MATH 235
Personal Notes

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Class 8: Examples of Matrix Representations, Introduction to Inner Product Spaces

INNER PRODUCT & INNER PRODUCT SPACES: $\langle v,w\rangle$ (D8.1)

Let V be a vector space over $\mathbb{F}$.
Then, we say the function $\langle\cdot,\cdot\rangle : V^2 \to \mathbb{F}$ is an "inner product" if

1. $\langle v,v\rangle \in \mathbb{R}$ & $\langle v,v\rangle \geq 0$ $\forall v\in V$;
2. $\langle v,v\rangle = 0 \iff v=0$ $\forall v \in V$; } Positivity
3. $\langle v_1+v_2, w\rangle = \langle v_1,w\rangle + \langle v_2,w\rangle$ $\forall v_1,v_2,w\in V$;
4. $\langle cv, w\rangle = c\langle v,w\rangle$ $\forall c\in\mathbb{F}$, $v,w\in V$; and } Linearity
5. $\langle w,v\rangle = \overline{\langle v,w\rangle}$ $\forall v,w\in V$. } Conjugate Symmetry

In this case, we call $\langle v,w\rangle$ the "inner product" of v & w.

We refer to V together with $\langle\cdot,\cdot\rangle$ as an "inner product space".

LENGTH [OF A VECTOR]: $\|v\|$ (D8.2)

Let V be an inner product space, and let $v\in V$.
Then, the "length" of v, denoted by "$\|v\|$", is defined to be equal to

$$\|v\| = \sqrt{\langle v,v\rangle}.$$

We can do this because $\langle v,v\rangle \in \mathbb{R}$ $\forall v\in V$.

ORTHOGONAL [VECTORS] (D8.3)

Let V be an IPS.
Then, we say $v,w\in V$ are "orthogonal" if $\langle v,w\rangle = 0$.

ORTHOGONAL [SETS] (D8.3)

Let $S\subseteq V$, where V is an IPS.
Then, we say S is "orthogonal" if $\langle v,w\rangle = 0$ $\forall v,w\in S$.

EXAMPLES OF IPS: PART I

The vector space $V=\mathbb{F}^n$ with inner product

$$\left\langle \begin{pmatrix} v_1 \\ \vdots \\ v_n\end{pmatrix}, \begin{pmatrix} w_1 \\ \vdots \\ w_n\end{pmatrix}\right\rangle = v_1\overline{w_1} + \cdots + v_n\overline{w_n}$$

is an inner product space. (E8.3)

The vector space $V = P_n(\mathbb{F})$ with inner product

$$\langle p,q\rangle = p(0)\overline{q(0)} + \cdots + p(n)\overline{q(n)}$$

is an inner product space. (E8.4)

CONJUGATE MATRIX: $\overline{A}$ (D8.4)

Let $A = (a_{ij}) \in M_{m\times n}(\mathbb{F})$.
Then, the "conjugate" of A, denoted by "$\overline{A}$", is equal to

$$\overline{A} = (\overline{a_{ij}}) \in M_{m\times n}(\mathbb{F}).$$

CONJUGATE TRANSPOSE MATRIX: $A^* = \overline{A}^T$ (D8.4)

Then, the "conjugate transpose" of A is defined to be the matrix

$$A^* = \overline{A}^T \in M_{n\times m}(\mathbb{F}).$$

STANDARD INNER PRODUCT ON $M_{m\times n}(\mathbb{F})$: $\langle A,B\rangle = tr(AB^*)$ (E8.5)

Let $V = M_{m\times n}(\mathbb{F})$.
Then, the "standard inner product" on V is given by

$$\langle A,B\rangle = tr(AB^*),$$

where $tr(A) = \sum_{i=1}^m a_{ii}$ for $A\in M_{m\times m}(\mathbb{F})$.

We can prove this is indeed an IPS.

Proof. Linearity is trivial (arises from fact that trace & matrix multiplication is linear).
Note that for $A=(a_{ij})$ & $B=(b_{ij})$, then $B^* = (\overline{b_{ji}})$.
Then

$$(AB^*)_{ii} = \sum_{k=1}^n a_{ik}(B^*)_{ki} = \sum_{k=1}^n a_{ik}\overline{b_{ik}}.$$

Hence

$$tr(AB^*) = \sum_{i=1}^m (AB^*)_{ii} = \sum_{i=1}^m\sum_{k=1}^n a_{ik}\overline{b_{ik}}.$$

In particular, this is "similar" to if we wrote the entries of A & B in $\mathbb{F}^{mn}$, and took the standard inner product of these vectors.
It trivially follows that this gives an inner product on V. □

INNER PRODUCT ON $C[a,b]$: $\langle f(x), g(x)\rangle = \int_a^b f(x)g(x)\,dx$ (E8.6)

We can show $C[a,b]$ with the function

$$\langle f(x), g(x)\rangle = \int_a^b f(x)g(x)\,dx$$

is an inner product space.

Proof. Linearity & conjugate symmetry (ie "normal" symmetry, since field $=\mathbb{R}$) follow pretty easily.
For positivity, note that

$$\langle f(x), f(x)\rangle = \int_a^b f(x)^2\,dx \geq \int_a^b 0\,dx = 0.$$

If $f\neq 0$, then trivially $\int_a^b f(x)^2\,dx > 0$ by EVT, completing the proof. ■

$T_w : V\to\mathbb{F}$ BY $T_w(v) = \langle v,w\rangle$ IS LINEAR (T8.2(1))

Let $w\in V$, and let $T_w : V\to\mathbb{F}$ by $T_w(v) = \langle v,w\rangle$ $\forall v\in V$.
Then necessarily T_w is linear.

SET OF VECTORS ORTHOGONAL TO w IS A SUBSPACE OF V (T8.2(2))

Let $w\in V$.
Then the set of vectors orthogonal to w is a subspace of V.

Proof. This follows from the fact that the set $= \ker(T_w)$. ■

$\|v\| \geq 0 \quad \forall v \in V, \quad v = 0 \iff \|v\| = 0$ (T8·3(1))

Let V be an IPS.

Then necessarily $\|v\| \geq 0 \ \forall v \in V$, and $\|v\| = 0$ if and only if $v = 0$.

Proof. This arises from properties of inner products.

$\|cv\| = |c| \cdot \|v\|$ (T8·3(2))

Let V be an IPS, and let $c \in \mathbb{F}$.

Then necessarily $\|cv\| = |c| \cdot \|v\|$.

Proof. This also arises from properties of inner products.

$|\langle v,w\rangle| \leq \|v\| \cdot \|w\|$; $|\langle v,w\rangle| = \|v\|\|w\| \iff v$ & w ARE LINEARLY DEPENDENT (THE CAUCHY-SCHWARTZ INEQUALITY) (T8·3(3))

Let $v, w \in V$.

Then necessarily $|\langle v,w\rangle| \leq \|v\| \cdot \|w\|$, and equality holds iff v and w are linearly dependent.

Proof. We show $|\langle v,w\rangle|^2 \leq \|v\|^2 \cdot \|w\|^2$.

First, if $w = 0$, the result is trivial. Otherwise, assume $w \neq 0$, and let $c = \frac{\langle v,w\rangle}{\|w\|^2}$.

By T8·3(1):

$$
\begin{aligned}
0 &\leq \|v - cw\|^2 \\
&= \langle v-cw, v-cw\rangle \\
&= \langle v, v-cw\rangle - c\langle w, v-cw\rangle \\
&= \langle v, v\rangle - \bar{c}\langle v,w\rangle - c\langle w,v\rangle + c\bar{c}\langle w,w\rangle \\
&= \|v\|^2 - \bar{c}\langle v,w\rangle - c\overline{\langle v,w\rangle} + |c|^2\|w\|^2 \\
&= \|v\|^2 - \frac{\overline{\langle v,w\rangle}}{\|w\|^2}\langle v,w\rangle - \frac{\langle v,w\rangle}{\|w\|^2}\overline{\langle v,w\rangle} + \frac{|\langle v,w\rangle|^2}{\|w\|^4}\|w\|^2 \\
&= \|v\|^2 - \frac{|\langle v,w\rangle|^2}{\|w\|^2} - \frac{|\langle v,w\rangle|^2}{\|w\|^2} + \frac{|\langle v,w\rangle|^2}{\|w\|^2}
\end{aligned}
$$

$$\therefore 0 \leq \|v\|^2 - \frac{|\langle v,w\rangle|^2}{\|w\|^2},$$

and so $|\langle v,w\rangle|^2 \leq \|v\|^2\|w\|^2$, as needed.

Then, note that

$$
\begin{aligned}
\|v - cw\| > 0 &\iff v - cw \neq 0 \text{ for some } c \in \mathbb{F} \\
&\iff v \neq cw \quad \text{''} \\
&\iff v \,\&\, w \text{ are lin ind,}
\end{aligned}
$$

and so $\|v - cw\| = 0 \iff v$ & w are lin dep. ☒

$\|v+w\| \leq \|v\| + \|w\| \quad \forall v,w \in V$ (TRIANGLE INEQUALITY) (T8·3 (4))

Let $v, w \in V$.

Then necessarily $\|v+w\| \leq \|v\| + \|w\|$.

Proof. Note that

$$
\begin{aligned}
\|v+w\|^2 &= \langle v+w, v+w\rangle \\
&= \langle v,v\rangle + \langle v,w\rangle + \langle w,v\rangle + \langle w,w\rangle \\
&= \|v\|^2 + \langle v,w\rangle + \overline{\langle v,w\rangle} + \|w\|^2 \\
&= \|v\|^2 + \|w\|^2 + 2\mathrm{Re}\langle v,w\rangle \\
&\leq \|v\|^2 + \|w\|^2 + 2\langle v,w\rangle \\
&\leq \|v\|^2 + 2\|v\|\|w\| + \|w\|^2 \quad \text{(by CSE)} \\
&= (\|v\| + \|w\|)^2,
\end{aligned}
$$

and the proof follows. ☒

Class 9: Orthogonal and Orthonormal Bases;The Gram-Schmidt Procedure

ORTHOGONAL & ORTHONORMAL BASIS (D9.1)

Let V be an IPS, and let $B \subseteq V$.
Then, we say B is an "orthogonal basis" for V if:

① B is a basis for V; and
② B is an orthogonal set of vectors.

We say B is an "orthonormal basis" for V if the above conditions are satisfied and $\|v\| = 1 \quad \forall v \in B$.

$S \subseteq V$ IS ORTHOGONAL & HAS NO ZERO VECTORS $\Rightarrow$ S IS LINEARLY INDEPENDENT (T9.1)

Let V be an IPS, and let $S \subseteq V$ be orthogonal and have no zero vectors.
Then necessarily S is linearly independent.

Proof. Let $c_1, \dots, c_n \in \mathbb{F}$, $v_1, \dots, v_n \in S$ s.t

$$c_1 v_1 + \dots + c_n v_n = 0.$$

Taking the inner product of each side with v_1, we see that

$$\begin{aligned} 0 &= \langle 0, v_1 \rangle \\ &= \langle c_1 v_1 + \dots + c_n v_n, v_1 \rangle \\ &= c_1 \langle v_1, v_1 \rangle + \dots + c_n \langle v_n, v_1 \rangle \\ &= c_1 \|v_1\|^2 + c_2(0) + \dots + c_n(0) \quad \text{(since } S \text{ is orthogonal)} \end{aligned}$$

$$\therefore \ 0 = c_1 \|v_1\|^2,$$

and so since $v_1 \neq 0$ it follows that $c_1 = 0$.
Repeating this argument by taking inner product with $v_2, \dots, v_n$ gives us that $c_1 = \dots = c_n = 0$, showing that the vectors are linearly independent. ∎

V HAS ORTHOGONAL ORDERED BASIS $B = \{v_1, \dots, v_n\} \Rightarrow$ $w = \sum_{i=1}^{n} \frac{\langle w, v_i \rangle}{\|v_i\|^2} v_i$ (T9.2)

Let V be a finite-dimensional IPS, and let V have an orthogonal ordered basis $B = \{v_1, \dots, v_n\}$.
Let $w \in V$ be arbitrary.
Then necessarily

$$w = \frac{\langle w, v_1 \rangle}{\|v_1\|^2} v_1 + \dots + \frac{\langle w, v_n \rangle}{\|v_n\|^2} v_n.$$

In other words,

$$[w]_B = \begin{pmatrix} \frac{\langle w, v_1 \rangle}{\|v_1\|^2} \\ \vdots \\ \frac{\langle w, v_n \rangle}{\|v_n\|^2} \end{pmatrix}.$$

Proof. Since B is a basis, $\exists c_1, \dots, c_n \ni$

$$w = c_1 v_1 + \dots + c_n v_n.$$

Taking IP of both sides w/ v_1 yields that

$$\begin{aligned} \langle w, v_1 \rangle &= c_1 \langle v_1, v_1 \rangle + \dots + c_n \langle v_n, v_1 \rangle \\ &= c_1 \|v_1\|^2 + c_2(0) + \dots + c_n(0) \\ &= c_1 \|v_1\|^2, \end{aligned}$$

and doing similarly for $v_2, \dots, v_n$ yields that

$$\langle w, v_i \rangle = c_i \|v_i\|^2 \quad \forall 1 \le i \le n.$$

Thus $c_i = \frac{\langle w, v_i \rangle}{\|v_i\|^2}$, which suffices to prove the claim. ∎

V HAS ORTHONORMAL ORDERED BASIS $B = \{v_1, \dots, v_n\} \Rightarrow$ $w = \sum_{i=1}^{n} \langle w, v_i \rangle v_i$ (C9.1)

Let V be a finite-dimensional IPS, and let V have an orthonormal ordered basis $B = \{v_1, \dots, v_n\}$.
Let $w \in V$ be arbitrary.
Then necessarily

$$w = \langle w, v_1 \rangle v_1 + \dots + \langle w, v_n \rangle v_n.$$

In other words,

$$[w]_B = \begin{pmatrix} \langle w, v_1 \rangle \\ \vdots \\ \langle w, v_n \rangle \end{pmatrix}$$

Proof. This follows almost immediately from T9.2.

$S=\{w_1,\dots,w_n\}$ IS LINEARLY INDEPENDENT; $v_i = v_i - \sum_{j=1}^{i-1} \frac{\langle w_i, v_j\rangle}{\|v_j\|^2} v_j \Rightarrow \{v_1,\dots,v_n\}$ IS ORTHOGONAL & $\{v_1,\dots,v_k\}$ IS AN ORTHOGONAL BASIS FOR $\text{Span}\{w_1,\dots,w_k\}$ (THE GRAM-SCHMIDT PROCEDURE) (L9.1)

Let V be an IPS, and let $S=\{w_1,\dots,w_n\}\subseteq V$ be linearly independent.
Define $\{v_1,\dots,v_n\}$ recursively by $v_1=w_1$, and

$$v_i = w_i - \frac{\langle w_i, v_1\rangle}{\|v_1\|^2} v_1 - \dots - \frac{\langle w_i, v_{i-1}\rangle}{\|v_{i-1}\|^2} v_{i-1} \quad \forall 1<i\le n.$$

Then
① $\{v_1,\dots,v_n\}$ is orthogonal; and
② $\{v_1,\dots,v_k\}$ is an orthogonal basis for $\text{Span}\{w_1,\dots,w_k\}$ for any $1\le k\le n$.

Proof. We prove this by induction.
(n=1) Since $w_1\neq 0$ (as S is lin ind), hence $\{v_1\}$ is orthogonal, and since $v_1=w_1$, so $\text{span}\{v_1\}=\text{span}\{w_1\}$, so the conclusions trivially follow.

(inductive) Suppose the claim is true for $1\le k<n$.
So $\{v_1,\dots,v_k\}$ is an orthogonal basis for $\text{Span}\{w_1,\dots,w_k\}$.
We want to show similarly $\{v_1,\dots,v_{k+1}\}$ is an orthogonal basis for $\text{Span}\{w_1,\dots,w_{k+1}\}$.
Since we know $\{v_1,\dots,v_k\}$ is orthogonal, we just need to check v_{n+1} is orthogonal to each v_i to verify $\{v_1,\dots,v_{k+1}\}$ is orthogonal.
Observe that

$$\langle v_{k+1}, v_i\rangle = \left\langle w_{k+1} - \frac{\langle w_{k+1}, v_1\rangle}{\|v_1\|^2} v_1 - \dots - \frac{\langle w_{k+1}, v_k\rangle}{\|v_k\|^2} v_k, v_i\right\rangle$$

$$= \langle w_{k+1}, v_i\rangle - \frac{\langle w_{k+1}, v_1\rangle}{\|v_1\|^2}\langle v_1, v_i\rangle - \dots - \frac{\langle w_{k+1}, v_i\rangle}{\|v_i\|^2}\langle v_i, v_i\rangle - \dots - \frac{\langle w_{k+1}, v_k\rangle}{\|v_k\|^2}\langle v_k, v_i\rangle$$

$$= \langle w_{k+1}, v_i\rangle - 0 - \dots - \frac{\langle w_{k+1}, v_i\rangle}{\|v_i\|^2}\|v_i\|^2 - \dots - 0$$

$$= \langle w_{k+1}, v_i\rangle - \langle w_{k+1}, v_i\rangle$$

$$= 0,$$

showing that v_{k+1} is orthogonal to each v_i, and so $\{v_1,\dots,v_{k+1}\}$ is orthogonal.

Next, we show $\text{Span}\{v_1,\dots,v_{k+1}\} = \text{Span}\{w_1,\dots,w_{k+1}\}$.
By hypothesis, $\text{Span}\{v_1,\dots,v_k\} = \text{Span}\{w_1,\dots,w_k\}$, and since

$$v_{k+1} = w_{k+1} - \frac{\langle w_{k+1}, v_1\rangle}{\|v_1\|^2} v_1 - \dots - \frac{\langle w_{k+1}, v_k\rangle}{\|v_k\|^2} v_k$$

shows that v_{k+1} is a lin comb of $v_1,\dots,v_k,w_{k+1}$. Since this is also trivially true for $v_1,\dots,v_k$ as well, thus any lin comb of $v_1,\dots,v_{k+1}$ is a lin comb of $v_1,\dots,v_k,w_{k+1}$, and so $\text{Span}\{v_1,\dots,v_{k+1}\}\subseteq \text{Span}\{v_1,\dots,v_k,w_{k+1}\}$.
Then, $\text{Span}\{v_1,\dots,v_k\} = \text{Span}\{w_1,\dots,w_k\} \Rightarrow$ $\text{Span}\{v_1,\dots,v_k,w_{k+1}\} = \text{Span}\{w_1,\dots,w_k,w_{k+1}\}$, and so $\text{Span}\{v_1,\dots,v_{k+1}\}\subseteq \text{Span}\{w_1,\dots,w_{k+1}\}$.
Conversely, for $1\le i\le k+1$, since w_i is a lin comb of $v_1,\dots,v_i$, hence any lin comb of $w_1,\dots,w_{k+1}$ is also a lin comb of $v_1,\dots,v_{k+1}$.
So $\text{span}\{w_1,\dots,w_{k+1}\}\subseteq \text{span}\{v_1,\dots,v_{k+1}\}$, and so $\text{Span}\{w_1,\dots,w_{k+1}\} = \text{Span}\{v_1,\dots,v_{k+1}\}$.
Since $\{w_1,\dots,w_{k+1}\}$ is lin ind, it follows $\{v_1,\dots,v_{k+1}\}$ is also lin ind, and so $\{v_1,\dots,v_{k+1}\}$ is an ortho basis for $\text{Span}\{w_1,\dots,w_{k+1}\}$, completing the inductive step.

$\dim V<\infty \Rightarrow V$ HAS AN ORTHOGONAL BASIS (T9.3)

Let V be a finite-dimensional IPS.
Then necessarily V has an orthogonal basis.

Proof. Since $\dim V<\infty$, V has a finite basis, say $\{w_1,\dots,w_n\}$.
Then, applying L9.1 to $\{w_1,\dots,w_n\}$ yields an orthogonal set $\{v_1,\dots,v_n\}$, for which $\{v_1,\dots,v_n\}$ is an orthogonal basis for $\text{Span}\{w_1,\dots,w_n\}=V$. ∎

$\dim V<\infty \Rightarrow V$ HAS AN ORTHONORMAL BASIS (C9.2)

Let V be a finite-dimensional IPS.
Then necessarily V has an orthonormal basis.

Proof. This follows by taking the basis obtained in T9.3 and scaling each vector down by its respective norm. ∎

Class 10:
Direct Sums of Subspaces and Orthogonal Projections

SUMS OF SUBSPACES: $W_1 + \cdots + W_n$ (D10.1)

Let $W_1, \ldots, W_n \leq V$.
Then, the "sum" of $W_1, \ldots, W_n$, denoted as "$W_1 + \cdots + W_n$", is the set
$$W_1 + \cdots + W_n = \{ w_1 + \cdots + w_n : w_i \in W_i,\ i = 1, \ldots, n \}.$$

Note the following:
① $W_1 + \cdots + W_k$ is a subspace of V, and the smallest subspace containing $W_1, \ldots, W_k$; and
② If $\operatorname{Span}\{S_i\} = W_i$ for each i, then $\operatorname{Span}\left(\bigcup_{i=1}^{k} S_i \right) = W_1 + \cdots + W_k$.

DIRECT SUM OF SUBSPACES (D10.2)

Let V be a vector space, and let $W_1, \ldots, W_k \leq V$.
Then, we say V is the "direct sum" of $W_1, \ldots, W_k$ if
① there exist unique vectors $w_i \in W_i$ with $v = w_1 + \cdots + w_k$ for each $v \in V$;
② $V = W_1 + \cdots + W_k$ and $W_i \cap (W_1 + \cdots + W_{i-1} + W_{i+1} + \cdots + W_k) = \{0\}$ for each $i = 1, \ldots, k$;
③ If B_i is a basis for W_i, then $\bigcup_{i=1}^{k} B_i$ is a basis for V.

Note the following conditions are equivalent.

In this case, we write that
$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_k.$$

DISTANCE [BETWEEN VECTORS]: $d(v,w)$ (D10.3)

Let V be an IPS, and let $v, w \in V$.
Then, the "distance" between v to w, denoted as "$d(v,w)$", is equal to
$$d(v,w) = \|v - w\|.$$

The distance function obeys the "usual" properties of distance (T10.3).

ORTHOGONAL PROJECTION MAP: $\operatorname{proj}_W(v)$ (D10.4)

Let V be an IPS, and let $W \leq V$ be finite-dimensional, with orthogonal basis $\{w_1, \ldots, w_n\}$.
Then, the "orthogonal projection map" of V onto W, denoted as "$\operatorname{proj}_W : V \to V$," is defined by
$$\operatorname{proj}_W(v) = \frac{\langle v, w_1 \rangle}{\|w_1\|^2} w_1 + \cdots + \frac{\langle v, w_n \rangle}{\|w_n\|^2} w_n.$$

$\operatorname{proj}_W(v)$ IS A LINEAR TRANSFORMATION (T10.4 (1))

Let V be an IPS, and let $W \leq V$ be finite-dimensional, with orthogonal basis $\{w_1, \ldots, w_n\}$.
Let $\operatorname{proj}_W : V \to V$ be the associated orthogonal projection mapping.
Then necessarily proj_W is a linear transformation.

Proof. See that
$$\begin{aligned}
\operatorname{proj}_W(c_1 v_1 + c_2 v_2) &= \frac{\langle c_1 v_1 + c_2 v_2, w_1 \rangle}{\|w_1\|^2} w_1 + \cdots + \frac{\langle c_1 v_1 + c_2 v_2, w_n \rangle}{\|w_n\|^2} w_n \\
&= \frac{c_1 \langle v_1, w_1 \rangle + c_2 \langle v_2, w_1 \rangle}{\|w_1\|^2} w_1 + \cdots + \frac{c_1 \langle v_1, w_n \rangle + c_2 \langle v_2, w_n \rangle}{\|w_n\|^2} w_n \\
&= c_1 \left[\frac{\langle v_1, w_1 \rangle}{\|w_1\|^2} w_1 + \cdots + \frac{\langle v_1, w_n \rangle}{\|w_n\|^2} w_n \right] + c_2 \left[\frac{\langle v_2, w_1 \rangle}{\|w_1\|^2} w_1 + \cdots + \frac{\langle v_2, w_n \rangle}{\|w_n\|^2} w_n \right] \\
&= c_1 \operatorname{proj}_W(v_1) + c_2 \operatorname{proj}_W(v_2). \quad \blacksquare
\end{aligned}$$

$d(v, \operatorname{proj}_W(u)) \leq d(v,w)\ \forall w \in W$ (T10.4 (2))

Let V be an IPS, and let $W \leq V$ be finite-dimensional, with orthogonal basis $\{w_1, \ldots, w_n\}$.
Let $\operatorname{proj}_W : V \to V$ be the associated orthogonal projection mapping.
Then necessarily for any $v \in V$, we have that
$$d(v, \operatorname{proj}_W(v)) \leq d(v,w) \quad \forall w \in W.$$
Moreover, note that
$$d(v,w) \text{ is smallest} \iff w = \operatorname{proj}_W(v). \quad \text{(T10.4 (3))}$$

Proof. Let $1 \leq i \leq n$. See that
$$\begin{aligned}
\langle v - \operatorname{proj}_W(v), w_i \rangle &= \left\langle v - \frac{\langle v, w_1 \rangle}{\|w_1\|^2} w_1 - \frac{\langle v, w_2 \rangle}{\|w_2\|^2} w_2 - \cdots - \frac{\langle v, w_k \rangle}{\|w_k\|^2} w_k, w_i \right\rangle \\
&= \langle v, w_i \rangle - \frac{\langle v, w_1 \rangle}{\|w_1\|^2} \langle w_1, w_i \rangle - \cdots - \frac{\langle v, w_k \rangle}{\|w_k\|^2} \langle w_k, w_i \rangle \\
&= \langle v, w_i \rangle - \frac{\langle v, w_i \rangle}{\|w_i\|^2} \|w_i\|^2 \\
&= \langle v, w_i \rangle - \langle v, w_i \rangle \\
&= 0,
\end{aligned}$$
and so $v - \operatorname{proj}_W(v)$ is orthogonal to all the vectors w_i, it follows that $v - \operatorname{proj}_W(v)$ is orthogonal to any $w \in W$.
Hence
$$\begin{aligned}
d(v,w)^2 &= \|v - w\|^2 \\
&= \langle v - w, v - w \rangle \\
&= \langle v - \operatorname{proj}_W(v) + \operatorname{proj}_W(v) - w,\ v - \operatorname{proj}_W(v) + \operatorname{proj}_W(v) - w \rangle \\
&= \langle v - \operatorname{proj}_W(v), v - \operatorname{proj}_W(v) \rangle + \langle v - \operatorname{proj}_W(v), \operatorname{proj}_W(v) - w \rangle \\
&\quad + \langle \operatorname{proj}_W(v) - w, v - \operatorname{proj}_W(v) \rangle + \langle \operatorname{proj}_W(v) - w, \operatorname{proj}_W(v) - w \rangle \\
&= \langle v - \operatorname{proj}_W(v), v - \operatorname{proj}_W(v) \rangle + 0 + 0 + \langle \operatorname{proj}_W(v) - w, \operatorname{proj}_W(v) - w \rangle \\
&= \|v - \operatorname{proj}_W(v)\|^2 + \|\operatorname{proj}_W(v) - w\|^2
\end{aligned}$$
$\therefore d(v,w)^2 \geq \|v - \operatorname{proj}_W(v)\|^2$,
with equality only when $\operatorname{proj}_W(v) - w = 0$, ie $w = \operatorname{proj}_W(v)$.
Thus $d(v,w) \geq \|v - \operatorname{proj}_W(v)\| = d(v, \operatorname{proj}_W(v))$, and the above observation also verifies uniqueness. $\blacksquare$

proj_W IS INDEPENDENT OF ORTHOGONAL BASIS FOR W (T10.4 (4))

Let V be an IPS, and let $W \leq V$ be finite-dimensional, with orthogonal basis $\{w_1, \ldots, w_n\}$.
Let $\operatorname{proj}_W : V \to V$ be the associated orthogonal projection mapping.
Let $\{x_1, \ldots, x_n\}$ be another orthogonal basis for W, with associated orthogonal projection proj_W'.
Then necessarily $\operatorname{proj}_W = \operatorname{proj}_W'$.

Proof. By T10.4(2) & (3), $\operatorname{proj}_W(v)$ is the unique vector in W closest to v.
Since proj_W' also satisfies this property, thus $\operatorname{proj}_W'(v) = \operatorname{proj}_W(v)$.
Since $v \in V$ was arbitrary, it follows that $\operatorname{proj}_W = \operatorname{proj}_W'$, as needed. $\blacksquare$

Class 11:

Orthogonal Complements and Polynomial Interpolation

ORTHOGONAL COMPLEMENTS: $S^\perp$ (D11.1)

Let V be an IPS, and let $S \subseteq V$.
Then, the "orthogonal complement" of S, denoted as "$S^\perp$" (read as "S perp") is the set of vectors orthogonal to every vector in S; ie

$$S^\perp = \{ v \in V \mid \langle v, w \rangle = 0 \quad \forall w \in S \}.$$

$\dim W < \infty \Rightarrow W \oplus W^\perp = V$ (T11.1(1))

Let $W \subseteq V$ be a finite-dimensional subspace.
Then necessarily $V = W \oplus W^\perp$.

Proof. See that for any $w \in W \cap W^\perp$, $\langle w, w \rangle = 0$ by def$^{\underline{n}}$, so $w = 0$ necessarily.
To show $V = W + W^\perp$, consider proj_W.
For any $v \in V$, $\mathrm{proj}_W(v)$ is a lin comb of vectors in W, and so belongs to W itself.
So

$$v = \underbrace{\mathrm{proj}_W(v)}_{\in W} + \underbrace{(v - \mathrm{proj}_W(v))}_{\in W^\perp},$$

since $v - \mathrm{proj}_W(v) \in W^\perp$ from the proof of T10.4(2).
Hence $V = W + W^\perp$, as needed. #

$\dim V < \infty$, B_1, B_2 ARE ORTHOGONAL BASES FOR $W, W^\perp$ RESP $\Rightarrow B_1 \cup B_2$ IS AN ORTHOGONAL BASIS FOR V; $\dim W + \dim W^\perp = \dim V$ (T11.1(2))

Let V be a finite-dimensional IPS, and let $W \subseteq V$.
Let B_1 be an orthogonal basis for W, and let B_2 be an orthogonal basis for $W^\perp$.
Then necessarily $B_1 \cup B_2$ is an orthogonal basis for V, and in particular,

$$\dim W + \dim W^\perp = \dim V.$$

Proof. Let $B_1 = \{w_1, \dots, w_k\}$ & $B_2 = \{x_1, \dots, x_\ell\}$. As $V = W + W^\perp$, thus $B_1 \cup B_2$ spans V.
To show $B_1 \cup B_2$ is an orthogonal basis, it suffices to show $B_1 \cup B_2$ is orthogonal, as then $B_1 \cup B_2$ would be lin ind by T9.1.
To show this, we just need to show $\langle w_i, x_j \rangle = 0$, as B_1 & B_2 are already orthogonal by construction.
But this follows from the fact that $w_i \in W$ & $x_j \in W^\perp$.
Thus $B_1 \cup B_2$ is an orthogonal basis for V, so that $\dim V = \dim W + \dim W^\perp$. ■

$\mathrm{Span}(S) = W \Rightarrow S^\perp = W^\perp$ (T11.1(3))

Let $W \subseteq V$, and let $\mathrm{Span}(S) = W$.
Then necessarily $S^\perp = W^\perp$.
In other words, to check if $v \in W^\perp$, it suffices to show v is orthogonal to every vector in S.

Proof. See that

$$\begin{aligned} v \in W^\perp &\Rightarrow v \text{ is ortho to all vectors in } W \\ &\Rightarrow v \text{ is ortho to all vectors in } S \text{ (as } S \subseteq W) \\ &\Rightarrow v \in S^\perp, \end{aligned}$$

so $W^\perp \subseteq S^\perp$.
Then, let $v \in S^\perp$. We want to show $\langle v, w \rangle = 0$ $\forall w \in W$.
By def$^{\underline{n}}$ of S, $\exists w_1, \dots, w_k \in S$, $c_1, \dots, c_k \in \mathbb{F}$ $\ni$

$$c_1 w_1 + \dots + c_k w_k = w.$$

As $\langle v, w_i \rangle = 0$ for each i (since $v \in S^\perp$), thus

$$\begin{aligned} \langle v, w \rangle &= \langle v, c_1 w_1 + \dots + c_k w_k \rangle \\ &= \bar{c}_1 \langle v, w_1 \rangle + \dots + \bar{c}_k \langle v, w_k \rangle \\ &= \bar{c}_1 (0) + \dots + \bar{c}_k (0) = 0, \end{aligned}$$

So $v \in W^\perp$, so that $S^\perp \subseteq W^\perp$, and so $S^\perp = W^\perp$, as needed. ■

$\dim V < \infty \Rightarrow (W^\perp)^\perp = W$; $S \subseteq V \Rightarrow (S^\perp)^\perp = \mathrm{Span}\, S$ (T11.1(4))

Let V be a finite-dimensional IPS, and let $W \subseteq V$ be a subspace.
Then necessarily $(W^\perp)^\perp = W$.
In general, if $S \subseteq V$, then $(S^\perp)^\perp = \mathrm{Span}\, S$.

Proof. Let $w \in W$. By def$^{\underline{n}}$, $\langle w, x \rangle = 0$ $\forall x \in W^\perp$, and so $w \in (W^\perp)^\perp$. Thus $W \subseteq (W^\perp)^\perp$.
By T11.1(2), $\dim W + \dim W^\perp = \dim V = \dim W^\perp + \dim (W^\perp)^\perp$, so that $\dim W = \dim (W^\perp)^\perp$. Hence $W = (W^\perp)^\perp$.
Then, let $S \subseteq V$ and let $W = \mathrm{Span}(S)$. By T11.1(3), necessarily $S^\perp = W^\perp$.
Taking orthogonal complements on both sides yields

$$(S^\perp)^\perp = (W^\perp)^\perp = W = \mathrm{Span}\, S,$$

as needed. ■

$\ker \mathrm{proj}_W = W^\perp$, $\mathrm{ran}\, \mathrm{proj}_W = W$ (T11.2)

Let V be an IPS, let W be a finite-dimensional subspace of V, and let $\mathrm{proj}_W : V \to V$ be the orthogonal projection onto W.
Then necessarily $\ker \mathrm{proj}_W = W^\perp$ and $\mathrm{ran}\, \mathrm{proj}_W = W$.

Proof. Let $\{w_1, \dots, w_k\}$ be an ortho basis for W.
Let $v \in \ker \mathrm{proj}_W$. See that

$$\mathrm{proj}_W(v) = \frac{\langle v, w_1 \rangle}{\|w_1\|^2} w_1 + \dots + \frac{\langle v, w_k \rangle}{\|w_k\|^2} w_k = 0.$$

By lin ind of $\{w_1, \dots, w_k\}$, hence $\frac{\langle v, w_i \rangle}{\|w_i\|^2} = 0$, and so each $\langle v, w_i \rangle = 0$, which suffices to show that $v \in W^\perp$.
Thus $\mathrm{proj}_W(v) = 0 \Leftrightarrow v \in W^\perp$, ie $\ker \mathrm{proj}_W = W^\perp$.
As $\mathrm{proj}_W(v) \in W$ $\forall v \in V$, thus $\mathrm{ran}\, \mathrm{proj}_W \subseteq W$. Now, let $w \in W$. We wish to show $w = \mathrm{proj}_W(w)$.
By T10.4, $d(\mathrm{proj}_W(w), w)$ is minimal. But $d(w, w) = 0$, so that showing that w is that "minimal element", so that $w = \mathrm{proj}_W(w)$, as needed. ■

LAGRANGE INTERPOLATION; FINDING A POLYNOMIAL TO APPROXIMATE DATA (T11.3)

Let $m \in \mathbb{N}$, and let $(x_1, y_1), \ldots, (x_{m+1}, y_{m+1}) \in \mathbb{R}^2$.
Let $\hat{B} = \{\hat{p}_1, \ldots, \hat{p}_{m+1}\} \subseteq P_m(\mathbb{R})$, where

$$\hat{p}_i = \frac{\prod_{j \neq i} (x - x_j)}{\prod_{j \neq i} (x_i - x_j)}, \quad 1 \leq i \leq m+1.$$

Then $\hat{B}$ is a basis for $P_m(\mathbb{R})$, and

$$p(x_i) = y_i \ \forall 1 \leq i \leq m+1 \iff p = y_1 \hat{p}_1 + \cdots + y_{m+1} \hat{p}_{m+1}.$$

Proof. See that

$$\hat{p}_i(x_k) = \frac{\prod_{j \neq i} (x_k - x_j)}{\prod_{j \neq i} (x_k - x_j)} = (\text{some stuff})(x_k - x_k) = 0 \text{ if } i \neq k,$$

and

$$\hat{p}_i(x_i) = \frac{\prod_{j \neq i} (x_i - x_j)}{\prod_{j \neq i} (x_i - x_j)} = 1.$$

Hence

$$p = y_1 \hat{p}_1 + \cdots + y_{m+1} \hat{p}_{m+1} \iff p(x_i) = y_1 \hat{p}_1(x_1) + \cdots + y_i \hat{p}_i(x_i) + \cdots + y_{m+1} \hat{p}_{m+1}(x_{m+1})$$
$$\iff p(x_i) = y_1(0) + \cdots + y_i(1) + \cdots + y_{m+1}(0)$$
$$\iff p(x_i) = y_i.$$

To show $\hat{B}$ is a basis for $P_m(\mathbb{R})$, we need only show it's lin ind, as $|\hat{B}| = m+1 = \dim P_m(\mathbb{R})$.
Let $c_1, \ldots, c_{m+1} \in \mathbb{R}$ such that

$$c_1 \hat{p}_1 + \cdots + c_{m+1} \hat{p}_{m+1} = 0.$$

For any $1 \leq i \leq m+1$ and evaluating both sides at x_i yields

$$0 = c_1 \hat{p}_1(x_i) + \cdots + c_i \hat{p}_i(x_i) + \cdots + c_{m+1} \hat{p}_{m+1}(x_i)$$
$$= 0 + \cdots + c_i(1) + \cdots + 0$$
$$\therefore 0 = c_i,$$

and so $c_1 = \cdots = c_{m+1} = 0$, showing lin ind, and we're done. ∎

COLUMN SPACE [OF A MATRIX]: Col(A) (D11.2)

1. Let $A \in M_{m \times n}(\mathbb{F})$.
Then, the "column space" of A, denoted as "Col(A)", is the set of vectors in $\mathbb{F}^m$ of the form Ax, where $x \in \mathbb{F}^n$.
Equivalently, Col(A) is the set of all linear combinations of columns of A.

2. In particular, Col(A) is a subspace of $\mathbb{F}^m$, and the columns of A span Col(A). (T11.4)

$A \in M_{m \times n}(\mathbb{R})$; $\mathrm{Col}(A) = \mathrm{Null}(A^T)$ (L11.1)

Let $A \in M_{m \times n}(\mathbb{R})$, and give $\mathbb{R}^m$ the standard inner product.
Then necessarily $\mathrm{Col}(A)^\perp = \mathrm{Null}(A^T)$.

Proof. Since Col(A) is spanned by A's columns, by T11.1(3) we know $y \in (\text{columns of } A)^\perp \Rightarrow y \in \mathrm{Col}(A)^\perp$.
Let $y \in \mathrm{Null}(A^T)$, so $A^T y = 0$. In particular,

$$A^T y = \begin{pmatrix} - a_1 - \\ \vdots \\ - a_n - \end{pmatrix} y = \begin{pmatrix} \langle a_1, y \rangle \\ \vdots \\ \langle a_n, y \rangle \end{pmatrix},$$

std inner product (ie dot product)

So $A^T y = 0 \Rightarrow \langle a_i, y \rangle = 0 \Rightarrow y$ is ortho to each a_i,
so $y \in \mathrm{Col}(A)^\perp$, so $\mathrm{Null}(A^T) \subseteq \mathrm{Col}(A)^\perp$.
Conversely, let $y \in \mathrm{Col}(A)^\perp$, so $\langle a_i, y \rangle = 0 \ \forall i$. By the computation above, $A^T y = 0$, so $y \in \mathrm{Null}(A^T)$. Hence $\mathrm{Col}(A)^\perp \subseteq \mathrm{Null}(A^T)$, and so $\mathrm{Col}(A)^\perp = \mathrm{Null}(A^T)$, as needed. ∎

$x \in \mathbb{R}^n$ MINIMIZES $\|Ax - b\| \iff A^T A x = A^T b$ (T11.5)

Let $A \in M_{m \times n}(\mathbb{R})$ and $b \in \mathbb{R}^m$.
Then $x \in \mathbb{R}^m$ minimizes $\|Ax - b\|$ if and only if $A^T A x = A^T b$.

Proof. See that

$$x \text{ minimizes } \|Ax - b\| \Rightarrow Ax \in \mathrm{proj}_{\mathrm{Col}(A)}(b) \quad (\text{since } Ax \in \mathrm{Col}(A) \text{ and by T10.4(2)})$$
$$\Rightarrow A^T(b - Ax) = 0 \quad (\text{by L11.1})$$
$$\Rightarrow A^T A x = A^T b,$$

and

$$A^T A x = A^T b \Rightarrow A^T(b - Ax) = 0$$
$$\Rightarrow b - Ax \in \mathrm{Null}(A^T) = \mathrm{Col}(A)^\perp \quad (\text{by L11.1})$$
$$\Rightarrow Ax = \mathrm{proj}_{\mathrm{Col}(A)}(b) \quad (\text{see reasoning on Pensall})$$
$$\Rightarrow \|Ax - b\| \text{ is minimized}$$
$$\Rightarrow x \text{ minimizes } \|Ax - b\|,$$

as needed. ∎

Class 12:

Linear Transformations on an Inner Product Space

[T:V→W] PRESERVES INNER PRODUCTS (D12.1)

Let $(V, \langle \cdot,\cdot \rangle)$ & $(W, [\cdot,\cdot])$ be IPS, and let $T: V \to W$ be linear.

Then, we say T "preserves inner products" if

$[T(v_1), T(v_2)] = T(\langle v_1, v_2 \rangle) \quad \forall v_1, v_2 \in V.$

In particular, we say T is an "isomorphism" of inner product spaces if T is also an isomorphism.

POLARIZATION IDENTITIES

The polarization identities state that

① V over $\mathbb{R}$ $\Rightarrow$ $\langle x,y \rangle = \frac{1}{4}\|x+y\|^2 + \frac{1}{4}\|x-y\|^2$; &

② V over $\mathbb{C}$ $\Rightarrow$ $\langle x,y \rangle = \frac{1}{4}\|x+y\|^2 + \frac{i}{4}\|x+iy\|^2 - \frac{1}{4}\|x-y\|^2 - \frac{i}{4}\|x-iy\|^2$.

Proof. This can be verified by expanding the norms in the RHS in terms of IPs. ☒

T PRESERVES INNER PRODUCTS ⟺ T PRESERVES NORMS (T12.1(1))

Let V, W be IPS, and let $T: V \to W$ be linear.

Then T preserves inner products iff T preserves norms;

ie $\|T(x)\|_W = \|x\|_V \quad \forall x \in V$.

Proof. If T preserves inner products, by defn of the norm, it also preserves norms.

Now, suppose T preserves norms. If V, W are over $\mathbb{R}$, then by the polarization identities:

$[T(x), T(y)] = \frac{1}{4}\|T(x)+T(y)\|_W^2 + \frac{1}{4}\|T(x)-T(y)\|_W^2$

$= \frac{1}{4}\|T(x+y)\|_W^2 + \frac{1}{4}\|T(x-y)\|_W^2$

$= \frac{1}{4}\|x+y\|_V^2 + \frac{1}{4}\|x-y\|_V^2$

$= \langle x,y \rangle$,

and the case where V, W are over $\mathbb{C}$ is similar, showing T preserves IPs as well. ☒

T PRESERVES INNER PRODUCTS ⇒ T IS 1-1 (T12.1(2))

Let $T: V \to W$ preserve inner products.

Then necessarily T is 1-1.

Proof. We will show $\ker T = \{0\}$.

If $v \in V \ni T(v) = 0$, then trivially

$[T(v), T(v)] = [0,0] = 0$.

Since T preserves IPs, we have that

$[T(v), T(v)] = \langle v, v \rangle = 0$, so $v = 0$ if $T(v) = 0$.

Thus $\ker T = \{0\}$, as needed. ☒

T IS AN ISOMORPHISM OF IPS, $\{v_1, \dots, v_n\}$ IS AN ORTHOGONAL/ORTHONORMAL BASIS FOR V ⇒ $\{T(v_1), \dots, T(v_n)\}$ IS AN ORTHOGONAL/ORTHONORMAL BASIS FOR W (T12.1(3))

Let $T: V \to W$ be an isomorphism of inner product spaces, and let $\{v_1, \dots, v_n\}$ be an orthogonal (or orthonormal) basis for V.

Then necessarily $\{T(v_1), \dots, T(v_n)\}$ is an orthogonal (or orthonormal) basis for W.

Proof. We first show $\{T(v_1), \dots, T(v_n)\}$ is orthogonal.

Since $\{v_1, \dots, v_n\}$ is orthogonal, thus $\langle v_i, v_j \rangle = 0 \quad \forall i \neq j$. As T preserves IPs, thus $[T(v_i), T(v_j)] = 0 \quad \forall i \neq j$, which shows $\{T(v_1), \dots, T(v_n)\}$ is orthogonal.

Then, as T is an isomorphism, thus T is 1-1, so $T(v_i) \neq 0 \quad \forall 1 \leq i \leq n$ and $\ker T = \{0\}$.

In particular, $\{T(v_1), \dots, T(v_n)\}$ is lin ind by T9.1. Since $T(V) = W$, thus $\dim V = \dim W$, and so it follows that $\{T(v_1), \dots, T(v_n)\}$ is a basis for W, which is what we wanted to prove. ☒

$\{v_1, \dots, v_n\}$ IS AN ORTHONORMAL BASIS FOR V, $\{T(v_1), \dots, T(v_n)\}$ IS AN ORTHONORMAL BASIS FOR W ⇒ T IS AN ISOMORPHISM OF IPS (T12.1(4))

Let $T: V \to W$ be linear, and let $\{v_1, \dots, v_n\}$ be an orthonormal basis for V such that $\{T(v_1), \dots, T(v_n)\}$ is an orthonormal basis for W.

Then necessarily T is an isomorphism of inner product spaces.

Proof. First, we show T preserves IPs.

Let $x_1, x_2 \in V$, say

$x_1 = c_1 v_1 + \dots + c_n v_n$ & $x_2 = d_1 v_1 + \dots + d_n v_n$.

Then

$T(x_1) = c_1 T(v_1) + \dots + c_n T(v_n)$ & $T(x_2) = d_1 T(v_1) + \dots + d_n T(v_n)$.

See that

$\langle x_1, x_2 \rangle = \langle c_1 v_1 + \dots + c_n v_n, d_1 v_1 + \dots + d_n v_n \rangle$

$= \sum_{i=1}^{n} \sum_{j=1}^{n} c_i \bar{d_j} \langle v_i, v_j \rangle$

$= \sum_{i=1}^{n} c_i \bar{d_i} \langle v_i, v_i \rangle$

$= \sum_{i=1}^{n} c_i \bar{d_i} (1) \qquad (\text{as } \|v_i\| = 1)$

$\therefore \langle x_1, x_2 \rangle = \sum_{i=1}^{n} c_i \bar{d_i}$,

and

$[T(x_1), T(x_2)] = [T(c_1 v_1 + \dots + c_n v_n), T(d_1 v_1 + \dots + d_n v_n)]$

$= [c_1 T(v_1) + \dots + c_n T(v_n), d_1 T(v_1) + \dots + d_n T(v_n)]$

$= \sum_{i=1}^{n} \sum_{j=1}^{n} c_i \bar{d_j} [T(v_i), T(v_j)]$

$= \sum_{i=1}^{n} c_i \bar{d_i} [T(v_i), T(v_i)]$

$= \sum_{i=1}^{n} c_i \bar{d_i} (1) \qquad (\text{as } \|T(v_i)\| = 1)$

$= \sum_{i=1}^{n} c_i \bar{d_i}$,

showing T preserves IPs.

By T12.1(2), thus T is 1-1. In particular, since $\dim V = \dim W$, thus T is also an isomorphism, and we're done. ☒

$B = (v_1, \dots, v_n)$ IS AN ORTHONORMAL BASIS FOR V ⇒ $[T]_B = (\langle T(v_j), v_i \rangle)_{ij} \in M_{n \times n}(\mathbb{F})$ (T12.2)

Let V be a finite-dimensional IPS, and in particular, let $B = (v_1, \dots, v_n)$ be an ordered orthonormal basis for V.

Let $A = [T]_B$. Then necessarily

$A_{ij} = \langle T(v_j), v_i \rangle \quad \forall 1 \leq i,j \leq n$.

Proof. By C9.1, we have that

$$[T(v_j)]_B = \begin{pmatrix} \langle T(v_j), v_1 \rangle \\ \vdots \\ \langle T(v_j), v_n \rangle \end{pmatrix} \quad \forall 1 \leq j \leq n.$$

Since $[T(v_j)]_B$ is the j^{th} column in $[T]_B$, it follows that the entry in the i^{th} row & j^{th} column of $[T]_B$ is $\langle T(v_j), v_i \rangle$, as needed. ☒

$B=(v_1,\dots,v_n)$ IS AN ORDERED ORTHONORMAL BASIS FOR $V \Rightarrow \langle x,y\rangle = [y]_B^*[x]_B$ (L12.1)

Let V be a finite-dimensional IPS, and in particular, let $B=(v_1,\dots,v_n)$ be an ordered orthonormal basis for V.
Let $x,y\in V$. Then necessarily

$$\langle x,y\rangle = [y]_B^*[x]_B.$$

Proof. Let $[x]_B = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ & $[y]_B = \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}$. Then

$$\begin{aligned}\langle x,y\rangle &= \langle c_1v_1+\dots+c_nv_n,\ d_1v_1+\dots+d_nv_n\rangle \\ &= \sum_{i=1}^n\sum_{j=1}^n c_i\overline{d_j}\langle v_i,v_j\rangle \\ &= \sum_{i=1}^n c_i\overline{d_i} \\ &= (\overline{d_1}\ \cdots\ \overline{d_n})\begin{pmatrix} c_1 \\ \vdots \\ c_n\end{pmatrix} \\ &= [y]_B^*[x]_B. \quad \blacksquare\end{aligned}$$

$B=(v_1,\dots,v_n)$ IS AN ARBITRARY ORDERED ORTHONORMAL BASIS FOR V; $T:V\to V$ IS AN IPS ISOMORPHISM $\Leftrightarrow [T]_B^* = [T]_B^{-1}$ (T12.3)

Let V be a finite-dimensional IPS, let $T:V\to V$ be linear, and let $B=(v_1,\dots,v_n)$ be an ordered orthonormal basis for V.
Then T is an inner product space isomorphism iff $[T]_B^* = [T]_B^{-1}$.

Proof. ($\Rightarrow$) By defn., $\langle T(x),T(y)\rangle = \langle x,y\rangle\ \forall x,y\in V$.

By L12.1, $\langle x,y\rangle = [y]_B^*[x]_B$ & $\langle T(x),T(y)\rangle = [T(y)]_B^*[T(x)]_B$.

We also know

$[T(x)]_B = [T]_B[x]_B$ & $[T(y)]_B = [T]_B[y]_B$.

Hence

$$\begin{aligned}\langle T(x),T(y)\rangle &= [T(y)]_B^*[T(x)]_B \\ &= ([T]_B[y]_B)^*([T]_B[x]_B) \\ &= [y]_B^*([T]_B^*[T]_B)[x]_B \\ &(= \langle x,y\rangle = [y]_B^*[x]_B).\end{aligned}$$

We claim this implies $[T]_B^*[T]_B = I_n$.
Indeed, let $[y]_B = e_i$ & $[x]_B = e_j$.
Then $([T]_B^*[T]_B)[x]_B$ picks out the j^{th} column of $[T]_B^*[T]_B$, & taking the product on the left with $[y]_B^*$ gives the (i,j) entry of $[T]_B^*[T]_B$.
On the other hand, $[y]^*[x]_B = e_i^*e_j = \begin{cases} 0, & i\neq j \\ 1, & i=j\end{cases}$, which suffices to show $[T]_B^*[T]_B = I_n$, and so $[T]_B^* = ([T]_B)^{-1}$, as needed. $\blacksquare$

($\Leftarrow$) We first show T preserves IPs. See that for $x,y\in V$, we have that

$$\begin{aligned}\langle T(x),T(y)\rangle &= [T(y)]_B^*[T(x)]_B \\ &= [y]_B^*([T]_B^*[T]_B)[x]_B \\ &= [y]_B^*(I_n)[x]_B \quad \text{(by assumption)} \\ &= [y]_B^*[x]_B = \langle x,y\rangle \text{ by L12.1.}\end{aligned}$$

Hence, by T12.1(2), T is 1-1. Since $T:V\to V$, thus T is an isomorphism of IPS, as needed. $\blacksquare$

UNITARY MATRICES: $A^* = A^{-1}$ (D12.2)

Let $A\in M_{n\times n}(\mathbb{F})$.
Then, we say A is "unitary" if $A^* = A^{-1}$.

ORTHOGONAL MATRICES: $A^T = A^{-1}$ (D12.2)

Let $A\in M_{n\times n}(\mathbb{F})$.
Then, we say A is "orthogonal" if $A^T = A^{-1}$.

Class 13: Diagonalization Review

$A = \begin{pmatrix} -a_1- \\ \vdots \\ -a_n- \end{pmatrix} \Rightarrow \exists D : (\mathbb{F}^n)^n \to \mathbb{F} \ni D(e_1, \dots, e_n) = 1$

& $D(a_1, \dots, a_{i-1}, a_i + cb_i, a_{i+1}, \dots, a_n) = D(a_1, \dots, a_n) + D(a_1, \dots, a_{i-1}, b_i, a_{i+1}, \dots, a_n)$ &

$a_i = a_j \Rightarrow D(a_1, \dots, a_i, \dots, a_j, \dots, a_n) = 0$ (T13.1)

💡[1] There exists a unique function $D : (\mathbb{F}^n)^n \to \mathbb{F}$ such that:

① If $e_1, \dots, e_n$ are the standard basis vectors, then $D(e_1, \dots, e_n) = 1$;

② D is "multilinear"; ie if we fix $1 \le i \le n$, $c \in \mathbb{F}$ & $b_i \in \mathbb{F}^n$, we have that

$D(a_1, \dots, a_{i-1}, a_i + b_i, a_{i+1}, \dots, a_n) = D(a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_n) + D(a_1, \dots, a_{i-1}, b_i, a_{i+1}, \dots, a_n)$; &

③ D is an "alternating function"; ie if $a_i = a_j$ for $i \ne j$, then $D(a_1, \dots, a_i, \dots, a_j, \dots, a_n) = 0$.

💡[2] In particular, $D(a_1, \dots, a_n)$ is the "determinant" of $A = \begin{pmatrix} -a_1- \\ \vdots \\ -a_n- \end{pmatrix}$.

* this is an alternative definition of determinants (aside from the cofactor definition).

DIAGONALIZABLE [LINEAR OPERATOR] (D13.2)

💡 Let V be finite-dimensional, and let $T : V \to V$ be linear. Then, we say T is "diagonalizable" if there exists an ordered basis B for V such that $[T]_B$ is a diagonal matrix.

DIAGONALIZABLE [MATRIX] (D13.2)

💡 Let $A \in M_{n\times n}(\mathbb{F})$. Then, we say A is "diagonalizable" if the matrix multiplication operator $T_A : \mathbb{F}^n \to \mathbb{F}^n$ by $T_A(x) = Ax \ \forall x \in \mathbb{F}^n$ is diagonalizable.

EIGENVECTOR & EIGENVALUE [OF LINEAR OPERATORS] (D13.3)

💡[1] Let $T : V \to V$ be linear. Then, we say $0 \ne v \in V$ is an "eigenvector" of T if there exists some $c \in \mathbb{F}$ such that $T(v) = cv$.

💡[2] In particular, we call $c \in \mathbb{F}$ an "eigenvalue" of T.

EIGENVECTOR & EIGENVALUE [OF MATRICES] (D13.3)

💡[1] Let $A \in M_{n\times n}(\mathbb{F})$. Then, the eigenvectors and eigenvalues of A are just the corresponding eigenvectors and eigenvalues of T_A.

💡[2] In other words, $0 \ne x \in \mathbb{F}^n$ is an eigenvector of A if $Ax = cx$, and in this case say that $c \in \mathbb{F}$ is the associated eigenvalue.

EIGENSPACE (T13.3)

💡[1] Let $T : V \to V$ be linear, and let $c \in \mathbb{F}$ be an eigenvalue of T. Then, the "eigenspace" associated to c, denoted as "E_c", is defined to be the set

$$E_c = \{ v \in V : T(v) = cv \}.$$

💡[2] Indeed, E_c is a subspace of V.

T IS INVERTIBLE <=> $[T]_B$ IS INVERTIBLE FOR ANY ORDERED BASIS B OF V (L13.1)

💡 Let $T : V \to V$ be linear on a finite-dimensional vector space V. Then necessarily T is invertible iff $[T]_B$ is invertible for every ordered basis B of V.

Proof. ($\Rightarrow$) Let B be an arbitrary basis of V. We know $\exists T^{-1} : V \to V \ni T^{-1} \circ T = T \circ T^{-1} = I$. Hence

$$I_n = [I]_B = [T \circ T^{-1}]_B = [T]_B [T^{-1}]_B,$$

showing $[T]_B$ is invertible.

($\Leftarrow$) Suppose T is not invertible. In particular T is not 1-1, so $\exists 0 \ne v \in V \ni T(v) = 0$. Take any ordered basis B. Translating into matrices, this

$$[T(v)]_B = [T]_B [v]_B = [0]_B = 0,$$

showing that $[v]_B \in \text{Null}([T]_B)$. Since $[v]_B \ne 0$, thus $[T]_B$ is not invertible. ☒

$c \in \mathbb{F}$ IS AN EIGENVALUE OF T <=> $\det [T - cI]_B = 0$ FOR SOME BASIS B OF V (L13.2)

💡 Let $\dim V < \infty$, and let $T : V \to V$ be linear. Then, $c \in \mathbb{F}$ is an eigenvalue of T iff $\det [T - cI]_B = 0$ for some choice of ordered basis B of V.

Proof. ($\Rightarrow$) Let $c \in \mathbb{F}$ be an eigenvalue of T, w/ eigenvector $v \in V$.

So $T(v) = cv$, and so $(T - cI)(v) = 0$.

In particular $c \in \ker(T - cI)$, and as $c \ne 0$, thus $T - cI$ is not 1-1.

Hence $[T - cI]_B$ is not invertible by L13.1 and so $\det [T - cI]_B = 0$.

($\Leftarrow$) If $\det [T - cI]_B = 0$, then by L13.1 $T - cI$ is not invertible.

In particular, there exists a $0 \ne v \in V \ni (T - cI)(v) = 0$. Hence

$$T(v) = cI(v) = cv,$$

showing that c is an eigenvalue of V. ☒

CHARACTERISTIC POLYNOMIAL [OF $T : V \to V$]: $C(t)$ (D13.4)

💡[1] Let $\dim V < \infty$, and let $T : V \to V$ be linear. Then, the "characteristic polynomial" of T is the polynomial

$$C(t) = \det [T - tI]_B,$$

where B is any ordered basis for V.

💡[2] In particular, if B' is another ordered basis for V, then necessarily

$$\det([T - tI]_B) = \det([T - tI]_{B'}). \quad \text{(L13.2)}$$

Proof. We know

$$[T]_{B'} = {}_{B'}[I]_B \cdot [T]_B \cdot ({}_{B'}[I]_B)^{-1}$$

Let $P = {}_{B'}[I]_B$. Note $[I]_B = [I]_{B'} = n = \dim V$.

Then, see that

$$\begin{aligned}
\det [T - tI]_{B'} &= \det([T]_{B'} - t[I]_{B'}) \\
&= \det([T]_{B'} - tI_n) \\
&= \det(P[T]_B P^{-1} - t(PI_nP^{-1})) \\
&= \det(P([T]_B - tI_n)P^{-1}) \\
&= \det(P)\det([T]_B - tI_n)\det(P^{-1}) \\
&= \det([T]_B - tI_n) \\
&= \det[T - tI_n]_B
\end{aligned}$$

as needed. ☒

ALGEBRAIC MULTIPLICITY [OF AN EIGENVALUE]: a_c (D13.5)

Let $\dim V < \infty$, and let $T: V \to V$ be linear, and let $c \in \mathbb{F}$ be an eigenvalue of T.
Then, the "algebraic multiplicity" of c, denoted "a_c", is the largest positive integer $k \in \mathbb{Z}^+$ such that $(t-c)^k$ is a factor of the characteristic polynomial $c(t)$.

GEOMETRIC MULTIPLICITY [OF AN EIGENVALUE]: g_c (D13.5)

Let $\dim V < \infty$, and let $T: V \to V$ be linear, and let $c \in \mathbb{F}$ be an eigenvalue of T.
Then, the "geometric multiplicity" of c, denoted by "g_c", is defined to be equal to

$$g_c = \dim E_c.$$

$1 \le g_c \le a_c$ (T13.4(1))

Let $T: V \to V$ be linear, and let $c \in \mathbb{F}$ be an eigenvalue of T.
Then necessarily $1 \le g_c \le a_c$.

T IS DIAGONALIZABLE $\iff g_c = a_c \ \forall c$ (T13.4(2))

Let $T: V \to V$ be linear.
Then T is diagonalizable iff $g_c = a_c$ for any eigenvalues $c \in \mathbb{F}$ of T.

$c_1, \dots, c_k$ ARE THE DISTINCT ROOTS OF $c(t) \Rightarrow E_{c_1} \oplus \cdots \oplus E_{c_k}$; T IS DIAGONALIZABLE $\iff E_{c_1} \oplus \cdots \oplus E_{c_k} = V$ (T13.4(3))

Let $T: V \to V$ be linear, with characteristic polynomial $c(t)$.
Let $c_1, \dots, c_k$ be the distinct roots of $c(t)$.
Then necessarily
1. $E_{c_1} \oplus \cdots \oplus E_{c_k}$ (ie sum is direct); and
2. T is diagonalizable iff $E_{c_1} \oplus \cdots \oplus E_{c_k} = V$.

Proof Let's first show 1. We do this by showing

$$E_{c_i} \cap (E_{c_1} + \cdots + E_{c_k}) = \{0\} \quad \forall 1 \le i \le k.$$

Let $v \in E_{c_i} \cap (E_{c_1} + \cdots + E_{c_k})$. In particular,

$$T(v) = c_i v \quad \& \quad v = w_1 + \cdots + w_{i-1} + w_{i+1} + \cdots + w_k$$

for some $w_j \in E_{c_j}$ for each j.
Suppose we chose $w_1, \dots, w_k$ so that the # of non-zero vectors is as minimal as possible.
If $\neg(w_1 = \cdots = w_k = 0)$, WLOG assume $w_k \neq 0$.
Then,

$$\begin{aligned} c_i v = T(v) &= T(w_1 + \cdots + w_{i-1} + w_{i+1} + \cdots + w_k) \\ &= T(w_1) + \cdots + T(w_{i-1}) + T(w_{i+1}) + \cdots + T(w_k) \\ &= c_1 w_1 + \cdots + c_{i-1} w_{i-1} + c_i w_i + \cdots + c_k w_k, \end{aligned}$$

but as $v = w_1 + \cdots + w_{i-1} + w_{i+1} + \cdots + w_k$, thus

$$c_k v = c_k w_1 + \cdots + c_k w_{i-1} + c_k w_{i+1} + \cdots + c_k w_k.$$

Hence

$$(c_i - c_k) v = (c_1 - c_k) w_1 + \cdots + (c_{i-1} - c_k) w_{i-1} + (c_{i+1} - c_k) w_{i+1} + \cdots + 0 w_k.$$

Dividing by $c_i - c_k$, we see that we have written v as a sum of fewer non-zero vectors from the other subspaces than we had before — a cont^{n} to our initial assumption.
Thus $w_1 = \cdots = w_{i-1} = w_{i+1} = \cdots = w_k = 0$, so $v = 0$, completing the proof that $E_{c_1} + \cdots + E_{c_k}$ is direct.

Next, since a basis for $E_{c_1} \oplus E_{c_2} \oplus \cdots \oplus E_{c_k}$ is built by combining bases from $E_{c_1}, \dots, E_{c_k}$, if $V = E_{c_1} \oplus \cdots \oplus E_{c_k}$, then we can obtain a basis for V by the above.
In particular, we can find a basis of eigenvectors of T for V, showing that T is diagonalizable.
Conversely, if T is diagonalizable, then there is a basis for V consisting of eigenvectors of T.
Each of these eigenvectors belongs to one of the spaces E_{c_i}, and so putting together bases for $E_{c_1}, \dots, E_{c_k}$ yields a basis of V, which suffices to show $E_{c_1} \oplus \cdots \oplus E_{c_k} = V$. ☐

Class 14: Orthogonal Diagonalization

$A \in M_{n\times n}(\mathbb{R})$ IS ORTHOGONAL $\Leftrightarrow$ COLUMNS OF A FORMS AN ORTHONORMAL BASIS FOR $\mathbb{R}^n$ WRT STD INNER PRODUCT (L14.1)

Let $A \in M_{n\times n}(\mathbb{R})$.
Then necessarily A is orthogonal iff the columns of A forms an orthonormal basis for $\mathbb{R}^n$ (with respect to the standard inner product).

Proof. Recall A is orthogonal $\Leftrightarrow A^T = A^{-1}$,
ie
$$I_n = A^T A.$$
Then, the (i,j) entry of A^TA is given by
$$\sum_{k=1}^n (A^T)_{ik} A_{kj} = \sum_{k=1}^n A_{ki}A_{kj} = \langle a_i, a_j\rangle,$$
where a_i is the i^{th} column of A.
Then, $A^TA = I_n$ iff $(A^TA)_{ii} = 1$ & $(A^TA)_{ij} = 0$ $\forall i \neq j$,
and so in particular,
$$\langle a_i, a_i\rangle = 1 \quad \& \quad \langle a_i, a_j\rangle = 0 \quad \forall i\neq j.$$
Hence the a_i forms an orthonormal set for $\mathbb{R}^n$,
and as $|\{a_1, \dots, a_n\}| = n = \dim \mathbb{R}^n$, this set is also a basis for $\mathbb{R}^n$, as needed. ☒

$A \in M_{n\times n}(\mathbb{C})$ IS UNITARY $\Leftrightarrow$ COLUMNS OF A FORMS AN ORTHONORMAL BASIS FOR $\mathbb{C}^n$ WRT STD INNER PRODUCT (L14.1)

Let $A \in M_{n\times n}(\mathbb{C})$.
Then, A is unitary iff the columns of A form an orthonormal basis for $\mathbb{C}^n$ with respect to its standard inner product.

Proof. See that A is unitary $\Leftrightarrow A^* = A^{-1}$,
ie $A^*A = I_n$,
or in other words
$$\sum_{k=1}^n (A^*)_{ik} A_{kj} = \sum_{k=1}^n \overline{A_{ki}}\, A_{kj},$$
which is exactly $\mathbb{C}$'s IP. The rest of the proof is like the real case. ☒

T_A IS DIAGONALIZABLE WRT ORDERED BASIS B $\Rightarrow$ $\exists$ UNITARY (ORTHOGONAL) P $\ni D = P^{-1}[T_A]_B P$; $P = (b_1 \cdots b_n)$

Let $A \in M_{n\times n}(\mathbb{F})$.
Suppose T_A is diagonalizable with respect to some orthonormal ordered basis B, so that $D = [T_A]_B$ is diagonal.
Then necessarily there exists an unitary (orthogonal if $\mathbb{F} = \mathbb{R}$) matrix $P \in M_{n\times n}(\mathbb{F})$ such that
$$D = P^{-1}AP,$$
and if $B = (v_1, \dots, v_n)$, then
$$P = (v_1 \cdots v_n) \in M_{n\times n}(\mathbb{F}).$$

Proof. Let S be the std ord basis of $\mathbb{F}^n$, so that
$$[T_A]_S = A.$$
Then
$$[T_A]_B = ({}_S[I_V]_B)^{-1}[T_A]_S({}_S[I_V]_B), \quad (*)$$
where $V = \mathbb{F}^n$. Then, see that
$${}_S[I_V]_B = ([v_1]_S \cdots [v_n]_S),$$
and as B is orthogonal & $[v_i]_S$ are just the "standard" representations of v_i for each i,
by L14.1 ${}_S[I_V]_B$ is unitary (ortho if $\mathbb{F} = \mathbb{R}$).
Letting $P = {}_S[I_V]_B$ and subbing back into $(*)$,
we see that
$$D = [T_A]_B = P^{-1}[T_A]_S P = P^{-1}AP, \quad \text{(since } S \text{ is the std ord basis of } \mathbb{F}^n\text{)}$$
as needed. ☒

ORTHOGONALLY SIMILAR (D14.1)

Let $A, B \in M_{n\times n}(\mathbb{R})$.
Then, we say A is "orthogonally similar" to B if there exists an orthogonal matrix $P \in M_{n\times n}(\mathbb{R})$ such that
$$B = P^{-1}AP = P^TAP \quad \text{(since } P \text{ is orthogonal).}$$

ORTHOGONALLY DIAGONALIZABLE (D14.1)

Let $A \in M_{n\times n}(\mathbb{R})$.
Then, we say A is "orthogonally diagonalizable" if A is orthogonally similar to some diagonal matrix $D \in M_{n\times n}(\mathbb{R})$.

UNITARILY SIMILAR (D14.1)

Let $A, B \in M_{n\times n}(\mathbb{C})$.
Then, we say A is "unitarily similar" to B if there exists an unitary matrix $P \in M_{n\times n}(\mathbb{C})$ such that
$$B = P^{-1}AP = P^*AP \quad \text{(since } P \text{ is unitary).}$$

UNITARILY DIAGONALIZABLE (D14.1)

Let $A \in M_{n\times n}(\mathbb{C})$.
Then, we say A is "unitarily diagonalizable" if A is unitarily similar to a diagonal matrix $D \in M_{n\times n}(\mathbb{C})$.

$A \in M_{n\times n}(\mathbb{R})$ IS ORTHOGONALLY DIAGONALIZABLE $\Rightarrow$ A IS SYMMETRIC (L14.2)

Let $A \in M_{n\times n}(\mathbb{R})$, and suppose A is orthogonally diagonalizable.
Then necessarily A is symmetric (ie $A^T = A$).

Proof. Since A is ortho diag, $\exists$ diag matrix $D \in M_{n\times n}(\mathbb{R})$ such that A is ortho similar to D; ie
$\exists$ ortho $P \in M_{n\times n}(\mathbb{F})$ $\ni$
$$D = P^{-1}AP.$$
Hence
$$A = PDP^{-1} = PDP^T.$$
Since D is diagonal, it is also symmetric.
Taking transposes of both sides yields that
$$A^T = (PDP^T)^T = PD^TP^T = PDP^T = A,$$
showing that $A^T = A$, as needed.

$A \in M_{n\times n}(\mathbb{R})$ IS SYMMETRIC $\Rightarrow$ EVERY EIGENVALUE OF A IS A REAL NUMBER (T14.1)

Let $A \in M_{n\times n}(\mathbb{R})$ be symmetric.
Then necessarily every eigenvalue of A is a real number.

Proof. Let $c \in \mathbb{C}$ be an eigenvalue of A, w/ corresponding eigenvector $x \in \mathbb{C}^n$, so that $Ax = cx$.
See that

$$
\begin{aligned}
c\langle x,x\rangle &= \langle cx, x\rangle \\
&= \langle Ax, x\rangle \\
&= x^*(Ax) \\
&= x^*(A^* x) \quad \text{(since } A \in M_{n\times n}(\mathbb{R}) \text{ \& } A \text{ is symmetric)} \\
&= (x^* A^*) x \\
&= (Ax)^* x \\
&= \langle x, Ax\rangle \\
&= \langle x, cx\rangle \\
&= \bar{c}\langle x,x\rangle,
\end{aligned}
$$

So in particular, $c = \bar{c}$. As $x \neq 0$, so $\langle x,x\rangle \neq 0$, so $c \in \mathbb{R}$, as needed. ∎

SELF-ADJOINT [MATRIX IN $\mathbb{C}$] (D14.2)

Let $A \in M_{n\times n}(\mathbb{C})$.
Then, we say A is "self-adjoint" if $A^* = A$.

$A \in M_{n\times n}(\mathbb{C})$ IS SELF-ADJOINT $\Rightarrow$ EVERY EIGENVALUE OF A IS A REAL NUMBER (C14.1)

Let $A \in M_{n\times n}(\mathbb{C})$ be self-adjoint.
Then necessarily every eigenvalue of A is a real number.

Proof. Same as T14.1, since $A^* = A$. ∎

A IS SYMMETRIC, (c_1, v_1), (c_2, v_2) ARE EIGENVALUES/VECTORS OF A $\Rightarrow$ v_1 & v_2 ARE ORTHOGONAL (T14.2)

Let $A \in M_{n\times n}(\mathbb{R})$ be symmetric, and let v_1 & v_2 be eigenvectors corresponding to the eigenvalues c_1 & c_2 of A respectively.
Then necessarily v_1 & v_2 are orthogonal with respect to the standard inner product of $\mathbb{R}^n$.

Proof. See that

$$
\begin{aligned}
c_1\langle v_1, v_2\rangle &= \langle c_1 v_1, v_2\rangle \\
&= \langle Av_1, v_2\rangle \\
&= v_2^T(Av_1) \quad \text{(as all entries in } \mathbb{R}\text{)} \\
&= v_2^T(A^T v_1) \quad \text{(} A \text{ is symmetric)} \\
&= (Av_2)^T v_1 \\
&= \langle v_1, Av_2\rangle \\
&= \langle v_1, c_2 v_2\rangle \\
&= c_2\langle v_1, v_2\rangle,
\end{aligned}
$$

and as $c_1 \neq c_2$, the equality holds iff $\langle v_1, v_2\rangle = 0$, showing the claim in question. ∎

A IS SELF-ADJOINT, (c_1, v_1), (c_2, v_2) ARE EIGENVALUES/VECTORS OF A $\Rightarrow$ v_1 & v_2 ARE ORTHOGONAL (C14.2)

Let $A \in M_{n\times n}(\mathbb{C})$ be self-adjoint, and let v_1 & v_2 be eigenvectors associated to the eigenvalues c_1 & c_2 of A respectively.
Then necessarily v_1 & v_2 are orthogonal with respect to the standard inner product on $\mathbb{C}^n$.

Proof. Almost identical to the proof for T14.2.