MATH 235
Personal Notes

Marcus Chan

UW '25

Taught by Nick Rollick

Reading 8: Examples of Matrix Representations, Introduction to Inner Product Spaces

INNER PRODUCT & INNER PRODUCT SPACES: $\langle v, w\rangle$ (D8.1)

1. Let V be a vector space over $\mathbb{F}$.
Then, we say the function $\langle \cdot,\cdot\rangle : V^2 \to \mathbb{F}$
is an "inner product" if

① $\langle v,v\rangle \in \mathbb{R}$ & $\langle v,v\rangle \geq 0$ $\forall v \in V$;
② $\langle v,v\rangle = 0 \iff v=0$ $\forall v\in V$; } Positivity
③ $\langle v_1+v_2, w\rangle = \langle v_1,w\rangle + \langle v_2,w\rangle$ $\forall v_1,v_2,w \in V$;
④ $\langle cv, w\rangle = c\langle v,w\rangle$ $\forall c\in\mathbb{F}$, $v,w\in V$; and } Linearity
⑤ $\langle w,v\rangle = \overline{\langle v,w\rangle}$ $\forall v,w\in V$. } Conjugate Symmetry

2. In this case, we call $\langle v,w\rangle$ the "inner product" of v & w.

3. We refer to V together with $\langle\cdot,\cdot\rangle$ as an "inner product space".

LENGTH [OF A VECTOR]: $\|v\|$ (D8.2)

Let V be an inner product space, and let $v\in V$.
Then, the "length" of v, denoted by "$\|v\|$",
is defined to be equal to
$$\|v\| = \sqrt{\langle v,v\rangle}.$$
We can do this because $\langle v,v\rangle \in \mathbb{R}$ $\forall v\in V$.

ORTHOGONAL [VECTORS] (D8.3)

Let V be an IPS.
Then, we say $v,w\in V$ are "orthogonal"
if $\langle v,w\rangle = 0$.

ORTHOGONAL [SETS] (D8.3)

Let $S \subseteq V$, where V is an IPS.
Then, we say S is "orthogonal" if
$\langle v,w\rangle = 0$ $\forall v,w\in S$.

EXAMPLES OF IPS: PART I

1. The vector space $V=\mathbb{F}^n$ with inner product
$$\left\langle \begin{pmatrix} v_1\\ \vdots \\ v_n\end{pmatrix}, \begin{pmatrix} w_1\\ \vdots \\ w_n\end{pmatrix}\right\rangle = v_1\overline{w_1} + \cdots + v_n\overline{w_n}$$
is an inner product space. (E8.3)

2. The vector space $V = P_n(\mathbb{F})$ with inner product
$$\langle p,q\rangle = p(0)\overline{q(0)} + \cdots + p(n)\overline{q(n)}$$
is an inner product space. (E8.4)

CONJUGATE MATRIX: $\overline{A}$ (D8.4)

Let $A=(a_{ij}) \in M_{m\times n}(\mathbb{F})$.
Then, the "conjugate" of A, denoted by "$\overline{A}$",
is equal to
$$\overline{A} = (\overline{a_{ij}}) \in M_{m\times n}(\mathbb{F}).$$

CONJUGATE TRANSPOSE MATRIX: $A^* = \overline{A^T}$ (D8.4)

Then, the "conjugate transpose" of A is defined to be the matrix
$$A^* = \overline{A^T} \in M_{n\times m}(\mathbb{F}).$$

STANDARD INNER PRODUCT ON $M_{m\times n}(\mathbb{F})$: $\langle A,B\rangle = tr(AB^*)$ (E8.5)

1. Let $V = M_{m\times n}(\mathbb{F})$.
Then, the "standard inner product" on V is given by
$$\langle A,B\rangle = tr(AB^*),$$
where $tr(A) = \sum_{i=1}^m a_{ii}$ for $A\in M_{m\times m}(\mathbb{F})$.

2. We can prove this is indeed an IPS.

Proof. Linearity is trivial (arises from fact that trace & matrix multiplication is linear).
Note that for $A=(a_{ij})$ & $B=(b_{ij})$, then $B^* = (\overline{b_{ji}})$.
Then
$$(AB^*)_{ii} = \sum_{k=1}^n a_{ik}(B^*)_{ki} = \sum_{k=1}^n a_{ik}\overline{b_{ik}}.$$
Hence
$$tr(AB^*) = \sum_{i=1}^m (AB^*)_{ii} = \sum_{i=1}^m\sum_{k=1}^n a_{ik}\overline{b_{ik}}.$$
In particular, this is "similar" to if we wrote the entries of A & B in $\mathbb{F}^{mn}$, and took the standard inner product of these vectors.
It trivially follows that this gives an inner product on V. □

INNER PRODUCT ON $C[a,b]$: $\langle f(x), g(x)\rangle = \int_a^b f(x)g(x)\,dx$ (E8.6)

We can show $C[a,b]$ with the function
$$\langle f(x), g(x)\rangle = \int_a^b f(x)g(x)\,dx$$
is an inner product space.

Proof. Linearity & conjugate symmetry (ie "normal" symmetry, since field $=\mathbb{R}$) follow pretty easily.
For positivity, note that
$$\langle f(x), f(x)\rangle = \int_a^b f(x)^2\,dx \geq \int_a^b 0\,dx = 0.$$
If $f\neq 0$, then trivially $\int_a^b f(x)^2\,dx > 0$ by EVT, completing the proof. ■

$T_w : V\to\mathbb{F}$ BY $T_w(v) = \langle v,w\rangle$ IS LINEAR (T8.2(1))

Let $w\in V$, and let $T_w : V\to\mathbb{F}$ by $T_w(v) = \langle v,w\rangle$ $\forall v\in V$.
Then necessarily T_w is linear.

SET OF VECTORS ORTHOGONAL TO w IS A SUBSPACE OF V (T8.2(2))

Let $w\in V$.
Then the set of vectors orthogonal to w is a subspace of V.

Proof. This follows from the fact that the set $=\ker(T_w)$. ■

$\|v\| \geq 0 \quad \forall v \in V, \qquad v=0 \iff \|v\|=0$ (T8·3(1))

Let V be an IPS.
Then necessarily $\|v\| \geq 0 \ \forall v \in V$, and $\|v\|=0$ if and only if $v=0$.

Proof. This arises from properties of inner products.

$\|cv\| = |c| \cdot \|v\|$ (T8·3(2))

Let V be an IPS, and let $c \in \mathbb{F}$.
Then necessarily $\|cv\| = |c| \cdot \|v\|$.

Proof. This also arises from properties of inner products.

$|\langle v,w\rangle| \leq \|v\| \cdot \|w\|$; $|\langle v,w\rangle| = \|v\|\|w\| \iff v$ & w ARE LINEARLY DEPENDENT (THE CAUCHY-SCHWARTZ INEQUALITY) (T8·3(3))

Let $v, w \in V$.
Then necessarily $|\langle v,w\rangle| \leq \|v\| \cdot \|w\|$, and equality holds iff v and w are linearly dependent.

Proof. We show $|\langle v,w\rangle|^2 \leq \|v\|^2 \cdot \|w\|^2$.
First, if $w=0$, the result is trivial. Otherwise, assume $w \neq 0$, and let $c = \frac{\langle v,w\rangle}{\|w\|^2}$.
By T8·3(1):

$$\begin{aligned}
0 &\leq \|v-cw\|^2 \\
&= \langle v-cw, v-cw\rangle \\
&= \langle v, v-cw\rangle - c\langle w, v-cw\rangle \\
&= \langle v,v\rangle - \bar{c}\langle v,w\rangle - c\langle w,v\rangle + c\bar{c}\langle w,w\rangle \\
&= \|v\|^2 - \bar{c}\langle v,w\rangle - c\overline{\langle v,w\rangle} + |c|^2\|w\|^2 \\
&= \|v\|^2 - \frac{\overline{\langle v,w\rangle}}{\|w\|^2}\langle v,w\rangle - \frac{\langle v,w\rangle}{\|w\|^2}\overline{\langle v,w\rangle} + \frac{|\langle v,w\rangle|^2}{\|w\|^4}\|w\|^2 \\
&= \|v\|^2 - \frac{|\langle v,w\rangle|^2}{\|w\|^2} - \frac{|\langle v,w\rangle|^2}{\|w\|^2} + \frac{|\langle v,w\rangle|^2}{\|w\|^2}
\end{aligned}$$

$\therefore \ 0 \leq \|v\|^2 - \frac{|\langle v,w\rangle|^2}{\|w\|^2}$,

and so $|\langle v,w\rangle|^2 \leq \|v\|^2\|w\|^2$, as needed.

Then, note that

$$\begin{aligned}
\|v-cw\| > 0 &\iff v-cw \neq 0 \quad \text{for some } c \in \mathbb{F} \\
&\iff v \neq cw \quad \text{''} \\
&\iff v \text{ \& } w \text{ are lin ind,}
\end{aligned}$$

and so $\|v-cw\|=0 \iff v$ & w are lin dep. ▨

$\|v+w\| \leq \|v\| + \|w\| \quad \forall v,w \in V$ (TRIANGLE INEQUALITY) (T8·3 (4))

Let $v,w \in V$.
Then necessarily $\|v+w\| \leq \|v\| + \|w\|$.

Proof. Note that

$$\begin{aligned}
\|v+w\|^2 &= \langle v+w, v+w\rangle \\
&= \langle v,v\rangle + \langle v,w\rangle + \langle w,v\rangle + \langle w,w\rangle \\
&= \|v\|^2 + \langle v,w\rangle + \overline{\langle v,w\rangle} + \|w\|^2 \\
&= \|v\|^2 + \|w\|^2 + 2\mathrm{Re}\langle v,w\rangle \\
&\leq \|v\|^2 + \|w\|^2 + 2\langle v,w\rangle \\
&\leq \|v\|^2 + 2\|v\|\|w\| + \|w\|^2 \quad \text{(by CSE)} \\
&= (\|v\| + \|w\|)^2,
\end{aligned}$$

and the proof follows. ▨

Reading 9: Orthogonal and Orthonormal Bases; The Gram-Schmidt Procedure

ORTHOGONAL & ORTHONORMAL BASIS (D9.1)

Let V be an IPS, and let $B \subseteq V$.
Then, we say B is an "orthogonal basis" for V if:

① B is a basis for V; and
② B is an orthogonal set of vectors.

We say B is an "orthonormal basis" for V if the above conditions are satisfied and $\|v\| = 1 \ \forall v \in B$.

$S \subseteq V$ IS ORTHOGONAL & HAS NO ZERO VECTORS $\Rightarrow$ S IS LINEARLY INDEPENDENT (T9.1)

Let V be an IPS, and let $S \subseteq V$ be orthogonal and have no zero vectors.
Then necessarily S is linearly independent.

Proof. Let $c_1, \ldots, c_n \in \mathbb{F}$, $v_1, \ldots, v_n \in S$ s.t

$$c_1 v_1 + \cdots + c_n v_n = 0.$$

Taking the inner product of each side with v_1, we see that

$$\begin{aligned} 0 &= \langle 0, v_1 \rangle \\ &= \langle c_1 v_1 + \cdots + c_n v_n, v_1 \rangle \\ &= c_1 \langle v_1, v_1 \rangle + \cdots + c_n \langle v_n, v_1 \rangle \\ &= c_1 \|v_1\|^2 + c_2(0) + \cdots + c_n(0) \quad \text{(since } S \text{ is orthogonal)} \end{aligned}$$

$\therefore \ 0 = c_1 \|v_1\|^2$,

and so since $v_1 \neq 0$ it follows that $c_1 = 0$.
Repeating this argument by taking inner product with $v_2, \ldots, v_n$ gives us that $c_1 = \cdots = c_n = 0$, showing that the vectors are linearly independent. ☒

V HAS ORTHOGONAL ORDERED BASIS $B = \{v_1, \ldots, v_n\} \Rightarrow w = \sum_{i=1}^{n} \frac{\langle w, v_i \rangle}{\|v_i\|^2} v_i$ (T9.2)

Let V be a finite-dimensional IPS, and let V have an orthogonal ordered basis $B = \{v_1, \ldots, v_n\}$.
Let $w \in V$ be arbitrary.
Then necessarily

$$w = \frac{\langle w, v_1 \rangle}{\|v_1\|^2} v_1 + \cdots + \frac{\langle w, v_n \rangle}{\|v_n\|^2} v_n.$$

In other words,

$$[w]_B = \begin{pmatrix} \frac{\langle w, v_1 \rangle}{\|v_1\|^2} \\ \vdots \\ \frac{\langle w, v_n \rangle}{\|v_n\|^2} \end{pmatrix}.$$

Proof. Since B is a basis, $\exists c_1, \ldots, c_n \ni$

$$w = c_1 v_1 + \cdots + c_n v_n.$$

Taking IP of both sides w/ v_1 yields that

$$\begin{aligned} \langle w, v_1 \rangle &= c_1 \langle v_1, v_1 \rangle + \cdots + c_n \langle v_n, v_1 \rangle \\ &= c_1 \|v_1\|^2 + c_2(0) + \cdots + c_n(0) \\ &= c_1 \|v_1\|^2, \end{aligned}$$

and doing similarly for $v_2, \ldots, v_n$ yields that

$$\langle w, v_i \rangle = c_i \|v_i\|^2 \quad \forall 1 \leq i \leq n.$$

Thus $c_i = \frac{\langle w, v_i \rangle}{\|v_i\|^2}$, which suffices to prove the claim. ☒

V HAS ORTHONORMAL ORDERED BASIS $B = \{v_1, \ldots, v_n\} \Rightarrow w = \sum_{i=1}^{n} \langle w, v_i \rangle v_i$ (C9.1)

Let V be a finite-dimensional IPS, and let V have an orthonormal ordered basis $B = \{v_1, \ldots, v_n\}$.
Let $w \in V$ be arbitrary.
Then necessarily

$$w = \langle w, v_1 \rangle v_1 + \cdots + \langle w, v_n \rangle v_n.$$

In other words,

$$[w]_B = \begin{pmatrix} \langle w, v_1 \rangle \\ \vdots \\ \langle w, v_n \rangle \end{pmatrix}$$

Proof. This follows almost immediately from T9.2.

$S=\{w_1,\dots,w_n\}$ IS LINEARLY INDEPENDENT; $v_i = v_i - \sum_{j=1}^{i-1} \frac{\langle w_i, v_j\rangle}{\|v_j\|^2} v_j \Rightarrow \{v_1,\dots,v_n\}$ IS ORTHOGONAL & $\{v_1,\dots,v_k\}$ IS AN ORTHOGONAL BASIS FOR $\operatorname{Span}\{w_1,\dots,w_k\}$ (THE GRAM-SCHMIDT PROCEDURE) (L9.1)

Let V be an IPS, and let $S=\{w_1,\dots,w_n\}\subseteq V$ be linearly independent.

Define $\{v_1,\dots,v_n\}$ recursively by $v_1 = w_1$, and

$$v_i = w_i - \frac{\langle w_i, v_1\rangle}{\|v_1\|^2} v_1 - \dots - \frac{\langle w_i, v_{i-1}\rangle}{\|v_{i-1}\|^2} v_{i-1} \quad \forall 1<i\leq n.$$

Then

① $\{v_1,\dots,v_n\}$ is orthogonal; and

② $\{v_1,\dots,v_k\}$ is an orthogonal basis for $\operatorname{Span}\{w_1,\dots,w_k\}$ for any $1\leq k\leq n$.

Proof. We prove this by induction.

(n=1) Since $w_1\neq 0$ (as S is lin ind), hence $\{v_1\}$ is orthogonal, and since $v_1=w_1$, so $\operatorname{span}\{v_1\}=\operatorname{span}\{w_1\}$, so the conclusions trivially follow.

(inductive) Suppose the claim is true for $1\leq k<n$. So $\{v_1,\dots,v_k\}$ is an orthogonal basis for $\operatorname{Span}\{w_1,\dots,w_k\}$. We want to show similarly $\{v_1,\dots,v_{k+1}\}$ is an orthogonal basis for $\operatorname{Span}\{w_1,\dots,w_{k+1}\}$.

Since we know $\{v_1,\dots,v_k\}$ is orthogonal, we just need to check v_{k+1} is orthogonal to each v_i to verify $\{v_1,\dots,v_{k+1}\}$ is orthogonal.

Observe that

$$\begin{aligned}
\langle v_{k+1}, v_i\rangle &= \left\langle w_{k+1} - \frac{\langle w_{k+1}, v_1\rangle}{\|v_1\|^2} v_1 - \dots - \frac{\langle w_{k+1}, v_k\rangle}{\|v_k\|^2} v_k, v_i\right\rangle \\
&= \langle w_{k+1}, v_i\rangle - \frac{\langle w_{k+1}, v_1\rangle}{\|v_1\|^2}\langle v_1, v_i\rangle \\
&\quad - \dots - \frac{\langle w_{k+1}, v_i\rangle}{\|v_i\|^2}\langle v_i, v_i\rangle \\
&\quad - \dots - \frac{\langle w_{k+1}, v_k\rangle}{\|v_k\|^2}\langle v_k, v_i\rangle \\
&= \langle w_{k+1}, v_i\rangle - 0 - \dots - \frac{\langle w_{k+1}, v_i\rangle}{\|v_i\|^2}\|v_i\|^2 - \dots - 0 \\
&= \langle w_{k+1}, v_i\rangle - \langle w_{k+1}, v_i\rangle \\
&= 0,
\end{aligned}$$

showing that v_{k+1} is orthogonal to each v_i, and so $\{v_1,\dots,v_{k+1}\}$ is orthogonal.

Next, we show $\operatorname{Span}\{v_1,\dots,v_{k+1}\} = \operatorname{Span}\{w_1,\dots,w_{k+1}\}$.

By hypothesis, $\operatorname{Span}\{v_1,\dots,v_k\} = \operatorname{Span}\{w_1,\dots,w_k\}$, and since

$$v_{k+1} = w_{k+1} - \frac{\langle w_{k+1}, v_1\rangle}{\|v_1\|^2} v_1 - \dots - \frac{\langle w_{k+1}, v_k\rangle}{\|v_k\|^2} v_k$$

shows that v_{k+1} is a lin comb of $v_1,\dots,v_k,w_{k+1}$. Since this is also trivially true for $v_1,\dots,v_k$ as well, thus any lin comb of $v_1,\dots,v_{k+1}$ is a lin comb of $v_1,\dots,v_k,w_{k+1}$, and so $\operatorname{Span}\{v_1,\dots,v_{k+1}\}\subseteq \operatorname{Span}\{v_1,\dots,v_k,w_{k+1}\}$.

Then, $\operatorname{Span}\{v_1,\dots,v_k\} = \operatorname{Span}\{w_1,\dots,w_k\} \Rightarrow$ $\operatorname{Span}\{v_1,\dots,v_k,w_{k+1}\} = \operatorname{Span}\{w_1,\dots,w_k,w_{k+1}\}$, and so $\operatorname{Span}\{v_1,\dots,v_{k+1}\}\subseteq \operatorname{Span}\{w_1,\dots,w_{k+1}\}$.

Conversely, for $1\leq i\leq k+1$, since w_i is a lin comb of $v_1,\dots,v_i$, hence any lin comb of $w_1,\dots,w_{k+1}$ is also a lin comb of $v_1,\dots,v_{k+1}$.

So $\operatorname{span}\{w_1,\dots,w_{k+1}\}\subseteq \operatorname{span}\{v_1,\dots,v_{k+1}\}$, and so $\operatorname{Span}\{w_1,\dots,w_{k+1}\} = \operatorname{Span}\{v_1,\dots,v_{k+1}\}$.

Since $\{w_1,\dots,w_{k+1}\}$ is lin ind, it follows $\{v_1,\dots,v_{k+1}\}$ is also lin ind, and so $\{v_1,\dots,v_{k+1}\}$ is an ortho basis for $\operatorname{Span}\{w_1,\dots,w_{k+1}\}$ completing the inductive step.

$\dim V<\infty \Rightarrow V$ HAS AN ORTHOGONAL BASIS (T9.3)

Let V be a finite-dimensional IPS. Then necessarily V has an orthogonal basis.

Proof. Since $\dim V<\infty$, V has a finite basis, say $\{w_1,\dots,w_n\}$. Then, applying L9.1 to $\{w_1,\dots,w_n\}$ yields an orthogonal set $\{v_1,\dots,v_n\}$, for which $\{v_1,\dots,v_n\}$ is an orthogonal basis for $\operatorname{Span}\{w_1,\dots,w_n\} = V$. ☒

$\dim V<\infty \Rightarrow V$ HAS AN ORTHONORMAL BASIS (C9.2)

Let V be a finite-dimensional IPS. Then necessarily V has an orthonormal basis.

Proof. This follows by taking the basis obtained in T9.3 and scaling each vector down by its respective norm. ☒