MATH 235
Personal Notes

Marcus Chan

UW '25

Taught by Nick Rollick

Class 8: Examples of Matrix Representations, Introduction to Inner Product Spaces

INNER PRODUCT & INNER PRODUCT SPACES: $\langle v, w\rangle$ (D8.1)

1. Let V be a vector space over $\mathbb{F}$.
Then, we say the function $\langle\cdot,\cdot\rangle : V^2 \to \mathbb{F}$
is an "inner product" if
① $\langle v,v\rangle \in \mathbb{R}$ & $\langle v,v\rangle \geq 0 \quad \forall v\in V$;
② $\langle v,v\rangle = 0 \iff v=0 \quad \forall v \in V$; } Positivity
③ $\langle v_1+v_2, w\rangle = \langle v_1,w\rangle + \langle v_2,w\rangle \quad \forall v_1,v_2,w\in V$;
④ $\langle cv, w\rangle = c\langle v,w\rangle \quad \forall c\in\mathbb{F},\ v,w\in V$; and } Linearity
⑤ $\langle w,v\rangle = \overline{\langle v,w\rangle} \quad \forall v,w \in V$. } Conjugate Symmetry

2. In this case, we call $\langle v,w\rangle$ the "inner product" of v & w.

3. We refer to V together with $\langle\cdot,\cdot\rangle$ as an "inner product space".

LENGTH [OF A VECTOR]: $\|v\|$ (D8.2)

Let V be an inner product space, and let $v\in V$.
Then, the "length" of v, denoted by "$\|v\|$",
is defined to be equal to
$$\|v\| = \sqrt{\langle v,v\rangle}.$$
We can do this because $\langle v,v\rangle \in \mathbb{R} \ \forall v \in V$.

ORTHOGONAL [VECTORS] (D8.3)

Let V be an IPS.
Then, we say $v,w\in V$ are "orthogonal"
if $\langle v,w\rangle = 0$.

ORTHOGONAL [SETS] (D8.3)

Let $S\subseteq V$, where V is an IPS.
Then, we say S is "orthogonal" if
$\langle v,w\rangle = 0 \quad \forall v,w \in S$.

EXAMPLES OF IPS: PART I

1. The vector space $V=\mathbb{F}^n$ with inner product
$$\left\langle \begin{pmatrix} v_1\\ \vdots \\ v_n\end{pmatrix}, \begin{pmatrix} w_1\\ \vdots\\ w_n\end{pmatrix}\right\rangle = v_1\overline{w_1} + \cdots + v_n\overline{w_n}$$
is an inner product space. (E8.3)

2. The vector space $V=P_n(\mathbb{F})$ with inner product
$$\langle p,q\rangle = p(0)\overline{q(0)} + \cdots + p(n)\overline{q(n)}$$
is an inner product space. (E8.4)

CONJUGATE MATRIX: $\bar{A}$ (D8.4)

Let $A=(a_{ij})\in M_{m\times n}(\mathbb{F})$.
Then, the "conjugate" of A, denoted by "$\bar{A}$",
is equal to
$$\bar{A} = (\overline{a_{ij}}) \in M_{m\times n}(\mathbb{F}).$$

CONJUGATE TRANSPOSE MATRIX: $A^* = \overline{A^T}$ (D8.4)

Then, the "conjugate transpose" of A is defined to be the matrix
$$A^* = \overline{A^T} \in M_{n\times m}(\mathbb{F}).$$

STANDARD INNER PRODUCT ON $M_{m\times n}(\mathbb{F})$: $\langle A,B\rangle = tr(AB^*)$ (E8.5)

1. Let $V=M_{m\times n}(\mathbb{F})$.
Then, the "standard inner product" on V is given by
$$\langle A,B\rangle = tr(AB^*),$$
where $tr(A) = \sum_{i=1}^m a_{ii}$ for $A\in M_{m\times m}(\mathbb{F})$.

2. We can prove this is indeed an IPS.

Proof. Linearity is trivial (arises from fact that trace & matrix multiplication is linear).
Note that for $A=(a_{ij})$ & $B=(b_{ij})$, then $B^* = (\overline{b_{ji}})$.
Then
$$(AB^*)_{ii} = \sum_{k=1}^n a_{ik}(B^*)_{ki} = \sum_{k=1}^n a_{ik}\overline{b_{ik}}.$$
Hence
$$tr(AB^*) = \sum_{i=1}^m (AB^*)_{ii} = \sum_{i=1}^m\sum_{k=1}^n a_{ik}\overline{b_{ik}}.$$
In particular, this is "similar" to if we wrote the entries of A & B in $\mathbb{F}^{mn}$, and took the standard inner product of these vectors.
It trivially follows that this gives an inner product on V. □

INNER PRODUCT ON $C[a,b]$: $\langle f(x), g(x)\rangle = \int_a^b f(x)g(x)\,dx$ (E8.6)

We can show $C[a,b]$ with the function
$$\langle f(x), g(x)\rangle = \int_a^b f(x)g(x)\,dx$$
is an inner product space.

Proof. Linearity & conjugate symmetry (ie "normal" symmetry, since field $=\mathbb{R}$) follow pretty easily.
For positivity, note that
$$\langle f(x), f(x)\rangle = \int_a^b f(x)^2\,dx \geq \int_a^b 0\,dx = 0.$$
If $f\neq 0$, then trivially $\int_a^b f(x)^2\,dx > 0$ by EVT, completing the proof. □

$T_w: V\to\mathbb{F}$ BY $T_w(v) = \langle v,w\rangle$ IS LINEAR (T8.2(1))

Let $w\in V$, and let $T_w: V\to\mathbb{F}$ by $T_w(v) = \langle v,w\rangle \ \forall v\in V$.
Then necessarily T_w is linear.

SET OF VECTORS ORTHOGONAL TO w IS A SUBSPACE OF V (T8.2(2))

Let $w\in V$.
Then the set of vectors orthogonal to w is a subspace of V.
Proof. This follows from the fact that the set $= \ker(T_w)$. □

$\|v\| \geq 0 \quad \forall v \in V, \quad v = 0 \iff \|v\| = 0$ (T8·3(1))

Let V be an IPS.
Then necessarily $\|v\| \geq 0 \ \forall v \in V$, and $\|v\| = 0$ if and only if $v = 0$.

Proof. This arises from properties of inner products.

$\|cv\| = |c| \cdot \|v\|$ (T8·3(2))

Let V be an IPS, and let $c \in \mathbb{F}$.
Then necessarily $\|cv\| = |c| \cdot \|v\|$.

Proof. This also arises from properties of inner products.

$|\langle v, w\rangle| \leq \|v\| \cdot \|w\|$; $|\langle v, w\rangle| = \|v\|\|w\| \iff v$ & w ARE LINEARLY DEPENDENT (THE CAUCHY-SCHWARTZ INEQUALITY) (T8·3(3))

Let $v, w \in V$.
Then necessarily $|\langle v,w\rangle| \leq \|v\| \cdot \|w\|$, and equality holds iff v and w are linearly dependent.

Proof. We show $|\langle v,w\rangle|^2 \leq \|v\|^2 \cdot \|w\|^2$.
First, if $w = 0$, the result is trivial. Otherwise, assume $w \neq 0$, and let $c = \frac{\langle v,w\rangle}{\|w\|^2}$.
By T8·3(1):

$$
\begin{aligned}
0 &\leq \|v - cw\|^2 \\
&= \langle v - cw, v - cw\rangle \\
&= \langle v, v - cw\rangle - c\langle w, v - cw\rangle \\
&= \langle v, v\rangle - \bar{c}\langle v, w\rangle - c\langle w, v\rangle + c\bar{c}\langle w, w\rangle \\
&= \|v\|^2 - \bar{c}\langle v,w\rangle - c\overline{\langle v,w\rangle} + |c|^2\|w\|^2 \\
&= \|v\|^2 - \frac{\overline{\langle v,w\rangle}}{\|w\|^2}\langle v,w\rangle - \frac{\langle v,w\rangle}{\|w\|^2}\overline{\langle v,w\rangle} + \frac{|\langle v,w\rangle|^2}{\|w\|^4}\|w\|^2 \\
&= \|v\|^2 - \frac{|\langle v,w\rangle|^2}{\|w\|^2} - \frac{|\langle v,w\rangle|^2}{\|w\|^2} + \frac{|\langle v,w\rangle|^2}{\|w\|^2}
\end{aligned}
$$

$\therefore 0 \leq \|v\|^2 - \frac{|\langle v,w\rangle|^2}{\|w\|^2}$,

and so $|\langle v,w\rangle|^2 \leq \|v\|^2\|w\|^2$, as needed.

Then, note that

$$
\begin{aligned}
\|v - cw\| > 0 &\iff v - cw \neq 0 \text{ for some } c \in \mathbb{F} \\
&\iff v \neq cw \quad \text{''} \\
&\iff v \text{ \& } w \text{ are lin ind,}
\end{aligned}
$$

and so $\|v - cw\| = 0 \iff v$ & w are lin dep. ∎

$\|v + w\| \leq \|v\| + \|w\| \quad \forall v, w \in V$ (TRIANGLE INEQUALITY) (T8·3 (4))

Let $v, w \in V$.
Then necessarily $\|v+w\| \leq \|v\| + \|w\|$.

Proof. Note that

$$
\begin{aligned}
\|v+w\|^2 &= \langle v+w, v+w\rangle \\
&= \langle v,v\rangle + \langle v,w\rangle + \langle w,v\rangle + \langle w,w\rangle \\
&= \|v\|^2 + \langle v,w\rangle + \overline{\langle v,w\rangle} + \|w\|^2 \\
&= \|v\|^2 + \|w\|^2 + 2\mathrm{Re}\langle v,w\rangle \\
&\leq \|v\|^2 + \|w\|^2 + 2\langle v,w\rangle \\
&\leq \|v\|^2 + 2\|v\|\|w\| + \|w\|^2 \quad \text{(by CSE)} \\
&= (\|v\| + \|w\|)^2,
\end{aligned}
$$

and the proof follows. ∎

Class 9: Orthogonal and Orthonormal Bases; The Gram-Schmidt Procedure

ORTHOGONAL & ORTHONORMAL BASIS (D9.1)

1. Let V be an IPS, and let $B \subseteq V$. Then, we say B is an "orthogonal basis" for V if:
   ① B is a basis for V; and
   ② B is an orthogonal set of vectors.
2. We say B is an "orthonormal basis" for V if the above conditions are satisfied and $\|v\| = 1 \ \forall v \in B$.

$S \subseteq V$ IS ORTHOGONAL & HAS NO ZERO VECTORS $\Rightarrow$ S IS LINEARLY INDEPENDENT (T9.1)

Let V be an IPS, and let $S \subseteq V$ be orthogonal and have no zero vectors.

Then necessarily S is linearly independent.

Proof. Let $c_1, \dots, c_n \in \mathbb{F}$, $v_1, \dots, v_n \in S$ s.t

$$c_1 v_1 + \dots + c_n v_n = 0.$$

Taking the inner product of each side with v_1, we see that

$$\begin{aligned} 0 &= \langle 0, v_1 \rangle \\ &= \langle c_1 v_1 + \dots + c_n v_n, v_1 \rangle \\ &= c_1 \langle v_1, v_1 \rangle + \dots + c_n \langle v_n, v_1 \rangle \\ &= c_1 \|v_1\|^2 + c_2(0) + \dots + c_n(0) \quad (\text{since } S \text{ is orthogonal}) \end{aligned}$$

$$\therefore 0 = c_1 \|v_1\|^2,$$

and so since $v_1 \neq 0$ it follows that $c_1 = 0$. Repeating this argument by taking inner product with $v_2, \dots, v_n$ gives us that $c_1 = \dots = c_n = 0$, showing that the vectors are linearly independent. ∎

V HAS ORTHOGONAL ORDERED BASIS $B = \{v_1, \dots, v_n\}$ $\Rightarrow$ $w = \sum_{i=1}^{n} \frac{\langle w, v_i \rangle}{\|v_i\|^2} v_i$ (T9.2)

Let V be a finite-dimensional IPS, and let V have an orthogonal ordered basis $B = \{v_1, \dots, v_n\}$.

Let $w \in V$ be arbitrary.

Then necessarily

$$w = \frac{\langle w, v_1 \rangle}{\|v_1\|^2} v_1 + \dots + \frac{\langle w, v_n \rangle}{\|v_n\|^2} v_n.$$

In other words,

$$[w]_B = \begin{pmatrix} \frac{\langle w, v_1 \rangle}{\|v_1\|^2} \\ \vdots \\ \frac{\langle w, v_n \rangle}{\|v_n\|^2} \end{pmatrix}$$

Proof. Since B is a basis, $\exists c_1, \dots, c_n \ni$

$$w = c_1 v_1 + \dots + c_n v_n.$$

Taking IP of both sides w/ v_1 yields that

$$\begin{aligned} \langle w, v_1 \rangle &= c_1 \langle v_1, v_1 \rangle + \dots + c_n \langle v_n, v_1 \rangle \\ &= c_1 \|v_1\|^2 + c_2(0) + \dots + c_n(0) \\ &= c_1 \|v_1\|^2, \end{aligned}$$

and doing similarly for $v_2, \dots, v_n$ yields that

$$\langle w, v_i \rangle = c_i \|v_i\|^2 \quad \forall 1 \leq i \leq n.$$

Thus $c_i = \frac{\langle w, v_i \rangle}{\|v_i\|^2}$, which suffices to prove the claim. ∎

V HAS ORTHONORMAL ORDERED BASIS $B = \{v_1, \dots, v_n\}$ $\Rightarrow$ $w = \sum_{i=1}^{n} \langle w, v_i \rangle v_i$ (C9.1)

Let V be a finite-dimensional IPS, and let V have an orthonormal ordered basis $B = \{v_1, \dots, v_n\}$.

Let $w \in V$ be arbitrary.

Then necessarily

$$w = \langle w, v_1 \rangle v_1 + \dots + \langle w, v_n \rangle v_n.$$

In other words,

$$[w]_B = \begin{pmatrix} \langle w, v_1 \rangle \\ \vdots \\ \langle w, v_n \rangle \end{pmatrix}$$

Proof. This follows almost immediately from T9.2.

$S = \{w_1, \dots, w_n\}$ IS LINEARLY INDEPENDENT; $v_i = v_i - \sum_{j=1}^{i-1} \frac{\langle w_i, v_j \rangle}{\|v_j\|^2} v_j \Rightarrow \{v_1, \dots, v_n\}$ IS ORTHOGONAL & $\{v_1, \dots, v_k\}$ IS AN ORTHOGONAL BASIS FOR $\text{Span}\{w_1, \dots, w_k\}$ (THE GRAM-SCHMIDT PROCEDURE) (L9.1)

Let V be an IPS, and let $S = \{w_1, \dots, w_n\} \subseteq V$ be linearly independent.

Define $\{v_1, \dots, v_n\}$ recursively by $v_1 = w_1$, and

$$v_i = w_i - \frac{\langle w_i, v_1 \rangle}{\|v_1\|^2} v_1 - \dots - \frac{\langle w_i, v_{i-1} \rangle}{\|v_{i-1}\|^2} v_{i-1} \quad \forall 1 < i \leq n.$$

Then

① $\{v_1, \dots, v_n\}$ is orthogonal; and

② $\{v_1, \dots, v_k\}$ is an orthogonal basis for $\text{Span}\{w_1, \dots, w_k\}$ for any $1 \leq k \leq n$.

Proof. We prove this by induction.

(n=1) Since $w_1 \neq 0$ (as S is lin ind), hence $\{v_1\}$ is orthogonal, and since $v_1 = w_1$, so $\text{span}\{v_1\} = \text{span}\{w_1\}$, so the conclusions trivially follow.

(inductive) Suppose the claim is true for $1 \leq k < n$.
So $\{v_1, \dots, v_k\}$ is an orthogonal basis for $\text{Span}\{w_1, \dots, w_k\}$.
We want to show similarly $\{v_1, \dots, v_{k+1}\}$ is an orthogonal basis for $\text{Span}\{w_1, \dots, w_{k+1}\}$.
Since we know $\{v_1, \dots, v_k\}$ is orthogonal, we just need to check v_{n+1} is orthogonal to each v_i to verify $\{v_1, \dots, v_{k+1}\}$ is orthogonal.
Observe that

$$\begin{aligned}
\langle v_{k+1}, v_i \rangle &= \left\langle w_{k+1} - \frac{\langle w_{k+1}, v_1 \rangle}{\|v_1\|^2} v_1 - \dots - \frac{\langle w_{k+1}, v_k \rangle}{\|v_k\|^2} v_k, v_i \right\rangle \\
&= \langle w_{k+1}, v_i \rangle - \frac{\langle w_{k+1}, v_1 \rangle}{\|v_1\|^2} \langle v_1, v_i \rangle \\
&\quad - \dots - \frac{\langle w_{k+1}, v_i \rangle}{\|v_i\|^2} \langle v_i, v_i \rangle \\
&\quad - \dots - \frac{\langle w_{k+1}, v_k \rangle}{\|v_k\|^2} \langle v_k, v_i \rangle \\
&= \langle w_{k+1}, v_i \rangle - 0 - \dots - \frac{\langle w_{k+1}, v_i \rangle}{\|v_i\|^2} \|v_i\|^2 - \dots - 0 \\
&= \langle w_{k+1}, v_i \rangle - \langle w_{k+1}, v_i \rangle \\
&= 0,
\end{aligned}$$

showing that v_{k+1} is orthogonal to each v_i, and so $\{v_1, \dots, v_{k+1}\}$ is orthogonal.

Next, we show $\text{Span}\{v_1, \dots, v_{k+1}\} = \text{Span}\{w_1, \dots, w_{k+1}\}$.
By hypothesis, $\text{Span}\{v_1, \dots, v_k\} = \text{Span}\{w_1, \dots, w_k\}$, and since

$$v_{k+1} = w_{k+1} - \frac{\langle w_{k+1}, v_1 \rangle}{\|v_1\|^2} v_1 - \dots - \frac{\langle w_{k+1}, v_k \rangle}{\|v_k\|^2} v_k$$

shows that v_{k+1} is a lin comb of $v_1, \dots, v_k, w_{k+1}$. Since this is also trivially true for $v_1, \dots, v_k$ as well, thus any lin comb of $v_1, \dots, v_{k+1}$ is a lin comb of $v_1, \dots, v_k, w_{k+1}$, and so $\text{Span}\{v_1, \dots, v_{k+1}\} \subseteq \text{Span}\{v_1, \dots, v_k, w_{k+1}\}$.
Then, $\text{Span}\{v_1, \dots, v_k\} = \text{Span}\{w_1, \dots, w_k\} \Rightarrow \text{Span}\{v_1, \dots, v_k, w_{k+1}\} = \text{Span}\{w_1, \dots, w_k, w_{k+1}\}$, and so $\text{Span}\{v_1, \dots, v_{k+1}\} \subseteq \text{Span}\{w_1, \dots, w_{k+1}\}$.
Conversely, for $1 \leq i \leq k+1$, since w_i is a lin comb of $v_1, \dots, v_i$, hence any lin comb of $w_1, \dots, w_{k+1}$ is also a lin comb of $v_1, \dots, v_{k+1}$.
So $\text{span}\{w_1, \dots, w_{k+1}\} \subseteq \text{span}\{v_1, \dots, v_{k+1}\}$, and so $\text{Span}\{w_1, \dots, w_{k+1}\} = \text{Span}\{v_1, \dots, v_{k+1}\}$.
Since $\{w_1, \dots, w_{k+1}\}$ is lin ind, it follows $\{v_1, \dots, v_{k+1}\}$ is also lin ind, and so $\{v_1, \dots, v_{k+1}\}$ is an ortho basis for $\text{Span}\{w_1, \dots, w_{k+1}\}$, completing the inductive step.

$\dim V < \infty \Rightarrow V$ HAS AN ORTHOGONAL BASIS (T9.3)

Let V be a finite-dimensional IPS.
Then necessarily V has an orthogonal basis.

Proof. Since $\dim V < \infty$, V has a finite basis, say $\{w_1, \dots, w_n\}$. Then, applying L9.1 to $\{w_1, \dots, w_n\}$ yields an orthogonal set $\{v_1, \dots, v_n\}$, for which $\{v_1, \dots, v_n\}$ is an orthogonal basis for $\text{Span}\{w_1, \dots, w_n\} = V$. ∎

$\dim V < \infty \Rightarrow V$ HAS AN ORTHONORMAL BASIS (C9.2)

Let V be a finite-dimensional IPS.
Then necessarily V has an orthonormal basis.

Proof. This follows by taking the basis obtained in T9.3 and scaling each vector down by its respective norm. ∎

Class 10: Direct Sums of Subspaces and Orthogonal Projections

SUMS OF SUBSPACES: $W_1 + \cdots + W_n$ (D10.1)

Let $W_1, \ldots, W_n \subseteq V$.

Then, the "sum" of $W_1, \ldots, W_n$, denoted as "$W_1 + \cdots + W_n$", is the set

$$W_1 + \cdots + W_n = \{ w_1 + \cdots + w_n : w_i \in W_i, \ i = 1, \ldots, n \}.$$

Note the following:

① $W_1 + \cdots + W_k$ is a subspace of V, and the smallest subspace containing $W_1, \ldots, W_k$; and

② If $\text{Span}\{S_i\} = W_i$ for each i, then $\text{Span}\left(\bigcup_{i=1}^{k} S_i\right) = W_1 + \cdots + W_k$.

DIRECT SUM OF SUBSPACES (D10.2)

Let V be a vector space, and let $W_1, \ldots, W_k \subseteq V$.

Then, we say V is the "direct sum" of $W_1, \ldots, W_k$ if

① there exist unique vectors $w_i \in W_i$ with $v = w_1 + \cdots + w_k$ for each $v \in V$;

② $V = W_1 + \cdots + W_k$ and $W_i \cap (W_1 + \cdots + W_{i-1} + W_{i+1} + \cdots + W_k) = \{0\}$ for each $i = 1, \ldots, k$;

③ If B_i is a basis for W_i, then $\bigcup_{i=1}^{k} B_i$ is a basis for V.

Note the following conditions are equivalent.

In this case, we write that

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_k.$$

DISTANCE [BETWEEN VECTORS]: $d(v,w)$ (D10.3)

Let V be an IPS, and let $v, w \in V$.

Then, the "distance" between v to w, denoted as "$d(v,w)$", is equal to

$$d(v,w) = \|v - w\|.$$

The distance function obeys the "usual" properties of distance (T10.3).

ORTHOGONAL PROJECTION MAP: $\text{proj}_W(v)$ (D10.4)

Let V be an IPS, and let $W \subseteq V$ be finite-dimensional, with orthogonal basis $\{w_1, \ldots, w_n\}$.

Then, the "orthogonal projection map" of V onto W, denoted as "$\text{proj}_W : V \to V$", is defined by

$$\text{proj}_W(v) = \frac{\langle v, w_1 \rangle}{\|w_1\|^2} w_1 + \cdots + \frac{\langle v, w_n \rangle}{\|w_n\|^2} w_n.$$

$\text{proj}_W(v)$ IS A LINEAR TRANSFORMATION (T10.4 (1))

Let V be an IPS, and let $W \subseteq V$ be finite-dimensional, with orthogonal basis $\{w_1, \ldots, w_n\}$.

Let $\text{proj}_W : V \to V$ be the associated orthogonal projection mapping.

Then necessarily proj_W is a linear transformation.

Proof. See that

$$\text{proj}_W(c_1 v_1 + c_2 v_2) = \frac{\langle c_1 v_1 + c_2 v_2, w_1 \rangle}{\|w_1\|^2} w_1 + \cdots + \frac{\langle c_1 v_1 + c_2 v_2, w_n \rangle}{\|w_n\|^2} w_n$$

$$= \frac{c_1 \langle v_1, w_1 \rangle + c_2 \langle v_2, w_1 \rangle}{\|w_1\|^2} w_1 + \cdots + \frac{c_1 \langle v_1, w_n \rangle + c_2 \langle v_2, w_n \rangle}{\|w_n\|^2} w_n$$

$$= c_1 \left[\frac{\langle v_1, w_1 \rangle}{\|w_1\|^2} w_1 + \cdots + \frac{\langle v_1, w_n \rangle}{\|w_n\|^2} w_n \right] + c_2 \left[\frac{\langle v_2, w_1 \rangle}{\|w_1\|^2} w_1 + \cdots + \frac{\langle v_2, w_n \rangle}{\|w_n\|^2} w_n \right]$$

$$= c_1 \text{proj}_W(v_1) + c_2 \text{proj}_W(v_2). \quad \blacksquare$$

$d(v, \text{proj}_W(u)) \leq d(v,w) \quad \forall w \in W$ (T10.4 (2))

Let V be an IPS, and let $W \subseteq V$ be finite-dimensional, with orthogonal basis $\{w_1, \ldots, w_n\}$.

Let $\text{proj}_W : V \to V$ be the associated orthogonal projection mapping.

Then necessarily for any $v \in V$, we have that

$$d(v, \text{proj}_W(v)) \leq d(v,w) \quad \forall w \in W.$$

Moreover, note that

$$d(v,w) \text{ is smallest} \iff w = \text{proj}_W(v). \quad \text{(T10.4 (3))}$$

Proof. Let $1 \leq i \leq n$. See that

$$\langle v - \text{proj}_W(v), w_i \rangle = \left\langle v - \frac{\langle v, w_1 \rangle}{\|w_1\|^2} w_1 - \frac{\langle v, w_2 \rangle}{\|w_2\|^2} w_2 - \cdots - \frac{\langle v, w_k \rangle}{\|w_k\|^2} w_k, w_i \right\rangle$$

$$= \langle v, w_i \rangle - \frac{\langle v, w_1 \rangle}{\|w_1\|^2} \langle w_1, w_i \rangle - \cdots - \frac{\langle v, w_k \rangle}{\|w_k\|^2} \langle w_k, w_i \rangle$$

$$= \langle v, w_i \rangle - \frac{\langle v, w_i \rangle}{\|w_i\|^2} \|w_i\|^2$$

$$= \langle v, w_i \rangle - \langle v, w_i \rangle$$

$$= 0,$$

and so $v - \text{proj}_W(v)$ is orthogonal to all the vectors w_i, it follows that $v - \text{proj}_W(v)$ is orthogonal to any $w \in W$.

Hence

$$d(v,w)^2 = \|v - w\|^2$$

$$= \langle v - w, v - w \rangle$$

$$= \langle v - \text{proj}_W(v) + \text{proj}_W(v) - w, \ v - \text{proj}_W(v) + \text{proj}_W(v) - w \rangle$$

$$= \langle v - \text{proj}_W(v), v - \text{proj}_W(v) \rangle + \langle v - \text{proj}_W(v), \text{proj}_W(v) - w \rangle + \langle \text{proj}_W(v) - w, v - \text{proj}_W(v) \rangle + \langle \text{proj}_W(v) - w, \text{proj}_W(v) - w \rangle$$

$$= \langle v - \text{proj}_W(v), v - \text{proj}_W(v) \rangle + 0 + 0 + \langle \text{proj}_W(v) - w, \text{proj}_W(v) - w \rangle$$

$$= \|v - \text{proj}_W(v)\|^2 + \|\text{proj}_W(v) - w\|^2$$

$$\therefore d(v,w)^2 \geq \|v - \text{proj}_W(v)\|^2,$$

with equality only when $\text{proj}_W(v) - w = 0$, ie $w = \text{proj}_W(v)$.

Thus $d(v,w) \geq \|v - \text{proj}_W(v)\| = d(v, \text{proj}_W(v))$, and the above observation also verifies uniqueness. $\blacksquare$

proj_W IS INDEPENDENT OF ORTHOGONAL BASIS FOR W (T10.4 (4))

Let V be an IPS, and let $W \subseteq V$ be finite-dimensional, with orthogonal basis $\{w_1, \ldots, w_n\}$.

Let $\text{proj}_W : V \to V$ be the associated orthogonal projection mapping.

Let $\{x_1, \ldots, x_n\}$ be another orthogonal basis for W, with associated orthogonal projection proj_W'.

Then necessarily $\text{proj}_W = \text{proj}_W'$.

Proof. By T10.4(2) & (3), $\text{proj}_W(v)$ is the unique vector in W closest to v.

Since proj_W' also satisfies this property, thus $\text{proj}_W'(v) = \text{proj}_W(v)$.

Since $v \in V$ was arbitrary, it follows that $\text{proj}_W = \text{proj}_W'$, as needed. $\blacksquare$

Class 11: Orthogonal Complements and Polynomial Interpolation

ORTHOGONAL COMPLEMENTS: $S^\perp$ (D11.1)

Let V be an IPS, and let $S \subseteq V$.
Then, the "orthogonal complement" of S, denoted as "$S^\perp$" (read as "S perp") is the set of vectors orthogonal to every vector in S; ie

$$S^\perp = \{ v \in V \mid \langle v, w \rangle = 0 \quad \forall w \in S \}.$$

$\dim W < \infty \Rightarrow W \oplus W^\perp = V$ (T11.1(1))

Let $W \subseteq V$ be a finite-dimensional subspace.
Then necessarily $V = W \oplus W^\perp$.

Proof. See that for any $w \in W \cap W^\perp$, $\langle w, w \rangle = 0$ by defn, so $w = 0$ necessarily.
To show $V = W + W^\perp$, consider proj_W.
For any $v \in V$, $\text{proj}_W(v)$ is a lin comb of vectors in W, and so belongs to W itself.
So

$$v = \underbrace{\text{proj}_W(v)}_{\in W} + \underbrace{(v - \text{proj}_W(v))}_{\in W^\perp},$$

since $v - \text{proj}_W(v) \in W^\perp$ from the proof of T10.4(2).
Hence $V = W + W^\perp$, as needed. #

$\dim V < \infty$, B_1, B_2 ARE ORTHOGONAL BASES FOR $W, W^\perp$ RESP $\Rightarrow$ $B_1 \cup B_2$ IS AN ORTHOGONAL BASIS FOR V; $\dim W + \dim W^\perp = \dim V$ (T11.1(2))

Let V be a finite-dimensional IPS, and let $W \subseteq V$.
Let B_1 be an orthogonal basis for W, and let B_2 be an orthogonal basis for $W^\perp$.
Then necessarily $B_1 \cup B_2$ is an orthogonal basis for V, and in particular,

$$\dim W + \dim W^\perp = \dim V.$$

Proof. Let $B_1 = \{w_1, \dots, w_k\}$ & $B_2 = \{x_1, \dots, x_\ell\}$. As $V = W + W^\perp$, thus $B_1 \cup B_2$ spans V.
To show $B_1 \cup B_2$ is an orthogonal basis, it suffices to show $B_1 \cup B_2$ is orthogonal, as then $B_1 \cup B_2$ would be lin ind by T9.1.
To show this, we just need to show $\langle w_i, x_j \rangle = 0$, as B_1 & B_2 are already orthogonal by construction.
But this follows from the fact that $w_i \in W$ & $x_j \in W^\perp$.
Thus $B_1 \cup B_2$ is an orthogonal basis for V, so that $\dim V = \dim W + \dim W^\perp$. ∎

$\text{Span}(S) = W \Rightarrow S^\perp = W^\perp$ (T11.1(3))

Let $W \subseteq V$, and let $\text{Span}(S) = W$.
Then necessarily $S^\perp = W^\perp$.
In other words, to check if $v \in W^\perp$, it suffices to show v is orthogonal to every vector in S.

Proof. See that

$$\begin{aligned} v \in W^\perp &\Rightarrow v \text{ is ortho to all vectors in } W \\ &\Rightarrow v \text{ is ortho to all vectors in } S \text{ (as } S \subseteq W) \\ &\Rightarrow v \in S^\perp, \end{aligned}$$

so $W^\perp \subseteq S^\perp$.
Then, let $v \in S^\perp$. We want to show $\langle v, w \rangle = 0$ $\forall w \in W$.
By defn of S, $\exists w_1, \dots, w_k \in S$, $c_1, \dots, c_k \in \mathbb{F}$ $\ni$

$$c_1 w_1 + \dots + c_k w_k = w.$$

As $\langle v, w_i \rangle = 0$ for each i (since $v \in S^\perp$), thus

$$\begin{aligned} \langle v, w \rangle &= \langle v, c_1 w_1 + \dots + c_k w_k \rangle \\ &= \overline{c_1} \langle v, w_1 \rangle + \dots + \overline{c_k} \langle v, w_k \rangle \\ &= \overline{c_1}(0) + \dots + \overline{c_k}(0) = 0, \end{aligned}$$

So $v \in W^\perp$, so that $S^\perp \subseteq W^\perp$, and so $S^\perp = W^\perp$, as needed. ∎

$\dim V < \infty \Rightarrow (W^\perp)^\perp = W$; $S \subseteq V \Rightarrow (S^\perp)^\perp = \text{Span}\, S$ (T11.1(4))

Let V be a finite-dimensional IPS, and let $W \subseteq V$ be a subspace.
Then necessarily $(W^\perp)^\perp = W$.
In general, if $S \subseteq V$, then $(S^\perp)^\perp = \text{Span}\, S$.

Proof. Let $w \in W$. By defn, $\langle w, x \rangle = 0$ $\forall x \in W^\perp$, and so $w \in (W^\perp)^\perp$. Thus $W \subseteq (W^\perp)^\perp$.
By T11.1(2), $\dim W + \dim W^\perp = \dim V = \dim W^\perp + \dim (W^\perp)^\perp$, so that $\dim W = \dim (W^\perp)^\perp$. Hence $W = (W^\perp)^\perp$.
Then, let $S \subseteq V$ and let $W = \text{Span}(S)$. By T11.1(3), necessarily $S^\perp = W^\perp$.
Taking orthogonal complements on both sides yields

$$(S^\perp)^\perp = (W^\perp)^\perp = W = \text{Span}\, S,$$

as needed. ∎

$\ker \text{proj}_W = W^\perp$, $\text{ran}\, \text{proj}_W = W$ (T11.2)

Let V be an IPS, let W be a finite-dimensional subspace of V, and let $\text{proj}_W : V \to V$ be the orthogonal projection onto W.
Then necessarily $\ker \text{proj}_W = W^\perp$ and $\text{ran}\, \text{proj}_W = W$.

Proof. Let $\{w_1, \dots, w_k\}$ be an ortho basis for W.
Let $v \in \ker \text{proj}_W$. See that

$$\text{proj}_W(v) = \frac{\langle v, w_1 \rangle}{\|w_1\|^2} w_1 + \dots + \frac{\langle v, w_k \rangle}{\|w_k\|^2} w_k = 0.$$

By lin ind of $\{w_1, \dots, w_k\}$, hence $\frac{\langle v, w_i \rangle}{\|w_i\|^2} = 0$, and so each $\langle v, w_i \rangle = 0$, which suffices to show that $v \in W^\perp$.
Thus $\text{proj}_W(v) = 0 \Leftrightarrow v \in W^\perp$, ie $\ker \text{proj}_W = W^\perp$.
As $\text{proj}_W(v) \in W$ $\forall v \in V$, thus $\text{ran}\, \text{proj}_W \subseteq W$. Now, let $w \in W$. We wish to show $w = \text{proj}_W(w)$.
By T10.4, $d(\text{proj}_W(w), w)$ is minimal. But $d(w, w) = 0$, so that showing that w is that "minimal element", so that $w = \text{proj}_W(w)$, as needed. ∎

LAGRANGE INTERPOLATION; FINDING A POLYNOMIAL TO APPROXIMATE DATA (T11.3)

Let $m \in \mathbb{N}$, and let $(x_1, y_1), \dots, (x_{m+1}, y_{m+1}) \in \mathbb{R}^2$.
Let $\hat{B} = \{\hat{p}_1, \dots, \hat{p}_{m+1}\} \subseteq P_m(\mathbb{R})$, where

$$\hat{p}_i = \frac{\prod_{j \neq i} (x - x_j)}{\prod_{j \neq i} (x_i - x_j)}, \quad 1 \leq i \leq m+1.$$

Then $\hat{B}$ is a basis for $P_m(\mathbb{R})$, and

$$p(x_i) = y_i \;\; \forall 1 \leq i \leq m+1 \iff p = y_1 \hat{p}_1 + \dots + y_{m+1} \hat{p}_{m+1}.$$

Proof. See that

$$\hat{p}_i(x_k) = \frac{\prod_{j \neq i} (x_k - x_j)}{\prod_{j \neq i} (x_k - x_j)} = (\text{some stuff})(x_k - x_k) = 0 \text{ if } i \neq k,$$

and

$$\hat{p}_i(x_i) = \frac{\prod_{j \neq i} (x_i - x_j)}{\prod_{j \neq i} (x_i - x_j)} = 1.$$

Hence

$$\begin{aligned} p = y_1 \hat{p}_1 + \dots + y_{m+1} \hat{p}_{m+1} \iff p(x_i) &= y_1 \hat{p}_1(x_1) + \dots + y_i \hat{p}_i(x_i) + \dots + y_{m+1} \hat{p}_{m+1}(x_{m+1}) \\ \iff p(x_i) &= y_1(0) + \dots + y_i(1) + \dots + y_{m+1}(0) \\ \iff p(x_i) &= y_i. \end{aligned}$$

To show $\hat{B}$ is a basis for $P_m(\mathbb{R})$, we need only show it's lin ind, as $|\hat{B}| = m+1 = \dim P_m(\mathbb{R})$.
Let $c_1, \dots, c_{m+1} \in \mathbb{R}$ such that

$$c_1 \hat{p}_1 + \dots + c_{m+1} \hat{p}_{m+1} = 0.$$

For any $1 \leq i \leq m+1$ and evaluating both sides at x_i yields

$$\begin{aligned} 0 &= c_1 \hat{p}_1(x_i) + \dots + c_i \hat{p}_i(x_i) + \dots + c_{m+1} \hat{p}_{m+1}(x_i) \\ &= 0 + \dots + c_i(1) + \dots + 0 \end{aligned}$$

$\therefore 0 = c_i$,

and so $c_1 = \dots = c_{m+1} = 0$, showing lin ind, and we're done. ◪

COLUMN SPACE [OF A MATRIX]: Col(A) (D11.2)

1. Let $A \in M_{m \times n}(\mathbb{F})$.
Then, the "column space" of A, denoted as "Col(A)", is the set of vectors in $\mathbb{F}^m$ of the form Ax, where $x \in \mathbb{F}^n$.
Equivalently, Col(A) is the set of all linear combinations of columns of A.

2. In particular, Col(A) is a subspace of $\mathbb{F}^m$, and the columns of A span Col(A). (T11.4)

$A \in M_{m \times n}(\mathbb{R})$; $\mathrm{Col}(A) = \mathrm{Null}(A^T)$ (L11.1)

Let $A \in M_{m \times n}(\mathbb{R})$, and give $\mathbb{R}^m$ the standard inner product.
Then necessarily $\mathrm{Col}(A)^\perp = \mathrm{Null}(A^T)$.

Proof. Since Col(A) is spanned by A's columns, by T11.1(3) we know $y \in (\text{columns of } A)^\perp \Rightarrow y \in \mathrm{Col}(A)^\perp$.
Let $y \in \mathrm{Null}(A^T)$, so $A^T y = 0$. In particular,

$$A^T y = \begin{pmatrix} — a_1 — \\ \vdots \\ — a_n — \end{pmatrix} y = \begin{pmatrix} \langle a_1, y \rangle \\ \vdots \\ \langle a_n, y \rangle \end{pmatrix},$$

std inner product (ie dot product)

So $A^T y = 0 \Rightarrow \langle a_i, y \rangle = 0 \Rightarrow y$ is ortho to each a_i,
so $y \in \mathrm{Col}(A)^\perp$, so $\mathrm{Null}(A^T) \subseteq \mathrm{Col}(A)^\perp$.
Conversely, let $y \in \mathrm{Col}(A)^\perp$, so $\langle a_i, y \rangle = 0 \;\forall i$. By the computation above, $A^T y = 0$, so $y \in \mathrm{Null}(A^T)$. Hence $\mathrm{Col}(A)^\perp \subseteq \mathrm{Null}(A^T)$, and so $\mathrm{Col}(A)^\perp = \mathrm{Null}(A^T)$, as needed. ◪

$x \in \mathbb{R}^n$ MINIMIZES $\|Ax - b\| \iff A^T A x = A^T b$ (T11.5)

Let $A \in M_{m \times n}(\mathbb{R})$ and $b \in \mathbb{R}^m$.
Then $x \in \mathbb{R}^m$ minimizes $\|Ax - b\|$ if and only if $A^T A x = A^T b$.

Proof. See that

$$\begin{aligned} x \text{ minimizes } \|Ax - b\| &\Rightarrow Ax \in \mathrm{proj}_{\mathrm{Col}(A)}(b) && (\text{since } Ax \in \mathrm{Col}(A) \text{ and by T10.4(2)}) \\ &\Rightarrow A^T(b - Ax) = 0 && (\text{by L11.1}) \\ &\Rightarrow A^T A x = A^T b, \end{aligned}$$

and

$$\begin{aligned} A^T A x = A^T b &\Rightarrow A^T(b - Ax) = 0 \\ &\Rightarrow b - Ax \in \mathrm{Null}(A^T) = \mathrm{Col}(A)^\perp && (\text{by L11.1}) \\ &\Rightarrow Ax = \mathrm{proj}_{\mathrm{Col}(A)}(b) && (\text{see reasoning on Pensall}) \\ &\Rightarrow \|Ax - b\| \text{ is minimized} \\ &\Rightarrow x \text{ minimizes } \|Ax - b\|, \end{aligned}$$

as needed. ◪

Class 12: Linear Transformations on an Inner Product Space

[T:V→W] PRESERVES INNER PRODUCTS (D12.1)

Let $(V, \langle\cdot,\cdot\rangle)$ & $(W, [\cdot,\cdot])$ be IPS, and let $T: V \to W$ be linear.
Then, we say T "preserves inner products" if
$$[T(v_1), T(v_2)] = T(\langle v_1, v_2\rangle) \quad \forall v_1, v_2 \in V.$$

In particular, we say T is an "isomorphism" of inner product spaces if T is also an isomorphism.

POLARIZATION IDENTITIES

The polarization identities state that

① V over $\mathbb{R}$ $\Rightarrow$ $\langle x,y\rangle = \frac{1}{4}\|x+y\|^2 + \frac{1}{4}\|x-y\|^2$; &

② V over $\mathbb{C}$ $\Rightarrow$ $\langle x,y\rangle = \frac{1}{4}\|x+y\|^2 + \frac{1}{4}\|x+iy\|^2 - \frac{1}{4}\|x-y\|^2 - \frac{1}{4}\|x-iy\|^2$.

Proof. This can be verified by expanding the norms in the RHS in terms of IPs. ∎

T PRESERVES INNER PRODUCTS ⇔ T PRESERVES NORMS (T12.1(1))

Let V, W be IPS, and let $T: V \to W$ be linear.
Then T preserves inner products iff T preserves norms; ie $\|T(x)\|_W = \|x\|_V$ $\forall x \in V$.

Proof. If T preserves inner products, by defn of the norm, it also preserves norms.
Now, suppose T preserves norms. If V, W are over $\mathbb{R}$, then by the polarization identities:
$$\begin{aligned}
[T(x), T(y)] &= \tfrac{1}{4}\|T(x)+T(y)\|_W^2 + \tfrac{1}{4}\|T(x)-T(y)\|_W^2 \\
&= \tfrac{1}{4}\|T(x+y)\|_W^2 + \tfrac{1}{4}\|T(x-y)\|_W^2 \\
&= \tfrac{1}{4}\|x+y\|_V^2 + \tfrac{1}{4}\|x-y\|_V^2 \\
&= \langle x, y\rangle,
\end{aligned}$$
and the case where V, W are over $\mathbb{C}$ is similar, showing T preserves IPs as well. ∎

T PRESERVES INNER PRODUCTS ⇒ T IS 1-1 (T12.1(2))

Let $T: V \to W$ preserve inner products.
Then necessarily T is 1-1.

Proof. We will show $\ker T = \{0\}$.
If $v \in V \ni T(v) = 0$, then trivially
$[T(v), T(v)] = [0,0] = 0$.
Since T preserves IPs, we have that
$[T(v), T(v)] = \langle v, v\rangle = 0$, so $v = 0$ if $T(v) = 0$.
Thus $\ker T = \{0\}$, as needed. ∎

T IS AN ISOMORPHISM OF IPS, {v₁,…,vₙ} IS AN ORTHOGONAL/ORTHONORMAL BASIS FOR V ⇒ {T(v₁),…,T(vₙ)} IS AN ORTHOGONAL/ORTHONORMAL BASIS FOR W (T12.1(3))

Let $T: V \to W$ be an isomorphism of inner product spaces, and let $\{v_1, \dots, v_n\}$ be an orthogonal (or orthonormal) basis for V.
Then necessarily $\{T(v_1), \dots, T(v_n)\}$ is an orthogonal (or orthonormal) basis for W.

Proof. We first show $\{T(v_1), \dots, T(v_n)\}$ is orthogonal.
Since $\{v_1, \dots, v_n\}$ is orthogonal, thus $\langle v_i, v_j\rangle = 0$ $\forall i \neq j$. As T preserves IPs, thus $[T(v_i), T(v_j)] = 0$ $\forall i \neq j$, which shows $\{T(v_1), \dots, T(v_n)\}$ is orthogonal.
Then, as T is an isomorphism, thus T is 1-1, so $T(v_i) \neq 0$ $\forall 1 \le i \le n$ and $\ker T = \{0\}$.
In particular, $\{T(v_1), \dots, T(v_n)\}$ is lin ind by T9.1. Since $T(V) = W$, thus $\dim V = \dim W$, and so it follows that $\{T(v_1), \dots, T(v_n)\}$ is a basis for W, which is what we wanted to prove. ∎

{v₁,…,vₙ} IS AN ORTHONORMAL BASIS FOR V, {T(v₁),…,T(vₙ)} IS AN ORTHONORMAL BASIS FOR W ⇒ T IS AN ISOMORPHISM OF IPS (T12.1(4))

Let $T: V \to W$ be linear, and let $\{v_1, \dots, v_n\}$ be an orthonormal basis for V such that $\{T(v_1), \dots, T(v_n)\}$ is an orthonormal basis for W.
Then necessarily T is an isomorphism of inner product spaces.

Proof. First, we show T preserves IPs.
Let $x_1, x_2 \in V$, say
$x_1 = c_1 v_1 + \dots + c_n v_n$ & $x_2 = d_1 v_1 + \dots + d_n v_n$.
Then
$T(x_1) = c_1 T(v_1) + \dots + c_n T(v_n)$ & $T(x_2) = d_1 T(v_1) + \dots + d_n T(v_n)$.
See that
$$\begin{aligned}
\langle x_1, x_2\rangle &= \langle c_1 v_1 + \dots + c_n v_n,\ d_1 v_1 + \dots + d_n v_n\rangle \\
&= \sum_{i=1}^n \sum_{j=1}^n c_i \bar{d_j} \langle v_i, v_j\rangle \\
&= \sum_{i=1}^n c_i \bar{d_i} \langle v_i, v_i\rangle \\
&= \sum_{i=1}^n c_i \bar{d_i} (1) \qquad (\text{as } \|v_i\| = 1)
\end{aligned}$$
$\therefore \langle x_1, x_2\rangle = \sum_{i=1}^n c_i \bar{d_i}$,
and
$$\begin{aligned}
[T(x_1), T(x_2)] &= [T(c_1 v_1 + \dots + c_n v_n),\ T(d_1 v_1 + \dots + d_n v_n)] \\
&= [c_1 T(v_1) + \dots + c_n T(v_n),\ d_1 T(v_1) + \dots + d_n T(v_n)] \\
&= \sum_{i=1}^n \sum_{j=1}^n c_i \bar{d_j} [T(v_i), T(v_j)] \\
&= \sum_{i=1}^n c_i \bar{d_i} [T(v_i), T(v_i)] \\
&= \sum_{i=1}^n c_i \bar{d_i} (1) \qquad (\text{as } \|T(v_i)\| = 1) \\
&= \sum_{i=1}^n c_i \bar{d_i},
\end{aligned}$$
showing T preserves IPs.
By T12.1(2), thus T is 1-1. In particular, since $\dim V = \dim W$, thus T is also an isomorphism, and we're done. ∎

B=(v₁,…,vₙ) IS AN ORTHONORMAL BASIS FOR V ⇒ $[T]_B = (\langle T(v_j), v_i\rangle)_{ij} \in M_{n\times n}(\mathbb{F})$ (T12.2)

Let V be a finite-dimensional IPS, and in particular, let $B = (v_1, \dots, v_n)$ be an ordered orthonormal basis for V.
Let $A = [T]_B$. Then necessarily
$$A_{ij} = \langle T(v_j), v_i\rangle \quad \forall 1 \le i, j \le n.$$

Proof. By C9.1, we have that
$$[T(v_j)]_B = \begin{pmatrix} \langle T(v_j), v_1\rangle \\ \vdots \\ \langle T(v_j), v_n\rangle \end{pmatrix} \quad \forall 1 \le j \le n.$$
Since $[T(v_j)]_B$ is the j^{th} column in $[T]_B$, it follows that the entry in the i^{th} row & j^{th} column of $[T]_B$ is $\langle T(v_j), v_i\rangle$, as needed. ∎

$B = (v_1, \dots, v_n)$ IS AN ORDERED ORTHONORMAL BASIS FOR $V \Rightarrow \langle x, y\rangle = [y]_B^* [x]_B$ (L12.1)

Let V be a finite-dimensional IPS, and in particular, let $B = (v_1, \dots, v_n)$ be an ordered orthonormal basis for V.

Let $x, y \in V$. Then necessarily

$$\langle x, y\rangle = [y]_B^* [x]_B.$$

Proof. Let $[x]_B = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ & $[y]_B = \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}$. Then

$$\begin{aligned}
\langle x, y\rangle &= \langle c_1 v_1 + \dots + c_n v_n,\ d_1 v_1 + \dots + d_n v_n\rangle \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} c_i \overline{d_j} \langle v_i, v_j\rangle \\
&= \sum_{i=1}^{n} c_i \overline{d_i} \\
&= (\overline{d_1} \ \cdots \ \overline{d_n}) \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \\
&= [y]_B^* [x]_B. \quad \blacksquare
\end{aligned}$$

$B = (v_1, \dots, v_n)$ IS AN ARBITRARY ORDERED ORTHONORMAL BASIS FOR V; $T: V \to V$ IS AN IPS ISOMORPHISM $\Leftrightarrow [T]_B^* = [T]_B^{-1}$ (T12.3)

Let V be a finite-dimensional IPS, let $T: V \to V$ be linear, and let $B = (v_1, \dots, v_n)$ be an ordered orthonormal basis for V.

Then T is an inner product space isomorphism iff $[T]_B^* = [T]_B^{-1}$.

Proof. ($\Rightarrow$) By defn, $\langle T(x), T(y)\rangle = \langle x, y\rangle \ \forall x, y \in V$.

By L12.1, $\langle x, y\rangle = [y]_B^* [x]_B$ & $\langle T(x), T(y)\rangle = [T(y)]_B^* [T(x)]_B$.

We also know

$[T(x)]_B = [T]_B [x]_B$ & $[T(y)]_B = [T]_B [y]_B$.

Hence

$$\begin{aligned}
\langle T(x), T(y)\rangle &= [T(y)]_B^* [T(x)]_B \\
&= ([T]_B [y]_B)^* ([T]_B [x]_B) \\
&= [y]_B^* ([T]_B^* [T]_B) [x]_B \\
&(= \langle x, y\rangle = [y]_B^* [x]_B).
\end{aligned}$$

We claim this implies $[T]_B^* [T]_B = I_n$.

Indeed, let $[y]_B = e_i$ & $[x]_B = e_j$.

Then $([T]_B^* [T]_B)[x]_B$ picks out the j^{th} column of $[T]_B^* [T]_B$, & taking the product on the left with $[y]_B^*$ gives the (i, j) entry of $[T]_B^* [T]_B$.

On the other hand, $[y]^* [x]_B = e_i^* e_j = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$, which suffices to show $[T]_B^* [T]_B = I_n$, and so $[T]_B^* = ([T]_B)^{-1}$, as needed. $\blacksquare$

($\Leftarrow$) We first show T preserves IPs. See that for $x, y \in V$, we have that

$$\begin{aligned}
\langle T(x), T(y)\rangle &= [T(y)]_B^* [T(x)]_B \\
&= [y]_B^* ([T]_B^* [T]_B) [x]_B \\
&= [y]_B^* (I_n) [x]_B \quad \text{(by assumption)} \\
&= [y]_B^* [x]_B = \langle x, y\rangle \text{ by L12.1.}
\end{aligned}$$

Hence, by T12.1(2), T is 1-1. Since $T: V \to V$, thus T is an isomorphism of IPS, as needed. $\blacksquare$

UNITARY MATRICES: $A^* = A^{-1}$ (D12.2)

Let $A \in M_{n \times n}(\mathbb{F})$.

Then, we say A is "unitary" if $A^* = A^{-1}$.

ORTHOGONAL MATRICES: $A^T = A^{-1}$ (D12.2)

Let $A \in M_{n \times n}(\mathbb{F})$.

Then, we say A is "orthogonal" if $A^T = A^{-1}$.