M3N10-Computational PDES-Project 2

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"The contexts of this report and associated codes are my own work unless otherwise stated" 08/01/2017

1. For the first question we are interested in solving the following the problem:

$$u_{xx} + u_{yy} = f(x, y) \qquad \text{with appropriate B.C} \tag{1}$$

For the remainder of the project the boundary condition are homogeneous Dirichlet so $u = 0$ on all the boundary's. We are finding the solution by discretizing the problem to a uniform grid in x and y. We are then approximating u_{xx} and u_{yy} using a centered approximation shown below across the grid:

$$u_{xx} \approx \frac{U_{nm+1} + U_{nm-1} - 2U_{nm}}{h_x^2} \qquad u_{yy} \approx \frac{U_{n+1m} + U_{n-1m} - 2U_{nm}}{h_y^2} \tag{2}$$

h_x and h_y is the distance between 2 adjacent points in the x and y directions respectively.

This then leads to a linear system $Au = b$. Which we wish to solve across the uniform grid, with known u at the boundaries. We solve this system iteratively by an implicit Gauss–Seidel method(GS) (assuming it is diagonally dominant) , used along with geometric multi-grid to increase the convergence of system.

Here is a bit of explanation of what we are solve for GS:

take (1) and sub in the approximations in (2), with f(x,y) approximated at the point (m,n):

$$\frac{U_{nm+1} + U_{nm-1} - 2U_{nm}}{h_x^2} + \frac{U_{n+1m} + U_{n-1m} - 2U_{nm}}{h_y^2} = f_{n,m} \tag{3}$$

rearranging to have U_{nm} on one side,

$$\frac{2U_{nm}}{h_x^2 + h_y^2} = \frac{U_{n+1m} + U_{n-1m}}{h_y^2} + \frac{U_{nm+1} + U_{nm-1}}{h_x^2} - f_{n,m} \tag{4}$$

$$U_{nm} = \frac{h_x^2 + h_y^2}{2}\left(\frac{U_{n+1m} + U_{n-1m}}{h_y^2} + \frac{U_{nm+1} + U_{nm-1}}{h_x^2} - f_{n,m}\right) \tag{5}$$

In the case we are going to look at $0 \leq x \leq 1$ and $0 \leq y \leq L$. If we let M be the number of points in the x direction and N number of point in the y direction. So we can simplify h_x and h_y:

$$h_x = \frac{1}{(M-1)} \qquad h_y = \frac{L}{(N-1)} \tag{6}$$

thus

$$U_{nm} = \frac{1}{2L^2(M-1)^2 + 2(N-1)^2}((N-1)^2(U_{n+1m}+U_{n-1m})+L^2(M-1)^2(U_{nm+1}+U_{nm-1})-L^2 f_{n,m}) \tag{7}$$

I have written code which iterates this expression.

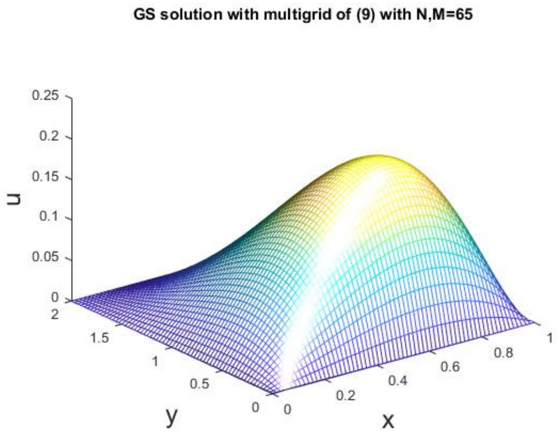
An exact solution of (1) with B.C $u = 0$ for $0 \leq x \leq 1$ and $0 \leq y \leq 2$ (L=2) is:

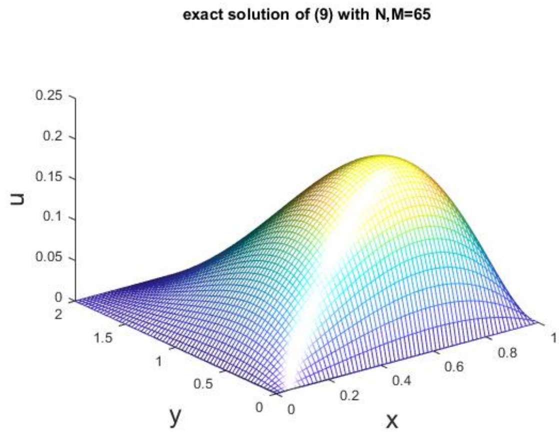
$$u = (x-1)xy(y-2)e^{x-y} \tag{8}$$

with $f(x, y)$ given by:

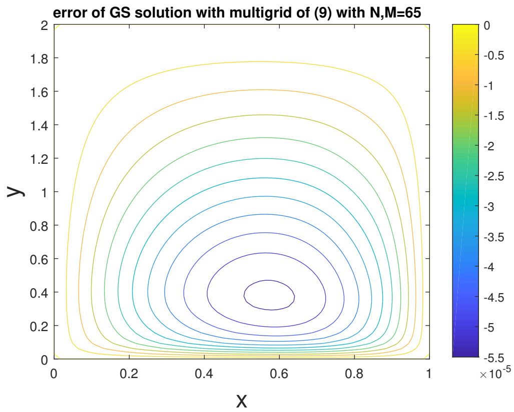
$$f(x, y) = ((y-2)yx(x+3) + (x-1)x(y^2 - 6y + 6))e^{x-y} \tag{9}$$

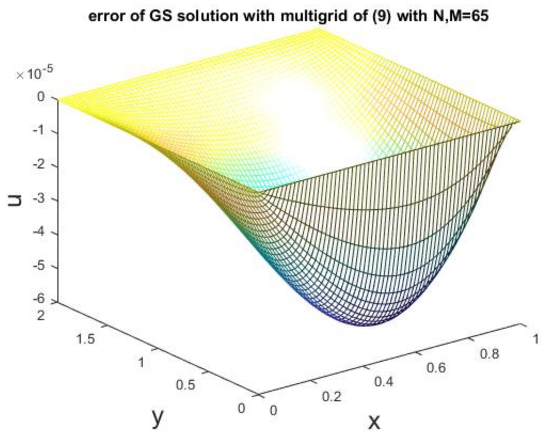
Below are the graphs of solving (1) with f given by (9) using GS along with geometric multigrid V-cycle which speeds up the convergence.



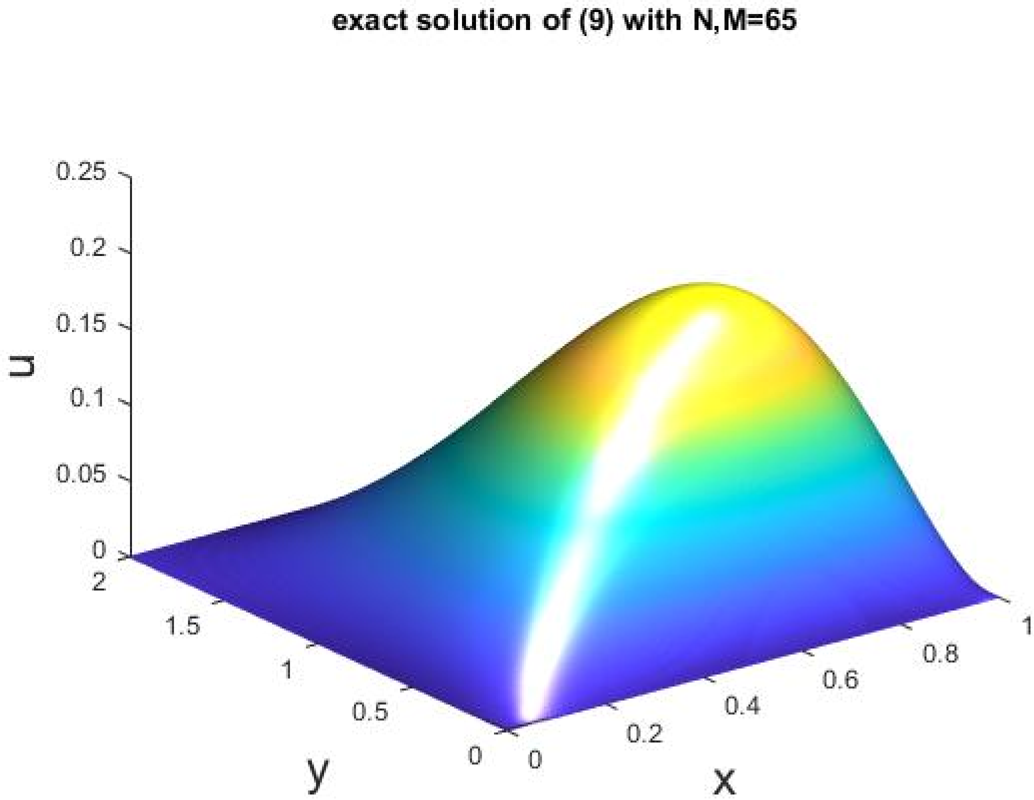


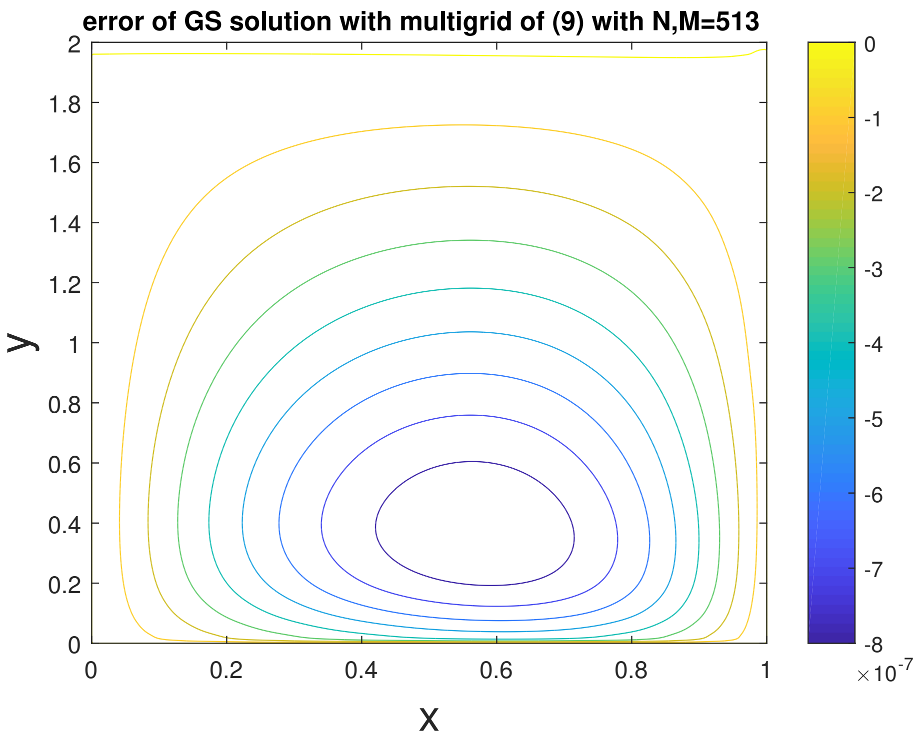
Below is a contour and garph showing the error an interesting point of note is the shape of the contours.





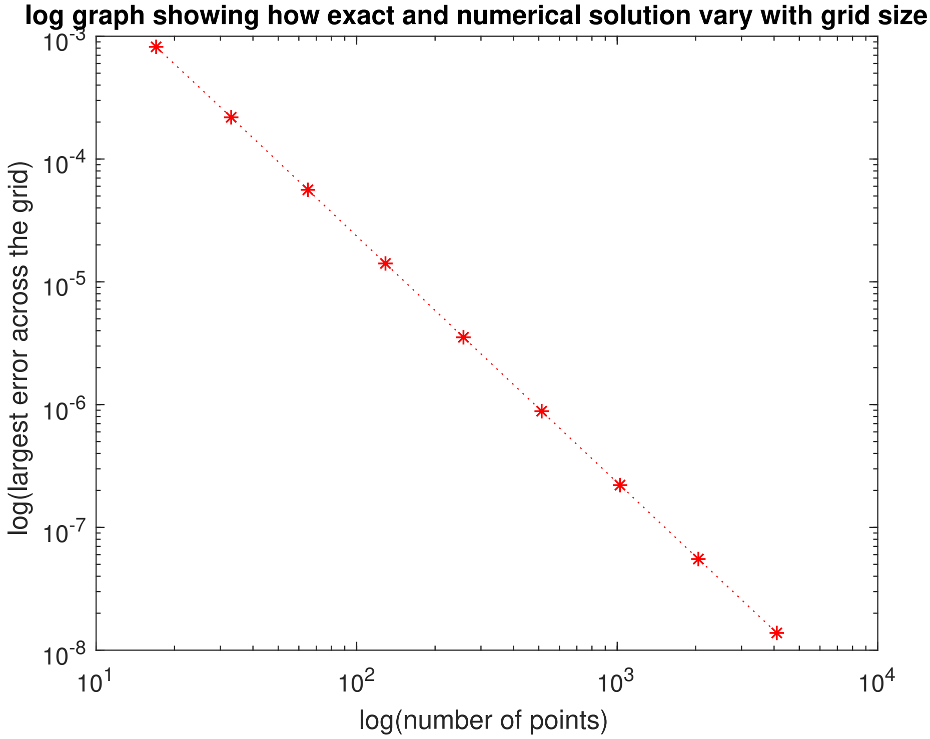
If we increase the number of points of the grid this appears to reduce the error and improves the clarity of solution. For example the graphs below are for N=513,M=513.





Again we get the same shape for the error contours but of order 10^{-7} instead of 10^{-5}. Which is reasonable as the relative error is now about 10^{-6}.

After a bit of investigation you can find a correlation between the error and the number of points used. Here the error is calculated by finding the magnitude of the largest difference between exact solution and numerical.



2. We now wish to solve the following P.D.E:

$$u_t = u_{xx} + u_{yy} + Q(x, y, u, t) \qquad \text{with appropriate B.C and} \quad u = u_o(x, y) \quad \text{at} \quad t = 0 \tag{10}$$

We will do this by using Crank-Nicholson method if Q depends on t evaluating it explicitly at $t + \frac{1}{2}k$, where k is the time step. We the solve the resulting linear system every time step using a similar GS method with multigrid used in 1.

$$\frac{U^{j+1}_{nm} - U^j_{nm}}{k} = 0.5[\delta^2 U^{j+1}_{nm} + \delta^2 U^j_{nm}] + Q_{nm}(U^j, t + 0.5k) \tag{11}$$

where

$$\delta^2 U^i_{nm} = \frac{U^i_{nm+1} + U^i_{nm-1} - 2U^i_{nm}}{h_x^2} + \frac{U^i_{n+1m} + U^i_{n-1m} - 2U^i_{nm}}{h_y^2} \tag{12}$$

As before to use the GS method we need to find an expression for U^{j+1}_{nm}. We always now the values of anything at the jth time level as we calculated it at the previous timestep.

$$[1 + k(\frac{1}{h_x^2} + \frac{1}{h_y^2})]U^{j+1}_{nm} = 0.5k[\frac{U^{j+1}_{nm+1} + U^{j+1}_{nm-1}}{h_x^2} + \frac{U^{j+1}_{n+1m} + U^{j+1}_{n-1m}}{h_y^2} + \delta^2 U^j_{nm}] + kQ_{nm} + U^j_{nm} \tag{13}$$

This is an expression for U^{j+1}_{nm} as its coefficient is just a constant depending on the grid and time step.

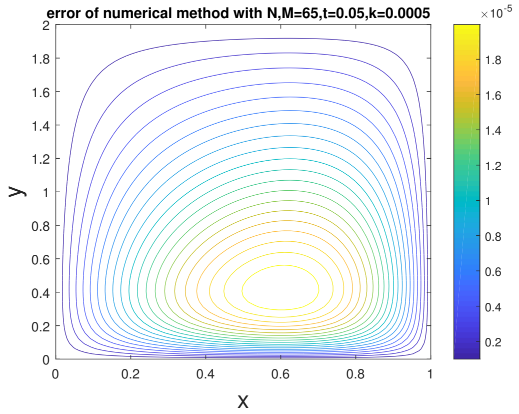
I have altered the code in 1. so that it implements the above and written a program which iterates the the multigrid process over time. We can test how accurate the numerical solution is when Q is dependent on time by comparing it to an exact solution. Such an exact solution is the following with Dirichlet B.C:

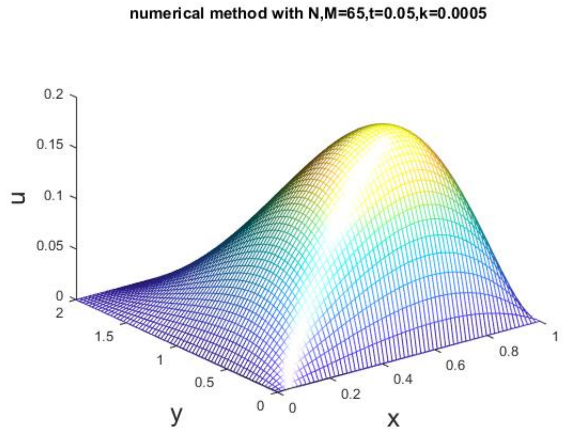
$$u = (x - 1)xy(y - 2)e^{x-y}e^{-t} \quad \text{with I.C} \quad u_o = u(x, y, t = 0) \tag{14}$$

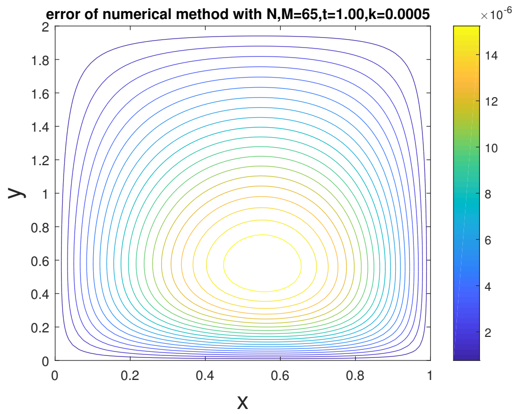
with Q(x,y,t) being the following:

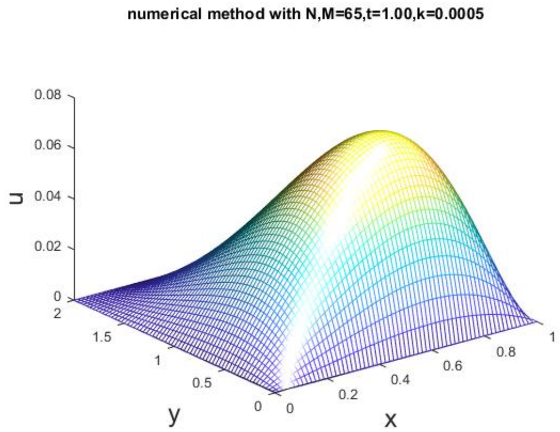
$$Q(x, y, t) = -x((3x + 1))y^2 + (2 - 10x)y + 6x - 6)e^{x-y}e^{-t} \tag{15}$$

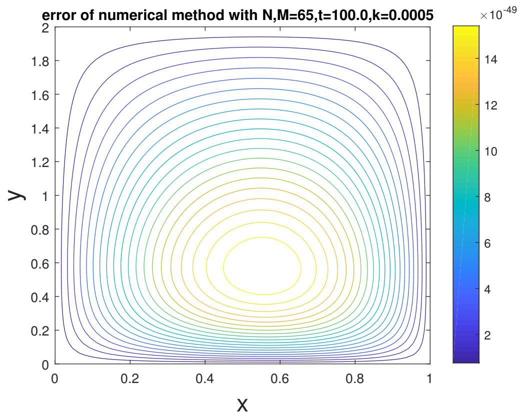
The following show the numerical results and the modulus of error compared to the exact solution, for various values of time. The initial condition is the same as solution in 1.

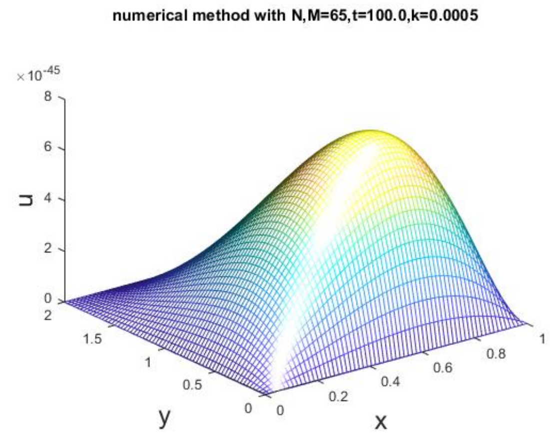






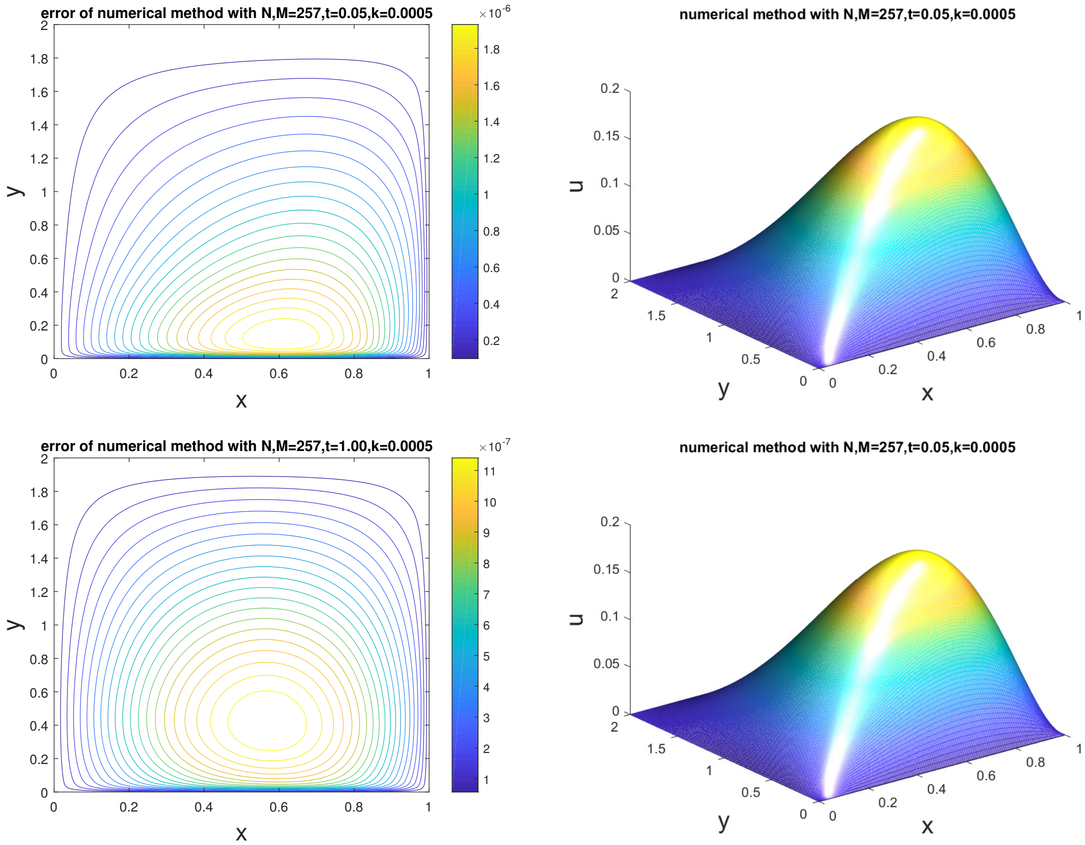






It is reassuring that the relative error does not change with time and remains proportional to the error of the multigrid method about 10^{-4} even when u gets very very small.

Just for comparison we shall show that using more points improves the quality of solution and reduces error.

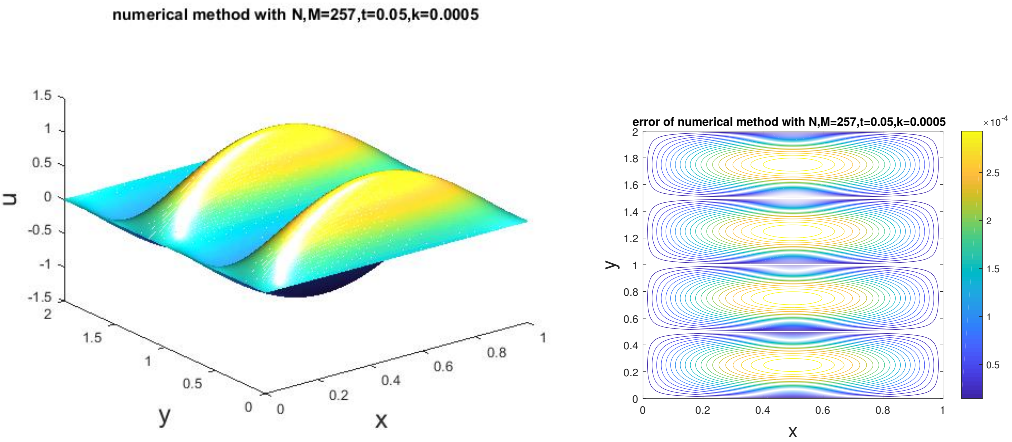


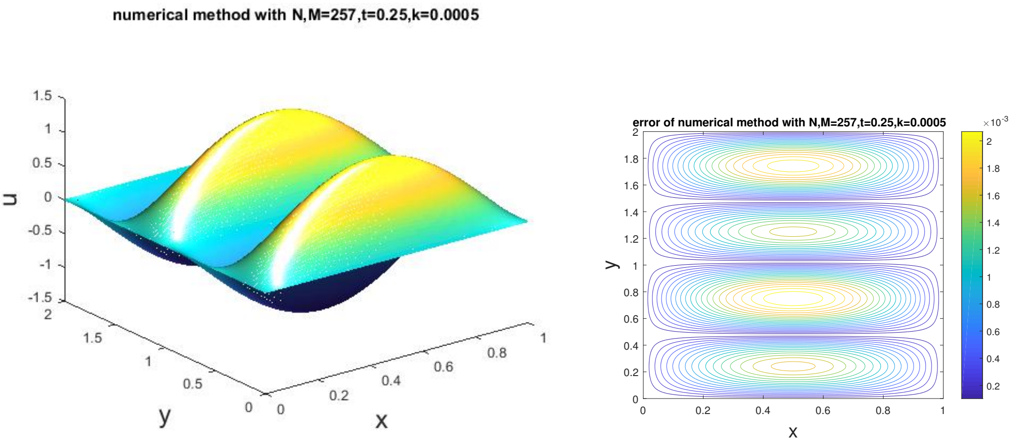
Now to test the method when Q depends on u.

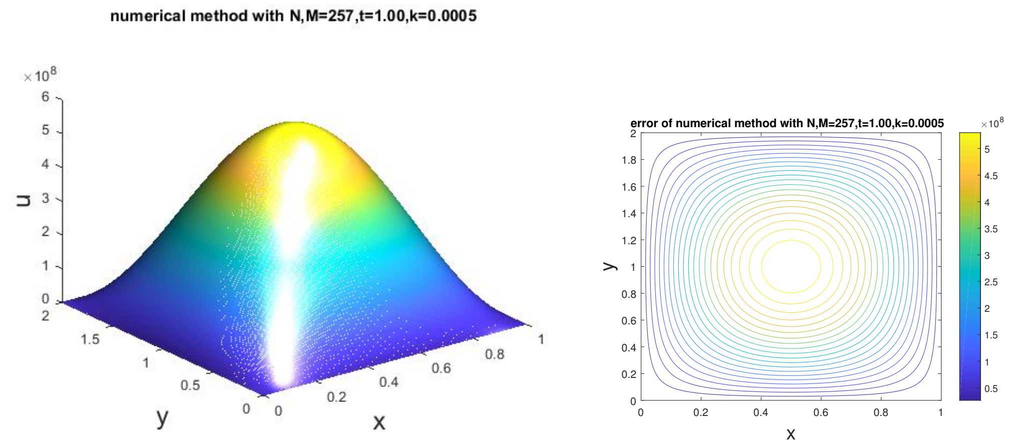
An exact solution is the following:

$$u = \sin(\pi x)\sin(2\pi y)e^t \quad \text{with I.C} \quad u_o = u(x, y, t = 0) \tag{16}$$

with Q given by $Q(u) = (1 + 5\pi^2)$. Now the graphs below show the results.







Now initially the graphs are very promising but as time seems to go on the errors start to accumulate, becoming the same order as the solution. This could be for a variety of reasons; could be because maybe u is not unique for this case or the exponential part of solution is causing the errors to increase. Or scheme doesn't converge.

A way round this could be to linearize Q, you would do this by instead of evaluating Q at U_{nm}^j you evaluate it as the following:

$$\frac{1}{2}[Q(U_{nm}^{j+1}) + Q(U_{nm}^j)]$$

Where $Q(U^{j+1})$ is written as a truncated power series:

$$Q(U^{j+1}) = Q(U^j) + \left.\frac{dQ}{du}\right|_{u=U^j}(U^{j+1} - U^j)$$

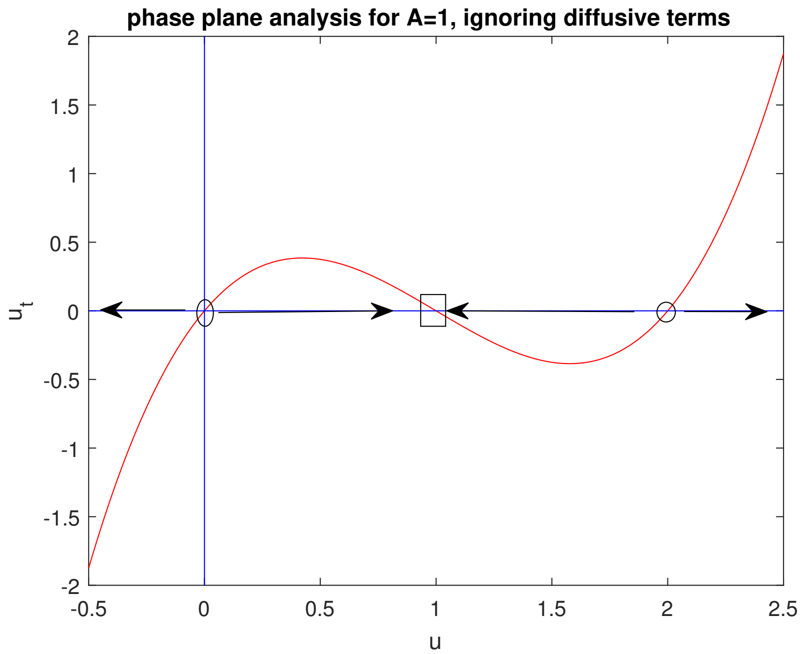
you substitute this Q into (11) but still evaluate it at $t + \frac{1}{2}k$ if it depends on t.

3. We are now interested in the following problem with Dirichlet boundary conditions(u=0 for so $0 \leq x \leq 1$, $0 \leq y \leq 2$) and $-100 < A < 100$:

$$u_t = u_{xx} + u_{yy} + Au(u-1)(u-2) \quad \text{with} \quad u = u_o(x, y) \quad \text{at} \quad t = 0 \tag{17}$$

Where u represents the density of a parasite population in an orchard, so let us assume throughout that $u \geq 0$.

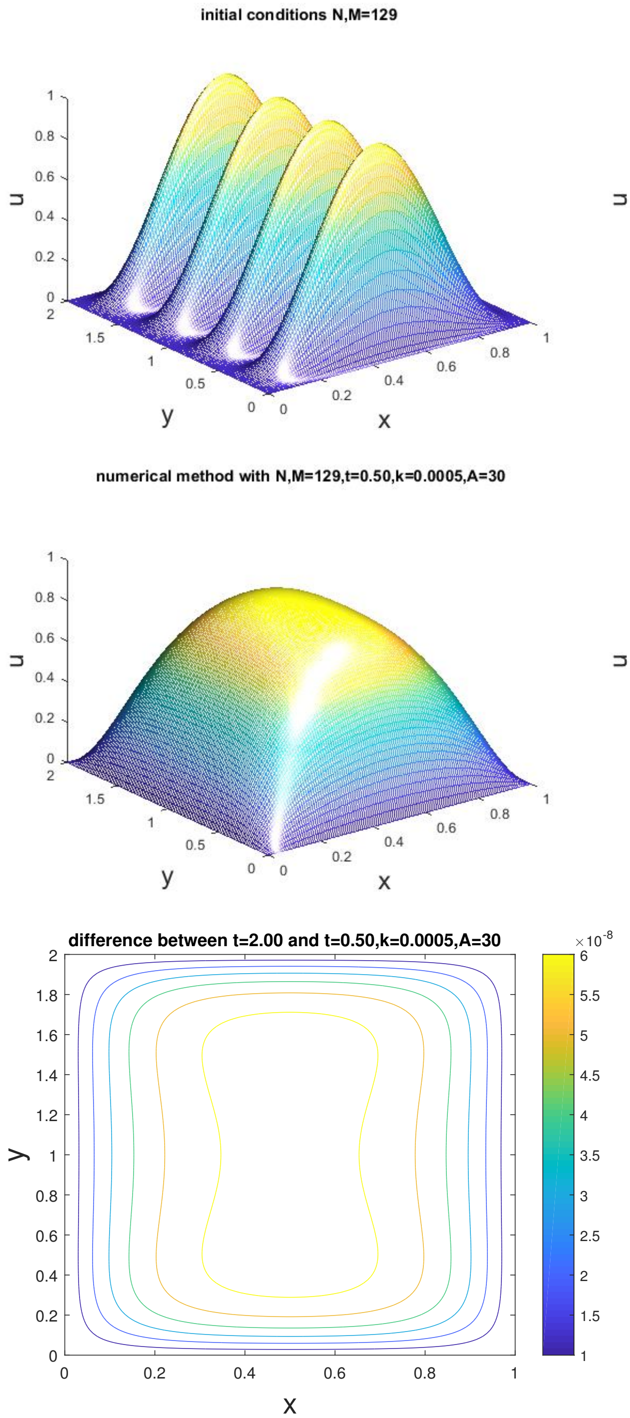
Let us first consider an A which is large and positive. If it is large enough enough we can ignore the diffusive terms $u_xx + u_yy$, and then do some phase plane analysis, which is what the below shows. with rectangles representing a stable node and ellipses unstable nodes.



Now as we can see if u is between 0 and 2 then it will converge to u=1. If it is greater then 2 it will go off to infinity. Now let us take into account the diffusive terms and consider this in 3D. u will effectively diffusive with surrounding u to try and reach an equilibrium. So these nodes are not strict in 3D. But if you have a point u much greater then 2 for example with, A large enough then it will cause all of the other surrounding points to infinity. So the initial conditions dictate the solution.

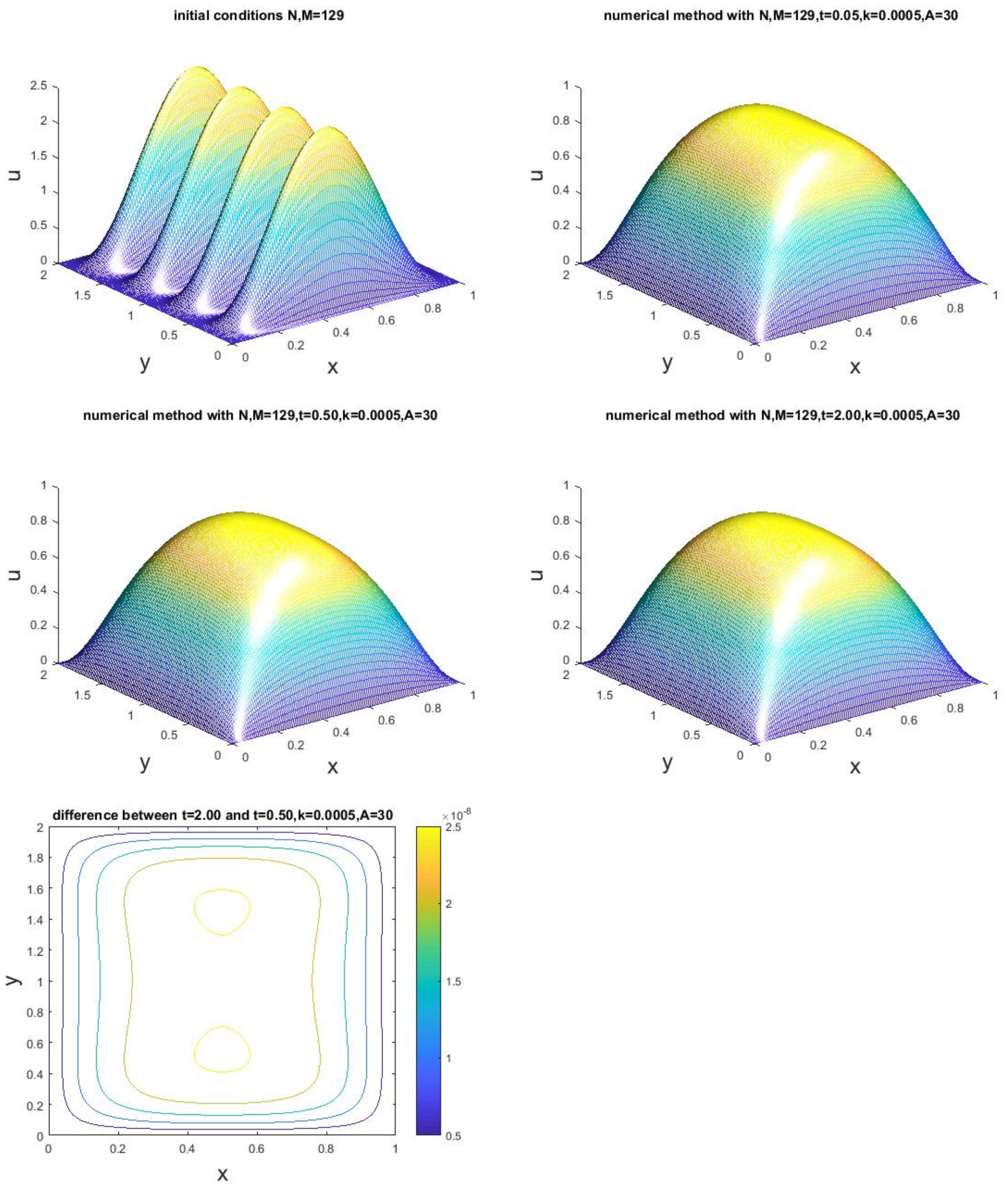
If initial conditions are roughly between 0 and 2 then the solution will be stable and converge. If there is a point much greater then 2, A large enough then solution explodes to infinity. As A is reduced the diffusive terms become more dominant and increase the likelihood of stability.

So for example let us take A=30 with initial conditions $u_o = (\sin(x\pi)\sin(2\pi y))^2$ which we know is positive and $0 \leq u_o \leq 1$.



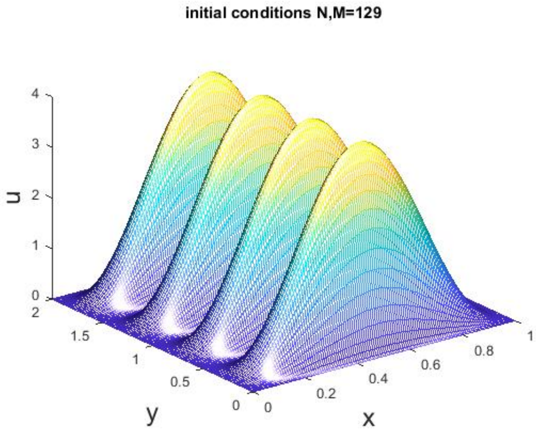
As we can see it does converge to a solution where u is close to 1 in the middle and 0 at the boundaries, with the diffusive terms creating a smooth dome like structure. This represents the parasite population being very dense in the middle and non existent at the boundaries. Which makes sense since there are no trees at the boundaries.

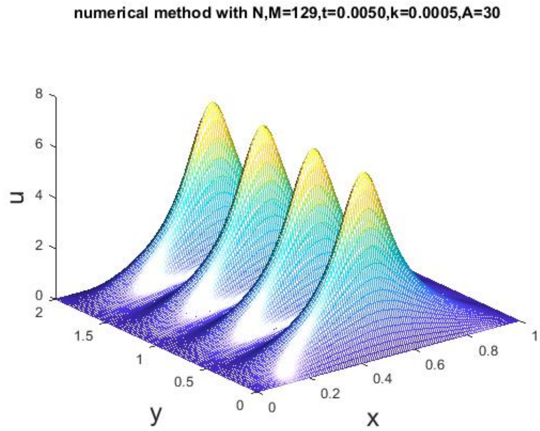
Now lets considerer it with different Initial conditions like $u_o = 2.5(\sin(x\pi)\sin(2\pi y))^2$.

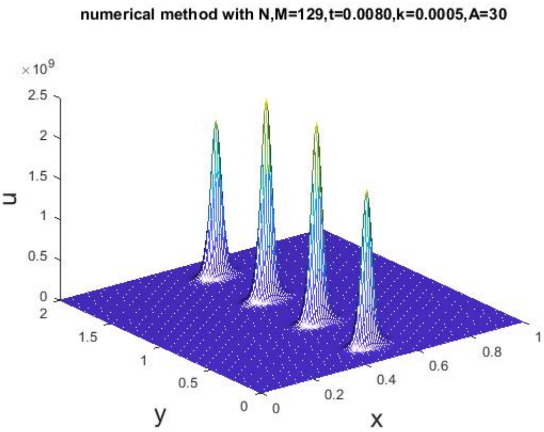


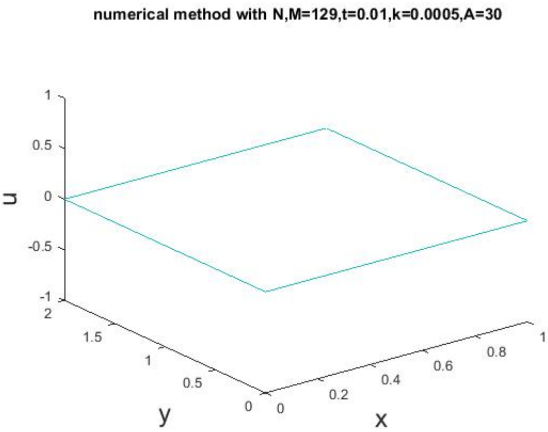
Interestingly the diffusive terms stop it from converging.

Now consider $u_o = 4(\sin(x\pi)\sin(2\pi y))^2$.



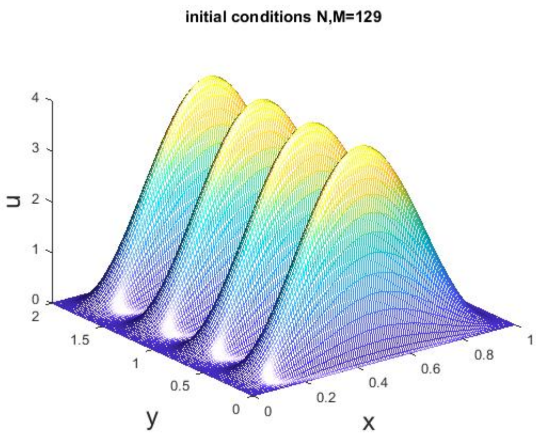


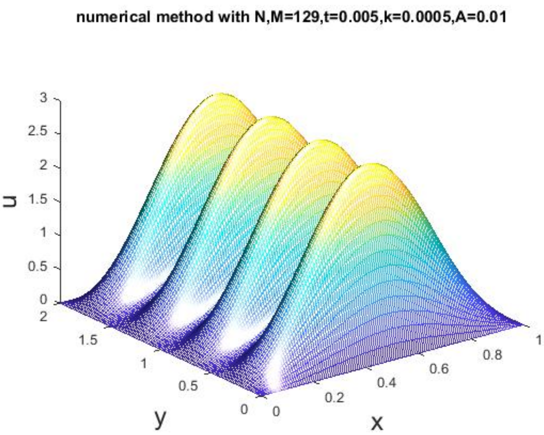


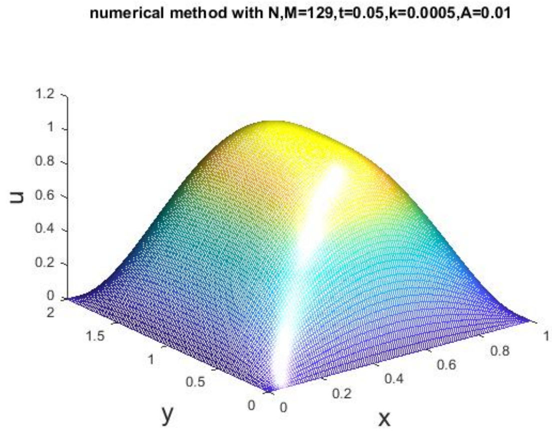


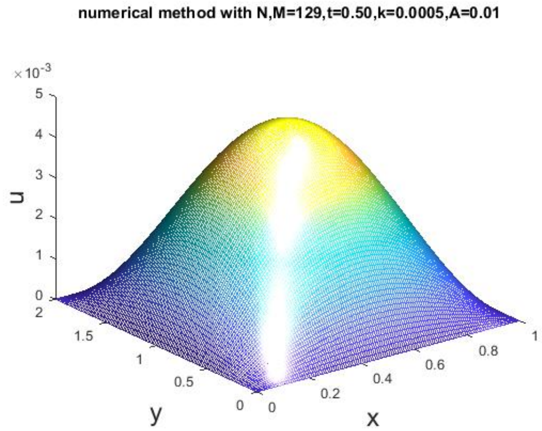
Clearly this just goes off to infinity as predicted.

Now let us make A=0.01 with the same initial conditions.



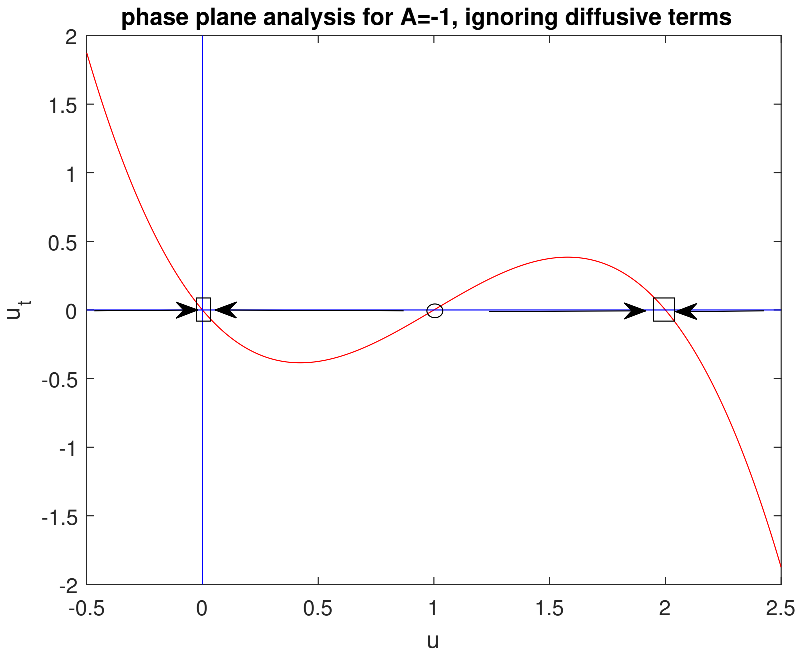






This interestingly converges to zero this is because the diffusive terms are more dominant then the 'source'. It still has that dome like structure.

We can do a similar analysis for A negative but to save on repetition, the below shows the phase plane analysis, which you can draw most of your conclusions from.



So A negative it convergences to a value around 2 or 0, depending on the initial conditions and the size of A.

As A tends to zero the solution tends to zero.

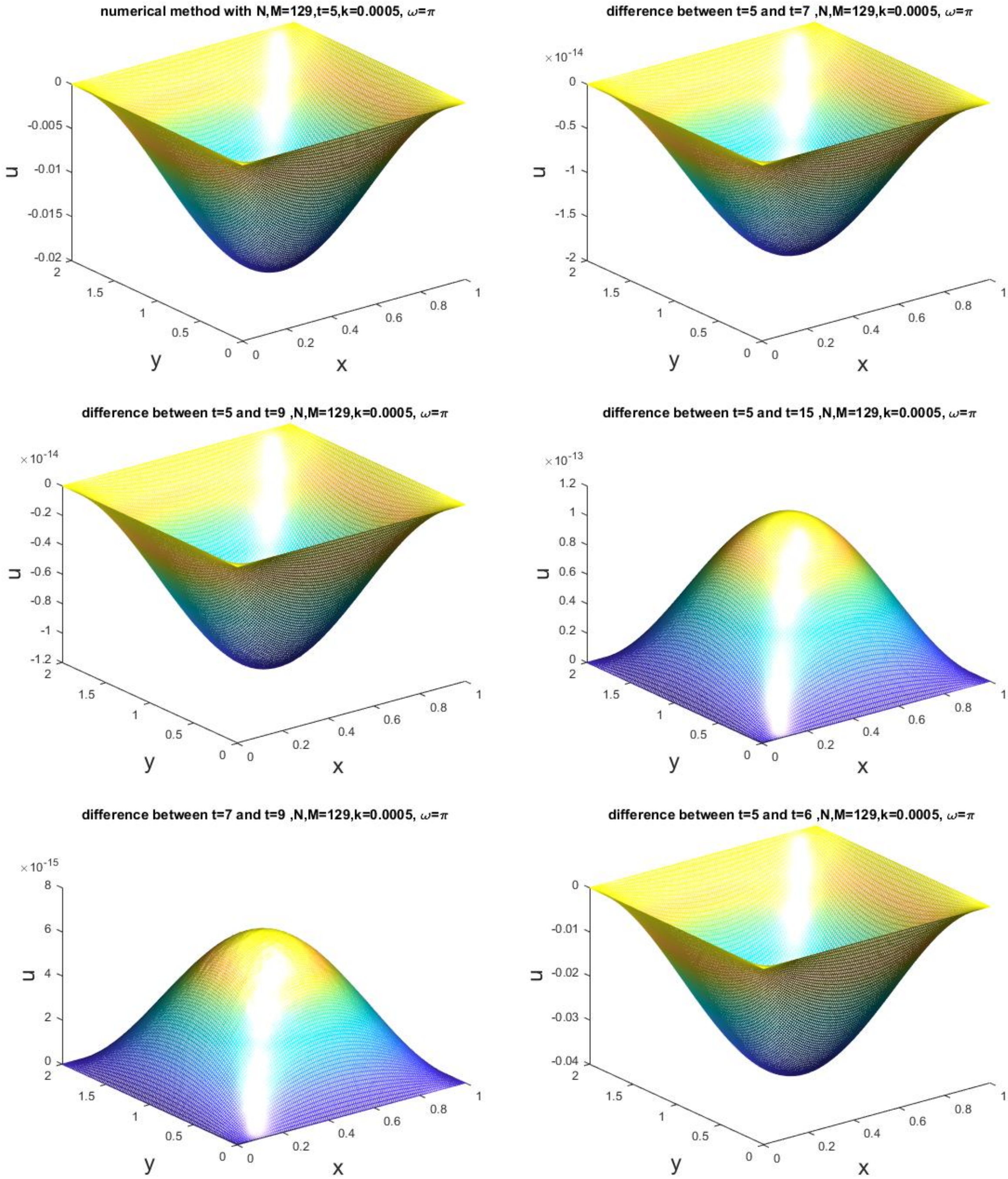
4. For the remainder of the question we shall investigate the following problem:

$$u_t = u_{xx} + u_{yy} + (x-1)xy(y-2)\cos(\omega t) \quad \text{with} \quad u = \sin(\pi x)\sin(2\pi y) \quad \text{at} \quad t = 0 \tag{18}$$

with Dirichlet boundary conditions(u=0 for so $0 \leq x \leq 1$, $0 \leq y \leq 2$) and ω a constant.

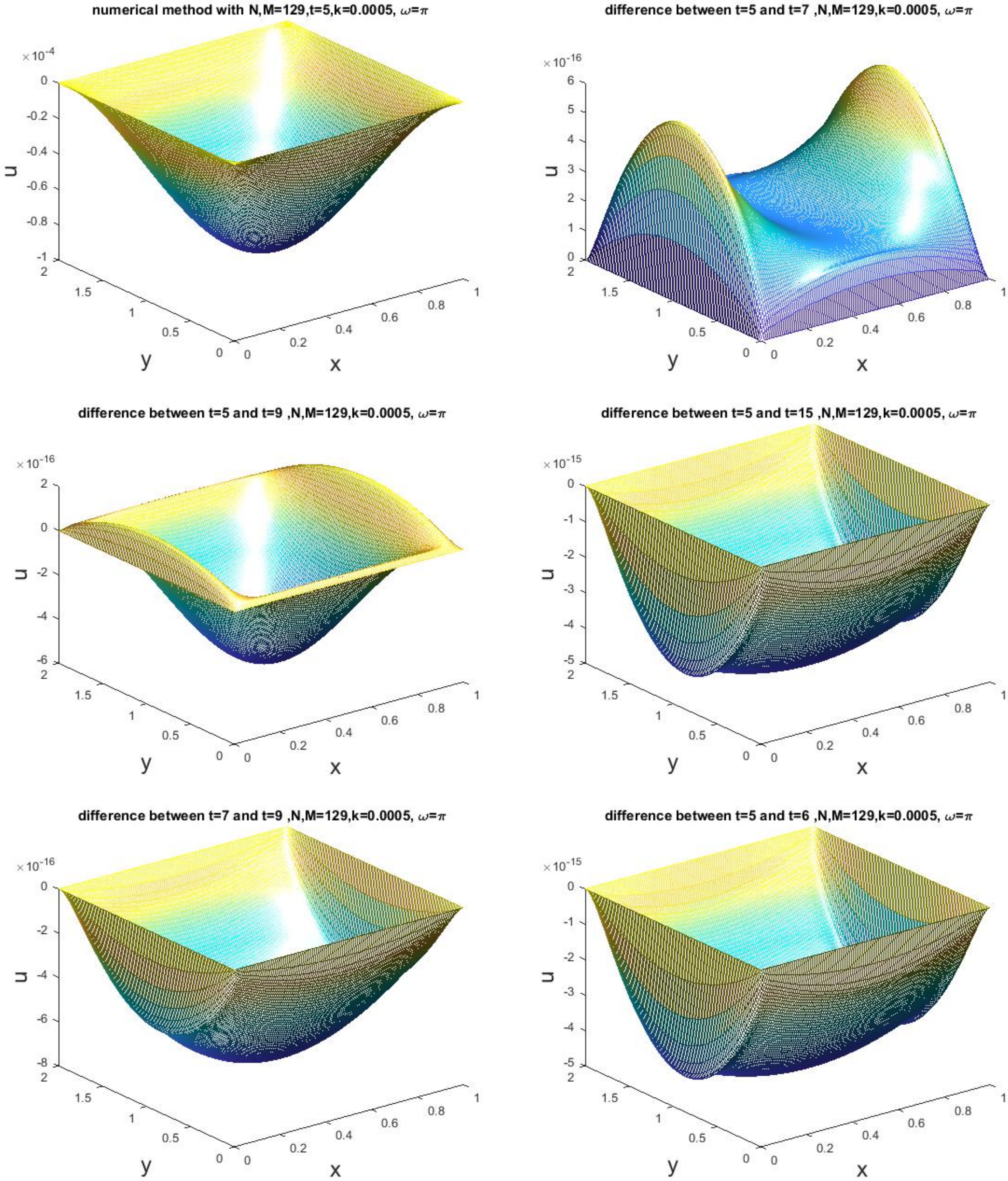
Now after a period of time we expect the solution u to converge periodical. By that I mean the the behavior of parasites should become oscillatory with time period $t = \frac{2\pi}{\omega}$.

Let us first consider $\omega = \pi$



As expected the the solution converges after a period of time, with it oscillating at a period of 2 seconds. For example if you look at the difference between t=5 and t=6 the error is massive compared to the others, which have a difference divisible by the time period. Another interesting point of note is the sign difference as it changes without structure.

Let us now consider $\omega = 1000\pi$



We now expect the time period to be 0.002, which it is. It just appears to take longer to reach a stable state. This could be due to the fact the system oscillates more, the diffusive terms are taking longer to stabilize the system. But interestingly the shape of the differences is not as consistent as before.

Another point of note is that irrelevant of initial conditions (unless trivial), it always seems to converge to this oscillating dome like structure. This represents how the parasites are more active in the middle of the orchard.