

# MATH 146

# Personal Notes

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# Chapter 1:

# Vector Spaces

(S1.1)

## KEY

S :	section	P :	proposition
D :	definition	A :	assignment
R :	remark		
E :	example		
T :	theorem		
L :	lemma		
C :	corollary		

Let  $\mathbb{F}$  be a field.

Then, we say  $V$  is a "vector space"

over  $\mathbb{F}$  if there exists

① an addition  $+ : (V \times V) \rightarrow V$  by  $+ (x, y) = x + y$ ; and

② a scalar multiplication  $\cdot : (\mathbb{F} \times V) \rightarrow V$  by  $\cdot (a, x) = ax$ ;

and the following conditions hold:

①  $V$  is an abelian group with respect

to addition; (VS 1 = commutativity; 2 = associativity; 3 = identity; 4 = inverse)

②  $\forall x \in V \quad \forall x \in V$  (VS 5)

③ multiplication is associative; ie  $a(bx) = (ab)x \quad \forall a, b \in \mathbb{F}, x \in V$ ;

and (VS 6)

④ the left and right distributive laws hold;

ie  $a(x+y) = ax+ay$  and  $(a+b)x = ax+bx \quad \forall a, b \in \mathbb{F}, x \in V$ . (D2)

(VS 7 = former, VS 8 = latter)

## $\mathbb{F}^n$ IS A VECTOR SPACE OVER $\mathbb{F}$ (E2(1))

We can show that the Cartesian product

$$\mathbb{F}^n = \{(a_1, a_2, \dots, a_n) : a_i \in \mathbb{F} \text{ for } i \in \{1, 2, \dots, n\}\}$$

is a vector space over  $\mathbb{F}$  with respect to the addition operation

$$(a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n) = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)$$

and multiplication operation

$$c(a_1, a_2, \dots, a_n) = (ca_1, ca_2, \dots, ca_n).$$

Proof. This follows from verifying each of the conditions above.  $\blacksquare$

Note that we generally say "the vector space  $\mathbb{F}^n$ " to refer to the vector space  $\mathbb{F}^n$  over  $\mathbb{F}$ . (R3(4))

## COLUMN VECTOR NOTATION (E2(2))

Note that we can also write elements of  $\mathbb{F}^n$  as "column vectors"; ie of the form

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

where  $a_1, a_2, \dots, a_n \in \mathbb{F}$ .

## $\mathbb{Q}^n$ IS A VECTOR SPACE OVER $\mathbb{Q}$ ,

## $\mathbb{R}^n$ IS A VECTOR SPACE OVER $\mathbb{R}$ , &

## $\mathbb{C}^n$ IS A VECTOR SPACE OVER $\mathbb{C}$ (R3(1))

We can show

①  $\mathbb{Q}^n$  is a vector space over  $\mathbb{Q}$ ;

②  $\mathbb{R}^n$  is a vector space over  $\mathbb{R}$ ; and

③  $\mathbb{C}^n$  is a vector space over  $\mathbb{C}$ .

Proof. This directly follows from the fact that  $\mathbb{Q}, \mathbb{R}$  and  $\mathbb{C}$  are fields (MATH 145), and substituting the respective fields into the above lemma.  $\blacksquare$

## $\mathbb{R}^n$ IS A VECTOR SPACE OVER $\mathbb{Q}$ , &

## $\mathbb{C}^n$ IS A VECTOR SPACE OVER $\mathbb{R}$ (R3(2))

Moreover, we can also show that

①  $\mathbb{R}^n$  is a vector space over  $\mathbb{Q}$ ; and

②  $\mathbb{C}^n$  is a vector space over  $\mathbb{R}$ .

Proof. Essentially, this stems from the fact that we can "multiply" vectors in  $\mathbb{R}^n$  by scalars in  $\mathbb{Q}$ , and vectors in  $\mathbb{C}^n$  by scalars in  $\mathbb{R}$ .

The formal proof is left to the reader.  $\blacksquare$

## MATRICES (D3(1))

Let  $\mathbb{F}$  be a field, and  $m, n \in \mathbb{Z}^+$ .

Then, we say  $A$  is an " $m \times n$  matrix" with entries from  $\mathbb{F}$  if it is of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

where  $a_{ij} \in \mathbb{F} \quad \forall i \in \{1, 2, \dots, m\}, j \in \{1, 2, \dots, n\}$ .

Alternatively, we can represent  $A$  via the notation

$$A = (a_{ij}), \quad i \in \{1, 2, \dots, m\}, j \in \{1, 2, \dots, n\}.$$

## ij-ENTRY OF A MATRIX (D3(2))

Given a  $m \times n$  matrix  $A$ , the " $ij$ -entry" of  $A$ , or " $a_{ij}$ ", is defined to be the entry in  $A$  at the  $i$ th row and  $j$ th column.

## ZERO MATRIX (D3(3))

The " $m \times n$  zero matrix", or more simply the "zero matrix", denoted as " $0$ ", is defined to be

$$0 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \underbrace{\qquad \qquad \qquad}_{m}$$

ie the  $m \times n$  matrix where which entry equals  $0$ .

## MATRIX EQUALITY (D3(4))

We say two matrices  $A$  and  $B$  are equal if and only if  $a_{ij} = b_{ij} \quad \forall i \in \{1, 2, \dots, m\}, j \in \{1, 2, \dots, n\}$ .

## MATRIX ADDITION (D3(5))

Let  $A$  and  $B$  be  $m \times n$  matrices with entries from some field  $\mathbb{F}$ .

Then, the "addition" of  $A$  and  $B$ , denoted by " $A+B$ ", is defined to be the matrix where

$$(a+b)_{ij} = a_{ij} + b_{ij} \quad \forall i \in \{1, 2, \dots, m\}, j \in \{1, 2, \dots, n\}.$$

## MATRIX SCALAR MULTIPLICATION (D3(6))

Let  $A$  be a  $m \times n$  matrix with entries from some field  $\mathbb{F}$ , and  $c \in \mathbb{F}$  be arbitrary.

Then the "scalar multiplication" of  $A$  by  $c$ , denoted by " $cA$ ", is defined to be the matrix where

$$(ca)_{ij} = c(a_{ij}) \quad \forall i \in \{1, 2, \dots, m\}, j \in \{1, 2, \dots, n\}.$$

## SPACE OF $m \times n$ MATRICES (E3)

Let  $\mathbb{F}$  be a field.

Then the "space of all  $m \times n$  matrices" with entries from  $\mathbb{F}$ , denoted by " $M_{m \times n}(\mathbb{F})$ ", is defined to be the set of all  $m \times n$  matrices with entries from  $\mathbb{F}$ .

Note that  $M_{m \times n}(\mathbb{F})$  is a vector space over  $\mathbb{F}$  with respect to the matrix addition and scalar multiplication operations.

Proof. This follows from verifying each of the conditions in D2.  $\blacksquare$

## FUNCTION SPACES (E4)

- $\exists$ : Let the set  $D \neq \emptyset$  be arbitrary, and let  $\mathbb{F}$  be a field.
- Then the space of all functions from  $D$  to  $\mathbb{F}$ , denoted by " $\mathbb{F}^D$ ", is defined to be the set of all functions of the form  $f: D \rightarrow \mathbb{F}$ .
- $\exists_2$ : Similarly, we can show that  $\mathbb{F}^D$  is a vector space over  $\mathbb{F}$  with respect to the operations of function addition

$$(f+g)(x) := f(x) + g(x) \quad \forall f, g \in \mathbb{F}^D, x \in D$$

and function scalar multiplication

$$(cf)(x) := c f(x) \quad \forall f \in \mathbb{F}^D, x \in D, c \in \mathbb{F}$$

Proof. Similar strategy to E3: verify each condition in D2 holds.

## POLYNOMIALS (D4)

### SET OF ALL POLYNOMIALS OF DEGREE AT MOST $n$ ( $D4(1)$ )

$\exists$ : Let  $\mathbb{F}$  be a field.

Then, we denote  $P_n(\mathbb{F})$  to be the set of all polynomials with coefficients from  $\mathbb{F}$  and of degree at most  $n$ ; ie

$$P_n(\mathbb{F}) = \left\{ \sum_{i=0}^n a_i x^i : a_i \in \mathbb{F}, i \in \{0, \dots, n\} \right\}.$$

### POLYNOMIAL SPACES (D4(2))

$\exists$ : Let  $\mathbb{F}$  be a field.

Then, we denote " $\mathbb{F}[x]$ " to be the set of all polynomials with coefficients from  $\mathbb{F}$ ; ie

$$\mathbb{F}[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i : a_i \in \mathbb{F}, i \in \mathbb{N} \cup \{0\} \right\}.$$

Then, we can show that  $\mathbb{F}[x]$  is a vector space over  $\mathbb{F}$  with respect to the operations of polynomial addition

$$(f+g)(x) = \sum_{i=0}^{\infty} (a_i + b_i) x^i \quad \forall f, g \in \mathbb{F}[x]$$

and polynomial scalar multiplication

$$(cf)(x) = \sum_{i=0}^{\infty} (ca_i) x^i \quad \forall f \in \mathbb{F}[x], c \in \mathbb{F}$$

Proof. Similar strategy to E4.

## BASIC PROPERTIES OF VECTOR SPACES (SI.2)

### CANCELLATION PROPERTY FOR VECTOR ADDITION (TI.1)

$\exists$ : Let  $V$  be a vector space.

Suppose there exists some  $x, y, z \in V$  such that

$$x+z = y+z$$

Then necessarily  $x=y$ .

Proof. Note that

$$\begin{aligned} x &= x+0 \\ &= x+(z+(-z)) \\ &= (x+z)+(-z) \\ &= (y+z)+(-z) \\ &= y+(z+(-z)) \end{aligned}$$

and so  $x=y$ , as required.  $\blacksquare$

### UNIQUENESS OF THE ZERO VECTOR IN VECTOR SPACES (CI.1.1 (1))

$\exists$ : Let  $V$  be a vector space.

Suppose  $0_1, 0_2 \in V$  are both zero vectors.

Then necessarily  $0_1 = 0_2$ .

Proof. This follows from the fact that  $V$  is an abelian group under addition.  $\blacksquare$

### UNIQUENESS OF ADDITIVE INVERSES IN VECTOR SPACES (CI.1.1 (2))

$\exists$ : Let  $V$  be a vector space.

Then for any  $x \in V$ , there exists one and only one vector  $y \in V$  that satisfies  $x+y=0$ .

Proof. This also follows from the fact that  $V$  is an abelian group under addition.  $\blacksquare$

### $0x=0 \quad \forall x \in V$ (TI.2 (1))

$\exists$ : Let  $V$  be a vector space over some field  $\mathbb{F}$ , and let  $0$  be the additive identity of  $\mathbb{F}$ .

Then, for any  $x \in V$ , necessarily  $0 \cdot x = 0$ .

Proof. This, again, follows from the fact that  $V$  is an abelian group under addition.  $\blacksquare$

### $a0=0 \quad \forall a \in \mathbb{F}$ (TI.2 (2))

$\exists$ : Let  $V$  be a vector space over some field  $\mathbb{F}$ , and let  $0$  be the zero vector of  $V$ .

Then, for any  $a \in \mathbb{F}$ , necessarily  $a \cdot 0 = 0$ .

Proof. This, again, follows from the fact that  $V$  is an abelian group under addition.  $\blacksquare$

### $(-a)x = -(ax) = a(-x) \quad \forall a \in \mathbb{F}, x \in V$ (TI.2 (3))

$\exists$ : Let  $V$  be a vector space over some field  $\mathbb{F}$ , and let  $a \in \mathbb{F}, x \in V$  be arbitrary.

Then necessarily  $(-a)x = -(ax) = a(-x)$ .

Proof. Proof is similar to the analog of this statement for rings (MATH145).  $\blacksquare$

# SUBSPACES (SI.3)

Let  $V$  be a vector space over some field  $\mathbb{F}$ . Then we say the subset  $W \subseteq V$  is a "subspace" of  $V$  if

- ①  $W \neq \emptyset$ ;

\* we usually check whether  $0 \in W$  to verify this claim. (R4)

- ② If  $x \in W$  and  $y \in W$ , then  $(x+y) \in W$ ; and

- ③ If  $c \in \mathbb{F}$  and  $x \in W$ , then  $cx \in W$ . (D6)

## SUBSPACES ARE VECTOR SPACES OVER $\mathbb{F}$ WITH RESPECT TO THE OPERATIONS OF $V$ (TI.3)

Let  $W$  be a subspace of a vector space  $V$  over some field  $\mathbb{F}$ .

Then  $W$  is also a vector space over  $\mathbb{F}$  under the operations of  $V$  restricted to  $W$ .

Proof. This follows from verifying the conditions in D2, taking into account the properties of subspaces.  $\square$

## $\{0\}$ AND $V$ ARE SUBSPACES OF $V$ (E8(1))

Let  $V$  be a vector space.

Then  $\{0\}$  and  $V$  itself are always subspaces of  $V$ .

Proof.  $\{0\}$  is vacuously a subspace, and  $V$  is trivially a subspace.  $\square$

## $P_2(\mathbb{R})$ IS A SUBSPACE OF $\mathbb{R}[x]$ (E8(2))

We can show that  $P_2(\mathbb{R})$  is a subspace of  $\mathbb{R}[x]$ .

Proof. This stems from the fact that:

- $P_2(\mathbb{R}) \subset \mathbb{R}[x]$  by definition;
- $0 \in P_2(\mathbb{R})$ ; and
- $P_2(\mathbb{R})$  is closed under the addition & scalar multiplication defined on  $\mathbb{R}[x]$ .  $\square$

$\{(a_{ij}) \in M_{n \times n}(\mathbb{F}) \mid \sum_{k=1}^n a_{kk} = 0\}$  IS A SUBSPACE

## OF $M_{n \times n}(\mathbb{F})$ (E8(3))

We can show that the set  $\{(a_{ij}) \in M_{n \times n}(\mathbb{F}) \mid \sum_{k=1}^n a_{kk} = 0\}$  is a subspace of  $M_{n \times n}(\mathbb{F})$ , where  $n \in \mathbb{N}$  is arbitrary.

Proof. Similar proof to the above.

$\{(a_{ij}) \in M_{n \times n}(\mathbb{F}) \mid \sum_{k=1}^n a_{kk} = 1\}$  IS NOT A

## SUBSPACE OF $M_{n \times n}(\mathbb{F})$ (E8(4))

We can show the set  $\{(a_{ij}) \in M_{n \times n}(\mathbb{F}) \mid \sum_{k=1}^n a_{kk} = 1\}$  is not a subspace of  $M_{n \times n}(\mathbb{F})$ .

Proof. Let  $a, b \in \{(a_{ij}) \in M_{n \times n}(\mathbb{F}) \mid \sum_{k=1}^n a_{kk} = 1\}$  be arbitrary.

Then, notice that

$$\begin{aligned} \sum_{k=1}^n (a+b)_{kk} &= \sum_{k=1}^n (a_{kk} + b_{kk}) \\ &= 1 + 1 \\ &= 2, \end{aligned}$$

so that  $a+b \notin \{(a_{ij}) \in M_{n \times n}(\mathbb{F}) \mid \sum_{k=1}^n a_{kk} = 1\}$ .  $\square$

## SUBSPACES OF $\mathbb{R}^2$ (E9(1))

Note that the subspaces of  $\mathbb{R}^2$  are

- ①  $\mathbb{R}^2$  itself;

- ②  $\{0_{\mathbb{R}^2}\} = \{(0,0)\}$ ; and

- ③ all lines in  $\mathbb{R}^2$  that pass through  $(0,0)$ .

## SUBSPACES OF $\mathbb{F}^2$ (E9(4a))

In general, for any field  $\mathbb{F}$ , the subspaces of

$$\mathbb{F}^2$$
 are

- ①  $\mathbb{F}^2$  itself;

- ②  $\{0\}$ ; and

- ③ all the "lines" in  $\mathbb{F}^2$  through  $0$ .

i.e. of the form  $\{(x,y) \in \mathbb{F}^2 \mid (x) = k(y), (y) \in \mathbb{F}^2\}$

## SUBSPACES OF $\mathbb{R}^3$ (E9(2))

Similarly, the subspaces of  $\mathbb{R}^3$  are

- ①  $\mathbb{R}^3$  itself;

- ②  $\{0_{\mathbb{R}^3}\} = \{(0,0,0)\}$ ;

- ③ all lines in  $\mathbb{R}^3$  that pass through  $(0,0,0)$ ; and

- ④ all planes in  $\mathbb{R}^3$  that pass through  $(0,0,0)$ .

## SUBSPACES OF $\mathbb{F}^3$ (E9(4b))

Similarly, for any field  $\mathbb{F}$ , the subspaces

$$\text{of } \mathbb{F}^3 \text{ are}$$

- ①  $\mathbb{F}^3$  itself;

- ②  $\{0\}$ ;

- ③ all the "lines" in  $\mathbb{F}^3$  through  $0$ ; and

i.e. of the form  $\{(x,y,z) \in \mathbb{F}^3 \mid (x) = k(y), (y) = l(z), (z) \in \mathbb{F}^3\}$  (E9(3))

- ④ all the "planes" in  $\mathbb{F}^3$  through  $0$ .

i.e. of the form  $\{(x,y,z) \in \mathbb{F}^3 \mid \exists a, b, c \in \mathbb{F} \text{ such that } ax+by+cz=0\}$ .

# LINEAR COMBINATIONS & SYSTEM OF LINEAR EQUATIONS (SI.4)

## LINEAR COMBINATION (D7(1))

$\exists_1$  Let  $V$  be a vector space over a field  $F$ , and let the subset  $S \subseteq V$  be such that  $S \neq \emptyset$ .

Then, we say a vector  $x \in V$  is a "linear combination" of vectors from  $S$  if there exists a finite number of vectors  $u_1, u_2, \dots, u_n \in S$  and scalars  $a_1, a_2, \dots, a_n \in F$  such that

$$x = a_1 u_1 + a_2 u_2 + \dots + a_n u_n,$$

where  $n \geq 1$ . (D7(1))

$\exists_2$  In this case, we also say that  $x$  is a linear combination of the vectors  $u_1, u_2, \dots, u_n$ .

## COEFFICIENTS OF A LINEAR COMBINATION (D7(2))

$\exists_1$  Let  $V$  be a vector space over some field  $F$ , and let the vector  $x \in V$  be a linear combination of the vectors  $u_1, u_2, \dots, u_n \in S$ , where  $S \subseteq V$  and  $S \neq \emptyset$ . Assume that  $x = a_1 u_1 + a_2 u_2 + \dots + a_n u_n$ , where  $a_1, a_2, \dots, a_n \in F$ .

Then we denote the scalars  $a_1, a_2, \dots, a_n \in F$  as the "coefficients" of the linear combination.

## SPAN (D7(3))

$\exists_1$  Let  $V$  be a vector space over some field  $F$ , and let the subset  $S \subseteq V$  be such that  $S \neq \emptyset$ .

Then, we define the "span" of  $S$ , denoted as "span( $S$ )", to be the set of all linear combinations of vectors in  $S$ .

$\exists_2$  Note that, for convenience, we define

$$\text{span}(\emptyset) = \{0\}.$$

## EXAMPLE 1: SPAN OF $(1,0,0)$ & $(0,1,0)$ IN $\mathbb{R}^3$ (E10(1))

$\exists_1$  Observe that in  $\mathbb{R}^3$ , the span of  $(1,0,0)$  &  $(0,1,0)$  in  $\mathbb{R}^3$  is

$$\{a(1,0,0) + b(0,1,0) : a, b \in \mathbb{R}\}$$

or more simply

$$\{(a, b, 0) : a, b \in \mathbb{R}\}.$$

## EXAMPLE 2: SPAN( $\{x^n : n \geq 1\}$ ) IN $\mathbb{Q}[x]$ (E10(2))

$\exists_1$  We can show that for the vector space  $\mathbb{Q}[x]$ , the span of  $S = \{x, x^2, \dots, x^n, \dots\}$  is the set of all polynomials in  $\mathbb{Q}[x]$  whose constant coefficient equals 0.

\* knowledge of elimination method is assumed.

## SPAN OF A FINITE AMOUNT OF VECTORS (E10(3.1))

$\exists_1$  Suppose  $V$  is a vector space over some field  $F$ , and let  $S \subseteq V$ . Further assume that

$$S = \{v_1, v_2, \dots, v_n\},$$

i.e. the size of  $S$  is finite.

Then, it follows that

$$\text{span}(S) = \{a_1 v_1 + a_2 v_2 + \dots + a_n v_n : a_i \in F \text{ for } i=1, 2, \dots, n\}.$$

## SPAN OF A COUNTABLE AMOUNT OF VECTORS (E10(3.2))

$\exists_1$  Suppose  $V$  is a vector space over some field  $F$ , and let  $S \subseteq V$ . Further assume that

$$S = \{v_1, v_2, \dots, v_n, \dots\},$$

i.e.  $|S| = |\mathbb{N}|$ .

Then, we can show that

$$\text{span}(S) = \text{span}(\{v_1\}) \cup \text{span}(\{v_1, v_2\}) \cup \dots \cup \text{span}(\{v_1, v_2, \dots, v_n\}) \cup \dots$$

or in other words, that

$$\text{span}(S) = \bigcup_{n=1}^{\infty} \text{span}(\{v_1, v_2, \dots, v_n\}).$$

## SPAN OF AN UNCOUNTABLE AMOUNT OF VECTORS (E10(3.3))

$\exists_1$  Suppose  $V$  is a vector space over some field  $F$ , and let  $S \subseteq V$ . Further assume that  $|S| > |\mathbb{N}|$ ; i.e. the size of  $S$  is uncountable. Then note that there are no "obvious" simplifications to the formula for  $\text{span}(S)$ .

## SPAN OF A SET IS ALWAYS A SUBSPACE OF THE PARENT VECTOR SPACE (T1.4)

$\exists_1$  Let  $V$  be a vector space over some field  $F$ , and let  $S \subseteq V$ . Then necessarily  $\text{span}(S)$  is a subspace of  $V$ .

Proof: This follows from verifying each subspace condition for  $\text{span}(S)$ .  $\square$

$\exists_2$  Moreover,  $\text{span}(S)$  is the "smallest possible" subspace of  $V$  that contains  $S$ , in the sense that

①  $S \subseteq \text{span}(S)$ ; and

② If  $W$  is any other subspace of  $V$  containing  $S$ , then  $\text{span}(S) \subseteq W$ .

## "GENERATES / SPANS" (D8)

$\exists_1$  Let  $V$  be a vector space, and let  $S \subseteq V$ .

Then, we say  $S$  "generates"  $V$ , or  $S$  "spans"  $V$ , if  $\text{span}(S) = V$ .

$\exists_2$  Note to prove  $\text{span}(S) = V$ , we just need to prove every vector in  $V$  can be written as a linear combination of vectors in  $S$ , since  $\text{span}(S) \subseteq V$  by definition.

(This follows from extensionality.) (R6)

# LINEAR INDEPENDENCE & DEPENDENCE (SI-5)

## LINEARLY DEPENDENT (D9(1))

$\text{💡}$  Let  $V$  be a vector space over some field  $F$ , and let  $S \subseteq V$ .

Then, we say  $S$  is "linearly dependent" if there exists a finite number of distinct vectors  $u_1, u_2, \dots, u_n \in S$  and scalars  $c_1, c_2, \dots, c_n \in F$ , where  $c_1, c_2, \dots, c_n$  are all not zero, such that

$$c_1 u_1 + c_2 u_2 + \dots + c_n u_n = 0.$$

$\text{💡}$  In this case, we also say the vectors of  $S$  are linearly dependent.

$\text{💡}$  Note that if  $S$  is finite, say  $S = \{u_1, u_2, \dots, u_n\}$ , then  $S$  is linearly dependent if and only if there exists a  $(c_1, c_2, \dots, c_n) \in F^n$ , where  $(c_1, c_2, \dots, c_n) \neq (0, 0, \dots, 0)$ , such that

$$c_1 u_1 + c_2 u_2 + \dots + c_n u_n = 0. \quad (\text{R7(4a)})$$

## LINEARLY INDEPENDENT (D9(2))

$\text{💡}$  Let  $V$  be a vector space over some field  $F$ , and let  $S \subseteq V$ .

Then, we say  $S$  is "linearly independent" if it is not "linearly dependent"; ie for every choice of distinct  $u_1, u_2, \dots, u_n \in S$ , if  $c_1, c_2, \dots, c_n \in F$  are scalars such that

$$c_1 u_1 + c_2 u_2 + \dots + c_n u_n = 0,$$

then necessarily  $c_1 = c_2 = \dots = c_n = 0$ .

$\text{💡}$  Similarly, if  $S$  is finite, say  $S = \{u_1, u_2, \dots, u_n\}$ , then  $S$  is linearly independent if and only if whenever  $(c_1, c_2, \dots, c_n) \in F^n$  are such that

$$c_1 u_1 + c_2 u_2 + \dots + c_n u_n = 0,$$

then necessarily  $c_1 = c_2 = \dots = c_n = 0$ .

## TRIVIAL REPRESENTATION OF 0 (R7(1))

$\text{💡}$  Note that for any vector space  $V$  and vectors  $u_1, u_2, \dots, u_n \in V$ , we denote the "trivial representation of  $0 \in V$ " as a linear combination of  $u_1, u_2, \dots, u_n$  by

$$0u_1 + 0u_2 + \dots + 0u_n = 0.$$

## EMPTY SET IS LINEARLY INDEPENDENT (R7(2))

$\text{💡}$  Note that the empty set,  $\emptyset$ , is vacuously linearly independent.

\* since linearly dependent sets must be non-empty by definition.

## $\{0\}$ IS LINEARLY DEPENDENT (R7(3))

$\text{💡}$  Note that the set  $\{0\}$  is linearly dependent, since  $1(0) = 0$  is a non-trivial representation of  $0$  as a linear combination of finitely many distinct vectors in  $S$ .

## $0 \in S \Rightarrow S$ IS LINEARLY DEPENDENT (R7(5))

$\text{💡}$  Note that any subset of a vector space that contains the zero vector is linearly dependent.

**EXAMPLE 1:**  $S = \{(0, 1, 1), (1, 0, 1), (1, 2, 3)\}$  IS LINEARLY DEPENDENT IN  $\mathbb{R}^3$  (E14)

$\text{💡}$  We can show that the set  $S = \{(0, 1, 1), (1, 0, 1), (1, 2, 3)\}$  is linearly dependent in  $\mathbb{R}^3$ .

Proof. We search for scalars  $a, b, c \in \mathbb{R}$ , not all 0, such that

$$a\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + b\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This reduces to solving the system

$$\begin{cases} b+c=0 \\ a+2c=0 \\ a+b+3c=0 \end{cases}$$

Simplifying, we get that

$$a = -2t, b = -t \text{ and } c = t,$$

where  $t \in \mathbb{R}$ .

For instance,  $(a, b, c) = (-2, -1, 1)$  is a solution in which not all of  $a, b, c$  are 0.

It follows that  $S$  is linearly dependent.  $\blacksquare$

**EXAMPLE 2:**  $S = \{1, x, x^2, x^3\}$  IS LINEARLY INDEPENDENT IN  $\mathbb{Z}_5[x]$  (E15)

$\text{💡}$  We can show that the set  $S = \{1, x, x^2, x^3\}$  is linearly independent in  $\mathbb{Z}_5[x]$ .

Proof. Note that if there exist  $a_0, a_1, a_2, a_3 \in \mathbb{Z}_5$  such that

$$a_0(1) + a_1x + a_2x^2 + a_3x^3 = 0,$$

then by definition necessarily  $a_0 = a_1 = a_2 = a_3 = 0$ , and this is sufficient to prove the claim.  $\blacksquare$

$S$  IS LINEARLY DEPENDENT  $\Leftrightarrow$

$S = \{0\}$  OR SOME VECTOR IN  $S$  IS A  
LINEAR COMBINATION OF OTHER VECTORS  
IN  $S$  (TI-S)

Let  $V$  be a vector space, and let  $S \subseteq V$ .  
Then  $S$  is linearly dependent if and only if  
 $S = \{0\}$  or some vector in  $S$  is a linear  
combination of other vectors in  $S$ .

Proof. We first prove the backward argument.

First, note we know why  $\{0\}$  is linearly  
dependent from a previous section.

So, suppose there exists a vector  $v \in S$   
such that

$$v = c_1 u_1 + c_2 u_2 + \dots + c_n u_n,$$

where  $c_i \in \mathbb{F}$  and  $u_i \in V$   $\forall i \in \{1, 2, \dots, n\}$ .

Without loss in generality, assume  $u_1, u_2, \dots, u_n$  are distinct.

By assumption, since  $v \notin \{u_1, u_2, \dots, u_n\}$ , necessarily

$u_1, u_2, \dots, u_n, v$  are distinct.

Finally, since

$$0 = (-1)v + c_1 u_1 + c_2 u_2 + \dots + c_n u_n,$$

and  $-1 \neq 0$ , it follows  $S$  is linearly dependent. \*

Next, we prove the forward argument.

Assume  $S$  is linearly dependent, so that there exist  
distinct  $u_1, u_2, \dots, u_n \in S$  and  $a_1, a_2, \dots, a_n \in \mathbb{F}$  (not all 0)  
such that

$$a_1 u_1 + a_2 u_2 + \dots + a_n u_n = 0.$$

Without loss in generality, assume  $a_1 \neq 0$   $\forall i \in \{1, 2, \dots, n\}$ .

Case 1:  $n=1$ .

Then  $a_1 u_1 = 0$ , and since  $a_1 \neq 0$  it follows that  $u_1 = 0$   
(since fields are integral domains, so the cancellation  
property applies.)

Hence  $0 \in S$ . If  $S = \{0\}$  we are done;  
otherwise, we can pick a  $v \in S \setminus \{0\}$ , and we  
can write  $0 = 0v$ , proving some vector in  $S$ , 0, can  
be written as a linear combination of another  
vector,  $v$ , in  $S$ .

Case 2:  $n > 1$ .

Then since  $a_1 \neq 0$ , we can solve for  $u_1$ :

$$u_1 = -a_1^{-1} [a_1 u_1 + a_2 u_2 + \dots + a_{n-1} u_{n-1}],$$

$$\therefore u_1 = (-a_1^{-1} a_1) u_1 + (-a_1^{-1} a_2) u_2 + \dots + (-a_1^{-1} a_{n-1}) u_{n-1},$$

showing  $u_1$  can be expressed as a linear

combination of other elements in  $S$ .

# BASES & DIMENSION (SI.6)

## BASIS (DIO)

Let  $V$  be a vector space.

Then, we say a subset  $S \subseteq V$  is a "basis" for  $V$  if  
①  $S$  is linearly independent; and  
②  $S$  spans  $V$ .

In this case, we also say that the vectors of  $S$  form a basis for  $V$ .

## STANDARD BASIS (C17)

In  $\mathbb{F}^n$ , define the "standard basis" for  $\mathbb{F}^n$  the subset

$$S = \{e_1, e_2, \dots, e_n\},$$

where  $e_j \in \mathbb{F}^n$  is the vector with  $j$ th coordinate 1 and other coordinates 0.

(It is easy to prove  $S$  is indeed a basis for  $\mathbb{F}^n$ )

In  $P_n(\mathbb{F})$ , define the "standard basis" for  $P_n(\mathbb{F})$  as the set

$$S = \{1, x, x^2, \dots, x^n\}.$$

(It is also easy to prove  $S$  is indeed a basis for  $P_n(\mathbb{F})$ ).

## UNIQUE REPRESENTATION OF ELEMENTS IN VECTOR SPACES UNDER A BASIS (TI.6)

Let  $\{v_1, v_2, \dots, v_n\}$  be a basis for a vector space  $V$ .

Then for every  $x \in V$ ,  $x$  can be uniquely represented as a linear combination of  $v_1, v_2, \dots, v_n$ ; ie there exists a unique  $n$ -tuple  $(a_1, a_2, \dots, a_n) \in \mathbb{F}^n$  such that

$$x = a_1 v_1 + a_2 v_2 + \dots + a_n v_n.$$

Proof. Existence: this follows from the fact that  $\{v_1, v_2, \dots, v_n\}$  spans  $V$  by definition.

Uniqueness: suppose there exists some  $b_1, b_2, \dots, b_n \in \mathbb{F}$  such that

$$x = a_1 v_1 + \dots + a_n v_n = b_1 v_1 + \dots + b_n v_n.$$

It follows that

$$0 = (a_1 - b_1)v_1 + \dots + (a_n - b_n)v_n,$$

and since  $\{v_1, v_2, \dots, v_n\}$  is linearly independent, necessarily  $a_i = b_i \quad \forall i \in \{1, 2, \dots, n\}$ .  $\square$

$V$  IS GENERATED BY  $S$ ,  $|S| = |\mathbb{N}|$

$\Rightarrow TCS$  IS ALSO A BASIS FOR  $V$  (TI.7)

Let  $V$  be a vector space, and assume that

$V$  is generated by a countable set  $S$ .

Then there exists a subset of  $S$  that is a basis for  $V$ .

Proof. If  $S = \emptyset$  or  $S = \{0\}$ , then  $\emptyset$  is a basis for  $V$  trivially.

Otherwise,  $S$  contains at least a non-zero vector.

Hence, we can write  $S$  as

$$S = \{v_1, v_2, \dots, v_n\} \text{ or } S = \{v_1, v_2, \dots\}.$$

By the WOP, we can pick the smallest index  $i \geq 1$  such that  $v_i \neq 0$ .

Then  $\{v_i\}$  is linearly independent.

Let  $i_2$  be the smallest index such that  $v_{i_2} \in \text{span}\{v_i\}$ .

Continue this "process" until we obtain the set

$$T = \{v_{i_k} \mid v_{i_k} \notin \text{span}\{v_{i_1}, \dots, v_{i_{k-1}}\}, k \geq 1\}.$$

Finally, we can prove  $T$  is a basis for  $V$ .

① Assume  $T$  is linearly dependent.

Then there exists  $a_1, a_2, \dots, a_k$ , all not 0, such that

$$a_1 v_{i_1} + \dots + a_k v_{i_k} = 0.$$

It follows that

$$v_{i_k} = -a_k^{-1} a_1 v_{i_1} - \dots - a_k^{-1} a_{k-1} v_{i_{k-1}},$$

contradicting the construction of  $T$ .

② We can prove by induction that  $\text{span}(S_k) = \text{span}(T_k) \quad \forall k \geq 1$ , where

$$S_k = \{v_1, v_2, \dots, v_k\} \text{ and } T_k = T \cap S_k = \{v_{i_q} \mid i_q \leq k\}.$$

Then, let  $x \in \text{span}(S)$ . Then  $x \in \text{span}(S_m)$  for some large  $m$ , so that  $x \in \text{span}(T_m) \subset \text{span}(T)$ .

Hence  $V \subseteq \text{span}(T)$ , and it follows that  $V = \text{span}(T)$ .  $\square$

## EVERY VECTOR SPACE HAS A BASIS

### (TI.8)

We can prove that every vector space has a basis.

(The proof uses Zorn's Lemma & maximal linearly independent subsets.)

## REPLACEMENT THEOREM (TI.9)

Suppose  $V$  is a vector space with a finite spanning set  $S$ . Let  $T$  be a linearly independent subset in  $V$ . Then

- ①  $|T| \leq |S|$ ; and
- ② There exists a set  $H \subseteq S$  containing exactly  $(|S|-|T|)$  vectors such that  $T \cup H$  generates  $V$ .

Proof. Let  $n = |S|$ , and let  $m = |T|$ . Then, when  $m=0$ , clearly  $m=0 \leq |S|$ . Next, assume the statement is true for some  $m \geq 0$ . This implies that if  $T_m \subseteq V$  is any linearly independent subset in  $V$  of size  $m$ , then  $m \leq n$  and there exists a set  $H_m \subseteq S$  containing exactly  $n-m$  vectors such that  $T_m \cup H_m$  generates  $V$ .

Let  $T_m = \{v_1, v_2, \dots, v_m\}$  and  $T = T_m \cup \{v_{m+1}\}$ , such that  $T$  is linearly independent and a subset of  $V$ .

Note that this implies  $T_m$  is also linearly independent.

Now, apply the induction hypothesis on  $T_m$  to get that  $n \geq m$ , and there exist  $(n-m)$  vectors  $w_{m+1}, \dots, w_n \in S$  such that

$\{v_1, \dots, v_m, w_{m+1}, \dots, w_n\}$  generates  $V$ .

Then, since  $n \geq m$ , either  $n=m$  or  $n > m$ .

If  $n=m$ ,  $\{v_1, \dots, v_m, w_{m+1}, \dots, w_n\} = \{v_1, \dots, v_m\}$ .

Thus,  $v_{m+1} \in \text{span}\{v_1, \dots, v_m\}$ , so by Theorem 1.5, the set  $\{v_1, \dots, v_m, v_{m+1}\}$  is linearly dependent.

But this is a contradiction; hence, it follows that  $n > m$ , so that  $n \geq m+1$ , proving ①.

Subsequently, write

$$v_{m+1} = a_1 v_1 + \dots + a_m v_m + a_{m+1} v_{m+1} + \dots + a_n w_n$$

for some scalars  $a_1, \dots, a_n \in \mathbb{F}$ .

Then, if  $a_{m+1} = \dots = a_n = 0$ , then we would get that  $v_{m+1} = a_1 v_1 + \dots + a_m v_m$ , which is a contradiction; hence, at least one of the scalars  $a_{m+1}, \dots, a_n$  must be non-zero.

Then, without loss in generality, assume  $a_{m+1} \neq 0$ .

It follows that

$$\begin{aligned} w_{m+1} &= -a_{m+1}^{-1} a_1 v_1 - \dots - a_{m+1}^{-1} a_m v_m - a_{m+1}^{-1} a_{m+1} v_{m+1} \\ &\quad - a_{m+1}^{-1} a_{m+2} w_{m+2} - \dots - a_{m+1}^{-1} a_n w_n. \end{aligned}$$

Let  $H = \{w_{m+2}, \dots, w_n\} \subset S$ . The above shows that

$w_{m+1} \in \text{span}(T \cup H)$ .

Moreover, since  $v_1, \dots, v_m \in T \subseteq \text{span}(T \cup H)$  and  $w_{m+2}, \dots, w_n \in H \subseteq \text{span}(T \cup H)$ , it follows that

$$V = \text{span}\{w_{m+1}, v_1, \dots, v_m, w_{m+2}, \dots, w_n\} \subseteq \text{span}(T \cup H).$$

But since  $\text{span}(T \cup H) \subseteq V$ , it follows that  $V = \text{span}(T \cup H)$ , completing the proof.  $\square$

## V IS FINITELY SPANNED $\Rightarrow$ ALL BASES OF V & H HAVE EQUAL CARDINALITIES (CI.9.1)

Suppose  $V$  is a finitely spanned vector space.

Then all bases of  $V$  are finite and have the same amount of elements.

Proof. Let  $S$  be a finite spanning set for  $V$ , and let  $B$  be an arbitrary basis for  $V$ . Then by definition,  $B$  is linearly independent.

By the Replacement Theorem,  $|B| \leq |S| < \infty$ .

Next, let  $B_1$  and  $B_2$  be two bases of  $V$ . Then, since  $B_1$  is linearly independent and  $B_2$

is a finite spanning set for  $V$ , by the Replacement Theorem necessarily  $|B_1| \leq |B_2|$ .

Similarly, since  $B_2$  is linearly independent and  $B_1$  is a finite spanning set for  $V$ , by the Replacement Theorem necessarily  $|B_2| \leq |B_1|$ .

It follows that  $|B_1| = |B_2|$ , and we are done.

## DIMENSION FINITE/INFINITE-DIMENSIONAL (DI.2)

We say a vector space  $V$  is "finite-dimensional" if it has a basis consisting of a finite number of vectors.

Otherwise, we say  $V$  is "infinite-dimensional".

## DIMENSION (DI.2)

Let  $V$  be a finite-dimensional vector space.

Then, the "dimension" of  $V$ , denoted as " $\dim V$ ", is defined to be the unique number of vectors in each basis for  $V$ .

By convention, we let  $\dim\{0\} = 0$ .

Examples:

- ①  $\dim \mathbb{F}^n = n$ ;
- ②  $\dim \mathbb{C}^n = 2n$ ;
- ③  $\dim M_{m \times n}(\mathbb{F}) = mn$ ; and
- ④  $\dim P_n(\mathbb{F}) = n+1$ . (E18)

## ANY FINITE SPANNING SET FOR $V$ CONTAINS AT LEAST $n$ VECTORS (C1.9.2(1))

Let  $V$  be a vector space with  $\dim V = n$ . Then if  $S$  is a finite spanning set for  $V$ , necessarily  $|S| \geq n$ .

Proof. By the Existence Theorem (T1.7), there exists a subset  $T$  of  $S$  that is a basis for  $V$ . Therefore  $|T| = \dim V = n$ , which implies that  $|S| \geq |T| = n$ .  $\square$

## $S$ GENERATES $V$ , $|V|=n \Rightarrow S$ IS A BASIS FOR $V$ (C1.9.2 (2))

Let  $V$  be a vector space with  $\dim V = n$ , and suppose  $S$  generates  $V$ , with  $|S|=n$ . Then  $S$  is a basis for  $V$ .

Proof. Again, by the Existence Theorem (T1.7), there exists some subset  $T \subseteq S$  such that  $T$  is a basis for  $V$ . By the above corollary,  $|T|=n$ , so that if  $|S|=n$ , necessarily  $S=T$ . It follows that  $S$  is a basis for  $V$ .  $\square$

## $S$ IS LINEARLY INDEPENDENT $\Rightarrow$ $S$ CONTAINS AT MOST $n$ VECTORS (C1.9.2(3))

Let  $V$  be a vector space, with  $\dim V = n$ . Suppose the subset  $S \subseteq V$  is linearly independent. Then  $S$  contains at most  $n$  vectors.

Proof. Applying the Replacement Theorem for the spanning set  $P$ , it follows that  $|S| \leq |P|$ , and since  $|P|=n$ , this tells us that  $|S| \leq n$ , as needed.  $\square$

## $S$ IS LINEARLY INDEPENDENT, $|S|=n$ $\Rightarrow S$ IS A BASIS FOR $V$ (C1.9.2 (4))

Let  $V$  be a vector space, with  $\dim V = n$ . Suppose the subset  $S \subseteq V$  is linearly independent and  $|V|=n$ . Then  $S$  is a basis for  $V$ .

Proof. Applying the Replacement Theorem for the spanning set  $P$  and the linearly independent set  $S$ , there must exist a subset  $H \subseteq P$  containing  $|P|-|S|=n-n=0$  vectors such that  $S \cup H$  generates  $V$ . But since  $|H|=0$ , hence  $H=\emptyset$ , so that  $S$  generates  $V$  (and hence is a basis for  $V$ ).  $\square$

## EVERY LINEARLY INDEPENDENT SUBSET OF $V$ CAN BE "EXTENDED" TO A BASIS OF $V$ (C1.9.2 (5))

Let  $V$  be a vector space, with  $\dim V = n$ . Suppose  $L = \{v_1, \dots, v_k\}$  is a linearly independent subset of  $V$ , where  $1 \leq k \leq n$ . Then there exists a HCV such that  $L \cup H$  is a basis of  $V$ .

Proof. If  $k=n$ , by C1.9.2(4)  $L$  is trivially a basis for  $V$ . If  $k < n$ , then by the Replacement Theorem for the spanning set  $P$  and  $L$ , there necessarily exists a subset  $H \subseteq P$  containing  $|P|-|L|=n-k$  vectors such that  $L \cup H$  generates  $V$ . By C1.9.2(1),  $|L \cup H| \geq n$ . But  $|L \cup H| \leq |L| + |H| = k + (n-k) = n$ , so that  $|L \cup H| = n$ . It follows by C1.9.2(2) that  $L \cup H$  is a basis for  $V$ .  $\square$

## $W$ IS A SUBSPACE OF $V$

$$\Rightarrow \dim W \leq \dim V ; \quad \dim W = \dim V \\ \Leftrightarrow W = V \quad (\text{C1.9.2 (6)})$$

Let  $W$  be a subspace of the vector space  $V$ . Then  $\dim W \leq \dim V$ , with equality occurring if and only if  $V=W$ .

Proof. If  $W=\{v\}$ , then  $\dim W=0 \leq \dim V$ . Otherwise,  $W$  contains a non-zero vector  $w_1$ . Then  $\{w_1\}$  is linearly independent. Continue to choose the vectors  $w_1, \dots, w_n \in W$  such that  $\{w_1, \dots, w_k\}$  is linearly independent. Note that this process cannot go on indefinitely, since  $\{w_1, \dots, w_k\}$  is also linearly independent in  $V$ . This implies that  $k \leq n$ . Next, by T1.5,  $W \subseteq \text{span}(\{w_1, \dots, w_k\}) = \text{span}(T)$ . Then, since  $T \subseteq W$ , necessarily  $\text{span}(T) \subseteq \text{span}(W) = W$ . It follows that  $W = \text{span}(T)$ , so that  $T$  is a basis (since it is also linearly independent), and  $\dim W = |T| = k \leq n = \dim V$ .

Note that if  $\dim V = n = \dim W$ , then a basis for  $W$  is also a linearly independent set containing  $n$  elements. Hence, by C1.9.2(4), that set is also a basis for  $V$ .  $\square$

## $W$ IS A SUBSPACE FOR $V \Rightarrow$ ANY BASIS OF $W$ CAN BE "EXTENDED" TO A BASIS IN $V$ (C1.9.2 (7))

Let  $W$  be a subspace of the vector space  $V$ , and let  $S$  be a basis of  $W$ . Then we can "extend"  $S$  to a basis in  $V$ .

Proof. By C1.9.2(6),  $\dim W \leq \dim V$ . Let  $T = \{w_1, \dots, w_n\}$  be a basis for  $W$ , so that  $T$  is linearly independent in  $W$ , which in turn implies  $T$  is linearly independent in  $V$ . So, by C1.9.2(5), we can "extend"  $T$  to a basis in  $V$ .  $\square$

# QUOTIENT SPACES (SI.7)

## COSET & REPRESENTATIVE (D13)

Let  $V$  be a vector space, and  $W$  be a subspace of  $V$ . Then, for a given  $x \in V$ , its corresponding "coset" of  $W$  in  $V$ , denoted as " $x+W$ ", is defined to be the set  $x+W = \{x+w : w \in W\}$ .

\* note that  $x+W \subseteq V$ .

In this case, we call " $x$ " a "representative" of the coset  $x+W$ .

## $x \equiv y \pmod{W}$ (D13)

Let  $V$  be a vector space, and let  $W$  be a subspace of  $V$ . Then, we write " $x \equiv y \pmod{W}$ " if and only if  $x-y \in W$ .

## $V/W$ (D13)

Let  $V$  be a vector space, and  $W$  a subspace of  $V$ . Then, we denote " $V/W$ " (ie " $V \pmod{W}$ ") as the set

$V/W = \{x+W : x \in V\}$ ;  
ie let  $V/W$  be the collection of cosets of  $W$  in  $V$ .

## $V/\{0\} = V$ (E19 (2))

For any vector space  $V$ , necessarily  $V/\{0\} = V$ .

Proof:  $V/\{0\} = \{0+x : x \in V\} = \{x : x \in V\} \therefore V/\{0\} = V$ .

## COSET TEST (P1)

Let  $W$  be a subspace of a vector space  $V$ , and let  $x, y \in V$  be arbitrary. Then  $x+W = y+W$  if and only if  $x-y \in W$ .

Proof: Similar to test for cosets in MATH 145.

## $\equiv \pmod{W}$ IS AN EQUIVALENCE RELATION ON $V$ (R8)

Note that the relation " $\equiv \pmod{W}$ " is an equivalence relation on  $V$ .

## ADDITION & MULTIPLICATION IN $V/W$ (D14)

Let  $V$  be a vector space over a field  $F$ , and let  $W$  be a subspace of  $V$ . Then, we can define an addition on  $V/W$  by

$(x+W) + (y+W) := ((x+y)+W)$ ;  
and a scalar multiplication on  $V/W$  by

$a(x+W) := (ax)+W$ ;

for any  $a \in F$  and  $x, y \in W$ .

Note that these addition and multiplication operations are well-defined. (L1)

Proof: Similar to proof for quotient groups/rings.

## $V/W$ IS A VECTOR SPACE

### (THE QUOTIENT SPACE OF $V$ BY $W$ ) (TI.10)

Let  $V$  be a vector space, and  $W$  a subspace of  $V$ .

Then the set  $V/W$  is a vector space over  $F$  with the operations of coset addition and scalar multiplication, denoted as "the quotient space of  $V$  by  $W$ ".

Proof: Verify all 8 conditions. (VS 1-8).

## BASIS FOR QUOTIENT SPACES (TI.11)

Let  $V$  be a vector space with  $\dim V = n$ , and let  $W$  be a subspace of  $V$  such that  $\dim W = k$ . Let  $\{v_1, \dots, v_n\}$  be a basis for  $V$ , such that  $\{v_1, \dots, v_k\}$  is a basis for  $W$ .

Then,

① The set  $\{v_{k+1}+W, \dots, v_n+W\}$  is a basis for  $V/W$ ; and

②  $\dim(V/W) = \dim V - \dim W$ .

Proof: To prove ①, we show  $\{v_{k+1}+W, \dots, v_n+W\}$  is both linearly independent and generates  $V/W$ , giving us our basis.

It follows that

$$\begin{aligned}\dim(V/W) &= |\{v_{k+1}+W, \dots, v_n+W\}| \\ &= n - (k+1) \\ &= n - k \\ \therefore \dim(V/W) &= \dim V - \dim W.\end{aligned}$$

$\dim V \geq \infty, \dim W \geq \infty \Rightarrow \dim V/W \geq \infty$  (R9)

Let  $V$  be an infinite-dimensional vector space, and let  $W$  be an infinite-dimensional subspace of  $V$ .

Then, note that it is not necessarily the case that  $\dim(V/W) \geq \infty$ .

Example: let  $V = \mathbb{F}^\infty$  &  $W = \{(0, x_2, \dots) : x_2 \in \mathbb{F}\}$ . Note that each element of  $V/W$  is simply "determined" by the value of the first coordinate  $x_1$ , so that  $\dim(V/W) = 1$ .

# SUMS & INTERNAL DIRECT SUMS OF SUBSPACES (SL8)

## SUM OF SUBSPACES (DIS)

Let  $V$  be a vector space over  $\mathbb{F}$ , and let  $W_1, W_2$  be subspaces of  $V$ . Then, we define the "sum" of  $W_1$  and  $W_2$ , denoted as  $W_1 + W_2$ , to be the set

$$W_1 + W_2 := \{v_1 + v_2 : v_1 \in W_1, v_2 \in W_2\}.$$

## INDEPENDENT/DISJOINT (DIS)

Let  $V$  be a vector space, and let  $W_1, W_2$  be subspaces of  $V$ . Then, we say  $W_1$  and  $W_2$  are "independent", or "disjoint", if and only if  $W_1 \cap W_2 = \{0\}$ .

## (INTERNAL) DIRECT SUM (DIS)

Let  $V$  be a vector space, and let  $W_1, W_2$  be independent subspaces of  $V$ .

Then, we define the "(internal) direct sum" of  $W_1$  and  $W_2$ , denoted as  $W_1 \oplus W_2$ , to be the set

$$W_1 \oplus W_2 = W_1 + W_2.$$

\* ie " $\oplus$ " is the notation for "+" used when  $W_1$  &  $W_2$  are independent.

Note that  $W_1 \oplus W_2$  is well-defined, as long as  $W_1 \cap W_2 = \{0\}$ . (R10)

## $W_1 + W_2$ IS THE "SMALLEST" SUBSPACE CONTAINING $W_1$ & $W_2$ (L2 (2))

Let  $V$  be a vector space, and let  $W_1, W_2$  be subspaces of  $V$ .

Then  $W_1 + W_2$  is necessarily the smallest subspace of  $V$  containing  $W_1$  and  $W_2$ .

Proof. First, we prove  $W_1 + W_2$  is a subspace of  $V$ .

Let  $(v_1 + v_2), (u_1 + u_2) \in W_1 + W_2$  and  $a \in \mathbb{F}$ , where  $v_1, v_2 \in W_1$  and  $u_1, u_2 \in W_2$ .

Then, since  $W_1$  and  $W_2$  are subspaces of  $W_1 + W_2$ , necessarily  $v_1 + u_1 \in W_1 + W_2$  and  $v_2 + u_2 \in W_1 + W_2$ .

so that

$$(v_1 + v_2) + (u_1 + u_2) = (v_1 + u_1) + (v_2 + u_2) \in W_1 + W_2.$$

Moreover, since  $av_1 \in W_1$  and  $av_2 \in W_2$ , necessarily

$$a(v_1 + v_2) = av_1 + av_2 \in W_1 + W_2.$$

proving  $W_1 + W_2$  is closed under addition and scalar multiplication.

Then, since  $v_1 = v_1 + 0 \in W_1 + W_2 \quad \forall v_1 \in W_1$  &  $v_2 = 0 + v_2 \in W_1 + W_2 \quad \forall v_2 \in W_2$ , it follows that

$$W_1 \subseteq W_1 + W_2 \text{ and } W_2 \subseteq W_1 + W_2.$$

Finally, let  $Y$  be a subspace of  $V$  that contains both  $W_1$  &  $W_2$ .

Since  $Y$  is closed under addition,  $v_1 + v_2 \in Y$

for every  $v_1 \in W_1$  and  $v_2 \in W_2$  necessarily.

It follows that  $W_1 + W_2 \subseteq Y$ , completing the proof.

$$V = W_1 \oplus W_2 \iff \forall v \in V : \exists \text{ unique } w_1 \in W_1,$$

$$w_2 \in W_2 \ni v = w_1 + w_2 \quad (\text{L2 (3)})$$

Let  $V$  be a vector space, and let  $W_1$  and  $W_2$  be subspaces of  $V$ .

Then  $W_1 \oplus W_2 = V$  if and only if for every vector  $v \in V$ , there exist unique elements  $w_1 \in W_1$  and  $w_2 \in W_2$  such that  $v = w_1 + w_2$ .

Proof. ( $\Rightarrow$ ) Since  $V = W_1 \oplus W_2$ , necessarily  $V = W_1 + W_2$ , and  $W_1 \cap W_2 = \{0\}$ .

let  $v \in V$ , and note that since  $V = W_1 + W_2$ , it implies that  $v \in W_1 + W_2$ .

So, by definition, there exist some  $w_1 \in W_1, w_2 \in W_2$  such that  $v = w_1 + w_2$ .

Next, suppose we have  $v = w'_1 + w'_2$  for some  $w'_1 \in W_1$  and  $w'_2 \in W_2$ .

$$\text{Then } 0 = (w_1 + w_2) - (w'_1 + w'_2) = (w_1 - w'_1) + (w_2 - w'_2).$$

Since  $w_1, w'_1 \in W_1$  &  $w_2, w'_2 \in W_2$ , necessarily  $w_1 - w'_1 \in W_1$  &  $w_2 - w'_2 \in W_2$  also, so that

$$(w_1 - w'_1) = w'_2 - w_2 \in W_1 \cap W_2 = \{0\},$$

Hence  $w_1 - w'_1 = w'_2 - w_2 = 0$ , implying that  $w_1 = w'_1$  &  $w_2 = w'_2$ , proving uniqueness. \*

( $\Leftarrow$ ) By assumption, every vector  $v \in V$  can be written as  $v = w_1 + w_2$  for some  $w_1 \in W_1$  &  $w_2 \in W_2$ . Hence  $V \subseteq W_1 + W_2$ , and by L2(2) necessarily  $W_1 + W_2 \subseteq V$ ; so  $V = W_1 + W_2$ .

Next, let  $x \in W_1 \cap W_2$ . Then  $-x \in W_1 \cap W_2$ .

Then, note that

$$0 = 0 + 0 = x + (-x) \in W_1 + W_2,$$

and due to the uniqueness assumption, necessarily  $x = 0$ .

Thus  $W_1 \cap W_2 = \{0\}$ , so that  $V = W_1 \oplus W_2$ . ■

$$\dim(W_1), \dim(W_2) < \infty \Rightarrow \dim(W_1 + W_2) < \infty \text{ &} \\ \dim(W_1) + \dim(W_2) = \dim(W_1 + W_2) + \dim(W_1 \cap W_2)$$

### (T1.12 (1))

Let  $V$  be a vector space over some field  $\mathbb{F}$ , and let  $W_1, W_2$  be finite dimensional subspaces of  $V$ .

Then necessarily  $W_1 + W_2$  is finite dimensional, and  $\dim(W_1) + \dim(W_2) = \dim(W_1 + W_2) + \dim(W_1 \cap W_2)$ .

Proof. First, note  $W_1 \cap W_2$  is a subspace of  $W_1$  (A2), so that  $\dim(W_1 \cap W_2) \leq \dim(W_1) < \infty$  (C1.9.2(6)).

Next, let  $\{u_1, u_2, \dots, u_k\}$  be a basis for  $W_1 \cap W_2$ .

Extend this basis to get the bases

$S_1 = \{u_1, \dots, u_k, v_1, \dots, v_m\}$  of  $W_1$  and  $S_2 = \{u_1, \dots, u_k, z_1, \dots, z_p\}$  of  $W_2$ , which we can always do by C1.9.2(5)).

Let  $S = \{u_1, \dots, u_k, v_1, \dots, v_m, z_1, \dots, z_p\}$ .

We claim  $S$  is a basis for  $W_1 + W_2$ .

Indeed, consider

$$a_1 u_1 + \dots + a_k u_k + b_1 v_1 + \dots + b_m v_m + c_1 z_1 + \dots + c_p z_p = 0 \quad \text{--- (2)}$$

for some scalars  $a_1, \dots, a_k, b_1, \dots, b_m, c_1, \dots, c_p$ .

Then

$$b_1 v_1 + \dots + b_m v_m = -a_1 u_1 - \dots - a_k u_k - c_1 z_1 - \dots - c_p z_p.$$

Since the RHS is a linear combination of vectors in  $W_2$ , the RHS  $\in W_2$ ; and since the LHS is a linear combination of vectors in  $W_1$ , the LHS  $\in W_1$ .

Thus  $b_1 v_1 + \dots + b_m v_m \in W_1 \cap W_2$ .

Next, since  $\{u_1, \dots, u_k\}$  is a basis for  $W_1 \cap W_2$ , there exist scalars  $d_1, \dots, d_k$  such that

$$b_1 v_1 + \dots + b_m v_m = d_1 u_1 + \dots + d_k u_k.$$

So

$$b_1 v_1 + \dots + b_m v_m - d_1 u_1 - \dots - d_k u_k = 0.$$

Since  $\{u_1, \dots, u_k, v_1, \dots, v_m\}$  is a basis for  $W_1$ , necessarily  $b_1 = \dots = b_m = d_1 = \dots = d_k = 0$ .

Substitute  $b_1 = \dots = b_m$  into (2) to get that

$$a_1 u_1 + \dots + a_k u_k + c_1 z_1 + \dots + c_p z_p = 0.$$

Then, since  $\{u_1, \dots, u_k, z_1, \dots, z_p\}$  is a basis for  $W_2$ , we have

$$a_1 = \dots = a_k = c_1 = \dots = c_p = 0,$$

proving  $S$  is linearly independent.

Subsequently, let  $x+y \in W_1 + W_2$  be arbitrary, where  $x \in W_1$  and  $y \in W_2$ .

Then, since  $S_1$  and  $S_2$  are bases for  $W_1$  and  $W_2$  respectively, we can write  $x$  and  $y$  as linear combinations of vectors in  $S_1$  and  $S_2$ , respectively:

$$x = a_1 u_1 + \dots + a_k u_k + b_1 v_1 + \dots + b_m v_m; \quad \text{--- (3)}$$

$$y = d_1 u_1 + \dots + d_k u_k + c_1 z_1 + \dots + c_p z_p;$$

where  $a_1, \dots, a_k, b_1, \dots, b_m, d_1, \dots, d_k, c_1, \dots, c_p \in \mathbb{F}$ .

Hence

$$x+y = (a_1+d_1)u_1 + \dots + (a_k+d_k)u_k + (b_1+c_1)z_1 + \dots + (b_m+c_p)z_p,$$

which is sufficient to show  $x+y \in \text{span}(S)$ .

Thus  $W_1 + W_2 \subseteq \text{span}(S)$ , and since  $\text{span}(S) \subseteq W_1 + W_2$ ,

by definition, it follows that  $W_1 + W_2 = \text{span}(S)$ ,

verifying that  $S$  is indeed a basis for

$W_1 + W_2$ .

In particular,

$$\begin{aligned} \dim(W_1 + W_2) + \dim(W_1 \cap W_2) &= |S| + k \\ &= m + p + k + k \\ &= (m+k) + (p+k) \\ &= \dim W_1 + \dim W_2. \end{aligned}$$

$$\dim(V) < \infty, \quad W_1 \oplus W_2 = V \Rightarrow \dim W_1 + \dim W_2 = \dim V$$

### (T1.12 (2))

Let  $V$  be a vector space over  $\mathbb{F}$ , and let  $W_1, W_2$  be finite-dimensional subspaces of  $V$ .

Suppose further that  $V$  itself is finite-dimensional, and  $W_1 \oplus W_2 = V$ .

Then necessarily  $\dim W_1 + \dim W_2 = \dim V$ .

Proof. Since  $W_1 \oplus W_2 = V$ , necessarily  $W_1 \cap W_2 = \{0\}$ .

So, by T1.12(1), it follows that

$$\begin{aligned} \dim W_1 + \dim W_2 &= \dim(W_1 + W_2) + \dim(W_1 \cap W_2) \\ &= \dim(V) + 0 \end{aligned}$$

$$\therefore \dim W_1 + \dim W_2 = \dim(V). \quad \blacksquare$$

## COMPLEMENTARY SUBSPACES (D15)

Let  $V$  be a vector space, and let  $W$  be a subspace of  $V$ .

Then a subspace  $W'$  of  $V$  is said to be a "complementary subspace" to  $W$  if  $W \oplus W' = V$ ; ie

$$\textcircled{1} \quad W \cap W' = \{0\}; \quad \text{and}$$

$$\textcircled{2} \quad W + W' = V.$$

$$\dim W + \dim W' = \dim V$$

Let  $V$  be a vector space, and let  $W$  be a subspace of  $V$ .

$W'$  be a complementary subspace to  $W$ .

Then necessarily  $\dim W + \dim W' = \dim V$ .

Proof. Follows directly from T1.12(2).

## EXISTENCE OF COMPLEMENTARY SUBSPACES (R11(1))

Let  $V$  be a vector space, and let  $W$

be a subspace of  $V$ .

Then there always exists a complementary subspace  $W'$  to  $W$  of  $V$  such that  $W \oplus W' = V$ .

Proof. First, note that every linearly independent set can be extended to a basis  $V$  that has a countable spanning set (A3).

Hence, every linearly independent subset of  $V$  can be extended to a basis for  $V$ .

It follows that every subspace  $W$  of  $V$  has a complementary subspace  $W'$ .  $\blacksquare$

## NON-UNIQUENESS OF COMPLEMENTARY SUBSPACES

### (R11(2))

Note that complementary subspaces of a given vector space  $V$  are not necessarily unique.

eg  $V = \mathbb{R}^3$ ,  $W = \{(1,0,0), (0,1,0)\}$ ,  $W'_1 = \{(0,0,1)\}$ ,  $W'_2 = \{(0,0,-1)\}$ ;

observe that both  $W'_1$  and  $W'_2$  are complementary subspaces to  $W$ .

# Chapter 2:

## Linear Transformations and Matrices

### LINEAR TRANSFORMATIONS (S2.1)

**💡** Let  $V$  and  $W$  be vector spaces over the same field  $\mathbb{F}$ .

Then, we say the function  $T: V \rightarrow W$  is a "linear transformation" from  $V$  to  $W$  if

$$(L1) \rightarrow T(x+y) = T(x) + T(y) \quad \forall x, y \in V; \text{ and}$$

$$(L2) \rightarrow T(cx) = cT(x) \quad \forall x \in V, c \in \mathbb{F}. \quad (D16)$$

**💡** In this case, we say the function  $T: V \rightarrow W$  is "linear".

**T IS LINEAR ( $\Rightarrow T(cx+ty) = cT(x) + T(y)$ ) (P2)**

**💡** Let the function  $T: V \rightarrow W$ , where  $V$  and  $W$  are vector spaces over the same field  $\mathbb{F}$ .

Then  $T$  is linear if and only if  $T(cx+ty) = cT(x) + T(y)$  for all  $x, y \in V$  and  $c \in \mathbb{F}$ .

**ZERO TRANSFORMATION (E23(1a))**

**💡** For any vector spaces  $V$  and  $W$ , the "zero transformation", given by " $T_0: V \rightarrow W$ ", is defined by  $T_0(x) = 0 \quad \forall x \in V$ .

**IDENTITY TRANSFORMATION (E23(1b))**

**💡** For any vector space  $V$ , the "identity transformation"  $I_V: V \rightarrow V$  is given by  $I_V(x) = x \quad \forall x \in V$ .

**T:  $V \rightarrow \mathbb{F}^n$  BY  $T(a_1v_1 + \dots + a_nv_n) := (a_1, \dots, a_n)$  (E23(3))**

**💡** Let  $V$  be a finite-dimensional vector space over  $\mathbb{F}$ , and let  $\{v_1, v_2, \dots, v_n\}$  be a basis for  $V$ . Then, the mapping

$T: V \rightarrow \mathbb{F}^n$  by  $T(a_1v_1 + \dots + a_nv_n) := (a_1, \dots, a_n)$

is linear.

**T:  $\mathbb{F}^n \rightarrow \mathbb{F}^k$ ,  $T(x_1, \dots, x_n) := (x_1, \dots, x_k)$  (E23(4))**

**💡** Let  $\mathbb{F}$  be a field, and suppose  $1 \leq k < n$ .

Then the projection mapping

$T: \mathbb{F}^n \rightarrow \mathbb{F}^k$  by  $T(x_1, \dots, x_n) := (x_1, \dots, x_k)$

is linear.

**T(0) = 0 (P3(1))**

**💡** Let  $T: V \rightarrow W$  be linear. Then necessarily  $T(0) = 0$ .

$$\text{Proof. } T(0) = T(0+0) = T(0) + T(0); \\ \text{Thus } 0 = T(0) + T(0) - T(0) = T(0). \quad \square$$

**T(x-y) = T(x) - T(y) (P3(2))**

**💡** Let  $T: V \rightarrow W$  be linear. Then necessarily  $T(x-y) = T(x) - T(y) \quad \forall x, y \in V$ .

$$\text{Proof. } T(x-y) = T(x) + T(-y) \\ = T(x) + (-1)T(y) \\ \therefore T(x-y) = T(x) - T(y). \quad \square$$

**$T(a_1x_1 + \dots + a_nx_n) = a_1T(x_1) + \dots + a_nT(x_n)$  (P3(3))**

**💡** Let  $T$  be linear, and  $a_1, \dots, a_n \in \mathbb{F}$  and  $x_1, \dots, x_n \in V$  be arbitrary.

Then necessarily

$$T(a_1x_1 + \dots + a_nx_n) = a_1T(x_1) + \dots + a_nT(x_n).$$

**$\{v_1, \dots, v_n\}$  IS A BASIS FOR  $V$ ,  $\{w_1, \dots, w_n\}$  ARE ELEMENTS FOR  $W \Rightarrow \exists$  A UNIQUE LINEAR MAPPING  $T: V \rightarrow W$   $\exists T(v_k) = w_k$  (T2.1)**

**💡** Let  $\{v_1, \dots, v_n\}$  be a basis for a vector space  $V$ , and let  $\{w_1, \dots, w_n\}$  be arbitrary elements of another vector space  $W$ .

Then there exists a unique linear mapping  $T: V \rightarrow W$  such that

$$T(v_1) = w_1, \dots, T(v_n) = w_n.$$

**Proof.** Let  $v \in V$  be arbitrary. Since  $\{v_1, v_2, \dots, v_n\}$  is a basis for  $V$ , there must exist  $a_1, \dots, a_n \in \mathbb{F}$  such that  $v = a_1v_1 + \dots + a_nv_n$ .

$$\text{Let } T(v) = T(a_1v_1 + \dots + a_nv_n) = a_1w_1 + \dots + a_nw_n \text{ (by P3(3)).}$$

Then, by construction, for any  $1 \leq k \leq n$ , we have

$$\begin{aligned} T(v_k) &= T(a_1v_1 + \dots + a_kv_k + a_{k+1}v_{k+1} + \dots + a_nv_n) \\ &= a_1w_1 + \dots + a_kw_k + a_{k+1}w_{k+1} + \dots + a_nw_n \\ &= w_k. \end{aligned}$$

Proving uniqueness.

Next, suppose there exists another linear mapping  $L: V \rightarrow W$  satisfying  $L(v_i) = w_i, \dots, L(v_n) = w_n$ .

Let  $v = a_1v_1 + \dots + a_nv_n$ , where  $v \in V$  and  $a_1, \dots, a_n \in \mathbb{F}$ .

Then

$$\begin{aligned} L(v) &= L(a_1v_1 + \dots + a_nv_n) \\ &= a_1L(v_1) + \dots + a_nL(v_n) \\ &= a_1w_1 + \dots + a_nw_n \\ &= a_1T(v_1) + \dots + a_nT(v_n) \\ &= T(a_1v_1 + \dots + a_nv_n) \end{aligned}$$

$$\therefore L(v) = T(v).$$

Hence  $L(v) = T(v) \quad \forall v \in V$ , so that  $T = L$ , proving uniqueness.  $\square$

**💡** It also follows that we have

$$T(a_1v_1 + \dots + a_nv_n) = a_1w_1 + \dots + a_nw_n. \quad (C2.1.1)$$

## NULL SPACE / KERNEL (D17(1))

Let  $V$  and  $W$  be vector spaces, and let  $T: V \rightarrow W$  be linear. Then the "null space" of  $T$ , or the "kernel" of  $T$ , denoted as " $N(T)$ ", is defined to be the set

$$N(T) := \{x \in V \mid T(x) = 0\}.$$

## RANGE / IMAGE (D17(2))

Let  $V$  and  $W$  be vector spaces, and let  $T: V \rightarrow W$  be linear.

Then the "range" of  $T$ , or the "image" of  $T$ , denoted as " $R(T)$ ", is defined to be the set

$$R(T) := \{T(x) : x \in V\}.$$

## $N(T)$ IS A SUBSPACE OF $V$ (T2.2)

Let  $T: V \rightarrow W$  be linear.

Then necessarily  $N(T)$  is a subspace of  $V$ .

## $R(T)$ IS A SUBSPACE OF $W$ (T2.2)

Let  $T: V \rightarrow W$  be linear.

Then necessarily  $R(T)$  is a subspace of  $W$ .

## $\{v_1, \dots, v_n\}$ IS A BASIS FOR $V \Rightarrow$

$$\text{span}(\{T(v_1), \dots, T(v_n)\}) = R(T) \quad (\text{T2.3})$$

Let  $V$  and  $W$  be vector spaces, and let  $T: V \rightarrow W$  be linear. Suppose the set  $\{v_1, \dots, v_n\}$  is a basis for  $V$ . Then necessarily  $\{T(v_1), \dots, T(v_n)\}$  generates  $R(T)$ .

## NULLITY (D18)

Let  $T: V \rightarrow W$  be linear, and suppose that  $\dim(N(T)) < \infty$ .

Then, we define the "nullity" of  $T$ , denoted by "nullity( $T$ )", to be equal to

$$\text{nullity}(T) = \dim(N(T)).$$

## RANK (D18)

Let  $T: V \rightarrow W$  be linear, and suppose that  $\dim(R(T)) < \infty$ .

Then, we define the "rank" of  $T$ , denoted by "rank( $T$ )", to be equal to

$$\text{rank}(T) = \dim(R(T)).$$

## RANK-NULLITY THEOREM (T2.4)

Let  $V$  and  $W$  be vector spaces, and let  $T: V \rightarrow W$  be linear with

$\dim(V) < \infty$ .

Then necessarily

$$\text{rank}(T) + \text{nullity}(T) = \dim(V).$$

Proof. Since  $N(T)$  is a subspace of  $V$  (T2.2)

and  $\dim V < \infty$ , by C19.2 (6) necessarily

$$\text{nullity}(T) \leq \dim(V) < \infty.$$

Then, let  $\text{nullity}(T) = k$ , and suppose that

$\{v_1, \dots, v_k\}$  is a basis for  $N(T)$ .

We know that we can "extend"  $\{v_1, \dots, v_k\}$  to get a basis for  $V$ ,  $\{v_1, \dots, v_n\}$ , so let us do so.

Next, we claim  $\{T(v_{k+1}), \dots, T(v_n)\}$  is a basis for  $R(T)$ .

First, we show  $\{T(v_{k+1}), \dots, T(v_n)\}$  spans  $R(T)$ .

By T2.2,  $R(T) = \text{span}(\{T(v_1), \dots, T(v_k), T(v_{k+1}), \dots, T(v_n)\})$ .

Then, since  $\{v_1, \dots, v_n\}$  is a basis for  $N(T)$ , necessarily

$$T(v_1) = \dots = T(v_k) = 0.$$

Hence,

$$R(T) = \text{span}(\{T(v_{k+1}), \dots, T(v_n)\}),$$

as needed.

Next, we show  $\{T(v_{k+1}), \dots, T(v_n)\}$  is linearly independent. Consider

$$c_{k+1}T(v_{k+1}) + \dots + c_nT(v_n) = 0, \quad \text{where } c_{k+1}, \dots, c_n \in \mathbb{C}$$

$$\Rightarrow T(c_{k+1}v_{k+1} + \dots + c_nv_n) = 0.$$

Hence  $c_{k+1}v_{k+1} + \dots + c_nv_n \in N(T)$ ; then, since  $\{v_1, \dots, v_n\}$  is a basis for  $N(T)$ , there exist  $d_1, \dots, d_n \in \mathbb{C}$  such that

$$c_{k+1}v_{k+1} + \dots + c_nv_n = d_1v_1 + \dots + d_nv_n.$$

$$\Rightarrow -d_1v_1 - \dots - d_nv_n + c_{k+1}v_{k+1} + \dots + c_nv_n = 0.$$

Since  $\{v_1, \dots, v_n\}$  is a basis for  $V$ , consequently

$$d_1 = \dots = d_n = c_{k+1} = \dots = c_n = 0,$$

showing  $\{T(v_{k+1}), \dots, T(v_n)\}$  is linearly independent.

Consequently,

$$\text{rank}(T) + \text{nullity}(T) = \dim(R(T)) + \dim(N(T))$$

$$= k + (n - (k+1) + 1)$$

$$= n$$

$$\therefore \text{rank}(T) + \text{nullity}(T) = \dim(V). \quad \blacksquare$$

## ONE-TO-ONE (1-1) (D19)

Let  $T: V \rightarrow W$  be linear.

Then, we say  $T$  is "one-to-one" if, for any  $x, y \in V$ ,  $T(x) = T(y)$  implies  $x = y$ .

## ONTO (D19)

Let  $T: V \rightarrow W$  be linear.

Then, we say  $T$  is "onto" if

$$R(T) = V.$$

## ISOMORPHISM (D19)

Let  $T: V \rightarrow W$  be linear.

Then, we say  $T$  is an "isomorphism" if it is both one-to-one and onto.

We say  $V$  is "isomorphic" to  $W$  if

an isomorphism  $T: V \rightarrow W$  exists, (D20)

and denote this by the notation

$$V \cong W.$$

# T IS 1-1 ( $\Leftrightarrow$ ) $N(T) = \{0\}$ (L3)

Let  $T: V \rightarrow W$  be linear.

Then  $T$  is one-to-one if and only if  $N(T) = \{0\}$ .

Proof. ( $\Rightarrow$ ) Suppose  $T$  is one-to-one.

Let  $x \in V$  be such that  $T(x) = 0$ .

Then since  $T(0) = 0 = T(x)$ , by definition

$x = 0$ , so that  $N(T) = \{0\}$ .

( $\Leftarrow$ ) Suppose  $N(T) = \{0\}$ . Consider  $x, y \in V$  such that  $T(x) = T(y)$ .

$$T(x-y) = T(x) - T(y) = 0,$$

so that  $x-y \in N(T)$ ;

hence  $x-y = 0$ , so that  $x=y$  (and hence

$T$  is 1-1).  $\square$

$\{v_1, \dots, v_n\}$  IS A BASIS FOR  $V \Rightarrow$

$T$  IS ISOMORPHIC ( $\Leftrightarrow$ )  $\{T(v_1), \dots, T(v_n)\}$  IS A BASIS FOR  $W$  (T2.5)

Let  $V$  and  $W$  be vector spaces over a field  $\mathbb{F}$ , with  $\dim V < \infty$ .

Let  $\{v_1, \dots, v_n\}$  be a basis for  $V$ , and let  $T: V \rightarrow W$  be linear.

Then  $T$  is an isomorphism if and only if  $\{T(v_1), \dots, T(v_n)\}$  is a basis for  $W$ .

Proof. ( $\Rightarrow$ ) Consider

$$c_1 T(v_1) + \dots + c_n T(v_n) = 0$$

$$\Rightarrow T(c_1 v_1 + \dots + c_n v_n) = 0.$$

Since  $T$  is one-to-one by definition, hence

$$c_1 v_1 + \dots + c_n v_n = 0,$$

and as  $\{v_1, \dots, v_n\}$  is a basis for  $V$ ,

necessarily  $c_1 = \dots = c_n = 0$ ;

hence  $\{T(v_1), \dots, T(v_n)\}$  is a basis for  $W$ .  $\#$

( $\Leftarrow$ ) If  $\{T(v_1), \dots, T(v_n)\}$  is a basis for  $W$ , by definition  $\{T(v_1), \dots, T(v_n)\}$  generates  $W$ .

$$\text{Thus } W = \text{span}(\{T(v_1), \dots, T(v_n)\}) = R(T).$$

where the second equality comes from T2.3

Then, since  $W = R(T)$ ,  $T$  is necessarily onto.

Then, since  $\{v_1, \dots, v_n\}$  is a basis for  $V$ , let  $x \in N(T)$ . Since  $\{v_1, \dots, v_n\}$  is a basis for  $V$ ,

there must exist some  $a_1, \dots, a_n \in \mathbb{F}$  such that

$$x = a_1 v_1 + \dots + a_n v_n.$$

Hence

$$0 = T(x) = a_1 T(v_1) + \dots + a_n T(v_n).$$

Since  $\{T(v_1), \dots, T(v_n)\}$  is a basis for  $W$  by assumption,

thus  $\{T(v_1), \dots, T(v_n)\}$  is linearly independent, so that

$$a_1 = \dots = a_n = 0,$$

and so

$$x = 0v_1 + \dots + 0v_n = 0.$$

Consequently  $N(T) = \{0\}$ , so that (by L3)  $T$  is 1-1.  $\square$

# CONSTRUCTING AN ISOMORPHISM FROM $V$ TO $W$

Let  $V$  and  $W$  be vector spaces.

Then, we can construct an isomorphism from  $V$  to  $W$  as follows:

① Choose a basis  $\{v_1, \dots, v_n\}$  for  $V$ , and a basis  $\{w_1, \dots, w_m\}$  for  $W$ .

② Let the linear transformation  $T: V \rightarrow W$  be such that  $T(v_k) = w_k \quad \forall k \in \{1, 2, \dots, n\}$ .

( $T$  exists; this follows from T2.1)

③ Then, by T2.5,  $T$  is also an isomorphism.

$V \cong W \Leftrightarrow \dim V = \dim W$  (T2.6)

Let  $V$  and  $W$  be two finite-dimensional vector spaces over a field  $\mathbb{F}$ .

Then  $V$  is isomorphic to  $W$  if and only if  $\dim V = \dim W$ .

$\dim V = \dim W < \infty$ ;  $T$  IS 1-1 ( $\Leftrightarrow$

$T$  IS ONTO ( $\Leftrightarrow$ )  $\text{rank}(T) = \dim(V)$  (T2.7)

Let  $V$  and  $W$  be two vector spaces over a field  $\mathbb{F}$ , and assume  $\dim V = \dim W < \infty$ .

Let  $T: V \rightarrow W$  be linear.

Then the following are equivalent to one another:

①  $T$  is one-to-one;

②  $T$  is onto; and

③  $\text{rank}(T) = \dim(V)$ .

# SET OF ALL LINEAR TRANSFORMATIONS (D21)

Let  $V$  and  $W$  be vector spaces over  $\mathbb{F}$ .

Then, we let  $\mathcal{L}(V, W) \subseteq W^V$  denote the set of all linear transformations  $T: V \rightarrow W$ .

$\mathcal{L}(V, W)$  IS A SUBSPACE OF  $W^V$  (T2.8)

Let  $V$  and  $W$  be vector spaces over some field  $\mathbb{F}$ .

Then necessarily  $\mathcal{L}(V, W)$  is a subspace of  $W^V$ .

Proof. Clearly  $\mathcal{L}(V, W) \subseteq W^V$ , so we only need

to show that it is non-empty and is closed under the addition & scalar multiplication operations of  $W^V$ .

Also note the zero transformation  $T_0: V \rightarrow W$  is in  $\mathcal{L}(V, W)$ , so that  $\mathcal{L}(V, W)$  is non-empty.

Next, assume  $T, U \in \mathcal{L}(V, W)$ . Note that for any  $x, y \in V$  &  $c \in \mathbb{F}$ :

$$\begin{aligned} (T+U)(cx+cy) &= T(cx+cy) + U(cx+cy) \\ &= cT(x) + T(y) + cU(x) + U(y) \\ &= c(T+U)(x) + (T+U)(y). \end{aligned}$$

showing  $T+U$  is linear (by P2), so that  $T+U \in \mathcal{L}(V, W)$

A similar argument shows  $cT \in \mathcal{L}(V, W)$  as well  $\forall c \in \mathbb{F}$ .

Thus  $\mathcal{L}(V, W)$  is a subspace of  $W^V$ , and we are done.  $\square$

# MORE ON MATRICES

## TRANSPOSITION OF A MATRIX

Let  $A \in M_{m \times n}(\mathbb{F})$  be arbitrary, and write

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}.$$

Then the "transposition" of  $A$ , denoted as " $A^T$ " (or " $A^t$ "), is defined to be the matrix

$$A^T = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{pmatrix} \in M_{n \times m}(\mathbb{F}).$$

## MATRIX VECTOR MULTIPLICATION (D22)

Let  $A \in M_{m \times n}(\mathbb{F})$  and  $x \in \mathbb{F}^n$  be arbitrary, where  $\mathbb{F}$  is some field.

We define " $Ax$ " to be equal to

$$Ax = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^n a_{1k} x_k \\ \sum_{k=1}^n a_{2k} x_k \\ \vdots \\ \sum_{k=1}^n a_{mk} x_k \end{pmatrix};$$

i.e. the  $i^{th}$  entry of  $Ax$  is obtained by multiplying the entries in the  $i^{th}$  row of  $A$  by the entries of  $x$ , and then summing up the resultant products.

$$L_A(x) = Ax \quad (\text{D23})$$

Let  $\mathbb{F}$  be a field, and let  $A \in M_{m \times n}(\mathbb{F})$  be arbitrary.

Then, we let the function  $L_A : \mathbb{F}^n \rightarrow \mathbb{F}^m$  be defined by  $L_A(x) = Ax \quad \forall x \in \mathbb{F}^n$ .

## " $a_j$ " MATRIX NOTATION

Let  $A \in M_{m \times n}(\mathbb{F})$ , and write

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}.$$

Then, we use the notation " $a_j$ " to denote

$$a_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix},$$

and we can also write  $A$  as

$$A = (a_1 \ a_2 \ \dots \ a_n).$$

$$Ax = x_1 a_1 + x_2 a_2 + \dots + x_n a_n \quad (\text{L4(1)})$$

Let  $A \in M_{m \times n}(\mathbb{F})$  be arbitrary, and write

$$A = (a_1 \ a_2 \ \dots \ a_n).$$

Then for any  $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{F}^n$ , we have

$$Ax = x_1 a_1 + x_2 a_2 + \dots + x_n a_n.$$

$$a_j = Ae_j \quad (\text{L4(2)})$$

Let  $A \in M_{m \times n}(\mathbb{F})$  be arbitrary, and write

$$A = (a_1 \ a_2 \ \dots \ a_n).$$

Suppose  $\{e_1, e_2, \dots, e_n\}$  are the standard basis vectors for  $\mathbb{F}^n$ .

Then necessarily  $Ae_j = a_j$ .

## MATRIX EQUALITY THEOREM (C2.8.1)

Let  $A, B \in M_{m \times n}(\mathbb{F})$  be arbitrary.

Then  $A=B$  if and only if  $Ax=Bx \quad \forall x \in \mathbb{F}^n$ .

Proof: ( $\Rightarrow$ ) is obvious.

( $\Leftarrow$ ) Suppose  $Ax=Bx \quad \forall x \in \mathbb{F}^n$ .

This implies  $Ae_j = Be_j \quad \forall j \in \{1, \dots, n\}$ , which tells us (by L4(2)) that  $a_j = b_j \quad \forall j \in \{1, \dots, n\}$ .

It follows that  $A=B$ , as needed.  $\blacksquare$

## $L_A$ IS A LINEAR TRANSFORMATION (T2.9)

Let  $A \in M_{m \times n}(\mathbb{F})$  be arbitrary.

Then  $L_A : \mathbb{F}^n \rightarrow \mathbb{F}^m$  is necessarily a linear transformation.

Proof: We prove  $L_A(cx+ty) = cL_A(x) + tL_A(y) \quad \forall x, y \in \mathbb{F}^n \& c, t \in \mathbb{F}$ ; the result follows from P2.

Write  $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$  and  $y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$ , and

$$A = (a_1 \ a_2 \ \dots \ a_n).$$

$$\begin{aligned} L_A(cx+ty) &= A(cx+ty) \\ &= (cx_1+y_1)a_1 + (cx_2+y_2)a_2 + \dots + (cx_n+y_n)a_n \quad (\text{by L4(1)}) \\ &= c(x_1a_1 + \dots + x_na_n) + (y_1a_1 + \dots + y_na_n) \\ &= c(Ax) + Ay \\ \therefore L_A(cx+ty) &= cL_A(x) + L_A(y), \quad \text{as needed. } \blacksquare \end{aligned}$$

$$L : M_{m \times n}(\mathbb{F}) \rightarrow \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m) \quad \text{BY } L(A) = L_A \quad \text{IS}$$

## A 1-1 LINEAR TRANSFORMATION (P4)

Let  $L : M_{m \times n}(\mathbb{F}) \rightarrow \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$  by  $L(A) = L_A \quad \forall A \in M_{m \times n}(\mathbb{F})$ , where  $\mathbb{F}$  is a field.

Then  $L$  is necessarily a one-to-one linear transformation.

Proof: We first show  $L$  is linear.

By P2, we just need to show  $L(cA+B) = cL(A) + L(B)$   $\forall A, B \in M_{m \times n}(\mathbb{F})$ , i.e.  $L_{cA+B} = cL_A + L_B$ .

To do this, let  $x \in \mathbb{F}^n$  be arbitrary.

Write  $A = (a_1 \ a_2 \ \dots \ a_n)$  and  $B = (b_1 \ b_2 \ \dots \ b_n)$ , so that  $cA+B = (ca_1+b_1 \ ca_2+b_2 \ \dots \ ca_n+b_n)$ .

So

$$\begin{aligned} L_{cA+B}(x) &= (ca_1+b_1)x \\ &= x_1(ca_1+b_1) + x_2(ca_2+b_2) + \dots + x_n(ca_n+b_n) \quad (\text{by L4(1)}) \\ &= c(x_1a_1 + \dots + x_na_n) + (x_1b_1 + \dots + x_nb_n) \\ &= c(Ax) + Bx \\ &= cL_A(x) + L_B(x) \\ \therefore L_{cA+B}(x) &= (cL_A + L_B)(x), \end{aligned}$$

and since  $x \in \mathbb{F}^n$  was arbitrary this is sufficient to prove  $L_{cA+B} = cL_A + L_B$ , as needed.  $\blacksquare$

Next, we prove  $L$  is 1-1.

Assume for some  $A, B \in M_{m \times n}(\mathbb{F})$ , we have  $L_A = L_B$ .

This means  $L_A(x) = L_B(x) \quad \forall x \in \mathbb{F}^n$ , or  $Ax = Bx \quad \forall x \in \mathbb{F}^n$ .

So by the Matrix Equality Theorem,  $A=B$ , which is sufficient to prove  $L$  is 1-1.  $\blacksquare$

# COORDINATES (S2.2)

## ORDERED BASIS (D24)

Let  $V$  be a vector space with  $\dim V < \infty$ .

Then, an "ordered basis" for  $V$  is a

basis  $\{v_1, \dots, v_n\}$  with a total order.

e.g.  $\{e_1, e_2, e_3\}$  is the standard ordered basis for  $\mathbb{R}^3$ , since we can define a "total order" by saying the indexes must be in "increasing order" (E30(c))

## COORDINATE VECTOR (D25)

Let  $\beta = \{u_1, \dots, u_n\}$  be an "ordered basis"

for a finite-dimensional vector space  $V$ .

By T1.6, we can write any  $x \in V$  in the form  $x = \sum_{k=1}^n a_k u_k$ , where  $a_1, \dots, a_n \in \mathbb{F}$ .

Then, we define the "coordinate vector" of  $x$  relative to  $\beta$ , denoted as " $[x]_\beta$ ", to be

$$[x]_\beta := \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{F}^n.$$

e.g. for  $V = P_2(\mathbb{R})$ ,  $\beta = \{1, x, x^2\}$ ,  $p(x) = 2 - 3x + 4x^2 \in V$ ,

$$[p(x)]_\beta = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}.$$

$[\ ]_\beta : V \rightarrow \mathbb{F}^n$  IS AN ISOMORPHISM (T2.10)

Let  $V$  be a vector space over some field  $\mathbb{F}$ ,

with  $\dim V = n$ , and let  $\beta$  be an ordered

basis for  $V$ .

Then, the map  $[\ ]_\beta : V \rightarrow \mathbb{F}^n$  is an isomorphism.

# MATRIX REPRESENTATION OF LINEAR TRANSFORMATIONS (S2.3)

Let  $V$  and  $W$  be finite-dimensional vector spaces over  $\mathbb{F}$ , and let  $T: V \rightarrow W$  be a linear transformation. Let  $\beta = \{v_1, \dots, v_n\}$  be an ordered basis for  $V$ , and let  $\gamma = \{w_1, \dots, w_m\}$  be an ordered basis for  $W$ . Then, the "matrix representation" of  $T$  in the ordered bases  $\beta$  and  $\gamma$ , denoted as  $[T]_{\beta}^{\gamma}$ , is defined as the matrix

$$[T]_{\beta}^{\gamma} := ([T(v_1)]_{\gamma} \ [T(v_2)]_{\gamma} \ \dots \ [T(v_n)]_{\gamma}).$$

In particular, if  $T: V \rightarrow V$  is linear and  $\beta$  is an ordered basis of the finite-dimensional vector space  $V$ , we denote

$$[T]_{\beta} := [T]_{\beta}^{\beta}.$$

Note that  $[T]_{\beta}^{\gamma} \in M_{m \times n}(\mathbb{F})$ , where  $m = \dim W$  and  $n = \dim V$ . (R12(1))

Also, we have

$$T(v_j) = \sum_{k=1}^n a_{kj} w_k,$$

where  $a_{kj}$  denotes the element at the  $k^{\text{th}}$  row and  $j^{\text{th}}$  column in the matrix  $[T]_{\beta}^{\gamma}$ . (R12(2))

eg If  $T: P_2(\mathbb{R}) \rightarrow \mathbb{R}^2$  by  $T(a+bx+cx^2) = \begin{pmatrix} a \\ b+4c \end{pmatrix}$ , we can verify  $T$  is linear.

Let  $\beta = \{1, (x+1), (x+1)^2\}$  and  $\gamma = \{(1), (-1)\}$ .

Then

$$\begin{aligned} [T]_{\beta}^{\gamma} &= ([T(1)]_{\gamma} \ [T(x+1)]_{\gamma} \ [T(x+1)^2]_{\gamma}) \\ &= \begin{pmatrix} 1 & 1 & 1 \\ 0+4(0) & 1+4(0) & 2+4(0) \end{pmatrix} \end{aligned}$$

$$\therefore [T]_{\beta}^{\gamma} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 6 \end{pmatrix}. \quad (\text{E32})$$

$$[L_A]_{\beta}^{\gamma} = A \quad (\text{E33})$$

Let  $A \in M_{m \times n}(\mathbb{F})$ , where  $\mathbb{F}$  is a field.

Let  $\beta$  be the standard ordered basis for  $\mathbb{F}^n$ , and  $\gamma$  the standard ordered basis for  $\mathbb{F}^m$ .

Then necessarily  $[L_A]_{\beta}^{\gamma} = A$ .

$$[T(x)]_{\gamma} = [T]_{\beta}^{\gamma} \cdot [x]_{\beta} \quad (\text{T2.11})$$

Let  $T: V \rightarrow W$  be linear, and let  $\beta = \{v_1, \dots, v_n\}$  and  $\gamma = \{w_1, \dots, w_m\}$  be ordered bases of  $V$  and  $W$  respectively.

Then necessarily  $[T(x)]_{\gamma} = [T]_{\beta}^{\gamma} \cdot [x]_{\beta} \quad \forall x \in V$ .

Proof. Let  $x \in V$  be arbitrary. Take  $x = \sum_{k=1}^n a_k v_k$ , where

$$[x]_{\beta} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}.$$

Then, since  $T$  is linear,

$$T(x) = T\left(\sum_{k=1}^n a_k v_k\right) = \sum_{k=1}^n a_k T(v_k).$$

Thus

$$[T(x)]_{\gamma} = \left[ \sum_{k=1}^n a_k T(v_k) \right]_{\gamma} = \sum_{k=1}^n a_k [T(v_k)]_{\gamma}. \quad (\text{by linearity of } [ ]_{\gamma})$$

Note that

$$\begin{aligned} \sum_{k=1}^n a_k [T(v_k)]_{\gamma} &= ([T(v_1)]_{\gamma} \ [T(v_2)]_{\gamma} \ \dots \ [T(v_n)]_{\gamma}) \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \\ &= [T]_{\beta}^{\gamma} \cdot [x]_{\beta}, \end{aligned}$$

so that

$$[T(x)]_{\gamma} = [T]_{\beta}^{\gamma} \cdot [x]_{\beta}, \quad \text{as needed.} \quad \blacksquare$$

# TRANSFORMATIONS (S2.3)

$[ ]_{\beta}^{\gamma}: \mathcal{L}(V, W) \rightarrow M_{m \times n}(\mathbb{F})$  IS AN ISOMORPHISM (P5)

Let  $V$  and  $W$  be finite-dimensional vector spaces over  $\mathbb{F}$ , and let  $\beta$  and  $\gamma$  be ordered bases of  $V$  and  $W$  respectively.

Then the map  $[ ]_{\beta}^{\gamma}: \mathcal{L}(V, W) \rightarrow M_{m \times n}(\mathbb{F})$  is an isomorphism, where  $m = \dim W$  and  $n = \dim V$ ; in other words,

① For any  $T, U \in \mathcal{L}(V, W)$  and  $c \in \mathbb{F}$ , we have that

$$[cT + U]_{\beta}^{\gamma} = c[T]_{\beta}^{\gamma} + [U]_{\beta}^{\gamma}; \quad \text{and}$$

② For any  $C \in M_{m \times n}(\mathbb{F})$ , there exists a unique  $T \in \mathcal{L}(V, W)$  such that  $[T]_{\beta}^{\gamma} = C$ .

Proof. We first prove ①.

Let  $\beta = \{v_1, \dots, v_n\}$ . Then

$$\begin{aligned} [T+U]_{\beta}^{\gamma} &= ([T(v_1)]_{\gamma} \ [T(v_2)]_{\gamma} \ \dots \ [T(v_n)]_{\gamma}) \\ &= ([T(v_1) + U(v_1)]_{\gamma} \ [T(v_2) + U(v_2)]_{\gamma} \ \dots \ [T(v_n) + U(v_n)]_{\gamma}) \\ &= (([T(v_1)]_{\gamma} + [U(v_1)]_{\gamma}) \ ([T(v_2)]_{\gamma} + [U(v_2)]_{\gamma}) \ \dots \ ([T(v_n)]_{\gamma} + [U(v_n)]_{\gamma})) \\ &= ([T(v_1)]_{\gamma} \ [T(v_2)]_{\gamma} \ \dots \ [T(v_n)]_{\gamma}) + ([U(v_1)]_{\gamma} \ [U(v_2)]_{\gamma} \ \dots \ [U(v_n)]_{\gamma}) \\ \therefore [T+U]_{\beta}^{\gamma} &= [T]_{\beta}^{\gamma} + [U]_{\beta}^{\gamma}, \end{aligned}$$

and a similar proof shows  $[cT]_{\beta}^{\gamma} = c[T]_{\beta}^{\gamma}$ , which is sufficient to show  $[cT+U]_{\beta}^{\gamma} = c[T]_{\beta}^{\gamma} + [U]_{\beta}^{\gamma}$ , and hence that the map  $[ ]_{\beta}^{\gamma}: \mathcal{L}(V, W) \rightarrow M_{m \times n}(\mathbb{F})$  is linear. \*

We next prove ②.

Suppose  $[T]_{\beta}^{\gamma} = [U]_{\beta}^{\gamma}$ , so that  $[T]_{\beta}^{\gamma}$  and  $[U]_{\beta}^{\gamma}$  have the same  $j^{\text{th}}$  column  $\forall j \in \{1, \dots, n\}$ .

This means  $[T(v_j)]_{\gamma} = [U(v_j)]_{\gamma}$ , and since  $[ ]_{\gamma}: W \rightarrow \mathbb{F}^n$  is a bijection (by T2.10) it follows that  $T(v_j) = U(v_j) \quad \forall j \in \{1, \dots, n\}$ . So, by T2.1,  $T = U$ , proving injectivity.

Then, let  $C = (c_1 \ c_2 \ \dots \ c_n) \in M_{m \times n}(\mathbb{F})$  be arbitrary.

For each  $j \in \{1, \dots, n\}$ , let  $w_j \in W$  be the unique vector satisfying  $[w_j]_{\gamma} = c_j$ .

By T2.10, there exists a unique linear transformation  $T: V \rightarrow W$  satisfying  $T(v_j) = w_j \quad \forall j \in \{1, \dots, n\}$ .

it follows this  $T$  satisfies  $[T]_{\beta}^{\gamma} = C$ , proving surjectivity. So we are done.  $\blacksquare$

$L: M_{m \times n}(\mathbb{F}) \rightarrow \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$  IS AN ISOMORPHISM (C2.1.1)

Recall that the map  $L: M_{m \times n}(\mathbb{F}) \rightarrow \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$  is defined by  $L(A) = L_A \quad \forall A \in M_{m \times n}(\mathbb{F})$ .

Then,  $L$  is necessarily an isomorphism.

Proof. We know  $L$  is already 1-1 & linear by P4, so we only need to prove it is onto.

Applying P5 to  $V = \mathbb{F}^n$  &  $W = \mathbb{F}^m$ , we get that  $\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m) \cong M_{m \times n}(\mathbb{F})$ , so that

$$\dim(\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)) = \dim(M_{m \times n}(\mathbb{F})) = mn \quad (\text{by T2.6}).$$

So  $L$  is a 1-1 linear transformation between vector spaces of the same finite dimension; it follows by T2.7 that  $L$  is onto.  $\blacksquare$

# MATRIX MULTIPLICATION & COMPOSITIONS OF LINEAR TRANSFORMATIONS (S2.4)

## MATRIX PRODUCT (D27)

Let  $\mathbb{F}$  be a field, and let  $A \in M_{m \times n}(\mathbb{F})$  and  $B \in M_{n \times p}(\mathbb{F})$  be arbitrary. Note the number of columns in  $A$  equals the number of rows in  $B$ ; this is required. Then, the matrix product of  $A$  and  $B$ , denoted by  $AB$ , is defined to be the  $m \times p$  matrix

$$AB = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mp} \end{pmatrix},$$

where  $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{t=1}^n a_{it}b_{tj}$ .

In other words,  $c_{ij}$  is the sum of products formed multiplying the entries in the  $i^{\text{th}}$  row of  $A$  with the  $j^{\text{th}}$  column of  $B$ .

\*An example is highlighted in blue;

$$c_{21} = a_{21}b_{11} + a_{22}b_{21} + \cdots + a_{2n}b_{n1}.$$

Note that  $c_j$  is the linear combination of the columns of  $A$  formed using the entries in the  $j^{\text{th}}$  column of  $B$  as coefficients. (R1B(3))

## ZERO MATRIX

The "zero matrix", denoted by the letter  $O$ , is defined to be the matrix with each entry being zero.

We write " $O_{mn}$ " to denote the  $m \times n$  zero matrix.

## IDENTITY MATRIX

The "nn identity matrix", denoted as  $I_n$ , is defined as the matrix  $(\delta_{ij})$  with

$$\delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i \neq j. \end{cases} \quad * \delta_{ij} \text{ is known as the "Kronecker delta".}$$

$$\text{eg } I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

## RIGHT MATRIX DISTRIBUTIVE LAW (LS(1))

For any  $A \in M_{m \times n}(\mathbb{F})$  and  $B, C \in M_{n \times p}(\mathbb{F})$ , we have

$$A(B+C) = AB + AC.$$

## LEFT MATRIX DISTRIBUTIVE LAW (LS(2))

Similarly, for any  $A \in M_{m \times n}(\mathbb{F})$  and  $D, E \in M_{q \times m}(\mathbb{F})$ , we have

$$(D+E)A = DA + EA.$$

## ASSOCIATIVITY OF MATRIX SCALAR MULTIPLICATION (LS(3))

For any  $\alpha \in \mathbb{F}$ ,  $A \in M_{m \times n}(\mathbb{F})$  and  $B \in M_{q \times m}(\mathbb{F})$ , we have

$$\alpha(AB) = (\alpha A)B = A(\alpha B).$$

$$(AB)^T = B^T A^T \quad (\text{LS}(4))$$

For any  $A \in M_{m \times n}(\mathbb{F})$  and  $B \in M_{n \times n}(\mathbb{F})$ , we have

$$(AB)^T = B^T A^T.$$

$$I_m A = AI_n \quad (\text{LS}(5))$$

For any  $A \in M_{m \times n}(\mathbb{F})$ , we have that  $I_m A = AI_n = A$ .

$$AO_{n \times p} = O_{m \times p}, \quad O_{q \times m} A = O_{q \times n} \quad (\text{LS}(6))$$

For any  $A \in M_{m \times n}(\mathbb{F})$ , we have

$$\textcircled{1} \quad AO_{n \times p} = O_{m \times p}; \quad \text{and}$$

$$\textcircled{2} \quad O_{q \times m} A = O_{q \times n}.$$

## COMPOSITION OF LINEAR TRANSFORMATIONS IS ALSO A LINEAR TRANSFORMATION (T2.12)

Let  $T: V \rightarrow W$  and  $U: W \rightarrow Z$  be linear transformations.

Then the composition  $(U \circ T): V \rightarrow Z$  is also a linear transformation.

\*we usually denote  $(U \circ T)$  as  $UT$ .

## MATRIX OF COMPOSITION OF LINEAR TRANSFORMATIONS (T2.13)

Let  $V, W$  and  $Z$  be finite-dimensional vector spaces having ordered bases  $\alpha = \{v_1, \dots, v_p\}$ ,  $\beta = \{w_1, \dots, w_n\}$  and  $\gamma = \{z_1, \dots, z_m\}$  respectively.

Let  $T: V \rightarrow W$  and  $U: W \rightarrow Z$  be linear transformations.

Denote  $A = [U]_{\beta}^{\gamma} \in M_{m \times n}(\mathbb{F})$ ,  $B = [T]_{\alpha}^{\beta} \in M_{n \times p}(\mathbb{F})$  and  $C = [UT]_{\alpha}^{\gamma} \in M_{m \times p}(\mathbb{F})$ .

Then necessarily  $C = AB$ ; ie  $[UT]_{\alpha}^{\gamma} = [U]_{\beta}^{\gamma} \cdot [T]_{\alpha}^{\beta}$ .

Proof. Note that both sides are  $m \times p$  matrices.

We show that the  $j^{\text{th}}$  columns of the LHS & RHS are equal  $\forall j \in \{1, \dots, p\}$ .

On one hand, the  $j^{\text{th}}$  column of  $[UT]_{\alpha}^{\gamma}$  is  $[(UT)(v_j)]_{\gamma}$ .

On the other hand,  $[T]_{\alpha}^{\beta} = B = (b_1, b_2, \dots, b_p)$ .

Hence, the  $j^{\text{th}}$  column of  $[U]_{\beta}^{\gamma} \cdot [T]_{\alpha}^{\beta}$  is  $[U]_{\beta}^{\gamma} \cdot b_j$ , which equals

$$\begin{aligned} [U]_{\beta}^{\gamma} \cdot b_j &= [U]_{\beta}^{\gamma} \cdot [T(v_j)]_{\beta} \\ &= [U(T(v_j))]_{\gamma} \quad (\text{by T2.11}) \\ &= [(UT)(v_j)]_{\gamma}. \end{aligned}$$

It follows that  $[UT]_{\alpha}^{\gamma}$  and  $[U]_{\beta}^{\gamma} \cdot [T]_{\alpha}^{\beta}$  have the same  $j^{\text{th}}$  columns; since  $j$  was arbitrary, it follows that  $[UT]_{\alpha}^{\gamma} = [U]_{\beta}^{\gamma} \cdot [T]_{\alpha}^{\beta}$ , as needed.  $\blacksquare$

$$L_{AB} = L_A L_B \quad (\text{C2.13.1 (1)})$$

Let  $A \in M_{m \times n}(\mathbb{F})$  and  $B = M_{n \times p}(\mathbb{F})$  be arbitrary.

Then necessarily  $L_{AB} = L_A L_B$ .

Proof. Let  $\alpha, \beta, \gamma$  denote the standard ordered bases for  $\mathbb{F}^p$ ,  $\mathbb{F}^n$  and  $\mathbb{F}^m$  respectively.

By E33,  $[L_A]_{\alpha}^{\gamma} = A$ ,  $[L_B]_{\beta}^{\gamma} = B$  and  $[L_{AB}]_{\alpha}^{\gamma} = AB$ .

On the other hand

$$[L_A L_B]_{\alpha}^{\gamma} = [L_A]_{\alpha}^{\beta} \cdot [L_B]_{\beta}^{\gamma} = AB = [L_{AB}]_{\alpha}^{\gamma} \quad (\text{by T2.13}).$$

Since the mapping  $[\cdot]_{\alpha}^{\gamma}$  is 1-1 (by C2.11.1), it follows that  $L_A L_B = L_{AB}$ , as needed.  $\blacksquare$

$$A(BC) = (AB)C \quad (\text{C2.13.1 (2)})$$

Assume the matrix product "A(BC)" is defined.

Then necessarily  $A(BC) = (AB)C$ .

Proof. By C2.13.1(1), we get that

$$L_{A(BC)} = L_A L_{BC} = L_A(L_B L_C) = (L_A L_B)L_C = L_{AB}L_C = L_{(AB)}C,$$

since function composition is associative.

Then, as  $L$  is 1-1 (by P4), it follows that

$$A(BC) = (AB)C, \text{ as needed. } \blacksquare$$

# INVERTIBILITY & ISOMORPHISMS (S2.5)

## INVERTIBLE MATRICES (D28)

- $\exists_1$  Let  $A \in M_{n \times n}(\mathbb{F})$  be arbitrary.  
Then, we say  $A$  is "invertible" if there exists a matrix  $B \in M_{n \times n}(\mathbb{F})$  such that  $AB = BA = I_n$ .  
Note that if such a matrix  $B$  exists, it is uniquely determined by  $A$ .  
Proof. Suppose  $B, C \in M_{n \times n}(\mathbb{F})$  are such that  $AB = BA = I_n$  &  $AC = CA = I_n$ . Then  
 $B = BI_n = B(AC) = (BA)C = I_n C = C$ , proving uniqueness.  $\square$

## INVERSE MATRICES (D28)

- $\exists_1$  Let  $A \in M_{n \times n}(\mathbb{F})$  be an invertible square matrix.  
Then the "inverse" of  $A$ , denoted as " $A^{-1}$ ", is the unique  $n \times n$  square matrix such that  $AA^{-1} = A^{-1}A = I_n$ .

## INVERTIBLE MAPPING (D29)

- $\exists_1$  Let  $T: V \rightarrow W$  be a linear mapping between two vector spaces  $V$  and  $W$ .  
Then, we say  $T$  is "invertible" if there exists a function  $U: W \rightarrow V$  such that  $UT = I_V$  and  $TU = I_W$ .

## INVERSE MAPPING (D29)

- $\exists_1$  Let  $T: V \rightarrow W$  be an invertible linear mapping.

Then the "inverse" of  $T$ , denoted as " $T^{-1}$ ", is the mapping  $T^{-1}: W \rightarrow V$  such that  $TT^{-1} = I_V$  and  $T^{-1}T = I_W$ .  
 $\exists_2$  Similarly, we can show  $T^{-1}$  is unique. (T2.14(1))  
Proof. Suppose there exist  $U_1, U_2: W \rightarrow V$  such that  $U_1T = I_V$ ,  $TU_1 = I_W$ ,  $U_2T = I_V$  &  $TU_2 = I_W$ . Then  
 $U_1 = U_1I_W = U_1(TU_2) = (U_1T)U_2 = I_VU_2 = U_2$ , proving uniqueness.  $\square$

## T IS LINEAR & INVERTIBLE $\Rightarrow$ T IS AN ISOMORPHISM (T2.14(2))

- $\exists_1$  Let  $T: V \rightarrow W$  be linear and invertible.  
Then  $T$  is necessarily an isomorphism.  
Proof. Suppose  $x, y \in V$  are such that  $T(x) = T(y)$ . Then observe that  
 $x = (T^{-1}T)(x) = T^{-1}(T(x)) = T^{-1}(T(y)) = y$ , proving injectivity.  
Then, let  $z \in W$ . Since  $TT^{-1} = I_W$ , we have  
 $z = I_W(z) = (TT^{-1})(z) = T(T^{-1}(z))$ . Since  $T^{-1}(z) \in V$ , it follows that  $T$  is surjective.  
Hence  $T$  is bijective, and since  $T$  is also linear, it follows that  $T$  is an isomorphism.  $\square$

## $T^{-1}$ IS ALSO LINEAR (T2.14(3))

- $\exists_1$  Let  $T: V \rightarrow W$  be linear and invertible.  
Then  $T^{-1}$  is necessarily also linear.  
Proof. Let  $y_1, y_2 \in V$  and  $c \in \mathbb{F}$  be arbitrary.  
Since  $T$  is bijective (by T2.14(2)), there exist unique  $x_1, x_2 \in V$  such that  $T(x_1) = y_1$  and  $T(x_2) = y_2$ . Then  
 $T^{-1}(cy_1 + y_2) = T^{-1}(cT(x_1) + T(x_2)) = T^{-1}(T(cx_1 + x_2))$   
 $= cx_1 + x_2 = cT^{-1}(y_1) + T^{-1}(y_2)$ , and it follows from P2 that  $T^{-1}$  is linear.  $\square$

**T IS AN ISOMORPHISM  $\Leftrightarrow [T]_\alpha^\beta$  IS INVERTIBLE (T2.15(1))**

- $\exists_1$  Let  $V$  and  $W$  be finite-dimensional vector spaces, and let  $\alpha$  and  $\beta$  be ordered bases of  $V$  and  $W$  respectively.

Let  $T: V \rightarrow W$  be linear.  
Then  $T$  is an isomorphism if and only if  $[T]_\alpha^\beta$  is an invertible matrix.

Proof. ( $\Rightarrow$ ) Suppose  $T$  is an isomorphism, so that  $V \cong W$ .

Then, by T2.6,  $\dim V = \dim W = n$ .

Let  $A := [T]_\alpha^\beta$ . By the above,  $A$  is a  $n \times n$  square matrix.

By T2.14(3),  $T^{-1}: W \rightarrow V$  is also linear.

Let  $B := [T^{-1}]_\beta^\alpha$ , which is also a  $n \times n$  matrix.

Also,

$$\begin{aligned} AB &= [T]_\alpha^\beta [T^{-1}]_\beta^\alpha = [TT^{-1}]_\alpha^\alpha = [I_n]_\alpha^\alpha \\ &= [I_n]_\alpha^\alpha \\ &= I_n. \end{aligned} \quad (\text{T2.13})$$

A similar proof shows  $BA = [I_n]_\alpha^\alpha = I_n$ . So, by D28,  $A$  is an invertible matrix, proving the forward argument.

( $\Leftarrow$ ) Suppose  $A = [T]_\alpha^\beta$  is an invertible matrix with inverse  $A^{-1}$ .

In particular,  $A$  must be square, say  $n \times n$ , so  $\dim V = \dim W = n$ .

Then, let  $x, y \in V$  such that  $T(x) = T(y)$ . By T2.11,

$$A[x]_\alpha = [T]_\alpha^\beta [x]_\alpha = [T(x)]_\beta = [T(y)]_\beta = [T]_\alpha^\beta [y]_\alpha = AC[y]_\alpha.$$

Thus  $A[x]_\alpha = A[y]_\alpha$ . It follows that

$$A^{-1}(A[x]_\alpha) = A^{-1}(A[y]_\alpha),$$

or  $[x]_\alpha = [y]_\alpha$  and so  $x = y$ , proving injectivity.

Then, as  $T$  is linear and  $\dim V = \dim W$ , by T2.7  $T$  is also onto.

Hence  $T$  is bijective, and since  $T$  is also linear, it follows that  $T$  is an isomorphism, proving the backward argument.  $\square$

$\exists_2$  In particular, if  $T$  is an isomorphism, then

$$[T^{-1}]_\beta^\alpha = ([T]_\alpha^\beta)^{-1}.$$

$A \in M_{n \times n}(\mathbb{F}) \Rightarrow (L_A \text{ IS AN ISOMORPHISM} \Leftrightarrow A \text{ IS INVERTIBLE}) \quad (\text{T2.15(2)})$

- $\exists_1$  Let  $V$  and  $W$  be finite-dimensional vector spaces, and let  $\alpha$  and  $\beta$  be ordered bases of  $V$  and  $W$  respectively.

Then for any  $A \in M_{n \times n}(\mathbb{F})$ , necessarily  $L_A$  is an isomorphism if and only if  $A$  is invertible.

Proof. By T2.15(1),  $L_A$  is an isomorphism if and only if  $[L_A]_{\alpha}^{\beta}$  is invertible, where  $\alpha$  is the standard ordered basis for  $\mathbb{F}^n$ .

By E33,  $[L_A]_{\alpha}^{\beta} = A$ , and this is sufficient to prove the claim.  $\square$

A IS INVERTIBLE  $\Rightarrow$  A<sup>-1</sup> IS INVERTIBLE

$(A^{-1})^{-1} = A$  (L6(1))

Let A be an invertible matrix.

Then A<sup>-1</sup> is also invertible, and (A<sup>-1</sup>)<sup>-1</sup> = A.

Proof. Since A is invertible, A<sup>-1</sup> exists.

In particular, A<sup>-1</sup>A = I<sub>n</sub>.

By uniqueness of matrix inverses, it follows that A = (A<sup>-1</sup>)<sup>-1</sup>, as needed.  $\square$

(cA)<sup>-1</sup> =  $\frac{1}{c}A^{-1}$  (L6(2))

Let A be an invertible matrix, and let c ∈ F.

Then necessarily (cA)<sup>-1</sup> =  $\frac{1}{c}A^{-1}$ .

(AT)<sup>-1</sup> = (A<sup>-1</sup>)<sup>T</sup> (L6(3))

Let A be an invertible matrix.

Then necessarily (AT)<sup>-1</sup> = (A<sup>-1</sup>)<sup>T</sup>.

(AB)<sup>-1</sup> = B<sup>-1</sup>A<sup>-1</sup> (L6(4))

Let A, B ∈ M<sub>n,n</sub>(F) be invertible matrices.

Then AB is also invertible, and necessarily

(AB)<sup>-1</sup> = B<sup>-1</sup>A<sup>-1</sup>.

Proof. (AB)(B<sup>-1</sup>A<sup>-1</sup>) = A(BB<sup>-1</sup>)A<sup>-1</sup> = AA<sup>-1</sup> = I<sub>n</sub>.

By uniqueness of matrix inverses, it follows that (AB)<sup>-1</sup> = B<sup>-1</sup>A<sup>-1</sup>.  $\square$

## THE CHANGE OF COORDINATE MATRIX (S2.6)

CHANGE OF COORDINATE MATRIX FROM α TO β (T2.17(1))

Let α and β be two ordered bases for a finite-dimensional vector space V.

Then the "change of coordinate matrix from α to β" is the matrix

$$Q = [I_V]_{\alpha}^{\beta}.$$

The matrix Q = [I\_V]\_{\alpha}^{\beta} is necessarily invertible.

Proof. Since I<sub>V</sub> is an isomorphism, by T2.15, Q is necessarily invertible.  $\square$

Also, note that if we let α = {v<sub>1</sub>, ..., v<sub>n</sub>} and β = {w<sub>1</sub>, ..., w<sub>n</sub>} and fix an x ∈ V, then

$$[I_V]_{\alpha}^{\beta} = ([v_1]_{\beta} \cdots [v_n]_{\beta}).$$

Then, by comparing the j<sup>th</sup> column on both sides, we have that

$$v_j = \sum_{i=1}^n Q_{ij} w_i. \quad (\text{R15})$$

$$[x]_{\beta} = Q[x]_{\alpha} \quad (\text{T2.17(2)})$$

Let α and β be two ordered bases of the finite-dimensional vector space V.

Let Q = [I<sub>V</sub>]<sub>α</sub><sup>β</sup> be the change of coordinate matrix from α to β.

Then necessarily for any x ∈ V, we have

$$[x]_{\beta} = Q[x]_{\alpha}.$$

Proof. By T2.11, we have

$$[x]_{\beta} = [I_V(x)]_{\beta} = [I_V]_{\alpha}^{\beta} [x]_{\alpha} = Q[x]_{\alpha},$$

as needed.  $\square$

AB IS INVERTIBLE  $\Rightarrow$  A & B ARE

INVERTIBLE (L6(5))

Let A, B ∈ M<sub>n,n</sub>(F) be such that AB is invertible.

Then necessarily A and B are also invertible matrices.

Proof. By T2.15, L<sub>AB</sub> is invertible. By T2.14, L<sub>AB</sub> is an isomorphism.

Thus, L<sub>AB</sub> is 1-1 and onto. Thus

L<sub>A</sub> is surjective and L<sub>B</sub> is injective.

Then, as L<sub>A</sub> and L<sub>B</sub> are both linear mappings from F<sup>n</sup> to itself, by T2.7 L<sub>A</sub> and L<sub>B</sub> are isomorphisms.

Hence A and B are invertible by T2.15(2), and we are done.  $\square$

## INVERSE MATRIX THEOREM, PART 1 (T2.16)

Let A ∈ M<sub>n,n</sub>(F). Then the following statements are equivalent:

① A is invertible;

② There exists a matrix C ∈ M<sub>n,n</sub>(F) such that AC = I<sub>n</sub>; and

③ There exists a matrix B ∈ M<sub>n,n</sub>(F) such that BA = I<sub>n</sub>.

Proof. This follows directly from the definition of inverse matrices.  $\square$

$$T: V \rightarrow V ; [T]_{\alpha} = Q^{-1}[T]_{\beta}Q \quad (\text{T2.18})$$

Let T: V → V be linear, where V is a finite-dimensional vector space.

Let α and β be two ordered bases of V, and let Q = [I<sub>V</sub>]<sub>α</sub><sup>β</sup>.

Then necessarily [T]<sub>α</sub> = Q<sup>-1</sup>[T]<sub>β</sub>Q.

Proof. By T2.13, we have

$$Q[T]_{\alpha} = [I_V]_{\alpha}^{\beta} [T]_{\beta}^{\alpha} = [I_V T]_{\alpha}^{\beta} = [T]_{\alpha}^{\beta},$$

and

$$[T]_{\beta} Q = [T]_{\beta}^{\beta} [I_V]_{\alpha}^{\beta} = [TI_V]_{\alpha}^{\beta} = [T]_{\alpha}^{\beta},$$

showing [T]<sub>β</sub>Q = Q[T]<sub>α</sub>.

Then, since Q is invertible, Q<sup>-1</sup> exists; hence

$$Q^{-1}[T]_{\beta}Q = Q^{-1}Q[T]_{\alpha} = [T]_{\alpha},$$

as needed.  $\square$

## SIMILAR MATRICES (D30)

Let A, B ∈ M<sub>n,n</sub>(F) be arbitrary.

Then, we say B is "similar" to A if there exists an invertible matrix Q such that

$$B = Q^{-1}AQ.$$

# Chapter 3:

# Elementary Matrix Operations

# and Systems of Linear Equations

## ELEMENTARY MATRIX OPERATIONS & ELEMENTARY MATRICES (S3.1)

### ELEMENTARY ROW/COLUMN OPERATIONS (D31)

Let  $A \in M_{m \times n}(F)$ . Then, we denote the following as "elementary row/column operations" on  $A$ :

- ① Interchanging any two rows/columns of  $A$ , denoted as " $R_i \leftrightarrow R_j$ ";

eg  $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{pmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix}$

- ② Multiplying any row/column by a non-zero scalar, denoted as " $R_i \leftarrow cR_i$ "; and

eg  $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \xrightarrow{R_2 \leftarrow 2R_2} \begin{pmatrix} 1 & 2 & 3 \\ 8 & 10 & 12 \\ 7 & 8 & 9 \end{pmatrix}$

- ③ Adding any scalar multiple of a row/column of  $A$  to another row/column, denoted as " $R_i \leftarrow R_i + cR_j$ ".

eg  $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 + 2R_1} \begin{pmatrix} 1 & 2 & 3 \\ 6 & 9 & 12 \\ 7 & 8 & 9 \end{pmatrix}$

### nxn ELEMENTARY MATRIX (D32)

An "nxn elementary matrix" is a matrix obtained by performing an elementary operation on  $I_n$ .

eg performing " $R_3 \leftarrow R_3 + 4R_1$ " on  $I_3$  results in

$$I' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix}. \quad (\text{E3.1})$$

Let  $A \in M_{m \times n}(F)$ , and suppose  $B$  is obtained from  $A$  by performing an elementary row operation. Then necessarily  $B = EA$ , where  $E$  is the nxn elementary matrix obtained from  $I_n$  by performing the said elementary row/column operation. (T3.1)

Conversely, if  $E$  is an  $m \times m$  elementary matrix, then  $EA$  is the matrix obtained from  $A$  by performing the same elementary row operation as that which produces  $E$  from  $I_m$ . (T3.1)

\* a similar result holds for elementary matrices formed by performing an elementary column operation, but in this case  $B = AE$ . (T3.2)

Proof. This can be proven by verifying each of the three elementary row/column operations "holds" under this transformation.

## ELEMENTARY MATRICES (S3.1)

### ELEMENTARY MATRICES ARE INVERTIBLE, & THE INVERSE OF AN ELEMENTARY MATRIX IS OF THE SAME "TYPE" (T3.3)

Note that any elementary matrix  $A \in M_{m \times n}(F)$  is invertible, and  $A^{-1}$  is also an elementary matrix with the same "type" as  $A$ .

Proof. Suppose  $A$  is an elementary matrix obtained from  $I_m$ . Then, we verify this theorem for each of the three operations:

- ①  $R_i \leftrightarrow R_j$ ;
- ②  $R_i \leftarrow c \cdot R_i$ ; and
- ③  $R_i \leftarrow R_i + cR_j$