

MATH 146

Personal Notes

Marcus Chan

Taught by Ross Willard

and Ciang Tran

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Chapter 1:

Vector Spaces

(S1.1)

KEY

S :	section	P :	proposition
D :	definition	A :	assignment
R :	remark		
E :	example		
T :	theorem		
L :	lemma		
C :	corollary		

Let \mathbb{F} be a field.

Then, we say V is a "vector space"

over \mathbb{F} if there exists

① an addition $+ : (V \times V) \rightarrow V$ by $+ (x, y) = x + y$; and

② a scalar multiplication $\cdot : (\mathbb{F} \times V) \rightarrow V$ by $\cdot (a, x) = ax$;

and the following conditions hold:

① V is an abelian group with respect

to addition; (VS 1 = commutativity; 2 = associativity; 3 = identity; 4 = inverse)

② $\forall x \in V \quad \forall x \in V$; (VS 5)

③ multiplication is associative; ie $a(bx) = (ab)x \quad \forall a, b \in \mathbb{F}, x \in V$;

and (VS 6)

④ the left and right distributive laws hold;

ie $a(x+y) = ax+ay$ and $(a+b)x = ax+bx \quad \forall a, b \in \mathbb{F}, x \in V$. (D2)

(VS 7 = former, VS 8 = latter)

\mathbb{F}^n IS A VECTOR SPACE OVER \mathbb{F} (E2(1))

We can show that the Cartesian product

$$\mathbb{F}^n = \{(a_1, a_2, \dots, a_n) : a_i \in \mathbb{F} \text{ for } i \in \{1, 2, \dots, n\}\}$$

is a vector space over \mathbb{F} with respect to the addition operation

$$(a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n) = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)$$

and multiplication operation

$$c(a_1, a_2, \dots, a_n) = (ca_1, ca_2, \dots, ca_n).$$

Proof. This follows from verifying each of the conditions above. \blacksquare

Note that we generally say "the vector space \mathbb{F}^n " to refer to the vector space \mathbb{F}^n over \mathbb{F} . (R3(4))

COLUMN VECTOR NOTATION (E2(2))

Note that we can also write elements of \mathbb{F}^n as "column vectors"; ie of the form

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

where $a_1, a_2, \dots, a_n \in \mathbb{F}$.

\mathbb{Q}^n IS A VECTOR SPACE OVER \mathbb{Q} ,

\mathbb{R}^n IS A VECTOR SPACE OVER \mathbb{R} , &

\mathbb{C}^n IS A VECTOR SPACE OVER \mathbb{C} (R3(1))

We can show

① \mathbb{Q}^n is a vector space over \mathbb{Q} ;

② \mathbb{R}^n is a vector space over \mathbb{R} ; and

③ \mathbb{C}^n is a vector space over \mathbb{C} .

Proof. This directly follows from the fact that \mathbb{Q}, \mathbb{R} and \mathbb{C} are fields (MATH 145), and substituting the respective fields into the above lemma. \blacksquare

\mathbb{R}^n IS A VECTOR SPACE OVER \mathbb{Q} , &

\mathbb{C}^n IS A VECTOR SPACE OVER \mathbb{R} (R3(2))

Moreover, we can also show that

① \mathbb{R}^n is a vector space over \mathbb{Q} ; and

② \mathbb{C}^n is a vector space over \mathbb{R} .

Proof. Essentially, this stems from the fact that we can "multiply" vectors in \mathbb{R}^n by scalars in \mathbb{Q} , and vectors in \mathbb{C}^n by scalars in \mathbb{R} .

The formal proof is left to the reader. \blacksquare

MATRICES (D3(1))

Let \mathbb{F} be a field, and $m, n \in \mathbb{Z}^+$.

Then, we say A is an " $m \times n$ matrix" with entries from \mathbb{F} if it is of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

where $a_{ij} \in \mathbb{F} \quad \forall i \in \{1, 2, \dots, m\}, j \in \{1, 2, \dots, n\}$.

Alternatively, we can represent A via the notation

$$A = (a_{ij}), \quad i \in \{1, 2, \dots, m\}, j \in \{1, 2, \dots, n\}.$$

ij-ENTRY OF A MATRIX (D3(2))

Given a $m \times n$ matrix A , the " ij -entry" of A , or " a_{ij} ", is defined to be the entry in A at the i th row and j th column.

ZERO MATRIX (D3(3))

The " $m \times n$ zero matrix", or more simply the "zero matrix", denoted as " 0 ", is defined to be

$$0 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \underbrace{\qquad\qquad\qquad}_{m}$$

ie the $m \times n$ matrix where which entry equals 0 .

MATRIX EQUALITY (D3(4))

We say two matrices A and B are equal if and only if $a_{ij} = b_{ij} \quad \forall i \in \{1, 2, \dots, m\}, j \in \{1, 2, \dots, n\}$.

MATRIX ADDITION (D3(5))

Let A and B be $m \times n$ matrices with entries from some field \mathbb{F} .

Then, the "addition" of A and B , denoted by " $A+B$ ", is defined to be the matrix where

$$(a+b)_{ij} = a_{ij} + b_{ij} \quad \forall i \in \{1, 2, \dots, m\}, j \in \{1, 2, \dots, n\}.$$

MATRIX SCALAR MULTIPLICATION (D3(6))

Let A be a $m \times n$ matrix with entries from some field \mathbb{F} , and $c \in \mathbb{F}$ be arbitrary.

Then the "scalar multiplication" of A by c , denoted by " cA ", is defined to be the matrix where

$$(ca)_{ij} = c(a_{ij}) \quad \forall i \in \{1, 2, \dots, m\}, j \in \{1, 2, \dots, n\}.$$

SPACE OF $m \times n$ MATRICES (E3)

Let \mathbb{F} be a field.

Then the "space of all $m \times n$ matrices" with entries from \mathbb{F} , denoted by " $M_{m \times n}(\mathbb{F})$ ", is defined to be the set of all $m \times n$ matrices with entries from \mathbb{F} .

Note that $M_{m \times n}(\mathbb{F})$ is a vector space over \mathbb{F} with respect to the matrix addition and scalar multiplication operations.

Proof. This follows from verifying each of the conditions in D2. \blacksquare

FUNCTION SPACES (E4)

Let the set $D \neq \emptyset$ be arbitrary, and let \mathbb{F} be a field.
Then the space of all functions from D to \mathbb{F} , denoted by " \mathbb{F}^D ", is defined to be the set of all functions of the form $f: D \rightarrow \mathbb{F}$.
Similarly, we can show that \mathbb{F}^D is a vector space over \mathbb{F} with respect to the operations of function addition.

$$(f+g)(x) := f(x) + g(x) \quad \forall f, g \in \mathbb{F}^D, x \in D$$

and function scalar multiplication

$$(cf)(x) := cf(x) \quad \forall f \in \mathbb{F}^D, x \in D, c \in \mathbb{F}.$$

Proof. Similar strategy to E3: verify each condition in D2 holds.

POLYNOMIALS (D4)

SET OF ALL POLYNOMIALS OF DEGREE AT MOST n ($D4(1)$)

Let \mathbb{F} be a field.

Then, we denote $P_n(\mathbb{F})$ to be the set of all polynomials with coefficients from \mathbb{F} and of degree at most n ; ie

$$P_n(\mathbb{F}) = \left\{ \sum_{i=0}^n a_i x^i : a_i \in \mathbb{F}, i \in \{0, \dots, n\} \right\}.$$

POLYNOMIAL SPACES (D4(2))

Let \mathbb{F} be a field.

Then, we denote " $\mathbb{F}[x]$ " to be the set of all polynomials with coefficients from \mathbb{F} ; ie

$$\mathbb{F}[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i : a_i \in \mathbb{F}, i \in \mathbb{N} \cup \{0\} \right\}.$$

Then, we can show that $\mathbb{F}[x]$ is a vector space over \mathbb{F} with respect to the operations of polynomial addition

$$(f+g)(x) = \sum_{i=0}^{\infty} (a_i + b_i) x^i \quad \forall f, g \in \mathbb{F}[x]$$

and polynomial scalar multiplication

$$(cf)(x) = \sum_{i=0}^{\infty} (ca_i) x^i \quad \forall f \in \mathbb{F}[x], c \in \mathbb{F}.$$

Proof. Similar strategy to E4.

BASIC PROPERTIES OF VECTOR SPACES (SI.2)

CANCELLATION PROPERTY FOR VECTOR ADDITION (TI.1)

Let V be a vector space.

Suppose there exists some $x, y, z \in V$ such that

$$x+z = y+z.$$

Then necessarily $x=y$.

Proof. Note that

$$\begin{aligned} x &= x+0 \\ &= x+(z+(-z)) \\ &= (x+z)+(-z) \\ &= (y+z)+(-z) \\ &= y+(z+(-z)) \end{aligned}$$

and so $x=y$, as required. \blacksquare

UNIQUENESS OF THE ZERO VECTOR IN VECTOR SPACES (CI.1.1 (1))

Let V be a vector space.

Suppose $0_1, 0_2 \in V$ are both zero vectors.

Then necessarily $0_1 = 0_2$.

Proof. This follows from the fact that V is an abelian group under addition. \blacksquare

UNIQUENESS OF ADDITIVE INVERSES IN VECTOR SPACES (CI.1.1 (2))

Let V be a vector space.

Then for any $x \in V$, there exists one and only one vector $y \in V$ that satisfies $x+y=0$.

Proof. This also follows from the fact that V is an abelian group under addition. \blacksquare

$0x=0 \quad \forall x \in V$ (TI.2 (1))

Let V be a vector space over some field \mathbb{F} , and let 0 be the additive identity of \mathbb{F} .

Then, for any $x \in V$, necessarily $0 \cdot x = 0$.

Proof. This, again, follows from the fact that V is an abelian group under addition. \blacksquare

$a0=0 \quad \forall a \in \mathbb{F}$ (TI.2 (2))

Let V be a vector space over some field \mathbb{F} , and let 0 be the zero vector of V .

Then, for any $a \in \mathbb{F}$, necessarily $a \cdot 0 = 0$.

Proof. This, again, follows from the fact that V is an abelian group under addition. \blacksquare

$(-a)x = -(ax) = a(-x) \quad \forall a \in \mathbb{F}, x \in V$ (TI.2 (3))

Let V be a vector space over some field \mathbb{F} , and let $a \in \mathbb{F}, x \in V$ be arbitrary.

Then necessarily $(-a)x = -(ax) = a(-x)$.

Proof. Proof is similar to the analog of this statement for rings (MATH145). \blacksquare

SUBSPACES (SI.3)

Let V be a vector space over some field \mathbb{F} . Then we say the subset $W \subseteq V$ is a "subspace" of V if

- ① $W \neq \emptyset$;

* we usually check whether $0 \in W$ to verify this claim. (R4)

- ② If $x \in W$ and $y \in W$, then $(x+y) \in W$; and

- ③ If $c \in \mathbb{F}$ and $x \in W$, then $cx \in W$. (D6)

SUBSPACES ARE VECTOR SPACES OVER \mathbb{F} WITH RESPECT TO THE OPERATIONS OF V (TI.3)

Let W be a subspace of a vector space V over some field \mathbb{F} .

Then W is also a vector space over \mathbb{F} under the operations of V restricted to W .

Proof. This follows from verifying the conditions in D2, taking into account the properties of subspaces. \square

$\{0\}$ AND V ARE SUBSPACES OF V (E8(1))

Let V be a vector space.

Then $\{0\}$ and V itself are always subspaces of V .

Proof. $\{0\}$ is vacuously a subspace, and V is trivially a subspace. \square

$P_2(\mathbb{R})$ IS A SUBSPACE OF $\mathbb{R}[x]$ (E8(2))

We can show that $P_2(\mathbb{R})$ is a subspace of $\mathbb{R}[x]$.

Proof. This stems from the fact that:

- $P_2(\mathbb{R}) \subset \mathbb{R}[x]$ by definition;
- $0 \in P_2(\mathbb{R})$; and
- $P_2(\mathbb{R})$ is closed under the addition & scalar multiplication defined on $\mathbb{R}[x]$. \square

$\{(a_{ij}) \in M_{n \times n}(\mathbb{F}) \mid \sum_{k=1}^n a_{kk} = 0\}$ IS A SUBSPACE

OF $M_{n \times n}(\mathbb{F})$ (E8(3))

We can show that the set $\{(a_{ij}) \in M_{n \times n}(\mathbb{F}) \mid \sum_{k=1}^n a_{kk} = 0\}$ is a subspace of $M_{n \times n}(\mathbb{F})$, where $n \in \mathbb{N}$ is arbitrary.

Proof. Similar proof to the above.

$\{(a_{ij}) \in M_{n \times n}(\mathbb{F}) \mid \sum_{k=1}^n a_{kk} = 1\}$ IS NOT A

SUBSPACE OF $M_{n \times n}(\mathbb{F})$ (E8(4))

We can show the set $\{(a_{ij}) \in M_{n \times n}(\mathbb{F}) \mid \sum_{k=1}^n a_{kk} = 1\}$ is not a subspace of $M_{n \times n}(\mathbb{F})$.

Proof. Let $a, b \in \{(a_{ij}) \in M_{n \times n}(\mathbb{F}) \mid \sum_{k=1}^n a_{kk} = 1\}$ be arbitrary.

Then, notice that

$$\begin{aligned} \sum_{k=1}^n (a+b)_{kk} &= \sum_{k=1}^n (a_{kk} + b_{kk}) \\ &= 1 + 1 \\ &= 2, \end{aligned}$$

so that $a+b \notin \{(a_{ij}) \in M_{n \times n}(\mathbb{F}) \mid \sum_{k=1}^n a_{kk} = 1\}$. \square

SUBSPACES OF \mathbb{R}^2 (E9(1))

Note that the subspaces of \mathbb{R}^2 are

- ① \mathbb{R}^2 itself;

- ② $\{0_{\mathbb{R}^2}\} = \{(0,0)\}$; and

- ③ all lines in \mathbb{R}^2 that pass through $(0,0)$.

SUBSPACES OF \mathbb{F}^2 (E9(4a))

In general, for any field \mathbb{F} , the subspaces of

$$\mathbb{F}^2$$
 are

- ① \mathbb{F}^2 itself;

- ② $\{0\}$; and

- ③ all the "lines" in \mathbb{F}^2 through 0 .

i.e. of the form $\{(x,y) \in \mathbb{F}^2 \mid (x) = k(y), (y) \in \mathbb{F}^2\}$

SUBSPACES OF \mathbb{R}^3 (E9(2))

Similarly, the subspaces of \mathbb{R}^3 are

- ① \mathbb{R}^3 itself;

- ② $\{0_{\mathbb{R}^3}\} = \{(0,0,0)\}$;

- ③ all lines in \mathbb{R}^3 that pass through $(0,0,0)$; and

- ④ all planes in \mathbb{R}^3 that pass through $(0,0,0)$.

SUBSPACES OF \mathbb{F}^3 (E9(4b))

Similarly, for any field \mathbb{F} , the subspaces

$$\text{of } \mathbb{F}^3 \text{ are}$$

- ① \mathbb{F}^3 itself;

- ② $\{0\}$;

- ③ all the "lines" in \mathbb{F}^3 through 0 ; and

i.e. of the form $\{(x,y,z) \in \mathbb{F}^3 \mid (x) = k(y), (y) = k(z), (z) \in \mathbb{F}^3\}$ (E9(3))

- ④ all the "planes" in \mathbb{F}^3 through 0 .

i.e. of the form $\{(x,y,z) \in \mathbb{F}^3 \mid \exists a, b, c \in \mathbb{F} \text{ such that } ax+by+cz=0\}$.

LINEAR COMBINATIONS & SYSTEM OF LINEAR EQUATIONS (SI.4)

LINEAR COMBINATION (D7(1))

* knowledge of elimination method is assumed.

\exists_1 Let V be a vector space over a field F , and let the subset $S \subseteq V$ be such that $S \neq \emptyset$.

Then, we say a vector $x \in V$ is a "linear combination" of vectors from S if there exists a finite number of vectors $u_1, u_2, \dots, u_n \in S$ and scalars $a_1, a_2, \dots, a_n \in F$ such that

$$x = a_1 u_1 + a_2 u_2 + \dots + a_n u_n,$$

where $n \geq 1$. (D7(1))

\exists_2 In this case, we also say that x is a linear combination of the vectors u_1, u_2, \dots, u_n .

COEFFICIENTS OF A LINEAR COMBINATION (D7(2))

\exists_1 Let V be a vector space over some field F , and let the vector $x \in V$ be a linear combination of the vectors $u_1, u_2, \dots, u_n \in S$, where $S \subseteq V$ and $S \neq \emptyset$. Assume that $x = a_1 u_1 + a_2 u_2 + \dots + a_n u_n$, where $a_1, a_2, \dots, a_n \in F$.

Then we denote the scalars $a_1, a_2, \dots, a_n \in F$ as the "coefficients" of the linear combination.

SPAN (D7(3))

\exists_1 Let V be a vector space over some field F , and let the subset $S \subseteq V$ be such that $S \neq \emptyset$.

Then, we define the "span" of S , denoted as "span(S)", to be the set of all linear combinations of vectors in S .

\exists_2 Note that, for convenience, we define

$$\text{span}(\emptyset) = \{\emptyset\}.$$

EXAMPLE 1: SPAN OF $(1,0,0)$ & $(0,1,0)$ IN \mathbb{R}^3 (E10(1))

\exists_1 Observe that in \mathbb{R}^3 , the span of $(1,0,0)$ & $(0,1,0)$ in \mathbb{R}^3 is

$$\{(a, 0, 0) + b(0, 1, 0) : a, b \in \mathbb{R}\}$$

or more simply

$$\{(a, b, 0) : a, b \in \mathbb{R}\}.$$

EXAMPLE 2: SPAN($\{x^n : n \geq 1\}$) IN $\mathbb{Q}[x]$ (E10(2))

\exists_1 We can show that for the vector space $\mathbb{Q}[x]$, the span of $S = \{x, x^2, \dots, x^n, \dots\}$ is the set of all polynomials in $\mathbb{Q}[x]$ whose constant coefficient equals 0.

SPAN OF A FINITE AMOUNT OF VECTORS (E10(3.1))

\exists_1 Suppose V is a vector space over some field F , and let $S \subseteq V$. Further assume that

$$S = \{v_1, v_2, \dots, v_n\},$$

i.e. the size of S is finite.

Then, it follows that

$$\text{span}(S) = \{a_1 v_1 + a_2 v_2 + \dots + a_n v_n : a_i \in F \text{ for } i=1, 2, \dots, n\}.$$

SPAN OF A COUNTABLE AMOUNT OF VECTORS (E10(3.2))

\exists_1 Suppose V is a vector space over some field F , and let $S \subseteq V$. Further assume that

$$S = \{v_1, v_2, \dots, v_n, \dots\},$$

i.e. $|S| = |\mathbb{N}|$.

Then, we can show that

$$\text{span}(S) = \text{span}(\{v_1\}) \cup \text{span}(\{v_1, v_2\}) \cup \dots \cup \text{span}(\{v_1, v_2, \dots, v_n\}) \cup \dots$$

or in other words, that

$$\text{span}(S) = \bigcup_{n=1}^{\infty} \text{span}(\{v_1, v_2, \dots, v_n\}).$$

SPAN OF AN UNCOUNTABLE AMOUNT OF VECTORS (E10(3.3))

\exists_1 Suppose V is a vector space over some field F , and let $S \subseteq V$. Further assume that $|S| > |\mathbb{N}|$; i.e. the size of S is uncountable. Then note that there are no "obvious" simplifications to the formula for $\text{span}(S)$.

SPAN OF A SET IS ALWAYS A SUBSPACE OF THE PARENT VECTOR SPACE (T1.4)

\exists_1 Let V be a vector space over some field F , and let $S \subseteq V$. Then necessarily $\text{span}(S)$ is a subspace of V .

Proof: This follows from verifying each subspace condition for $\text{span}(S)$. \square

\exists_2 Moreover, $\text{span}(S)$ is the "smallest possible" subspace of V that contains S , in the sense that

① $S \subseteq \text{span}(S)$; and

② If W is any other subspace of V containing S , then $\text{span}(S) \subseteq W$.

"GENERATES / SPANS" (D8)

\exists_1 Let V be a vector space, and let $S \subseteq V$.

Then, we say S "generates" V , or S "spans" V , if $\text{span}(S) = V$.

\exists_2 Note to prove $\text{span}(S) = V$, we just need to prove every vector in V can be written as a linear combination of vectors in S , since $\text{span}(S) \subseteq V$ by definition.

(This follows from extensionality.) (R6)

LINEAR INDEPENDENCE & DEPENDENCE (SI-5)

LINEARLY DEPENDENT (D9(1))

💡 Let V be a vector space over some field F , and let $S \subseteq V$.

Then, we say S is "linearly dependent" if there exists a finite number of distinct vectors $u_1, u_2, \dots, u_n \in S$ and scalars $c_1, c_2, \dots, c_n \in F$, where c_1, c_2, \dots, c_n are all not zero, such that

$$c_1 u_1 + c_2 u_2 + \dots + c_n u_n = 0.$$

💡 In this case, we also say the vectors of S are linearly dependent.

💡 Note that if S is finite, say $S = \{u_1, u_2, \dots, u_n\}$, then S is linearly dependent if and only if there exists a $(c_1, c_2, \dots, c_n) \in F^n$, where $(c_1, c_2, \dots, c_n) \neq (0, 0, \dots, 0)$, such that

$$c_1 u_1 + c_2 u_2 + \dots + c_n u_n = 0. \quad (\text{R7(4a)})$$

LINEARLY INDEPENDENT (D9(2))

💡 Let V be a vector space over some field F , and let $S \subseteq V$.

Then, we say S is "linearly independent" if it is not "linearly dependent"; ie for every choice of distinct $u_1, u_2, \dots, u_n \in S$, if $c_1, c_2, \dots, c_n \in F$ are scalars such that

$$c_1 u_1 + c_2 u_2 + \dots + c_n u_n = 0,$$

then necessarily $c_1 = c_2 = \dots = c_n = 0$.

💡 Similarly, if S is finite, say $S = \{u_1, u_2, \dots, u_n\}$, then S is linearly independent if and only if whenever $(c_1, c_2, \dots, c_n) \in F^n$ are such that

$$c_1 u_1 + c_2 u_2 + \dots + c_n u_n = 0,$$

then necessarily $c_1 = c_2 = \dots = c_n = 0$.

TRIVIAL REPRESENTATION OF 0 (R7(1))

💡 Note that for any vector space V and vectors $u_1, u_2, \dots, u_n \in V$, we denote the "trivial representation of $0 \in V$ " as a linear combination of u_1, u_2, \dots, u_n by

$$0u_1 + 0u_2 + \dots + 0u_n = 0.$$

EMPTY SET IS LINEARLY INDEPENDENT (R7(2))

💡 Note that the empty set, \emptyset , is vacuously linearly independent.

* since linearly dependent sets must be non-empty by definition.

$\{0\}$ IS LINEARLY DEPENDENT (R7(3))

💡 Note that the set $\{0\}$ is linearly dependent, since $1(0) = 0$ is a non-trivial representation of 0 as a linear combination of finitely many distinct vectors in S .

$0 \in S \Rightarrow S$ IS LINEARLY DEPENDENT (R7(5))

💡 Note that any subset of a vector space that contains the zero vector is linearly dependent.

EXAMPLE 1: $S = \{(0, 1, 1), (1, 0, 1), (1, 2, 3)\}$ IS LINEARLY DEPENDENT IN \mathbb{R}^3 (E14)

💡 We can show that the set $S = \{(0, 1, 1), (1, 0, 1), (1, 2, 3)\}$ is linearly dependent in \mathbb{R}^3 .

Proof. We search for scalars $a, b, c \in \mathbb{R}$, not all 0, such that

$$a\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + b\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This reduces to solving the system

$$\begin{cases} b+c=0 \\ a+2c=0 \\ a+b+3c=0 \end{cases}$$

Simplifying, we get that

$$a = -2t, b = -t \text{ and } c = t,$$

where $t \in \mathbb{R}$.

For instance, $(a, b, c) = (-2, -1, 1)$ is a solution in which not all of a, b, c are 0.

It follows that S is linearly dependent. \blacksquare

EXAMPLE 2: $S = \{1, x, x^2, x^3\}$ IS LINEARLY INDEPENDENT IN $\mathbb{Z}_5[x]$ (E15)

💡 We can show that the set $S = \{1, x, x^2, x^3\}$ is linearly independent in $\mathbb{Z}_5[x]$.

Proof. Note that if there exist $a_0, a_1, a_2, a_3 \in \mathbb{Z}_5$ such that

$$a_0(1) + a_1x + a_2x^2 + a_3x^3 = 0,$$

then by definition necessarily $a_0 = a_1 = a_2 = a_3 = 0$, and this is sufficient to prove the claim. \blacksquare

S IS LINEARLY DEPENDENT \Leftrightarrow

$S = \{0\}$ OR SOME VECTOR IN S IS A
LINEAR COMBINATION OF OTHER VECTORS
IN S (TI-S)

Let V be a vector space, and let $S \subseteq V$.
Then S is linearly dependent if and only if
 $S = \{0\}$ or some vector in S is a linear
combination of other vectors in S .

Proof. We first prove the backward argument.

First, note we know why $\{0\}$ is linearly
dependent from a previous section.

So, suppose there exists a vector $v \in S$
such that

$$v = c_1 u_1 + c_2 u_2 + \dots + c_n u_n,$$

where $c_i \in \mathbb{F}$ and $u_i \in V$ $\forall i \in \{1, 2, \dots, n\}$.

Without loss in generality, assume u_1, u_2, \dots, u_n are distinct.

By assumption, since $v \notin \{u_1, u_2, \dots, u_n\}$, necessarily

u_1, u_2, \dots, u_n, v are distinct.

Finally, since

$$0 = (-1)v + c_1 u_1 + c_2 u_2 + \dots + c_n u_n,$$

and $-1 \neq 0$, it follows S is linearly dependent. *

Next, we prove the forward argument.

Assume S is linearly dependent, so that there exist
distinct $u_1, u_2, \dots, u_n \in S$ and $a_1, a_2, \dots, a_n \in \mathbb{F}$ (not all 0)
such that

$$a_1 u_1 + a_2 u_2 + \dots + a_n u_n = 0.$$

Without loss in generality, assume $a_1 \neq 0$ $\forall i \in \{1, 2, \dots, n\}$.

Case 1: $n=1$.

Then $a_1 u_1 = 0$, and since $a_1 \neq 0$ it follows that $u_1 = 0$
(since fields are integral domains, so the cancellation
property applies.)

Hence $0 \in S$. If $S = \{0\}$ we are done;
otherwise, we can pick a $v \in S \setminus \{0\}$, and we
can write $0 = 0v$, proving some vector in S , 0, can
be written as a linear combination of another
vector, v , in S .

Case 2: $n > 1$.

Then since $a_1 \neq 0$, we can solve for u_1 :

$$u_1 = -a_1^{-1} [a_1 u_1 + a_2 u_2 + \dots + a_{n-1} u_{n-1}],$$

$$\therefore u_1 = (-a_1^{-1} a_1) u_1 + (-a_1^{-1} a_2) u_2 + \dots + (-a_1^{-1} a_{n-1}) u_{n-1},$$

showing u_1 can be expressed as a linear

combination of other elements in S .

BASES & DIMENSION (SI.6)

BASIS (D10)

Let V be a vector space.

Then, we say a subset $S \subseteq V$ is a "basis" for V if
① S is linearly independent; and
② S spans V .

In this case, we also say that the vectors of S form a basis for V .

STANDARD BASIS (C17)

In \mathbb{F}^n , define the "standard basis" for \mathbb{F}^n the subset

$$S = \{e_1, e_2, \dots, e_n\},$$

where $e_j \in \mathbb{F}^n$ is the vector with j th coordinate 1 and other coordinates 0.

(It is easy to prove S is indeed a basis for \mathbb{F}^n)

In $P_n(\mathbb{F})$, define the "standard basis" for $P_n(\mathbb{F})$ as the set

$$S = \{1, x, x^2, \dots, x^n\}.$$

(It is also easy to prove S is indeed a basis for $P_n(\mathbb{F})$).

UNIQUE REPRESENTATION OF ELEMENTS IN VECTOR SPACES UNDER A BASIS (T1.6)

Let $\{v_1, v_2, \dots, v_n\}$ be a basis for a vector space V .

Then for every $x \in V$, x can be uniquely represented as a linear combination of v_1, v_2, \dots, v_n ; ie there exists a unique n -tuple $(a_1, a_2, \dots, a_n) \in \mathbb{F}^n$ such that

$$x = a_1 v_1 + a_2 v_2 + \dots + a_n v_n.$$

Proof. Existence: this follows from the fact that $\{v_1, v_2, \dots, v_n\}$ spans V by definition.

Uniqueness: suppose there exists some $b_1, b_2, \dots, b_n \in \mathbb{F}$ such that

$$x = a_1 v_1 + \dots + a_n v_n = b_1 v_1 + \dots + b_n v_n.$$

It follows that

$$0 = (a_1 - b_1)v_1 + \dots + (a_n - b_n)v_n,$$

and since $\{v_1, v_2, \dots, v_n\}$ is linearly independent, necessarily $a_i = b_i \quad \forall i \in \{1, 2, \dots, n\}$. \square

V IS GENERATED BY S , $|S| = |\mathbb{N}|$

$\Rightarrow TCS$ IS ALSO A BASIS FOR V (T1.7)

Let V be a vector space, and assume that

V is generated by a countable set S .

Then there exists a subset of S that is a basis for V .

Proof. If $S = \emptyset$ or $S = \{0\}$, then \emptyset is a basis for V trivially.

Otherwise, S contains at least a non-zero vector.

Hence, we can write S as

$$S = \{v_1, v_2, \dots, v_n\} \text{ or } S = \{v_1, v_2, \dots\}.$$

By the WOP, we can pick the smallest index $i \geq 1$ such that $v_i \neq 0$.

Then $\{v_i\}$ is linearly independent.

Let i_2 be the smallest index such that $v_{i_2} \in \text{span}\{v_i\}$.

Continue this "process" until we obtain the set

$$T = \{v_{i_k} \mid v_{i_k} \notin \text{span}\{v_{i_1}, \dots, v_{i_{k-1}}\}, k \geq 1\}.$$

Finally, we can prove T is a basis for V .

① Assume T is linearly dependent.

Then there exists a_1, a_2, \dots, a_k , all not 0, such that

$$a_1 v_{i_1} + \dots + a_k v_{i_k} = 0.$$

It follows that

$$v_{i_k} = -a_1^{-1} a_1 v_{i_1} - \dots - a_{k-1}^{-1} a_{k-1} v_{i_{k-1}},$$

contradicting the construction of T .

② We can prove by induction that $\text{span}(S_k) = \text{span}(T_k) \quad \forall k \geq 1$, where

$$S_k = \{v_1, v_2, \dots, v_k\} \text{ and } T_k = T \cap S_k = \{v_{i_q} \mid i_q \leq k\}.$$

Then, let $x \in \text{span}(S)$. Then $x \in \text{span}(S_m)$ for some large m , so that $x \in \text{span}(T_m) \subset \text{span}(T)$.

Hence $V \subseteq \text{span}(T)$, and it follows that $V = \text{span}(T)$. \square

EVERY VECTOR SPACE HAS A BASIS

(T1.8)

We can prove that every vector space has a basis.

(The proof uses Zorn's Lemma & maximal linearly independent subsets.)

REPLACEMENT THEOREM (TI.9)

Suppose V is a vector space with a finite spanning set S . Let T be a linearly independent subset in V . Then

- ① $|T| \leq |S|$; and
- ② There exists a set $H \subseteq S$ containing exactly $(|S|-|T|)$ vectors such that $T \cup H$ generates V .

Proof. Let $n = |S|$, and let $m = |T|$. Then, when $m=0$, clearly $m=0 \leq |S|$. Next, assume the statement is true for some $m \geq 0$. This implies that if $T_m \subseteq V$ is any linearly independent subset in V of size m , then $m \leq n$ and there exists a set $H_m \subseteq S$ containing exactly $n-m$ vectors such that $T_m \cup H_m$ generates V .

Let $T_m = \{v_1, v_2, \dots, v_m\}$ and $T = T_m \cup \{v_{m+1}\}$, such that T is linearly independent and a subset of V .

Note that this implies T_m is also linearly independent.

Now, apply the induction hypothesis on T_m to get that $n \geq m$, and there exist $(n-m)$ vectors $w_{m+1}, \dots, w_n \in S$ such that

$\{v_1, \dots, v_m, w_{m+1}, \dots, w_n\}$ generates V .

Then, since $n \geq m$, either $n=m$ or $n > m$.

If $n=m$, $\{v_1, \dots, v_m, w_{m+1}, \dots, w_n\} = \{v_1, \dots, v_m\}$.

Thus, $v_{m+1} \in \text{span}\{v_1, \dots, v_m\}$, so by Theorem 1.5, the set $\{v_1, \dots, v_m, v_{m+1}\}$ is linearly dependent.

But this is a contradiction; hence, it follows that $n > m$, so that $n \geq m+1$, proving ①.

Subsequently, write

$$v_{m+1} = a_1 v_1 + \dots + a_m v_m + a_{m+1} v_{m+1} + \dots + a_n w_n$$

for some scalars $a_1, \dots, a_n \in \mathbb{F}$.

Then, if $a_{m+1} = \dots = a_n = 0$, then we would get that $v_{m+1} = a_1 v_1 + \dots + a_m v_m$, which is a contradiction; hence, at least one of the scalars a_{m+1}, \dots, a_n must be non-zero.

Then, without loss in generality, assume $a_{m+1} \neq 0$.

It follows that

$$\begin{aligned} w_{m+1} &= -a_{m+1}^{-1} a_1 v_1 - \dots - a_{m+1}^{-1} a_m v_m - a_{m+1}^{-1} a_{m+1} v_{m+1} \\ &\quad - a_{m+1}^{-1} a_{m+2} w_{m+2} - \dots - a_{m+1}^{-1} a_n w_n. \end{aligned}$$

Let $H = \{w_{m+2}, \dots, w_n\} \subset S$. The above shows that

$w_{m+1} \in \text{span}(T \cup H)$.

Moreover, since $v_1, \dots, v_m \in T \subseteq \text{span}(T \cup H)$ and $w_{m+2}, \dots, w_n \in H \subseteq \text{span}(T \cup H)$, it follows that

$$V = \text{span}\{w_{m+1}, v_1, \dots, v_m, w_{m+2}, \dots, w_n\} \subseteq \text{span}(T \cup H).$$

But since $\text{span}(T \cup H) \subseteq V$, it follows that $V = \text{span}(T \cup H)$, completing the proof. \square

V IS FINITELY SPANNED \Rightarrow ALL BASES OF V & H HAVE EQUAL CARDINALITIES (CI.9.1)

Suppose V is a finitely spanned vector space.

Then all bases of V are finite and have the same amount of elements.

Proof. Let S be a finite spanning set for V , and let B be an arbitrary basis for V . Then by definition, B is linearly independent.

By the Replacement Theorem, $|B| \leq |S| < \infty$.

Next, let B_1 and B_2 be two bases of V . Then, since B_1 is linearly independent and B_2

is a finite spanning set for V , by the

Replacement Theorem necessarily $|B_1| \leq |B_2|$.

Similarly, since B_2 is linearly independent and B_1 is a finite spanning set for V , by the

Replacement Theorem necessarily $|B_2| \leq |B_1|$.

It follows that $|B_1| = |B_2|$, and we are done.

DIMENSION FINITE/INFINITE-DIMENSIONAL (DI.2)

We say a vector space V is "finite-dimensional" if it has a basis consisting of a finite number of vectors.

Otherwise, we say V is "infinite-dimensional".

DIMENSION (DI.2)

Let V be a finite-dimensional vector space.

Then, the "dimension" of V , denoted as "dim V ", is defined to be the unique number of vectors in each basis for V .

By convention, we let $\dim\{0\} = 0$.

Examples:

- ① $\dim \mathbb{F}^n = n$;
- ② $\dim \mathbb{C}^n = 2n$;
- ③ $\dim M_{m \times n}(\mathbb{F}) = mn$; and
- ④ $\dim P_n(\mathbb{F}) = n+1$. (E18)

ANY FINITE SPANNING SET FOR V CONTAINS AT LEAST n VECTORS (C1.9.2(1))

Let V be a vector space with $\dim V = n$. Then if S is a finite spanning set for V , necessarily $|S| \geq n$.

Proof. By the Existence Theorem (T1.7), there exists a subset T of S that is a basis for V . Therefore $|T| = \dim V = n$, which implies that $|S| \geq |T| = n$. \square

S GENERATES V , $|V|=n \Rightarrow S$ IS A BASIS FOR V (C1.9.2 (2))

Let V be a vector space with $\dim V = n$, and suppose S generates V , with $|S|=n$. Then S is a basis for V .

Proof. Again, by the Existence Theorem (T1.7), there exists some subset $T \subseteq S$ such that T is a basis for V . By the above corollary, $|T|=n$, so that if $|S|=n$, necessarily $S=T$. It follows that S is a basis for V . \square

S IS LINEARLY INDEPENDENT \Rightarrow S CONTAINS AT MOST n VECTORS (C1.9.2(3))

Let V be a vector space, with $\dim V = n$. Suppose the subset $S \subseteq V$ is linearly independent. Then S contains at most n vectors.

Proof. Applying the Replacement Theorem for the spanning set P , it follows that $|S| \leq |P|$, and since $|P|=n$, this tells us that $|S| \leq n$, as needed. \square

S IS LINEARLY INDEPENDENT, $|S|=n$ $\Rightarrow S$ IS A BASIS FOR V (C1.9.2 (4))

Let V be a vector space, with $\dim V = n$. Suppose the subset $S \subseteq V$ is linearly independent and $|V|=n$. Then S is a basis for V .

Proof. Applying the Replacement Theorem for the spanning set P and the linearly independent set S , there must exist a subset $H \subseteq P$ containing $|P|-|S|=n-n=0$ vectors such that $S \cup H$ generates V . But since $|H|=0$, hence $H=\emptyset$, so that S generates V (and hence is a basis for V). \square

EVERY LINEARLY INDEPENDENT SUBSET OF V CAN BE "EXTENDED" TO A BASIS OF V (C1.9.2 (5))

Let V be a vector space, with $\dim V = n$. Suppose $L = \{v_1, \dots, v_k\}$ is a linearly independent subset of V , where $1 \leq k \leq n$. Then there exists a HCV such that $L \cup H$ is a basis of V .

Proof. If $k=n$, by C1.9.2(4) L is trivially a basis for V . If $k < n$, then by the Replacement Theorem for the spanning set P and L , there necessarily exists a subset $H \subseteq P$ containing $|P|-|L|=n-k$ vectors such that $L \cup H$ generates V . By C1.9.2(1), $|L \cup H| \geq n$. But $|L \cup H| \leq |L| + |H| = k + (n-k) = n$, so that $|L \cup H| = n$. It follows by C1.9.2(2) that $L \cup H$ is a basis for V . \square

W IS A SUBSPACE OF V

$$\Rightarrow \dim W \leq \dim V ; \dim W = \dim V \\ \Leftrightarrow W = V \quad (\text{C1.9.2 (6)})$$

Let W be a subspace of the vector space V . Then $\dim W \leq \dim V$, with equality occurring if and only if $V=W$.

Proof. If $W=\{v\}$, then $\dim W=0 \leq \dim V$. Otherwise, W contains a non-zero vector w_1 . Then $\{w_1\}$ is linearly independent. Continue to choose the vectors $w_1, \dots, w_n \in W$ such that $\{w_1, \dots, w_k\}$ is linearly independent. Note that this process cannot go on indefinitely, since $\{w_1, \dots, w_k\}$ is also linearly independent in V . This implies that $k \leq n$. Next, by T1.5, $W \subseteq \text{span}(\{w_1, \dots, w_k\}) = \text{span}(T)$. Then, since $T \subseteq W$, necessarily $\text{span}(T) \subseteq \text{span}(W) = W$. It follows that $W = \text{span}(T)$, so that T is a basis (since it is also linearly independent), and $\dim W = |T| = k \leq n = \dim V$.

Note that if $\dim V = n = \dim W$, then a basis for W is also a linearly independent set containing n elements. Hence, by C1.9.2(4), that set is also a basis for V . \square

W IS A SUBSPACE FOR $V \Rightarrow$ ANY BASIS OF W CAN BE "EXTENDED" TO A BASIS IN V (C1.9.2 (7))

Let W be a subspace of the vector space V , and let S be a basis of W . Then we can "extend" S to a basis in V .

Proof. By C1.9.2(6), $\dim W \leq \dim V$. Let $T = \{w_1, \dots, w_n\}$ be a basis for W , so that T is linearly independent in W , which in turn implies T is linearly independent in V . So, by C1.9.2(5), we can "extend" T to a basis in V . \square

QUOTIENT SPACES (SI.7)

COSET & REPRESENTATIVE (D13)

Let V be a vector space, and W be a subspace of V . Then, for a given $x \in V$, its corresponding "coset" of W in V , denoted as " $x+W$ ", is defined to be the set $x+W = \{x+w : w \in W\}$.

* note that $x+W \subseteq V$.

In this case, we call " x " a "representative" of the coset $x+W$.

$x \equiv y \pmod{W}$ (D13)

Let V be a vector space, and let W be a subspace of V . Then, we write " $x \equiv y \pmod{W}$ " if and only if $x-y \in W$.

V/W (D13)

Let V be a vector space, and W a subspace of V . Then, we denote " V/W " (ie " $V \pmod{W}$ ") as the set

$V/W = \{x+W : x \in V\}$;
ie let V/W be the collection of cosets of W in V .

$V/\{0\} = V$ (E19 (2))

For any vector space V , necessarily

$$\begin{aligned} V/\{0\} &= V. \\ \text{Proof: } V/\{0\} &= \{0+x : x \in V\} \\ &= \{x : x \in V\} \\ &\therefore V/\{0\} = V. \end{aligned}$$

COSET TEST (P1)

Let W be a subspace of a vector space V , and let $x, y \in V$ be arbitrary. Then $x+W = y+W$ if and only if $x-y \in W$.

Proof: Similar to test for cosets in MATH 145.

$\equiv \pmod{W}$ IS AN EQUIVALENCE RELATION ON V (R8)

Note that the relation " $\equiv \pmod{W}$ " is an equivalence relation on V .

ADDITION & MULTIPLICATION IN V/W (D14)

Let V be a vector space over a field F , and let W be a subspace of V . Then, we can define an addition on V/W by

$$(x+W) + (y+W) := ((x+y)+W);$$

and a scalar multiplication on V/W by

$$a(x+W) := (ax)+W;$$

for any $a \in F$ and $x, y \in W$.

Note that these addition and multiplication operations are well-defined. (L1)

Proof: Similar to proof for quotient groups/rings.

V/W IS A VECTOR SPACE

(THE QUOTIENT SPACE OF V BY W) (T1.10)

Let V be a vector space, and W a subspace of V . Then the set V/W is a vector space over F with the operations of coset addition and scalar multiplication, denoted as "the quotient space of V by W ".

Proof: Verify all 8 conditions. (VS 1-8). \square

BASIS FOR QUOTIENT SPACES (T1.11)

Let V be a vector space with $\dim V = n$, and let W be a subspace of V such that $\dim W = k$. Let $\{v_1, \dots, v_n\}$ be a basis for V , such that $\{v_1, \dots, v_k\}$ is a basis for W .

Then,

- ① The set $\{v_{k+1}+W, \dots, v_n+W\}$ is a basis for V/W ; and
- ② $\dim(V/W) = \dim V - \dim W$.

Proof: To prove ①, we show $\{v_{k+1}+W, \dots, v_n+W\}$ is both linearly independent and generates V/W , giving us our basis.

It follows that

$$\begin{aligned} \dim(V/W) &= |\{v_{k+1}+W, \dots, v_n+W\}| \\ &= n - (k+1) \\ &= n - k \\ \therefore \dim(V/W) &= \dim V - \dim W. \end{aligned}$$

$$\dim V \geq \infty, \dim W \geq \infty \Rightarrow \dim V/W \geq \infty \quad (\text{R9})$$

Let V be an infinite-dimensional vector space, and let W be an infinite-dimensional subspace of V .

Then, note that it is not necessarily the case that $\dim(V/W) \geq \infty$.

Example: let $V = \mathbb{F}^\infty$ & $W = \{(0, x_2, \dots) : x_2 \in \mathbb{F}\}$. Note that each element of V/W is simply "determined" by the value of the first coordinate x_1 , so that $\dim(V/W) = 1$.

SUMS & INTERNAL DIRECT SUMS OF SUBSPACES (SL8)

SUM OF SUBSPACES (DIS)

Let V be a vector space over \mathbb{F} , and let W_1, W_2 be subspaces of V . Then, we define the "sum" of W_1 and W_2 , denoted as $W_1 + W_2$, to be the set

$$W_1 + W_2 := \{v_1 + v_2 : v_1 \in W_1, v_2 \in W_2\}.$$

INDEPENDENT/DISJOINT (DIS)

Let V be a vector space, and let W_1, W_2 be subspaces of V . Then, we say W_1 and W_2 are "independent", or "disjoint", if and only if $W_1 \cap W_2 = \{0\}$.

(INTERNAL) DIRECT SUM (DIS)

Let V be a vector space, and let W_1, W_2 be independent subspaces of V .

Then, we define the "(internal) direct sum" of W_1 and W_2 , denoted as $W_1 \oplus W_2$, to be the set

$$W_1 \oplus W_2 = W_1 + W_2.$$

* ie " \oplus " is the notation for "+" used when W_1 & W_2 are independent.

Note that $W_1 \oplus W_2$ is well-defined, as long as $W_1 \cap W_2 = \{0\}$. (R10)

$W_1 + W_2$ IS THE "SMALLEST" SUBSPACE CONTAINING W_1 & W_2 (L2 (2))

Let V be a vector space, and let W_1, W_2 be subspaces of V .

Then $W_1 + W_2$ is necessarily the smallest subspace of V containing W_1 and W_2 .

Proof. First, we prove $W_1 + W_2$ is a subspace of V .

Let $(v_1 + v_2), (u_1 + u_2) \in W_1 + W_2$ and $a \in \mathbb{F}$, where $v_1, v_2 \in W_1$ and $u_1, u_2 \in W_2$.

Then, since W_1 and W_2 are subspaces of $W_1 + W_2$, necessarily $v_1 + u_1 \in W_1 + W_2$ and $v_2 + u_2 \in W_1 + W_2$.

so that

$$(v_1 + v_2) + (u_1 + u_2) = (v_1 + u_1) + (v_2 + u_2) \in W_1 + W_2.$$

Moreover, since $av_1 \in W_1$ and $av_2 \in W_2$, necessarily

$$a(v_1 + v_2) = av_1 + av_2 \in W_1 + W_2.$$

proving $W_1 + W_2$ is closed under addition and scalar multiplication.

Then, since $v_1 = v_1 + 0 \in W_1 + W_2 \quad \forall v_1 \in W_1$ & $v_2 = 0 + v_2 \in W_1 + W_2 \quad \forall v_2 \in W_2$, it follows that

$$W_1 \subseteq W_1 + W_2 \text{ and } W_2 \subseteq W_1 + W_2.$$

Finally, let Y be a subspace of V that contains both W_1 & W_2 .

Since Y is closed under addition, $v_1 + v_2 \in Y$

for every $v_1 \in W_1$ and $v_2 \in W_2$ necessarily.

It follows that $W_1 + W_2 \subseteq Y$, completing the proof.

$$V = W_1 \oplus W_2 \iff \forall v \in V : \exists \text{ unique } w_1 \in W_1,$$

$$w_2 \in W_2 \ni v = w_1 + w_2 \quad (\text{L2 (3)})$$

Let V be a vector space, and let W_1 and W_2 be subspaces of V .

Then $W_1 \oplus W_2 = V$ if and only if for every vector $v \in V$, there exist unique elements $w_1 \in W_1$ and $w_2 \in W_2$ such that $v = w_1 + w_2$.

Proof. (\Rightarrow) Since $V = W_1 \oplus W_2$, necessarily $V = W_1 + W_2$, and $W_1 \cap W_2 = \{0\}$.

let $v \in V$, and note that since $V = W_1 + W_2$, it implies that $v \in W_1 + W_2$.

So, by definition, there exist some $w_1 \in W_1, w_2 \in W_2$ such that $v = w_1 + w_2$.

Next, suppose we have $v = w'_1 + w'_2$ for some $w'_1 \in W_1$ and $w'_2 \in W_2$. Then

$$0 = (w_1 + w_2) - (w'_1 + w'_2) = (w_1 - w'_1) + (w_2 - w'_2).$$

Since $w_1, w'_1 \in W_1$ & $w_2, w'_2 \in W_2$, necessarily $w_1 - w'_1 \in W_1$ & $w_2 - w'_2 \in W_2$ also, so that

$$(w_1 - w'_1) = w'_2 - w_2 \in W_1 \cap W_2 = \{0\},$$

Hence $w_1 - w'_1 = w'_2 - w_2 = 0$, implying that $w_1 = w'_1$ & $w_2 = w'_2$, proving uniqueness. *

(\Leftarrow) By assumption, every vector $v \in V$ can be written as $v = w_1 + w_2$ for some $w_1 \in W_1$ & $w_2 \in W_2$.

Hence $V \subseteq W_1 + W_2$, and by L2(2) necessarily $W_1 + W_2 \subseteq V$; so $V = W_1 + W_2$.

Next, let $x \in W_1 \cap W_2$. Then $-x \in W_1 \cap W_2$.

Then, note that

$$0 = 0 + 0 = x + (-x) \in W_1 + W_2,$$

and due to the uniqueness assumption, necessarily $x = 0$.

Thus $W_1 \cap W_2 = \{0\}$, so that $V = W_1 \oplus W_2$. \blacksquare

$$\dim(W_1), \dim(W_2) < \infty \Rightarrow \dim(W_1 + W_2) < \infty \text{ &} \\ \dim(W_1) + \dim(W_2) = \dim(W_1 + W_2) + \dim(W_1 \cap W_2)$$

(T1.12 (1))

Let V be a vector space over some field \mathbb{F} , and let W_1, W_2 be finite dimensional subspaces of V .

Then necessarily $W_1 + W_2$ is finite dimensional, and $\dim(W_1) + \dim(W_2) = \dim(W_1 + W_2) + \dim(W_1 \cap W_2)$.

Proof. First, note $W_1 \cap W_2$ is a subspace of W_1 (A2), so that $\dim(W_1 \cap W_2) \leq \dim(W_1) < \infty$ (C1.9.2(6)).

Next, let $\{u_1, u_2, \dots, u_k\}$ be a basis for $W_1 \cap W_2$.

Extend this basis to get the bases

$S_1 = \{u_1, \dots, u_k, v_1, \dots, v_m\}$ of W_1 and $S_2 = \{u_1, \dots, u_k, z_1, \dots, z_p\}$ of W_2 , which we can always do by C1.9.2(5)).

Let $S = \{u_1, \dots, u_k, v_1, \dots, v_m, z_1, \dots, z_p\}$.

We claim S is a basis for $W_1 + W_2$.

Indeed, consider

$$a_1 u_1 + \dots + a_k u_k + b_1 v_1 + \dots + b_m v_m + c_1 z_1 + \dots + c_p z_p = 0 \quad \text{--- (2)}$$

for some scalars $a_1, \dots, a_k, b_1, \dots, b_m, c_1, \dots, c_p$.

Then

$$b_1 v_1 + \dots + b_m v_m = -a_1 u_1 - \dots - a_k u_k - c_1 z_1 - \dots - c_p z_p.$$

Since the RHS is a linear combination of vectors in W_2 , the RHS $\in W_2$; and since the LHS is a linear combination of vectors in W_1 , the LHS $\in W_1$.

Thus $b_1 v_1 + \dots + b_m v_m \in W_1 \cap W_2$.

Next, since $\{u_1, \dots, u_k\}$ is a basis for $W_1 \cap W_2$, there exist scalars d_1, \dots, d_k such that

$$b_1 v_1 + \dots + b_m v_m = d_1 u_1 + \dots + d_k u_k.$$

So

$$b_1 v_1 + \dots + b_m v_m - d_1 u_1 - \dots - d_k u_k = 0.$$

Since $\{u_1, \dots, u_k, v_1, \dots, v_m\}$ is a basis for W_1 , necessarily $b_1 = \dots = b_m = d_1 = \dots = d_k = 0$.

Substitute $b_1 = \dots = b_m$ into (2) to get that

$$a_1 u_1 + \dots + a_k u_k + c_1 z_1 + \dots + c_p z_p = 0.$$

Then, since $\{u_1, \dots, u_k, z_1, \dots, z_p\}$ is a basis for W_2 , we have

$$a_1 = \dots = a_k = c_1 = \dots = c_p = 0,$$

proving S is linearly independent.

Subsequently, let $x+y \in W_1 + W_2$ be arbitrary, where $x \in W_1$ and $y \in W_2$.

Then, since S_1 and S_2 are bases for W_1 and W_2 respectively, we can write x and y as linear combinations of vectors in S_1 and S_2 , respectively:

$$x = a_1 u_1 + \dots + a_k u_k + b_1 v_1 + \dots + b_m v_m; \quad \text{--- (3)}$$

$$y = d_1 u_1 + \dots + d_k u_k + c_1 z_1 + \dots + c_p z_p;$$

where $a_1, \dots, a_k, b_1, \dots, b_m, d_1, \dots, d_k, c_1, \dots, c_p \in \mathbb{F}$.

Hence

$$x+y = (a_1+d_1)u_1 + \dots + (a_k+d_k)u_k + (b_1+c_1)z_1 + \dots + (b_m+c_p)z_p,$$

which is sufficient to show $x+y \in \text{span}(S)$.

Thus $W_1 + W_2 \subseteq \text{span}(S)$, and since $\text{span}(S) \subseteq W_1 + W_2$,

by definition, it follows that $W_1 + W_2 = \text{span}(S)$,

verifying that S is indeed a basis for

$W_1 + W_2$.

In particular,

$$\begin{aligned} \dim(W_1 + W_2) + \dim(W_1 \cap W_2) &= |S| + k \\ &= m + p + k + k \\ &= (m+k) + (p+k) \\ &= \dim W_1 + \dim W_2. \end{aligned}$$

$$\dim(V) < \infty, \quad W_1 \oplus W_2 = V \Rightarrow \dim W_1 + \dim W_2 = \dim V$$

(T1.12 (2))

Let V be a vector space over \mathbb{F} , and let W_1, W_2 be finite-dimensional subspaces of V .

Suppose further that V itself is finite-dimensional, and $W_1 \oplus W_2 = V$.

Then necessarily $\dim W_1 + \dim W_2 = \dim V$.

Proof. Since $W_1 \oplus W_2 = V$, necessarily $W_1 \cap W_2 = \{0\}$.

So, by T1.12(1), it follows that

$$\begin{aligned} \dim W_1 + \dim W_2 &= \dim(W_1 + W_2) + \dim(W_1 \cap W_2) \\ &= \dim(V) + 0 \end{aligned}$$

$$\therefore \dim W_1 + \dim W_2 = \dim(V). \quad \blacksquare$$

COMPLEMENTARY SUBSPACES (D15)

Let V be a vector space, and let W be a subspace of V .

Then a subspace W' of V is said to be a "complementary subspace" to W if $W \oplus W' = V$; ie

$$\textcircled{1} \quad W \cap W' = \{0\}; \quad \text{and}$$

$$\textcircled{2} \quad W + W' = V.$$

$$\dim W + \dim W' = \dim V$$

Let V be a vector space, and let W be a subspace of V .

W' be a complementary subspace to W . Let W' be a complementary subspace to W .

Then necessarily $\dim W + \dim W' = \dim V$.

Proof. Follows directly from T1.12(2).

EXISTENCE OF COMPLEMENTARY SUBSPACES (R11(1))

Let V be a vector space, and let W

be a subspace of V .

Then there always exists a complementary subspace

W' to W of V such that $W \oplus W' = V$.

Proof. First, note that every linearly independent set can be extended to a basis V that has a countable spanning set (A3).

Hence, every linearly independent subset of V

can be extended to a basis for V .

It follows that every subspace W of V has a complementary subspace W' . \blacksquare

NON-UNIQUENESS OF COMPLEMENTARY SUBSPACES (R11(2))

Note that complementary subspaces of a given

vector space V are not necessarily unique.

eg $V = \mathbb{R}^3$, $W = \{(1,0,0), (0,1,0)\}$, $W'_1 = \{(0,0,1)\}$,

$W'_2 = \{(0,0,-1)\}$;

observe that both W'_1 and W'_2 are complementary subspaces to W .

Chapter 2:

Linear Transformations and Matrices

LINEAR TRANSFORMATIONS (S2.1)

💡 Let V and W be vector spaces over the same field \mathbb{F} .

Then, we say the function $T: V \rightarrow W$ is a "linear transformation" from V to W if

$$(L1) \rightarrow T(x+y) = T(x) + T(y) \quad \forall x, y \in V; \text{ and}$$

$$(L2) \rightarrow T(cx) = cT(x) \quad \forall x \in V, c \in \mathbb{F}. \quad (D16)$$

💡 In this case, we say the function $T: V \rightarrow W$ is "linear".

T IS LINEAR ($\Rightarrow T(cx+ty) = cT(x) + T(y)$) (P2)

💡 Let the function $T: V \rightarrow W$, where V and W are vector spaces over the same field \mathbb{F} .

Then T is linear if and only if $T(cx+ty) = cT(x) + T(y)$ for all $x, y \in V$ and $c \in \mathbb{F}$.

ZERO TRANSFORMATION (E23(1a))

💡 For any vector spaces V and W , the "zero transformation", given by " $T_0: V \rightarrow W$ ", is defined by $T_0(x) = 0 \quad \forall x \in V$.

IDENTITY TRANSFORMATION (E23(1b))

💡 For any vector space V , the "identity transformation" $I_V: V \rightarrow V$ is given by

$$I_V(x) = x \quad \forall x \in V.$$

T: $V \rightarrow \mathbb{F}^n$ BY $T(a_1v_1 + \dots + a_nv_n) := (a_1, \dots, a_n)$ (E23(3))

💡 Let V be a finite-dimensional vector space over \mathbb{F} , and let $\{v_1, v_2, \dots, v_n\}$ be a basis for V .

Then, the mapping

$$T: V \rightarrow \mathbb{F}^n \text{ by } T(a_1v_1 + \dots + a_nv_n) := (a_1, \dots, a_n)$$

is linear.

T: $\mathbb{F}^n \rightarrow \mathbb{F}^k$, $T(x_1, \dots, x_n) := (x_1, \dots, x_k)$ (E23(4))

💡 Let \mathbb{F} be a field, and suppose $1 \leq k < n$.

Then the projection mapping

$$T: \mathbb{F}^n \rightarrow \mathbb{F}^k \text{ by } T(x_1, \dots, x_n) := (x_1, \dots, x_k)$$

is linear.

T(0) = 0 (P3(1))

💡 Let $T: V \rightarrow W$ be linear. Then necessarily $T(0) = 0$.

$$\text{Proof. } T(0) = T(0+0) = T(0) + T(0); \\ \text{Thus } 0 = T(0) + T(0) - T(0) = T(0). \quad \blacksquare$$

T(x-y) = T(x) - T(y) (P3(2))

💡 Let $T: V \rightarrow W$ be linear. Then necessarily $T(x-y) = T(x) - T(y) \quad \forall x, y \in V$.

$$\text{Proof. } T(x-y) = T(x) + T(-y) \\ = T(x) + (-1)T(y) \\ \therefore T(x-y) = T(x) - T(y). \quad \blacksquare$$

T(a₁x₁ + ... + a_nx_n) = a₁T(x₁) + ... + a_nT(x_n) (P3(3))

💡 Let T be linear, and $a_1, \dots, a_n \in \mathbb{F}$ and $x_1, \dots, x_n \in V$ be arbitrary.

Then necessarily

$$T(a_1x_1 + \dots + a_nx_n) = a_1T(x_1) + \dots + a_nT(x_n).$$

{v₁, ..., v_n} IS A BASIS FOR V, **{w₁, ..., w_n}** ARE ELEMENTS FOR W $\Rightarrow \exists$ A UNIQUE LINEAR MAPPING

T: V \rightarrow W $\ni T(v_k) = w_k$ (T2.1)

💡 Let **{v₁, ..., v_n}** be a basis for a vector space V , and let **{w₁, ..., w_n}** be arbitrary elements of another vector space W .

Then there exists a unique linear mapping $T: V \rightarrow W$ such that

$$T(v_1) = w_1, \dots, T(v_n) = w_n.$$

Proof. Let $v \in V$ be arbitrary. Since **{v₁, v₂, ..., v_n}** is a basis for V , there must exist $a_1, \dots, a_n \in \mathbb{F}$ such that

$$v = a_1v_1 + \dots + a_nv_n. \quad (\text{by P3(3)})$$

$$\text{Let } T(v) = T(a_1v_1 + \dots + a_nv_n) = a_1w_1 + \dots + a_nw_n.$$

Then, by construction, for any $1 \leq k \leq n$, we have

$$\begin{aligned} T(v_k) &= T(a_1v_1 + \dots + a_nv_n + 0v_{k-1} + 0v_{k+1} + \dots + 0v_n) \\ &= a_1w_1 + \dots + a_kw_k + a_{k+1}w_{k+1} + \dots + a_nw_n \\ &= w_k. \end{aligned}$$

Proving uniqueness.

Next, suppose there exists another linear mapping $L: V \rightarrow W$ satisfying $L(v_i) = w_i, \dots, L(v_n) = w_n$.

Let $v = a_1v_1 + \dots + a_nv_n$, where $v \in V$ and $a_1, \dots, a_n \in \mathbb{F}$.

Then

$$\begin{aligned} L(v) &= L(a_1v_1 + \dots + a_nv_n) \\ &= a_1L(v_1) + \dots + a_nL(v_n) \\ &= a_1w_1 + \dots + a_nw_n \\ &= a_1T(v_1) + \dots + a_nT(v_n) \\ &= T(a_1v_1 + \dots + a_nv_n) \end{aligned}$$

$$\therefore L(v) = T(v).$$

Hence $L(v) = T(v) \quad \forall v \in V$, so that $T = L$, proving uniqueness. \blacksquare

💡 It also follows that we have

$$T(a_1v_1 + \dots + a_nv_n) = a_1w_1 + \dots + a_nw_n. \quad (C2.1.1)$$

NULL SPACE / KERNEL (D17(1))

Let V and W be vector spaces, and let $T: V \rightarrow W$ be linear. Then the "null space" of T , or the "kernel" of T , denoted as " $N(T)$ ", is defined to be the set

$$N(T) := \{x \in V \mid T(x) = 0\}.$$

RANGE / IMAGE (D17(2))

Let V and W be vector spaces, and let $T: V \rightarrow W$ be linear.

Then the "range" of T , or the "image" of T , denoted as " $R(T)$ ", is defined to be the set

$$R(T) := \{T(x) : x \in V\}.$$

$N(T)$ IS A SUBSPACE OF V (T2.2)

Let $T: V \rightarrow W$ be linear.

Then necessarily $N(T)$ is a subspace of V .

$R(T)$ IS A SUBSPACE OF W (T2.2)

Let $T: V \rightarrow W$ be linear.

Then necessarily $R(T)$ is a subspace of W .

$\{v_1, \dots, v_n\}$ IS A BASIS FOR $V \Rightarrow$

$$\text{span}(\{T(v_1), \dots, T(v_n)\}) = R(T) \quad (\text{T2.3})$$

Let V and W be vector spaces, and let $T: V \rightarrow W$ be linear. Suppose the set $\{v_1, \dots, v_n\}$ is a basis for V . Then necessarily $\{T(v_1), \dots, T(v_n)\}$ generates $R(T)$.

NULLITY (D18)

Let $T: V \rightarrow W$ be linear, and suppose that $\dim(N(T)) < \infty$.

Then, we define the "nullity" of T , denoted by "nullity(T)", to be equal to

$$\text{nullity}(T) = \dim(N(T)).$$

RANK (D18)

Let $T: V \rightarrow W$ be linear, and suppose that $\dim(R(T)) < \infty$.

Then, we define the "rank" of T , denoted by "rank(T)", to be equal to

$$\text{rank}(T) = \dim(R(T)).$$

RANK-NULLITY THEOREM (T2.4)

Let V and W be vector spaces, and let $T: V \rightarrow W$ be linear with

$\dim(V) < \infty$.

Then necessarily

$$\text{rank}(T) + \text{nullity}(T) = \dim(V).$$

Proof. Since $N(T)$ is a subspace of V (T2.2) and $\dim V < \infty$, by C19.2 (6) necessarily $\text{nullity}(T) \leq \dim(V) < \infty$.

Then, let $\text{nullity}(T) = k$, and suppose that $\{v_1, \dots, v_k\}$ is a basis for $N(T)$.

We know that we can "extend" $\{v_1, \dots, v_k\}$ to get a basis for V , $\{v_1, \dots, v_n\}$, so let us do so.

Next, we claim $\{T(v_{k+1}), \dots, T(v_n)\}$ is a basis for $R(T)$.

First, we show $\{T(v_{k+1}), \dots, T(v_n)\}$ spans $R(T)$.

By T2.2, $R(T) = \text{span}(\{T(v_1), \dots, T(v_k), T(v_{k+1}), \dots, T(v_n)\})$.

Then, since $\{v_1, \dots, v_n\}$ is a basis for $N(T)$, necessarily

$$T(v_1) = \dots = T(v_k) = 0.$$

Hence,

$$R(T) = \text{span}(\{T(v_{k+1}), \dots, T(v_n)\}),$$

as needed.

Next, we show $\{T(v_{k+1}), \dots, T(v_n)\}$ is linearly independent. Consider

$$c_{k+1}T(v_{k+1}) + \dots + c_nT(v_n) = 0, \quad \text{where } c_{k+1}, \dots, c_n \in \mathbb{C}$$

$$\Rightarrow T(c_{k+1}v_{k+1} + \dots + c_nv_n) = 0.$$

Hence $c_{k+1}v_{k+1} + \dots + c_nv_n \in N(T)$; then, since $\{v_1, \dots, v_n\}$ is a basis for $N(T)$, there exist $d_1, \dots, d_n \in \mathbb{C}$ such that

$$c_{k+1}v_{k+1} + \dots + c_nv_n = d_1v_1 + \dots + d_nv_n.$$

$$\Rightarrow -d_1v_1 - \dots - d_nv_n + c_{k+1}v_{k+1} + \dots + c_nv_n = 0.$$

Since $\{v_1, \dots, v_n\}$ is a basis for V , consequently

$$d_1 = \dots = d_n = c_{k+1} = \dots = c_n = 0,$$

showing $\{T(v_{k+1}), \dots, T(v_n)\}$ is linearly independent.

Consequently,

$$\text{rank}(T) + \text{nullity}(T) = \dim(R(T)) + \dim(N(T))$$

$$= k + (n - (k+1) + 1)$$

$$= n$$

$$\therefore \text{rank}(T) + \text{nullity}(T) = \dim(V). \quad \blacksquare$$

ONE-TO-ONE (1-1) (D19)

Let $T: V \rightarrow W$ be linear.

Then, we say T is "one-to-one" if, for any $x, y \in V$, $T(x) = T(y)$ implies $x = y$.

ONTO (D19)

Let $T: V \rightarrow W$ be linear.

Then, we say T is "onto" if

$$R(T) = W.$$

ISOMORPHISM (D19)

Let $T: V \rightarrow W$ be linear.

Then, we say T is an "isomorphism" if it is both one-to-one and onto.

We say V is "isomorphic" to W if

an isomorphism $T: V \rightarrow W$ exists, (D20)

and denote this by the notation

$$V \cong W.$$

T IS 1-1 (\Leftrightarrow) $N(T) = \{0\}$ (L3)

Let $T: V \rightarrow W$ be linear.

Then T is one-to-one if and only if $N(T) = \{0\}$.

Proof. (\Rightarrow) Suppose T is one-to-one.

Let $x \in V$ be such that $T(x) = 0$.

Then since $T(0) = 0 = T(x)$, by definition

$x = 0$, so that $N(T) = \{0\}$.

(\Leftarrow) Suppose $N(T) = \{0\}$. Consider $x, y \in V$ such that $T(x) = T(y)$.

$$T(x-y) = T(x) - T(y) = 0,$$

so that $x-y \in N(T)$,

hence $x-y = 0$, so that $x=y$ (and hence

T is 1-1). \square

$\{v_1, \dots, v_n\}$ IS A BASIS FOR $V \Rightarrow$

T IS ISOMORPHIC (\Leftrightarrow) $\{T(v_1), \dots, T(v_n)\}$ IS A BASIS FOR W (T2.5)

Let V and W be vector spaces over a field \mathbb{F} , with $\dim V < \infty$.

Let $\{v_1, \dots, v_n\}$ be a basis for V , and let $T: V \rightarrow W$ be linear.

Then T is an isomorphism if and only if $\{T(v_1), \dots, T(v_n)\}$ is a basis for W .

Proof. (\Rightarrow) Consider

$$c_1 T(v_1) + \dots + c_n T(v_n) = 0$$

$$\Rightarrow T(c_1 v_1 + \dots + c_n v_n) = 0.$$

Since T is one-to-one by definition, hence

$$c_1 v_1 + \dots + c_n v_n = 0,$$

and as $\{v_1, \dots, v_n\}$ is a basis for V ,

necessarily $c_1 = \dots = c_n = 0$;

hence $\{T(v_1), \dots, T(v_n)\}$ is a basis for W . $\#$

(\Leftarrow) If $\{T(v_1), \dots, T(v_n)\}$ is a basis for W , by definition $\{T(v_1), \dots, T(v_n)\}$ generates W .

$$\text{Thus } W = \text{span}(\{T(v_1), \dots, T(v_n)\}) = R(T).$$

where the second equality comes from T2.3

Then, since $W = R(T)$, T is necessarily onto.

Then, since $\{v_1, \dots, v_n\}$ is a basis for V , let $x \in N(T)$. Since $\{v_1, \dots, v_n\}$ is a basis for V ,

there must exist some $a_1, \dots, a_n \in \mathbb{F}$ such that

$$x = a_1 v_1 + \dots + a_n v_n.$$

Hence

$$0 = T(x) = a_1 T(v_1) + \dots + a_n T(v_n).$$

Since $\{T(v_1), \dots, T(v_n)\}$ is a basis for W by assumption,

thus $\{T(v_1), \dots, T(v_n)\}$ is linearly independent, so that

$$a_1 = \dots = a_n = 0,$$

and so

$$x = 0v_1 + \dots + 0v_n = 0.$$

Consequently $N(T) = \{0\}$, so that (by L3) T is 1-1. \square

CONSTRUCTING AN ISOMORPHISM FROM V TO W

Let V and W be vector spaces.

Then, we can construct an isomorphism from V to W as follows:

① Choose a basis $\{v_1, \dots, v_n\}$ for V , and a basis $\{w_1, \dots, w_m\}$ for W .

② Let the linear transformation $T: V \rightarrow W$ be such that $T(v_k) = w_k \quad \forall k \in \{1, 2, \dots, n\}$.

(T exists; this follows from T2.1)

③ Then, by T2.5, T is also an isomorphism.

$V \cong W \Leftrightarrow \dim V = \dim W$ (T2.6)

Let V and W be two finite-dimensional vector spaces over a field \mathbb{F} .

Then V is isomorphic to W if and only if $\dim V = \dim W$.

$\dim V = \dim W < \infty$; T IS 1-1 (\Leftrightarrow

T IS ONTO (\Leftrightarrow) $\text{rank}(T) = \dim(V)$ (T2.7)

Let V and W be two vector spaces over a field \mathbb{F} , and assume $\dim V = \dim W < \infty$.

Let $T: V \rightarrow W$ be linear.

Then the following are equivalent to one another:

① T is one-to-one;

② T is onto; and

③ $\text{rank}(T) = \dim(V)$.

SET OF ALL LINEAR TRANSFORMATIONS (D21)

Let V and W be vector spaces over \mathbb{F} .

Then, we let $\mathcal{L}(V, W) \subseteq W^V$ denote the set of all linear transformations $T: V \rightarrow W$.

$\mathcal{L}(V, W)$ IS A SUBSPACE OF W^V (T2.8)

Let V and W be vector spaces over some field \mathbb{F} .

Then necessarily $\mathcal{L}(V, W)$ is a subspace of W^V .

Proof. Clearly $\mathcal{L}(V, W) \subseteq W^V$, so we only need

to show that it is non-empty and is closed under the addition & scalar multiplication operations of W^V .

Also note the zero transformation $T_0: V \rightarrow W$ is in $\mathcal{L}(V, W)$, so that $\mathcal{L}(V, W)$ is non-empty.

Next, assume $T, U \in \mathcal{L}(V, W)$. Note that for any $x, y \in V$ & $c \in \mathbb{F}$:

$$\begin{aligned} (T+U)(cx+cy) &= T(cx+cy) + U(cx+cy) \\ &= cT(x) + T(y) + cU(x) + U(y) \\ &= c(T+U)(x) + (T+U)(y). \end{aligned}$$

showing $T+U$ is linear (by P2), so that $T+U \in \mathcal{L}(V, W)$

A similar argument shows $cT \in \mathcal{L}(V, W)$ as well $\forall c \in \mathbb{F}$.

Thus $\mathcal{L}(V, W)$ is a subspace of W^V , and we are done. \square

MORE ON MATRICES

TRANSPOSITION OF A MATRIX

Let $A \in M_{m \times n}(\mathbb{F})$ be arbitrary, and write

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}.$$

Then the "transposition" of A , denoted as " A^T " (or " A^t "), is defined to be the matrix

$$A^T = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{pmatrix} \in M_{n \times m}(\mathbb{F}).$$

MATRIX VECTOR MULTIPLICATION (D22)

Let $A \in M_{m \times n}(\mathbb{F})$ and $x \in \mathbb{F}^n$ be arbitrary, where \mathbb{F} is some field.

We define " Ax " to be equal to

$$Ax = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^n a_{1k} x_k \\ \sum_{k=1}^n a_{2k} x_k \\ \vdots \\ \sum_{k=1}^n a_{mk} x_k \end{pmatrix};$$

i.e. the i^{th} entry of Ax is obtained by multiplying the entries in the i^{th} row of A by the entries of x , and then summing up the resultant products.

$$L_A(x) = Ax \quad (\text{D23})$$

Let \mathbb{F} be a field, and let $A \in M_{m \times n}(\mathbb{F})$ be arbitrary.

Then, we let the function $L_A : \mathbb{F}^n \rightarrow \mathbb{F}^m$ be defined by $L_A(x) = Ax \quad \forall x \in \mathbb{F}^n$.

" a_j " MATRIX NOTATION

Let $A \in M_{m \times n}(\mathbb{F})$, and write

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}.$$

Then, we use the notation " a_j " to denote

$$a_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix},$$

and we can also write A as

$$A = (a_1 \ a_2 \ \dots \ a_n).$$

$$Ax = x_1 a_1 + x_2 a_2 + \dots + x_n a_n \quad (\text{L4(1)})$$

Let $A \in M_{m \times n}(\mathbb{F})$ be arbitrary, and write

$$A = (a_1 \ a_2 \ \dots \ a_n).$$

Then for any $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{F}^n$, we have

$$Ax = x_1 a_1 + x_2 a_2 + \dots + x_n a_n.$$

$$a_j = Ae_j \quad (\text{L4(2)})$$

Let $A \in M_{m \times n}(\mathbb{F})$ be arbitrary, and write

$$A = (a_1 \ a_2 \ \dots \ a_n).$$

Suppose $\{e_1, e_2, \dots, e_n\}$ are the standard basis vectors for \mathbb{F}^n .

Then necessarily $Ae_j = a_j$.

MATRIX EQUALITY THEOREM (C2.8.1)

Let $A, B \in M_{m \times n}(\mathbb{F})$ be arbitrary.

Then $A=B$ if and only if $Ax=Bx \quad \forall x \in \mathbb{F}^n$.

Proof: (\Rightarrow) is obvious.

(\Leftarrow) Suppose $Ax=Bx \quad \forall x \in \mathbb{F}^n$.

This implies $Ae_j = Be_j \quad \forall j \in \{1, \dots, n\}$, which tells us (by L4(2)) that $a_j = b_j \quad \forall j \in \{1, \dots, n\}$.

It follows that $A=B$, as needed. \blacksquare

L_A IS A LINEAR TRANSFORMATION (T2.9)

Let $A \in M_{m \times n}(\mathbb{F})$ be arbitrary.

Then $L_A : \mathbb{F}^n \rightarrow \mathbb{F}^m$ is necessarily a linear transformation.

Proof: We prove $L_A(cx+ty) = cL_A(x) + tL_A(y) \quad \forall x, y \in \mathbb{F}^n \& c, t \in \mathbb{F}$; the result follows from P2.

Write $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$, and

$$A = (a_1 \ a_2 \ \dots \ a_n).$$

$$\begin{aligned} L_A(cx+ty) &= A(cx+ty) \\ &= (cx_1+y_1)a_1 + (cx_2+y_2)a_2 + \dots + (cx_n+y_n)a_n \quad (\text{by L4(1)}) \\ &= c(x_1a_1 + \dots + x_na_n) + (y_1a_1 + \dots + y_na_n) \\ &= c(Ax) + Ay \\ \therefore L_A(cx+ty) &= cL_A(x) + L_A(y), \quad \text{as needed. } \blacksquare \end{aligned}$$

$$L : M_{m \times n}(\mathbb{F}) \rightarrow \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m) \quad \text{BY } L(A) = L_A \quad \text{IS}$$

A 1-1 LINEAR TRANSFORMATION (P4)

Let $L : M_{m \times n}(\mathbb{F}) \rightarrow \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ by $L(A) = L_A \quad \forall A \in M_{m \times n}(\mathbb{F})$, where \mathbb{F} is a field.

Then L is necessarily a one-to-one linear transformation.

Proof: We first show L is linear.

By P2, we just need to show $L(cA+B) = cL(A) + L(B)$ $\forall A, B \in M_{m \times n}(\mathbb{F})$, i.e. $L_{cA+B} = cL_A + L_B$.

To do this, let $x \in \mathbb{F}^n$ be arbitrary.

Write $A = (a_1 \ a_2 \ \dots \ a_n)$ and $B = (b_1 \ b_2 \ \dots \ b_n)$, so that $cA+B = (ca_1+b_1 \ ca_2+b_2 \ \dots \ ca_n+b_n)$.

So

$$\begin{aligned} L_{cA+B}(x) &= (ca_1+b_1)x \\ &= x_1(ca_1+b_1) + x_2(ca_2+b_2) + \dots + x_n(ca_n+b_n) \quad (\text{by L4(1)}) \\ &= c(x_1a_1 + \dots + x_na_n) + (x_1b_1 + \dots + x_nb_n) \\ &= c(Ax) + Bx \\ &= cL_A(x) + L_B(x) \\ \therefore L_{cA+B}(x) &= (cL_A + L_B)(x), \end{aligned}$$

and since $x \in \mathbb{F}^n$ was arbitrary this is sufficient to prove $L_{cA+B} = cL_A + L_B$, as needed. \blacksquare

Next, we prove L is 1-1.

Assume for some $A, B \in M_{m \times n}(\mathbb{F})$, we have $L_A = L_B$.

This means $L_A(x) = L_B(x) \quad \forall x \in \mathbb{F}^n$, or $Ax = Bx \quad \forall x \in \mathbb{F}^n$.

So by the Matrix Equality Theorem, $A=B$, which is sufficient to prove L is 1-1. \blacksquare

COORDINATES (S2.2)

ORDERED BASIS (D24)

Let V be a vector space with $\dim V < \infty$.

Then, an "ordered basis" for V is a

basis $\{v_1, \dots, v_n\}$ with a total order.

e.g. $\{e_1, e_2, e_3\}$ is the standard ordered basis for \mathbb{R}^3 , since we can define a "total order" by saying the indexes must be in "increasing order" (E30(c))

COORDINATE VECTOR (D25)

Let $\beta = \{u_1, \dots, u_n\}$ be an "ordered basis"

for a finite-dimensional vector space V .

By T1.6, we can write any $x \in V$ in the form $x = \sum_{k=1}^n a_k u_k$, where $a_1, \dots, a_n \in \mathbb{F}$.

Then, we define the "coordinate vector" of x relative to β , denoted as " $[x]_\beta$ ", to be

$$[x]_\beta := \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{F}^n.$$

e.g. for $V = P_2(\mathbb{R})$, $\beta = \{1, x, x^2\}$, $p(x) = 2 - 3x + 4x^2 \in V$,

$$[p(x)]_\beta = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}.$$

$[\]_\beta : V \rightarrow \mathbb{F}^n$ IS AN ISOMORPHISM (T2.10)

Let V be a vector space over some field \mathbb{F} ,

with $\dim V = n$, and let β be an ordered

basis for V .

Then, the map $[\]_\beta : V \rightarrow \mathbb{F}^n$ is an isomorphism.

MATRIX REPRESENTATION OF LINEAR TRANSFORMATIONS (S2.3)

Let V and W be finite-dimensional vector spaces over \mathbb{F} , and let $T: V \rightarrow W$ be a linear transformation.

Let $\beta = \{v_1, \dots, v_n\}$ be an ordered basis for V , and let $\gamma = \{w_1, \dots, w_m\}$ be an ordered basis for W . Then, the "matrix representation" of T in the ordered bases β and γ , denoted as $[T]_{\beta}^{\gamma}$, is defined as the matrix

$$[T]_{\beta}^{\gamma} := ([T(v_1)]_{\gamma} \ [T(v_2)]_{\gamma} \ \dots \ [T(v_n)]_{\gamma}).$$

In particular, if $T: V \rightarrow V$ is linear and β is an ordered basis of the finite-dimensional vector space V , we denote

$$[T]_{\beta} := [T]_{\beta}^{\beta}.$$

Note that $[T]_{\beta}^{\gamma} \in M_{m \times n}(\mathbb{F})$, where $m = \dim W$ and $n = \dim V$. (R12(1))

Also, we have

$$T(v_j) = \sum_{k=1}^n a_{kj} w_k,$$

where a_{kj} denotes the element at the k^{th} row and j^{th} column in the matrix $[T]_{\beta}^{\gamma}$. (R12(2))

eg If $T: P_2(\mathbb{R}) \rightarrow \mathbb{R}^2$ by $T(a+bx+cx^2) = \begin{pmatrix} a \\ b+4c \end{pmatrix}$, we can verify T is linear.

Let $\beta = \{1, (x+1), (x+1)^2\}$ and $\gamma = \{(1), (-1)\}$.

Then

$$\begin{aligned} [T]_{\beta}^{\gamma} &= ([T(1)]_{\gamma} \ [T(x+1)]_{\gamma} \ [T(x+1)^2]_{\gamma}) \\ &= \begin{pmatrix} 1 & 1 & 1 \\ 0+4(0) & 1+4(0) & 2+4(0) \end{pmatrix} \end{aligned}$$

$$\therefore [T]_{\beta}^{\gamma} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 6 \end{pmatrix}. \quad (\text{E32})$$

$$[L_A]_{\beta}^{\gamma} = A \quad (\text{E33})$$

Let $A \in M_{m \times n}(\mathbb{F})$, where \mathbb{F} is a field.

Let β be the standard ordered basis for \mathbb{F}^n , and γ the standard ordered basis for \mathbb{F}^m .

Then necessarily $[L_A]_{\beta}^{\gamma} = A$.

$$[T(x)]_{\gamma} = [T]_{\beta}^{\gamma} \cdot [x]_{\beta} \quad (\text{T2.11})$$

Let $T: V \rightarrow W$ be linear, and let $\beta = \{v_1, \dots, v_n\}$ and $\gamma = \{w_1, \dots, w_m\}$ be ordered bases of V and W respectively.

Then necessarily $[T(x)]_{\gamma} = [T]_{\beta}^{\gamma} \cdot [x]_{\beta} \quad \forall x \in V$.

Proof. Let $x \in V$ be arbitrary. Take $x = \sum_{k=1}^n a_k v_k$, where

$$[x]_{\beta} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}.$$

Then, since T is linear,

$$T(x) = T\left(\sum_{k=1}^n a_k v_k\right) = \sum_{k=1}^n a_k T(v_k).$$

Thus

$$[T(x)]_{\gamma} = \left[\sum_{k=1}^n a_k T(v_k) \right]_{\gamma} = \sum_{k=1}^n a_k [T(v_k)]_{\gamma}. \quad (\text{by linearity of } []_{\gamma})$$

Note that

$$\begin{aligned} \sum_{k=1}^n a_k [T(v_k)]_{\gamma} &= ([T(v_1)]_{\gamma} \ [T(v_2)]_{\gamma} \ \dots \ [T(v_n)]_{\gamma}) \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \\ &= [T]_{\beta}^{\gamma} \cdot [x]_{\beta}, \end{aligned}$$

so that

$$[T(x)]_{\gamma} = [T]_{\beta}^{\gamma} \cdot [x]_{\beta}, \quad \text{as needed.} \quad \blacksquare$$

TRANSFORMATIONS (S2.3)

$[]_{\beta}^{\gamma}: \mathcal{L}(V, W) \rightarrow M_{m \times n}(\mathbb{F})$ IS AN ISOMORPHISM (P5)

Let V and W be finite-dimensional vector spaces over \mathbb{F} , and let β and γ be ordered bases of V and W respectively.

Then the map $[]_{\beta}^{\gamma}: \mathcal{L}(V, W) \rightarrow M_{m \times n}(\mathbb{F})$ is an isomorphism, where $m = \dim W$ and $n = \dim V$; in other words,

① For any $T, U \in \mathcal{L}(V, W)$ and $c \in \mathbb{F}$, we have that

$$[cT + U]_{\beta}^{\gamma} = c[T]_{\beta}^{\gamma} + [U]_{\beta}^{\gamma}; \quad \text{and}$$

② For any $C \in M_{m \times n}(\mathbb{F})$, there exists a unique $T \in \mathcal{L}(V, W)$ such that $[T]_{\beta}^{\gamma} = C$.

Proof. We first prove ①.

Let $\beta = \{v_1, \dots, v_n\}$. Then

$$\begin{aligned} [T+U]_{\beta}^{\gamma} &= ([T(v_1)]_{\gamma} \ [T(v_2)]_{\gamma} \ \dots \ [T(v_n)]_{\gamma}) \\ &= ([T(v_1) + U(v_1)]_{\gamma} \ [T(v_2) + U(v_2)]_{\gamma} \ \dots \ [T(v_n) + U(v_n)]_{\gamma}) \\ &= (([T(v_1)]_{\gamma} + [U(v_1)]_{\gamma}) \ ([T(v_2)]_{\gamma} + [U(v_2)]_{\gamma}) \ \dots \ ([T(v_n)]_{\gamma} + [U(v_n)]_{\gamma})) \\ &= ([T(v_1)]_{\gamma} \ [T(v_2)]_{\gamma} \ \dots \ [T(v_n)]_{\gamma}) + ([U(v_1)]_{\gamma} \ [U(v_2)]_{\gamma} \ \dots \ [U(v_n)]_{\gamma}) \\ \therefore [T+U]_{\beta}^{\gamma} &= [T]_{\beta}^{\gamma} + [U]_{\beta}^{\gamma}, \end{aligned}$$

and a similar proof shows $[cT]_{\beta}^{\gamma} = c[T]_{\beta}^{\gamma}$, which is sufficient to show $[cT+U]_{\beta}^{\gamma} = c[T]_{\beta}^{\gamma} + [U]_{\beta}^{\gamma}$, and hence that the map $[]_{\beta}^{\gamma}: \mathcal{L}(V, W) \rightarrow M_{m \times n}(\mathbb{F})$ is linear. *

We next prove ②.

Suppose $[T]_{\beta}^{\gamma} = [U]_{\beta}^{\gamma}$, so that $[T]_{\beta}^{\gamma}$ and $[U]_{\beta}^{\gamma}$ have the same j^{th} column $\forall j \in \{1, \dots, n\}$.

This means $[T(v_j)]_{\gamma} = [U(v_j)]_{\gamma}$, and since $[]_{\gamma}: W \rightarrow \mathbb{F}^n$ is a bijection (by T2.10) it follows that $T(v_j) = U(v_j) \quad \forall j \in \{1, \dots, n\}$. So, by T2.1, $T = U$, proving injectivity.

Then, let $C = (c_1 \ c_2 \ \dots \ c_n) \in M_{m \times n}(\mathbb{F})$ be arbitrary.

For each $j \in \{1, \dots, n\}$, let $w_j \in W$ be the unique vector satisfying $[w_j]_{\gamma} = c_j$.

By T2.10, there exists a unique linear transformation $T: V \rightarrow W$ satisfying $T(v_j) = w_j \quad \forall j \in \{1, \dots, n\}$.

It follows this T satisfies $[T]_{\beta}^{\gamma} = C$, proving surjectivity. So we are done. \blacksquare

$L: M_{m \times n}(\mathbb{F}) \rightarrow \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ IS AN ISOMORPHISM (C2.1.1)

Recall that the map $L: M_{m \times n}(\mathbb{F}) \rightarrow \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ is defined by $L(A) = L_A \quad \forall A \in M_{m \times n}(\mathbb{F})$.

Then, L is necessarily an isomorphism.

Proof. We know L is already 1-1 & linear by P4, so we only need to prove it is onto.

Applying P5 to $V = \mathbb{F}^n$ & $W = \mathbb{F}^m$, we get that $\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m) \cong M_{m \times n}(\mathbb{F})$, so that

$$\dim(\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)) = \dim(M_{m \times n}(\mathbb{F})) = mn \quad (\text{by T2.6}).$$

So L is a 1-1 linear transformation between vector spaces of the same finite dimension; it follows by T2.7 that L is onto. \blacksquare

MATRIX MULTIPLICATION & COMPOSITIONS OF LINEAR TRANSFORMATIONS (S2.4)

MATRIX PRODUCT (D27)

Let \mathbb{F} be a field, and let $A \in M_{m \times n}(\mathbb{F})$ and $B \in M_{n \times p}(\mathbb{F})$ be arbitrary. Note the number of columns in A equals the number of rows in B ; this is required. Then, the matrix product of A and B , denoted by AB , is defined to be the $m \times p$ matrix

$$AB = \left(\begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{array} \right) \left(\begin{array}{cccc} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{array} \right) = \left(\begin{array}{cccc} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mp} \end{array} \right),$$

where $c_{ij} = a_{11}b_{1j} + a_{12}b_{2j} + \cdots + a_{1n}b_{nj} = \sum_{t=1}^n a_{it}b_{tj}$.

In other words, c_{ij} is the sum of products formed multiplying the entries in the i^{th} row of A with the j^{th} column of B .

*An example is highlighted in blue;

$$c_{21} = a_{21}b_{11} + a_{22}b_{21} + \cdots + a_{2n}b_{n1}.$$

Note that c_{ij} is the linear combination of the columns of A formed using the entries in the j^{th} column of B as coefficients. (R1B(3))

ZERO MATRIX

The "zero matrix", denoted by the letter O , is defined to be the matrix with each entry being zero.

We write " O_{mn} " to denote the $m \times n$ zero matrix.

IDENTITY MATRIX

The "nn identity matrix", denoted as I_n , is defined as the matrix (δ_{ij}) with

$$\delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i \neq j. \end{cases} \quad * \delta_{ij} \text{ is known as the "Kronecker delta".}$$

$$\text{eg } I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

RIGHT MATRIX DISTRIBUTIVE LAW (LS(1))

For any $A \in M_{m \times n}(\mathbb{F})$ and $B, C \in M_{n \times p}(\mathbb{F})$, we have

$$A(B+C) = AB + AC.$$

LEFT MATRIX DISTRIBUTIVE LAW (LS(2))

Similarly, for any $A \in M_{m \times n}(\mathbb{F})$ and $D, E \in M_{q \times m}(\mathbb{F})$, we have

$$(D+E)A = DA + EA.$$

ASSOCIATIVITY OF MATRIX SCALAR MULTIPLICATION (LS(3))

For any $\alpha \in \mathbb{F}$, $A \in M_{m \times n}(\mathbb{F})$ and $B \in M_{q \times m}(\mathbb{F})$, we have

$$\alpha(AB) = (\alpha A)B = A(\alpha B).$$

$$(AB)^T = B^T A^T \quad (\text{LS}(4))$$

For any $A \in M_{m \times n}(\mathbb{F})$ and $B \in M_{n \times n}(\mathbb{F})$, we have

$$(AB)^T = B^T A^T.$$

$$I_m A = AI_n \quad (\text{LS}(5))$$

For any $A \in M_{m \times n}(\mathbb{F})$, we have that $I_m A = AI_n = A$.

$$AO_{n \times p} = O_{m \times p}, \quad O_{q \times m} A = O_{q \times n} \quad (\text{LS}(6))$$

For any $A \in M_{m \times n}(\mathbb{F})$, we have

$$\textcircled{1} \quad AO_{n \times p} = O_{m \times p}; \quad \text{and}$$

$$\textcircled{2} \quad O_{q \times m} A = O_{q \times n}.$$

COMPOSITION OF LINEAR TRANSFORMATIONS IS ALSO A LINEAR TRANSFORMATION (T2.12)

Let $T: V \rightarrow W$ and $U: W \rightarrow Z$ be linear transformations.

Then the composition $(U \circ T): V \rightarrow Z$ is also a linear transformation.

*we usually denote $(U \circ T)$ as UT .

MATRIX OF COMPOSITION OF LINEAR TRANSFORMATIONS (T2.13)

Let V, W and Z be finite-dimensional vector spaces having ordered bases $\alpha = \{v_1, \dots, v_p\}$, $\beta = \{w_1, \dots, w_n\}$ and $\gamma = \{z_1, \dots, z_m\}$ respectively.

Let $T: V \rightarrow W$ and $U: W \rightarrow Z$ be linear transformations.

Denote $A = [U]_{\beta}^{\gamma} \in M_{m \times n}(\mathbb{F})$, $B = [T]_{\alpha}^{\beta} \in M_{n \times p}(\mathbb{F})$ and $C = [UT]_{\alpha}^{\gamma} \in M_{m \times p}(\mathbb{F})$.

Then necessarily $C = AB$; ie $[UT]_{\alpha}^{\gamma} = [U]_{\beta}^{\gamma} \cdot [T]_{\alpha}^{\beta}$.

Proof. Note that both sides are $m \times p$ matrices.

We show that the j^{th} columns of the LHS & RHS are equal $\forall j \in \{1, \dots, p\}$.

On one hand, the j^{th} column of $[UT]_{\alpha}^{\gamma}$ is $[(UT)(v_j)]_{\gamma}$.

On the other hand, $[T]_{\alpha}^{\beta} = B = (b_1, b_2, \dots, b_p)$.

Hence, the j^{th} column of $[U]_{\beta}^{\gamma} \cdot [T]_{\alpha}^{\beta}$ is $[U]_{\beta}^{\gamma} \cdot b_j$, which equals

$$\begin{aligned} [U]_{\beta}^{\gamma} \cdot b_j &= [U]_{\beta}^{\gamma} \cdot [T(v_j)]_{\beta} \\ &= [U(T(v_j))]_{\gamma} \quad (\text{by T2.11}) \\ &= [UT(v_j)]_{\gamma}. \end{aligned}$$

It follows that $[UT]_{\alpha}^{\gamma}$ and $[U]_{\beta}^{\gamma} \cdot [T]_{\alpha}^{\beta}$ have the same j^{th} columns; since j was arbitrary, it follows that

$$[UT]_{\alpha}^{\gamma} = [U]_{\beta}^{\gamma} \cdot [T]_{\alpha}^{\beta}, \quad \text{as needed.} \quad \blacksquare$$

$$L_{AB} = L_A L_B \quad (\text{C2.13.1 (1)})$$

Let $A \in M_{m \times n}(\mathbb{F})$ and $B = M_{n \times p}(\mathbb{F})$ be arbitrary.

Then necessarily $L_{AB} = L_A L_B$.

Proof. Let α, β, γ denote the standard ordered bases for \mathbb{F}^p , \mathbb{F}^n and \mathbb{F}^m respectively.

$$\text{By E33, } [L_A]_{\beta}^{\gamma} = A, \quad [L_B]_{\alpha}^{\beta} = B \quad \text{and} \quad [L_{AB}]_{\alpha}^{\gamma} = AB.$$

On the other hand

$$[L_A L_B]_{\alpha}^{\gamma} = [L_A]_{\beta}^{\gamma} \cdot [L_B]_{\alpha}^{\beta} = AB = [L_{AB}]_{\alpha}^{\gamma} \quad (\text{by T2.13}).$$

Since the mapping $[\cdot]_{\alpha}^{\gamma}$ is 1-1 (by C2.11.1), it follows that $L_A L_B = L_{AB}$, as needed. \blacksquare

$$A(BC) = (AB)C \quad (\text{C2.13.1 (2)})$$

Assume the matrix product "A(BC)" is defined.

Then necessarily $A(BC) = (AB)C$.

Proof. By C2.13.1(1), we get that

$$L_{A(BC)} = L_A L_{BC} = L_A(L_B L_C) = (L_A L_B)L_C = L_{AB}L_C = L_{(AB)}C,$$

since function composition is associative.

Then, as L is 1-1 (by P4), it follows that

$$A(BC) = (AB)C, \quad \text{as needed.} \quad \blacksquare$$