MATH 146
Personal Notes

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UW Math '25

Chapter 1: Vector Spaces

(S1.1)

Let $\mathbb{F}$ be a field.
Then, we say V is a "vector space" over $\mathbb{F}$ if there exists

① an addition $+ : (V \times V) \to V$ by $+(x,y) = x+y$; and

② a scalar multiplication $\times : (\mathbb{F} \times V) \to V$ by $\times(a,x) = ax$;

and the following conditions hold:

① V is an abelian group with respect to addition; (VS 1 = commutativity; 2 = associativity; 3 = identity; 4 = inverse)

② $1_{\mathbb{F}} x = x \quad \forall x \in V$; (VS 5)

③ multiplication is associative; ie $a(bx) = (ab)x \quad \forall a,b \in \mathbb{F}, x \in V$; and (VS 6)

④ the left and right distributive laws hold; ie $a(x+y) = ax + ay$ and $(a+b)x = ax + bx \quad \forall a,b \in \mathbb{F}, x \in V$. (D2)
(VS 7 = former, VS 8 = latter)

$\mathbb{F}^n$ IS A VECTOR SPACE OVER $\mathbb{F}$ (E2(1))

We can show that the Cartesian product

$$\mathbb{F}^n = \{(a_1, a_2, \dots, a_n) : a_i \in \mathbb{F} \ \forall i \in \{1,2,\dots,n\}\}$$

is a vector space over $\mathbb{F}$ with respect to the addition operation

$$(a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n) = (a_1+b_1, a_2+b_2, \dots, a_n+b_n)$$

and multiplication operation

$$c(a_1, a_2, \dots, a_n) = (ca_1, ca_2, \dots, ca_n).$$

Proof. This follows from verifying each of the conditions above. ☒

Note that we generally say "the vector space $\mathbb{F}^n$" to refer to the vector space $\mathbb{F}^n$ over $\mathbb{F}$. (R3(4))

COLUMN VECTOR NOTATION (E2(2))

Note that we can also write elements of $\mathbb{F}^n$ as "column vectors"; ie of the form

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \quad \text{where } a_1, a_2, \dots, a_n \in \mathbb{F}.$$

$\mathbb{Q}^n$ IS A VECTOR SPACE OVER $\mathbb{Q}$, $\mathbb{R}^n$ IS A VECTOR SPACE OVER $\mathbb{R}$, & $\mathbb{C}^n$ IS A VECTOR SPACE OVER $\mathbb{C}$ (R3(1))

We can show
① $\mathbb{Q}^n$ is a vector space over $\mathbb{Q}$;
② $\mathbb{R}^n$ is a vector space over $\mathbb{R}$; and
③ $\mathbb{C}^n$ is a vector space over $\mathbb{C}$.

Proof. This directly follows from the fact that $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are fields (MATH 145), and substituting the respective fields into the above lemma. ☒

$\mathbb{R}^n$ IS A VECTOR SPACE OVER $\mathbb{Q}$, & $\mathbb{C}^n$ IS A VECTOR SPACE OVER $\mathbb{R}$ (R3(2))

Moreover, we can also show that
① $\mathbb{R}^n$ is a vector space over $\mathbb{Q}$; and
② $\mathbb{C}^n$ is a vector space over $\mathbb{R}$.

Proof. Essentially, this stems from the fact that we can "multiply" vectors in $\mathbb{R}^n$ by scalars in $\mathbb{Q}$, and vectors in $\mathbb{C}^n$ by scalars in $\mathbb{R}$.
The formal proof is left to the reader. ☒

KEY
S : section
D : definition
R : remark
E : example
T : theorem
L : lemma
C : corollary

MATRICES (D3(1))

Let $\mathbb{F}$ be a field, and $m,n \in \mathbb{Z}^+$.
Then, we say A is an "$m \times n$ matrix" with entries from $\mathbb{F}$ if it is of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \\ a_{m1} & a_{m2} & & a_{mn} \end{pmatrix},$$

where $a_{ij} \in \mathbb{F} \quad \forall i \in \{1,2,\dots,m\}, j \in \{1,2,\dots,n\}$.

Alternatively, we can represent A via the notation

$$A = (a_{ij}), \quad \text{ie } \{1,2,\dots,m\}, \ j \in \{1,2,\dots,n\}.$$

ij-ENTRY OF A MATRIX (D3(2))

Given a $m \times n$ matrix A, the "ij-entry" of A, or "a_{ij}", is defined to be the entry in A at the i^{th} row and j^{th} column.

ZERO MATRIX (D3(3))

The "$m \times n$ zero matrix", or more simply the "zero matrix", denoted as "O", is defined to be

$$O = \begin{array}{c} \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & 0 \end{pmatrix} \Big\updownarrow m \\ \overleftrightarrow{\quad n \quad} \end{array},$$

ie the $m \times n$ matrix where which entry equals 0.

MATRIX EQUALITY (D3(4))

We say two matrices A and B are equal if and only if $a_{ij} = b_{ij} \quad \forall i \in \{1,2,\dots,m\}, j \in \{1,2,\dots,n\}$.

MATRIX ADDITION (D3(5))

Let A and B be $m \times n$ matrices with entries from some field $\mathbb{F}$.
Then, the "addition" of A and B, denoted by "$A+B$", is defined to be the matrix where

$$(a+b)_{ij} = a_{ij} + b_{ij} \quad \forall i \in \{1,2,\dots,m\}, j \in \{1,2,\dots,n\}.$$

MATRIX SCALAR MULTIPLICATION (D3(6))

Let A be a $m \times n$ matrix with entries from some field $\mathbb{F}$, and $c \in \mathbb{F}$ be arbitrary.
Then the "scalar multiplication" of A by c, denoted by "cA", is defined to be the matrix where

$$(ca)_{ij} = c(a_{ij}) \quad \forall i \in \{1,2,\dots,m\}, j \in \{1,2,\dots,n\}.$$

SPACE OF $m \times n$ MATRICES (E3)

Let $\mathbb{F}$ be a field.
Then the "space of all $m \times n$ matrices" with entries from $\mathbb{F}$, denoted by "$M_{m \times n}(\mathbb{F})$", is defined to be the set of all $m \times n$ matrices with entries from $\mathbb{F}$.

Note that $M_{m \times n}(\mathbb{F})$ is a vector space over $\mathbb{F}$ with respect to the matrix addition and scalar multiplication operations.

Proof. This follows from verifying each of the conditions in D2. ☒

FUNCTION SPACES (E4)

Let the set $D \neq \emptyset$ be arbitrary, and let $\mathbb{F}$ be a field.

Then the "space of all functions" from D to $\mathbb{F}$, denoted by "$\mathbb{F}^D$", is defined to be the set of all functions of the form $f: D \to \mathbb{F}$.

Similarly, we can show that $\mathbb{F}^D$ is a vector space over $\mathbb{F}$ with respect to the operations of function addition

$$(f+g)(x) := f(x) + g(x) \quad \forall f,g \in \mathbb{F}^D,\ x \in \mathbb{F}$$

and function scalar multiplication

$$(cf)(x) := cf(x) \quad \forall f \in \mathbb{F}^D,\ x,c \in \mathbb{F}.$$

Proof. Similar strategy to E3: verify each condition in D2 holds.

POLYNOMIALS (D4)

SET OF ALL POLYNOMIALS OF DEGREE AT MOST n (D4 (1))

Let $\mathbb{F}$ be a field.
Then, we denote $P_n(\mathbb{F})$ to be the set of all polynomials with coefficients from $\mathbb{F}$ and of degree at most n; ie

$$P_n(\mathbb{F}) = \left\{ \sum_{i=0}^{n} a_n x^n \ : \ a_j \in \mathbb{F} \ \forall j \in \{0,1,\dots,n\} \right\}.$$

POLYNOMIAL SPACES (D4 (2))

Let $\mathbb{F}$ be a field.
Then, we denote "$\mathbb{F}[x]$" to be the set of all polynomials with coefficients from $\mathbb{F}$; ie

$$\mathbb{F}[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \ : \ a_j \in \mathbb{F} \ \forall j \in \mathbb{N} \cup \{0\} \right\}.$$

Then, we can show that $\mathbb{F}[x]$ is a vector space over $\mathbb{F}$ with respect to the operations of polynomial addition

$$(f+g)(x) = \sum_{i=0}^{n} (a_n + b_n) x^n \quad \forall f,g \in \mathbb{F}[x]$$

and polynomial scalar multiplication

$$cf(x) = \sum_{i=0}^{n} (ca_n) x^n \quad \forall f \in \mathbb{F}[x],\ c \in \mathbb{F}.$$

Proof. Similar strategy to E4.

BASIC PROPERTIES OF VECTOR SPACES (S1.2)

CANCELLATION PROPERTY FOR VECTOR ADDITION (T1.1)

Let V be a vector space.
Suppose there exists some $x,y,z \in V$ such that $x+z = y+z$.
Then necessarily $x = y$.

Proof. Note that

$$\begin{aligned} x &= x + 0 \\ &= x + (z + (-z)) \\ &= (x+z) + (-z) \\ &= (y+z) + (-z) \\ &= y + (z + (-z)) \end{aligned}$$

and so $x = y$, as required. $\boxtimes$

UNIQUENESS OF THE ZERO VECTOR IN VECTOR SPACES (C1.1.1 (1))

Let V be a vector space.
Suppose $0_1, 0_2 \in V$ are both zero vectors.
Then necessarily $0_1 = 0_2$.

Proof. This follows from the fact that V is an abelian group under addition. $\boxtimes$

UNIQUENESS OF ADDITIVE INVERSES IN VECTOR SPACES (C1.1.1 (2))

Let V be a vector space.
Then for any $x \in V$, there exists one and only one vector $y \in V$ that satisfies $x+y=0$.

Proof. This also follows from the fact that V is an abelian group under addition. $\boxtimes$

$0x = 0 \quad \forall x \in V$ (T1.2 (1))

Let V be a vector space over some field $\mathbb{F}$, and let 0 be the additive identity of $\mathbb{F}$.
Then, for any $x \in V$, necessarily $0 \cdot x = 0$.

Proof. This, again, follows from the fact that V is an abelian group under addition. $\boxtimes$

$a0 = 0 \quad \forall a \in \mathbb{F}$ (T1.2 (2))

Let V be a vector space over some field $\mathbb{F}$, and let 0 be the zero vector of V.
Then, for any $a \in \mathbb{F}$, necessarily $a\ 0 = 0$.

Proof. This, again, follows from the fact that V is an abelian group under addition. $\boxtimes$

$(-a)x = -(ax) = a(-x) \quad \forall a \in \mathbb{F},\ x \in V$ (T1.2 (3))

Let V be a vector space over some field $\mathbb{F}$, and let $a \in \mathbb{F}$, $x \in V$ be arbitrary.
Then necessarily $(-a)x = -(ax) = a(-x)$.

Proof. Proof is similar to the analog of this statement for rings (MATH 145). $\boxtimes$

SUBSPACES (S1.3)

Let V be a vector space over some field $\mathbb{F}$.
Then we say the subset $W \subseteq V$ is a "subspace" of V if

① $W \neq \emptyset$;
*we usually check whether $0 \in W$ to verify this claim. (R4)

② If $x \in W$ and $y \in W$, then $(x+y) \in W$; and

③ If $c \in \mathbb{F}$ and $x \in W$, then $cx \in W$. (D6)

SUBSPACES ARE VECTOR SPACES OVER $\mathbb{F}$ WITH RESPECT TO THE OPERATIONS OF V (T1.3)

Let W be a subspace of a vector space V over some field $\mathbb{F}$.
Then W is also a vector space over $\mathbb{F}$ under the operations of V restricted to W.

Proof. This follows from verifying the conditions in D2, taking into account the properties of subspaces. ▨

$\{0\}$ AND V ARE SUBSPACES OF V (E8 (1))

Let V be a vector space.
Then $\{0\}$ and V itself are always subspaces of V.

Proof. $\{0\}$ is vacuously a subspace, and V is trivially a subspace. ▨

$P_2(\mathbb{R})$ IS A SUBSPACE OF $\mathbb{R}[x]$ (E8 (2))

We can show that $P_2(\mathbb{R})$ is a subspace of $\mathbb{R}[x]$.

Proof. This stems from the fact that:
- $P_2(\mathbb{R}) \subset \mathbb{R}[x]$ by definition;
- $0 \in P_2(\mathbb{R})$; and
- $P_2(\mathbb{R})$ is closed under the addition & scalar multiplication defined on $\mathbb{R}[x]$. ▨

$\{(a_{ij}) \in M_{n\times n}(\mathbb{F}) \mid \sum_{k=1}^{n} a_{kk} = 0\}$ IS A SUBSPACE OF $M_{n\times n}(\mathbb{F})$ (E8 (3))

We can show that the set $\{(a_{ij}) \in M_{n\times n}(\mathbb{F}) \mid \sum_{k=1}^{n} a_{kk} = 0\}$ is a subspace of $M_{n\times n}(\mathbb{F})$, where $n \in \mathbb{N}$ is arbitrary.

Proof. Similar proof to the above.

$\{(a_{ij}) \in M_{n\times n}(\mathbb{F}) \mid \sum_{k=1}^{n} a_{kk} = 1\}$ IS <u>NOT</u> A SUBSPACE OF $M_{n\times n}(\mathbb{F})$ (E8 (4))

We can show the set $\{(a_{ij}) \in M_{n\times n}(\mathbb{F}) \mid \sum_{k=1}^{n} a_{kk} = 1\}$ is <u>not</u> a subspace of $M_{n\times n}(\mathbb{F})$.

Proof. Let $a, b \in \{(a_{ij}) \in M_{n\times n}(\mathbb{F}) \mid \sum_{k=1}^{n} a_{kk} = 1\}$ be arbitrary.
Then, notice that

$$\begin{aligned} \sum_{k=1}^{n} (a+b)_{kk} &= \sum_{k=1}^{n} (a_{kk} + b_{kk}) \\ &= 1+1 \\ &= 2, \end{aligned}$$

So that $a+b \notin \{(a_{ij}) \in M_{n\times n}(\mathbb{F}) \mid \sum_{k=1}^{n} a_{kk} = 1\}$. ▨

SUBSPACES OF $\mathbb{R}^2$ (E9 (1))

Note that the subspaces of $\mathbb{R}^2$ are
① $\mathbb{R}^2$ itself;
② $\{0_{\mathbb{R}^2}\} = \{(0,0)\}$; and
③ all <u>lines</u> in $\mathbb{R}^2$ that pass through $\underline{(0,0)}$.

SUBSPACES OF $\mathbb{F}^2$ (E9 (4a))

In general, for any field $\mathbb{F}$, the subspaces of $\mathbb{F}^2$ are
① $\mathbb{F}^2$ itself;
② $\{0\}$; and
③ all the "lines" in $\mathbb{F}^2$ through 0.
ie of the form $\{(x,y) \in \mathbb{F}^2 \mid \binom{x}{y} = k\binom{x_1}{y_1},\ \binom{x_1}{y_1} \in \mathbb{F}^2\}$

SUBSPACES OF $\mathbb{R}^3$ (E9 (2))

Similarly, the subspaces of $\mathbb{R}^3$ are
① $\mathbb{R}^3$ itself;
② $\{0_{\mathbb{R}^3}\} = \{(0,0,0)\}$;
③ all <u>lines</u> in $\mathbb{R}^3$ that pass through $\underline{(0,0,0)}$; and
④ all <u>planes</u> in $\mathbb{R}^3$ that pass through $\underline{(0,0,0)}$.

SUBSPACES OF $\mathbb{F}^3$ (E9 (4b))

Similarly, for any field $\mathbb{F}$, the subspaces of $\mathbb{F}^3$ are
① $\mathbb{F}^3$ itself;
② $\{0\}$;
③ all the "lines" in $\mathbb{F}^3$ through 0; and
ie of the form $\{(x,y,z) \in \mathbb{F}^3 \mid \begin{pmatrix}x\\y\\z\end{pmatrix} = k\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix},\ \begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix} \in \mathbb{F}^3\}$ (E9(3))
④ all the "planes" in $\mathbb{F}^3$ through 0.
ie of the form $\{(x,y,z) \in \mathbb{F}^3 \mid \exists a,b,c \in \mathbb{F}$ such that $ax+by+cz=0.\}$

LINEAR COMBINATIONS & SYSTEM OF LINEAR EQUATIONS (S1.4)

* knowledge of elimination method is assumed.

LINEAR COMBINATION (D7(1))

1. Let V be a vector space over a field $\mathbb{F}$, and let the subset $S \subseteq V$ be such that $S \neq \emptyset$.

Then, we say a vector $x \in V$ is a "linear combination" of vectors from S if there exists a finite number of vectors $u_1, u_2, \ldots, u_n \in S$ and scalars $a_1, a_2, \ldots, a_n \in \mathbb{F}$ such that

$$x = a_1 u_1 + a_2 u_2 + \cdots + a_n u_n,$$

where $n \geq 1$. (D7(1))

2. In this case, we also say that x is a linear combination of the vectors $u_1, u_2, \ldots, u_n$.

COEFFICIENTS OF A LINEAR COMBINATION (D7 (2))

1. Let V be a vector space over some field $\mathbb{F}$, and let the vector $x \in V$ be a linear combination of the vectors $u_1, u_2, \ldots, u_n \in S$, where $S \subseteq V$ and $S \neq \emptyset$.

Assume that $x = a_1 u_1 + a_2 u_2 + \cdots + a_n u_n$, where $a_1, a_2, \ldots, a_n \in \mathbb{F}$.

Then we denote the scalars $a_1, a_2, \ldots, a_n \in \mathbb{F}$ as the "coefficients" of the linear combination.

SPAN (D7(3))

1. Let V be a vector space over some field $\mathbb{F}$, and let the subset $S \subseteq V$ be such that $S \neq \emptyset$.

Then, we define the "span" of S, denoted as "$\text{span}(S)$", to be the set of all linear combinations of vectors in S.

2. Note that, for convenience, we define

$$\text{span}(\emptyset) = \{0\}.$$

EXAMPLE 1: SPAN OF (1,0,0) & (0,1,0) IN $\mathbb{R}^3$ (E10 (1))

1. Observe that in $\mathbb{R}^3$, the span of $(1,0,0)$ & $(0,1,0)$ in $\mathbb{R}^3$ is

$$\{a(1,0,0) + b(0,1,0) : a, b \in \mathbb{R}\}$$

or more simply

$$\{(a, b, 0) : a, b \in \mathbb{R}\}.$$

EXAMPLE 2: SPAN($\{x^n : n \geq 1\}$) IN $\mathbb{Q}[x]$ (E10(2))

1. We can show that for the vector space $\mathbb{Q}[x]$, the span of $S = \{x, x^2, \ldots, x^n, \ldots\}$ is the set of all polynomials in $\mathbb{Q}[x]$ whose constant coefficient equals 0.

SPAN OF A FINITE AMOUNT OF VECTORS (E10(3.1))

1. Suppose V is a vector space over some field $\mathbb{F}$, and let $S \subseteq V$. Further assume that

$$S = \{v_1, v_2, \ldots, v_n\},$$

ie the size of S is finite.

Then, it follows that

$$\text{span}(S) = \{a_1 v_1 + a_2 v_2 + \cdots a_n v_n : a_i \in \mathbb{F} \ \forall i \in \{1, 2, \ldots, n\}\}.$$

SPAN OF A COUNTABLE AMOUNT OF VECTORS (E10 (3.2))

1. Suppose V is a vector space over some field $\mathbb{F}$, and let $S \subseteq V$. Further assume that

$$S = \{v_1, v_2, \ldots, v_n, \ldots\},$$

ie $|S| = |\mathbb{N}|$.

Then, we can show that

$$\text{span}(S) = \text{span}(\{v_1\}) \cup \text{span}(\{v_1, v_2\}) \cup \cdots \cup \text{span}(\{v_1, v_2, \ldots, v_n\}) \cup \cdots$$

or in other words, that

$$\text{span}(S) = \bigcup_{n=1}^{\infty} \text{span}\{v_1, v_2, \ldots, v_n\}.$$

SPAN OF AN UNCOUNTABLE AMOUNT OF VECTORS (E10 (3.3))

1. Suppose V is a vector space over some field $\mathbb{F}$, and let $S \subseteq V$. Further assume that $|S| > |\mathbb{N}|$; ie the size of S is uncountable.

Then note that there are no "obvious" simplifications to the formula for $\text{span}(S)$.

SPAN OF A SET IS ALWAYS A SUBSPACE OF THE PARENT VECTOR SPACE (T1.4)

1. Let V be a vector space over some field $\mathbb{F}$, and let $S \subseteq V$.

Then necessarily $\text{span}(S)$ is a subspace of V.

Proof. This follows from verifying each subspace condition for $\text{span}(S)$. ■

2. Moreover, $\text{span}(S)$ is the "smallest possible" subspace of V that contains S, in the sense that

① $S \subseteq \text{span}(S)$; and

② If W is any other subspace of V containing S, then $\text{span}(S) \subseteq W$.

"GENERATES/SPANS" (D8)

1. Let V be a vector space, and let $S \subseteq V$.

Then, we say S "generates" V, or S "spans" V, if $\text{span}(S) = V$.

2. Note to prove $\text{span}(S) = V$, we just need to prove every vector in V can be written as a linear combination of vectors in S, since $\text{span}(S) \subseteq V$ by definition. (This follows from extensionality.) (R6)

LINEAR INDEPENDENCE & DEPENDENCE (S1·5)

LINEARLY DEPENDENT (D9 (1))

Let V be a vector space over some field $\mathbb{F}$, and let $S \subseteq V$.
Then, we say S is "linearly dependent" if there exists a finite number of distinct vectors $u_1, u_2, \ldots, u_n \in S$ and scalars $c_1, c_2, \ldots, c_n \in \mathbb{F}$, where $c_1, c_2, \ldots, c_n$ are <u>all not zero</u>, such that

$$c_1 u_1 + c_2 u_2 + \cdots + c_n u_n = 0.$$

In this case, we also say the vectors of S are linearly dependent.

Note that if S is finite, say $S = \{u_1, u_2, \ldots, u_n\}$, then S is linearly dependent <u>if and only if</u> there exists a $(c_1, c_2, \ldots, c_n) \in \mathbb{F}^n$, where $(c_1, c_2, \ldots, c_n) \neq (0, 0, \ldots, 0)$, such that

$$c_1 u_1 + c_2 u_2 + \cdots + c_n u_n = 0. \quad \text{(R7 (4a))}$$

LINEARLY INDEPENDENT (D9 (2))

Let V be a vector space over some field $\mathbb{F}$, and let $S \subseteq V$.
Then, we say S is "linearly independent" if it is <u>not</u> "linearly dependent"; ie for every choice of distinct $u_1, u_2, \ldots, u_n \in S$, if $c_1, c_2, \ldots, c_n \in \mathbb{F}$ are scalars such that

$$c_1 u_1 + c_2 u_2 + \cdots + c_n u_n = 0,$$

then necessarily $c_1 = c_2 = \cdots = c_n = 0$.

Similarly, if S is finite, say $S = \{u_1, u_2, \ldots, u_n\}$, then S is linearly independent <u>if and only if</u> whenever $(c_1, c_2, \ldots, c_n) \in \mathbb{F}^n$ are such that

$$c_1 u_1 + c_2 u_2 + \cdots + c_n u_n = 0,$$

then necessarily $c_1 = c_2 = \cdots = c_n = 0$.

TRIVIAL REPRESENTATION OF 0 (R7 (1))

Note that for any vector space V and vectors $u_1, u_2, \ldots, u_n \in V$, we denote the "trivial representation of $0 \in V$" as a linear combination of $u_1, u_2, \ldots, u_n$ by

$$0u_1 + 0u_2 + \cdots + 0u_n = 0.$$

EMPTY SET IS LINEARLY INDEPENDENT (R7 (2))

Note that the empty set, $\emptyset$, is <u>vacuously</u> linearly independent.

* since linearly dependent sets must be non-empty by definition.

$\{0\}$ IS LINEARLY DEPENDENT (R7 (3))

Note that the set $\{0\}$ is linearly dependent, since $1(0) = 0$ is a <u>non-trivial</u> representation of 0 as a linear combination of finitely many distinct vectors in S.

$0 \in S \Rightarrow S$ IS LINEARLY DEPENDENT (R7 (5))

Note that any subset of a vector space that contains the zero vector is linearly dependent.

EXAMPLE 1: $S = \{(0,1,1), (1,0,1), (1,2,3)\}$ IS LINEARLY DEPENDENT IN $\mathbb{R}^3$ (E14)

We can show that the set $S = \{(0,1,1), (1,0,1), (1,2,3)\}$ is linearly dependent in $\mathbb{R}^3$.

<u>Proof</u>. We search for scalars $a, b, c \in \mathbb{R}$, not all 0, such that

$$a\begin{pmatrix}0\\1\\1\end{pmatrix} + b\begin{pmatrix}1\\0\\1\end{pmatrix} + c\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

This reduces to solving the system

$$\begin{cases} b + c = 0 \\ a + 2c = 0 \\ a + b + 3c = 0. \end{cases}$$

Simplifying, we get that

$$a = -2t, \quad b = -t \quad \text{and} \quad c = t,$$

where $t \in \mathbb{R}$.
For instance, $(a,b,c) = (-2,-1,1)$ is a solution in which not all of a, b, c are 0.
It follows that S is linearly dependent. ☒

EXAMPLE 2: $S = \{1, x, x^2, x^3\}$ IS LINEARLY INDEPENDENT IN $\mathbb{Z}_5[x]$ (E15)

We can show that the set $S = \{1, x, x^2, x^3\}$ is linearly independent in $\mathbb{Z}_5[x]$.

<u>Proof</u>. Note that if there exist $a_0, a_1, a_2, a_3 \in \mathbb{Z}_5$ such that

$$a_0(1) + a_1 x + a_2 x^2 + a_3 x^3 = 0,$$

then by definition necessarily $a_0 = a_1 = a_2 = a_3 = 0$, and this is sufficient to prove the claim. ☒

S IS LINEARLY DEPENDENT $\Leftrightarrow$ $S=\{0\}$ OR SOME VECTOR IN S IS A LINEAR COMBINATION OF OTHER VECTORS IN S (T1.5)

Let V be a vector space, and let $S \subseteq V$. Then S is linearly dependent if and only if $S=\{0\}$ or some vector in S is a linear combination of other vectors in S.

Proof. We first prove the backward argument.

First, note we know why $\{0\}$ is linearly dependent from a previous section.

So, suppose there exists a vector $v \in S$ such that

$$v = c_1 u_1 + c_2 u_2 + \cdots + c_n u_n,$$

where $c_i \in \mathbb{F}$ and $u_i \in V$ $\forall i \in \{1,2,\ldots,n\}$.

Without loss in generality, assume $u_1, u_2, \ldots, u_n$ are distinct.

By assumption, since $v \notin \{u_1, u_2, \ldots, u_n\}$, necessarily $u_1, u_2, \ldots, u_n, v$ are distinct.

Finally, since

$$0 = (-1)v + c_1 u_1 + c_2 u_2 + \cdots + c_n u_n,$$

and $-1 \neq 0$, it follows S is linearly dependent. $\divideontimes$

Next, we prove the forward argument.

Assume S is linearly dependent, so that there exist distinct $u_1, u_2, \ldots, u_n \in S$ and $a_1, a_2, \ldots, a_n \in \mathbb{F}$ (not all 0) such that

$$a_1 u_1 + a_2 u_2 + \cdots + a_n u_n = 0.$$

Without loss in generality, assume $a_i \neq 0$ $\forall i \in \{1,2,\ldots,n\}$.

<u>Case 1</u>: $n=1$.

Then $a_1 u_1 = 0$, and since $a_1 \neq 0$ it follows that $u_1 = 0$ (since fields are integral domains, so the cancellation property applies.)

Hence $0 \in S$. If $S=\{0\}$ we are done; otherwise, we can pick a $v \in S\setminus\{0\}$, and we can write $0 = 0v$, proving some vector in S, 0, can be written as a linear combination of another vector, v, in S.

<u>Case 2</u>: $n>1$.

Then since $a_n \neq 0$, we can solve for u_n:

$$u_n = -a_n^{-1}[a_1 u_1 + a_2 u_2 + \cdots + a_{n-1} u_{n-1}],$$

$$\therefore u_n = (-a_n^{-1} a_1) u_1 + (-a_n^{-1} a_2) u_2 + \cdots + (-a_n^{-1} a_{n-1}) u_{n-1},$$

showing u_n can be expressed as a linear combination of other elements in S.

BASES & DIMENSION (S1.6)

BASIS (D10)

Let V be a vector space.
Then, we say a subset $S \subseteq V$ is a "basis" for V if
① S is linearly independent; and
② S spans V.

In this case, we also say that the vectors of S form a basis for V.

STANDARD BASIS (E17)

In $\mathbb{F}^n$, define the "standard basis" for $\mathbb{F}^n$ the subset

$$S = \{e_1, e_2, \dots, e_n\},$$

where $e_j \in \mathbb{F}^n$ is the vector with j^{th} coordinate 1 and other coordinates 0.

(It is easy to prove S is indeed a basis for $\mathbb{F}^n$.)

In $P_n(\mathbb{F})$, define the "standard basis" for $P_n(\mathbb{F})$ as the set

$$S = \{1, x, x^2, \dots, x^n\}.$$

(It is also easy to prove S is indeed a basis for $P_n(\mathbb{F})$).

UNIQUE REPRESENTATION OF ELEMENTS IN VECTOR SPACES UNDER A BASIS (T1.6)

Let $\{v_1, v_2, \dots, v_n\}$ be a basis for a vector space V.
Then for every $x \in V$, x can be uniquely represented as a linear combination of $v_1, v_2, \dots, v_n$;
ie there exists a unique n-tuple $(a_1, a_2, \dots, a_n) \in \mathbb{F}^n$ such that

$$x = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n.$$

Proof. Existence: this follows from the fact that $\{v_1, v_2, \dots, v_n\}$ spans V by definition.
Uniqueness: suppose there exists some $b_1, b_2, \dots, b_n \in \mathbb{F}$ such that

$$x = a_1 v_1 + \cdots + a_n v_n = b_1 v_1 + \cdots + b_n v_n.$$

It follows that

$$0 = (a_1 - b_1) v_1 + \cdots + (a_n - b_n) v_n,$$

and since $\{v_1, v_2, \dots, v_n\}$ is linearly independent, necessarily $a_k = b_k$ $\forall k \in \{1, 2, \dots, n\}$. ▨

V IS GENERATED BY S, $|S| = |\mathbb{N}|$ ⇒ T⊂S IS ALSO A BASIS FOR V (T1.7)

Let V be a vector space, and assume that V is generated by a countable set S.
Then there exists a subset of S that is a basis for V.

Proof. If $S = \emptyset$ or $S = \{0\}$, then $\emptyset$ is a basis for V trivially.
Otherwise, S contains at least a non-zero vector.
Hence, we can write S as

$$S = \{v_1, v_2, \dots, v_n\} \quad \text{or} \quad S = \{v_1, v_2, \dots\}.$$

By the WOP, we can pick the smallest index $i_1 \geq 1$ such that $v_{i_1} \neq 0$.
Then $\{v_{i_1}\}$ is linearly independent.
Let i_2 be the smallest index such that $v_{i_2} \in S$ and $v_{i_2} \notin \operatorname{span}(\{v_{i_1}\})$.
Continue this "process" until we obtain the set

$$T = \{v_{i_k} \in S \mid v_{i_k} \notin \operatorname{span}(\{v_{i_1}, \dots, v_{i_{k-1}}\}),\ k \geq 1\}.$$

Finally, we can prove T is a basis for V.

① Assume T is linearly dependent.
Then there exists $a_1, a_2, \dots, a_k$, all not 0, such that

$$a_1 v_{i_1} + \cdots + a_k v_{i_k} = 0.$$

It follows that

$$v_{i_k} = -a_k^{-1} a_1 v_{i_j} - \cdots - a_k^{-1} a_{k-1} v_{i_{k-1}},$$

contradicting the construction of T.

② We can prove by induction that $\operatorname{span}(S_k) = \operatorname{span}(T_k)$ $\forall k \geq 1$, where

$$S_k = \{v_1, v_2, \dots, v_k\} \quad \text{and} \quad T_k = T \cap S_k = \{v_{i_q} \mid i_q \leq k\}.$$

Then, let $x \in V = \operatorname{span}(S)$. Then $x \in \operatorname{span}(S_m)$ for some large m, so that $x \in \operatorname{span}(T_m) \subset \operatorname{span}(T)$.
Hence $V \subseteq \operatorname{span}(T)$, and it follows that $V = \operatorname{span}(T)$. ▨

EVERY VECTOR SPACE HAS A BASIS (T1.8)

We can prove that every vector space has a basis.
(The proof uses Zorn's lemma & maximal linearly independent subsets.)

REPLACEMENT THEOREM (T1.9)

Suppose V is a vector space with a finite spanning set S.
Let T be a linearly independent subset in V. Then

① $|T| \leq |S|$; and

② There exists a set $H \subseteq S$ containing exactly $(|S| - |T|)$ vectors such that $T \cup H$ generates V.

Proof. Let $n = |S|$, and let $m = |T|$.

Then, when $m = 0$, clearly $m = 0 \leq |S|$.

Next, assume the statement is true for some $m \geq 0$.

This implies that if $T_m \subset V$ is any linearly independent subset in V of size m, then $m \leq n$ and there exists a set $H_m \subseteq S$ containing exactly $n-m$ vectors such that $T_m \cup H_m$ generates V.

Let $T_m = \{v_1, v_2, \dots, v_m\}$ and $T = T_m \cup \{v_{m+1}\}$, such that T is linearly independent and a subset of V.

Note that this implies T_m is also linearly independent.

Now, apply the induction hypothesis on T_m to get that $n \geq m$, and there exist $(n-m)$ vectors $w_{m+1}, \dots, w_n \in S$ such that $\{v_1, \dots, v_m, w_{m+1}, \dots, w_n\}$ generates V.

Then, since $n \geq m$, either $n = m$ or $n > m$.

If $n = m$, $\{v_1, \dots, v_m, w_{m+1}, \dots, w_n\} = \{v_1, \dots, v_m\}$.

Thus, $v_{m+1} \in \operatorname{span}\{v_1, \dots, v_m\}$, so by Theorem 1.5, the set $\{v_1, \dots, v_m, v_{m+1}\}$ is linearly dependent.

But this is a contradiction; hence, it follows that $n > m$, so that $n \geq m+1$, proving ①.

Subsequently, write

$$v_{m+1} = a_1 v_1 + \cdots + a_m v_m + a_{m+1} v_{m+1} + \cdots + a_n w_n$$

for some scalars $a_1, \dots, a_n \in \mathbb{F}$.

Then, if $a_{m+1} = \cdots = a_n = 0$, then we would get that $v_{m+1} = a_1 v_1 + \cdots + a_m v_m$, which is a contradiction; hence, at least one of the scalars $a_{m+1}, \dots, a_n$ must be non-zero.

Then, without loss in generality, assume $a_{m+1} \neq 0$.

It follows that

$$w_{m+1} = -a_{m+1}^{-1} a_1 v_1 - \cdots - a_{m+1}^{-1} a_m v_m - a_{m+1}^{-1} a_{m+1} v_{m+1} - a_{m+1}^{-1} a_{m+2} w_{m+2} - \cdots - a_{m+1}^{-1} a_n w_n.$$

Let $H = \{w_{m+2}, \dots, w_n\} \subset S$. The above shows that $w_{m+1} \in \operatorname{span}(T \cup H)$.

Moreover, since $v_1, \dots, v_m \in T \subset T \cup H$ and $w_{m+2}, \dots, w_n \in H \subset T \cup H$, it follows that

$$V = \operatorname{span}\{w_{m+1}, v_1, \dots, v_m, w_{m+2}, \dots, w_n\} \subseteq \operatorname{span}(T \cup H).$$

But since $\operatorname{span}(T \cup H) \subseteq V$, it follows that $V = \operatorname{span}(T \cup H)$, completing the proof. ▨

V IS FINITELY SPANNED ⟹ ALL BASES OF V HAVE EQUAL CARDINALITIES (C1.9.1)

Suppose V is a finitely spanned vector space.
Then all bases of V are finite and have the same amount of elements.

Proof. Let S be a finite spanning set for V, and let B be an arbitrary basis for V.
Then by definition, B is linearly independent.
By the Replacement Theorem, $|B| \leq |S| < \infty$.
Next, let B_1 and B_2 be two bases of V.
Then, since B_1 is linearly independent and B_2 is a finite spanning set for V, by the Replacement Theorem necessarily $|B_1| \leq |B_2|$.
Similarly, since B_2 is linearly independent and B_1 is a finite spanning set for V, by the Replacement Theorem necessarily $|B_2| \leq |B_1|$.
It follows that $|B_1| = |B_2|$, and we are done.

DIMENSION

FINITE/INFINITE - DIMENSIONAL (D12)

1. We say a vector space V is "finite-dimensional" if it has a basis consisting of a finite number of vectors.

2. Otherwise, we say V is "infinite-dimensional".

DIMENSION (D12)

1. Let V be a finite-dimensional vector space.
Then, the "dimension" of V, denoted as "$\dim V$", is defined to be the unique number of vectors in each basis for V.

2. By convention, we let $\dim\{0\} = 0$.

3. Examples:

① $\dim \mathbb{F}^n = n$;

② $\dim \mathbb{C}^n = 2n$;

③ $\dim M_{m \times n}(\mathbb{F}) = mn$; and

④ $\dim P_n(\mathbb{F}) = n+1$. (E18)

ANY FINITE SPANNING SET FOR V CONTAINS AT LEAST n VECTORS (C1.9.2 (1))

Let V be a vector space with $\dim V = n$.
Then if S is a finite spanning set for V, necessarily $|S| \geq n$.

Proof. By the Existence Theorem (T1.7), there exists a subset T of S that is a basis for V.
Therefore $|T| = \dim V = n$, which implies that $|S| \geq |T| = n$. ▨

S GENERATES V, |V| = n ⇒ S IS A BASIS FOR V (C1.9.2 (2))

Let V be a vector space with $\dim V = n$, and suppose S generates V, with $|S| = n$.
Then S is a basis for V.

Proof. Again, by the Existence Theorem (T1.7), there exists some subset $T \subseteq S$ such that T is a basis for V.
By the above corollary, $|T| = n$, so that if $|S| = n$, necessarily $S = T$.
It follows that S is a basis for V. ▨

S IS LINEARLY INDEPENDENT ⇒ S CONTAINS AT MOST n VECTORS (C1.9.2 (3))

Let V be a vector space, with $\dim V = n$.
Suppose the subset $S \subseteq V$ is linearly independent.
Then S contains at most n vectors.

Proof. Applying the Replacement Theorem for the spanning set β, it follows that $|S| \leq |\beta|$, and since $|\beta| = n$, this tells us that $|S| \leq n$, as needed. ▨

S IS LINEARLY INDEPENDENT, |S| = n ⇒ S IS A BASIS FOR V (C1.9.2 (4))

Let V be a vector space, with $\dim V = n$.
Suppose the subset $S \subseteq V$ is linearly independent and $|V| = n$.
Then S is a basis for V.

Proof. Applying the Replacement Theorem for the spanning set β and the linearly independent set S, there must exist a subset $H \subseteq \beta$ containing $|\beta| - |S| = n - n = 0$ vectors such that $S \cup H$ generates V.
But since $|H| = 0$, hence $H = \emptyset$, so that S generates V (and hence is a basis for V.) ▨

EVERY LINEARLY INDEPENDENT SUBSET OF V CAN BE "EXTENDED" TO A BASIS OF V (C1.9.2 (5))

Let V be a vector space, with $\dim V = n$.
Suppose $L = \{v_1, \dots, v_k\}$ is a linearly independent subset of V, where $1 \leq k \leq n$.
Then there exists a $H \subset V$ such that $L \cup H$ is a basis of V.

Proof. If $k = n$, by C1.9.2(4) L is trivially a basis for V.
If $k < n$, then by the Replacement Theorem for the spanning set β and L, there necessarily exists a subset $H \subset \beta$ containing $|\beta| - |L| = n - k$ vectors such that $L \cup H$ generates V.
By C1.9.2(1), $|L \cup H| \geq n$. But
$$|L \cup H| \leq |L| + |H| = k + (n-k) = n,$$
so that $|L \cup H| = n$.
It follows by C1.9.2(2) that $L \cup H$ is a basis for V. ▨

W IS A SUBSPACE OF V ⇒ dim W ≤ dim V ; dim W = dim V ⟺ W = V (C1.9.2 (6))

Let W be a subspace of the vector space V.
Then $\dim W \leq \dim V$, with equality occurring if and only if $V = W$.

Proof. If $W = \{0\}$, then $\dim W = 0 \leq \dim V$.
Otherwise, W contains a non-zero vector w_1.
Then $\{w_1\}$ is linearly independent.
Continue to choose the vectors $w_1, \dots, w_n \in W$ such that $\{w_1, \dots, w_k\}$ is linearly independent.
Note that this process cannot go on indefinitely, since $\{w_1, \dots, w_k\}$ is also linearly independent in V.
This implies that $k \leq n$.
Next, by T1.5, $W \subset \operatorname{span}(\{w_1, \dots, w_k\}) = \operatorname{span}(T)$.
Then, since $T \subseteq W$, necessarily $\operatorname{span}(T) \subset \operatorname{span}(W) = W$.
It follows that $W = \operatorname{span}(T)$, so that T is a basis (since it is also linearly independent), and
$$\dim W = |T| = k \leq n = \dim V.$$
Note that if $\dim V = n = \dim W$, then a basis for W is also a linearly independent set containing n elements.
Hence, by C1.9.2(4), that set is also a basis for V. ▨

W IS A SUBSPACE FOR V ⇒ ANY BASIS OF W CAN BE "EXTENDED" TO A BASIS IN V (C1.9.2 (7))

Let W be a subspace of the vector space V, and let S be a basis of W.
Then we can "extend" S to a basis in V.

Proof. By C1.9.2(6), $\dim W \leq \dim V$.
Let $T = \{w_1, \dots, w_k\}$ be a basis for W, so that T is linearly independent in W, which in turn implies T is linearly independent in V.
So, by C1.9.2(5), we can "extend" T to a basis in V. ▨