every linear operator on a complex vector space has at least one eigenvector and, in most cases, these form a basis.
- Linear operators over real vector spaces need not have eigenvectors (e.g. rotation of the plane $\mathbb{R}^2$ by an angle θ has no eigenvector unless $\theta = 0$ or π).
- Real matrices that are *positive* (having only positive components) are guaranteed to have at least one positive eigenvector.

2.5.4.2 The Effect of Multiplication by a Positive Matrix

TODO: Artin[134]

2.5.4.3 Determining the Eigenvectors

The process of finding eigenvectors is to first determine the eigenvalues and then calculate the eigenvectors that correspond to those eigenvalues. Let I be the identity operator. Then,

$$T(\vec{\boldsymbol{v}}) = c\vec{\boldsymbol{v}} \iff T(\vec{\boldsymbol{v}}) - c\vec{\boldsymbol{v}} = \vec{\mathbf{0}} \iff [T - cI](\vec{\boldsymbol{v}}) = \vec{\mathbf{0}}$$

where the expression $T - cI$ is a linear combination of linear transformations and so is also a linear transformation. Furthermore, it is an operator as both its terms are operators (in fact they need to have the same dimensions in order for the expression to make sense).
Two things are clear from this expression:

(i) The matrix of the linear operator $T - cI$ is $A - cI$ where I is the identity matrix.

(ii) The eigenvector $\vec{\boldsymbol{v}}$ is in the kernel of $T - cI$ and so is in the nullspace of $A - cI$.

Proposition 2.5.22. *A linear operator T has a nontrivial kernel iff 0 is an eigenvalue of T.*

Proof. This follows from the fact that, if we let $\vec{\boldsymbol{v}} \neq \vec{\mathbf{0}} \in ker\, T$, then

$$T(\vec{\boldsymbol{v}}) = \vec{\mathbf{0}} = 0\vec{\boldsymbol{v}} = c\vec{\boldsymbol{v}}$$

for $c = 0$ so that a nontrivial kernel implies that 0 is an eigenvalue. Conversely, if 0 is an eigenvalue we must have $0\vec{\boldsymbol{v}} = \vec{\mathbf{0}} = T(\vec{\boldsymbol{v}})$ and, since if a vector $\vec{\boldsymbol{v}}$ is an eigenvector, by definition, $\vec{\boldsymbol{v}} \neq \vec{\mathbf{0}}$, this therefore implies that the kernel contains a nonzero vector. □

Corollary 2.5.10. *A linear operator T has all the properties in Proposition 2.5.11 iff 0 is an eigenvalue of T.*

Proposition 2.5.23. *The eigenvalues of a linear operator T are the scalars $c \in \mathbb{F}$ such that the linear operator $[T - cI]$ is singular.*

Proof. The eigenvalues of a linear operator T are the scalars $c \in \mathbb{F}$ such that there exists a nonzero vector $\vec{\boldsymbol{v}}$ with $[T - cI](\vec{\boldsymbol{v}}) = \vec{\mathbf{0}}$. If such a vector exists then the kernel of $T - cI$ is nontrivial and so, by Proposition 2.5.11, $T - cI$ is singular. □

Corollary 2.5.11. *If $A - cI$ is the matrix of $T - cI$, **the eigenvalues of T are the scalars $c \in \mathbb{F}$ such that $det(A - cI) = 0$.***

Corollary 2.5.12. *The eigenvalues of $A - cI$ are the same as the eigenvalues of $cI - A$.*

Proof. If A is a $n \times n$ matrix representing the operator T then,

$$det(-A) = (-1)^n det(A).$$

So, if $det(A) = 0$ then $det(-A) = 0$ also. Therefore,

$$det(A - cI) = 0 \iff det(cI - A) = 0.$$ □

2.5.4.4 The Characteristic Polynomial

Notation. It is customary to use either the variable t or λ to denote the eigenvalue in the characteristic polynomial.

Definition 119. The **characteristic polynomial** of a linear operator T is the polynomial,

$$p(t) = det(tI - A) = \sum s(j_1, \cdots, j_n) a_{1j_1} \cdots a_{nj_n}$$

where the sum is defined over all permutations $j_1, \cdots, j_n$ of $\{1, \ldots, n\}$ and $s(j_1, \cdots, j_n)$ is the sign of the permutation.

The determinant is an expression in which every term is the product of a component in every column and row of the matrix with no column or row

appearing more than once in each term,

$$tI - A = \begin{bmatrix} (t - a_{11}) & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & (t - a_{22}) & \cdots & -a_{2n} \\ \vdots & & & \vdots \\ -a_{n1} & \cdots & \cdots & (t - a_{nn}) \end{bmatrix}.$$

It can be seen in the matrix of $tI - A$ that the highest power of t will be obtained in the term of the determinant that forms the product of all the diagonal terms $a_{11} \cdots a_{nn}$ which occurs when $j_1, \cdots, j_n = 1, \ldots, n$. This term of the determinant will be the product of precisely n terms containing the eigenvalue t. Therefore the result is a polynomial of degree n in the eigenvalue t.

Theorem 2.5.5. *The eigenvalues of a linear operator are the roots of its characteristic polynomial.*

Proof. If $p(t)$ is the characteristic polynomial of a linear operator T then the values of t for which $p(t) = 0$ are the values of t such that $det(tI - A) = 0$ and these are precisely the eigenvalues. □

Proposition 2.5.24. *The eigenvalues of an upper or lower triangular matrix are its diagonal entries.*

Proof. The determinant of a triangular matrix is equal to the product of its diagonal entries and if A is a triangular matrix then $tI - A$ is also triangular. Therefore, the characteristic polynomial is simply,

$$p(t) = (t - a_{11}) \cdots (t - a_{nn})$$

and so the eigenvalues are the diagonal entries $a_{11}, \ldots, a_{nn}$. □

Proposition 2.5.25. *A positive matrix (a matrix whose entries are all positive) has at least one eigenvector with positive coordinates.*

An abstract vector does not have coordinates so when we refer to a "positive" vector with positive-valued coordinates, this is with respect to a particular basis. In this context, the basis in question is the basis with respect to which the matrix outputs the transformed vectors.

Proof. TODO: review: this "proof" is not really a proof, more an example using a 2x2 matrix. □

Proposition 2.5.26. *The characteristic polynomial of a linear operator does not depend on the basis with respect to which the matrix of the operator is defined.*

Proof. For two similar matrices representing the same linear operator T we have,

$$A' = PAP^{-1}$$

where P is the matrix of change of basis between the bases of A and A'. If we form the characteristic polynomial of A',

$$\begin{aligned} & tI - A' = tI - PAP^{-1} \\ \iff \quad & = PtIP^{-1} - PAP^{-1} && PtIP^{-1} = tPP^{-1} = tI \\ \iff \quad & = P(tI - A)P^{-1} && \text{by distributivity of matrix multiplication.} \end{aligned}$$

Then,

$$\begin{aligned} & det(tI - A') = det(P(tI - A)P^{-1}) \\ \iff \quad & = det\, P \cdot det(tI - A) \cdot det\, P^{-1} \\ \iff \quad & = det(tI - A). \end{aligned}$$

This result, $det(tI - A') = det(tI - A)$, must hold for all t and therefore implies that, for p, p' characteristic polynomials of A and A' respectively,

$$\forall t \in \mathbb{F} \; . \; p(t) = p'(t).$$

This implies that the characteristic polynomials are equal. □

Proposition 2.5.27. *The characteristic polynomial $p(t)$ of a matrix A has the form*

$$p(t) = t^n - (tr\, A)t^{n-1} + \cdots + (-1)^n(det\, A),$$

where $tr\, A$ is the trace of A (see: 2.3.2.1):

$$tr\, A = a_{11} + \cdots + a_{nn}.$$

Proof. Calculation of the characteristic polynomial of a matrix A is calculation of $p(t) = det(tI - A)$ the determinant of the matrix $tI - A$ which takes the form,

$$tI - A = \begin{bmatrix} (t - a_{11}) & \cdots & \cdots & \vdots \\ \vdots & (t - a_{22}) & \cdots & \vdots \\ \vdots & & & \vdots \\ \vdots & \cdots & (t - a_{(n-1)(n-1)}) & -a_{(n-1)n} \\ \vdots & \cdots & -a_{n(n-1)} & (t - a_{nn}) \end{bmatrix}.$$

This calculation proceeds with terms of products of elements from each row and a permutation of the column indices (see: 2.3.5.2) of which we examine the first two terms.

The first term is for the identity permutation $j_1, \ldots, j_n = 1, \ldots, n$ along the diagonal:

$$\begin{aligned} a_{11} \cdots a_{nn} &= (t - a_{11}) \cdots (t - a_{nn}) \\ &= t^n - (a_{11} + \cdots + a_{nn})t^{n-1} \\ &\quad + (a_{11}a_{22} + a_{11}a_{33} + \cdots + a_{(n-1)(n-1)}a_{nn})t^{n-2} \\ &\quad \cdots + (-1)^n(a_{11} \cdots a_{nn}) \end{aligned}$$

The second term is the permutation one swap away from identity

$$j_1, \ldots, j_n = 1, \ldots, n, (n-1)$$

which is an odd permutation, so the sign is -1:

$$\begin{aligned} &(-1)a_{11} \cdots a_{(n-2)(n-2)}a_{(n-1)n}a_{n(n-1)} \\ &= (-1)(t - a_{11}) \cdots (t - a_{(n-2)(n-2)})(-a_{(n-1)n})(-a_{n(n-1)}) \end{aligned}$$

$$= -a_{(n-1)n}a_{n(n-1)}t^{n-2} + a_{(n-1)n}a_{n(n-1)}(a_{11} + \cdots + a_{(n-2)(n-2)})t^{n-3}$$
$$\cdots - (-1)^n(a_{11} \cdots a_{(n-2)(n-2)}a_{(n-1)n}a_{n(n-1)})$$

From these first two terms we can discern enough about the general pattern of the characteristic polynomial to see that the first two terms in t^n and t^{n-1} are produced by the first permutation and take the form

$$t^n - (a_{11} + \cdots + a_{nn})t^{n-1} = t^n - (tr\, A)t^{n-1}$$

as required. We can also see that the final terms of each permutation — those that involve no powers of t — are going to sum up to the value of $(-1)^n(det\, A)$. Therefore the characteristic polynomial takes the form,

$$p(t) = t^n - (tr\, A)t^{n-1} + \cdots + (-1)^n(det\, A)$$

as claimed. □

Corollary 2.5.13. *The trace of a matrix of a linear operator is independent of the basis with respect to which the matrix is defined.*

Proof. Let T be a linear operator and A be the matrix of T with respect to a basis B. Then Proposition 2.5.26 tells us that any matrix of the same linear operator defined with respect to some other basis (i.e. a similar matrix to A) has the same characteristic polynomial. Proposition 2.5.27 tells us that the coefficients of this characteristic polynomial include the trace of the matrix A. Therefore, if A' is a matrix of T defined against the basis B' and P is the change of basis matrix such that $B'P = B$ then,

$$A' = PAP^{-1} \iff p'(t) = p(t) \iff tr\, A' = tr\, A.$$ □

As a result of this we can refer to the characteristic polynomial, determinant and trace of a linear operator T without reference to a particular matrix or basis.

Proposition 2.5.28. *Let T be a linear operator on a finite-dimensional complex vector space V. There is a basis B of V such that the matrix A of T is upper triangular.*

Proof. By Proposition 2.5.15, T has at least one eigenvector. We can extend this eigenvector to a basis of V, say,

$$B' = \{\vec{\boldsymbol{v}}'_1, \ldots, \vec{\boldsymbol{v}}'_n\}.$$

Then the first column of the matrix A' of T with respect to B' will be

$$(c_1, 0, \ldots, 0)^T$$

where c_1 is the eigenvalue of $\vec{\boldsymbol{v}}'_1$. Therefore A' has the form

$$A' = \begin{bmatrix} c_1 & \cdots \\ 0 & D \end{bmatrix}$$

where D is a $(n-1) \times (n-1)$ matrix and, if $P = [B']^{-1} I = [B']^{-1}$ is the change of basis matrix, then

$$A' = PAP^{-1}.$$

Now we can use induction on the dimension of the matrix n and the induction hypothesis will be that there exists some upper triangular

$$Q' = QDQ^{-1}.$$

Define

$$Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix}.$$

Then

$$(Q_1 P)A(Q_1 P)^{-1} = Q_1 PAP^{-1} {Q_1}^{-1} = Q_1 A' {Q_1}^{-1}$$

takes the form

$$\begin{bmatrix} c_1 & \cdots \\ 0 & QDQ^{-1} \end{bmatrix}$$

which is upper triangular. This proves the induction step. □

Note that this proof is over the complex number field but the same proof would work over any field that contains all the roots of the characteristic polynomial.

Theorem 2.5.6. *Let T be a linear operator on a vector space V of dimension n over a field F. Assume that its characteristic polynomial has n* ***distinct*** *roots in F. Then there is a basis for V with respect to which the matrix of T is diagonal.*

Proof. If the characteristic polynomial has n distinct roots then there are n distinct eigenvalues along with their associated eigenvectors. By Proposition 2.5.16, these eigenvectors form a linearly independent set. Since the dimension of the space V is n, by Theorem 2.4.2, these eigenvectors form a basis of V. Then, by Proposition 2.5.20, the matrix of T with respect to this basis is diagonal. □

Note that:

- *The diagonal entries of the matrix of a linear operator with respect to a basis of eigenvectors are the eigenvalues. For this reason, the set of values is wholly determined by the linear operator although the order is determined on the order of the vectors in the basis set (which is not significant);*
- *If a matrix A is found to be similar to a diagonal matrix B via a change of basis expressed in the matrix P then, by Theorem 2.3.3,*

$$A^m = (P^{-1}BP)^m = P^{-1}B^mP.$$

Corollary 2.5.14. *If a linear operator on a vector space of dimension n over a field F has a characteristic polynomial with n distinct roots then its determinant is equal to the product of its eigenvalues and its trace is equal to the sum of its eigenvalues.*

Proof. By Proposition 2.3.21 and Corollary 2.5.13, the determinant and trace are independent of the basis with respect to which the matrix is defined and so the existence of a basis with respect to which the matrix is diagonal means that we can look at the determinant and trace of the diagonal matrix. □

Proposition 2.5.29. *Let T be a linear operator on a finite-dimensional vector space V over a field F, $q(t)$ an arbitrary polynomial over F, and $\vec{\boldsymbol{v}}$ an eigenvector of T with eigenvalue c. Then*

$$q(T)\vec{\boldsymbol{v}} = q(c)\vec{\boldsymbol{v}}.$$

Proof. The polynomial $q(t)$ has the general form,

$$q(t) = a_n t^n + \cdots + a_1 t + a_0.$$

Since T is an operator we can use 2.5.3 to define,

$$q(T) = a_n T^n + \cdots + a_1 T + a_0 I.$$

Then,

$$\begin{aligned} q(T)\vec{\boldsymbol{v}} &= a_n T^n\, \vec{\boldsymbol{v}} + \cdots + a_1 T\vec{\boldsymbol{v}} + a_0\vec{\boldsymbol{v}} \\ &= a^n c^n\, \vec{\boldsymbol{v}} + \cdots + a_1 c\vec{\boldsymbol{v}} + a_0\vec{\boldsymbol{v}} \\ &= (a^n c^n + \cdots + a_1 c + a_0)\vec{\boldsymbol{v}} \\ &= q(c)\vec{\boldsymbol{v}}. \quad \square \end{aligned}$$

Corollary 2.5.15. *If $p(t)$ is the characteristic polynomial of a **diagonalizable** linear operator T over a finite-dimensional vector space, then $p(T)$ is 0, the zero operator T_0.*

Proof. As above, for an eigenvector $\vec{\boldsymbol{v}}$ with eigenvalue c,

$$p(T)\vec{\boldsymbol{v}} = p(c)\vec{\boldsymbol{v}}.$$

But now, $p(t)$ is the characteristic polynomial of T and c, as an eigenvalue of T, *is a root of the characteristic polynomial*. So, for any eigenvector of T corresponding to the eigenvalue c,

$$p(T)\vec{\boldsymbol{v}} = p(c)\vec{\boldsymbol{v}} = 0\vec{\boldsymbol{v}} = 0.$$

Since T is diagonalizable, there exists a basis of the space V comprised solely of eigenvectors. Therefore, for any vector $\vec{\boldsymbol{v}} \in V$, we can express it as a linear combination of eigenvectors each of which will be mapped to the zero vector by $p(T)$. $\square$

Proposition 2.5.30. *Let T be a linear operator on a finite-dimensional vector space V, and let W be a T-invariant subspace of V. Then the characteristic polynomial of the restriction of T to W, T_w, divides the characteristic polynomial of T.*

Proof. Let $B_w = \{\vec{\boldsymbol{w}}_1, \ldots, \vec{\boldsymbol{w}}_k\}$ be a basis of W and extend it to a basis of V, $B = \{\vec{\boldsymbol{w}}_1, \ldots, \vec{\boldsymbol{w}}_k, \vec{\boldsymbol{v}}_1, \ldots, \vec{\boldsymbol{v}}_{n-k}\}$. Then the matrix of T with respect to the basis B has the form,

$$A = \begin{bmatrix} A_w & C \\ 0 & D \end{bmatrix}$$

where $A_w = T_w(B_w)$ is the matrix of T_w w.r.t. to the basis B_w.
So, the characteristic polynomial $p(t)$ of A is

$$\begin{aligned} p(t) = \det(A - tI_n) &= \begin{vmatrix} A_w - tI_k & C \\ 0 & D - tI_{n-k} \end{vmatrix} \\ &= \det(A_w - tI_k) \cdot \det(D - tI_{n-k}) \\ &= q(t) \cdot \det(D - tI_{n-k}) \end{aligned}$$

where $q(t)$ is the characterstic polynomial of T_w, the restriction of T to W. $\square$

Proposition 2.5.31. *Let T be a linear operator on a finite-dimensional vector space V, and let W denote the T-cyclic subspace of V generated by $\vec{\boldsymbol{v}} \in V$. Suppose that $\dim W = k \geq 1$ (and hence $\vec{\boldsymbol{v}} \neq \vec{\boldsymbol{0}}$). Then,*

(i) $\{\vec{\boldsymbol{v}}, T\vec{\boldsymbol{v}}, T^2\,\vec{\boldsymbol{v}}, \ldots, T^{k-1}\,\vec{\boldsymbol{v}}\}$ *is a basis for W.*

(ii) If

$$T^k\,\vec{\boldsymbol{v}} = -a_0\vec{\boldsymbol{v}} - a_1 T\vec{\boldsymbol{v}} - \cdots - a_{k-1}T^{k-1}\,\vec{\boldsymbol{v}},$$

then the characteristic polynomial of T_w is

$$p(t) = (-1)^k(t^k + a_{k-1}t^{k-1} + \cdots + a_1 t + a_0).$$

Proof. Let $B_x = \{\, T^i\,\vec{\boldsymbol{v}} \mid i \in \mathbb{N},\ 0 \leq i \leq x \,\}$. Since V is finite-dimensional, there exists some $m \in \mathbb{N}$ such that

$$T^m\,\vec{\boldsymbol{v}} \in \operatorname{span} B_{m-1}.$$

Let $j \in \mathbb{N}$ be the lowest such value so that $B_{j-1} = \{ T^i \vec{\boldsymbol{v}} \mid i \in \mathbb{N},\ 0 \leq i \leq j-1 \}$ is linearly independent. If we can show that B_{j-1} spans W then, by Theorem 2.4.2, B_{j-1} is a basis of W.
We can show inductively that for any $m \in \mathbb{N}$, $T^m \vec{\boldsymbol{v}} \in \operatorname{span} B_{j-1}$. Begin by observing that, clearly, $T^m \vec{\boldsymbol{v}}$ for any $0 \leq m \leq j-1$ is a member of B_{j-1} and is, therefore, trivially in the span. Then, for some $m > j-1$ we have that, by the induction hypothesis, $T^{m-1} \vec{\boldsymbol{v}}$ is in the span of B_{j-1} so,

$$T^{m-1} \vec{\boldsymbol{v}} = a_0 \vec{\boldsymbol{v}} + a_1 T \vec{\boldsymbol{v}} + \cdots + a_{j-1} T^{j-1} \vec{\boldsymbol{v}}$$

for some $a_0, a_1, \ldots, a_{j-1} \in \mathbb{F}$. Then,

$$\begin{aligned} T^m \vec{\boldsymbol{v}} &= T(a_0 \vec{\boldsymbol{v}} + a_1 T \vec{\boldsymbol{v}} + \cdots + a_{j-1} T^{j-1} \vec{\boldsymbol{v}}) \\ &= a_0 T \vec{\boldsymbol{v}} + a_1 T^2 \vec{\boldsymbol{v}} + \cdots + a_{j-1} T^j \vec{\boldsymbol{v}} \end{aligned}$$

which is in the span of B_{j-1} because $T^j \vec{\boldsymbol{v}}$ is in the span by construction. Therefore, B_{j-1} is a basis of W and it follows then that it must have length k so that $j = k$.

Note that we could also have attempted to prove this by invoking the division theorem (Theorem 1.2.2) to observe that, if $j \in \mathbb{N}$ is the lowest natural number such that $T^j \vec{\boldsymbol{v}} \in \operatorname{span} B_{j-1}$, then, for any $m > j \in \mathbb{N}$, there exist $q \in \mathbb{Z}, r \in \mathbb{N}$ with $0 \leq r < j$, such that

$$m = qj + r \implies T^m \vec{\boldsymbol{v}} = (T^j \vec{\boldsymbol{v}})^q (T^r \vec{\boldsymbol{v}}).$$

Here, $T^j \vec{\boldsymbol{v}}$ is in the span of B_{j-1} by hypothesis and $T^r \vec{\boldsymbol{v}}$ is an element of B_{j-1} and so, trivially, in the span. However, this still leaves the question of whether $(T^j \vec{\boldsymbol{v}})^q$ is in the span of B_{j-1} – which is precisely what we're trying to prove!

To build the characteristic polynomial of T_w, observe that the regular structure of the basis B_{k-1} results in a regular structure of the matrix of the transformation w.r.t. this basis (remember that, by Proposition 2.5.26, the characterstic polynomial is the same regardless of the basis w.r.t. which we specify the matrix). If A_w is the matrix of T_w w.r.t. to the basis B_{k-1} then (see: 2.5.2.2),

$$A_w = [B_{k-1}]^{-1} [T(B_{k-1})]$$

where $[B_{k-1}]$ denotes the matrix whose columns are the elements of B_{k-1} and $[T(B_{k-1})]$ denotes the matrix whose columns are the elements of B_{k-1}

transformed by T. So, the columns of $[T(B_{k-1})]$ are $T\vec{\boldsymbol{v}}, T^2\,\vec{\boldsymbol{v}}, \ldots, T^k\,\vec{\boldsymbol{v}}$ which, when expressed w.r.t. the basis B_{k-1} are going to be,

$$(0,1,0,\ldots,0),\ (0,0,1,0\ldots,0),\ldots,(-a_0,-a_1,\ldots,-a_{k-1}).$$

So, the matrix A_w is going to take the form,

$$A_w = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 & -a_{k-2} \\ 0 & 0 & \cdots & 1 & -a_{k-1} \end{bmatrix}.$$

Note that A_w is square $k \times k$ which is as expected because T_w is $W \longmapsto W$.

To show that this results in the desired characteristic polynomial we use induction on k, the dimension of A_w.

For $k = 2$,

$$\begin{aligned} A_w = \begin{bmatrix} 0 & -a_{k-2} \\ 1 & -a_{k-1} \end{bmatrix} &\rightsquigarrow \begin{vmatrix} -t & -a_{k-2} \\ 1 & -a_{k-1}-t \end{vmatrix} \\ &= (-t)^k + (-t)^{k-1}(-a_{k-1}) + (-1)^{k-1}(-a_{k-2})(1) \\ &= (-1)^k(t^k + a_{k-1}t^{k-1} + a_0). \end{aligned}$$

which has the desired form $(-1)^k(t^k + a_{k-1}t^{k-1} + \cdots + a_1 t + a_0)$.

For $k = n > 2 \in \mathbb{N}$, we assume that matrices of size $(n-1) \times (n-1)$ have the desired form of characteristic polynomial,

$$p_{n-1}(t) = (-1)^{n-1}(t^{n-1} + a_{n-2}t^{n-2} + \cdots + a_1 t + a_0).$$

Then,

$$\det A_w = \begin{vmatrix} -t & 0 & \cdots & 0 & -a_0 \\ 1 & -t & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & -t & -a_{k-2} \\ 0 & 0 & \cdots & 1 & -a_{k-1} - t \end{vmatrix}$$

$$\begin{aligned} &= (-t)p_{n-1}(t) + (-1)^{k-1}(-a_0)(1) \\ &= (-t)[(-1)^{k-1}(t^{k-1} + a_{k-1}t^{k-1} + \cdots + a_2 t + a_1)] + (-1)^{k-1}(-a_0) \\ &= (-1)^k(t^k + a_{k-1}t^k + \cdots + a_2 t^2 + a_1 t) + (-1)^k a_0 \\ &= (-1)^k(t^k + a_{k-1}t^k + \cdots + a_2 t^2 + a_1 t + a_0). \quad \square \end{aligned}$$

Theorem 2.5.7. ***(Cayley-Hamilton Theorem.)*** *If $p(t)$ is the characteristic polynomial of* ***any*** *linear operator T over a finite-dimensional vector space, then $p(T)$ is 0, the zero operator T_0.*

Proof. Let $T : V \longmapsto V$ be a linear operator over a finite-dimensional vector space. Then for an arbitrary $\vec{\boldsymbol{v}} \neq \vec{\mathbf{0}} \in V$ there exists an associated T-cyclic subspace W with basis $\{\vec{\boldsymbol{v}}, T\vec{\boldsymbol{v}}, \ldots, T^{k-1}\,\vec{\boldsymbol{v}}\}$ and we have

$$T^k\,\vec{\boldsymbol{v}} = -a_0\vec{\boldsymbol{v}} - a_1 T\vec{\boldsymbol{v}} - \cdots - a_{k-1}T^{k-1}\,\vec{\boldsymbol{v}}$$

for some scalars $a_0, a_1, \ldots, a_{k-1}$.

By Proposition 2.5.31 then, the characteristic polynomial of W is

$$p_w(t) = (-1)^k(t^k + a_{k-1}t^{k-1} + \cdots + a_1 t + a_0)$$

and then,

$$p_w(T)\vec{\boldsymbol{v}} = (-1)^k(T^k\,\vec{\boldsymbol{v}} + a_{k-1}T^{k-1}\,\vec{\boldsymbol{v}} + \cdots + a_1 T\vec{\boldsymbol{v}} + a_0\vec{\boldsymbol{v}}) = \vec{\mathbf{0}}.$$

But also, Proposition 2.5.30 tells us that $p_w(t)$ divides $p(t)$, the characteristic polynomial of T. Therefore,

$$p(T)\vec{\boldsymbol{v}} = (p_w(T) \cdot q(T))\vec{\boldsymbol{v}} = (q(T) \cdot p_w(T))\vec{\boldsymbol{v}} = q(T)(p_w(T)\vec{\boldsymbol{v}}) = q(T)(\vec{\mathbf{0}}) = \vec{\mathbf{0}}.$$

Note the commutativity of the polynomial factors for polynomials in a single linear operator (see: 2.5.3).

In this way, $p(T)$ has been shown to map any arbitrary non-zero vector in the space to the zero vector and, since it is a linear operator (by the definition of a polynomial of a linear operator 2.5.3), it must also map the zero vector to the zero vector. It therefore follows that $p(T) = 0 = T_0$, the zero operator. $\square$

Proposition 2.5.32. *Let T be a linear operator over a finite-dimensional vector space V and let*

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_n$$

where each W_i, for $1 \leq i \leq n$, is a T-invariant subspace of V. Then, if $p(t)$ is the characteristic polynomial of T while $p_i(t)$ is the characteristic polynomial of T_{W_i} we have,

$$p(t) = p_1(t)p_2(t)\cdots p_n(t).$$

Proof. This proof proceeds by induction on the number of invariant subspaces n with base case $n = 2$.

For $n = 2$, we have $W_1 \oplus W_2 = V$ and if B_1 is a basis for W_1 and B_2 is a basis of W_2 then

$$B = B_1 \cup B_2$$

is a basis of V. If we form the matrix of T w.r.t. B then we get

$$A = \begin{bmatrix} A_{W_1} & 0 \\ 0 & A_{W_2} \end{bmatrix}.$$

Let $k_1 = \dim W_1, k_2 = \dim W_2$, then

$$\begin{aligned} & \det(A - tI) = \det(A - tI_{k_1}) \cdot \det(A - tI_{k_2}) \\ \iff \quad & p(t) = p_1(t)p_2(t). \end{aligned}$$

So the proposition is proven for the base case $n = 2$.

For the induction step, assume that the proposition holds for some $n-1 \geq 2 \in \mathbb{N}$. Then

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_n = (W_1 \oplus W_2 \oplus \cdots \oplus W_{n-1}) \oplus W_n.$$

If we let

$$W' = W_1 \oplus W_2 \oplus \cdots \oplus W_{n-1}$$

then W' is a T-invariant subspace of V whose characteristic polynomial, by the induction hypothesis, takes the form

$$p_{W'}(t) = p_1(t)p_2(t)\cdots p_{n-1}(t)$$

but we also have

$$W' \oplus W_n = V$$

which, by the base case proof means that the characteristic polynomial of T can be expressed as

$$p(t) = p_{W'}(t)p_n(t) = p_1(t)p_2(t)\cdots p_{n-1}(t)p_n(t).$$

□

2.5.4.5 Examples of Diagonalization using the Characteristic Polynomial

(64) Let the real-valued matrix A be,

$$A = \begin{bmatrix} 4 & 0 & 4 \\ 0 & 4 & 4 \\ 4 & 4 & 8 \end{bmatrix}.$$

Constructing the characteristic polynomial,

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 4-\lambda & 0 & 4 \\ 0 & 4-\lambda & 4 \\ 4 & 4 & 8-\lambda \end{vmatrix} \\ &= (4-\lambda)\begin{vmatrix} 4-\lambda & 4 \\ 4 & 8-\lambda \end{vmatrix} + 4\begin{vmatrix} 0 & 4-\lambda \\ 4 & 4 \end{vmatrix} \end{aligned}$$

$$
\begin{aligned}
&= (4-\lambda)((4-\lambda)(8-\lambda) - 16 - 16) \\
&= (4-\lambda)(\lambda^2 - 12\lambda) \\
&= (4-\lambda)\lambda(\lambda - 12).
\end{aligned}
$$

So the eigenvalues are 4,0 and 12. To find an eigenvector for 4 we need to solve the equation $(A - 4I)\vec{\boldsymbol{x}} = \vec{\mathbf{0}}$ so we construct the matrix,

$$
A - 4I = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 0 & 4 \\ 4 & 4 & 4 \end{bmatrix}
$$

and then find the nullspace of this matrix using row reduction,

$$
\begin{aligned}
\begin{bmatrix} 0 & 0 & 4 \\ 0 & 0 & 4 \\ 4 & 4 & 4 \end{bmatrix} &= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

which gives $x_3 = 0$ and one free variable x_2. So

$$
\vec{\boldsymbol{x}} = \begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \qquad t \neq 0 \in \mathbb{R}.
$$

Note that $t \neq 0$ because eigenvectors are nonzero by definition.

(65) Let $T : \mathbb{R}^4 \longmapsto \mathbb{R}^4$ be defined as,

$$
T(a, b, c, d) = (a + b + 2c - d,\ b + d,\ 2c - d,\ c + d).
$$

The subspace $W = \{\, (x, y, 0, 0) \mid x, y \in \mathbb{R} \,\}$ is a T-invariant subspace of $\mathbb{R}^4$ as,

$$
T(x, y, 0, 0) = (x + y,\ y,\ 0,\ 0).
$$

Clearly, the standard basis of the xy-plane, $\{\vec{\boldsymbol{e_1}}, \vec{\boldsymbol{e_2}}\}$ is a basis of W. This basis of W can be extended to the standard basis of $\mathbb{R}^4$ and the matrix of T w.r.t. this basis is

$$A = \begin{bmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

which contains the matrix for the restriction of T to W w.r.t. to the standard basis of the xy-plane in the top left corner,

$$A_w = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

So, we can see that the characteristic polynomial of A_w divides that of A as,

$$\det(A - tI_4) = \det(A_w - tI_2) \cdot \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix}.$$

(66) Continuing the example 60, we have $T : \mathbb{R}^3 \longmapsto \mathbb{R}^3$ defined by,

$$T(a, b, c) = (-b + c,\ a + c,\ 3c)$$

and the T-cyclic subspace generated by $\vec{\boldsymbol{e_1}}$,

$$W = \text{span}\{\vec{\boldsymbol{e_1}}, \vec{\boldsymbol{e_2}}\} = \{\ (x, y, 0)\ \mid\ x, y \in \mathbb{R}\ \}.$$

The restriction of T to W is

$$T_w = T(a, b) = (-b, a).$$

If we form the matrix of T_w

$$A_w = [T_w(\{\vec{\boldsymbol{e_1}}, \vec{\boldsymbol{e_2}}\})] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

then the characteristic polynomial generated by the matrix method is

$$\begin{vmatrix} -t & -1 \\ 1 & -t \end{vmatrix} = t^2 + 1.$$

Alternatively, we could apply Proposition 2.5.31 which tells us that, since in W we have

$$T^2\,\vec{\boldsymbol{e_1}} = T(T\vec{\boldsymbol{e_1}}) = T\vec{\boldsymbol{e_2}} = -\vec{\boldsymbol{e_1}}$$

then the characteristic polynomial of T_w is

$$p(t) = (-1)^2(t^2+1) = t^2+1.$$

2.5.4.6 Repeated Eigenvalues

It's not every matrix that is diagonalizable because we can only form a diagonal matrix when there is a complete set of eigenvectors of the matrix that form a basis of the domain of the matrix. However, if we refer to the Fundamental Theorem of Algebra Theorem 2.2.2 we see that there will always be (when counted with multiplicity) n roots of a degree-n non-constant polynomial with complex coefficients. So, if we consider our matrix to be defined over the complex field, then we will always have, counted with multiplicity, n roots of a characteristic polynomial of a $n \times n$ matrix. The reason is that, when we don't have a complete eigenbasis, we have one or more roots of the characteristic polynomial with a multiplicity greater than one – or, in other words, repeated roots.

Consider, for example, the matrix

$$A = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}$$

that results in the characteristic polynomial

$$t^2 - 2at + a^2 = 0 \implies (t-a)(t-a) = (t-a)^2 = 0.$$

This implies that we have a single eigenvalue $t = a$, repeated twice. In this case, we cannot form an eigenbasis of the space and a diagonalized matrix of A. However, we can still find two eigenvectors, both of which will be in the

eigenspace of the eigenvalue $t = a$. The reasoning is as follows: If we find the eigenvector $\vec{\boldsymbol{v}}_1$ for the eigenvalue in the usual way,

$$(A - aI)\vec{\boldsymbol{v}}_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \vec{\boldsymbol{v}}_1 = \vec{\mathbf{0}} \implies \vec{\boldsymbol{v}}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

But we also have a second eigenvector $\vec{\boldsymbol{v}}_2$ such that,

$$(A - aI)^2 \vec{\boldsymbol{v}}_2 = (A - aI)(A - aI)\vec{\boldsymbol{v}}_2 = \vec{\mathbf{0}}.$$

We can leverage the first eigenvector to solve this:

$$(A - aI)\vec{\boldsymbol{v}}_2 = \vec{\boldsymbol{v}}_1 \implies (A - aI)[(A - aI)\vec{\boldsymbol{v}}_2] = (A - aI)\vec{\boldsymbol{v}}_1 = 0.$$

So, if we can find a vector $\vec{\boldsymbol{v}}_2$ such that,

$$(A - aI)\vec{\boldsymbol{v}}_2 = \vec{\boldsymbol{v}}_1 \iff A\vec{\boldsymbol{v}}_2 = a\vec{\boldsymbol{v}}_2 + \vec{\boldsymbol{v}}_1$$

then $\vec{\boldsymbol{v}}_2$ is also a kind of eigenvector with eigenvalue a. This kind of eigenvector is known as a *generalized eigenvector*.

In the example here it is easy to see that,

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \vec{\boldsymbol{v}}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \implies \vec{\boldsymbol{v}}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and together, $\vec{\boldsymbol{v}}_1$ and $\vec{\boldsymbol{v}}_2$ form a basis of the two-dimensional domain of the matrix A. In fact,

$$A = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & 0 \\ 1 & a \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

or, alternatively,

$$A = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Proposition 2.5.33. *The multiplicities of the eigenvalues of a linear operator sum to the dimension of the vector space over which it is defined.*

Proof. It is easy to see that, if the characteristic polynomial of a linear operator factorizes to

$$p(t) = (t - \lambda_1)^{m_1}(t - \lambda_2)^{m_2} \cdots (t - \lambda_k)^{m_k}$$

then the highest power of t is $t^{m_1+m_2+\cdots+m_k}$ and so the degree of the polynomial is $\sum_i m_i$, the sum of the multiplicities of the eigenvalues.
Furthermore, it is also clear from the definition of the characteristic polynomial (2.5.4.4) that – since the variable t only appears in the diagonal elements of the matrix of the operator – the degree of the characteristic polynomial is equal to the number of diagonal elements in the matrix; that's to say, the dimension of the space. □

Proposition 2.5.34. *The dimension of an eigenspace is less than or equal to the multiplicity of the corresponding eigenvalue.*

Proof. Let T be a linear operator over a finite-dimensional vector space V of dimension n and let λ be an eigenvalue of T. Further, let E_λ be the eigenspace corresponding to the eigenvalue λ and let $d = \dim E_\lambda$ and m be the multiplicity of λ.
Then, there exists a d-length basis for E_λ. Let this basis be $\{\vec{\boldsymbol{x}}_1, \ldots, \vec{\boldsymbol{x}}_d\}$ and extend it to a basis of V,

$$B = \{\vec{\boldsymbol{x}}_1, \ldots, \vec{\boldsymbol{x}}_d, \vec{\boldsymbol{x}}_{d+1}, \ldots, \vec{\boldsymbol{x}}_n\}.$$

If we construct the matrix of T w.r.t. to this basis of V, we see that the block form of the matrix comes out,

$$[T]_B = [T(B)] = \begin{bmatrix} \lambda I_d & A \\ 0 & C \end{bmatrix}$$

so that the characteristic polynomial of T (which is the same against any basis by Proposition 2.5.26),

$$\begin{aligned} p(t) &= \det((\lambda - t)I_d) \cdot \det(C - tI_{n-d}) \\ &= (t - \lambda)^d \cdot \det(C - tI_{n-d}). \end{aligned}$$

Therefore, the multiplicity of λ is at least d and so we have $m \geq d$ as required. □

Proposition 2.5.35. *A linear operator over a finite-dimensional vector space such that the characteristic polynomial factorizes, is diagonalizable iff for each eigenvalue, the dimension of the eigenspace is equal to the multiplicity of the eigenvalue.*

Proof. Let T be a linear operator over a finite-dimensional vector space V of dimension n such that the characteristic polynomial of T contains the eigenvalue λ with multiplicity m.

Matrix-based proof

Assume T is diagonalizable so that there exists a basis such that the matrix of T w.r.t. to this basis, is diagonal. Since the eigenvalue λ in the characteristic polynomial has multiplicity m then there must m columns of the matrix that have λ as the diagonal element. Therefore the basis vectors that are eigenvectors of λ are the vectors corresponding to these m columns. The eigenspace of λ is therefore m-dimensional.

Conversely, assume that, for every eigenvalue λ with multiplicity m, the dimension of the eigenspace of λ is m. Then the number of columns with the sole non-zero element being λ in the diagonal position is m. Since this applies to every eigenvalue, the total number of columns of the matrix with the only non-zero element being the eigenvalue in the diagonal position is the sum of the multiplicities which, by the definition of the characteristic polynomial, is the number of columns in the matrix and hence, the dimension of V.

Non matrix-based proof

Suppose that T is diagonalizable and let B be a basis of V consisting of eigenvectors of T. Let the distinct eigenvalues of T be $\{ \lambda_i \mid 1 \leq i \leq k \}$ and let $B_i = B \cap E_{\lambda_i}$ be the subset of the basis B consisting of eigenvectors of the eigenvalue λ_i.
If we further let $n_i = |B_i|$ be the number of eigenvectors of λ_i in the basis B then, because B_i is a linearly independent set in E_{λ_i}, it follows that

$$\dim E_{\lambda_i} \geq n_i.$$

Also, by Proposition 2.5.34, if m_i is the multiplicity of λ_i, then

$$\dim E_{\lambda_i} \leq m_i.$$

Since B is a basis for V,

$$\dim V = n \implies |B| = n = \sum_i n_i$$

and also, by Proposition 2.5.33,

$$\sum_i m_i = \dim V = n.$$

Thus we have,

$$n = \sum_i n_i \leq \sum_i \dim E_{\lambda_i} \leq \sum_i m_i = n.$$

We can therefore conclude that

$$\sum_i \dim E_{\lambda_i} = n.$$

Furthermore, if there were any eigenspace E_{λ_i} such that $\dim E_{\lambda_i} < m_i$ then there would have to be another such that the dimension is greater than its multiplicity so that the sum of all the dimensions of the eigenspaces equals the sum of the multiplicities. But, by Proposition 2.5.34, there cannot be an eigenspace with dimension greater than its multiplicity. That's to say,

$$\begin{aligned} & \Big(\forall i \,.\, (m_i - \dim E_{\lambda_i}) \geq 0\Big) \wedge \left(\sum_i (m_i - \dim E_{\lambda_i}) = 0\right) \\ \implies \qquad & \forall i \,.\, (m_i - \dim E_{\lambda_i}) = 0. \end{aligned}$$

Therefore, we conclude that,

$$\forall i \,.\, \dim E_{\lambda_i} = m_i.$$

Conversely, suppose that $\forall i \,.\, \dim E_{\lambda_i} = m_i$. Then, for each eigenspace E_{λ_i}, there exists a basis B_i such that $|B_i| = m_i$. Furthermore, since the eigenspaces are linearly independent (Proposition 2.5.16), they form a direct sum

$$W = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_k}$$

where $\dim W = \sum_i m_i = n$.
Therefore, the union of the bases

$$B = \bigcup_i B_i$$

is a linearly independent set of length n and is, therefore, a basis for V consisting of eigenvectors of T. It follows, by the definition, that T is diagonalizable. $\square$

2.5.4.7 Generalized Eigenvectors

Definition 120. The **algebraic multiplicity** of an eigenvalue is the number of times that it appears as a root of the characteristic polynomial.

The **geometric multiplicity** of an eigenvalue is the dimension of its associated eigenspace.

An eigenvalue is said to be **defective** if its geometric multiplicity is less than its algebraic multiplicity.

Definition 121. A **generalized eigenvector of rank (or index) r** of a matrix A corresponding to an eigenvalue λ is defined as a vector $\vec{\boldsymbol{v}}$ such that,

$$(A - \lambda I)^r \, \vec{\boldsymbol{v}} = \vec{\mathbf{0}} \quad \text{and} \quad (A - \lambda I)^{r-1} \, \vec{\boldsymbol{v}} \neq \vec{\mathbf{0}}.$$

An eigenvector is a particular case of a generalized eigenvector with rank 1 as,

$$(A - \lambda I)^1 \, \vec{\boldsymbol{v}} = \vec{\mathbf{0}} \iff (A - \lambda I) \, \vec{\boldsymbol{v}} = \vec{\mathbf{0}}$$

and we have the restriction that an eigenvector cannot be $\vec{\mathbf{0}}$ so also,

$$(A - \lambda I)^0 \, \vec{\boldsymbol{v}} = \vec{\boldsymbol{v}} \neq \vec{\mathbf{0}}.$$

Definition 122. The generalized eigenspace of an eigenvalue λ of a linear operator T over a vector space V is the set

$$K_\lambda = \{ \, \vec{\boldsymbol{v}} \in V \mid \exists k \in \mathbb{N} \text{ s.t. } (T - \lambda I)^k \, \vec{\boldsymbol{v}} = \vec{\mathbf{0}} \, \}.$$

Proposition 2.5.36. *The generalized eigenspace of an eigenvalue λ of a linear operator T over a vector space V is a T-invariant subspace of V that contains the eigenspace of λ.*

Proof. Let K_λ be the generalized eigenspace of an eigenvalue λ. K_λ is a subspace of the vector space V because:

- It contains the zero vector because $(T - \lambda I)^k \vec{\mathbf{0}} = \vec{\mathbf{0}}$ for any $k \in \mathbb{N}$;
- If $\vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2$ are members of K_λ such that

  $$(T - \lambda I)^p \vec{\boldsymbol{v}}_1 = \vec{\mathbf{0}} = (T - \lambda I)^q \vec{\boldsymbol{v}}_2$$

  for some $p, q \in \mathbb{N}$, then

  $$(T - \lambda I)^{p+q} (\vec{\boldsymbol{v}}_1 + \vec{\boldsymbol{v}}_2) = (T - \lambda I)^q (\vec{\mathbf{0}}) + (T - \lambda I)^p (\vec{\mathbf{0}}) = \vec{\mathbf{0}}.$$

K_λ is T-invariant because if $\vec{\boldsymbol{v}}$ is a member of K_λ with index p then,

$$(T - \lambda I)^p \vec{\boldsymbol{v}} = \vec{\mathbf{0}} \iff (T - \lambda I)^p T\vec{\boldsymbol{v}} = T[(T - \lambda I)^p \vec{\boldsymbol{v}}] = T\vec{\mathbf{0}} = \vec{\mathbf{0}}.$$

Therefore $\vec{\boldsymbol{v}} \in K_\lambda \implies T\vec{\boldsymbol{v}} \in K_\lambda$.

Finally, K_λ clearly contains the eigenspace of λ as this is just

$$\{ \vec{\boldsymbol{v}} \in V \mid (T - \lambda I)^k \vec{\boldsymbol{v}} = \vec{\mathbf{0}} \}$$

where $k = 1$. □

Proposition 2.5.37. *If T is a linear operator on a vector space V with eigenvalue λ then T-invariance is equivalent to $(T - \lambda I)$-invariance.*

Proof. Let W be a T-invariant subspace of V. Then, for any $\vec{\boldsymbol{w}} \in W$,

$$T\vec{\boldsymbol{w}} \in W,\ \vec{\boldsymbol{w}} \in W \implies T\vec{\boldsymbol{w}} - \lambda\vec{\boldsymbol{w}} = (T - \lambda I)\vec{\boldsymbol{w}} \in W.$$

Conversely, if W is a $(T - \lambda I)$-invariant subspace of V, then, for any $\vec{\boldsymbol{w}} \in W$,

$$(T - \lambda I)\vec{\boldsymbol{w}} \in W,\ \vec{\boldsymbol{w}} \in W \implies T\vec{\boldsymbol{w}} - \lambda\vec{\boldsymbol{w}} + \lambda\vec{\boldsymbol{w}} = T\vec{\boldsymbol{w}} \in W.$$

□

Proposition 2.5.38. *If T is a linear operator on a vector space V and $p(t)$ is a polynomial with coefficients in the same field as V, then T-invariance is equivalent to $p(T)$-invariance.*

Proof. TODO: This is easily shown by expanding the polynomial to monomial terms that are a linear combination of the basis vectors of a T-cyclic space. □

Proposition 2.5.39. *If a vector $\vec{\boldsymbol{v}}_r$ is a generalized eigenvector of rank r corresponding to the eigenvalue λ of the linear transformation T then,*

$$\vec{\boldsymbol{v}} = (T - \lambda I)^{r-1} \vec{\boldsymbol{v}}_r$$

is an eigenvector of T with eigenvalue λ.

Proof. By the definition of a generalized eigenvector of rank r we have,

$$(T - \lambda I)^r \vec{\boldsymbol{v}}_r = \vec{\boldsymbol{0}} \iff (T - \lambda I)(T - \lambda I)^{r-1} \vec{\boldsymbol{v}}_r = \vec{\boldsymbol{0}}$$

but also

$$(T - \lambda I)^{r-1} \vec{\boldsymbol{v}}_r \neq \vec{\boldsymbol{0}}$$

which also implies that

$$(T - \lambda I)^m \vec{\boldsymbol{v}}_r \neq \vec{\boldsymbol{0}}$$

for any $m < r - 1$. It therefore follows that also,

$$(T - \lambda I) \vec{\boldsymbol{v}}_r \neq \vec{\boldsymbol{0}}.$$

So we must have,

$$(T - \lambda I)^{r-1} \vec{\boldsymbol{v}}_r = \vec{\boldsymbol{v}} \text{ s.t. } (T - \lambda I)\vec{\boldsymbol{v}} = \vec{\boldsymbol{0}}.$$

Therefore $\vec{\boldsymbol{v}} = (T - \lambda I)^{r-1} \vec{\boldsymbol{v}}_r$ is an eigenvector of the transformation T. □

Proposition 2.5.40. *The generalized eigenspaces of distinct eigenvalues are linearly independent spaces.*

Proof. Let T be a linear operator on a (not necessarily finite) vector space defined over a field $\mathbb{F}$, and let $\{ \lambda_i \mid 1 \leq i \leq k \}$ be a (not necessarily exhaustive) set of distinct eigenvalues of T. Let K_{λ_i} be the generalized eigenspace corresponding to the eigenvalue λ_i and let $\vec{\boldsymbol{v}}_i \in K_{\lambda_i}$ be an arbitrary vector in the generalized eigenspace of λ_i. The proposition is proven if it is shown that

$$\vec{\boldsymbol{v}}_1 + \vec{\boldsymbol{v}}_2 + \cdots + \vec{\boldsymbol{v}}_k = \vec{\boldsymbol{0}} \implies \vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2, \ldots, \vec{\boldsymbol{v}}_k = \vec{\boldsymbol{0}}.$$

Following a proof by induction on the number of distinct eigenvalues k, if we take $k = 1$ as the base case, the proposition holds trivially as

$$\vec{\boldsymbol{v}}_1 = \vec{\boldsymbol{0}} \implies \vec{\boldsymbol{v}}_1 = \vec{\boldsymbol{0}}.$$

For the induction step, assume that the proposition holds for some $k - 1 \geq 1 \in \mathbb{N}$. Now identify two possible cases: $\vec{\boldsymbol{v}}_k = \vec{\boldsymbol{0}}$ and $\vec{\boldsymbol{v}}_k \neq \vec{\boldsymbol{0}}$.

Assume $\vec{\boldsymbol{v}}_k = \vec{\boldsymbol{0}}$. Then we have,

$$\vec{\boldsymbol{v}}_1 + \vec{\boldsymbol{v}}_2 + \cdots + \vec{\boldsymbol{v}}_{k-1} + \vec{\boldsymbol{v}}_k = \vec{\boldsymbol{v}}_1 + \vec{\boldsymbol{v}}_2 + \cdots + \vec{\boldsymbol{v}}_{k-1} = \vec{\boldsymbol{0}}$$

which satisfies the proposition by the induction hypothesis so that

$$\vec{\boldsymbol{v}}_1 + \vec{\boldsymbol{v}}_2 + \cdots + \vec{\boldsymbol{v}}_{k-1} = \vec{\boldsymbol{0}} \implies \vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2, \ldots, \vec{\boldsymbol{v}}_{k-1} = \vec{\boldsymbol{0}}.$$

Since, also, $\vec{\boldsymbol{v}}_k = \vec{\boldsymbol{0}}$ by assumption, the proposition holds.

Conversely, assume that $\vec{\boldsymbol{v}}_k \neq \vec{\boldsymbol{0}}$ and that we have a linear relation so that,

$$\vec{\boldsymbol{v}}_1 + \vec{\boldsymbol{v}}_2 + \cdots + \vec{\boldsymbol{v}}_k = \vec{\boldsymbol{0}} \implies \vec{\boldsymbol{v}}_k = -(\vec{\boldsymbol{v}}_1 + \vec{\boldsymbol{v}}_2 + \cdots + \vec{\boldsymbol{v}}_{k-1}).$$

Let $r_i \in \mathbb{N}$ be the rank of each $\vec{\boldsymbol{v}}_i$ so that,

$$(T - \lambda_i I)^{r_i} \vec{\boldsymbol{v}}_i = \vec{\boldsymbol{0}}$$

and, using Proposition 2.5.39, define the eigenvectors $\vec{\boldsymbol{x}}_i$,

$$\vec{\boldsymbol{x}}_i = (T - \lambda_i I)^{r_i - 1} \vec{\boldsymbol{v}}_i.$$

The linear relation implies,

$$\begin{aligned}
& \vec{\boldsymbol{v}}_1 + \vec{\boldsymbol{v}}_2 + \cdots + \vec{\boldsymbol{v}}_k = \vec{\mathbf{0}} \\
\implies \quad & (T - \lambda_1 I)^{r_1} \vec{\boldsymbol{v}}_1 + (T - \lambda_1 I)^{r_1} \vec{\boldsymbol{v}}_2 + \cdots + (T - \lambda_1 I)^{r_1} \vec{\boldsymbol{v}}_k = \vec{\mathbf{0}} \\
\implies \quad & \vec{\mathbf{0}} + (T - \lambda_1 I)^{r_1} \vec{\boldsymbol{v}}_2 + \cdots + (T - \lambda_1 I)^{r_1} \vec{\boldsymbol{v}}_k = \vec{\mathbf{0}} \\
\implies \quad & (T - \lambda_1 I)^{r_1} (T - \lambda_2 I)^{r_2} \vec{\boldsymbol{v}}_2 + \cdots + (T - \lambda_1 I)^{r_1} (T - \lambda_2 I)^{r_2} \vec{\boldsymbol{v}}_k = \vec{\mathbf{0}} \\
\implies \quad & (T - \lambda_1 I)^{r_1} (T - \lambda_2 I)^{r_2} \cdots (T - \lambda_k I)^{r_k} \vec{\boldsymbol{v}}_k = \vec{\mathbf{0}}.
\end{aligned}$$

If we define the polynomial with coefficients in $\mathbb{F}$,

$$p(t) = (t - \lambda_1)^{r_1} (t - \lambda_2)^{r_2} \cdots (t - \lambda_{k-1})^{r_{k-1}}$$

then, using 2.5.3, the linear relation implies that

$$(T - \lambda_1)^{r_1} (T - \lambda_2)^{r_2} \cdots (T - \lambda_k)^{r_k} \vec{\boldsymbol{v}}_k = \vec{\mathbf{0}} \implies p(T)\vec{\boldsymbol{v}}_k = \vec{\mathbf{0}}.$$

However, we also have the eigenvector $\vec{\boldsymbol{x}}_k$,

$$\vec{\boldsymbol{x}}_k = (T - \lambda_k I)^{r_k - 1} \vec{\boldsymbol{v}}_k$$

and by Proposition 2.5.29,

$$p(T)\vec{\boldsymbol{x}}_k = (\lambda_k - \lambda_1)(\lambda_k - \lambda_2) \cdots (\lambda_k - \lambda_{k-1}) \neq \vec{\mathbf{0}}.$$

And so, for the eigenvector $\vec{\boldsymbol{x}}_k$,

$$\begin{aligned}
& p(T)\vec{\boldsymbol{x}}_k = p(T)(T - \lambda_k I)^{r_k - 1} \vec{\boldsymbol{v}}_k \neq \vec{\mathbf{0}} \\
\iff \quad & (T - \lambda_k I)^{r_k - 1} p(T)\vec{\boldsymbol{v}}_k \neq \vec{\mathbf{0}} \quad \text{commutativity by 2.5.3} \\
\implies \quad & p(T)\vec{\boldsymbol{v}}_k \neq \vec{\mathbf{0}}
\end{aligned}$$

which contradicts the implications of the linear relation. □

Proposition 2.5.41. *Let T be a linear operator on a finite-dimensional vector space V with distinct eigenvalues $\{ \lambda_i \mid 1 \leq i \leq k \}$. For each i, let S_i be a linearly independent subset of the generalized eigenspace K_{λ_i}. Then,*

$$S_i \cap S_j = \emptyset \quad \textit{for} \quad i \neq j$$

and $S = S_1 \cup S_2 \cup \cdots \cup S_k$ is a linearly independent subset of V.

Proof. Suppose that $\vec{\boldsymbol{x}} \in S_i \cap S_j$ for some $i \neq j$. Let $\vec{\boldsymbol{y}} = -\vec{\boldsymbol{x}}$. Then $\vec{\boldsymbol{x}} \in K_{\lambda_i}$, $\vec{\boldsymbol{y}} \in K_{\lambda_j}$, and $\vec{\boldsymbol{x}} + \vec{\boldsymbol{y}} = \vec{\boldsymbol{0}}$. By Proposition 2.5.40, $\vec{\boldsymbol{x}} = \vec{\boldsymbol{0}}$, contrary to the fact that $\vec{\boldsymbol{x}}$ lies in a linearly independent set. Thus $S_i \cap S_j = \emptyset$.

Now suppose that for each i

$$S_i = \{\vec{\boldsymbol{x}}_{i1}, \vec{\boldsymbol{x}}_{i2}, \ldots, \vec{\boldsymbol{x}}_{in_i}\}.$$

Then

$$S = \{\, \vec{\boldsymbol{x}}_{ij} \mid 1 \leq j \leq n_i, 1 \leq i \leq k \,\}.$$

Consider any scalars a_{ij} such that

$$\sum_{i=1}^{k} \sum_{j=1}^{n_i} a_{ij} \vec{\boldsymbol{x}}_{ij} = \vec{\boldsymbol{0}}.$$

For each i let

$$\vec{\boldsymbol{y}}_i = \sum_{j=1}^{n_i} a_{ij} \vec{\boldsymbol{x}}_{ij}.$$

Then $\vec{\boldsymbol{y}}_i \in K_{\lambda_i}$ for each i and $\vec{\boldsymbol{y}}_1 + \vec{\boldsymbol{y}}_2 + \cdots + \vec{\boldsymbol{y}}_k = \vec{\boldsymbol{0}}$. Therefore, by Proposition 2.5.40, $\vec{\boldsymbol{y}}_i = \vec{\boldsymbol{0}}$ for all i. But for all i, S_i is linearly independent by hypothesis. Thus, for each i, it follows that $a_{ij} = 0$ for all j. We conclude that S is linearly independent. □

Chains/Cycles of Generalized Eigenvectors

Definition 123. Let T be a linear transformation of which λ is an eigenvalue and $\vec{\boldsymbol{v}}$ is a generalized eigenvector of rank r. Then the set

$$\{\, (T - \lambda I)^{r-1} \vec{\boldsymbol{v}},\ (T - \lambda I)^{r-2} \vec{\boldsymbol{v}},\ \ldots,\ \vec{\boldsymbol{v}} \,\}$$

is known as a **chain** or **cycle** of generalized eigenvectors of length r. The elements $(T-\lambda I)^{r-1} \vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{v}}$ are known as the **initial vector** and **end vector** respectively of the chain or cycle.

Note that a chain or cycle of generalized eigenvectors is a subset of a generalized eigenspace. There may be more than one chain/cycle in a given generalized eigenspace.

Theorem 2.5.8. *The generalized eigenvectors in a chain are linearly independent.*

Proof. Prove by induction on the length of the chain.

Base case: length = 2

A chain of length 2 corresponding to the eigenvector λ of a linear transformation T, has the form

$$C_2 = \{(T - \lambda I)\vec{\boldsymbol{v}},\, \vec{\boldsymbol{v}}\}$$

where $\vec{\boldsymbol{v}}$ is a generalized eigenvector of rank 2. Then also, by the definition of the generalized eigenvector of rank 2,

$$(T - \lambda I)^2\, \vec{\boldsymbol{v}} = \vec{\mathbf{0}} \quad \text{and} \quad (T - \lambda I)\vec{\boldsymbol{v}} \neq \vec{\mathbf{0}}.$$

Assume, for contradiction, that the chain is not linearly independent. Then there exists a linear relation between the elements of the chain,

$$\alpha_1(T - \lambda I)\, \vec{\boldsymbol{v}} + \alpha_2 \vec{\boldsymbol{v}} = \vec{\mathbf{0}}$$

where $\alpha_1, \alpha_2 \neq 0$ are constant scalars. This further implies, defining $\alpha = -\frac{\alpha_1}{\alpha_2}$, that

$$\vec{\boldsymbol{v}} = \alpha(T - \lambda I)\, \vec{\boldsymbol{v}} \implies (T - \lambda I)\vec{\boldsymbol{v}} = \alpha(T - \lambda I)^2\, \vec{\boldsymbol{v}} = \vec{\mathbf{0}}.$$

But $(T - \lambda I)\vec{\boldsymbol{v}} = \vec{\mathbf{0}}$ contradicts the definition of $\vec{\boldsymbol{v}}$ as a generalized eigenvector of rank 2. It follows then, that a chain of length 2 is linearly independent.

Induction step: length = k

A chain of length k corresponding to the eigenvector λ of a linear transformation T, has the form

$$C_k = \{(T - \lambda I)^{k-1}\, \vec{\boldsymbol{v}},\, (T - \lambda I)^{k-2}\, \vec{\boldsymbol{v}},\, \ldots,\, (T - \lambda I)\vec{\boldsymbol{v}},\, \vec{\boldsymbol{v}}\}$$

where $\vec{\boldsymbol{v}}$ is a generalized eigenvector of rank k. Then also, by the definition of the generalized eigenvector of rank k,

$$(T - \lambda I)^k\, \vec{\boldsymbol{v}} = \vec{\mathbf{0}} \quad \text{and} \quad (T - \lambda I)^{k-1}\, \vec{\boldsymbol{v}} \neq \vec{\mathbf{0}}.$$

It follows from the fact that $\vec{\boldsymbol{v}}$ is a generalized eigenvector of rank k that $\vec{\boldsymbol{w}} = (T - \lambda I)\vec{\boldsymbol{v}}$ is a generalized eigenvector of rank $k-1$ because,

$$(T-\lambda I)^{k-1}\,\vec{\boldsymbol{w}} = (T-\lambda I)^{k}\,\vec{\boldsymbol{v}} = \vec{\mathbf{0}} \quad \text{and} \quad (T-\lambda I)^{k-2}\,\vec{\boldsymbol{w}} = (T-\lambda I)^{k-1}\,\vec{\boldsymbol{v}} \neq \vec{\mathbf{0}}.$$

Therefore

$$C_{k-1} = \{(T-\lambda I)^{k-1}\,\vec{\boldsymbol{v}},\, (T-\lambda I)^{k-2}\,\vec{\boldsymbol{v}},\, \ldots,\, (T-\lambda I)\vec{\boldsymbol{v}}\}$$

is a $(k-1)$-length chain and is, by the induction hypothesis, linearly independent.

It follows that C_k will be linearly independent iff $\vec{\boldsymbol{v}}$ is not in the span of C_{k-1}. Assume for contradiction, that $\vec{\boldsymbol{v}}$ is indeed in the span of C_{k-1}. Then $\vec{\boldsymbol{v}}$ can be expressed as a linear combination of the elements of C_{k-1},

$$\vec{\boldsymbol{v}} = \alpha_1(T-\lambda I)^{k-1}\,\vec{\boldsymbol{v}} + \alpha_2(T-\lambda I)^{k-2}\,\vec{\boldsymbol{v}} + \cdots + \alpha_{k-1}(T-\lambda I)\vec{\boldsymbol{v}}.$$

But this implies that,

$$\begin{aligned}(T-\lambda I)^{k-1}\,\vec{\boldsymbol{v}} &= \alpha_1(T-\lambda I)^{k-2}((T-\lambda I)^{k}\,\vec{\boldsymbol{v}}) \\ &\quad + \alpha_2(T-\lambda I)^{k-3}((T-\lambda I)^{k}\,\vec{\boldsymbol{v}}) + \cdots + \alpha_{k-1}((T-\lambda I)^{k}\,\vec{\boldsymbol{v}}) \\ &= \alpha_1(T-\lambda I)^{k-2}\,(\vec{\mathbf{0}}) + \alpha_2(T-\lambda I)^{k-3}(\vec{\mathbf{0}}) + \cdots + \alpha_{k-1}(\vec{\mathbf{0}}) \\ &= \vec{\mathbf{0}}.\end{aligned}$$

Since $(T-\lambda I)^{k-1}\,\vec{\boldsymbol{v}} = \vec{\mathbf{0}}$ contradicts the definition of $\vec{\boldsymbol{v}}$ as a generalized eigenvector of rank k, we can deduce that $\vec{\boldsymbol{v}}$ cannot be in the span of C_{k-1}. □

Theorem 2.5.9. *The generalized eigenvectors in a chain form the basis of a T-invariant space.*

Proof. Let C_k be a chain/cycle of length k corresponding to the eigenvector λ of a linear transformation T so that,

$$C_k = \{(T-\lambda I)^{k-1}\,\vec{\boldsymbol{v}},\, (T-\lambda I)^{k-2}\,\vec{\boldsymbol{v}},\, \ldots,\, (T-\lambda I)\vec{\boldsymbol{v}},\, \vec{\boldsymbol{v}}\}.$$

By Theorem 2.5.8 we have that C_k is linearly independent so the proposition will be proven if we can show that the space spanned by C_k is T-invariant.

Note that it may be tempting to use the proof used in Proposition 2.5.36: For any $\vec{\boldsymbol{x}} \in C_k$ we have some $m \leq k \in \mathbb{N}$ such that

$$(T - \lambda I)^m \vec{\boldsymbol{x}} = \vec{\mathbf{0}}.$$

Therefore, for any $n \in \mathbb{N}$,

$$(T - \lambda I)^m T^n \vec{\boldsymbol{x}} = T^n (T - \lambda I)^m \vec{\boldsymbol{x}} = T^n (\vec{\mathbf{0}}) = \vec{\mathbf{0}}.$$

*But this only proves that $T^n \vec{\boldsymbol{x}}$ is a generalized eigenvector with eigenvalue λ, it does **not** prove that it is in the span of the chain/cycle C_k. Remember, there may be more than one chain/cycle in a generalized eigenspace.*

$$\begin{aligned} T\vec{\boldsymbol{v}} &= (T - \lambda I)\vec{\boldsymbol{v}} + \lambda\vec{\boldsymbol{v}}, \\ T(T - \lambda I)\vec{\boldsymbol{v}} &= (T - \lambda I)^2 \vec{\boldsymbol{v}} + \lambda T\vec{\boldsymbol{v}} - \lambda^2\vec{\boldsymbol{v}} \\ &= (T - \lambda I)^2 \vec{\boldsymbol{v}} + \lambda(T - \lambda I)\vec{\boldsymbol{v}}. \end{aligned}$$

For any $m \in \mathbb{N}$ such that $1 \leq m \leq k - 1$,

$$\begin{aligned} T(T - \lambda I)^m \vec{\boldsymbol{v}} &= (T - \lambda I)(T - \lambda I)^m \vec{\boldsymbol{v}} + \lambda(T - \lambda I)^m \vec{\boldsymbol{v}} \\ &= (T - \lambda I)^{m+1} \vec{\boldsymbol{v}} + \lambda(T - \lambda I)^m \vec{\boldsymbol{v}}. \end{aligned}$$

When $m = k - 1$ we have the special case that,

$$\begin{aligned} T(T - \lambda I)^{k-1} \vec{\boldsymbol{v}} &= (T - \lambda I)^k \vec{\boldsymbol{v}} + \lambda(T - \lambda I)^{k-1} \vec{\boldsymbol{v}} \\ &= \vec{\mathbf{0}} + \lambda(T - \lambda I)^{k-1} \vec{\boldsymbol{v}} \end{aligned}$$

which reflects the fact that $(T - \lambda I)^{k-1} \vec{\boldsymbol{v}}$ is an eigenvector.

So, we have shown that $T\vec{\boldsymbol{x}}$ for any $\vec{\boldsymbol{x}} \in C_k$ is in the span of C_k. For any $n > 1 \in \mathbb{N}$,

$$\begin{aligned} & T^n (T - \lambda I)^m \vec{\boldsymbol{v}} = (T - \lambda I)T^{n-1} (T - \lambda I)^m \vec{\boldsymbol{v}} + \lambda(T - \lambda I)^m \vec{\boldsymbol{v}} \\ \iff \quad & = T^{n-1} (T - \lambda I)^{m+1} \vec{\boldsymbol{v}} + \lambda(T - \lambda I)^m \vec{\boldsymbol{v}} \end{aligned}$$

In the case that $m + 1 = k$,

$$T^{n-1} (T - \lambda I)^{m+1} \vec{\boldsymbol{v}} + \lambda(T - \lambda I)^m \vec{\boldsymbol{v}} = \vec{\mathbf{0}} + \lambda(T - \lambda I)^m \vec{\boldsymbol{v}}$$

which is clearly in the span of C_k. Otherwise,

$$\begin{aligned} & T^{n-1}\,(T-\lambda I)^{m+1}\,\vec{\boldsymbol{v}} + \lambda(T-\lambda I)^m\,\vec{\boldsymbol{v}} \in \operatorname{span} C_k \\ \iff \quad & T^{n-1}\,(T-\lambda I)^{m+1}\,\vec{\boldsymbol{v}} \in \operatorname{span} C_k. \end{aligned}$$

Therefore, by induction, the space spanned by C_k is shown to be T-invariant. □

Proposition 2.5.42. *Let T be a linear operator on a finite-dimensional vector space V and let Z_i for $1 \leq i \leq q$ be cycles of generalized eigenvectors of T corresponding to a single common eigenvalue λ.*
If $\vec{\boldsymbol{y}}_i$ is the initial vector in cycle i then, if the set $\{\,\vec{\boldsymbol{y}}_i \mid 1 \leq i \leq q\,\}$ is linearly independent then the sets Z_i are disjoint and their union

$$Z = \bigcup_{i=1}^{q} Z_i$$

is linearly independent.

Proof. To show that the cycles Z_i are disjoint, assume for contradiction that, for $i \neq j$,

$$\begin{aligned} & \exists \vec{\boldsymbol{x}} \in Z_i \cap Z_j \\ \implies \quad & \exists r_i, r_j \in \mathbb{N}\ .\ [\vec{\boldsymbol{y}}_i = (T-\lambda I)^{r_i-1}\,\vec{\boldsymbol{x}}] \wedge [\vec{\boldsymbol{y}}_j = (T-\lambda I)^{r_j-1}\,\vec{\boldsymbol{x}}]. \end{aligned}$$

If we now assume w.l.o.g. that $r_j \geq r_i$ so that $r = r_j - r_i \geq 0$, we have

$$\vec{\boldsymbol{y}}_j = (T-\lambda I)^r\,\vec{\boldsymbol{y}}_i = (T-\lambda I)^{r-1}\,(T-\lambda I)\vec{\boldsymbol{y}}_i = \vec{\boldsymbol{0}}$$

where $(T-\lambda I)\vec{\boldsymbol{y}}_i = \vec{\boldsymbol{0}}$ because $\vec{\boldsymbol{y}}_i$ is an eigenvector by hypothesis. But $\vec{\boldsymbol{y}}_j$ is also an eigenvector by hypothesis and therefore, by definition, cannot be $\vec{\boldsymbol{0}}$. It therefore follows that

$$\nexists\, \vec{\boldsymbol{x}} \in Z_i \cap Z_j \implies Z_i \cap Z_j = \emptyset$$

and the cycles Z_i are disjoint.

To show that Z is linearly independent, we will use induction on $n = |Z|$, the cardinality of the set Z. If $n = 1$ then the proposition holds trivially because

the cycles, by definition, contain only non-zero vectors.

Assume that, for some $n > 1$, the proposition holds for any number less than n. We will show that Z is a basis of W, the space that it generates. Since W is clearly $(T - \lambda I)$-invariant, we can define the restriction of $(T - \lambda I)$ to W and denote it $(T - \lambda I)_W$.

Firstly, note that the image under $(T - \lambda I)$ of Z_i, is equal to Z_i but with the end vector swapped for $\vec{\mathbf{0}}$. So, if we define,

$$Z_i' = (T - \lambda I)Z_i \setminus \{\vec{\mathbf{0}}\}$$

then their union

$$Z' = \bigcup_{i=1}^{q} Z_i'$$

contains all the non-zero images under $(T - \lambda I)$ of the vectors in Z and therefore spans the range of $(T - \lambda I)_W$. Furthermore, this is also a disjoint union because, for each i, $Z_i' \subset Z_i$ and the sets Z_i are disjoint. So we can deduce that

$$\begin{aligned}
\left|Z'\right| &= \sum_{i=1}^{q} \left|Z_i'\right| \\
&= \sum_{i=1}^{q} (|Z_i| - 1) \\
&= \left(\sum_{i=1}^{q} |Z_i|\right) - q \\
&= |Z| - q \\
&= n - q.
\end{aligned}$$

Since the cardinality of Z' is less than n and it consists of cycles of generalized eigenvectors with the same set of linearly independent eigenvector initial vectors as Z, we can use the induction hypothesis to deduce that Z' is linearly independent. Since Z' spans the range of $(T - \lambda I)_W$ and is linearly independent, it is therefore a basis of the range of $(T - \lambda I)_W$ which, in turn, must therefore be of dimension $n - q$. Meanwhile, the kernel of $(T - \lambda I)_W$ is the set of eigenvectors $\{\, \vec{\boldsymbol{y}}_i \mid 1 \leq i \leq q \,\}$ and so has dimension q.

The space W is generated by Z – a finite set of vectors – and is thus finite-dimensional. So we can employ the dimension formula (Theorem 2.5.1) to deduce that,

$$\dim W = \text{rank}(T - \lambda I)_W + \text{nullity}(T - \lambda I)_W = (n - q) + q = n.$$

Since, by the induction hypothesis, $|Z| = n$ and this is equal to the dimension of W, the space that Z generates, it follows that Z is a basis of the space W. Therefore Z is linearly independent. □

2.5.4.8 Jordan Canonical Form

Definition 124. A **Jordan block** is a matrix (appearing as a block within a larger matrix) with the form,

$$\begin{bmatrix} \lambda & 1 & & & & \\ & \lambda & 1 & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ & & & & \ddots & \ddots \\ & & & & & \lambda & 1 \end{bmatrix}$$

or its transpose (depending on the ordering of the basis vectors).

Definition 125. A **Jordan** matrix or **Jordan form** or **Jordan Canonical form** matrix, is a matrix consisting solely of jordan blocks.

Definition 126. Let T be a linear operator on a vector space V such that the characteristic polynomial of T factorizes. A **Jordan Canonical Basis** is a basis for V that is a disjoint union of cycles of generalized eigenvectors of T.

Preliminary to Proof of Existence Jordan Canonical Basis

If $T(x,y) = (ax + y, ay)$ then, w.r.t. the standard basis, T is represented in matrix form as,

$$\begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}.$$

The x-axis is T-invariant because $T(x,0) = (ax, 0)$. So, the characteristic polynomial of the restriction of T to the x-axis divides the characteristic polynomial of T. The characteristic polynomial of T is $p(t) = (a-t)^2$ so that $(T - aI)^2 \vec{\boldsymbol{v}} = \vec{\mathbf{0}}$ for any $\vec{\boldsymbol{v}}$ in the space.

Let $W = R(T - aI)$ and define the restriction of T to W as T_W so that, for $\vec{\boldsymbol{w}} \in W$,

$$(T - aI)\vec{\boldsymbol{w}} = \vec{\mathbf{0}}.$$

What this looks like in matrix form is

$$(T - aI) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \textit{and} \quad W = R(T - aI) = \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \textit{for} \quad \alpha \in \mathbb{F}.$$

So, clearly, the jordan basis for T_W is the single vector $(1,0)^T = \vec{\boldsymbol{e_1}}$ and this is equal to the nullspace $N(T - aI)$ meaning that we have a cycle of length 2 associated with the eigenvalue a.

If, on the other hand, we had $T(x,y) = (ax, ay)$ with matrix w.r.t. the standard basis,

$$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix},$$

then $W = R(T - aI) = \vec{\mathbf{0}}$ with basis $\emptyset = \{\}$ and the nullspace $N(T - aI)$ is the whole space (in this case the xy-plane). So, in this case, we have no cycles of length > 1 associated with the eigenvalue a and, instead, we have two lone eigenvectors.

The converse case is $T(x,y) = (ax, by)$ with matrix w.r.t. the standard basis,

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}.$$

Then

$$(T - aI) = \begin{bmatrix} 0 & 0 \\ 0 & b-a \end{bmatrix} \quad \textit{and} \quad W = R(T-aI) = \alpha \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \textit{for} \quad \alpha \in \mathbb{F}.$$

In this case, the nullspace $N(T-aI) \not\subseteq R(T-aI)$ and there are no cycles of length > 1; just one lone eigenvector associated with the eigenvalue a. A basis of the space is completed by another eigenvector for a different eigenvalue b.

Theorem 2.5.10. *(**Jordan Canonical Basis.**) Let T be a linear operator on a vector space V such that the characteristic polynomial of T factorizes. Then there exists a Jordan canonical basis for T.*

Proof. Let V have dimension n. The proof will follow an induction on n.

For $n = 1$, the proposition holds trivially as any vector in the space is an eigenvector that spans the space.

Assume that the proposition holds for any vector space of dimension less than $n = \dim V > 1$. Let $\{ \lambda_i \mid 1 \leq i \leq k \}$ be the set of distinct eigenvalues of T. We arbitrarily select the first eigenvalue λ_1 and define (where R denotes the range),

$$W = R(T - \lambda_1 I),\ r = \operatorname{rank}(T - \lambda_1 I) = \dim W.$$

By Proposition 2.5.12, W is $(T-\lambda_1 I)$-invariant and therefore, by Proposition 2.5.37, W is also T-invariant. So we can define T_W, the restriction of T to W.

For all $\vec{\boldsymbol{v}} \in V$ we have,

$$(T - \lambda_1 I)^{m_1} (T - \lambda_2 I)^{m_2} \cdots (T - \lambda_k I)^{m_k}\, \vec{\boldsymbol{v}} = \vec{\mathbf{0}}$$

and, since

$$\forall \vec{\boldsymbol{w}} \in W\ .\ \exists \vec{\boldsymbol{v}} \in V\ \textit{s.t.}\ \vec{\boldsymbol{w}} = (T - \lambda_1 I)\vec{\boldsymbol{v}},$$

for all $\vec{\boldsymbol{w}} \in W$ we have,

$$(T - \lambda_1 I)^{m_1 - 1} (T - \lambda_2 I)^{m_2} \cdots (T - \lambda_k I)^{m_k}\, \vec{\boldsymbol{w}} = \vec{\mathbf{0}}.$$

Then, by Proposition 2.5.30, the characteristic polynomial of T_W divides that of T and so, we can deduce, also factorizes. Furthermore, T_W is a linear operator over an r-dimensional vector space where $r < n$ because, by the definition of λ_1 as an eigenvalue of T, $(T - \lambda_1 I)$ must have a non-trivial nullspace. So T_W is a linear operator over a vector space of dimension less than n such that the characteristic polynomial of T_W factorizes. Therefore, we can use the induction hypothesis to assert the existence of a jordan canonical basis for T_W – which we shall denote J_W.

Let S_i for $1 \leq i \leq k$ be the generalized eigenvectors in J_W corresponding to the eigenvalue λ_i. Since S_i is a subset of a jordan basis, it is therefore a linearly independent set of disjoint cycles of generalized eigenvectors. Since it is the subset corresponding to λ_i, it is therefore a linearly independent set of disjoint cycles of generalized eigenvectors corresponding to λ_i.

Let $\{ Z_j \mid 1 \leq j \leq p \}$ be the set of cycles of generalized eigenvectors whose disjoint union is equal to S_1. For each cycle, let

$$Z_j' = \{\vec{\boldsymbol{y}}_j\} \cup Z_j \quad \text{for some} \quad \vec{\boldsymbol{y}}_j \in V$$

such that $(T - \lambda_1 I)\vec{\boldsymbol{y}}_j$ is the end vector of Z_j. Such a $\vec{\boldsymbol{y}}_j$ is guaranteed to exist because

$$Z_j \subseteq W = R(T - \lambda_1 I).$$

Then Z_j' is also a cycle of generalized eigenvectors of T corresponding to λ_1. Now, let $I = \{ z_j \mid 1 \leq j \leq p \}$ be the set of initial vectors of the cycles Z_j. Since $I \subseteq J_W$ and J_W is a basis and hence linearly independent, I is linearly independent. Furthermore, I is a subset of the nullspace $N(T - \lambda_1 I)$.

> *The nullspace $N(T - \lambda_1 I)$ comprises the eigenvectors of T for the eigenvalue λ_1 but these are only in the space W – the range of $T - \lambda_1 I$ – if they are the initial vectors in a cycle of length > 1. Hence $I \subseteq N(T - \lambda_1 I)$.*

Next, we extend I to a basis for the nullspace which, by the dimension formula for finite-dimensional vector spaces (Theorem 2.5.1), has dimension $n - r$,

$$I_N = \{z_1, \ldots, z_p, z_{p+1}, \ldots, z_{n-r}\}.$$

Here,

$$p = \dim(R(T - \lambda_1 I) \cap N(T - \lambda_1 I)).$$

If there are no cycles (of length > 1) for the eigenvalue λ_1, then $p = 0$ and the basis of the nullspace of $N(T - \lambda_1 I)$ is comprised solely of $n - r$ eigenvectors. Otherwise $z_1, \ldots, z_p$ are the eigenvectors that are initial vectors of cycles of generalized eigenvectors for λ_1 and they are extended to a basis for $N(T - \lambda_1 I)$ by adding the eigenvectors that are not in cycles (of length > 1).

If $p < n - r$ (i.e. if there are eigenvectors that are not in cycles of length > 1), then $z_{p+1}, \ldots, z_{n-r}$ consists of eigenvectors not in W. These eigenvectors can be considered as cycles of length 1 and used to extend the cycles Z'_j such that,

$$Z'_j = \{z_j\} \quad \text{for} \quad p + 1 \leq j \leq n - r.$$

So, $Z' = \{ Z'_j \mid 1 \leq j \leq n - r \}$ is a collection of disjoint cycles of generalized eigenvectors corresponding to λ_1. Let

$$S'_1 = \bigcup_{j=1}^{n-r} Z'_j = Z'_1 \cup Z'_2 \cup \cdots \cup Z'_{n-r}.$$

Then, since the initial vectors of the cycles in Z' form a linearly independent set, by Proposition 2.5.42, S'_1 is a linearly independent disjoint union of the cycles. Furthermore,

$$\left|S'_1\right| = |S_1| + (n - r).$$

If we define,

$$J = S'_1 \cup S_2 \cup \cdots \cup S_k$$

then J is linearly independent by Proposition 2.5.41 and

$$|J| = |J_W| + (n - r) = r + (n - r) = n.$$

Therefore J is a basis of V comprising disjoint cycles of generalized eigenvectors of T and, as such, is a jordan canonical basis for T. □

Proposition 2.5.43. *Let T be a linear operator on a vector space V such that the characteristic polynomial of T factorizes. Then, for each eigenvalue, the dimension of its generalized eigenspace is equal to its multiplicity.*

Proof. Let $\{ \lambda_i \mid 1 \leq k \leq k \}$ be the distinct eigenvalues of T with the corresponding multiplicities $\{ m_i \mid 1 \leq k \leq k \}$. By Theorem 2.5.10, there exists a jordan basis for T. Let this basis be J and the matrix of T w.r.t. J,

$$[T]_J = [J]^{-1}[T(J)].$$

Since $[T]_J$ is in Jordan Normal Form, it is upper-triangular and, therefore, the multiplicity of any eigenvalue λ_i is equal to the number of columns of the matrix that have λ_i as the diagonal element. This, in turn, is equal to the number of vectors in the jordan basis J that are generalized eigenvectors corresponding to the eigenvalue λ_i.

So if, for each i,

$$U_i = K_{\lambda_i} \cap J$$

is the subset of the jordan basis that comprises generalized eigenvectors corresponding to the eigenvalue λ_i, then

$$\forall i \ . \ |U_i| = m_i.$$

Since U_i is a linearly independent set in K_{λ_i} we must have,

$$|U_i| \leq \dim K_{\lambda_i} \implies m_i \leq \dim K_{\lambda_i}$$

and by Proposition 2.5.33, we have that

$$\sum_i m_i = \dim V.$$

Furthermore, since the spaces K_{λ_i} are linearly independent and the direct sum of the spaces is a subspace of V, we must have,

$$\sum_i \dim K_{\lambda_i} \leq \dim V.$$

Therefore,

$$\begin{aligned} \dim V = \sum_i m_i \leq \sum_i \dim K_{\lambda_i} \leq \dim V \\ \implies \sum_i m_i = \sum_i \dim K_{\lambda_i} = \dim V. \end{aligned}$$

So we can deduce,

$$\begin{aligned} \forall i (\dim K_{\lambda_i} - m_i \geq 0) \ \wedge \ \sum_i \dim K_{\lambda_i} - m_i = 0 \\ \implies \forall i (\dim K_{\lambda_i} - m_i = 0). \end{aligned}$$

Therefore, for each i, $\dim K_{\lambda_i} = m_i$ as required. □

Corollary 2.5.16. *If T is a linear operator on a vector space V such that the characteristic polynomial of T factorizes, then T is diagonalizable iff, for each eigenvalue, its eigenspace is equal to its generalized eigenspace.*

Proof. For any linear operator such as T, by Proposition 2.5.43, for each distinct eigenvalue λ of T with multiplicity m,

$$\dim K_\lambda = m.$$

Also, by Proposition 2.5.35, T is diagonalizable iff,

$$\dim E_\lambda = m.$$

So we can put these two together to say that T is diagonalizable iff

$$\dim E_\lambda = \dim K_\lambda.$$

Furthermore, since E_λ is a subspace of K_λ (Proposition 2.5.36),

$$\dim E_\lambda = \dim K_\lambda \implies E_\lambda = K_\lambda.$$

□

Corollary 2.5.17. *If T is a linear operator on a vector space V such that the characteristic polynomial of T factorizes, then for every eigenvalue λ with multiplicity m,*

$$K_\lambda = N((T - \lambda I)^m)$$

where N denotes the nullspace.

Proof. If $\vec{\boldsymbol{v}} \in N((T - \lambda I)^m)$ then, by the definition of the generalized eigenspace (2.5.4.7), $\vec{\boldsymbol{v}} \in K_\lambda$. Therefore $N((T - \lambda I)^m) \subseteq K_\lambda$.

Conversely, if $\vec{\boldsymbol{v}} \in K_\lambda$ then there exists some minimial $p \in \mathbb{N}$ such that,

$$(T - \lambda I)^p \, \vec{\boldsymbol{v}} = \vec{\mathbf{0}}.$$

If we consider the p-length cycle whose end vector is this $\vec{\boldsymbol{v}}$,

$$C = \{(T - \lambda I)^{p-1} \, \vec{\boldsymbol{v}}, \ldots, (T - \lambda I)\vec{\boldsymbol{v}}, \vec{\boldsymbol{v}}\},$$

then, by Theorem 2.5.8, the vectors in the cycle C are linearly independent. By Proposition 2.5.43, the dimension of K_λ is m and, since the dimension is

an upper bound on the length of a linearly independent set in the space, we must have $p \leq m$ and so,

$$(T - \lambda I)^m \vec{\boldsymbol{v}} = (T - \lambda I)^{m-p} (T - \lambda I)^p \vec{\boldsymbol{v}} = (T - \lambda I)^{m-p} \vec{\mathbf{0}} = \vec{\mathbf{0}}.$$

It follows then, that $\vec{\boldsymbol{v}} \in N((T - \lambda I)^m)$ and so $K_\lambda \subseteq N((T - \lambda I)^m)$. □

Corollary 2.5.18. *If T is a linear operator on a vector space V such that the characteristic polynomial of T factorizes, then V is a direct sum of the generalized eigenspaces of T.*

Proof. By Proposition 2.5.43, for each distinct eigenvalue λ_i with multiplicity m_i,

$$\dim K_{\lambda_i} = m_i.$$

By Proposition 2.5.40, the spaces K_{λ_i} for the different λ_i, are linearly independent. Therefore if we let B_i be a basis for K_{λ_i} then

$$B = \bigcup_i B_i$$

is a disjoint union and the set B is linearly independent because, for $\vec{\boldsymbol{b}}_i \in B_i$,

$$\begin{aligned} & \vec{\boldsymbol{b}}_1 + \vec{\boldsymbol{b}}_2 + \cdots + \vec{\boldsymbol{b}}_k = \vec{\mathbf{0}} \\ \iff \quad & \vec{\boldsymbol{b}}_1 = \vec{\boldsymbol{b}}_2 = \cdots = \vec{\boldsymbol{b}}_k = \vec{\mathbf{0}} \end{aligned}$$

and for each $\vec{\boldsymbol{b}}_i$,

$$\vec{\boldsymbol{b}}_i = \alpha_1 \vec{\boldsymbol{b}}_{i1} + \cdots + \alpha_k \vec{\boldsymbol{b}}_{ik} = \vec{\mathbf{0}} \implies \alpha_1, \ldots \alpha_k = 0.$$

So B is a linearly independent set of vectors in the space V with

$$|B| = \sum_i |B_i| = \sum_i \dim K_{\lambda_i} = \sum_i m_i.$$

By Proposition 2.5.33, we have,

$$\sum_i m_i = \dim V$$

and so, B is a basis for V.

Therefore, by Proposition 2.4.20, we have,

$$K_{\lambda_1} \oplus K_{\lambda_2} \oplus \cdots \oplus K_{\lambda_k} = V.$$ □

Generalized Eigenvectors and Jordan Blocks

Consider a simple case of repeated eigenvalues: a matrix A that simply represents a uniform scaling by a, for some constant a,

$$A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}.$$

In this case, clearly, the only eigenvalue is $\lambda = a$. From which we obtain,

$$(A - aI) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

which has a 2-dimensional nullspace equal to the entire vector space over which it operates. So we can use the standard basis as the two eigenvectors. If the diagonalized matrix is D and the change of basis matrix to the eigenbasis is P (which in this case is the identity) then $AP = PD$ which is the matrix equation,

$$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}.$$

On the other hand, if the diagonal entries of A are distinct so that the matrix becomes a non-uniform, rectangular scaling,

$$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

we have, in this case, eigenvalues: $\lambda_1 = a$, $\lambda_2 = b$. From which we obtain,

$$(A - aI) = \begin{bmatrix} 0 & 0 \\ 0 & b - a \end{bmatrix} \quad \text{and} \quad (A - bI) = \begin{bmatrix} a - b & 0 \\ 0 & 0 \end{bmatrix}$$

which, each have a 1-dimensional nullspace, the direct sum of them forming the entire vector space. In fact, the eigenvectors are the two standard basis vectors and, similarly to the previous case, we have the following matrix equation,

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}.$$

However, in the case,

$$A = \begin{bmatrix} a & 1 \\ 0 & b \end{bmatrix}$$

we have the same eigenvalues $\lambda_1 = a$, $\lambda_2 = b$, but the eigenvector of $\lambda_2 = b$ is not the same.

$$(A - aI) = \begin{bmatrix} 0 & 1 \\ 0 & b-a \end{bmatrix} \quad \text{and} \quad (A - bI) = \begin{bmatrix} a-b & 1 \\ 0 & 0 \end{bmatrix}$$

In this case, the eigenvector of $\lambda_2 = b$ is

$$\begin{bmatrix} \frac{1}{b-a} \\ 1 \end{bmatrix}.$$

The reason is that, here, we have a stretching of b in the second dimension (say, the y-direction) that also has a component of 1 in the x-direction – which is being stretched by a. So, in order to have a vector whose components are *both* stretched by b we need the x component to obey,

$$1 + ax = bx \iff x = 1/(b-a).$$

TODO: relate this to modular arithmetic and resonance

Here the matrix equation is,

$$\begin{bmatrix} a & 1 \\ 0 & b \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{b-a} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{b-a} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}.$$

Now, if we allow b to go to a then the first component of the eigenvector is going to go to infinity but we are also approaching,

$$\begin{bmatrix} a & 1 \\ 0 & b \end{bmatrix} \to \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \rightsquigarrow (A - aI) = (A - bI) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

which is a repeated eigenvalue with a generalized eigenvector. Specifically, the eigenvector is $(1, 0)^T$ and the generalized eigenvector satisfies,

$$(A - aI)\vec{\boldsymbol{v}} = (1, 0)^T \implies \vec{\boldsymbol{v}} = (0, 1)^T$$

so that

$$(A - aI)^2 \vec{\boldsymbol{v}} = (A - aI)(1, 0)^T = \vec{\mathbf{0}}.$$

*Notice also, that since A is a 2×2 matrix, in accordance with **??**, we have,*

$$(A - aI)^2 = 0.$$

So, in matrix equations, we have

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

If we try to use the generalized eigenvector in the same way as a "normal" eigenvector and form the diagonalized matrix in the normal way then it doesn't work as intended because, clearly

$$\begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}.$$

but, in fact,

$$\begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}.$$

The modification to the diagonal matrix reflects the fact that, for the generalized eigenvector,

$$\begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + a \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ a \end{bmatrix}.$$

The matrix,

$$\begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}$$

is a *Jordan block* for the eigenvalue a.

Examples of Jordan Canonical Basis

(67) Let $T : P_2(\mathbb{C}) \longmapsto P_2(\mathbb{C})$ be a linear operator over the degree-2 complex polynomials and let T be defined by

$$T(p) = -p - p'.$$

If we use the standard basis for $P_2(\mathbb{C})$, $\{1, z, z^2\}$ then, the matrix of T w.r.t. this basis,

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -2 \\ 0 & 0 & -1 \end{bmatrix}.$$

Clearly, the characteristic polynomial is $p(t) = -(t+1)^3$ and there is a single eigenvalue, -1, with multiplicity 3. The generalized eigenspace is the whole space of dimension 3, equal to the multiplicity (as predicted by Proposition 2.5.43), and any basis of the space $P_2(\mathbb{C})$ is also a basis of the generalized eigenspace.

Note, though, that using, for example, the standard basis of $P_2(\mathbb{C})$ would not produce a Jordan Canonical Form matrix.

The eigenspace, meanwhile, is the nullspace $N(T + I)$ where

$$A + I = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

and, in terms of the linear map,

$$(T + I)(p) = -p - p' + p = -p'.$$

So the nullspace is

$$N(A + I) = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{for} \quad t \in \mathbb{C}$$

and

$$N(T + I) = \{\, p(z) \in P_2(\mathbb{C}) \mid p' = 0 \,\} = \{\, p(z) = \alpha_0 \mid \alpha_0 \in \mathbb{C} \,\}.$$

That's to say, the eigenspace is the space of constant polynomials.

If we look for a cycle with initial vector $(1,0,0)^T$ then we want,

$$(A+I)^2\,\vec{\boldsymbol{v}} = \begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}0&0&2\\0&0&0\\0&0&0\end{bmatrix}\vec{\boldsymbol{v}} \implies \vec{\boldsymbol{v}} = \begin{bmatrix}0\\0\\\frac{1}{2}\end{bmatrix}$$

and

$$(T+I)^2\,(p) = (T+I)(-p') = p'' = 1$$
$$\implies \qquad p(z) = \frac{1}{2}z^2 + \alpha z + \beta \quad \text{for} \quad \alpha,\beta \in \mathbb{C}.$$

So the simplest cycle is: $\frac{1}{2}z^2, -z, 1$ and we can choose as a jordan basis,

$$J = \{2, -2z, z^2\}.$$

Then the matrix of T w.r.t. this basis is in Jordan Canonical Form,

$$[T]_J = [J]^{-1}[T(J)] = \begin{bmatrix}\frac{1}{2}&0&0\\0&-\frac{1}{2}&0\\0&0&1\end{bmatrix}\begin{bmatrix}-2&2&0\\0&2&-2\\0&0&-1\end{bmatrix} = \begin{bmatrix}-1&1&0\\0&-1&1\\0&0&-1\end{bmatrix}.$$

2.5.5 Projections

2.5.5.1 Linear Operators as Projections

Definition 127. **(Projection)** A linear operator $T : V \longmapsto V$ is a *projection* on $W_1 \subseteq V$ if

(i) there exists a subspace $W_2 \subseteq V$ such that $W_1 \oplus W_2 = V$;

(ii) for $\vec{\boldsymbol{v}} = \vec{\boldsymbol{w}}_1 + \vec{\boldsymbol{w}}_2 \in V$ where $\vec{\boldsymbol{w}}_1 \in W_1$, $\vec{\boldsymbol{w}}_2 \in W_2$, we have

$$T\vec{\boldsymbol{v}} = T(\vec{\boldsymbol{w}}_1 + \vec{\boldsymbol{w}}_2) = \vec{\boldsymbol{w}}_1.$$

The projection of V onto W_1 — often denoted P_{W_1} — is described as *parallel* to W_2 because the displacement vector, from the original vector to its projected image,

$$(\vec{\boldsymbol{w}}_1 + \vec{\boldsymbol{w}}_2) - T(\vec{\boldsymbol{w}}_1 + \vec{\boldsymbol{w}}_2) = (\vec{\boldsymbol{w}}_1 + \vec{\boldsymbol{w}}_2) - \vec{\boldsymbol{w}}_1 = \vec{\boldsymbol{w}}_2$$

is in W_2.

If we have,

$$W_1^{\perp} = W_2$$

then the projection is further described as an *orthogonal projection*.

Proposition 2.5.44. *If T is a projection in a vector space V then*

$$V = R(T) \oplus N(T)$$

where $R(T)$ and $N(T)$ denote the range and nullspace of T respectively.

Proof. If T is a projection then, by the definition,

$$W_1 \oplus W_2 = V \quad \text{and} \quad \forall \vec{\boldsymbol{w}}_1 \in W_1, \vec{\boldsymbol{w}}_2 \in W_2 \,.\, T(\vec{\boldsymbol{w}}_1 + \vec{\boldsymbol{w}}_2) = \vec{\boldsymbol{w}}_1.$$

Then $R(T) \subseteq W_1$ and because, for every $\vec{\boldsymbol{w}}_1 \in W_1$ there exists a $\vec{\boldsymbol{v}} = \vec{\boldsymbol{w}}_1 + \vec{\boldsymbol{w}}_2 \in V$ such that $T(\vec{\boldsymbol{v}}) = \vec{\boldsymbol{w}}_1$, also $W_1 \subseteq R(T)$. Therefore, $R(T) = W_1$.

Furthermore,

$$T(\vec{\boldsymbol{v}}) = T(\vec{\boldsymbol{w}}_1 + \vec{\boldsymbol{w}}_2) = \vec{\mathbf{0}} = \vec{\boldsymbol{w}}_1 \implies \vec{\boldsymbol{v}} = \vec{\boldsymbol{w}}_1 + \vec{\boldsymbol{w}}_2 = \vec{\mathbf{0}} + \vec{\boldsymbol{w}}_2 = \vec{\boldsymbol{w}}_2 \in W_2$$

which implies that $N(T) \subseteq W_2$. Since also $T(\vec{\boldsymbol{w}}_2) = T(\vec{\mathbf{0}} + \vec{\boldsymbol{w}}_2) = \vec{\mathbf{0}}$ then we also have

$$W_2 \subseteq N(T)$$

and so $W_2 = N(T)$.

Therefore $V = R(T) \oplus N(T)$ as required. □

Proposition 2.5.45. *If T is a linear operator over a vector space V and T is a projection of V onto W, then W is T-invariant and the restriction of T to W is the identity operator on W,*

$$T_W = I_W.$$

Proof. If T is a projection of V onto W then, for all $\vec{\boldsymbol{v}} \in V$ there is some $\vec{\boldsymbol{w}} \in W$ such that,

$$T\vec{\boldsymbol{v}} = \vec{\boldsymbol{w}}.$$

Therefore, since $W \subseteq V$ we also have, for all $\vec{\boldsymbol{w}}_1 \in W$ and some $\vec{\boldsymbol{w}}_2 \in W$,

$$T\vec{\boldsymbol{w}}_1 = \vec{\boldsymbol{w}}_2 \in W \implies TW \subseteq W.$$

So W is shown to be T-invariant.

Since W is T-invariant, we can define the restriction of T to W,

$$T_w : W \longmapsto W \text{ s.t. } T_w\vec{\boldsymbol{w}}_1 = T\vec{\boldsymbol{w}}_1 = \vec{\boldsymbol{w}}_2.$$

But also, since T is a projection onto W, it is also idempotent,

$$T^2 = T \implies T^2\,\vec{\boldsymbol{v}} = T\vec{\boldsymbol{v}} \implies {T_w}^2\,\vec{\boldsymbol{w}} = T^2\,\vec{\boldsymbol{w}} = T\vec{\boldsymbol{w}} = T_w\vec{\boldsymbol{w}}.$$

Putting these two together we obtain, for any $\vec{\boldsymbol{w}}_1, \vec{\boldsymbol{w}}_2 \in W$ such that $T_w\vec{\boldsymbol{w}}_1 = \vec{\boldsymbol{w}}_2$,

$${T_w}^2\,\vec{\boldsymbol{w}}_1 = T_w\vec{\boldsymbol{w}}_1 = \vec{\boldsymbol{w}}_2 = T_w(T_w\vec{\boldsymbol{w}}_1) = T_w\vec{\boldsymbol{w}}_2 = {T_w}^2\,\vec{\boldsymbol{w}}_2.$$

We have obtained

$$T_w\vec{\boldsymbol{w}}_2 = \vec{\boldsymbol{w}}_2$$

which implies that $T_w = I_w$, the identity operator on W. □

Proposition 2.5.46. *A linear operator T on a finite-dimensional vector space V is a projection iff it is idempotent. That's to say,*

$$T^2 = T.$$

Proof. Let $W_1 \oplus W_2 = V$ and T be a projection such that for $\vec{\boldsymbol{w}}_1 \in W_1, \vec{\boldsymbol{w}}_2 \in W_2$,

$$T(\vec{\boldsymbol{w}}_1 + \vec{\boldsymbol{w}}_2) = \vec{\boldsymbol{w}}_1.$$

Then T is clearly idempotent because

$$T(T(\vec{\boldsymbol{w}}_1 + \vec{\boldsymbol{w}}_2)) = T(\vec{\boldsymbol{w}}_1) = \vec{\boldsymbol{w}}_1 = T(\vec{\boldsymbol{w}}_1 + \vec{\boldsymbol{w}}_2) \implies T^2 = T.$$

Conversely, let T be a linear operator on V such that $T^2 = T$. Then, for any $\vec{\boldsymbol{v}} \in V$,

$$\begin{aligned}
& T(T\vec{\boldsymbol{v}}) = T\vec{\boldsymbol{v}} \\
\iff \quad & T(T\vec{\boldsymbol{v}}) - T\vec{\boldsymbol{v}} = \vec{\boldsymbol{0}} \\
\iff \quad & T(T\vec{\boldsymbol{v}} - \vec{\boldsymbol{v}}) = \vec{\boldsymbol{0}} \\
\iff \quad & T\vec{\boldsymbol{v}} - \vec{\boldsymbol{v}} \in N(T) \\
\iff \quad & \exists \vec{\boldsymbol{u}} \in N(T) \text{ s.t. } \vec{\boldsymbol{v}} = T\vec{\boldsymbol{v}} + \vec{\boldsymbol{u}}.
\end{aligned}$$

Since $T\vec{\boldsymbol{v}} \in R(T)$, this shows that

$$R(T) + N(T) = V.$$

For any $\vec{\boldsymbol{u}} \in R(T) \cap N(T)$, since $\vec{\boldsymbol{u}} \in R(T)$ then there exists some $\vec{\boldsymbol{v}} \in V$ such that $\vec{\boldsymbol{u}} = T\vec{\boldsymbol{v}}$ and also, since $\vec{\boldsymbol{u}} \in N(T)$, we have $T\vec{\boldsymbol{u}} = \vec{\boldsymbol{0}}$. Putting these together with the idempotence of T,

$$\vec{\boldsymbol{0}} = T\vec{\boldsymbol{u}} = T(T\vec{\boldsymbol{u}}) = T(T\vec{\boldsymbol{v}}) = T\vec{\boldsymbol{v}} = \vec{\boldsymbol{u}}.$$

Therefore, we can deduce that $R(T) \cap N(T) = \{\vec{\boldsymbol{0}}\}$ and so,

$$R(T) \oplus N(T) = V.$$

We can also deduce, using the idempotence, that for any $\vec{\boldsymbol{w}} = T\vec{\boldsymbol{v}} \in R(T)$,

$$T(\vec{\boldsymbol{w}}) = T(T\vec{\boldsymbol{v}}) = T^2\vec{\boldsymbol{v}} = T\vec{\boldsymbol{v}} = \vec{\boldsymbol{w}} \implies \forall \vec{\boldsymbol{w}} \in R(T) \,.\, T\vec{\boldsymbol{w}} = \vec{\boldsymbol{w}}.$$

So, if we let $W_1 = R(T), W_2 = N(T)$ so that

$$W_1 \oplus W_2 = V$$

and, for any $\vec{\boldsymbol{v}} \in V$, there exists $\vec{\boldsymbol{w}}_1 \in W_1, \vec{\boldsymbol{w}}_2 \in W_2$ such that $\vec{\boldsymbol{v}} = \vec{\boldsymbol{w}}_1 + \vec{\boldsymbol{w}}_2$, then

$$T\vec{\boldsymbol{v}} = T(\vec{\boldsymbol{w}}_1 + \vec{\boldsymbol{w}}_2) = T(\vec{\boldsymbol{w}}_1) + T(\vec{\boldsymbol{w}}_2) = \vec{\boldsymbol{w}}_1 + \vec{\mathbf{0}} = \vec{\boldsymbol{w}}_1.$$

Therefore, T is a projection as required. □

Proposition 2.5.47. *A projection on a finite-dimensional inner product space on the reals with the standard inner product is orthogonal iff its matrix is symmetric.*

Proof. Let P be a projection on V, a finite-dimensional inner product space over the reals with the standard inner product. For convenience, let P denote both the projection as a linear transformation and the matrix representing it.

If P is symmetric then

$$\begin{aligned} & P = P^T && \text{by symmetry} \\ \implies \quad & N(P) = N(P^T) = R(P)^\perp. && \text{by Proposition 2.6.8} \end{aligned}$$

Since P is a projection, by Proposition 2.5.44, we also have

$$V = R(P) \oplus N(P) = R(P) \oplus R(P)^\perp$$

and so symmetry of matrix P implies that P is an orthogonal projection.

Conversely, if P is an orthogonal projection then

$$V = R(P) \oplus N(P) = R(P) \oplus R(P)^\perp$$

so define

$$W_1 = R(P) \quad \text{and} \quad W_2 = N(P) = R(P)^\perp$$

and then, for any $\vec{\boldsymbol{v}}, \vec{\boldsymbol{v}}' \in V$ there exists some

$$\vec{\boldsymbol{w}}_1, \vec{\boldsymbol{w}}_2, \vec{\boldsymbol{w}}_1', \vec{\boldsymbol{w}}_2' \text{ s.t. } \vec{\boldsymbol{v}} = \vec{\boldsymbol{w}}_1 + \vec{\boldsymbol{w}}_2 \quad \text{and} \quad \vec{\boldsymbol{v}}' = \vec{\boldsymbol{w}}_1' + \vec{\boldsymbol{w}}_2'.$$

Then,

$$\begin{aligned}
\langle P\vec{\boldsymbol{v}},\, \vec{\boldsymbol{v}}'\rangle &= \langle P(\vec{\boldsymbol{w}}_1 + \vec{\boldsymbol{w}}_2),\, \vec{\boldsymbol{w}}_1' + \vec{\boldsymbol{w}}_2'\rangle \\
&= \langle P(\vec{\boldsymbol{w}}_1),\, \vec{\boldsymbol{w}}_1'\rangle + \langle P(\vec{\boldsymbol{w}}_1),\, \vec{\boldsymbol{w}}_2'\rangle + \\
&\qquad \langle P(\vec{\boldsymbol{w}}_2),\, \vec{\boldsymbol{w}}_1'\rangle + \langle P(\vec{\boldsymbol{w}}_2),\, \vec{\boldsymbol{w}}_2'\rangle \\
&= \langle \vec{\boldsymbol{w}}_1,\, \vec{\boldsymbol{w}}_1'\rangle + \langle \vec{\boldsymbol{w}}_1,\, \vec{\boldsymbol{w}}_2'\rangle + \langle \vec{\boldsymbol{0}},\, \vec{\boldsymbol{w}}_1'\rangle + \langle \vec{\boldsymbol{0}},\, \vec{\boldsymbol{w}}_2'\rangle \quad \because P \text{ is projection onto } W_1 \\
&= \langle \vec{\boldsymbol{w}}_1,\, \vec{\boldsymbol{w}}_1'\rangle + 0 + \langle \vec{\boldsymbol{0}},\, \vec{\boldsymbol{w}}_1'\rangle + \langle \vec{\boldsymbol{0}},\, \vec{\boldsymbol{w}}_2'\rangle \qquad \because W_1 \perp W_2 \\
&= \langle \vec{\boldsymbol{w}}_1,\, \vec{\boldsymbol{w}}_1'\rangle + 0 + 0 + 0 \\
&= \langle \vec{\boldsymbol{w}}_1,\, \vec{\boldsymbol{w}}_1'\rangle \\
&= \langle \vec{\boldsymbol{w}}_1 + \vec{\boldsymbol{w}}_2,\, P(\vec{\boldsymbol{w}}_1' + \vec{\boldsymbol{w}}_2')\rangle = \langle \vec{\boldsymbol{v}},\, P\vec{\boldsymbol{v}}'\rangle.
\end{aligned}$$

By Proposition 4.2.5 then, P is symmetric.

We could also have said,

$$\begin{aligned}
& (I-P)\vec{\boldsymbol{v}} = \vec{\boldsymbol{v}} - P\vec{\boldsymbol{v}} = \vec{\boldsymbol{w}}_1 + \vec{\boldsymbol{w}}_2 - \vec{\boldsymbol{w}}_1 = \vec{\boldsymbol{w}}_2 \\
\implies \quad & \langle P\vec{\boldsymbol{v}},\, (I-P)\vec{\boldsymbol{v}}'\rangle = 0 \\
\implies \quad & v^T P^T (I-P)\vec{\boldsymbol{v}}' = 0
\end{aligned}$$

which, since it applies to any $\vec{\boldsymbol{v}}, \vec{\boldsymbol{v}}' \in V$, by Proposition 2.3.5, implies that $P^T(I-P) = 0$ and so

$$\begin{aligned}
& P^T(I-P) = P^T - P^T P = 0 \\
\iff \quad & P^T = P^T P \\
\iff \quad & P = (P^T P)^T \\
\iff \quad & P = P^T P \\
\iff \quad & P = P^T.
\end{aligned}$$

□

Proposition 2.5.48. *Let V be a finite-dimensional inner product space over the reals with the standard inner product. Let U be a subspace of V, P the orthogonal projection of V onto U, and $\vec{\boldsymbol{v}} \in V$ is an arbitrary vector.*

Then the closest point in U to $\vec{\boldsymbol{v}}$ is given by the orthogonal projection $P\vec{\boldsymbol{v}}$ of the vector onto the space U.

That's to say, for all $\vec{\boldsymbol{u}} \in U$,

$$\|\vec{\boldsymbol{v}} - \vec{\boldsymbol{u}}\| \geq \|\vec{\boldsymbol{v}} - P\vec{\boldsymbol{v}}\|.$$

Proof. Since P is the orthogonal projection of V onto U, by Proposition 2.5.44, we have

$$R(P) \oplus N(P) = U + U^{\perp} = V.$$

So, for any $\vec{\boldsymbol{v}} \in V$,

$$\vec{\boldsymbol{v}} = P\vec{\boldsymbol{v}} + (\vec{\boldsymbol{v}} - P\vec{\boldsymbol{v}})$$

where $P\vec{\boldsymbol{v}} \in U$ and $\vec{\boldsymbol{v}} - P\vec{\boldsymbol{v}} \in U^{\perp}$.

We can see that $\vec{\boldsymbol{v}} - P\vec{\boldsymbol{v}} \in U^{\perp}$ by observing that, because of the idempotence of P, we have

$$P(\vec{\boldsymbol{v}} - P\vec{\boldsymbol{v}}) = P\vec{\boldsymbol{v}} - P^2\vec{\boldsymbol{v}} = P\vec{\boldsymbol{v}} - P\vec{\boldsymbol{v}} = \vec{\boldsymbol{0}} \implies \vec{\boldsymbol{v}} - P\vec{\boldsymbol{v}} \in N(P) = U^{\perp}.$$

Then,

$$\|\vec{\boldsymbol{v}} - \vec{\boldsymbol{u}}\| = \|P\vec{\boldsymbol{v}} + (\vec{\boldsymbol{v}} - P\vec{\boldsymbol{v}}) - \vec{\boldsymbol{u}}\| = \|(\vec{\boldsymbol{v}} - P\vec{\boldsymbol{v}}) + (P\vec{\boldsymbol{v}} - \vec{\boldsymbol{u}})\|$$

where $P\vec{\boldsymbol{v}} - \vec{\boldsymbol{u}} \in U$ because $P\vec{\boldsymbol{v}} \in R(P) = U$. Since the vectors $\vec{\boldsymbol{v}} - P\vec{\boldsymbol{v}}$ and $P\vec{\boldsymbol{v}} - \vec{\boldsymbol{u}}$ are orthogonal, by the Generalised Pythagoras Theorem (Theorem 2.6.1), we have

$$\|\vec{\boldsymbol{v}} - \vec{\boldsymbol{u}}\|^2 = \|(\vec{\boldsymbol{v}} - P\vec{\boldsymbol{v}})\|^2 + \|(P\vec{\boldsymbol{v}} - \vec{\boldsymbol{u}})\|^2.$$

By positive definiteness of the norm (2.6.1.3),

$$\begin{aligned} & \|(P\vec{\boldsymbol{v}} - \vec{\boldsymbol{u}})\|^2 \geq 0 \\ \implies \quad & \|\vec{\boldsymbol{v}} - \vec{\boldsymbol{u}}\|^2 \geq \|(\vec{\boldsymbol{v}} - P\vec{\boldsymbol{v}})\|^2 \\ \implies \quad & \|\vec{\boldsymbol{v}} - \vec{\boldsymbol{u}}\| \geq \|(\vec{\boldsymbol{v}} - P\vec{\boldsymbol{v}})\|. \quad \square \end{aligned}$$

Proposition 2.5.49. *Let $A \in \mathbb{R}^{m\times n}$ be a matrix of rank n (i.e. full column rank). Then the matrix*

$$P = A(A^TA)^{-1}A^T$$

represents the orthogonal projection of $\mathbb{R}^m$ onto the range of A.

Proof.

$$\begin{aligned} P^2 &= (A(A^TA)^{-1}A^T)(A(A^TA)^{-1}A^T) \\ &= A(A^TA)^{-1}(A^TA)(A^TA)^{-1}A^T \\ &= A(A^TA)^{-1}A^T = P. \end{aligned}$$

Therefore P is idempotent and, by Proposition 2.5.46 then, P is a projection.

$$\begin{aligned} P^T &= (A(A^TA)^{-1}A^T)^T \\ &= A((A^TA)^{-1})^TA^T \\ &= A(A^TA)^{T-1}A^T \qquad \text{by Proposition 2.3.3} \\ &= A(A^TA)^{-1}A^T = P. \end{aligned}$$

Therefore P is symmetric and, since it is also a projection, by Proposition 2.5.47, P is an orthogonal projection.

$$P\vec{\boldsymbol{x}} = A(A^TA)^{-1}A^T\vec{\boldsymbol{x}} = A((A^TA)^{-1}A^T\vec{\boldsymbol{x}}) = A\vec{\boldsymbol{v}}$$

for some $\vec{\boldsymbol{v}} \in V$. Therefore $R(P) \subseteq R(A)$.

$$A\vec{\boldsymbol{x}} = A(A^TA)^{-1}(A^TA)\vec{\boldsymbol{x}} = A(A^TA)^{-1}A^T(A\vec{\boldsymbol{x}}) = A(A^TA)^{-1}A^T\vec{\boldsymbol{v}}$$

for some $\vec{\boldsymbol{v}} \in V$. Therefore $R(A) \subseteq R(P)$.

So $R(P) = R(A)$ and P is an orthogonal projection onto the range of A as required. □

Corollary 2.5.19. ***(Normal Equation)*** *If a design matrix X contains the samples of a dataset to which a model $\vec{\boldsymbol{\theta}}$ is to be fitted using least squares with the labels being the matrix Y, then the best fit model $\hat{\vec{\boldsymbol{\theta}}}$ is given by*

$$\hat{\vec{\boldsymbol{\theta}}} = (X^T X)^{-1} X^T Y.$$

Proof. Fitting the model using the least squares method means finding the value of $\vec{\boldsymbol{\theta}}$ that minimises the squared error. That is

$$\arg\min_{\vec{\boldsymbol{\theta}}} \left\| Y - X\vec{\boldsymbol{\theta}} \right\|^2.$$

By Proposition 2.5.48, the value of $X\vec{\boldsymbol{\theta}}$ that miminises this cost function is obtained by the orthogonal projection of Y onto the range of X, which by Proposition 2.5.49, is

$$X\hat{\vec{\boldsymbol{\theta}}} = P_X Y = X(X^T X)^{-1} X^T Y$$

and so, resolving for the argument $\hat{\vec{\boldsymbol{\theta}}}$ we have

$$\hat{\vec{\boldsymbol{\theta}}} = X^{-1} P_X Y = X^{-1}(X(X^T X)^{-1} X^T Y) = (X^T X)^{-1} X^T Y. \qquad \square$$

2.5.5.2 Reduced-Row Echelon Form as Projection

TODO: the reduced row echelon form of a matrix A is a projection onto the range of A

---

(68) If we attempt to project the x-axis of $\mathbb{F}^3$ onto the z-axis,

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}0\\0\\1\end{bmatrix},\ T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\0\\0\end{bmatrix},\ T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

then this leads to the matrix,

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

But this is **not** a projection because $A^2 = 0 \neq A$. The z-axis is not invariant under A; in fact, A maps the z-axis to the origin.

If we try to fix this by also satisfying,

$$T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

we get the matrix,

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

This matrix A, *does* satisfy $A^2 = A$ and *is* therefore a projection of the x-axis onto the z-axis.

(69) Following on from example 75: Suppose

$$U = \text{Lin}\left\{\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\} \quad \text{and} \quad W = U^{\perp} = \text{Lin}\left\{\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right\}.$$

Then, by Proposition 2.6.6,

$$U \oplus W = \mathbb{R}^3$$

so we can define a function P_U, the projection of $\mathbb{R}^3$ onto U parallel to W.

For any $\vec{\boldsymbol{x}} = (x, y, z)^T \in \mathbb{R}^3$, we can express it w.r.t. a basis that is the disjoint union of bases of U and W,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \alpha_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

So we have,

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

which gives

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} x + 2y - z \\ 3x + 3z \\ -2x + 2y + 2z \end{bmatrix}$$

and so the expression of the vector in $\mathbb{R}^3$ in this basis is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{6} \left((x + 2y - z) \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + (3x + 3z) \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + (-2x + 2y + 2z) \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right)$$

and the projection onto U is therefore

$$\begin{aligned} P_U((x, y, z)^T) &= \frac{1}{6} \left(\begin{bmatrix} (x + 2y - z) \\ 2(x + 2y - z) \\ -(x + 2y - z) \end{bmatrix} + \begin{bmatrix} (3x + 3z) \\ 0 \\ (3x + 3z) \end{bmatrix} \right) \\ &= \frac{1}{6} \left(\begin{bmatrix} 2(2x + y + z) \\ 2(x + 2y - z) \\ 2(x - y + 2z) \end{bmatrix} \right) \\ &= \frac{1}{3} \left(\begin{bmatrix} 2x + y + z \\ x + 2y - z \\ x - y + 2z \end{bmatrix} \right). \end{aligned}$$

We can instead calculate this by taking the matrix whose columns are the basis of U,

$$A = [U] = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 1 \end{bmatrix}$$

and using Proposition 2.5.49 to determine the orthogonal projection onto U as

$$P_U = A(A^T A)^{-1} A^T.$$

Since

$$A^T A = \begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix} \implies A^T A^{-1} = \begin{bmatrix} \frac{1}{6} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

we then have,

$$\begin{aligned} P_U &= \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 1 \end{bmatrix} \frac{1}{6} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix} \\ &= \frac{1}{6} \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix} \\ &= \frac{1}{6} \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & -2 \\ 2 & -2 & 4 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}. \end{aligned}$$

(70) Suppose we are trying to fit a linear model $Y = \theta_0 + \theta_1 X$ to the following dataset.

X	Y
0	1
3	4
6	5

We are trying to find a least squares solution (values of θ_0, θ_1 that minimise the squared error) to the system

$$\begin{aligned} \theta_0 &= 1 \\ \theta_0 + 3\theta_1 &= 4 \\ \theta_0 + 6\theta_1 &= 5. \end{aligned}$$

It's easy to see there is no exact solution (i.e. no solution with 0 squared error).

If we describe this system in matrix form we have

$$X\vec{\boldsymbol{\theta}} = Y \iff \begin{bmatrix} 1 & 0 \\ 1 & 3 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} \theta_0 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}.$$

Using Corollary 2.5.19 to get the best fit value of $\vec{\boldsymbol{\theta}}$ we have,

$$\hat{\vec{\boldsymbol{\theta}}} = (X^T X)^{-1} X^T Y = \begin{bmatrix} \frac{4}{3} \\ \frac{2}{3} \end{bmatrix}.$$

So the best fit model found is

$$Y = \frac{4}{3} + \frac{2}{3}X.$$

(71) The least squares fitting method can be used to fit non-linear models also. In this case, the non-linear model becomes linearized by the encoding of the features. For example, TODO: LSE Further Linear Algebra [76]

2.5.6 Generalised Inverses

2.5.6.1 Left and Right Inverses

Definition 128. Let A be an $m \times n$ matrix. Then the $m \times n$ matrix L is a *left inverse* for A iff

$$LA = I_n$$

while the $n \times m$ matrix R is a *right inverse* for A iff

$$AR = I_m.$$

Proposition 2.5.50. *Let A be an $m \times n$ matrix. Then A has a **left** inverse iff:*

(i) The nullspace of A is the trivial nullspace $\{\vec{\mathbf{0}}\}$;

(ii) A is an injection: The equation $A\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}}$ has either no solution or one unique solution;

(iii) A has rank n.

Proof.

Let L be a left inverse of A.

(i) Since, for any $\vec{\boldsymbol{x}} \in N(A)$,

$$A\vec{\boldsymbol{x}} = \vec{\mathbf{0}} \iff LA\vec{\boldsymbol{x}} = L\vec{\mathbf{0}} = \vec{\mathbf{0}} \iff I_n\vec{\boldsymbol{x}} = \vec{\boldsymbol{x}} = \vec{\mathbf{0}},$$

we deduce that $N(A) = \{\vec{\mathbf{0}}\}$.

(ii) It follows from the fact that the nullspace of A is the trivial kernel that A is an injection (Corollary 2.5.3).

(iii) It also follows from the fact that the nullspace of A is the trivial kernel that the nullity of A is 0 and so the dimension formula for finite-dimensional linear transformations (Theorem 2.5.1) tells us that

$$n = \operatorname{rank} A + \operatorname{nullity} A = \operatorname{rank} A + 0 \implies \operatorname{rank} A = n.$$

It follows from the fact that A has full column rank, by Proposition 2.6.31, A^TA is invertible and so

$$L = (A^TA)A^T$$

is a left inverse of A. □

Corollary 2.5.20. *If A is an $m \times n$ matrix with $m < n$ then A cannot have a left inverse.*

Corollary 2.5.21. *If A is an $n \times n$ square matrix with rank n then there is a single unique left inverse equal to the right inverse. That's to say,*

$$L = A^{-1} = R.$$

Proposition 2.5.51. *Let A be an $m \times n$ matrix. Then A has a **right** inverse iff:*

(i) $A\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}}$ *has at least one solution for every* $\vec{\boldsymbol{b}} \in \mathbb{F}^m$*;*

(ii) A is a surjection: The range of A is $\mathbb{F}^m$*;*

(iii) A has rank m.

Proof.

Let R be a right inverse of A.

(i) Since, for every $\vec{\boldsymbol{b}} \in \mathbb{F}^m$

$$A\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}} \iff A\vec{\boldsymbol{x}} = I_m\vec{\boldsymbol{b}} = (AR)\vec{\boldsymbol{b}} \iff A\vec{\boldsymbol{x}} = A(R\vec{\boldsymbol{b}}),$$

we can deduce that there exists at least one solution, namely $\vec{\boldsymbol{x}} = R\vec{\boldsymbol{b}}$. (Note that we can't deduce that this is the only solution because A is not, in general, invertible.)

(ii) It follows from the fact that, for every $\vec{\boldsymbol{b}} \in \mathbb{F}^m$ there exists at least one $\vec{\boldsymbol{x}} = R\vec{\boldsymbol{b}} \in \mathbb{F}^n$ such that $A\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}}$ that A is surjective and its range is all of $\mathbb{F}^m$.

(iii) Since the rank of A is, by definition, the dimension of its range, it follows then, from the fact that the range of A is $\mathbb{F}^m$ that the rank of A is m.

It follows from the fact that A has full row rank that A^T has full column rank and so, by Proposition 2.6.31, $(A^T)^T A^T = AA^T$ is invertible and so

$$R = (AA^T)A$$

is a right inverse of A. □

Corollary 2.5.22. *If A is an $m \times n$ matrix with $m > n$ then A cannot have a right inverse.*

Corollary 2.5.23. *If A is an $n \times n$ square matrix with rank n then there is a single unique right inverse equal to the left inverse. That's to say,*

$$R = A^{-1} = L.$$

---

(72) TODO: LSE FLA Example 6.2

2.5.6.2 Weak and Strong Generalised Inverses

Definition 129. **(Weak Generalised Inverse)** Let A be an $m \times n$ matrix. A *weak generalised inverse* (WGI) of A, denoted by A^g, is any $n \times m$ matrix such that

$$AA^gA = A.$$

Proposition 2.5.52. *Both left and right inverses are WGIs.*

Proof. Let A be an $m \times n$ matrix.

If L is a left inverse of A then

$$ALA = AI_n = A.$$

If R is a right inverse of A then

$$ARA = I_mA = A.$$

□

2.6 Inner Product Spaces

2.6.1 Inner Products and Norms

Definition 130. **(Inner Product Space)** Let $\mathbb{F}$ denote either the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$.

An **inner product space** is a vector space V over the field $\mathbb{F}$ together with a map

$$\langle \cdot, \cdot \rangle : V \times V \longmapsto \mathbb{F}$$

called an **inner product** that satisfies the following conditions for all vectors $\vec{\boldsymbol{x}}, \vec{\boldsymbol{y}}, \vec{\boldsymbol{z}} \in V$ and all scalars $\alpha, \beta \in \mathbb{F}$.

(i) **Conjugate Symmetry:** $\langle \vec{\boldsymbol{x}}, \vec{\boldsymbol{y}} \rangle = \overline{\langle \vec{\boldsymbol{y}}, \vec{\boldsymbol{x}} \rangle}$,

(ii) **Linearity in the first argument:**
$$\langle \alpha\vec{\boldsymbol{x}} + \beta\vec{\boldsymbol{y}}, \vec{\boldsymbol{z}} \rangle = \alpha\langle \vec{\boldsymbol{x}}, \vec{\boldsymbol{z}} \rangle + \beta\langle \vec{\boldsymbol{y}}, \vec{\boldsymbol{z}} \rangle,$$

(iii) **Positive definiteness:** $\langle \vec{\boldsymbol{x}}, \vec{\boldsymbol{x}} \rangle \geq 0$ with $\langle \vec{\boldsymbol{x}}, \vec{\boldsymbol{x}} \rangle = 0 \iff \vec{\boldsymbol{x}} = \vec{\boldsymbol{0}}$.

Proposition 2.6.1. *For a vector space V over the field $\mathbb{F}$, whether $\mathbb{F}$ is the real number field $\mathbb{R}$ or the complex number field $\mathbb{C}$, we have, for all $\vec{\boldsymbol{x}} \in \mathbb{F}$,*

$$\langle \vec{\boldsymbol{x}}, \vec{\boldsymbol{x}} \rangle \in \mathbb{R}.$$

Proof. Property (i) of the inner product (2.6.1) — conjugate symmetry — implies that

$$\langle \vec{\boldsymbol{x}}, \vec{\boldsymbol{x}} \rangle = \overline{\langle \vec{\boldsymbol{x}}, \vec{\boldsymbol{x}} \rangle}$$

and, by Proposition 1.2.22, it then follows that

$$\mathrm{Im}(\langle \vec{\boldsymbol{x}}, \vec{\boldsymbol{x}} \rangle) = 0.$$

Therefore $\langle \vec{\boldsymbol{x}}, \vec{\boldsymbol{x}} \rangle \in \mathbb{R}$. □

*This is why the inner product can be described as "positive definite" even on complex fields where there is no concept of positive and negative because the complex numbers are not ordered (ref: **??**).*

Proposition 2.6.2. *The inner product is **conjugate linear** in its second argument. That's to say, if V is an inner product space over the field $\mathbb{F}$ and $\vec{\boldsymbol{x}}, \vec{\boldsymbol{y}}, \vec{\boldsymbol{z}} \in V$ and $\alpha, \beta \in \mathbb{F}$, then*

$$\langle \vec{\boldsymbol{x}},\, \alpha\vec{\boldsymbol{y}} + \beta\vec{\boldsymbol{z}} \rangle = \overline{\alpha}\langle \vec{\boldsymbol{x}},\, \vec{\boldsymbol{y}} \rangle + \overline{\beta}\langle \vec{\boldsymbol{x}},\, \vec{\boldsymbol{z}} \rangle.$$

Proof. Using the definition of the inner product in 2.6.1:

$$\begin{aligned}
\langle \vec{\boldsymbol{x}},\, \alpha\vec{\boldsymbol{y}} + \beta\vec{\boldsymbol{z}} \rangle &= \overline{\langle \alpha\vec{\boldsymbol{y}} + \beta\vec{\boldsymbol{z}},\, \vec{\boldsymbol{x}} \rangle} && \text{by conjugate symmetry} \\
&= \overline{\alpha\langle \vec{\boldsymbol{y}},\, \vec{\boldsymbol{x}} \rangle + \beta\langle \vec{\boldsymbol{z}},\, \vec{\boldsymbol{x}} \rangle} && \text{by linearity of 1st argument} \\
&= \overline{\alpha}\overline{\langle \vec{\boldsymbol{y}},\, \vec{\boldsymbol{x}} \rangle} + \overline{\beta}\overline{\langle \vec{\boldsymbol{z}},\, \vec{\boldsymbol{x}} \rangle} && \text{by complex conjugate properties: 1.2.22} \\
&= \overline{\alpha}\langle \vec{\boldsymbol{x}},\, \vec{\boldsymbol{y}} \rangle + \overline{\beta}\langle \vec{\boldsymbol{x}},\, \vec{\boldsymbol{z}} \rangle && \text{by conjugate symmetry.} \quad \square
\end{aligned}$$

Proposition 2.6.3. *For any inner product space, for $\vec{\boldsymbol{x}}$ in the space,*

$$\langle \vec{\mathbf{0}},\, \vec{\boldsymbol{x}} \rangle = \langle \vec{\boldsymbol{x}},\, \vec{\mathbf{0}} \rangle = 0.$$

Proof.

$$\begin{aligned}
& \langle \vec{\mathbf{0}},\, \vec{\boldsymbol{x}} \rangle = \langle \vec{\mathbf{0}} + \vec{\mathbf{0}},\, \vec{\boldsymbol{x}} \rangle && \text{by props. of } \vec{\mathbf{0}} \text{ vector} \\
\iff \quad & \langle \vec{\mathbf{0}},\, \vec{\boldsymbol{x}} \rangle = \langle \vec{\mathbf{0}},\, \vec{\boldsymbol{x}} \rangle + \langle \vec{\mathbf{0}},\, \vec{\boldsymbol{x}} \rangle && \text{by prop. (ii) of inner product} \\
\iff \quad & \langle \vec{\mathbf{0}},\, \vec{\boldsymbol{x}} \rangle - \langle \vec{\mathbf{0}},\, \vec{\boldsymbol{x}} \rangle = \langle \vec{\mathbf{0}},\, \vec{\boldsymbol{x}} \rangle && \text{by props. of } \mathbb{R} \\
\iff \quad & 0 = \langle \vec{\mathbf{0}},\, \vec{\boldsymbol{x}} \rangle.
\end{aligned}$$

$$\begin{aligned}
& \langle \vec{\boldsymbol{x}},\, \vec{\mathbf{0}} \rangle = \langle \vec{\boldsymbol{x}},\, \vec{\mathbf{0}} + \vec{\mathbf{0}} \rangle \\
\iff \quad & \langle \vec{\boldsymbol{x}},\, \vec{\mathbf{0}} \rangle = \langle \vec{\boldsymbol{x}},\, \vec{\mathbf{0}} \rangle + \langle \vec{\boldsymbol{x}},\, \vec{\mathbf{0}} \rangle \\
\iff \quad & 0 = \langle \vec{\boldsymbol{x}},\, \vec{\mathbf{0}} \rangle.
\end{aligned}$$

$\square$

(73) It is possible to define an inner product over the space of degree-n real polynomials $P_n(\mathbb{R})$ by, for $p, q \in P_n(\mathbb{R})$,

$$\langle p,\, q\rangle = \sum_{i=1}^{n+1} p(x_i)q(x_i)$$

for any distinct $x_1, x_2, \ldots, x_{n+1} \in \mathbb{R}$. This inner product satisfies

$$\langle \vec{\boldsymbol{v}},\, \vec{\boldsymbol{v}}\rangle = 0 \iff \vec{\boldsymbol{v}} = \vec{\mathbf{0}}$$

because, for $p(x_i)^2$ to equal 0 at $n+1$ distinct x_i values, it must have $n+1$ roots – which is not possible unless p is the zero function (refer: Proposition 2.3.35).

Similarly, we could define the following inner product over the space of degree-n complex polynomials $P_n(\mathbb{C})$,

$$\langle p,\, q\rangle = \sum_{i=1}^{n+1} p(x_i)\,\overline{q(x_i)}.$$

2.6.1.1 Orthogonality

Definition 131. **(Orthogonality)** The vectors $\vec{\boldsymbol{x}}$ and $\vec{\boldsymbol{y}}$ in an inner product space are said to be **orthogonal** — denoted $\vec{\boldsymbol{x}} \perp \vec{\boldsymbol{y}}$ — iff $\langle \vec{\boldsymbol{x}},\, \vec{\boldsymbol{y}}\rangle = 0$.

Note that orthogonality is w.r.t. a particular inner product but, in practice, the inner product is often not specified, in which case the orthogonality is w.r.t. the standard inner product (2.6.1.2).

Definition 132. **(Orthogonal Complement)** Let V be an inner product space and $S \subset V$. Then the **orthogonal complement** of S is defined as,

$$S^{\perp} = \{\, \vec{\boldsymbol{v}} \in V \mid \forall \vec{\boldsymbol{x}} \in S \,.\, \vec{\boldsymbol{v}} \perp \vec{\boldsymbol{x}} \,\}.$$

Proposition 2.6.4. *If a set of non-zero vectors in an inner product space is pairwise orthogonal, then it is also linearly independent.*

That's to say, if $\vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2, \ldots, \vec{\boldsymbol{v}}_k \in V$ are non-zero vectors in an inner product space such that, for each $1 \leq i \neq j \leq k$, we have $\vec{\boldsymbol{v}}_i \perp \vec{\boldsymbol{v}}_j$, then $S = \{\vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2, \ldots, \vec{\boldsymbol{v}}_k\}$ is linearly independent.

Proof. Assume there exists a linear dependence relation between some subset of S,

$$\alpha_1\vec{\boldsymbol{x}}_1 + \alpha_2\vec{\boldsymbol{x}}_2 + \cdots + \alpha_m\vec{\boldsymbol{x}}_m = \vec{\boldsymbol{0}}$$

for $\vec{\boldsymbol{x}}_1, \vec{\boldsymbol{x}}_2, \ldots, \vec{\boldsymbol{x}}_m \in S$ and $\alpha_1, \alpha_2, \ldots, \alpha_m \neq 0$. Then, any vector in the dependence relation can be expressed as a linear combination of the others,

$$\vec{\boldsymbol{x}}_1 = -\frac{1}{\alpha_1}(\alpha_2\vec{\boldsymbol{x}}_2 + \cdots + \alpha_m\vec{\boldsymbol{x}}_m).$$

Now, since the elements of S are pairwise orthogonal, we have

$$\langle \vec{\boldsymbol{x}}_i,\, \vec{\boldsymbol{x}}_j \rangle = 0 \quad \text{for} \quad 1 \leq i \neq j \leq m.$$

Therefore,

$$\begin{aligned}
& \langle \vec{\boldsymbol{x}}_1,\, \vec{\boldsymbol{x}}_2 \rangle = 0 && \\
\iff \quad & \langle -\frac{1}{\alpha_1}(\alpha_2\vec{\boldsymbol{x}}_2 + \cdots + \alpha_m\vec{\boldsymbol{x}}_m),\, \vec{\boldsymbol{x}}_2 \rangle = 0 && \\
\iff \quad & -\frac{1}{\alpha_1}\langle (\alpha_2\vec{\boldsymbol{x}}_2 + \cdots + \alpha_m\vec{\boldsymbol{x}}_m),\, \vec{\boldsymbol{x}}_2 \rangle = 0 && \text{by 2.6.1} \\
\iff \quad & \alpha_2\langle \vec{\boldsymbol{x}}_2,\, \vec{\boldsymbol{x}}_2 \rangle + \alpha_3\langle \vec{\boldsymbol{x}}_3,\, \vec{\boldsymbol{x}}_2 \rangle + \cdots + \alpha_m\langle \vec{\boldsymbol{x}}_m,\, \vec{\boldsymbol{x}}_2 \rangle = 0 && \text{by 2.6.1} \\
\iff \quad & \alpha_2\langle \vec{\boldsymbol{x}}_2,\, \vec{\boldsymbol{x}}_2 \rangle + 0 + \cdots + 0 = 0 && \text{by pairwise orthogonality} \\
\iff \quad & \alpha_2\langle \vec{\boldsymbol{x}}_2,\, \vec{\boldsymbol{x}}_2 \rangle = 0 && \\
\iff \quad & \vec{\boldsymbol{x}}_2 = \vec{\boldsymbol{0}}. && \text{by 2.6.1}
\end{aligned}$$

But all the vectors in S are non-zero by construction so this is a contradiction. □

Proposition 2.6.5. *For any subset S of an inner product space V over a field $\mathbb{F}$, $S^\perp$ is a subspace of V.*

Proof. This follows from the definition of the orthogonal complement (2.6.1.1), the linearity of the inner product (which guarantees closure of $S^\perp$ under linear combinations), and the fact $\vec{\mathbf{0}}$ is orthogonal to every vector and so is a member of $S^\perp$. □

Proposition 2.6.6. *For any subspace S of a finite-dimensional inner product space V on a field $\mathbb{F}$, the orthogonal complement $S^\perp$ is a complement space of S in V (see: 2.4.6.7). In other words,*

$$S \oplus S^\perp = V.$$

Proof. Let $B_s = \{\vec{\boldsymbol{v}}_1, \ldots, \vec{\boldsymbol{v}}_k\}$ be a basis of the subspace S. We can extend this to a basis of V and then use Gram-Schmidt (2.6.2.4) to orthogonalise this to an orthonormal basis of V

$$O_v = \{\vec{\boldsymbol{u}}_1, \ldots, \vec{\boldsymbol{u}}_k, \vec{\boldsymbol{u}}_{k+1}, \ldots, \vec{\boldsymbol{u}}_n\}$$

By the Gram-Schmidt process, the first k vectors of the orthonormal basis O_v are a basis for $\text{Lin}\{\vec{\boldsymbol{v}}_1, \ldots, \vec{\boldsymbol{v}}_k\}$ and so, a basis for S. The remaining vectors in O_v,

$$B_w = \{\vec{\boldsymbol{u}}_{k+1}, \ldots, \vec{\boldsymbol{u}}_n\},$$

form a basis of a space W such that, for any $\vec{\boldsymbol{w}} \in W$ and $\vec{\boldsymbol{v}} \in S$,

$$\begin{aligned}
& \vec{\boldsymbol{v}} + \vec{\boldsymbol{w}} = 0 \\
\iff \quad & \alpha_1\vec{\boldsymbol{u}}_1 + \cdots + \alpha_k\vec{\boldsymbol{u}}_k + \alpha_{k+1}\vec{\boldsymbol{u}}_{k+1} + \cdots + \alpha_n\vec{\boldsymbol{u}}_n = 0 \\
\iff \quad & \alpha_1, \ldots, \alpha_k, \alpha_{k+1}, \ldots, \alpha_n = 0. \quad \because O_v \text{ is a basis}
\end{aligned}$$

Therefore, if $\vec{\boldsymbol{x}} \in W \cap S$ then there exists some $\vec{\boldsymbol{w}} \in W$ and $\vec{\boldsymbol{v}} \in S$ such that,

$$\begin{aligned}
& \vec{\boldsymbol{x}} = \vec{\boldsymbol{v}} = \vec{\boldsymbol{w}} \\
\iff \quad & \vec{\boldsymbol{v}} - \vec{\boldsymbol{w}} = \vec{\mathbf{0}} \\
\iff \quad & \vec{\boldsymbol{v}} = \vec{\boldsymbol{w}} = \vec{\mathbf{0}}.
\end{aligned}$$

Since $\vec{\mathbf{0}}$ is clearly in both W and S because any linear span contains $\vec{\mathbf{0}}$, we have,

$$W \cap S = \{\vec{\mathbf{0}}\}$$

and then the sum of W and S is a direct sum,

$$W + S = W \oplus S.$$

Furthermore, for any $\vec{\boldsymbol{w}} \in W$ and $\vec{\boldsymbol{v}} \in S$,

$$\langle \vec{\boldsymbol{w}},\, \vec{\boldsymbol{v}} \rangle = \langle \alpha_1 \vec{\boldsymbol{u}}_1 + \cdots + \alpha_k \vec{\boldsymbol{u}}_k,\, \alpha_{k+1} \vec{\boldsymbol{u}}_{k+1} + \cdots + \alpha_n \vec{\boldsymbol{u}}_n \rangle = 0$$

because, using the linearity of the inner product (item 2.6.1), we get an expression of the form,

$$\langle \vec{\boldsymbol{w}},\, \vec{\boldsymbol{v}} \rangle = \sum_{i=1}^{k} \sum_{j=k+1}^{n} \langle \alpha_i \vec{\boldsymbol{u}}_i,\, \alpha_j \vec{\boldsymbol{u}}_j \rangle$$

where every term has $i \neq j$ and so the orthogonality of the basis O_v implies that every term is 0. So every vector in W is orthogonal to every vector in S and thus,

$$W \subseteq S^{\perp}.$$

Conversely, if a vector

$$\vec{\boldsymbol{x}} = \alpha_1 \vec{\boldsymbol{u}}_1 + \cdots + \alpha_k \vec{\boldsymbol{u}}_k + \alpha_{k+1} \vec{\boldsymbol{u}}_{k+1} + \cdots + \alpha_n \vec{\boldsymbol{u}}_n \in V$$

is in $S^{\perp}$ then it must be orthogonal to every vector

$$\vec{\boldsymbol{v}} = \beta_1 \vec{\boldsymbol{u}}_1 + \cdots + \beta_k \vec{\boldsymbol{u}}_k \in S$$

so we must have, for all $\vec{\boldsymbol{v}} \in S$

$$\langle \vec{\boldsymbol{x}},\, \vec{\boldsymbol{v}} \rangle = 0.$$

But, by the linearity of the inner product and the orthonormality of O_v,

$$\langle \vec{\boldsymbol{x}},\, \vec{\boldsymbol{v}} \rangle = \alpha_1 \overline{\beta}_1 + \cdots + \alpha_k \overline{\beta}_k.$$

Since this has to apply for a given $\vec{\boldsymbol{x}}$ and for every $\vec{\boldsymbol{v}} \in S$ we have,

$$\forall \beta_1, \ldots, \beta_k \in \mathbb{F} \,.\, \alpha_1 \overline{\beta}_1 + \cdots + \alpha_k \overline{\beta}_k = 0 \implies \alpha_1, \ldots, \alpha_k = 0.$$

Therefore,

$$\vec{\boldsymbol{x}} = \alpha_{k+1} \vec{\boldsymbol{u}}_{k+1} + \cdots + \alpha_n \vec{\boldsymbol{u}}_n$$

and so

$$S^{\perp} \subseteq W. \qquad \square$$

Proposition 2.6.7. *If S is a subspace of a finite-dimensional inner product space V on a field $\mathbb{F}$, then*

$$(S^\perp)^\perp = S.$$

Proof. By the definition of the orthogonal complement (2.6.1.1),

$$S^\perp = \{\, \vec{\boldsymbol{v}} \in V \mid \forall \vec{\boldsymbol{w}} \in S \,.\, \langle \vec{\boldsymbol{v}}, \vec{\boldsymbol{w}} \rangle = 0 \,\}$$

and

$$(S^\perp)^\perp = \{\, \vec{\boldsymbol{u}} \in V \mid \forall \vec{\boldsymbol{v}} \in S^\perp \,.\, \langle \vec{\boldsymbol{u}}, \vec{\boldsymbol{v}} \rangle = 0 \,\}.$$

Since the inner product is conjugate symmetric (2.6.1), if $\langle \vec{\boldsymbol{x}}, \vec{\boldsymbol{y}} \rangle \in \mathbb{R}$ then

$$\langle \vec{\boldsymbol{x}}, \vec{\boldsymbol{y}} \rangle = \overline{\langle \vec{\boldsymbol{y}}, \vec{\boldsymbol{x}} \rangle} \iff \overline{\langle \vec{\boldsymbol{x}}, \vec{\boldsymbol{y}} \rangle} = \langle \vec{\boldsymbol{x}}, \vec{\boldsymbol{y}} \rangle = \langle \vec{\boldsymbol{y}}, \vec{\boldsymbol{x}} \rangle.$$

Applying this symmetry to the definition of $S^\perp$ we deduce that

$$\begin{aligned}
& \forall \vec{\boldsymbol{v}} \in S^\perp \,.\, \forall \vec{\boldsymbol{w}} \in S \,.\, \langle \vec{\boldsymbol{w}}, \vec{\boldsymbol{v}} \rangle = 0 \\
\iff \quad & \forall \vec{\boldsymbol{w}} \in S \subseteq V \,.\, \forall \vec{\boldsymbol{v}} \in S^\perp \,.\, \langle \vec{\boldsymbol{w}}, \vec{\boldsymbol{v}} \rangle = 0 \\
\iff \quad & \forall \vec{\boldsymbol{w}} \in S \,.\, \vec{\boldsymbol{w}} \in (S^\perp)^\perp \\
\iff \quad & S \subseteq (S^\perp)^\perp.
\end{aligned}$$

For the converse proposition that $(S^\perp)^\perp \subseteq S$, we begin by forming orthonormal bases of S and $S^\perp$ (which is always possible by Gram-Schmidt 2.6.2.4), and observing that, by Proposition 2.6.6,

$$S \oplus S^\perp = V.$$

Therefore, if the orthonormal basis of S is

$$O_S = \{\vec{\boldsymbol{x}}_1, \ldots, \vec{\boldsymbol{x}}_k\}$$

and that of $S^\perp$ is

$$O_{S^\perp} = \{\vec{\boldsymbol{x}}_{k+1}, \ldots, \vec{\boldsymbol{x}}_n\},$$

then any vector $\vec{\boldsymbol{v}} \in V$ can be expressed as

$$\vec{\boldsymbol{v}} = (\alpha_1 \vec{\boldsymbol{x}}_1 + \cdots + \alpha_k \vec{\boldsymbol{x}}_k) + (\alpha_{k+1} \vec{\boldsymbol{x}}_{k+1} + \cdots + \alpha_n \vec{\boldsymbol{x}}_n).$$

If $\vec{\boldsymbol{v}} \in (S^\perp)^\perp$ then, for all $\vec{\boldsymbol{w}} \in S^\perp$,

$$\begin{aligned} & \langle \vec{\boldsymbol{v}},\, \vec{\boldsymbol{w}} \rangle = 0 \\ \iff \quad & \langle \vec{\boldsymbol{v}},\, \beta_{k+1}\vec{\boldsymbol{x}}_{k+1} + \cdots + \beta_n \vec{\boldsymbol{x}}_n \rangle = 0 \\ \iff \quad & \alpha_{k+1}\beta_{k+1} + \cdots + \alpha_n \beta_n = 0. \quad \text{by orthogonality} \end{aligned}$$

Since this must be the case for all $\vec{\boldsymbol{w}} \in S^\perp$,

$$\alpha_{k+1}, \ldots, \alpha_n = 0.$$

This implies that $\vec{\boldsymbol{v}} \in S$ and so $(S^\perp)^\perp \subseteq S$. □

Proposition 2.6.8. *The nullspace of a real matrix $A \in \mathbb{R}^{m\times n}$ is the orthogonal complement of the rowspace of A.*

Proof. Following the definition of the orthogonal complement and denoting the range of A as $R(A)$ and the nullspace as $N(A)$,

$$N(A) = \{\, \vec{\boldsymbol{v}} \in \mathbb{R}^n \mid A\vec{\boldsymbol{v}} = \vec{\boldsymbol{0}} \,\} \quad \text{and} \quad R(A^T) = \{\, A^T\vec{\boldsymbol{v}} \mid \vec{\boldsymbol{v}} \in \mathbb{R}^m \,\}.$$

It follows then that, for any $\vec{\boldsymbol{x}} \in N(A)$ and for all $\vec{\boldsymbol{y}} \in R(A^T)$

$$\begin{aligned} \langle \vec{\boldsymbol{x}},\, \vec{\boldsymbol{y}} \rangle &= \langle \vec{\boldsymbol{x}},\, A^T\vec{\boldsymbol{v}} \rangle && \text{for some } \vec{\boldsymbol{v}} \in \mathbb{R}^m \\ &= \vec{\boldsymbol{x}}^T A^T \vec{\boldsymbol{v}} && \text{by defn. of standard real inner product} \\ &= (A\vec{\boldsymbol{x}})^T \vec{\boldsymbol{v}} \\ &= (\vec{\boldsymbol{0}})\vec{\boldsymbol{v}} && \because \vec{\boldsymbol{x}} \in N(A) \\ &= 0 \end{aligned}$$

which implies that $N(A) \subseteq R(A^T)^\perp$.

Conversely, for any $\vec{\boldsymbol{x}} \in R(A^T)^\perp$ and for all $\vec{\boldsymbol{y}} \in R(A^T)$,

$$\begin{aligned} & \langle \vec{\boldsymbol{x}},\, \vec{\boldsymbol{y}} \rangle = 0 && \text{by defn. of orthogonal complement} \\ \iff \quad & \langle \vec{\boldsymbol{x}},\, A^T\vec{\boldsymbol{v}} \rangle = 0 && \text{for some } \vec{\boldsymbol{v}} \in \mathbb{R}^m \\ \iff \quad & \vec{\boldsymbol{x}}^T A^T \vec{\boldsymbol{v}} = 0 \\ \iff \quad & (A\vec{\boldsymbol{x}})^T \vec{\boldsymbol{v}} = 0 \end{aligned}$$

which, since it holds for all $\vec{\boldsymbol{y}} \in R(A^T)$ and therefore for all $\vec{\boldsymbol{v}} \in \mathbb{R}^m$, must hold for $\vec{\boldsymbol{v}} = A\vec{\boldsymbol{x}} \in \mathbb{R}^m$ and so,

$$(A\vec{\boldsymbol{x}})^T\vec{\boldsymbol{v}} = 0 = (A\vec{\boldsymbol{x}})^T(A\vec{\boldsymbol{v}}) = \langle A\vec{\boldsymbol{x}},\, A\vec{\boldsymbol{x}}\rangle.$$

Now, the positive definiteness property of the inner product (2.6.1) implies that $A\vec{\boldsymbol{x}} = 0$ and so $\vec{\boldsymbol{x}} \in N(A)$. This, in turn, implies that $R(A^T)^\perp \subseteq N(A)$.

Therefore $N(A) = R(A^T)^\perp$. □

Proposition 2.6.9. *The nullspace of a matrix $A \in \mathbb{F}^{m\times n}$ is the orthogonal complement of the range of the hermitian conjugate of A.*

Proof. Following the definition of the orthogonal complement and denoting the range of A as $R(A)$ and the nullspace as $N(A)$,

$$N(A) = \{\, \vec{\boldsymbol{v}} \in V \mid A\vec{\boldsymbol{v}} = \vec{\boldsymbol{0}} \,\} \quad \text{and} \quad R(A^*) = \{\, A^*\vec{\boldsymbol{v}} \mid \vec{\boldsymbol{v}} \in V \,\}.$$

It follows then that, for any $\vec{\boldsymbol{x}} \in N(A)$ and for all $\vec{\boldsymbol{y}} \in R(A^*)$

$$\begin{aligned}
\langle \vec{\boldsymbol{x}},\, \vec{\boldsymbol{y}}\rangle &= \langle \vec{\boldsymbol{x}},\, A^*\vec{\boldsymbol{v}}\rangle && \text{for some } \vec{\boldsymbol{v}} \in \mathbb{F}^m \\
&= \vec{\boldsymbol{x}}^T A^*\vec{\boldsymbol{v}} && \text{by defn. of standard hermitian inner product} \\
&= (A\vec{\boldsymbol{x}})^*\vec{\boldsymbol{v}} \\
&= (\vec{\boldsymbol{0}})\vec{\boldsymbol{v}} && \because \vec{\boldsymbol{x}} \in N(A) \\
&= 0
\end{aligned}$$

which implies that $N(A) \subseteq R(A^*)^\perp$.

Conversely, for any $\vec{\boldsymbol{x}} \in R(A^*)^\perp$ and for all $\vec{\boldsymbol{y}} \in R(A^*)$,

$$\begin{aligned}
& \langle \vec{\boldsymbol{x}},\, \vec{\boldsymbol{y}}\rangle = 0 && \text{by defn. of orthogonal complement} \\
\iff \quad & \langle \vec{\boldsymbol{x}},\, A^*\vec{\boldsymbol{v}}\rangle = 0 && \text{for some } \vec{\boldsymbol{v}} \in \mathbb{F}^m \\
\iff \quad & \vec{\boldsymbol{x}}^T A^*\vec{\boldsymbol{v}} = 0 \\
\iff \quad & (A\vec{\boldsymbol{x}})^*\vec{\boldsymbol{v}} = 0
\end{aligned}$$

which, since it holds for all $\vec{\boldsymbol{y}} \in R(A^*)$ and therefore for all $\vec{\boldsymbol{v}} \in \mathbb{F}^m$, must hold for $\vec{\boldsymbol{v}} = A\vec{\boldsymbol{x}} \in \mathbb{F}^m$ and so,

$$(A\vec{\boldsymbol{x}})^*\vec{\boldsymbol{v}} = 0 = (A\vec{\boldsymbol{x}})^*(A\vec{\boldsymbol{v}}) = \langle A\vec{\boldsymbol{x}},\, A\vec{\boldsymbol{x}}\rangle.$$

Now, the positive definiteness property of the inner product (2.6.1) implies that $A\vec{\boldsymbol{x}} = 0$ and so $\vec{\boldsymbol{x}} \in N(A)$. This, in turn, implies that $R(A^*)^\perp \subseteq N(A)$.

Therefore $N(A) = R(A^*)^\perp$. □

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(74) The classic example is of a vector in $\mathbb{R}^3$ and the plane defined by the orthogonal complement of the vector. For example, if $\vec{\boldsymbol{v}} = (1, 2, -1)^T$ then,

$$S^\perp = \{\, (x, y, z)^T \mid x + 2y - z = 0 \,\}.$$

(75) Suppose we want to find the orthogonal complement of the plane created by the linear span of two non-colinear vectors in $\mathbb{R}^3$ with the standard inner product. Let

$$\vec{\boldsymbol{v}} = (1, 2, -1)^T \quad \text{and} \quad \vec{\boldsymbol{w}} = (1, 0, 1)^T.$$

Then the orthogonal complement $\text{Lin}\{\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}\}^\perp$ is the nullspace of the matrix whose rows are $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{w}}$,

$$A\vec{\boldsymbol{x}} = \vec{\mathbf{0}} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix} \vec{\boldsymbol{x}}.$$

Using **??**, the nullspace of the matrix A is

$$t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \quad \text{for} \quad t \in \mathbb{R}.$$

So the orthogonal complement is

$$\text{Lin}\{\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}\}^\perp = \{\, (-x, y, z)^T \mid x, y, z \in \mathbb{R}^3 \,\}$$

2.6.1.2 The Standard Inner Product

Definition 133. The **Standard Inner Product** for real vector spaces is the Euclidean dot product 2.4.8.

For complex vector spaces the **Standard Complex Inner Product** is similar to the dot product but with conjugation of the second argument:

For $\vec{\boldsymbol{x}}, \vec{\boldsymbol{y}} \in \mathbb{C}^n$,

$$\langle \vec{\boldsymbol{x}},\, \vec{\boldsymbol{y}} \rangle = \vec{\boldsymbol{y}}^* \vec{\boldsymbol{x}} = x_1 \overline{y_1} + \cdots + x_n \overline{y_n}$$

where $\vec{\boldsymbol{y}}^* = \overline{\vec{\boldsymbol{y}}}^T$ is the hermitian conjugate (2.3.1.7).
Some authors have the standard complex inner product as $\langle \vec{\boldsymbol{x}},\, \vec{\boldsymbol{y}} \rangle = \vec{\boldsymbol{x}}^T \overline{\vec{\boldsymbol{y}}}$ but this is for compatibility with an inner product that is linear in the second argument (and conjugate linear in the first).
This inner product exhibits conjugate symmetry because, using properties of the complex conjugate (1.2.22) and modulus (1.2.21),

$$\begin{aligned}
\langle \vec{\boldsymbol{x}},\, \vec{\boldsymbol{y}} \rangle &= x_1 \overline{y_1} + \cdots + x_n \overline{y_n} \\
&= \overline{\overline{x_1} y_1} + \cdots + \overline{\overline{x_n} y_n} \\
&= \overline{\overline{x_1} y_1 + \cdots + \overline{x_n} y_n} \\
&= \overline{y_1 \overline{x_1} + \cdots + y_n \overline{x_n}} \\
&= \overline{\langle \vec{\boldsymbol{y}},\, \vec{\boldsymbol{x}} \rangle}.
\end{aligned}$$

And it is positive definite because,

$$\begin{aligned}
\langle \vec{\boldsymbol{x}},\, \vec{\boldsymbol{x}} \rangle &= x_1 \overline{x_1} + \cdots + x_n \overline{x_n} \\
&= |x_1|^2 + \cdots + |x_n|^2
\end{aligned}$$

and for each $|x_i|^2$,

$$x_i = 0 \implies |x_i|^2 = 0 \quad \text{and} \quad x_i \neq 0 \implies |x_i|^2 > 0.$$

Proposition 2.6.10. *If the inner product $\langle \cdot, \cdot \rangle$ is the standard inner product then, for any matrix A and vectors $\vec{\boldsymbol{x}}, \vec{\boldsymbol{y}}$,*

$$\langle \vec{\boldsymbol{x}}, A\vec{\boldsymbol{y}} \rangle = \langle A^* \vec{\boldsymbol{x}}, \vec{\boldsymbol{y}} \rangle.$$

Proof.

$$\langle \vec{\boldsymbol{x}}, A\vec{\boldsymbol{y}} \rangle = \vec{\boldsymbol{x}}^* A\vec{\boldsymbol{y}} = (A^* \vec{\boldsymbol{x}})^* \vec{\boldsymbol{y}} = \langle A^* \vec{\boldsymbol{x}}, \vec{\boldsymbol{y}} \rangle.$$

□

2.6.1.3 Normed Spaces

Definition 134. An inner product induces a **norm** on a vector space V such that the norm of a vector $\vec{\boldsymbol{v}} \in V$ — denoted $\|\vec{\boldsymbol{v}}\|$ — is defined as

$$\|\vec{\boldsymbol{v}}\| = \sqrt{\langle \vec{\boldsymbol{v}}, \vec{\boldsymbol{v}} \rangle}.$$

Since the inner product is positive definite we have,

$$\langle \vec{\boldsymbol{v}}, \vec{\boldsymbol{v}} \rangle \geq 0 \in \mathbb{R} \implies \|\vec{\boldsymbol{v}}\| \geq 0 \in \mathbb{R}$$

and also,

$$(\langle \vec{\boldsymbol{v}}, \vec{\boldsymbol{v}} \rangle = 0 \iff \vec{\boldsymbol{v}} = \vec{\boldsymbol{0}}) \implies (\|\vec{\boldsymbol{v}}\| = 0 \iff \vec{\boldsymbol{v}} = \vec{\boldsymbol{0}}).$$

Therefore, the norm is a real-valued positive definite function

$$f : V \longmapsto \mathbb{R}.$$

A vector space equipped with a norm is called a **normed** vector space.

Definition 135. A **unit vector** is a vector in a normed vector space that has norm equal to 1.

If a vector $\vec{\boldsymbol{v}} \in V$ of arbitrary norm is divided by its norm (i.e. scalar multiplied by the reciprocal of its norm), then

$$\vec{\boldsymbol{u}} = \frac{1}{\|\vec{\boldsymbol{v}}\|}\vec{\boldsymbol{v}}$$

is a unit vector colinear with $\vec{\boldsymbol{v}}$. This process is known as **normalization** of the vector $\vec{\boldsymbol{v}}$.

Proposition 2.6.11. *If $\vec{\boldsymbol{x}}$ is a vector in a normed space over a field $\mathbb{F}$ and $\alpha \in \mathbb{F}$ then,*

$$\|\alpha\vec{\boldsymbol{x}}\| = |\alpha|\|\vec{\boldsymbol{x}}\|\,.$$

Proof.

$$\|\alpha\vec{\boldsymbol{x}}\| = \sqrt{\langle\alpha\vec{\boldsymbol{x}},\, \alpha\vec{\boldsymbol{x}}\rangle} = \sqrt{\alpha\overline{\alpha}\langle\vec{\boldsymbol{x}},\, \vec{\boldsymbol{x}}\rangle} = \sqrt{|\alpha|^2}\sqrt{\langle\vec{\boldsymbol{x}},\, \vec{\boldsymbol{x}}\rangle} = |\alpha|\|\vec{\boldsymbol{x}}\|\,.$$

□

Generalized Geometry

In a bare inner product space there are no points or coordinates and no metric defining distance between points or coordinates. Nevertheless, we can define the angle between two vectors in terms of the inner product and the induced norm.

Definition 136. **(Angle in Real Vector Space)** Let V be an inner product space over the reals $\mathbb{R}$ and $\vec{\boldsymbol{x}}, \vec{\boldsymbol{y}} \in V$. Then the angle θ between the vectors $\vec{\boldsymbol{x}}$ and $\vec{\boldsymbol{y}}$ is defined as,

$$\cos\theta = \frac{\langle \vec{\boldsymbol{x}},\, \vec{\boldsymbol{y}} \rangle}{\|\vec{\boldsymbol{x}}\|\|\vec{\boldsymbol{y}}\|} \iff \theta = \cos^{-1}\left(\frac{\langle \vec{\boldsymbol{x}},\, \vec{\boldsymbol{y}} \rangle}{\|\vec{\boldsymbol{x}}\|\|\vec{\boldsymbol{y}}\|}\right).$$

In the case that the vectors are orthogonal we have $\langle \vec{\boldsymbol{x}},\, \vec{\boldsymbol{y}} \rangle = 0$ so that,

$$\theta = \cos^{-1}(0) = \pm\frac{\pi}{2}.$$

Definition 137. **(Angle in Complex Vector Space)** In a complex vector space the geometrical significance of the "angle" as defined above is not so clear. As a result other concepts of angle are often used in complex spaces (see: Scharnhorst,K. - Angles in Complex Vector Spaces). One such alternative is typically referred to as the **Euclidean Angle** and defined by,

$$\theta = \cos^{-1}\left(\frac{\mathrm{Re}(\langle \vec{\boldsymbol{x}},\, \vec{\boldsymbol{y}} \rangle)}{\|\vec{\boldsymbol{x}}\|\|\vec{\boldsymbol{y}}\|}\right)$$

where the inner product used is the complex standard inner product 2.6.1.2. This angle treats the complex scalars as two real numbers and then takes the angle between two such-defined real vectors.

Another commonly used definition of angle in complex spaces is the **Hermitian Angle** defined by,

$$\theta = \cos^{-1}\left(\frac{|\langle \vec{\boldsymbol{x}},\, \vec{\boldsymbol{y}} \rangle|}{\|\vec{\boldsymbol{x}}\|\|\vec{\boldsymbol{y}}\|}\right)$$

which is the ratio of the orthogonal projection (using the complex inner product) of $\vec{\boldsymbol{x}}$ onto $\vec{\boldsymbol{y}}$ divided by the norm of $\vec{\boldsymbol{x}}$ (or the reverse).

In a Euclidean space if two parametric lines have the same direction vector,

$$l_1(t_1) = \vec{\boldsymbol{p}}_1 + t_1\vec{\boldsymbol{v}},\ \ l_2(t_2) = \vec{\boldsymbol{p}}_2 + t_2\vec{\boldsymbol{v}}$$

then the distance between them remains constant at all points on the lines. This is Euclid's 5th postulate, known as The Parallel Postulate *(wikipedia).*

In this case, the distance is defined as,

$$d(t_1) = \min_{t_2} \left\| (\vec{\boldsymbol{p}}_2 + t_2\vec{\boldsymbol{v}}) - (\vec{\boldsymbol{p}}_1 + t_1\vec{\boldsymbol{v}}) \right\|$$

where the norm used here is the Euclidean norm (2.4.8.7). This Euclidean norm is defined as,

$$\|\vec{\boldsymbol{v}}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

for $\vec{\boldsymbol{v}} \in \mathbb{R}^n$. Each component of $\vec{\boldsymbol{v}}$ represents a scaling of a standard basis vector $v_i\vec{\boldsymbol{e_i}}$ whose resultant length — and therefore, norm — will be v_i. The vector $\vec{\boldsymbol{v}}$ is the sum of these component vectors,

$$\vec{\boldsymbol{v}} = \sum_{i=1}^{n} v_i\vec{\boldsymbol{e_i}} = v_1\vec{\boldsymbol{e_1}} + v_2\vec{\boldsymbol{e_2}} + \cdots + v_n\vec{\boldsymbol{e_n}}$$

and, since the standard basis vectors are orthogonal and so subtend right angles with each other, we can use pythagoras theorem to deduce that

$$\begin{aligned} \|\vec{\boldsymbol{v}}\|^2 &= \|v_1\vec{\boldsymbol{e_1}}\|^2 + \|v_2\vec{\boldsymbol{e_2}}\|^2 + \cdots + \|v_n\vec{\boldsymbol{e_n}}\|^2 \\ &= v_1^2 + v_2^2 + \cdots + v_n^2. \end{aligned}$$

That's to say, the Euclidean norm, which is also the metric (i.e. definition of distance) in Euclidean space, relies on the assumption of an orthogonal basis. Meanwhile, the constancy of the metric value w.r.t. the coordinates (i.e. that the distance between parallel lines is constant) makes Euclidean space "flat".

If we consider WGS-84 GPS coordinate space as an example of a non-Euclidean space: the axes — latitude and longitude — form an orthogonal basis at the equator and also, even though the longitude lines converge at the poles, they remain orthogonal to the lines of latitude. So, even though the coordinate bases are orthogonal everywhere, the metric function that returns the distance between two coordinates in the space, depends on the latitude. It is for this reason that it is a non-Euclidean space.

In general, an inner product space may not be a metric space, may not be a coordinate space, and — if we are working with coordinates w.r.t. a basis — the basis may not be orthogonal.

If the metric is equal to the norm then parallel vectors will remain the same distance apart? (i.e. the space is flat?) Maybe because the distance between two coordinate points remains the same? Between two coordinate points we can define a vector whose length is the vector norm, if that is also the distance, then the distance depends in a constant way on the coordinates and so the space is not "curved". (?) If we take vectors defined as displacements in the gps coordinate system then the length of the vector depends only on the difference in the coordinates not in where the starting point is (as we would expect because vectors do not have a particular starting point) but the distance represented by the displacement depends on the latitude of the starting point.

Lemma 2.6.1. *Let V be an inner product space and $\vec{\boldsymbol{x}}, \vec{\boldsymbol{y}} \in V$. Then,*

$$\|\vec{\boldsymbol{x}} + \vec{\boldsymbol{y}}\|^2 = \langle \vec{\boldsymbol{x}},\, \vec{\boldsymbol{x}} \rangle + \langle \vec{\boldsymbol{y}},\, \vec{\boldsymbol{y}} \rangle + 2\,\mathrm{Re}(\langle \vec{\boldsymbol{x}},\, \vec{\boldsymbol{y}} \rangle).$$

Proof.

$$\begin{aligned}
\|\vec{\boldsymbol{x}} + \vec{\boldsymbol{y}}\|^2 &= \langle \vec{\boldsymbol{x}} + \vec{\boldsymbol{y}},\, \vec{\boldsymbol{x}} + \vec{\boldsymbol{y}} \rangle \\
&= \langle \vec{\boldsymbol{x}},\, \vec{\boldsymbol{x}} + \vec{\boldsymbol{y}} \rangle + \langle \vec{\boldsymbol{y}},\, \vec{\boldsymbol{x}} + \vec{\boldsymbol{y}} \rangle \\
&= \langle \vec{\boldsymbol{x}},\, \vec{\boldsymbol{x}} \rangle + \langle \vec{\boldsymbol{x}},\, \vec{\boldsymbol{y}} \rangle + \langle \vec{\boldsymbol{y}},\, \vec{\boldsymbol{x}} \rangle + \langle \vec{\boldsymbol{y}},\, \vec{\boldsymbol{y}} \rangle \\
&= \langle \vec{\boldsymbol{x}},\, \vec{\boldsymbol{x}} \rangle + \langle \vec{\boldsymbol{x}},\, \vec{\boldsymbol{y}} \rangle + \overline{\langle \vec{\boldsymbol{x}},\, \vec{\boldsymbol{y}} \rangle} + \langle \vec{\boldsymbol{y}},\, \vec{\boldsymbol{y}} \rangle \quad \text{by 2.6.1 prop. (i)} \\
&= \langle \vec{\boldsymbol{x}},\, \vec{\boldsymbol{x}} \rangle + \langle \vec{\boldsymbol{y}},\, \vec{\boldsymbol{y}} \rangle + 2\,\mathrm{Re}(\langle \vec{\boldsymbol{x}},\, \vec{\boldsymbol{y}} \rangle). \qquad \because z + \overline{z} = 2\,\mathrm{Re}(z)
\end{aligned}$$

□

Theorem 2.6.1. ***(Generalized Pythagoras Theorem.)*** *Let V be an inner product space and $\vec{\boldsymbol{x}}, \vec{\boldsymbol{y}} \in V$. If $\vec{\boldsymbol{x}}$ and $\vec{\boldsymbol{y}}$ are orthogonal then,*

$$\|\vec{\boldsymbol{x}} + \vec{\boldsymbol{y}}\|^2 = \|\vec{\boldsymbol{x}}\|^2 + \|\vec{\boldsymbol{y}}\|^2 .$$

Proof.
If $\vec{\boldsymbol{x}}$ and $\vec{\boldsymbol{y}}$ are orthogonal then $\langle \vec{\boldsymbol{x}}, \vec{\boldsymbol{y}} \rangle = 0$. Then the Lemma 2.6.1 gives us,

$$\|\vec{\boldsymbol{x}} + \vec{\boldsymbol{y}}\|^2 = \langle \vec{\boldsymbol{x}}, \vec{\boldsymbol{x}} \rangle + \langle \vec{\boldsymbol{y}}, \vec{\boldsymbol{y}} \rangle + 0 = \|\vec{\boldsymbol{x}}\|^2 + \|\vec{\boldsymbol{y}}\|^2 . \qquad \square$$

Theorem 2.6.2. ***(Cauchy-Schwarz Inequality.)*** *Let $\vec{\boldsymbol{x}}, \vec{\boldsymbol{y}} \in V$ be vectors in an inner product space over the field $\mathbb{F}$ where $\mathbb{F}$ is either the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$. Then,*

$$|\langle \vec{\boldsymbol{x}}, \vec{\boldsymbol{y}} \rangle| \leq \|\vec{\boldsymbol{x}}\| \|\vec{\boldsymbol{y}}\|$$

where the equality holds iff $\vec{\boldsymbol{x}}$ and $\vec{\boldsymbol{y}}$ are linearly dependent.

Proof.
Assume that $\vec{\boldsymbol{x}} = \vec{\mathbf{0}}$ then,

$$\begin{aligned} & \left|\langle \vec{\mathbf{0}}, \vec{\boldsymbol{y}} \rangle\right| \leq \left\|\vec{\mathbf{0}}\right\| \|\vec{\boldsymbol{y}}\| \\ \iff \quad & |0| \leq (0)\|\vec{\boldsymbol{y}}\| \quad \text{by 2.6.3 and inner product prop. (iii)} \\ \iff \quad & 0 = 0. \end{aligned}$$

If $\vec{\boldsymbol{x}} \neq 0$ and $\vec{\boldsymbol{y}} = \vec{\mathbf{0}}$ we obtain the same result as, by 2.6.3, $\langle \vec{\boldsymbol{x}}, \vec{\mathbf{0}} \rangle = 0$ also. Clearly also, the same result holds if both are zero. So, if at least one of $\vec{\boldsymbol{x}}$ and $\vec{\boldsymbol{y}}$ is $\vec{\mathbf{0}}$ then we have,

$$|\langle \vec{\boldsymbol{x}}, \vec{\boldsymbol{y}} \rangle| = \|\vec{\boldsymbol{x}}\| \|\vec{\boldsymbol{y}}\| .$$

Conversely, assume that $\vec{\boldsymbol{x}}$ and $\vec{\boldsymbol{y}}$ are both non-zero. We can find the component of $\vec{\boldsymbol{y}}$ that is orthogonal to $\vec{\boldsymbol{x}}$ by forming a linear combination of $\vec{\boldsymbol{x}}$ and $\vec{\boldsymbol{y}}$ that is orthogonal to $\vec{\boldsymbol{x}}$,

$$\langle \alpha_1 \vec{\boldsymbol{x}} + \alpha_2 \vec{\boldsymbol{y}}, \vec{\boldsymbol{x}} \rangle = 0 \quad \text{for} \quad \alpha_1, \alpha_2 \in \mathbb{F}.$$

Solving for the scalars α_1, α_2,

$$\begin{aligned} & \langle \alpha_1 \vec{\boldsymbol{x}} + \alpha_2 \vec{\boldsymbol{y}}, \vec{\boldsymbol{x}} \rangle = 0 \\ \iff \quad & \alpha_1 \langle \vec{\boldsymbol{x}}, \vec{\boldsymbol{x}} \rangle + \alpha_2 \langle \vec{\boldsymbol{y}}, \vec{\boldsymbol{x}} \rangle = 0 \\ \iff \quad & \alpha_1 = -\frac{\alpha_2 \langle \vec{\boldsymbol{y}}, \vec{\boldsymbol{x}} \rangle}{\langle \vec{\boldsymbol{x}}, \vec{\boldsymbol{x}} \rangle}. \end{aligned}$$

If we set $\alpha_2 = 1$ and define $\alpha = \frac{\langle \vec{y}, \vec{x} \rangle}{\langle \vec{x}, \vec{x} \rangle}$ then,

$$\vec{z} = \vec{y} - \frac{\langle \vec{y}, \vec{x} \rangle}{\langle \vec{x}, \vec{x} \rangle} \vec{x} = \vec{y} - \alpha \vec{x}$$

is orthogonal to $\vec{x}$.

If we take the inner product of $\vec{z}$ with itself,

$$\begin{aligned}
& \langle \vec{z}, \vec{z} \rangle = \langle \vec{y} - \alpha \vec{x}, \vec{y} - \alpha \vec{x} \rangle \\
\iff \quad & \langle \vec{z}, \vec{z} \rangle = \langle \vec{y}, \vec{y} - \alpha \vec{x} \rangle - \alpha \langle \vec{x}, \vec{z} \rangle \\
\iff \quad & \langle \vec{z}, \vec{z} \rangle = \langle \vec{y}, \vec{y} - \alpha \vec{x} \rangle && \because \vec{z} \perp \vec{x} \\
\iff \quad & \langle \vec{z}, \vec{z} \rangle = \langle \vec{y}, \vec{y} \rangle - \overline{\alpha} \langle \vec{y}, \vec{x} \rangle && \text{by 2.6.2} \\
\iff \quad & \langle \vec{z}, \vec{z} \rangle = \langle \vec{y}, \vec{y} \rangle - \frac{\overline{\langle \vec{y}, \vec{x} \rangle}}{\langle \vec{x}, \vec{x} \rangle} \langle \vec{y}, \vec{x} \rangle \\
\iff \quad & \langle \vec{x}, \vec{x} \rangle \langle \vec{z}, \vec{z} \rangle = \langle \vec{x}, \vec{x} \rangle \langle \vec{y}, \vec{y} \rangle - \overline{\langle \vec{y}, \vec{x} \rangle} \langle \vec{y}, \vec{x} \rangle \\
\iff \quad & \langle \vec{x}, \vec{x} \rangle \langle \vec{z}, \vec{z} \rangle = \langle \vec{x}, \vec{x} \rangle \langle \vec{y}, \vec{y} \rangle - \langle \vec{x}, \vec{y} \rangle \overline{\langle \vec{x}, \vec{y} \rangle} \\
\iff \quad & \|\vec{x}\|^2 \|\vec{z}\|^2 = \|\vec{x}\|^2 \|\vec{y}\|^2 - |\langle \vec{x}, \vec{y} \rangle|^2 .
\end{aligned}$$

Since the norm is positive definite (2.6.1.3),

$$\begin{aligned}
& \|\vec{x}\|^2 \|\vec{z}\|^2 \geq 0 \\
\iff \quad & \|\vec{x}\|^2 \|\vec{y}\|^2 - |\langle \vec{x}, \vec{y} \rangle|^2 \geq 0 \\
\iff \quad & \|\vec{x}\|^2 \|\vec{y}\|^2 \geq |\langle \vec{x}, \vec{y} \rangle|^2 \\
\iff \quad & \|\vec{x}\| \|\vec{y}\| \geq |\langle \vec{x}, \vec{y} \rangle| .
\end{aligned}$$

The equality case happens when $\|\vec{x}\|^2 \|\vec{z}\|^2 = 0$ and, since $\vec{x}$ is non-zero by hypothesis, this implies that $\vec{z} = \vec{0}$. It then follows that

$$\vec{z} = \vec{y} - \alpha \vec{x} = \vec{0} \iff \vec{y} = \alpha \vec{x}$$

meaning that $\vec{x}$ and $\vec{y}$ are colinear and so, linearly dependent. (The vector $\vec{z}$ was constructed to be the component of $\vec{y}$ that is orthogonal to $\vec{x}$ and so, if $\vec{x}$

and $\vec{\boldsymbol{y}}$ are colinear, $\vec{\boldsymbol{z}}$ will be $\vec{\mathbf{0}}$.) Furthermore, the converse implication — that equality implies linear dependence — holds because, using Proposition 2.6.11,

$$|\langle \vec{\boldsymbol{x}},\, \vec{\boldsymbol{y}} \rangle| = |\langle \alpha\vec{\boldsymbol{y}},\, \vec{\boldsymbol{y}} \rangle| = \left|\alpha \, \|\vec{\boldsymbol{y}}\|^2\right| = |\alpha| \|\vec{\boldsymbol{y}}\| \|\vec{\boldsymbol{y}}\| = \|\alpha\vec{\boldsymbol{y}}\| \|\vec{\boldsymbol{y}}\| \,.$$

If we have instead $\vec{\boldsymbol{y}} = \alpha\vec{\boldsymbol{x}}$, then the result is the same because $|\overline{\alpha}| = |\alpha|$.

In the case where $\mathbb{F} = \mathbb{R}$ we can use the following proof.

For all $\alpha \in \mathbb{R}$ we have,

$$\begin{aligned}
& \|\alpha\vec{\boldsymbol{x}} + \vec{\boldsymbol{y}}\|^2 \geq 0 \\
\iff \quad & \langle \alpha\vec{\boldsymbol{x}} + \vec{\boldsymbol{y}},\, \alpha\vec{\boldsymbol{x}} + \vec{\boldsymbol{y}} \rangle \geq 0 \\
\iff \quad & \langle \alpha\vec{\boldsymbol{x}},\, \alpha\vec{\boldsymbol{x}} + \vec{\boldsymbol{y}} \rangle + \langle \vec{\boldsymbol{y}},\, \alpha\vec{\boldsymbol{x}} + \vec{\boldsymbol{y}} \rangle \geq 0 \\
\iff \quad & \langle \alpha\vec{\boldsymbol{x}},\, \alpha\vec{\boldsymbol{x}} \rangle + \langle \alpha\vec{\boldsymbol{x}},\, \vec{\boldsymbol{y}} \rangle + \langle \vec{\boldsymbol{y}},\, \alpha\vec{\boldsymbol{x}} \rangle + \langle \vec{\boldsymbol{y}},\, \vec{\boldsymbol{y}} \rangle \geq 0 \\
\iff \quad & \alpha^2 \langle \vec{\boldsymbol{x}},\, \vec{\boldsymbol{x}} \rangle + 2\alpha \langle \vec{\boldsymbol{x}},\, \vec{\boldsymbol{y}} \rangle + \langle \vec{\boldsymbol{y}},\, \vec{\boldsymbol{y}} \rangle \geq 0 \quad \textit{by inner product properties.}
\end{aligned}$$

We can re-arrange this result as a real-valued quadratic in α,

$$p(\alpha) = \langle \vec{\boldsymbol{x}},\, \vec{\boldsymbol{x}} \rangle \alpha^2 + 2\langle \vec{\boldsymbol{x}},\, \vec{\boldsymbol{y}} \rangle \alpha + \langle \vec{\boldsymbol{y}},\, \vec{\boldsymbol{y}} \rangle$$

which has discriminant

$$\begin{aligned}
d &= (2\langle \vec{\boldsymbol{x}},\, \vec{\boldsymbol{y}} \rangle)^2 - 4\langle \vec{\boldsymbol{x}},\, \vec{\boldsymbol{x}} \rangle \langle \vec{\boldsymbol{y}},\, \vec{\boldsymbol{y}} \rangle \\
&= 4(\, \langle \vec{\boldsymbol{x}},\, \vec{\boldsymbol{y}} \rangle^2 - \langle \vec{\boldsymbol{x}},\, \vec{\boldsymbol{x}} \rangle \langle \vec{\boldsymbol{y}},\, \vec{\boldsymbol{y}} \rangle \,).
\end{aligned}$$

Since we have that $\forall \alpha \in \mathbb{R} \,.\, p(\alpha) \geq 0$ we can deduce that the discriminant $d \leq 0$ and so,

$$\begin{aligned}
& 4(\, \langle \vec{\boldsymbol{x}},\, \vec{\boldsymbol{y}} \rangle^2 - \langle \vec{\boldsymbol{x}},\, \vec{\boldsymbol{x}} \rangle \langle \vec{\boldsymbol{y}},\, \vec{\boldsymbol{y}} \rangle \,) \leq 0 \\
\iff \quad & \langle \vec{\boldsymbol{x}},\, \vec{\boldsymbol{y}} \rangle^2 \leq \langle \vec{\boldsymbol{x}},\, \vec{\boldsymbol{x}} \rangle \langle \vec{\boldsymbol{y}},\, \vec{\boldsymbol{y}} \rangle \\
\iff \quad & -\sqrt{\langle \vec{\boldsymbol{x}},\, \vec{\boldsymbol{x}} \rangle \langle \vec{\boldsymbol{y}},\, \vec{\boldsymbol{y}} \rangle} \leq \langle \vec{\boldsymbol{x}},\, \vec{\boldsymbol{y}} \rangle \leq \sqrt{\langle \vec{\boldsymbol{x}},\, \vec{\boldsymbol{x}} \rangle \langle \vec{\boldsymbol{y}},\, \vec{\boldsymbol{y}} \rangle} \\
\iff \quad & |\langle \vec{\boldsymbol{x}},\, \vec{\boldsymbol{y}} \rangle| \leq \|\vec{\boldsymbol{x}}\| \|\vec{\boldsymbol{y}}\| \,. \quad \square
\end{aligned}$$

Corollary 2.6.1. *Let $\vec{\boldsymbol{x}}, \vec{\boldsymbol{y}} \in V$ be vectors in an inner product space over the field $\mathbb{F}$ where $\mathbb{F}$ is either the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$. Then,*

$$\mathrm{Re}(\langle \vec{\boldsymbol{x}}, \vec{\boldsymbol{y}} \rangle), \mathrm{Im}(\langle \vec{\boldsymbol{x}}, \vec{\boldsymbol{y}} \rangle) \leq \|\vec{\boldsymbol{x}}\| \|\vec{\boldsymbol{y}}\| \,.$$

Proof. This is a consequence of Cauchy-Schwarz and Proposition 1.2.21 property (iii),

$$\mathrm{Re}(\langle \vec{\boldsymbol{x}}, \vec{\boldsymbol{y}} \rangle), \mathrm{Im}(\langle \vec{\boldsymbol{x}}, \vec{\boldsymbol{y}} \rangle) \leq |\langle \vec{\boldsymbol{x}}, \vec{\boldsymbol{y}} \rangle| \leq \|\vec{\boldsymbol{x}}\| \|\vec{\boldsymbol{y}}\| \,.$$

□

Theorem 2.6.3. ***(Triangle Inequality for Norms.)*** *Let V be an inner product space and $\vec{\boldsymbol{x}}, \vec{\boldsymbol{y}} \in V$. Then,*

$$\|\vec{\boldsymbol{x}} + \vec{\boldsymbol{y}}\| \leq \|\vec{\boldsymbol{x}}\| + \|\vec{\boldsymbol{y}}\| \,.$$

Proof.
This follows from the Lemma 2.6.1 and a corollary of Cauchy-Schwarz 2.6.1:

$$\begin{aligned} \|\vec{\boldsymbol{x}} + \vec{\boldsymbol{y}}\|^2 &= \langle \vec{\boldsymbol{x}}, \vec{\boldsymbol{x}} \rangle + \langle \vec{\boldsymbol{y}}, \vec{\boldsymbol{y}} \rangle + 2\,\mathrm{Re}(\langle \vec{\boldsymbol{x}}, \vec{\boldsymbol{y}} \rangle) \\ &\leq \langle \vec{\boldsymbol{x}}, \vec{\boldsymbol{x}} \rangle + \langle \vec{\boldsymbol{y}}, \vec{\boldsymbol{y}} \rangle + 2\|\vec{\boldsymbol{x}}\| \|\vec{\boldsymbol{y}}\| \qquad \text{by Cauchy-Schwarz} \\ &= (\|\vec{\boldsymbol{x}}\| + \|\vec{\boldsymbol{y}}\|)^2. \quad \square \end{aligned}$$

Compare with the Triangle Inequality for simple signed magnitudes: 1.2.3.1.

Summary of Norm Properties

Proposition 2.6.12. *A norm is a real-valued function that:*

(i) is positive definite;

(ii) is homomorphic w.r.t. scalar multiplication;

(iii) exhibits the triangle inequality.

Proof.

(i) By the definition of the induced norm, 2.6.1.3, the norm is positive definite.

(ii) If we consider a field $\mathbb{F}$ to be a 1-dimensional vector space, then the norm of a vector $\vec{\boldsymbol{v}} = \alpha \in \mathbb{F}$ in such a space is

$$\|\vec{\boldsymbol{v}}\| = \sqrt{\langle \vec{\boldsymbol{v}},\, \vec{\boldsymbol{v}} \rangle} = \sqrt{\alpha^2} = |\alpha| \,.$$

Therefore Proposition 2.6.11 implies that

$$\|\alpha \vec{\boldsymbol{x}}\| = |\alpha| \|\vec{\boldsymbol{x}}\| = \|\alpha\| \|\vec{\boldsymbol{x}}\| \,.$$

(iii) By Theorem 2.6.3, norms exhibit the triangle inequality.

□

Corollary 2.6.2. *The absolute value function in* $\mathbb{R}$ *and the modulus function in* $\mathbb{C}$ *are norms in their respective fields.*

Proof. This is implied by the reasoning above in (ii). We can also see it by examining the properties of the absolute value and modulus functions that they are also:

(i) positive definite by definition (1.2.3, 1.2.21);

(ii) homomorphic w.r.t. scalar multiplication (1.2.16, 1.2.21);

(iii) exhibit the triangle inequality (1.2.3.1, 1.2.21).

□

2.6.2 Orthonormal Bases and Orthogonal Operators

Definition 138. A set of pairwise mutually orthogonal (see: 2.6.1.1) unit vectors in an inner product space is called an **orthonormal set**.

2.6.2.1 Orthonormal Bases

Definition 139. A basis consisting of an orthonormal set of vectors is known as an **orthonormal basis**.

Proposition 2.6.13. *Let* $U = \{\vec{\boldsymbol{u}}_1, \ldots, \vec{\boldsymbol{u}}_n\}$ *be an orthonormal basis. Then, for any* $\vec{\boldsymbol{u}}_i, \vec{\boldsymbol{u}}_j \in U$, $i \neq j$,

$$\langle \vec{\boldsymbol{u}}_i, \vec{\boldsymbol{u}}_j \rangle = 0 \quad \textit{and} \quad \langle \vec{\boldsymbol{u}}_i, \vec{\boldsymbol{u}}_i \rangle = 1 = \langle \vec{\boldsymbol{u}}_j, \vec{\boldsymbol{u}}_j \rangle.$$

Proof. The elements of U are pairwise mutually orthogonal so — by the definition of orthogonal vectors (2.6.1.1) — $\langle \vec{\boldsymbol{u}}_i, \vec{\boldsymbol{u}}_j \rangle = 0$ for $i \neq j$. Furthermore,

$$\langle \vec{\boldsymbol{u}}_i, \vec{\boldsymbol{u}}_i \rangle = \|\vec{\boldsymbol{u}}_i\|^2 = 1$$

because the length of unit vectors is 1. □

Corollary 2.6.3. *A set of pairwise orthogonal vectors in an inner product space* V *with cardinality* $\dim V$ *is a basis of* V.

Proof. This is a consequence of Proposition 2.6.4. □

2.6.2.2 Orthonormal Bases in Coordinate Spaces

Proposition 2.6.14. *Let $B = \{\vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2, \ldots, \vec{\boldsymbol{v}}_n\}$ be an orthonormal basis of an inner product space V and let $\vec{\boldsymbol{w}} \in V$. Let the coordinates of $\vec{\boldsymbol{w}}$ w.r.t. the basis B be α_i for $1 \leq i \leq n$. Then the coordinates α_i are given by*

$$\alpha_i = \langle \vec{\boldsymbol{w}},\, \vec{\boldsymbol{v}}_i \rangle.$$

Proof.

$$\begin{aligned}
\langle \vec{\boldsymbol{w}},\, \vec{\boldsymbol{v}}_i \rangle &= \langle \alpha_1 \vec{\boldsymbol{v}}_1 + \alpha_2 \vec{\boldsymbol{v}}_2 + \cdots + \alpha_n \vec{\boldsymbol{v}}_n,\, \vec{\boldsymbol{v}}_i \rangle \\
&= \alpha_1 \langle \vec{\boldsymbol{v}}_1,\, \vec{\boldsymbol{v}}_i \rangle + \alpha_2 \langle \vec{\boldsymbol{v}}_2,\, \vec{\boldsymbol{v}}_i \rangle + \cdots + \alpha_n \langle \vec{\boldsymbol{v}}_n,\, \vec{\boldsymbol{v}}_i \rangle \\
&= \alpha_i \langle \vec{\boldsymbol{v}}_i,\, \vec{\boldsymbol{v}}_i \rangle && \because \text{pairwise orthogonal} \\
&= \alpha_i. && \because \|\vec{\boldsymbol{v}}_i\| = 1 \quad \square
\end{aligned}$$

Proposition 2.6.15. *Let $p : V \times V \longmapsto \mathbb{R}$ be a non-standard inner product defined on a vector space V and $B \subset V$ be an orthonormal basis w.r.t. the inner product p. Let $\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}} \in V$ and let $\vec{\boldsymbol{v}}_B, \vec{\boldsymbol{w}}_B \in V$ be vectors $\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}$ expressed in coordinates w.r.t. the basis B. Then the inner product*

$$p(\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}) = \vec{\boldsymbol{v}}_B \cdot \vec{\boldsymbol{w}}_B,$$

p is equal to the standard inner product (2.6.1.2) on the vectors expressed w.r.t. the orthonormal basis.

Proof. Let the inner product p be denoted by the usual inner product notation $\langle \vec{\boldsymbol{u}}_1,\, \vec{\boldsymbol{u}}_2 \rangle = p(\vec{\boldsymbol{u}}_1, \vec{\boldsymbol{u}}_2)$ and let the orthonormal basis be $B = \{\vec{\boldsymbol{b}}_1, \ldots, \vec{\boldsymbol{b}}_n\}$. Then,

$$\begin{aligned}
\langle \vec{\boldsymbol{v}}_B,\, \vec{\boldsymbol{w}}_B \rangle &= \langle \alpha_1 \vec{\boldsymbol{b}}_1 + \cdots + \alpha_n \vec{\boldsymbol{b}}_n,\, \beta_1 \vec{\boldsymbol{b}}_1 + \cdots + \beta_n \vec{\boldsymbol{b}}_n \rangle \\
&= \langle \alpha_1 \vec{\boldsymbol{b}}_1,\, \beta_1 \vec{\boldsymbol{b}}_1 \rangle + \cdots + \langle \alpha_n \vec{\boldsymbol{b}}_n,\, \beta_n \vec{\boldsymbol{b}}_n \rangle && \because i \neq j \implies \langle \vec{\boldsymbol{b}}_i,\, \vec{\boldsymbol{b}}_j \rangle = 0 \\
&= \alpha_1 \overline{\beta_1} \langle \vec{\boldsymbol{b}}_1,\, \vec{\boldsymbol{b}}_1 \rangle + \cdots + \alpha_n \overline{\beta_n} \langle \vec{\boldsymbol{b}}_n,\, \vec{\boldsymbol{b}}_n \rangle \\
&= \alpha_1 \overline{\beta_1} + \cdots + \alpha_n \overline{\beta_n}. && \because \langle \vec{\boldsymbol{b}}_i,\, \vec{\boldsymbol{b}}_i \rangle = 1 \quad \square
\end{aligned}$$

Example

(76) Consider an inner product defined over $\mathbb{R}^2$ given by

$$\langle \vec{\boldsymbol{x}},\, \vec{\boldsymbol{y}} \rangle = \vec{\boldsymbol{x}}^T A \vec{\boldsymbol{y}} \quad \text{where} \quad A = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}.$$

This is an inner product because:

Symmetry

Since A is symmetric and the inner product — being a 1×1 matrix — is also symmetric we have,

$$\langle \vec{\boldsymbol{x}},\, \vec{\boldsymbol{y}} \rangle = \langle \vec{\boldsymbol{x}},\, \vec{\boldsymbol{y}} \rangle^T = (\vec{\boldsymbol{x}}^T A \vec{\boldsymbol{y}})^T = \vec{\boldsymbol{y}}^T A^T \vec{\boldsymbol{x}} = \vec{\boldsymbol{y}}^T A \vec{\boldsymbol{x}} = \langle \vec{\boldsymbol{y}},\, \vec{\boldsymbol{x}} \rangle.$$

Linearity in first argument

By the linearity of matrix multiplication,

$$\langle \alpha \vec{\boldsymbol{x}} + \vec{\boldsymbol{z}},\, \vec{\boldsymbol{y}} \rangle = (\alpha \vec{\boldsymbol{x}} + \vec{\boldsymbol{z}}) A \vec{\boldsymbol{y}} = \alpha \vec{\boldsymbol{x}} A \vec{\boldsymbol{y}} + \vec{\boldsymbol{z}} A \vec{\boldsymbol{y}} = \alpha \langle \vec{\boldsymbol{x}},\, \vec{\boldsymbol{y}} \rangle + \langle \vec{\boldsymbol{z}},\, \vec{\boldsymbol{y}} \rangle.$$

Reflexive positive definiteness and only 0 for $\vec{\mathbf{0}}$

If we define $\vec{\boldsymbol{x}} = (x_1, x_2)^T \in \mathbb{R}^2$ then,

$$\langle \vec{\boldsymbol{x}},\, \vec{\boldsymbol{x}} \rangle = 5x_1^2 + 4x_1 x_2 + x_2^2 = (\sqrt{5} x_1 + \frac{2x_2}{\sqrt{5}}) + \frac{x_2^2}{5}$$

which expression clearly implies that

$$\langle \vec{\boldsymbol{x}},\, \vec{\boldsymbol{x}} \rangle \geq 0 \quad \text{and} \quad \vec{\boldsymbol{x}} = \vec{\mathbf{0}} \implies \langle \vec{\boldsymbol{x}},\, \vec{\boldsymbol{x}} \rangle = 0.$$

On the other hand, if $\langle \vec{\boldsymbol{x}},\, \vec{\boldsymbol{x}} \rangle = 0$, since we have,

$$(\sqrt{5} x_1 + \frac{2x_2}{\sqrt{5}}) \geq 0 \quad \text{and} \quad \frac{x_2^2}{5} \geq 0$$

then we can deduce that

$$\begin{aligned} & (\sqrt{5} x_1 + \frac{2x_2}{\sqrt{5}}) = 0 = \frac{x_2^2}{5} \\ \implies \quad & x_2 = 0 \\ \implies \quad & x_1 = 0 \\ \therefore \quad & \vec{\boldsymbol{x}} = \vec{\mathbf{0}}. \end{aligned}$$

Having established that we have defined an inner product on $\mathbb{R}^2$, we can use it to determine orthogonality of vectors in the space. If we let $\vec{\boldsymbol{v}} = (1,1)^T$ then the space of vectors orthogonal to $\vec{\boldsymbol{v}}$ is the orthogonal complement (2.6.1.1) to the span of $\vec{\boldsymbol{v}}$,

$$O = (\mathrm{Lin}\{\vec{\boldsymbol{v}}\})^{\perp} = \{\, \vec{\boldsymbol{w}} \in \mathbb{R}^2 \;\mid\; \langle \vec{\boldsymbol{w}},\, \vec{\boldsymbol{v}} \rangle = 0 \,\}.$$

Since,

$$\begin{aligned}
& \langle \vec{\boldsymbol{w}},\, \vec{\boldsymbol{v}} \rangle = 0 \\
\iff \quad & \vec{\boldsymbol{w}}^T \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0 \\
\iff \quad & \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0 \\
\iff \quad & 7w_1 + 3w_2 = 0
\end{aligned}$$

then the set O is

$$O = \{\, (w_1, w_2) \in \mathbb{R}^2 \;\mid\; 7w_1 + 3w_2 = 0 \,\}.$$

The set O is clearly a 1-d vector space and so we can select any vector from it as a basis of it,

$$O = t \begin{bmatrix} -3 \\ 7 \end{bmatrix} \quad \text{for} \quad t \in R.$$

Then we have an orthonormal basis (orthogonal w.r.t. this inner product) if we normalize the two basis vectors,

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 7 \end{bmatrix} \right\}.$$

So we need to calculate the norms,

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 7 + 3 = 10,$$

$$\begin{bmatrix} -3 & 7 \end{bmatrix} \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 7 \end{bmatrix} = 3 + 7 = 10.$$

It is a coincidence that these values have turned out to be equal.

So the set

$$B = \left\{ \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{10}} \begin{bmatrix} -3 \\ 7 \end{bmatrix} \right\}$$

is an orthonormal basis for the inner product space defined by $\mathbb{R}^2$ equipped with the inner product $\vec{\boldsymbol{x}}^T A \vec{\boldsymbol{y}}$.

If we now express a couple of vectors w.r.t. the orthonormal basis B,

$$[B]^{-1} = \frac{1}{\sqrt{10}} \begin{bmatrix} 7 & 3 \\ -1 & 1 \end{bmatrix}.$$

$$\vec{\boldsymbol{y}} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \ \vec{\boldsymbol{y}}_B = \frac{1}{\sqrt{10}} \begin{bmatrix} 7 & 3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \frac{1}{\sqrt{10}} \begin{bmatrix} 33 \\ 1 \end{bmatrix}.$$

$$\vec{\boldsymbol{z}} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}, \ \vec{\boldsymbol{z}}_B = \frac{1}{\sqrt{10}} \begin{bmatrix} 7 & 3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{10}} \begin{bmatrix} 45 \\ -5 \end{bmatrix}.$$

then their inner product

$$\begin{aligned} \langle \vec{\boldsymbol{y}}, \vec{\boldsymbol{z}} \rangle &= \begin{bmatrix} 3 & 4 \end{bmatrix} A \begin{bmatrix} 6 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 4 \end{bmatrix} \begin{bmatrix} 32 \\ 13 \end{bmatrix} \\ &= 32 \times 3 + 13 \times 4 = 148, \end{aligned}$$

while the dot product of the vectors w.r.t. the orthonormal basis is

$$\begin{aligned} \vec{\boldsymbol{y}}_B \cdot \vec{\boldsymbol{z}}_B &= \frac{1}{\sqrt{10}} \begin{bmatrix} 33 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{10}} \begin{bmatrix} 45 \\ -5 \end{bmatrix} \\ &= \frac{1}{10}(33 \times 45 + 1 \times -5) = 148. \end{aligned}$$

Intuition of Orthogonal Bases

If we consider what an orthonormal basis would look like in coordinate vectors: Obviously, the standard basis is an orthonormal basis as,

$$\begin{bmatrix}1\\0\\0\end{bmatrix} \cdot \begin{bmatrix}0\\1\\0\end{bmatrix} = 0$$

and clearly for any i, j such that $i \neq j$, $\vec{\boldsymbol{e_i}} \cdot \vec{\boldsymbol{e_j}} = 0$ and $\vec{\boldsymbol{e_i}}, \vec{\boldsymbol{e_j}}$ have unit length.

We can form other orthonormal bases in $\mathbb{R}^3$ though. The vectors,

$$\begin{bmatrix}1\\1\\0\end{bmatrix} \cdot \begin{bmatrix}-1\\1\\0\end{bmatrix} = 0$$

are orthogonal but do not have unit length. We can make them unit length, though, by dividing them by $\sqrt{2}$ so that,

$$\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\end{bmatrix}, \begin{bmatrix}-1/\sqrt{2}\\1/\sqrt{2}\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$$

is also an orthonormal basis of $\mathbb{R}^3$.

Another orthonormal basis of $\mathbb{R}^3$ is

$$\begin{bmatrix}\cos\theta\\\sin\theta\\0\end{bmatrix}, \begin{bmatrix}-\sin\theta\\\cos\theta\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$$

for any angle θ measured anticlockwise from the x-axis. Actually this is the general case of which the previous example is a special case when $\theta = \pi/2$.

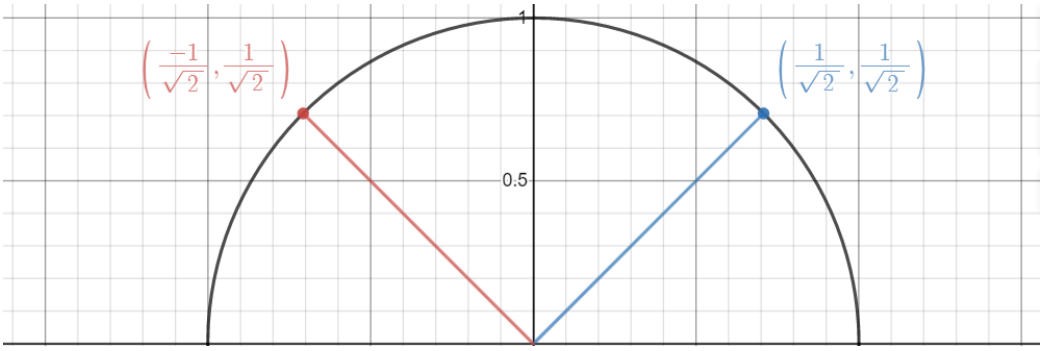


Figure 2.5: $(\cos\pi/4, \sin\pi/4)^T = (1/\sqrt{2}, 1/\sqrt{2})^T$, $(-\sin\pi/4, \cos\pi/4)^T = (-1/\sqrt{2}, 1/\sqrt{2})^T$

But for any value of the angle θ these remain orthogonal as can be seen if we generate the basis vectors for $\theta = \pi/3$.

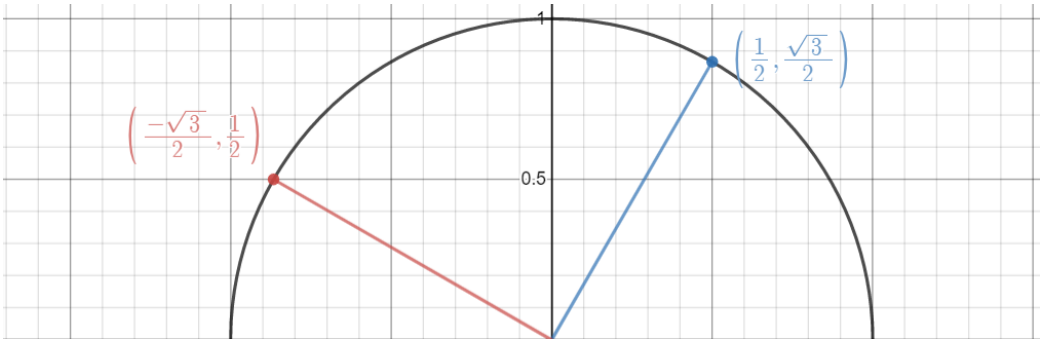


Figure 2.6: $(\cos\pi/3, \sin\pi/3)^T = (1/2, \sqrt{3}/2)^T$, $(-\sin\pi/3, \cos\pi/3)^T = (-\sqrt{3}/2, 1/2)^T$

2.6.2.3 Unitary and Orthogonal Operators

Definition 140. A matrix whose columns form an orthonormal basis is called an **orthogonal matrix**.

The operation of left multiplication by such a matrix is called an **orthogonal operator**.

Proposition 2.6.16. *A matrix $A \in \mathbb{R}^{n\times n}$ is orthogonal iff $A^T = A^{-1}$.*

Proof. A is an orthogonal matrix so, by definition, its columns form an orthogonal basis. Then,

$$A^TA = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & & & \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}^2 + \cdots + a_{n1}^2 & a_{11}a_{12} + \cdots + a_{n1}a_{n2} & \cdots & \\ a_{12}a_{11} + \cdots + a_{n2}a_{n1} & a_{12}^2 + \cdots + a_{n2}^2 & \cdots & \\ \vdots & & & \\ \cdots & \cdots & \cdots & a_{1n}^2 + \cdots + a_{nn}^2 \end{bmatrix}$$

Along the main diagonal the components take the form

$$a_{1j}^2 + a_{2j}^2 + \cdots + a_{nj}^2 = a_j \cdot a_j$$

where a_j is the jth column of the matrix A. But the columns of A are vectors in an orthonormal basis and so $a_j \cdot a_j = 1$.
Furthermore, the off-diagonal values take the form

$$a_{1j}a_{1j'} + \cdots + a_{nj}a_{nj'} = a_j \cdot a_{j'}$$

for $j \neq j'$. Since the columns are from an orthonormal basis we know that $a_j \cdot a_{j'} = 0$.

Therefore the resultant matrix looks like,

$$A^TA = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I_n.$$

Clearly, the same effect would be seen for AA^T and so,

$$A^TA = AA^T = I \iff A^T = A^{-1}.$$

Conversely, we can reverse the logic and it is easy to see that if $A^TA = I$ then,

(i) since the diagonal entries are all 1, if $\vec{\boldsymbol{u}}_i$ is a column of A, then $\vec{\boldsymbol{u}}_i \cdot \vec{\boldsymbol{u}}_i = 1$;

(ii) since the off-diagonal entries are all 0, if $\vec{\boldsymbol{u}}_i, \vec{\boldsymbol{u}}_j$ with $i \neq j$ are distinct columns of A, then $\vec{\boldsymbol{u}}_i \cdot \vec{\boldsymbol{u}}_j = 0$.

Therefore, the columns of A form an orthogonal basis and thus, by definition, A is an orthogonal matrix. □

This provides another way of seeing Proposition 2.6.14: If $[B]$ is the matrix whose columns are the vectors of an orthonormal basis then — just as with any change of basis — the vector $\vec{\boldsymbol{x}}$ w.r.t. the standard basis, has coordinates w.r.t. the basis B given by,

$$[B]^{-1}\vec{\boldsymbol{x}}.$$

But, since $[B]$ is an orthogonal matrix,

$$[B]^{-1}\vec{\boldsymbol{x}} = [B]^T\vec{\boldsymbol{x}}$$

which is the vector whose i-th row component is the dot product of the i-th column of $[B]$ and $\vec{\boldsymbol{x}}$.

Proposition 2.6.17. *A matrix $A \in \mathbb{C}^{n\times n}$ is unitary iff its columns are an orthonormal basis of $\mathbb{C}^n$.*

Proof. Assume A is a unitary matrix in $\mathbb{C}^{n\times n}$ and let its rows be denoted a_i and its columns be denoted a_j, for $1 \leq i, j \leq n$. The unitary property of A tells us that,

$$A^*A = \overline{A}^T A = I.$$

Each row of the matrix $\overline{A}^T$ has the form $\overline{a_j}$, the conjugate of the j-th row of A. Therefore, by matrix multiplication, the diagonal entries of the matrix product $\overline{A}^T A$ are given by the expression $\overline{a_j} \cdot a_j$ while every entry off the main diagonal is given by the expression $\overline{a_m} \cdot a_j$ for some $m \neq j$. Since the product is equal to the identity matrix — which has all diagonal entries equal to 1 and all off-diagonal entries equal to 0 — we must have, for all $1 \leq m \neq j \leq n$

$$\overline{a_j} \cdot a_j = 1 \wedge \overline{a_m} \cdot a_j = 0.$$

Therefore, by 2.6.2, the columns a_j are an orthonormal set. Since each columns is an n-vector in $\mathbb{C}^n$ and there are n orthonormal columns, by 2.6.3, they are a basis of $\mathbb{C}^n$.

Conversely, assume that the columns of a matrix $A \in \mathbb{C}^{n\times n}$ are an orthonormal basis of $\mathbb{C}^n$. Then, applying the logic above in reverse we have that,

$$A^*A = I$$

and so A is unitary. (TODO: does this prove the inverse exists on both sides?) □

Proposition 2.6.18. *In a real vector space, left multiplication by an orthogonal matrix preserves the dot product. In other words, for all vectors $\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}$, A is an orthogonal matrix if and only if,*

$$A\vec{\boldsymbol{v}} \cdot A\vec{\boldsymbol{w}} = \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}}.$$

Proof. Using Proposition 2.6.16 and the matrix formula for the dot product we can deduce that, for all vectors $\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}$,

$$\begin{aligned} A\vec{\boldsymbol{v}} \cdot A\vec{\boldsymbol{w}} &= (A\vec{\boldsymbol{v}})^T A\vec{\boldsymbol{w}} \\ &= \vec{\boldsymbol{v}}^T A^T A\vec{\boldsymbol{w}} \\ &= \vec{\boldsymbol{v}}^T \vec{\boldsymbol{w}} && A \text{ is orthogonal so } A^TA = I \\ &= \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}} \end{aligned}$$

which is to say that an orthogonal matrix preserves the dot product.

Conversely, if we assume that A preserves the dot product then, for all vectors $\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}$,

$$\begin{aligned}
& A\vec{\boldsymbol{v}} \cdot A\vec{\boldsymbol{w}} = \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}} \\
\iff \quad & (A\vec{\boldsymbol{v}})^T A\vec{\boldsymbol{w}} = \vec{\boldsymbol{v}}^T \vec{\boldsymbol{w}} \\
\iff \quad & \vec{\boldsymbol{v}}^T A^T A\vec{\boldsymbol{w}} = \vec{\boldsymbol{v}}^T \vec{\boldsymbol{w}} \\
\iff \quad & \vec{\boldsymbol{v}}^T A^T A\vec{\boldsymbol{w}} - \vec{\boldsymbol{v}}^T \vec{\boldsymbol{w}} = 0 \qquad \text{both terms are scalars} \\
\iff \quad & \vec{\boldsymbol{v}}^T (A^T A - I)\vec{\boldsymbol{w}} = 0.
\end{aligned}$$

Now for any arbitrary matrix B,

$$\vec{\boldsymbol{e_i}}^T B \vec{\boldsymbol{e_j}} = b_{ij}$$

where b_{ij} is the (i, j)th element of B. Then, for

$$\vec{\boldsymbol{e_i}}^T B \vec{\boldsymbol{e_j}} = 0$$

to be true *for all possible* $\vec{\boldsymbol{e_i}}, \vec{\boldsymbol{e_j}}$ would require that,

$$\forall i, j \;.\; b_{ij} = 0 \iff B = [0]$$

where $[0]$ is the zero matrix. Therefore,

$$\forall \vec{\boldsymbol{v}}, \vec{\boldsymbol{w}} \;.\; \vec{\boldsymbol{v}}^T (A^T A - I)\vec{\boldsymbol{w}} = 0 \iff A^T A - I = [0] \iff A^T A = I$$

which, by Proposition 2.6.16, implies that A is orthogonal. So if A preserves the dot product then it is orthogonal. □

Proposition 2.6.19. *A matrix $A \in \mathbb{C}^{n \times n}$ is unitary iff it preserves the standard complex inner product. That's to say,*

$$\forall \vec{\boldsymbol{x}}, \vec{\boldsymbol{y}} \in \mathbb{C}^n \;.\; (A\vec{\boldsymbol{x}})^* A\vec{\boldsymbol{y}} = \vec{\boldsymbol{x}}^* \vec{\boldsymbol{y}}.$$

Proof. If we assume that A is unitary then, by properties of the hermitian conjugate (2.3.1.7),

$$(A\vec{\boldsymbol{x}})^* A\vec{\boldsymbol{y}} = \vec{\boldsymbol{x}}^* (A^* A)\vec{\boldsymbol{y}} = \vec{\boldsymbol{x}}^* \vec{\boldsymbol{y}}$$

which shows that A preserves the standard complex inner product.

A possible alternative proof?

Conversely, if we assume that A preserves the standard complex inner product then,

$$\begin{aligned} & (A\vec{\boldsymbol{x}})^* A\vec{\boldsymbol{y}} = \vec{\boldsymbol{x}}^*\vec{\boldsymbol{y}} \\ \iff \quad & \vec{\boldsymbol{x}}^*(A^*A)\vec{\boldsymbol{y}} = \vec{\boldsymbol{x}}^*\vec{\boldsymbol{y}} \qquad \textit{by hermitian conjugate properties} \\ \iff \quad & \vec{\boldsymbol{x}}^*(A^*A)\vec{\boldsymbol{y}} = \vec{\boldsymbol{x}}^* I\vec{\boldsymbol{y}}. \end{aligned}$$

and this implies, by 2.3.5, that $A^*A = I$. This therefore shows that A is unitary. (TODO: does this prove the inverse exists on both sides?)

Conversely, if we assume that A preserves the standard complex inner product then,

$$\forall \vec{\boldsymbol{x}}, \vec{\boldsymbol{y}} \in \mathbb{C}^n \; . \; (A\vec{\boldsymbol{x}})^* A\vec{\boldsymbol{y}} = \vec{\boldsymbol{x}}^*\vec{\boldsymbol{y}}.$$

Since this applies to all vectors in $\mathbb{C}^n$, it must apply to the standard basis vectors of $\mathbb{C}^n$,

$$(A\vec{\boldsymbol{e_p}})^* A\vec{\boldsymbol{e_q}} = {a_p}^* a_q = \vec{\boldsymbol{e_p}}^*\vec{\boldsymbol{e_q}}$$

where a_i denotes the i-th column of the matrix A. Since the expression $\vec{\boldsymbol{e_p}}^*\vec{\boldsymbol{e_q}}$ is 0 for all $p \neq q$ and 1 for $p = q$ we deduce that the columns of A are such that ${a_p}^* a_q$ is 0 for all $p \neq q$ and 1 for $p = q$. The columns of A are, therefore, an orthonormal set and so also an orthonormal basis of $\mathbb{C}^n$. It follows then, that A is unitary. □

Proposition 2.6.20. *The determinant of any orthogonal matrix is 1 or -1.*

Proof. If a matrix A is orthogonal then $A^T A = I$ which implies that,

$$det(A^T)det(A) = det(I) = 1.$$

By Proposition 2.3.32, $det(A^T) = det(A)$ so,

$$det(A^T)det(A) = det(A)^2 = 1 \iff \sqrt{det(A)} = 1 \iff det(A) = \pm 1. \quad \square$$

*If an orthogonal operator has determinant equal to 1 it is described as **orientation preserving** and if it is equal to -1 it is described as **orientation***

reversing.

Proposition 2.6.21. *The orthogonal matrices form a subgroup of* $GL_n(\mathbb{F})$.

Proof. Let $S = \{ A \in GL_n(\mathbb{F}) \mid A^T A = I \}$. Then,

- S is nonempty because $I \in S$.
- For $B, C \in S$,

  $$(BC)^T(BC) = C^T B^T (BC) = C^T (B^T B) C = I$$

  so $BC \in S$.
- For $B \in S$, by Proposition 2.6.16, $B^{-1} = B^T$ and

  $$(B^T)^T B^T = BB^T = BB^{-1} = I$$

  so S contains inverses.

Therefore $S \leq GL_n(\mathbb{F})$. □

The subgroup of the general linear group formed by the orthogonal matrices is called the ***orthogonal group*** *and is denoted* O_n.

Example

(77) The matrix

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$$

is a unitary matrix because

$$P^* = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}$$

so that

$$PP^* = \frac{1}{2}\begin{bmatrix}1 & i\\ i & 1\end{bmatrix}\begin{bmatrix}1 & -i\\ -i & 1\end{bmatrix} = I.$$

Note that P is not a hermitian matrix (2.3.2.7), but would be if one of the diagonal values were multiplied by -1, e.g. the matrix

$$Q = \begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}}\\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -i\\ i & 1\end{bmatrix}$$

is hermitian because $Q^* = Q$.

2.6.2.4 Gram-Schmidt Orthonormalisation

Definition 141. The process known as **Orthonormalisation** takes a set of linearly independent vectors and returns an orthonormal set of the same cardinality as the original set and in the same space spanned by the original set.
Since the orthonormal set has the same cardinality as the original set, by Corollary 2.6.3, it is an orthonormal basis for the space spanned by the original set.

The Gram-Schmidt algorithm is as follows:

Input: A k-length linearly independent set of vectors $S = \{\vec{\boldsymbol{v}}_1, \ldots, \vec{\boldsymbol{v}}_k\} \subset V$ where V is an inner product space, and an initially-empty set $O = \{\}$.
Output: A k-length orthonormal set of vectors $O = \{\vec{\boldsymbol{u}}_1, \ldots, \vec{\boldsymbol{u}}_k\}$ s.t. $\operatorname{Lin} O = \operatorname{Lin} S$.

for $i = 1$ to k **do**
  find $\vec{\boldsymbol{w}}$, a basis for the orthogonal complement of O in $O \cup \{\vec{\boldsymbol{v}}_i\}$.
  $\vec{\boldsymbol{u}}_i \leftarrow \frac{\vec{\boldsymbol{w}}}{\|\vec{\boldsymbol{w}}\|}$
end for

The process of finding the orthogonal complement on iteration number i is to find the set of vectors

$$O^{\perp} = \{\, \vec{\boldsymbol{w}} \in O \cup \{\vec{\boldsymbol{v}}_i\} \mid \forall \vec{\boldsymbol{u}} \in O \,.\, \vec{\boldsymbol{w}} \perp \vec{\boldsymbol{u}} \,\}.$$

For $\vec{\boldsymbol{w}} \in O^{\perp}$ we need, for every $\vec{\boldsymbol{u}}_j \in O$,

$$\begin{aligned}
& \langle \vec{\boldsymbol{w}},\, \vec{\boldsymbol{u}}_j \rangle = 0 && \\
\iff \quad & \langle \alpha_1 \vec{\boldsymbol{u}}_1 + \cdots + \alpha_{i-1} \vec{\boldsymbol{u}}_{i-1} + \alpha_i \vec{\boldsymbol{v}}_i,\, \vec{\boldsymbol{u}}_j \rangle = 0 && \\
\iff \quad & \langle \alpha_j \vec{\boldsymbol{u}}_j + \alpha_i \vec{\boldsymbol{v}}_i,\, \vec{\boldsymbol{u}}_j \rangle = 0 && \text{by orthogonality} \\
\iff \quad & \alpha_j + \alpha_i \langle \vec{\boldsymbol{v}}_i,\, \vec{\boldsymbol{u}}_j \rangle = 0 && \because \left\| \vec{\boldsymbol{u}}_j \right\| = 1 \\
\iff \quad & \alpha_j = -\alpha_i \langle \vec{\boldsymbol{v}}_i,\, \vec{\boldsymbol{u}}_j \rangle. &&
\end{aligned}$$

So, if we set $\alpha_i = 1$ then we get

$$\vec{\boldsymbol{w}} = \vec{\boldsymbol{v}}_i - \langle \vec{\boldsymbol{v}}_i,\, \vec{\boldsymbol{u}}_1 \rangle \vec{\boldsymbol{u}}_1 - \langle \vec{\boldsymbol{v}}_i,\, \vec{\boldsymbol{u}}_2 \rangle \vec{\boldsymbol{u}}_2 - \cdots - \langle \vec{\boldsymbol{v}}_i,\, \vec{\boldsymbol{u}}_{i-1} \rangle \vec{\boldsymbol{u}}_{i-1}.$$

Clearly, our only degree of freedom in the definition of the vector $\vec{\boldsymbol{w}}$ is the value of α_i. However, if we were to use some other value, $\alpha_i = \beta$, then the result would be $\beta \vec{\boldsymbol{w}}$, a scaling of $\vec{\boldsymbol{w}}$ by β. As a result, this is clearly a 1-dimensional space in which any vector will suffice as a basis. So we can use $\vec{\boldsymbol{w}}$ as the basis vector and then, it is a simple matter to normalize it and add it to the set O.

2.6.2.5 Unitary and Orthogonal Diagonalisation

Definition 142. **(Unitary Diagonalisation)** A matrix is said to be *unitarily diagonalisable* if there is a unitary matrix that diagonalises it. That's to say, the matrix A is unitarily diagonalisable iff there exists some unitary matrix P such that

$$P^*AP = D$$

is diagonal.

Note that since P is unitary (2.3.2.8), this definition implies the non-unitary diagonlisation condition,

$$P^*AP = P^{-1}AP = D$$

with the difference being that, for the diagonalising matrix P,

$$P^{-1} = \overline{P}^T = P^*.$$

Definition 143. **(Orthogonal Diagonalisation)** A real matrix is said to be *orthogonally diagonalisable* if there is an orthogonal matrix that diagonalises it. That's to say, the matrix A is orthogonally diagonalisable iff there exists some real orthogonal matrix P such that

$$P^{-1}AP = P^TAP = D$$

is diagonal.

Proposition 2.6.22. *The matrix $A \in \mathbb{C}^{n\times n}$ is unitarily diagonalisable iff there exists an orthonormal eigenbasis of A.*

Proof. If the matrix A is unitarily diagonalisable then there exists a unitary matrix $P \in \mathbb{C}^{n\times n}$ such that,

$$D = P^*AP = P^{-1}AP$$

is diagonal. This implies that the columns of P are an eigenbasis for the matrix A (2.5.4) and, since P is unitary, by Proposition 2.6.17, the columns are also an orthonormal basis of $\mathbb{C}^n$.

Conversely, if there exists an orthonormal basis of $\mathbb{C}^n$ consisting solely of eigenvectors of A then there exists a matrix $P \in \mathbb{C}^{n\times n}$ whose columns are the vectors of this basis. Then by the definition of the eigenbasis 2.5.4,

$$D = P^{-1}AP$$

is diagonal. Since the columns of P are an orthonormal basis, then P is unitary and so we have,

$$D = P^{-1}AP = P^*AP$$

which is the property for A to be unitarily diagonalisable. □

Corollary 2.6.4. *A real matrix is orthogonally diagonalisable iff there exists an orthonormal eigenbasis of A.*

Proof. This is a special case of Proposition 2.6.22. If the matrix $A \in \mathbb{C}^{n\times n}$ has only real-valued entries then we also have $A \in \mathbb{R}^{n\times n} \subset \mathbb{C}^{n\times n}$. In this case, orthogonal diagonalisability means that there is a real orthogonal matrix that diagonlises A, which implies that there is an orthonormal set of eigenvectors of A that forms a basis of $\mathbb{R}^n$.
Conversely, if there exists an orthonormal eigenbasis for A in $\mathbb{R}^n$ then there is a diagonalising matrix whose columns are this orthonormal basis and the matrix will therefore be orthogonal. □

Proposition 2.6.23. *The eigenvectors corresponding to distinct eigenvalues of a Hermitian matrix are orthogonal w.r.t. to the standard inner product.*

Proof. Let A be a hermitian matrix (2.3.2.7). Let λ_1 and λ_2 be distinct eigenvalues of A and let $\vec{\boldsymbol{v}}_1$ be an eigenvector corresponding to λ_1 and $\vec{\boldsymbol{v}}_2$ an

eigenvector corresponding to λ_2. Then,

$$\begin{aligned}
& \vec{v}_1^* A \vec{v}_2 = \vec{v}_1^* A^* \vec{v}_2 && \because A \text{ is hermitian} \\
\iff \quad & \vec{v}_1^* (A\vec{v}_2) = (A\vec{v}_1)^* \vec{v}_2 && \text{by props. of hermitian conjugate 2.3.1.7} \\
\iff \quad & \vec{v}_1^* (\lambda_2 \vec{v}_2) = (\lambda_1 \vec{v}_1)^* \vec{v}_2 && \text{by eigenvalue property} \\
\iff \quad & \vec{v}_1^* \lambda_2 \vec{v}_2 = \overline{\lambda}_1 \vec{v}_1^* \vec{v}_2 && \text{by 2.3.8} \\
\iff \quad & \lambda_2 \vec{v}_1^* \vec{v}_2 = \lambda_1 \vec{v}_1^* \vec{v}_2 && \text{by ??} \\
\iff \quad & (\lambda_2 - \lambda_1) \vec{v}_1^* \vec{v}_2 = 0 \\
\therefore \quad & \vec{v}_1^* \vec{v}_2 = 0. && \because \lambda_1 \neq \lambda_2 \implies \lambda_2 - \lambda_1 \neq 0
\end{aligned}$$

This last result says that the standard complex inner product of $\vec{v}_1$ and $\vec{v}_2$ is zero, which is to say that $\vec{v}_1$ and $\vec{v}_2$ are orthogonal w.r.t. to the standard complex inner product. □

TODO: does this result imply that the eigenvectors are orthogonal w.r.t. every basis and every inner product or only the standard basis and inner product?

Proposition 2.6.24. *A normal matrix is hermitian iff it has only real-valued eigenvalues.*

Proof. Let A be a normal matrix with only real-valued eigenvalues. By Proposition 2.6.25, A is unitarily diagonalisable so there exists a unitary matrix P such that,

$$D = P^* A P$$

is a diagonal matrix whose diagonal entries are the eigenvalues of A. Since, by hypothesis, these are all real numbers, we have $D^* = D$. Using this fact and the rearranged relation with the diagonal matrix $A = PDP^*$,

$$\begin{aligned}
A^* &= (PDP^*)^* \\
&= PD^*P^* && \text{by props. of hermitian conjugate 2.3.1.7} \\
&= PDP^* = A.
\end{aligned}$$

Conversely, let A be a hermitian matrix (2.3.2.7) and let λ be an arbitrary

eigenvalue of A with $\vec{\boldsymbol{v}}$ as an eigenvector corresponding to λ. Then,

$$\begin{aligned}
& \vec{\boldsymbol{v}}^* A\vec{\boldsymbol{v}} = \vec{\boldsymbol{v}}^* A^* \vec{\boldsymbol{v}} && \because A \text{ is hermitian} \\
\iff \quad & \vec{\boldsymbol{v}}^* (A\vec{\boldsymbol{v}}) = (A\vec{\boldsymbol{v}})^* \vec{\boldsymbol{v}} && \text{by props. of hermitian conjugate 2.3.1.7} \\
\iff \quad & \vec{\boldsymbol{v}}^* (\lambda\vec{\boldsymbol{v}}) = (\lambda\vec{\boldsymbol{v}})^* \vec{\boldsymbol{v}} && \text{by eigenvalue property} \\
\iff \quad & \vec{\boldsymbol{v}}^* (\lambda\vec{\boldsymbol{v}}) = \overline{\lambda}\vec{\boldsymbol{v}}^* \vec{\boldsymbol{v}} && \text{by 2.3.8} \\
\iff \quad & \lambda\|\vec{\boldsymbol{v}}\|^2 = \overline{\lambda}\|\vec{\boldsymbol{v}}\|^2 && \text{by defn. of induced norm 2.6.1.3} \\
\iff \quad & \lambda = \overline{\lambda} && \because \vec{\boldsymbol{v}} \neq \vec{\boldsymbol{0}} \implies \|\vec{\boldsymbol{v}}\| > 0 \text{ so has inverse} \\
\therefore \quad & \lambda \in \mathbb{R}. \quad \square
\end{aligned}$$

Corollary 2.6.5. *A real symmetric matrix has only real-valued eigenvalues.*

Proof. This follows from Proposition 2.6.24 and the fact that for a symmetric matrix $A \in \mathbb{R}^{n\times n} \subset \mathbb{C}^{n\times n}$, the operation of complex conjugation on the real entries of A has no effect so that $\overline{A} = A$ and, by the symmetric property, we have $A^T = A$. So, putting these together we obtain,

$$A^* = \overline{A}^T = A^T = A$$

which says that if A is real and symmetric, then it is hermitian. $\square$

Proposition 2.6.25. *A matrix is unitarily diagonalisable iff it is normal.*

Proof. If a matrix A is unitarily diagonalisable then there exists a unitary matrix P such that

$$D = P^* A P$$

is diagonal. Then it also follows that $A = PDP^*$ and so,

$$\begin{aligned}
AA^* &= (PDP^*)\,(PDP^*)^* \\
&= PDP^*\,PD^*P^* \\
&= PDD^*P^* && \because P \text{ is unitary} \\
&= PD^*DP^* && \because D \text{ is diagonal} \implies D \text{ is normal} \\
&= PD^*P^*\,PDP^* && \because P \text{ is unitary} \\
&= (PDP^*)^*\,(PDP^*) = A^*A.
\end{aligned}$$

Conversely, assume that a matrix A is normal. By the Schur decomposition (Theorem 2.3.2), we can write any matrix as UTU^*, where U is unitary and T is upper-triangular. Letting $A = UTU^*$ and using the normal property gives,

$$\begin{aligned}
& AA^* = A^*A \\
\iff \quad & (UTU^*)(UTU^*)^* = (UTU^*)^*(UTU^*) \\
\iff \quad & UTU^*\,UT^*U^* = UT^*U^*\,UTU^* \\
\iff \quad & UTT^*U^* = UT^*TU^* && \because U \text{ is unitary} \\
\iff \quad & TT^* = T^*T.
\end{aligned}$$

So the upper-triangular matrix T is also normal. By Proposition 2.3.18 then, T is diagonal. Therefore,

$$U^*AU = T$$

is diagonal and A is unitarily diagonalisable. □

Corollary 2.6.6. *A real matrix is orthogonally diagonalisable iff it is symmetric.*

Proof. If a matrix A is orthogonally diagonalisable then there exists an orthogonal matrix P such that,

$$D = P^{-1}AP$$

is diagonal. Since the matrix D is diagonal, it is symmetrical and equal to its transpose so we can reason,

$$D^T = D \implies (P^{-1}AP)^T = P^{-1}AP.$$

Furthermore, P is an orthogonal matrix so, by Proposition 2.6.16, $P^T = P^{-1}$. Therefore,

$$\begin{aligned}
& (P^{-1}AP)^T = P^{-1}AP \\
\iff \quad & P^TA^T(P^{-1})^T = P^{-1}AP \\
\iff \quad & P^{-1}A^T(P^T)^T = P^{-1}AP \\
\iff \quad & P^{-1}A^TP = P^{-1}AP \\
\iff \quad & A^T = A
\end{aligned}$$

and A is symmetric.

Conversely, let A be a real symmetric matrix in complex space. Since A is real and symmetric, it is hermitian and normal (Proposition 2.3.13,

Proposition 2.3.17). The hermitian property of A implies, by Proposition 2.6.24, that all its eigenvalues are real-valued. The normal property of A, by Proposition 2.6.25, implies that it is unitarily diagonalisable; that's to say there exists a unitary matrix U and a diagonal matrix D such that,

$$U^*AU = D.$$

Since all the eigenvalues of A are real, by Proposition 2.5.18, we can choose real eigenvectors corresponding to them. So, by construction, U is real. Since U is real and unitary, it is orthogonal (Proposition 2.3.16). Therefore A is orthogonally diagonalisable. □

---

(78) The matrix

$$A = \begin{bmatrix} 1 & 2+i \\ 2-i & 5 \end{bmatrix}$$

is hermitian and therefore also normal and so can be unitarily diagonalised. The characteristic polynomial is

$$\lambda^2 - 6\lambda + 5 - 5 = \lambda^2 - 6\lambda = \lambda(\lambda - 6) = 0$$

so the eigenvalues are $\lambda \in \{0, 6\}$. Note that, as predicted by **??**, the eigenvalues ae real. After row-reduction the determined eigenvectors are:

- For $\lambda_1 = 0$,
$$\vec{\boldsymbol{v}}_1 = \begin{bmatrix} 2+i \\ -1 \end{bmatrix}.$$
- For $\lambda_2 = 6$,
$$\vec{\boldsymbol{v}}_2 = \begin{bmatrix} 2+i \\ 5 \end{bmatrix}.$$

The vectors $\vec{\boldsymbol{v}}_1$ and $\vec{\boldsymbol{v}}_2$ are orthogonal w.r.t. the standard complex inner product. Their norms are $\sqrt{6}$ and $\sqrt{30}$ respectively so that the associated unitary matrix is

$$P = \begin{bmatrix} \frac{2+i}{\sqrt{6}} & \frac{2+i}{\sqrt{30}} \\ \frac{-1}{\sqrt{6}} & \frac{5}{\sqrt{30}} \end{bmatrix}$$

and

$$P^*AP = \begin{bmatrix} \frac{2-i}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \\ \frac{2-i}{\sqrt{30}} & \frac{5}{\sqrt{30}} \end{bmatrix} \begin{bmatrix} 1 & 2+i \\ 2-i & 5 \end{bmatrix} \begin{bmatrix} \frac{2+i}{\sqrt{6}} & \frac{2+i}{\sqrt{30}} \\ \frac{-1}{\sqrt{6}} & \frac{5}{\sqrt{30}} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 6 \end{bmatrix}.$$

2.6.2.6 Spectral Decomposition

Definition 144. **(Spectral Decomposition)** If an operator on a finite-dimensional inner product space is normal (2.3.2.9) then it can be written as a linear combination of pairwise orthogonal projections, called its *spectral decomposition* or *eigendecomposition*.

The spectral decomposition is a special case of both the Schur Decomposition (Theorem 2.3.2) and the Singular Value Decomposition.

Theorem 2.6.4. ***(Spectral Decomposition)*** *Let A be a normal matrix and let $\{\vec{\boldsymbol{x}}_1, \vec{\boldsymbol{x}}_2, \ldots, \vec{\boldsymbol{x}}_n\}$ be an orthonormal basis of eigenvectors of A with corresponding eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. Then,*

$$A = \lambda_1 \vec{\boldsymbol{x}}_1 \vec{\boldsymbol{x}}_1^* + \lambda_2 \vec{\boldsymbol{x}}_2 \vec{\boldsymbol{x}}_2^* + \cdots + \lambda_n \vec{\boldsymbol{x}}_n \vec{\boldsymbol{x}}_n^*.$$

Proof. Let

$$B = \lambda_1 \vec{\boldsymbol{x}}_1 \vec{\boldsymbol{x}}_1^* + \lambda_2 \vec{\boldsymbol{x}}_2 \vec{\boldsymbol{x}}_2^* + \cdots + \lambda_n \vec{\boldsymbol{x}}_n \vec{\boldsymbol{x}}_n^*.$$

For each eigenvalue λ_i of A with associated eigenvector $\vec{\boldsymbol{x}}_i$,

$$\begin{aligned}
& A\vec{\boldsymbol{x}}_i = B\vec{\boldsymbol{x}}_i \\
\iff \quad & \lambda_i \vec{\boldsymbol{x}}_i = (\lambda_1 \vec{\boldsymbol{x}}_1 \vec{\boldsymbol{x}}_1^* + \cdots + \lambda_n \vec{\boldsymbol{x}}_n \vec{\boldsymbol{x}}_n^*)\vec{\boldsymbol{x}}_i \\
\iff \quad & \lambda_i \vec{\boldsymbol{x}}_i = (\lambda_i \vec{\boldsymbol{x}}_i \vec{\boldsymbol{x}}_i^*)\vec{\boldsymbol{x}}_i && \because \forall i \neq j \,.\, \vec{\boldsymbol{x}}_i \perp \vec{\boldsymbol{x}}_j \implies \vec{\boldsymbol{x}}_j^* \vec{\boldsymbol{x}}_i = 0 \\
\iff \quad & \lambda_i \vec{\boldsymbol{x}}_i = (\lambda_i \vec{\boldsymbol{x}}_i) \|\vec{\boldsymbol{x}}_i\|^2 \\
\iff \quad & \lambda_i \vec{\boldsymbol{x}}_i = \lambda_i \vec{\boldsymbol{x}}_i.
\end{aligned}$$

This result implies that the expression B is a matrix that has the same eigenvalues and corresponding eigenvectors as A. Therefore, by diagonalisability of A and Proposition 2.5.19, $A = B$. □

Proposition 2.6.26. *Let A be a normal matrix and let*

$$A = \lambda_1 \vec{\boldsymbol{x}}_1 \vec{\boldsymbol{x}}_1^* + \lambda_2 \vec{\boldsymbol{x}}_2 \vec{\boldsymbol{x}}_2^* + \cdots + \lambda_n \vec{\boldsymbol{x}}_n \vec{\boldsymbol{x}}_n^*$$

be the spectral decomposition of A. Then each $\vec{\boldsymbol{x}}_i \vec{\boldsymbol{x}}_i^*$ is an $n \times n$ matrix representing pairwise orthogonal projections onto the respective eigenspace of the eigenvalue λ_i.

Proof.

- Each matrix $\vec{\boldsymbol{x}}_i \vec{\boldsymbol{x}}_i^*$ is real and symmetric which implies that they are hermitian.
- Mutual orthogonality w.r.t. standard complex inner product is easy to show.
- $$(\vec{\boldsymbol{x}}_i \vec{\boldsymbol{x}}_i^*)(\vec{\boldsymbol{x}}_i \vec{\boldsymbol{x}}_i^*)\vec{\boldsymbol{v}} = \vec{\boldsymbol{x}}_i(\vec{\boldsymbol{x}}_i^* \vec{\boldsymbol{x}}_i)\vec{\boldsymbol{x}}_i^* \vec{\boldsymbol{v}} = \vec{\boldsymbol{x}}_i(1)\vec{\boldsymbol{x}}_i^* \vec{\boldsymbol{v}} = \vec{\boldsymbol{x}}_i \vec{\boldsymbol{x}}_i^* \vec{\boldsymbol{v}}$$

  means that they are idempotent.
- These properties imply that they are orthogonal projections.
- They clearly project onto the associated eigenspace because the eigenvectors $\vec{\boldsymbol{x}}_i$ are an eigenbasis so that for any $\vec{\boldsymbol{v}}$ in the space,

  $$\vec{\boldsymbol{v}} = \alpha_1 \vec{\boldsymbol{x}}_1 + \cdots + \alpha_n \vec{\boldsymbol{x}}_n$$

  and so the action of a matrix $\vec{\boldsymbol{x}}_i \vec{\boldsymbol{x}}_i^*$ on a vector in the space is,

  $$x_i \vec{\boldsymbol{x}}_i^* \vec{\boldsymbol{v}} = x_i \vec{\boldsymbol{x}}_i^* (\alpha_1 \vec{\boldsymbol{x}}_1 + \cdots + \alpha_n \vec{\boldsymbol{x}}_n) = \vec{\boldsymbol{x}}_i \vec{\boldsymbol{x}}_i^* \alpha_i \vec{\boldsymbol{x}}_i = \alpha_i \vec{\boldsymbol{x}}_i.$$

TODO: complete this after studying orthogonal projections □

Proposition 2.6.27. *If matrices $E_1, E_2, \ldots, E_n$ are orthogonal projections then, for any real scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ and any positive integer m,*

$$(\alpha_1 E_1 + \alpha_2 E_2 + \cdots + \alpha_n E_n)^m = \alpha_1^m E_1 + \alpha_2^m E_2 + \cdots + \alpha_n^m E_n.$$

Proof. TODO: proof □

2.6.3 Quadratic Forms

Definition 145. **(Quadratic Form)** A *quadratic form* in $n \geq 2$ variables is a polynomial over the reals in which every term has degree 2 in the variables. In matrix form, a quadratic form q is an expression,

$$q = \vec{\boldsymbol{x}}^T A \vec{\boldsymbol{x}}$$

where $A \in \mathbb{R}^{n \times n}$ is symmetric and $\vec{\boldsymbol{x}} \in \mathbb{R}^n$.

The matrix A is said to be *positive definite* if the quadratic form q is positive definite and, similarly if it is positive negative, semi-definite, etc.

Proposition 2.6.28. *Let A be a symmetric real matrix. Then,*

- *if all eigenvalues of A are positive then A is positive definite;*
- *if all eigenvalues of A are non-negative then A is positive semi-definite;*
- *if all eigenvalues of A are negative then A is negative definite;*
- *if all eigenvalues of A are non-positive then A is negative semi-definite;*
- *if the eigenvalues of A are of mixed sign then A is indefinite;*
-

Proof. Since A is a real symmetric matrix it is orthogonally diagonalisable (Corollary 2.6.6) and there exists an invertible matrix P and a diagonal matrix D such that,

$$A = P^{-1} D P = P^T D P.$$

So, the quadratic form $q(\vec{\boldsymbol{x}}) = \vec{\boldsymbol{x}}^T A \vec{\boldsymbol{x}}$ can be rewritten

$$q(\vec{\boldsymbol{x}}) = \vec{\boldsymbol{x}}^T A \vec{\boldsymbol{x}} = \vec{\boldsymbol{x}}^T P^T D P \vec{\boldsymbol{x}} = (P\vec{\boldsymbol{x}})^T D (P\vec{\boldsymbol{x}}).$$

If we let $\vec{\boldsymbol{y}} = P\vec{\boldsymbol{x}}$ and $D = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ then

$$q(\vec{\boldsymbol{x}}) = \vec{\boldsymbol{y}}^T D \vec{\boldsymbol{y}} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2 = \sum_{i=1}^{n} \lambda_i y_i^2.$$

Written in this way, it's clear to see that the quadratic form is the sum of terms that are the product of an eigenvalue and a square of a real number. The square of a real number is positive definite and so every term will be positive definite if every eigenvalue is positive and positive semi-definite if every eigenvalue is non-negative. Similar reasoning follows also for negative definite and negative semi-definite. □

Proposition 2.6.29. *A matrix is positive definite iff all its principal minors (2.3.5.3) are positive.*

Proof. see Artin[252-262]. □

Proposition 2.6.30. *A matrix is positive negative iff its principal minors (2.3.5.3) of odd order are negative and those of even order are positive.*

Proof. see Artin[252-262]. □

- *If the principal minors don't fall into either pattern of all positive or alternating signs and the determinant of the matrix is non-zero, then we can conclude that the matrix is indefinite.*
- *If the determinant of the matrix is 0 then we only know that at least one of the eigenvalues is 0.*
- *There is no simple check for positive or negative semi-definiteness (see example 79).*

Proposition 2.6.31. *Let $A \in \mathbb{F}^{m\times n}$ be a matrix with full column rank (i.e.* $\operatorname{rank} A = n$*) defined in an inner product space equipped with the standard complex inner product (2.6.1). Then A^*A is:*

(i) Hermitian,

(ii) Invertible.

Proof.

(i) $(A^*A)^* = A^*A$

(ii) **alternative proof 1**

Let

$$A^*A\vec{\boldsymbol{x}} = \vec{\mathbf{0}} \implies A\vec{\boldsymbol{x}} \in N(A^*).$$

By, Proposition 2.6.9, we have

$$N(A^*) = R((A^*)^*)^\perp = R(A)^\perp.$$

That's to say, the nullspace of A^* is the orthogonal complement of the range of A. Therefore,

$$A\vec{\boldsymbol{x}} \in N(A^*) \iff A\vec{\boldsymbol{x}} \in R(A)^\perp.$$

But $R(A) = \{\, A\vec{\boldsymbol{v}} \mid \vec{\boldsymbol{v}} \in \mathbb{F}^n \,\}$ so also $A\vec{\boldsymbol{x}} \in R(A)$. It follows then that

$$A\vec{\boldsymbol{x}} \in R(A) \cap R(A)^\perp.$$

By Proposition 2.6.6,

$$R(A) \oplus R(A)^\perp = \mathbb{F}^m \implies R(A) \cap R(A)^\perp = \{\vec{\mathbf{0}}\}$$

so

$$A\vec{\boldsymbol{x}} = \vec{\mathbf{0}}.$$

But, by hypothesis, A is full column rank and so

$$A\vec{\boldsymbol{x}} = \vec{\mathbf{0}} \iff \vec{\boldsymbol{x}} = \vec{\mathbf{0}}.$$

alternative proof 2

Let

$$\vec{\boldsymbol{x}} \in N(A^*A) \implies A^*A\vec{\boldsymbol{x}} = \vec{\mathbf{0}}.$$

Then,

$$\vec{\boldsymbol{x}}^*A^*A\vec{\boldsymbol{x}} = 0 = (A\vec{\boldsymbol{x}})^*A\vec{\boldsymbol{x}} = \langle A\vec{\boldsymbol{x}},\, A\vec{\boldsymbol{x}}\rangle = \|A\vec{\boldsymbol{x}}\|^2.$$

Then, the positive definiteness property of the inner product allows us to deduce that

$$\|A\vec{\boldsymbol{x}}\|^2 = 0 \implies A\vec{\boldsymbol{x}} = 0 \implies \vec{\boldsymbol{x}} \in N(A).$$

But A has full column rank and so $N(A) = \{\vec{\mathbf{0}}\}$ which implies that $\vec{\boldsymbol{x}} = \vec{\mathbf{0}}$.

conclusion

Since we have shown that

$$A^*A\vec{\boldsymbol{x}} = \vec{\mathbf{0}} \implies \vec{\boldsymbol{x}} = \vec{\mathbf{0}}$$

it follows that

$$N(A^*A) = \{\vec{\mathbf{0}}\}$$

which, by Corollary 2.5.3, implies that A^*A is injective. Since A^*A is $n \times n$ and injective then it is also surjective (TODO: review this) and is therefore bijective and invertible.

□

Corollary 2.6.7. *Let $A \in \mathbb{R}^{m\times n}$ be a real matrix with full column rank (i.e.* $\operatorname{rank} A = n$*). Then A^TA is:*

(i) Positive definite;

(ii) Symmetric;

(iii) Invertible.

Proof. TODO: prove this □

---

(79) Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & t \end{bmatrix}.$$

where $t \in \mathbb{R}$ is any real number.

The eigenvalues of A are $\lambda \in \{0, 2, t\}$. The principal minors of A are

$$a_{11} = 1, \qquad \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0, \qquad \det A = 0.$$

Notice that the values of the principal minors do not depend on the value of t. So t could be any real number — positive or negative or zero — and the principal minors would come out the same. Clearly, in such a case, the principal minors do not inform us about the definiteness of the matrix.

2.7 Affine Spaces and Transformations

2.7.0.1 Affine Spaces

Definition 146. An **affine space** is a generalization of a Euclidean space in which there is no particular point designated as the origin. As a result vectors can be viewed as displacements rather than points.

Let V be a vector space and P be a set of points. Then we can form an affine space over V and P by defining the vectors in V as displacements connecting members of P such that, for $Q_1, Q_2 \in P$, $\vec{\boldsymbol{v}} \in V$,

$$Q_1 + \vec{\boldsymbol{v}} = Q_2 \iff Q_2 - Q_1 = \vec{\boldsymbol{v}} \iff Q_1 - Q_2 = -\vec{\boldsymbol{v}}.$$

Definition 147. A **frame** of an affine space is an extension of a basis of its underlying vector space to include a point designated as an origin. If $\vec{\boldsymbol{v}}_1, \ldots, \vec{\boldsymbol{v}}_n$ is a basis of a vector space V and Q is a point in the set of points P, then $F = (\vec{\boldsymbol{v}}_1, \ldots, \vec{\boldsymbol{v}}_n, Q)$ is a frame of the affine space over V and P.

Definition 148. The **dimension** of an affine space is the dimension of the underlying vector space.

Proposition 2.7.1. *Any linear combination of points in an affine space where the coefficients sum to 0 results in a vector.*

Proof. Let S be a sum of n points $Q_i \in P$ in an affine space associated with a vector space V such that,

$$S = \sum_{i=0}^{n} \alpha_i Q_i \quad \text{and} \quad \sum_{i=0}^{n} \alpha_i = 0.$$

Then, if we take the partial sum of the first two points,

$$S_2 = \alpha_1 Q_1 + \alpha_2 Q_2 = \alpha_1(Q_1 - Q_2) + (\alpha_1 + \alpha_2)Q_2$$

and then the next partial sum of the first three points,

$$\begin{aligned} S_3 &= \alpha_1(Q_1 - Q_2) + (\alpha_1 + \alpha_2)Q_2 + \alpha_3 Q_3 \\ &= \alpha_1(Q_1 - Q_2) + (\alpha_1 + \alpha_2)(Q_2 - Q_3) + (\alpha_1 + \alpha_2 + \alpha_3)Q_3 \end{aligned}$$

we can see that, by induction, the nth sum is,

$$\begin{aligned} S = \alpha_1(Q_1 - Q_2) \\ &+ (\alpha_1 + \alpha_2)(Q_2 - Q_3) \\ &+ (\alpha_1 + \alpha_2 + \alpha_3)(Q_3 - Q_4) \\ &\vdots \\ &+ (\alpha_1 + \cdots + \alpha_{n-1})(Q_{n-1} - Q_n) \\ &+ (\alpha_1 + \cdots + \alpha_n)Q_n. \end{aligned}$$

But we have $\alpha_1 + \cdots + \alpha_n = 0$ so the final term is 0. As a result S is a summation of terms of the form,

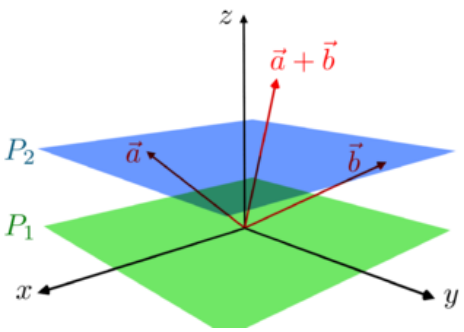
$$(\alpha_1 + \cdots + \alpha_{i-1})(Q_{i-1} - Q_i)$$

where $Q_{i-1} - Q_i$ is a vector. Therefore S is a linear combination of vectors in V and is therefore also a vector in V. □

Corollary 2.7.1. *Any linear combination of points in an affine space where the coefficients sum to 1 results in a point.*

Proof. In the preceding proof if we had, instead, $\alpha_1 + \cdots + \alpha_n = 1$ then the final term would equal Q_n and the resulting summation would be a vector plus the point Q_n. Therefore the sum is a point. □

2.7.0.2 Intuition of Affine Spaces



A translated linear subspace of a vector space like P_2 above that no longer passes through the origin is referred to as an **affine subspace**. It is not a vector space as $\vec{\mathbf{0}} \notin P_2$ and $\vec{\boldsymbol{a}}, \vec{\boldsymbol{b}} \in P_2$ but $\vec{\boldsymbol{a}} + \vec{\boldsymbol{b}} \notin P_2$. However, if we instead consider displacements between points — e.g. $\vec{\boldsymbol{b}} - \vec{\boldsymbol{a}}$ — then we see the relationship between affine spaces and vector spaces: $\vec{\boldsymbol{b}} - \vec{\boldsymbol{a}} \in P_1$. The displacements between points in P_2 form a linear subspace. So we can define an affine space A based on the set of points in P_2 and the vectors in P_1.

2.7.0.3 Affine Combinations

Definition 149. An **affine combination** of vectors is a combination such that the coefficients sum to 1.

The definition of an affine combination differs from that of a convex combination in that the coefficients of a convex combination are additionally required to be non-negative.

In an affine space there is no particular point designated as the origin but we can describe vector displacements between points as an ordered pair of points, for example, $(p, a) = \vec{\boldsymbol{pa}}$. Affine combinations of displacements agree on the resulting point with linear combinations in the corresponding Euclidean space. For example, imagine a point p in an affine space is at coordinates $(-1, 4)$ in the corresponding Euclidean space and similarly points a and b are at $(3, 4)$ and $(6, 1)$ respectively. Then, if we take an affine combination of the displacements to a and b the resulting point is independent of the chosen origin point.

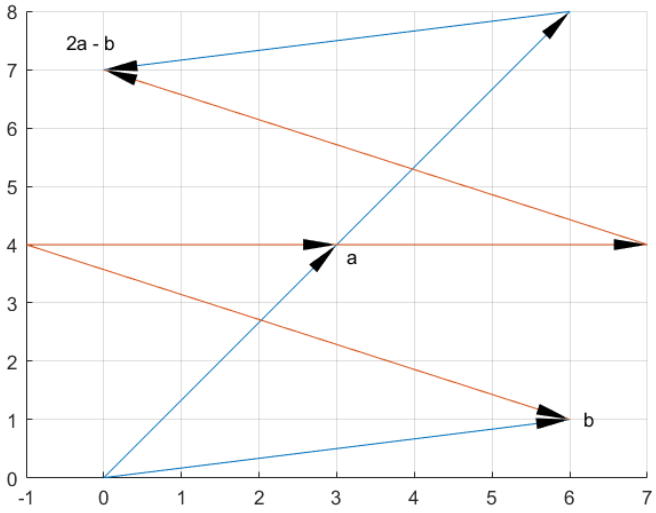


Figure 2.7: The diagram shows the affine combination $2\vec{\boldsymbol{a}} - \vec{\boldsymbol{b}} = 2\vec{\boldsymbol{pa}} - \vec{\boldsymbol{pb}}$ where $\vec{\boldsymbol{a}}$ denotes the usual position vector in the Euclidean space.

2.7.0.4 Affine Transformations

Definition 150. Let $T : A_1 \longmapsto A_2$ be a mapping between the affine spaces A_1 and A_2. Then T is an **affine transformation** if:

- T maps vectors to vectors and points to points.
- T is a linear transformation over the vectors in the underlying vector space.
- $T(Q + \vec{\boldsymbol{v}}) = T(Q) + T(\vec{\boldsymbol{v}})$.

*In an affine space a **translation** can be regarded as a change of frame in which we **change the origin point** while the vector basis may remain unchanged.*

Proposition 2.7.2. *Affine transformations preserve parallelism.*

Proof. Let A be an affine space defined over a set of points P and a vector space V and let $Q_1, Q_2 \in P$ and $\vec{\boldsymbol{v}} \in V$. Then, for $s, t \in \mathbb{F}$,

$$l_1 = Q_1 + t\vec{\boldsymbol{v}} \quad \text{and} \quad l_2 = Q_2 + s\vec{\boldsymbol{v}}$$

are parallel lines in A. Let T be an arbitrary affine transformation over A. Then,

$$l_1' = T(l_1) = T(Q_1) + tT(\vec{\boldsymbol{v}}) \quad \text{and} \quad l_2' = T(l_2) = T(Q_2) + sT(\vec{\boldsymbol{v}})$$

are also parallel. □

Proposition 2.7.3. *An affine transformation that preserves the dot product is left multiplication by an orthogonal matrix.*

Proof. Firstly, note that if a transformation m preserves the dot product and also fixes the standard basis vectors $\vec{\boldsymbol{e_i}}$ then,

$$m(\vec{\boldsymbol{e_i}}) = \vec{\boldsymbol{e_i}} \quad \text{and} \quad x_i = \vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{e_i}} = m(\vec{\boldsymbol{x}}) \cdot m(\vec{\boldsymbol{e_i}}) = m(\vec{\boldsymbol{x}}) \cdot \vec{\boldsymbol{e_i}} = m(\vec{\boldsymbol{x}})_i.$$

Therefore, such a transformation m is the identity transformation.
Now, assume a transformation m' preserves the dot product (but does not necessarily fix the standard basis vectors) in $\mathbb{R}^n$ and let

$$B' = \{m'(\vec{\boldsymbol{e_1}}), \ldots, m'(\vec{\boldsymbol{e_n}})\}$$

be the transformed standard basis vectors. Then, if $[B']$ is the matrix whose columns are the elements of B' then, because m' preserves the dot product, $[B']$ is an orthogonal matrix as, by Proposition 2.6.21, is $[B']^{-1}$. So, composing this with m',

$$m'' = [B']^{-1} m'$$

is a transformation that both preserves the dot product and fixes the standard basis vectors. Therefore we have,

$$m'' = I_n = [B']^{-1} m' \iff [B'] = m'$$

so that m' is left multiplication by $[B']$, an orthogonal matrix. □

Properties of affine transformations

We can characterize affine transformations according to their form and behaviour in the Euclidean spaces $\mathbb{R}^2$ and $\mathbb{R}^3$. Specifically, whether the transformation:

1. Fixes the origin: $T(\vec{\mathbf{0}}) = \vec{\mathbf{0}}$
2. Preserves the dot product: $T(\vec{\boldsymbol{v}}) \cdot T(\vec{\boldsymbol{w}}) = \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}}$, matrix is orthogonal
3. Preserves distances: $\|T(\vec{\boldsymbol{v}}) - T(\vec{\boldsymbol{w}})\| = \|\vec{\boldsymbol{v}} - \vec{\boldsymbol{w}}\|$
4. Preserves angles: matrix is scalar multiple of orthogonal matrix
5. Preserves orientation: transformation does not include a reflection, determinant of matrix is positive
6. Preserves parallelism: transformation exhibits affine property (linearity under affine combinations)

As we can see here, the property of preserving parallelism depends only on the affine property so clearly, all affine transformations exhibit this behaviour. As a consequence of preserving parallelism, affine transformations preserve the dimension of affine subspaces (points, lines, planes, etc.). They do not preserve distances between points however, but they do preserve ratios of distances between points lying on a straight line.

Classes of affine transformations

The most common subclassifications of affine transformations are:

- Linear: preserves the origin.
- Conformal: preserves angles.
- Isometry (also known as a congruent transformation): preserves distances in metric spaces and so also implicity, angles. In a Euclidean space these transformations are known as a Euclidean isometries or rigid transformations.
- Rigid Motion: Euclidean isometry / rigid transformation that also preserves orientation.

2.7.0.5 Isometries

A Euclidean isometry or rigid motion, for example, carries a triangle to a congruent triangle. So, it preserves distances and angles but not necessarily orientation (a reflection flips the orientation).
The composition of two rigid motions is a rigid motion and the inverse of a rigid motion is also a rigid motion. Therefore, the rigid motions of $\mathbb{R}^n$ form a group under composition of operations. This group is called the *group of motions* and denoted M_n.

Proposition 2.7.4. *A rigid motion that fixes the origin preserves the dot product.*

Proof. Let m be a rigid motion that fixes the origin. Then m is an isometry and so preserves distances and also m maps the origin to the origin. So we have,

$$\|m(\vec{\boldsymbol{v}}) - m(\vec{\boldsymbol{w}})\| = \|\vec{\boldsymbol{v}} - \vec{\boldsymbol{w}}\| \quad \text{and} \quad m(\vec{\mathbf{0}}) = \vec{\mathbf{0}}.$$

We can rewrite the isometry property as,

$$\begin{aligned} & \sqrt{(m(\vec{\boldsymbol{v}}) - m(\vec{\boldsymbol{w}})) \cdot (m(\vec{\boldsymbol{v}}) - m(\vec{\boldsymbol{w}}))} = \sqrt{(\vec{\boldsymbol{v}} - \vec{\boldsymbol{w}}) \cdot (\vec{\boldsymbol{v}} - \vec{\boldsymbol{w}})} \\ \iff \quad & (m(\vec{\boldsymbol{v}}) - m(\vec{\boldsymbol{w}})) \cdot (m(\vec{\boldsymbol{v}}) - m(\vec{\boldsymbol{w}})) = (\vec{\boldsymbol{v}} - \vec{\boldsymbol{w}}) \cdot (\vec{\boldsymbol{v}} - \vec{\boldsymbol{w}}). \end{aligned}$$

Now, if we take the case where $\vec{\boldsymbol{w}} = \vec{\mathbf{0}} = m(\vec{\boldsymbol{w}})$,

$$\begin{aligned} & (m(\vec{\boldsymbol{v}}) - \vec{\mathbf{0}}) \cdot (m(\vec{\boldsymbol{v}}) - \vec{\mathbf{0}}) = (\vec{\boldsymbol{v}} - \vec{\mathbf{0}}) \cdot (\vec{\boldsymbol{v}} - \vec{\mathbf{0}}) \\ \iff \quad & m(\vec{\boldsymbol{v}}) \cdot m(\vec{\boldsymbol{v}}) = \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{v}}. \end{aligned}$$

and we can deduce that $m(\vec{\boldsymbol{x}}) \cdot m(\vec{\boldsymbol{x}}) = \vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{x}}$ for any vector $\vec{\boldsymbol{x}}$. Using this along with the properties of the dot product we obtain,

$$\begin{aligned} & (m(\vec{\boldsymbol{v}}) - m(\vec{\boldsymbol{w}})) \cdot (m(\vec{\boldsymbol{v}}) - m(\vec{\boldsymbol{w}})) = (\vec{\boldsymbol{v}} - \vec{\boldsymbol{w}}) \cdot (\vec{\boldsymbol{v}} - \vec{\boldsymbol{w}}) \\ \iff \quad & m(\vec{\boldsymbol{v}}) \cdot m(\vec{\boldsymbol{v}}) + m(\vec{\boldsymbol{w}}) \cdot m(\vec{\boldsymbol{w}}) - 2m(\vec{\boldsymbol{v}}) \cdot m(\vec{\boldsymbol{w}}) = \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{v}} + \vec{\boldsymbol{w}} \cdot \vec{\boldsymbol{w}} - 2\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}} \\ \iff \quad & -2m(\vec{\boldsymbol{v}}) \cdot m(\vec{\boldsymbol{w}}) = -2\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}} \\ \iff \quad & m(\vec{\boldsymbol{v}}) \cdot m(\vec{\boldsymbol{w}}) = \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}}. \end{aligned}$$

Therefore, m preserves the dot product and, by Proposition 2.7.3, is left multiplication by an orthogonal matrix. □

Corollary 2.7.2. *A rigid motion that fixes the origin is left multiplication by an orthogonal matrix and, therefore, also a linear operator.*

Proposition 2.7.5. *Left multiplication by any orthogonal matrix is a Euclidean isometry (a rigid motion) that fixes the origin.*

Proof. By, Proposition 2.6.18, left multiplication by an orthogonal matrix preserves the dot product. So, if m is an affine transformation such that $m(\vec{\boldsymbol{x}}) = A\vec{\boldsymbol{x}}$ where A is an orthogonal matrix then,

$$\begin{aligned} & \|m(\vec{\boldsymbol{v}} - \vec{\boldsymbol{w}})\|^2 = m(\vec{\boldsymbol{v}} - \vec{\boldsymbol{w}}) \cdot m(\vec{\boldsymbol{v}} - \vec{\boldsymbol{w}}) = (\vec{\boldsymbol{v}} - \vec{\boldsymbol{w}}) \cdot (\vec{\boldsymbol{v}} - \vec{\boldsymbol{w}}) = \|\vec{\boldsymbol{v}} - \vec{\boldsymbol{w}}\|^2 \\ \iff \quad & \|m(\vec{\boldsymbol{v}} - \vec{\boldsymbol{w}})\| = \|\vec{\boldsymbol{v}} - \vec{\boldsymbol{w}}\| \end{aligned}$$

But also, left multiplication by a matrix is a linear operator so $m(\vec{\boldsymbol{v}} - \vec{\boldsymbol{w}}) = m(\vec{\boldsymbol{v}}) - m(\vec{\boldsymbol{w}})$ meaning that,

$$\|m(\vec{\boldsymbol{v}} - \vec{\boldsymbol{w}})\| = \|m(\vec{\boldsymbol{v}}) - m(\vec{\boldsymbol{w}})\| = \|\vec{\boldsymbol{v}} - \vec{\boldsymbol{w}}\|$$

which is the isometry property for m. □

Linear

Isometries that fix the origin are linear operators and rigid motions in Euclidean space.

Translation

If T is a translation then, in general, $T(\vec{\mathbf{0}}) \neq \vec{\mathbf{0}}$ so translations do not fix the origin and, as a result, are **not** linear transformations. However, translations preserve distances and angles (and orientation because there is no reflection) so they are rigid motions. For example:

(80) Let $\vec{\boldsymbol{v}} = (v_1, \ldots, v_n)$ be any fixed vector in $\mathbb{R}^n$. Then translation by $\vec{\boldsymbol{v}}$ is the map,

$$t_v(\vec{\boldsymbol{x}}) = \vec{\boldsymbol{x}} + \vec{\boldsymbol{v}} = \begin{bmatrix} x_1 + v_1 \\ \vdots \\ x_n + v_n \end{bmatrix}$$

which can be seen to be isometric (a rigid transformation) by,

$$\begin{aligned} & t_v(\vec{\boldsymbol{x}}) - t_v(\vec{\boldsymbol{y}}) = (\vec{\boldsymbol{x}} + \vec{\boldsymbol{v}}) - (\vec{\boldsymbol{y}} + \vec{\boldsymbol{v}}) = \vec{\boldsymbol{x}} - \vec{\boldsymbol{y}} \\ \implies \qquad & \|t_v(\vec{\boldsymbol{x}}) - t_v(\vec{\boldsymbol{y}})\| = \|\vec{\boldsymbol{x}} - \vec{\boldsymbol{y}}\| . \end{aligned}$$

Proposition 2.7.6. *Every rigid motion m is the composition of an orthogonal linear operator and a translation. In other words, for some orthogonal matrix A and fixed vector $\vec{\boldsymbol{v}}$, it takes the form,*

$$m(\vec{\boldsymbol{x}}) = A\vec{\boldsymbol{x}} + \vec{\boldsymbol{v}}.$$

Proof. Let $\vec{\boldsymbol{v}} = m(\vec{\mathbf{0}})$ and $t_v(\vec{\boldsymbol{x}}) = \vec{\boldsymbol{x}} + \vec{\boldsymbol{v}}$ with inverse $t_{-v}(\vec{\boldsymbol{x}}) = \vec{\boldsymbol{x}} - \vec{\boldsymbol{v}}$. Then composing this with m, the resulting transformation,

$$(t_{-v} \circ m)(\vec{\boldsymbol{x}}) = m(\vec{\boldsymbol{x}}) - \vec{\boldsymbol{v}}$$

continues to be isometric — because it is the composition of isometric transformations — and it fixes the origin because $(t_{-v} \circ m)(\vec{\mathbf{0}}) = m(\vec{\mathbf{0}}) - \vec{\boldsymbol{v}} = m(\vec{\mathbf{0}}) - m(\vec{\mathbf{0}}) = \vec{\mathbf{0}}$. It is therefore, by Corollary 2.7.2, left multiplication by an orthogonal matrix. So we can represent it as,

$$(t_{-v} \circ m)(\vec{\boldsymbol{x}}) = t_{-v}(m(\vec{\boldsymbol{x}})) = A\vec{\boldsymbol{x}}.$$

Since $t_{-v} = {t_v}^{-1}$ we can apply t_v to both sides of the equation,

$$m(\vec{\boldsymbol{x}}) = t_v(A\vec{\boldsymbol{x}}) = A\vec{\boldsymbol{x}} + \vec{\boldsymbol{v}}.$$

The obtained representation is uniquely determined by m as $\vec{\boldsymbol{v}} = m(\vec{\mathbf{0}})$ is clearly unique and then the translation t_{-v} is uniquely determined by $\vec{\boldsymbol{v}}$ and then $A = (t_{-v} \circ m)$ is unique for a given $\vec{\boldsymbol{v}}$ and m. □

For a rigid motion $m(\vec{\boldsymbol{x}}) = A\vec{\boldsymbol{x}} + \vec{\boldsymbol{v}}$, m is orientation-preserving if the matrix A is orientation-preserving and orientation-reversing if A is orientation-reversing.

Rotation

Rotations preserve distances, angles and orientation and so are rigid motions. Rotations also fix a vector which is known as the axis of rotation. If the axis of rotation contains the origin then they fix the origin and so are linear operators.

Theorem 2.7.1. *The rotations of $\mathbb{R}^2$ and $\mathbb{R}^3$ about the origin are the linear operators whose matrices with respect to the standard basis are orthogonal and have determinant 1.*

Proof. A rotation about the origin m involves rotating the standard basis vectors through an angle θ. It is in the definition of this rotation that the image of the standard basis vectors continue to subtend the same angle, $\pi/2$. Therefore, the rotation must preserve angles. It is also part of the definition that the image under rotation is not scaled so the rotation must preserve distances and must be a congruent transformation. Since the axis of rotation passes through the origin the origin is unchanged by this rotation and so these rotations are rigid motions that fix the origin and have the form,

$$m(\vec{\boldsymbol{x}}) = A\vec{\boldsymbol{x}}$$

where A is an orthogonal matrix. Additionally, rotations do not change the orientation of an shape and so their matrices have determinant 1. □

The rotation matrices — orthogonal matrices with determinant 1 — form a subgroup of the group O_n of orthogonal matrices called the ***special orthogonal group*** *and denoted SO_n.*

Proposition 2.7.7. *Every member of the special orthogonal group $A \in SO_2$ is the matrix of a rotation.*

Proof. Let $A \in SO_2$. Then A is a 2×2 orthogonal matrix with determinant 1. Let $\vec{\boldsymbol{v}}_1$ be the first column of A which, since A is orthogonal, is a unit vector. Now assume that R is the matrix of a rotation whose first column is $\vec{\boldsymbol{v}}_1$ — which is possible because $\vec{\boldsymbol{v}}_1$ is a unit vector so R can be orthogonal. Then the matrix

$$B = R^{-1}A$$

fixes $\vec{\boldsymbol{e_1}}$ and also, as the composition of two orthogonal vectors, is orthogonal. Therefore the second column of B is a unit vector orthogonal to $\vec{\boldsymbol{e_1}}$ which could be $\vec{\boldsymbol{e_2}}$ or $-\vec{\boldsymbol{e_2}}$.
However, R is an orthogonal matrix with determinant 1 and so is a member of SO_2 which means that $R^{-1}A = B$ is also in SO_2.

This, in turn, means that B has determinant 1 which implies that the second column of B is not $-\vec{\boldsymbol{e_2}}$ and is, therefore, $\vec{\boldsymbol{e_2}}$. So, we have obtained the result that $B = I = R^{-1}A$ which implies that $R = A$. □

Rotating $\mathbb{R}^2$ about the origin

Rotating the 2-d plane about the origin means that the axis of rotation is just the origin (so the fixed vector is $\vec{\mathbf{0}}$).
For example:

(81) A rotation ρ_θ of the plane through an angle θ is a linear operator on $\mathbb{R}^2$ whose matrix with respect to the standard basis is

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

We can see that this is a rotation if we take $\vec{\boldsymbol{x}} = (x_1, x_2)^T \in \mathbb{R}^2$ and write it in polar coordinates,

$$\vec{\boldsymbol{x}} = (r, \alpha).$$

So, relating the polar and rectangular coordinates,

$$\vec{\boldsymbol{x}} = (r\cos\alpha, r\sin\alpha)^T.$$

When we left-multiply by R,

$$\begin{aligned} R\vec{\boldsymbol{x}} &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} r\cos\alpha \\ r\sin\alpha \end{bmatrix} \\ &= \begin{bmatrix} r\cos\alpha\cos\theta - r\sin\alpha\sin\theta \\ r\cos\alpha\sin\theta + r\sin\alpha\cos\theta \end{bmatrix} \\ &= \begin{bmatrix} r\cos(\alpha+\theta) \\ r\sin(\alpha+\theta) \end{bmatrix}. \end{aligned}$$

Note that R is orthogonal because

$$\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \cdot \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} = -\cos\theta\sin\theta + \sin\theta\cos\theta = 0$$

and $det\, R = 1$,

$$\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1.$$

Rotating $\mathbb{R}^3$ about the origin

Definition 151. Define ρ as a rotation in $\mathbb{R}^3$ around the origin if:

(i) ρ is a rigid motion (orientation-preserving Euclidean isometry) that fixes the origin;

(ii) ρ also fixes a nonzero vector $\vec{\boldsymbol{v}}$;

(iii) ρ operates as a rotation on the plane P orthogonal to $\vec{\boldsymbol{v}}$.

Note that this definition could be described as selecting a 2-dimensional subspace of $\mathbb{R}^3$ and performing a 2-dimensional rotation on it as if it were $\mathbb{R}^2$.

Condition (i) implies, by Corollary 2.7.2, that ρ is left multiplication by an orthogonal matrix. Condition (ii) states that ρ has an eigenvector $\vec{\boldsymbol{v}}$ with eigenvalue 1. Then, because ρ preserves angles, the plane P referenced in condition (iii) that is orthogonal to the eigenvector $\vec{\boldsymbol{v}}$, must map to a plane that is orthogonal to the map of $\vec{\boldsymbol{v}}$ in the image of ρ. But $\vec{\boldsymbol{v}}$ is fixed by ρ and is unchanged in the image. Also $\vec{\boldsymbol{v}}$ uniquely identifies a plane orthogonal to it. Therefore the plane P is unchanged in the image also. In other words, P is an invariant subspace. So, condition (iii) says that the restriction of ρ to this invariant subspace is a rotation.

For example:

(82) A rotation of $\mathbb{R}^3$ about the origin can be described by a pair $(\vec{\boldsymbol{v}}, \theta)$ consisting of a unit vector $\vec{\boldsymbol{v}}$, a vector of length 1, which lies in the axis of rotation, and a nonzero angle θ, the angle of rotation. The two pairs $(\vec{\boldsymbol{v}}, \theta)$ and $(-\vec{\boldsymbol{v}}, -\theta)$ represent the same rotation. We also consider the identity map to be a rotation, though its axis is indeterminate.
The matrix representing a rotation through the angle θ about the vector $\vec{\boldsymbol{e_1}}$ is obtained easily from the 2×2 rotation matrix. It is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}.$$

Multiplication by A fixes the first coordinate x_1 of a vector and operates by rotation on $(x_2, x_3)^T$. All rotations of $\mathbb{R}^3$ are linear operators, but their matrices can be fairly complicated.

Proposition 2.7.8. *Every element of SO_3 has eigenvalue 1.*

Proof. Let $A \in SO_3$. Then A is an orthogonal 3×3 matrix with determinant equal to 1. Reasoning from orthogonality of A we have,

$$\begin{aligned} & A^T A = I \\ \iff \quad & A^T A - A^T = I - A^T \\ \iff \quad & A^T(A - I) = I - A^T \\ \iff \quad & A^T(A - I) = (I - A)^T. \qquad \text{by Proposition 2.3.4} \end{aligned}$$

If we take the determinants of both sides of this equation we obtain,

$$\begin{aligned} & det(A^T) \cdot det(A - I) = det((I - A)^T) \qquad \text{by Proposition 2.3.30} \\ \iff \quad & det(A) \cdot det(A - I) = det(I - A) \qquad \text{by Proposition 2.3.32} \\ \iff \quad & det(A - I) = det(I - A). \qquad det(A) = 1 \end{aligned}$$

But the dimension of A being 3 implies that

$$det(-A) = (-1)^3 det(A) = -det(A)$$

so that,

$$det(A - I) = det(I - A) \iff det(A - I) = 0.$$

Therefore A has the eigenvalue 1. □

Proposition 2.7.9. *The elements of SO_3 are precisely the rotations about the origin of $\mathbb{R}^3$.*

Proof. Let $\rho : \mathbb{R}^3 \longmapsto \mathbb{R}^3$ be defined as $\rho(\vec{\boldsymbol{x}}) = A\vec{\boldsymbol{x}}$ where $A \in SO_3$. Then,

- by Proposition 2.7.5 and orthogonality of $A \in SO_3$, left multiplication by A is a rigid motion that fixes the origin. So ρ is isometric and fixes the origin;

- Proposition 2.7.8 shows that every $A \in SO_3$ has eigenvalue 1 which implies that ρ fixes a nonzero vector;
- if we let $\vec{\boldsymbol{v}}$ be the nonzero vector fixed by ρ (i.e. its eigenvector with eigenvalue 1), then we can normalize it to find the unit vector parallel to $\vec{\boldsymbol{v}}$, say $\vec{\boldsymbol{u}}_1$. Next we can find two unit vectors orthogonal to $\vec{\boldsymbol{u}}_1$ — say $\vec{\boldsymbol{u}}_2$ and $\vec{\boldsymbol{u}}_3$ — and these must be a basis for the plane orthogonal to $\vec{\boldsymbol{v}}$. Furthermore, if we select $\vec{\boldsymbol{u}}_2$ and $\vec{\boldsymbol{u}}_3$ to be orthogonal to each other than $B = \{\vec{\boldsymbol{u}}_1, \vec{\boldsymbol{u}}_2, \vec{\boldsymbol{u}}_3\}$ is an orthonormal basis of $\mathbb{R}^3$.
  Now if we define $P = [B]^{-1}$ then,

$$A' = P^{-1}AP$$

  is similar to the matrix A and so has the same determinant, 1. Furthermore, because B is an orthonormal basis, the matrices $[B]$, $[B]^{-1} = P$ are orthogonal. Since both P and A are orthogonal, $P^{-1}AP = A'$ is orthogonal also. Since A' is orthogonal and has determinant equal to 1, it is a member of SO_3.
  If we examine the structure of A', we see that the first column of A' is $\vec{\boldsymbol{v}}_1$ — the unit vector in the direction of $\vec{\boldsymbol{v}}$. Since $\vec{\boldsymbol{v}}$ is an eigenvector of ρ with eigenvalue 1, the first column of A' is $\vec{\boldsymbol{e_1}}$ and since A' is orthogonal, the other columns are orthogonal to the first. So the block structure of A' looks like,

$$A' = \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix}$$

  where R is a 2×2 matrix.
  We know that the determinant of A' is 1 and this implies that the determinant of R is also 1. Furthermore, R must also be orthogonal and so $R \in SO_2$. So, by Proposition 2.7.7, R is a rotation. Therefore R represents a rotation of the plane orthogonal to $\vec{\boldsymbol{v}}$ and this implies that ρ rotates the plane orthogonal to $\vec{\boldsymbol{v}}$ as required.

□

Uniform Scaling

Uniform scaling is a scalar multiple of an orthogonal matrix and is therefore a linear operator which means that it fixes the origin. It also preserves angles but not distances between points.

2.7.0.6 Conformal Transformations

Non-uniform Scaling

Non-uniform scaling, however, its matrix is not a scalar multiple of an orthogonal matrix and, as such, it does not preserve angles. An important example is:

(83) **Mercator projection**:

This is a map projection that was designed so that rhumb lines (lines of constant bearing over the surface of the earth) are straight lines on the map. To achieve this the projection ensures that a square on the surface of the earth presents as a square on the map.
Then, modelling the earth as a sphere of radius R, if lines of latitude are horizontal grid lines across the map, then each actual line of latitude with circumference $2\pi R\cos\phi$ where ϕ is the angle of latitude will present on the map as the same length as the equator, which in reality is $2\pi R$. So they appear to be a line of length $\cos\phi$ times longer than they actually are, i.e. they are stretched by $\sec\phi$.
So, the map projection will stretch the width of a square on the surface of the earth by $\sec\phi$ and to maintain it as a square, it is necessary to stretch the height of the square by the same amount, $\sec\phi$. The actual square on the surface of the earth has height (approximately for a small square) $R\Delta\phi$ so, on the map we need a height $\Delta y \propto \Delta\phi\sec\phi$. Therefore, we have,

$$\frac{dy}{d\phi} = \sec\phi \implies y = \ln(\tan\phi + \sec\phi) + c$$

where c is a constant that we can set to 0. See https://www.math.ubc.ca/~israel/m103/mercator/mercator.html for a description of this derivation. For more information on conformal map projections generally see: Map Projection, York University, Toronto and Conformal Cartographic Representations - University of Barcelona.

Reflection

Reflection usually refers to a Euclidean Isometry (rigid transformation over a Euclidean space) that fixes a hyperplane (so a line in $\mathbb{R}^2$ and a plane in $\mathbb{R}^3$) but does not preserve orientation so is not a rigid motion. However, it may

also refer to a transformation that fixes an affine space of lower dimension than a hyperplane — for example, reflection in a point — in which case it does preserve orientation and is, therefore, a rigid motion (in fact, reflection in the origin in $\mathbb{R}^2$ is equal to rotation by π). Reflection may fix the origin or may not, depending on whether or not the origin is contained in the affine space fixed by the reflection.

2.7.0.7 Non-Rigid non-Conformal Transformations

Shear

Shear neither preserves distances nor angles. It does preserve parallelism though (as do all affine transformations) and it also fixes the origin, so it is a linear operator.

For more in-depth treatment of affine spaces and transformations see:

- *First two lectures of University of Texas - Multivariable Analysis.*
- *https://www.maa.org/sites/default/files/pdf/pubs/books/meg/meg_ch12.pdf*

Chapter 3

Analysis

3.1 Real Analysis

3.1.1 Sets of Real Numbers

3.1.1.1 Supremum and Infimum

Definition 152. **(Bounded Set)** An *upper bound* on a set A is a value x such that,

$$\forall a \in A,\, a \leq x$$

and a *lower bound* is similarly defined as a value y such that,

$$\forall a \in A,\, a \geq y.$$

A set is said to be *upper-bounded* if there exists some upper-bound on the set and is said to be *lower-bounded* if there exists some lower bound on the set. If there exists both upper and lower bounds then the set is said to be *bounded*.

Axiom 3.1.1. ***(Continuum Property)*** *Every non-empty set of real numbers that is bounded above has a least upper bound and every non-empty set of real numbers that is bounded below has a greatest upper bound.*

Definition 153. **(Supremum)** The *supremum* of an upper-bounded set A is a value σ_A such that σ_A is an upper bound on A and,

$$\sigma'_A < \sigma_A \iff \exists a \in A \text{ s.t. } a > \sigma'_A$$

which is to say that if $\sigma'_A < \sigma_A$ then σ'_A is not an upper bound on A and, if σ'_A is not an upper bound on A then it must be less than σ_A since σ_A is an upper bound on A.

An alternative, equivalent definition is: For σ_A, an upper bound on A,

$$\forall \epsilon > 0,\, \exists a \in A \text{ s.t. } a > \sigma_A - \epsilon.$$

Rearranging the alternative definition we obtain,

$$\begin{aligned} & \forall \epsilon > 0 \,.\, \exists a \in A \text{ s.t. } a + \epsilon > \sigma_A \\ \iff \quad & \forall \epsilon > 0 \,.\, \exists a \in A \text{ s.t. } \epsilon > \sigma_A - a \end{aligned}$$

which shows us that for any positive epsilon there needs to be an a close enough to the value of σ_A that the difference in their values is less than epsilon. Since a can approach arbitrarily close to σ_A this is achievable for any positive epsilon. This property seems to be equivalent to the fact that σ_A is a limit point of A (see: Topology).

Definition 154. The **infimum** of a lower-bounded set A is defined similarly to the supremum: as a value τ_A such that τ_A is a lower bound on A and,

$$\tau'_A > \tau_A \iff \exists a \in A \text{ s.t. } a < \tau'_A$$

or alternatively,

$$\forall \epsilon > 0, \exists a \in A \text{ s.t. } a < \tau_A + \epsilon.$$

Notation. The supremum of A is denoted $\sup A$ and the infimum is denoted $\inf A$.

Proposition 3.1.1. *If a bounded set $A \subset \mathbb{R}$ has the property that,*

$$\forall x, y \in A \,.\, |x - y| < 1$$

then it follows that,

$$(\sup A - \inf A) \leq 1.$$

Proof.

Let $\sigma_A = \sup A$ and $\tau_A = \inf A$. Then,

$$\forall \epsilon_1 > 0 \,.\, \exists x \in A \text{ s.t. } x - \epsilon_1 < \tau_A \quad \text{and} \quad \forall \epsilon_2 > 0 \,.\, \exists y \in A \text{ s.t. } y + \epsilon_2 > \sigma_A.$$

If we let

$$0 < \epsilon < \min\{\epsilon_1, \epsilon_2\}$$

then $\forall \epsilon_1, \epsilon_2 > 0$. $\exists x, y \in A$ such that

$$(x - \epsilon < \tau_A) \ \wedge (y + \epsilon > \sigma_A)$$

and so

$$\sigma_A - \tau_A < (y + \epsilon) - (x - \epsilon) = (y - x) + 2\epsilon. \qquad (*)$$

If we then, further constrict the value of ϵ so that

$$0 < \epsilon < \min \left\{ \epsilon_1, \epsilon_2, \frac{\sigma_A - \tau_A}{2} \right\}$$

then equation (*) becomes

$$\sigma_A - \tau_A < (y - x) + 2\epsilon < (y - x) + (\sigma_A - \tau_A)$$

so that $y - x > 0$.

Now suppose, for contradiction, that $(\sigma_A - \tau_A) > 1$. Then we can say that,

$$\exists r > 0 \ . \ (\sigma_A - \tau_A) = 1 + r.$$

If we then, once again, further constrict ϵ such that,

$$0 < \epsilon < \min \left\{ \epsilon_1, \epsilon_2, \frac{\sigma_A - \tau_A}{2}, \frac{r}{2} \right\}$$

then equation (*) further becomes,

$$\begin{aligned} & \sigma_A - \tau_A < (y - x) + 2\epsilon \\ \iff \quad & \sigma_A - \tau_A < (y - x) + r \\ \iff \quad & 1 + r < (y - x) + r \\ \iff \quad & 1 < y - x. \end{aligned}$$

Since $y - x > 0$ we, therefore, have

$$|y - x| > 1$$

which contradicts the set property that $\forall x, y \in A$, $|x - y| < 1$. So this shows that $(\sigma_A - \tau_A) \leq 1$. $\square$

Proposition 3.1.2. *Let $A \subset \mathbb{R}$ be a bounded set and let B be the set defined by*

$$B = \{\, b \mid b = f(a),\ a \in A \,\}$$

where the function f is some strictly monotonic function. Then it follows that,

$$\sup B = f(\sup A).$$

Proof. A is bounded and so $\sigma_A = \sup A$ exists. So, using the supremum properties we have,

$$\begin{aligned}
& \forall a \in A \,.\, a \leq \sigma_A \\
\iff \quad & \forall a \in A \,.\, f(a) \leq f(\sigma_A) && \text{by monotonicity of f} \\
\iff \quad & \forall b \in B \,.\, b \leq f(\sigma_A)
\end{aligned}$$

which is to say that $\sigma_B = f(\sigma_A)$ is an upper bound on B.
Furthermore, using the other supremum property, we have that,

$$\begin{aligned}
& \sigma'_A < \sigma_A \implies \exists a \in A \text{ s.t. } a > \sigma'_A \\
\iff \quad & f(\sigma'_A) < f(\sigma_A) \implies \exists a \in A \text{ s.t. } f(a) > f(\sigma'_A) && \text{by strict monotonicity of f} \\
\iff \quad & \sigma'_B < \sigma_B \implies \exists b \in B \text{ s.t. } b > \sigma'_B.
\end{aligned}$$

Therefore σ_B satisfies both requirements of the supremum and we have shown that,

$$\sup B = f(\sup A).$$

□

3.1.2 Sequences

Definition 155. **(Sequence)** A (real-valued) *sequence* is a function from the naturals to the reals,

$$f : \mathbb{N} \longmapsto \mathbb{R}.$$

The value $f(n)$ is called the n^{th} *term* of the sequence.

A sequence may be defined either:

- Explicitly, by defining the function f as an expression of the term number n,

  $$f(n) = h(n)$$

  where h is some real-valued function of arity 1.

- Inductively, by specifying one or more initial terms and a recurrence that specifies term n as a function of one or more previous terms,

  $$f(1) = \alpha_1, \ldots, f(k) = \alpha_k,$$

  $$f(n) = h(f(n-1), \ldots, f(n-k))$$

  where $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$ and h is some real-valued function of arity k.

Notation. A sequence may be denoted $(f(n))_{n=1}^{\infty}$ or $(f(n))$, where f is the function that defines the sequence or, a more informal notation is common where $(a_n)_{n=1}^{\infty}$ or (a_n) (or sometimes $\{a_n\}$) is used to denote the sequence with a_n referring to the n-th term.

Definition 156. **(Bounded Sequence)** If a_n is a sequence and $S = \{\, a_n \mid n \in \mathbb{N} \,\}$ then a_n is said to be **bounded below** if S has a lower bound and **bounded above** if S has an upper bound, and **bounded** if it is bounded above and below.

Definition 157. An **increasing** sequence is a sequence a_n such that,

$$\forall n \in \mathbb{N} \; . \; a_{n+1} \geq a_n$$

and **decreasing** if,

$$\forall n \in \mathbb{N} \; . \; a_{n+1} \leq a_n$$

and **monotonic** if either increasing or decreasing.

3.1.2.1 Limits of Sequences

Definition 158. **(Sequence Convergence)** A sequence a_n is said to **tend** to L or have the **limit** L iff,

$$\forall \epsilon > 0 \in \mathbb{R}, \exists N \in \mathbb{N} \text{ s.t. } \forall n > N, |a_n - L| < \epsilon.$$

Definition 159. The interval $(L - \epsilon, L + \epsilon)$ is called the **ϵ-neighbourhood of L**.

Definition 160. **(Sequence Divergence)** A sequence a_n is said to **tend to infinity** iff,

$$\forall M > 0 \in \mathbb{R}, \exists N \in \mathbb{N} \text{ s.t. } \forall n > N, a_n > M$$

and **tend to minus-infinity** iff,

$$\forall M < 0 \in \mathbb{R}, \exists N \in \mathbb{N} \text{ s.t. } \forall n > N, a_n < M.$$

Definition 161. A sequence that has a limit is called **convergent** and otherwise is called **divergent**. Note that **divergent** sequences include both sequences that remain bounded but oscillate without converging and those that tend to infinity (or minus-infinity).

A mnemonic for the taxonomy of convergence/divergence behaviours.

convergent	*divergent*		
bounded	*bounded*	*unbounded*	
	oscillatory	*oscillatory*	*non-oscillatory*
		unbounded oscillatory	$\to \infty$ *or* $\to -\infty$

Note that if we were to use a less formal description of a limit, for example, if we say that a sequence tends to some value L when the terms of the sequence gets closer and closer to L, then we encounter the following problems:

- *that the sequence gets closer and closer to many numbers so that this does not specify a single specific limit.*
- *that the sequence can have a limit but it's not the case that every term is closer than the previous term to the limit. For example,*

$$a_{2k} = 1/k,\ a_{2k-1} = \frac{1}{k+1}$$

tends to 0 but $a_{2k} > a_{2k-1}$.

Proposition 3.1.3. *A sequence has at most one limit. In other words, a sequence can only converge, if at all, to a single unique value.*

Proof. Suppose L_1 and L_2 are both limits of the sequence a_n and that

$$L_1 \neq L_2 \implies |L_2 - L_1| = d > 0.$$

By the definition of the limit of a sequence 3.1.2.1,

$$\forall \epsilon_1 > 0 \in \mathbb{R} \,.\, \exists N_1 \in \mathbb{N} \,.\, \forall n > N_1 \,.\, |a_n - L_1| < \epsilon_1,$$

$$\forall \epsilon_2 > 0 \in \mathbb{R} \,.\, \exists N_2 \in \mathbb{N} \,.\, \forall n > N_2 \,.\, |a_n - L_2| < \epsilon_2.$$

Let $\epsilon_1 = \epsilon_2 = \epsilon$ so that, for all ϵ,

$$\exists N = \max\{N_1, N_2\} \,.\, \forall n > N \,.\, |a_n - L_1| < \epsilon \ \wedge\ |a_n - L_2| < \epsilon.$$

Then,

$$|L_2 - L_1| = |(L_2 - a_n) + (a_n - L_1)| \leq |L_2 - a_n| + |L_1 - a_n| < 2\epsilon.$$

So, if we restrict the value of ϵ so that

$$0 < \epsilon < \frac{d}{2}$$

then

$$|L_2 - L_1| < 2\epsilon < d.$$

But this is a contradiction of the hypothesis that $|L_2 - L_1| = d$ and we can therefore deduce that $L_1 = L_2$. □

Lemma 3.1.1. *Any finite set of elements from an ordered field has a minimum and a maximum.*

Proof. This can be proven quite easily using induction. Taking the base case of a set of cardinality one, clearly there is a maximum and a minimum both of which are the sole element of the set. Then, the induction step is to say, given a set S that has a maximum, s_{max}, and a minimum, s_{min}, if we add a new element e, then if e is greater than s_{max} it is the maximum of the new set and if it is less than s_{min} it is the minimum of the new set. Otherwise, the previous maximum and minimum also pertain to the new set. Therefore, adding a new element to a set that has a maximum and a minimum creates a new set with a maximum and a minimum.

This is in fact the Well-Ordering Principle (see: 1.1.1.4).

□

Proposition 3.1.4. *Any convergent sequence is bounded.*

Proof. Let a_n be an arbitrary convergent sequence so that,

$$\forall \epsilon > 0 \in \mathbb{R} \,.\, \exists N \in \mathbb{N} \,.\, \forall n > N \,.\, |a_n - L| < \epsilon.$$

Now choose some value of ϵ and then there exists some $N \in \mathbb{N}$ such that $\forall n > N$,

$$\begin{aligned} & |a_n - L| < \epsilon \\ \iff \quad & -\epsilon < a_n - L < \epsilon \\ \iff \quad & L - \epsilon < a_n < L + \epsilon. \end{aligned}$$

That's to say, the sequence for values of $n > N$ is bound within the ϵ-neighbourhood of L.

Meanwhile the sequence for values of $n \leq N$ is a finite set of values in $\mathbb{R}$, an ordered field. So, by the *Well-Ordering Principle* (see: 1.1.1.4), it has a minimum and a maximum, which we denote m_1 and M_1 respectively,

$$m_2 = \min\{\, a_n \mid n \leq N \,\} \quad \text{and} \quad M_1 = \max\{\, a_n \mid n \leq N \,\}.$$

On the other hand, for the values of the sequence for $n > N$,

$$m_2 = \min\{\, a_n \mid n > N \,\} = L - \epsilon \quad \text{and} \quad M_2 = \max\{\, a_n \mid n > N \,\} = L + \epsilon.$$

So clearly then, all values in the whole sequence are bounded below by the minimum of these two minima,

$$S_{min} = \min\{\, a_n \mid n \in N \,\} = \min\{m_1, m_2\},$$

and above by the maximum of these maxima,

$$S_{max} = \max\{\, a_n \mid n \in N \,\} = \max\{M_1, M_2\}.$$

Therefore, the sequence a_n is bounded. □

Proposition 3.1.5. *Any increasing sequence that is bounded above has a limit.*

Proof. Let a_n be an increasing sequence that is bounded above. Then,

$$\forall n \in \mathbb{N} \,.\, a_{n+1} \geq a_n$$

and let $S = \{\, a_n \mid n \in \mathbb{N} \,\}$. Since a_n is bounded above, by the Continuum Property (Axiom 3.1.1.1) it has a supremum. Let $\sigma = \sup S$ be the supremum so that,

$$\forall a_n \in S \,.\, a_n \leq \sigma \quad \text{and} \quad \forall \epsilon > 0 \in \mathbb{R} \,.\, \exists a_n \in S \,.\, a_n > \sigma - \epsilon.$$

So if we choose any fixed value of ϵ then there exists some $a_n \in S$ such that

$$a_n > \sigma - \epsilon \iff a_n - \sigma > -\epsilon \iff \sigma - a_n < \epsilon.$$

Furthermore, since σ is an upper bound on the values a_n, the expression $\sigma - a_n$ is positive and so this last result implies

$$\sigma - a_n = |a_n - \sigma| < \epsilon.$$

Since the sequence is increasing and $a_{n+1} \geq a_n$,

$$-a_{n+1} \leq -a_n \iff \sigma - a_{n+1} \leq \sigma - a_n < \epsilon \implies \sigma - a_{n+1} < \epsilon.$$

Again, because σ is an upper bound on the values of the sequence we have,

$$\sigma - a_{n+1} = |a_{n+1} - \sigma| < \epsilon.$$

Therefore, we have shown that,

$$|a_n - \sigma| < \epsilon \implies |a_{n+1} - \sigma| < \epsilon$$

and so we can reason inductively that, if we let $N = n$, then

$$\forall n > N \; . \; |a_n - \sigma| < \epsilon.$$

This result was shown for an arbitrary, strictly positive, value of ϵ so we therefore have,

$$\forall \epsilon > 0 \in \mathbb{R} \; . \; \exists N \in \mathbb{N} \; . \; \forall n > N \; . \; |a_n - \sigma| < \epsilon$$

which, by the definition of the limit of a sequence (3.1.2.1), is to say that the sequence a_n converges to the limit σ. □

Corollary 3.1.1. *Any increasing sequence that is bounded above converges to the supremum of its elements (terms, values, etc.).*

Corollary 3.1.2. *A decreasing sequence that is bounded below converges to the infimum of its elements.*

Proposition 3.1.6. *If a sequence a_n converges to L and $a_n \geq M$ for all $n \in \mathbb{N}$ then,*

$$L \geq M.$$

Proof. By the definition of the limit of a sequence (3.1.2.1),

$$\forall \epsilon > 0 \in \mathbb{R} \,.\, \exists N \,.\, \forall n > N \in \mathbb{N} \,.\, |a_n - L| < \epsilon.$$

Suppose, for contradiction, that $L < M$. Then there exists $\epsilon = M - L > 0 \in \mathbb{R}$ and since $a_n \geq M$, we have for all n,

$$a_n \geq M \iff a_n - L \geq M - L = \epsilon.$$

But the definition of the limit of a_n implies that

$$-\epsilon < a_n - L < \epsilon$$

and so, $L < M$ and $a_n \geq M$ together, contradict the convergence of a_n to L. It follows that $L \geq M$. □

Proposition 3.1.7. *If a sequence (a_n) converges to L and a sequence (b_n) converges to M then,*

$$\forall n \in \mathbb{N} \,.\, a_n \leq b_n \implies L \leq M.$$

Proof. Assume for contradiction that $L > M$. Then there exists some $\epsilon = \frac{L-M}{4} > 0 \in \mathbb{R}$ so that,

$$\exists N_1 \,.\, \forall n > N_1 \in \mathbb{N} \,.\, |a_n - L| < \epsilon = \frac{L-M}{4}$$

and also

$$\exists N_2 \,.\, \forall n > N_2 \in \mathbb{N} \,.\, |b_n - M| < \epsilon = \frac{L-M}{4}$$

so we can take $N = \max\{N_1, N_2\}$ and then, for all $n > N \in \mathbb{N}$,

$$|(a_n - b_n) + (M - L)| = |(a_n - L) + (M - b_n)| \leq |a_n - L| + |b_n - M| < 2\epsilon.$$

It follows then that,

$$\begin{aligned} & |(a_n - b_n) + (M - L)| < \frac{L-M}{2} \\ \iff \quad & \frac{M-L}{2} < (a_n - b_n) + (M - L) < \frac{L-M}{2} \\ \iff \quad & \frac{1}{2}(L-M) < a_n - b_n < \frac{3}{2}(L-M). \end{aligned}$$

But the assumption $L > M$ implies that $L - M > 0$ so the above result implies that $a_n - b_n > 0$ also. This contradicts the construction of the sequences in which $a_n \leq b_n$ for all n. $\square$

Proposition 3.1.8. *Let a_n and b_n be convergent sequences with limits a and b, respectively. Let C be a real number and let k be a positive integer. Then as $n \to \infty$,*

a) $Ca_n \to Ca$

b) $|a_n| \to |a|$

c) $a_n + b_n \to a + b$

d) $a_n b_n \to ab$

e) ${a_n}^k \to a^k$

f) if, for all n, $b_n \neq 0$ and $b \neq 0$, then $\frac{1}{b_n} \to \frac{1}{b}$.

Proof. We prove each property individually in the given order.

Proof of (a) $Ca_n \to Ca$

If $C = 0$ then $Ca_n = 0 = Ca$ for all n and the proposition holds trivially. If $C \neq 0$ then, since $a_n \to a$,

$$\forall \epsilon' > 0 \,.\, \exists N \,.\, \forall n > N \in \mathbb{N} \,.\, |a_n - a| < \epsilon'.$$

Now let $\epsilon = |C|\,\epsilon'$. Then,

$$\begin{aligned} & \forall \epsilon' > 0 \,.\, \exists N \,.\, \forall n > N \in \mathbb{N} \,.\, |C|\,|a_n - a| < |C|\,\epsilon' = \epsilon \\ \iff \quad & \forall \epsilon > 0 \,.\, \exists N \,.\, \forall n > N \in \mathbb{N} \,.\, |Ca_n - Ca| < \epsilon. \end{aligned} \qquad |x|\,|y| = |xy| \text{ 1.2.16}$$

Proof of (b) $|a_n| \to |a|$

$$\begin{aligned} & |a_n| = |a_n - a + a| \leq |a_n - a| + |a| && \text{by triangle inequality 1.2.3.1} \\ \iff & |a_n| - |a| \leq |a_n - a| \,. && (1) \end{aligned}$$

$$\begin{aligned} & |a| = |a - a_n + a_n| \leq |a - a_n| + |a_n| && \text{by triangle inequality 1.2.3.1} \\ \iff & |a| - |a_n| \leq |a - a_n| = |a_n - a| \\ \iff & |a_n| - |a| \geq -|a_n - a| \,. && (2) \end{aligned}$$

Putting (1) and (2) together we have,

$$\begin{aligned} & -|a_n - a| \leq |a_n| - |a| \leq |a_n - a| && \text{by triangle inequality 1.2.3.1} \\ \iff & ||a_n| - |a|| \leq |a_n - a| \,. \end{aligned}$$

The fact that a_n converges to a implies that $|a_n - a|$ converges to zero. Since it is an upper bound on the value of $||a_n| - |a||$, the value $||a_n| - |a||$ must also converge to zero. Specifically any value of n, ϵ such that $|a_n - a| < \epsilon$ will also satisfy $||a_n| - |a|| \leq |a_n - a| < \epsilon$.

Proof of (c) $a_n + b_n \to a + b$

Using, again, the "triangle inequality" 1.2.3.1,

$$|(a_n + b_n) - (a + b)| = |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b| \,.$$

So, $|a_n - a| + |b_n - b|$ is an upper bound on the value of $|(a_n + b_n) - (a + b)|$. If we take any arbitrary $\epsilon > 0$ then,

$$\exists N_1 \,.\, \forall n > N_1 \in \mathbb{N} \,.\; |a_n - a| < \frac{\epsilon}{2}$$

and

$$\exists N_2 \,.\, \forall n > N_2 \in \mathbb{N} \,.\; |b_n - b| < \frac{\epsilon}{2}.$$

Then, if we take $N = \max\{N_1, N_2\}$, we have,

$$\forall n > N \in \mathbb{N} \,.\; |(a_n + b_n) - (a + b)| \leq |a_n - a| + |b_n - b| < \epsilon.$$

Proof of (d) $a_n b_n \to ab$

Again using the triangle inequality 1.2.3.1,

$$\begin{aligned} & |a_n b_n - ab| = |a_n b_n - ab_n + ab_n - ab| \leq |b_n(a_n - a)| + |a(b_n - b)| \\ \iff & |a_n b_n - ab| \leq |b_n|\,|a_n - a| + |a|\,|b_n - b|\,. \end{aligned}$$

Since b_n converges, by Proposition 3.1.4, it is bounded. Therefore, $|b_n|$ has some upper bound which we shall call B_U. Then, if we take some arbitrary value of $\epsilon > 0$,

$$\exists N_1 \in \mathbb{N}\ .\ \forall n > N_1\ .\ |a_n - a| < \frac{\epsilon}{2B_U}$$

and

$$\exists N_2 \in \mathbb{N}\ .\ \forall n > N_2\ .\ |b_n - b| < \frac{\epsilon}{2\,|a|}.$$

Now if we let $N = \max\{N_1, N_2\}$ then

$$\begin{aligned} & \forall n > N\ .\ B_U\,|a_n - a| + |a|\,|b_n - b| < \epsilon \\ \iff & \forall n > N\ .\ |a_n b_n - ab| < \epsilon. \end{aligned}$$

Proof of (e) ${a_n}^k \to a^k$

Using (d) - and because k is a positive integer - we can do induction on the power k.

Base cases 0 and 1 are clearly true as $k = 0$ results in a_n being the constant 1 for all n and so trivially converges to $a = 1$; and $k = 1$ results in the same sequence as a_n.

So, we perform the induction step for $k \geq 2$. Then, by the induction hypothesis, ${a_n}^{k-1} \to a^{k-1}$. But ${a_n}^k = {a_n}^{k-1} a_n$ and, by (d) and the induction hypothesis, we have that ${a_n}^{k-1} a_n \to a^{k-1} a = a^k$. Therefore, ${a_n}^k \to a^k$.

Proof of (f) $\forall n\ .\ b_n, b \neq 0 \implies \frac{1}{b_n} \to \frac{1}{b}$

Again invoking Proposition 3.1.4 and letting the lower bound on the sequence b_n be B_L,

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \left|\frac{b - b_n}{b_n b}\right| = \left|\frac{b_n - b}{b_n b}\right| = \frac{|b_n - b|}{|b_n|\,|b|} \leq \frac{1}{B_L\,|b|}\,|b_n - b|\,.$$

Now, since $\frac{1}{B_L|b|}$ is a constant we can define the constant $C = \frac{1}{B_L|b|}$ and then we see that in (a) we have already proven that $C\,|b_n - b|$ converges to 0. In (a) we used that to prove that $Cb_n \to Cb$ but here it proves that $\frac{1}{b_n} \to \frac{1}{b}$. $\square$

Lemma 3.1.2. *For all $n \in \mathbb{N}$ and any fixed $h > 0 \in \mathbb{R}$,*

$$(1+h)^n \geq 1 + hn.$$

Proof. Clearly the hypothesis holds for $n = 0$

$$(1+h)^0 = 1 = 1 + 0,$$

and for $n = 1$

$$(1+h)^1 = 1 + h.$$

Assume the hypothesis is correct for some $n - 1 > 1$. Then,

$$\begin{aligned}
(1+h)^n &= (1+h)(1+h)^{n-1} \\
&\geq (1+h)(1+(n-1)h) \quad \text{by induction hypothesis} \\
&= 1 + nh + (n-1)h^2 \\
&> 1 + nh. \qquad \because (n-1), h > 0
\end{aligned}$$

□

Proposition 3.1.9. *If x_n is a sequence and $|x_n|$ tends to infinity as $n \to \infty$ then the sequence diverges.*

Proof. Assume for contradiction that x_n converges to some finite limit L. Then,

$$\forall \epsilon > 0 \in \mathbb{R} \,.\, \exists N_1 \,.\, \forall n > N_1 \,.\, |x_n - L| < \epsilon.$$

But, since $|x_n|$ tends to infinity as $n \to \infty$, we also have,

$$\forall M > 0 \in \mathbb{R} \,.\, \exists N_2 \,.\, \forall n > N_2 \, |x_n| > M.$$

Therefore, if we choose any such arbitrary value of ϵ and let $M = \epsilon + |L|$, then there exists some $N = \max\{N_1, N_2\}$ such that, for all $n > N$,

$$|x_n| > M \implies |x_n| > \epsilon + |L|$$

and also

$$|x_n| = |x_n - L + L| \leq |x_n - L| + |L| < \epsilon + |L| \,.$$

This is clearly a contradiction and so we conclude that x_n does not converge to a finite limit. □

Proposition 3.1.10. *If x_n is a sequence and $|x_n|$ tends to 0 as $n \to \infty$ then the sequence converges to 0. That's to say,*

$$\lim_{n\to\infty} |x_n| = 0 \implies \lim_{n\to\infty} x_n = 0.$$

Proof. This follows from the fact that $|x_n| = ||x_n|| = ||x_n| - 0|$ so that

$$|x_n| < \epsilon \iff ||x_n| - 0| < \epsilon.$$

□

*Note that this **only** applies when the limit is 0. If we had $\lim_{n\to\infty} |x_n| = 1$ for instance, then it could be that the sequence was divergently oscillating between 1 and -1.*

Proposition 3.1.11. *A sequence of the form,*

$$x_n = ax_{n-1}$$

with initial value x_0, can only diverge or converge to 0 or x_0.

Proof. Consider the sequence $|x_n|$,

$$|x_n| = |a^n x_0| = |a^n|\,|x_0| = |a|^n\,|x_0|\,.$$

Now we divide the possible values of $|a|$ into 3 cases:

(i) $|a| > 1$

Observe that

$$|a| > 1 \implies \exists h > 0 \in R \text{ s.t. } |a| = 1 + h.$$

By Lemma 3.1.2, for all $n \in \mathbb{N}$,

$$|a|^n = (1+h)^n \geq 1 + hn.$$

If we choose any arbitrarily large $M \in \mathbb{R}$ then, because

$$1 + hn > M \iff hn > M - 1 \iff n > \frac{M-1}{h},$$

there exists

$$N = \frac{M-1}{h} \in \mathbb{R}$$

such that

$$\forall n > N \in \mathbb{N} \,.\, 1 + hn > M \implies |a|^n > M.$$

Therefore $|a|^n$ tends to infinity and so also

$$|a|^n \, |x_0| = |x_n|$$

tends to infinity as $n \to \infty$. So, by Proposition 3.1.9, we can conclude that x_n diverges.

(ii) $|a| = 1$

In this case, either:

- $a = 1$ and for all $n \in \mathbb{N}$, $x_n = x_0$ so that the sequence converges to x_0; or else
- $a = -1$ and the sequence alternates between 1 and -1 and so diverges.

(iii) $0 < |a| < 1$

Here we have,

$$0 < |a| < 1 \implies \exists h > 0 \in R \text{ s.t. } |a| = \frac{1}{1+h}.$$

By Lemma 3.1.2, for all $n \in \mathbb{N}$,

$$(1+h)^n \geq 1 + hn \implies |a|^n < \frac{1}{1+hn}.$$

For any arbitrary $\epsilon > 0 \in \mathbb{R}$ then,

$$\frac{1}{1+hn} < \epsilon \iff 1 + hn > \frac{1}{\epsilon} \iff n > \frac{1-\epsilon}{h\epsilon}$$

and so, there exists

$$N = \frac{1-\epsilon}{h\epsilon}$$

such that,

$$\forall n > N \,.\, \frac{1}{1+hn} < \epsilon \implies |a|^n = \Big||a|^n - 0\Big| < \epsilon$$

which implies that $|a|^n$ converges to 0 as $n \to \infty$. Therefore, applying property (a) of Proposition 3.1.8,

$$\lim_{n\to\infty} |a|^n = 0 \implies \lim_{n\to\infty} |a|^n x_0 = x_0(\lim_{n\to\infty} |a|^n) = 0$$

so $|x_n|$ also converges to 0 as $n \to \infty$. It follows then, by Proposition 3.1.10, that x_n also converges to 0.

□

Theorem 3.1.1. ***(Sandwich Theorem)*** *Let a_n, b_n, c_n be sequences such that,*

$$\textit{for all } n,\ a_n \le b_n \le c_n \quad \textit{and} \quad \lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n.$$

Then $\lim_{n\to\infty} b_n = L$.

Proof. $\lim_{n\to\infty} a_n = L$ means that, for any $\epsilon > 0$

$$\begin{aligned} & \exists N_1 \,.\, \forall n > N_1 \in \mathbb{N} \,.\, |a_n - L| < \epsilon \\ \iff \quad & \exists N_1 \,.\, \forall n > N_1 \in \mathbb{N} \,.\, -\epsilon < a_n - L < \epsilon \\ \iff \quad & \exists N_1 \,.\, \forall n > N_1 \in \mathbb{N} \,.\, L - \epsilon < a_n < L + \epsilon. \end{aligned}$$

By the same reasoning we also have, for the same value of ϵ,

$$\exists N_2 \,.\, \forall n > N_2 \in \mathbb{N} \,.\, L - \epsilon < c_n < L + \epsilon.$$

So, if we let $N = \max\{N_1, N_2\}$ then we have,

$$\forall n > N \in \mathbb{N} \,.\, L - \epsilon < a_n, c_n < L + \epsilon$$

and since we also know that $a_n \le b_n \le c_n$ it follows that,

$$\forall n > N \in \mathbb{N} \,.\, L - \epsilon < a_n \le b_n \le c_n < L + \epsilon.$$

This shows that,

$$\forall \epsilon > 0 \,.\, \exists N \,.\, \forall n > N \in \mathbb{N} \,.\, |b_n - L| < \epsilon.$$

□

(84) **Non-convergent Oscillation**

The sequence $a_n = (-1)^n$ is divergent despite always remaining bounded within the interval $[-1, 1]$ as it neither converges to 1 or to -1.

To prove this formally we can assume for contradiction that a_n converges to some finite limit L and then take $0 < \epsilon < 1 \in \mathbb{R}$. Then, by the definition of the limit, for any such value of ϵ,

$$\exists N \,.\, \forall n > N \in \mathbb{N} \,.\, |a_n - L| < \epsilon.$$

But for all n,

$$|a_{n+1} - a_n| = 2$$

and so

$$2 = |a_{n+1} - a_n| = |a_{n+1} - L + L - a_n| \leq |a_{n+1} - L| + |a_n - L|$$

but also

$$|a_{n+1} - L| + |a_n - L| < 2\epsilon < 2.$$

So we have obtained a contradiction.

We could also have proved this by using the fact that a_n and a_{n+1} need to be converging to the same limit and then show that, in fact, they converge to two different limits: 1 for even n and -1 for odd n.

(85) **Limit of an Infinite Recurrence**

Take the (convergent) sequence given by,

$$a_1 = 1,\ a_{n+1} = \frac{a_n}{2} + \frac{3}{2a_n}\ \ (n \geq 1).$$

Assume there is an equilibrium value, a^*, then

$$a^* = \frac{a^*}{2} + \frac{3}{2a^*}$$

$$\iff \qquad 2a^{*2} = a^{*2} + 3$$

$$\iff \qquad a^{*2} = 3$$

$$\iff \qquad a^* = \sqrt{3}. \qquad \forall n, a_n \geq 0$$

So $\sqrt{3}$ is the steady-state value that this recurrence converges to as $n \to \infty$. If the recurrence didn't converge then the assumption of an equilibrium value would result in a contradiction. Note, however, that the fact that there is an equilibrium value does *not*, by itself, prove that this sequence converges (although this sequence does).
That's to say, given that this sequence is convergent, its limit will be the equilibrium value but, if it weren't convergent, then the equilibrium value would never be obtained.

(86) $\boldsymbol{|a| < 1 \implies \lim_{n\to\infty} a^n = 0.}$

First of all, note that if $|a| = 0$ then $a = 0 = a^n$ for all n and so the limit holds trivially. For this reason, from here on, we will consider only the case where $a \neq 0$.
There are 3 parts to this proof:

1. $|a| < 1 \implies \lim_{n\to\infty} |a|^n = 0$,
2. $|a|^n = |a^n|$,
3. $\lim_{n\to\infty} |a^n| = 0 \implies \lim_{n\to\infty} a^n = 0$.

1. $\boldsymbol{|a| < 1 \implies \lim_{n\to\infty} |a|^n = 0}$

It would be natural to prove this by showing that,

$$\forall \epsilon > 0 \,.\, \exists N \,.\, \forall n > N \in \mathbb{N} \,.\, \left||a|^n - 0\right| = |a|^n < \epsilon.$$

We can show this by "reverse engineering" the value of N from the requirement that $|a|^n$ be less than ϵ,

$$|a|^n < \epsilon \iff n \ln |a| < \ln \epsilon \iff n > \frac{\ln \epsilon}{\ln |a|}$$

with the last step changing the direction of the inequality because we divide by $\ln |a|$ which - remembering that $|a| < 1$ - is a negative value. So, in this way, we have shown that $N = \frac{\ln \epsilon}{\ln |a|}$

is a general formula that relates a value of N with the required property with any arbitrary ϵ.
However, this proof is not valid because it uses the concept of the logarithm which requires a lot of analysis that has not been proven at this stage. Since we are trying to build the fundamental basis of analysis, at this point we can only use concepts that are pre-requisites (axiomatic) in analysis or have been proven at this stage.

Without using logs we can still prove this in (at least) a few ways. One way is to use Proposition 3.1.11 with $x_0 = 1$.
Another way is to reason as follows: Let x_n be the sequence $x_n = |a|^n$. Then, because $0 < |a| < 1$,

$$x_{n+1} = x_n \cdot x = |a|^n |a| < |a|^n = x_n$$

so that x_n is a decreasing sequence. Additionally, $\forall n \in \mathbb{R} \; . \; |a|^n \geq 0$ so 0 is a lower bound on the sequence. Therefore, the sequence converges to a limit (note we haven't yet established that 0 is the limit - only that it is a candidate).

At this point, the logical thing to do is to try to show that 0 is the supremum of the sequence of values to apply Corollary 3.1.2 but this again leads to the problem of not being able to use logs.

Instead we use the recurrent nature of the sequence,

$$x_{n+1} = |a|^{n+1} = |a|^n |a| \, ,$$

and reason that, if $x_n \to L$, then $x_{n+1} \to L$ also. But, putting these two facts together, along with property (d) of limits of

sequences (Proposition 3.1.8), we have,

$$\begin{aligned} L &= \lim_{n\to\infty} |a|^{n+1} \\ &= \lim_{n\to\infty} |a|^n |a| \\ &= (\lim_{n\to\infty} |a|^n)(\lim_{n\to\infty} |a|) \quad \text{by prop. (d) of 3.1.8} \\ &= |a| (\lim_{n\to\infty} |a|^n) \\ &= |a| L. \end{aligned}$$

So,

$$L = |a| L \iff L(1 - |a|) = 0$$

and, since we know that $|a| \neq 1$, therefore L must be 0.

A third way is to use the Sandwich Theorem Theorem 3.1.1 by observing that,

$$0 < |a| < 1 \implies \exists h > 0 \in R \text{ s.t. } |a| = \frac{1}{1+h}.$$

By Lemma 3.1.2, for all $n \in \mathbb{N}$,

$$(1+h)^n \geq 1 + hn \implies |a|^n < \frac{1}{1+hn}.$$

Then we have the sequence

$$x^n = \frac{1}{(1+h)^n} \leq \frac{1}{1+hn}$$

such that,

$$0 < x^n \leq \frac{1}{1+hn}.$$

Since h is some fixed value, clearly,

$$\lim_{n\to\infty} \frac{1}{1+hn} = 0$$

and, obviously, the limit of the constant 0 is always 0 so, by the Sandwich Theorem,

$$\lim_{n\to\infty} x^n = 0.$$

2. $\boldsymbol{|a|^n = |a^n|}$

   The exponent here is an integer because it is the natural number n used as the sequence index. For integer exponents, the proposition $|a|^n = |a^n|$ can be shown by induction on Proposition 1.2.16. If we had to show this for real-valued exponents, an approach something like 2.2.1.1 would need to be used.

3. $\boldsymbol{\lim_{n\to\infty} |a^n| = 0 \implies \lim_{n\to\infty} a^n = 0}$

   This follows directly from Proposition 3.1.10.

3.1.2.2 Subsequences

Definition 162. **(Subsequence)** Let $(a_n)_{n\in\mathbb{N}}$ be a sequence and let f be a strictly increasing function $\mathbb{N} \longmapsto \mathbb{N}$. Then the sequence $(a_{f(n)})_{n\in\mathbb{N}}$ is called a *subsequence* of the sequence $(a_n)_{n\in\mathbb{N}}$.

Equivalently, if $A = \{\, f(n) \mid n \in \mathbb{N} \,\} \subseteq \mathbb{N}$ then the subsequence $(a_{f(n)})_{n\in\mathbb{N}}$ can also be referred to as $(a_n)_{n\in A}$.

Proposition 3.1.12. *If a_n is a sequence that tends to a limit, then any subsequence of it tends to the same limit.*

Proof. Firstly, notice that if the nth index of some subsequence is k_n then $k_n \geq n$ (because the subsequence can only skip terms of the original - it can't add in terms). So then, if we have a sequence a_n that tends to a limit a and an arbitrary subsequence a_{k_n} then,

$$\forall \epsilon > 0 \,.\, \exists N \,.\, \forall n > N \in \mathbb{N} \,.\, |a_n - a| < \epsilon \implies |a_{k_n} - a| < \epsilon$$

because $k_n \geq n > N$. □

Proposition 3.1.13. *Every sequence has a monotonic subsequence.*

Proof. Either there is an infinite number of terms that are greater than all the following terms or there is not. If there is not, then after the last such term all terms have a term that follows them that is greater than or equal to them.
In the first case we have a strict monotonic decreasing sequence and in the second we have a non-strict monotonic increasing sequence. □

It might be tempting to reason: Pick a term a_{k_1} and either it is a max. or a min. or neither of the remaining sequence $(a_n)_{n>k_1\in\mathbb{N}}$. If it is a max. then select as a_{k_2} the max. of the remaining terms and continue likewise to obtain a monotonically decreasing sequence. If it is a min. then do the same in reverse to obtain an increasing sequence. If it is neither then simply move k_1 to the max. of the remaining sequence and then proceed to obtain a

decreasing sequence.

But the sequence need not have a maximum or a minimum! A sequence may not be bounded above or below. For example:

$$a_n = (-1)^n \left\lfloor \frac{n+1}{2} \right\rfloor$$

has values: $0, -1, 1, -2, 2, -3, 3, \ldots$

Theorem 3.1.2. ***The Bolzano-Weierstrass Theorem***
Every bounded sequence has a convergent subsequence.

Proof. By the definitions of a bounded sequence and a subsequence (3.1.2, 3.1.2.2) we can see that, because a subsequence of a bounded sequence has terms that are a subset of the terms of the original sequence, the bounds of the original sequence must also be bounds of the subsequence. By Proposition 3.1.13, every sequence has a monotonic subsequence so a bounded sequence must therefore have a monotonic, bounded subsequence which, by Proposition 3.1.5, must therefore be convergent. □

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(87) **Let $(a_n)_{n\in\mathbb{N}}$ be a sequence, and let $(b_n)_{n\in\mathbb{N}}$ be the sequence defined by $b_n = |a_n|$ for $n \in \mathbb{N}$. Which of the following two statements implies the other?**

Answer: a_n converges $\implies$ b_n converges also but b_n converges $\not\implies$ a_n converges.
The first implication is because,

$$\begin{aligned} & |a_n| = |a_n - a + a| \leq |a_n - a| + |a| \\ \iff \quad & |a_n| - |a| \leq |a_n - a| \end{aligned}$$

$$|a| = |a - a_n + a_n| \leq |a_n - a| + |a_n|$$

$$\iff \qquad |a| - |a_n| \leq |a_n - a|$$
$$\iff \qquad |a_n| - |a| \geq -|a_n - a|$$

which both together imply that $||a_n| - |a|| \leq |a_n - a|$.
The latter non-implication is easy to see if one thinks of a sequence that consists of two subsequences that converge to 2 and -2. Then, their absolute value would converge to 2 but their values do not converge. Remember Proposition 3.1.12, for a sequence to converge to a limit, every subsequence of it must converge to the same limit.

(88) **What is the behaviour as $n \to \infty$ of the following:**

(i) $\frac{2n^3+1}{n+1}\left(\frac{3}{4}\right)^n$

As $n \to \infty$, clearly, 0 is a lower bound on the value. This means that the expression cannot be tending to a negative number so, when we are also able to upper bound it by an expression that tends to 0 as follows:

$$0 < \frac{2n^3+1}{n+1}\left(\frac{3}{4}\right)^n < \frac{3n^3}{n}\left(\frac{3}{4}\right)^n = 3n^2\left(\frac{3}{4}\right)^n \to 0,$$

then we can apply the Sandwich Theorem (Theorem 3.1.1) to conclude that the original expression tends to 0 also.

(ii) $\frac{2^{2n}+n}{n^3 3^n+1}$

$$\frac{2^{2n}+n}{n^3 3^n+1} = \frac{4^n+n}{n^3 3^n+1} > \frac{4^n}{2n^3 3^n} = \frac{(4/3)^n}{2n^3} \to \infty$$

(89) **Let (a_n) be a sequence of non-negative numbers. Prove that if $a_n \to L$ as $n \to \infty$ then $\sqrt{a_n} \to \sqrt{L}$ as $n \to \infty$.**

I can't see anything wrong with proving this proposition using property (e) of Proposition 3.1.8 as follows:

$$\begin{aligned}
& \lim_{n\to\infty}(\sqrt{a_n})^2 = \left[\lim_{n\to\infty}\sqrt{a_n}\right]^2 \\
\iff \quad & \lim_{n\to\infty} a_n = \left[\lim_{n\to\infty}\sqrt{a_n}\right]^2 \quad \because \forall n \,.\, a_n \geq 0 \\
\iff \quad & \sqrt{\lim_{n\to\infty} a_n} = \lim_{n\to\infty}\sqrt{a_n} \\
\iff \quad & \sqrt{L} = \lim_{n\to\infty}\sqrt{a_n}.
\end{aligned}$$

But, for some reason LSE Abstract Maths pages 169,175 insist that it shouldn't be proved in this manner and that instead we should prove it directly from the definition of the limit. Question: Is there anything wrong this proof?

Proof. We are told that $a_n \to L$ as $n \to \infty$ so we have,

$$\forall \epsilon' > 0 \,.\, \exists N \,.\, \forall n > N \in \mathbb{N} \,.\, |a_n - L| < \epsilon'.$$

We are also told that (a_n) is non-negative so $L \geq 0$.

There are two methods of proof that work for $L > 0$ but not for $L = 0$, so first we need to prove the $L = 0$ case. If $L = 0$ then $|a_n| < \epsilon'$ for $n > N$. But,

$$|a_n| = (\sqrt{a_n})^2 < \epsilon' \iff \sqrt{a_n} < (\epsilon')^2$$

so that taking $\epsilon = (\epsilon')^2$ we obtain,

$$\forall \epsilon > 0 \,.\, \exists N \,.\, \forall n > N \in \mathbb{N} \,.\, \left|\sqrt{a_n}\right| < \epsilon.$$

The remaining possibility is that $L > 0$. Notice that the expression we're looking to bound can be rewritten as follows:

$$\left|\sqrt{a_n} - \sqrt{L}\right| = \left|(\sqrt{a_n} - \sqrt{L})\frac{\sqrt{a_n} + \sqrt{L}}{\sqrt{a_n} + \sqrt{L}}\right| = \left|\frac{a_n - L}{\sqrt{a_n} + \sqrt{L}}\right|.$$

Furthermore, since $a_n \to L$ and $L > 0$, clearly $0 < L/2 < L$ and there will be some $\epsilon < L/2$ such that,

$$L - L/2 < L - \epsilon < a_n < L + \epsilon < L + L/2 \iff |a_n - L| < \epsilon < L/2.$$

This means that, inside this ϵ-neighbourhood of L, we can find a constant lower bound on $\sqrt{a_n} + \sqrt{L}$ as,

$$\sqrt{a_n} + \sqrt{L} > \sqrt{L/2} + \sqrt{L} = C$$

for constant C. Now we have

$$\left|\sqrt{a_n} - \sqrt{L}\right| = \left|\frac{a_n - L}{\sqrt{a_n} + \sqrt{L}}\right| < \frac{|a_n - L|}{C} < \frac{\epsilon'}{C}$$

Note that this is where this proof fails for $L = 0$ because that would result in $C = 0$.

so we can take $\epsilon = \epsilon'/C$ to obtain $|a_n - L| < \epsilon$ as required. □

Proof. This is an alternative proof of the $L > 0$ case using the fact that

$$\sqrt{x + y} \leq \sqrt{x} + \sqrt{y}.$$

Choose ϵ such that $0 < \epsilon \leq \sqrt{L}$ and take N so that, for $n > N$, $|a_n - L| < \epsilon^2$, or in other words $L - \epsilon^2 < a_n < L + \epsilon^2$. Then we have

$$\sqrt{L} - \epsilon \leq \sqrt{L - \epsilon^2} < \sqrt{a_n} < \sqrt{L + \epsilon^2} \leq \sqrt{L} + \epsilon$$

Note that this proof fails here for $L = 0$ because if $\epsilon > \sqrt{L}$ then $\sqrt{L - \epsilon^2}$ is complex.

which places $\sqrt{a_n}$ in ϵ-neighbourhood of L, so we're done. □

(90) **Let a_n be a positive decreasing sequence. Show that, if there exist numbers N and α such that**

$$0 < \frac{a_{n+1}}{a_n} < \alpha < 1 \qquad \forall n > N$$

then $a_n \to 0$ as $n \to \infty$. But if, on the other hand, we have

$$0 < \frac{a_{n+1}}{a_n} < 1 \qquad \forall n > N$$

then we cannot conclude that $a_n \to 0$.

The basic principle here is that, if a convergent sequence converges to a non-zero limit, then the ratio of consecutive terms must tend to 1. If the ratio of consecutive terms remains below 1 then the sequence must go to zero.

If $\frac{a_{n+1}}{a_n} < \alpha$ then α is an upper bound on the ratio of consecutive terms so we can deduce that,

$$a_{n+1} < \alpha a_n < a_n \qquad \text{since } \alpha < 1$$

and that, if we let a_N be the value of the sequence at $n = N$ and consider the sequence for $n > N$,

$$a_n < \alpha^{n-N} a_N$$

which is of the form,

$$a_n < \alpha^n C$$

for constant C. So we have bound the values of the sequence below $\alpha^n C$ which clearly goes to 0 as $n \to \infty$. Since we are told that the sequence is positive so that a_n is also bounded below by 0, we can conclude, by Sandwich Theorem, that $a_n \to 0$ as $n \to \infty$. $\square$

An alternative way of showing the same thing is to use the algebra of limits and the fact that, in a convergent sequence, any subsequence must also converge to the same limit (Proposition 3.1.12) to deduce that if $a_n \to L$ then we must also have $a_{n+1} \to L$. Then, if $L \neq 0$ we can apply the algebra of limits to obtain,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{L}{L} = 1$$

which contradicts $\frac{a_{n+1}}{a_n} < \alpha < 1$. Therefore, $L = 0$. $\square$

On the other hand, if $\frac{a_{n+1}}{a_n} \to 1$ as $n \to \infty$ then the sequence may converge to 0 or to some other value. For example:

(i) $\boldsymbol{a_n = \frac{1}{n} + 1}$

$$\frac{a_{n+1}}{a_n} = \frac{n(n+2)}{(n+1)^2} = \frac{n^2+2n}{n^2+2n+1} \to 1 \text{ as } n \to \infty$$

and clearly,

$$\lim_{n\to\infty} \frac{1}{n} + 1 = 1.$$

But also,

(ii) $\boldsymbol{a_n = \frac{1}{n}}$

$$\frac{a_{n+1}}{a_n} = \frac{n}{n+1} \to 1 \text{ as } n \to \infty$$

even though

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

(91) The sequence $n^{(-1)^n}$ has a subsequence that goes infinite (when the exponent is positive) and another which converges to 0 (when the exponent is negative).

3.1.3 Series

Definition 163. **(Series)** Let $(a_n)_{n=1}^{\infty}$ be a sequence and let, for $n \in \mathbb{N}$,

$$s_n = a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

be the n-th partial sum of the terms of the sequence $(a_n)_{n=1}^{\infty}$. Then, the sequence $(s_n)_{n=1}^{\infty}$ is called a *series*, denoted $\sum a_n$, with n-th term a_n and n-th partial sum s_n.

Definition 164. **(Series Convergence)** A series $\sum a_n$ is *convergent* iff the sequence of partial sums $(s_n)_{n=1}^{\infty}$ is convergent and divergent iff the sequence of partial sums is divergent.

If this sequence converges to a finite value L we say that the *series converges to* L or the series *has sum* L.

Definition 165. **(Series Rearrangement)** A *rearrangement* of a series $\sum a_n$ is a series $\sum a_{f(n)}$ where f is a permutation of $\mathbb{N}$. In other words, the rearrangement contains the same terms as the original series but appearing in a different order.

Definition 166. **(Non-negative Series)** A series is described as *non-negative* (or *positive*) if all its terms are non-negative. That's to say, the sequence $\sum a_n$ is non-negative iff

$$\forall n \in \mathbb{N} \,.\, a_n \geq 0.$$

Definition 167. **(Alternating Series)** A series is described as *alternating* if its terms are alternately positive and negative.

Notation. The series $(s_n)_{n=1}^{\infty}$ with n-th term a_n is typically denoted $\sum a_n$ but some authors use the notation $\sum_{n=1}^{\infty} a_n$ which has the disadvantage that it may be confused with the sum to infinity of the series (which may or may not exist depending on whether the series converges).

A series may also be finite but finite series necessarily converge and so are not a subject of study in Analysis (but may be studied in Calculus).

Proposition 3.1.14. *If a series $\sum a_n$ converges then the terms $a_n \to 0$ as $n \to \infty$.*

Proof. Let $f : \mathbb{N} \longmapsto \mathbb{N}$ be defined as

$$f(n) = n - 1.$$

Then f is a strictly increasing function over the naturals and so, by the definition 3.1.2.2, $(s_{f(n)})$ is a subsequence of the sequence of partial sums (s_n) of the series $\sum a_n$. By Proposition 3.1.12 then, the subsequence $(s_{f(n)})$ tends to the same limit as (s_n).
Since both sequences tend to some finite limit L, we can use Proposition 3.1.23 to evaluate the limit,

$$\lim_{n\to\infty}(s_{f(n)} - s_n) = L - L = 0.$$

Since $f(n) = n - 1$, it follows that

$$(s_{f(n)} - s_n) = (s_{n-1} - s_n) = a_n$$

and so

$$\lim_{n\to\infty}(s_{f(n)} - s_n) = \lim_{n\to\infty} a_n = 0. \qquad \square$$

Proposition 3.1.15. ***(Algebra of Series)*** *Let $\sum a_n$ be a series that converges to L and $\sum b_n$ be a series that converges to M. Then for any real number C,*

(i) $\sum_{n=1}^{\infty} Ca_n = CL$

(ii) $\sum_{n=1}^{\infty}(a_n + b_n) = L + M$

Proof.

(i) Let

$$s_n = \sum_{i=1}^{n} a_i$$

be the n-th partial sum of $\sum a_n$. Then

$$t_n = \sum_{i=1}^{n} Ca_i = C \sum_{i=1}^{n} a_i = Cs_n.$$

Using Proposition 3.1.23, we can therefore deduce

$$\lim_{n\to\infty} s_n = L \implies \lim_{n\to\infty} t_n = \lim_{n\to\infty} Cs_n = CL.$$

(ii) Let

$$s_n = \sum_{i=1}^{n} a_i, t_n = \sum_{i=1}^{n} b_i$$

be the n-th partial sums of $\sum a_n$ and $\sum b_n$ respectively. Then

$$u_n = \sum_{i=1}^{n} a_i + b_i = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i = s_n + t_n$$

and we can use Proposition 3.1.23 to deduce

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} s_n + t_n = L + M.$$

□

Note that linear combinations of series conform to the algebra of (finite) limits of sequences Proposition 3.1.23; which is to be expected as the series is a sum of the terms of a sequence. Also, as expected, due to the lack of distributivity of addition over multiplication, the series algebra does not work for products — e.g. if $\sum a_n$ and $\sum b_n$ converge then it does ***not*** *follow that the product series $\sum a_n b_n$ converges (see: 92).*

3.1.3.1 Non-negative Series

Proposition 3.1.16. *If a series is non-negative then its partial sums are monotonically increasing.*

Proof. Let s_n be the n-th partial sum of the sequence $\sum a_n$. For all n,

$$a_n \geq 0 \implies s_n - s_{n-1} \geq 0 \iff s_n \geq s_{n-1}.$$

Therefore, the sequence of partial sums $(s_n)_{n=1}^{\infty}$ is monontonically increasing. □

Proposition 3.1.17. *If a series $\sum a_n$ is non-negative (3.1.3), then either the series converges or it goes to infinity. That's to say, if the series diverges, then the n-th partial sum $s_n \to \infty$ as $n \to \infty$.*

Proof. By Proposition 3.1.16, the sequence of n-th partial sums (s_n) is increasing. In addition, the sequence either has an upper bound or it does not. By Proposition 3.1.5, if it has an upper bound then it converges which, by the definition 3.1.3, is to say that the sequence $\sum a_n$ converges.

Alternatively, if does not have an upper bound then

$$\forall K \in \mathbb{R} \,.\, \exists N \,.\, \forall n > N \in \mathbb{N} \,.\, s_n > K$$

which, by the definition 3.1.2.1, is to say that $\lim_{n\to\infty} s_n = \infty$. □

Well-known Non-negative Series

Theorem 3.1.3. ***(Geometric Series)*** *Let $a, r \in \mathbb{R}$. Then the series $\sum ar^{n-1}$,*

(i) converges to $\frac{1}{a-r}$ if $|r| < 1$;

(ii) diverges if $|r| \geq 1$.

Proof.
The sum of a geometric progression, for $r = 1$ is given by $s_n = na$ and, for $r \neq 1$, is given by

$$s_n = \frac{ar^n - a}{r - 1} = \frac{a}{r - 1}(r^n - 1).$$

So, the limit of this sum as $n \to \infty$ is given by,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a}{r-1}(r^n - 1) = \frac{a}{r-1} \lim_{n\to\infty}(r^n - 1).$$

Then, by Proposition 3.1.11,

(i) if $|r| < 1$, then

$$\lim_{n\to\infty} r^n = 0 \implies \lim_{n\to\infty}(r^n - 1) = -1 \implies \lim_{n\to\infty} s_n = -\frac{a}{r-1} = \frac{a}{1-r}.$$

(ii) if $|r| \geq 1$, then either:

- $|r| > 1$ and so

$$\lim_{n\to\infty} r^n \text{ diverges} \implies \lim_{n\to\infty} s_n \text{ diverges},$$

if $r < -1$, the sequence (r^n) oscillates unboundedly

- $|r| = 1 \iff r \in \{-1, 1\}$ and, if $r = 1$ then

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} na = a \lim_{n\to\infty} n = \infty$$

or else $r = -1$ and $\lim_{n\to\infty} r^n = \lim_{n\to\infty}(-1)^n$ which alternates between -1 and 1 and so diverges in a bounded oscillatory fashion.

□

Theorem 3.1.4. ***Harmonic Series*** *The harmonic series $\sum \frac{1}{n}$ diverges.*

Proof.
Let $s_n = \sum_{i=1}^{n} \frac{1}{i}$ be the n-th partial sum of the series $\sum \frac{1}{n}$ and let $f : \mathbb{N} \longmapsto \mathbb{N}$ be the monotonically increasing function,

$$f(n) = 2^{n-1}n$$

so that $(s_{f(n)})$ is a subsequence of (s_n).

Suppose, for contradiction, that (s_n) converges to a finite limit L. Then, by Proposition 3.1.12, $(s_{f(n)})$ also converges to L and so, by Proposition 3.1.14,

$$\lim_{n\to\infty}(s_{f(n)} - s_{f(n-1)}) = 0. \qquad (*)$$

If we expand out the expression $s_{f(n)} - s_{f(n-1)}$, we can deduce that, for all $n \in \mathbb{N}$,

$$\begin{aligned} s_{f(n)} - s_{f(n-1)} &= s_{2m} - s_m && \text{for some } m \in \mathbb{N} \\ &= \sum_{i=1}^{2m} \frac{1}{i} - \sum_{i=1}^{m} \frac{1}{i} \\ &= \sum_{i=m+1}^{2m} \frac{1}{i} \\ &= \frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{2m} \\ &> \frac{1}{2m} + \frac{1}{2m} + \cdots + \frac{1}{2m} = \frac{m}{2m} = \frac{1}{2}. \end{aligned}$$

But by Proposition 3.1.6 we have

$$\forall n \,.\, (s_{f(n)} - s_{f(n-1)}) \geq \frac{1}{2} \implies \lim_{n\to\infty}(s_{f(n)} - s_{f(n-1)}) \geq \frac{1}{2}$$

which contradicts (*).

Therefore, the series diverges. □

Compare with

$$\int_a^{ab} \frac{1}{t}\,\mathrm{d}t = \int_1^b \frac{a}{au}\,\mathrm{d}u = \int_1^b \frac{1}{u}\,\mathrm{d}u = \ln(b)$$

which is to say that the value of integral of $1/t$ *between* a *and* ab *doesn't depend on the value of* a*. In other words, if the interval of integration is from some* x *to* cx *for constant* c*, then the value of the integral is* $\ln(c)$*.*

Proposition 3.1.18. *The series* $\sum \frac{1}{n^k}$*,*

(i) diverges if $k \leq 1$*;*

(ii) converges if $k > 1$*.*

Proof.
Let (s_n) be the sequence of partial sums of $\sum 1/n^k$.

(i) Assume $k \leq 1 \in \mathbb{R}$. Then, for all $n \in \mathbb{N}$,

$$n^k < n \iff \frac{1}{n^k} > \frac{1}{n}.$$

Furthermore, since both $\sum 1/n$ and $\sum 1/n^k$ are non-negative series, by Proposition 3.1.17, they can only converge to a finite limit or go to infinity. By Theorem 3.1.4, the harmonic series $\sum 1/n$ goes to infinity. Therefore, $\sum 1/n^k$ must be unbounded and also go to infinity.

(ii) Assume $k > 1$. The series $\sum 1/n^k$ is non-negative and so it is monotonically increasing (Proposition 3.1.16). It follows then that

$$2^n - 1 \geq n \implies s_{2^n-1} \geq s_n$$

and so the subsequence (s_{2^n-1}) is an upper bound on the sequence (s_n). If (s_{2^n-1}) is upper bounded then this will imply that (s_n) is upper bounded and then, by Proposition 3.1.5, (s_n) must converge.

Consider the partial sums of (s_{2^n-1}),

$$\begin{aligned} s_{2^n-1} &= \sum_{i=1}^{2^n-1} \frac{1}{i^k} \\ &= \frac{1}{1^k} + \frac{1}{2^k} + \cdots + \frac{1}{(2^n-1)^k} \\ &= 1 + \left(\frac{1}{2^k} + \frac{1}{3^k}\right) + \left(\frac{1}{4^k} + \frac{1}{5^k} + \frac{1}{6^k} + \frac{1}{7^k}\right) + \cdots \\ &\quad + \left(\frac{1}{(2^{n-1})^k} + \frac{1}{(2^{n-1}+1)^k} + \cdots + \frac{1}{(2^n-1)^k}\right) \end{aligned}$$

where we have grouped terms into groups of $1, 2, 4, 8, \ldots, 2^{n-1}$ terms. The sum of each such group is of the form, for $m \in \mathbb{N} \cup \{0\}$,

$$\frac{1}{(2^m+0)^k} + \frac{1}{(2^m+1)^k} + \cdots + \frac{1}{(2^m+(2^m-1))^k} \leq \frac{2^m}{(2^m)^k} = \frac{1}{2^{m(k-1)}}.$$

Therefore, if we let $t_n =$

$$\begin{aligned} s_{2^n-1} &\leq \sum_{m=0}^{n-1} \frac{1}{2^{m(k-1)}} \\ &= \frac{\frac{1}{2^{n(k-1)}} - 1}{\frac{1}{2^{(k-1)}} - 1} \end{aligned}$$

and the limit as $n \to \infty$ of this upper bound on s_{2^n-1} is

$$\lim_{n\to\infty} \frac{\frac{1}{2^{n(k-1)}} - 1}{\frac{1}{2^{(k-1)}} - 1} = \frac{0-1}{\frac{1}{2^{(k-1)}} - 1} = \frac{1}{1 - \frac{1}{2^{(k-1)}}} = \frac{1}{1-2^{(1-k)}}.$$

So we have found a convergent sequence that is an upper bound on (s_{2^n-1}) and so (s_{2^n-1}) is upper bound and, therefore, (s_n) converges.

□

Convergence Tests for Non-negative Series

Theorem 3.1.5. ***(Comparison Test)*** *Suppose (a_n) and (b_n) are non-negative sequences. Then:*

(i) If $a_n \leq b_n$ for all $n \in \mathbb{N}$, then

- *if $\sum b_n$ converges then $\sum a_n$ converges and*

$$\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n,$$

- *if $\sum a_n$ diverges then $\sum b_n$ diverges;*

(ii) If there exists some N such that $a_n \leq b_n$ for all $n > N \in \mathbb{N}$, then

- *if $\sum b_n$ converges then $\sum a_n$ converges,*
- *if $\sum a_n$ diverges then $\sum b_n$ diverges;*

(iii) If

$$L = \lim_{n\to\infty} \frac{a_n}{b_n}$$

is finite and non-zero, then

$$\sum a_n \text{ converges } \iff \sum b_n \text{ converges.}$$

Proof.
Let s_n and t_n be the n-th partial sums of $\sum a_n$ and $\sum b_n$ respectively. Since the terms a_n and b_n are non-negative, by Proposition 3.1.16, the sequences (s_n) and (t_n) are monotonically increasing.

(i) Assume $a_n \leq b_n$ for all $n \in \mathbb{N}$. Then

$$s_n = \sum_{i=1}^{n} a_n \leq \sum_{i=1}^{n} b_n = t_n$$

which is to say that t_n is an upper bound for s_n. Since $\sum b_n$ converges, by definition t_n converges. Since t_n converges, by Proposition 3.1.4 it is bounded, and in particular, upper bounded. The upper bound on t_n must transitively be an upper bound on s_n also. Then, since (s_n)

is monotonically increasing, by Proposition 3.1.5, (s_n) is convergent. Furthermore, by Corollary 3.1.1,

$$\lim_{n\to\infty} t_n = \sup\{\, t_n \mid n \in \mathbb{N} \,\} \leq \sup\{\, s_n \mid n \in \mathbb{N} \,\} = \lim_{n\to\infty} s_n.$$

If, on the other hand, $\sum a_n$ diverges, by the increasing property and Proposition 3.1.17, it must go to infinity (which is to say that the partial sums s_n go to infinity). Therefore, the partial sums t_n must also go to infinity and so, by definition, $\sum b_n$ diverges also.

(ii) Assume there exists some N such that $a_n \leq b_n$ for all $n > N \in \mathbb{N}$. Then

$$\sum_{i=N+1}^{n} a_n \leq \sum_{i=N+1}^{n} b_n.$$

Let

$$s_N = \sum_{i=1}^{N} a_n \quad \text{and} \quad t_N = \sum_{i=1}^{N} b_n.$$

So we have,

$$s_n = s_N + \sum_{i=N+1}^{n} a_n \quad \text{and} \quad t_n = t_N + \sum_{i=N+1}^{n} b_n$$

and then it follows that

$$\begin{aligned}
& \sum_{i=N+1}^{n} a_n \leq \sum_{i=N+1}^{n} b_n \\
\iff \quad & s_N + \sum_{i=N+1}^{n} a_n \leq s_N + \sum_{i=N+1}^{n} b_n \\
\iff \quad & s_n \leq t_n - t_N + s_N \\
\iff \quad & s_n \leq t_n + (s_N - t_N).
\end{aligned}$$

In a similar fashion to (i): Convergence of the series $\sum b_n$ implies that the sequence (t_n) converges, which by Proposition 3.1.4, implies that it is upper bounded. Since $s_N - t_N$ is some constant, the upper bound on t_n means that s_n also has an upper bound, and, being monotonically

increasing, by Proposition 3.1.5, is therefore convergent.

If $\sum a_n$ diverges on the other hand, then by Proposition 3.1.17, we have

$$\lim_{n\to\infty} s_n = \infty \implies \lim_{n\to\infty}[t_n + (s_N - t_N)] = \infty \implies \lim_{n\to\infty} t_n = \infty$$

which is to say that $\sum b_n$ diverges.

(iii) Suppose we have

$$L = \lim_{n\to\infty} \frac{a_n}{b_n}.$$

Then, by the definition of the limit of sequences (3.1.2.1), for all $\epsilon > 0 \in \mathbb{R}$, there exists some N such that,

$$\forall n > N \; . \; \left|\frac{a_n}{b_n} - L\right| < \epsilon$$

which is to say

$$L - \epsilon < \frac{a_n}{b_n} < \epsilon + L.$$

Let $K \in \mathbb{R}$ be a constant such that $K = \epsilon + L$. Then,

$$a_n < Kb_n \quad \text{and} \quad b_n < \frac{1}{K} a_n$$

for all $n > N \in \mathbb{N}$. If $\sum b_n$ converges then by the algebra of series (Proposition 3.1.15), $\sum Kb_n$ also converges and we can apply (ii) to deduce that $\sum a_n$ converges also. Similarly, if $\sum a_n$ converges then $\sum \frac{1}{K} a_n$ converges and applying (ii) we obtain the result that $\sum b_n$ converges also.

□

Theorem 3.1.6. ***(Ratio Test)*** *Let $\sum a_n$ be a non-negative series with the property that*

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

where L may be infinity. Then,

(i) $L < 1 \implies \sum a_n$ *converges,*

(ii) $L > 1 \implies \sum a_n$ *diverges.*

Proof.

(i) In the case that $L < 1$, there exists some N such that, for all $n > N \in \mathbb{N}$ and $\epsilon < 1 - L$,

$$\left| \frac{a_{n+1}}{a_n} - L \right| < \epsilon \iff L - \epsilon < \frac{a_{n+1}}{a_n} < L + \epsilon.$$

Let $r = \epsilon + L$. Then,

$$\frac{a_{n+1}}{a_n} < r \iff a_{n+1} < r a_n$$

with $0 < r < 1$. Therefore, the sum of the terms after N,

$$\sum_{i=N+1}^{\infty} a_i < \sum_{i=1}^{\infty} r^i a_N$$

which is a geometric series with absolute common ratio less than one. Therefore, by Theorem 3.1.3, the sum of the terms after N is convergent and finite and since, the sum of the finite number of terms upto N is necessarily finite also, we conclude that the series converges. Since this is a convergent upper bound on $\sum a_n$, by Theorem 3.1.5, $\sum a_n$ converges.

(ii) In the case that $L > 1$, we can use a similar argument to show that the sequence $\sum a_n$ is lower bounded by a geometric sequence with absolute common ratio greater than 1 so that Theorem 3.1.3 tells us that this lower bound diverges. Then, again applying the Theorem 3.1.5, $\sum a_n$ diverges.

□

Note that the Ratio Test makes no inference if the limit of the ratio of consecutive terms is equal to 1. In this case the series may or may not converge. For example, $\sum 1/n$ *and* $\sum 1/n^2$ *both have a ratio of consecutive*

terms that tends to 1 but the first diverges while the second converges.

Theorem 3.1.7. ***(Root Test)*** *Let $\sum a_n$ be a non-negative series with the property that*

$$L = \lim_{n\to\infty} {a_n}^{1/n}$$

where L may be infinity. Then,

(i) $L < 1 \implies \sum a_n$ *converges,*

(ii) $L > 1 \implies \sum a_n$ *diverges.*

Proof.

(i) Assume $L < 1$ so that for some $0 < r < 1$,

$$\lim_{n\to\infty} {a_n}^{1/n} < 1 \implies \exists N \,.\, \forall n > N \,.\, {a_n}^{1/n} < r$$

which is to say that $a_n < r^n$ and therefore the series $\sum a_n$ is upper bounded by the series $\sum r^n$. Since $|r| < 1$, by Theorem 3.1.3, the series $\sum r^n$ converges and so by Theorem 3.1.5, $\sum a_n$ converges also.

(ii) Similarly, if $L > 1$ we can lower bound the series $\sum a_n$ with a geometric series with common ratio $|r| > 1$ and, using Theorem 3.1.3, deduce that this lower bound series diverges which, since it is a non-negative series, by Proposition 3.1.17, means that the partial sums go to infinity. Therefore the partial sums of $\sum a_n$ also go to infinity.

□

Theorem 3.1.8. ***(Integral Test)*** *Let $a \geq 1 \in \mathbb{R}$ and $f : \mathbb{R} \longmapsto \mathbb{R}$ be a positive, decreasing, integrable function on $[a, \infty)$. Then the series $\sum f(n)$ converges iff the improper integral $\int_a^\infty f(t)\,\mathrm{d}t$ exists.*

Proof. Denote n-th partial sum of $\sum f(n)$, s_n. Then

$$s_n = \sum_{i=1}^{n} f(i).$$

By the definition of the Riemmann Integral 191 and the upper and lower estimates used in the definition 4.6.1.1, since f is a decreasing function, the upper estimate of the integral $\int_a^b f(t)\,\mathrm{d}t$ — for some $b > a \in \mathbb{R}$ — over the partition $P = \{a+1, a+2, \ldots, b\}$ is

$$U(P) = \sum_{i=a}^{b-1} (x_{i+1} - x_i) \max_{x_i \leq x \leq x_{i+1}} f(x).$$

But for this partition P we have $(x_{i+1} - x_i) = 1$ and since f is a decreasing function, for all $a \leq i \leq b-1$,

$$\max_{x_i \leq x \leq x_{i+1}} f(x) = \max_{i \leq x \leq i+1} f(x) = f(i).$$

So the upper estimate becomes, letting $n = b - 1$,

$$U(P) = \sum_{i=a}^{b-1} f(i) = s_n - s_a$$

the n^{th} partial sum minus the a^{th} partial sum of the series $\sum f(n)$. Since the upper estimate is, by definition, an upper bound on the value of the integral we therefore have,

$$s_n - s_a \geq \int_a^b f(t)\,\mathrm{d}t \iff s_n \geq \int_a^b f(t)\,\mathrm{d}t + s_a.$$

Since f is a positive function, the integral $\int_a^b f(t)\,\mathrm{d}t$ is positive and so, if b is allowed to go to infinity, then the value of the integral must, by Proposition 3.1.17, either converge to a finite limit or tend to infinity.
If the integral goes to infinity (i.e. the improper integral doesn't exist) then s_n, being an upper bound on its value, must also go to infinity. On the other hand, if the integral exists (i.e. it converges to a finite limit) then, by the definition of the Riemann Integral, the upper and lower estimates must converge. Therefore (s_n) converges. □

3.1.3.2 Absolute and Conditional Convergence

Definition 168. **(Series Absolute Convergence)** If a series $\sum a_n$ is such that $\sum |a_n|$ is a convergent series, then the series $\sum a_n$ is described as *absolutely convergent*.

Clearly, for non-negative series in which, by definition, $\forall n \,.\, a_n = |a_n|$, convergence implies absolute convergence.

Definition 169. **(Series Conditional Convergence)** If a series $\sum a_n$ is such that $\sum |a_n|$ is a divergent series but the original series $\sum a_n$ is a convergent series, then the series $\sum a_n$ is described as *conditionally convergent*.

Proposition 3.1.19. *Absolute convergence implies conditional convergence.*

Proof. Let $\sum a_n$ be an absolutely convergent sequence and consider the sequence $\sum |a_n| - a_n$. For each term, since a_n either equals $|a_n|$ or $-|a_n|$, then

$$|a_n| - a_n \in \{0, 2\,|a_n|\}.$$

Therefore, the series $\sum |a_n| - a_n$ is non-negative and upper-bounded by the series $\sum 2\,|a_n|$. Since by construction, $\sum |a_n|$ converges, we can apply the algebra of series (Proposition 3.1.15) to deduce that $\sum 2\,|a_n| = 2\sum |a_n|$ and thus, that $\sum 2\,|a_n|$ is also convergent. Therefore, by Proposition 3.1.5, $\sum |a_n| - a_n$ converges.

Now we have two convergent series: $\sum |a_n|$ and $\sum |a_n| - a_n$. Since they both converge, we can apply the algebra of series (Proposition 3.1.15) again to determine that

$$\sum |a_n| - \sum |a_n| - a_n = \sum a_n$$

also converges. □

*Note that the triangle inequality tells us that $\sum |a_n|$ is an upper bound on $\sum a_n$ but because $\sum a_n$ is not a non-negative sequence, convergence of $\sum |a_n|$ does **not** imply that $\sum a_n$ converges: it may diverge to negative infinity or*

oscillate boundedly.

TODO: complete this proof

Proposition 3.1.20. *A convergent series that is not non-negative (conditionally converges?) doesn't necessarily converge to same limit after rearrangement*

Proof. □

Proposition 3.1.21. *If a series is non-negative and convergent (i.e. absolutely convergent) then its value is invariant to rearrangements.*

Proof. This proof requires the Cauchy Criterion of Convergence: see: math.stackexchange. □

Theorem 3.1.9. ***(LAST: Leibniz Alternating Series Test)*** *Let*

$$\sum a_n = (-1)^{n+1} b_n.$$

If

- $b_n \geq 0$,
- (b_n) *is monotonically decreasing, and*
- $\lim_{n\to\infty} b_n = 0$,

then $\sum a_n$ *converges.*

Proof.
Let s_n be the n-th partial sum of the series $\sum a_n$. Then

$$s_n = b_1 - b_2 + b_3 - b_4 + \cdots = (b_1 - b_2) + (b_3 - b_4) + \cdots = c_1 + c_2 + \cdots$$

where, because (b_n) is a positive, monotonically decreasing sequence, the sequence (c_n) is positive. So the even-numbered partial sums take the form

$$s_{2n} = c_1 + c_2 + \cdots + c_n$$

and the subsequence (s_{2n}) is therefore positive and monotonically increasing.

In addition, if we group the terms differently then the n-th partial sum,

$$s_n = b_1 - (b_2 - b_3) - (b_4 - b_5) - \cdots = b_1 - d_1 - d_2 - \cdots$$

where the sequence (d_n) is positive. Therefore $s_n \leq b_1$. TODO: check this proof against rearrangements □

Theorem 3.1.10. ***(Riemann Rearrangement Theorem)*** *TODO: complete this*

Proof. □

Tests for Absolute Convergence

TODO: tests for absolute convergence

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(92) The series $\sum(-1)^n/\sqrt{n}$ converges but if the partial sums are squared, $\sum(-1)^{2n}/n$ diverges. So this is an example of a non-linear combination of series.

(93) Consider the series

$$\sum \frac{n+2}{n^2}.$$

If we examine the limit of the ratio of consecutive terms (i.e. perform the Ratio Test Theorem 3.1.6) we find,

$$\begin{aligned}\frac{(n+1)+2}{(n+1)^2} \cdot \frac{n^2}{n+2} &= \frac{n^3+3n^2}{(n^2+2n+1)(n+2)} \\ &= \frac{n^3+3n^2}{n^3+4n^2+5n+2} \\ &= \frac{1+3/n}{1+4/n+5/n^2+2/n^3} \to 1.\end{aligned}$$

So the Ratio Test is inconclusive. But we can, instead, use the Comparison Test (Theorem 3.1.5) by examining the dominant behaviours of the numerator and denominator of the terms of the series, we can reason that it's asymptotic behaviour is likely similar to $\sum 1/n$, the Harmonic Series (Theorem 3.1.4). Performing the Comparison Test we obtain,

$$\frac{n+2}{n^2} \cdot \frac{n}{1} = \frac{n+2}{n} = \frac{1+2/n}{1} \to 1.$$

So the limit of the ratio of the terms of the sequence with the terms of the sequence of the harmonic series also tends to 1. In this case we can draw a conclusion: that the asymptotic behaviour of the series under test is the same as that of the harmonic series. We therefore deduce that the series diverges.

(94) Consider the series

$$\sum \frac{1}{n \log n}.$$

Here $\log x$ *refers to the natural log (see: math.stackexchange).*

We have

$$\frac{1}{n^2} < \frac{1}{n \log n} < \frac{1}{n}$$

so we can't use a comparison with $1/n^r$ for some r. So we use the Integral Test Theorem 3.1.8 as follows.

$$\begin{aligned} \int_2^n \frac{1}{t \log t} \, \mathrm{d}t &= \int_{\log 2}^{\log n} \frac{1}{u} \, \mathrm{d}u \qquad\qquad u = \log t \implies \mathrm{d}t = t \, \mathrm{d}u \\ &= [\log u]_{\log 2}^{\log n} \\ &= \log \log n - \log \log 2. \end{aligned}$$

Since $\lim_{n \to \infty} \log n = \infty$, as $n \to \infty$, $\log n$ is growing without bound. Therefore, $\log \log n$ is also growing without bound and so the integral diverges.

3.1.4 Functions

Definition 170. **(Bounded Function)** For a subset X of the domain of a function f, we say that f is *bounded* on X if there exists M such that $|f(x)| \leq M$ for each $x \in X$.

Definition 171. **(Bound of a Function)** We define the supremum (or maximum) of f on X as $\sup\{\, f(x) \mid x \in X \,\}$ (or $\max\{\, f(x) \mid x \in X \,\}$ if it exists).

Proposition 3.1.22. *If two functions $f(x)$ and $g(x)$ are bounded on the interval $X \subseteq \mathbb{R}$ then their product $f(x)g(x)$ is bounded but, in general, their ratio $f(x)/g(x)$ is not. That's to say,*

$$f(x) \text{ and } g(x) \text{ are bounded} \implies f(x)g(x) \text{ is bounded}$$

$$f(x) \text{ and } g(x) \text{ are bounded} \not\implies \frac{f(x)}{g(x)} \text{ is bounded.}$$

Proof.

By the definition of a bounded function (3.1.4) we have some $M, N \in \mathbb{R}$ such that, for all $x \in X$,

$$|f(x)| \leq M \ \wedge \ |g(x)| \leq N.$$

It follows then that, for all $x \in X$,

$$\begin{aligned} & |f(x)|\,|g(x)| \leq MN \\ \iff \quad & |f(x)g(x)| \leq MN \quad \text{by Proposition 1.2.16.} \end{aligned}$$

However, since

$$\begin{aligned} & |g(x)| \leq N \\ \iff \quad & \frac{1}{N} \leq \frac{1}{|g(x)|}, \end{aligned}$$

we don't have an upper bound for the expression $\frac{1}{|g(x)|}$ and so we cannot determine any upper bound for

$$f(x)\frac{1}{|g(x)|} = \frac{f(x)}{|g(x)|}.$$

In particular, since we have

$$0 \leq |g(x)|,$$

then for any $K \in \mathbb{R}$,

$$|g(x)| < \frac{|f(x)|}{K} \iff \frac{|f(x)|}{|g(x)|} > K$$

which shows that, because $|g(x)|$ can be an arbitrarily small positive value, the value of the ratio $f(x)/g(x)$ is unbounded. □

3.1.4.1 Limits of Functions

Definition 172. **(Function tends to a value)** Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a function. We say that L is the *limit of* $f(x)$ *as* x *approaches* a if, for each $\epsilon > 0$ there exists $\delta > 0$ such that,

$$0 < |x - a| < \delta \implies |f(x) - L| < \epsilon.$$

Definition 173. **(Function tends to infinity)** Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a function. We say that $f(x)$ *tends to infinity as* x *approaches* a if, for each K there exists $\delta > 0$ such that,

$$0 < |x - a| < \delta \implies f(x) > K.$$

Similarly, we say that $f(x)$ *tends to minus infinity as* x *approaches* a if, for each K there exists $\delta > 0$ such that,

$$0 < |x - a| < \delta \implies f(x) < K.$$

***Important**: Note that the limit of the function $f(x)$ as $x \to a$ does not depend on the value $f(a)$ of the function at $x = a$. For this reason, the above definitions specify $0 < |x - a| < \delta$ rather than simply $|x - a| < \delta$.*

Definition 174. **(One-Sided Limit)** Let $f : \mathbb{R} \to \mathbb{R}$ be a function. We say that L is the *limit of $f(x)$ as x approaches a from the left (or from below)*, denoted by $\lim_{x \to a^-} f(x) = L$, if for each $\epsilon > 0$, there exists $\delta > 0$ such that,

$$a - \delta < x < a \implies |f(x) - L| < \epsilon.$$

Definition 175. **(Limit at Infinity)** Let $f : \mathbb{R} \to \mathbb{R}$ be a function. We say that L is the *limit of $f(x)$ as x approaches ∞*, denoted by $\lim_{x \to \infty} f(x) = L$, if for each $\epsilon > 0$, there exists $M > 0$ such that,

$$x \geq M \implies |f(x) - L| < \epsilon.$$

Proposition 3.1.23. ***(Algebra of Finite Limits of Functions)*** *Let $f, g : \mathbb{R} \longmapsto \mathbb{R}$ be two functions and c be any real number. Suppose that $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$. Then,*

(i) $\lim_{x \to a}(cf)(x) = cL$

(ii) $\lim_{x \to a}(|f|)(x) = |L|$

(iii) $\lim_{x \to a}(f + g)(x) = L + M$

(iv) $\lim_{x \to a}(f - g)(x) = L - M$

(v) $\lim_{x \to a} f(x)g(x) = LM$

(vi) $\lim_{x \to a}(f/g)(x) = L/M$ *provided* $g(x) \neq 0$ *for any x in the neighbourhood of a*

(vii) $\lim_{x \to a} f(x)^c = L^c$.

Proof. First we need to prove a couple of more basic statements about limits as preliminaries.

- $\boldsymbol{\lim_{x\to a} c = c}$:

  $$\forall \epsilon > 0 \,.\, \exists \delta \,.\, 0 < |x - a| < \delta \implies |c - c| < \epsilon$$

  is clearly satisfied for any interval of x-values (since the constant function doesn't depend on x). So, the proposition is trivially satisfied by any ϵ, δ.

- $\boldsymbol{\lim_{x\to a} x = a}$:

  $$\forall \epsilon > 0 \,.\, \exists \delta \,.\, 0 < |x - a| < \delta \implies |x - a| < \epsilon$$

  is also trivially satisfied for any $\epsilon = \delta$.

(i) $\boldsymbol{\lim_{x\to a}(cf)(x) = cL}$:

Let

$$\epsilon > 0,\ \epsilon_1 = \frac{\epsilon}{|c|}.$$

By hypothesis,

$$\forall \epsilon_1 > 0 \,.\, \exists \delta_1 \,.\, 0 < |x - a| < \delta_1 \implies |f(x) - L| < \epsilon_1.$$

Furthermore,

$$\begin{aligned} & |f(x) - L| < \epsilon_1 \\ \iff \quad & |c|\,|f(x) - L| < |c|\,\epsilon_1 \\ \iff \quad & |cf(x) - cL| < |c|\,\epsilon_1 = |c|\,\frac{\epsilon}{|c|} = \epsilon. \end{aligned}$$

by Proposition 1.2.16

(ii) $\boldsymbol{\lim_{x\to a}(|f|)(x) = |L|}$:

Using properties of absolute value (see: 1.2.3),

$$|f(x) - L| \geq \Big||f(x)| - |L|\Big|$$

so that

$$|f(x) - L| < \epsilon \implies \Big||f(x)| - |L|\Big| < \epsilon.$$

This means that the limit hypothesis on $f(x)$ implies that

$$\lim_{x \to a}(|f|)(x) = |L|$$

as required.

(iii) $\boldsymbol{\lim_{x \to a}(f + g)(x) = L + M}$:

Let

$$\epsilon > 0,\ \epsilon_1 = \frac{\epsilon}{2}.$$

By hypothesis we have,

$$\forall \epsilon_1 > 0\ .\ \exists \delta_1\ .\ 0 < |x - a| < \delta_1 \implies |f(x) - L| < \epsilon_1,$$

$$\forall \epsilon_1 > 0\ .\ \exists \delta_2\ .\ 0 < |x - a| < \delta_2 \implies |g(x) - M| < \epsilon_1.$$

Let

$$\delta = \min\{\delta_1, \delta_2\}$$

so that, for $0 < |x - a| < \delta$ we have,

$$|f(x) - L| < \epsilon_1 \quad \text{and} \quad |g(x) - M| < \epsilon_1.$$

If we sum the two expressions and employ the absolute value properties (see: 1.2.3) we obtain,

$$\begin{aligned} & |f(x) - L| + |g(x) - M| < 2\epsilon_1 = \epsilon \\ \implies \quad & |(f(x) - L) + (g(x) - M)| < \epsilon \qquad\qquad |x + y| \leq |x| + |y| \\ \iff \quad & |(f(x) + g(x)) - (L + M)| < \epsilon. \end{aligned}$$

(iv) $\boldsymbol{\lim_{x \to a}(f - g)(x) = L - M}$:

Arguing similarly to (iii), again, for x in the δ-neighbourhood of a, we have both

$$|f(x) - L| < \epsilon_1 \quad \text{and} \quad |g(x) - M| < \epsilon_1.$$

This time, we are going to subtract the two expressions and use a fact derived from the absolute value properties (see: 1.2.3) that,

$$||x| - |y|| \leq |x - y| \leq |x| + |y|\,.$$

$$\begin{aligned} & |(f(x) - L) - (g(x) - M)| \leq |f(x) - L| + |g(x) - M| < 2\epsilon_1 \\ \iff \quad & |(f(x) - L) - (g(x) - M)| < 2\epsilon_1 = \epsilon \\ \iff \quad & |(f(x) - g(x)) - (L - M)| < \epsilon. \end{aligned}$$

Note that the maximum "error" is the sum of the individual errors $\epsilon_1 + \epsilon_2$ just as it is for (iii).

(v) $\mathbf{\lim_{x\to a} f(x)g(x) = LM}$:

First, we prove a lemma that will be useful for the second part of this proof.

Lemma 3.1.3.

$$\lim_{x\to a}(f(x) - L)(g(x) - M) = 0$$

Proof. Let

$$\epsilon > 0,\ \epsilon_1 = \sqrt{\epsilon}.$$

By hypothesis we have,

$$\forall \epsilon_1 > 0\ .\ \exists \delta_1\ .\ 0 < |x - a| < \delta_1 \implies |f(x) - L| < \epsilon_1,$$

$$\forall \epsilon_1 > 0\ .\ \exists \delta_2\ .\ 0 < |x - a| < \delta_2 \implies |g(x) - M| < \epsilon_1.$$

Let

$$\delta = \min\{\delta_1, \delta_2\}$$

so that, for $0 < |x - a| < \delta$ we have,

$$|f(x) - L| < \epsilon_1 \quad \text{and} \quad |g(x) - M| < \epsilon_1.$$

Then,

$$|f(x) - L|\,|g(x) - M| < {\epsilon_1}^2 = \epsilon.$$

By Proposition 1.2.16,

$$|f(x) - L|\,|g(x) - M| = |(f(x) - L)(g(x) - M)|$$

so we have,

$$\forall \epsilon > 0\ .\ \exists \delta\ .\ 0 < |x - a| < \delta \implies |(f(x) - L)(g(x) - M) - 0| < \epsilon.$$

Which is to say,

$$\lim_{x \to a} (f(x) - L)(g(x) - M) = 0. \qquad \square$$

The second part begins by observing that,

$$f(x)g(x) = (f(x) - L)(g(x) - M) + Mf(x) + Lg(x) - LM.$$

If we take the limit of both sides of this equation as $x \to a$ then,

$$\begin{aligned}
\lim_{x \to a} f(x)g(x) &= \lim_{x \to a} [(f(x) - L)(g(x) - M) \\
&\qquad + Mf(x) + Lg(x) - LM] \\
&= \lim_{x \to a} (f(x) - L)(g(x) - M) \\
&\qquad + M \lim_{x \to a} f(x) + L \lim_{x \to a} g(x) - LM \\
&= 0 + ML + LM - LM = LM.
\end{aligned}$$

(vi) $\boldsymbol{\lim_{x \to a}(f/g)(x) = L/M}$ provided $g(x) \neq 0$ for any x in the neighbourhood of a:

First we prove the following lemma:

Lemma 3.1.4.

$$\lim_{x \to a} \frac{1}{g(x)} = \frac{1}{M}$$

Proof. Firstly observe that,

$$\frac{1}{g(x)} - \frac{1}{M} = \frac{M - g(x)}{Mg(x)}$$

and that

$$\begin{aligned} & |M| = |M - g(x) + g(x)| \leq |M - g(x)| + |g(x)| \\ \iff \quad & |g(x)| \geq |M| - |M - g(x)| \\ \iff \quad & \frac{1}{|g(x)|} \leq \frac{1}{|M| - |g(x) - M|}. \end{aligned}$$

By hypothesis there exists some δ_1 such that,

$$0 < |x - a| < \delta_1 \implies |g(x) - M| < \frac{|M|}{2} \implies \frac{1}{|M| - |g(x) - M|} < \frac{2}{|M|}.$$

Also, for any arbitrary $\epsilon > 0$ there exists some δ_2 such that,

$$0 < |x - a| < \delta_1 \implies |g(x) - M| < \frac{|M|^2}{2}\epsilon$$

which means that,

$$\frac{|M - g(x)|}{|Mg(x)|} = |g(x) - M| \cdot \frac{1}{|M|} \cdot \frac{1}{g(x)} < \frac{|M|^2}{2}\epsilon \cdot \frac{1}{|M|} \cdot \frac{2}{|M|} = \epsilon. \quad \square$$

Now we can use (v) to deduce that,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = (\lim_{x \to a} f(x)) \left(\lim_{x \to a} \frac{1}{g(x)} \right) = L \cdot \frac{1}{M} = \frac{L}{M}.$$

(vii) $\boldsymbol{\lim_{x \to a} f(x)^c = L^c}$:

We need to prove this in progressive stages. First for a natural number power, then for any rational number and then for irrational numbers.

Lemma 3.1.5.

$$\lim_{x \to a} f(x)^n = L^n \qquad n \in \mathbb{N}$$

Proof. Prove by induction on n. Base case $n = 2$:

$$\begin{aligned}\lim_{x\to a} f(x)^2 &= \lim_{x\to a} f(x)f(x) \\ &= (\lim_{x\to a} f(x))(\lim_{x\to a} f(x)) && \text{by (v)} \\ &= \left[\lim_{x\to a} f(x)\right]^2.\end{aligned}$$

Induction step:

$$\begin{aligned}\lim_{x\to a} f(x)^{n+1} &= \lim_{x\to a} f(x)^n f(x) \\ &= (\lim_{x\to a} f(x)^n)(\lim_{x\to a} f(x)) \\ &= \left[\lim_{x\to a} f(x)\right]^n (\lim_{x\to a} f(x)) && \text{by induction hypothesis} \\ &= \left[\lim_{x\to a} f(x)\right]^{n+1}. && \square\end{aligned}$$

Lemma 3.1.6.

$$\lim_{x\to a} f(x)^{1/n} = L^{1/n} \qquad n \in \mathbb{N}$$

Proof.

$$\begin{aligned} & \left[\lim_{x\to a} f(x)^{1/n}\right]^n = \lim_{x\to a}(f(x)^{1/n})^n && \text{by Lemma 3.1.5} \\ \iff \quad & \left[\lim_{x\to a} f(x)^{1/n}\right]^n = \lim_{x\to a} f(x) \\ \iff \quad & \lim_{x\to a} f(x)^{1/n} = \left[\lim_{x\to a} f(x)\right]^{1/n}. && \square\end{aligned}$$

Lemma 3.1.7.

$$\lim_{x\to a} f(x)^q = L^q \qquad q \in \mathbb{Q}$$

Proof. Firstly consider the case that $q \geq 0$. Since q is a rational number, it can be expressed as m/n for $m, n \in \mathbb{N}$. Then,

$$\lim_{x\to a} f(x)^q = \lim_{x\to a} f(x)^{m/n}$$

$$
\begin{aligned}
&= \lim_{x\to a}(f(x)^{1/n})^m \\
&= \left[\lim_{x\to a} f(x)^{1/n}\right]^m && \text{by Lemma 3.1.5} \\
&= \left[\left[\lim_{x\to a} f(x)\right]^{1/n}\right]^m && \text{by Lemma 3.1.6} \\
&= \left[\lim_{x\to a} f(x)\right]^{m/n} = \left[\lim_{x\to a} f(x)\right]^q.
\end{aligned}
$$

Now consider the case where $q < 0$. Then we can express q as $-m/n$ for $m, n \in \mathbb{N}$. By (vi),

$$
\begin{aligned}
\lim_{x\to a} f(x)^q &= \lim_{x\to a} f(x)^{-m/n} \\
&= \lim_{x\to a}(f(x)^{-1})^{m/n} \\
&= \lim_{x\to a}(f(x)^{m/n})^{-1} \\
&= \left[\lim_{x\to a} f(x)^{m/n}\right]^{-1} && \text{by (vi)} \\
&= \left[\lim_{x\to a} f(x)\right]^{-q}. && \square
\end{aligned}
$$

What follows is a tentative, speculative proof for irrational powers and will hopefully be confirmed or corrected at a later date.

If we view an arbitrary irrational number r as a Cauchy sequence then it is a convergent sum of an infinite series of rational terms. So, for $q_1, q_2, \cdots \in \mathbb{Q}$

$$x^r = x^{(q_1+q_2+\cdots)} = x^{q_1}x^{q_2}\cdots$$

where

$$\lim_{i\to\infty} q_i = 0 \implies \lim_{i\to\infty} x^{q_i} = 1.$$

Therefore,

$$
\begin{aligned}
\lim_{x\to a} f(x)^r &= \lim_{x\to a} f(x)^{(q_1+q_2+\cdots)} \\
&= \lim_{x\to a} f(x)^{q_1} f(x)^{q_2}\cdots \\
&= \left[\lim_{x\to a} f(x)\right]^{q_1}\left[\lim_{x\to a} f(x)\right]^{q_2}\cdots
\end{aligned}
$$

$$= \left[\lim_{x\to a} f(x)\right]^{(q_1+q_2+\cdots)}$$
$$= \left[\lim_{x\to a} f(x)\right]^{r}.$$

□

Proposition 3.1.24. ***(Algebra of Infinite Limits of Functions)*** *Let $f, g, h : \mathbb{R} \longmapsto \mathbb{R}$ be functions and c be any real number. Suppose that $\lim_{x\to a} f(x) = \infty$, $\lim_{x\to a} g(x) = -\infty$, and $\lim_{x\to a} h(x) = 0$. Then,*

(i) $c > 0 \implies \lim_{x\to a}(cf)(x) = \infty$ *and* $c < 0 \implies \lim_{x\to a}(cf)(x) = -\infty$

(ii) $\lim_{x\to a}(|f|)(x) = \infty$ *and* $\lim_{x\to a}(|g|)(x) = \infty$

(iii) $\lim_{x\to a} \frac{1}{f(x)} = 0$ *and* $\lim_{x\to a} \frac{1}{g(x)} = 0$

(iv) $\lim_{x\to a} \frac{1}{|h(x)|} = \infty$

Proof.
By the definition 3.1.4.1,

$$\forall K \in \mathbb{R} \,.\, \exists \delta > 0 \,.\, |x - a| < \delta \implies f(x) > K,$$
$$\forall M \in \mathbb{R} \,.\, \exists \gamma > 0 \,.\, |x - a| < \gamma \implies g(x) < M.$$

(i) $c > 0 \implies \lim_{x\to a}(cf)(x) = \infty$ and $c < 0 \implies \lim_{x\to a}(cf)(x) = -\infty$:

If $c > 0$ then

$$f(x) > K \implies cf(x) > K$$

and so also

$$\lim_{x\to a} f(x) = \infty \implies \lim_{x\to a} cf(x) = \infty.$$

Conversely, if $c < 0$ then

$$f(x) > K \implies cf(x) < K$$

and

$$c < 0, \lim_{x\to a} f(x) = \infty \implies \lim_{x\to a} cf(x) = -\infty.$$

(ii) $\lim_{x\to a}(|f|)(x) = \infty$ and $\lim_{x\to a}(|g|)(x) = \infty$:

Clearly, since for any ordered field $|a| \geq a$, we have

$$f(x) > K \implies |f(x)| > K$$

so that

$$\lim_{x\to a} f(x) = \infty \implies \lim_{x\to a} |f(x)| = \infty.$$

In the case of g,

$$\begin{aligned} \exists\gamma_1 > 0 \,.\, |x-a| < \gamma_1 &\implies g(x) < M \\ &\implies -g(x) > -M \\ &\implies |g(x)| > -M \quad \because \forall a \in \mathbb{R} \,.\, |a| \geq a \end{aligned}$$

and also

$$\begin{aligned} \exists\gamma_2 > 0 \,.\, |x-a| < \gamma_2 &\implies g(x) < -M \\ &\implies -g(x) > M \\ &\implies |g(x)| > M. \quad \because \forall a \in \mathbb{R} \,.\, |a| \geq a \end{aligned}$$

Therefore, $\forall K \in \mathbb{R}$

$$\exists\gamma \,.\, |x-a| < \gamma \implies |g(x)| > K.$$

(iii) $\lim_{x\to a} \frac{1}{f(x)} = 0$ and $\lim_{x\to a} \frac{1}{g(x)} = 0$:

Assume $K > 0 \in \mathbb{R}$. There exists some δ such that

$$0 < |x-a| < \delta \implies f(x) > K.$$

Therefore, $f(x) > 0$ and

$$f(x) > K \implies \frac{1}{K} > \frac{1}{f(x)}.$$

It follows that, for all $\epsilon = \frac{1}{K} > 0 \in \mathbb{R}$, there exists some δ such that

$$\left|\frac{1}{f(x)} - 0\right| = \left|\frac{1}{f(x)}\right| < \epsilon.$$

Conversely, assume $K < 0 \in \mathbb{R}$. There exists some δ such that

$$0 < |x - a| < \delta \implies g(x) < K.$$

Therefore, $g(x) < 0$ and

$$\begin{aligned}
& g(x) < K \\
\iff \quad & 1 > \frac{K}{g(x)} \\
\iff \quad & \frac{1}{K} < \frac{1}{g(x)} \\
\iff \quad & -\frac{1}{K} > -\frac{1}{g(x)} \\
\iff \quad & -\frac{1}{K} > 0 - \frac{1}{g(x)} \\
\iff \quad & -\frac{1}{K} > \left| 0 - \frac{1}{g(x)} \right|. \quad \because g(x) < 0 \implies -\tfrac{1}{g(x)} > 0
\end{aligned}$$

So, for all $\epsilon = -\frac{1}{K} > 0 \in \mathbb{R}$, there exists some δ such that

$$\left| \frac{1}{g(x)} - 0 \right| = \left| \frac{1}{g(x)} \right| < \epsilon.$$

(iv) $\lim_{x \to a} \frac{1}{|h(x)|} = \infty$:

For any $\epsilon > 0 \in \mathbb{R}$, there exists some δ such that

$$0 < |x - a| < \delta \implies |h(x) - 0| = |h(x)| < \epsilon.$$

So, since $|h(x)|$ and ϵ are both positive,

$$\frac{1}{|h(x)|} > \frac{1}{\epsilon}$$

and, therefore, for any $K = \frac{1}{\epsilon} > 0 \in \mathbb{R}$, there exists some δ such that

$$\frac{1}{|h(x)|} > K.$$

Note that determining $\lim_{x\to a} \frac{1}{h(x)}$ *(without the absolute) would require knowing the sign of* $h(x)$ *as it is going to 0.*

□

Note: Remaining indeterminate forms are:

1. $\infty - \infty$
2. $\infty \times -\infty$
3. $0 \times \infty$
4. $\frac{0}{0}$
5. $\frac{\infty}{\infty}$

which are handled by algebraic manipulation and L'Hôpital's rule (see: 4.1.1).

---

(95) **Using the definition of the limit we can deduce that if**

$$\boldsymbol{f : \mathbb{R} \mapsto \mathbb{R} \text{ s.t. } f(x) = x^2 + x}$$

then

$$\boldsymbol{\lim_{x\to 2} f(x) = 6}.$$

Let $0 < |x - 2| < \delta$ and consider some arbitrary $\epsilon > 0$. Then we have,

$$\left|(x^2 + x) - 6\right| = |(x - 2)(x + 3)| \leq |x - 2|\,|x + 3| < \delta\,|x + 3|\,.$$

It might be tempting at this point to say that, since we are examining the behaviour when x *approaches 2, we can assume* $x \approx 2 \iff (x + 3) \approx 5$ *and then we can set* $\delta = \frac{\epsilon}{6}$*, but we need to find explicit bounds for expressions.*

We can use the triangle inequality (1.2.3.1) to determine that,

$$|x+3| = |x-2+5| \leq |x-2| + 5 < \delta + 5$$

so that

$$\Big|(x^2+x)-6\Big| < \delta\,|x+3| < \delta(\delta+5) = \delta^2 + 5\delta.$$

At this point we could start solving the quadratic in δ and looking for a way to appropriately select one of the two solutions but it's clearly better to find a simpler way.
What we have determined so far is that: for x in the δ-neighbourhood of 2, the expression $\delta^2 + 5\delta$ represents an upper bound on the expression we are trying to bound, $\Big|(x^2+x)-6\Big|$. Therefore any value greater than $\delta^2 + 5\delta$ is also a valid upper bound.
Since we are interested in the behaviour as $x \to 2$, we want small values of δ, and if $\delta < 1$ then $\delta^2 < \delta$ so

$$\delta < 1 \implies \delta^2 + 5\delta < 6\delta$$

and 6δ is also an upper bound.

However, although we are interested in small values of δ, the definition of the limit (3.1.4.1),

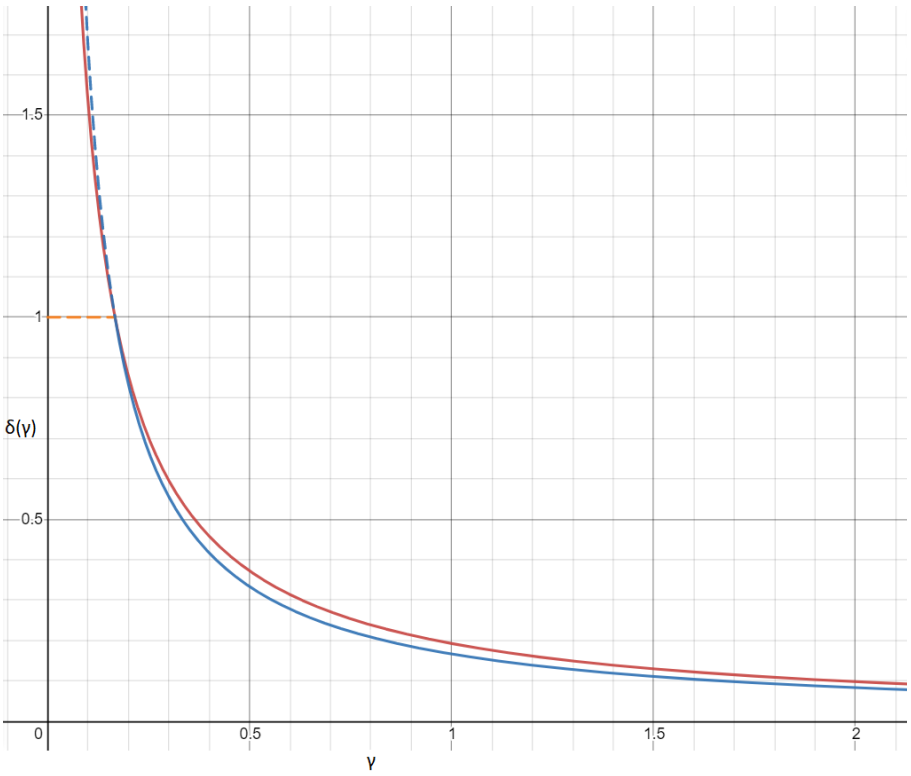
$$\forall \epsilon > 0 \in \mathbb{R}\ .\ \exists \delta \in \mathbb{R}\ .\ 0 < |x-a| < \delta \implies |f(x) - L| < \epsilon,$$

requires the limit condition to be true for all $\epsilon > 0$. If we create the simple bijective relation between δ and ϵ,

$$\epsilon(\delta) = 6\delta \iff \delta(\epsilon) = \frac{\epsilon}{6},$$

then for $\epsilon > 6$ we will have $\delta > 1$ and, as a result, 6δ will no longer be an upper bound.

Below is a graph of δ as a function of $\gamma = \frac{1}{\epsilon}$ so that $\epsilon \to 0$ is represented by $\gamma \to \infty$.
The red line is the inverse of the function $\delta^2 + 5\delta$ (for positive values) that we have established as an upper bound on the value of $\Big|(x^2+x)-6\Big|$



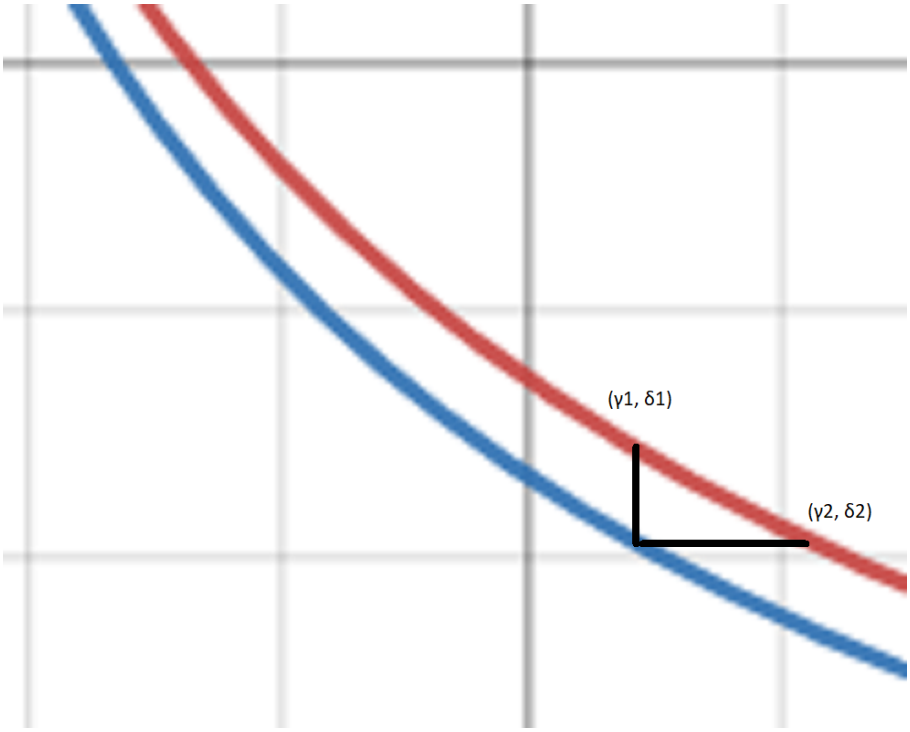
and the blue line is the inverse of 6δ (so $\delta = \frac{\epsilon}{6} = \frac{1}{6\gamma}$).

As we can see, for values of $\epsilon < 6 \iff \gamma > \frac{1}{6}$, the blue line is underneath the red line which tells us that if we use those values for δ then we will definitely satisfy the limit condition. This can be seen more clearly in the below maginification of an area of the graph.

By using the function represented by the blue line, for the value $\gamma = \gamma_1$ (which corresponds to some value of ϵ) we return — instead of δ_1 — the value δ_2 with $\delta_2 < \delta_1$. The red line tells us that δ_2 actually suffices for a bound $\gamma_2 > \gamma_1$. We therefore have

$$|x - 2| < \delta_2 \implies \left|(x^2 + x) - 6\right| < \frac{1}{\gamma_2} < \frac{1}{\gamma_1}$$

which shows that the blue line gives a tighter bound than necessary and so is also valid. Referring back to the main graph, we can see that



the blue dotted line shows that the blue line crosses the red line at $\epsilon = 6 \iff \gamma = \frac{1}{6}$ and so is no longer valid for achieving the required ϵ-neighbourhood.

However, instead, we can use the orange dotted line as values of δ for the remaining values of γ. This corresponds to the following function,

$$\delta(\epsilon) = \begin{cases} \frac{\epsilon}{6} & \epsilon \leq 6 \\ 1 & \epsilon > 6 \end{cases}$$

$$= \min\left\{\frac{\epsilon}{6}, 1\right\}.$$

If we use this function to define the δ for any ϵ then, since *both* 1 and $\frac{\epsilon}{6}$ are an upper bound on the value of δ (which means that even if ϵ is arbitrarily large, δ is still upper-bounded by 1), we can therefore reason that,

$$\left|(x^2 + x) - 6\right| < \delta(\delta + 5) \leq 6\delta \leq 6\frac{\epsilon}{6} = \epsilon.$$

3.1.4.2 Continuity

Definition 176. **(Continuity at a point)** A function f is *continuous at a point* a if

- $f(a)$ is defined,
- $\lim_{x\to a} f(x) = f(a)$.

More formally, the definition of $\lim_{x\to a} f(x) = L$ is,

$$\forall \epsilon > 0 \;.\; \exists \delta \text{ s.t. } 0 < |x - a| < \delta \implies |f(x) - L| < \epsilon.$$

But if $f(a)$ is defined, then when $|x - a| = 0$ (i.e. when $x = a$) we have

$$f(x) = f(a) \implies |f(x) - f(a)| = 0 < \epsilon.$$

Therefore, if $f(a)$ is defined and $\lim_{x\to a} f(x) = f(a)$, then

$$\forall \epsilon > 0 \;.\; \exists \delta > 0 \text{ s.t. } |x - a| < \delta \implies |f(x) - f(a)| < \epsilon.$$

Definition 177. **(Continuous Function)** A function is *continuous* if it is *continuous at every point.*

Definition 178. **(Continuity on a closed interval)** A function is *continuous on the closed interval* $[a, b]$ if it is

- continuous at every point in the open interval (a, b),
- $\lim_{x\to a^+} f(x) = f(a)$,
- $\lim_{x\to b^-} f(x) = f(b)$.

Definition 179. **(Left and Right-Continuity)** A function is *left continuous* or *continuous on the left* at a if,

$$\lim_{x \to a-} f(x) = f(a).$$

Obviously, from the other side, a function can be *right continuous.*

Proposition 3.1.25. *Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions that are continuous at $a \in \mathbb{R}$ and c be any real number. Then $|f|, (cf), (f-g), (f+g), (f(x)g(x))$ are all continuous at a, and (f/g) is continuous provided $g(x) \neq 0$ for any x in some neighbourhood of a.*

Proof. This follows from the algebra of finite limits of functions given in Proposition 3.1.23. □

Corollary 3.1.3. *It follows then that every polynomial is continuous. This can be seen as the most simple polynomial is a constant - which is clearly continuous; also $f(x) = x$ is clearly continuous. Then powers of x are continuous as they are products of continuous functions and when multiplied by coefficients this is a constant multiplying a continuous function so the resultant function is continuous. Then any polynomial is a summation of such terms so the result is continuous as the sum of continuous functions is continuous.*

Proposition 3.1.26. *If g is a function which is continuous at a, and f is a function which is continuous at $g(a)$. Then $(f \circ g)$ is continuous at a.*

Proof. Continuity of f at $g(a)$ guarantees that for any $\epsilon > 0$ there exists some $\delta' > 0$ such that $|x' - g(a)| < \delta' \implies |f(x') - f(g(a))| < \epsilon$ and continuity of g at a guarantees that for any $\delta' > 0$ there exists some $\delta > 0$ such that $|x - a| < \delta \implies |g(x) - g(a)| = |x' - g(a)| < \delta'$. □

Corollary 3.1.4. *If g is a function which is continuous at a, and f is a function which is continuous at $g(a)$. Then,*

$$\lim_{x \to a}(f \circ g)(x) = \lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x)).$$

3.1.4.3 Functions over sequences

The following proposition gives an alternative definition of continuity over convergent sequences rather than explicit subsets of $\mathbb{R}$.

Proposition 3.1.27. *A function f is continuous at a if and only if for each sequence (x_n) such that $\lim_{n\to\infty} x_n = a$ we have $\lim_{n\to\infty} f(x_n) = f(a)$.*

> *Before the proof, an important point to note is that this theorem applies to "each sequence" with the described limit. This is important as it is possible to find an individual sequence such that the inference is not valid. For example, the constant sequence $\forall n \in \mathbb{N} \,.\, x_n = a$ clearly tends to a as $n \to \infty$ but this would not imply continuity of f as the limit of $f(x_n)$ for such a sequence would amount to saying that $f(a) = f(a)$. The definition of continuity is assertion of the equality of the limit of f over values in the neighbourhood of a (but not at a itself) with the value of f at a. So, continuity is implied by the fact that the limit of $f(x_n)$ for the constant sequence $x_n = a$ is equal to the limit of $f(x_n)$ for all other sequences x_n whose limit is a.*
> *Another way of looking at it is that f is continuous at a because the limit there equals $f(a)$ however the argument converges to a.*

Proof. Breaking it down into two propositions we have,

$$(\forall x_n \text{ s.t. } \lim_{n\to\infty} x_n = a) \ \lim_{n\to\infty} f(x_n) = f(a) \tag{P_1}$$

$$\forall \epsilon > 0 \,.\, \exists \delta > 0 \,.\, |x - a| < \delta \implies |f(x) - f(a)| < \epsilon \tag{P_2}$$

and we need to show that $P_1 \iff P_2$.

Unpacking P_1 we have a function f such that

$$\forall \epsilon > 0 \,.\, \exists N \,.\, \forall n > N \in \mathbb{N} \,.\, |f(x_n) - f(a)| < \epsilon \tag{1}$$

where x_n is a sequence such that

$$\forall \delta > 0 \,.\, \exists N' \,.\, \forall n > N' \in \mathbb{N} \,.\, |x_n - a| < \delta. \tag{2}$$

Firstly, it is easily shown that $P_2 \implies P_1$. We begin by assuming P_2 which was that,

$$\forall \epsilon > 0 \,.\, \exists \delta > 0 \,.\, |x - a| < \delta \implies |f(x) - f(a)| < \epsilon.$$

We can choose any ϵ and there exists some δ such that P_2 holds. Then, as we can see in (2), P_1 tells us that, for this value of δ,

$$\exists N' \,.\, \forall n > N' \in \mathbb{N} \,.\, |x_n - a| < \delta$$

and P_2 tells us that,

$$|x_n - a| < \delta \implies |f(x_n) - f(a)| < \epsilon.$$

Putting the two together we get,

$$\forall \epsilon > 0 \,.\, \exists \delta > 0 \,.\, \exists N' \,.\, \forall n > N' \in \mathbb{N} \,.\, |x_n - a| < \delta \implies |f(x_n) - f(a)| < \epsilon$$

which implies that

$$\forall \epsilon > 0 \,.\, \exists N \,.\, \forall n > N \in \mathbb{N} \,.\, |f(x_n) - f(a)| < \epsilon.$$

So we have shown that P_2 implies that (2) implies (1) which is $P_2 \implies P_1$ as required.

To prove the converse $P_1 \implies P_2$ we will use a proof by contradiction.
So, we are assuming P_1 but also assuming, for contradiction, that P_2 is false. Then we are negating the statement,

$$\forall \epsilon > 0 \,.\, \exists \delta > 0 \,.\, |x - a| < \delta \implies |f(x) - f(a)| < \epsilon$$

so we are asserting that,

$$\exists \epsilon > 0 \,.\, \nexists \delta > 0 \,.\, |x - a| < \delta \implies |f(x) - f(a)| < \epsilon$$

which is equivalent to

$$\exists \epsilon > 0 \,.\, \forall \delta > 0 \,.\, |x - a| < \delta \not\implies |f(x) - f(a)| < \epsilon$$

or alternatively,

$$\exists \epsilon > 0 \,.\, \forall \delta > 0 \,.\, \exists x \,.\, (|x - a| < \delta) \wedge (|f(x) - f(a)| \geq \epsilon). \qquad (*)$$

So (*) says that $\neg P_2$ is equivalent to the statement that there exists some $\epsilon > 0$ such that for all $\delta > 0$ there will be some x in the δ-neighbourhood of a such that $|f(x) - f(a)| \geq \epsilon$. To prove that this implies $\neg P_1$ we will need to show that it follows that there exists some sequence x_n such that $x_n \to a$ as $n \to \infty$ but also that $\lim_{n\to\infty} f(x_n) \neq f(a)$.

Now, if we choose a value of δ that depends on a natural number n in such a way that $\delta \to 0$ as $n \to \infty$ – for example, if $\delta = 1/n$ – then the δ-neighbourhoods around a will get smaller as $n \to \infty$. Then we can select an x from the δ-neighbourhood that corresponds to a particular value of n and call it x_n and, in this way, we create a sequence x_n that converges to a. So we have $\lim_{n\to\infty} x_n = a$.

But now, if we fix the value of ϵ to a value that satisfies (*) then (*) tells us that, for every δ there is an x_n in the δ-neighbourhood of a with $|f(x_n) - f(a)| \geq \epsilon$. We can choose this value for x_n and so, construct a sequence x_n that converges to a but

$$\lim_{n\to\infty} f(x_n) \neq f(a).$$

This contradicts hypothesis P_1 and so $\neg P_2 \implies \neg P_1$. □

Corollary 3.1.5. *For a function f that is continuous at a,*

$$\lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right).$$

Proof. This is just a rewriting of Proposition 3.1.27. □

3.1.4.4 Continuous Functions on Closed Intervals

3.1.4.5 Examples of continuity

(96) The indicator function for rational numbers within the reals, known as the **Dirichlet Function**,

$$f(x) = \begin{cases} 0 & x \text{ is irrational} \\ 1 & x \text{ is rational} \end{cases}.$$

This function is nowhere continuous because between every two irrational numbers there is a rational number (and vice-versa) so $f(x)$ is flipping between 0 and 1 in every neighbourhood of every point however small a neighbourhood we consider. So this function never converges anywhere but *is* bounded because it only ever takes values of 1 or 0 so, clearly, its maximum is 1 and minimum is 0.

(97) The function $f(x) = 1/x$ over the interval $(0, 1]$ is continuous at every point but unbounded as it goes to infinity as $x \to 0$. If we consider the same function over the closed interval $[0, 1]$ then we no longer have continuity over this interval as there is a singularity at $x = 0$.

However, if we define the function at $x = 0$ so that we have a new function,

$$g(x) = \begin{cases} 1/x & x \neq 0 \\ 1 & x = 0 \end{cases}$$

then the function g is unbounded on the closed interval $[0, 1]$ despite also being defined at every point in the interval. The reason is that we have removed the singularity at 0 by defining a finite value at the point. This creates a jump-discontinuity at 0 so we have sacrificed continuity in order to achieve this.

3.1.4.6 Extreme Value Theorem

TODO: consider rewrite:
If a function f is continous on the closed interval $[a, b]$ then it attains a maximum and minimum on the interval.

Theorem 3.1.11. *Let f be continous on $[a, b]$. Then f is bounded on $[a, b]$ and it achieves its maximum; that's to say, the supremum is equal to the maximum.*

Note that, even if f is defined at every point in $[a, b]$, if it is not continuous then it may not be bounded. There do exist functions that are not continuous but bounded (for example the Dirichlet Function 96) but there also exist functions that are not continuous and unbounded such as the reciprocal function 97. So, functions that are not continuous on a closed interval may

be bounded or not; and functions that are continuous on an open interval also may or may not be bounded (again, the reciprocal function is an example of a function that is continuous on an open interval but not bounded); but here we will show that functions that are continuous on a closed interval must be bounded on that interval.

Proof. It may be tempting to begin trying to prove this by reasoning as follows.

> That f is continuous on $[a, b]$ means that, for any $c \in (a, b)$,
>
> $$\forall \epsilon > 0 \,.\, \exists \delta > 0 \,.\, |x - c| < \delta \implies |f(x) - f(c)| < \epsilon.$$
>
> This means that the value of $f(x)$ must be finite everywhere in (a, b) as, choosing any fixed point c in the open interval, $|x - c|$ is finite and, therefore, less than some δ thus implying that $|f(x) - f(c)| < \epsilon$ for some finite $\epsilon \ldots$

However this is **dead wrong!** If we take the example of the reciprocal function (97) over the interval $(0, 1]$: If we take x-values approaching 0, the value of $f(x)$ grows unbounded. For any given x-value it will be less than some finite ϵ but we can always find another x-value with a greater value of $f(x)$. So, there is no maximum ϵ and so, also no maximum value of $f(x)$ on the interval.

Another tempting way to prove this is as follows.

> A closed interval is an interval such that every sequence of values in the interval converges to a point in the interval.
> That f is continuous on $[a, b]$ implies that for every sequence, x_n, of values in $[a, b]$ that converges to some point in $c \in [a, b]$, $\lim_{n\to\infty} f(x_n) = f(c)$. Furthermore, that the interval is closed implies that every sequence of values in the interval converges to a point in the interval. Therefore, we can conclude that $f(x)$ at every point in $[a, b]$ exists and is finite so that f is bounded and obtains a maximum on the interval.

There are two issues with this:

1. The statement about the nature of a closed interval that the proof relies upon has not been proven.

2. That $f(x)$ is defined and finite for all $x \in [a, b]$ is taken as proof that the function is bounded and obtains a maximum in the interval.

To deal with issue (1) – if we're not going to prove the proposition about closed intervals as a pre-requisite of the proof – we need to develop a proof that doesn't rely on this characteristic of closed intervals. To deal with (2) meanwhile, we need to explicitly prove boundedness and that f obtains a maximum.
The proof given in LSE Abstract Mathematics course material follows.

> Suppose first that f is unbounded above. For each $n \in N$, let x_n be a point in $[a, b]$ such that $f(x_n) > n$. The sequence (x_n) is bounded, so has a convergent subsequence (x_{k_n}), tending to some limit c (by Theorem 10.11). Necessarily $c \in [a, b]$. Since f is continuous at c, $f(x_{k_n}) \to f(c)$ as $n \to \infty$. But this contradicts the construction of the sequence (x_n), since $f(x_{k_n}) > n \to \infty$. So f is bounded above. Let $M = \sup\{\, f(x) \mid x \in [a, b] \,\}$. For each $n \in N$, let x_n be a point in $[a, b]$ such that $f(x_n) > M - \frac{1}{n}$. Again take a convergent subsequence (x_{k_n}) of (x_n), tending to some limit $c \in [a, b]$. Arguing as before, we see $f(c) = M$.

This proof says: Assume that f is unbounded on the interval. Then we can construct a sequence of x-values such that, for the nth value x_n, $f(x_n) > n$. This is possible because, if f is unbounded, for any value of n, there is some subinterval of x-values such that for each of them $f(x) > n$ and any interval of the real numbers contains an infinite number of real numbers and so, a sequence x_n with $n \to \infty$. Note that, at this point, x_n is an arbitrary sequence which is not necessarily convergent (it could bounce around the interval).
Then, we notice that this sequence x_n is necessarily bounded (as it is a subinterval of $[a, b]$) and so we invoke the Bolzano-Weierstrass Theorem, Theorem 3.1.2 (called Theorem 10.11 in the quoted proof), to deduce that it has a convergent subsequence which we call x_{k_n} and call its limit c.
At this point we can use the continuity of f on the interval to deduce that $f(x_{k_n}) \to f(c)$ as $n \to \infty$ by which we obtain a contradiction to the condition we set on the values of x_{k_n} when we constructed the sequence – namely that $f(x_{k_n}) > n$. Notice that this is constructing a sequence x_{k_n} such that $f(x_{k_n})$

grows without bound as $n \to \infty$ and then saying, "but the limit of x_{k_n} as $n \to \infty$ is c which is inside the interval and so (by continuity) the limit of $f(x_{k_n})$ as $n \to \infty$ is $f(c)$ – a fixed finite value". This is the point where the fact that the interval $[a, b]$ is closed comes into play – if the interval were not closed it would be possible that c was not inside the interval and then we would not be able to invoke continuity to assert that the limit of $f(x_{k_n})$ over this sequence was finite.
So, now we have shown that if a function is continuous over a closed interval then assuming that the function is unbounded produces a contradiction and, therefore, we can conclude that it is, in fact, bounded.
The last thing that needs to be proven is that f obtains its maximum in the interval. Having shown that the function is bounded on the interval we now know that there exists

$$M = \sup\{\, f(x) \mid x \in [a, b] \,\}$$

and we need to show that there is such an x-value in $[a, b]$ that $f(x) = M$. In an open interval this might not be the case as the supremum of the function on the interval might occur as the limit of f over a sequence of x-values converging to a point that lies outside the interval. So, in this case, we construct a convergent sequence such that $f(x_{k_n}) \to M$ as $n \to \infty$ by selecting x_n such that $f(x_n) = M - \frac{1}{n}$. This is possible because the definition of the supremum says that, because it is the *lowest* upper bound,

$$\forall \epsilon > 0 \,.\, \exists f(x_n) \,.\, f(x_n) > M - \epsilon$$

and so we are letting $\epsilon = \frac{1}{n}$. Then, as previously, we take a convergent subsequence of this sequence and name it x_{k_n}. So, as before, we have a sequence converging on some point, we'll call it $c \in [a, b]$, at which f is continuous so that $f(x_{k_n}) \to M$ as $n \to \infty$ implies that $M = f(c)$. This, in turn, means that f obtains a maximum in the interval. □

3.1.4.7 Intermediate Value Theorem

Theorem 3.1.12. *Let f be continuous on $[a, b]$ with $f(a) < f(b)$. Then, for all K s.t. $f(a) < K < f(b)$, there exists some $c \in (a, b)$ with $f(c) = K$.*

Note that this theorem is ***not*** *written for an interval such that* $f(a) \leq f(b)$ *because if* $f(a) = f(b)$ *then the only* K *s.t.* $f(a) \leq K \leq f(b)$ *is* $f(a) = K = f(b)$*. But now it is* ***not*** *true to say that there exists some* $c \in (a, b)$ *with* $f(c) = K$ *as there is no reason why the value of* $f(x)$ *at the bounds of the interval should be repeated somewhere in the interior of the interval.*

Question: What is wrong with the following proof?

Proof. Suppose $\nexists\, x \in [a, b]$ s.t. $f(a) < f(x) < f(b)$. Since we know that at $x = a$ the value is $f(a)$ and at $x = b$ the value is $f(b)$ with $f(b) > f(a)$, we can deduce that,

$$\exists c \in [a, b] \text{ s.t. } \lim_{x \to c^-} f(x) = f(a) \wedge \lim_{x \to c^+} f(x) = f(b).$$

But this contradicts the hypothesis that f is continuous on the entire interval. Therefore, there must exist some

$$x \in [a, b] \text{ s.t. } f(a) < f(x) < f(b).$$

But then we can recursively apply this same logic to the intervals $[f(a), f(x)]$ and $[f(x), f(b)]$ to show that there must be values of f between them also. Since it is always possible for us to apply the reasoning to any interval however small, we can conclude that between any distinct values of the function $f(x_1) < f(x_2)$, there must be an intermediate value of the function $f(x_1) < f(x_3) < f(x_2)$. $\square$

We begin by proving a special case from which the general proof will follow.

Lemma 3.1.8. *Let* f *be continuous on* $[a, b]$ *with* $f(a) < 0 < f(b)$*. Then there exists some* $c \in (a, b)$ *with* $f(c) = 0$*.*

Proof. A first attempt at this proof is given below.

> Continuity at the interval bounds a and b means that, for some $\epsilon_1, \epsilon_2 > 0$,
>
> $$\exists \delta_1 \,.\, 0 \leq x - a < \delta_1 \implies |f(x) - f(a)| < \epsilon_1,$$

$$\exists \delta_2 \,.\, 0 \leq b - x < \delta_2 \implies \big|f(x) - f(b)\big| < \epsilon_2.$$

So, we have a lower neighbourhood around $f(a)$ and an upper neighbourhood around $f(b)$ as follows,

$$f(a) - \epsilon_1 < f(x) < f(a) + \epsilon_1,$$

$$f(b) - \epsilon_2 < f(x) < f(b) + \epsilon_2$$

If $\epsilon_1 > \big|f(a)\big|$ and $\epsilon_2 > \big|f(b)\big|$ then both neighbourhoods contain $f(x) = 0$. Therefore, there is some interval of x such that $f(x)$ lies inside both the lower and upper neighbourhood – in the overlap of the two. In the overlap we have,

$$f(b) - \epsilon_2 < f(x) < f(a) + \epsilon_1.$$

Now if we let ϵ_1 vary freely but make ϵ_2 a function of ϵ_1 like so,

$$\epsilon_2 = f(a) + f(b) + \epsilon_1$$

then we still have $\epsilon_1 > \big|f(a)\big|$ and $\epsilon_2 > \big|f(b)\big|$ as,

$$\epsilon_1 > \big|f(a)\big| \iff -\epsilon_1 < f(a) < \epsilon_1 \iff 0 < f(a) + \epsilon_1 < 2\epsilon_1$$

so $\epsilon_2 > f(b)$ and, conversely, if we assume that $\epsilon_2 > \big|f(b)\big|$ then,

$$\epsilon_2 > \big|f(b)\big| \iff -\epsilon_2 < f(b) < \epsilon_2 \iff -\epsilon_2 - f(b) < 0 < \epsilon_2 - f(b)$$

$$\epsilon_1 = \epsilon_2 - f(a) - f(b) > -f(a) = \big|f(a)\big| \qquad \text{since } f(a) < 0.$$

Therefore,

$$\epsilon_2 = f(a) + f(b) + \epsilon_1 \implies [\epsilon_1 > \big|f(a)\big| \iff \epsilon_2 > \big|f(b)\big|].$$

Now we have,

$$f(b) - \epsilon_2 = -f(a) - \epsilon_1 < f(x) < f(a) + \epsilon_1$$

which is equivalent to

$$\big|f(x)\big| < f(a) + \epsilon_1 = \epsilon_3.$$

Now, since $f(a) + \epsilon_1 > 0$ we also have $\epsilon_3 > 0$ and it can become arbitrarily small by making $\big|f(a)\big| - \epsilon_1$ arbitrarily small. So we have shown that continuity over the closed interval and $f(a) < f(b)$ imply that we can find subintervals of $[a, b]$ such that $f(x)$ is constricted to ever-decreasing neighbourhoods of 0. In other words, for some c s.t. $a < c < b$, $f(x) \to 0$ as $x \to c$ and since f is continuous on the interval this implies that $f(c) = 0$ also.

This is not bad but suffers from vagueness in a couple of areas: the overlap of the lower and upper neighbourhoods probably needs to be more precisely defined and, certainly, the final part of the proof stating that confining $f(x)$ to ever-decreasing neighbourhoods of 0 implies that there is some c such that $f(c) = 0$ needs to be drawn much more explicitly.

Here is the proof given in LSE Abstract Mathematics.

> We construct a sequence of intervals $[a_n, b_n]$ such that
>
> 1. $f(a_n) < 0,\ f(b_n) > 0$ for each n
> 2. $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$ for each n.
>
> We start by letting $[a_1, b_1] = [a, b]$. Then for each $n \geq 1$, we define $[a_{n+1}, b_{n+1}]$ as follows.
>
> Let $c_n = (a_n + b_n)/2$, be the midpoint of the previous interval. If $f(c_n) = 0$, then the theorem is proved and so we need not continue constructing intervals!
>
> Otherwise, if $f(c_n) < 0$, we define $a_{n+1} = c_n$ and $b_{n+1} = b_n$. And if $f(c_n) > 0$, we define $b_{n+1} = c_n$ and $a_{n+1} = a_n$. Note that the condition 1. is satisfied by choosing our intervals in this manner. Moreover, note that the $(n + 1)$st interval is half the size of the nth interval and so $b_{n+1} - a_{n+1} \leq (b_1 - a_1)/2^n$. It follows that
>
> $$\lim_{n\to\infty} (b_n - a_n) = 0. \tag{3}$$
>
> Finally, note that (a_n) is increasing and bounded above (by b_1) and so it has a limit; similarly (b_n) is decreasing and bounded below and so has a limit. Thus by (3) (and algebra of limits) these limits are equal to, say, c. Thus by continuity (using Proposition 3.1.27),
>
> $$f(c) = \lim_{n\to\infty} f(b_n) \geq 0,$$
>
> where the last inequality follows from the fact that each $f(b_n) \geq 0$ (in fact > 0). Similarly,
>
> $$f(c) = \lim_{n\to\infty} f(a_n) \leq 0.$$
>
> Thus $f(c)$ must be equal to zero, and the proof is complete.

□

Clearly, the general proof of the Intermediate Value Theorem follows naturally from this because,

- If $f(a) > f(b)$ then we can consider the function $g(x) = -f(x)$ so that we have $g(a) < g(b)$,
- If we have K s.t. $f(a) < K < f(b)$ with $K \neq 0$ we can consider $g(x) = f(x) - K$ so that we have $g(a) < 0 < g(b)$.

So, the general problem posed in the Intermediate Value Theorem is reducible to the case we have proved.

Corollary 3.1.6. *Suppose that the real function f is continuous on the closed interval $[a, b]$ and that f maps $[a, b]$ into $[a, b]$. Then there is $c \in [a, b]$ with $f(c) = c$.*

Proof. Let $h(x) = f(x) - x$ so that $f(c) = c$ if and only if $h(c) = 0$. Then also we have,

$$a \leq f(x) \leq b \implies h(a) \geq 0,\ h(b) \leq 0.$$

But this means that, either one of $h(a)$ or $h(b)$ is equal to 0 or neither are. In the case that one of them is equal to 0 then we have found our c such that $f(c) = c$. Otherwise, if neither is equal to 0, then we must have $h(a) > 0$ and $h(b) < 0$. So, we may apply the Intermediate Value Theorem to conclude that there exists $c \in (a, b)$ such that $h(c) = 0$ which is to say $f(c) = c$. □

3.1.4.8 Examples of reasoning with the Intermediate Value Theorem

(98) *Suppose the real function f is continuous, positive and unbounded on $\mathbb{R}$ and that $\inf\{\, f(x) \mid x \in \mathbb{R} \,\} = 0$. Use the Intermediate Value Theorem to prove that the range of f is $(0, \infty)$.*

Let $y \in (0, 1)$. We show that there is some $c \in \mathbb{R}$ such that $f(c) = y$. This shows that the range is the whole of $(0, 1)$. (The fact that it is no larger follows from the given fact that f is positive.)
Now, $\inf f(\mathbb{R}) = \inf\{\, f(x) \mid x \in \mathbb{R} \,\} = 0$, so, since $y > 0$, there must be some $y_1 \in f(\mathbb{R})$ with $y_1 < y$. This means there is some $x_1 \in \mathbb{R}$

such that $y_1 = f(x_1) < y$.
Similarly, because f is unbounded, which means $f(\mathbb{R})$ is unbounded, there must be some $y_2 \in f(\mathbb{R})$ with $y_2 > y$ and there will be some $x_2 \in \mathbb{R}$ such that $y_2 = f(x_2) > y$.
Then y lies between $f(x_1)$ and $f(x_2)$ and, since f is continuous, the Intermediate Value Theorem shows that there is some c between x_1 and x_2 with $f(c) = y$.

(99) *Suppose the real function g is continuous on $\mathbb{R}$ and that g maps $[a, b]$ into $[d, e]$ and maps $[d, e]$ into $[a, b]$ where $a < b,\ d < e$. By considering the function*

$$k(x) = g(g(x)),$$

prove that there are $p, q \in \mathbb{R}$ such that

$$g(p) = q,\ g(q) = p.$$

Hence show that there is $c \in \mathbb{R}$ such that $g(c) = c$.

The function k, being a composition of continuous functions, is continuous and also maps $[a, b]$ into $[a, b]$ so that we can use Corollary 3.1.6 to deduce that there exists $c \in [a, b]$ such that $k(c) = c$. If we let $p = c$ and $q = g(p)$ then we have,

$$k(p) = p \iff g(g(p)) = p \iff g(q) = p.$$

Now we can employ the same trick again by defining

$$h(x) = g(x) - x$$

and then we have

$$h(p) = g(p) - p = q - p,\ h(q) = g(q) - q = p - q$$

so that $h(p) = -h(q)$ and, therefore, $h(x)$ changes sign between p and q. (Note that we don't know which of p and q is the lower end and upper end of the interval but we know that between the two values the function changes sign.) Then, applying the Intermediate Value Theorem we have some c between p and q such that,

$$h(c) = 0 \iff g(c) - c = 0 \iff g(c) = c.$$

3.1.4.9 Relationship Between Sequences and Functions

Let $f : \mathbb{R} \mapsto \mathbb{R},\ f(x) = y$. Now imagine that we take regular intervals on the domain, say, interval 1. We name the x-values at the upper bound of these intervals $x_1, x_2, \ldots$ for $x = 1, 2, \ldots$. Then, the corresponding y-values, $y = f(x_1), f(x_2), \ldots$ can be named $y_1, y_2, \ldots$. In this way, we have defined a sequence, $y_n = f(x_n)$ where $x_n \in \mathbb{N}$ (note that this is a different notation from that used before where a sequence $x_n = f(n)$ for $n \in \mathbb{N}$).
Looking at the derivative of the function,

$$\frac{dy}{dx} \approxeq \frac{\Delta y}{\Delta x} = \frac{f(x_{n+1}) - f(x_n)}{1} = f(x_{n+1}) - f(x_n).$$

While the ratio of consecutive terms is,

$$\frac{y_{n+1}}{y_n} = \frac{y_n + \Delta y}{y_n} = \frac{f(x_{n+1})}{f(x_n)} = \frac{f(x_n) + (f(x_{n+1}) - f(x_n))}{f(x_n)}.$$

Note, also, that

$$\frac{y_n + \Delta y}{y_n} = 1 + \frac{\Delta y}{y_n} \approxeq 1 + \frac{dy/dx}{y} = 1 + \frac{d}{dx} \ln y$$

which is to say that $\frac{\Delta y}{y_n}$ is the discrete form of the log derivative. Clearly, when $\Delta y > 0$, we can put this in the form

$$\frac{y_{n+1}}{y_n} = 1 + \frac{\Delta y}{y_n} = 1 + \frac{\Delta y / y_n}{1} = \frac{1 + h}{1}.$$

If $\Delta y < 0$ however,

$$\frac{y_n + \Delta y}{y_n} = \frac{1}{y_n/(y_n + \Delta y)} = \frac{1}{\frac{y_n + \Delta y - \Delta y}{y_n + \Delta y}} = \frac{1}{1 + \frac{-\Delta y}{y_n + \Delta y}} = \frac{1}{1 + h}.$$

If $\frac{\Delta y}{y_n}$ does not go to 0 then the ratio of consecutive terms stays below 1 and the sequence converges to 0. If it does go to 0 then the ratio of consecutive terms goes to 1 and the sequence may converge to 0 or to some non-zero value. These cases appear (proof?) to be distinguishable by looking at how fast $\frac{\Delta y}{y_n}$ goes to 0. For example,

(i) $\boldsymbol{a_n = \frac{1}{n} + 1}$

$$\lim_{n\to\infty} \frac{1}{n} + 1 = 1$$

$$\frac{\Delta a}{a_n} = \frac{-1}{(n+1)^2}$$

(ii) $\boldsymbol{a_n = \frac{1}{n}}$

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

$$\frac{\Delta a}{a_n} = \frac{-1}{n+1}$$

Maybe the fact that $\frac{\Delta y}{y_n}$ goes to 0 faster as $n \to \infty$ in the first case indicates that it converges before it gets to 0?

3.1.5 Differentiation

Chapter 4

Calculus

4.1 Limits

4.1.1 Calculating Limits

(1) If the expression is a composition of functions that tend to finite limits (with the exception of a denominator tending to 0) then we can use the algebra of limits (Proposition 3.1.23).

(2) Otherwise we can try to use the algebra of infinite limits (Proposition 3.1.24).

(3) Else look for algebraic manipulations to reduce the expression to one that can be solved using (1) or (2).

(4) Else use L'Hôpital's Rule.

For example, the forms:

1. $(\infty) - (\infty)$, $(\infty) \times (-\infty)$: These are case (3) in which we look for algebraic manipulations that reduce the problem to case (1) or (2).
2. $\frac{0}{0}$, $\frac{\infty}{\infty}$: Case (4), L'Hôpital's Rule
3. $(0) \times (\infty)$: This case can be reduced to (4) by converting the product to a ratio.
4. $\frac{0}{\infty}$: This case can be reduced to (1) by expressing it as $(0) \times (\frac{1}{\infty})$ and then treating the second operand (the ratio) as case (2) to obtain the form $(0) \times (0) = 0$.

---

(100) $\lim_{x\to\infty}\left[\left(x^3+x^2\right)^{\frac{1}{3}}-x\right]$:

First, note that

$$\lim_{x\to\infty}\left(x^3+x^2\right)^{\frac{1}{3}}=\infty \quad \text{and} \quad \lim_{x\to\infty} x=\infty.$$

So the expression we are trying to find the limit of takes the form $(\infty)-(\infty)$ and so this is case (3) and we need to look for algebraic manipulations that can transform it into one of the other cases.

We can use the difference of cubes (Theorem 2.2.1) as follows.

$$\begin{aligned}
&\left(x^3+x^2\right)^{\frac{1}{3}}-x \\
&=\left(x^3+x^2\right)^{\frac{1}{3}}-x\cdot\frac{\left(x^3+x^2\right)^{\frac{2}{3}}+x^2+x\left(x^3+x^2\right)^{\frac{1}{3}}}{\left(x^3+x^2\right)^{\frac{2}{3}}+x^2+x\left(x^3+x^2\right)^{\frac{1}{3}}} \\
&=\frac{\left(x^3+x^2\right)-x^3}{\left(x^3+x^2\right)^{\frac{2}{3}}+x^2+x\left(x^3+x^2\right)^{\frac{1}{3}}} && \text{by Theorem 2.2.1} \\
&=\frac{x^2}{\left(x^3+x^2\right)^{\frac{2}{3}}+x^2+x\left(x^3+x^2\right)^{\frac{1}{3}}} \\
&=\frac{\frac{1}{x^2}}{\frac{1}{x^2}}\cdot\frac{x^2}{\left(x^3+x^2\right)^{\frac{2}{3}}+x^2+x\left(x^3+x^2\right)^{\frac{1}{3}}} \\
&=\frac{1}{\left(1+\frac{1}{x}\right)^{\frac{2}{3}}+1+\frac{1}{x}\left(x^3+x^2\right)^{\frac{1}{3}}} \\
&=\frac{1}{\left(1+\frac{1}{x}\right)^{\frac{2}{3}}+1+\left(1+\frac{1}{x}\right)^{\frac{1}{3}}}.
\end{aligned}$$

Now we have reduced it to an expression that is a composition of functions that have finite limits so it can now be solved as an instance

of case (1).

$$\lim_{x\to\infty}\left[\left(x^3+x^2\right)^{\frac{1}{3}}-x\right]$$

$$=\lim_{x\to\infty}\frac{1}{\left(1+\frac{1}{x}\right)^{\frac{2}{3}}+1+\left(1+\frac{1}{x}\right)^{\frac{1}{3}}}$$

$$=\frac{\lim_{x\to\infty}1}{\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{\frac{2}{3}}+\lim_{x\to\infty}1+\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{\frac{1}{3}}}\quad\text{by Proposition 3.1.23}$$

$$=\frac{1}{1+1+1}=\frac{1}{3}.$$

4.1.1.1 L'Hôpital's Rule

see: UTexas

- L'Hôpital's Rule does not apply if the limit of derivatives does not exist (though some sources say that the limit being infinite is an 'existing' limit in this case - which would leave only oscillatory divergent functions as not having limits - needs confirmation)
- math.stackexchange discussion
- Example of L'Hôpital's Rule failing to find a simple limit:

$$\lim_{x\to\infty}\frac{\sqrt{4x^2+3}}{x+3}$$

4.1.2 Classes of Asymptotic Behaviour

We can identify three classes of asymptotic behaviour of a function $f(x)$ as $x\to a$ (where a may be infinity), distinguished by the value of

$$L=\lim_{x\to a}f(x).$$

These are functions $f(x)$ such that:

(i) $L = 0$;

(ii) $0 < L < \infty$;

(iii) $L = \infty$.

Functions of classes (i) and (ii) can always be combined using Proposition 3.1.23 apart from the specific exemption if a function of class (i) is a denominator. Linear combinations including one function of class (iii) are dominated by it. For example, if A is some number, we have,

$$A + \infty = \infty = \infty - A$$

and

$$A - \infty = -\infty = -A - \infty.$$

Products and ratios involving functions of classes (i) and (iii) are resolved by looking at the derivatives of the functions – i.e. using L'Hôpital's Rule.

4.1.2.1 Orders of Growth

see: wikipedia - Asymptotic Analysis

Let a test value T be defined as,

$$T = \lim_{t\to\infty} \frac{f(t)}{g(t)}.$$

Since,

$$\frac{f(t)}{g(t)} - 1 = \frac{f(t) - g(t)}{g(t)}$$

we therefore have

$$\lim_{t\to\infty} \frac{f(t)}{g(t)} = 1 \implies \lim_{t\to\infty} \frac{f(t)}{g(t)} - 1 = \lim_{t\to\infty} \frac{f(t) - g(t)}{g(t)} = 0.$$

This will be the case if the function,

$$h(x) = f(x) - g(x)$$

has a lower order of growth then $g(x)$. This can be seen most easily with polynomials. Say, for example,

$$f(x) = \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0 \quad \text{and} \quad g(x) = \beta_3 x^3 + \beta_2 x^2 + \beta_1 x + \beta_0$$

then we have

$$h(x) = (\alpha_3 - \beta_3)x^3 + (\alpha_2 - \beta_2)x^2 + (\alpha_1 - \beta_1)x + (\alpha_0 - \beta_0).$$

If $\alpha_3 = \beta_3$ then the order of h is lower than that of g and so $\frac{h(x)}{g(x)} \to 0$ which means that $T = 1$ and f and g have the same order.

In general, if f and g have the same degree, then the degree of h is less than or equal to the degree of g and so T will be 0 — in the case that the degree of h is lower than that of g — and some finite non-zero number in the case that the degree of h is the same as that of g.

On the other hand, if the degree of f is greater than that of g then the degree of h will also be greater than that of g and so $\frac{h(x)}{g(x)} \to \infty$ and the value of T will be infinite.

4.2 Homogeneity

4.2.1 Homogeneous Functions

Definition 180. A **homogeneous** function is a multivariate function $f(x_1, \ldots, x_n)$ such that,

$$f(\lambda x_1, \ldots, \lambda x_n) = \lambda^d f(x_1, \ldots, x_n) \qquad d \in \mathbb{Z},\ \lambda \in \mathbb{R}.$$

The integer power d is known as the **degree** so that f is described as **homogeneous of degree** d.

If $f : V \longmapsto W$ is a function between two vector spaces over a field $\mathbb{F}$, then f is said to be homogeneous of degree d if

$$f(\lambda \vec{\boldsymbol{v}}) = \lambda^d f(\vec{\boldsymbol{v}})$$

for all non-zero $\lambda \in \mathbb{F}$ and $\vec{\boldsymbol{v}} \in V$.

When defined on vector spaces over the reals, a more restricted definition is often used requiring only that,

$$f(\lambda \vec{\boldsymbol{v}}) = \lambda^d f(\vec{\boldsymbol{v}})$$

for all $\lambda > 0$.

Proposition 4.2.1. *If a linear map is homogeneous then it preserves the origin.*

Proof. Let $T : V \longmapsto W$ be a linear map between vector spaces over the

field $\mathbb{F}$ that is homogeneous of degree d. Then,

$$\begin{aligned}
\lambda^d T(\vec{\boldsymbol{v}}) &= T(\lambda\vec{\boldsymbol{v}}) && \text{by homogeneity of } T \\
&= T(\lambda(\vec{\boldsymbol{v}} + \vec{\mathbf{0}})) && \text{by vector axioms} \\
&= T(\lambda\vec{\boldsymbol{v}} + \lambda\vec{\mathbf{0}}) && \text{by vector axioms} \\
&= T(\lambda\vec{\boldsymbol{v}}) + T(\lambda\vec{\mathbf{0}}) && \text{by linearity of } T \\
&= T(\lambda\vec{\boldsymbol{v}}) + T(\vec{\mathbf{0}}) && \text{by vector axioms} \\
&= \lambda^d T(\vec{\boldsymbol{v}}) + T(\vec{\mathbf{0}}) && \text{by homogeneity of } T.
\end{aligned}$$

Therefore $T(\vec{\mathbf{0}}) = \vec{\mathbf{0}}$. □

Proposition 4.2.2. *Any linear map is homogeneous of degree 1.*

Proof. Let $T : V \longmapsto W$ be a linear map between vector spaces over the field $\mathbb{F}$. Then by the definition of a linear map,

$$T(\lambda\vec{\boldsymbol{v}}) = \lambda\, T(\vec{\boldsymbol{v}}) = \lambda^1\, T(\vec{\boldsymbol{v}})$$

for any $\vec{\boldsymbol{v}} \in V$ and $\lambda \in \mathbb{F}$. □

Corollary 4.2.1. *Any linear map is homogeneous of degree 1 and preserves the origin.*

Proof. By Proposition 4.2.2, any linear map is homogeneous of degree 1 and, by Proposition 4.2.1 it, therefore, preserves the origin. □

4.2.2 Homogeneous Polynomials

Definition 181. A **homogeneous polynomial or monomial** is a polynomial or monomial that, when considered as a function of its variables, forms a homogeneous function.

Proposition 4.2.3. *Any monomial is homogeneous with degree equal to the sum of the powers of the variables.*

Proof. Using the definition of a monomial (wikipedia) we form a function whose implementation is a monomial expression,

$$f(x_1, x_2, \ldots, x_n) = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}.$$

Then,

$$\begin{aligned} f(\lambda x_1, \lambda x_2, \ldots, \lambda x_n) &= (\lambda x_1)^{\alpha_1} (\lambda x_2)^{\alpha_2} \ldots (\lambda x_n)^{\alpha_n} \\ &= \lambda^{\alpha_1+\alpha_2+\cdots+\alpha_n} x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n} \\ &= \lambda^{\alpha_1+\alpha_2+\cdots+\alpha_n} f(x_1, x_2, \ldots, x_n). \end{aligned}$$

So f is homogeneous of degree $\alpha_1 + \cdots + \alpha_n$. □

Proposition 4.2.4. *A polynomial comprised of monomial terms of degree d is homogeneous of degree d.*

Proof. Let

$$p(x_1, \ldots, x_n) = m_1(x_1, \ldots, x_n) + \cdots + m_n(x_1, \ldots, x_n)$$

be a polynomial whose terms are given by the monomial functions $m_1, \ldots, m_n$. If each of the monomial functions m_i has degree d then each is homogeneous of degree d and,

$$\begin{aligned} p(\lambda x_1, \ldots, \lambda x_n) &= \lambda^d m_1(x_1, \ldots, x_n) + \cdots + \lambda^d m_n(x_1, \ldots, x_n) \\ &= \lambda^d [m_1(x_1, \ldots, x_n) + \cdots + m_n(x_1, \ldots, x_n)] \\ &= \lambda^d p(x_1, \ldots, x_n). \quad \square \end{aligned}$$

Proposition 4.2.5. *A homogeneous polynomial of degree d can be converted into a homogeneous polynomial of degree 1 by raising to the power $1/d$.*

Proof. Let $p(x_1, \ldots, x_n)$ be a homogeneous polynomial of degree d. Then by homogeneity of p,

$$p(\lambda x_1, \ldots, \lambda x_n) = \lambda^d p(x_1, \ldots, x_n)$$

and we define a function q such that,

$$q(x_1, \ldots, x_n) = [p(x_1, \ldots, x_n)]^{\frac{1}{d}}.$$

Now,

$$\begin{aligned} q(\lambda x_1, \ldots, \lambda x_n) &= [p(\lambda x_1, \ldots, \lambda x_n)]^{\frac{1}{d}} \\ &= [\lambda^d p(x_1, \ldots, x_n)]^{\frac{1}{d}} \\ &= \lambda [p(x_1, \ldots, x_n)]^{\frac{1}{d}} \\ &= \lambda q(x_1, \ldots, x_n). \quad \square \end{aligned}$$

Note that this seems obvious when considered in the abstract as in the proof here but, in practice, may not be obvious. For example, the implication is that,

$$f(x, y, z) = (x^k + y^k + z^k)^{\frac{1}{k}}$$

has the property that,

$$f(\lambda x, \lambda y, \lambda z) = \lambda f(x, y, z).$$

Proposition 4.2.6. *An arbitrary polynomial in n variables can be converted into a homogeneous polynomial of an arbitrary degree d in $n + 1$ variables.*

Proof. Let $p(x_1, \ldots, x_n)$ be an arbitrary non-homogeneous polynomial. Then p can be expressed as the sum of monomial functions,

$$p(x_1, \ldots, x_n) = \sum_i m_i(x_1, \ldots, x_n)$$

where each monomial function m_i has degree d_i. By Proposition 4.2.3, we have,

$$d_i = \sum_{j=1}^{n} \alpha_j$$

where the α_j are the exponents of each of the n variables in the monomial term m_i.

If we now define another polynomial function q such that,

$$\begin{aligned} q(x_0, x_1, \ldots, x_n) &= {x_0}^d \, p\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right) \\ &= {x_0}^d \sum_i m_i \left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right) \\ &= {x_0}^d \sum_i \left(\frac{1}{x_0}\right)^{d_i} m_i(x_1, \ldots, x_n) \\ &= \sum_i {x_0}^{d-d_i} \, m_i(x_1, \ldots, x_n) \end{aligned}$$

we can see now that the function q has the property that,

$$\begin{aligned} q(\lambda x_0, \lambda x_1, \ldots, \lambda x_n) &= \sum_i (\lambda x_0)^{d-d_i} \, m_i(\lambda x_1, \ldots, \lambda x_n) \\ &= \sum_i (\lambda^{d-d_i} \, {x_0}^{d-d_i}) \, \lambda^{d_i} \, m_i(x_1, \ldots, x_n) \\ &= \lambda^d \sum_i {x_0}^{d-d_i} \, m_i(x_1, \ldots, x_n) \\ &= \lambda^d \, q(x_0, x_1, \ldots, x_n) \quad \square \end{aligned}$$

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4.2.3 Homogeneous Coordinates

We can consider coordinates as a function C from the domain of all possible coordinate values to a set of possible points P,

$$C : \mathbb{R}^n \longmapsto P.$$

Then, *homogeneous* coordinates describe a function H that does this non-injectively. Specifically,

$$H : \mathbb{R}^{n+1} \longmapsto P, \quad H(x_0, x_1, \ldots, x_n) = C\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right)$$

implementing a function that is homogeneous of degree 0 so that,

$$\begin{aligned} H(\lambda x_0, \lambda x_1, \ldots, \lambda x_n) &= C\left(\frac{\lambda x_1}{\lambda x_0}, \ldots, \frac{\lambda x_n}{\lambda x_0}\right) \\ &= C\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right) \\ &= H(x_0, x_1, \ldots, x_n). \end{aligned}$$

So the coordinates are invariant under uniform scaling of their components. For this reason, they also represent a projective space and are also known as *projective* coordinates (wikipedia).

Projective coordinates can represent translation as a linear transformation:

$$\begin{bmatrix} a & 0 & t_x \\ 0 & b & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ax + t_x \\ by + t_y \\ 1 \end{bmatrix} = \begin{bmatrix} ax \\ by \\ 0 \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ 1 \end{bmatrix}$$

which, in "normal" coordinates, would be,

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}.$$

4.2.4 Homogeneous Equations and Systems

Equations may be considered as functions in two ways: as an explicit or as an implicit function. The explicit form involves selecting a single variable as the dependent variable which will be the function and explicitly defining it in terms of the other variables, e.g.

$$ax + by + cz + d = 0 \rightsquigarrow z(x, y) = -\left(\frac{a}{c}\right)x - \left(\frac{b}{c}\right)y + \left(\frac{d}{c}\right).$$

(Note that it is not always possible to find an explicit expression such as this.)

The implicit form on the other hand involves considering the equation as a constant-valued function of all of its variables, e.g.,

$$ax + by + cz + d = 0 \rightsquigarrow f(x, y, z) = ax + by + cz = -d.$$

Clearly, this is always possible.

The examples here show linear equations but there is no reason why they can't be nonlinear, e.g.

$$f(x, y, z) = \cos x + y^2 + \frac{1}{z} = -d.$$

In the case of a linear equation, the constant-value implicit function form of the equation can be considered as a linear map over its variables,

$$\begin{aligned} & \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = b \\ \iff \quad & \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = b \\ \iff \quad & A\vec{\boldsymbol{x}} = b. \end{aligned}$$

Definition 182. A **homogeneous equation** is one which, when expressed as a constant-valued implicit function of all of its variables, is homogeneous **of degree zero**.

Proposition 4.2.7. *A linear equation that is also homogeneous has no constant term.*

Proof. Let $f(x_1, \ldots, x_n) = C$ be a constant-valued implicit function representing an equation over a field $\mathbb{F}$. If the equation is homogeneous then, by the definition, this function f must be homogeneous of degree zero.

Therefore,

$$f(\lambda x_1, \ldots, \lambda x_n) = \lambda^0 f(x_1, \ldots, x_n) = \lambda^0 C = C \tag{1}$$

for some constants $C, \lambda \neq 0 \in \mathbb{F}, d \in \mathbb{Z}$. But since the underlying equation is linear then the function f is a linear transformation of its variables considered as a vector,

$$f(x_1, \ldots, x_n) = A\vec{\boldsymbol{x}} = C.$$

But, by Proposition 4.2.2, the linear map is homogeneous of degree 1. So, we have,

$$f(\lambda x_1, \ldots, \lambda x_n) = A\lambda\vec{\boldsymbol{x}} = \lambda A\vec{\boldsymbol{x}} = \lambda C \tag{2}$$

From (1) and (2) we have that,

$$f(\lambda x_1, \ldots, \lambda x_n) = C = \lambda C$$

with $\lambda \neq 0$ from the definition of homogeneity. It follows from field axioms, therefore, that $C = 0$. □

Proposition 4.2.8. *A linear equation that is also homogeneous always has the trivial solution.*

Proof. Let $f(x_1, \ldots, x_n) = A\vec{\boldsymbol{x}} = C$ be a constant-valued implicit function representing a linear homogeneous equation over a field $\mathbb{F}$. Using Proposition 4.2.7, we know that $C = 0$ and using Corollary 4.2.1 we have that $A\vec{\mathbf{0}} = \vec{\mathbf{0}}$. It follows then, that $\vec{\mathbf{0}}$ is a solution to the equation. □

Proposition 4.2.9. *A linear equation that is also homogeneous has solutions that form a linear space.*

Proof. TODO: todo □

Proposition 4.2.10. *A linear equation that is non-homogeneous has solutions that form an affine space.*

Proof. TODO: todo □

Systems

Definition 183. A linear system of equations is described as **homogeneous** if all its equations are homogeneous.

It follows that a *linear* system of equations is *homogeneous* if the constant terms are all zero,

$$\begin{array}{cccccc} a_{11}x_1 + & a_{12}x_2 + \cdots + & a_{1n}x_n = 0 \\ a_{21}x_1 + & a_{22}x_2 + \cdots + & a_{2n}x_n = 0 \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}x_1 + & a_{m2}x_2 + \cdots + & a_{mn}x_n = 0. \end{array}$$

which is equivalent to the matrix equation,

$$A\vec{\boldsymbol{x}} = \vec{\mathbf{0}}.$$

It further follows that such systems, and matrix equations, always have the trivial solution $\vec{\boldsymbol{x}} = \vec{\mathbf{0}}$.

In fact, solutions to homogeneous systems form a linear space whereas solutions to non-homogeneous systems form an affine space.

4.2.5 Differentiation of Homogeneous Functions

Lemma 4.2.1. *If a function f is homogeneous of degree d then it can be expressed as a polynomial whose terms contain powers of variables such that the powers sum to d.*

Proof. Assume f can be expressed as a multivariate polynomial. Then $f(x_1, \ldots, x_n)$ can be expressed as the sum of terms of the form,

$$\alpha x_1^{i_1} \cdots x_n^{i_n}$$

for some constant $\alpha \in \mathbb{R}$. Therefore, for each term of $f(\lambda x_1, \ldots, \lambda x_n)$ we have,

$$\alpha(\lambda x_1)^{i_1} \cdots (\lambda x_n)^{i_n} = \alpha \lambda^{i_1} x_1^{i_1} \cdots \lambda^{i_n} x_n^{i_n} = \lambda^{(i_1 + \cdots + i_n)}(\alpha x_1^{i_1} \cdots x_n^{i_n}).$$

Since f is homogeneous of degree d we also have,

$$f(\lambda x_1, \ldots, \lambda x_n) = \lambda^d f(x_1, \ldots, x_n)$$

where each term of $\lambda^d f(x_1, \ldots, x_n)$ has the form,

$$\lambda^d(\alpha x_1^{i_1} \cdots x_n^{i_n})$$

which means that, for every term of the expression for $f(\lambda x_1, \ldots, \lambda x_n)$, we must have,

$$\lambda^{(i_1 + \cdots + i_n)} = \lambda^d \iff i_1 + \cdots + i_n = d. \qquad \square$$

Proposition 4.2.11. ***Euler's Theorem of Homogeneous Functions****: If $f(x_1, \ldots, x_n)$ is a homogeneous function of degree d then,*

$$d \cdot f(x_1, \ldots, x_n) = x_1 \frac{\partial f}{\partial x_1} + \cdots + x_n \frac{\partial f}{\partial x_n}.$$

Proof. Assume f can be expressed as a multivariate polynomial. Then $f(x_1, \ldots, x_n)$ can be expressed as the sum of terms of the form,

$$\alpha x_1^{i_1} \cdots x_n^{i_n}$$

for some constant $\alpha \in \mathbb{R}$. If we take the partial derivative of f with respect to x_1, each term of the result will take the form,

$$i_1 \alpha x_1^{(i_1-1)} x_2^{i_2} \cdots x_n^{i_n}$$

and each term of the partial derivative with respect to x_2 will have the form,

$$i_2 \alpha x_1^{i_1} x_2^{(i_2-1)} \cdots x_n^{i_n}$$

and the n-th partial derivative will have terms,

$$i_n \alpha x_1^{i_1} x_2^{i_2} \cdots x_n^{(i_n-1)}.$$

Now if we look at the terms of the expression $x_1 f_{x_1}$,

$$x_1 \cdot i_1 \alpha x_1^{(i_1-1)} x_2^{i_2} \cdots x_n^{i_n} = i_1(\alpha x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n})$$

we see that the terms are the same as the terms of the original function $f(x_1, \ldots, x_n)$ except multiplied by the power of x_1 in that term. Therefore, each term of the expression $x_1 f_{x_1} + \cdots + x_n f_{x_n}$ has the form,

$$(i_1 + i_2 + \cdots + i_n)(\alpha x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}) = d(\alpha x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n})$$

where we have used Lemma 4.2.1 to determine that,

$$i_1 + \cdots + i_n = d.$$

Therefore,

$$x_1 f_{x_1} + \cdots + x_n f_{x_n} = d \cdot f(x_1, \ldots, x_n). \qquad \square$$

4.3 Taylor Series

Derivation of Taylor's Theorem

Rolle's Theorem

Taken from https://en.wikipedia.org/wiki/Rolle%27s_theorem#Standard_version_of_the_theorem

If a real-valued function f is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b) and $f(a) = f(b)$, then there exists at least one $c \in (a, b)$ such that $f'(c) = 0$.

Mean Value Theorem

Based on https://en.wikipedia.org/wiki/Mean_value_theorem#Proof

If a real-valued function f is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b) and $f(a) \neq f(b)$, then we can define a number $M \in \mathbb{R}$ such that,

$$f(b) = f(a) + M(b - a)$$

then let $g(x) = f(x) - f(a) - M(x - a)$ so that $g'(x) = f'(x) - M$. Now, since by the definition of M, $g(a) = g(b) = 0$, we can apply Rolle's theorem so that,

$$\begin{aligned} & g'(c) = 0 \text{ for some } c \in (a, b) \\ \implies & 0 = f'(c) - M \\ \iff & M = f'(c) \\ \therefore & f(b) = f(a) + f'(c)(b - a) \text{ for some } c \in (a, b) \end{aligned}$$

Taylor's Theorem

Taken from *Walter Rudin, Principles of Mathematical Analysis.*

Suppose f is a real-valued function on $[a, b]$, n, is a positive integer, $f^(n - 1)$

is continuous on $[a, b]$, $f^(n)$ exists for every $t \in (a, b)$. Let α, β be distinct points of [a, b], and define

$$P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t - \alpha)^k.$$

Then there exists a point between α and β such that

$$f(\beta) = P(\beta) + \frac{f^{(n)}(x)}{n!}(\beta - \alpha)^n.$$

Note that if $n = 0$ this degenerates to the Mean Value Theorem:

$$\begin{aligned} f(\beta) &= \sum_{k=0}^{0} \frac{f^{(k)}(\alpha)}{k!}(t - \alpha)^k + \frac{f^{(1)}(x)}{1!}(\beta - \alpha)^1 \\ &= \frac{f^{(0)}(\alpha)}{0!}(t - \alpha)^0 + \frac{f^{(1)}(x)}{1!}(\beta - \alpha)^1 \\ &= f(\alpha) + f'(x)(\beta - \alpha) \end{aligned}$$

*

Proof Let M be the number defined by

$$f(\beta) = P(\beta) + M(\beta - \alpha)^n$$

and put

$$g(t) = f(t) - P(t) - M(t - \alpha)^n \qquad (a \leq t \leq b).$$

We have to show that $n!M = f^{(n)}(x)$ for some x between α and β. Taking the nth derivative of $g(t)$,

$$g^{(n)}(t) = f^{(n)}(t) - n!M \qquad (a < t < b).$$

The proof will be complete if we can show that $g^{(n)}(x) = 0$ for some x between α and β. Since $P^{(k)}(\alpha) = f^{(k)}(\alpha)$ for $k = 0, \ldots, n - 1$ we have

$$g(\alpha) = g'(\alpha) = \ldots = g^{(n-1)}(\alpha) = 0.$$

Our choice of M shows that $g(\beta) = 0$, so that $g'(x_1) = 0$ for some $x_1 \in (\alpha, \beta)$ by the Mean Value Theorem. Since $g'(\alpha) = 0$ we conclude similarly that $g''(x_2) = 0$ for some $x_2 \in (\alpha, \beta)$. After n steps we arrive at the conclusion that $g^{(n)}(x_n) = 0$ for some $x_n \in (\alpha, x_{n-1})$, that is, between α and β.

Examples of Taylor Series

*

$\cos x$ for x close to 0 Note: Taylor series at zero are also called Maclaurin series.

$$\begin{aligned}\cos x &= \cos 0 + (-\sin 0)x + \frac{(-\cos 0)}{2!}x^2 + \dots \\ &= 1 - \frac{x^2}{2} + \dots\end{aligned}$$

*

$\cos 2h$ for h close to 0

$$\cos 2h \approx 1 - \frac{(2h)^2}{2} = 1 - 2h^2$$

Note that if we choose to differentiate wrt. h rather than (2h) then we get the same result,

$$\begin{aligned}\cos 2h &= \cos 0 + (-2\sin 0)h + \frac{(-4\cos 0)}{2!}h^2 + \dots \\ &= 1 - 2h^2 + \dots\end{aligned}$$

Finite Maclaurin Series and the Binomial Theorem

4.4 The Number e

Definition 184. The **natural logarithm** is defined as,

$$\ln x = \int_1^x \frac{1}{t}\,\mathrm{d}t.$$

Definition 185. The number **e** can be defined in various ways. Some of the most common of these are:

(i)
$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

(ii)
$$e = \sum_{n=0}^{\infty}\frac{1}{n!} = \frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\cdots$$

(iii) e is the unique number such that,

$$\ln e = 1$$

Proposition 4.4.1. *The definition of the natural log implies:*

(i) $\ln 1 = 0$*;*

(ii) $\frac{\mathrm{d}}{\mathrm{d}x}\ln x = 1/x$*;*

(iii) $\ln x = \ln y \implies x = y$.

Proof. The proofs of each of the properties are the following.

(i) $\mathbf{\ln 1 = 0:}$

By the properties of integrals,

$$\ln 1 = \int_1^1 \frac{1}{t}\,\mathrm{d}t = 0.$$

(ii) $\mathbf{\frac{d}{dx}\ln x = 1/x:}$

This is a consequence of the Fundamental Theorem of Calculus and,

$$\ln x = \int_1^x \frac{1}{t}\,\mathrm{d}t.$$

(iii) $\mathbf{\ln x = \ln y \implies x = y:}$

This is a consequence of the previous property that,

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln x = \frac{1}{x}.$$

For $x > 0$, the function $1/x$ is strictly positive. This, in turn, means that the function,

$$f(x) = \int_1^x \frac{1}{t}\,\mathrm{d}t$$

is strictly monotonically increasing for all $x > 0$. Therefore,

$$x > y \implies f(x) > f(y)$$

and so,

$$x \neq y \implies f(x) \neq f(y).$$ □

Proposition 4.4.2. *Let* $f : \mathbb{R} \longmapsto \mathbb{R}$ *be defined as*

$$f(x) = e^x.$$

Then the function f *is the inverse of the natural log function* ln.

Proof. Using the definition of the natural log and properties of integrals we have,

$$\begin{aligned}
\ln e^x &= \int_1^{e^x} \frac{1}{t}\,\mathrm{d}t \\
&= \int_1^{e} \frac{1}{t}\,\mathrm{d}t + \int_e^{e^2} \frac{1}{t}\,\mathrm{d}t + \cdots + \int_{e^{x-1}}^{e^x} \frac{1}{t}\,\mathrm{d}t \\
&= \sum_{i=1}^{x} \int_{e^{i-1}}^{e^i} \frac{1}{t}\,\mathrm{d}t \\
&= \sum_{i=1}^{x} \int_1^{e} \frac{e^{i-1}}{t}\,\mathrm{d}u = \sum_{i=1}^{x} \int_1^{e} \frac{1}{u}\,\mathrm{d}u && u = t/(e^{i-1}),\ e^{i-1}\,\mathrm{d}u = \mathrm{d}t \\
&= \sum_{i=1}^{x} \ln e = \sum_{i=1}^{x} 1 = x.
\end{aligned}$$

Conversely, using similar logic,

$$\begin{aligned}
\ln e^{\ln x} &= \int_1^{e^{\ln x}} \frac{1}{t}\,\mathrm{d}t \\
&= \sum_{i=1}^{\ln x} \ln e = \sum_{i=1}^{\ln x} 1 = \ln x.
\end{aligned}$$

Then, by the injectivity of the natural log (property (iii) of Proposition 4.4.1),

$$\ln e^{\ln x} = \ln x \implies e^{\ln x} = x.$$

So, we have shown that,

$$\ln e^x = x = e^{\ln x}$$

which implies that the functions are inverses. □

Proposition 4.4.3. *For any $x, r \in \mathbb{R}$,*

$$\ln x^r = r \ln x.$$

Proof. By Proposition 4.4.2, the functions e^x and $\ln x$ are inverses. Therefore,

$$\ln x^r = \ln\left(e^{\ln x}\right)^r = \ln e^{r \ln x} = r \ln x. \qquad \square$$

Proposition 4.4.4. *Definitions (i) and (ii) of the number e are equivalent. That's to say,*

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n \iff e = \sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots$$

Proof. Consider, for finite n, the binomial expansion,

$$\begin{aligned}
\left(1 + \frac{1}{n}\right)^n &= \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} \\
&= \frac{1}{0!} + \frac{n}{1!}\left(\frac{1}{n}\right) + \frac{(n)(n-1)}{2!}\left(\frac{1}{n^2}\right) + \frac{(n)(n-1)(n-2)}{3!}\left(\frac{1}{n^3}\right) + \\
&\qquad \cdots + \frac{(n)(n-1)\cdots(1)}{n!}\left(\frac{1}{n^n}\right) \\
&= \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!}\left(\frac{n-1}{n}\right) + \frac{1}{3!}\left(\frac{(n-1)(n-2)}{n^2}\right) + \\
&\qquad \cdots + \frac{1}{n!}\left(\frac{(n-1)(n-2)\cdots(1)}{n^{n-1}}\right) \\
&= \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \\
&\qquad \cdots + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{n-1}{n}\right)
\end{aligned}$$

which tends, as $n \to \infty$, to

$$\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots \qquad \square$$

The actual proof of this requires limit inferior and superior (see wikipedia) because we haven't shown that the expression derived from the binomial expression converges. The full proof can be found in Artin[73] and wikipedia.

Proposition 4.4.5. *Definitions (i) and (iii) of the number e are equivalent. That's to say,*

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \iff e \text{ is the unique number such that } \ln e = 1.$$

Proof. Assume that,

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

Then, taking logs of both sides,

$$\begin{aligned}
\ln e &= \ln \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \\
&= \lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right)^n && \text{by Proposition 3.1.26} \\
&= \lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{1/n} && \text{using Proposition 4.4.3} \\
&= \lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n}\right) - \ln 1}{1/n} \\
&= \lim_{h\to 0} \frac{\ln(1+h) - \ln 1}{h} \\
&= \left.\frac{\mathrm{d}(\ln x)}{\mathrm{d}x}\right|_{x=1} = \left.\frac{1}{x}\right|_{x=1} = 1.
\end{aligned}$$

This shows that

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \implies \ln e = 1.$$

This number is unique by property (iii) of the natural log in Proposition 4.4.1.

Conversely, if we assume that e is the unique number such that $\ln e = 1$ then the fact already shown, that

$$\ln \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = 1$$

implies that

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e. \qquad \square$$

4.4.0.1 e^x

Taking the limit definition of e (definition (i)) and raising it to the power of x, by (vi) of Proposition 3.1.23, we have

$$e^x = \left[\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n\right]^x = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{xn}.$$

If we expand this out using binomial theorem we get,

$$\begin{aligned}
\left(1 + \frac{1}{n}\right)^{xn} &= \sum_{k=0}^{xn} \binom{xn}{k} \frac{1}{n^k} \\
&= 1 + (xn)\left(\frac{1}{n}\right) + \frac{(xn)(xn-1)}{2!}\left(\frac{1}{n^2}\right) + \frac{(xn)(xn-1)(xn-2)}{3!}\left(\frac{1}{n^3}\right) + \cdots \\
&= 1 + x + \frac{1}{2!}\left(\frac{x(xn-1)}{n}\right) + \frac{1}{3!}\left(\frac{x(xn-1)(xn-2)}{n^2}\right) + \cdots \\
&= 1 + x + \frac{x}{2!}\left(x - \frac{1}{n}\right) + \frac{x}{3!}\left(x - \frac{1}{n}\right)\left(x - \frac{2}{n}\right) + \cdots
\end{aligned}$$

If we take the limit of this expression as $n \to \infty$ then we get,

$$e^x = \frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

But also, if we take the expression (which arises if we infinitely compound interest at an interest rate of x),

$$\left(1 + \frac{x}{n}\right)^n$$

and again expand it using binomial theorem,

$$\begin{aligned}
\left(1 + \frac{x}{n}\right)^n &= \sum_{k=0}^{n} \binom{n}{k} \frac{x^k}{n^k} \\
&= 1 + \frac{nx}{n} + \frac{n(n-1)}{2!}\frac{x^2}{n^2} + \frac{n(n-1)(n-2)}{3!}\frac{x^3}{n^3} + \cdots \\
&= 1 + x + \frac{x^2}{2!}\frac{n-1}{n} + \frac{x^3}{3!}\frac{(n-1)(n-2)}{n^2} + \cdots \\
&= 1 + x + \frac{x^2}{2!}\left(1 - \frac{1}{n}\right) + \frac{x^3}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \cdots
\end{aligned}$$

again if we take the limit of this expression as $n \to \infty$ then we get,

$$\frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x.$$

4.4.0.2 The Matrix Exponential

Definition 186. If A is a matrix in $\mathbb{F}^{n\times n}$, then the **matrix exponential e^A** is defined as

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$$

which is a matrix in $\mathbb{F}^{n\times n}$.

Theorem 4.4.1. *The series* $e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$ *converges (absolutely) for all complex matrices* A*.*

Proof. Let the norm $\|A\| = \max_{i,j} \left|A_{ij}\right|$ be the maximum absolute value of the entries of the matrix A. Since, for $n \times n$ matrices A and B, we have

$$\begin{aligned}
\left|(AB)_{ij}\right| &= \left|\sum_{k=1}^{n} A_{ik}B_{kj}\right| \\
&\leq \sum_{k=1}^{n} \left|A_{ik}B_{kj}\right| && \because \text{triangle inequality 1.2.3.1} \\
&= \sum_{k=1}^{n} \left|A_{ik}\right| \left|B_{kj}\right| && \because \text{??} \\
&\leq \sum_{k=1}^{n} \|A\|\|B\| = n\|A\|\|B\|
\end{aligned}$$

we can, therefore, deduce that

$$\|AB\| \leq n\|A\|\|B\| \implies \left\|A^k\right\| \leq n^{k-1}\|A\|^k .$$

□

Proposition 4.4.6. *If* A *is similar to a matrix* $\tilde{A} = P^{-1}AP$ *then* $P^{-1}e^AP = e^{\tilde{A}}$*.*

Proof. Using Theorem 2.3.3 and the distributivity of matrix multiplication over addition,

$$\begin{aligned}
P^{-1}e^AP &= P^{-1}IP + P^{-1}AP + \frac{1}{2!}P^{-1}A^2P + \frac{1}{3!}P^{-1}A^3P + \cdots \\
&= I + \tilde{A} + \frac{1}{2!}\tilde{A}^2 + \frac{1}{3!}\tilde{A}^3 + \cdots = e^{\tilde{A}}. \quad \square
\end{aligned}$$

Corollary 4.4.1. *If* A *is diagonalizable into* $D = P^{-1}AP$ *then* $P^{-1}e^AP = e^D$ *and so* $e^A = Pe^DP^{-1}$*.*

Proposition 4.4.7. *If A and B commute then*

$$e^{A+B} = e^A e^B = e^B e^A.$$

Proof. TODO: proof Because matrix addition is commutative we have,

$$e^A e^B = e^{A+B} = e^{B+A} = e^B e^A.$$

□

Corollary 4.4.2. *The inverse of e^A is e^{-A}.*

Proof. An arbitrary square matrix A commutes with itself so $A(-A) = (-1)A^2 = (-A)A$ and A also commutes with $-A$. Therefore,

$$e^A\, e^{-A} = e^{A-A} = e^0 = I$$

where the 0 is an $n \times n$ matrix of zeroes. □

Proposition 4.4.8. *If t is a variable scalar and A a square matrix, the function e^{tA} is a differentiable function of t and its derivative is Ae^{tA}.*

Proof. TODO: proof of differentiability of e^{tA} □

Theorem 4.4.2. *Let A be a real or complex $n \times n$ matrix. The columns of the matrix e^{tA} form a basis for the vector space of solutions of the differential equation*

$$\frac{\mathrm{d}X(t)}{\mathrm{d}t} = AX.$$

Proof. TODO: proof that matrix exponential solves system of diff eqs □

If we look at the entries of e^A using the notation $(e^A)_{ij}$ to indicate the ij-th entry,

$$(e^A)_{ij} = I_{ij} + A_{ij} + \frac{1}{2!}(A^2)_{ij} + \frac{1}{3!}(A^3)_{ij} + \cdots$$

we can see that – *if A is diagonal* – each entry has the value $e^{A_{ij}}$. That's to say, the new matrix has entries equal to e to the power of the corresponding entry in the original matrix A. But this is *only* the case if A is diagonal.

If A is triangular the calculation may also be manageable. For example, let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

so that

$$A^2 = \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix}, A^3 = \begin{bmatrix} 1 & 7 \\ 0 & 8 \end{bmatrix}, \ldots, A^n = \begin{bmatrix} 1 & 2^n - 1 \\ 0 & 2^n \end{bmatrix}.$$

This results in a matrix exponential

$$e^A = \begin{bmatrix} 1 & e^2 - e \\ 0 & e^2 \end{bmatrix}.$$

4.5 Differentiation

4.5.1 Differentiation as a Linear Transformation

Notation. The derivative of $y(x)$ with respect to x will be denoted $\mathrm{D}_x\, y$ and of $f(x)$ with $\mathrm{D}_x\, f$.

Definition 187. The set $P_n \subset \mathbb{R}^{\mathbb{R}}$ of all univariate real-valued polynomials $p : \mathbb{R} \longmapsto \mathbb{R}$ of degree n is defined as,

$$P_n = \{\, p \in \mathbb{R}^{\mathbb{R}} \mid p(x) = \sum_{i=0}^{n} \theta_i x^i, \quad \theta_i \in \mathbb{R} \,\}.$$

Or alternatively, in vector notation,

$$P_n = \{\, p \in \mathbb{R}^{\mathbb{R}} \mid p(x) = \vec{\boldsymbol{\theta}}^T \vec{\boldsymbol{x}}, \quad \vec{\boldsymbol{\theta}} \in \mathbb{R}^{n+1} \,\}$$

where $\vec{\boldsymbol{x}}$ is the standard basis of the space of degree-n polynomials,

$$(1, x, \ldots, x^n)^T.$$

4.5.1.1 Linear Algebra of First-order Differential Equations

First Order

$$\frac{dy}{dx} = ax + b$$

$$\begin{aligned}
\iff \qquad & \int \frac{dy}{dx}\,\mathrm{d}x = \int ax + b\,\mathrm{d}x \\
\iff \qquad & y = a'x^2 + bx + c. \qquad a' = a/2,\ \text{c is any constant}
\end{aligned}$$

Integrating we see that a first order differential equation only determines the function upto a constant value. In order to determine a specific function we need a relation between a value of x and a value of y (i.e. a point, in graphical terms). Typically this is described as an initial condition.

Second Order

$$\begin{aligned}
& \frac{d^2y}{dx^2} = ax + b \\
\iff \qquad & \int \frac{d^2y}{dx^2}\,\mathrm{d}x = \int ax + b\,\mathrm{d}x \\
\iff \qquad & \frac{dy}{dx} = a'x^2 + bx + c \qquad a' = a/2,\ \text{c is any constant} \\
\iff \qquad & \int \frac{dy}{dx}\,\mathrm{d}x = \int a'x^2 + bx + c\,\mathrm{d}x \\
\iff \qquad & y = a''x^3 + b'x^2 + cx + d. \qquad a'' = a/6,\ b' = b/2,\ \text{d is any constant}
\end{aligned}$$

Integrating a second order equation twice we see that we introduced two constants of integration, c, d, and the last two terms $cx + d$ are undetermined. So a second order equation of this type has only determined a function upto a first-degree polynomial (a line). To determine a specific function, in this case, we require two relations between x and y (two points determine a line).

The derivative $\mathrm{D}_x\, p(x)$, of the univariate degree-n polynomial $p \in P_n$ can be described as a linear transformation as,

$$\mathrm{D}_x\, p(x) = (A\vec{\boldsymbol{\theta}})^T\vec{\boldsymbol{x}} = \vec{\boldsymbol{x}}^T A\vec{\boldsymbol{\theta}}$$

where A is the $n \times (n+1)$ matrix,

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & n \end{bmatrix}.$$

The matrix A clearly has rank n and a 1-dimensional kernel so D_x transforms from $n+1$-space to n-space. The nullspace of A being,

$$nullspace(A) = t \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad t \in \mathbb{R}$$

the interpretation of which is that the derivative of constant polynomials is zero.

In general the differential equation,

$$\mathrm{D}_x\, y(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_{n-1} x^{n-1} \tag{4.1}$$

has the vector form,

$$\vec{\boldsymbol{x}}^T A\vec{\boldsymbol{\theta}} = \vec{\boldsymbol{x}}^T \vec{\boldsymbol{\alpha}} \iff A\vec{\boldsymbol{\theta}} = \vec{\boldsymbol{\alpha}} \tag{4.2}$$

where the solution is

$$y(x) = \theta_0 + \theta_1 x + \cdots + \theta_n x^n = \vec{\boldsymbol{x}}^T \vec{\boldsymbol{\theta}}.$$

Note that we can say

$$\vec{\boldsymbol{x}}^T A\vec{\boldsymbol{\theta}} = \vec{\boldsymbol{x}}^T \vec{\boldsymbol{\alpha}} \iff A\vec{\boldsymbol{\theta}} = \vec{\boldsymbol{\alpha}}$$

because $\vec{\boldsymbol{x}}^T$ is a basis and therefore a linearly independent set. By linear independence, the equation on the left implies that the coefficients are equal.

This is equivalent to analysing P_n, the $(n+1)$-dimensional vector space of polynomials, by working in the coordinate space formed by the coefficients. This space is $\mathbb{R}^{n+1}$ which is isomorphic to any $(n+1)$-dimensional vector space by Proposition 2.4.24.

The matrix form of Equation 4.2 is,

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 2 & \cdots & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & n-1 & 0 \\ 0 & 0 & 0 & \cdots & \cdots & n \end{bmatrix} \begin{bmatrix} \theta_0 \\ \theta_1 \\ \vdots \\ \vdots \\ \vdots \\ \theta_n \end{bmatrix} = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{n-1} \end{bmatrix}. \tag{4.3}$$

Constructing the augmented matrix and using Gaussian Elimination we obtain,

$$\left[\begin{array}{cccccc|c} 0 & 1 & 0 & \cdots & \cdots & 0 & \alpha_0 \\ 0 & 0 & 2 & \cdots & \cdots & 0 & \alpha_1 \\ \vdots & & & & & & \vdots \\ 0 & 0 & 0 & \cdots & n-1 & 0 & \alpha_{n-2} \\ 0 & 0 & 0 & \cdots & \cdots & n & \alpha_{n-1} \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cccccc|c} 0 & 1 & 0 & \cdots & \cdots & 0 & \alpha_0 \\ 0 & 0 & 1 & \cdots & \cdots & 0 & \alpha_1/2 \\ \vdots & & & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & \alpha_{n-2}/n-1 \\ 0 & 0 & 0 & \cdots & \cdots & 1 & \alpha_{n-1}/n \end{array}\right]$$

so that θ_0 is a free variable and we can read off the other values of the coefficients θ_i corresponding to the columns of the matrix as follows:

$$\begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \vdots \\ \theta_{n-1} \\ \theta_n \end{bmatrix} = \begin{bmatrix} t \\ \alpha_0 \\ \alpha_1/2 \\ \vdots \\ \alpha_{n-2}/n-1 \\ \alpha_{n-1}/n \end{bmatrix} \qquad \text{for } t \in \mathbb{R} \tag{4.4}$$

and this implies that the solution to the differential equation is:

$$y(x) = t + \alpha_0 x + \frac{\alpha_1}{2}x^2 + \cdots + \frac{\alpha_{n-1}}{n}x^n, \qquad t \in \mathbb{R}. \tag{4.5}$$

We can rewrite the result in Equation 4.4 as,

$$\begin{bmatrix} 0 \\ \alpha_0 \\ \alpha_1/2 \\ \vdots \\ \alpha_{n-2}/n-1 \\ \alpha_{n-1}/n \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \qquad \text{for } t \in \mathbb{R} \tag{4.6}$$

which makes it clear that we have a particular solution (with $t = 0$) plus a 1-dimensional nullspace. However, in practical applications modeled by differential equations, there will typically be an initial condition — an initial value of y such as $y(0)$ for example — and this will be used to determine a particular solution. In this situation (known as IVP or Initial Value Problems) the particular solution involves finding a value of the parameter t that fits the initial condition. For this reason, the solution in Equation 4.10 is referred to as the general solution of the differential equation while the particular solution has a particular value of the parameter t.

4.5.1.2 Linear Algebra of Second-order Differential Equations

The most simple type of second-order linear differential equation looks like,

$$\mathrm{D}_x^2\, y(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_{n-2} x^{n-2}. \tag{4.7}$$

We can add an extra row of zeros to the matrix of D_x to make it square so that we can take powers of it,

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & n \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

and output an extra coefficient with value zero.

Then the vector form of Equation 4.7 is

$$A^2\vec{\boldsymbol{\theta}} = \vec{\boldsymbol{\alpha}} \tag{4.8}$$

and the matrix form is

$$\begin{bmatrix} 0 & 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 6 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & n(n-1) \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \theta_0 \\ \theta_1 \\ \vdots \\ \vdots \\ \vdots \\ \theta_n \end{bmatrix} = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{n-2} \\ 0 \\ 0 \end{bmatrix} \tag{4.9}$$

which gives the solution to the differential equation as:

$$y(x) = s + tx + \frac{\alpha_0}{2}x^2 + \frac{\alpha_1}{6}x^3 + \cdots + \frac{\alpha_{n-2}}{n(n-1)}x^n, \qquad s, t \in \mathbb{R}. \tag{4.10}$$

So, here the kernel is 2-dimensional and we need two initial values to determine a particular solution.

4.5.1.3 Eigenvectors of the Differentiation Operator

If we had the differential equation,

$$\mathrm{D}_x\, y(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \cdots \tag{4.11}$$

so that the derivative of $y(x)$ is an infinite power series (not an infinite polynomial as polynomials are by definition finite), then the general solution would take the form, for $t \in \mathbb{R}$,

$$y(x) = t + \alpha_0 x + \frac{\alpha_1}{2}x^2 + \frac{\alpha_2}{3}x^3 + \cdots \tag{4.12}$$

So, if we had

$$\alpha_0 = \alpha_1,\ \frac{\alpha_1}{2} = \alpha_2,\ \frac{\alpha_2}{3} = \alpha_3$$

then the general solution would be, for $t \in \mathbb{R}$,

$$y(x) = t + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \cdots \tag{4.13}$$

which is a family of functions which includes the derivative $\mathrm{D}_x\, y(x)$ — the derivative being the member of this set of functions with $t = \alpha_0$.

Note that, for polynomials, the derivative is a transformation between finite-dimensional vector spaces and so the Dimension Formula (Theorem 2.5.1) of linear transformations applies. We can see this in that the derivative has a one-dimensional kernel (the set of constant polynomials) and the derivative maps from P_n to P_{n-1}, the image having one less dimension then the domain space because there is a one-dimensional kernel.
However, the coordinate space of the power series above is isomorphic to an infinite-dimensional vector space where the Dimension Formula no longer applies (see Theorem 2.4.4). So when we differentiate it we get a result of the same dimensionality despite there being a non-trivial one-dimensional kernel.

In order to achieve this we need the coefficients to form the series,

$$\alpha_0,\ \frac{\alpha_0}{1},\ \frac{\alpha_0}{1\times 2},\ \frac{\alpha_0}{1\times 2\times 3},\ \cdots$$

so that

$$\begin{aligned} y(x) &= \alpha_0 + \frac{\alpha_0}{1}x + \frac{\alpha_0}{1\times 2}x^2 + \frac{\alpha_0}{1\times 2\times 3}x^3 + \dots \\ &= \alpha_0 + \frac{\alpha_0}{1!}x + \frac{\alpha_0}{2!}x^2 + \frac{\alpha_0}{3!}x^3 + \dots \\ &= \alpha_0\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right) = \alpha_0 e^x. \end{aligned}$$

So, the general solution is,

$$y(x) = te^x \qquad t \in \mathbb{R}$$

and the particular solution, for $t = \alpha_0$, is

$$y(x) = \alpha_0 e^x.$$

Theorem 4.5.1. *The derivative of $e^{f(x)}$ is $f'(x)e^{f(x)}$ where f' is the derivative of the function f.*

Proof.

$$e^{f(x)} = 1 + \frac{f(x)}{1!} + \frac{(f(x))^2}{2!} + \frac{(f(x))^3}{3!} + \cdots$$

$$\implies \quad \frac{\mathrm{d}}{\mathrm{d}x}e^{f(x)} = f'(x) + f'(x)\frac{f(x)}{1!} + f'(x)\frac{(f(x))^2}{2!} + f'(x)\frac{(f(x))^3}{3!} + \cdots$$

$$= f'(x)e^{f(x)}.$$

□

Corollary 4.5.1. *Any function of the form $Ae^{f(x)}$ is an eigenfunction of the differentiation operator with eigenvalue equal to $f'(x)$.*

Proof.

$$\frac{\mathrm{d}}{\mathrm{d}x}Ae^{f(x)} = A\frac{\mathrm{d}}{\mathrm{d}x}e^{f(x)} = Af'(x)e^{f(x)} = f'(x)(Ae^{f(x)}).$$ □

Corollary 4.5.2. *The set of functions $e^{\int f'(x)\,\mathrm{d}x}$ form an eigenspace of the differentiation operator with eigenvalue $f'(x)$.*

Proposition 4.5.1. *For any $a, x \in \mathbb{R}$,*

$$\frac{\mathrm{d}}{\mathrm{d}x}a^x = (\ln a)a^x.$$

Proof.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}a^x &= \frac{\mathrm{d}}{\mathrm{d}x}e^{(\ln a)x} && \text{by Proposition 4.4.2} \\
&= (\ln a)e^{(\ln a)x} && \text{by Theorem 4.5.1} \\
&= (\ln a)a^x && \text{by Proposition 4.4.2}
\end{aligned}$$

4.5.1.4 Problems with this approach to Differentiation

We can describe polynomial functions as finite vectors defined against the basis of monomials $(1, x, x^2, \dots)$ but to describe transcendental functions such as e^x in the same manner we would need an infinite linear combination of monomials which cannot be generally defined to create a vector space of such vectors.

4.5.1.5 The Polynomial Differential Operator

Definition 188. The n-th degree **polynomial differential operator** is defined as

$$L = a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0$$

so that, for example, a second-degree operator may be used to define the homogeneous differential equation

$$a_2 y'' + a_1 y' + a_0 y = 0$$

as

$$Ly = 0 = a_2 D^2 y + a_1 Dy + a_0 y.$$

We can form polynomials of the differential operator D because, as a linear operator, we can use 2.5.3. However, it should be noted that, if u, v are both functions of t, $u(t), v(t)$,

$$Duv = v \implies Du = 1.$$

This seems obvious because,

$$Duv = (Du)v + u(Dv)$$

but some implications may not be so obvious. For example,

$$(D - a)uv = v \implies (D - a)u = 1.$$

4.6 Integration

4.6.1 Univariate Integration

4.6.1.1 The Riemann Integral

Definition 189. In the context of the Riemann Integral, a **partition** of an interval $[a, b] \in \mathbb{R}$ is a set,

$$P = \{\, x_i \mid 0 \leq i \leq n \,\} \quad \text{where} \quad a \leq x_i < x_{i+1} \leq b.$$

That's to say,

$$P = \{x_0, x_1, \ldots, x_n\} \quad \text{where} \quad a = x_0 < x_1 < \cdots < x_n = b.$$

Definition 190. The **lower estimate** of the area under the curve of a function $f(x)$ with respect to a particular partition is defined as,

$$L(P) = \sum_{i=0}^{n-1} (x_{i+1} - x_i) \min_{x_i \leq x \leq x_{i+1}} f(x)$$

where each $x_i \in P$. The **upper estimate** is similarly defined as,

$$U(P) = \sum_{i=0}^{n-1} (x_{i+1} - x_i) \max_{x_i \leq x \leq x_{i+1}} f(x).$$

Definition 191. Let $P(n)$ be a partition of cardinality $n+1$ over the interval $[a,b]$. If, for a function $f(x)$,

$$\lim_{n\to\infty} L(P(n)) = \lim_{n\to\infty} U(P(n)) = I$$

then the **Riemann Integral** of $f(x)$ over $P(n)$ is defined as,

$$\int_a^b f(x)\,\mathrm{d}x = I.$$

Proposition 4.6.1. *Let f and g be integrable functions on an interval $[a,b]$ such that*

$$\forall x \in [a,b]\ .\ f(x) \leq g(x).$$

Then,

$$\int_a^b f(x)\,\mathrm{d}x \leq \int_a^b g(x)\,\mathrm{d}x.$$

Proof.

Since $\forall x \in [a,b]\ .\ f(x) \leq g(x)$ then, for any $a \leq x_1 < x_2 \leq b$,

$$\min_{x_1\leq x\leq x_2} f(x) \leq \min_{x_1\leq x\leq x_2} g(x) \quad \text{and} \quad \max_{x_1\leq x\leq x_2} f(x) \leq \max_{x_1\leq x\leq x_2} g(x).$$

Therefore, if $L_f(P(n)), U_f(P(n))$ is the lower and upper estimates (4.6.1.1) of the area under the curve of $f(x)$ and similarly $L_g(P(n)), U_g(P(n))$, then we must have,

$$\lim_{n\to\infty} L_f(P(n)) \leq \lim_{n\to\infty} L_g(P(n)) \quad \text{and} \quad \lim_{n\to\infty} U_f(P(n)) \leq \lim_{n\to\infty} U_g(P(n)).$$

It follows then, that the Riemann Integral (191) of $f(x)$ over the interval $[a,b]$ must be less than or equal to that of $g(x)$ over the same interval. □

Corollary 4.6.1. *Let f and g be integrable functions on an interval $[a,b]$ such that*

$$\forall x \in [a,b]\ .\ f(x) \geq g(x).$$

Then,

$$\int_a^b f(x)\,\mathrm{d}x \geq \int_a^b g(x)\,\mathrm{d}x.$$

Proof.
By Proposition 4.6.1, since

$$\forall x \in [a,b]\,.\, f(x) \geq g(x) \iff \forall x \in [a,b]\,.\, -f(x) \leq -g(x)$$

we therefore have,

$$\begin{aligned}
& \int_a^b -f(x)\,\mathrm{d}x \leq \int_a^b -g(x)\,\mathrm{d}x \\
\iff \quad & -\int_a^b f(x)\,\mathrm{d}x \leq -\int_a^b g(x)\,\mathrm{d}x \qquad \text{by linearity of integration} \\
\iff \quad & \int_a^b f(x)\,\mathrm{d}x \geq \int_a^b g(x)\,\mathrm{d}x. \quad \square
\end{aligned}$$

---

(101) Suppose $f(x) = e^x$ and we define a partition over the interval $[0,1]$,

$$P(n) = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\}.$$

Then the lower estimate of the area under the curve of this function is given by,

$$\begin{aligned}
L(P(n)) &= \sum_{i=0}^{n-1} (x_{i+1} - x_i) \min_{x_i \leq x \leq x_{i+1}} f(x) \\
&= \frac{1}{n} \sum_{i=0}^{n-1} \min_{\frac{i}{n} \leq x \leq \frac{i+1}{n}} e^x \\
&= \frac{1}{n} \sum_{i=0}^{n-1} e^{\frac{i}{n}} = \frac{1}{n}(1 + e^{\frac{1}{n}} + \cdots + e^{\frac{n-1}{n}}) \\
&= \frac{1}{n} \left(\frac{e-1}{e^{\frac{1}{n}} - 1} \right).
\end{aligned}$$

Meanwhile, the upper estimate is given by,

$$\begin{aligned} U(P(n)) &= \frac{1}{n}\sum_{i=0}^{n-1} \max_{\frac{i}{n}\le x\le \frac{i+1}{n}} e^x \\ &= \frac{1}{n}\sum_{i=0}^{n-1} e^{\frac{i+1}{n}} = \frac{1}{n}(e^{\frac{1}{n}} + \cdots + e^{\frac{n-1}{n}} + e) \\ &= \frac{e^{\frac{1}{n}}}{n}\left(\frac{e-1}{e^{\frac{1}{n}}-1}\right). \end{aligned}$$

Taking the limits as $n \to \infty$,

$$\begin{aligned} \lim_{n\to\infty} L(P(n)) &= \lim_{n\to\infty} \frac{1}{n}\left(\frac{e-1}{e^{\frac{1}{n}}-1}\right) \\ &= (e-1)\lim_{n\to\infty} \frac{1}{n}\left(\frac{1}{e^{\frac{1}{n}}-1}\right) \\ &= (e-1)\lim_{n\to\infty} \frac{1/n}{e^{\frac{1}{n}}-1} \\ &= (e-1)\lim_{n\to\infty} \frac{-1/n^2}{(-1/n^2)e^{\frac{1}{n}}} \qquad \text{by L'Hôpital} \\ &= (e-1)\lim_{n\to\infty} e^{-\frac{1}{n}} = (e-1)(1) = e-1, \end{aligned}$$

$$\begin{aligned} \lim_{n\to\infty} U(P(n)) &= \lim_{n\to\infty} \frac{e^{\frac{1}{n}}}{n}\left(\frac{e-1}{e^{\frac{1}{n}}-1}\right) \\ &= (e-1)\lim_{n\to\infty} \frac{e^{\frac{1}{n}}}{n}\left(\frac{1}{e^{\frac{1}{n}}-1}\right) \\ &= (e-1)\left(\lim_{n\to\infty} e^{\frac{1}{n}}\right)\left(\lim_{n\to\infty} \frac{1}{n}\left[\frac{1}{e^{\frac{1}{n}}-1}\right)\right] \\ &= (e-1)(1)(1) = e-1. \end{aligned}$$

So, we see that

$$\lim_{n\to\infty} L(P(n)) = \lim_{n\to\infty} U(P(n)) = e - 1$$

and this is the value of the riemann integral for e^x over the interval $[0, 1]$. Note that it is in agreement with the analytic solution,

$$\begin{aligned}\int_0^1 e^x &= [e^x]_0^1 \\ &= e^1 - e^0 = e - 1.\end{aligned}$$

4.6.1.2 FTC and the Chain Rule

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{p(t)}^{q(t)} f(x)\,\mathrm{d}x \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\int_0^{q(t)} f(x)\,\mathrm{d}x - \frac{\mathrm{d}}{\mathrm{d}t}\int_0^{p(t)} f(x)\,\mathrm{d}x \\ &= \frac{\mathrm{d}q}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}q}\int_0^{q} f(x)\,\mathrm{d}x - \frac{\mathrm{d}p}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}p}\int_0^{p} f(x)\,\mathrm{d}x \\ &= \frac{\mathrm{d}q}{\mathrm{d}t} f(q) - \frac{\mathrm{d}p}{\mathrm{d}t} f(p).\end{aligned}$$

4.6.1.3 Definite and Indefinite Integration

Indefinite integration determines an antiderivative up to a constant so that, if $F(x)$ is some antiderivative of $f(x)$ and C is a constant, then

$$\int f(x)\,\mathrm{d}x = F(x) + C.$$

This expresses the fact that any value of C would produce a valid antiderivative of $f(x)$. In fact, these are the fibres – the cosets of the kernel – of the

differentiation linear transformation. The kernel of differentiation is the set of constant-valued functions,

$$f(x) = C$$

for any $C \in \mathbb{R}$.

In the case of a definite integral, however, the constant cancels out:

$$\int_a^b f(x)\,\mathrm{d}x = [F(x) + C]_a^b = (F(b) + C) - (F(a) + C) = F(b) - F(a)$$

so that we are able to resolve the value of the definite integral to a specific constant value. In this case, the result is the sum of two elements of the kernel of the differentiation transform and so the result is also in the kernel.

However, if we consider a function of a variable, t, such that

$$h(t) = \int_a^t f(x)\,\mathrm{d}x$$

then $h(t)$ is also an antiderivative of $f(t)$ but also,

$$h(t) = \int_a^t f(x)\,\mathrm{d}x = [F(x) + C]_a^t = F(t) - F(a) = F(t) + C_2$$

where $C_2 = -F(a)$ is a constant. So, the definite integral – specifically, the initial value, a – has allowed us to resolve a particular antiderivative from the set of functions produced by the possible values of C. That's to say, the initial value a specified for us a particular element in the kernel of the differentiation operator – namely, $C_2 = -F(a)$.

Furthermore, since the definite integral is a particular antiderivative of $f(x)$ we can also relate the indefinite and definite integrals as follows,

$$\int f(x)\,\mathrm{d}x = \int_a^t f(x)\,\mathrm{d}x + C.$$

This amounts to – expressed in terms of group theory – the statement that, if G is a group with kernel K and $x, k \in G,\ k \in K$, then

$$xK = xkK$$

where

$$xK = \int f(x)\,\mathrm{d}x = F(x) + C$$
$$xk = \int_a^t f(x)\,\mathrm{d}x = F(t) - F(a)$$
$$xkK = \int_a^t f(x)\,\mathrm{d}x + C = F(t) - F(a) + C.$$

4.6.1.4 Change of Variable

There are two types of variable substitution.

- Substitution: $x = g(u)$,

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^b f(g(u))\,\mathrm{d}(g(u)) = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(u))g'(u)\,\mathrm{d}u.$$

  For example, making the substitution $x = \sin\theta$ in,

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x &= \int_0^1 \frac{1}{\sqrt{1-\sin^2\theta}}\,\mathrm{d}(\sin\theta)\\ &= \int_{\sin^{-1}(0)}^{\sin^{-1}(1)} \frac{1}{\sqrt{1-\sin^2\theta}}\cos\theta\,\mathrm{d}\theta\\ &= \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-\sin^2\theta}}\cos\theta\,\mathrm{d}\theta\\ &= \int_0^{\frac{\pi}{2}} \mathrm{d}\theta = \frac{\pi}{2}.\end{aligned}$$

- "Reverse" Substitution: $u = g(x)$,

$$\int_a^b f(x)\,\mathrm{d}x = \int_{g(a)}^{g(b)} f(x)\frac{1}{g'(x)}\,\mathrm{d}u.$$

  For example, making the substitution $u = \frac{x}{a}$ in,

$$\int_a^{at} f(x)\,\mathrm{d}x = \int_1^t f(au)a\,\mathrm{d}u = a\int_1^t f(au)\,\mathrm{d}u.$$

4.6.1.5 Improper Integrals

Definition 192. **(Improper Integral)** An integral is called *improper* if the interval it is defined over is half-open. That's to say, if either of the interval bounds a or b in the integral

$$\int_a^b f(t)\,\mathrm{d}t$$

is infinity or the function is undefined at that point. The value of such an integral is defined as, if we take the case where the interval is open at the upper bound,

$$\int_a^b f(t)\,\mathrm{d}t = \lim_{x\to b}\int_a^x f(t)\,\mathrm{d}t.$$

More precisely, if the integrand function is undefined at the point $t = b$ then

$$\int_a^b f(t)\,\mathrm{d}t = \lim_{x\to b^-}\int_a^x f(t)\,\mathrm{d}t$$

and if the function is undefined at the point $t = a$ then

$$\int_a^b f(t)\,\mathrm{d}t = \lim_{x\to a^+}\int_x^b f(t)\,\mathrm{d}t.$$

Meanwhile if $b = \infty$ then

$$\int_a^b f(t)\,\mathrm{d}t = \int_a^\infty f(t)\,\mathrm{d}t = \lim_{x\to\infty}\int_a^x f(t)\,\mathrm{d}t$$

and similarly if $a = -\infty$ then

$$\int_a^b f(t)\,\mathrm{d}t = \int_{-\infty}^b f(t)\,\mathrm{d}t = \lim_{x\to-\infty}\int_x^b f(t)\,\mathrm{d}t.$$

An integral over an interval that is open at both ends such as,

$$\int_{-\infty}^\infty f(t)\,\mathrm{d}t$$

is defined as the sum of two improper integrals over the half-open intervals,

$$\int_{-\infty}^{\infty} f(t)\,\mathrm{d}t = \int_{0}^{\infty} f(t)\,\mathrm{d}t + \int_{-\infty}^{0} f(t)\,\mathrm{d}t.$$

If either of the two summand integrals diverges, then their sum is defined as divergent. The result of this is that, even if the integrand is an odd function so that for all $a > 0 \in \mathbb{R}$,

$$\int_{-a}^{a} f(t)\,\mathrm{d}t = 0,$$

the integral

$$\int_{-\infty}^{\infty} f(t)\,\mathrm{d}t$$

may (or may not) be defined as divergent. The reason for this is that, although if the upper and lower bounds go to infinity at the same rate (as in the case of a and $-a$), then we indeed have a limit of 0, this is not true if the bounds are allowed to vary independently. If the bounds vary independently then the value of the integral depends on the rate at which the bounds go to infinity. This causes a problem for the definition of the Riemann Integral and so these integrals are commonly defined to be divergent (see example 106).
However, in some circumstances (most notably distribution theory), another definition of the integral may be used: the Lebesgue Integral with the Cauchy Principal Value (see: wikipedia).

Theorem 4.6.1. ***(Direct Comparison Test)*** *If $0 \leq f(t) \leq g(t)$ for all $t > T \in \mathbb{R}$ then*

$$\int_{T}^{\infty} g(t)\,\mathrm{d}t \text{ converges } \implies \int_{T}^{\infty} f(t)\,\mathrm{d}t \text{ converges}$$

and if $0 \leq g(t) \leq f(t)$ for all $t > T \in \mathbb{R}$ then

$$\int_{T}^{\infty} g(t)\,\mathrm{d}t \text{ diverges } \implies \int_{T}^{\infty} f(t)\,\mathrm{d}t \text{ diverges.}$$

Proof.

By Proposition 4.6.1, we have that, if $0 \leq f(t) \leq g(t)$ for all $t > T \in \mathbb{R}$ then,

$$\int_{T}^{\infty} f(t)\,\mathrm{d}t \leq \int_{T}^{\infty} g(t)\,\mathrm{d}t$$

and so, if the integral of g over this interval is finite then so must the integral of f be also.

Also, by Corollary 4.6.1, if $0 \leq g(t) \leq f(t)$ for all $t > T \in \mathbb{R}$ then,

$$\int_T^\infty f(t)\,\mathrm{d}t \geq \int_T^\infty g(t)\,\mathrm{d}t$$

and so, if the integral of g over this interval is infinite then so must the integral of f be also. □

TODO: note about functions compared in DCT needing to be positive.

Corollary 4.6.2. *If f is an integrable function and is lower bounded on $[a, \infty)$ by $c > 0 \in \mathbb{R}$ such that*

$$\forall x \in [a, b]\ .\ f(x) \geq c,$$

then

$$\int_a^\infty f(x)\,\mathrm{d}x \text{ diverges.}$$

Proof. Since

$$\int_a^\infty c\,\mathrm{d}x = c\int_a^\infty \mathrm{d}x = c[x]_a^\infty = \infty$$

diverges, Theorem 4.6.1 tells us that the integral of f also diverges. □

Corollary 4.6.3. *If f and g are non-negative integrable functions and on $[a, \infty)$ we have*

$$\int_a^\infty f(x)\,\mathrm{d}x \text{ converges}$$

and, for some $M \geq 0$,

$$g(x) \leq M,$$

then

$$\int_a^\infty f(x)g(x)\,\mathrm{d}x \text{ converges.}$$

Proof. If the integral of f over the interval converges then it must evaluate to a finite value and we can therefore also deduce that

$$M\int_a^\infty f(x)\,\mathrm{d}x = \int_a^\infty Mf(x)\,\mathrm{d}x \text{ converges.}$$

Since, for $x \in [a, \infty)$,

$$f(x)g(x) \leq Mf(x)$$

we can therefore use Theorem 4.6.1 to reason that

$$\int_a^\infty f(x)g(x)\,\mathrm{d}x \text{ converges.} \qquad \square$$

Proposition 4.6.2.

$$\int_1^\infty f(x)\,\mathrm{d}x \textit{ converges } \implies \int_1^\infty f(x^2)\,\mathrm{d}x \textit{ converges}$$

but

$$\int_1^\infty f(x^2)\,\mathrm{d}x \textit{ converges } \not\implies \int_1^\infty f(x)\,\mathrm{d}x \textit{ converges.}$$

Proof.

$$\begin{aligned}\int_1^\infty f(x)\,\mathrm{d}x &= \int_1^\infty f(u^2)\,\mathrm{d}(u^2)\\ &= \int_1^\infty f(u^2)2u\,\mathrm{d}u\\ &\therefore \int_1^\infty f(u^2)2u\,\mathrm{d}u \text{ converges.}\end{aligned}$$

Since, for $u \in [1, \infty)$, we have $f(u^2)2u \geq f(u^2)$, by Theorem 4.6.1,

$$\int_1^\infty f(u^2)\,\mathrm{d}u \text{ converges.}$$

Clearly then also, the convergence of $f(x^2)$ over the same interval doesn't allow us to draw any such similar conclusion about the convergence of $f(x)$. For example, let f be the function $f(t) = 1/t$. Then,

$$\int_1^\infty f(x^2)\,\mathrm{d}x \text{ converges}$$

but

$$\int_1^\infty f(x)\,\mathrm{d}x \text{ diverges.} \qquad \square$$

Theorem 4.6.2. ***(Limit Comparison Test)*** *If $f(t)$ and $g(t)$ are both positive for all $t \in [a, b)$ then, for the test value*

$$T = \lim_{t \to b} \frac{f(t)}{g(t)},$$

- *if $T = 1$ then*

$$\int_a^b f(t)\,\mathrm{d}t \text{ converges} \iff \int_a^b g(t)\,\mathrm{d}t \text{ converges;}$$

- *if $T = 0$ then*

$$\int_a^b g(t)\,\mathrm{d}t \text{ converges} \implies \int_a^b f(t)\,\mathrm{d}t \text{ converges.}$$

- *if $T = \infty$ then*

$$\int_a^b g(t)\,\mathrm{d}t \text{ diverges} \implies \int_a^b f(t)\,\mathrm{d}t \text{ diverges.}$$

TODO: Why do the functions f and g have to be positive?

Proof.

If $T = 1$ then the numerator and denominator are growing at the same rate as $t \to b$ and so convergence of one implies convergence of the other. If, on the other hand, $T = 0$ then the numerator is growing more slowly than the denominator and so if the denominator converges then the numerator must also converge. Conversely, if $T = \infty$ then the numerator is growing faster than the denominator and so, if the denominator diverges then the numerator must also diverge. □

In Theorem 4.6.2, the functions f and g must be positive because this makes the integral strictly increasing as t goes to the limit point and so it either converges or goes infinite. If the integrand function takes negative values also then the value of the integral may be oscillating as t goes to the limit point and so it could be divergent even though bound. For example,

$$\lim_{t \to \infty} \frac{\sin x}{x^2}$$

TODO: this example doesn't work? have I thought about this correctly?

TODO: work out how to write this up better

The strategy for using the limit comparison test is:

1. *Look for an integrable function that can be limit tested against the function under test to obtain the desired result;*
2. *Integrate the integrable function to show that it is convergent/divergent depending on requirement.*

TODO: note or example(s) to show that convergence of integrals is convergence of series not sequences.

---

(102) $\boldsymbol{\int_1^\infty \frac{1}{x+1}\,\mathrm{d}x}$:

This integral does not converge despite the integrand $\frac{1}{x+1} \leq \frac{1}{x}$ for all x over the integration. However, $\int_1^\infty \frac{1}{x^{3/2}}\,\mathrm{d}x$ does converge.

(103) $\boldsymbol{\int_1^\infty \frac{\pi}{2} - \tan^{-1}(x)\,\mathrm{d}x}$:

This integral diverges because, if we use L'Hopital's rule to take the limit

$$\begin{aligned}\lim_{x\to\infty} \frac{\frac{\pi}{2} - \tan^{-1}(x)}{\frac{1}{x}} &= \lim_{x\to\infty} \frac{-\frac{1}{1+x^2}}{-\frac{1}{x^2}} \\ &= \lim_{x\to\infty} \frac{x^2}{1+x^2} \\ &= \lim_{x\to\infty} \frac{1}{\frac{1}{x^2}+1} = 1.\end{aligned}$$

So, the integral exhibits the same behaviour at infinity as $\int_1^\infty \frac{1}{x}\,\mathrm{d}x$ which diverges. TODO: add reference

(104) $\boldsymbol{\int_2^\infty \frac{1}{x(\ln(x))^r}\,\mathrm{d}x}$:

This integral converges for all $r > 1 \in \mathbb{R}$ because

$$\int_2^\infty \frac{1}{x(\ln(x))^r}\,\mathrm{d}x = \int_{\ln 2}^\infty \frac{1}{(\ln(x))^r}\,\mathrm{d}(\ln x)$$

and so, for $u(x) = \ln(x)$ we have

$$\int_{\ln 2}^\infty \frac{1}{u^r}\,\mathrm{d}u$$

which, by TODO: add reference, converges for $r > 1$.

(105) $\boldsymbol{\int_1^\infty \frac{1}{\sqrt{x^4+x^3}-x^2}\,\mathrm{d}x}$:

Observe that

$$\begin{aligned}
\frac{1}{\sqrt{x^4+x^3}-x^2} &= \frac{1}{\sqrt{x^4+x^3}-x^2}\cdot\frac{\sqrt{x^4+x^3}+x^2}{\sqrt{x^4+x^3}+x^2} \\
&= \frac{\sqrt{x^4+x^3}+x^2}{x^3} \\
&= \frac{\sqrt{x^4(1+\frac{1}{x})}+x^2}{x^3} \\
&= \frac{x^2((1+\frac{1}{x})+1)}{x^3} \\
&= \frac{(1+\frac{1}{x})+1}{x}.
\end{aligned}$$

We can guess that this will behave asymptotically similarly to $\frac{2}{x}$ and so we perform the ratio test TODO: add reference,

$$\frac{\frac{(1+\frac{1}{x})+1}{x}}{\frac{2}{x}} = \frac{(1+\frac{1}{x})+1}{2} \to 1 \quad \text{as} \quad x \to \infty.$$

Therefore, the integral behaves similarly to

$$\int_1^\infty \frac{2}{x}\,\mathrm{d}x = 2\int_1^\infty \frac{1}{x}\,\mathrm{d}x$$

which diverges by the divergence of $\int_1^\infty \frac{1}{x}\,\mathrm{d}x$ (TODO: add reference).

(106) $\boldsymbol{\int_{-\infty}^{\infty} \frac{x}{1+x^2}\,\mathrm{d}x}$:

This is an example of a symmetric improper integral of an odd function such that, for all $a > 0 \in \mathbb{R}$,

$$\int_{-a}^{a} \frac{x}{1+x^2}\,\mathrm{d}x = 0.$$

But this integral is defined to be the sum of the limits,

$$\lim_{a_1 \to -\infty,\, a_2 \to \infty} \left(\int_{a_1}^{0} \frac{x}{1+x^2}\,\mathrm{d}x + \int_{0}^{a_2} \frac{x}{1+x^2}\,\mathrm{d}x \right)$$

and since both the summands are infinite and have no defined limit, the sum is also undefined. Therefore, the Riemann Integral is undefined (or divergent).

4.6.2 Multivariate Integration

4.6.2.1 Partial Differentiation of Integrals

$$\frac{\partial}{\partial x}\left(\int f(x,y)\,\mathrm{d}y\right) = \int \frac{\partial f(x,y)}{\partial x}\,\mathrm{d}y$$

4.6.2.2 Integration of Partial Derivatives

Suppose we have a function $f(x,y)$. Then,

$$\int \frac{\partial f(x,y)}{\partial x}\,\mathrm{d}x = g(x,y) + C(y) \quad \text{and} \quad \int \frac{\partial f(x,y)}{\partial y}\,\mathrm{d}x = g(x,y) + C(x).$$

So, the antiderivatives are only determined upto a function of the other variable. In this case, we can recover the original function $f(x,y)$ if we have both partial derivatives by setting the antiderivatives equal,

$$g_1(x,y) + C_1(y) = g_2(x,y) + C_2(x).$$

Any cross terms (terms containing both x and y) in f will appear in both $g_1(x,y)$ and $g_2(x,y)$ and the remaining term in g_1 will be equal to C_2 while the remaining term in g_2 will be equal to C_1.

Example

(107) Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be such that for all $(x,y) \in \mathbb{R}^2$ the following equalities hold:

$$\frac{\partial \varphi}{\partial x}(x,y) = \underbrace{e^x \sin(y)}_{f(x,y)} \wedge \frac{\partial \varphi}{\partial y}(x,y) = \underbrace{e^x \cos(y)}_{g(x,y)}.$$

Integrating f with respect to x we get

$$\varphi(x,y) = \int e^x \sin(y) dx = e^x \sin(y) + \psi(y),$$

for some differentiable function $\psi : \mathbb{R} \to \mathbb{R}$. Differentiating with respect to y it follows that

$$e^x \cos(y) + \psi'(y) = g(x,y) = e^x \cos(y).$$

Therefore ψ' is always 0 and it follows that there exists $C \in \mathbb{R}$ such that for all $u \in \mathbb{R}$ we have $\psi(u) = C$ – that is, ψ is constant.
This gives you $\varphi(x, y) = e^x \sin(y) + C \in \mathbb{R}$, for some $C \in \mathbb{R}$. Taking $C \in \mathbb{R}$ arbitrarily will give you a possible φ.

In the case of definite integration though, whether the variables are independent is important (?). For example, suppose $y = y(x)$. Refer: https://math.stackexchange.com/questions/754742/integrating-a-partial-derivative and https://math.stackexchange.com/questions/2714061/integrating-partial-derivatives?noredirect=1&lq=1.

- If $y = y(x)$ then the result of the integral may not be meaningful.
- If we have both of the partial derivatives (wrt. x and y), then we can recover the original function regardless of whether the variables are independent or not.

4.7 Difference and Differential Equations

4.7.1 First-order Difference Equations

*A **difference equation** is also known as a **recurrence equation**.*

Definition 193. Let y_t be the t-th value in a sequence (typically t represents time). Then,

$$y_t = ay_{t-1} + b, \qquad t \geq 1$$

is called a **first-order linear difference equation with constant coefficients**. The value y_0 is called an initial condition.

A solution to such an equation is an explicit – or a closed-form (see: wikipedia) – expression for y_t in terms of t and y_0.

If $b = 0$ we have,

$$y_t = ay_{t-1} \iff y_t - ay_{t-1} = 0$$

which is known as a **homogeneous** first-order linear difference equation with constant coefficients.

Proposition 4.7.1. *A first-order linear difference equation with constant coefficients of the form $y_t = ay_{t-1} + b$ where $a = 1$ is a arithmetic progression.*

Proof. Let $y_t = y_{t-1} + b$ then,

$$y_1 = y_0 + b,\ y_2 = y_1 + b = (y_0 + b) + b = y_0 + 2b,\ y_3 = y_0 + 3b, \ldots$$

so we have $\boldsymbol{y_t = y_0 + tb}$. If we describe this as an arithmetic progression we have,

$$x_n = a + nd$$

where the zeroth term a corresponds to y_0, the common difference d corresponds to b and, clearly, t and n are both term indices. □

Proposition 4.7.2. *A first-order linear difference equation with constant coefficients of the form $y_t = ay_{t-1} + b$ where $b = 0$ is a geometric progression.*

Proof. Let $y_t = ay_{t-1} + 0$ then,

$$y_1 = ay_0,\ y_2 = ay_1 = a(ay_0) = a^2y_0,\ y_3 = a^3y_0, \ldots$$

so we have $\boldsymbol{y_t = a^t y_0}$. If we describe this as a geometric progression we have,

$$x_n = ar^n$$

where the zeroth term a corresponds to y_0, the common ratio r corresponds to a, and n is the term index corresponding to t. □

Proposition 4.7.3. *A first-order linear difference equation with constant coefficients of the form $y_t = ay_{t-1} + b$ where $a \neq 1$ and $b \neq 0$ has solution*

$$\begin{aligned} y_t &= a^t y_0 + b(a^{t-1} + a^{t-2} + \cdots + a + 1) \\ &= a^t y_0 + b\sum_{i=0}^{t-1} a^i. \end{aligned}$$

Proof.

$$\begin{aligned} y_t &= ay_{t-1} + b \\ &= a(ay_{t-2} + b) + b = a^2y_{t-2} + ab + b \\ &= a^2(a(y_{t-3} + b)) + ab + b = a^3y_{t-3} + a^2b + ab + b \\ &= a^3y_{t-3} + b(a^2 + a + 1) \\ &= a^t y_0 + b(a^{t-1} + \cdots + a + 1). \end{aligned}$$

□

Proposition 4.7.4. *A first-order linear difference equation with constant coefficients of the form $y_t = ay_{t-1} + b$ where $a \neq 1$ and $b \neq 0$ has solution*

$$y_t = a^t \left(y_0 - \frac{b}{1-a} \right) + \frac{b}{1-a}$$

where the value

$$\frac{b}{1-a} = y^* = ay^* + b$$

is the ***equilibrium*** *or* ***steady-state*** *value of the recurrence.*

Proof. From Proposition 4.7.3 we know that the solution to the given recurrence is

$$y_t = a^t y_0 + b \sum_{i=0}^{t-1} a^i.$$

The summation in the second term is the sum of a geometric progression,

$$\begin{aligned} \sum_{i=0}^{t-1} a^i = 1 + a + \cdots + a^{t-1} &= \frac{a^t - 1}{a - 1} \\ &= \frac{a^t}{a-1} - \frac{1}{a-1} \\ &= \frac{1}{1-a} - \frac{a^t}{1-a}. \end{aligned}$$

So we see that the sum of a geometric progression can be separated into two terms: one term depends on the number of elements in the progression (here t), and the other term does not. This is the explanation of the sum of convergent geometric series: if $|a| < 1$ then, when $t \to \infty$, the t-dependent term disappears leaving only the steady-state term.

The solution to the recurrence therefore becomes

$$\begin{aligned} y_t &= a^t y_0 + b \left(\frac{1}{1-a} - \frac{a^t}{1-a} \right) \\ &= a^t \left(y_0 - \frac{b}{1-a} \right) + \frac{b}{1-a}. \end{aligned}$$

□

4.7.1.1 First-order Linear Difference Equations as Affine Transformations

If we define a vectorized linear difference equation with constant coefficients,

$$\vec{\boldsymbol{y}}_t = A\vec{\boldsymbol{y}}_{t-1} + \vec{\boldsymbol{b}}$$

where A is a matrix, then clearly $\vec{\boldsymbol{y}}_t$ is an affine transformation of $\vec{\boldsymbol{y}}_{t-1}$.

We can linearize this by adding an extra dimension that takes the value 1 like so:

$$\begin{bmatrix} \vec{\boldsymbol{y}}_t \\ 1 \end{bmatrix} = \begin{bmatrix} A & \vec{\boldsymbol{b}} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{\boldsymbol{y}}_{t-1} \\ 1 \end{bmatrix}.$$

A minimal, one-by-one matrix is just a scalar (see: 56) and a minimal vector can be just a field element (see: 2.4.1.1) so we can define $A = a$ such that the linearized equation becomes

$$\begin{aligned} \begin{bmatrix} y_t \\ 1 \end{bmatrix} &= \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}^t \begin{bmatrix} y_0 \\ 1 \end{bmatrix}. \end{aligned}$$

Eigenvectors of the Transformation

In order to easily take powers of the transformation we find the eigenvectors.

Let M be the transformation matrix,

$$M = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$$

and, for an eigenvector $\vec{\boldsymbol{v}}$,

$$\begin{aligned} & & M\vec{\boldsymbol{v}} &= \lambda\vec{\boldsymbol{v}} \\ \iff & & \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \vec{\boldsymbol{v}} &= \lambda\vec{\boldsymbol{v}} \\ \iff & & \left(\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} - \lambda I \right) \vec{\boldsymbol{v}} &= \vec{\boldsymbol{0}} \end{aligned}$$

$$\iff \qquad \begin{bmatrix} a-\lambda & b \\ 0 & 1-\lambda \end{bmatrix} \vec{\boldsymbol{v}} = \vec{\mathbf{0}}$$

$$\begin{vmatrix} a-\lambda & b \\ 0 & 1-\lambda \end{vmatrix} = 0$$

$$\iff \qquad (a-\lambda)(1-\lambda) = 0$$

$$\iff \qquad \lambda \in \{1, a\}.$$

So, for eigenvalue 1:

$$\begin{bmatrix} a-1 & b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\iff \qquad (a-1)v_1 + bv_2 = 0$$

$$\iff \qquad (a-1)v_1 = -b \qquad \text{letting } v_2 = 1$$

$$\iff \qquad v_1 = \frac{-b}{a-1} = \frac{b}{1-a}$$

$$\therefore \qquad \vec{\boldsymbol{v}} = \begin{bmatrix} \frac{b}{1-a} \\ 1 \end{bmatrix}.$$

Note that $\frac{b}{1-a}$ is the steady-state solution of the difference equation.

For eigenvalue a:

$$\begin{bmatrix} 0 & b \\ 0 & 1-a \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\iff \qquad v_2 = 0$$

$$\therefore \qquad \vec{\boldsymbol{v}} = c \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ for } c \in \mathbb{R}.$$

Note that the second component of this vector adds the translation of the affine transformation so this is telling us that if $b = 0$ then any vector is an eigenvector with eigenvalue a. This corresponds to the homogeneous solution.

Diagonalization

So, if B is the set of two eigenvectors

$$\left\{ \begin{bmatrix} \frac{b}{1-a} \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

and we take this as a basis, then the change of basis matrix to this basis is

$$P = [B]^{-1} = \begin{bmatrix} \frac{b}{1-a} & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & \frac{-b}{1-a} \end{bmatrix}$$

and the diagonal matrix that represents the transformation with respect to this basis is

$$\begin{aligned} D &= PMP^{-1} \\ &= \begin{bmatrix} 0 & 1 \\ 1 & \frac{-b}{1-a} \end{bmatrix} \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{b}{1-a} & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ 1 & \frac{-b}{1-a} \end{bmatrix} \begin{bmatrix} \frac{b}{1-a} & a \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}. \end{aligned}$$

Since D is diagonal we have

$$D^t = \begin{bmatrix} 1 & 0 \\ 0 & a^t \end{bmatrix}.$$

So, in order to calculate,

$$M^t = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}^t$$

we can calculate,

$$D^t = (PMP^{-1})^t = PM^tP^{-1}$$

$$\iff \qquad P^{-1}D^tP = M^t.$$

$$\therefore \qquad M^t = \begin{bmatrix} \frac{b}{1-a} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & a^t \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & \frac{-b}{1-a} \end{bmatrix}$$

$$\iff \qquad M^t = \begin{bmatrix} \frac{b}{1-a} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ a^t & \frac{-a^tb}{1-a} \end{bmatrix}$$

$$\iff \qquad M^t = \begin{bmatrix} a^t & \frac{b(1-a^t)}{1-a} \\ 0 & 1 \end{bmatrix}.$$

So, the solution to the first-order linear recurrence is found to be

$$\begin{bmatrix} y_t \\ 1 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}^t \begin{bmatrix} y_0 \\ 1 \end{bmatrix} = \begin{bmatrix} a^t & \frac{b(1-a^t)}{1-a} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_0 \\ 1 \end{bmatrix}$$

so that

$$y_t = a^t y_0 + \frac{b(1-a^t)}{1-a} = a^t \left(y_0 - \frac{b}{1-a} \right) + \frac{b}{1-a}.$$

4.7.1.2 Difference as Rate of change

Geometric progression: Say we have a first-order linear difference equation with constant coefficients of the form $y_t = ay_{t-1} + b$ where $b = 0$ so that the terms follow a geometric progression:

$$y_1 = ay_0,\ y_2 = a^2y_0,\ y_3 = a^3y_0, \ldots$$

Then the rate of change is

$$\frac{\Delta y}{\Delta t} = y_t - y_{t-1} = ay_{t-1} - y_{t-1} = (a-1)y_{t-1}.$$

Note that the ratio of consecutive terms remains constant,

$$\frac{y_t}{y_{t-1}} = a$$

which is the "geometric" characteristic of a geometric progression, but the rate of change is proportional to the value.

The rate of change of the rate of change is, therefore,

$$\frac{\Delta^2 y}{\Delta t^2} = (a-1)\frac{\Delta y_{t-1}}{\Delta t} = (a-1)\frac{\Delta y}{\Delta t} = (a-1)^2 y_{t-1}.$$

4.7.1.3 Examples of first-order linear difference equations w/ const. coefficients

(108) Let y_t be an account balance after t years and r be the annual interest rate paid on the account. Suppose also, that the interest is compounded n times per year. Then the formula for y_t is,

$$y_t = \left(1 + \frac{r}{n}\right)^n y_{t-1} = y_0 \left(1 + \frac{r}{n}\right)^{nt}.$$

The rate of change per year is,

$$\frac{\Delta y}{\Delta t} = y_t - y_{t-1} = \left(1 + \frac{r}{n}\right)^n y_{t-1} - y_{t-1} = \left(\left(1 + \frac{r}{n}\right)^n - 1\right) y_{t-1}$$

as expected for a geometric progression.

If, instead, we let t be continuous we can find the instantaneous rate of change by considering the values for $t, t + \Delta t$,

$$\begin{aligned}\frac{\Delta y}{\Delta t} = \frac{y_{(t+\Delta t)} - y_t}{\Delta t} &= \frac{y_0 \left(1 + \frac{r}{n}\right)^{n(t+\Delta t)} - y_0 \left(1 + \frac{r}{n}\right)^{nt}}{\Delta t} \\ &= \frac{y_0 \left(1 + \frac{r}{n}\right)^{nt} \left(\left(1 + \frac{r}{n}\right)^{n\Delta t} - 1\right)}{\Delta t}\end{aligned}$$

and then letting $\Delta t \to 0$ to obtain,

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}t} &= \lim_{\Delta t \to 0} \frac{y_0 \left(1+\frac{r}{n}\right)^{nt} \left(\left(1+\frac{r}{n}\right)^{n\Delta t} - 1\right)}{\Delta t} \\
&= y_0 \left(1+\frac{r}{n}\right)^{nt} \lim_{\Delta t \to 0} \frac{\left(1+\frac{r}{n}\right)^{n\Delta t} - 1}{\Delta t} \\
&= y_0 \left(1+\frac{r}{n}\right)^{nt} \lim_{\Delta t \to 0} \left[\frac{\mathrm{d}}{\mathrm{d}(\Delta t)} \left(1+\frac{r}{n}\right)^{n\Delta t}\right] && \text{by L'Hôpital's rule} \\
&= y_0 \left(1+\frac{r}{n}\right)^{nt} \lim_{\Delta t \to 0} \left[n \ln\left(1+\frac{r}{n}\right) \left(1+\frac{r}{n}\right)^{n\Delta t}\right] && \text{by Proposition 4.5.1} \\
&= n y_0 \ln\left(1+\frac{r}{n}\right) \left(1+\frac{r}{n}\right)^{nt}
\end{aligned}$$

But the question is: How meaningful is this really? If the interest is being compounded only n times per year and these are the only moments when the account balance changes, then change happens at certain discrete moments rather than continuously so is it really meaningful to talk about the instantaneous rate of change? We can recover the discrete-time rate of change per year from this instantaneous one by integrating over a year.

Now suppose that n, the number of times per year the interest is compounded, goes to infinity. Then (see 4.4.0.1) we have,

$$y_t = y_0 e^{rt}.$$

For discrete time we have,

$$\frac{\Delta y}{\Delta t} = y_0 e^{rt} - y_0 e^{r(t-1)} = y_0 e^{r(t-1)}(e^r - 1) = y_{t-1}(e^r - 1)$$

which should be no surprise as we still have a geometric progression as

$$\frac{y_t}{y_{t-1}} = e^r.$$

If we now let t be continuous as before then, by a similar logic using L'Hôpital's rule, the instantaneous rate of change obtained is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = ry_0e^{rt}.$$

Note that this is wholly consistent with the discrete time version as we can obtain the discrete time rate of change per year from this instantaneous rate of change by integrating over a year as follows,

$$\begin{aligned}
\int_{t-1}^{t} ry_0e^{rx}\,\mathrm{d}x &= ry_0\int_{t-1}^{t} e^{rx}\,\mathrm{d}x \\
&= ry_0\left[\frac{e^{rx}}{r}\right]_{t-1}^{t} \\
&= y_0\left[e^{rx}\right]_{t-1}^{t} \\
&= y_0(e^{rt} - e^{r(t-1)}) \\
&= y_0e^{r(t-1)}(e^r - 1).
\end{aligned}$$

Now suppose b is deposited at the end of each year so that,

$$\begin{aligned}
y_t &= \left(1+\frac{r}{n}\right)^n y_{t-1} + b \\
&= \left(1+\frac{r}{n}\right)^n \left[\left(1+\frac{r}{n}\right)^n y_{t-2} + b\right] + b \\
&= \left(1+\frac{r}{n}\right)^{nt} y_0 + b\sum_{i=0}^{t-1}\left(1+\frac{r}{n}\right)^{ni}.
\end{aligned}$$

But if we, again, let the number of compounds n, go to infinity, then,

$$\begin{aligned}
y_t &= e^r y_{t-1} + b \\
&= e^r(e^r y_{t-2} + b) + b \\
&= y_0e^{rt} + b\sum_{i=0}^{t-1} e^{ri} \\
&= y_0e^{rt} + b\left(\frac{e^{rt}-1}{e^r-1}\right)
\end{aligned}$$

$$= y_0 e^{rt} + b\frac{e^{rt}}{e^r - 1} - b\frac{1}{e^r - 1}$$
$$= \left(y_0 - \frac{b}{1 - e^r}\right) e^{rt} + \frac{b}{1 - e^r}$$

where $\frac{b}{1-e^r}$ is the steady-state value.

(109) Let M_t be an account balance at the end of year t. Let $Q(t)$ be an amount that is deposited into the account (or withdrawn if the value is negative) one time, at the end of year t and let the interest rate be a fixed rate of I. Then,

$$\begin{aligned} M_t &= IM_{t-1} + Q(t) \\ &= I(IM_{t-2} + Q(t-1)) + Q(t) \\ &= I(I(IM_{t-3} + Q(t-2)) + Q(t-1)) + Q(t) \\ &= M_0 I^t + \sum_{i=0}^{t-1} Q(t-i) I^i \end{aligned}$$

and the rate of change is,

$$M_t - M_{t-1} = M_0 I^{t-1}(I-1) + Q(1)I^{t-1} = I^{t-1}(M_0(I-1) + Q(1))$$

which is proportional to I^{t-1}.

If the interest rate I, is also a function of t, then – if we define $I(0) = 1$ – we end up with:

$$M_t = M_0 \prod_{i=0}^{t} I(i) + \sum_{i=0}^{t-1} Q(t-i) \prod_{j=0}^{i} I(i)$$

which is getting pretty messy. At this point it becomes easier to work with continuous time.

4.7.2 Second-order Difference Equations

Definition 194. A recurrence equation of the form

$$y_t = ay_{t-1} + by_{t-2} + c$$

where $a, b, c \in \mathbb{R}$, is known as a **second-order linear recurrence with constant coefficients**.

Following the same approach as with the first-order equations we have,

$$\begin{bmatrix} y_t \\ y_{t-1} \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ y_{t-2} \\ 1 \end{bmatrix}.$$

Determining eigenvalues we get,

$$\begin{vmatrix} a-\lambda & b & c \\ 1 & -\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$\iff \quad (1-\lambda)((-\lambda)(a-\lambda)-b) = 0$$

$$\therefore \quad \lambda \in \left\{1, \frac{a+\sqrt{a^2+4b}}{2}, \frac{a-\sqrt{a^2+4b}}{2}\right\}.$$

Due to the translation represented by the lone 1 in the final row of the matrix, there will always be the eigenvalue 1. This corresponds to the steady state value which may be thought of as the eigenvector of the translation. The other two eigenvalues are the eigenvectors of the linear transformation part of the affine transformation. The block structure is as follows.

$$\begin{bmatrix} \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} & \begin{bmatrix} c \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 \end{bmatrix} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ y_{t-2} \\ 1 \end{bmatrix} = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ y_{t-2} \end{bmatrix} + \begin{bmatrix} c \\ 0 \end{bmatrix}$$

$$= A\vec{\boldsymbol{y}} + \vec{\boldsymbol{t}}.$$

Steady-state value

Considering the case of the eigenvalue $\lambda = 1$ we have,

$$\begin{bmatrix} a-1 & b & c \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ y_{t-2} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

So, to find the nullspace of the matrix we can find the row-reduced echelon form,

$$\begin{bmatrix} a-1 & b & c \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & \frac{b}{a-1} & \frac{c}{a-1} \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & \frac{b}{a-1} & \frac{c}{a-1} \\ 0 & -1-\frac{b}{a-1} & -\frac{c}{a-1} \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & \frac{b}{a-1} & \frac{c}{a-1} \\ 0 & 1 & \frac{c}{a+b-1} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 0 & \frac{c}{a+b-1} \\ 0 & 1 & \frac{c}{a+b-1} \\ 0 & 0 & 0 \end{bmatrix}$$

where the last step is because

$$\begin{aligned} \frac{c}{a-1} - \left(\frac{b}{a-1}\right)\frac{c}{a+b-1} &= \frac{c}{a-1}\left(1 - \frac{b}{a+b-1}\right) \\ &= \frac{c}{a-1}\left(\frac{a-1}{a+b-1}\right). \end{aligned}$$

So the nullspace is

$$t \begin{bmatrix} \frac{-c}{a+b-1} \\ \frac{-c}{a+b-1} \\ 1 \end{bmatrix}$$

for any $t \in \mathbb{R}$ and the steady-state value is

$$\frac{-c}{a+b-1}.$$

This is also sometimes referred to as a ***particular solution of the non-homogeneous equation****.*

Eigenvalues of the linear transformation

Let the discriminant $d = \sqrt{a^2 + 4b}$. Then the eigenvalues of the linear transformation are

$$\frac{a+d}{2} \quad \text{and} \quad \frac{a-d}{2}.$$

In the case of the eigenvalue $\frac{a+d}{2}$ we have,

$$\begin{bmatrix} \frac{a-d}{2} & b & c \\ 1 & -(\frac{a+d}{2}) & 0 \\ 0 & 0 & 1-(\frac{a+d}{2}) \end{bmatrix} \begin{bmatrix} y_{t-1} \\ y_{t-2} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Again using row reduction to find the nullspace:

$$\begin{bmatrix} \frac{a-d}{2} & b & c \\ 1 & -(\frac{a+d}{2}) & 0 \\ 0 & 0 & 1-(\frac{a+d}{2}) \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -(\frac{a+d}{2}) & \frac{2c}{a-d} \\ 1 & -(\frac{a+d}{2}) & 0 \\ 0 & 0 & 1-(\frac{a+d}{2}) \end{bmatrix} \quad \text{using } \frac{2b}{a-d}\frac{a+d}{a+d} = -(\frac{a+d}{2})$$

$$\rightsquigarrow \quad \begin{bmatrix} 1 & -(\frac{a+d}{2}) & \frac{2c}{a-d} \\ 0 & 0 & \frac{-2c}{a-d} \\ 0 & 0 & 1-(\frac{a+d}{2}) \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -(\frac{a+d}{2}) & \frac{2c}{a-d} \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\rightsquigarrow \quad \begin{bmatrix} 1 & -(\frac{a+d}{2}) & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So the nullspace is

$$t \begin{bmatrix} \frac{a+d}{2} \\ 1 \\ 0 \end{bmatrix}$$

for any $t \in \mathbb{R}$ and the value

$$\frac{a+d}{2} = \frac{a + \sqrt{a^2 + 4b}}{2}$$

is the solution to the homogeneous equation corresponding to the eigenvalue $(a+d)/2$. It's also not hard to see that the other eigenvalue $(a-d)/2$ results in the solution to the homogeneous equation

$$\frac{a-d}{2} = \frac{a-\sqrt{a^2+4b}}{2}.$$

Another common way of finding these solutions in this case is to form what is known as the ***auxiliary equation*** *as:*

$$\begin{aligned}
& & y_t - ay_{t-1} - by_{t-2} &= 0 & \\
\rightsquigarrow & & m^t - am^{t-1} - bm^{t-2} &= 0 & \text{for some } m \neq 0 \in \mathbb{R} \\
\iff & & m^{t-2}(m^2 - am - b) &= 0 & \\
\iff & & m^2 - am - b &= 0. & \text{since } m \neq 0
\end{aligned}$$

Note that this is the characteristic polynomial of the linear transformation part of the matrix:

$$(-\lambda)(a-\lambda) - b = \lambda^2 - a\lambda - b.$$

In this case, determining the final equation for y_t using matrix powers results in a very complex matrix and formula which is only reasonably performed on a computer. But we can infer that the formula will involve the eigenvalues of the linear transformation, raised to the power t, and the steady-state value,

$$y_t = C_1\frac{(a+d)^t}{2^t} + C_2\frac{(a-d)^t}{2^t} + \frac{c}{a+b-1}.$$

Then we can use the initial values to solve for the constants C_1, C_2,

$$y_0 = C_1 + C_2 + \frac{c}{a+b-1} \quad \text{and} \quad y_1 = C_1\frac{a+d}{2} + C_2\frac{a-d}{2} + \frac{c}{a+b-1}.$$

The values

$$\frac{a+d}{2} = \frac{a+\sqrt{a^2+4b}}{2} \quad \text{and} \quad \frac{a-d}{2} = \frac{a-\sqrt{a^2+4b}}{2}$$

may be two distinct real values, one real value (if $a^2+4b=0$) or two distinct complex values (if $a^2+4b<0$).

In the case of one real value: the formula becomes,

$$y_t = (C_1 + C_2 t)\frac{(a+d)^t}{2^t} + \frac{c}{a+b-1}.$$

The explanation of this appears to be that the matrix $A - \lambda I$ has cyclic order 2. TODO: ? eh?

In the case of two complex values: expressing the eigenvalues in exponential form we have,

$$\begin{aligned}\frac{a}{2} \pm \frac{\sqrt{-a^2-4b}}{2}i &= \sqrt{\frac{a^2}{4} + \frac{-a^2-4b}{4}}\,\exp\left(i\arctan \pm\frac{\sqrt{-a^2-4b}}{a}\right)\\ &= \sqrt{-b}\,\exp\left(i\arctan \pm\sqrt{-1-\frac{4b}{a^2}}\right).\end{aligned}$$

Let

$$\theta = \arctan\sqrt{-1-\frac{4b}{a^2}} \quad \text{and} \quad -\theta = \arctan -\sqrt{-1-\frac{4b}{a^2}}.$$

Then the eigenvalues raised to the power of t are:

$$(\sqrt{-b})^t\, e^{i\theta t} \quad \text{and} \quad (\sqrt{-b})^t\, e^{-i\theta t}$$

which means that the formula becomes,

$$\begin{aligned} y_t &= C_1(\sqrt{-b})^t\,(\cos(\theta t) + i\sin(\theta t))\\ &\quad + C_2(\sqrt{-b})^t\,(\cos(-\theta t) + i\sin(-\theta t))\\ &\quad + \frac{c}{a+b-1}\\ &= C_1(\sqrt{-b})^t\,(\cos(\theta t) + i\sin(\theta t))\\ &\quad + C_2(\sqrt{-b})^t\,(\cos(\theta t) - i\sin(\theta t))\\ &\quad + \frac{c}{a+b-1}\\ &= (\sqrt{-b})^t\,(C_1 + C_2)\cos(\theta t)\end{aligned}$$

$$
\begin{aligned}
&+ i(\sqrt{-b})^t \, (C_1 - C_2) \sin(\theta t) \\
&+ \frac{c}{a+b-1} \\
&= (\sqrt{-b})^t \, (C_3 \cos(\theta t) + iC_4 \sin(\theta t)) + \frac{c}{a+b-1}.
\end{aligned}
$$

But, since we are looking for real-valued solutions and any linear combination of the homogeneous solutions is also a homogeneous solution, we can generate other, real-valued homogeneous solutions from linear combinations of these ones. In fact, we can just divide the imaginary solution by i so that our real-valued solution is:

$$
(\sqrt{-b})^t \, (C_3 \cos(\theta t) + C_4 \sin(\theta t)) + \frac{c}{a+b-1}.
$$

TODO: eh? if divide by i then the real part will be divided by i also(answer: 2.4.28)

4.7.3 Markov Chains

4.7.3.1 Markov Matrices (a.k.a. Stochastic Matrices)

This section taken from Harvard.

Definition 195. An $n \times n$ matrix is called a **Markov** or **Stochastic** matrix if all entries are nonnegative and the sum of each column vector is equal to 1.

The matrix

$$A = \begin{bmatrix} 1/2 & 1/3 \\ 1/2 & 2/3 \end{bmatrix}$$

is a Markov matrix.

Many authors write the transpose of the matrix and apply the matrix to the right of a row vector.

Let's call a vector with nonnegative entries p_k for which all the p_k add up to 1 a stochastic vector. For a stochastic matrix, every column is a stochastic vector.

Theorem 4.7.1. *If p is a stochastic vector and A is a stochastic matrix, then Ap is a stochastic vector.*

Proof. Let $v_1, \ldots, v_n$ be the column vectors of A. Then

$$Ap = \begin{bmatrix} p_1 \\ p_2 \\ \cdots \\ p_n \end{bmatrix} = p_1 v_1 + \ldots + v_n v_n$$

If we sum this up we get $p_1 + p_2 + \ldots + p_n = 1$. □

Theorem 4.7.2. *A Markov matrix A always has an eigenvalue* 1. *All other eigenvalues are in absolute value smaller or equal to* 1.

Proof. For the transpose matrix A^T, the sum of the row vectors is equal to 1. The matrix A^T therefore has the eigenvector

$$\begin{bmatrix} 1 \\ 1 \\ \dots \\ 1 \end{bmatrix}.$$

Because A and A^T have the same determinant, also $A - \lambda I_n$ and $A^T - \lambda I_n$ have the same determinant so that the eigenvalues of A and A^T are the same (TODO: we can reference the proof that a matrix is similar to its transpose but does this explanation also make sense?). With A^T having an eigenvalue 1 also A has an eigenvalue 1 Assume now that v is an eigenvector with an eigenvalue $|\lambda| > 1$. Then $A^n v = |\lambda|^n v$ has exponentially growing length for $n \to \infty$. This implies that there is for large n one coefficient $[A^n]_{ij}$ which is larger than 1. But A^n is a stochastic matrix (see homework) and has all entries ≤ 1. The assumption of an eigenvalue larger than 1 can not be valid. □

For there to be a long-term distribution of the markov chain it is necessary that the eigenvalue 1 in the markov matrix have multiplicity 1. Otherwise, there may be more than one eigenvector corresponding with eigenvalue 1 or the matrix may not be diagonalizable. In the case that it is diagonalizable, there will only be one eigenvector with eigenvalue 1 that is also a distribution vector (stochastic vector).

4.7.4 Differential Equations

Differential equations are most often used to describe the evolving state of dynamical systems – that is, systems whose future state is a function of its current state. Therefore, that portion of the future state that is dependent on the previous state is compounded in a similar fashion to compound interest. For this reason, the solutions to differential equations typically involve the exponential function.

4.7.4.1 Types of Differential Equations

Definition 196. A **differential equation** – that is, a single equation as opposed to a system of equations – is an equation that relates a single dependent variable's derivatives to each other and may or may not explicitly include the independent variable. A common convention is for the dependent variable to be y and the independent variable to be t – reflecting the fact that it is often modelling time – but x is often used also.

Definition 197. A differential equation that does not explicitly include the independent variable is known as an **autonomous** equation. It represents a time-invariant system if the independent variable is viewed as time. So, if $y(t) = g(t)$ is a solution then $y(t) = g(t + c)$, for constant c, is also a solution.

Definition 198. A **first-order** differential equation is an equation in which only derivatives upto and including the first derivative of the dependent variable appear. That's to say, an equation that relates the dependent variable to its first derivative and, potentially, to the independent variable.

Definition 199. Similarly, a **second-order** differential equation relates the dependent variable to both its first and second derivatives as well as, potentially, to the independent variable. Higher-order differential equations also exist – but are less common – with the order being given by the highest derivative present in the equation.

Definition 200. A **linear** differential equation is an equation containing only degree-one monomial terms in the derivatives of the dependent variable. So, the equation is linear in the derivatives of the dependent variable although it may also contain any function of the independent variable.

Definition 201. A **nonlinear** differential equation is an equation that contains nonlinear terms of the derivatives of the dependent variable.

Definition 202. A **separable** first-order equation is one where the first-derivative of the dependent variable may be expressed as a single term. That's to say, we can put the equation in the form,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t)g(y).$$

Note that the definition of a separable equations means that – if A, B, C, D are constants – the following equation is separable,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ABt^2y^2 + ADt^2 + BCy^2 + CD$$

because it can be factorized into,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = (At^2 + C)(By^2 + D) = f(t)g(y).$$

Definition 203. A **partial** differential equation is a differential equation that includes at least one partial derivative. Otherwise, a differential equation is known as an **ordinary** differential equation. The two terms are frequently abbreviated to ODE and PDE.

4.7.4.2 Solutions

Definition 204. A **solution** to a differential equation – also called a **general solution** – is a set of functions that share the relation expressed in the differential equation.

This is similar to the solution to an indefinite integral being a set of antiderivatives. However, an indefinite integral determines an antiderivative upto an additive constant but the general solution of a differential equation may only determine a function upto a multiplicative constant.

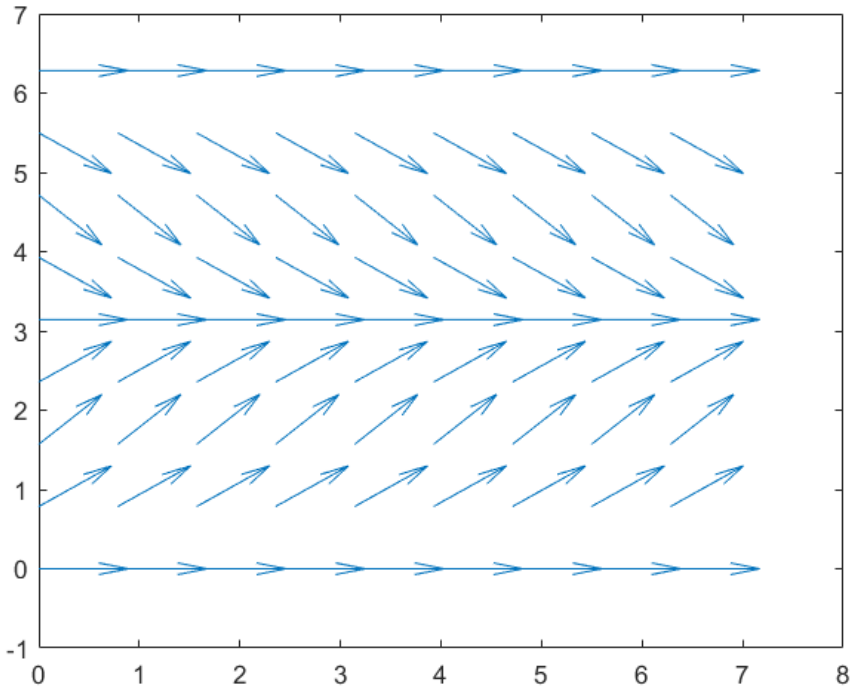
Definition 205. A **solution to the initial value problem** of a differential equation is a specific function where the initial value has determined a particular member of the set of functions in the general solution. That's to say, it is a function that **both** exhibits the relation in the differential equation and satisfies the initial condition(s) of the initial value problem.

A general solution of a differential equation defines a direction field where at every co-ordinate – (x, y) or (t, y) – the gradient is defined by the equation $\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y)$. The solution to an IVP is a path through this direction field beginning at a point (t_0, y_0) representing the initial conditions.

4.7.4.3 Equilibrium Points / Steady-states

Definition 206. A **steady-state** solution is an IVP solution that is constant across all values of the independent variable. Therefore, the initial conditions of the IVP must be this constant value.
For example, if $y(t) = C$ for some constant C, is a steady-state solution then, for the initial condition $y(t_0) = C$, the constant function $y(t) = C$ is a solution to the IVP. Since this function is constant for all values of t, the derivative $\frac{\mathrm{d}y}{\mathrm{d}t} = 0$ also at all values of t.

Figure 4.1: The direction field of $y' = \sin(y)$. The field contains steady-states at $y = 0, \pi, \frac{pi}{2}$.



Although a function $y(t) = C$ is only a solution to the IVP $y(t_0) = C$ other solutions satisfying other initial conditions may converge to the value C. In this way, the steady-state may be the long-term behaviour of IVP solutions from a whole set of initial conditions.

If we take the general form of a first-order differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t)g(y) + h(t),$$

then, for some value of the function y_s to be a steady-state solution we need, for all t,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t)g(y_s) + h(t) = 0$$

$$\iff \qquad g(y_s) = -\frac{h(t)}{f(t)}.$$

For example, the equation $y' = \frac{4t}{y} - 2t$ has a steady-state at $y = 2$.

Definition 207. A steady-state is called **stable** if the set of initial conditions that converge to it is greater than its value alone. In other words, if $y(t) = C$ is a steady-state, then it is stable if the set of all initial values of the function $y_0 = y(t_0)$ whose IVP solutions converge to it,

$$S = \{\, y_0 \mid y(t_0) = y_0 \implies \lim_{t\to\infty} y(t) = C \,\}$$

is a proper superset of $\{C\}$ – i.e. $\{C\} \subsetneq S$.

Conversely, a steady-state is called **unstable** if the set of initial conditions that converge to it is only the value itself. That's to say,

$$S = \{C\}\,.$$

A steady-state can also be **semi-stable** if it is stable from one side and unstable from the other.

*Stable steady-states happen when greater values of the function decrease to the steady-state value and lesser values of the function increase to the steady-state value. An **unstable** steady-state, conversely, is one where values of the function that are a little lesser and greater move away from the steady-state value. The states of real-world systems modelled by these unstable steady-states may sometimes not, in practice, be referred to as equilibria due to their instability.*

Examples of types of steady-state

(110) A ball rolling into a dip in the ground stays there whereas a ball at the top of a hump in the ground will likely roll off. If a dip in the ground is described by the curve $y = x^2$ so that the stable equilibrium is at $x = 0$ then, if the ball is pushed and rolls up the hill on the right side, the potential energy gained is proportional to x^2 and the tendency to return to the equilibrium is dependent on the rate at which the potential energy is released as the ball rolls back to the bottom, which is $\frac{\mathrm{d}x^2}{\mathrm{d}x} = 2x$. Since this is acting to reduce the value of the displacement x its sign is negative so $-2x$. Clearly the converse is the situation for a hump in the ground. So, in this model, the dip in the ground has a gradient field with a maximum at the equilibrium (similar to the curve of $y = -x^2$) while the hump in the ground has a gradient field with a minimum at the equilibrium (similar to the curve of $y = x^2$).

(111) The equation $y' = \sin(y)$ in figure 4.1 shows a stable equilibrium at $y = \pi$ and unstable equilibria at $y = 0, 2\pi$. This is because $\frac{\partial}{\partial y}\sin(y) = \cos(y)$ which is negative around $y = \pi$ – so this is a maximum and therefore, stable – and positive around the values $y = 0, 2\pi$ – so these are minimums and therefore, unstable.

(112) Another example: The equation $y' = \frac{4t}{y} - 2t$ has a **stable** steady-state at $y = 2$ for positive t but at the same y-value the steady-state is unstable for negative values of t. This is because $\frac{\partial}{\partial y}\left(\frac{4t}{y} - 2t\right) = -\frac{4t}{y^2} = -t$ at $y = 2$ which means that the steady-state is a maximum and stable for positive t and a minimum and unstable for negative t.

> *Note that, if t is modelling time, negative values of t may not be meaningful.*

(113) The equation $y' = y^2$ has a semi-stable steady-state at $y = 0$: solution curves below it converge to it, but those above it diverge. Why? At $y = 0$ we have $\frac{\partial}{\partial y}\, y^2 = 2y = 0$ so this is an inflection point. We look at before and after and see that both have positive values of the gradient y'.

(114) A corner case: The equation $y' = 2\sqrt{y}$. This has a steady-state at $y = 0$ which is unstable from above because $\frac{\partial}{\partial y}\left(2\sqrt{y}\right) = \frac{1}{\sqrt{y}}$ is positive for positive y so this is a minimum when viewed from above. But the partial derivative has an infinite discontinuity as $y = 0$ and doesn't exist as a real number for negative y. Likewise the gradient y' is not a real number for negative y and so is undefined in this situation.

4.7.5 First-order Linear ODEs

The form $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)y$

The most simple form has a single y-term whose coefficient may be a function of x. This form is separable as,

$$\begin{aligned}
& \frac{\mathrm{d}y}{\mathrm{d}x} = f(x)y \\
\iff \quad & \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = f(x) \\
\iff \quad & \int \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = \int f(x)\,\mathrm{d}x \\
\iff \quad & \ln|y| = F(x) + c && y \neq 0,\ F \text{ is an antiderivative of } f \\
\iff \quad & |y| = e^{F(x)} \cdot e^{c} \\
\iff \quad & y = ke^{F(x)} && k \in \mathbb{R}.
\end{aligned}$$

Check solution:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)ke^{F(x)} = f(x)y.$$

Note that the solution has the form,

$$y = ke^{F(x)}$$

where $F(x)$ is an antiderivative of $f(x)$, the coefficient of y in the original differential equation. Since the antiderivative is unique upto a constant factor, the other possible antiderivatives are achieved by the value of the coefficient k because,

$$e^{F(x)+c} = e^{F(x)} \cdot e^{c} = ke^{F(x)}.$$

I.V.P. Solution

If we have an initial value for the function – say $y(t_0) = y_0$ – then we need

$$y(t_0) = y_0 = y_0(1) = y_0e^0 = y_0e^{\int_{t_0}^{t} f(x)\,\mathrm{d}x}.$$

That's to say, with the addition of an initial value, the general solution,

$$y(t) = e^{\int f(t)\,\mathrm{d}t} = ke^{F(t)}$$

becomes a complete solution,

$$y(t) = y_0 e^{\int_{t_0}^{t} f(x)\,\mathrm{d}x}.$$

We can see the equivalence by setting the constant k in the general solution to $y_0/e^{F(t_0)}$,

$$y(t) = \frac{y_0}{e^{F(t_0)}} e^{F(t)} = y_0 \frac{e^{F(t)}}{e^{F(t_0)}} = y_0 e^{(F(t)-F(t_0))} = y_0 e^{\int_{t_0}^{t} f(x)\,\mathrm{d}x}.$$

Separable Equations

Any equation of the form $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)y$ is separable into the form,

$$f(x)\,\mathrm{d}x = g(y)\,\mathrm{d}y$$

and can then be solved by integrating both sides. As a result, the solution moves on level sets of $\int f(x)\,\mathrm{d}x - \int g(y)\,\mathrm{d}y$. This value, then, is invariant (i.e. constant) across all solutions and so, tends to represent a conserved quantity in physical systems. Therefore, these type of equations, when modelling physical systems, could be described as modelling closed systems with no external influence.

Examples

(115) Say we have an IVP (Initial Value Problem) such that our independent variable is t beginning at 0, with $y(0) = y_0$, and

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t)y.$$

Then if we look at what happens at integral intervals of t we can see that:

$$y(0) = y_0$$

$$
\begin{aligned}
y(1) &= y_0 e^{\int_0^1 f(t)dt} \\
y(2) &= y_0 e^{\int_0^1 f(t)dt} e^{\int_1^2 f(t)dt} = y_0 e^{\int_0^2 f(t)dt} \\
&\vdots
\end{aligned}
$$

So, in general, at time t,

$$y(t) = y_0 e^{\int_0^t f(u)du}.$$

The form $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)y + g(x)$

This form is not separable as it is. But if we multiply both sides by $e^{-F(x)}$, where $F(x)$ is an antiderivative of $f(x)$, then

$$
\begin{aligned}
& & \frac{\mathrm{d}y}{\mathrm{d}x} &= f(x)y + g(x) \\
\iff & & e^{-F(x)}\frac{\mathrm{d}y}{\mathrm{d}x} &= e^{-F(x)}f(x)y + e^{-F(x)}g(x) \\
\iff & e^{-F(x)}\frac{\mathrm{d}y}{\mathrm{d}x} - e^{-F(x)}f(x)y &= e^{-F(x)}g(x) & \\
\iff & \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-F(x)}y\right) &= e^{-F(x)}g(x) & \\
\iff & \int \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-F(x)}y\right)\mathrm{d}x &= \int e^{-F(x)}g(x)\,\mathrm{d}x & \\
\iff & & e^{-F(x)}y &= \int e^{-F(x)}g(x)\,\mathrm{d}x \\
\iff & & y &= e^{F(x)}\int \frac{g(x)}{e^{F(x)}}\,\mathrm{d}x
\end{aligned}
$$

where the constant of integration of the integral on the left has been absorbed into the integral on the right.

Note, also, that

$$ke^{F(x)}\int \frac{g(x)}{ke^{F(x)}}\,\mathrm{d}x = ke^{F(x)}\frac{1}{k}\int \frac{g(x)}{e^{F(x)}}\,\mathrm{d}x = e^{F(x)}\int \frac{g(x)}{e^{F(x)}}\,\mathrm{d}x.$$

So, the solution has the form,

$$y = ke^{F(x)} + h(x)e^{F(x)}$$

where $h(x)$ is an antiderivative of $g(x)/e^{F(x)}$ and k is the constant of integration.

Check solution:

$$\begin{aligned} y = e^{F(x)}\int \frac{g(x)}{e^{F(x)}}\,\mathrm{d}x \implies \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= f(x)\left(e^{F(x)}\int \frac{g(x)}{e^{F(x)}}\,\mathrm{d}x\right) + e^{F(x)}\frac{g(x)}{e^{F(x)}} \\ &= f(x)\left(e^{F(x)}\int \frac{g(x)}{e^{F(x)}}\,\mathrm{d}x\right) + g(x) \\ &= f(x)y + g(x). \end{aligned}$$

and also,

$$\begin{aligned} y = ke^{F(x)} + h(x)e^{F(x)} \implies \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= f(x)ke^{F(x)} + f(x)h(x)e^{F(x)} + g(x)e^{-F(x)}e^{F(x)} \\ &= f(x)ke^{F(x)} + f(x)h(x)e^{F(x)} + g(x) \\ &= f(x)y + g(x) \end{aligned}$$

where we have used the fact that $h(x)$ is an antiderivative of $g(x)e^{-F(x)}$.

I.V.P. Solution

For the solution of the IVP we are going to want a definite integral. If the exponent in the integrating factor has to be an antiderivative of $f(t)$, as a

definite integral, we can use $e^{\int_a^t f(x)\,\mathrm{d}x}$ where a is any constant real number and t is our independent variable.

Note that the function,

$$F(t) = \int_a^t f(x)\,\mathrm{d}x$$

has derivative $F'(t) = f(t)$, by the FTC, and has $F(a) = 0$.

We can set the value of a to fit the initial conditions. For example, if $y(t_0) = y_0$ and $y(t) = y_0 e^{\int_a^t f(x)\,\mathrm{d}x}$ as in the separable case, then we can set $a = t_0$ so that the initial condition is met.

Now suppose we have an non-separable differential equation with an initial value $y(t_0) = y_0$. When we integrate using our integrating factor we want definite integration starting at t_0, so,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t)y + g(t)$$

$$\iff \quad e^{-\int_a^t f(x)\,\mathrm{d}x}\frac{\mathrm{d}y}{\mathrm{d}t} - e^{-\int_a^t f(x)\,\mathrm{d}x} f(t)y = e^{-\int_a^t f(x)\,\mathrm{d}x} g(t)$$

$$\iff \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-\int_a^t f(x)\,\mathrm{d}x} y\right) = e^{-\int_a^t f(x)\,\mathrm{d}x} g(t)$$

$$\iff \quad \int_{t_0}^t \frac{\mathrm{d}}{\mathrm{d}u}\left(e^{-\int_a^u f(x)\,\mathrm{d}x} y(u)\right)\mathrm{d}u = \int_{t_0}^t e^{-\int_a^u f(x)\,\mathrm{d}x} g(u)\,\mathrm{d}u$$

$$\iff \quad e^{-\int_a^t f(x)\,\mathrm{d}x} y(t) - e^{-\int_a^{t_0} f(x)\,\mathrm{d}x} y(t_0) = \int_{t_0}^t e^{-\int_a^u f(x)\,\mathrm{d}x} g(u)\,\mathrm{d}u$$

$$\iff \quad e^{-\int_a^t f(x)\,\mathrm{d}x} y(t) = \int_{t_0}^t e^{-\int_a^u f(x)\,\mathrm{d}x} g(u)\,\mathrm{d}u + e^{-\int_a^{t_0} f(x)\,\mathrm{d}x} y(t_0)$$

$$\iff \quad y(t) = e^{\int_a^t f(x)\,\mathrm{d}x}\Big(\int_{t_0}^t e^{-\int_a^u f(x)\,\mathrm{d}x} g(u)\,\mathrm{d}u + e^{-\int_a^{t_0} f(x)\,\mathrm{d}x} y(t_0)\Big)$$

$$\iff \quad y(t) = \int_{t_0}^t e^{\int_u^t f(x)\,\mathrm{d}x} g(u)\,\mathrm{d}u + e^{\int_{t_0}^t f(x)\,\mathrm{d}x} y(t_0).$$

So the resultant solution form is:

$$y(t) = y_0 e^{\int_{t_0}^{t} f(x)\,\mathrm{d}x} + \int_{t_0}^{t} e^{\int_{u}^{t} f(x)\,\mathrm{d}x} g(u)\,\mathrm{d}u$$

for $y(t_0) = y_0$.

Examples

(116) $\boldsymbol{x\frac{\mathrm{d}y}{\mathrm{d}x} - 2y = 6}$:

The first step is to rearrange it to obtain an expression for the derivative.

$$\begin{aligned} & x\frac{\mathrm{d}y}{\mathrm{d}x} - 2y = 6 \\ \iff \qquad & \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{x}y + \frac{6}{x}. \end{aligned}$$

Then we can apply the derived formula using the antiderivative $F(x) = 2\ln x$.

$$\begin{aligned} y &= e^{2\ln x} \int \frac{6/x}{e^{2\ln x}}\,\mathrm{d}x \\ &= 6x^2 \int \frac{1}{x^3}\,\mathrm{d}x \\ &= 6x^2 \left(-\frac{1}{2x^2} + c\right) \\ &= -3 + c'x^2. \qquad\qquad c' = 6c \end{aligned}$$

We can confirm this result by performing the calculation.

$$e^{-2\ln x}\frac{\mathrm{d}y}{\mathrm{d}x} = e^{-2\ln x}\frac{2}{x}y + e^{-2\ln x}\frac{6}{x}$$

$$
\begin{aligned}
\iff \qquad & x^{-2}\frac{\mathrm{d}y}{\mathrm{d}x} - x^{-2}\frac{2}{x}y = x^{-2}\frac{6}{x} \\
\iff \qquad & x^{-2}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{2}{x^3}y = \frac{6}{x^3} \\
\iff \qquad & \frac{\mathrm{d}}{\mathrm{d}x}(x^{-2}y) = \frac{6}{x^3} \\
\iff \qquad & \int \frac{\mathrm{d}}{\mathrm{d}x}(x^{-2}y)\,\mathrm{d}x = 6\int \frac{1}{x^3}\,\mathrm{d}x \\
\iff \qquad & x^{-2}y = 6(-\frac{1}{2x^2} + c) \\
\iff \qquad & x^{-2}y = -3\frac{1}{x^2} + c' && c' = 6c \\
\iff \qquad & y = -3 + c'x^2.
\end{aligned}
$$

(117) Consider a bank account with variable interest. Let $M(t)$ be the amount of money in the account at time t, measured in years (though the specific unit isn't important conceptually), and let $I(t)$ be the rate of interest at time t: for example, 3% interest corresponds to $I(t) = 0.03$. Finally, let $Q(t)$ be the amount of money put in (or negative for removing money) in year t. Thus, $M(t)$ obeys the differential equation

$$\frac{\mathrm{d}M}{\mathrm{d}t} = I(t)M(t) + Q(t).$$

This illustrates a common trend: the first term indicates how it would grow independent of external forces, and the second term represents external influences.
This can be solved in different ways:

First, suppose $Q(t) = 0$ and $I(t) = I$ is constant. Then $M(t) = M(0)e^{It}$.

On the other hand, if $I(t)$ can vary, then $M(t) = M(0)e^{\int_0^t I(u)\,\mathrm{d}u}$ as explained in 115.

When $Q(t)$ isn't zero, we can begin to understand it by considering the case where money is only put in once at the end of each year — so we have $Q(0), Q(1), \ldots$ — giving the solution:

$$M(t) = M(0)e^{\int_0^t I(u)\,\mathrm{d}u} + Q(1)e^{\int_1^t I(u)\,\mathrm{d}u} + Q(2)e^{\int_2^t I(u)\,\mathrm{d}u} + \cdots$$

In the case of continuous deposit and withdrawal from the account we end up with:

$$M(t) = M(0)e^{\int_0^t I(u)\,\mathrm{d}u} + \int_0^t Q(x)e^{\int_x^t I(u)\,\mathrm{d}u}\,\mathrm{d}x.$$

We can relate this to the general formula for this form of differential equation using 4.6.1.3 as follows:

$$\begin{aligned} M(t) &= e^{\int I(t)\,\mathrm{d}t} \int Q(t)e^{-\int I(u)\,\mathrm{d}u}\,\mathrm{d}t \\ &= e^{\int_0^t I(u)\,\mathrm{d}u} \left(\int_0^t Q(x)e^{-\int_0^x I(u)\,\mathrm{d}u}\,\mathrm{d}x + C \right) \\ &= Ce^{\int_0^t I(u)\,\mathrm{d}u} + \int_0^t Q(x)e^{\int_x^t I(u)\,\mathrm{d}u}\,\mathrm{d}x \end{aligned}$$

which gives us $M(0) = C$ when we substitute in $t = 0$.

4.7.5.1 Number of Solutions

First-order linear differential equations always have a single solution. The linear algebra of first-order linear equations certainly supports a single unique solution: the matrix is non-singular. However, the proof of this lies in the explicit formula derived above,

$$y(t) = e^{\int_{t_0}^t f(x)\,\mathrm{d}x} \int_{t_0}^t g(x)e^{\int_x^t f(u)\,\mathrm{d}u}\,\mathrm{d}x.$$

4.7.6 First-order Nonlinear ODEs

For first-order nonlinear odes, separable equations can be solved in exactly the same manner as separable linear odes. Non-separable equations, however, cannot be solved as with linear equations.

4.7.6.1 Number of Solutions

For nonlinear differential equations, nothing general can be said about the number of *global* solutions. There could be 0, 1, or many. For *local* solutions over a restricted domain, two things can be said though:

Let the function $f(t, y)$ be differentiable on the interval $[t_0, t_1]$ and let $y' = f(t, y)$ with the initial condition $y(t_0) = y_0$. Then,

(i) There is at least one solution over some interval $[t_0, t_2]$ for $t_0 \leq t_2 \leq t_1$;

(ii) If $\frac{\partial f}{\partial y}$ is continuous over some interval $[t_0, t_2]$ for $t_0 \leq t_2 \leq t_1$, then there is one unique solution over the interval $[t_0, t_2]$.

$\frac{\partial f}{\partial y}$ represents the way that the derivative of y changes with the value of y. If it has an infinite discontinuity then this would seem to suggest an infinite sensitivity to initial conditions at the point of the singularity. Of course, any type of discontinuity may be a modelling problem.

Examples

(118) Continuing the example 113, suppose we have the equation $y' = y^2$ and the initial value $y(0) = 1$. This is an autonomous equation and, as such, is separable. So,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y^2$$

$$\iff \qquad \frac{1}{y^2}\,\mathrm{d}y = \mathrm{d}x$$

$$\iff \qquad \frac{-1}{y} = x + C$$

$$\iff \qquad y = \frac{-1}{x+C}.$$

Applying the initial value we have,

$$y(0) = \frac{-1}{0+C} = 1$$
$$\iff \qquad C = -1.$$

Therefore the solution is $y(x) = \frac{1}{1-x}$. But this is **not** a *global* solution! There is a singularity at $x = 1$ so this formula only provides *local* solutions over intervals of x that do not include 1.

(i) There is a solution $y(x) = \frac{1}{1-x}$ over any interval that does not include $x = 1$.

(ii) $\frac{\partial f}{\partial y} = 2y$ is continuous everywhere so solutions of this equation over an interval are unique. Note that there is a steady-state at $y = 0$ but it is unreachable from the initial condition of $y(0) = 1$.

(119) Continuing the example 114, suppose we have the equation $y' = 2\sqrt{y}$ and the initial value $y(0) = 0$. This is also an autonomous equation and so is separable.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\sqrt{y}$$

$$\iff \qquad \frac{1}{\sqrt{y}}\,\mathrm{d}y = 2\,\mathrm{d}x$$

$$\iff \qquad 2\sqrt{y} = 2x + C$$

$$\begin{aligned} &\iff && \sqrt{y} = x + D \\ &\iff && y = (x+D)^2 = x^2 + 2Dx + D^2. \end{aligned}$$

If we apply the initial value then we can determine that

$$y(0) = D^2 = 0 \implies D = 0.$$

This gives us the solution $y(x) = x^2$. But there is another solution: There is a steady-state at $y = 0$ and, since the initial condition is $y(0) = 0$, the initial condition is in the steady-state. Therefore $y(x) = 0$ is also a solution.

(i) There are 2 solutions $y(x) = x^2$ and $y(x) = 0$ over all intervals of x.

(ii) $\frac{\partial f}{\partial y} = \frac{1}{\sqrt{y}}$ is discontinuous at $y = 0$ – in fact, it is an infinite discontinuity – so, since $y = 0$ is the initial value, solutions over any interval with this initial value will not be unique. (And, in this case, there are 2 solutions).

(120) Consider a ball falling under the influence of gravity but resisted by air-resistance. (The example uses a spherical object because other shapes would be likely to have a much more complicated influence of air-resistance on them.) Let $v(t)$ be the velocity at a time t and g be the acceleration due to gravity.

We can model the air-resistance as $-av^2$ for some constant a. This model is quadratic in the velocity of the falling object because: the faster the object is falling through the air, the greater the resisting force produced by the air (by Newton's Second Law) but, also, the greater the amount of air that the object is coming into contact with in a given time interval. Furthermore, since the air-resistance is acting in the opposite direction to the movement of the object it has the opposite sign to the velocity.

Therefore, our final model of the falling ball is,

$$\frac{\mathrm{d}v}{\mathrm{d}t} = g - av^2.$$

This is a separable equation so we can proceed to solve it by the re-arrangement,

$$\frac{1}{g - av^2}\,\mathrm{d}v = \mathrm{d}t.$$

We have two (at least) possible approaches to the analytical solution of the integration,

$$\int \frac{1}{g - av^2}\,\mathrm{d}v.$$

We can use trigonometry functions or partial fractions. Using trig. we proceed as,

$$\begin{aligned}
\int \frac{1}{g - av^2}\,\mathrm{d}v &= \frac{1}{g}\int \frac{1}{1 - (a/g)v^2}\,\mathrm{d}v \\
&= \frac{1}{g}\int \frac{1}{1 - \sin^2\theta}\,\mathrm{d}(\sqrt{g/a}\sin\theta) \\
&= \frac{1}{g}\int \frac{\sqrt{g/a}\cos\theta}{1 - \sin^2\theta}\,\mathrm{d}\theta \\
&= \frac{\sqrt{g/a}}{g}\int \frac{\cos\theta}{\cos^2\theta}\,\mathrm{d}\theta \\
&= \frac{1}{\sqrt{ag}}\int \sec\theta\,\mathrm{d}\theta \\
&= \frac{1}{\sqrt{ag}}\ln(\sec\theta + \tan\theta) && \text{w/o the constant} \\
&= \frac{1}{\sqrt{ag}}\ln\left(\frac{1}{\sqrt{1 - (a/g)v^2}} + \sqrt{\frac{(a/g)v^2}{1 - (a/g)v^2}}\right) \\
&= \frac{1}{\sqrt{ag}}\ln\left(\frac{1 + \sqrt{\frac{a}{g}}\,v}{\sqrt{1 - (\frac{a}{g})v^2}}\right) \\
&= \frac{1}{\sqrt{ag}}\ln\left(\frac{\sqrt{g} + \sqrt{a}\,v}{\sqrt{g - av^2}}\right)
\end{aligned}$$

$$= \frac{1}{2\sqrt{ag}} \ln\left(\frac{g + 2\sqrt{ag}\, v + av^2}{g - av^2}\right)$$

$$= \frac{1}{2\sqrt{ag}} \ln\left(\frac{(\sqrt{g} + \sqrt{a}\, v)^2}{(\sqrt{g} + \sqrt{a}\, v)(\sqrt{g} - \sqrt{a}\, v)}\right)$$

$$= \frac{1}{2\sqrt{ag}} \ln\left(\frac{\sqrt{g} + \sqrt{a}\, v}{\sqrt{g} - \sqrt{a}\, v}\right).$$

Alternatively, using partial fractions we can proceed as,

$$\begin{aligned}
\int \frac{1}{g - av^2}\, \mathrm{d}v &= \int \frac{A}{\sqrt{g} + \sqrt{a}v} + \frac{B}{\sqrt{g} - \sqrt{a}v}\, \mathrm{d}v \\
&= \int \frac{1}{2\sqrt{g}(\sqrt{g} + \sqrt{a}v)} + \frac{1}{2\sqrt{g}(\sqrt{g} - \sqrt{a}v)}\, \mathrm{d}v \\
&= \frac{1}{2\sqrt{g}} \int \frac{1}{\sqrt{g} + \sqrt{a}v} + \frac{1}{\sqrt{g} - \sqrt{a}v}\, \mathrm{d}v \\
&= \frac{1}{2\sqrt{g}}\left(\frac{1}{\sqrt{a}} \ln(\sqrt{g} + \sqrt{a}v) + \frac{-1}{\sqrt{a}} \ln(\sqrt{g} - \sqrt{a}v)\right) \\
&= \frac{1}{2\sqrt{ag}}(\ln(\sqrt{g} + \sqrt{a}v) - \ln(\sqrt{g} - \sqrt{a}v)) \\
&= \frac{1}{2\sqrt{ag}} \ln\left(\frac{\sqrt{g} + \sqrt{a}v}{\sqrt{g} - \sqrt{a}v}\right).
\end{aligned}$$

So, either way, we arrive at,

$$\frac{1}{2\sqrt{ag}} \ln\left(\frac{\sqrt{g} + \sqrt{a}v}{\sqrt{g} - \sqrt{a}v}\right) = t + C$$

$$\iff \qquad \frac{\sqrt{g} + \sqrt{a}v}{\sqrt{g} - \sqrt{a}v} = Ae^{2\sqrt{ag}\, t} \qquad A = e^{2\sqrt{ag}\, C}$$

$$\iff \quad \sqrt{g} + \sqrt{a}v(1 + Ae^{2\sqrt{ag}\, t}) = \sqrt{g}Ae^{2\sqrt{ag}\, t}$$

$$\begin{aligned}
&\iff && \sqrt{a}v(1 + Ae^{2\sqrt{ag}\,t}) = \sqrt{g}(Ae^{2\sqrt{ag}\,t} - 1) \\
&\iff && v = \frac{\sqrt{g}}{\sqrt{a}}\left(\frac{Ae^{2\sqrt{ag}\,t} - 1}{1 + Ae^{2\sqrt{ag}\,t}}\right) \\
&\iff && v = \sqrt{\frac{g}{a}}\left(\frac{Ae^{2\sqrt{ag}\,t} - 1}{Ae^{2\sqrt{ag}\,t} + 1}\right).
\end{aligned}$$

If the initial condition is $v(0) = 0$ – that's to say that the body starts at rest – then we can resolve the value of the constant $A = 1$. This then gives us the solution,

$$v(t) = \sqrt{\frac{g}{a}}\left(\frac{e^{2\sqrt{ag}\,t} - 1}{e^{2\sqrt{ag}\,t} + 1}\right) = \sqrt{\frac{g}{a}}\tanh(\sqrt{ag}\,t).$$

The terminal velocity is given by allowing $t \to \infty$ and the result is that $v \to \sqrt{\frac{g}{a}}$. As expected, the terminal velocity decreases with increasing values of the constant a.

Also, as expected from looking at the original equation and the partial derivative w.r.t. v, we have obtained a single unique solution – which is good because it could represent a problem for Newtonian physics if we didn't.

4.7.7 Autonomous, Separable and Exact ODEs

Autonomous Equations and Exact Equations

Definition 208. An **autonomous** equation refers to a differential equation with no **explicit** dependence on the independent variable (typically in dynamical systems, t for time). These differential equations express change in a system based solely on the current value of the system. For example:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -ky \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}t} = ry(1-y).$$

Autonomous equations are always separable.

On the other hand, an equation of the form,

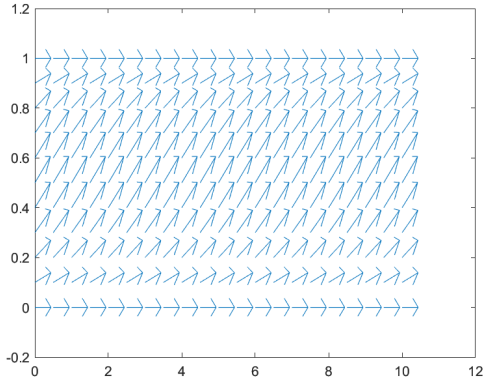
$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t)y(1-y)$$

for example, would express that the growth rate intrinsic to the system is varying over time. Equations of this form are not autonomous but are separable.

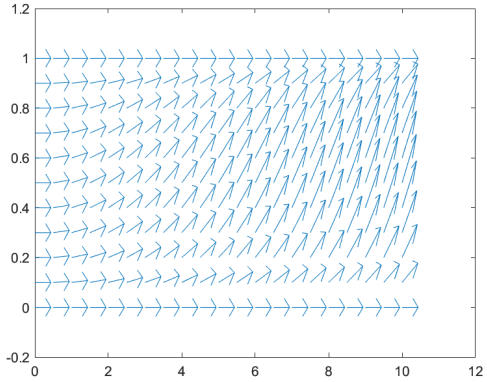
Meanwhile, an equation of the form,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -ky + f(x)$$

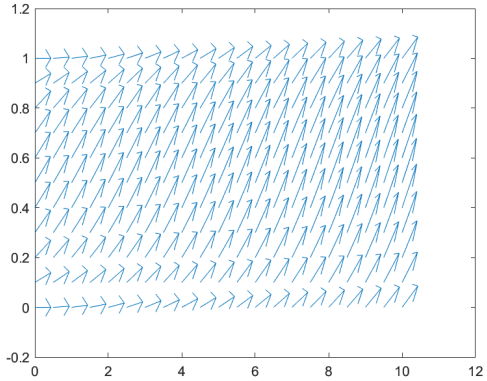
for example, would express that the rate of change of the system is being influenced by something that varies with the independent variable and is, in some way, extrinsic to the system (because it depends only on the independent variable and not on the state of the system). Equations of this form are neither autonomous nor separable.



(a) autonomous: $y' = y(1-y)$ - note that the field is invariant to a shift along the x-axis.



(b) separable: $y' = f(x)y(1-y)$ where $f(x) = 0.2x$



(c) non-separable: $y' = y(1-y) + f(x)$ where $f(x) = 0.02x$

Figure 4.1: autonomous, separable and non-separable direction fields

Definition 209. An **exact** equation refers to a differential equation whose solutions all conserve the value of some property. All separable equations are exact equations because,

$$f(y)\,\mathrm{d}y = g(x)\,\mathrm{d}x \implies \int f(y)\,\mathrm{d}y - \int g(x)\,\mathrm{d}x = 0.$$

However, there are also exact equations that are not separable such as,

$$y' \sin x + y \cos x + 2x = 0 \implies (y \sin x + x^2)' = 0 \implies y = \frac{C - x^2}{\sin x}.$$

Proposition 4.7.5. *Let y be any function satisfying the differential equation,*

$$A(x,y) + B(x,y)\,\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

with

$$\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}\,.$$

Then all such functions y preserve some invariant property $\Psi(x,y) = C$ for constant C.

Proof. The function Ψ may consist of terms of only x, terms of only y, terms containing both x and y and a constant term. So we can describe it as,

$$\Psi(x,y) = p(x)q(y) + r(x) + s(y) + k$$

where p, q, r, s are arbitrary functions and k is a constant. But, since Ψ will be a constant – and we are only interested in the fact that it is constant, not the value of the constant – we can ignore the constant term k because it will just change the value of the constant that Ψ is equal to. In other words, we can describe Ψ as,

$$\Psi(x,y) = p(x)q(y) + r(x) + s(y) = C$$

with k absorbed into the value of C.

Then, taking partial derivatives with respect to x and y we have,

$$\frac{\partial \Psi}{\partial x} = \frac{\mathrm{d}p}{\mathrm{d}x}\,q(y) + \frac{\mathrm{d}r}{\mathrm{d}x} \quad \text{and} \quad \frac{\partial \Psi}{\partial y} = p(x)\,\frac{\mathrm{d}q}{\mathrm{d}y} + \frac{\mathrm{d}s}{\mathrm{d}y}\,.$$

Furthermore, since Ψ is a constant function we know that,

$$\frac{\mathrm{d}\Psi}{\mathrm{d}x} = 0 = \frac{\partial\Psi}{\partial x} + \frac{\partial\Psi}{\partial y}\,\frac{\mathrm{d}y}{\mathrm{d}x}\,.$$

This has the required form $A(x,y) + B(x,y)\,\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ with $\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}$. So, if we have a differential equation of the given form and we postulate the existence of a constant function Ψ such that $\frac{\partial\Psi}{\partial x} = A(x,y)$ and $\frac{\partial\Psi}{\partial y} = B(x,y)$, then,

$$\begin{aligned}\int A(x,y)\,\mathrm{d}x = \int \frac{\partial\Psi}{\partial x}\,\mathrm{d}x &= \int \left(\frac{\mathrm{d}p}{\mathrm{d}x}\,q(y) + \frac{\mathrm{d}r}{\mathrm{d}x}\right)\mathrm{d}x \\ &= p(x)q(y) + r(x) + C_1(y),\end{aligned}$$

$$\begin{aligned}\int B(x,y)\,\mathrm{d}y = \int \frac{\partial\Psi}{\partial y}\,\mathrm{d}y &= \int \left(p(x)\,\frac{\mathrm{d}q}{\mathrm{d}y} + \frac{\mathrm{d}s}{\mathrm{d}y}\right)\mathrm{d}y \\ &= p(x)q(y) + s(y) + C_2(x).\end{aligned}$$

Comparing these two results,

$$\Psi(x,y) = p(x)q(y) + r(x) + C_1(y) = p(x)q(y) + s(y) + C_2(x),$$

we can see that the function $\Psi(x,y)$ that we are looking for is $\Psi(x,y) = p(x)q(y) + r(x) + s(y)$.

Another way that we could have resolved Ψ is to take the first integral,

$$\int A(x,y)\,\mathrm{d}x = \int \frac{\partial\Psi}{\partial x}\,\mathrm{d}x = p(x)q(y) + r(x) + C_1(y)$$

and say, if $\Psi(x,y) = p(x)q(y) + r(x) + C_1(y)$ then,

$$\frac{\partial\Psi}{\partial y} = p(x)\,\frac{\mathrm{d}q}{\mathrm{d}y} + \frac{\mathrm{d}C_1}{\mathrm{d}y}$$

and compare this with $B(x,y)$ to resolve the function C_1,

$$\begin{aligned} & p(x)\,\frac{\mathrm{d}q}{\mathrm{d}y} + \frac{\mathrm{d}C_1}{\mathrm{d}y} = B(x,y) = p(x)\,\frac{\mathrm{d}q}{\mathrm{d}y} + \frac{\mathrm{d}s}{\mathrm{d}y} \\ \iff \quad & \frac{\mathrm{d}C_1}{\mathrm{d}y} = \frac{\mathrm{d}s}{\mathrm{d}y} \\ \iff \quad & C_1(y) = s(y) + k \qquad \text{integrating both sides wrt. } y\end{aligned}$$

where k is a constant of integration which, in this case, can be ignored. □

Proposition 4.7.6. *All separable equations are exact equations.*

Proof. A separable differential equation is one that may be put in the form,

$$f(y)\,\mathrm{d}y = g(x)\,\mathrm{d}x.$$

We can re-arrange this,

$$\begin{aligned} & f(y)\,\mathrm{d}y = g(x)\,\mathrm{d}x \\ \iff \qquad & f(y)\,\frac{\mathrm{d}y}{\mathrm{d}x} - g(x) = 0. \end{aligned}$$

Now, taking $A(x,y) = -g(x)$ and $B(x,y) = f(y)$ we have the form $A(x,y) + B(x,y)\,\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ with $\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x} = 0$ so this is an exact equation. □

All autonomous equations are separable (though the reverse is not generally true) and all separable equations are exact equations (although some exact equations are not separable). So we have,

$$\{\textit{Autonomous}\} \subset \{\textit{Separable}\} \subset \{\textit{Exact}\}.$$

Examples of Exact Equations

(121) Suppose we have the equation

$$(3x^2 + 2xy) + (2y + x^2)\,\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

We have,

$$\frac{\partial(3x^2 + 2xy)}{\partial y} = 2x = \frac{\partial(2y + x^2)}{\partial x}$$

so this is an exact equation. We can solve for the constant function preserved by all solutions by setting

$$\int 3x^2 + 2xy \, \mathrm{d}x = \int 2y + x^2 \, \mathrm{d}y$$

$$\iff \quad x^3 + x^2 y + C_1(y) = y^2 + x^2 y + C_2(x)$$

$$\iff \quad x^3 + C_1(y) = y^2 + C_2(x)$$

$$\therefore \quad \Psi(x, y) = x^2 y + y^2 + x^3 = C \qquad \text{for constant C.}$$

So, all solutions $y(x)$ of this differential equation preserve the value of Ψ and so, if we have an initial condition – say $y(0) = 1$ – then we can use this to find the value of $C = \Psi(0, 1)$ and this value will be preserved for all values of x and y.

$$\Psi(0, 1) = (0^2)(1) + (1)^2 + (0^3) = 1 = x^2 y + y^2 + x^3 \qquad \forall x, y \in \mathbb{R}.$$

Now we can solve for y by treating x as a constant and arranging the equation as a quadratic in y:

$$y^2 + x^2 y + (x^3 - 1) = 0$$

and then using the quadratic formula for y,

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-x^2 \pm \sqrt{x^4 - 4x^3 - 4}}{2}.$$

(122) Suppose we have the equation

$$(3x + 2y) + (\frac{2y}{x} + x) \frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

This is equation is *not* exact because,

$$\frac{\partial(3x + 2y)}{\partial y} = 2 \neq \frac{\partial(\frac{2y}{x} + x)}{\partial x} = \frac{-2y}{x^2} + 1.$$

However, if we multiply the equation by x then we get the equation in the previous example,

$$x(3x + 2y) + x(\frac{2y}{x} + x) \frac{\mathrm{d}y}{\mathrm{d}x} = (3x^2 + 2xy) + (2y + x^2) \frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

So there are times when multiplying by an integrating factor can result in an exact equation.

(123) Consider a body falling toward earth from a significant distance so that the change in acceleration due to gravity as the body gets closer to the earth is significant. Acceleration due to gravity is proportional to $1/r^2$ where r is the distance of the body from the centre of the earth – remember the mass of the body is a multiplier of the force of gravity but also of the inertia so in the expression for the resultant acceleration the mass cancels out – so we can model this situation as,

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = -\frac{c}{r^2}$$

where c is a positive constant. Note that the expression on the right hand side is negative because r is always positive but, in the scenario, is decreasing so $\frac{\mathrm{d}r}{\mathrm{d}t}$ is negative, and it is getting more negative as r gets smaller, so the second derivative is also negative.

If we use classical Newtonian mechanics then we can model r as a displacement s from the center of the earth. Then the velocity $v = \frac{\mathrm{d}s}{\mathrm{d}t}$ is negative as the body is falling towards the earth. We also have,

$$a = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}s} \cdot \frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}s} \cdot v.$$

Interestingly, we can see that,

$$\frac{\mathrm{d}v}{\mathrm{d}s} = \frac{a}{v} = \frac{\mathrm{d}v/\,\mathrm{d}t}{v}$$

which is the log-derivative of v representing the relative infinitesimal change (wikipedia) of the velocity and also,

$$\frac{\mathrm{d}v}{\mathrm{d}s} = \frac{\mathrm{d}(\mathrm{d}s/\,\mathrm{d}t)}{\mathrm{d}s}$$

the way that the rate of change of the displacement changes with the displacement.

If we consider the first-order equation

$$\frac{\mathrm{d}v}{\mathrm{d}s}\, v = -\frac{c}{s^2},$$

this has the form $y\,\frac{\mathrm{d}y}{\mathrm{d}t} = f(t)$ and so, is separable.

$$\begin{aligned}
& \frac{\mathrm{d}v}{\mathrm{d}s}\, v = -\frac{c}{s^2} \\
\iff \qquad & \int v\,\mathrm{d}v = -c\int \frac{1}{s^2}\,\mathrm{d}s \\
\iff \qquad & \frac{v^2}{2} = -c\left(\frac{-1}{s}\right) + C \\
\iff \qquad & \frac{v^2}{2} = \frac{c}{s} + C \\
\iff \qquad & \frac{v^2}{2} - \frac{c}{s} = C.
\end{aligned}$$

So the quantity $\frac{v^2}{2} - \frac{c}{s}$ is preserved by all solutions of this differential equation. The gravitational potential energy of a body displaced from the earth is given (by Newton's 3rd law) as the work done if the body were to fall all the way to the earth's centre,

$$E_p = F \cdot s = m\, a(s)\, \cdot s$$

where m is the mass of the body, s is the displacement of the body from the centre of the earth and a is the acceleration due to gravity as a function of the displacement. Substituting in the function for the acceleration due to gravity as a function of the distance from the earth's centre we have,

$$E_p = m\left(-\frac{c}{s^2}\right) s = -\frac{mc}{s}.$$

The kinetic energy of the falling body is given by,

$$E_k = \frac{mv^2}{2}.$$

Since the falling movement of the body is the potential energy turning to kinetic energy, the sum $E_p + E_k$ is conserved. So, for some constant K we have,

$$\frac{mv^2}{2} - \frac{mc}{s} = K$$

$$\iff \quad m\left(\frac{v^2}{2} - \frac{c}{s}\right) = K$$

$$\iff \quad \frac{v^2}{2} - \frac{c}{s} = C \qquad \text{for } C = K/m.$$

(124) Imagine a pendulum swinging on a rod (or a rope that remains taut) of length L. Let θ be the angle the rod makes with the vertical and g the acceleration due to gravity. The gravitational force acting on the pendulum has a component that is balanced by the tension in the rod maintaining the pendulum swinging along the circumference of a circle. This component acts in a direction perpendicular to the circumference of the circle in which the pendulum mass moves and along the radius of the circle of movement. The other component of g, orthogonal to this one, acts along the tangent of the circumference of movement of the pendulum mass. It is this component that creates the motion of the pendulum mass. (If this is not clear: Penn State page on pendulum oscillation.)

We have:

- Acceleration due to gravity with a component perpendicular to the pendulum rod given by: $a = -g \sin\theta$.
- Angular velocity of the pendulum mass: $\omega = \dfrac{\mathrm{d}\theta}{\mathrm{d}t}$.
- Tangential velocity of the pendulum mass: $v_t = L\,\dfrac{\mathrm{d}\theta}{\mathrm{d}t}$.

So, using $F = m\frac{\mathrm{d}v}{\mathrm{d}t}$, the equation describing the acceleration of the pendulum mass is,

$$\begin{aligned} & m\,\frac{\mathrm{d}v_t}{\mathrm{d}t} = -mg\sin\theta \\ \iff \quad & \frac{\mathrm{d}v_t}{\mathrm{d}t} = -g\sin\theta \\ \iff \quad & L\,\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -g\sin\theta. \end{aligned}$$

This is a second-order nonlinear equation. For small oscillations we can use the small-angle approximation of sine to linearize it (see example 127). If the oscillations are not small though, the small-angle approximation becomes too inaccurate to be useful and we must solve the nonlinear equation.

We can simplify it into a first-order nonlinear equation by describing the tangential velocity in relation to the angle θ and eliminating time from the model,

$$v = L\omega = L\,\frac{\mathrm{d}\theta}{\mathrm{d}t} \iff \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{v}{L},$$

$$L\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}\theta}\cdot\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}\theta}\cdot\frac{v}{L} = -g\sin\theta.$$

So we end up with the *separable* first-order nonlinear equation,

$$\begin{aligned} & v\,\frac{\mathrm{d}v}{\mathrm{d}\theta} = -gL\sin\theta \\ \iff \quad & \int v\,\mathrm{d}v = -gL\int \sin\theta\,\mathrm{d}\theta \\ \iff \quad & v^2 = 2gL\cos\theta + C \\ \iff \quad & v(\theta) = \pm\sqrt{2gL\cos\theta + C}. \end{aligned}$$

Note that we have ended up with an expression for the tangential velocity as a function of the angle θ *only*. Also worth noting is that the constant quantity,

$$v^2 - 2gL\cos\theta = C$$

represents the conserved energy of the pendulum system – the v^2 term being proportional to the kinetic energy and the $2gL\cos\theta$ term being proportional to the potential energy.

If we consider an oscillation with maximum angle θ_{max} then the pendulum is momentarily at rest when $\theta = \theta_{max}$. It is convenient to use this as the initial condition of the angle of the pendulum $\theta_0 = \theta_{max}$ so that the pendulum begins at rest and so $v(\theta_0) = 0$ and then we can resolve the value of the constant,

$$\begin{aligned} v(\theta_0) &= \pm\sqrt{2gL\cos\theta_0 + C} = 0 \\ \therefore \qquad C &= -2gL\cos\theta_0. \end{aligned}$$

So, for the initial condition, we have resolved the tangential velocity w.r.t. to the angle of displacement of the pendulum as,

$$v(\theta) = \pm\sqrt{2gL(\cos\theta - \cos\theta_0)}.$$

We could also have obtained this result with definite integration from θ_0 to θ,

$$\begin{aligned} & \int_{v(\theta_0)}^{v(\theta)} v\,\mathrm{d}v = -gL\int_{\theta_0}^{\theta}\sin t\,\mathrm{d}t \\ \iff \qquad & \frac{1}{2}(v(\theta)^2 - v(\theta_0)^2) = gL(\cos\theta - \cos\theta_0) \\ \iff \qquad & v(\theta)^2 = 2gL(\cos\theta - \cos\theta_0) \qquad \because v(\theta_0) = 0. \end{aligned}$$

To recover the time information into the solution we can bring back the definition of the velocity as a function of time *as well as* the angle of displacement,

$$v = L\,\frac{\mathrm{d}\theta}{\mathrm{d}t}\,.$$

Substituting this back into the solution we get,

$$L\frac{\mathrm{d}\theta}{\mathrm{d}t} = \pm\sqrt{2gL(\cos\theta - \cos\theta_0)}$$

$$\iff \qquad \mathrm{d}t = \sqrt{\frac{L}{2g}}\frac{1}{\sqrt{\cos\theta - \cos\theta_0}}\,\mathrm{d}\theta.$$

So, to find the time taken when the pendulum swings from its central position, with $\theta = 0$, to its amplitude, with $\theta_{max} = \theta_0$, we can integrate the expression on the right of this equation between θ_0 and 0. To get the whole time period of the oscillation we can multiply this time by 4,

$$T = 4\sqrt{\frac{L}{2g}}\int_{\theta_0}^{0}\frac{1}{\sqrt{\cos\theta - \cos\theta_0}}\,\mathrm{d}\theta.$$

This integral is a type of improper integral called an elliptic integral and it doesn't have an analytic solution. However, for small-angle oscillations we can use the small-angle approximation for cosine,

$$\cos x = 1 - \frac{x^2}{2},$$

to obtain an approximate value of the integral,

$$\int_{\theta_0}^{0}\frac{1}{\sqrt{(1-\frac{\theta^2}{2})-(1-\frac{\theta_0^2}{2})}}\,\mathrm{d}\theta$$

$$=\int_{\theta_0}^{0}\frac{1}{\sqrt{\frac{1}{2}(\theta_0^2-\theta^2)}}\,\mathrm{d}\theta$$

$$=\sqrt{2}\int_{\theta_0}^{0}\frac{1}{\sqrt{\theta_0^2-\theta^2}}\,\mathrm{d}\theta$$

$$=\frac{\sqrt{2}}{\theta_0}\int_{\theta_0}^{0}\frac{1}{\sqrt{1-\left(\frac{\theta}{\theta_0}\right)^2}}\,\mathrm{d}\theta$$

$$=\frac{\sqrt{2}}{\theta_0}\int_{\theta_0}^{0}\frac{1}{\sqrt{1-\sin^2\alpha}}\,\mathrm{d}(\theta_0\sin\alpha)$$

$$
\begin{aligned}
&= \frac{\sqrt{2}}{\theta_0} \int_{\sin^{-1}(1)}^{\sin^{-1}(0)} \frac{\theta_0 \cos\alpha}{\cos\alpha}\, \mathrm{d}\alpha \\
&= \sqrt{2}(\sin^{-1}(0) - \sin^{-1}(1)) \\
&= \sqrt{2}\left(-\frac{\pi}{2}\right) = -\frac{\pi}{\sqrt{2}}.
\end{aligned}
$$

In this case we can ignore the minus sign – it's merely a factor of the choice that $\sin^{-1}$ has to make in order to be a function; returning $\frac{\pi}{2}$ for $\sin^{-1}(1)$ instead of $-\frac{\pi}{2}$. So, the time period becomes,

$$T = 4\sqrt{\frac{L}{2g}}\left(\frac{\pi}{\sqrt{2}}\right) = 2\pi\sqrt{\frac{L}{g}}.$$

Note that the amplitude of the pendulum's oscillation θ_0 cancelled out in the calculation of the time period and that the resultant expression for the *time period of small oscillations* of a pendulum depends only on the length of the string and the acceleration due to gravity.

Homogeneous Differential Equations

Definition 210. A **homogeneous differential equation** is an equation of the form,

$$f(x,y)\frac{\mathrm{d}y}{\mathrm{d}x} = g(x,y)$$

where the functions f, g are both homogeneous of degree d.

The functions f, g need not be linear and so these differential equations are not necessarily linear.

The key insight here is that, due to the homogeneous function property (see: 4.2), if we define the function y to be $y(x) = x \cdot v(x)$ — which we may do because, due to the non-trivial kernel of differentiation, we are only able to determine a solution to a differential equation upto a constant term so we can set the constant term to 0 for convenience during the calculation — then,

$$f(\lambda x, \lambda y) = \lambda^d f(x, y) \implies f(x, xv) = x^d g(v).$$

When this is applied in a differential equation of the above form we obtain,

$$\begin{aligned}
& f(x,y)\frac{\mathrm{d}y}{\mathrm{d}x} = g(x,y) \\
\iff \quad & x^d f_1(v)\left(v + x\frac{\mathrm{d}v}{\mathrm{d}x}\right) = x^d f_2(v) && y = xv,\ y' = v + xv' \\
\iff \quad & f_1(v)\left(v + x\frac{\mathrm{d}v}{\mathrm{d}x}\right) = f_2(v) \\
\iff \quad & v + x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{f_2(v)}{f_1(v)} = f_3(v) \\
\iff \quad & x\frac{\mathrm{d}v}{\mathrm{d}x} = f_3(v) - v = f_4(v) \\
\iff \quad & \frac{1}{f_4(v)}\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{x} \\
\iff \quad & \int \frac{1}{f_4(v)}\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = \int \frac{1}{x}\,\mathrm{d}x = \ln|x| + c \\
\iff \quad & \int \frac{1}{f_4(v)}\,\mathrm{d}v = \ln|x| + c \\
\iff \quad & \frac{1}{f_4'(v)}\ln|f_4(v)| = \ln|x| + c.
\end{aligned}$$

4.7.8 Comparison of Differential Equations with Difference Equations

TODO: Comparison of Differential Equations with Difference Equations

4.7.9 Second-order Linear ODEs

4.7.9.1 Number of Solutions

Solutions to linear equations form a linear vector space so that if $y_1(t)$, $y_2(t)$ are solutions to an equation then any linear combination,

$$y(t) = \alpha y_1(t) + \beta y_2(t)$$

is also a solution.

4.7.9.2 Examples of Second-order Linear

(125) **The Logistic:** The logistic model of population growth takes $P(t)$ to be the population at time t. The model takes some additional factor that illustrates that a space can only hold a fixed carrying capacity of the population, producing the equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = cP(t)\left(1 - \frac{P(t)}{A}\right).$$

From looking at the differential equation

Whatever the value of the derivative, we can see that the scalar c will increase its magnitude thereby amplifying changes. So, c is the growth rate and larger values of c cause the population to change more rapidly.

The steady-states of $P(t)$ are at the values such that the derivative is 0. Therefore,

$$cP(t)\left(1 - \frac{P(t)}{A}\right) = 0 \implies P(t) \in \{0, 1\}.$$

Looking at the steady-states one-by-one:

- $\boldsymbol{P(t) = 0}$: If $P(t)$ climbs above 0 then the derivative will be positive and so the function will be increasing away from the steady-state at 0. This is therefore an *unstable* steady-state.

Note that, since $P(t)$ is a population, negative values don't make sense.

- $\boldsymbol{P(t) = A}$: If $P(t)$ is less than A then the derivative will be positive and so the function will be increasing toward the steady-state at A. This is therefore a *stable* steady-state from below. Also, if $P(t)$ is greater than A, the derivative is negative and so the function is decreasing towards the steady-state value at A – so this a *stable* steady-state from above also. However, if the initial population value is less than A, then the population will never arrive at values above A and this is the way that this function is normally used – taking values between 0 and A. From either side, the larger the value of the growth rate c, the faster the function will approach the steady-state.

Finding the solution

The equation is separable so we have,

$$\frac{\mathrm{d}P}{\mathrm{d}t} = cP\left(1 - \frac{P}{A}\right)$$

$$\iff \quad \frac{\mathrm{d}P}{\mathrm{d}t} = \left(\frac{c}{A}\right)P(A - P)$$

$$\iff \quad \frac{1}{P(A-P)}\,\mathrm{d}P = \left(\frac{c}{A}\right)\mathrm{d}t$$

$$\iff \quad \int \frac{1}{AP} + \frac{1}{A(A-P)}\,\mathrm{d}P = \int \left(\frac{c}{A}\right)\mathrm{d}t \qquad \text{by partial fractions}$$

$$\iff \quad \int \frac{1}{P} + \frac{1}{(A-P)}\,\mathrm{d}P = \int c\,\mathrm{d}t \qquad \text{by partial fractions}$$

$$\iff \quad \ln P - \ln(A-P) = ct + D \qquad D \text{ is const. of integration}$$

$$\iff \quad \ln\left(\frac{P}{A-P}\right) = ct + D$$

$$\iff \quad \frac{P}{A-P} = Be^{ct} \qquad B = e^{D}$$

$$\iff \qquad P = Be^{ct}(A - P)$$

$$\iff \qquad P(1 + Be^{ct}) = ABe^{ct}$$

$$\iff \qquad P = A\left(\frac{Be^{ct}}{1 + Be^{ct}}\right).$$

Note that:

- *Another common way to write the logistic function is to divide by Be^{ct} to get:*
$$A\left(\frac{1}{Ee^{-ct} + 1}\right)$$
where $E = \frac{1}{B}$.

- *As has been noted: $0 \leq P(t) \leq A$. So,*
$$0 \leq A\left(\frac{Be^{ct}}{1 + Be^{ct}}\right) \leq A$$
$$\iff \qquad 0 \leq \left(\frac{Be^{ct}}{1 + Be^{ct}}\right) \leq 1.$$

The value,
$$\frac{Be^{ct}}{1 + Be^{ct}} = \frac{1}{Ee^{-ct} + 1}$$
is a proportion, a value in $[0, 1]$.

- *Since we have,*
$$P(t) = A\theta$$
where θ is a proportion and the derivative is given by,
$$\frac{\mathrm{d}P(t)}{\mathrm{d}t} = cA\theta(1 - \theta) = cA\theta - cA\theta^2$$
we can see that if θ is small then the derivative will be approximately,
$$cA\theta = cP(t).$$
For this reason, when θ is small – which happens when t is small – the growth of the function is approximately exponential.

(126) **Simple Harmonic Motion:** $y'' = -ky$ TODO: S.H.M.

(127) This is a second-order non-linear equation but, for small angular displacements, we can linearize this equation using the small-angle approximation for sine $\sin\theta \approx \theta$ giving:

$$\begin{aligned} & L\tfrac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -g\theta \\ \iff \quad & \tfrac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\frac{g}{L}\theta \\ \iff \quad & \tfrac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{g}{L}\theta = 0. \end{aligned}$$

The auxiliary equation of this equation is $z^2 + \frac{g}{L} = 0$ – the roots of which are clearly complex and given by,

$$z = \pm\frac{\sqrt{-4(g/L)}}{2} = \pm\sqrt{-\frac{g}{L}} = \pm\sqrt{\frac{g}{L}}i.$$

TODO: 2nd order linearized pendulum equation

4.7.10 Second-order Nonlinear ODEs

4.7.10.1 Examples of Second-order Nonlinear

4.7.11 Systems of ODEs

Definition 211. A **system** of differential equations is a collection of related differential equations that must be solved simultaneously.

For example, a system of two linear first-order equations takes the form,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f_1(t, x, y)$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = f_2(t, x, y).$$

Definition 212. A **solution** to a system of differential equations means to have found an analytic expression for each of the dependent variables in the system in terms of the independent variable only. So, in the system,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f_1(t, x, y)$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = f_2(t, x, y).$$

a solution would be to find $x(t) = g_1(t)$ and $y(t) = g_2(t)$ that satisfy the system.

4.7.11.1 Degrees of Freedom

If a system has m equations of order n, then the system has $m \times n$ degrees of freedom and can be transformed into other equivalent systems with the same number of degrees of freedom by creating or eliminating variables that depend only on a single derivative.

For example, the system of two first-order equations (2×1 system),

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x + y^2$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = -x$$

can be transformed into one second-order equation (1×2 system) by observing that $\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{\mathrm{d}^2y}{\mathrm{d}t^2}$ so that,

$$\begin{aligned} & \frac{\mathrm{d}x}{\mathrm{d}t} = x + y^2 \\ \iff \quad & -\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = -\frac{\mathrm{d}y}{\mathrm{d}t} + y^2 \\ \iff \quad & \frac{\mathrm{d}^2y}{\mathrm{d}t^2} = \frac{\mathrm{d}y}{\mathrm{d}t} - y^2. \end{aligned}$$

However, the 2×1 system,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x + y^2$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = -x + y$$

cannot be transformed to a single equation.

4.7.11.2 Linear Systems

Definition 213. A **vector-valued function** can be viewed either as a function that returns a vector or as a vector whose entries are functions. These views are equivalent because we can view each element of the vector as being generated by a separate function (branch) within the function.

For example, let $X : \mathbb{R} \longmapsto \mathbb{R}^2$. Then we can represent $X(t)$ as

$$X(t) = \vec{\boldsymbol{x}}$$

for $\vec{\boldsymbol{x}} \in \mathbb{R}^2$, or as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

Similarly, a **matrix-valued function** $A : \mathbb{R} \longmapsto \mathbb{R}^{2\times 2}$ can be represented as

$$A(t) = A$$

for $A \in \mathbb{R}^{2\times 2}$ or by

$$\begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix}.$$

The concepts of limits and differentiation are extended to vector-valued and matrix-valued functions by applying them to each entry individually and the result is only defined if the result is defined for each entry. So,

$$\lim_{t\to a} A(t) = \begin{bmatrix} \lim_{t\to a} a_{11}(t) & \lim_{t\to a} a_{12}(t) \\ \lim_{t\to a} a_{21}(t) & \lim_{t\to a} a_{22}(t) \end{bmatrix}$$

and

$$\frac{\mathrm{d}A(t)}{\mathrm{d}t} = \begin{bmatrix} \dfrac{\mathrm{d}}{\mathrm{d}t} a_{11}(t) & \dfrac{\mathrm{d}}{\mathrm{d}t} a_{12}(t) \\ \dfrac{\mathrm{d}}{\mathrm{d}t} a_{21}(t) & \dfrac{\mathrm{d}}{\mathrm{d}t} a_{22}(t) \end{bmatrix}.$$

Proposition 4.7.7. *Let A and E be $n \times n$ matrices such that the columns of E are an eigenbasis of A and $D = E^{-1}AE$ is diagonal. Then* ***all*** *the solutions of the system of equations,*

$$\frac{\mathrm{d}X(t)}{\mathrm{d}t} = AX(t)$$

have the form $X(t) = c_1\vec{\boldsymbol{v}}_1 e^{\lambda_1 t} + c_2\vec{\boldsymbol{v}}_2 e^{\lambda_2 t} + \cdots + c_n\vec{\boldsymbol{v}}_n e^{\lambda_n t}$ *where* $\{\, \lambda_i \mid 1 \leq i \leq n \,\}$ *are the eigenvalues along the diagonal of the matrix D and* $\{\, \vec{\boldsymbol{v}}_i \mid 1 \leq i \leq n \,\}$ *are the corresponding eigenvectors.*

Proof. Let $\tilde{X}(t) = E^{-1}X(t)$. Then, dropping the variable t for convenience of notation,

$$\begin{aligned} \frac{\mathrm{d}\tilde{X}}{\mathrm{d}t} &= E^{-1}\,\frac{\mathrm{d}X}{\mathrm{d}t} \\ &= E^{-1}AX \\ &= E^{-1}AE\tilde{X} \\ \therefore\quad \frac{\mathrm{d}\tilde{X}}{\mathrm{d}t} &= D\tilde{X}. \end{aligned}$$

The system has been converted into an equivalent diagonal system. Since D is the diagonal matrix of eigenvalues, this last equation has the form,

$$\begin{bmatrix} \tilde{x}_1'(t) \\ \tilde{x}_2'(t) \\ \vdots \\ \tilde{x}_n'(t) \end{bmatrix} = \begin{bmatrix} \lambda_1\tilde{x}_1(t) \\ \lambda_2\tilde{x}_2(t) \\ \vdots \\ \lambda_n\tilde{x}_n(t) \end{bmatrix}$$

with general solution,

$$\tilde{X}(t) = \begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{bmatrix}$$

for undetermined constants $\{ c_i \mid 1 \leq i \leq n \}$. To convert this solution back into a solution of the original equation we have to multiply it by the basis of eigenvectors,

$$X(t) = E\tilde{X}(t)$$

so that,

$$X(t) = c_1\vec{\boldsymbol{v}}_1 e^{\lambda_1 t} + c_2\vec{\boldsymbol{v}}_2 e^{\lambda_2 t} + \cdots + c_n\vec{\boldsymbol{v}}_n e^{\lambda_n t}$$

where $\{ \vec{\boldsymbol{v}}_i \mid 1 \leq i \leq n \}$ are the eigenvectors in E. □

Intuition

Say we have a homogeneous linear system of differential equations,

$$\frac{\mathrm{d}\vec{\boldsymbol{x}}}{\mathrm{d}t} = A\vec{\boldsymbol{x}}$$

where matrix A has the eigenvector $\vec{\boldsymbol{v}}$ with eigenvalue λ so that,

$$A\vec{\boldsymbol{v}} = \lambda\vec{\boldsymbol{v}}.$$

In fact, the eigenvector $\vec{\boldsymbol{v}}$ is the basis of an *eigenspace* such that any scalar multiple $\alpha\vec{\boldsymbol{v}}$ is also an eigenvector with eigenvalue λ. In the context of the system of differential equations, the scalar α and the eigenvalue λ can be a function of the independent variable $\alpha(t)$, $\lambda(t)$. So we have,

$$A(\alpha(t)\vec{\boldsymbol{v}}) = \lambda(t)(\alpha(t)\vec{\boldsymbol{v}}).$$

> *It is common to treat real-valued functions as though they were real numbers, treating them like scalars in a field and assuming that field operations over them are meaningful. This is an implicit application of pointwise (ref: wikipedia) definition of operations over functions and works well enough for continuous real-valued functions and the standard field operations but care should be taken to note that its meaningfulness may breakdown outside of this context.*

On the other side of the equation we have a derivative that must be satisfied also. We have the fact that the exponential function of an antiderivative of λ is an eigenfunction of the differentiation operator with eigenvalue λ so,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\int \lambda \,\mathrm{d}t}\right) = \lambda e^{\int \lambda \,\mathrm{d}t}.$$

As an eigenfunction (an eigenvector of an operator over function spaces), the same logic applies about scalar multiples. Additionally, λ may also be a function of the independent variable. So we have,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\beta e^{\int \lambda \,\mathrm{d}t}\right) = \lambda(t)\beta e^{\int \lambda \,\mathrm{d}t}.$$

If β is a function of t then we would have,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\beta(t) e^{\int \lambda \,\mathrm{d}t}\right) = \lambda(t)\beta(t) e^{\int \lambda \,\mathrm{d}t} + \beta'(t) e^{\int \lambda \,\mathrm{d}t}.$$

This leads to the formula for the solutions of non-separable linear differential equations,

$$y' = f(t)y + g(t) \implies y(t) = \left(\int g(t) e^{-\int f(t)\,\mathrm{d}t}\,\mathrm{d}t\right) e^{\int f(t)\,\mathrm{d}t}$$

where $\lambda(t) = f(t)$.

What we are trying to find is the intersection between a set of antiderivatives and a set of solutions to a linear matrix equation. So we leverage the similar form of the eigenfunction of the derivative operator and the eigenvectors of the matrix to find the intersection,

$$\{\, \beta e^{\int \lambda(t)\,\mathrm{d}t} \mid t \in \mathbb{R} \,\} \cap \{\, \alpha(t)\vec{\boldsymbol{v}} \mid A\vec{\boldsymbol{v}} = \lambda(t)\vec{\boldsymbol{v}},\ t \in \mathbb{R} \,\}.$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{\boldsymbol{v}} e^{\int \lambda(t)\,\mathrm{d}t}\right) = A\vec{\boldsymbol{v}} e^{\int \lambda(t)\,\mathrm{d}t}$$

$$\iff \quad \lambda(t)\vec{\boldsymbol{v}} e^{\int \lambda(t)\,\mathrm{d}t} = \lambda(t)\vec{\boldsymbol{v}} e^{\int \lambda(t)\,\mathrm{d}t}$$

for constant $\beta = \vec{\boldsymbol{v}}$. If $\beta(t) = \vec{\boldsymbol{v}}$ is a function of t then we have,

$$\lambda(t)\vec{\boldsymbol{v}} e^{\int \lambda(t)\,\mathrm{d}t} + \frac{\mathrm{d}\vec{\boldsymbol{v}}}{\mathrm{d}t} e^{\int \lambda(t)\,\mathrm{d}t} = A\,y(t) + Q$$

where Q is a translation matrix implementing a term in the differential equation that doesn't depend on the current state of the system. This represents external input to the system as opposed to the response of the system itself.

That's to say: The homogeneous part of the linear equation – the linear transformation of the system state $A\,y(t)$ – is the vector displacement in the affine transformation and represents the system's internal tendency to transition to a new state based on its current state. Meanwhile, the translation term is the point used as a new origin and represents the system's behaviour even if the current system state is $\vec{\mathbf{0}}$.

The Matrix Exponential

TODO: proofs of matrix exponential propositions TODO: do these only apply to constant-coefficient equations?

Complex Eigenvalues

Consider taking eigenvalues of a 2×2 matrix,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightsquigarrow \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = \lambda^2 - (a+d)\lambda + (ad - bc).$$

The discriminant of the characteristic polynomial is,

$$D = (a+d)^2 - 4(ad - bc)$$

which is the square of the sum of the diagonal entries (the trace) minus 4 times the determinant. When the square of the trace is greater than 4 times the determinant, the eigenvalues will be real; when the square of the trace is less than 4 times the determinant, the eigenvalues will be complex; and when they are equal, there will be repeated eigenvalues.

The characteristic polynomial gives us $\lambda = \frac{a+d}{2} \pm \frac{\sqrt{D}}{2}$. In the case where the discriminant $D < 0$ this becomes,

$$\lambda_1 = \frac{(a+d) + \sqrt{-D}\,i}{2} \quad \text{and} \quad \lambda_2 = \frac{(a+d) - \sqrt{-D}\,i}{2}.$$

So the general solution is going to take the form,

$$c_1 \vec{\boldsymbol{v}}_1 e^{\lambda_1 t} + c_2 \vec{\boldsymbol{v}}_2 e^{\lambda_2 t}.$$

For the eigenvalue $\lambda_1 = \frac{a+d}{2} + \frac{\sqrt{D}}{2}$ we have,

$$\begin{bmatrix} a - (\frac{a+d}{2} + \frac{\sqrt{D}}{2}) & b \\ c & d - (\frac{a+d}{2} + \frac{\sqrt{D}}{2}) \end{bmatrix} = \begin{bmatrix} \frac{(a-d)-\sqrt{D}}{2} & b \\ c & \frac{-(a-d)-\sqrt{D}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{(a-d)-\sqrt{D}}{2} & b \\ c & -\frac{(a-d)+\sqrt{D}}{2} \end{bmatrix}$$

and the other eigenvalue λ_2 gives,

$$\begin{bmatrix} \frac{(a-d)+\sqrt{D}}{2} & b \\ c & -\frac{(a-d)-\sqrt{D}}{2} \end{bmatrix}.$$

In the case of complex eigenvalues, if we let $z = \frac{(a-d)-\sqrt{-D}\,i}{2}$ and use the usual notation $\overline{z}$ for the complex conjugate, then we have,

$$\begin{bmatrix} z & b \\ c & -\overline{z} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \overline{z} & b \\ c & -z \end{bmatrix}.$$

Row reduction on these results in,

$$\begin{bmatrix} 1 & \frac{b}{z} \\ 0 & \frac{-\overline{z}z-bc}{z} \end{bmatrix} = \begin{bmatrix} 1 & \frac{b}{|z|}\overline{z} \\ 0 & \frac{-\overline{z}z-bc}{z} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & \frac{b}{\overline{z}} \\ 0 & \frac{-z\overline{z}-bc}{\overline{z}} \end{bmatrix} = \begin{bmatrix} 1 & \frac{b}{|z|}z \\ 0 & \frac{-z\overline{z}-bc}{\overline{z}} \end{bmatrix}$$

where

$$\begin{aligned} |z| &= \frac{(a-d)^2 - D}{4} \\ &= \frac{(a-d)^2 - (a+d)^2 + 4(ad-bc)}{4} \\ &= \frac{-4ad + 4(ad-bc)}{4} && (a-d)^2 - (a+d)^2 = -4ad \\ &= \frac{-4bc)}{4} = -bc. \end{aligned}$$

so that

$$\begin{aligned} -\overline{z}z - bc = -z\overline{z} - bc &= -|z| - bc \\ &= -(-bc) - bc = 0. \end{aligned}$$

So the eigenvectors are

$$\vec{\boldsymbol{v}}_1 = -\frac{b}{|z|}\begin{bmatrix}\overline{z}\\1\end{bmatrix} = \frac{1}{c}\begin{bmatrix}\overline{z}\\1\end{bmatrix} \quad \text{and} \quad \vec{\boldsymbol{v}}_2 = -\frac{b}{|z|}\begin{bmatrix}z\\1\end{bmatrix} = \frac{1}{c}\begin{bmatrix}z\\1\end{bmatrix}$$

meaning that we have the general solution,

$$\begin{aligned}
&\frac{1}{c}\left(c_1\begin{bmatrix}\overline{z}\\1\end{bmatrix}e^{\lambda_1 t} + c_2\begin{bmatrix}z\\1\end{bmatrix}e^{\lambda_2 t}\right)\\
&= \frac{1}{c}\left(c_1\begin{bmatrix}\overline{z}e^{\lambda_1 t}\\e^{\lambda_1 t}\end{bmatrix} + c_2\begin{bmatrix}ze^{\lambda_2 t}\\e^{\lambda_2 t}\end{bmatrix}\right)\\
&= \frac{1}{c}\begin{bmatrix}c_1\,\overline{z}\,e^{\lambda_1 t} + c_2\,z\,e^{\lambda_2 t}\\c_1\,e^{\lambda_1 t} + c_2\,e^{\lambda_2 t}\end{bmatrix}
\end{aligned}$$

Since the eigenvalues λ_1, λ_2 are conjugates, their exponentials $e^{\lambda_1 t}, e^{\lambda_2 t}$ are also conjugates. So, if we define $w = e^{\lambda_1 t}$, we have the form (ignoring the constant $\frac{1}{c}$),

$$\begin{bmatrix}c_1\,\overline{z}\,w + c_2\,z\,\overline{w}\\c_1\,w + c_2\,\overline{w}\end{bmatrix}.$$

Using **??**, we have that $\overline{\overline{z}\,w} = z\,\overline{w}$ so that both elements of our solution vector have the form $c_1 u + c_2\overline{u}$ for some $u \in \mathbb{C}$.

$$\begin{aligned}
& c_1 u + c_2\overline{u} = c_1(a+bi) + c_2(a-bi)\\
\iff \quad & = (c_1+c_2)a + (c_1-c_2)bi\\
\iff \quad & = c_3 a + c_4 bi.
\end{aligned}$$

This means that we can also express the general solution using only one of the eigenvalues/vectors as,

$$c_3\ \mathrm{Re}(\vec{\boldsymbol{v}}_1\,e^{\lambda_1 t}) + c4\ \mathrm{Im}(\vec{\boldsymbol{v}}_1\,e^{\lambda_1 t}).$$

(see: 2.4.28)

IVP

If you have real-valued initial conditions then, for each variable in the system,

you will always have something of the form,

$$r = c\,(z_v \cdot z_e)$$

where r is the real-valued initial value of the first variable; z_v is the complex-valued first component of the eigenvector; z_e is the complex number formed by taking the exponent of the complex-valued eigenvalue and c is a constant to be determined.

As a result, if the initial conditions are real-valued, then that will always force the value of the constant c to be a value that cancels out the i in the general solution. Which, in turn, means that the values of the solution function will be real-valued for all values of t.

Examples

(128) Consider, for example, the matrix,

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \rightsquigarrow \begin{vmatrix} -\lambda & -1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1.$$

So, the characteristic polynomial leads to the eigenvalues $\lambda = \pm i$ and the corresponding eigenvectors

$$\langle i, 1\rangle, \langle -i, 1\rangle.$$

Which leads to the general solution,

$$\begin{aligned} \vec{\boldsymbol{x}}(t) &= c_1 \begin{bmatrix} i \\ 1 \end{bmatrix} e^{it} + c_2 \begin{bmatrix} -i \\ 1 \end{bmatrix} e^{-it} \\ &= \begin{bmatrix} c_1 i(\cos t + i \sin t) - c_2 i(\cos t - i \sin t) \\ c_i(\cos t + i \sin t) + c_2(\cos t - i \sin t) \end{bmatrix} \\ &= \begin{bmatrix} c_1 i \cos t - c_1 \sin t - c_2 i \cos t - c_2 \sin t \\ c_i \cos t + c_1 i \sin t + c_2 \cos t - c_2 i \sin t \end{bmatrix} \\ &= \begin{bmatrix} (c_1 - c_2) i \cos t - (c_1 + c_2) \sin t \\ (c_1 + c_2) \cos t + (c_1 - c_2) i \sin t. \end{bmatrix} \end{aligned}$$

If we have the initial conditions $\vec{\boldsymbol{x}}(0) = \langle 1, 0 \rangle$ then,

$$\begin{aligned}
\begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} (c_1 - c_2)i\cos 0 - (c_1 + c_2)\sin 0 \\ (c_1 + c_2)\cos 0 + (c_1 - c_2)i\sin 0 \end{bmatrix} \\
&= \begin{bmatrix} (c_1 - c_2)i \\ (c_1 + c_2) \end{bmatrix}
\end{aligned}$$

which implies that

$$\begin{aligned}
& (c_1 - c_2 = \frac{1}{i} = -i) \wedge (c_1 = -c_2) \\
\implies \quad & c_1 - c_2 = -c_2 - c_2 = -2c_2 = -i \\
\implies \quad & c_2 = \frac{i}{2} \\
\implies \quad & c_1 = -\frac{i}{2}.
\end{aligned}$$

So we have, $c_1 + c_2 = 0,\ c_1 - c_2 = -\frac{2i}{2} = -i$. This leads to the solution to the IVP,

$$\begin{aligned}
\vec{\boldsymbol{x}}(t) &= \begin{bmatrix} (-i)i\cos t - (0)\sin t \\ (0)\cos t + (-i)i\sin t. \end{bmatrix} \\
\iff \quad &= \begin{bmatrix} \cos t \\ \sin t. \end{bmatrix}
\end{aligned}$$

To see it another way,

$$\begin{aligned}\vec{\boldsymbol{x}}(t) &= c_1 \begin{bmatrix} i \\ 1 \end{bmatrix} e^{it} + c_2 \begin{bmatrix} -i \\ 1 \end{bmatrix} e^{-it} \\ &= \begin{bmatrix} c_1 i e^{it} - c_2 i e^{-it} \\ c_i e^{it} + c_2 e^{-it} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{i}{2} i e^{it} - \frac{i}{2} i e^{-it} \\ -\frac{i}{2} e^{it} + \frac{i}{2} e^{-it} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} e^{it} + \frac{1}{2} e^{-it} \\ -\frac{i}{2} e^{it} + \frac{i}{2} e^{-it} \end{bmatrix}.\end{aligned}$$

Now,

$$\frac{e^{it} + e^{-it}}{2} = \frac{\cos t + i \sin t + \cos t - i \sin t}{2} = \frac{2 \cos t}{2} = \cos t$$

and,

$$\frac{-ie^{it} + ie^{-it}}{2} = \frac{-i(\cos t + i \sin t) + i(\cos t - i \sin t)}{2} = \frac{2 \sin t}{2} = \sin t.$$

Repeated Eigenvalues

In the case of a 2×2 system, a repeated eigenvalue will occur when the discriminant

$$D = (a + d)^2 - 4(ad - bc) = 0.$$

Actually the eigenvalues are,

$$\begin{aligned}\lambda_1, \lambda_2 &= \frac{a + d}{2} \pm \frac{\sqrt{(a + d)^2 - 4(ad - bc)}}{2} \\ &= \frac{a + d}{2} \pm \frac{1}{2} \cdot \sqrt{4 \left[\frac{(a + d)^2}{4} - (ad - bc) \right]} \\ &= \frac{a + d}{2} \pm \sqrt{\left(\frac{a + d}{2} \right)^2 - (ad - bc)}\end{aligned}$$

and will be equal when

$$\left(\frac{a+d}{2}\right)^2 - (ad - bc) = 0.$$

That's to say, when the mean of the diagonal elements is equal to the determinant.

An obvious case where this will happen is a scaling matrix, a scalar multiple of the identity matrix,

$$\begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}$$

where $\alpha \neq 1$ is the scale factor. The mean of the diagonal elements here is clearly α and the determinant of the matrix is α^2 so our eigenvalues will be,

$$\lambda_1, \lambda_2 = \frac{2\alpha}{2} \pm \frac{\sqrt{(2\alpha)^2 - 4(\alpha^2 - 0)}}{2} = \alpha.$$

The characteristic polynomial is

$$(\alpha - \lambda)^2$$

and the system being modelled is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

or

$$D\vec{\boldsymbol{x}} = A\vec{\boldsymbol{x}}$$

Another case would be

$$\begin{bmatrix} \alpha + 1 & -1 \\ 1 & \alpha - 1 \end{bmatrix}$$

so that the characterstic polynomial comes out as

$$\begin{aligned} & ((\alpha + 1) - \lambda)((\alpha - 1) - \lambda) + 1 = 0 \\ \iff & \lambda^2 - 2\alpha\lambda + \alpha^2 = 0 \\ \iff & (\lambda - \alpha)^2 = 0 \end{aligned}$$

so that, using the polynomial differential operator subsubsection 4.5.1.5,

$$(A - \lambda I)^2 \vec{\boldsymbol{x}} = 0 = (D - \lambda)^2 \vec{\boldsymbol{x}}$$

and here, for the eigenvalue $\lambda = \alpha$,

$$\begin{bmatrix} (\alpha+1)-\lambda & -1 \\ 1 & (\alpha-1)-\lambda \end{bmatrix}^2 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and the eigenvector for λ is clearly $(1,1)^T$.

So, the eigenvalue λ has given us the solution $c_1(1,1)^T e^{\lambda t}$ such that

$$(A-\lambda I)^2 c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{\lambda t} = (D-\lambda)^2 c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{\lambda t} = 0.$$

Now, since

$$A \begin{bmatrix} c_1 e^{\lambda t} \\ c_1 e^{\lambda t} \end{bmatrix} = \lambda \begin{bmatrix} c_1 e^{\lambda t} \\ c_1 e^{\lambda t} \end{bmatrix} = D \begin{bmatrix} c_1 e^{\lambda t} \\ c_1 e^{\lambda t} \end{bmatrix},$$

if we denote $\vec{\boldsymbol{v}}_e = (c_1 e^{\lambda t}, c_1 e^{\lambda t})^T$ then

$$\begin{aligned} & A\vec{\boldsymbol{v}}_e = \lambda \vec{\boldsymbol{v}}_e = D\vec{\boldsymbol{v}}_e \\ \iff \quad & A\vec{\boldsymbol{v}}_e - \lambda \vec{\boldsymbol{v}}_e = D\vec{\boldsymbol{v}}_e - \lambda \vec{\boldsymbol{v}}_e = 0 \\ \iff \quad & (A-\lambda I)\vec{\boldsymbol{v}}_e = (D-\lambda)\vec{\boldsymbol{v}}_e = 0 \end{aligned}$$

which means that $\vec{\boldsymbol{v}}_e$ satisfies

$$(A-\lambda I)^2 \vec{\boldsymbol{v}}_e = (D-\lambda)^2 \vec{\boldsymbol{v}}_e = 0$$

because

$$(A-\lambda I)^2 \vec{\boldsymbol{v}}_e = (A-\lambda I)((A-\lambda I)\vec{\boldsymbol{v}}_e) = (A-\lambda I)(0) = 0$$

and, also of course,

$$(D-\lambda)^2 \vec{\boldsymbol{v}}_e = 0 = (D-\lambda)((D-\lambda)\vec{\boldsymbol{v}}_e) = (D-\lambda)(0) = 0.$$

But we have a system with 2 degrees of freedom and so we need to find another linearly independent solution to get the general solution. If we can find a solution $\vec{\boldsymbol{v}}_{e'}$ such that $(D-\lambda)^2 \vec{\boldsymbol{v}}_{e'} = 0$ but $(D-\lambda)\vec{\boldsymbol{v}}_{e'} \neq 0$ then it will be linearly independent to $\vec{\boldsymbol{v}}_e$ because any constant multiple of $\vec{\boldsymbol{v}}_e$ is also a solution to $(D-\lambda)\vec{\boldsymbol{v}} = 0$.

The solution is to have a $\vec{\boldsymbol{v}}_{e'} = \alpha(t)e^{\lambda t}$ such that $D^2\alpha(t) = 0$. This way we have (dropping the function argument of α and using α' to denote its derivative),

$$\begin{aligned}(D-\lambda)((D-\lambda)\vec{\boldsymbol{v}}_{e'}) &= (D-\lambda)[\alpha' e^{\lambda t} + \lambda\alpha e^{\lambda t} - \lambda\alpha e^{\lambda t}] \\ &= (D-\lambda)[\alpha' e^{\lambda t}] \\ &= \alpha'' e^{\lambda t} + \lambda\alpha' e^{\lambda t} - \lambda\alpha' e^{\lambda t} \\ &= \alpha'' e^{\lambda t} = 0 \qquad\qquad \alpha'' = D^2\alpha = 0.\end{aligned}$$

Solving for the Non-homogeneous System

The previous technique for repeated eigenvalues can also be adapted to give solutions to the non-homogeneous solution. If we have an equation of the form,

$$y' = ay + b$$

and we use a function of the form $\alpha(t)e^{at}$ then

$$y' - ay = \alpha' e^{at} + a\alpha e^{at} - a\alpha e^{at} = \alpha' e^{at}.$$

So, clearly, if $\alpha' e^{at} = b$ then we have a solution. Therefore,

$$\alpha' = be^{-at} \implies \alpha(t) = \int be^{-at}\,\mathrm{d}t.$$

This is another way of deriving the formula for solutions of non-separable first-order ordinary equations,

$$y(t) = e^{at}\int be^{-at}\,\mathrm{d}t.$$

If the external driver signal is equal to the natural response of the system then we have

$$Dy = \lambda(t)y + g(t) \implies (D-\lambda(t))\,\alpha(t)\,e^{\int\lambda(t)\,\mathrm{d}t} = g(t)$$

but with $g(t) = e^{\int\lambda(t)\,\mathrm{d}t}$. *In this case,*

$$D\alpha(t) = g(t)e^{-\int\lambda(t)\,\mathrm{d}t} = 1 \implies \alpha(t) = t + C.$$

*This is **resonance**.*