MATH 148 Personal Notes

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Chapter 1: The Riemann Integral

CODE KEY

- D : definition
- N : note
- R : remark
- L : lemma
- E : example
- C : corollary
- T : theorem
- NT : notation

PARTITION OF $[a,b]$ (D1.1)

D_1 A "partition" of the closed interval $[a,b]$ is any set $X = \{x_0, x_1, \dots, x_n\}$ such that

$$a = x_0 < x_1 < \cdots < x_n = b.$$

SUBINTERVAL (D1.1)

D_1 Let $X = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a,b]$.

Then a "sub-interval" of $[a,b]$ is any interval of the form $[x_{k-1}, x_k]$, where $k \in \{1, 2, \dots, n\}$, and denote them by

$$\Delta_k x = x_k - x_{k-1}.$$

D_2 Note that

$$\Delta_1 x + \Delta_2 x + \cdots + \Delta_n x = \sum_{k=1}^{n} \Delta_k x = b - a.$$

SIZE (D1.1)

D_1 Let X be a partition of $[a,b]$. Then the "size" of X, denoted as $|X|$, is defined to be

$$|X| = \max(\{\Delta_k x \mid 1 \leq k \leq n\}).$$

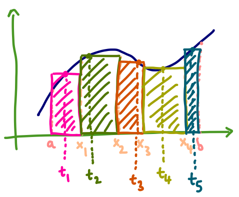
RIEMANN SUM (D1.2)

D_1 Let X be a partition of $[a,b]$, and let $f: [a,b] \to \mathbb{R}$ be bounded.

Then, a "Riemann sum" for f on X is a sum of the form

$$S = \sum_{k=1}^{n} f(t_k) \Delta_k x,$$

where $t_k \in [x_{k-1}, x_k]$ $\forall k \in \{1, 2, \dots, n\}$.



S = sum of areas of rectangles

= ▨ + ▨ + ▨ + ▨ + ▨.

*note that t_1, t_2, t_3, t_4 & t_5 are at arbitrary positions in their respective intervals!

(also note distance between consecutive x_n need not be constant).

SAMPLE POINTS (D1.2)

D_1 Let $S = \sum_{k=1}^{n} f(t_k) \Delta_k x$ be a Riemann sum for some bounded function $f: [a,b] \to \mathbb{R}$ on a partition X of $[a,b]$.

Then we say t_k is a "sample point" of S for any $k \in \{1, 2, \dots, n\}$.

RIEMANN INTEGRAL

(RIEMANN) INTEGRABLE (D1.3)

D_1 Let $f: [a,b] \to \mathbb{R}$ be bounded.

Then, we say f is "Riemann integrable", or just "integrable", on $[a,b]$ if there exists an $I \in \mathbb{R}$ such that for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any partition X of $[a,b]$ with $|X| < \delta$, we have that

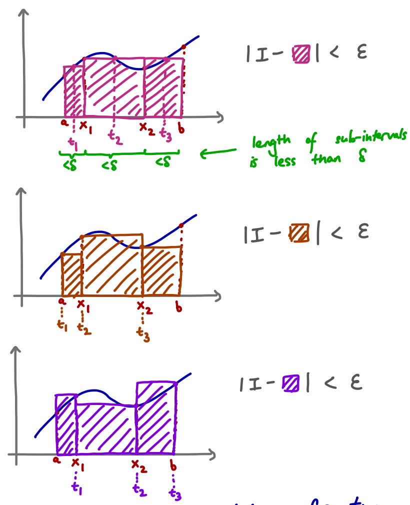
$$|S - I| < \varepsilon$$

for any Riemann sum S for f on X.

D_2 In other words, we have that

$$\left| \sum_{k=1}^{n} f(t_k) \Delta_k(x) - I \right| < \varepsilon$$

irregardless of our choices for $t_k \in [x_{k-1}, x_k]$.



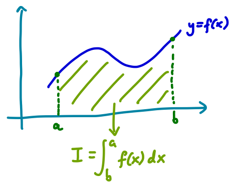
* regardless of our choices for t_k, we must always get $|$sum of areas $- I| < \varepsilon$!

(RIEMANN) INTEGRAL (D1.3)

D_1 We say the "Riemann integral" of f on $[a,b]$ is defined to be the number $I \in \mathbb{R}$ described above, and write

$$I = \int_a^b f = \int_a^b f(x)\, dx.$$

*I represents the "area under the graph".



D_2 We can prove I is unique.

Proof. Suppose I & J are two such numbers.

Let $\varepsilon > 0$ be arbitrary. Then, choose a $\delta_1 > 0$ such that for any partition X with $|X| < \delta_1$, we have $|S - I| < \frac{\varepsilon}{2}$ for every Riemann sum S on X.

Similarly, choose a $\delta_2 > 0$ such that for any partition X with $|X| < \delta_2$, we have $|S - J| < \frac{\varepsilon}{2}$ for every Riemann sum S on X.

Then, let $\delta = \min(\delta_1, \delta_2)$, and let X be any partition of $[a,b]$ with $|X| < \delta$.

For each $k \in \{1, 2, \dots, n\}$, choose a $t_k \in [x_{k-1}, x_k]$.

Let $S = \sum_{k=1}^{n} f(t_k) \Delta_k x$.

Then, by the Triangle Inequality, we have that

$$|I - J| \leq |I - S| + |S - J| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

But since $\varepsilon > 0$ was arbitrary, it follows that $I = J$, proving uniqueness. ▨

SOME FUNCTIONS ARE NOT INTEGRABLE (E1.4)

We can show that certain functions are not integrable on a specific closed interval.

Example: $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$ is not integrable on $[0,1]$.

Proof. Suppose f is integrable on $[0,1]$. Write $I = \int_0^1 f(x)\,dx$.

Let $\varepsilon = \frac{1}{2}$. Then, by definition, we can choose a $\delta > 0$ such that for every partition X with $|X| < \delta$, we have that $|S - I| < \frac{1}{2}$ for every Riemann sum S for f on X.

Then, choose some partition X with $|X| < \delta$.

Denote $S_1 = \sum_{k=1}^{n} f(t_k)\Delta_k x$ and $S_2 = \sum_{k=1}^{n} f(s_k)\Delta_k x$,

where $t_k \in \mathbb{Q}$ and $s_k \notin \mathbb{Q}$ and $t_k, s_k \in [x_{k-1}, x_k]$ for each $k \in \{1, 2, \dots, n\}$.

Note that $|S_1 - I| < \frac{1}{2}$ and $|S_2 - I| < \frac{1}{2}$, by our previous assumption that $\varepsilon = \frac{1}{2}$.

Subsequently, since $t_k \in \mathbb{Q}$ and $s_k \in \mathbb{R} \setminus \mathbb{Q}$ $\forall k \in \{1, 2, \dots, n\}$, it follows that $f(t_k) = 1$ and $f(s_k) = 0$; hence, we must get that

$$S_1 = \sum_{k=1}^{n} f(t_k)\Delta_k x = \sum_{k=1}^{n} \Delta_k x = 1 - 0 = 1$$

and

$$S_2 = \sum_{k=1}^{n} f(s_k)\Delta_k x = 0.$$

Thus, since $|S_1 - I| < \frac{1}{2}$ and $|S_2 - I| < \frac{1}{2}$, we must finally deduce that

$$|1 - I| < \frac{1}{2} \quad \text{and} \quad |I| < \frac{1}{2},$$

so that

$$\frac{1}{2} < I < \frac{3}{2} \quad \text{and} \quad -\frac{1}{2} < I < \frac{1}{2},$$

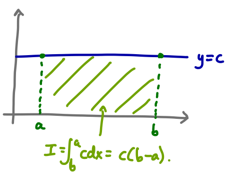
which is clearly impossible.

Therefore f is not integrable on $[0,1]$, which we wanted to show. ∎

THE INTEGRAL OF THE CONSTANT FUNCTION (E1.5)

The constant function $f(x) = c$ is always integrable on any interval $[a,b]$, and

$$\int_a^b c\,dx = c(b-a).$$



Proof. Let S be a Riemann sum for f on a partition X of $[a,b]$.

Then,

$$S = \sum_{k=1}^{n} f(t_k)\Delta_k x, \quad t_k \in [x_{k-1}, x_k] \ \forall k \in \{1, 2, \dots, n\}$$

$$= \sum_{k=1}^{n} c\,\Delta_k x$$

$$= c \sum_{k=1}^{n} \Delta_k x$$

$\therefore S = c(b-a)$.

But since S was arbitrary, it follows that

$$I = \int_a^b c\,dx = c(b-a),$$

as needed. ∎

THE INTEGRAL OF THE IDENTITY FUNCTION (E1.6)

The identity function $f(x) = x$ is also integrable on any interval $[a,b]$, and

$$\int_a^b x\,dx = \frac{1}{2}(b^2 - a^2).$$

Proof. Let $\varepsilon > 0$ be arbitrary, and let $\delta = \frac{2\varepsilon}{b-a}$.

Let X be any partition of $[a,b]$ with $|X| < \delta$.

Then, the Riemann sum S for f on X is equal to

$$S = \sum_{k=1}^{n} f(t_k)\Delta_k x = \sum_{k=1}^{n} t_k \Delta_k x,$$

where $t_k \in [x_{k-1}, x_k]$ $\forall k \in \{1, 2, \dots, n\}$.

Next, notice that

$$\sum_{k=1}^{n} (x_k + x_{k-1})\Delta_k x = \sum_{k=1}^{n} (x_k + x_{k-1})(x_k - x_{k-1})$$

$$= \sum_{k=1}^{n} (x_k^2 - x_{k-1}^2)$$

$$= (\cancel{x_1^2} - x_0^2) + (\cancel{x_2^2} - \cancel{x_1^2}) + \cdots + (x_n^2 - \cancel{x_{n-1}^2})$$

$$= x_n^2 - x_0^2$$

$$\therefore \sum_{k=1}^{n} (x_k + x_{k-1})\Delta_k x = b^2 - a^2.$$

Moreover, $t_k \in [x_{k-1}, x_k]$ implies that

$$\left|t_k - \tfrac{1}{2}(x_k + x_{k-1})\right| \le \tfrac{1}{2}(x_k - x_{k-1}) = \tfrac{1}{2}\Delta_k x,$$

and consequently it follows that

$$\left|S - \tfrac{1}{2}(b^2 - a^2)\right| = \left|\sum_{k=1}^{n} t_k \Delta_k x - \tfrac{1}{2}\sum_{k=1}^{n} (x_k + x_{k-1})\Delta_k x\right|$$

$$= \left|\sum_{k=1}^{n} \left(t_k - \tfrac{1}{2}(x_k + x_{k-1})\right)\Delta_k x\right|$$

$$\le \sum_{k=1}^{n} \left|t_k - \tfrac{1}{2}(x_k + x_{k-1})\right|\Delta_k x$$

$$\le \sum_{k=1}^{n} \left(\tfrac{1}{2}\Delta_k x\right)\Delta_k x$$

$$\le \sum_{k=1}^{n} \tfrac{1}{2}\delta(b-a) \quad \text{(since } \Delta_k x < \delta \text{ and } \Delta_k x = b-a \text{ by definition)}$$

$$= \varepsilon, \quad \text{(since } \delta = \tfrac{2\varepsilon}{b-a}\text{)}$$

So that $\left|S - \frac{1}{2}(b^2 - a^2)\right| \le \varepsilon$.

But as $\varepsilon > 0$ was arbitrary, this tells us that

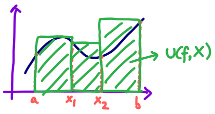
$$I = \int_a^b x\,dx = \frac{1}{2}(b^2 - a^2),$$

which we wanted to prove. ∎

UPPER & LOWER RIEMANN SUMS (D1.7)

Let X be a partition of $[a,b]$, and let $f:[a,b]\to\mathbb{R}$ be bounded.

Then, the "upper Riemann sum" for f on X, denoted by $U(f,X)$, is defined to be

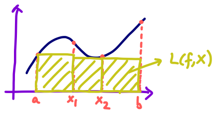


$$U(f,X)=\sum_{k=1}^{n} M_k\Delta_k x,$$

where

$$M_k=\sup(\{f(t):\ t\in[x_{k-1},x_k]\})\quad \forall k\in\{1,2,\dots,n\}.$$

Similarly, the "lower Riemann sum" for f on X, denoted by $L(f,X)$, is defined to be



$$L(f,X)=\sum_{k=1}^{n} m_k\Delta_k x,$$

where

$$m_k=\inf(\{f(t):\ t\in[x_{k-1},x_k]\}\quad \forall k\in\{1,2,\dots,n\}).$$

U(f,X) & L(f,X) ARE NOT ALWAYS RIEMANN SUMS (R1.8)

Note that, in general, $U(f,X)$ and $L(f,X)$ are not always Riemann sums, as we do not always have that $M_k=f(t_k)$ or $m_k=f(s_k)$ $\forall t_k,s_k\in[x_{k-1},x_k]$, where $k\in\{1,2,\dots,n\}$.

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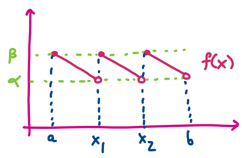


$m_2=\inf(\{f(t):\ t\in[x_1,x_2]\})$

$\therefore m_2=\alpha$

But $f(t)\neq\alpha$ $\forall t\in[x_1,x_2]$!

(So $L(f,X)$ is not a Riemann sum.)

U(f,X) & L(f,X) ARE RIEMANN SUMS IF f IS STRICTLY MONOTONIC (R1.8)

Let X be a partition of $[a,b]$, and $f:[a,b]\to\mathbb{R}$ be bounded.

Assume f is strictly monotonic (ie increasing or decreasing).

Then $U(f,X)$ and $L(f,X)$ are Riemann sums for f on X.

Proof. If f is increasing, then $M_k=f(x_k)$ and $m_k=f(x_{k-1})$ $\forall k\in\{1,2,\dots,n\}$, so that $U(f,X)$ and $L(f,X)$ are indeed Riemann sums.

Similarly, if f is decreasing, then $M_k=f(x_{k-1})$ and $m_k=f(x_k)$ $\forall k\in\{1,2,\dots,n\}$ so that $U(f,X)$ and $L(f,X)$ are indeed Riemann sums. ∎

U(f,X) & L(f,X) ARE RIEMANN SUMS IF f IS CONTINUOUS (R1.8)

Let X be a partition of $[a,b]$, and let $f:[a,b]\to\mathbb{R}$ be bounded.

Assume f is continuous on $[a,b]$.

Then $U(f,X)$ and $L(f,X)$ are Riemann sums for f on X.

Proof. By the Extreme Value Theorem, there necessarily exists some $t_k,s_k\in[x_{k-1},x_k]$ such that $f(s_k)\leqslant f(t)\leqslant f(t_k)$ $\forall t\in[x_{k-1},x_k]$ for each $k\in\{1,2,\dots,n\}$.

Let $m_k=f(s_k)$ and $M_k=f(t_k)$. Then since $t_k,s_k\in[x_{k-1},x_k]$, it follows that $U(f,X)$ and $L(f,X)$ are indeed Riemann sums for f on X. ∎

U(f,X) IS THE LARGEST RIEMANN SUM & L(f,X) IS THE SMALLEST RIEMANN SUM FOR f ON X (N1.9)

Let $\mathcal{T}$ be the set of all Riemann sums for a bounded function $f:[a,b]\to\mathbb{R}$ on a partition X of $[a,b]$.

Then $U(f,X)=\sup(\mathcal{T})$ and $L(f,X)=\inf(\mathcal{T})$.

In particular, we have that

$$L(f,X)\leqslant S\leqslant U(f,X)$$

for every $S\in\mathcal{T}$.

Proof. We prove the former statement, since the proof for the latter is similar.

Then, note for any $S\in\mathcal{T}$, we have that

$$S=\sum_{k=1}^{n} f(t_k)\Delta_k x\leqslant\sum_{k=1}^{n} M_k\Delta_k x=U(f,X),$$

by construction of M_k.

Hence $U(f,X)$ is an upper bound for $\mathcal{T}$, and so necessarily $U(f,X)\geqslant\sup(\mathcal{T})$.

Next, let $\varepsilon>0$ be arbitrary.

Then, since $M_k=\sup(\{f(t)\mid t\in[x_{k-1},x_k]\})$, we can choose a $t_k\in[x_{k-1},x_k]$ with $M_k-f(t_k)<\frac{\varepsilon}{b-a}$, for every $k\in\{1,2,\dots,n\}$.

Hence, it follows that there exists a $S\in\mathcal{T}$ such that

$$\begin{aligned}U(f,X)-S&=\sum_{k=1}^{n}M_k\Delta_k x-\sum_{k=1}^{n}f(t_k)\Delta_k x\\&=\sum_{k=1}^{n}(M_k-f(t_k))\Delta_k x\\&<\sum_{k=1}^{n}\frac{\varepsilon}{b-a}\Delta_k x\\&=\frac{\varepsilon}{b-a}(b-a)\end{aligned}$$

$\therefore U(f,X)-S<\varepsilon,$

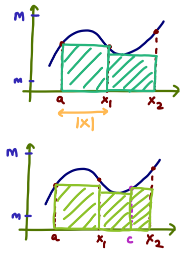
and since $\varepsilon>0$ was arbitrary it follows that

$$\sup(\mathcal{T})=U(f,X),$$

as needed. ∎

$0 \le L(f, X \cup \{c\}) - L(f,X) \le (M-m)|X|,$
$0 \le U(f, X) - U(f, X \cup \{c\}) \le (M-m)|X|$ (L1.10)

Let $f: [a,b] \to \mathbb{R}$ be bounded with upper bound M and lower bound m.
Let X and $Y = X \cup \{c\}$ be partitions of $[a,b]$, where $c \notin X$.
Then
$$0 \le L(f,Y) - L(f,X) \le (M-m)|X|,$$
and
$$0 \le U(f,X) - U(f,Y) \le (M-m)|X|.$$



$(0 \le \square - \square \le (M-m)|X|)$

*similar illustration for second statement

Proof. We prove the first statement, as the proof for the second statement is similar.

Say $X = \{x_0, x_1, \dots, x_n\}$ and $c \in [x_{k-1}, x_k]$ for some $k \in \{1, 2, \dots, n\}$, so that
$$Y = \{x_0, x_1, \dots, x_{k-1}, c, x_k, \dots, x_n\}.$$

Then
$$L(f,Y) - L(f,X) = [r(c - x_{k-1}) + s(x_k - c)] - m_k(x_k - x_{k-1}),$$
where $r = \inf(\{f(t) \mid t \in [x_{k-1}, c]\})$, $s = \inf(\{f(t) \mid t \in [c, x_k]\})$
and $m_k = \inf(\{f(t) \mid t \in [x_{k-1}, x_k]\})$.

Next, since $m_k = \min(r, s)$, it follows that $r \ge m_k$ & $s \ge m_k$ so that
$$L(f,Y) - L(f,X) \ge m_k(c - x_{k-1}) + m_k(x_k - c) - m_k(x_k - x_{k-1}) = 0,$$
establishing the first inequality.

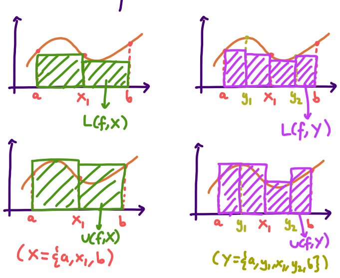
Then, since $r \le M$ and $s \le M$, & $m_k \ge m$ by construction, it also follows that
$$\begin{aligned} L(f,Y) - L(f,X) &\le M(c - x_{k-1}) + M(x_k - c) - m(x_k - x_{k-1}) \\ &= (M-m)(x_k - x_{k-1}) \\ &\le (M-m)|X|. \end{aligned}$$

Therefore, we have that
$$0 \le L(f,Y) - L(f,X) \le (M-m)|X|,$$
which we wanted to prove. ∎

$L(f,X) \le L(f,Y) \le U(f,Y) \le U(f,X)$; $X \subseteq Y$ (N1.11)

Let X and Y be partitions of $[a,b]$, such that $X \subseteq Y$.
Let $f: [a,b] \to \mathbb{R}$ be bounded.
Then, we always have that
$$L(f,X) \le L(f,Y) \le U(f,Y) \le U(f,X).$$

Proof. If Y is obtained by adding one point to X, then this follows from the above lemma.
In general, Y can be obtained by adding finitely many points to X, one point at a time. ∎



$L(f,X) \le L(f,Y) \le U(f,Y) \le U(f,Y)$

$L(f,X) \le U(f,Y)$ (N1.12)

Let X and Y be any partitions of $[a,b]$.
Then necessarily $L(f,X) \le U(f,Y)$.

Proof. Let $Z = X \cup Y$. Then by the above note,
$$L(f,X) \le L(f,Z) \le U(f,Z) \le U(f,Y),$$
and the proof follows from here. ∎

UPPER & LOWER INTEGRALS (D1.13)

1. Let $f:[a,b]\to\mathbb{R}$ be bounded.
Then, the "upper integral" of f on $[a,b]$, denoted by $U(f)$, is defined to be
$$U(f) = \inf(\{U(f,X) \mid X \text{ is a partition of } [a,b]\}).$$

2. Similarly, the "lower integral" of f on $[a,b]$, denoted by $L(f)$, is defined to be
$$L(f) = \sup(\{L(f,X) \mid X \text{ is a partition of } [a,b]\}).$$

3. Note that $U(f)$ and $L(f)$ always exist even if f is not integrable. (N1.14)

$L(f) \leqslant U(f)$ (N1.15)

Let $f:[a,b]\to\mathbb{R}$ be a bounded function.
Then $L(f)\leqslant U(f)$.

Proof. Let $\varepsilon>0$ be arbitrary.
Choose partitions X_1 and X_2 such that
$L(f)-L(f,X_1)<\frac{\varepsilon}{2}$ and $U(f,X_2)-U(f)<\frac{\varepsilon}{2}$.
Then
$$\begin{aligned} U(f)-L(f) &= (U(f)-U(f,X_2)) + (U(f,X_2)-L(f,X_1)) + (L(f,X_1)-L(f)) \\ &> -\tfrac{\varepsilon}{2}+0-\tfrac{\varepsilon}{2} \\ &= -\varepsilon. \end{aligned}$$
Hence $L(f)-U(f)<\varepsilon$, and since $\varepsilon>0$ was arbitrary, this in turn implies that $L(f)\leqslant U(f)$. ☒

EQUIVALENT DEFINITIONS OF INTEGRABILITY (T1.16)

Let $f:[a,b]\to\mathbb{R}$ be bounded.
Then the following statements are equivalent:

① $L(f)=U(f)$;

② For any $\varepsilon>0$, there exists a partition X such that $U(f,X)-L(f,X)<\varepsilon$; and

③ f is integrable on $[a,b]$.

Proof. First, we show ① ⇒ ②.
Suppose $L(f)=U(f)$. Let $\varepsilon>0$ be arbitrary.
Then, choose partitions X_1 and X_2 so that
$L(f)-L(f,X_1)<\frac{\varepsilon}{2}$ and $U(f,X_2)-U(f)<\frac{\varepsilon}{2}$.
Let $X=X_1\cup X_2$.
Next, since $L(f,X_1)\leqslant L(f,X)\leqslant L(f)$ (as $X_1\subseteq X$),
it follows that $L(f)-L(f,X)\leqslant L(f)-L(f,X_1)<\frac{\varepsilon}{2}$,
and since $U(f)\leqslant U(f,X)\leqslant U(f,X_2)$ (as $X_2\subseteq X$), it follows that
$U(f,X)-U(f)<\frac{\varepsilon}{2}$ also.
Hence
$$\begin{aligned} U(f,X)-L(f,X) &= [U(f,X)-U(f)] + [U(f)-L(f)] + [L(f)-L(f,X)] \\ &< \tfrac{\varepsilon}{2}+0+\tfrac{\varepsilon}{2} \\ &= \varepsilon, \end{aligned}$$
which is sufficient to show that ② is true. ※

Subsequently, we show ② ⇒ ①.
Suppose for any $\varepsilon>0$, there exists a partition X such that $U(f,X)-L(f,X)<\varepsilon$.
Fix $\varepsilon>0$, and choose X so that $U(f,X)-L(f,X)<\varepsilon$.
Then
$$\begin{aligned} U(f)-L(f) &= [U(f)-U(f,X)] + [U(f,X)-L(f,X)] + [L(f,X)-L(f)] \\ &< 0+\varepsilon+0 \\ &= \varepsilon. \end{aligned}$$
Since $0\leqslant U(f)-L(f)<\varepsilon$ $\forall\varepsilon>0$, this tells us that $U(f)=L(f)$, proving ①. ※

Next, we show ③ ⇒ ②.
Suppose f is integrable on $[a,b]$, with $I=\int_a^b f(x)\,dx$.
Let $\varepsilon>0$. Then, choose a $\delta>0$ such that for every partition X with $|X|<\delta$, we have that $|S-I|<\frac{\varepsilon}{4}$ for every Riemann sum S for f on X.
Let S_1 and S_2 be Riemann sums for f on X such that $|U(f,X)-S_1|<\frac{\varepsilon}{4}$ and $|S_2-L(f,X)|<\frac{\varepsilon}{4}$.
Then, by the Triangle Inequality,
$$\begin{aligned} |U(f,X)-L(f,X)| &\leqslant |U(f,X)-S_1| + |S_1-I| + |I-S_2| + |S_2-L(f,X)| \\ &< \tfrac{\varepsilon}{4}+\tfrac{\varepsilon}{4}+\tfrac{\varepsilon}{4}+\tfrac{\varepsilon}{4} \\ &= \varepsilon, \end{aligned}$$
which is sufficient to prove ②. ※

Lastly, we prove ① ⇒ ③.
Suppose $L(f)=U(f)$, and let $I=L(f)=U(f)$.
Then, let $\varepsilon>0$.
Choose a partition X_0 of $[a,b]$ so that
$L(f)-L(f,X_0)<\frac{\varepsilon}{2}$ and $U(f,X_0)-U(f)<\frac{\varepsilon}{2}$.
Say $X_0=\{x_0,x_1,\dots,x_n\}$, and set $\delta=\frac{\varepsilon}{2(n-1)(M-m)}$,
where M and m are upper and lower bounds for f on $[a,b]$.
Let X be any partition of $[a,b]$ with $|X|<\delta$,
and $Y=X_0\cup X$.
Note that Y is obtained from X by adding at most $n-1$ points, and that each time we add a point, the size of the new partition is at most $|X|<\delta$.
Hence
$$0\leqslant U(f,X)-U(f,Y)\leqslant(n-1)(M-m)|X|<(n-1)(M-m)\delta=\tfrac{\varepsilon}{2}$$
and
$$0\leqslant L(f,Y)-L(f,X)\leqslant(n-1)(M-m)|X|<(n-1)(M-m)\delta=\tfrac{\varepsilon}{2}$$
by the first lemma on the previous page.
Next, let S be any Riemann sum for f on X.
Note that $L(f,X_0)\leqslant L(f,Y)\leqslant L(f)=U(f)\leqslant U(f,Y)\leqslant U(f,X_0)$
and $L(f,X)\leqslant S\leqslant U(f,X)$, so that
$$\begin{aligned} S-I &\leqslant U(f,X)-I \\ &= U(f,X)-U(f) \\ &= (U(f,X)-U(f,Y)) + (U(f,Y)-U(f)) \\ &\leqslant (U(f,X)-U(f,Y)) + (U(f,X_0)-U(f)) \\ &< \tfrac{\varepsilon}{2}+\tfrac{\varepsilon}{2} = \varepsilon, \end{aligned}$$
and
$$\begin{aligned} I-S &\leqslant I-L(f,X) \\ &= L(f)-L(f,X) \\ &= (L(f)-L(f,Y)) + (L(f,Y)-L(f,X)) \\ &\leqslant (L(f)-L(f,X_0)) + (L(f,Y)-L(f,X)) \\ &< \tfrac{\varepsilon}{2}+\tfrac{\varepsilon}{2} = \varepsilon, \end{aligned}$$
and since $\varepsilon>0$ was arbitrary this is sufficient to prove ③. ☒

INTEGRALS OF CONTINUOUS FUNCTIONS

CONTINUOUS FUNCTIONS ARE ALWAYS INTEGRABLE (T1.17)

Let $f: [a,b] \to \mathbb{R}$ be continuous.
Then f is integrable on $[a,b]$.

Proof. First, note f is uniformly continuous on $[a,b]$.
Hence, we can choose a $\delta > 0$ so that for all $x, y \in [a,b]$, we have that $|x-y| < \delta$ implies $|f(x) - f(y)| < \frac{\varepsilon}{b-a}$.

Let X be any partition of $[a,b]$ with $|X| < \delta$.

Then, by the Extreme Value Theorem, there exists some $t_k, s_k \in [x_{k-1}, x_k]$ such that $m_k = f(s_k) \leq t \leq M_k = f(t_k)$ $\forall t \in [x_{k-1}, x_k]$, where $k \in \{1, 2, \ldots, n\}$.

Finally, since $|t_k - s_k| \leq |x_k - x_{k-1}| \leq |X| = \delta$, it follows that $|M_k - m_k| = |f(t_k) - f(s_k)| < \frac{\varepsilon}{b-a}$ (since f is uniformly continuous).

Thus

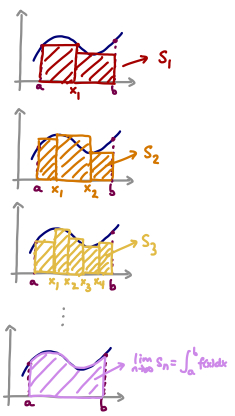
$$\begin{aligned} U(f,X) - L(f,X) &= \sum_{k=1}^{n} (M_k - m_k)\Delta_k x \\ &< \frac{\varepsilon}{b-a} \sum_{k=1}^{n} \Delta_k x \\ &= \varepsilon, \end{aligned}$$

and as $\varepsilon > 0$ was arbitrary this tells us that $U(f,X) = L(f,X)$, which by the equivalent definitions of integrability implies that f is integrable on $[a,b]$. ▨

SEQUENTIAL CHARACTERISATION OF INTEGRATION (N1.18)

Let $f: [a,b] \to \mathbb{R}$ be integrable on $[a,b]$, and let $\{X_n\}$ be a sequence of partitions of $[a,b]$ with $\lim_{n\to\infty} |X_n| = 0$.
For any given $n \in \mathbb{N}$, let S_n be any Riemann sum for f on X_n.
Then the sequence $\{S_n\}$ necessarily converges, with

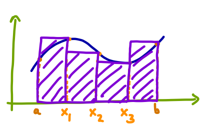
$$\lim_{n\to\infty} S_n = \int_a^b f(x)\,dx.$$



Proof. Denote $I = \int_a^b f(x)\,dx$.
Then, given a $\varepsilon > 0$, choose a $\delta > 0$ so that for every partition X of $[a,b]$ with $|X| < \delta$, we have that $|S - I| < \varepsilon$ for every Riemann sum S for f on X.
Choose a $N \in \mathbb{N}$ so that if $n > N$, $|X_n| < \delta$. (We can do this since $\{|X_n|\} \to 0$.)
It follows that if $n > N$, then $|S_n - I| < \varepsilon$,
and as $\varepsilon > 0$ was arbitrary this is sufficient to prove that $\lim_{n\to\infty} S_n = \int_a^b f(x)\,dx$. ▨

Let $f: [a,b] \to \mathbb{R}$ be integrable on $[a,b]$.
Then, if we let X_n be the partition of $[a,b]$ into n equal-sized sub-intervals, and S_n be the Riemann sum on X_n using right-endpoints, it follows from the above that

$$\begin{aligned} \int_a^b f(x)\,dx &= \lim_{n\to\infty} \sum_{k=1}^{n} f(x_{n,k})\Delta_{n,k} x \\ &= \lim_{n\to\infty} \left(\frac{b-a}{n}\right) \sum_{k=1}^{n} f\left(a + \frac{b-a}{n}k\right). \end{aligned} \quad \text{(N1.19)}$$



Note $x_1 - a = x_2 - x_1 = x_3 - x_2 = b - x_3$,
so that
$x_n - x_{n-1} = \frac{b-a}{4}$ for each $n \in \{1,2,3,4\}$.
Hence

$$\begin{aligned} ▨ &= \left(\frac{b-a}{4}\right)[f(x_1) + f(x_2) + f(x_3) + f(x_4)] \\ &= \left(\frac{b-a}{4}\right) \sum_{k=1}^{4} f(x_k). \end{aligned}$$

EXAMPLE: INTEGRAL OF $f(x) = 2^x$ (E1.20)

We can use the previous derived results to evaluate integrals of specific continuous functions;
eg $\int_0^2 2^x\,dx$.

Let $f(x) = 2^x$. Note f is continuous, and hence integrable (on $[0,2]$).
Then, using the formula from N1.19, we have that

$$\begin{aligned} \int_0^2 2^x\,dx &= \lim_{n\to\infty} \sum_{k=1}^{n} f(x_{n,k})\Delta_{n,k} x \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} f\left(\tfrac{2k}{n}\right)\left(\tfrac{2}{n}\right) \quad (\text{since } |[0,2]| = 2) \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} 2^{\frac{2k}{n}} \cdot \frac{2}{n} \\ &= \lim_{n\to\infty} \frac{2 \cdot 4^{\frac{1}{n}}}{n} \cdot \frac{4-1}{4^{\frac{1}{n}} - 1} \quad (\text{by the formula for the sum of a geometric sequence}) \\ &= \lim_{n\to\infty} (6 \cdot 4^{\frac{1}{n}}) \lim_{n\to\infty} \frac{1}{n(4^{\frac{1}{n}} - 1)} \\ &= 6 \lim_{n\to\infty} \frac{(\frac{1}{n})}{4^{\frac{1}{n}} - 1} \\ &= 6 \lim_{x\to 0^+} \frac{x}{4^x - 1} \\ &= 6 \lim_{x\to 0^+} \frac{1}{\ln(4) \cdot 4^x} \quad (\text{by L'Hôpital's rule, since } \tfrac{x}{4^x - 1} = \tfrac{0}{0} \text{ if } x = 0) \\ &= \frac{6}{\ln(4)} \end{aligned}$$

$$\therefore \int_0^2 2^x\,dx = \frac{3}{\ln(2)}.$$

SUMMATION FORMULAS (L1.21)

Note that
① $\sum_{i=1}^{n} 1 = n$;
② $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$;
③ $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$; and
④ $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$.

Proof. ① is trivial, so we prove ② first.
Consider $\sum_{k=1}^{n} (k^2 - (k-1)^2)$.
On one hand,

$$\sum_{k=1}^{n} (k^2 - (k-1)^2) = (1^2 - 0^2) + (2^2 - 1^2) + \cdots + (n^2 - (n-1)^2) = n^2,$$

but on the other hand,

$$\sum_{k=1}^{n} (k^2 - (k-1)^2) = \sum_{k=1}^{n} (k^2 - (k^2 - 2k + 1)) = 2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1.$$

Hence $n^2 = 2\sum_{k=1}^{n} k - n$,
so that

$$\sum_{k=1}^{n} k = \frac{1}{2}(n^2 + n) = \frac{n(n+1)}{2}. \quad \#$$

Next, we prove ③.
Consider $\sum_{k=1}^{n} (k^3 - (k-1)^3)$.
On one hand,

$$\sum_{k=1}^{n} (k^3 - (k-1)^3) = (1^3 - 0^3) + (2^3 - 1^3) + \cdots + (n^3 - (n-1)^3) = n^3.$$

On the other hand,

$$\begin{aligned} \sum_{k=1}^{n} (k^3 - (k-1)^3) &= \sum_{k=1}^{n} (k^3 - (k^3 - 3k^2 + 3k - 1)) \\ &= 3\sum_{k=1}^{n} k^2 - 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 \\ &= 3\sum_{k=1}^{n} k^2 - 3\frac{n(n+1)}{2} + n. \end{aligned}$$

Equating these, we get that

$$n^3 = 3\sum_{k=1}^{n} k^2 - 3\frac{n(n+1)}{2} + n$$

which eventually simplifies to

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}. \quad \#$$

Lastly, we prove ④.
Consider $\sum_{k=1}^{n} (k^4 - (k-1)^4)$.
On one hand,

$$\sum_{k=1}^{n} (k^4 - (k-1)^4) = (1^4 - 0^4) + (2^4 - 1^4) + \cdots + (n^4 - (n-1)^4) = n^4,$$

and on the other hand,

$$\begin{aligned} \sum_{k=1}^{n} (k^4 - (k-1)^4) &= 4\sum_{k=1}^{n} k^3 - 6\sum_{k=1}^{n} k^2 + 4\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1 \\ &= 4\sum_{k=1}^{n} k^3 - 6\frac{n(n+1)(2n+1)}{6} + 4\frac{n(n+1)}{2} - n. \end{aligned}$$

Hence

$$n^4 = 4\sum_{k=1}^{n} k^3 - 6\frac{n(n+1)(2n+1)}{6} + 4\frac{n(n+1)}{2} - n,$$

which simplifies to

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}. \quad \#$$

USING SUMMATION FORMULAS TO CALCULATE INTEGRALS (E1.22)

We can use summation formulae to calculate integrals of certain functions;

eg $\int_1^3 (x+2x^3)\,dx$.

Note that

$$\begin{aligned}\int_1^3 (x+2x^3)\,dx &= \lim_{n\to\infty}\sum_{k=1}^{n} f(x_{n,k})\,\Delta_{n,k}x \\ &= \lim_{n\to\infty}\sum_{k=1}^{n} f(1+\tfrac{2}{n}k)\left(\tfrac{2}{n}\right) \quad (\text{since } |[1,3]|=2) \\ &= \lim_{n\to\infty}\sum_{k=1}^{n}\left((1+\tfrac{2}{n}k)+2(1+\tfrac{2}{n}k)^3\right)\left(\tfrac{2}{n}\right) \\ &= \lim_{n\to\infty}\sum_{k=1}^{n}\left(\tfrac{6}{n}+\tfrac{28}{n^2}k+\tfrac{48}{n^3}k^2+\tfrac{32}{n^4}k^3\right) \\ &= \lim_{n\to\infty}\left(\tfrac{6}{n}\sum_{k=1}^{n}1+\tfrac{28}{n^2}\sum_{k=1}^{n}k+\tfrac{48}{n^3}\sum_{k=1}^{n}k^2+\tfrac{32}{n^4}\sum_{k=1}^{n}k^3\right) \\ &= \lim_{n\to\infty}\left(\tfrac{6}{n}\cdot n+\tfrac{28}{n^2}\cdot\tfrac{n(n+1)}{2}+\tfrac{48}{n^3}\cdot\tfrac{n(n+1)(2n+1)}{6}+\tfrac{32}{n^4}\cdot\tfrac{n^2(n+1)^2}{4}\right) \\ &= 6+\tfrac{28}{2}+\tfrac{48}{6}+\tfrac{32}{4}\end{aligned}$$

$\therefore \int_1^3 (x+2x^3)\,dx = 44$.

BASIC PROPERTIES OF INTEGRALS

LINEARITY (T1.23)

Let f and g be integrable on $[a,b]$.
Let $c\in\mathbb{R}$ be arbitrary.
Then $(f+g)$ and $c\cdot f$ are both integrable on $[a,b]$, and

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g$$

and

$$\int_a^b cf = c\int_a^b f.$$

Proof. Note that

$$\begin{aligned}\int_a^b f+\int_a^b g &= \lim_{n\to\infty}\sum_{k=1}^{n} f(x_{n,k})\,\Delta_{n,k}x + \lim_{n\to\infty}\sum_{k=1}^{n} g(x_{n,k})\,\Delta_{n,k}x \\ &= \lim_{n\to\infty}\frac{b-a}{n}\sum_{k=1}^{n}(f+g)(x_{n,k}) \\ &= \int_a^b (f+g),\end{aligned}$$

and that

$$\begin{aligned}c\int_a^b f &= c\lim_{n\to\infty}\sum_{k=1}^{n} f(x_{n,k})\,\Delta_{n,k}x \\ &= \lim_{n\to\infty}\frac{b-a}{n}\sum_{k=1}^{n}(cf)(x_{n,k}) \\ &= \int_a^b cf. \quad \blacksquare\end{aligned}$$

COMPARISON (T1.24)

Let f and g be integrable on $[a,b]$.
Suppose $f(x)\leq g(x)$ $\forall x\in[a,b]$.
Then

$$\int_a^b f \leq \int_a^b g.$$

Proof. Note that

$$\begin{aligned}\int_a^b f &= \lim_{n\to\infty}\sum_{k=1}^{n} f(x_{n,k})\,\Delta_{n,k}x \\ &\leq \lim_{n\to\infty}\sum_{k=1}^{n} g(x_{n,k})\,\Delta_{n,k}x \quad (\text{since } f(x)\leq g(x)\ \forall x\in[a,b]) \\ &= \int_a^b g. \quad \blacksquare\end{aligned}$$

ADDITIVITY (T1.25)

Let $a<b<c$, and $f:[a,c]\to\mathbb{R}$ be bounded.
Then f is integrable on $[a,c]$ if and only if f is integrable on both $[a,b]$ and $[b,c]$,
and in this case

$$\int_a^b f + \int_b^c f = \int_a^c f.$$

Proof. First, suppose f is integrable on $[a,c]$.
Choose a partition X of $[a,c]$ such that $U(f,X)-L(f,X)<\varepsilon$.
Say that $b\in[x_{k-1},x_k]$, and let $Y=\{x_0,x_1,\ldots,x_{k-1},b\}$
and $Z=\{b,x_k,x_{k+1},\ldots,x_n\}$, so that Y and Z are partitions on $[a,b]$ and $[b,c]$ respectively.

Then

$$\begin{aligned}U(f,Y)-L(f,Y) &\leq U(f,X\cup\{b\})-L(f,X\cup\{b\}) \quad (\text{by N1.11}) \\ &\leq U(f,X)-L(f,X) \quad (\text{by N1.11 also}) \\ &< \varepsilon,\end{aligned}$$

and

$$\begin{aligned}U(f,Z)-L(f,Z) &\leq U(f,X\cup\{b\})-L(f,X\cup\{b\}) \\ &\leq U(f,X)-L(f,X) \\ &< \varepsilon,\end{aligned}$$

which is sufficient to show f is integrable on both $[a,b]$ and $[b,c]$.

Conversely, suppose f is integrable on both $[a,b]$ and $[b,c]$.
Choose partitions Y of $[a,b]$ & Z of $[b,c]$ so that $U(f,Y)-L(f,Y)<\frac{\varepsilon}{2}$ and $U(f,Z)-L(f,Z)<\frac{\varepsilon}{2}$.
Then $X=Y\cup Z$ is a partition of $[a,c]$, and

$$U(f,X)-L(f,X) = [U(f,Y)+U(f,Z)]-(L(f,Y)+L(f,Z)] < \varepsilon,$$

which tells us $U(f,X)=L(f,X)$ (since $\varepsilon>0$ was arbitrary) and consequently (by the Equivalent Definitions of Integrability) that f is integrable on $[a,c]$.

Finally, suppose f is integrable on $[a,c]$, and hence also on $[a,b]$ and $[b,c]$.
Let $I_1=\int_a^b f$, $I_2=\int_b^c f$ and $I=\int_a^c f$.
Let $\varepsilon>0$ be arbitrary. Then, choose a $\delta>0$ so that for all partitions X_1,X_2 and X of $[a,b]$, $[b,c]$ and $[a,c]$ respectively, if $|X_1|,|X_2|,|X|<\delta$, then $|S_1-I_1|,\ |S_2-I_2|,\ |S-I|<\frac{\varepsilon}{3}$ for all Riemann sums S_1,S_2,S for f on X_1,X_2 & X respectively.
Choose partitions X_1 and X_2 of $[a,b]$ and $[b,c]$ with $|X_1|<\delta$ and $|X_2|<\delta$.
Choose Riemann sums S_1 and S_2 for f on X_1 and X_2.
Let $X=X_1\cup X_2$, and note that $|X|<\delta$ and $S=S_1+S_2$ is a Riemann sum for f on X.
Then necessarily

$$\begin{aligned}|I-(I_1+I_2)| &= |(I-S)+(S_1-I_1)+(S_2-I_2)| \\ &\leq |I-S|+|S_1-I_1|+|S_2-I_2| \quad (\text{by the Triangle Inequality}) \\ &\leq \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3} \\ &= \varepsilon,\end{aligned}$$

and since $\varepsilon>0$ was arbitrary this is sufficient to prove that $I=I_1+I_2$. $\blacksquare$

PIECEWISE CONTINUOUS FUNCTIONS ARE INTEGRABLE (C1.26)

Let $X = \{x_0, x_1, \cdots, x_n\}$ be a partition of $[a,b]$,
and let $g_k : [x_{k-1}, x_k] \to \mathbb{R}$ be continuous $\forall k \in \{1, 2, \cdots, n\}$.
Let $f : [a,b] \to \mathbb{R}$ be a function with $f(t) = g_k(t)$ $\forall t \in (x_{k-1}, x_k)$.
Then f is integrable on $[a,b]$ with

$$\int_a^b f(x)\,dx = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} g_k(x)\,dx.$$

Proof. This follows from the additivity and linearity properties of integrals. ☒

$\int_a^a f = 0$, $\int_b^a f = -\int_a^b f$ (D1.27)

For any function f and $a \in \mathbb{R}$,

$$\int_a^a f = 0.$$

Additionally, if $\int_a^b f(x)\,dx$ exists, then

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

Note that this definition can be used to extend the scope of the Additivity Theorem to the case where $a, b, c \in \mathbb{R}$ are not in increasing order. (N1.28)

ESTIMATION (T1.29)

Let f be integrable on $[a,b]$.
Then $|f|$ is also integrable on $[a,b]$,
and

$$\left| \int_a^b f \right| \leq \int_a^b |f|.$$

Proof. Let $\varepsilon > 0$ be arbitrary.
Choose a partition X of $[a,b]$ such that $U(f,X) - L(f,X) < \varepsilon$.
Denote $M_k(f) = \sup\{f(t) \mid t \in [x_{k-1}, x_k]\}$ and
$M_k(|f|) = \sup\{|f(t)| \mid t \in [x_{k-1}, x_k]\}$ $\forall k \in \{1, 2, \cdots, n\}$,
with similar definitions for $m_k(f)$ and $m_k(|f|)$.
Then,

① if $0 \leq m_k(f) \leq M_k(f)$, $M_k(|f|) = M_k(f)$ and $m_k(|f|) = m_k(f)$;

② if $m_k(f) \leq 0 \leq M_k(f)$, $M_k(|f|) = \max(\{M_k(f), -m_k(f)\})$ and $m_k(|f|) \geq 0$,
so that $M_k(|f|) - m_k(|f|) \leq \max(\{M_k(f), -m_k(f)\}) \leq M_k(f) - m_k(f)$;
&

③ if $m_k(f) \leq M_k(f) \leq 0$, $M_k(|f|) = -m_k(f)$ and $m_k(|f|) = -M_k(f)$,
so that $M_k(|f|) - m_k(|f|) = M_k(f) - m_k(f)$.

In any one of these cases, we have that

$$M_k(|f|) - m_k(|f|) \leq M_k(f) - m_k(f),$$

and so

$$\begin{aligned} U(|f|, X) - L(|f|, X) &= \sum_{k=1}^{n} (M_k(|f|) - m_k(|f|))\,\Delta_k x \\ &\leq \sum_{k=1}^{n} (M_k(f) - m_k(f))\,\Delta_k x \\ &= U(f,X) - L(f,X) \\ &< \varepsilon, \end{aligned}$$

which is sufficient to prove that $|f|$ is integrable on $[a,b]$.

Next, let $\varepsilon > 0$ be arbitrary.
Choose a partition X on $[a,b]$ and choose values $t_k \in [x_{k-1}, x_k]$ $\forall k \in \{1, 2, \cdots, n\}$ so that

$$\left| \sum_{k=1}^{n} f(t_k)\,\Delta_k x - \int_a^b f \right| < \frac{\varepsilon}{2} \quad \text{and} \quad \left| \sum_{k=1}^{n} |f(t_k)|\,\Delta_k x - \int_a^b |f| \right| < \frac{\varepsilon}{2}.$$

Note by the Triangle Inequality that

$$\sum_{k=1}^{n} f(t_k)\,\Delta_k x \leq \sum_{k=1}^{n} |f(t_k)|\,\Delta_k x,$$

so that

$$\begin{aligned} \left|\int_a^b f\right| - \int_a^b |f| &= \left(\left|\int_a^b f\right| - \left|\sum_{k=1}^{n} f(t_k)\,\Delta_k x\right| \right) + \left(\left|\sum_{k=1}^{n} f(t_k)\,\Delta_k x\right| - \sum_{k=1}^{n} |f(t_k)|\,\Delta_k x \right) \\ &\quad + \left(\sum_{k=1}^{n} |f(t_k)|\,\Delta_k x - \int_a^b |f| \right) \\ &< \frac{\varepsilon}{2} + 0 + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, this tells us that

$$\left|\int_a^b f\right| - \int_a^b |f| \leq 0,$$

as required. ☒

THE FUNDAMENTAL THEOREM OF CALCULUS

First, note that for any function F, defined on an interval containing $[a,b]$, we write

$$[F(x)]_a^b = F(b) - F(a). \quad (\text{NT } 1.30)$$

Let f be integrable on $[a,b]$.
Define $F: [a,b] \to \mathbb{R}$ by

$$F(x) = \int_a^x f(t)\,dt.$$

Then F is continuous on $[a,b]$.
Moreover, if f is continuous at a point $x \in [a,b]$, then F is differentiable at x and

$$F'(x) = f(x). \quad (\text{T}1.31)$$

Proof. Let m be an upper bound for $|f|$ on $[a,b]$.
Then, for any $a \leq x, y \leq b$, we have

$$\begin{aligned}
|F(y) - F(x)| &= \left|\int_a^y f - \int_a^x f\right| \\
&= \left|\int_x^y f\right| \quad (\text{by additivity / linearity}) \\
&\leq \left|\int_x^y |f|\right| \quad (\text{by estimation}) \\
&\leq \left|\int_x^y m\right| \\
&= m|y-x|,
\end{aligned}$$

so that given an $\varepsilon > 0$, we can choose a $\delta = \frac{\varepsilon}{m}$ to get that $|y-x| < \delta$ implies that
$|F(y) - F(x)| \leq m|y-x| < m\delta = \varepsilon$,
showing F is continuous (indeed, uniformly continuous) on $[a,b]$.

Subsequently, suppose f is continuous at some $x \in [a,b]$.
Then, for any $a \leq x, y \leq b$ with $x \neq y$, we have that

$$\begin{aligned}
\left|\frac{F(y) - F(x)}{y-x} - f(x)\right| &= \left|\frac{\int_a^y f - \int_a^x f}{y-x} - f(x)\right| \\
&= \left|\frac{\int_x^y f}{y-x} - \frac{\int_x^y f(x)\,dt}{y-x}\right| \\
&= \frac{1}{|y-x|}\left|\int_x^y (f(t) - f(x))\,dt\right| \\
&\leq \frac{1}{|y-x|}\left|\int_x^y |f(t) - f(x)|\,dt\right|.
\end{aligned}$$

Let $\varepsilon > 0$ be arbitrary. Since f is continuous at x, it follows that we can choose a $\delta > 0$ so that if $|y-x| < \delta$, then $|f(y) - f(x)| < \varepsilon$.
So, if $0 < |y-x| < \delta$, then

$$\begin{aligned}
\left|\frac{F(y) - F(x)}{y-x} - f(x)\right| &\leq \frac{1}{|y-x|}\left|\int_x^y |f(t) - f(x)|\,dt\right| \\
&\leq \frac{1}{|y-x|}\left|\int_x^y \varepsilon\,dt\right| \\
&= \frac{1}{|y-x|}\,\varepsilon\,|y-x| \\
&= \varepsilon,
\end{aligned}$$

showing that $F'(x)$ exists and $F'(x) = f(x)$ (as $\varepsilon > 0$ was arbitrary). ☒

Let f be integrable on $[a,b]$, and F be differentiable on $[a,b]$ with $F' = f$.
Then

$$\int_a^b f = [F(x)]_a^b = F(b) - F(a). \quad (\text{T}1.31)$$

Proof. Let $\varepsilon > 0$ be arbitrary.
Choose a $\delta > 0$ so that for every partition X of $[a,b]$ with $|X| < \delta$, we have that

$$\left|\int_a^b f - \sum_{k=1}^{n} f(t_k)\Delta_k x\right| < \varepsilon$$

for every choice of sample points $t_k \in [x_{k-1}, x_k]$.
Then, choose sample points $t_k \in [x_{k-1}, x_k]$ so that

$$F'(t_k) = \frac{F(x_k) - F(x_{k-1})}{x_k - x_{k-1}},$$

which we can do by the Mean Value Theorem.
This implies that $f(t_k)\Delta_k x = F(x_k) - F(x_{k-1})$.
Hence

$$\begin{aligned}
\sum_{k=1}^{n} f(t_k)\Delta_k x &= \sum_{k=1}^{n}(F(x_k) - F(x_{k-1})) \\
&= (\cancel{F(x_1)} - F(x_0)) + (\cancel{F(x_2)} - \cancel{F(x_1)}) + \cdots + (F(x_n) - \cancel{F(x_{n-1})}) \\
&= F(x_n) - F(x_0) \\
&= F(b) - F(a),
\end{aligned}$$

and consequently

$$\left|\int_a^b f - (F(b) - F(a))\right| < \varepsilon.$$

But since $\varepsilon > 0$ was arbitrary, it follows that

$$\int_a^b f = F(b) - F(a),$$

as needed. ☒

ANTIDERIVATIVE (D1.32)

We say F is an "antiderivative" for f on some interval $[a,b]$ if $F' = f$ on $[a,b]$.

In this case, we write

① $\int f = F + c, \ c \in \mathbb{R}$; or

② $\int f(x)\,dx = F(x) + c, \ c \in \mathbb{R}$. (N1.34)

Note that if $G' = F' = f$ on $[a,b]$, then necessarily $(G-F)' = 0$, so that $G-F$ is constant on the interval;
ie $G = F + c$ for some $c \in \mathbb{R}$. (N1.33)

EXAMPLE: $\int_0^{\sqrt{3}} \frac{dx}{1+x^2}$ (E1.35)

We can use the Fundamental Theorem of Calculus to calculate integrals of specific functions;
eg $\int_0^{\sqrt{3}} \frac{dx}{1+x^2}$.

Since $\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2}$, it follows that

$$\begin{aligned}
\int_0^{\sqrt{3}} \frac{dx}{1+x^2} &= [\tan^{-1}(x)]_0^{\sqrt{3}} \\
&= \tan^{-1}(\sqrt{3}) - \tan^{-1}(0)
\end{aligned}$$

$$\therefore \int_0^{\sqrt{3}} \frac{dx}{1+x^2} = \frac{\pi}{3}.$$

Chapter 2:
Methods of Integration

BASIC INTEGRALS (N2.1)

Here is a list of basic integrals:

1. $\int x^p \, dx = \frac{x^{p+1}}{p+1} + c, \quad p \neq -1$
2. $\int \frac{1}{x} \, dx = \ln(x) + c$
3. $\int e^x \, dx = e^x + c$
4. $\int a^x \, dx = \frac{a^x}{\ln(a)} + c$
5. $\int \ln(x) \, dx = x\ln(x) - x + c$
6. $\int \sin(x) \, dx = -\cos(x) + c$
7. $\int \cos(x) \, dx = \sin(x) + c$
8. $\int \sec^2(x) \, dx = \tan(x) + c$
9. $\int \sec(x)\tan(x) \, dx = \sec(x) + c$
10. $\int \tan(x) \, dx = \ln|\cos(x)| + c$
11. $\int \sec(x) \, dx = \ln|\sec(x) + \tan(x)| + c$
12. $\int \frac{1}{1+x^2} \, dx = \tan^{-1}(x) + c$
13. $\int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1}(x) + c$
14. $\int \frac{1}{x\sqrt{x^2-1}} \, dx = \sec^{-1}(x) + c.$

Proof. Each of these could be verified by taking the derivative of the RHS, and confirming it matches with the function in the integral. The proof then follows from the Fundamental Theorem of Calculus. ∎

EXAMPLE 1: $\int_1^4 \frac{x^2-5}{\sqrt{x}} \, dx$ (E2.2)

We can solve the integral $\int_1^4 \frac{x^2-5}{\sqrt{x}} \, dx$ using the Fundamental Theorem of Calculus.

$$\int_1^4 \frac{x^2-5}{\sqrt{x}} \, dx = \int_1^4 x^{\frac{3}{2}} - 5x^{-\frac{1}{2}} \, dx$$
$$= \left[\frac{2}{5}x^{\frac{5}{2}} - 10x^{\frac{1}{2}}\right]_1^4$$
$$= \left(\frac{64}{5} - 20\right) - \left(\frac{2}{5} - 10\right)$$
$$\therefore \int_1^4 \frac{x^2-5}{\sqrt{x}} \, dx = \frac{12}{5}.$$

EXAMPLE 2: $\int_{\pi/6}^{\pi/3} \sin(2x) + \cos(3x) \, dx$ (E2.3)

We can also solve the integral $\int_{\pi/6}^{\pi/3} \sin(2x) + \cos(3x) \, dx$ using the Fundamental Theorem of Calculus.

First, note $\frac{d}{dx}(\cos(2x)) = -2\sin(2x)$ and $\frac{d}{dx}(\sin(3x)) = 3\cos(3x)$, it follows that $\frac{d}{dx}\left(-\frac{1}{2}\cos(2x)\right) = \sin(2x)$ and $\frac{d}{dx}\left(\frac{1}{3}\sin(3x)\right) = \cos(3x)$.

Hence

$$\int_{\pi/6}^{\pi/3} \sin(2x) + \cos(3x) \, dx = \left[-\frac{1}{2}\cos(2x) + \frac{1}{3}\sin(3x)\right]_{\pi/6}^{\pi/3}$$
$$= \left(\frac{1}{4} + 0\right) - \left(-\frac{1}{4} + \frac{1}{3}\right)$$
$$= \frac{1}{6}.$$

SUBSTITUTION / CHANGE OF VARIABLES (T2.4)

Let $u=g(x)$ be differentiable on an interval, and let $f(u)$ be continuous on the range of $g(x)$.

Then
$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$$
and
$$\int_{x=a}^{x=b} f(g(x))\,g'(x)\,dx = \int_{u=g(a)}^{u=g(b)} f(u)\,du.$$

Proof. Let $F(u)$ be an antiderivative of $f(u)$, so that $F'(u)=f(u)$ and $\int f(u)\,du = F(u)+c$.

Then, by the Chain Rule, we know that
$$\frac{d}{dx}F(g(x)) = F'(g(x))\,g'(x) = f(g(x))\,g'(x),$$
and so, by the Fundamental Theorem of Calculus,
$$\int f(g(x))\,g'(x)\,dx = F(g(x))+c = F(u)+c = \int f(u)\,du$$
and
$$\int_{x=a}^{x=b} f(g(x))\,g'(x)\,dx = \left[F(g(x))\right]_{x=a}^{x=b} = F(g(b)) - F(g(a)) = \left[F(u)\right]_{u=g(a)}^{u=g(b)}$$
$$\therefore \int_{x=a}^{x=b} f(g(x))\,g'(x)\,dx = \int_{u=g(a)}^{u=g(b)} f(u)\,du. \quad \blacksquare$$

Note that if $f(u)=g(x)$, we often write $f'(u)\,du = g'(x)\,dx$. (NT2.5)

EXAMPLE 1: $\int \sqrt{2x+3}\,dx$ (E2.6)

Substitution can be used to compute integrals such as $\int \sqrt{2x+3}\,dx$.

Let $u=2x+3$, so that $du=2\,dx$. (using the notation from above).

Then
$$\int \sqrt{2x+3}\,dx = \int u^{\frac{1}{2}}\left(\frac{du}{2}\right) = \frac{1}{3}u^{\frac{3}{2}} + c$$
$$\therefore \int \sqrt{2x+3}\,dx = \frac{1}{3}(2x+3)^{\frac{3}{2}} + c.$$

EXAMPLE 2: $\int xe^{x^2}dx$ (E2.7)

The integral $\int xe^{x^2}dx$ can also be solved using substitution.

Let $u=x^2$ so that $du=2x\,dx$.

Then
$$\int xe^{x^2}dx = \frac{1}{2}\int e^u\,du = \frac{1}{2}e^u + c$$
$$\therefore \int xe^{x^2}dx = \frac{1}{2}e^{x^2} + c.$$

EXAMPLE 3: $\int \frac{\ln(x)}{x}\,dx$ (E2.8)

Substitution can also be used to solve integrals like $\int \frac{\ln(x)}{x}\,dx$.

Let $u=\ln(x)$ so that $du = \frac{1}{x}\,dx$.

Then
$$\int \frac{\ln(x)}{x}\,dx = \int u\,du = \frac{u^2}{2} + c$$
$$\therefore \int \frac{\ln(x)}{x}\,dx = \frac{(\ln(x))^2}{2} + c.$$

EXAMPLE 4: $\int \tan(x)\,dx$ (E2.9)

We can use substitution to solve more complicated integrals like $\int \tan(x)\,dx$.

First, note $\tan(x) = \frac{\sin(x)}{\cos(x)}$.

Then, let $u=\cos(x)$, so that $du=-\sin(x)\,dx$.

It follows that
$$\int \tan(x)\,dx = \int \frac{\sin(x)\,dx}{\cos(x)} = \int \frac{-du}{u} = -\ln|u| + c$$
$$\therefore \int \tan(x)\,dx = -\ln|\cos(x)| + c.$$

EXAMPLE 5: $\int \frac{dx}{x+\sqrt{x}}$ (E2.10)

Sometimes, we might have to do two substitutions to calculate some integrals;

eg $\int \frac{dx}{x+\sqrt{x}}$

First, let $u=\sqrt{x}$, so that $x=u^2$ and $2u\,du=dx$.

Then
$$\int \frac{dx}{x+\sqrt{x}} = \int \frac{2u\,du}{u^2+u} = \int \frac{2\,du}{u+1}.$$

Next, let $v=u+1$, so that $dv=du$. It follows that
$$\int \frac{dx}{x+\sqrt{x}} = \int \frac{2\,du}{u+1} = \int \frac{2\,dv}{v} = 2\ln|v| + c = 2\ln|u+1| + c$$
$$\therefore \int \frac{dx}{x+\sqrt{x}} = 2\ln|\sqrt{x}+1| + c$$

EXAMPLE 6: $\int_0^2 \frac{x\,dx}{\sqrt{2x^2+1}}$ (E2.11)

When doing substitution, we need to change the values of the "endpoints" accordingly;

eg $\int_0^2 \frac{x}{\sqrt{2x^2+1}}\,dx$.

Let $u=2x^2+1$, so that $du=4x\,dx$.

Note that $u=1$ and $u=9$ when $x=0$ and $x=2$ respectively.

Then
$$\int_{x=0}^{x=2} \frac{x}{\sqrt{2x^2+1}}\,dx = \int_{u=1}^{u=9} \frac{\frac{1}{4}du}{\sqrt{u}} = \int_{u=1}^{u=9} \frac{1}{4}u^{-\frac{1}{2}}\,du = \left[\frac{1}{2}u^{\frac{1}{2}}\right]_1^9 = \frac{3}{2} - \frac{1}{2}$$
$$\therefore \int_{x=0}^{x=2} \frac{x}{\sqrt{2x^2+1}}\,dx = 1.$$

EXAMPLE 7: $\int_0^1 \frac{dx}{1+3x^2}$ (E2.12)

Sometimes, we might have to make a weird substitution to solve an integral;

eg $\int_0^1 \frac{dx}{1+3x^2}$.

Let $u=\sqrt{3}x$, so that $du=\sqrt{3}\,dx$.

Note that $x=0 \Rightarrow u=0$, and $x=1 \Rightarrow u=\sqrt{3}$.

Then
$$\int_{x=0}^{x=1} \frac{dx}{1+3x^2} = \int_{u=0}^{u=\sqrt{3}} \frac{1}{\sqrt{3}} \cdot \frac{du}{1+u^2} = \left[\frac{1}{\sqrt{3}}\tan^{-1}(u)\right]_0^{\sqrt{3}} = \frac{1}{\sqrt{3}}\left(\frac{\pi}{3} - 0\right)$$
$$\therefore \int_{x=0}^{x=1} \frac{dx}{1+3x^2} = \frac{\pi}{3\sqrt{3}}.$$

INTEGRATION BY PARTS (T2.13)

Let $f(x)$ and $g(x)$ be differentiable in an interval.

Then
$$\int f(x)\,g'(x)\,dx = f(x)g(x) - \int g(x) f'(x)\,dx,$$
so that
$$\int_{x=a}^{x=b} f(x)\,g'(x)\,dx = \Big[f(x)g(x) - \int g(x) f'(x)\,dx\Big]_{x=a}^{x=b}.$$

Proof. By the Product Rule,
$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x).$$
Hence, by the Fundamental Theorem of Calculus,
$$\int (f'(x)g(x) + f(x)g'(x))\,dx = f(x)g(x) + c,$$
which can be rewritten as
$$\int f(x)\,g'(x)\,dx = f(x)g(x) - \int g(x) f'(x)\,dx.$$
(the arbitrary constant c is not needed since there is an integral on both sides of the equation.) $\blacksquare$

If we let $u = f(x)$, $du = f'(x)\,dx$, $v = g(x)$ and $dv = g'(x)\,dx$, the above formula becomes
$$\int u\,dv = uv - \int v\,du. \qquad \text{(NT2.14)}$$

POLYNOMIAL × TRIGONOMETRIC OR EXPONENTIAL FUNCTION

If the integral involves a polynomial multiplied by an exponential function or a trigonometric function, try integrating by parts with u equal to the polynomial. (N2.15)

* note: multiple applications of integration by parts may be required if the degree of the polynomial is high.

EXAMPLE 1: $\int x\sin(x)\,dx$ (E2.16)

We employ the above strategy to evaluate the integral $\int x\sin(x)\,dx$.

Integrate by parts using $\begin{pmatrix} u = x & v = -\cos(x) \\ du = 1\,dx & dv = \sin(x)\,dx \end{pmatrix}$

$$\int x\sin(x)\,dx = -x\cos(x) - \int -\cos(x)\,dx$$
$$= -x\cos(x) + \int \cos(x)\,dx$$
$$\therefore \int x\sin(x)\,dx = -x\cos(x) + \sin(x) + c.$$

EXAMPLE 2: $\int (x^2+1)e^{2x}\,dx$ (E2.17)

Similarly, we can use the above strategy to evaluate the integral $\int (x^2+1)e^{2x}\,dx$.

First, integrate by parts using $\begin{pmatrix} u = x^2+1 & v = \frac{1}{2}e^{2x} \\ du = 2x\,dx & dv = e^{2x}\,dx \end{pmatrix}$ to get

$$\int (x^2+1)e^{2x}\,dx = \tfrac{1}{2}(x^2+1)e^{2x} - \int \tfrac{1}{2}e^{2x}(2x\,dx)$$
$$= \tfrac{1}{2}(x^2+1)e^{2x} - \int x e^{2x}\,dx.$$

To find $\int xe^{2x}$, we integrate by parts again, this time using $\begin{pmatrix} u_2 = x & v_2 = \frac{1}{2}e^{2x}\,dx \\ du_2 = 1\,dx & dv_2 = e^{2x}\,dx \end{pmatrix}$:

$$\int (x^2+1)e^{2x}\,dx = \tfrac{1}{2}(x^2+1)e^{2x} - \int xe^{2x}\,dx$$
$$= \tfrac{1}{2}(x^2+1)e^{2x} - \Big(\tfrac{1}{2}xe^{2x} - \int \tfrac{1}{2}e^{2x}\,dx\Big)$$
$$= \tfrac{1}{2}(x^2+1)e^{2x} - \tfrac{1}{2}xe^{2x} + \tfrac{1}{4}e^{2x} + c$$
$$\therefore \int (x^2+1)e^{2x}\,dx = \tfrac{1}{4}(2x^2-2x+3)e^{2x} + c.$$

POLYNOMIAL × LOGARITHMIC OR INVERSE TRIGONOMETRIC FUNCTION

If the integral involves a polynomial multiplied by a logarithmic or inverse trigonometric function, try integrating by parts with u equal to the logarithmic / inverse trigonometric function. (N2.15)

EXAMPLE 1: $\int \ln(x)\,dx$ (E2.18)

We can use the above strategy to evaluate the integral $\int \ln(x)\,dx$.

Integrate by parts using $\begin{pmatrix} u = \ln(x) & v = x \\ du = \frac{1}{x}dx & dv = 1\,dx \end{pmatrix}$ to get

$$\int \ln(x)\,dx = x\ln(x) - \int x\cdot\tfrac{1}{x}\,dx$$
$$= x\ln(x) - \int 1\,dx$$
$$\therefore \int \ln(x)\,dx = x\ln(x) - x + c.$$

EXAMPLE 2: $\int_1^4 \sqrt{x}\ln(x)\,dx$ (E2.19)

The above strategy can even be used when the polynomial contains terms with non-integer powers, eg $\int_1^4 \sqrt{x}\ln(x)\,dx$.

Integrate by parts using $\begin{pmatrix} u = \ln(x) & v = \frac{2}{3}x^{\frac{3}{2}} \\ du = \frac{1}{x}dx & dv = x^{\frac{1}{2}}\,dx \end{pmatrix}$ to get

$$\int_1^4 \sqrt{x}\ln(x)\,dx = \Big[\tfrac{2}{3}x^{\frac{3}{2}}\ln(x) - \int \tfrac{2}{3}x^{\frac{1}{2}}\,dx\Big]_1^4$$
$$= \Big[\tfrac{2}{3}x^{\frac{3}{2}}\ln(x) - \tfrac{4}{9}x^{\frac{3}{2}}\Big]_1^4$$
$$= \Big(\tfrac{16}{3}\ln(4) - \tfrac{32}{9}\Big) - \Big(\tfrac{2}{3}\ln(1) - \tfrac{4}{9}\Big)$$
$$\therefore \int_1^4 \sqrt{x}\ln(x)\,dx = \tfrac{16}{3}\ln(4) - \tfrac{28}{9}.$$

EXPONENTIAL × SINE/COSINE FUNCTION

If the integral involves an exponential function times a sine/cosine function, try integrating by parts twice, letting u be the exponential function both times. (N2.15)

EXAMPLE: $\int e^x\sin(x)\,dx$ (E2.20)

We can use the above strategy to evaluate the integral $\int e^x\sin(x)\,dx$.

Proof. Let $I = \int e^x\sin(x)\,dx$.

Integrate by parts twice, first using $\begin{pmatrix} u_1 = e^x & v_1 = -\cos(x) \\ du_1 = e^x dx & dv_1 = \sin(x)\,dx \end{pmatrix}$, and then with $\begin{pmatrix} u_2 = e^x & v_2 = \sin(x) \\ du_2 = e^x dx & dv_2 = \cos(x)\,dx \end{pmatrix}$ to get

$$I = \int e^x\sin(x)\,dx = -e^x\cos(x) + \int e^x\cos(x)\,dx$$
$$= -e^x\cos(x) + \Big(e^x\sin(x) - \int e^x\sin(x)\,dx\Big)$$
$$\therefore I = -e^x\cos(x) + e^x\sin(x) - I.$$

Hence $2I = -e^x\cos(x) + e^x\sin(x) + c$,

so that $I = \tfrac{1}{2}(\sin(x) - \cos(x))e^x + d.$

OTHER SORTS OF PROBLEMS

EXAMPLE 1: $\int \sin^n(x)\,dx$ (E2.21)

We can use integration by parts to get a general formula for $\int \sin^n(x)\,dx$ in terms of $\int \sin^{n-2}(x)\,dx$.

Let $I = \int \sin^n(x)\,dx = \int \sin^{n-1}(x)\sin(x)\,dx$

Integrate by parts using $\begin{pmatrix} u = \sin^{n-1}(x) & v = -\cos(x) \\ du = (n-1)(\sin^{n-2}(x))(\cos(x))dx & dv = \sin(x)\,dx \end{pmatrix}$

to get

$$I = \int \sin^n(x)\,dx = -\sin^{n-1}(x)\cos(x) - \int -\cos(x)(n-1)(\sin^{n-2}(x))(\cos(x))\,dx$$

$$= -\sin^{n-1}(x)\cos(x) + \int (n-1)(\cos^2(x))(\sin^{n-2}(x))\,dx$$

$$= -\sin^{n-1}(x)\cos(x) + \int (n-1)(1-\sin^2(x))(\sin^{n-2}(x))\,dx$$

$$\therefore\ I = -\sin^{n-1}(x)\cos(x) + (n-1)\int \sin^{n-2}(x)\,dx - (n-1)I.$$

Hence

$$(n-1)I + I = nI = -\sin^{n-1}(x)\cos(x) + (n-1)\int \sin^{n-2}(x)\,dx,$$

so that

$$I = -\frac{1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx.$$

In particular, we can use the attained above formula to evaluate $\int \sin^2(x)\,dx$ and $\int \sin^4(x)\,dx$.

In particular, when $n=2$, we get

$$\int \sin^2(x)\,dx = -\frac{1}{2}\sin(x)\cos(x) + \frac{1}{2}\int 1\,dx$$

$$\therefore \int \sin^2(x)\,dx = -\frac{1}{2}\sin(x)\cos(x) + \frac{1}{2}x + c.$$

When $n=4$, we get

$$\int \sin^4(x)\,dx = -\frac{1}{4}\sin^3(x)\cos(x) + \frac{3}{4}\int \sin^2(x)\,dx$$

$$= -\frac{1}{4}\sin^3(x)\cos(x) + \frac{3}{4}\left(-\frac{1}{2}\sin(x)\cos(x) + \frac{1}{2}x\right) + c$$

$$\therefore \int \sin^4(x)\,dx = -\frac{1}{4}\sin^3(x)\cos(x) - \frac{3}{8}\sin(x)\cos(x) + \frac{3}{8}x + c.$$

EXAMPLE 2: $\int \sec^n(x)\,dx$ (E2.22)

In a similar manner to the above, we can use integration by parts to attain a general formula for $\int \sec^n(x)\,dx$ in terms of $\int \sec^{n-2}(x)\,dx$.

Let $I = \int \sec^n(x)\,dx = \int \sec^{n-2}(x)\sec^2(x)\,dx.$

Integrate by parts using $\begin{pmatrix} u = \sec^{n-2}(x) & v = \tan(x) \\ du = (n-2)(\sec^{n-3}(x))(\sec(x)\tan(x)) & dv = \sec^2(x)\,dx \\ \quad = (n-2)(\sec^{n-2}(x))(\tan(x)) & \end{pmatrix}$

to get

$$I = \int \sec^n(x)\,dx = \sec^{n-2}(x)\tan(x) - \int (n-2)(\sec^{n-2}(x))(\tan^2(x))\,dx$$

$$= \sec^{n-2}(x)\tan(x) - \int (n-2)(\sec^{n-2}(x))(\sec^2 x - 1)\,dx$$

$$\therefore\ I = \sec^{n-2}(x)\tan(x) - (n-2)I + (n-2)\int \sec^{n-2}(x)\,dx.$$

Hence

$$(n-1)I = \sec^{n-2}(x)\tan(x) + (n-2)\int \sec^{n-2}(x)\,dx,$$

so that

$$I = \frac{1}{n-1}\sec^{n-2}(x)\tan(x) + \frac{n-2}{n-1}\int \sec^{n-2}(x)\,dx.$$

We can use the above formula to evaluate the integral $\int \sec^3(x)\,dx$.

In particular, when $n=3$, we have that

$$\int \sec^3(x)\,dx = \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\int \sec(x)\,dx$$

$$\therefore \int \sec^3(x)\,dx = \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln|\sec(x)+\tan(x)| + c.$$

TRIGONOMETRIC INTEGRALS

$\int f(\sin(x))\cos^{2n+1}(x)\,dx$ OR $\int f(\cos(x))\sin^{2n+1}(x)\,dx$

1. To find $\int f(\sin(x))\cos^{2n+1}(x)\,dx$, write $\cos^{2n+1}(x) = (1-\sin^2(x))^n\cos(x)$ and then try the substitution $u=\sin(x)$, $du=\cos(x)\,dx$. (N2.23 (1))
2. Similarly, to find $\int f(\cos(x))\sin^{2n+1}(x)\,dx$, write $\sin^{2n+1}(x) = (1-\cos^2(x))^n\sin(x)$ and then try the substitution $u=\cos(x)$, $du=-\sin(x)\,dx$. (N2.23 (2))

EXAMPLE: $\int_0^{\pi/3} \frac{\sin^3(x)}{\cos^2(x)}\,dx$ (E2.24)

We can use the above strategy to solve the integral $\int_0^{\pi/3}\frac{\sin^3(x)}{\cos^2(x)}\,dx$.

Make the substitution $u=\cos(x)$, so that $du=-\sin(x)\,dx$.
Then

$$\int_0^{\pi/3}\frac{\sin^3(x)}{\cos^2(x)}\,dx = \int_{x=0}^{x=\frac{\pi}{3}}\frac{(1-\cos^2x)\sin(x)\,dx}{\cos^2(x)}$$
$$= \int_{u=1}^{u=\frac{1}{2}} -\frac{(1-u^2)\,du}{u^2}$$
$$= \int_{u=1}^{u=\frac{1}{2}} -\frac{1}{u^2}+1\,du$$
$$= \left[\frac{1}{u}+u\right]_1^{\frac{1}{2}}$$
$$= \left(2+\tfrac{1}{2}\right)-(1+1)$$
$$\therefore \int_0^{\pi/3}\frac{\sin^3(x)}{\cos^2(x)}\,dx = \frac{1}{2}.$$

$\int \sin^{2m}(x)\cos^{2n}(x)\,dx$

1. To find $\int\sin^{2m}(x)\cos^{2n}(x)$, try using the trigonometric identities $\sin^2(\theta) = \frac{1}{2}-\frac{1}{2}\cos(2\theta)$ and $\cos^2(\theta) = \frac{1}{2}+\frac{1}{2}\cos(2\theta)$.
2. Alternatively, write $\cos^{2n}(x) = (1-\sin^2(x))^n$ and use the formula from E2.21. (N2.23 (3))

EXAMPLE: $\int_0^{\pi/4}\sin^6(x)\,dx$ (E2.25)

We can use either strategy 1 or 2 to evaluate the integral $\int_0^{\pi/4}\sin^6(x)\,dx$.

We use strategy 1.
Note that

$$\int_0^{\pi/4}\sin^6(x)\,dx = \int_0^{\pi/4}\left(\tfrac{1}{2}-\tfrac{1}{2}\cos(2x)\right)^3dx \quad \text{(using the half-angle formula)}$$
$$= \int_0^{\pi/4}\frac{1}{8}-\frac{3}{8}\cos(2x)+\frac{3}{8}\cos^2(2x)-\frac{1}{8}\cos^3(2x)\,dx$$
$$= \int_0^{\pi/4}\frac{1}{8}-\frac{3}{8}\cos(2x)+\frac{3}{8}\left(\tfrac{1}{2}+\tfrac{1}{2}\cos(4x)\right)-\frac{1}{8}(1-\sin^2(2x))\cos(2x)\,dx$$
$$= \int_0^{\pi/4}\frac{5}{16}-\frac{1}{2}\cos(2x)+\frac{3}{16}\cos(4x)+\frac{1}{8}\sin^2(2x)\cos(2x)\,dx$$
$$= \left[\frac{5}{16}x-\frac{1}{4}\sin(2x)+\frac{3}{64}\sin(4x)+\frac{1}{48}\sin^3(2x)\right]_0^{\pi/4}$$
$$= \frac{5\pi}{64}-\frac{1}{4}+\frac{1}{48}$$
$$\therefore \int_0^{\pi/4}\sin^6(x)\,dx = \frac{5\pi}{64}-\frac{11}{48}.$$

$\int f(\tan(x))\sec^{2n+2}(x)\,dx$

To find $\int f(\tan(x))\sec^{2n+2}(x)\,dx$, write $\sec^{2n+2}(x) = (1+\tan^2(x))^n\sec^2(x)$ and try the substitution $u=\tan(x)$, $du=\sec^2(x)$. (N2.23 (4))

EXAMPLE 1: $\int_0^{\pi/4}\tan^4(x)\,dx$ (E2.26)

The above strategy can be used to solve the integral $\int_0^{\pi/4}\tan^4(x)\,dx$.

Note first that

$$\int_0^{\pi/4}\tan^4(x)\,dx = \int_0^{\pi/4}\tan^2(x)\sec^2(x)-\tan^2x\,dx$$
$$= \int_0^{\pi/4}\tan^2(x)\sec^2(x)-\sec^2(x)+1\,dx.$$

To find $\int\tan^2(x)\sec^2(x)\,dx$, make the substitution $u=\tan(x)$, $du=\sec^2(x)\,dx$ to get that

$$\int\tan^2(x)\sec^2(x)\,dx = \int u^2\,du$$
$$= \frac{u^3}{3}+c$$
$$\therefore \int\tan^2(x)\sec^2(x)\,dx = \frac{\tan^3(x)}{3}+c.$$

It follows that

$$\int_0^{\pi/4}\tan^4(x) = \left[\frac{\tan^3(x)}{3}-\tan(x)+x\right]_0^{\pi/4}$$
$$= \frac{1}{3}-1+\frac{\pi}{4}$$
$$\therefore \int_0^{\pi/4}\tan^4(x) = -\frac{2}{3}+\frac{\pi}{4}.$$

EXAMPLE 2: $\int_0^{\pi/4}\frac{\sec^4(x)}{\sqrt{\tan(x)+1}}\,dx$ (E2.27)

We can again use the above strategy to evaluate the integral $\int_0^{\pi/4}\frac{\sec^4(x)}{\sqrt{\tan(x)+1}}\,dx$.

Make the substitution $u=\tan(x)$, so that $du=\sec^2(x)\,dx$.
Then

$$\int_0^{\pi/4}\frac{\sec^4(x)}{\sqrt{\tan(x)+1}}\,dx = \int_{x=0}^{x=\frac{\pi}{4}}\frac{(\tan^2(x)+1)\sec^2(x)\,dx}{\sqrt{\tan(x)+1}}$$
$$= \int_{u=0}^{u=1}\frac{(u^2+1)}{\sqrt{u+1}}\,du.$$

Next, make the substitution $v=u+1$, so that $du=dv$.
Then

$$\int_0^{\pi/4}\frac{\sec^4(x)}{\sqrt{\tan(x)+1}}\,dx = \int_{u=0}^{u=1}\frac{(u^2+1)}{\sqrt{u+1}}\,du$$
$$= \int_{v=1}^{v=2}\frac{((v-1)^2+1)}{\sqrt{v}}\,dv$$
$$= \int_1^2 v^{\frac{3}{2}}-2v^{\frac{1}{2}}+2v^{-\frac{1}{2}}\,dv$$
$$= \left[\frac{2}{5}v^{\frac{5}{2}}-\frac{4}{3}v^{\frac{3}{2}}+4v^{\frac{1}{2}}\right]_1^2$$
$$= \left(\frac{2}{5}(4\sqrt{2})-\frac{4}{3}(2\sqrt{2})+4(\sqrt{2})\right)-\left(\frac{2}{5}-\frac{4}{3}+4\right)$$
$$\therefore \int_0^{\pi/4}\frac{\sec^4(x)}{\sqrt{\tan(x)+1}}\,dx = \frac{44\sqrt{2}-46}{15}.$$

$\int f(\sec(x))\tan^{2n+1}(x)\,dx$

To solve $\int f(\sec(x))\tan^{2n+1}(x)\,dx$, write $\tan^{2n+1}(x) = \frac{(\sec^2(x)-1)^n}{\sec(x)}\sec(x)\tan(x)$ and try the substitution $u=\sec(x)$, $du=\sec(x)\tan(x)\,dx$. (N2.23 (5))

$\int \sec^{2n+1}(x)\tan^{2n}(x)\,dx$

To solve $\int\sec^{2n+1}(x)\tan^{2n}(x)\,dx$, write $\tan^{2n}(x) = (\sec^2(x)-1)^n$ and use the formula from E2.22. (N2.23 (6))

$\int \sin(ax)\sin(bx)\,dx$, $\int \cos(ax)\cos(bx)\,dx$ OR $\int \sin(ax)\cos(bx)\,dx$ (N2.28)

To evaluate $\int \sin(ax)\sin(bx)\,dx$, $\int \cos(ax)\cos(bx)\,dx$ or $\int \sin(ax)\cos(bx)\,dx$, use the identities

① $\cos(A-B) - \cos(A+B) = 2\sin(A)\sin(B)$;

② $\cos(A-B) + \cos(A+B) = 2\cos(A)\cos(B)$; or

③ $\sin(A-B) + \sin(A+B) = 2\sin(A)\cos(B)$.

EXAMPLE: $\int_0^{\pi/6} \cos(3x)\cos(2x)\,dx$ (E2.29)

We can employ the above strategy to evaluate the integral $\int_0^{\pi/6} \cos(3x)\cos(2x)\,dx$.

By ② in the above, we have that

$$\begin{aligned} 2\cos(3x)\cos(2x) &= \cos(3x-2x) + \cos(3x+2x) \\ &= \cos(x) + \cos(5x). \end{aligned}$$

Hence

$$\begin{aligned} \int_0^{\pi/6} \cos(2x)\cos(3x)\,dx &= \int_0^{\pi/6} \frac{1}{2}(\cos(x) + \cos(5x))\,dx \\ &= \left[\frac{1}{2}\sin(x) + \frac{1}{10}\sin(5x)\right]_0^{\pi/6} \\ &= \frac{1}{4} + \frac{1}{20} \end{aligned}$$

$$\therefore \int_0^{\pi/6} \cos(2x)\cos(3x)\,dx = \frac{3}{10}.$$

WEIERSTRASS SUBSTITUTION (N2.30)

1. The Weierstrass substitution is letting $u = \tan\left(\frac{x}{2}\right)$, so that $x = 2\tan^{-1}(u)$, $dx = \frac{2}{1+u^2}\,du$.

2. Additionally, it implies $\sin\left(\frac{x}{2}\right) = \frac{u}{\sqrt{1+u^2}}$ & $\cos\left(\frac{x}{2}\right) = \frac{1}{\sqrt{1+u^2}}$, so that

① $$\begin{aligned} \sin(x) &= 2\sin\left(\tfrac{x}{2}\right)\cos\left(\tfrac{x}{2}\right) \\ &= 2\left(\frac{u}{\sqrt{1+u^2}}\right)\left(\frac{1}{\sqrt{1+u^2}}\right) \end{aligned}$$

$\therefore \sin(x) = \frac{2u}{1+u^2}$; and

② $$\begin{aligned} \cos(x) &= \cos^2\left(\tfrac{x}{2}\right) - \sin^2\left(\tfrac{x}{2}\right) \\ &= \left(\frac{1}{\sqrt{1+u^2}}\right)^2 - \left(\frac{u}{\sqrt{1+u^2}}\right)^2 \end{aligned}$$

$\therefore \cos(x) = \frac{1-u^2}{1+u^2}$.

EXAMPLE: $\int \frac{dx}{1-\cos(x)}$ (E2.31)

The Weierstrass substitution can be used to solve some integrals; eg $\int \frac{dx}{1-\cos(x)}$.

Let $u = \tan\left(\frac{x}{2}\right)$, so that $dx = \frac{2}{1+u^2}\,du$, and $\cos(x) = \frac{1-u^2}{1+u^2}$.

Then

$$\begin{aligned} \int \frac{dx}{1-\cos(x)} &= \int \frac{1}{1-\left(\frac{1-u^2}{1+u^2}\right)}\left(\frac{2}{1+u^2}\,du\right) \\ &= \int \frac{2}{1+u^2-(1-u^2)}\,du \\ &= \int \frac{du}{u^2} \\ &= -\frac{1}{u} + c \end{aligned}$$

$$\therefore \int \frac{dx}{1-\cos(x)} = -\cot\left(\frac{x}{2}\right) + c.$$

INVERSE TRIGONOMETRIC SUBSTITUTION

$\int f\left(\sqrt{a^2-b^2(x+c)^2}\right)dx$

For an integral involving $\sqrt{a^2-b^2(x+c)^2}$, try the substitution $\theta=\sin^{-1}\left(\frac{b(x+c)}{a}\right)$, so that

① $a\sin\theta = b(x+c)$;

② $a\cos\theta = \sqrt{a^2-b^2(x+c)^2}$; and

③ $a\cos\theta\, d\theta = b\,dx$. (N2.32 (2))

EXAMPLE 1: $\int_0^1 \frac{dx}{(4-3x^2)^{3/2}}$ (E2.33)

The above method can be used to evaluate the integral $\int_0^1 \frac{dx}{(4-3x^2)^{3/2}}$.

Let $2\sin\theta=\sqrt{3}x$, so that $2\cos\theta=\sqrt{4-3x^2}$ and $2\cos\theta\, d\theta=\sqrt{3}\,dx$.

Then

$$\int_{x=0}^{x=1}\frac{dx}{(4-3x^2)^{\frac{3}{2}}} = \int_{\theta=0}^{\theta=\frac{\pi}{3}}\frac{\frac{2}{\sqrt{3}}\cos\theta\, d\theta}{(2\cos\theta)^3}$$

$$= \int_0^{\pi/3}\frac{1}{4\sqrt{3}}\sec^2\theta\, d\theta$$

$$= \left[\frac{1}{4\sqrt{3}}\tan\theta\right]_0^{\pi/3}$$

$$\therefore \int_{x=0}^{x=1}\frac{dx}{(4-3x^2)^{\frac{3}{2}}} = \frac{1}{4}.$$

EXAMPLE 2: $\int_2^3 (4x-x^2)^{\frac{3}{2}}dx$ (E2.36)

The above strategy can also be applied to more complex integrals, like $\int_2^3 (4x-x^2)^{\frac{3}{2}}dx$.

Let $2\sin\theta=x-2$, so that $2\cos\theta=\sqrt{4x-x^2}$ and $2\cos\theta\, d\theta=dx$. Then

$$\int_{x=2}^{x=3}(4x-x^2)^{\frac{3}{2}}dx = \int_{\theta=0}^{\theta=\pi/6}(2\cos\theta)^3(2\cos\theta\, d\theta)$$

$$= \int_0^{\pi/6}16\cos^4\theta\, d\theta$$

$$= \int_0^{\pi/6}4(1+\cos2\theta)^2\, d\theta$$

$$= \int_0^{\pi/6}4+8\cos2\theta+4\cos^2 2\theta\, d\theta$$

$$= \int_0^{\pi/6}4+8\cos2\theta+2+2\cos4\theta\, d\theta$$

$$= \left[6\theta+4\sin2\theta+\tfrac{1}{2}\sin4\theta\right]_0^{\pi/6}$$

$$\therefore \int_{x=2}^{x=3}(4x-x^2)^{\frac{3}{2}}dx = \pi+\frac{9\sqrt{3}}{4}.$$

$\int f\left(\sqrt{a^2+b^2(x+c)^2}\right)dx$

For an integral involving $\sqrt{a^2+b^2(x+c)^2}$ (or $\frac{1}{\sqrt{a^2+b^2(x+c)^2}}$), try the substitution $\theta=\tan^{-1}\left(\frac{b(x+c)}{a}\right)$, so that

① $a\tan\theta = b(x+c)$;

② $a\sec\theta = \sqrt{a^2+b^2(x+c)^2}$; and

③ $a\sec^2\theta\, d\theta = b\,dx$. (N2.32(1))

EXAMPLE: $\int_1^{\sqrt{3}}\frac{dx}{x^2\sqrt{x^2+3}}$ (E2.34)

We can use the above strategy to evaluate the integral $\int_1^{\sqrt{3}}\frac{dx}{x^2\sqrt{x^2+3}}$.

Let $\sqrt{3}\tan\theta=x$, so that $\sqrt{3}\sec\theta=\sqrt{x^2+3}$ and $\sqrt{3}\sec^2\theta\, d\theta=dx$. Then

$$\int_{x=1}^{x=\sqrt{3}}\frac{dx}{x^2\sqrt{x^2+3}} = \int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{6}}\frac{\sqrt{3}\sec^2\theta\, d\theta}{3\tan^2\theta(\sqrt{3}\sec\theta)}$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{6}}\frac{1}{3}\frac{\sec\theta}{\tan^2\theta}d\theta$$

$$\therefore \int_{x=1}^{x=\sqrt{3}}\frac{dx}{x^2\sqrt{x^2+3}} = \int_{\frac{\pi}{4}}^{\frac{\pi}{6}}\frac{1}{3}\frac{\cos\theta}{\sin^2\theta}d\theta$$

Then, let $u=\sin\theta$, so that $du=\cos\theta\, d\theta$.

It follows that

$$\int_{x=1}^{x=\sqrt{3}}\frac{dx}{x^2\sqrt{x^2+3}} = \int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{6}}\frac{1}{3}\frac{\cos\theta}{\sin^2\theta}d\theta$$

$$= \int_{u=\frac{1}{\sqrt{2}}}^{u=\frac{1}{2}}\frac{1}{3}\frac{1}{u^2}du$$

$$= \left[-\frac{1}{3u}\right]_{1/\sqrt{2}}^{1/2}$$

$$\therefore \int_{x=1}^{x=\sqrt{3}}\frac{dx}{x^2\sqrt{x^2+3}} = \frac{2-\sqrt{2}}{3}.$$

$\int f\left(\sqrt{b^2(x+c)^2-a^2}\right)dx$

For an integral involving $\sqrt{b^2(x+c)^2-a^2}$, try the substitution $\theta=\sec^{-1}\left(\frac{b(x+c)}{a}\right)$, so that

① $a\sec\theta = b(x+c)$;

② $a\tan\theta = \sqrt{b^2(x+c)^2-a^2}$; and

③ $a\sec\theta\tan\theta = b\,dx$. (N2.32 (3))

EXAMPLE: $\int_2^4\frac{\sqrt{x^2-4}}{x^2}dx$ (E2.35)

The above strategy can be used to evaluate the integral $\int_2^4\frac{\sqrt{x^2-4}}{x^2}dx$.

Let $2\sec\theta=x$, so that $2\tan\theta=\sqrt{x^2-4}$ and $2\sec\theta\tan\theta\, d\theta=dx$.

Then

$$\int_{x=2}^{x=4}\frac{\sqrt{x^2-4}}{x^2} = \int_{\theta=0}^{\theta=\frac{\pi}{3}}\frac{\tan^2\theta\sec\theta\, d\theta}{\sec^2\theta}$$

$$= \int_0^{\pi/3}\frac{\tan^2\theta}{\sec\theta}d\theta$$

$$= \int_0^{\pi/3}\frac{\sec^2\theta-1}{\sec\theta}d\theta$$

$$= \int_0^{\pi/3}(\sec\theta-\cos\theta)\,d\theta$$

$$= \left[\ln|\sec\theta+\tan\theta|-\sin\theta\right]_0^{\pi/3}$$

$$\therefore \int_{x=2}^{x=4}\frac{\sqrt{x^2-4}}{x^2} = \ln(2+\sqrt{3})-\frac{\sqrt{3}}{2}.$$

PARTIAL FRACTIONS (N2.37)

We can find the integral of a rational function $\frac{f(x)}{g(x)}$ (where f & g are polynomials) as follows:

① Use long division to find polynomials $q(x)$ and $r(x)$ such that

$$f(x) \equiv g(x)q(x) + r(x),$$

where $\deg(r) < \deg(g)$.

* if $\deg(f) < \deg(g)$, then $q(x)=0$ and $r(x)=f(x)$.

② Then, note that $\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}$,

and it follows that

$$\int \frac{f(x)}{g(x)}\,dx = \int q(x) + \frac{r(x)}{g(x)}\,dx.$$

③ Next, factor $g(x)$ into linear and irreducible quadratic factors.

*we can always do this! (MATH 145 R34)

④ Finally, split $\frac{r(x)}{g(x)}$ into its "partial fraction decomposition";

ie write $\frac{r(x)}{g(x)}$ as a sum of terms

so that

i) for each linear factor $(ax+b)^k$, we have the k terms

$$\frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k};$$ and

ii) for each irreducible quadratic factor $(ax^2+bx+c)^k$, we have the k terms

$$\frac{B_1x + C_1}{(ax^2+bx+c)} + \frac{B_2x + C_2}{(ax^2+bx+c)^2} + \cdots + \frac{B_kx + C_k}{(ax^2+bx+c)^k}.$$

eg if $g(x) = x(x-1)^3(x^2+2x+3)^2$, then we would write

$$\frac{r(x)}{g(x)} = \left(\frac{A}{x}\right) + \left(\frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3}\right) + \left(\frac{E}{x^2+2x+3} + \frac{F}{(x^2+2x+3)^2}\right),$$

and then solve for the various constants. (E2.38)

⑤ From here, we can solve the integral.

EXAMPLE 1: $\int_2^3 \frac{x-7}{(x-1)^2(x+2)}\,dx$ (E2.39)

The above strategy can be used to solve the integral $\int_2^3 \frac{x-7}{(x-1)^2(x+2)}\,dx$.

First, we need to find A, B, C such that

$$\frac{x-7}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2},$$

or

$$x-7 = A(x-1)(x+2) + B(x+2) + C(x-1)^2.$$

Equating coefficients, we get that

$$\begin{cases} A+C=0 \\ A+B-2C=1 \\ -2A+2B+C=-7. \end{cases}$$

Solving this system gives us that $A=1$, $B=-2$ & $C=-1$.

Hence,

$$\int_2^3 \frac{x-7}{(x-1)^2(x+2)}\,dx = \int_2^3 \frac{1}{x-1} - \frac{2}{(x-1)^2} - \frac{1}{x+2}\,dx$$

$$= \left[\ln(x-1) - \frac{2}{x-1} - \ln(x+2)\right]_2^3$$

$$\therefore \int_2^3 \frac{x-7}{(x-1)^2(x+2)}\,dx = \ln\left(\frac{8}{5}\right) - 1.$$

EXAMPLE 2: $\int_1^{\sqrt{3}} \frac{x^4-x^3+1}{x^3+x}\,dx$ (E2.40)

As mentioned in step ① of the method, sometimes long division is needed before partial fraction decomposition can be carried out;

eg $\int_1^{\sqrt{3}} \frac{x^4-x^3+1}{x^3+x}\,dx$.

First, use polynomial long division to get that

$$\frac{x^4-x^3+1}{x^3+x} = (x-1) + \frac{-x^2+x+1}{x^3+x}.$$

Then, note that to get

$$\frac{-x^2+x+1}{x^3+x} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

we need

$$-x^2+x+1 = A(x^2+1) + (Bx+C)(x).$$

Equating coefficients gives $A+B=-1$, $C=1$ and $A=1$.

Solving these equations gives $A=1$, $B=-2$ and $C=1$.

Thus

$$\int_1^{\sqrt{3}} \frac{x^4-x^3+1}{x^3+x}\,dx = \int_1^{\sqrt{3}} x-1 + \frac{1}{x} - \frac{2x}{x^2+1} + \frac{1}{x^2+1}\,dx$$

$$= \left[\frac{1}{2}x^2 - x + \ln(x) - \ln(x^2+1) + \tan^{-1}(x)\right]_1^{\sqrt{3}}$$

$$\therefore \int_1^{\sqrt{3}} \frac{x^4-x^3+1}{x^3+x}\,dx = 2-\sqrt{3} + \ln\left(\frac{\sqrt{3}}{2}\right) + \frac{\pi}{12}.$$

EXAMPLE 3: $\int_1^2 \frac{x^5+x^4-2x^3-2x^2-5x-25}{x^2(x^2-2x+5)^2}\,dx$ (E2.41)

Partial fraction decomposition can also be applied in tandem with substitution to solve integrals;

eg $\int_1^2 \frac{x^5+x^4-2x^3-2x^2-5x-25}{x^2(x^2-2x+5)^2}\,dx$.

Let I be the above integral.

To get

$$\frac{x^5+x^4-2x^3-2x^2-5x-25}{x^2(x^2+2x+5)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+2x+5} + \frac{Ex+F}{(x^2+2x+5)^2}$$

we need

$$x^5+x^4-2x^3-2x^2-5x-25 = Ax(x^2-2x+5)^2 + B(x^2-2x+5)^2 + (Cx+D)(x^2+2x+5)(x^2) + (Ex+F)(x^2).$$

Comparing coefficients, we get that $A+C=1$; $-4A+B-2C+D=1$; $14A-4B+5C-2D+E=-2$; $-20A+14B+5D+F=-2$; $25A-20B=-5$; and $25B=-25$.

Solving these equations gives $A=-1$, $B=-1$, $C=2$, $D=2$, $E=2$ and $F=-18$.

Hence

$$I = \int_1^2 -\frac{1}{x} - \frac{1}{x^2} + \frac{2x+2}{x^2-2x+5} + \frac{2x-18}{(x^2-2x+5)^2}\,dx$$

$$= \int_1^2 -\frac{1}{x} - \frac{1}{x^2} + \frac{(2x-2)+4}{x^2-2x+5} + \frac{(2x-2)-16}{(x^2-2x+5)^2}\,dx$$

$$I = \int_1^2 -\frac{1}{x} - \frac{1}{x^2} + \frac{2x-2}{x^2-2x+5} + \frac{4}{x^2-2x+5} + \frac{2x-2}{(x^2-2x+5)^2} - \frac{16}{(x^2-2x+5)^2}\,dx.$$

To compute $\int \frac{2x-2}{x^2-2x+5}\,dx$ and $\int \frac{2x-2}{(x^2-2x+5)^2}\,dx$, make the substitution $u = x^2-2x+5$, so that $du = (2x-2)\,dx$.

Then

$$\int \frac{2x-2}{x^2-2x+5}\,dx = \int \frac{du}{u} = \ln|u| + c = \ln|x^2-2x+5| + c;$$

and

$$\int \frac{2x-2}{(x^2-2x+5)^2}\,dx = \int \frac{du}{u^2} = \frac{-1}{u} + c = \frac{-1}{x^2-2x+5} + c.$$

To compute $\int \frac{4\,dx}{x^2-2x+5}$ and $\int \frac{16\,dx}{(x^2-2x+5)^2}$, make the substitution $2\tan\theta = x-1$, so that $2\sec\theta = \sqrt{x^2-2x+5}$ and $2\sec^2\theta\,d\theta = dx$.

Then

$$\int \frac{4\,dx}{x^2-2x+5} = \int \frac{4\cdot 2\sec^2\theta\,d\theta}{(2\sec\theta)^2} = \int 2\,d\theta = 2\theta + c = 2\tan^{-1}\left(\frac{x-1}{2}\right) + c$$

and

$$\int \frac{16\,dx}{(x^2-2x+5)^2} = \int \frac{16\cdot 2\sec^2\theta\,d\theta}{(2\sec\theta)^4} = \int \frac{2\,d\theta}{\sec^2\theta} = \int 2\cos^2\theta\,d\theta = \int 1+\cos(2\theta)\,d\theta$$

$$= \theta + \frac{1}{2}\sin(2\theta) + c = \theta + \sin\theta\cos\theta + c = \tan^{-1}\left(\frac{x-1}{2}\right) + \frac{2(x-1)}{x^2-2x+5} + c.$$

Thus

$$I = \left[-\ln(x) + \frac{1}{x} + \ln(x^2-2x+5) + 2\tan^{-1}\left(\frac{x-1}{2}\right) - \frac{1}{x^2-2x+5} - \tan^{-1}\left(\frac{x-1}{2}\right) - \frac{2(x-1)}{x^2-2x+5}\right]_1^2$$

$$\therefore I = \ln\left(\frac{5}{8}\right) - \frac{17}{20} + \tan^{-1}\left(\frac{1}{2}\right).$$

EXAMPLE 4: $\int \frac{\sec^3(x)}{\sec(x)-1}\,dx$ (E2.42)

Partial fraction decomposition can also be applied even if the function is not rational (at first);

eg $\int \frac{\sec^3(x)}{\sec(x)-1}\,dx$.

First, note that

$$\int \frac{\sec^3(x)}{\sec(x)-1}\,dx = \int \frac{\sec^3(x)}{\sec(x)-1}\cdot\frac{\sec(x)+1}{\sec(x)+1}\,dx$$
$$= \int \frac{\sec^4(x)+\sec^3(x)}{\sec^2(x)-1}\,dx$$
$$= \int \frac{\sec^4(x)+\sec^3(x)}{\tan^2(x)}\,dx$$

$$\therefore \int \frac{\sec^3(x)}{\sec(x)-1}\,dx = \int \frac{\sec^4(x)}{\tan^2(x)}\,dx + \int \frac{\sec^3(x)}{\tan^2(x)}.$$

To find $\int \frac{\sec^4(x)}{\tan^2(x)}\,dx$, make the substitution $u=\tan(x)$, so that $du=\sec^2(x)\,dx$. Then

$$\int \frac{\sec^4(x)}{\tan^2(x)}\,dx = \int \frac{(\tan^2(x)+1)\sec^2(x)}{\tan^2(x)}\,dx$$
$$= \int \frac{(u^2+1)}{u^2}\,du$$
$$= \int 1+\frac{1}{u^2}\,du$$
$$= u-\frac{1}{u}+c$$

$$\therefore \int \frac{\sec^4(x)}{\tan^2(x)}\,dx = \tan(x)-\cot(x)+c;$$

To find $\int \frac{\sec^3(x)}{\tan^2(x)}\,dx$, make the substitution $v=\sin(x)$, so that $dv=\cos(x)\,dx$. Then

$$\int \frac{\sec^3(x)}{\tan^2(x)}\,dx = \int \frac{dx}{\cos(x)\sin^2(x)}$$
$$= \int \frac{\cos(x)\,dx}{(1-\sin^2(x))\sin^2(x)}$$
$$= \int \frac{dv}{(1-v^2)v^2}.$$

Then, note that $\frac{1}{(1-v^2)v^2} = \frac{1/2}{1-v}+\frac{1/2}{1+v}+\frac{0}{v}+\frac{1}{v^2}$ (by partial fraction decomposition), so that

$$\int \frac{\sec^3(x)}{\tan^2(x)}\,dx = \int \frac{1/2}{1-v}+\frac{1/2}{1+v}+\frac{1}{v^2}\,dv$$
$$= -\frac{1}{2}\ln|1-v| + \frac{1}{2}\ln|1+v| - \frac{1}{v}+c$$

$$\therefore \int \frac{\sec^3(x)}{\tan^2(x)}\,dx = -\frac{1}{2}\ln|1-\sin(x)|+\frac{1}{2}\ln|1+\sin(x)| - \csc(x)+c.$$

Finally, it follows that

$$\int \frac{\sec^3(x)}{\sec(x)-1}\,dx = \int \frac{\sec^4(x)}{\tan^2(x)}\,dx + \int \frac{\sec^3(x)}{\tan^2(x)}\,dx$$
$$= \tan(x)-\cot(x)-\frac{1}{2}\ln|1-\sin(x)|+\frac{1}{2}\ln|1+\sin(x)|-\csc(x)+c$$

$$\therefore \int \frac{\sec^3(x)}{\sec(x)-1}\,dx = \tan(x)-\cot(x)+\ln|\sec(x)+\tan(x)|-\csc(x)+c.$$

APPROXIMATE INTEGRATION (D2.43 (1))

💡 Let f be integrable on $[a,b]$.
Then, we can approximate the integral of f on $[a,b]$ by any Riemann sum

$$I = \int_a^b f(x)\,dx \cong \sum_{k=1}^{n} f(c_k)\,\Delta_k x,$$

where $a = x_0 < x_1 < \cdots < x_n = b$, $\Delta_k x = x_{k-1} - x_k$
and $c_k \in [x_{k-1}, x_k]$ $\forall k \in \{1, 2, \ldots, n\}$.

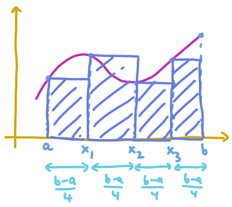
LEFT ENDPOINT APPROXIMATION (D2.43 (2))

💡 Let f be integrable on $[a,b]$.
Then, the "n^{th} left endpoint approximation" for $I = \int_a^b f$, denoted by L_n, is defined to be

$$L_n = \sum_{k=1}^{n} f(x_{k-1})\,\Delta_k x;$$

ie

$$L_n = \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + \tfrac{b-a}{n}(k-1)\right).$$



*the subintervals are equally sized!

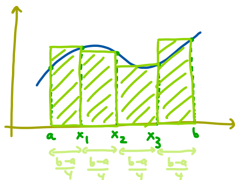
RIGHT ENDPOINT APPROXIMATION (D2.43 (3))

💡 Let f be integrable on $[a,b]$.
Then, the "n^{th} right endpoint approximation" for $I = \int_a^b f$, denoted by R_n, is defined to be

$$R_n = \sum_{k=1}^{n} f(x_k)\,\Delta_k x;$$

ie

$$R_n = \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + \tfrac{b-a}{n}k\right).$$



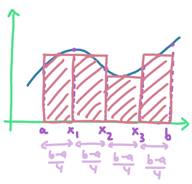
MIDPOINT APPROXIMATION (D2.43 (4))

💡 Let f be integrable on $[a,b]$.
Then, the "n^{th} midpoint approximation" for $I = \int_a^b f$, denoted by M_n, is defined to be

$$M_n = \sum_{k=1}^{n} f\left(\frac{x_{k-1}+x_k}{2}\right)\Delta_k x;$$

ie

$$M_n = \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + \tfrac{b-a}{n}\left(\tfrac{2k-1}{2}\right)\right).$$



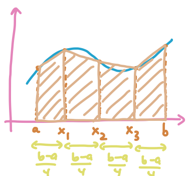
TRAPEZOIDAL APPROXIMATION (D2.44)

💡$_1$ Let f be integrable on $[a,b]$.
Then, the "n^{th} trapezoidal approximation" for $I = \int_a^b f$, denoted by T_n, is defined by

$$T_n = \sum_{k=1}^{n} \frac{f(x_{k-1}) + f(x_k)}{2}\,\Delta_k x;$$

ie

$$T_n = \frac{b-a}{n} \sum_{k=1}^{n} \frac{f\left(a+\frac{b-a}{n}(k-1)\right) + f\left(a+\frac{b-a}{n}k\right)}{2}.$$



*note each "area" is a trapezoid!

💡$_2$ Note that $T_n = \dfrac{L_n + R_n}{2}$.

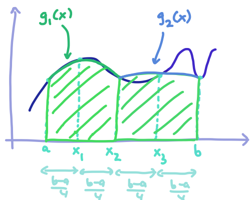
SIMPSON APPROXIMATION (D2.45)

💡$_1$ Let f be integrable on $[a,b]$.
Then, for some $n \in 2\mathbb{Z}^+$, the "nth Simpson approximation" for $I = \int_a^b f$, denoted by S_n, is defined to be

$$S_n = \sum_{k=1}^{n/2} \int_{x_{2k-2}}^{x_{2k}} g(x)\,dx,$$

where $x_m = a + \frac{b-a}{n}m$ $\forall m \in \{0, 1, \ldots, n\}$, and
$g(x) = g_k(x)$ $\forall k \in \{1, 2, \ldots, \frac{n}{2}\}$,
where for each k, g_k is a quadratic polynomial such that

$$\begin{cases} g_k(x_{2k-2}) = f(x_{2k-2}); \\ g_k(x_{2k-1}) = f(x_{2k-1}); \quad \text{and} \\ g_k(x_{2k}) = f(x_{2k}). \end{cases}$$



💡$_2$ We can prove that

$$S_n = \sum_{k=1}^{n/2} \frac{f(x_{2k-2}) + 4f(x_{2k-1}) + f(x_{2k})}{3}\,\Delta_k x;$$

ie

$$S_n = \frac{b-a}{n} \sum_{k=1}^{n/2} \frac{f\left(a+\frac{b-a}{n}(2k-2)\right) + 4f\left(a+\frac{b-a}{n}(2k-1)\right) + f\left(a+\frac{b-a}{n}(2k)\right)}{3}.$$

Proof. First, note if $h(x) = Ax^2 + Bx + C$ satisfies $h(-1) = u$, $h(0) = v$ and $h(1) = w$, then necessarily

$$\begin{cases} A - B + C = u; \\ C = v; \quad \text{and} \\ A + B + C = w. \end{cases}$$

Solving these equations yields that $A = \frac{u-2v+w}{2}$, $B = \frac{w-u}{2}$ and $C = w$, so that

$$\int_{-1}^{1} h(x)\,dx = \int_{-1}^{1} \frac{u-2v+w}{2}x^2 + \frac{w-u}{2}x + v\,dx$$

$$= \left[\frac{u-2v+w}{6}x^3 + \frac{w-u}{4}x^2 + vx\right]_{-1}^{1}$$

$$= \frac{u-2v+w}{3} + 2v$$

$$\therefore \int_{-1}^{1} h(x)\,dx = \frac{u+4v+w}{3}.$$

Then, by shifting and scaling, it follows that

$$\int_{x_{2k-2}}^{x_{2k}} g_k(x)\,dx = \frac{f(x_{2k-2}) + 4f(x_{2k-1}) + f(x_{2k})}{3}\left(\frac{b-a}{n}\right). \quad \blacksquare$$

ERROR BOUNDS FOR APPROXIMATE INTEGRATION (T2.46)

Let f be integrable on $[a,b]$, and suppose the higher order derivatives of f exist. Denote $I = \int_a^b f(x)\,dx$. Then

① $|L_n - I| \leq \frac{(b-a)^2}{2n} \max_{a \leq x \leq b} |f'(x)|$;

② $|R_n - I| \leq \frac{(b-a)^2}{2n} \max_{a \leq x \leq b} |f'(x)|$;

③ $|T_n - I| \leq \frac{(b-a)^3}{12n^2} \max_{a \leq x \leq b} |f''(x)|$;

④ $|M_n - I| \leq \frac{(b-a)^3}{24n^2} \max_{a \leq x \leq b} |f''(x)|$; and

⑤ $|S_n - I| \leq \frac{(b-a)^5}{180n^4} \max_{a \leq x \leq b} |f''''(x)|$.

EXAMPLE : ERROR BOUNDS OF APPROXIMATIONS OF $\int_0^{4\pi/3} \sin^2(x)\,dx$ (E2.47)

We can use the above theorem to find the bounds on the errors for L_8, R_8, M_8, T_8 & S_8 on $I = \int_0^{4\pi/3} \sin^2(x)\,dx$.

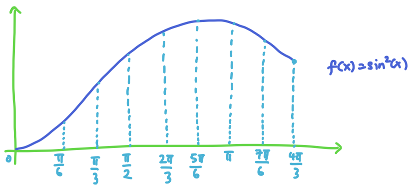
First, note that

$$I = \int_0^{4\pi/3} \sin^2(x)\,dx = \int_0^{4\pi/3} \frac{1}{2} - \frac{1}{2}\cos(2x)\,dx$$

$$= \left[\frac{1}{2}x - \frac{1}{4}\sin(2x)\right]_0^{4\pi/3}$$

$$\therefore I = \frac{4\pi}{3} - \frac{\sqrt{3}}{8}.$$

Next, when we divide the interval $[0, \frac{4\pi}{3}]$ into 8 equal sub-intervals, the size of each subinterval is $\frac{\pi}{6}$ and the endpoints of the sub-intervals are $0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}$ and $\frac{4\pi}{3}$.



For convenience, let $f(x) = \sin^2(x)$.

Thus, the approximations are

① $L_8 = \frac{b-a}{8} \sum_{k=1}^{8} f(x_{k-1})$

$= \frac{1}{8}\left(\frac{4\pi}{3} - 0\right)\left(f(0) + f(\frac{\pi}{6}) + f(\frac{\pi}{3}) + f(\frac{\pi}{2}) + f(\frac{2\pi}{3}) + f(\frac{5\pi}{6}) + f(\pi) + f(\frac{7\pi}{6})\right)$

$= \frac{1}{8}\left(\frac{4\pi}{3}\right)\left(0 + \frac{1}{4} + \frac{3}{4} + 1 + \frac{3}{4} + \frac{1}{4} + 0 + \frac{1}{4}\right)$

$\therefore L_8 = \frac{13\pi}{24}$;

② $R_8 = \frac{b-a}{8} \sum_{k=1}^{8} f(x_{k-1})$

$= \frac{1}{8}\left(\frac{4\pi}{3}\right)\left(f(\frac{\pi}{6}) + f(\frac{\pi}{3}) + f(\frac{\pi}{2}) + f(\frac{2\pi}{3}) + f(\frac{5\pi}{6}) + f(\pi) + f(\frac{7\pi}{6}) + f(\frac{4\pi}{3})\right)$

$= \frac{1}{8}\left(\frac{4\pi}{3}\right)\left(\frac{1}{4} + \frac{3}{4} + 1 + \frac{3}{4} + \frac{1}{4} + 0 + \frac{1}{4} + \frac{3}{4}\right)$

$\therefore R_8 = \frac{2\pi}{3}$;

③ $T_8 = \frac{1}{2}(L_8 + R_8)$

$= \frac{1}{2}\left(\frac{13\pi}{24} + \frac{2\pi}{3}\right)$

$\therefore T_8 = \frac{29\pi}{48}$;

④ $M_8 = \frac{b-a}{8} \sum_{k=1}^{8} f\left(\frac{x_{k-1} + x_k}{2}\right)$

$= \frac{1}{8}\left(\frac{4\pi}{3}\right)\left(f(\frac{\pi}{12}) + f(\frac{\pi}{4}) + f(\frac{5\pi}{12}) + f(\frac{7\pi}{12}) + f(\frac{3\pi}{4}) + f(\frac{11\pi}{12}) + f(\frac{13\pi}{12}) + f(\frac{15\pi}{12})\right)$

$= \frac{\pi}{6}\left(\frac{2-\sqrt{3}}{4} + \frac{1}{2} + \frac{2+\sqrt{3}}{4} + \frac{2+\sqrt{3}}{4} + \frac{1}{2} + \frac{2-\sqrt{3}}{4} + \frac{2-\sqrt{3}}{4} + \frac{1}{2}\right)$

*using the identity $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$ to figure out the values of $f(\frac{n\pi}{12})$.

$= \frac{\pi}{6}\left(4 - \frac{\sqrt{3}}{4}\right)$; and

⑤ $S_8 = \frac{b-a}{8} \sum_{k=1}^{8} \frac{f(x_{2k-2}) + 4f(x_{2k-1}) + f(x_{2k})}{3}$

$= \frac{1}{24}\left(\frac{4\pi}{3}\right)\left(f(0) + 4f(\frac{\pi}{6}) + 2f(\frac{\pi}{3}) + 4f(\frac{\pi}{2}) + 2f(\frac{2\pi}{3}) + 4f(\frac{5\pi}{6}) + 2f(\pi) + 4f(\frac{7\pi}{6}) + f(\frac{4\pi}{3})\right)$

$= \frac{\pi}{18}\left(0 + 1 + \frac{3}{2} + 4 + \frac{3}{2} + 1 + 0 + 1 + \frac{3}{4}\right)$

$\therefore S_8 = \frac{43\pi}{72}$.

Then, for $f(x) = \sin^2(x)$; note that

① $f'(x) = 2\sin(x)\cos(x) = \sin(2x)$;

② $f''(x) = 2\cos(2x)$;

③ $f'''(x) = -4\sin(2x)$; and

④ $f''''(x) = -8\cos(2x)$.

It follows that

① $\max_{0 \leq x \leq \frac{4\pi}{3}} |f'(x)| = \max_{0 \leq x \leq \frac{4\pi}{3}} |\sin(2x)| = 1$;

② $\max_{0 \leq x \leq \frac{4\pi}{3}} |f''(x)| = \max_{0 \leq x \leq \frac{4\pi}{3}} |2\cos(2x)| = 2$; and

③ $\max_{0 \leq x \leq \frac{4\pi}{3}} |f''''(x)| = \max_{0 \leq x \leq \frac{4\pi}{3}} |-8\cos(2x)| = 8$.

Finally, by the above theorem, we get that

① $|L_8 - I| \leq \frac{1}{16}\left(\frac{4\pi}{3}\right)^2(1) = \frac{\pi^2}{9}$;

② $|R_8 - I| \leq \frac{1}{16}\left(\frac{4\pi}{3}\right)^2(1) = \frac{\pi^2}{9}$;

③ $|T_8 - I| \leq \frac{1}{12 \cdot 6^2}\left(\frac{4\pi}{3}\right)^3(2) = \frac{8\pi^3}{729}$;

④ $|M_8 - I| \leq \frac{1}{24 \cdot 6^2}\left(\frac{4\pi}{3}\right)^3(2) = \frac{4\pi^3}{729}$; and

⑤ $|S_8 - I| \leq \frac{1}{180 \cdot 6^4}\left(\frac{4\pi}{3}\right)^5(8) = \frac{2^7\pi^5}{5 \cdot 3^{11}}$.

IMPROPER INTEGRATION

IMPROPER INTEGRATION ON $[a,b)$ (D2.48 (1))

1. Suppose that $f:[a,b)\to\mathbb{R}$ is integrable on every closed interval contained in $[a,b)$.
Then the "improper integral of f" on $[a,b)$ is defined to be

$$\int_a^b f = \lim_{t\to b^-}\int_a^t f.$$

2. We say f is "improperly integrable" on $[a,b)$, or that the improper integral of f on $[a,b)$ "converges", if $\int_a^b f$ exists and is finite.

3. We also allow the case where $b=\infty$, and in this case we have

$$\int_a^\infty f = \lim_{t\to\infty}\int_a^t f.$$

IMPROPER INTEGRATION ON $(a,b]$ (D2.48 (2))

1. Suppose that $f:(a,b]\to\mathbb{R}$ is integrable on every closed interval contained in $(a,b]$.
Then, the "improper integral of f" on $(a,b]$ is defined to be

$$\int_a^b f = \lim_{t\to a^+}\int_t^b f.$$

2. Similarly, we say f is "improperly integrable" on $(a,b]$, or that the improper integral of f on $(a,b]$ "converges", if $\int_a^b f$ exists and is finite.

3. We also allow the case where $a=-\infty$, and in this case we have

$$\int_{-\infty}^b f = \lim_{t\to-\infty}\int_t^b f.$$

IMPROPER INTEGRATION ON (a,b) (D2.48 (3))

1. Suppose that $f:(a,b)\to\mathbb{R}$ is integrable on every closed interval in (a,b).
Suppose further that for any point $c\in(a,b)$, the integrals $\int_a^c f$ and $\int_c^b f$ both exist and can be added.
Then the "improper integral of f" on (a,b) is defined to be

$$\int_a^b f = \int_a^c f + \int_c^b f,$$

*the choice of c does not matter!

where $c\in(a,b)$ is arbitrary.

2. We say f is "improperly integrable" on (a,b) when both $\int_a^c f$ and $\int_c^b f$ are finite.

EVALUATING IMPROPER INTEGRALS

We write

1. $[F(x)]_{a^+}^{b^-} = \lim_{x \to b^-} F(x) - \lim_{x \to a^+} F(x)$;
2. $[F(x)]_{a^+}^{b} = F(b) - \lim_{x \to a^+} F(x)$; and
3. $[F(x)]_{a}^{b^-} = \lim_{x \to b^-} F(x) - F(a)$. (NT2.49)

Suppose that $f : (a,b) \to \mathbb{R}$ is integrable on every closed interval contained in (a,b), and assume that F is differentiable with $F' = f$ on (a,b).

Then

$$\int_a^b f = [F(x)]_{a^+}^{b^-}. \qquad \text{(N2.50)}$$

(A similar result holds for functions defined on $[a,b)$ and $(a,b]$).

Proof. Choose some $c \in (a,b)$. Then, by the Fundamental Theorem of Calculus,

$$\begin{aligned}
\int_a^b f &= \int_a^c f + \int_c^b f \\
&= \lim_{s \to a^+} \int_s^c f + \lim_{t \to b^-} \int_c^t f \\
&= \lim_{s \to a^+} (F(c) - F(s)) + \lim_{t \to b^-} (F(t) - F(c)) \\
&= \lim_{t \to b^-} F(t) - \lim_{s \to a^+} F(s)
\end{aligned}$$

$$\therefore \int_a^b f = [F(x)]_{a^+}^{b^-}. \qquad \blacksquare$$

EXAMPLE 1: $\int_0^1 \frac{dx}{x}$ (E2.51 (1))

The above strategy can help us evaluate the integral $\int_0^1 \frac{dx}{x}$.

$$\begin{aligned}
\int_0^1 \frac{dx}{x} &= [\ln(x)]_{0^+}^1 \\
&= 0 - (-\infty)
\end{aligned}$$

$$\therefore \int_0^1 \frac{dx}{x} = \infty.$$

EXAMPLE 2: $\int_0^1 \frac{dx}{\sqrt{x}}$ (E2.51 (2))

Similarly, we can evaluate $\int_0^1 \frac{dx}{\sqrt{x}}$ by the above method.

$$\begin{aligned}
\int_0^1 \frac{dx}{\sqrt{x}} &= [2\sqrt{x}]_{0^+}^1 \\
&= 2 - 0
\end{aligned}$$

$$\therefore \int_0^1 \frac{dx}{\sqrt{x}} = 2.$$

EXAMPLE 3: $\int_0^1 \frac{dx}{x^p}$ CONVERGES $\iff p < 1$ (E2.52)

By extension of the previous two examples, we can in fact show $\int_0^1 \frac{dx}{x^p}$ converges if and only if $p < 1$.

Proof. The case with $p = 1$ was dealt in E2.50.

If $p > 1$, then $p - 1 > 0$, so that

$$\int_0^1 \frac{dx}{x^p} = \left[\frac{-1}{(p-1)x^{p-1}}\right]_{0^+}^1 = \left(-\frac{1}{p-1}\right) - (-\infty) = \infty,$$

and if $p < 1$, then $1 - p > 0$, so that

$$\int_0^1 \frac{dx}{x^p} = \left[\frac{x^{1-p}}{1-p}\right]_{0^+}^1 = \left(\frac{1}{1-p}\right) - (0) = \frac{1}{1-p},$$

and these deductions are sufficient to prove the claim. $\blacksquare$

EXAMPLE 4: $\int_1^\infty \frac{dx}{x^p}$ CONVERGES $\iff p > 1$ (E2.53)

Similarly, we can prove $\int_1^\infty \frac{dx}{x^p}$ converges if and only if $p > 1$.

Proof. When $p = 1$, then

$$\int_1^\infty \frac{dx}{x^p} = \int_1^\infty \frac{dx}{x} = [\ln(x)]_1^\infty = \infty - 0 = \infty.$$

When $p > 1$, then $p - 1 > 0$, so that

$$\int_1^\infty \frac{dx}{x^p} = \left[\frac{-1}{(p-1)x^{p-1}}\right]_1^\infty = (0) - \left(-\frac{1}{p-1}\right) = \frac{1}{p-1}.$$

When $p < 1$, then $1 - p > 0$, so that

$$\int_1^\infty \frac{dx}{x^p} = \left[\frac{x^{1-p}}{1-p}\right]_1^\infty = (\infty) - \left(\frac{1}{1-p}\right) = \infty,$$

and these deductions are sufficient to prove the claim. $\blacksquare$

EXAMPLE 5: $\int_0^\infty e^{-x}\,dx$ (E2.54)

In a similar manner, we can evaluate the integral $\int_0^\infty e^{-x}\,dx$.

$$\begin{aligned}
\int_0^\infty e^{-x}\,dx &= [-e^{-x}]_0^\infty \\
&= 0 - (-1)
\end{aligned}$$

$$\therefore \int_0^\infty e^{-x}\,dx = 1.$$

EXAMPLE 6: $\int_0^1 \ln(x)\,dx$ (E2.55)

Similarly, we can evaluate the integral $\int_0^1 \ln(x)\,dx$ using the strategy in N2.50.

$$\begin{aligned}
\int_0^1 \ln(x)\,dx &= [x\ln(x) - x]_{0^+}^1 \\
&= (-1) - \lim_{x \to 0^+} (x\ln(x)) \\
&= -1 - \lim_{x \to 0^+} \frac{\ln(x)}{\left(\frac{1}{x}\right)} \\
&= -1 - \lim_{x \to 0^+} \frac{\left(\frac{1}{x}\right)}{\left(-\frac{1}{x^2}\right)} \qquad \left(\text{by L'Hôpital's Rule, since } \frac{\ln(0)}{\left(\frac{1}{0}\right)} = \frac{\infty}{\infty}\right) \\
&= -1 - \lim_{x \to 0^+} (-x) \\
&= -1 - (0)
\end{aligned}$$

$$\therefore \int_0^1 \ln(x)\,dx = -1.$$

COMPARISON FOR IMPROPER INTEGRALS (T2.56)

Let f and g be integrable on any closed intervals contained in (a,b), and suppose further that

$0 \leqslant f(x) \leqslant g(x) \quad \forall x \in (a,b)$.

Suppose g is improperly integrable on (a,b).
Then so is f, and

$$\int_a^b f \leqslant \int_a^b g.$$

On the other hand, if $\int_a^b f$ diverges, then $\int_a^b g$ diverges as well.

(Similar results hold for functions f & g defined on half-open intervals)

EXAMPLE 1: $\int_0^{\pi/2} \sqrt{\sec(x)}\, dx$ CONVERGES (E2.57)

Using comparison, we can show that $\int_0^{\pi/2} \sqrt{\sec(x)}\, dx$ converges.

Proof. First, note $\forall x \in [0, \frac{\pi}{2})$, we have that $\cos(x) \geqslant 1 - \frac{2}{\pi}x$,
so that $\sec(x) \leqslant \frac{1}{1-\frac{2}{\pi}x}$, and hence $\sqrt{\sec(x)} \leqslant \sqrt{\frac{1}{1-\frac{2}{\pi}x}}$.

Let $u = 1 - \frac{2}{\pi}x$, so that $du = -\frac{2}{\pi}dx$. Then

$$\int_{x=0}^{x=\frac{\pi}{2}} \frac{1}{\sqrt{1-\frac{2}{\pi}x}}\, dx = \int_{u=1}^{u=0} -\frac{\pi}{2} u^{-\frac{1}{2}}\, du = \left[-\pi u^{\frac{1}{2}}\right]_1^0$$

$$\therefore \int_{x=0}^{x=\frac{\pi}{2}} \frac{1}{\sqrt{1-\frac{2}{\pi}x}}\, dx = \pi,$$

which is clearly finite.

It follows by comparison that $\int_0^{\pi/2} \sqrt{\sec(x)}\, dx$ converges. ■

EXAMPLE 2: $\int_0^\infty e^{-x^2} dx$ CONVERGES (E2.58)

Similarly, we can show $\int_0^\infty e^{-x^2} dx$ converges using comparison.

Proof. First, note for $\forall u \in [0, \infty)$, $e^u \geqslant 1+u$; hence
$e^{x^2} \geqslant 1 + x^2 > 0$, so that $e^{-x^2} \leqslant \frac{1}{1+x^2}$.

Then, since

$$\int_0^\infty \frac{dx}{1+x^2} = \left[\tan^{-1}(x)\right]_0^\infty \leqslant \frac{\pi}{2},$$

which is finite, we see that $\int_0^\infty e^{-x^2} dx$ converges by comparison. ■

ESTIMATION FOR IMPROPER INTEGRALS (T2.59)

Let $f : (a,b) \to \mathbb{R}$ be integrable on any closed interval contained within (a,b).

Suppose $|f|$ is improperly integrable on (a,b).
Then so is f, and in this case

$$\left| \int_a^b f \right| \leqslant \int_a^b |f|$$

(Similar results hold for functions defined on half-open intervals).

EXAMPLE: $\int_0^\infty \frac{\sin(x)}{x} dx$ CONVERGES (E2.60)

Using estimation, we can show that $\int_0^\infty \frac{\sin(x)}{x} dx$ converges.

Proof. We show $\int_0^1 \frac{\sin(x)}{x} dx$ and $\int_1^\infty \frac{\sin(x)}{x} dx$ converge.

First, since $\lim_{x \to 0^+} \frac{\sin(x)}{x} = 1$ by the Fundamental Trigonometric Limit, the function $f(x) = \begin{cases} 1, & x = 0 \\ \frac{\sin(x)}{x}, & x > 0 \end{cases}$ is continuous on $[0,1]$, and so by T1.17 $f(x)$ is also integrable on $[0,1]$.

Then, by the Fundamental Theorem of Calculus, $\int_r^1 f(x)\, dx$ is continuous for $r \in [0,1]$, and so

$$\int_0^1 \frac{\sin(x)}{x} dx = \lim_{r \to 0^+} \int_r^1 \frac{\sin(x)}{x} dx = \lim_{r \to 0^+} \int_r^1 f(x) dx = \int_0^1 f(x)\, dx,$$

which is finite, so $\int_0^1 \frac{\sin(x)}{x} dx$ converges as well.

Next, integrate by parts using $\begin{pmatrix} u = \frac{1}{x} & v = \frac{1}{x} \\ du = -\frac{1}{x^2}dx & dv = \frac{-1}{x^2}dx \end{pmatrix}$ to get

$$\int_1^\infty \frac{\sin(x)}{x} dx = \left[\frac{-\cos(x)}{x}\right]_1^\infty - \int_1^\infty \frac{\cos(x)}{x^2} dx$$

$$\therefore \int_1^\infty \frac{\sin(x)}{x} dx = \cos(1) - \int_1^\infty \frac{\cos(x)}{x^2} dx.$$

Then, since $\left|\frac{\cos(x)}{x^2}\right| \leqslant \frac{1}{x^2}$ and $\int_1^\infty \frac{dx}{x^2}$ converges, necessarily $\int_1^\infty \left|\frac{\cos(x)}{x^2}\right| dx$ converges too by comparison.

Hence, by estimation, $\int_1^\infty \frac{\cos(x)}{x^2} dx$ also converges.

Finally, since

$$\int_0^\infty \frac{\sin(x)}{x} dx = \int_0^1 \frac{\sin(x)}{x} dx + \int_1^\infty \frac{\sin(x)}{x} dx,$$

and both $\int_0^1 \frac{\sin(x)}{x} dx$ and $\int_1^\infty \frac{\sin(x)}{x}$ are finite, it follows that $\int_0^\infty \frac{\sin(x)}{x} dx$ converges, and we are done. ■