

MATH 148

Personal Notes

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Chapter 1:

The Riemann Integral

CODE KEY

D :	definition
N :	note
R :	remark
L :	lemma
E :	example
C :	corollary
T :	theorem
NT :	notation

PARTITION OF $[a, b]$

(DI.1)

\exists : A "partition" of the closed interval $[a, b]$ is any set $X = \{x_0, x_1, \dots, x_n\}$ such that $a = x_0 < x_1 < \dots < x_n = b$.

SUBINTERVAL (DI.1)

\exists : Let $X = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$.

Then a "sub-interval" of $[a, b]$ is any interval of the form $[x_{k-1}, x_k]$, where $k \in \{1, 2, \dots, n\}$, and denote them by

$$\Delta_k x = x_k - x_{k-1}.$$

\exists : Note that

$$\Delta_1 x + \Delta_2 x + \dots + \Delta_n x = \sum_{k=1}^n \Delta_k x = b - a.$$

SIZE (DI.1)

\exists : Let X be a partition of $[a, b]$.

Then the "size" of X , denoted as $|X|$, is defined to be

$$|X| = \max(\{\Delta_k x \mid 1 \leq k \leq n\}).$$

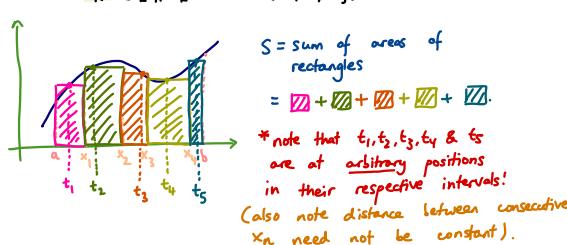
RIEMANN SUM (DI.2)

\exists : Let X be a partition of $[a, b]$, and let $f: [a, b] \rightarrow \mathbb{R}$ be bounded.

Then, a "Riemann sum" for f on X is a sum of the form

$$S = \sum_{k=1}^n f(t_k) \Delta_k x,$$

where $t_k \in [x_{k-1}, x_k] \quad \forall k \in \{1, 2, \dots, n\}$.



SAMPLE POINTS (DI.2)

\exists : Let $S = \sum_{k=1}^n f(t_k) \Delta_k x$ be a Riemann sum for some bounded function $f: [a, b] \rightarrow \mathbb{R}$ on a partition X of $[a, b]$.

Then we say t_k is a "sample point" of S for any $k \in \{1, 2, \dots, n\}$.

RIEMANN INTEGRAL

(RIEMANN) INTEGRABLE (DI.3)

\exists : Let $f: [a, b] \rightarrow \mathbb{R}$ be bounded.

Then, we say f is "Riemann integrable", or just "integrable", on $[a, b]$ if there exists an $I \in \mathbb{R}$ such that for any $\epsilon > 0$, there exists a $\delta > 0$ such that for any partition X of $[a, b]$ with $|X| < \delta$, we have that

$$|S - I| < \epsilon$$

for any Riemann sum S for f on X .

\exists : In other words, we have that

$$\left| \sum_{k=1}^n f(t_k) \Delta_k x - I \right| < \epsilon$$

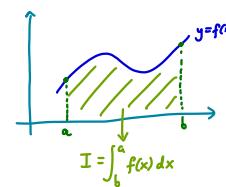
irregardless of our choices for $t_k \in [x_{k-1}, x_k]$.

(RIEMANN) INTEGRAL (DI.3)

\exists : We say the "Riemann integral" of f on $[a, b]$ is defined to be the number $I \in \mathbb{R}$ described above, and write

$$I = \int_a^b f = \int_a^b f(x) dx.$$

* I represents the "area under the graph".



\exists : We can prove I is unique.

Proof: Suppose I & J are two such numbers.

Let $\epsilon > 0$ be arbitrary. Then, choose a $\delta_1 > 0$ such that for any partition X with $|X| < \delta_1$, we have $|S - I| < \frac{\epsilon}{2}$ for every Riemann sum S on X .

Similarly, choose a $\delta_2 > 0$ such that for any partition X with $|X| < \delta_2$, we have $|S - J| < \frac{\epsilon}{2}$ for every Riemann sum S on X .

Then, let $\delta = \min(\delta_1, \delta_2)$, and let X be any partition of $[a, b]$ with $|X| < \delta$.

For each $k \in \{1, 2, \dots, n\}$, choose a $t_k \in [x_{k-1}, x_k]$.

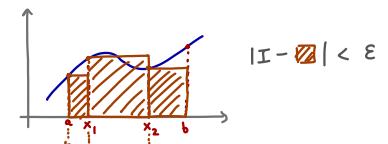
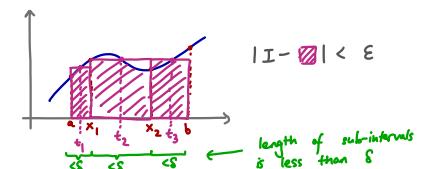
$$S = \sum_{k=1}^n f(t_k) \Delta_k x.$$

Then, by the Triangle Inequality, we have that

$$|I - J| \leq |I - S| + |S - J| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

But since $\epsilon > 0$ was arbitrary, it follows that

$$I = J, \text{ proving uniqueness. } \blacksquare$$



* regardless of our choices for t_k , we must always get that $|S - I| < \epsilon$!

SOME FUNCTIONS ARE NOT INTEGRABLE (E1.4)

We can show that certain functions are not integrable on a specific closed interval.

Example: $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$ is not integrable on $[0, 1]$.

Proof. Suppose f is integrable on $[0, 1]$.

$$\text{Write } I = \int_0^1 f(x) dx.$$

Let $\epsilon = \frac{1}{2}$. Then, by definition, we can choose a $\delta > 0$ such that for every partition X with $|X| < \delta$, we have that $|S - I| < \frac{1}{2}$ for every Riemann sum S for f on X .

Then, choose some partition X with $|X| < \delta$.

$$\text{Denote } S_1 = \sum_{k=1}^n f(t_k) \Delta_k x \text{ and } S_2 = \sum_{k=1}^n f(s_k) \Delta_k x,$$

where $t_k \in \mathbb{Q}$ and $s_k \notin \mathbb{Q}$ and $t_k, s_k \in [x_{k-1}, x_k]$ for each $k \in \{1, 2, \dots, n\}$.

Note that $|S_1 - I| < \frac{1}{2}$ and $|S_2 - I| < \frac{1}{2}$, by our previous assumption that $\epsilon = \frac{1}{2}$.

Subsequently, since $t_k \in \mathbb{Q}$ and $s_k \notin \mathbb{R} \setminus \mathbb{Q}$ $\forall k \in \{1, 2, \dots, n\}$, it follows that $f(t_k) = 1$ and $f(s_k) = 0$;

hence, we must get that

$$S_1 = \sum_{k=1}^n f(t_k) \Delta_k x = \sum_{k=1}^n \Delta_k x = 1 - 0 = 1$$

and

$$S_2 = \sum_{k=1}^n f(s_k) \Delta_k x = 0.$$

Thus, since $|S_1 - I| < \frac{1}{2}$ and $|S_2 - I| < \frac{1}{2}$, we must finally deduce that

$$|I - I| < \frac{1}{2} \quad \text{and} \quad |I| < \frac{1}{2},$$

so that

$$\frac{1}{2} < I < \frac{3}{2} \quad \text{and} \quad -\frac{1}{2} < I < \frac{1}{2},$$

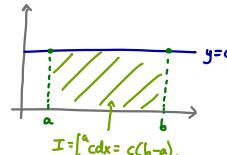
which is clearly impossible.

Therefore f is not integrable on $[0, 1]$, which we wanted to show. \square

THE INTEGRAL OF THE CONSTANT FUNCTION (E1.5)

The constant function $f(x) = c$ is always integrable on any interval $[a, b]$, and

$$\int_a^b c dx = c(b-a).$$



Proof. Let S be a Riemann sum for f on a partition X of $[a, b]$.

Then,

$$S = \sum_{k=1}^n f(t_k) \Delta_k x, \quad t_k \in [x_{k-1}, x_k] \quad \forall k \in \{1, 2, \dots, n\}$$

$$= \sum_{k=1}^n c \Delta_k x$$

$$= c \sum_{k=1}^n \Delta_k x$$

$$\therefore S = c(b-a).$$

But since S was arbitrary, it follows that

$$I = \int_a^b c dx = c(b-a),$$

as needed. \square

THE INTEGRAL OF THE IDENTITY FUNCTION (E1.6)

The identity function $f(x) = x$ is also integrable on any interval $[a, b]$, and

$$\int_a^b x dx = \frac{1}{2}(b^2 - a^2).$$

Proof. Let $\epsilon > 0$ be arbitrary, and let $\delta = \frac{2\epsilon}{b-a}$.

Let X be any partition of $[a, b]$ with $|X| < \delta$.

Then, the Riemann sum S for f on X is equal to

$$S = \sum_{k=1}^n f(t_k) \Delta_k x = \sum_{k=1}^n t_k \Delta_k x,$$

where $t_k \in [x_{k-1}, x_k] \quad \forall k \in \{1, 2, \dots, n\}$.

Next, notice that

$$\begin{aligned} \sum_{k=1}^n (x_k + x_{k-1}) \Delta_k x &= \sum_{k=1}^n (x_k + x_{k-1})(x_k - x_{k-1}) \\ &= \sum_{k=1}^n (x_k^2 - x_{k-1}^2) \\ &= (x_n^2 - x_0^2) + (x_2^2 - x_1^2) + \dots + (x_n^2 - x_{n-1}^2) \\ \therefore \sum_{k=1}^n (x_k + x_{k-1}) \Delta_k x &= b^2 - a^2. \end{aligned}$$

Moreover, $t_k \in [x_{k-1}, x_k]$ implies that

$$|t_k - \frac{1}{2}(x_k + x_{k-1})| \leq \frac{1}{2}(x_k - x_{k-1}) = \frac{1}{2}\Delta_k x,$$

and consequently it follows that

$$\begin{aligned} |S - \frac{1}{2}(b^2 - a^2)| &= \left| \sum_{k=1}^n t_k \Delta_k x - \frac{1}{2} \sum_{k=1}^n (x_k + x_{k-1}) \Delta_k x \right| \\ &= \left| \sum_{k=1}^n (t_k - \frac{1}{2}(x_k + x_{k-1})) \Delta_k x \right| \\ &\leq \sum_{k=1}^n |t_k - \frac{1}{2}(x_k + x_{k-1})| \Delta_k x \\ &\leq \sum_{k=1}^n \left(\frac{1}{2} \Delta_k x \right) \Delta_k x \\ &\leq \sum_{k=1}^n \frac{1}{2} \delta (b-a) \quad (\text{since } \Delta_k x < \delta \text{ and } \Delta_k x = b-a \text{ by definition}) \\ &= \epsilon, \quad (\text{since } \delta = \frac{2\epsilon}{b-a}) \end{aligned}$$

So that $|S - \frac{1}{2}(b^2 - a^2)| \leq \epsilon$.

But as $\epsilon > 0$ was arbitrary, this tells us that

$$I = \int_a^b x dx = \frac{1}{2}(b^2 - a^2),$$

which we wanted to prove. \square

UPPER & LOWER RIEMANN SUMS (DI-7)

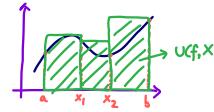
Let X be a partition of $[a, b]$, and let $f: [a, b] \rightarrow \mathbb{R}$ be bounded.

Then, the "upper Riemann sum" for f on X , denoted by $U(f, X)$, is defined to be

$$U(f, X) = \sum_{k=1}^n M_k \Delta_k x,$$

where

$$M_k = \sup\{f(t) : t \in [x_{k-1}, x_k]\} \quad \forall k \in \{1, 2, \dots, n\}.$$



Similarly, the "lower Riemann sum" for f on X , denoted by $L(f, X)$, is defined to be

$$L(f, X) = \sum_{k=1}^n m_k \Delta_k x,$$

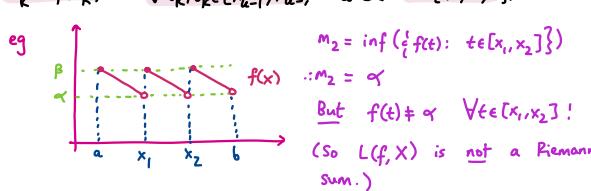
where

$$m_k = \inf\{f(t) : t \in [x_{k-1}, x_k]\} \quad \forall k \in \{1, 2, \dots, n\}.$$

$U(f, X)$ & $L(f, X)$ ARE NOT ALWAYS RIEMANN SUMS (RI-8)

Note that, in general $U(f, X)$ and $L(f, X)$

are not always Riemann sums, as we do not always have that $M_k = f(t_k)$ or $m_k = f(s_k)$ $\forall t_k, s_k \in [x_{k-1}, x_k]$, where $k \in \{1, 2, \dots, n\}$.



$U(f, X)$ & $L(f, X)$ ARE RIEMANN SUMS IF f IS STRICTLY MONOTONIC (RI-8)

Let X be a partition of $[a, b]$, and $f: [a, b] \rightarrow \mathbb{R}$ be bounded.

Assume f is strictly monotonic (ie increasing or decreasing).

Then $U(f, X)$ and $L(f, X)$ are Riemann sums for f on X .

Proof. If f is increasing, then

$M_k = f(x_k)$ and $m_k = f(x_{k-1})$ $\forall k \in \{1, 2, \dots, n\}$, so that $U(f, X)$ and $L(f, X)$ are indeed Riemann sums.

Similarly, if f is decreasing, then

$M_k = f(x_{k-1})$ and $m_k = f(x_k)$ $\forall k \in \{1, 2, \dots, n\}$, so that $U(f, X)$ and $L(f, X)$ are indeed Riemann sums. \square

$U(f, X)$ & $L(f, X)$ ARE RIEMANN SUMS IF f IS CONTINUOUS (RI-8)

Let X be a partition of $[a, b]$, and let $f: [a, b] \rightarrow \mathbb{R}$ be bounded.

Assume f is continuous on $[a, b]$.

Then $U(f, X)$ and $L(f, X)$ are Riemann sums for f on X .

Proof. By the Extreme Value Theorem,

there necessarily exists some $t_k, s_k \in [x_{k-1}, x_k]$ such that $f(s_k) \leq f(t) \leq f(t_k)$ $\forall t \in [x_{k-1}, x_k]$ for each $k \in \{1, 2, \dots, n\}$.

Let $M_k = f(s_k)$ and $m_k = f(t_k)$. Then since $t_k, s_k \in [x_{k-1}, x_k]$, it follows that $U(f, X)$ and $L(f, X)$ are indeed Riemann sums for f on X . \square

$U(f, X)$ IS THE LARGEST RIEMANN SUM & $L(f, X)$ IS THE SMALLEST RIEMANN SUM FOR f ON X (NI-9)

Let T be the set of all Riemann sums for a bounded function $f: [a, b] \rightarrow \mathbb{R}$ on a partition X of $[a, b]$.

Then $U(f, X) = \sup(T)$ and $L(f, X) = \inf(T)$.

In particular, we have that

$$L(f, X) \leq S \leq U(f, X)$$

for every $S \in T$.

Proof. We prove the former statement, since the proof for the latter is similar.

Then, note for any $S \in T$, we have that

$$S = \sum_{k=1}^n f(t_k) \Delta_k x \leq \sum_{k=1}^n M_k \Delta_k x = U(f, X),$$

by construction of M_k .

Hence $U(f, X)$ is an upper bound for T , and so necessarily $U(f, X) \geq \sup(T)$.

Next, let $\epsilon > 0$ be arbitrary.

Then, since $M_k = \sup\{f(t) : t \in [x_{k-1}, x_k]\}$, we can choose a $t_k \in [x_{k-1}, x_k]$ with $M_k - f(t_k) < \frac{\epsilon}{b-a}$, for every $k \in \{1, 2, \dots, n\}$.

Hence, it follows that there exists a $S \in T$ such that

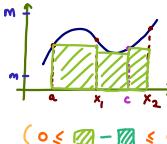
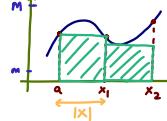
$$\begin{aligned} U(f, X) - S &= \sum_{k=1}^n M_k \Delta_k x - \sum_{k=1}^n f(t_k) \Delta_k x \\ &= \sum_{k=1}^n (M_k - f(t_k)) \Delta_k x \\ &< \sum_{k=1}^n \frac{\epsilon}{b-a} \Delta_k x \\ &= \frac{\epsilon}{b-a} (b-a) \end{aligned}$$

$\therefore U(f, X) - S < \epsilon$,

and since $\epsilon > 0$ was arbitrary it follows that $\sup(T) = U(f, X)$, as needed. \square

$$0 \leq L(f, X) \leq (M-m)|X|,$$

$$0 \leq U(f, X) - U(f, X \cup \{c\}) \leq (M-m)|X| \quad (\text{LI.10})$$



$(0 \leq \square - \square \leq (M-m)|X|)$
Similar illustration for second statement.

Proof. We prove the first statement, as the proof for the second statement is similar.

Say $X = \{x_0, x_1, \dots, x_n\}$ and $c \in [x_{k-1}, x_k]$

for some $k \in \{2, \dots, n\}$, so that

$$Y = \{x_0, x_1, \dots, x_{k-1}, c, x_k, \dots, x_n\}.$$

Then

$$L(f, Y) - L(f, X) = [r(c-x_{k-1}) + s(x_k-c)] - m_k(x_k-x_{k-1}),$$

where $r = \inf\{f(t) \mid t \in [x_{k-1}, c]\}$, $s = \inf\{f(t) \mid t \in [c, x_k]\}$

and $m_k = \inf\{f(t) \mid t \in [x_{k-1}, x_k]\}$.

Next, since $m_k = \min(r, s)$, it follows that $r \geq m_k$ & $s \geq m_k$
so that

$$L(f, Y) - L(f, X) \geq m_k(c-x_{k-1}) + m_k(x_k-c) - m_k(x_k-x_{k-1}) = 0,$$

establishing the first inequality.

Then, since $r \leq M$ and $s \leq M$, & $m_k \geq m$ by construction, it also follows that

$$\begin{aligned} L(f, Y) - L(f, X) &\leq M(c-x_{k-1}) + M(x_k-c) - m(x_k-x_{k-1}) \\ &= (M-m)(x_k-x_{k-1}) \\ &\leq (M-m)|X|. \end{aligned}$$

Therefore, we have that

$$0 \leq L(f, Y) - L(f, X) \leq (M-m)|X|,$$

which we wanted to prove. \blacksquare

$$L(f, X) \leq L(f, Y) \leq U(f, Y) \leq U(f, X); \quad X \subseteq Y \quad (\text{NI.11})$$

Q: Let X and Y be partitions of $[a, b]$,

such that $X \subseteq Y$.

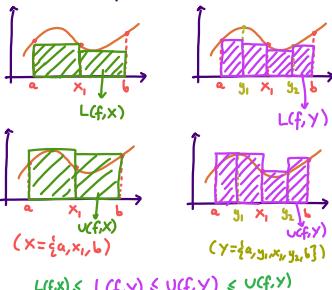
Let $f: [a, b] \rightarrow \mathbb{R}$ be bounded.

Then, we always have that

$$L(f, X) \leq L(f, Y) \leq U(f, Y) \leq U(f, X).$$

Proof. If Y is obtained by adding one point to X , then this follows from the above lemma.

In general, Y can be obtained by adding finitely many points to X , one point at a time. \blacksquare



$$L(f, X) \leq L(f, Y) \leq U(f, Y) \leq U(f, X)$$

Q: Let X and Y be any partitions of $[a, b]$.

Then necessarily $L(f, X) \leq U(f, Y)$.

Proof. Let $Z = X \cup Y$. Then by the above note,

$$\begin{aligned} L(f, X) &\leq L(f, Z) \leq U(f, Z) \leq U(f, Y), \\ \text{and the proof follows from here. } \blacksquare \end{aligned}$$

UPPER & LOWER INTEGRALS (DI.13)

Let $f: [a,b] \rightarrow \mathbb{R}$ be bounded.

Then, the "upper integral" of f on $[a,b]$,

denoted by $U(f)$, is defined to be

$$U(f) = \inf \{ U(f, X) \mid X \text{ is a partition of } [a,b] \}.$$

Similarly, the "lower integral" of f on $[a,b]$,

denoted by $L(f)$, is defined to be

$$L(f) = \sup \{ L(f, X) \mid X \text{ is a partition of } [a,b] \}.$$

Note that $U(f)$ and $L(f)$ always exist even if f is not integrable. (NI.14)

$L(f) \leq U(f)$ (NI.15)

Let $f: [a,b] \rightarrow \mathbb{R}$ be a bounded function.

$$\text{Then } L(f) \leq U(f).$$

Proof. Let $\epsilon > 0$ be arbitrary.

Choose partitions X_1 and X_2 such that

$$L(f) - L(f, X_1) < \frac{\epsilon}{2} \quad \text{and} \quad U(f, X_2) - U(f) < \frac{\epsilon}{2}.$$

Then

$$\begin{aligned} U(f) - L(f) &= (U(f) - U(f, X_2)) + (U(f, X_2) - L(f, X_2)) + (L(f, X_2) - L(f)) \\ &> -\frac{\epsilon}{2} + 0 - \frac{\epsilon}{2} \\ &= -\epsilon. \end{aligned}$$

Hence $L(f) - U(f) < \epsilon$, and since $\epsilon > 0$ was arbitrary, this in turn implies that $L(f) \leq U(f)$. \blacksquare

EQUIVALENT DEFINITIONS OF INTEGRABILITY (TI.16)

Let $f: [a,b] \rightarrow \mathbb{R}$ be bounded.

Then the following statements are equivalent:

$$\textcircled{1} \quad L(f) = U(f);$$

\textcircled{2} For any $\epsilon > 0$, there exists a partition X such that $U(f, X) - L(f, X) < \epsilon$; and

\textcircled{3} f is integrable on $[a,b]$.

Proof. First, we show \textcircled{1} \Rightarrow \textcircled{2}.

Suppose $L(f) = U(f)$. Let $\epsilon > 0$ be arbitrary.

Then, choose partitions X_1 and X_2 so that

$$L(f) - L(f, X_1) < \frac{\epsilon}{2} \quad \text{and} \quad U(f, X_2) - U(f) < \frac{\epsilon}{2}.$$

Let $X = X_1 \cup X_2$.

Next, since $L(f, X_1) \leq L(f, X) \leq L(f)$ (as $X_1 \subseteq X$),

it follows that $L(f) - L(f, X) \leq L(f) - L(f, X_1) < \frac{\epsilon}{2}$,

and since $U(f) \leq U(f, X) \leq U(f, X_2)$ (as $X_2 \subseteq X$), it follows that $U(f, X) - U(f) < \frac{\epsilon}{2}$ also.

Hence

$$\begin{aligned} U(f, X) - L(f, X) &= [U(f, X_2) - U(f)] + [U(f, X_1) - U(f)] + [U(f) - L(f, X_1)] \\ &< \frac{\epsilon}{2} + 0 + \frac{\epsilon}{2} \\ &= \epsilon, \end{aligned}$$

which is sufficient to show that \textcircled{2} is true. *

Subsequently, we show \textcircled{2} \Rightarrow \textcircled{1}.

Suppose for any $\epsilon > 0$, there exists a partition X such that

$$U(f, X) - L(f, X) < \epsilon.$$

Fix $\epsilon > 0$, and choose X so that $U(f, X) - L(f, X) < \epsilon$.

Then

$$\begin{aligned} U(f) - L(f) &= [U(f) - U(f, X)] + [U(f, X) - L(f, X)] + [L(f, X) - L(f)] \\ &< 0 + \epsilon + 0 \\ &= \epsilon. \end{aligned}$$

Since $0 \leq U(f) - L(f) < \epsilon \forall \epsilon > 0$, this tells us that

$$U(f) = L(f), \text{ proving } \textcircled{1}. *$$

Next, we show \textcircled{3} \Rightarrow \textcircled{2}.

Suppose f is integrable on $[a,b]$, with $I = \int_a^b f(x) dx$.

Let $\epsilon > 0$. Then, choose a $\delta > 0$ such that

for every partition X with $|X| < \delta$, we have that $|S - I| < \frac{\epsilon}{4}$ for every Riemann sum S for f on X .

Let S_1 and S_2 be Riemann sums for f on X

such that $|U(f, X) - S_1| < \frac{\epsilon}{4}$ and $|S_2 - L(f, X)| < \frac{\epsilon}{4}$.

Then, by the Triangle Inequality,

$$\begin{aligned} |U(f, X) - L(f, X)| &\leq |U(f, X) - S_1| + |S_1 - I| + |I - S_2| + |S_2 - L(f, X)| \\ &< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} \\ &= \epsilon, \end{aligned}$$

which is sufficient to prove \textcircled{2}. *

Lastly, we prove \textcircled{1} \Rightarrow \textcircled{3}.

Suppose $L(f) = U(f)$, and let $I = L(f) = U(f)$.

Then, let $\epsilon > 0$.

Choose a partition X_0 of $[a,b]$ so that

$$L(f) - L(f, X_0) < \frac{\epsilon}{2} \quad \text{and} \quad U(f, X_0) - U(f) < \frac{\epsilon}{2}.$$

Say $X_0 = \{x_0, x_1, \dots, x_n\}$, and set $\delta = \frac{\epsilon}{2(n-1)(M-m)}$,

where M and m are upper and lower bounds for f on $[a,b]$.

Let X be any partition of $[a,b]$ with $|X| < \delta$, and $y = x_0 \cup X$.

Note that y is obtained from X by adding at most $n-1$ points, and that each time we add a point, the size of the new partition is at most $|x| < \delta$.

Hence

$$\begin{aligned} 0 &\leq U(f, X) - U(f, y) \leq (n-1)(M-m)|x| < (n-1)(M-m)\delta = \frac{\epsilon}{2} \quad \text{and} \\ 0 &\leq L(f, y) - L(f, X) \leq (n-1)(M-m)|x| < (n-1)(M-m)\delta = \frac{\epsilon}{2} \end{aligned}$$

by the first lemma on the previous page.

Next, let S be any Riemann sum for f on X .

Note that $L(f, X_0) \leq L(f, y) \leq L(f) = U(f) \leq U(f, y) \leq U(f, X)$

and $L(f, X) \leq S \leq U(f, X)$, so that

$$\begin{aligned} S - I &\leq U(f, X) - I \\ &= U(f, X) - U(f) \\ &= (U(f, X) - U(f, y)) + (U(f, y) - U(f)) \\ &\leq (U(f, X) - U(f, y)) + (U(f, y) - L(f, X_0)) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

and

$$\begin{aligned} I - S &\leq I - L(f, X_0) \\ &= L(f) - L(f, X_0) \\ &= (L(f) - L(f, y)) + (L(f, y) - L(f, X_0)) \\ &\leq (L(f) - L(f, y)) + (L(f, y) - L(f, X)) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

and since $\epsilon > 0$ was arbitrary this is sufficient to prove \textcircled{3}. \blacksquare

INTEGRALS OF CONTINUOUS FUNCTIONS

CONTINUOUS FUNCTIONS ARE ALWAYS INTEGRABLE (T1.17)

Let $f: [a, b] \rightarrow \mathbb{R}$ be continuous.
Then f is integrable on $[a, b]$.

Proof. First, note f is uniformly continuous on $[a, b]$.

Hence, we can choose a $\delta > 0$ so that for all $x, y \in [a, b]$, we have that $|x - y| < \delta$ implies $|f(x) - f(y)| < \frac{\epsilon}{b-a}$.

Let X be any partition of $[a, b]$ with $|X| < \delta$.

Then, by the Extreme Value Theorem, there exists some $t_k, s_k \in [x_{k-1}, x_k]$ such that $m_k = f(s_k) \leq t \leq M_k = f(t_k) \quad \forall t \in [x_{k-1}, x_k]$, where $k \in \{1, 2, \dots, n\}$.

Finally, since $|t_k - s_k| \leq |x_k - x_{k-1}| \leq |X| = \delta$, it follows that $|M_k - m_k| = |f(t_k) - f(s_k)| < \frac{\epsilon}{b-a}$ (since f is uniformly continuous).

Thus

$$\begin{aligned} U(f, X) - L(f, X) &= \sum_{k=1}^n (M_k - m_k) \Delta_{x_k} \\ &< \frac{\epsilon}{b-a} \sum_{k=1}^n \Delta_{x_k} \\ &= \epsilon, \end{aligned}$$

and as $\epsilon > 0$ was arbitrary this tells us that $U(f, X) = L(f, X)$, which by the equivalent definitions of integrability implies that f is integrable on $[a, b]$. \blacksquare

SEQUENTIAL CHARACTERISATION OF INTEGRATION (N1.18)

Let $f: [a, b] \rightarrow \mathbb{R}$ be integrable on $[a, b]$, and let $\{X_n\}$ be a sequence of partitions of $[a, b]$ with $\lim_{n \rightarrow \infty} |X_n| = 0$.

For any given $n \in \mathbb{N}$, let S_n be any Riemann sum for f on X_n .

Then the sequence $\{S_n\}$ necessarily converges, with

$$\lim_{n \rightarrow \infty} S_n = \int_a^b f(x) dx.$$

Proof. Denote $I = \int_a^b f(x) dx$.

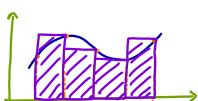
Then, given a $\epsilon > 0$, choose a $\delta > 0$ so that for every partition X of $[a, b]$ with $|X| < \delta$, we have that $|S - I| < \epsilon$ for every Riemann sum S for f on X .

Choose a $N \in \mathbb{N}$ so that if $n > N$, $|X_n| < \delta$. (We can do this since $\{|X_n|\} \rightarrow 0$.)

It follows that if $n > N$, then $|S_n - I| < \epsilon$, and as $\epsilon > 0$ was arbitrary this is sufficient to prove that $\lim_{n \rightarrow \infty} S_n = \int_a^b f(x) dx$. \blacksquare

Let $f: [a, b] \rightarrow \mathbb{R}$ be integrable on $[a, b]$. Then, if we let X_n be the partition of $[a, b]$ into n equal-sized sub-intervals, and S_n be the Riemann sum on X_n using right-endpoints, it follows from the above that

$$\begin{aligned} \int_a^b f(x) dx &= \lim_{n \rightarrow \infty} \sum_{k=1}^n f(x_{n,k}) \Delta_{x_k} \\ &= \lim_{n \rightarrow \infty} \left(\frac{b-a}{n} \right) \sum_{k=1}^n f(a + \frac{b-a}{n} k). \quad (\text{N1.19}) \end{aligned}$$



Note $x_1 - a = x_2 - x_1 = x_3 - x_2 = b - x_3$, so that $x_n - x_1 = \frac{b-a}{4}$ for each $n \in \{2, 3, 4\}$. Hence

$$\begin{aligned} \boxed{I} &= \left(\frac{b-a}{4} \right) [f(x_1) + f(x_2) + f(x_3) + f(x_4)] \\ &= \left(\frac{b-a}{4} \right) \sum_{k=1}^4 f(x_k). \end{aligned}$$

EXAMPLE: INTEGRAL OF $f(x) = 2^x$ (E1.20)

We can use the previous derived results to evaluate integrals of specific continuous functions; eg $\int_0^2 2^x dx$.

Let $f(x) = 2^x$. Note f is continuous, and hence integrable (on $[0, 2]$).

Then, using the formula from N1.19, we have that

$$\begin{aligned} \int_0^2 2^x dx &= \lim_{n \rightarrow \infty} \sum_{k=1}^n f(x_{n,k}) \Delta_{x_k} \\ &= \lim_{n \rightarrow \infty} \sum_{k=1}^n f\left(\frac{2k}{n}\right) \left(\frac{2}{n}\right) \quad (\text{since } |[0, 2]| = 2) \\ &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{2^{\frac{2k}{n}}}{n} \cdot \frac{2}{n} \\ &= \lim_{n \rightarrow \infty} \frac{2 \cdot 4^{\frac{n}{2}}}{n} \cdot \frac{4-1}{4^n-1} \quad (\text{by the formula for the sum of a geometric sequence}) \\ &= \lim_{n \rightarrow \infty} \left(6^{\frac{1}{n}}\right) \lim_{n \rightarrow \infty} \frac{1}{n(4^{\frac{n}{2}}-1)} \\ &= 6 \lim_{n \rightarrow \infty} \frac{1}{4^{\frac{n}{2}}-1} \\ &= 6 \lim_{x \rightarrow 0^+} \frac{x}{4^x-1} \\ &= 6 \lim_{x \rightarrow 0^+} \frac{1}{1-4^{-x}} \quad (\text{by L'Hopital's rule, since } \frac{x}{4^x-1} \rightarrow 0 \text{ if } x=0) \\ &= \frac{6}{\ln(4)} \\ \therefore \int_0^2 2^x dx &= \underline{\underline{\frac{3}{\ln(2)}}}. \end{aligned}$$

SUMMATION FORMULAS (L1.21)

Note that

$$\textcircled{1} \quad \sum_{i=1}^n 1 = n;$$

$$\textcircled{2} \quad \sum_{i=1}^n i = \frac{n(n+1)}{2};$$

$$\textcircled{3} \quad \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}; \quad \text{and}$$

$$\textcircled{4} \quad \sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}.$$

Proof: $\textcircled{1}$ is trivial, so we prove $\textcircled{2}$ first.

Consider $\sum_{k=1}^n (k^2 - (k-1)^2)$.

On one hand,

$$\begin{aligned} \sum_{k=1}^n (k^2 - (k-1)^2) &= (2^2 - 1^2) + (3^2 - 2^2) + \cdots + (n^2 - (n-1)^2) \\ &= n^2, \end{aligned}$$

but on the other hand,

$$\begin{aligned} \sum_{k=1}^n (k^2 - (k-1)^2) &= \sum_{k=1}^n (k^2 - (k^2 - 2k + 1)) \\ &= 2 \sum_{k=1}^n k - \sum_{k=1}^n 1. \end{aligned}$$

Hence $n^2 = 2 \sum_{k=1}^n k - n$,

$$\text{so that } \sum_{k=1}^n k = \frac{1}{2}(n^2 + n) = \frac{n(n+1)}{2}. \quad \text{*}$$

Next, we prove $\textcircled{3}$.

Consider $\sum_{k=1}^n (k^3 - (k-1)^3)$.

On one hand,

$$\begin{aligned} \sum_{k=1}^n (k^3 - (k-1)^3) &= (1^3 - 0^3) + (2^3 - 1^3) + \cdots + (n^3 - (n-1)^3) \\ &= n^3. \end{aligned}$$

On the other hand,

$$\begin{aligned} \sum_{k=1}^n (k^3 - (k-1)^3) &= \sum_{k=1}^n (k^3 - (k^3 - 3k^2 + 3k - 1)) \\ &= 3 \sum_{k=1}^n k^2 - 3 \sum_{k=1}^n k + \sum_{k=1}^n 1 \\ &= 3 \sum_{k=1}^n k^2 - 3 \frac{n(n+1)}{2} + n. \end{aligned}$$

Equating these, we get that

$$n^3 = 3 \sum_{k=1}^n k^2 - 3 \frac{n(n+1)}{2} + n$$

which eventually simplifies to

$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}. \quad \text{*}$$

Lastly, we prove $\textcircled{4}$.

Consider $\sum_{k=1}^n (k^4 - (k-1)^4)$.

On one hand,

$$\begin{aligned} \sum_{k=1}^n (k^4 - (k-1)^4) &= (1^4 - 0^4) + (2^4 - 1^4) + \cdots + (n^4 - (n-1)^4) \\ &= n^4, \end{aligned}$$

and on the other hand,

$$\begin{aligned} \sum_{k=1}^n (k^4 - (k-1)^4) &= \sum_{k=1}^n (k^4 - (k^4 - 4k^3 + 6k^2 - 4k + 1)) \\ &= 4 \sum_{k=1}^n k^3 - 6 \sum_{k=1}^n k^2 + 4 \sum_{k=1}^n k - \sum_{k=1}^n 1 \\ &= 4 \sum_{k=1}^n k^3 - 6 \frac{n(n+1)(2n+1)}{6} + 4 \frac{n(n+1)}{2} - n. \end{aligned}$$

Hence

$$n^4 = 4 \sum_{k=1}^n k^3 - 6 \frac{n(n+1)(2n+1)}{6} + 4 \frac{n(n+1)}{2} - n,$$

which simplifies to

$$\sum_{k=1}^n k^3 = \frac{n^2(n+1)^2}{4}. \quad \text{*}$$

USING SUMMATION FORMULAS TO CALCULATE INTEGRALS (E1.22)

Q: We can use summation formulae to calculate integrals of certain functions;

$$\text{eg } \int_1^3 (x+2x^3) dx.$$

Note that

$$\begin{aligned} \int_1^3 (x+2x^3) dx &= \lim_{n \rightarrow \infty} \sum_{k=1}^n f(x_{n,k}) \Delta_{n,k} x \\ &= \lim_{n \rightarrow \infty} \sum_{k=1}^n f\left(1 + \frac{2}{n} k\right) \left(\frac{2}{n}\right) \quad (\text{since } |[1,3]|=2) \\ &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(f\left(1 + \frac{2}{n} k\right) + 2\left(1 + \frac{2}{n} k\right)^3\right) \left(\frac{2}{n}\right) \\ &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\frac{6}{n} + \frac{28}{n^2} k + \frac{48}{n^3} k^2 + \frac{32}{n^4} k^3\right) \\ &= \lim_{n \rightarrow \infty} \left(\frac{6}{n} \sum_{k=1}^n 1 + \frac{28}{n^2} \sum_{k=1}^n k + \frac{48}{n^3} \sum_{k=1}^n k^2 + \frac{32}{n^4} \sum_{k=1}^n k^3\right) \\ &= \lim_{n \rightarrow \infty} \left(\frac{6}{n} \cdot n + \frac{28}{n^2} \cdot \frac{n(n+1)}{2} + \frac{48}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} + \frac{32}{n^4} \cdot \frac{n^2(n+1)^2}{4}\right) \\ &= 6 + \frac{28}{2} + \frac{48}{6} + \frac{32}{4} \\ \therefore \int_1^3 (x+2x^3) dx &= 44. \end{aligned}$$

BASIC PROPERTIES OF INTEGRALS

LINEARITY (TI.23)

Q: Let f and g be integrable on $[a,b]$.

Let $c \in \mathbb{R}$ be arbitrary.

Then $(f+g)$ and cf are both integrable on $[a,b]$, and

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g$$

and

$$\int_a^b cf = c \int_a^b f.$$

Proof. Note that

$$\begin{aligned} \int_a^b f + \int_a^b g &= \lim_{n \rightarrow \infty} \sum_{k=1}^n f(x_{n,k}) \Delta_{n,k} x + \lim_{n \rightarrow \infty} \sum_{k=1}^n g(x_{n,k}) \Delta_{n,k} x \\ &= \lim_{n \rightarrow \infty} \frac{b-a}{n} \sum_{k=1}^n (f+g)(x_{n,k}) \\ &= \int_a^b (f+g), \end{aligned}$$

and that

$$\begin{aligned} \int_a^b cf &= c \lim_{n \rightarrow \infty} \sum_{k=1}^n f(x_{n,k}) \Delta_{n,k} x \\ &= \lim_{n \rightarrow \infty} \frac{b-a}{n} \sum_{k=1}^n (cf)(x_{n,k}) \\ &= \int_a^b cf. \quad \blacksquare \end{aligned}$$

ADDITIONIVITY (TI.25)

Q: Let $a < b < c$, and $f: [a,c] \rightarrow \mathbb{R}$ be bounded.

Then f is integrable on $[a,c]$ if and only if f is integrable on both $[a,b]$ and $[b,c]$,

and in this case

$$\int_a^b f + \int_b^c f = \int_a^c f.$$

Proof. First, suppose f is integrable on $[a,c]$.

Choose a partition X of $[a,c]$ such that $U(f,X) - L(f,X) < \epsilon$.

Say that $b \in [x_{k-1}, x_k]$, and let $Y = \{x_0, x_1, \dots, x_{k-1}, b\}$

and $Z = \{b, x_k, x_{k+1}, \dots, x_m\}$, so that Y and Z are partitions on $[a,b]$ and $[b,c]$ respectively.

Then $U(f,Y) - L(f,Y) \leq U(f, X \cup \{b\}) - L(f, X \cup \{b\})$ (by NI.11)

$$\leq U(f,X) - L(f,X) \quad (\text{by NI.11 also}) \\ < \epsilon,$$

$$\text{and } U(f,Z) - L(f,Z) \leq U(f, X \cup \{b\}) - L(f, X \cup \{b\}) \\ \leq U(f,X) - L(f,X) \\ < \epsilon,$$

which is sufficient to show f is integrable on both $[a,b]$ and $[b,c]$.

Conversely, suppose f is integrable on both $[a,b]$ and $[b,c]$.

Choose partitions Y of $[a,b]$ & Z of $[b,c]$ so that

$$U(f,Y) - L(f,Y) < \frac{\epsilon}{2} \quad \text{and} \quad U(f,Z) - L(f,Z) < \frac{\epsilon}{2}.$$

Then $X = Y \cup Z$ is a partition of $[a,c]$, and

$$U(f,X) - L(f,X) = [U(f,Y) + U(f,Z)] - [L(f,Y) + L(f,Z)] < \epsilon,$$

which tells us $U(f,X) = L(f,X)$ (since $\epsilon > 0$ was arbitrary) and consequently (by the equivalent definitions of Integrability) that f is integrable on $[a,c]$.

COMPARISON (TI.24)

Q: Let f and g be integrable on $[a,b]$.

Suppose $f(x) \leq g(x) \quad \forall x \in [a,b]$.

Then

$$\int_a^b f \leq \int_a^b g.$$

Proof. Note that

$$\begin{aligned} \int_a^b f &= \lim_{n \rightarrow \infty} \sum_{k=1}^n f(x_{n,k}) \Delta_{n,k} x \\ &\leq \lim_{n \rightarrow \infty} \sum_{k=1}^n g(x_{n,k}) \Delta_{n,k} x \quad (\text{since } f(x) \leq g(x) \quad \forall x \in [a,b]) \\ &= \int_a^b g. \quad \blacksquare \end{aligned}$$

Finally, suppose f is integrable on $[a,c]$, and hence also on $[a,b]$ and $[b,c]$.

Let $I_1 = \int_a^b f$, $I_2 = \int_b^c f$ and $I = \int_a^c f$.

Let $\epsilon > 0$ be arbitrary. Then, choose a $\delta > 0$ so that for all partitions X_1, X_2 and X of $[a,b]$, $[b,c]$ and $[a,c]$ respectively, if $|X_1|, |X_2|, |X| < \delta$,

then $|I_1 - I_1|, |I_2 - I_2|, |I - I| < \frac{\epsilon}{3}$ for all Riemann sums S_1, S_2, S for f on X_1, X_2 & X respectively.

Choose partitions X_1 and X_2 of $[a,b]$ and $[b,c]$ with $|X_1| < \delta$ and $|X_2| < \delta$.

Choose Riemann sums S_1 and S_2 for f on X_1 and X_2 .

Let $X = X_1 \cup X_2$, and note that $|X| < \delta$ and $S = S_1 + S_2$ is a Riemann sum for f on X .

Then necessarily

$$\begin{aligned} |I - (I_1 + I_2)| &= |(I - S) + (S_1 - I_1) + (S_2 - I_2)| \\ &\leq |I - S| + |S_1 - I_1| + |S_2 - I_2| \quad (\text{by the Triangle Inequality}) \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon, \end{aligned}$$

and since $\epsilon > 0$ was arbitrary this is sufficient to prove that $I = I_1 + I_2$. \blacksquare

PIECEWISE CONTINUOUS FUNCTIONS ARE INTEGRABLE (C1.26)

\therefore Let $X = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$, and let $g_k : [x_{k-1}, x_k] \rightarrow \mathbb{R}$ be continuous $\forall k \in \{1, 2, \dots, n\}$. Let $f : [a, b] \rightarrow \mathbb{R}$ be a function with $f(t) = g_k(t) \quad \forall t \in (x_{k-1}, x_k)$. Then f is integrable on $[a, b]$ with

$$\int_a^b f(x) dx = \sum_{k=1}^n \int_{x_{k-1}}^{x_k} g_k(x) dx.$$

Proof. This follows from the additivity and linearity properties of integrals. \blacksquare

$$\int_a^a f = 0, \quad \int_b^a f = - \int_a^b f \quad (\text{D1.27})$$

\therefore For any function f and $a \in \mathbb{R}$,

$$\int_a^a f = 0.$$

Additionally, if $\int_a^b f(x) dx$ exists, then

$$\int_a^b f(x) dx = - \int_b^a f(x) dx.$$

\therefore Note that this definition can be used to extend the scope of the Additivity Theorem to the case where $a, b, c \in \mathbb{R}$ are not in increasing order. (N1.28)

ESTIMATION (C1.29)

\therefore Let f be integrable on $[a, b]$. Then $|f|$ is also integrable on $[a, b]$, and

$$|\int_a^b f| \leq \int_a^b |f|.$$

Proof. Let $\epsilon > 0$ be arbitrary.

Choose a partition X of $[a, b]$ such that

$$U(f, X) - L(f, X) < \epsilon.$$

Denote $M_k(f) = \sup\{f(t) \mid t \in [x_{k-1}, x_k]\}$ and

$$M_k(|f|) = \sup\{|f(t)| \mid t \in [x_{k-1}, x_k]\} \quad \forall k \in \{1, 2, \dots, n\},$$

with similar definitions for $m_k(f)$ and $m_k(|f|)$.

Then,

$$\begin{aligned} \textcircled{1} \quad \text{if } 0 \leq m_k(f) \leq M_k(f), \quad M_k(|f|) = m_k(f) \text{ and} \\ m_k(|f|) = m_k(f); \end{aligned}$$

$$\begin{aligned} \textcircled{2} \quad \text{if } m_k(f) \leq 0 \leq M_k(f), \quad M_k(|f|) = \max\{m_k(f), -m_k(f)\} \\ \text{and } m_k(|f|) \geq 0, \\ \text{so that } M_k(|f|) - m_k(|f|) \leq \max\{m_k(f), -m_k(f)\} \leq M_k(f) - m_k(f); \end{aligned}$$

$$\textcircled{3} \quad \text{if } m_k(f) \leq M_k(f) \leq 0, \quad M_k(|f|) = -m_k(f) \text{ and } m_k(|f|) = -M_k(f), \\ \text{so that } M_k(|f|) - m_k(|f|) = M_k(f) - m_k(f).$$

In any one of these cases, we have that

$$M_k(|f|) - m_k(|f|) \leq M_k(f) - m_k(f),$$

and so

$$\begin{aligned} U(|f|, X) - L(|f|, X) &= \sum_{k=1}^n (M_k(|f|) - m_k(|f|)) \Delta_k x \\ &\leq \sum_{k=1}^n (M_k(f) - m_k(f)) \Delta_k x \\ &= U(f, X) - L(f, X) \\ &< \epsilon, \end{aligned}$$

which is sufficient to prove that $|f|$ is integrable on $[a, b]$.

Next, let $\epsilon > 0$ be arbitrary.

Choose a partition X on $[a, b]$ and choose values $t_k \in [x_{k-1}, x_k] \quad \forall k \in \{1, 2, \dots, n\}$ so that

$$\left| \sum_{k=1}^n f(t_k) \Delta_k x - \int_a^b f \right| < \frac{\epsilon}{2} \quad \text{and} \quad \left| \sum_{k=1}^n |f(t_k)| \Delta_k x - \int_a^b |f| \right| < \frac{\epsilon}{2}.$$

Note by the Triangle Inequality that

$$\sum_{k=1}^n f(t_k) \Delta_k x \leq \sum_{k=1}^n |f(t_k)| \Delta_k x,$$

so that

$$\begin{aligned} \left| \int_a^b f \right| - \int_a^b |f| &= \left(\left| \int_a^b f \right| - \left| \sum_{k=1}^n f(t_k) \Delta_k x \right| \right) + \left(\left| \sum_{k=1}^n f(t_k) \Delta_k x \right| - \sum_{k=1}^n |f(t_k)| \Delta_k x \right) \\ &\quad + \left(\sum_{k=1}^n |f(t_k)| \Delta_k x - \int_a^b |f| \right) \\ &< \frac{\epsilon}{2} + 0 + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Since $\epsilon > 0$ was arbitrary, this tells us that

$$\left| \int_a^b f \right| - \int_a^b |f| \leq 0,$$

as required. \blacksquare

THE FUNDAMENTAL THEOREM OF CALCULUS

\exists_1 : First, note that for any function F , defined on an interval containing $[a, b]$, we write

$$[F(x)]_a^b = F(b) - F(a). \quad (\text{NTI-30})$$

\exists_2 : Let f be integrable on $[a, b]$.

Define $F: [a, b] \rightarrow \mathbb{R}$ by

$$F(x) = \int_a^x f(t) dt.$$

Then F is continuous on $[a, b]$.

Moreover, if f is continuous at a point $x \in [a, b]$, then F is differentiable at x and

$$F'(x) = f(x). \quad (\text{TI-31})$$

Proof. Let M be an upper bound for $|f(t)|$ on $[a, b]$.

Then, for any $a \leq x, y \leq b$, we have

$$\begin{aligned} |F(y) - F(x)| &= \left| \int_a^y f - \int_a^x f \right| \\ &= \left| \int_x^y f \right| \quad (\text{by additivity/linearity}) \\ &\leq \left| \int_x^y M \right| \quad (\text{by estimation}) \\ &\leq \left| \int_x^y M \right| \\ &= M|y-x|, \end{aligned}$$

so that given an $\epsilon > 0$, we can choose a $\delta = \frac{\epsilon}{M}$ to get that $|y-x| < \delta$ implies that

$|F(y) - F(x)| \leq M|y-x| < M\delta = \epsilon$, showing F is continuous (indeed, uniformly continuous) on $[a, b]$.

Subsequently, suppose f is continuous at some $x \in [a, b]$. Then, for any $a \leq x, y \leq b$ with $x \neq y$, we have

$$\begin{aligned} \left| \frac{F(y) - F(x)}{y-x} - f(x) \right| &= \left| \frac{\int_a^y f - \int_a^x f}{y-x} - f(x) \right| \\ &= \left| \frac{\int_x^y f}{y-x} - \frac{\int_x^y f(t) dt}{y-x} \right| \\ &= \frac{1}{y-x} \left| \int_x^y (f(t) - f(x)) dt \right| \\ &\leq \frac{1}{|y-x|} \left| \int_x^y (f(t) - f(x)) dt \right|. \end{aligned}$$

Let $\epsilon > 0$ be arbitrary. Since f is continuous at x , it follows that we can choose a $\delta > 0$ so that if $|y-x| < \delta$, then $|f(y) - f(x)| < \epsilon$.

So, if $0 < |y-x| < \delta$, then

$$\begin{aligned} \left| \frac{F(y) - F(x)}{y-x} - f(x) \right| &\leq \frac{1}{|y-x|} \left| \int_x^y (f(t) - f(x)) dt \right| \\ &\leq \frac{1}{|y-x|} \left| \int_x^y \epsilon dt \right| \\ &= \frac{1}{|y-x|} \epsilon |y-x| \\ &= \epsilon, \end{aligned}$$

showing that $F'(x)$ exists and $F'(x) = f(x)$ (as $\epsilon > 0$ was arbitrary). \square

\exists_3 : Let f be integrable on $[a, b]$, and F be differentiable on $[a, b]$ with $F' = f$.

Then

$$\int_a^b f = [F(x)]_a^b = F(b) - F(a). \quad (\text{TI-31})$$

Proof. Let $\epsilon > 0$ be arbitrary.

Choose a $\delta > 0$ so that for every partition X of $[a, b]$ with $|X| < \delta$, we have that

$$\left| \int_a^b f - \sum_{k=1}^n f(t_k) \Delta x_k \right| < \epsilon$$

for every choice of sample points $t_k \in [x_{k-1}, x_k]$.

Then, choose sample points $t_k \in [x_{k-1}, x_k]$ so that

$$F'(t_k) = \frac{F(x_k) - F(x_{k-1})}{x_k - x_{k-1}},$$

which we can do by the Mean Value Theorem.

This implies that $f(t_k) \Delta x_k = F(x_k) - F(x_{k-1})$.

Hence

$$\begin{aligned} \sum_{k=1}^n f(t_k) \Delta x_k &= \sum_{k=1}^n (F(x_k) - F(x_{k-1})) \\ &= (F(x_n) - F(x_0)) + (F(x_0) - F(x_1)) + \dots + (F(x_n) - F(x_{n-1})) \\ &= F(x_n) - F(x_0) \\ &= F(b) - F(a), \end{aligned}$$

and consequently

$$\left| \int_a^b f - (F(b) - F(a)) \right| < \epsilon.$$

But since $\epsilon > 0$ was arbitrary, it follows that

$$\int_a^b f = F(b) - F(a),$$

as needed. \square

ANTIDERIVATIVE (DI-32)

\exists_1 : We say F is an "antiderivative" for f on some interval $[a, b]$ if $F' = f$ on $[a, b]$.

\exists_2 : In this case, we write

$$\textcircled{1} \quad \int f = F + c, \quad c \in \mathbb{R}; \quad \text{or}$$

$$\textcircled{2} \quad \int f(x) dx = F(x) + c, \quad c \in \mathbb{R}. \quad (\text{NI-34})$$

\exists_3 : Note that if $G = F' = f$ on $[a, b]$, then necessarily $(G-F)' = 0$, so that $G-F$ is constant on the interval; ie $G = F+c$ for some $c \in \mathbb{R}$. (NI-33)

EXAMPLE: $\int_0^{\sqrt{3}} \frac{dx}{1+x^2} \quad (\text{EI-35})$

\exists_1 : We can use the Fundamental Theorem of Calculus to calculate integrals of specific functions;

$$\text{eg } \int_0^{\sqrt{3}} \frac{dx}{1+x^2}.$$

Since $\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2}$, it follows that

$$\begin{aligned} \int_0^{\sqrt{3}} \frac{dx}{1+x^2} &= [\tan^{-1}(x)]_0^{\sqrt{3}} \\ &= \tan^{-1}(\sqrt{3}) - \tan^{-1}(0) \end{aligned}$$

$$\therefore \int_0^{\sqrt{3}} \frac{dx}{1+x^2} = \frac{\pi}{3}.$$

Chapter 2:

Methods of Integration

BASIC INTEGRALS (N2.1)

Q: Here is a list of basic integrals:

- | | |
|--|---|
| ① $\int x^p dx = \frac{x^{p+1}}{p+1} + C, \quad p \neq -1$ | ② $\int sec^2(x) dx = tan(x) + C$ |
| ③ $\int \frac{1}{x} dx = ln(x) + C$ | ④ $\int sec(x) tan(x) dx = sec(x) + C$ |
| ⑤ $\int e^x dx = e^x + C$ | ⑥ $\int tan(x) dx = ln sec(x) + C$ |
| ⑦ $\int a^x dx = \frac{a^x}{ln(a)} + C$ | ⑧ $\int sec(x) dx = ln sec(x) + tan(x) + C$ |
| ⑨ $\int ln(x) dx = x ln(x) - x + C$ | ⑩ $\int \frac{1}{1+x^2} dx = tan^{-1}(x) + C$ |
| ⑪ $\int sin(x) dx = -cos(x) + C$ | ⑫ $\int \frac{1}{\sqrt{1-x^2}} dx = sin^{-1}(x) + C$ |
| ⑬ $\int cos(x) dx = sin(x) + C$ | ⑭ $\int \frac{1}{x\sqrt{x^2-1}} dx = sec^{-1}(x) + C$ |

Proof. Each of these could be verified by taking the derivative of the RHS, and confirming it matches with the function in the integral.

The proof then follows from the Fundamental Theorem of Calculus. ☐

EXAMPLE 1: $\int_1^4 \frac{x^2-5}{\sqrt{x}} dx$ (E2.2)

Q: We can solve the integral $\int_1^4 \frac{x^2-5}{\sqrt{x}} dx$ using the Fundamental Theorem of Calculus.

$$\begin{aligned} \int_1^4 \frac{x^2-5}{\sqrt{x}} dx &= \int_1^4 x^{\frac{3}{2}} - 5x^{\frac{1}{2}} dx \\ &= \left[\frac{2}{5}x^{\frac{5}{2}} - 10x^{\frac{3}{2}} \right]_1^4 \\ &= \left(\frac{64}{5} - 32 \right) - \left(\frac{2}{5} - 10 \right) \end{aligned}$$

$$\therefore \int_1^4 \frac{x^2-5}{\sqrt{x}} dx = \frac{12}{5}.$$

EXAMPLE 2: $\int_{\pi/6}^{\pi/3} \sin(2x) + \cos(3x) dx$ (E2.3)

Q: We can also solve the integral $\int_{\pi/6}^{\pi/3} \sin(2x) + \cos(3x) dx$ using the Fundamental Theorem of Calculus.

First, note $\frac{d}{dx}(\cos(2x)) = -2\sin(2x)$ and $\frac{d}{dx}(\sin(3x)) = 3\cos(3x)$, it follows that $\frac{d}{dx}(-\frac{1}{2}\cos(2x)) = \sin(2x)$ and $\frac{d}{dx}(\frac{1}{3}\sin(3x)) = \cos(3x)$.

Hence

$$\begin{aligned} \int_{\pi/6}^{\pi/3} \sin(2x) + \cos(3x) dx &= \left[-\frac{1}{2}\cos(2x) + \frac{1}{3}\sin(3x) \right]_{\pi/6}^{\pi/3} \\ &= \left(\frac{1}{4} + 0 \right) - \left(-\frac{1}{4} + \frac{1}{3} \right) \\ &= \frac{1}{6}. \end{aligned}$$

SUBSTITUTION / CHANGE OF VARIABLES (T2.4)

E1: Let $u=g(x)$ be differentiable on an interval, and let $f(u)$ be continuous on the range of $g(x)$.

Then $\int f(g(x)) g'(x) dx = \int f(u) du$

and $\int_{x=a}^{x=b} f(g(x)) g'(x) dx = \int_{u=g(a)}^{u=g(b)} f(u) du.$

Proof. (if $F(u)$ be an antiderivative of $f(u)$, so that $F'(u) = f(u)$ and $\int f(u) du = F(u) + C$.)

Then, by the Chain Rule, we know that

$$\frac{d}{dx} F(g(x)) = F'(g(x)) g'(x) = f(g(x)) g'(x),$$

and so, by the Fundamental Theorem of Calculus,

$$\int f(g(x)) g'(x) dx = F(g(x)) + C = F(u) + C = \int f(u) du$$

and

$$\begin{aligned} \int_{x=a}^{x=b} f(g(x)) g'(x) dx &= [F(g(x))]_{x=a}^{x=b} \\ &= F(g(b)) - F(g(a)) \\ &= [F(u)]_{u=g(a)}^{u=g(b)} \\ \therefore \int_{x=a}^{x=b} f(g(x)) g'(x) dx &= \int_{u=g(a)}^{u=g(b)} f(u) du. \end{aligned}$$

E2: Note that if $f(u) = g(x)$, we often write $f'(u) du = g'(x) dx$. (NT2.5)

EXAMPLE 1: $\int \sqrt{2x+3} dx$ (E2.6)

E3: Substitution can be used to compute integrals such as $\int \sqrt{2x+3} dx$.

Let $u=2x+3$, so that $du=2dx$. (using the notation from above).

Then

$$\begin{aligned} \int \sqrt{2x+3} dx &= \int u^{\frac{1}{2}} \left(\frac{du}{2}\right) \\ &= \frac{1}{2} u^{\frac{3}{2}} + C \\ \therefore \int \sqrt{2x+3} dx &= \frac{1}{3} (2x+3)^{\frac{3}{2}} + C. \end{aligned}$$

EXAMPLE 2: $\int x e^{x^2} dx$ (E2.7)

E4: The integral $\int x e^{x^2} dx$ can also be solved using substitution.

Let $u=x^2$ so that $du=2x dx$.

Then

$$\begin{aligned} \int x e^{x^2} dx &= \frac{1}{2} \int e^u du \\ &= \frac{1}{2} e^u + C \\ \therefore \int x e^{x^2} dx &= \frac{1}{2} e^{x^2} + C. \end{aligned}$$

EXAMPLE 3: $\int \frac{\ln(x)}{x} dx$ (E2.8)

E5: Substitution can also be used to solve integrals like $\int \frac{\ln(x)}{x} dx$.

Let $u=\ln(x)$ so that $du=\frac{1}{x} dx$.

Then

$$\begin{aligned} \int \frac{\ln(x)}{x} dx &= \int u du \\ &= \frac{u^2}{2} + C \\ \therefore \int \frac{\ln(x)}{x} dx &= \frac{(\ln(x))^2}{2} + C. \end{aligned}$$

EXAMPLE 4: $\int \tan(x) dx$ (E2.9)

E6: We can use substitution to solve more complicated integrals like $\int \tan(x) dx$.

First, note $\tan(x) = \frac{\sin(x)}{\cos(x)}$.

Then, let $u=\cos(x)$, so that $du=-\sin(x) dx$.

It follows that

$$\begin{aligned} \int \tan(x) dx &= \int \frac{\sin(x) dx}{\cos(x)} \\ &= \int \frac{-du}{u} \\ &= -\ln|u| + C \\ \therefore \int \tan(x) dx &= -\ln|\cos(x)| + C. \end{aligned}$$

EXAMPLE 5: $\int \frac{dx}{x+\sqrt{x}}$ (E2.10)

E7: Sometimes, we might have to do two substitutions to calculate some integrals;

eg $\int \frac{dx}{x+\sqrt{x}}$

First, let $u=\sqrt{x}$, so that $x=u^2$ and $2udu=dx$.

$$\text{Then } \int \frac{dx}{x+\sqrt{x}} = \int \frac{2udu}{u^2+u} = \int \frac{2du}{u+1}.$$

Next, let $v=u+1$, so that $dv=du$. It follows

$$\begin{aligned} \int \frac{dx}{x+\sqrt{x}} &= \int \frac{2du}{u+1} \\ &= \int \frac{2dv}{v} \\ &= 2\ln|v| + C \\ &= 2\ln|u+1| + C \\ \therefore \int \frac{dx}{x+\sqrt{x}} &= 2\ln|\sqrt{x}+1| + C \end{aligned}$$

EXAMPLE 6: $\int_0^2 \frac{x dx}{\sqrt{2x^2+1}}$ (E2.11)

E8: When doing substitution, we need to change the values of the "endpoints" accordingly;

eg $\int_0^2 \frac{x}{\sqrt{2x^2+1}} dx$.

Let $u=2x^2+1$, so that $du=4x dx$.

Note that $u=1$ and $u=9$ when $x=0$ and $x=2$ respectively.

Then

$$\begin{aligned} \int_{x=0}^{x=2} \frac{x}{\sqrt{2x^2+1}} dx &= \int_{u=1}^{u=9} \frac{\frac{1}{4} du}{\sqrt{u}} \\ &= \int_{u=1}^{u=9} \frac{1}{4} u^{-\frac{1}{2}} du \\ &= \left[\frac{1}{2} u^{\frac{1}{2}} \right]_1^9 \\ &= \frac{3}{2} - \frac{1}{2} \\ \therefore \int_{x=0}^{x=2} \frac{x}{\sqrt{2x^2+1}} dx &= 1. \end{aligned}$$

EXAMPLE 7: $\int_0^1 \frac{dx}{1+3x^2}$ (E2.12)

E9: Sometimes, we might have to make a weird substitution to solve an integral;

eg $\int_0^1 \frac{dx}{1+3x^2}$.

Let $u=\sqrt{3}x$, so that $du=\sqrt{3} dx$.

Note that $x=0 \Rightarrow u=0$, and $x=1 \Rightarrow u=\sqrt{3}$.

Then

$$\begin{aligned} \int_{x=0}^{x=1} \frac{dx}{1+3x^2} &= \int_{u=0}^{u=\sqrt{3}} \frac{1}{1+u^2} \cdot \frac{du}{\sqrt{3}} \\ &= \left[\frac{1}{\sqrt{3}} \tan^{-1}(u) \right]_0^{\sqrt{3}} \\ &= \frac{1}{\sqrt{3}} \left(\frac{\pi}{3} - 0 \right) \\ \therefore \int_{x=0}^{x=1} \frac{dx}{1+3x^2} &= \frac{\pi}{3\sqrt{3}}. \end{aligned}$$

INTEGRATION BY PARTS (T2.13)

Let $f(x)$ and $g(x)$ be differentiable in an interval.

Then

$$\int f(x) g'(x) dx = f(x)g(x) - \int g(x) f'(x) dx,$$

so that

$$\int_{x=a}^{x=b} f(x) g'(x) dx = [f(x)g(x) - \int g(x) f'(x) dx]_{x=a}^{x=b}.$$

Proof. By the Product Rule,

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x).$$

Hence, by the Fundamental Theorem of Calculus,

$$\int (f'(x)g(x) + f(x)g'(x)) dx = f(x)g(x) + C,$$

which can be rewritten as

$$\int f(x)g'(x) dx = f(x)g(x) - \int g(x)f'(x) dx.$$

(the arbitrary constant C is not needed since there is an integral on both sides of the equation.) \square

If we let $u = f(x)$, $du = f'(x) dx$, $v = g(x)$ and $dv = g'(x) dx$, the above formula becomes

$$\int u dv = uv - \int v du. \quad (\text{NT2.14})$$

POLYNOMIAL X TRIGONOMETRIC OR EXPONENTIAL FUNCTION

If the integral involves a polynomial multiplied by an exponential function or a trigonometric function, try integrating by parts with u equal to the polynomial. (N2.15)

* note: multiple applications of integration by parts may be required if the degree of the polynomial is high.

EXAMPLE 1: $\int x \sin(x) dx$ (E2.16)

We employ the above strategy to evaluate the integral $\int x \sin(x) dx$.

Integrate by parts using $(u=x \quad v=-\cos(x) \quad du=1 dx \quad dv=\sin(x) dx)$

$$\begin{aligned} \int x \sin(x) dx &= -x \cos(x) - \int -\cos(x) dx \\ &= -x \cos(x) + \int \cos(x) dx \end{aligned}$$

$$\therefore \int x \sin(x) dx = -x \cos(x) + \sin(x) + C.$$

EXAMPLE 2: $\int (x^2+1)e^{2x} dx$ (E2.17)

Similarly, we can use the above strategy to evaluate the integral $\int (x^2+1)e^{2x} dx$. First, integrate by parts using $(u=x^2+1 \quad v=\frac{1}{2}e^{2x} \quad du=2x dx \quad dv=e^{2x} dx)$ to get

$$\begin{aligned} \int (x^2+1)e^{2x} dx &= \frac{1}{2}(x^2+1)e^{2x} - \int \frac{1}{2}e^{2x}(2x) dx \\ &= \frac{1}{2}(x^2+1)e^{2x} - \int xe^{2x} dx. \end{aligned}$$

To find $\int xe^{2x} dx$, we integrate by parts again, this time using $(u=x \quad v=e^{2x} dx \quad du=1 dx \quad dv=\frac{1}{2}e^{2x} dx)$:

$$\begin{aligned} \int (x^2+1)e^{2x} dx &= \frac{1}{2}(x^2+1)e^{2x} - \int xe^{2x} dx \\ &= \frac{1}{2}(x^2+1)e^{2x} - \left(\frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x} dx \right) \\ &= \frac{1}{2}(x^2+1)e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + C \end{aligned}$$

$$\therefore \int (x^2+1)e^{2x} dx = \frac{1}{4}(2x^2-2x+3)e^{2x} + C.$$

POLYNOMIAL X LOGARITHMIC OR INVERSE TRIGONOMETRIC FUNCTION

If the integral involves a polynomial multiplied by a logarithmic or inverse trigonometric function, try integrating by parts with u equal to the logarithmic/inverse trigonometric function. (N2.15)

EXAMPLE 1: $\int \ln(x) dx$ (E2.18)

We can use the above strategy to evaluate the integral $\int \ln(x) dx$.

Integrate by parts using $(u=\ln(x) \quad v=x \quad du=\frac{1}{x} dx \quad dv=1 dx)$ to get

$$\begin{aligned} \int \ln(x) dx &= x \ln(x) - \int x \cdot \frac{1}{x} dx \\ &= x \ln(x) - \int 1 dx \\ \therefore \int \ln(x) dx &= x \ln(x) - x + C. \end{aligned}$$

EXAMPLE 2: $\int_1^4 \sqrt{x} \ln(x) dx$ (E2.19)

The above strategy can even be used when the polynomial contains terms with non-integer powers,

eg $\int_1^4 \sqrt{x} \ln(x) dx$. Integrate by parts using $(u=\ln(x) \quad v=\frac{2}{3}x^{\frac{3}{2}} \quad du=\frac{1}{x} dx \quad dv=x^{\frac{1}{2}} dx)$ to get

$$\begin{aligned} \int_1^4 \sqrt{x} \ln(x) dx &= \left[\frac{2}{3}x^{\frac{3}{2}} \ln(x) - \int \frac{2}{3}x^{\frac{1}{2}} dx \right]_1^4 \\ &= \left[\frac{2}{3}x^{\frac{3}{2}} \ln(x) - \frac{4}{9}x^{\frac{3}{2}} \right]_1^4 \\ &= \left(\frac{16}{3} \ln(4) - \frac{32}{9} \right) - \left(\frac{2}{3} \ln(1) - \frac{4}{9} \right) \\ \therefore \int_1^4 \sqrt{x} \ln(x) dx &= \frac{16}{3} \ln(4) - \frac{28}{9}. \end{aligned}$$

EXPONENTIAL X SINE/COSINE FUNCTION

If the integral involves an exponential function times a sine/cosine function, try integrating by parts twice, letting u be the exponential function both times. (N2.15)

EXAMPLE: $\int e^x \sin(x) dx$ (E2.20)

We can use the above strategy to evaluate the integral $\int e^x \sin(x) dx$.

Proof. Let $I = \int e^x \sin(x) dx$.

Integrate by parts twice, first using $(u_1=e^x \quad v_1=-\cos(x) \quad du_1=e^x dx \quad dv_1=\sin(x) dx)$, and then with $(u_2=e^x \quad v_2=\sin(x) \quad du_2=e^x dx \quad dv_2=\cos(x) dx)$ to get

$$\begin{aligned} I &= \int e^x \sin(x) dx = -e^x \cos(x) + \int e^x \cos(x) dx \\ &= -e^x \cos(x) + (e^x \sin(x) - \int e^x \sin(x) dx) \\ \therefore I &= -e^x \cos(x) + e^x \sin(x) - I. \end{aligned}$$

Hence $2I = -e^x \cos(x) + e^x \sin(x) + C$,

so that $I = \frac{1}{2}(\sin(x) - \cos(x))e^x + C$.

OTHER SORTS OF PROBLEMS

EXAMPLE 1: $\int \sin^n(x) dx$ (E2.21)

\therefore we can use integration by parts to get a general formula for $\int \sin^n(x) dx$ in terms of $\int \sin^{n-2}(x) dx$.

$$\begin{aligned} \text{Let } I &= \int \sin^n(x) dx = \int \sin^{n-1}(x) \sin(x) dx \\ \text{Integrate by parts using } &\left(u = \sin^{n-1}(x) \quad v = -\cos(x) \right. \\ &\left. du = (n-1)(\sin^{n-2}(x))\cos(x) dx \quad dv = \sin(x) dx \right) \\ \text{to get} \\ I &= \int \sin^n(x) dx = -\sin^{n-1}(x)\cos(x) - \int -\cos(x)(n-1)(\sin^{n-2}(x))(\cos(x))dx \\ &= -\sin^{n-1}(x)\cos(x) + \int (n-1)(\cos^2(x))(\sin^{n-2}(x))dx \\ &= -\sin^{n-1}(x)\cos(x) + \int (n-1)(1-\sin^2(x))(\sin^{n-2}(x))dx \\ \therefore I &= -\sin^{n-1}(x)\cos(x) + (n-1) \int \sin^{n-2}(x) dx - (n-1)I. \end{aligned}$$

Hence

$$(n-1)I + I = nI = -\sin^{n-1}(x)\cos(x) + (n-1) \int \sin^{n-2}(x) dx,$$

so that

$$I = -\frac{1}{n} \sin^{n-1}(x)\cos(x) + \frac{n-1}{n} \int \sin^{n-2}(x) dx.$$

\therefore In particular, we can use the attained above formula to evaluate $\int \sin^2(x) dx$ and $\int \sin^4(x) dx$.

In particular, when $n=2$, we get

$$\begin{aligned} \int \sin^2(x) dx &= -\frac{1}{2} \sin(x)\cos(x) + \frac{1}{2} \int 1 dx \\ \therefore \int \sin^2(x) dx &= -\frac{1}{2} \sin(x)\cos(x) + \frac{1}{2}x + C. \end{aligned}$$

when $n=4$, we get

$$\begin{aligned} \int \sin^4(x) dx &= -\frac{1}{4} \sin^3(x)\cos(x) + \frac{3}{4} \int \sin^2(x) dx \\ &= -\frac{1}{4} \sin^3(x)\cos(x) + \frac{3}{4} \left(-\frac{1}{2} \sin(x)\cos(x) + \frac{1}{2}x \right) + C \\ \therefore \int \sin^4(x) dx &= -\frac{1}{4} \sin^3(x)\cos(x) - \frac{3}{8} \sin(x)\cos(x) + \frac{3}{8}x + C. \end{aligned}$$

EXAMPLE 2: $\int \sec^n(x) dx$ (E2.22)

\therefore In a similar manner to the above, we can use integration by parts to attain a general formula for $\int \sec^n(x) dx$ in terms of $\int \sec^{n-2}(x) dx$.

$$\text{Let } I = \int \sec^n(x) dx = \int \sec^{n-2}(x) \sec^2(x) dx.$$

$$\begin{aligned} \text{Integrate by parts using } &\left(u = \sec^{n-2}(x) \quad v = \tan(x) \right. \\ &\left. du = (n-2)(\sec^{n-3}(x))(\sec(x)\tan(x)) \quad dv = \sec^2(x) dx \right) \\ \text{to get} \\ &= (n-2)(\sec^{n-2}(x))(\tan(x)) \end{aligned}$$

$$\begin{aligned} I &= \int \sec^n(x) dx = \sec^{n-2}(x)\tan(x) - \int (n-2)(\sec^{n-2}(x))(\tan^2(x))dx \\ &= \sec^{n-2}(x)\tan(x) - \int (n-2)(\sec^{n-2}(x))(\sec^2 x - 1)dx \\ \therefore I &= \sec^{n-2}(x)\tan(x) - (n-2)I + (n-2) \int \sec^{n-2}(x) dx. \end{aligned}$$

Hence

$$(n-1)I = \sec^{n-2}(x)\tan(x) + (n-2) \int \sec^{n-2}(x) dx,$$

so that

$$I = \frac{1}{n-1} \sec^{n-2}(x)\tan(x) + \frac{n-2}{n-1} \int \sec^{n-2}(x) dx.$$

\therefore We can use the above formula to evaluate the integral $\int \sec^3(x) dx$.

In particular when $n=3$, we have that

$$\begin{aligned} \int \sec^3(x) dx &= \frac{1}{2} \sec(x)\tan(x) + \frac{1}{2} \int \sec(x) dx \\ \therefore \int \sec^3(x) dx &= \frac{1}{2} \sec(x)\tan(x) + \frac{1}{2} \ln |\sec(x) + \tan(x)| + C. \end{aligned}$$

TRIGONOMETRIC INTEGRALS

$$\int f(\sin(x)) \cos^{2n+1}(x) dx \quad \text{OR}$$

$$\int f(\cos(x)) \sin^{2n+1}(x) dx$$

💡 To find $\int f(\sin(x)) \cos^{2n+1}(x) dx$, write $\cos^{2n+1}(x) = (1-\sin^2(x))^n \cos(x)$ and then try the substitution $u=\sin(x)$, $du=\cos(x)dx$. (N2.23 (1))

💡 Similarly, to find $\int f(\cos(x)) \sin^{2n+1}(x) dx$, write $\sin^{2n+1}(x) = (1-\cos^2(x))^n \sin(x)$ and then try the substitution $u=\cos(x)$, $du=-\sin(x)dx$. (N2.23 (2))

$$\text{EXAMPLE : } \int_0^{\pi/3} \frac{\sin^3(x)}{\cos^2(x)} dx \quad (\text{E2.24})$$

We can use the above strategy to solve the integral $\int_0^{\pi/3} \frac{\sin^3(x)}{\cos^2(x)} dx$.

Make the substitution $u=\cos(x)$, so that $du=-\sin(x)dx$. Then

$$\begin{aligned} \int_0^{\pi/3} \frac{\sin^3(x)}{\cos^2(x)} dx &= \int_{x=0}^{x=\frac{\pi}{3}} \frac{(1-\cos^2(x))\sin(x)}{\cos^2(x)} dx \\ &= \int_{u=1}^{u=\frac{1}{2}} \frac{(1-u^2) du}{u^2} \\ &= \int_{u=1}^{u=\frac{1}{2}} -\frac{1}{u^2} + 1 du \\ &= \left[\frac{1}{u} + u \right]_1^{\frac{1}{2}} \\ &= (2 + \frac{1}{2}) - (1+1) \end{aligned}$$

$$\therefore \int_0^{\pi/3} \frac{\sin^3(x)}{\cos^2(x)} dx = \frac{1}{2}.$$

$$\int \sin^{2m}(x) \cos^{2n}(x) dx$$

💡 To find $\int \sin^{2m}(x) \cos^{2n}(x) dx$, try using the trigonometric identities $\sin^2(\theta) = \frac{1}{2} - \frac{1}{2}\cos(2\theta)$ and $\cos^2(\theta) = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$.

💡 Alternatively, write $\cos^{2n}(x) = (1-\sin^2(x))^n$ and use the formula from E2.21. (N2.23 (3))

$$\text{EXAMPLE : } \int_0^{\pi/4} \sin^6(x) dx \quad (\text{E2.25})$$

We can use either strategy 1 or 2 to evaluate the integral $\int_0^{\pi/4} \sin^6(x) dx$.

We use strategy 1.

Note that

$$\begin{aligned} \int_0^{\pi/4} \sin^6(x) dx &= \int_0^{\pi/4} \left(\frac{1}{2} - \frac{1}{2}\cos(2x) \right)^3 dx \quad (\text{using the half-angle formula}) \\ &= \int_0^{\pi/4} \frac{1}{8} - \frac{3}{8}\cos(2x) + \frac{3}{8}\cos^2(2x) - \frac{1}{8}\cos^3(2x) dx \\ &= \int_0^{\pi/4} \frac{1}{8} - \frac{2}{8}\cos(2x) + \frac{3}{8}\left(\frac{1}{2} + \frac{1}{2}\cos(4x)\right) - \frac{1}{8}(1-\sin^2(2x))\cos(2x) dx \\ &= \int_0^{\pi/4} \frac{5}{16} - \frac{1}{2}\cos(2x) + \frac{3}{16}\cos(4x) + \frac{1}{8}\sin^2(2x)\cos(2x) dx \\ &= \left[\frac{5}{16}x - \frac{1}{4}\sin(2x) + \frac{3}{64}\sin(4x) + \frac{1}{48}\sin^3(2x) \right]_0^{\pi/4} \\ &= \frac{5\pi}{64} - \frac{1}{4} + \frac{1}{48} \end{aligned}$$

$$\therefore \int_0^{\pi/4} \sin^6(x) dx = \frac{5\pi}{64} - \frac{11}{48}.$$

$$\int f(\tan(x)) \sec^{2n+2}(x) dx$$

💡 To find $\int f(\tan(x)) \sec^{2n+2}(x) dx$, write $\sec^{2n+2}(x) = (1+\tan^2(x))^n \sec^2(x)$ and try the substitution $u=\tan(x)$, $du=\sec^2(x)$. (N2.23 (4))

$$\text{EXAMPLE 1 : } \int_0^{\pi/4} \tan^4(x) dx \quad (\text{E2.26})$$

💡 The above strategy can be used to solve the integral $\int_0^{\pi/4} \tan^4(x) dx$.

Note first that

$$\begin{aligned} \int_0^{\pi/4} \tan^4(x) dx &= \int_0^{\pi/4} \tan^2(x) \sec^2(x) - \tan^2(x) dx \\ &= \int_0^{\pi/4} \tan^2(x) \sec^2(x) - \sec^2(x) + 1 dx. \end{aligned}$$

To find $\int \tan^2(x) \sec^2(x) dx$, make the substitution $u=\tan(x)$, $du=\sec^2(x)dx$ to get that

$$\begin{aligned} \int \tan^2(x) \sec^2(x) dx &= \int u^2 du \\ &= \frac{u^3}{3} + C \end{aligned}$$

$$\therefore \int \tan^2(x) \sec^2(x) dx = \frac{\tan^3(x)}{3} + C.$$

It follows that

$$\begin{aligned} \int_0^{\pi/4} \tan^4(x) dx &= \left[\frac{\tan^3(x)}{3} - \tan(x) + x \right]_0^{\pi/4} \\ &= \frac{1}{3} - 1 + \frac{\pi}{4} \\ \therefore \int_0^{\pi/4} \tan^4(x) dx &= -\frac{2}{3} + \frac{\pi}{4}. \end{aligned}$$

$$\text{EXAMPLE 2 : } \int_0^{\pi/4} \frac{\sec^4(x)}{\sqrt{\tan(x)+1}} dx \quad (\text{E2.27})$$

💡 We can again use the above strategy to evaluate the integral $\int_0^{\pi/4} \frac{\sec^4(x)}{\sqrt{\tan(x)+1}} dx$.

Make the substitution $u=\tan(x)$, so that $du=\sec^2(x)dx$.

Then

$$\begin{aligned} \int_0^{\pi/4} \frac{\sec^4(x)}{\sqrt{\tan(x)+1}} dx &= \int_{x=0}^{x=\frac{\pi}{4}} \frac{(\tan^2(x)+1)\sec^2(x)}{\sqrt{\tan(x)+1}} dx \\ &= \int_{u=0}^{u=1} \frac{(u^2+1)}{\sqrt{u+1}} du. \end{aligned}$$

Next, make the substitution $v=u+1$, so that $du=dv$.

Then

$$\begin{aligned} \int_0^{\pi/4} \frac{\sec^4(x)}{\sqrt{\tan(x)+1}} dx &= \int_{u=0}^{u=1} \frac{u^2+1}{\sqrt{u+1}} du \\ &= \int_{v=1}^{v=2} \frac{(v-1)^2+1}{\sqrt{v}} dv \\ &= \int_1^2 \sqrt{\frac{2}{v}-2\sqrt{\frac{1}{v}}+2} dv \\ &= \left[\frac{2}{3}v^{\frac{3}{2}} - \frac{4}{3}\sqrt{v} + 4\sqrt{v} \right]_1^2 \\ &= \left(\frac{2}{3}(4\sqrt{2}) - \frac{4}{3}(2\sqrt{2}) + 4(\sqrt{2}) \right) - \left(\frac{2}{3} - \frac{4}{3} + 4 \right) \\ \therefore \int_0^{\pi/4} \frac{\sec^4(x)}{\sqrt{\tan(x)+1}} dx &= \frac{44\sqrt{2}-46}{15}. \end{aligned}$$

$$\int f(\sec(x)) \tan^{2n+1}(x) dx$$

💡 To solve $\int f(\sec(x)) \tan^{2n+1}(x) dx$, write $\tan^{2n+1}(x) = \frac{(\sec^2(x)-1)^n}{\sec(x)} \sec(x) \tan(x)$ and try the substitution $u=\sec(x)$, $du=\sec(x)\tan(x)dx$. (N2.23 (5))

$$\int \sec^{2n+1}(x) \tan^{2n}(x) dx$$

💡 To solve $\int \sec^{2n+1}(x) \tan^{2n}(x) dx$, write $\tan^{2n}(x) = (\sec^2(x)-1)^n$ and use the formula from E2.22. (N2.23 (6))

$\int \sin(ax) \sin(bx) dx$, $\int \cos(ax) \cos(bx) dx$

OR $\int \sin(ax) \cos(bx) dx$ (N2.28)

To evaluate $\int \sin(ax) \sin(bx) dx$, $\int \cos(ax) \cos(bx) dx$ or

$\int \sin(ax) \cos(bx) dx$, use the identities

$$\textcircled{1} \quad \cos(A-B) - \cos(A+B) = 2\sin(A)\sin(B);$$

$$\textcircled{2} \quad \cos(A-B) + \cos(A+B) = 2\cos(A)\cos(B); \text{ or}$$

$$\textcircled{3} \quad \sin(A-B) + \sin(A+B) = 2\sin(A)\cos(B).$$

EXAMPLE : $\int_0^{\pi/6} \cos(3x) \cos(2x) dx$ (E2.29)

We can employ the above strategy to evaluate the integral $\int_0^{\pi/6} \cos(3x) \cos(2x) dx$.

By $\textcircled{2}$ in the above, we have that

$$2\cos(3x) \cos(2x) = \cos(3x-2x) + \cos(3x+2x) \\ = \cos(x) + \cos(5x).$$

Hence

$$\int_0^{\pi/6} \cos(2x) \cos(3x) dx = \int_0^{\pi/6} \frac{1}{2}(\cos(x) + \cos(5x)) dx \\ = \left[\frac{1}{2}\sin(x) + \frac{1}{10}\sin(5x) \right]_0^{\pi/6} \\ = \frac{1}{4} + \frac{1}{20}$$

$$\therefore \int_0^{\pi/6} \cos(2x) \cos(3x) dx = \frac{3}{10}.$$

WEIERSTRASS SUBSTITUTION (N2.30)

The Weierstrass substitution is letting $u = \tan(\frac{x}{2})$,

so that $x = 2\tan^{-1}(u)$, $dx = \frac{2}{1+u^2} du$.

Additionally, it implies $\sin(\frac{x}{2}) = \frac{u}{\sqrt{1+u^2}}$ & $\cos(\frac{x}{2}) = \frac{1}{\sqrt{1+u^2}}$

so that

$$\textcircled{1} \quad \sin(x) = 2\sin(\frac{x}{2})\cos(\frac{x}{2}) \\ = 2\left(\frac{u}{\sqrt{1+u^2}}\right)\left(\frac{1}{\sqrt{1+u^2}}\right)$$

$$\therefore \sin(x) = \frac{2u}{1+u^2}; \text{ and}$$

$$\textcircled{2} \quad \cos(x) = \cos^2(\frac{x}{2}) - \sin^2(\frac{x}{2}) \\ = \left(\frac{1}{\sqrt{1+u^2}}\right)^2 - \left(\frac{u}{\sqrt{1+u^2}}\right)^2$$

$$\therefore \cos(x) = \frac{1-u^2}{1+u^2}.$$

EXAMPLE: $\int \frac{dx}{1-\cos(x)}$ (E2.31)

The Weierstrass substitution can be used to solve some integrals;

e.g $\int \frac{dx}{1-\cos(x)}$.

Let $u = \tan(\frac{x}{2})$, so that $dx = \frac{2}{1+u^2} du$, and

$$\cos(x) = \frac{1-u^2}{1+u^2}.$$

Then

$$\int \frac{dx}{1-\cos(x)} = \int \frac{1}{1-\left(\frac{1-u^2}{1+u^2}\right)} \left(\frac{2}{1+u^2} du\right)$$

$$= \int \frac{2}{1+u^2-(1-u^2)} du$$

$$= \int \frac{du}{u^2}$$

$$= -\frac{1}{u} + C$$

$$\therefore \int \frac{dx}{1-\cos(x)} = -\cot(\frac{x}{2}) + C.$$

INVERSE TRIGONOMETRIC SUBSTITUTION

$$\int f(\sqrt{a^2 - b^2(x+c)^2}) dx$$

For an integral involving $\sqrt{a^2 - b^2(x+c)^2}$, try the substitution $\theta = \sin^{-1}(\frac{b(x+c)}{a})$, so that

- ① $a \sin \theta = b(x+c)$;
- ② $a \cos \theta = \sqrt{a^2 - b^2(x+c)^2}$; and
- ③ $a \cos \theta d\theta = b dx$. (N2.32 (2))

EXAMPLE 1: $\int_0^1 \frac{dx}{(4-3x^2)^{3/2}}$ (E2.33)

The above method can be used to evaluate the integral $\int_0^1 \frac{dx}{(4-3x^2)^{3/2}}$. Let $2\sin\theta = \sqrt{3}x$, so that $2\cos\theta = \sqrt{4-3x^2}$ and $2\cos\theta d\theta = \sqrt{3} dx$.

Then

$$\begin{aligned} \int_{x=0}^{x=1} \frac{dx}{(4-3x^2)^{3/2}} &= \int_{\theta=0}^{\theta=\pi/2} \frac{\frac{2}{\sqrt{3}} \cos \theta d\theta}{(2\cos\theta)^3} \\ &= \int_0^{\pi/3} \frac{1}{4\sqrt{3}} \sec^2 \theta d\theta \\ &= \left[\frac{1}{4\sqrt{3}} \tan \theta \right]_0^{\pi/3} \\ \therefore \int_{x=0}^{x=1} \frac{dx}{(4-3x^2)^{3/2}} &= \frac{1}{4}. \end{aligned}$$

EXAMPLE 2: $\int_2^3 (4x-x^2)^{3/2} dx$ (E2.36)

The above strategy can also be applied to more complex integrals, like $\int_2^3 (4x-x^2)^{3/2} dx$.

Let $2\sin\theta = x-2$, so that $2\cos\theta = \sqrt{4x-x^2}$ and $2\cos\theta d\theta = dx$. Then

$$\begin{aligned} \int_{x=2}^{x=3} (4x-x^2)^{3/2} dx &= \int_{\theta=0}^{\theta=\pi/6} (2\cos\theta)^3 (2\cos\theta d\theta) \\ &= \int_0^{\pi/6} 16 \cos^4 \theta d\theta \\ &= \int_0^{\pi/6} 4(1+\cos 2\theta)^2 d\theta \\ &= \int_0^{\pi/6} 4 + 8\cos 2\theta + 4\cos^2 2\theta d\theta \\ &= \int_0^{\pi/6} 4 + 8\cos 2\theta + 2 + 2\cos 4\theta d\theta \\ &= [6\theta + 4\sin 2\theta + \frac{1}{2}\sin 4\theta]_0^{\pi/6} \\ \therefore \int_{x=2}^{x=3} (4x-x^2)^{3/2} dx &= \pi + \frac{9\sqrt{3}}{4}. \end{aligned}$$

$$\int f(\sqrt{a^2 + b^2(x+c)^2}) dx$$

For an integral involving $\sqrt{a^2 + b^2(x+c)^2}$ (or $\sqrt{a^2+b^2(x+c)^2}$), try the substitution $\theta = \tan^{-1}(\frac{b(x+c)}{a})$, so that

- ① $a \tan \theta = b(x+c)$;
- ② $a \sec \theta = \sqrt{a^2 + b^2(x+c)^2}$; and
- ③ $a \sec^2 \theta d\theta = b dx$. (N2.32 (1))

EXAMPLE: $\int_1^{\sqrt{3}} \frac{dx}{x^2 \sqrt{x^2+3}}$ (E2.34)

We can use the above strategy to evaluate the integral $\int_1^{\sqrt{3}} \frac{dx}{x^2 \sqrt{x^2+3}}$. Let $\sqrt{3} \tan\theta = x$, so that $\sqrt{3} \sec\theta = \sqrt{x^2+3}$ and $\sqrt{3} \sec^2 \theta d\theta = dx$. Then

$$\begin{aligned} \int_{x=1}^{x=\sqrt{3}} \frac{dx}{x^2 \sqrt{x^2+3}} &= \int_{\theta=\pi/6}^{\theta=\pi/2} \frac{\sqrt{3} \sec^2 \theta d\theta}{3 \tan^2 \theta (\sqrt{3} \sec \theta)} \\ &= \int_{\theta=\pi/6}^{\pi/2} \frac{1}{3} \frac{\sec \theta}{\tan^2 \theta} d\theta \\ \therefore \int_{x=1}^{x=\sqrt{3}} \frac{dx}{x^2 \sqrt{x^2+3}} &= \int_{\theta=\pi/6}^{\pi/2} \frac{1}{3} \frac{\cos \theta}{\sin^2 \theta} d\theta \end{aligned}$$

Then, let $u = \sin\theta$, so that $du = \cos\theta d\theta$. It follows that

$$\begin{aligned} \int_{x=1}^{x=\sqrt{3}} \frac{dx}{x^2 \sqrt{x^2+3}} &= \int_{\theta=\pi/6}^{\theta=\pi/2} \frac{1}{3} \frac{\cos \theta}{\sin^2 \theta} d\theta \\ &= \int_{u=\frac{1}{2}}^{\frac{1}{\sqrt{3}}} \frac{1}{3} \frac{1}{u^2} du \\ &= \left[-\frac{1}{3u} \right]_{1/2}^{1/\sqrt{3}} \\ \therefore \int_{x=1}^{x=\sqrt{3}} \frac{dx}{x^2 \sqrt{x^2+3}} &= \frac{2-\sqrt{2}}{3}. \end{aligned}$$

$$\int f(\sqrt{b^2(x+c)^2 - a^2}) dx$$

For an integral involving $\sqrt{b^2(x+c)^2 - a^2}$, try the substitution $\theta = \sec^{-1}(\frac{b(x+c)}{a})$, so that

- ① $a \sec \theta = b(x+c)$;
- ② $a \tan \theta = \sqrt{b^2(x+c)^2 - a^2}$; and
- ③ $a \sec \theta \tan \theta = b dx$. (N2.32 (3))

EXAMPLE: $\int_2^4 \frac{\sqrt{x^2-4}}{x^2} dx$ (E2.35)

The above strategy can be used to evaluate the integral $\int_2^4 \frac{\sqrt{x^2-4}}{x^2} dx$.

Let $2\sec\theta = x$, so that $2\tan\theta = \sqrt{x^2-4}$ and $2\sec\theta \tan\theta d\theta = dx$.

Then

$$\begin{aligned} \int_{x=2}^{x=4} \frac{\sqrt{x^2-4}}{x^2} dx &= \int_{\theta=0}^{\theta=\pi/3} \frac{\tan^2 \theta \sec \theta}{\sec^2 \theta} d\theta \\ &= \int_0^{\pi/3} \frac{\tan^2 \theta}{\sec \theta} d\theta \\ &= \int_0^{\pi/3} \frac{\sec^2 \theta - 1}{\sec \theta} d\theta \\ &= \int_0^{\pi/3} (\sec \theta - \cos \theta) d\theta \\ &= \left[\ln |\sec \theta + \tan \theta| - \sin \theta \right]_0^{\pi/3} \\ \therefore \int_{x=2}^{x=4} \frac{\sqrt{x^2-4}}{x^2} dx &= \ln(2+\sqrt{3}) - \frac{\sqrt{3}}{2}. \end{aligned}$$

PARTIAL FRACTIONS (N2.37)

We can find the integral of a rational function $\frac{f(x)}{g(x)}$ (where f & g are polynomials) as follows:

① Use long division to find polynomials $q(x)$ and $r(x)$ such that

$$f(x) = g(x)q(x) + r(x),$$

where $\deg(r) < \deg(g)$.

* if $\deg(f) < \deg(g)$, then $q(x)=0$ and $r(x)=f(x)$.

② Then, note that $\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}$, and it follows that

$$\int \frac{f(x)}{g(x)} dx = \int q(x) + \frac{r(x)}{g(x)} dx.$$

③ Next, factor $g(x)$ into linear and irreducible quadratic factors. *we can always do this! (MATH 145 R34)

④ Finally, split $\frac{r(x)}{g(x)}$ into its "partial fraction decomposition";

i.e. write $\frac{r(x)}{g(x)}$ as a sum of terms so that

i) for each linear factor $(ax+b)^k$, we have the k terms

$$\frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \dots + \frac{A_k}{(ax+b)^k}; \text{ and}$$

ii) for each irreducible quadratic factor $(ax^2+bx+c)^k$, we have the k terms

$$\frac{B_1x+C_1}{(ax^2+bx+c)} + \frac{B_2x+C_2}{(ax^2+bx+c)^2} + \dots + \frac{B_kx+C_k}{(ax^2+bx+c)^k}.$$

eg if $g(x) = x(x-1)^3(x^2+2x+3)^2$, then we would write $\frac{r(x)}{g(x)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3} + \frac{Ex+F}{(x^2+2x+3)} + \frac{Gx+H}{(x^2+2x+3)^2}$, and then solve for the various constants. (E2.38)

⑤ From here, we can solve the integral.

EXAMPLE 1: $\int_2^3 \frac{x-7}{(x-1)^2(x+2)} dx$ (E2.39)

The above strategy can be used to solve the integral $\int_2^3 \frac{x-7}{(x-1)^2(x+2)} dx$.

First, we need to find A, B, C such that $\frac{x-7}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$,

or $x-7 = A(x-1)(x+2) + B(x+2) + C(x-1)^2$.

Equating coefficients, we get that

$$\begin{cases} A+C=0 \\ A+B-2C=1 \\ -2A+2B+C=-7 \end{cases}$$

Solving this system gives us that $A=1, B=-2$ & $C=-1$.

Hence,

$$\begin{aligned} \int_2^3 \frac{x-7}{(x-1)^2(x+2)} dx &= \int_2^3 \left(\frac{1}{x-1} - \frac{2}{(x-1)^2} - \frac{1}{x+2} \right) dx \\ &= \left[\ln(x-1) - \frac{2}{x-1} - \ln(x+2) \right]_2^3 \end{aligned}$$

$$\therefore \int_2^3 \frac{x-7}{(x-1)^2(x+2)} dx = \ln\left(\frac{8}{5}\right) - 1.$$

EXAMPLE 2: $\int_1^{\sqrt{3}} \frac{x^4-x^3+1}{x^3+x} dx$ (E2.40)

As mentioned in step ① of the method, sometimes long division is needed before partial fraction decomposition can be carried out;

$$\text{eg } \int_1^{\sqrt{3}} \frac{x^4-x^3+1}{x^3+x} dx.$$

First, use polynomial long division to get that

$$\frac{x^4-x^3+1}{x^3+x} = (x-1) + \frac{-x^2+x+1}{x^3+x}.$$

Then, note that to get

$$\frac{-x^2+x+1}{x^3+x} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

we need

$$-x^2+x+1 = A(x^2+1) + (Bx+C)x.$$

Equating coefficients gives $A+B=-1, C=1$ and $A=1$.

Solving these equations gives $A=1, B=-2$ and $C=1$.

Thus

$$\begin{aligned} \int_1^{\sqrt{3}} \frac{x^4-x^3+1}{x^3+x} dx &= \int_1^{\sqrt{3}} \left(x-1 + \frac{1}{x} - \frac{2x}{x^2+1} + \frac{1}{x^2+1} \right) dx \\ &= \left[\frac{1}{2}x^2 - x + (\ln x) - (\ln(x^2+1)) + \tan^{-1}(x) \right]_1^{\sqrt{3}} \end{aligned}$$

$$\therefore \int_1^{\sqrt{3}} \frac{x^4-x^3+1}{x^3+x} dx = 2 - \sqrt{3} + (\ln(\frac{\sqrt{3}}{2}) + \frac{\pi}{12}).$$

EXAMPLE 3: $\int_1^2 \frac{x^5+x^4-2x^3-2x^2-5x-25}{x^2(x^2-2x+5)^2} dx$ (E2.41)

Partial fraction decomposition can also be applied in tandem with substitution to solve integrals;

$$\text{eg } \int_1^2 \frac{x^5+x^4-2x^3-2x^2-5x-25}{x^2(x^2-2x+5)^2} dx.$$

Let I be the above integral.

To get

$$\frac{x^5+x^4-2x^3-2x^2-5x-25}{x^2(x^2-2x+5)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2-2x+5} + \frac{Ex+F}{(x^2-2x+5)^2}$$

we need

$$x^5+x^4-2x^3-2x^2-5x-25 = Ax(x^2-2x+5)^2 + B(x^2-2x+5)^2 + (Cx+D)(x^2-2x+5)(x^2) + (Ex+F)(x^2).$$

Comparing coefficients, we get that $A+C=1; -4A+B-2C+D=1; 14A-4B+8C-2D+E=-2; -20A+14B+SD+F=-2; 25A-20B=-5$; and $25B=-25$.

Solving these equations gives $A=-1, B=-1, C=2, D=2, E=2$ and $F=-10$.

Hence

$$\begin{aligned} I &= \int_1^2 \left(-\frac{1}{x} - \frac{1}{x^2} + \frac{2x+2}{x^2-2x+5} + \frac{2x-18}{(x^2-2x+5)^2} \right) dx \\ &= \int_1^2 \left(-\frac{1}{x} - \frac{1}{x^2} + \frac{(2x-2)+4}{x^2-2x+5} + \frac{(2x-2)-16}{(x^2-2x+5)^2} \right) dx \end{aligned}$$

$$I = \int_1^2 \left(-\frac{1}{x} - \frac{1}{x^2} + \frac{2x-2}{x^2-2x+5} + \frac{4}{x^2-2x+5} + \frac{2x-2-16}{(x^2-2x+5)^2} \right) dx.$$

To compute $\int \frac{2x-2}{x^2-2x+5} dx$ and $\int \frac{2x-2}{(x^2-2x+5)^2} dx$, make the substitution $u = x^2-2x+5$, so that $du = (2x-2)dx$.

$$\text{Then } \int \frac{2x-2}{x^2-2x+5} dx = \int \frac{du}{u} = \ln|u| + c = \ln|x^2-2x+5| + c;$$

$$\text{and } \int \frac{2x-2}{(x^2-2x+5)^2} dx = \int \frac{du}{u^2} = \frac{-1}{u} + c = \frac{-1}{x^2-2x+5} + c.$$

To compute $\int \frac{4dx}{x^2-2x+5}$ and $\int \frac{16dx}{(x^2-2x+5)^2}$, make the substitution $2\tan\theta = x-1$, so that $2\sec^2\theta d\theta = dx$.

$$\text{Then } \int \frac{4dx}{x^2-2x+5} = \int \frac{4 \cdot 2\sec^2\theta d\theta}{(2\sec\theta)^2} = \int 2d\theta = 2\theta + c = 2\tan^{-1}\left(\frac{x-1}{2}\right) + c$$

and

$$\int \frac{16dx}{(x^2-2x+5)^2} = \int \frac{16 \cdot 2\sec^2\theta d\theta}{(2\sec\theta)^4} = \int \frac{2d\theta}{\sec^2\theta} = \int 2\cos^2\theta d\theta = \int (1+\cos(2\theta))d\theta$$

$$= \theta + \frac{1}{2}\sin(2\theta) + c = \theta + \sin\theta\cos\theta + c = \tan^{-1}\left(\frac{x-1}{2}\right) + \frac{2(x-1)}{x^2-2x+5} + c.$$

Thus

$$I = \left[-\ln(x) + \frac{1}{x} + \ln(x^2-2x+5) + 2\tan^{-1}\left(\frac{x-1}{2}\right) - \frac{1}{x^2-2x+5} - \frac{2(x-1)}{x^2-2x+5} \right]_1^2$$

$$\therefore I = \ln\left(\frac{8}{5}\right) - \frac{17}{20} + \tan^{-1}\left(\frac{1}{2}\right).$$

EXAMPLE 4: $\int \frac{\sec^3(x)}{\sec(x)-1} dx$ (E2.42)

Q: Partial fraction decomposition can also be applied even if the function is not rational (at first);

$$\text{eg } \int \frac{\sec^3(x)}{\sec(x)-1} dx.$$

First, note that

$$\begin{aligned}\int \frac{\sec^3(x)}{\sec(x)-1} dx &= \int \frac{\sec^3(x)}{\sec(x)-1} \cdot \frac{\sec(x)+1}{\sec(x)+1} dx \\ &= \int \frac{\sec^4(x) + \sec^3(x)}{\sec^2(x) - 1} dx \\ &= \int \frac{\sec^4(x) + \sec^3(x)}{\tan^2(x)} dx\end{aligned}$$

$$\therefore \int \frac{\sec^3(x)}{\sec(x)-1} dx = \int \frac{\sec^4(x)}{\tan^2(x)} dx + \int \frac{\sec^3(x)}{\tan^2(x)} dx.$$

To find $\int \frac{\sec^4(x)}{\tan^2(x)} dx$, make the substitution $u = \tan(x)$, so that $du = \sec^2(x) dx$. Then

$$\begin{aligned}\int \frac{\sec^4(x)}{\tan^2(x)} dx &= \int \frac{(\tan^2(u)+1) \sec^2(x)}{\tan^2(u)} du \\ &= \int \frac{(u^2+1)}{u^2} du \\ &= \int 1 + \frac{1}{u^2} du \\ &= u - \frac{1}{u} + c\end{aligned}$$

$$\therefore \int \frac{\sec^4(x)}{\tan^2(x)} dx = \tan(x) - \cot(x) + c;$$

To find $\int \frac{\sec^3(x)}{\tan^2(x)} dx$, make the substitution $v = \sin(x)$,

so that $dv = \cos(x) dx$. Then

$$\begin{aligned}\int \frac{\sec^3(x)}{\tan^2(x)} dx &= \int \frac{dx}{\cos(x)\sin^2(x)} \\ &= \int \frac{\cos(x) dx}{(1-\sin^2(x))\sin^2(x)} \\ &= \int \frac{dv}{(1-v^2)v^2}.\end{aligned}$$

Then, note that $\frac{1}{(1-v^2)v^2} = \frac{1/2}{1-v} + \frac{1/2}{1+v} + \frac{0}{v} + \frac{1}{v^2}$ (by partial fraction decomposition), so that

$$\begin{aligned}\int \frac{\sec^3(x)}{\tan^2(x)} dx &= \int \frac{1/2}{1-v} + \frac{1/2}{1+v} + \frac{1}{v^2} dv \\ &= -\frac{1}{2} \ln|1-v| + \frac{1}{2} \ln|1+v| - \frac{1}{v} + c\end{aligned}$$

$$\therefore \int \frac{\sec^3(x)}{\tan^2(x)} dx = -\frac{1}{2} \ln|1-\sin(x)| + \frac{1}{2} \ln|1+\sin(x)| - \csc(x) + c.$$

Finally, it follows that

$$\begin{aligned}\int \frac{\sec^3(x)}{\sec(x)-1} dx &= \int \frac{\sec^4(x)}{\tan^2(x)} dx + \int \frac{\sec^3(x)}{\tan^2(x)} dx \\ &= \tan(x) - \cot(x) - \frac{1}{2} \ln|1-\sin(x)| + \frac{1}{2} \ln|1+\sin(x)| - \csc(x) + c\end{aligned}$$

$$\therefore \int \frac{\sec^3(x)}{\sec(x)-1} dx = \tan(x) - \cot(x) + \ln|\sec(x) + \tan(x)| - \csc(x) + c.$$

APPROXIMATE INTEGRATION (D2.43 (1))

Let f be integrable on $[a, b]$. Then, we can approximate the integral of f on $[a, b]$ by any Riemann sum.

$$I = \int_a^b f(x) dx \approx \sum_{k=1}^n f(c_k) \Delta_k x,$$

where $a = x_0 < x_1 < \dots < x_n = b$, $\Delta_k x = x_{k-1} - x_k$ and $c_k \in [x_{k-1}, x_k] \quad \forall k \in \{1, 2, \dots, n\}$.

LEFT ENDPOINT APPROXIMATION (D2.43 (2))

Let f be integrable on $[a, b]$.

Then, the " n th left endpoint approximation" for $I = \int_a^b f$, denoted by L_n , is defined to be

$$L_n = \sum_{k=1}^n f(x_{k-1}) \Delta_k x;$$

i.e.

$$L_n = \frac{b-a}{n} \sum_{k=1}^n f(a + \frac{b-a}{n}(k-1)).$$

*the subintervals are equally sized!

RIGHT ENDPOINT APPROXIMATION (D2.43 (3))

Let f be integrable on $[a, b]$.

Then, the " n th right endpoint approximation" for $I = \int_a^b f$, denoted by R_n , is defined to be

$$R_n = \sum_{k=1}^n f(x_k) \Delta_k x;$$

i.e.

$$R_n = \frac{b-a}{n} \sum_{k=1}^n f(a + \frac{b-a}{n}k).$$

MIDPOINT APPROXIMATION (D2.43 (4))

Let f be integrable on $[a, b]$.

Then, the " n th midpoint approximation" for $I = \int_a^b f$, denoted by M_n , is defined to be

$$M_n = \sum_{k=1}^n f(\frac{x_{k-1} + x_k}{2}) \Delta_k x;$$

i.e.

$$M_n = \frac{b-a}{n} \sum_{k=1}^n f(a + \frac{b-a}{n}(\frac{k-1}{2})).$$

TRAPEZOIDAL APPROXIMATION (D2.44)

Let f be integrable on $[a, b]$.

Then, the " n th trapezoidal approximation" for $I = \int_a^b f$, denoted by T_n , is defined by

$$T_n = \sum_{k=1}^n \frac{f(x_{k-1}) + f(x_k)}{2} \Delta_k x;$$

i.e.

$$T_n = \frac{b-a}{n} \sum_{k=1}^n \frac{f(a + \frac{b-a}{n}(k-1)) + f(a + \frac{b-a}{n}k)}{2}.$$

Note that $T_n = \frac{L_n + R_n}{2}$.

SIMPSON APPROXIMATION (D2.45)

Let f be integrable on $[a, b]$.

Then, for some $n \in 2\mathbb{Z}^+$, the " n th Simpson approximation" for $I = \int_a^b f$, denoted by S_n , is defined to be

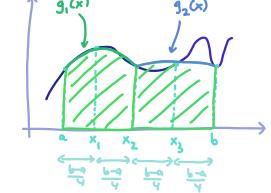
$$S_n = \sum_{k=1}^{n/2} \int_{x_{2k-2}}^{x_{2k}} g(x) dx,$$

where $x_m = a + \frac{b-a}{n}m \quad \forall m \in \{0, 1, \dots, n\}$, and

$$g(x) = g_k(x) \quad \forall k \in \{1, 2, \dots, \frac{n}{2}\},$$

where for each k , g_k is a quadratic polynomial such that

$$\begin{cases} g_k(x_{2k-2}) = f(x_{2k-2}); \\ g_k(x_{2k-1}) = f(x_{2k-1}); \\ g_k(x_{2k}) = f(x_{2k}). \end{cases}$$



We can prove that

$$S_n = \sum_{k=1}^{n/2} \frac{f(x_{2k-2}) + 4f(x_{2k-1}) + f(x_{2k})}{3} \Delta_k x;$$

i.e.

$$S_n = \frac{b-a}{n} \sum_{k=1}^{n/2} \frac{f(a + \frac{b-a}{n}(2k-2)) + 4f(a + \frac{b-a}{n}(2k-1)) + f(a + \frac{b-a}{n}(2k))}{3}.$$

Proof. First, note if $h(x) = Ax^2 + Bx + C$ satisfies $h(-1) = u$, $h(0) = v$ and $h(1) = w$, then necessarily

$$\begin{cases} A - B + C = u; \\ C = v; \\ A + B + C = w. \end{cases}$$

Solving these equations yields that $A = \frac{u-2v+w}{2}$, $B = \frac{w-u}{2}$ and $C = v$, so that

$$\begin{aligned} \int_{-1}^1 h(x) dx &= \int_{-1}^1 \frac{u-2v+w}{2} x^2 + \frac{w-u}{2} x + v dx \\ &= \left[\frac{u-2v+w}{6} x^3 + \frac{w-u}{4} x^2 + vx \right]_{-1}^1 \\ &= \frac{u-2v+w}{3} + 2v \end{aligned}$$

∴ $\int_{-1}^1 h(x) dx = \frac{u+4v+w}{3}$.

Then, by shifting and scaling, it follows that

$$\int_{x_{2k-2}}^{x_{2k}} g_k(x) dx = \frac{f(x_{2k-2}) + 4f(x_{2k-1}) + f(x_{2k})}{3} \left(\frac{b-a}{n} \right).$$

ERROR BOUNDS FOR APPROXIMATE INTEGRATION (T2.46)

Let f be integrable on $[a, b]$, and suppose the higher order derivatives of f exist.

Denote $I = \int_a^b f(x) dx$. Then

$$\textcircled{1} |L_n - I| \leq \frac{(b-a)^2}{2n} \max_{a \leq x \leq b} |f'(x)| ;$$

$$\textcircled{2} |R_n - I| \leq \frac{(b-a)^2}{2n} \max_{a \leq x \leq b} |f'(x)| ;$$

$$\textcircled{3} |T_n - I| \leq \frac{(b-a)^3}{12n^2} \max_{a \leq x \leq b} |f''(x)| ;$$

$$\textcircled{4} |M_n - I| \leq \frac{(b-a)^3}{24n^2} \max_{a \leq x \leq b} |f''(x)| ; \text{ and}$$

$$\textcircled{5} |S_n - I| \leq \frac{(b-a)^5}{180n^4} \max_{a \leq x \leq b} |f'''(x)| .$$

EXAMPLE : ERROR BOUNDS OF APPROXIMATIONS OF

$$\int_0^{4\pi/3} \sin^2(x) dx \quad (\text{E2.47})$$

We can use the above theorem to find the bounds on the errors for L_8, R_8, M_8, T_8 & S_8 on

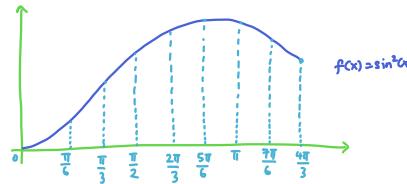
$$I = \int_0^{4\pi/3} \sin^2(x) dx .$$

First, note that

$$\begin{aligned} I &= \int_0^{4\pi/3} \sin^2(x) dx = \int_0^{4\pi/3} \frac{1}{2} - \frac{1}{2} \cos(2x) dx \\ &= \left[\frac{1}{2}x - \frac{1}{4} \sin(2x) \right]_0^{4\pi/3} \\ \therefore I &= \frac{4\pi}{3} - \frac{\sqrt{3}}{8} . \end{aligned}$$

Next, when we divide the interval $[0, \frac{4\pi}{3}]$ into 8 equal sub-intervals, the size of each subinterval is $\frac{\pi}{6}$ and the endpoints of the sub-intervals are

$$0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6} \text{ and } \frac{4\pi}{3} .$$



For convenience, let $f(x) = \sin^2(x)$.

Thus, the approximations are

$$\begin{aligned} \textcircled{1} L_8 &= \frac{b-a}{8} \sum_{k=1}^8 f(x_{k-1}) \\ &= \frac{1}{8} \left(\frac{4\pi}{3} - 0 \right) (f(0) + f(\frac{\pi}{6}) + f(\frac{\pi}{3}) + f(\frac{\pi}{2}) + f(\frac{2\pi}{3}) + f(\frac{5\pi}{6}) + f(\pi) + f(\frac{7\pi}{6}) + f(\frac{4\pi}{3})) \\ &= \frac{1}{8} \left(\frac{4\pi}{3} \right) \left(0 + \frac{1}{4} + \frac{3}{4} + 1 + \frac{3}{4} + \frac{1}{4} + 0 + \frac{1}{4} \right) \end{aligned}$$

$$\therefore L_8 = \frac{13\pi}{24} ;$$

$$\begin{aligned} \textcircled{2} R_8 &= \frac{b-a}{8} \sum_{k=1}^8 f(x_k) \\ &= \frac{1}{8} \left(\frac{4\pi}{3} \right) (f(\frac{\pi}{6}) + f(\frac{\pi}{3}) + f(\frac{\pi}{2}) + f(\frac{2\pi}{3}) + f(\frac{5\pi}{6}) + f(\pi) + f(\frac{7\pi}{6}) + f(\frac{4\pi}{3})) \\ &= \frac{1}{8} \left(\frac{4\pi}{3} \right) \left(\frac{1}{4} + \frac{3}{4} + 1 + \frac{3}{4} + \frac{1}{4} + 0 + \frac{1}{4} + \frac{3}{4} \right) \end{aligned}$$

$$\therefore R_8 = \frac{2\pi}{3} ;$$

$$\begin{aligned} \textcircled{3} T_8 &= \frac{1}{2}(L_8 + R_8) \\ &= \frac{1}{2} \left(\frac{13\pi}{24} + \frac{2\pi}{3} \right) \end{aligned}$$

$$\therefore T_8 = \frac{29\pi}{48} ;$$

$$\begin{aligned} \textcircled{4} M_8 &= \frac{b-a}{8} \sum_{k=1}^8 f\left(\frac{x_{k-1}+x_k}{2}\right) \\ &= \frac{1}{8} \left(\frac{4\pi}{3} \right) (f(\frac{\pi}{12}) + f(\frac{\pi}{6}) + f(\frac{\pi}{12}) + f(\frac{2\pi}{12}) + f(\frac{3\pi}{12}) + f(\frac{4\pi}{12}) + f(\frac{5\pi}{12}) + f(\frac{6\pi}{12})) \end{aligned}$$

*using the identity $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$ to figure out the values of $f(\frac{k\pi}{12})$.

$$= \frac{\pi}{6} \left(4 + \frac{-\sqrt{3}}{4} \right) ; \text{ and}$$

$$\begin{aligned} \textcircled{5} S_8 &= \frac{b-a}{8} \sum_{k=1}^8 \frac{f(x_{2k-2}) + 4f(x_{2k-1}) + f(x_{2k})}{3} \\ &= \frac{1}{24} \left(\frac{4\pi}{3} \right) (f(0) + 4f(\frac{\pi}{6}) + 2f(\frac{\pi}{3}) + 4f(\frac{\pi}{2}) + 2f(\frac{2\pi}{3}) + 4f(\frac{5\pi}{6}) + 2f(\pi) + 4f(\frac{7\pi}{6}) + f(\frac{4\pi}{3})) \end{aligned}$$

$$= \frac{\pi}{18} (0 + 1 + \frac{3}{2} + 4 + \frac{3}{2} + 1 + 0 + 1 + \frac{3}{2})$$

$$\therefore S_8 = \frac{43\pi}{72} .$$

Then, for $f(x) = \sin^2(x)$; note that

$$\textcircled{1} f'(x) = 2 \sin(x) \cos(x) = \sin(2x) ;$$

$$\textcircled{2} f''(x) = 2 \cos(2x) ;$$

$$\textcircled{3} f'''(x) = -4 \sin(2x) ; \text{ and}$$

$$\textcircled{4} f''''(x) = -8 \cos(2x) .$$

It follows that

$$\textcircled{1} \max_{0 \leq x \leq \frac{4\pi}{3}} |f'(x)| = \max_{0 \leq x \leq \frac{4\pi}{3}} |\sin(2x)| = 1 ;$$

$$\textcircled{2} \max_{0 \leq x \leq \frac{4\pi}{3}} |f''(x)| = \max_{0 \leq x \leq \frac{4\pi}{3}} |2 \cos(2x)| = 2 ; \text{ and}$$

$$\textcircled{3} \max_{0 \leq x \leq \frac{4\pi}{3}} |f'''(x)| = \max_{0 \leq x \leq \frac{4\pi}{3}} |-4 \sin(2x)| = 8 .$$

Finally, by the above theorem, we get that

$$\textcircled{1} |L_8 - I| \leq \frac{1}{16} \left(\frac{4\pi}{3} \right)^2 (1) = \frac{\pi^2}{9} ;$$

$$\textcircled{2} |R_8 - I| \leq \frac{1}{16} \left(\frac{4\pi}{3} \right)^2 (1) = \frac{\pi^2}{9} ;$$

$$\textcircled{3} |T_8 - I| \leq \frac{1}{12 \cdot 6^2} \left(\frac{4\pi}{3} \right)^3 (2) = \frac{8\pi^3}{729} ;$$

$$\textcircled{4} |M_8 - I| \leq \frac{1}{24 \cdot 6^2} \left(\frac{4\pi}{3} \right)^3 (2) = \frac{4\pi^3}{729} ; \text{ and}$$

$$\textcircled{5} |S_8 - I| \leq \frac{1}{180 \cdot 6^4} \left(\frac{4\pi}{3} \right)^5 (8) = \frac{2^7 \pi^5}{5 \cdot 3^4} .$$

IMPROPER INTEGRATION

IMPROPER INTEGRATION ON $[a, b]$ (D2.48 (1))

\exists_1 : Suppose that $f: [a, b] \rightarrow \mathbb{R}$ is integrable on every closed interval contained in $[a, b]$.

Then the "improper integral of f " on $[a, b]$ is defined to be

$$\int_a^b f = \lim_{t \rightarrow b^-} \int_a^t f.$$

\exists_2 : We say f is "improperly integrable" on $[a, b]$, or that the improper integral of f on $[a, b]$ "converges", if $\int_a^b f$ exists and is finite.

\exists_3 : We also allow the case where $b = \infty$, and in this case we have

$$\int_a^\infty f = \lim_{t \rightarrow \infty} \int_a^t f.$$

IMPROPER INTEGRATION ON $(a, b]$ (D2.48 (2))

\exists_1 : Suppose that $f: (a, b] \rightarrow \mathbb{R}$ is integrable on every closed interval contained in $(a, b]$.

Then, the "improper integral of f " on $(a, b]$ is defined to be

$$\int_a^b f = \lim_{t \rightarrow a^+} \int_t^b f.$$

\exists_2 : Similarly, we say f is "improperly integrable" on $(a, b]$, or that the improper integral of f on $(a, b]$ "converges", if $\int_a^b f$ exists and is finite.

\exists_3 : We also allow the case where $a = -\infty$, and in this case we have

$$\int_{-\infty}^b f = \lim_{t \rightarrow -\infty} \int_t^b f.$$

IMPROPER INTEGRATION ON (a, b) (D2.48 (3))

\exists_1 : Suppose that $f: (a, b) \rightarrow \mathbb{R}$ is integrable on every closed interval in (a, b) .

Suppose further that for any point $c \in (a, b)$, the integrals $\int_a^c f$ and $\int_c^b f$ both exist and can be added.

Then the "improper integral of f " on (a, b) is defined to be

$$\int_a^b f = \int_a^c f + \int_c^b f, \quad * \text{the choice of } c \text{ does not matter!}$$

where $c \in (a, b)$ is arbitrary.

\exists_2 : We say f is "improperly integrable" on (a, b) when both $\int_a^c f$ and $\int_c^b f$ are finite.

EVALUATING IMPROPER INTEGRALS

\therefore we write

$$\textcircled{1} \quad [F(x)]_{a+}^{b-} = \lim_{x \rightarrow b^-} F(x) - \lim_{x \rightarrow a^+} F(x);$$

$$\textcircled{2} \quad [F(x)]_{a+}^{b-} = F(b) - \lim_{x \rightarrow a^+} F(x); \text{ and}$$

$$\textcircled{3} \quad [F(x)]_{a+}^{b-} = \lim_{x \rightarrow b^-} F(x) - F(a). \quad (\text{NTZ-49})$$

\therefore Suppose that $f: (a, b) \rightarrow \mathbb{R}$ is integrable on every closed interval contained in (a, b) , and assume that F is differentiable with $F' = f$ on (a, b) . Then

$$\int_a^b f = [F(x)]_{a+}^{b-}. \quad (\text{N2.50})$$

(A similar result holds for functions defined on $[a, b)$ and $(a, b]$).

Proof. Choose some $c \in (a, b)$. Then, by the Fundamental Theorem of Calculus,

$$\begin{aligned} \int_a^b f &= \int_a^c f + \int_c^b f \\ &= \lim_{s \rightarrow a^+} \int_s^c f + \lim_{t \rightarrow b^-} \int_c^t f \\ &= \lim_{s \rightarrow a^+} (F(c) - F(s)) + \lim_{t \rightarrow b^-} (F(t) - F(c)) \\ &= \lim_{t \rightarrow b^-} F(t) - \lim_{s \rightarrow a^+} F(s) \\ \therefore \int_a^b f &= [F(x)]_{a+}^{b-}. \quad \blacksquare \end{aligned}$$

EXAMPLE 1: $\int_0^1 \frac{dx}{x}$ (E2.51 (1))

\therefore The above strategy can help us evaluate the integral $\int_0^1 \frac{dx}{x}$.

$$\begin{aligned} \int_0^1 \frac{dx}{x} &= [\ln(x)]_{0+}^1 \\ &= 0 - (-\infty) \\ \therefore \int_0^1 \frac{dx}{x} &= \infty. \end{aligned}$$

EXAMPLE 2: $\int_0^1 \frac{dx}{\sqrt{x}}$ (E2.51 (2))

\therefore Similarly, we can evaluate $\int_0^1 \frac{dx}{\sqrt{x}}$ by the above method.

$$\begin{aligned} \int_0^1 \frac{dx}{\sqrt{x}} &= [2\sqrt{x}]_{0+}^1 \\ &= 2 - 0 \\ \therefore \int_0^1 \frac{dx}{\sqrt{x}} &= 2. \end{aligned}$$

EXAMPLE 3: $\int_0^1 \frac{dx}{x^p}$ CONVERGES $\Leftrightarrow p < 1$ (E2.52)

\therefore By extension of the previous two examples, we can in fact show $\int_0^1 \frac{dx}{x^p}$ converges if and only if $p < 1$.

Proof. The case with $p=1$ was dealt in E2.50.

If $p > 1$, then $p-1 > 0$, so that

$$\int_0^1 \frac{dx}{x^p} = \left[\frac{-1}{(p-1)x^{p-1}} \right]_{0+}^1 = \left(-\frac{1}{p-1} \right) - (-\infty) = \infty,$$

and if $p < 1$, then $1-p > 0$, so that

$$\int_0^1 \frac{dx}{x^p} = \left[\frac{x^{1-p}}{1-p} \right]_{0+}^1 = \left(\frac{1}{1-p} \right) - (0) = \frac{1}{1-p},$$

and these deductions are sufficient to prove the claim. \blacksquare

EXAMPLE 4: $\int_1^\infty \frac{dx}{x^p}$ CONVERGES $\Leftrightarrow p > 1$ (E2.53)

\therefore Similarly, we can prove $\int_1^\infty \frac{dx}{x^p}$ converges if and only if $p > 1$.

Proof. When $p=1$, then

$$\int_1^\infty \frac{dx}{x^p} = \int_1^\infty \frac{dx}{x} = [\ln(x)]_1^\infty = \infty - 0 = \infty.$$

When $p > 1$, then $p-1 > 0$, so that

$$\int_1^\infty \frac{dx}{x^p} = \left[\frac{-1}{(p-1)x^{p-1}} \right]_1^\infty = (0) - \left(-\frac{1}{p-1} \right) = \frac{1}{p-1}.$$

When $p < 1$, then $1-p > 0$, so that

$$\int_1^\infty \frac{dx}{x^p} = \left[\frac{x^{1-p}}{1-p} \right]_1^\infty = (\infty) - \left(\frac{1}{1-p} \right) = \infty,$$

and these deductions are sufficient to prove the claim. \blacksquare

EXAMPLE 5: $\int_0^\infty e^{-x} dx$ (E2.54)

\therefore In a similar manner, we can evaluate the integral $\int_0^\infty e^{-x} dx$.

$$\begin{aligned} \int_0^\infty e^{-x} dx &= [-e^{-x}]_0^\infty \\ &= 0 - (-1) \end{aligned}$$

$$\therefore \int_0^\infty e^{-x} dx = 1.$$

EXAMPLE 6: $\int_0^1 \ln(x) dx$ (E2.55)

\therefore Similarly, we can evaluate the integral $\int_0^1 \ln(x) dx$ using the strategy in N2.50.

$$\begin{aligned} \int_0^1 \ln(x) dx &= [x \ln(x) - x]_{0+}^1 \\ &= (-1) - \lim_{x \rightarrow 0^+} (x \ln(x)) \\ &= -1 - \lim_{x \rightarrow 0^+} \frac{\ln(x)}{x} \\ &= -1 - \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{(-\frac{1}{x^2})} \quad (\text{by L'Hopital's Rule, since } \frac{\ln(x)}{x} \rightarrow \frac{\infty}{\infty}) \\ &= -1 - \lim_{x \rightarrow 0^+} (-x) \\ &= -1 - (0) \\ \therefore \int_0^1 \ln(x) dx &= -1. \end{aligned}$$

COMPARISON FOR IMPROPER INTEGRALS (T2.56)

Let f and g be integrable on any closed intervals contained in (a, b) , and suppose further that

$$0 \leq f(x) \leq g(x) \quad \forall x \in (a, b).$$

Suppose g is improperly integrable on (a, b) .

Then so is f , and

$$\int_a^b f \leq \int_a^b g.$$

On the other hand, if $\int_a^b f$ diverges, then $\int_a^b g$ diverges as well.

(Similar results hold for functions f & g defined on half-open intervals)

EXAMPLE 1: $\int_0^{\pi/2} \sqrt{\sec(x)} dx$ CONVERGES (E2.57)

Using comparison, we can show that $\int_0^{\pi/2} \sqrt{\sec(x)} dx$ converges.

Proof. First, note $\forall x \in [0, \frac{\pi}{2}]$, we have that $\cos(x) \geq 1 - \frac{2}{\pi}x$, so that $\sec(x) \leq \frac{1}{1 - \frac{2}{\pi}x}$, and hence $\sqrt{\sec(x)} \leq \sqrt{\frac{1}{1 - \frac{2}{\pi}x}}$.

Let $u = 1 - \frac{2}{\pi}x$, so that $du = -\frac{2}{\pi}dx$. Then

$$\begin{aligned} \int_{x=0}^{x=\frac{\pi}{2}} \frac{1}{\sqrt{1-\frac{2}{\pi}x}} dx &= \int_{u=1}^{u=0} -\frac{\pi}{2} u^{-\frac{1}{2}} du \\ &= [-\pi u^{\frac{1}{2}}]_1^0, \end{aligned}$$

which is clearly finite.

It follows by comparison that $\int_0^{\pi/2} \sqrt{\sec(x)} dx$ converges. \square

EXAMPLE 2: $\int_0^{\infty} e^{-x^2} dx$ CONVERGES (E2.58)

Similarly, we can show $\int_0^{\infty} e^{-x^2} dx$ converges using comparison.

Proof. First, note for $x \in [0, \infty)$, $e^x \geq 1+x$; hence $e^{x^2} \geq 1+x^2 > 0$, so that $e^{-x^2} \leq \frac{1}{1+x^2}$.

Then, since

$$\int_0^{\infty} \frac{dx}{1+x^2} = [\tan^{-1}(x)]_0^{\infty} \leq \frac{\pi}{2},$$

which is finite, we see that $\int_0^{\infty} e^{-x^2} dx$ converges by comparison. \square

ESTIMATION FOR IMPROPER INTEGRALS (T2.59)

Let $f: (a, b) \rightarrow \mathbb{R}$ be integrable on any closed interval contained within (a, b) .

Suppose $|f|$ is improperly integrable on (a, b) .

Then so is f , and in this case

$$|\int_a^b f| \leq \int_a^b |f|$$

(Similar results hold for functions defined on half-open intervals).

EXAMPLE: $\int_0^{\infty} \frac{\sin(x)}{x} dx$ CONVERGES (E2.60)

Using estimation, we can show that $\int_0^{\infty} \frac{\sin(x)}{x} dx$ converges.

Proof. We show $\int_0^1 \frac{\sin(x)}{x} dx$ and $\int_1^{\infty} \frac{\sin(x)}{x} dx$ converge.

First, since $\lim_{x \rightarrow 0^+} \frac{\sin(x)}{x} = 1$ by the Fundamental Trigonometric Limit, the function $f(x) = \begin{cases} 1, & x=0 \\ \frac{\sin(x)}{x}, & x>0 \end{cases}$ is continuous on $[0, 1]$,

and so by T1.17 $f(x)$ is also integrable on $[0, 1]$.

Then, by the Fundamental Theorem of Calculus, $\int_r^1 f(x) dx$ is continuous for $r \in [0, 1]$, and so

$$\int_0^1 \frac{\sin(x)}{x} dx = \lim_{r \rightarrow 0^+} \int_r^1 \frac{\sin(x)}{x} dx = \lim_{r \rightarrow 0^+} \int_r^1 f(x) dx = \int_0^1 f(x) dx,$$

which is finite, so $\int_0^1 \frac{\sin(x)}{x} dx$ converges as well.

Next, integrate by parts using $\begin{pmatrix} u = \frac{1}{x} & v = \frac{1}{x} \\ du = -\frac{1}{x^2} dx & dv = \frac{1}{x^2} dx \end{pmatrix}$ to get

$$\begin{aligned} \int_1^{\infty} \frac{\sin(x)}{x} dx &= \left[-\frac{\cos(x)}{x} \right]_1^{\infty} - \int_1^{\infty} \frac{\cos(x)}{x^2} dx \\ &\therefore \int_1^{\infty} \frac{\sin(x)}{x} dx = \cos(1) - \int_1^{\infty} \frac{\cos(x)}{x^2} dx. \end{aligned}$$

Then, since $|\frac{\cos(x)}{x^2}| \leq \frac{1}{x^2}$ and $\int_1^{\infty} \frac{dx}{x^2}$ converges, necessarily $\int_1^{\infty} |\frac{\cos(x)}{x^2}| dx$ converges too by comparison.

Hence, by estimation, $\int_1^{\infty} \frac{\cos(x)}{x^2} dx$ also converges.

Finally, since

$$\int_0^{\infty} \frac{\sin(x)}{x} dx = \int_0^1 \frac{\sin(x)}{x} dx + \int_1^{\infty} \frac{\sin(x)}{x} dx,$$

and both $\int_0^1 \frac{\sin(x)}{x} dx$ and $\int_1^{\infty} \frac{\sin(x)}{x} dx$ are finite, it follows that $\int_0^{\infty} \frac{\sin(x)}{x} dx$ converges, and we are done. \square

Chapter 3:

Applications of the Definite Integral

AREA BETWEEN CURVES (D3.2)

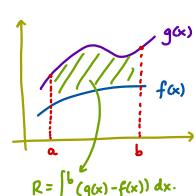
💡 Let $f, g: [a, b] \rightarrow \mathbb{R}$ be integrable with $f(x) \leq g(x) \quad \forall x \in [a, b]$.

Then, we define the "area" of the region R given by

$$a \leq x \leq b, \quad f(x) \leq y \leq g(x)$$

to be

$$A = \int_a^b (g(x) - f(x)) dx.$$



EXAMPLE 1: AREA OF REGION x -AXIS AND $y=1-x^2$ (E3.3)

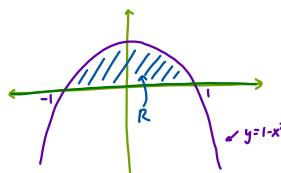
💡 We can use the above formula to calculate the area of the region between the x -axis and the parabola $y=1-x^2$.

Note that the region R is given by $-1 \leq x \leq 1, \quad 0 \leq y \leq 1-x^2$,

so the area is

$$A = \int_{-1}^1 (1-x^2) dx = \left[x - \frac{x^3}{3} \right]_{-1}^1 = \frac{4}{3}.$$

BETWEEN



EXAMPLE 2: AREA OF REGION BETWEEN $y=x^2+3x+2$ & $y=x^3-3x+2$ (E3.4)

💡 Similarly, we can use the method above to calculate the area of the region between the curves $y=x^2+3x+2$ and $y=x^3-3x+2$.

(at $f(x)=x^2+3x+2$ and $g(x)=x^3-3x+2$).

Then, note that

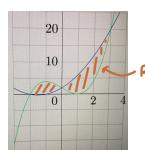
$$\begin{aligned} f(x)-g(x) &= (x^2+3x+2) - (x^3-3x+2) \\ &= -x^3+x^2+6x \end{aligned}$$

$$= -x(x-3)(x+2)$$

and so $f(x) \geq g(x)$ when $x=0, x=3$ and $x=-2$.

Moreover, $f(x) \geq g(x) \quad \forall x \in (-\infty, -2] \cup [0, 3]$ and

$f(x) \leq g(x) \quad \forall x \in [-2, 0] \cup [3, \infty)$.



Then, from the diagram, observe that

$$\begin{aligned} A &= \int_{-2}^0 (g(x)-f(x)) dx + \int_0^3 (g(x)-f(x)) dx \\ &= \int_{-2}^0 (x^3-x^2-6x) dx + \int_0^3 (-x^3+x^2+6x) dx \\ &= \left[\frac{1}{4}x^4 - \frac{1}{3}x^3 - 3x^2 \right]_{-2}^0 + \left[-\frac{1}{4}x^4 + \frac{1}{3}x^3 + 3x^2 \right]_0^3 \\ &= \frac{253}{12}. \end{aligned}$$

EXAMPLE 3: AREA OF A CIRCLE OF RADIUS r (E3.5)

💡 We can use a similar method to calculate the area of a circle with radius r .

Observe that the area of a circle with radius r is simply 4 times the area of the region given by $0 \leq x \leq r, \quad 0 \leq y \leq \sqrt{r^2-x^2}$,

so that the area of the circle is

$$A = 4 \int_0^r \sqrt{r^2-x^2} dx.$$

Make the substitution $r\sin\theta = x$, so that $r\cos\theta = \sqrt{r^2-x^2}$ and $r\cos\theta d\theta = dx$ to get that

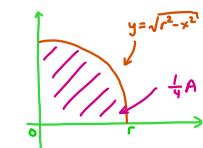
$$A = 4 \int_{\theta=0}^{\pi/2} r\cos\theta (r\cos\theta d\theta)$$

$$= 4r^2 \int_0^{\pi/2} (\frac{1}{2} - \frac{1}{2}\cos 2\theta) d\theta \quad (\text{by the double angle formula})$$

$$= r^2 \int_0^{\pi/2} (2 - 2\cos 2\theta) d\theta$$

$$= r^2 [2\theta - 2\sin 2\theta]_0^{\pi/2}$$

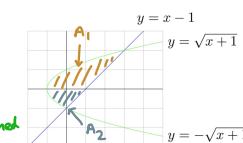
$$\therefore A = \pi r^2.$$



EXAMPLE 4: AREA OF REGION BETWEEN $y=x-1$ & $y^2=x+1$ (E3.6)

💡 We can find the area of the region between the curves $y=x-1$ and $y^2=x+1$ using a similar method.

To find the area of the region, we consider two parts separately:



(\square) This is the region defined by

$$-1 \leq x \leq 0, \quad \sqrt{x+1} \leq y \leq \sqrt{x+1}.$$

It follows that

$$A_1 = \int_{-1}^0 \sqrt{x+1} - (-\sqrt{x+1}) dx$$

$$= \int_{-1}^0 2(\sqrt{x+1})^2 dx$$

$$= \left[\frac{2}{3}(x+1)^{\frac{3}{2}} \right]_{-1}^0$$

$$\therefore A_1 = \frac{4}{3}.$$

(\square) This is the region defined by

$$0 \leq x \leq 3, \quad x-1 \leq y \leq \sqrt{x+1}.$$

It follows that

$$A_2 = \int_0^3 \sqrt{x+1} - (x-1) dx$$

$$= \left[\frac{2}{3}(x+1)^{\frac{3}{2}} - \frac{1}{2}x^2 + x \right]_0^3$$

$$= \frac{19}{6}.$$

Hence, the total area of the region is given by

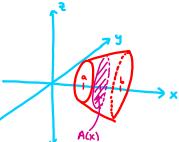
$$A = A_1 + A_2$$

$$= \frac{4}{3} + \frac{19}{6}$$

$$\therefore A = \frac{9}{2}.$$

VOLUME BY CROSS-SECTION

Suppose that a solid S lies in space between $x=a$ and $x=b$, and its cross-sectional area at "position" x is equal to $A(x)$, and assume that $A(x)$ is integrable on $[a,b]$. We can approximate the volume of S as follows:



① Choose a partition of $[a,b]$

$$a < x_0 < x_1 < \dots < x_n = b,$$

with corresponding sample points

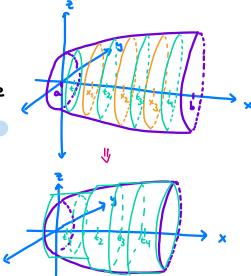
$$c_k \in [x_{k-1}, x_k] \quad \forall k \in \{1, 2, \dots, n\}.$$

② Divide the solid into strips, where the k^{th} strip lies between $x=x_{k-1}$ and $x=x_k$ and has thickness

$$\Delta x_k = x_k - x_{k-1}.$$

③ Then, note that the total volume of S is equal to

$$V = \sum_{k=1}^n \Delta x_k V_k \approx \sum_{k=1}^n A(c_k) \Delta x_k. \quad (\text{N3.7})$$



VOLUME BY CROSS-SECTION (E3.8)

Suppose that a solid S lies in space between $x=a$ and $x=b$, and that its cross-sectional area at x is equal to $A(x)$, where $A(x)$ is integrable on $[a,b]$.

Then, we define the "volume" of S to be

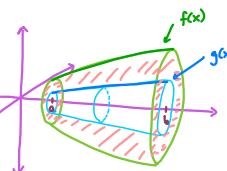
$$V = \lim_{n \rightarrow \infty} \sum_{k=1}^n A(c_k) \Delta x_k,$$

or alternatively,

$$V = \int_a^b A(x) dx.$$

VOLUME OF AN "ANNULUS SOLID" (E3.9)

Let f and g be integrable on $[a,b]$ with $0 \leq f(x) \leq g(x) \quad \forall x \in [a,b]$. Let R be the region in the xy -plane given by $a \leq x \leq b, f(x) \leq y \leq g(x)$ and let S be the solid obtained by revolving R about the x -axis.



Then, the area function at position x is given by

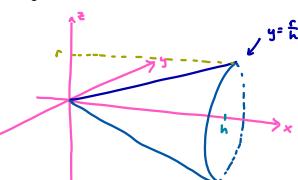
$$A(x) = \pi([g(x)]^2 - [f(x)]^2),$$

so that the volume of S is

$$V = \int_a^b A(x) dx = \int_a^b \pi([g(x)]^2 - [f(x)]^2) dx.$$

VOLUME OF A CONE (E3.10)

Note that a cone with base radius r and height h can be obtained by revolving the triangular region R given by $0 \leq x \leq h, 0 \leq y \leq \frac{r}{h}x$ about the x -axis.



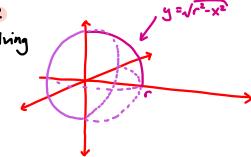
It follows that the volume of the cone is

$$V = \int_0^h \pi\left(\frac{r}{h}x\right)^2 dx = \frac{\pi r^2}{h^2} \left[\frac{x^3}{3}\right]^h_0$$

$$\therefore V = \frac{1}{3}\pi r^2 h.$$

VOLUME OF A SPHERE (E3.11)

Similarly, we can find the volume of one half of a sphere by revolving the region R given by $0 \leq x \leq r, 0 \leq y \leq \sqrt{r^2 - x^2}$ around the x -axis.



Hence, the volume of the sphere is

$$V = 2 \int_0^r \pi(\sqrt{r^2 - x^2})^2 dx \\ = 2\pi \left[r^2 x - \frac{1}{3}x^3\right]^r_0 \\ \therefore V = \frac{4}{3}\pi r^3.$$

VOLUME OF A "FOOTBALL" (E3.12)

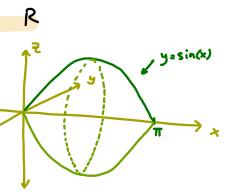
We can use the same formula to calculate the volume of the "football-shaped" solid S which is obtained by revolving the region R given by

$$0 \leq x \leq \pi, \quad 0 \leq y \leq \sin(x)$$

around the x -axis.

The volume of the solid is hence given by

$$V = \int_0^\pi \pi \sin^2(x) dx = \int_0^\pi \pi \left(\frac{1}{2} - \frac{1}{2} \cos(2x)\right) dx \\ = \pi \left[\frac{1}{2}x - \frac{1}{4}\sin(2x)\right]^{\pi}_0 \\ \therefore V = \frac{\pi^2}{2}.$$



VOLUME OF GABRIEL'S HORN (E3.13)

"Gabriel's horn" refers to the solid S obtained by revolving the region R given by

$$1 \leq x < \infty, \quad 0 \leq y \leq \frac{1}{x}$$

around the x -axis.

Then, the volume of Gabriel's horn is just

$$V = \int_1^\infty \pi\left(\frac{1}{x^2}\right) dx = \pi\left[-\frac{1}{x}\right]_1^\infty = \pi.$$

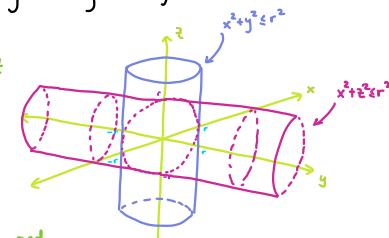
Note that Gabriel's horn has infinite surface area, but only finite volume. (P3.14)



VOLUME OF INTERSECTION OF TWO CYLINDERS (E3.15)

Finally, we can use our formula to calculate the volume of the solid given by $x^2 + y^2 \leq r^2, x^2 + z^2 \leq r^2$.

To find the cross-section at $x \in [-r, r]$, note that the region of said cross-section is given by $y^2 \leq r^2 - x^2$ & $z^2 \leq r^2 - x^2$, or equivalently $|y| \leq \sqrt{r^2 - x^2}$ and $|z| \leq \sqrt{r^2 - x^2}$.



Hence, the cross-section at x is the square given by $|y| \leq \sqrt{r^2 - x^2}$ and $|z| \leq \sqrt{r^2 - x^2}$, so that

$$A(x) = [2\sqrt{r^2 - x^2}]^2 = 4(r^2 - x^2).$$

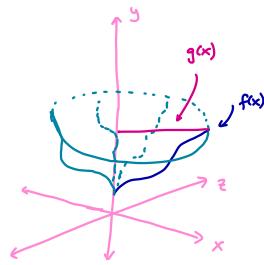
(since the square has side length $2\sqrt{r^2 - x^2}$)

It follows that the volume of S is

$$V = \int_{-r}^r A(x) dx = \int_{-r}^r 4(r^2 - x^2) dx = 4[r^2 x - \frac{1}{3}x^3]_{-r}^r = \frac{16}{3}r^3.$$

VOLUME BY CYLINDRICAL SHELLS (N3.16)

Suppose that f and g are integrable on $[a, b]$, with $f(x) \leq g(x) \forall x \in [a, b]$. Let R be the region in the xy -plane given by $a \leq x \leq b, f(x) \leq y \leq g(x)$



and let S be the solid obtained by revolving R about the y -axis.

We can find the volume of S by the following:

(1) Choose a partition of $[a, b]$

$$a = x_0 < x_1 < \dots < x_n = b,$$

with corresponding sample points

$$c_k \in [x_{k-1}, x_k] \quad \forall k \in \{1, 2, \dots, n\}.$$

(2) Divide the region into "strips", where the k th strip R_k is given by

$$x_{k-1} \leq x \leq x_k, \quad f(x) \leq y \leq g(x).$$

(3) Revolve each strip R_k around the y -axis to create cylindrical "shells".

(4) Note that these shells approximate the total volume of the solid.

(5) Then, the volume of the k th shell is

$$\Delta_k V \approx 2\pi c_k (g(c_k) - f(c_k)) \Delta_k x,$$

so the total volume of the solid is

$$V \approx \sum_{k=1}^n \Delta_k V = \sum_{k=1}^n 2\pi c_k (g(c_k) - f(c_k)) \Delta_k x.$$

VOLUME (BY CYLINDRICAL SHELLS) (D3.17)

Suppose that f and g are integrable on $[a, b]$ with $f(x) \leq g(x) \forall x \in [a, b]$, and let R be the region in the xy -plane given by $a \leq x \leq b, f(x) \leq y \leq g(x)$.

Let S be the solid by revolving R about the y -axis.

Then, the "volume" of S is defined to be

$$V = \int_a^b 2\pi x (f(x) - g(x)) dx.$$

VOLUME OF A "DISCUS" (E3.19)

Let S be the solid obtained by revolving the region R given by $0 \leq x \leq \frac{\pi}{2}, -\cos(x) \leq y \leq \cos(x)$ about the y -axis.

The volume of S is

$$V = \int_0^{\pi/2} 2\pi x [\cos(x) - (-\cos(x))] dx$$

$$\therefore V = \int_0^{\pi/2} 4\pi x \cos(x) dx.$$

To solve this integral, integrate by parts using

$$\begin{pmatrix} u=x & v=\sin(x) \\ du=1dx & dv=\cos(x)dx \end{pmatrix} \text{ to get that}$$

$$V = 4\pi \left[x\sin(x) - \int \sin(x) dx \right]_0^{\pi/2}$$

$$= 4\pi \left[x\sin(x) - \cos(x) \right]_0^{\pi/2}$$

$$\therefore V = 2\pi^2 - 4\pi.$$

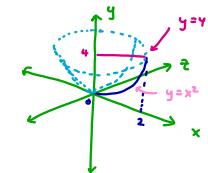
VOLUME OF A "BOWL" (E3.20)

We can use similar reasoning to find the capacity of a "bowl", which is the solid obtained by revolving the parabola $y=x^2$ with $0 \leq x \leq 2$ about the y -axis.

The volume of the bowl is given by

$$V = \int_0^2 2\pi x (4-x^2) dx \\ = \pi [4x^2 - \frac{1}{3}x^4]_0^2$$

$$\therefore V = 8\pi.$$



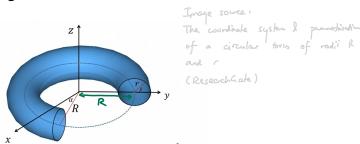
VOLUME OF A TORUS (E3.21)

A "torus" can be obtained by revolving

the disc D given by

$$(x-R)^2 + y^2 \leq r^2$$

about the y -axis.



The disc D is given by

$$R-r \leq x \leq R+r, \quad -\sqrt{r^2-(x-R)^2} \leq y \leq \sqrt{r^2-(x-R)^2},$$

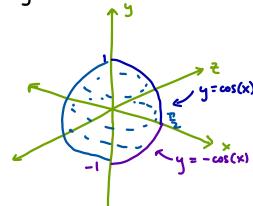
so that the volume of the torus is

$$V = \int_{R-r}^{R+r} 2\pi x \cdot 2\sqrt{r^2-(x-R)^2} dx.$$

(let $r\sin\theta = x-R$, so that $r\cos\theta = \sqrt{r^2-(x-R)^2}$ and $r\cos\theta dx$, to get

$$\begin{aligned} V &= \int_{x=R-r}^{x=R+r} 4\pi x \sqrt{r^2-(x-R)^2} dx = \int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} 4\pi(R+r\sin\theta)(r\cos\theta)(r\cos\theta d\theta) \\ &= 4\pi r^2 \int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} R\cos^2\theta + r\sin\theta\cos^2\theta d\theta \\ &= 4\pi r^2 \left[R(\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta) + \frac{r}{3}\cos^3\theta \right]_{-\pi/2}^{\pi/2} \\ \therefore V &= 2\pi^2 r^2 R. \end{aligned}$$

Image source:
The coordinate system is perpendicular to a circular torus of radii R and r (Revolution)



ARCLENGTH (N3.22)

E₁ Let f be differentiable on $[a, b]$, or let f be differentiable on (a, b) and continuous on $[a, b]$.

Let C be the curve $y = f(x)$ with $a \leq x \leq b$.

We can approximate the length of C as follows.

① Choose a partition of $[a, b]$
 $a = x_0 < x_1 < \dots < x_n = b$.

② By the Mean Value Theorem, there exist sample points $c_k \in [x_{k-1}, x_k]$ such that

$$f'(c_k) = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} = \frac{\Delta_k y}{\Delta_k x}.$$

③ Let C_k be the part of the curve C from $x_{k-1} \leq x \leq x_k$, and let D_k be the line segment from $(x_{k-1}, f(x_{k-1}))$ to $(x_k, f(x_k))$.

④ Then, note that the total length of C_k is approximately equal to the length of D_k ; ie

$$\begin{aligned}\Delta_k L &\approx |D_k| = \sqrt{(\Delta_k x)^2 + (\Delta_k y)^2} \\ &= \sqrt{1 + \left(\frac{\Delta_k y}{\Delta_k x}\right)^2} \cdot \Delta_k x \\ \therefore \Delta_k L &= \sqrt{1 + [f'(c_k)]^2} \cdot \Delta_k x,\end{aligned}$$

so that the total length of C is approximately

$$L \approx \sum_{k=1}^n \Delta_k L = \sum_{k=1}^n \sqrt{1 + [f'(c_k)]^2} \cdot \Delta_k x.$$

LENGTH (D3.23)

E₂ Let f be differentiable on $[a, b]$, or let f be differentiable on (a, b) and continuous on $[a, b]$.

Then, we define the "length", or "arclength", of the curve $y = f(x)$ from $x=a$ to $x=b$ to be

$$L = \int_a^b \sqrt{1 + f'(x)^2} dx.$$

RECTIFIABLE (D3.23)

E₃ We say that f (from $x=a$ to $x=b$) is "rectifiable" if its arclength from $x=a$ to $x=b$ is finite.

EXAMPLE 1: LENGTH OF $y=x^2$ (E3.24)

E₄ Using the above formula, we can find the length of the curve $y=x^2$ with $0 \leq x \leq 2$.

let $f(x) = x^2$, so that $f'(x) = 2x$. Then, the length of the curve is

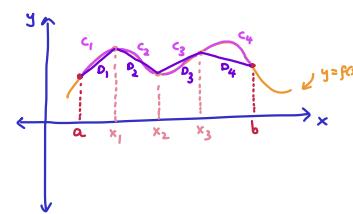
$$L = \int_0^2 \sqrt{1 + f'(x)^2} dx = \int_0^2 \sqrt{1 + 4x^2} dx.$$

To solve this integral, let $2x = \tan \theta$ so $\sec \theta = \sqrt{1+4x^2}$ & $\sec^2 \theta d\theta = 2dx$ to get

$$\begin{aligned}\int \sqrt{1+4x^2} dx &= \int \frac{1}{2} \sec^3 \theta d\theta \\ &= \frac{1}{4} \sec \theta \tan \theta + \frac{1}{4} \ln |\sec \theta + \tan \theta| + C \\ &= \frac{1}{2} x \sqrt{1+4x^2} + \frac{1}{4} \ln |2x + \sqrt{1+4x^2}| + C.\end{aligned}$$

If follows that

$$\begin{aligned}L &= \int_0^2 \sqrt{1+4x^2} dx = \left[\frac{1}{2} x \sqrt{1+4x^2} + \frac{1}{4} \ln |2x + \sqrt{1+4x^2}| \right]_0^2 \\ &= \sqrt{17} + \frac{1}{4} \ln(4 + \sqrt{17}).\end{aligned}$$



SURFACE AREA (N3.26)

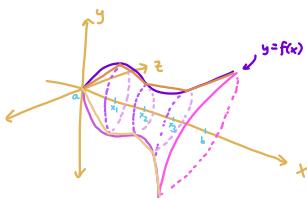
Let f be differentiable on $[a, b]$, or let f be differentiable on (a, b) and continuous on $[a, b]$. Let C be the curve in the xy -plane given by $y = f(x)$ with $a \leq x \leq b$, and let S be the surface obtained by revolving C about the x -axis.

We can approximate the area of S by the following:

- ① Choose a partition of $[a, b]$
 $a = x_0 < x_1 < \dots < x_n < b$.
- ② By the Mean Value Theorem, there exists sample points $c_k \in [x_{k-1}, x_k]$ such that

$$f'(c_k) = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} = \frac{\Delta_k y}{\Delta_k x}.$$

- ③ Let C_k be the part of C with $x_{k-1} \leq x \leq x_k$, and let S_k denote the "slice" of S which is obtained by revolving C_k around the x -axis.
- ④ Let D_k be the line segment from $(x_{k-1}, f(x_{k-1}))$ to $(x_k, f(x_k))$, and let T_k be the slice of a cone obtained by revolving D_k about the x -axis.



- ⑤ Then, the area of the slice S_k is approximately equal to the area of T_k . So that

$$\Delta_k A \approx \pi(f(x_{k-1}) + f(x_k)) \Delta_k L \quad (\text{using the formula for the area of a cone (N2.35)})$$

$$= \pi(f(x_{k-1}) + f(x_k)) \sqrt{1 + f'(c_k)^2} \Delta_k x$$

$$\therefore \Delta_k A \approx 2\pi f(c_k) \sqrt{1 + f'(c_k)^2} \Delta_k x,$$

and it follows the area of S is approximately equal to

$$A \approx \sum_{k=1}^n \Delta_k A \approx \sum_{k=1}^n 2\pi f(c_k) \sqrt{1 + f'(c_k)^2} \Delta_k x.$$

AREA (D3.27)

Let f be differentiable on $[a, b]$, or let f be differentiable on (a, b) and continuous on $[a, b]$. Let C be the curve given by $y = f(x)$ with $a \leq x \leq b$, and S the surface obtained by revolving C about the x -axis.

Then, we define the "area" of S to be

$$A = \int_a^b 2\pi f(x) \sqrt{1 + f'(x)^2} dx.$$

* if the curve is revolved about the y -axis instead, replace the $f(x)$ with x . (D3.29)

AREA OF A SPHERE (E3.30)

Note that a sphere can be obtained by revolving the curve $y = \sqrt{r^2 - x^2}$ with $-r \leq x \leq r$ about the x -axis.

Let $f(x) = \sqrt{r^2 - x^2}$, so that $f'(x) = -\frac{x}{\sqrt{r^2 - x^2}}$, so that

$$\sqrt{1 + f'(x)^2} = \sqrt{1 + \frac{x^2}{r^2 - x^2}} = \sqrt{\frac{r^2}{r^2 - x^2}} = \frac{r}{\sqrt{r^2 - x^2}}.$$

So, the area of the sphere is

$$\begin{aligned} A &= \int_{-r}^r 2\pi f(x) \sqrt{1 + f'(x)^2} dx \\ &= \int_{-r}^r 2\pi \sqrt{r^2 - x^2} \cdot \frac{r}{\sqrt{r^2 - x^2}} dx \\ &= \int_{-r}^r 2\pi r dx \\ \therefore A &= 4\pi r^2. \end{aligned}$$

AREA OF A TORUS (E3.31)

Similarly, half of a "torus" can be obtained by revolving the curve $y = \sqrt{r^2 - (R-x)^2}$ with $R-r \leq x \leq R+r$ about the y -axis.

Let $f(x) = \sqrt{r^2 - (R-x)^2}$, so that $f'(x) = \frac{-(R-x)}{\sqrt{r^2 - (R-x)^2}}$, so that

$$\sqrt{1 + f'(x)^2} = \sqrt{1 + \frac{(R-x)^2}{r^2 - (R-x)^2}} = \sqrt{\frac{r^2}{r^2 - (R-x)^2}} = \frac{r}{\sqrt{r^2 - (R-x)^2}}.$$

It follows that the surface area of the torus is

$$A = 2 \int_{R-r}^{R+r} 2\pi x \cdot \frac{r}{\sqrt{r^2 - (R-x)^2}} dx = 4\pi r \int_{R-r}^{R+r} \frac{x dx}{\sqrt{r^2 - (R-x)^2}}.$$

Make the substitution $r\sin\theta = x - R$, so that $\cos\theta = \sqrt{r^2 - (x-R)^2}$ and $r\cos\theta d\theta = dx$. Then

$$\begin{aligned} A &= 4\pi r \int_{x=R-r}^{x=R+r} \frac{x dx}{\sqrt{r^2 - (x-R)^2}} = 4\pi r \int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \frac{(R+r\sin\theta) \cdot r\cos\theta}{r\cos\theta} d\theta \\ &= 4\pi r \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (R+r\sin\theta) d\theta \\ &= 4\pi r [R\theta - r\cos\theta]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\ \therefore A &= 4\pi^2 r R. \end{aligned}$$

MASS & DENSITY (E3.32)

Suppose a rod lies along the x-axis from $x=a$ to $x=b$, with linear density (mass per unit length) equal to $p(x)$, where $p(x)$ is integrable on $[a,b]$. We can approximate the mass of the rod as follows:

- ① Choose a partition of $[a,b]$
 $a = x_0 < x_1 < \dots < x_n = b$
with corresponding sample points
 $c_k \in [x_{k-1}, x_k]$.
- ② Then, the mass of the part of the rod between $x=x_{k-1}$ and $x=x_k$ is approximately

$$\Delta_k M \approx p(c_k) \Delta_k x,$$

so that the total mass of the rod is

$$M = \sum_{k=1}^n \Delta_k M = \sum_{k=1}^n p(c_k) \Delta_k x.$$

MASS (E3.32)

Suppose a rod lies along the x-axis from $x=a$ to $x=b$, with linear density $p(x)$, where $p(x)$ is integrable on $[a,b]$.

Then the "mass" of the rod is

$$M = \int_a^b p(x) dx.$$

MASS OF A SPHERE WITH VARYING DENSITY (E3.33)

Suppose that a ball of radius R has varying density, such that the density of each point r units away from the origin is equal to $p(r)$, where $p(r)$ is integrable on $[0,R]$. We approximate the mass of the ball as follows:

Choose a partition $0 = r_0 < r_1 < \dots < r_n = R$ of $[0,R]$, with sample points $c_k \in [r_{k-1}, r_k]$.

Divide the sphere into spherical shells using concentric spheres of radius r_k .

Then, the volume of the k th spherical shell is

$$\Delta_k V \approx 4\pi c_k^2 \Delta r,$$

so the shell's mass is

$$\Delta_k M \approx p(c_k) \Delta_k V \approx p(c_k) 4\pi c_k^2 \Delta r.$$

Hence, the total mass of the sphere is

$$M = \int_a^b 4\pi r^2 p(r) dr.$$

FORCE

PRESSURE EXAMPLE (E3.34)

3.34 Example: A tank is in the shape of the parabolic sheet given by $y = x^2$, $-2 \leq x \leq 2$, $-5 \leq z \leq 5$ together with the two ends given by $-2 \leq x \leq 2$, $x^2 \leq y \leq 4$ with $z = \pm 5$ (where the x -axis is pointing upwards). The tank is filled with a liquid of density ρ . The pressure $P(h)$ (force per unit area) exerted by the liquid on each wall at all points which lie at a depth h is given by

$$P = \rho gh$$

Along one of the ends of the tank, consider a thin horizontal slice at position y of thickness Δy .

The slice is at a "depth" of $h = 4-y$, so the pressure at all points on the slice is $P = \rho gh = \rho g(4-y)$.

The width of the slice is equal to $2\sqrt{y}$.

So the area of the slice is

$$\Delta A = 2\sqrt{y} \Delta y,$$

so the force exerted by the water on the slice is

$$\Delta F = P \Delta A = \rho g(4-y) \cdot 2\sqrt{y} \Delta y.$$

Hence, the total force exerted on the end of the tank is

$$\begin{aligned} F &= \int_0^4 \rho g(4-y) \cdot 2\sqrt{y} dy = \rho g \int_0^4 8y^{\frac{1}{2}} - 2y^{\frac{3}{2}} dy \\ &= \rho g \left[\frac{16}{3}y^{\frac{3}{2}} - \frac{4}{5}y^{\frac{5}{2}} \right]_0^4 \\ &\therefore F = \frac{256}{15} \rho g. \end{aligned}$$

CHARGED ROD (COULOMB'S LAW) (E3.35)

3.35 Example: A charged rod, of charge Q (with its charge evenly distributed along its length) lies along the x -axis from $x=0$ to $x=2$. A small object of charge q lies at position $(x,y) = (2,1)$. Find the force exerted by the rod on the object. Use the fact that the force exerted by one small object of charge q_1 at position p_1 on another of charge q_2 at position p_2 is equal to

$$F = \frac{k q_1 q_2}{|u|^2} \cdot \frac{u}{|u|}$$

where k is a constant and u is the direction vector from p_1 to p_2 , that is $u = p_2 - p_1$.

First, consider a small slice of rod at position x , of thickness Δx .

Since the rod has length 2, the charge per unit length is $\frac{Q}{2}$, so that the charge on the slice of the rod is $\Delta Q = \frac{Q}{2} \Delta x$.

Then, the distance from the slice, which is at position $(x,0)$, to the small object (at $(2,1)$) is

$$r = |u| = \sqrt{(2-x)^2 + 1},$$

so that

$$|\Delta F| = \frac{k q \cdot \frac{Q}{2} \Delta x}{(2-x)^2 + 1}.$$

Next, by similar triangles, the x & y -components of the force exerted by the slice on the object are given by

$$\Delta F_x = \frac{2-x}{\sqrt{(2-x)^2 + 1}} \Delta F = \frac{k q Q (2-x) \Delta x}{2((2-x)^2 + 1)^{3/2}}$$

$$\& \Delta F_y = \frac{1}{\sqrt{(2-x)^2 + 1}} \Delta F = \frac{k q Q \Delta x}{2((2-x)^2 + 1)^{3/2}}.$$

It follows that the x and y -components of the total force are

$$F_x = \int_0^2 \frac{k q Q (2-x)}{2((2-x)^2 + 1)^{3/2}} dx$$

and

$$F_y = \int_0^2 \frac{k q Q}{2((2-x)^2 + 1)^{3/2}} dx.$$

Solving, we get that the total force exerted by the rod on the object, expressed as a vector, is

$$F = (F_x, F_y) = \frac{1}{2} k q Q \left(1 - \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right).$$

WORK

TANK EXAMPLE (E3.36)

3.36 Example: A tank is in the shape of the parabolic sheet given by $y = x^2$, $-2 \leq x \leq 2$, $-5 \leq z \leq 5$ together with the two ends given by $-2 \leq x \leq 2$, $x^2 \leq y \leq 4$ with $z = \pm 5$ (where the y -axis is pointing vertically). The tank is filled with a liquid of density ρ . Find the work required to pump all the liquid out of the tank, bringing it all to the level of the top of the tank. Use the fact that the work required to raise a small object of mass m from height h_1 to height h_2 is equal to

$$W = mgh$$

where $h = h_2 - h_1$.

Consider a thin slice of liquid at position y of thickness Δy .

The slice is in the shape of a thin rectangle of length $l=10$, width $w=2\sqrt{y}$ and thickness Δy , so it follows that its volume is

$$\Delta V = 20\sqrt{y} \Delta y.$$

Hence, its mass is

$$\Delta M = \rho \Delta V = 20\rho\sqrt{y} \Delta y.$$

Then, all of the water in this slice must be raised from height $h_1=4-y$ to height $h_2=4$, so that the work done "raising" the water in this slice is

$$\Delta W = gh \Delta M = 20\rho g(4-y)\sqrt{y} \Delta y.$$

It follows that the total work required to pump all of the water in the tank is

$$W = \int_0^4 20\rho g(4-y)\sqrt{y} dy = 20\rho g \left[\frac{8}{3}y^{\frac{3}{2}} - \frac{2}{5}y^{\frac{5}{2}} \right]_0^4$$

$$\therefore W = \frac{2560}{3} \rho g.$$

CHAIN EXAMPLE (E3.37)

3.37 Example: A chain, of length π and mass M , lies along the x -axis. Find the work required to lift the chain and lie it along the top half of the circle $x^2 + (y-1)^2 = 1$ (where the y -axis points upwards).

Note that for a thin slice of the chain (when it is lying on the top half of the circle) at position θ of thickness $\Delta\theta$, the mass of the slice is

$$\Delta M = \frac{M}{\pi} \Delta\theta,$$

and the height above the x -axis is

$$y = 1 + \sin\theta,$$

so that the work done is lifting the slice from the x -axis is

$$\Delta W = gy \Delta M = \frac{gM}{\pi} (1 + \sin\theta) \Delta\theta.$$

It follows that the total work done is

$$W = \int_0^\pi \frac{gM}{\pi} (1 + \sin\theta) d\theta = \frac{gM}{\pi} [\theta - \cos\theta]_0^\pi = \frac{gM}{\pi} (\pi + 2).$$

