Pure Mathematics

0.0.1 Vector Spaces

(tags: vector spaces, polynomials, periodic functions)

0.0.2 Vector Space properties

(tags: vector spaces, polynomials)

0.0.3 Definition of a Vector Space

(tags: vector spaces)

A field F is a subfield of $\mathbb{C}$ if the following properties hold:

- If $a, b \in F$, then $a + b \in F$.
- If $a \in F$, then $-a \in F$.
- If $a, b \in F$, then $ab \in F$.
- If $a \in F$ and $a \neq 0$, then $a^{-1} \in F$.
- $1 \in F$.

Note that using the first, second and last of these axioms we can deduce that $1 - 1 = 0$ is an element of F.

Let F denote a field which is a subfield of $\mathbb{C}$ and V denote a vector space over F.

Definition. ***Addition, Scalar Multiplication***

- *An* ***addition*** *on a set V is a function that assigns an element $u+v \in V$ to each pair of elements $u, v \in V$.*

- *A* ***scalar multiplication*** *on a set V is a function that assigns an element $\lambda v \in V$ to each $\lambda \in F$ and each $v \in V$.*

Note that both functions are closed over V.

Definition. ***A vector space*** *is a set V along with an addition on V and a scalar multiplication on V such that the following properties hold:*

commutativity $\vec{\boldsymbol{u}} + \vec{\boldsymbol{v}} = \vec{\boldsymbol{v}} + \vec{\boldsymbol{u}}$ *for all* $\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}} \in V$;

associativity $(\vec{\boldsymbol{u}} + \vec{\boldsymbol{v}}) + \vec{\boldsymbol{w}} = \vec{\boldsymbol{u}} + (\vec{\boldsymbol{v}} + \vec{\boldsymbol{w}})$ *and* $(ab)\vec{\boldsymbol{v}} = a(b\vec{\boldsymbol{v}})$ *for all* $\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}, \vec{\boldsymbol{w}} \in V$ *and all* $a, b \in F$;

additive identity *there exists an element* $\vec{\mathbf{0}} \in V$ *such that* $\vec{\boldsymbol{v}} + \vec{\mathbf{0}} = \vec{\boldsymbol{v}}$ *for all* $\vec{\boldsymbol{v}} \in V$;

additive inverse *for every* $\vec{\boldsymbol{v}} \in V$ *there exists* $\vec{\boldsymbol{w}} \in V$ *such that* $\vec{\boldsymbol{v}} + \vec{\boldsymbol{w}} = \vec{\mathbf{0}}$;

multiplicative identity $1\vec{\boldsymbol{v}} = \vec{\boldsymbol{v}}$ *for all* $\vec{\boldsymbol{v}} \in V$;

distributive properties $a(\vec{\boldsymbol{u}} + \vec{\boldsymbol{v}}) = a\vec{\boldsymbol{u}} + a\vec{\boldsymbol{v}}$ *and* $(a+b)\vec{\boldsymbol{u}} = a\vec{\boldsymbol{u}} + b\vec{\boldsymbol{u}}$ *for all* $a, b \in F$ *and* $\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}} \in V$;

0.0.4 Derived properties of a Vector Space

(tags: vector spaces)

Proposition 1. *A vector space contains a unique additive identity element.*

Proof. If $\vec{\mathbf{0}}'$ is also an additive identity then by the additive identity property,

$$\vec{\mathbf{0}} + \vec{\mathbf{0}}' = \vec{\mathbf{0}}$$

but since $\vec{\mathbf{0}}$ is also an additive identity,

$$\vec{\mathbf{0}}' + \vec{\mathbf{0}} = \vec{\mathbf{0}}'$$

Then, by the commutativity of vector addition,

$$\vec{\mathbf{0}} = \vec{\mathbf{0}} + \vec{\mathbf{0}}' = \vec{\mathbf{0}}' + \vec{\mathbf{0}} = \vec{\mathbf{0}}' \qquad \square$$

Proposition 2. *A vector space contains a unique additive inverse for each element.*

Proof. If $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{w}}$ are both additive inverses of $\vec{\boldsymbol{u}}$ then, by the additive inverse property we have,

$$\vec{\boldsymbol{u}} + \vec{\boldsymbol{v}} = \vec{\mathbf{0}} \text{ and also } \vec{\boldsymbol{u}} + \vec{\boldsymbol{w}} = \vec{\mathbf{0}}$$

using the uniqueness of the additive identity,

$$\vec{\boldsymbol{u}} + \vec{\boldsymbol{v}} = \vec{\mathbf{0}} = \vec{\boldsymbol{u}} + \vec{\boldsymbol{w}}$$

Then, if we add one of the additive inverses of $\vec{\boldsymbol{u}}$ to both sides,

$$\vec{\boldsymbol{u}} + \vec{\boldsymbol{v}} + \vec{\boldsymbol{v}} = \vec{\boldsymbol{u}} + \vec{\boldsymbol{w}} + \vec{\boldsymbol{v}}$$

and use the associativity of vector addition,

$$\begin{aligned} (\vec{\boldsymbol{u}} + \vec{\boldsymbol{v}}) + \vec{\boldsymbol{v}} &= (\vec{\boldsymbol{u}} + \vec{\boldsymbol{v}}) + \vec{\boldsymbol{w}} \\ \vec{\mathbf{0}} + \vec{\boldsymbol{v}} &= \vec{\mathbf{0}} + \vec{\boldsymbol{w}} \\ \vec{\boldsymbol{v}} &= \vec{\boldsymbol{w}} \end{aligned} \qquad \square$$

Because additive inverses are unique we can use the notation $-\vec{\boldsymbol{v}}$ to denote the additive inverse of $\vec{\boldsymbol{v}}$. Then we define $\vec{\boldsymbol{w}} - \vec{\boldsymbol{v}}$ to mean $\vec{\boldsymbol{w}} + -\vec{\boldsymbol{v}}$.

Definition. *Vector Subtraction*

$$\vec{\boldsymbol{u}} - \vec{\boldsymbol{v}} := \vec{\boldsymbol{u}} + -\vec{\boldsymbol{v}}$$

Proposition 3. $0\vec{\boldsymbol{v}} = \vec{\mathbf{0}}$ *for every* $\vec{\boldsymbol{v}} \in V$.

Note that this proposition is asserting something about scalar multiplication and the additive identity of V. The only part of the definition of a vector space that connects scalar multiplication and vector addition is the distributive property. Therefore the distributive property must be used in this proof.

Proof. Firstly take,

$$\vec{\boldsymbol{v}} + 0\vec{\boldsymbol{v}} = 0\vec{\boldsymbol{v}} + 1\vec{\boldsymbol{v}}$$

and then use the properties of the underlying field to say

$$(0+1)\vec{\boldsymbol{v}} = 1\vec{\boldsymbol{v}} = \vec{\boldsymbol{v}}$$

Now we have shown that,

$$\vec{\boldsymbol{v}} + 0\vec{\boldsymbol{v}} = \vec{\boldsymbol{v}}$$

which, by the definition and uniqueness of the additive identity, shows that $0\vec{\boldsymbol{v}} = \vec{\mathbf{0}}$. But if we want to continue algebraically we can now add the additive inverse to both sides,

$$(\vec{\boldsymbol{v}} + -\vec{\boldsymbol{v}}) + 0\vec{\boldsymbol{v}} = (\vec{\boldsymbol{v}} + -\vec{\boldsymbol{v}})$$
$$\vec{\mathbf{0}} + 0\vec{\boldsymbol{v}} = 0\vec{\boldsymbol{v}} = \vec{\mathbf{0}} \qquad \square$$

Another, simpler proof exists.

Proof. Using the underlying field properties and the distributivity of scalar vector multiplication,

$$0\vec{\boldsymbol{v}} = (0+0)\vec{\boldsymbol{v}} = 0\vec{\boldsymbol{v}} + 0\vec{\boldsymbol{v}}$$

and then adding the additive inverse to both sides,

$$(0\vec{\boldsymbol{v}} + -(0\vec{\boldsymbol{v}})) = (0\vec{\boldsymbol{v}} + -(0\vec{\boldsymbol{v}})) + 0\vec{\boldsymbol{v}}$$
$$\vec{\mathbf{0}} = \vec{\mathbf{0}} + 0\vec{\boldsymbol{v}} = 0\vec{\boldsymbol{v}} \qquad \square$$

Proposition 4. *$a\vec{\mathbf{0}} = \vec{\mathbf{0}}$ for every $a \in F$.*

Proof. Using the distributivity of scalar multiplication of vectors and the additive identity,

$$a\vec{\mathbf{0}} = a(\vec{\mathbf{0}} + \vec{\mathbf{0}}) = a\vec{\mathbf{0}} + a\vec{\mathbf{0}}$$

Then, adding the additive inverse to both sides,

$$(a\vec{\mathbf{0}} + -(a\vec{\mathbf{0}})) = a\vec{\mathbf{0}} + (a\vec{\mathbf{0}} + -(a\vec{\mathbf{0}}))$$
$$\vec{\mathbf{0}} = a\vec{\mathbf{0}} + \vec{\mathbf{0}} = a\vec{\mathbf{0}} \qquad \square$$

Proposition 5. $(-1)\vec{\boldsymbol{v}} = -\vec{\boldsymbol{v}}$ *for every* $\vec{\boldsymbol{v}} \in V$.

Proof. Using the distributivity of scalar multiplication of vectors and the underlying field properties we have,

$$(-1)\vec{\boldsymbol{v}} + \vec{\boldsymbol{v}} = (-1)\vec{\boldsymbol{v}} + 1\vec{\boldsymbol{v}} = (-1+1)\vec{\boldsymbol{v}} = 0\vec{\boldsymbol{v}} = \vec{\boldsymbol{0}}$$

Now we could add the additive inverse to both sides to show that,

$$\begin{aligned}(-1)\vec{\boldsymbol{v}} + (\vec{\boldsymbol{v}} + -\vec{\boldsymbol{v}}) &= \vec{\boldsymbol{0}} + -\vec{\boldsymbol{v}} \\ (-1)\vec{\boldsymbol{v}} + \vec{\boldsymbol{0}} &= \vec{\boldsymbol{0}} + -\vec{\boldsymbol{v}} \\ (-1)\vec{\boldsymbol{v}} &= \vec{\boldsymbol{v}}\end{aligned}$$

□

But we already have,

$$(-1)\vec{\boldsymbol{v}} + \vec{\boldsymbol{v}} = \vec{\boldsymbol{0}}$$

and this, by the definition of the additive inverse, proves that $(-1)\vec{\boldsymbol{v}}$ is an additive inverse of $\vec{\boldsymbol{v}}$. Since we have previously proven the uniqueness of the additive inverse in Proposition 2 we can conclude, in fact, that $(-1)\vec{\boldsymbol{v}} = -\vec{\boldsymbol{v}}$ the unique additive inverse of v.

0.0.5 The notation F^S

(tags: vector spaces)

If S is a set then F^S denotes the set of functions $S \mapsto F$.

Addition is defined as, for $f, g, (f+g) \in F^S$,

$$(f+g)(x) = f(x) + g(x)$$

for all $x \in S$.

Scalar multiplication is defined as, for $\lambda \in F, \lambda f \in F^S$,

$$(\lambda f)(x) = \lambda f(x)$$

for all $x \in S$.

Example: If S is the interval $[0, 1]$ and $F = \mathbb{R}$ then $\mathbb{R}^{[0,1]}$ is the set of real-valued functions on the interval $[0, 1]$. $\mathbb{R}^{[0,1]}$ is a vector space with additive identity $0 : [0, 1] \mapsto \mathbb{R}$ defined as $0(x) = 0$ and the additive inverse of some function $f \in \mathbb{R}^{[0,1]}$ is the function defined as $(-f)(x) = -f(x)$.
Any *non-empty* set S in conjunction with a subset of $\mathbb{C}$ would similarly produce a vector space. In fact, the vector space F^n can be thought of as the space of functions from the set $\{1, 2, 3, \ldots, n\}$ to F. For example, vectors in 3-dimensional space can be viewed as:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \equiv f : \{1, 2, 3\} \mapsto \mathbb{R} \text{ with } f(t) = \begin{cases} x & t = 1 \\ y & t = 2 \\ z & t = 3 \end{cases}$$

0.0.6 Polynomials as a vector space

(tags: vector spaces, polynomials)

A very important example involves treating a polynomial as a vector. A function $p : F \mapsto F$ is called a polynomial with coefficients in F if there exist $a_0, \ldots, a_m \in F$ such that,

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$$

for all $z \in F$.
Then we can define a vector space, $P(F)$, to be the set of all polynomials with coefficients in F.

Addition on $P(F)$ is defined as,

$$(p + q)(z) = p(z) + q(z) \qquad \text{for } p, q \in P(F), z \in F$$

whose associativity is clear from the definition and the commutativity can be shown by,

$$\begin{aligned} ((p + q) + r)(z) &= (p + q)(z) + r(z) \\ &= p(z) + q(z) + r(z) \\ &= p(z) + (q + r)(z) \\ &= (p + (q + r))(z) \end{aligned}$$

Scalar multiplication on $P(F)$ is defined as,

$$(ap)(z) = ap(z) \qquad \text{for } p \in P(F), a, z \in F$$

whose associativity can be shown by substituting (ab) for a in the definition,

$$[(ab)p](z) = (ab)p(z)$$

Then, by the associativity of the multiplication of the elements of the field F we have,

$$(ab)p(z) = a[b(p(z)]$$

then we use the definition in reverse,

$$a[b(p(z)]) = a[(bp)(z)] = [a(bp)](z)$$

(compare with $(ab)\vec{\boldsymbol{v}} = a(b\vec{\boldsymbol{v}})$)

modeling Concretely, each $p(z) \in P(F)$ is a vector that could be modeled, say, as

$$\vec{\boldsymbol{p}} = \{\, (a_0, a_1, \ldots, a_m) \mid p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m \in P(F) \,\}$$

0.0.7 Subspaces of vector spaces

(tags: vector spaces, polynomials, periodic functions)

0.0.8 Definition of a Subspace

(tags: vector spaces)

Definition. *A set U is a subspace of V if it is a subset of V and if the same addition and multiplication over U forms a vector space.*

Considering the required properties of a vector space, we can see that commutativity and associativity of the addition; associativity of the scalar multiplication; and distributivity of the scalar multiplication over the addition; will all be satisfied as we have the same addition and multiplication over a subset of the elements in V. That's to say, the vector space properties ensure that these properties hold $\forall \vec{\boldsymbol{v}} \in V$ and we have $\forall \vec{\boldsymbol{u}} \in U, \vec{\boldsymbol{u}} \in V$.
Furthermore, the multiplicative identity also holds $\forall \vec{\boldsymbol{v}} \in V$ so will also hold for every element of U.

So what remains to be proven to satisfy the requirements of a subspace?

- Existence of the additive identity
- Existence of an additive inverse for every element of U
- Closure of the addition and scalar multiplication over U

Note, however that - having proved in Proposition 5 that multiplication by -1 gives the additive inverse - closure of the scalar multiplication over U also implies the presence in U of the additive inverse of every element of U. So, actually, what we need to prove for U to be a subspace is only,

- $\vec{\mathbf{0}} \in U$
- Closure of the addition and scalar multiplication over U

0.0.9 A subspace of the polynomials

(tags: vector spaces, polynomials)

An example of a subspace of the polynomials, $P(F)$ is,

$$\{\, p \in P(F) \mid p(3) = 0 \,\}$$

Members of this subspace include:

- $p(z) = 3 - z$
- $p(z) = 9 - z^2$

- $p(z) = 3 - z + 3z^2 - z^3$
- $p(z) = 12z - 4z^2$
- ...etc.

To verify this we need to show that addition and multiplication are closed over this set and that $\vec{\mathbf{0}}$ is a member of the set. It's easy to see that $\vec{\mathbf{0}}$ is a member of the set as,

$$p(3) = 0 + 0(3) + 0(3)^2 + \cdots + 0(3)^m = 0$$

as required. Scalar multiplication is closed as,

$$ap(3) = a(0) = 0$$

whereas addition can be shown to be closed as,

$$(q + r)(3) = q(3) + r(3) = 0 + 0 = 0$$

Note that for values of $z \neq 3$, the closure of these functions is the same as for the general case of $P(F)$.

0.0.10 Sums and Direct Sums

(tags: vector spaces)

Definition. *If $U_1, U_2, \ldots, U_m$ are subspaces of V then their sum is defined as*

$$U_1 + U_2 + \cdots + U_m = \{ \vec{\boldsymbol{u_1}} + \vec{\boldsymbol{u_2}} + \cdots + \vec{\boldsymbol{u_m}} \mid \vec{\boldsymbol{u_1}} \in U_1, \vec{\boldsymbol{u_2}} \in U_2, \ldots, \vec{\boldsymbol{u_m}} \in U_m \}$$

The sum of the subspaces of V is also a subspace of V because,

- Closure of addition

$$\begin{aligned} &(\vec{\boldsymbol{u_1}} + \vec{\boldsymbol{u_2}} + \cdots + \vec{\boldsymbol{u_m}}) + (\vec{\boldsymbol{u'_1}} + \vec{\boldsymbol{u'_2}} + \cdots + \vec{\boldsymbol{u'_m}}) \\ =&(\vec{\boldsymbol{u_1}} + \vec{\boldsymbol{u'_1}}) + (\vec{\boldsymbol{u_2}} + \vec{\boldsymbol{u'_2}}) + \cdots + (\vec{\boldsymbol{u_m}} + \vec{\boldsymbol{u'_m}}) \\ =&\vec{\boldsymbol{v_1}} + \vec{\boldsymbol{v_2}} + \cdots + \vec{\boldsymbol{v_m}} \qquad \text{where } \vec{\boldsymbol{v_1}} \in U_1, \vec{\boldsymbol{v_2}} \in U_2, \ldots, \vec{\boldsymbol{v_m}} \in U_m \end{aligned}$$

- Closure of scalar multiplication

$$\begin{aligned} & a(\vec{\boldsymbol{u_1}} + \vec{\boldsymbol{u_2}} + \cdots + \vec{\boldsymbol{u_m}}) && \text{where } a \in F \\ =\, & a\vec{\boldsymbol{u_1}} + a\vec{\boldsymbol{u_2}} + \cdots + a\vec{\boldsymbol{u_m}} \\ =\, & \vec{\boldsymbol{v_1}} + \vec{\boldsymbol{v_2}} + \cdots + \vec{\boldsymbol{v_m}} && \text{where } \vec{\boldsymbol{v_1}} \in U_1, \vec{\boldsymbol{v_2}} \in U_2, \ldots, \vec{\boldsymbol{v_m}} \in U_m \end{aligned}$$

- Existence of $\vec{\mathbf{0}}$

$$\begin{aligned} & U_1, U_2, \ldots, U_m \text{ are subspaces} \\ \implies & \vec{\mathbf{0}} \in U_1, \vec{\mathbf{0}} \in U_2, \ldots, \vec{\mathbf{0}} \in U_m \\ \implies & \vec{\mathbf{0}} + \vec{\mathbf{0}} + \cdots + \vec{\mathbf{0}} \in U_1 + U_2 + \cdots + U_m \end{aligned}$$

  Note though, that this may not be the only way of producing $\vec{\mathbf{0}}$ from the sum of vectors of these subspaces. That's to say, there could be some $(\vec{\boldsymbol{u_1}} + \vec{\boldsymbol{u_2}} + \cdots + \vec{\boldsymbol{u_m}}) = \vec{\mathbf{0}}$ and this is a key difference from direct sums.

Proposition 6. *$U_1 + U_2 + \cdots + U_m$ is the smallest subspace of V containing $U_1, U_2, \ldots, U_m$.*

Proof. $U_1+U_2+\cdots+U_m$ is a subspace of V that contains $U_1, U_2, \ldots, U_m$ because we can obtain U_i by setting all the u_j for $j \neq i$ to $\vec{\mathbf{0}}$.

If a subspace of V contains $U_1, U_2, \ldots, U_m$ then, by the closure of addition, it must also contain $U_1 + U_2 + \cdots + U_m$.

Therefore the smallest subspace of V that contains $U_1, U_2, \ldots, U_m$ is $U_1 + U_2 + \cdots + U_m$. $\square$

Definition. *If $U_1, U_2, \ldots, U_m$ are subspaces of V then their **direct sum** is defined as,*

$$U_1 \oplus U_2 \oplus \cdots \oplus U_m = \{\, \vec{\boldsymbol{u_1}} + \vec{\boldsymbol{u_2}} + \cdots + \vec{\boldsymbol{u_m}} \mid \vec{\boldsymbol{u_1}} \in U_1, \vec{\boldsymbol{u_2}} \in U_2, \ldots, \vec{\boldsymbol{u_m}} \in U_m \,\}$$

such that,

$$\vec{\boldsymbol{u_1}} + \vec{\boldsymbol{u_2}} + \cdots + \vec{\boldsymbol{u_m}} = \vec{\mathbf{0}} \implies \vec{\boldsymbol{u_1}} = \vec{\mathbf{0}}, \vec{\boldsymbol{u_2}} = \vec{\mathbf{0}}, \ldots, \vec{\boldsymbol{u_m}} = \vec{\mathbf{0}}.$$

That the unique way of obtaining $\vec{\mathbf{0}}$ is for all of the vectors from each of the subspaces to be $\vec{\mathbf{0}}$ is equivalent to there only being a single unique way of obtaining each resultant vector from an addition of the vectors from the individual subspaces. This can be seen as,

$$\begin{aligned}
\vec{u_1} + \vec{u_2} + \cdots + \vec{u_m} &= \vec{u'_1} + \vec{u'_2} + \cdots + \vec{u'_m} \\
(\vec{u_1} + \vec{u_2} + \cdots + \vec{u_m}) - (\vec{u'_1} + \vec{u'_2} + \cdots + \vec{u'_m}) &= \vec{\mathbf{0}} \\
(\vec{u_1} - \vec{u'_1}) + (\vec{u_2} - \vec{u'_2}) + \cdots + (\vec{u_m} - \vec{u'_m}) &= \vec{\mathbf{0}}
\end{aligned}$$

Therefore, since vector spaces always contain $\vec{\mathbf{0}}$ and so we will always have the representation,

$$\vec{\mathbf{0}} + \vec{\mathbf{0}} + \cdots + \vec{\mathbf{0}} = \vec{\mathbf{0}}$$

if this is the unique representation of $\vec{\mathbf{0}}$ then it follows that,

$$\begin{aligned}
(\vec{u_1} - \vec{u'_1}) = \vec{\mathbf{0}}, (\vec{u_2} - \vec{u'_2}) = \vec{\mathbf{0}}, \ldots, (\vec{u_m} - \vec{u'_m}) = \vec{\mathbf{0}} \\
\implies \vec{u_1} = \vec{u'_1}, \vec{u_2} = \vec{u'_2}, \ldots, \vec{u_m} = \vec{u'_m}
\end{aligned}$$

which means that these are the same representation. And this clearly holds in reverse also as, if there is a single way of representing each resultant vector then there must be a single way of representing $\vec{\mathbf{0}}$ and due to the definition of a vector space we must always have the representation of all $\vec{\mathbf{0}}$. Therefore, this is the only representation of $\vec{\mathbf{0}}$.

Note that this is a condition on the contents of the subspaces and not on the way that the addition is performed. So, the difference between vector space sum $(U_1 + U_2)$ and vector space direct sum $(U_1 \oplus U_2)$ is not in the operator itself but in the operands they operate over.

For two subspaces, say, U_1, U_2 this condition on the subspaces reduces to the requirement that $U_1 \cap U_2 = \{\vec{\mathbf{0}}\}$ which can be seen as,

$$\begin{aligned}
\vec{u_1} + \vec{u_2} &= \vec{\mathbf{0}} \\
\vec{u_1} + -\vec{u_1} + \vec{u_2} &= \vec{\mathbf{0}} + -\vec{u_1} \\
\vec{u_2} &= -\vec{u_1} \\
\implies -\vec{u_1} \in U_2 &\implies \vec{u_1} \in U_2
\end{aligned}$$

So, for two subspaces, obtaining $\vec{\mathbf{0}}$ as the sum of vectors from the subspaces implies a vector in common between them. So, for $\vec{\mathbf{0}} + \vec{\mathbf{0}}$ to be the only way of obtaining $\vec{\mathbf{0}}$ implies that $\vec{\mathbf{0}}$ is the only vector in common.

However, for more than two subspaces, say U_1, U_2, U_3, the situation is different as we could have,

$$\begin{aligned} \vec{\boldsymbol{u_1}} + \vec{\boldsymbol{u_2}} + \vec{\boldsymbol{u_3}} &= \vec{\mathbf{0}} \\ \iff \vec{\boldsymbol{u_1}} + -\vec{\boldsymbol{u}}_\mathbf{1} + \vec{\boldsymbol{u_2}} + -\vec{\boldsymbol{u}}_\mathbf{2} + \vec{\boldsymbol{u_3}} &= \vec{\mathbf{0}} + -\vec{\boldsymbol{u}}_\mathbf{1} + -\vec{\boldsymbol{u}}_\mathbf{2} \\ \iff \vec{\boldsymbol{u_3}} &= -\vec{\boldsymbol{u}}_\mathbf{1} + -\vec{\boldsymbol{u}}_\mathbf{2} \end{aligned}$$

which does not imply any vectors held in common.

0.0.11 Span, Dimension and Bases

(tags: vector spaces)

Definition. *The* span *of a list of vectors* $\vec{\boldsymbol{v_1}}, \vec{\boldsymbol{v_2}}, \dots, \vec{\boldsymbol{v_k}}$ *- written* $\operatorname{span}(\vec{\boldsymbol{v_1}}, \vec{\boldsymbol{v_2}}, \dots, \vec{\boldsymbol{v_k}})$ *- is defined as*

$$\{\, \alpha_1\vec{\boldsymbol{v_1}} + \alpha_2\vec{\boldsymbol{v_2}} + \cdots + \alpha_k\vec{\boldsymbol{v_k}} \;\mid\; \alpha_1, \alpha_2, \dots, \alpha_k \in F \,\}$$

Proposition 7. *The span of a list of vectors is the smallest subspace containing those vectors.*

Note that a vector space over $\mathbb{R}$ or $\mathbb{C}$ is an uncountable set as - while the dimensions of the vector space may be finite - closure under scalar multiplication means that the vectors in the space are continuously valued as the field providing the scalars is continuously valued.
This means that the notion of the *smallest* subspace cannot refer to the cardinality of the set and must refer to ordering based on subset. So, the smallest subspace containing a list of vectors is a subspace that contains the list of vectors and, of which, there is no proper subset which also contains the list of vectors.

Proof.

$$\begin{aligned}\text{Let } S &:= \mathit{span}(\vec{\boldsymbol{v_1}}, \vec{\boldsymbol{v_2}}, \dots, \vec{\boldsymbol{v_k}}) \\ &:= \{\, \alpha_1\vec{\boldsymbol{v_1}} + \alpha_2\vec{\boldsymbol{v_2}} + \cdots + \alpha_k\vec{\boldsymbol{v_k}} \mid \alpha_1, \alpha_2, \dots, \alpha_k \in F \,\} \\ \text{and let } V &:= \text{ the smallest vector space containing } \vec{\boldsymbol{v_1}}, \vec{\boldsymbol{v_2}}, \dots, \vec{\boldsymbol{v_k}}.\end{aligned}$$

then S contains every linear combination of $\vec{\boldsymbol{v_1}}, \vec{\boldsymbol{v_2}}, \dots, \vec{\boldsymbol{v_k}}$ and nothing else and so is a vector space containing $\vec{\boldsymbol{v_1}}, \vec{\boldsymbol{v_2}}, \dots, \vec{\boldsymbol{v_k}}$,

$$V \subseteq S$$

Additionally, any vector space containing the vectors $\vec{\boldsymbol{v_1}}, \vec{\boldsymbol{v_2}}, \dots, \vec{\boldsymbol{v_k}}$ must contain all their linear combinations, $\mathit{span}(\vec{\boldsymbol{v_1}}, \vec{\boldsymbol{v_2}}, \dots, \vec{\boldsymbol{v_k}})$,

$$S \subseteq V$$

Therefore there is no proper subset of $\mathit{span}(\vec{\boldsymbol{v_1}}, \vec{\boldsymbol{v_2}}, \dots, \vec{\boldsymbol{v_k}})$ that is also a vector space containing $\vec{\boldsymbol{v_1}}, \vec{\boldsymbol{v_2}}, \dots, \vec{\boldsymbol{v_k}}$, and so $\mathit{span}(\vec{\boldsymbol{v_1}}, \vec{\boldsymbol{v_2}}, \dots, \vec{\boldsymbol{v_k}})$ is the smallest vector space containing $\vec{\boldsymbol{v_1}}, \vec{\boldsymbol{v_2}}, \dots, \vec{\boldsymbol{v_k}}$,

$$(V \subseteq S) \wedge (S \subseteq V) \iff V = S \qquad \square$$

Proposition 8. *Length of every linearly independent list in a space is less than or equal to the length of a spanning list in the same space.*

Proof. Let $U = \vec{\boldsymbol{u_1}}, \vec{\boldsymbol{u_2}}, \dots, \vec{\boldsymbol{u_m}}$ be a linearly independent list of vectors in V and $W = \vec{\boldsymbol{w_1}}, \vec{\boldsymbol{w_2}}, \dots, \vec{\boldsymbol{w_n}}$ be a spanning list of vectors in V.
If we take $\vec{\boldsymbol{u_1}}$ from U and add it to W then - since the other vectors in W are a spanning list - W must be linearly dependent. That's to say,

$$\begin{aligned}\exists\, \alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R} \cdot \alpha_1\vec{\boldsymbol{w_1}} + \cdots + \alpha_n\vec{\boldsymbol{w_n}} &= \vec{\boldsymbol{u_1}} \\ \iff \alpha_1\vec{\boldsymbol{w_1}} + \cdots + \alpha_n\vec{\boldsymbol{w_n}} - \vec{\boldsymbol{u_1}} &= -\alpha_i\vec{\boldsymbol{w_i}} \\ \iff \frac{-\alpha_1}{\alpha_i}\vec{\boldsymbol{w_1}} + \cdots + \frac{-\alpha_n}{\alpha_i}\vec{\boldsymbol{w_n}} + \frac{1}{\alpha_i}\vec{\boldsymbol{u_1}} &= \vec{\boldsymbol{w_i}}\end{aligned}$$

So, $\vec{\boldsymbol{w_i}}$ is in the span of $\vec{\boldsymbol{u_1}}, \vec{\boldsymbol{w_2}}, \dots, \vec{\boldsymbol{w_n}}$ and we can drop $\vec{\boldsymbol{w_i}}$ from the list, W, and it will still span the vector space.
We can keep doing this with the remaining vectors in U - each time the vector

to be removed will be some $\vec{\boldsymbol{w}}_{\boldsymbol{i}}$ because all the $\vec{\boldsymbol{u}}_{\boldsymbol{i}}$ are linearly independent - and all the while W remains a spanning list. We continue until we have replaced (potentially) all n vectors in W, which would happen if $m > n$. At this point we would have the spanning list $W = \vec{\boldsymbol{u}}_{\boldsymbol{1}}, \vec{\boldsymbol{u}}_{\boldsymbol{2}}, \ldots, \vec{\boldsymbol{u}}_{\boldsymbol{n}}$ and $(m-n)$ remaining vectors in U.
Now, since W spans the space, the $(m-n)$ vectors that remain in U will be in the span of W. But, all the vectors that originally came from U were linearly independent, so it is impossible for any vectors in U to be in the span of W (which now comprises only vectors that originally came from U). We therefore conclude that there can be no remaining vectors in U and, consequently that m cannot be greater than n, i.e. $m \leq n$. $\square$

If we look for quadratic polynomials, $p\,(x)$, that pass throught the 3 points (1, 3), (3, 1) and (5, 2):

Then the first has roots at $x = 1, 3$ and passes through the point (5, 2). So, we have:

$$p(1) = p(3) = 0, p(5) = 2$$

meaning that $(x-1)$and $(x-3)$are factors. Therefore,

$$\begin{aligned} p(x) &= \alpha(x-1)(x-3) \\ &= \alpha(x^2-4x+3) \end{aligned}$$

$$
\begin{aligned}
& p(5) &&= 2 \\
\implies\quad & \alpha(5^2 - 4(5) + 3) &&= 2 \\
\iff\quad & 8\alpha &&= 2 \\
\iff\quad & \alpha &&= \tfrac{1}{4}
\end{aligned}
$$

$$
\therefore p(x) = \frac{1}{4}(x^2 - 4x + 3)
$$

The second has roots at $x = 1, 5$ and passes throught the point (3, 1):

$$
\begin{aligned}
p(x) &= \alpha(x-1)(x-5) \\
&= \alpha(x^2 - 6x + 5)
\end{aligned}
$$

$$\begin{aligned} & p(3) &&= 1 \\ \implies & \alpha(3^2 - 6(3) + 5) &&= 1 \\ \iff & -4\alpha &&= 1 \\ \iff & \alpha &&= -\tfrac{1}{4} \end{aligned}$$

$$\therefore p(x) = -\frac{1}{4}(x^2 - 6x + 5)$$

The third has roots at $x = 3, 5$ and passes through the point (1, 3):

$$\begin{aligned} p(x) &= \alpha(x-3)(x-5) \\ &= \alpha(x^2 - 8x + 15) \end{aligned}$$

$$\begin{array}{rcl} & p(1) & = 3 \\ \implies & \alpha(1^2 - 8(1) + 15) & = 3 \\ \iff & 8\alpha & = 3 \\ \iff & \alpha & = \frac{3}{8} \end{array}$$

$$\therefore p(x) = \frac{3}{8}(x^2 - 8x + 15)$$

Adding them together we get,

$$\begin{aligned} & \tfrac{1}{4}(x^2 - 4x + 3) - \tfrac{1}{4}(x^2 - 6x + 5) + \tfrac{3}{8}(x^2 - 8x + 15) \\ = & (\tfrac{1}{4} - \tfrac{1}{4} + \tfrac{3}{8})x^2 + (-1 + \tfrac{6}{4} - 3)x + (\tfrac{3}{4} - \tfrac{5}{4} + \tfrac{45}{8}) \\ = & \tfrac{3}{8}x^2 - \tfrac{10}{4}x + \tfrac{41}{8} \end{aligned}$$