

# MATH 146

# Personal Notes

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# Chapter 1:

# Vector Spaces

(S1.1)

## KEY

S :	section	P :	proposition
D :	definition	A :	assignment
R :	remark		
E :	example		
T :	theorem		
L :	lemma		
C :	corollary		

Let  $\mathbb{F}$  be a field.

Then, we say  $V$  is a "vector space"

over  $\mathbb{F}$  if there exists

① an addition  $+ : (V \times V) \rightarrow V$  by  $+ (x, y) = x + y$ ; and

② a scalar multiplication  $\cdot : (\mathbb{F} \times V) \rightarrow V$  by  $\cdot (a, x) = ax$ ;

and the following conditions hold:

①  $V$  is an abelian group with respect

to addition; (VS 1 = commutativity; 2 = associativity; 3 = identity; 4 = inverse)

②  $\forall x \in V \quad \forall x \in V$ ; (VS 5)

③ multiplication is associative; ie  $a(bx) = (ab)x \quad \forall a, b \in \mathbb{F}, x \in V$ ;

and (VS 6)

④ the left and right distributive laws hold;

ie  $a(x+y) = ax+ay$  and  $(a+b)x = ax+bx \quad \forall a, b \in \mathbb{F}, x \in V$ . (D2)

(VS 7 = former, VS 8 = latter)

## $\mathbb{F}^n$ IS A VECTOR SPACE OVER $\mathbb{F}$ (E2(1))

We can show that the Cartesian product

$$\mathbb{F}^n = \{(a_1, a_2, \dots, a_n) : a_i \in \mathbb{F} \text{ for } i \in \{1, 2, \dots, n\}\}$$

is a vector space over  $\mathbb{F}$  with respect to the addition operation

$$(a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n) = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)$$

and multiplication operation

$$c(a_1, a_2, \dots, a_n) = (ca_1, ca_2, \dots, ca_n).$$

Proof. This follows from verifying each of the conditions above.  $\blacksquare$

Note that we generally say "the vector space  $\mathbb{F}^n$ " to refer to the vector space  $\mathbb{F}^n$  over  $\mathbb{F}$ . (R3(4))

## COLUMN VECTOR NOTATION (E2(2))

Note that we can also write elements of  $\mathbb{F}^n$  as "column vectors"; ie of the form

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

where  $a_1, a_2, \dots, a_n \in \mathbb{F}$ .

## $\mathbb{Q}^n$ IS A VECTOR SPACE OVER $\mathbb{Q}$ ,

## $\mathbb{R}^n$ IS A VECTOR SPACE OVER $\mathbb{R}$ , &

## $\mathbb{C}^n$ IS A VECTOR SPACE OVER $\mathbb{C}$ (R3(1))

We can show

①  $\mathbb{Q}^n$  is a vector space over  $\mathbb{Q}$ ;

②  $\mathbb{R}^n$  is a vector space over  $\mathbb{R}$ ; and

③  $\mathbb{C}^n$  is a vector space over  $\mathbb{C}$ .

Proof. This directly follows from the fact that  $\mathbb{Q}, \mathbb{R}$  and  $\mathbb{C}$  are fields (MATH 145), and substituting the respective fields into the above lemma.  $\blacksquare$

## $\mathbb{R}^n$ IS A VECTOR SPACE OVER $\mathbb{Q}$ , &

## $\mathbb{C}^n$ IS A VECTOR SPACE OVER $\mathbb{R}$ (R3(2))

Moreover, we can also show that

①  $\mathbb{R}^n$  is a vector space over  $\mathbb{Q}$ ; and

②  $\mathbb{C}^n$  is a vector space over  $\mathbb{R}$ .

Proof. Essentially, this stems from the fact that we can "multiply" vectors in  $\mathbb{R}^n$  by scalars in  $\mathbb{Q}$ , and vectors in  $\mathbb{C}^n$  by scalars in  $\mathbb{R}$ .

The formal proof is left to the reader.  $\blacksquare$

## MATRICES (D3(1))

Let  $\mathbb{F}$  be a field, and  $m, n \in \mathbb{Z}^+$ .

Then, we say  $A$  is an " $m \times n$  matrix" with entries from  $\mathbb{F}$  if it is of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

where  $a_{ij} \in \mathbb{F} \quad \forall i \in \{1, 2, \dots, m\}, j \in \{1, 2, \dots, n\}$ .

Alternatively, we can represent  $A$  via the notation

$$A = (a_{ij}), \quad i \in \{1, 2, \dots, m\}, j \in \{1, 2, \dots, n\}.$$

## ij-ENTRY OF A MATRIX (D3(2))

Given a  $m \times n$  matrix  $A$ , the " $ij$ -entry" of  $A$ , or " $a_{ij}$ ", is defined to be the entry in  $A$  at the  $i$ th row and  $j$ th column.

## ZERO MATRIX (D3(3))

The " $m \times n$  zero matrix", or more simply the "zero matrix", denoted as " $0$ ", is defined to be

$$0 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \underbrace{\qquad\qquad\qquad}_{m}$$

ie the  $m \times n$  matrix where which entry equals  $0$ .

## MATRIX EQUALITY (D3(4))

We say two matrices  $A$  and  $B$  are equal if and only if  $a_{ij} = b_{ij} \quad \forall i \in \{1, 2, \dots, m\}, j \in \{1, 2, \dots, n\}$ .

## MATRIX ADDITION (D3(5))

Let  $A$  and  $B$  be  $m \times n$  matrices with entries from some field  $\mathbb{F}$ .

Then, the "addition" of  $A$  and  $B$ , denoted by " $A+B$ ", is defined to be the matrix where

$$(a+b)_{ij} = a_{ij} + b_{ij} \quad \forall i \in \{1, 2, \dots, m\}, j \in \{1, 2, \dots, n\}.$$

## MATRIX SCALAR MULTIPLICATION (D3(6))

Let  $A$  be a  $m \times n$  matrix with entries from some field  $\mathbb{F}$ , and  $c \in \mathbb{F}$  be arbitrary.

Then the "scalar multiplication" of  $A$  by  $c$ , denoted by " $ca$ ", is defined to be the matrix where

$$(ca)_{ij} = c(a_{ij}) \quad \forall i \in \{1, 2, \dots, m\}, j \in \{1, 2, \dots, n\}.$$

## SPACE OF $m \times n$ MATRICES (E3)

Let  $\mathbb{F}$  be a field.

Then the "space of all  $m \times n$  matrices" with entries from  $\mathbb{F}$ , denoted by " $M_{m \times n}(\mathbb{F})$ ", is defined to be the set of all  $m \times n$  matrices with entries from  $\mathbb{F}$ .

Note that  $M_{m \times n}(\mathbb{F})$  is a vector space over  $\mathbb{F}$  with respect to the matrix addition and scalar multiplication operations.

Proof. This follows from verifying each of the conditions in D2.  $\blacksquare$

## FUNCTION SPACES (E4)

- $\exists$ : Let the set  $D \neq \emptyset$  be arbitrary, and let  $\mathbb{F}$  be a field.
- Then the space of all functions from  $D$  to  $\mathbb{F}$ , denoted by " $\mathbb{F}^D$ ", is defined to be the set of all functions of the form  $f: D \rightarrow \mathbb{F}$ .
- $\exists_2$ : Similarly, we can show that  $\mathbb{F}^D$  is a vector space over  $\mathbb{F}$  with respect to the operations of function addition

$$(f+g)(x) := f(x) + g(x) \quad \forall f, g \in \mathbb{F}^D, x \in D$$

and function scalar multiplication

$$(cf)(x) := c f(x) \quad \forall f \in \mathbb{F}^D, x \in D, c \in \mathbb{F}$$

Proof. Similar strategy to E3: verify each condition in D2 holds.

## POLYNOMIALS (D4)

### SET OF ALL POLYNOMIALS OF DEGREE AT MOST $n$ ( $D4(1)$ )

$\exists$ : Let  $\mathbb{F}$  be a field.

Then, we denote  $P_n(\mathbb{F})$  to be the set of all polynomials with coefficients from  $\mathbb{F}$  and of degree at most  $n$ ; ie

$$P_n(\mathbb{F}) = \left\{ \sum_{i=0}^n a_i x^i : a_i \in \mathbb{F}, i \in \{0, \dots, n\} \right\}.$$

### POLYNOMIAL SPACES (D4(2))

$\exists$ : Let  $\mathbb{F}$  be a field.

Then, we denote " $\mathbb{F}[x]$ " to be the set of all polynomials with coefficients from  $\mathbb{F}$ ; ie

$$\mathbb{F}[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i : a_i \in \mathbb{F}, i \in \mathbb{N} \cup \{0\} \right\}.$$

Then, we can show that  $\mathbb{F}[x]$  is a vector space over  $\mathbb{F}$  with respect to the operations of polynomial addition

$$(f+g)(x) = \sum_{i=0}^{\infty} (a_i + b_i) x^i \quad \forall f, g \in \mathbb{F}[x]$$

and polynomial scalar multiplication

$$(cf)(x) = \sum_{i=0}^{\infty} (ca_i) x^i \quad \forall f \in \mathbb{F}[x], c \in \mathbb{F}$$

Proof. Similar strategy to E4.

## BASIC PROPERTIES OF VECTOR SPACES (SI.2)

### CANCELLATION PROPERTY FOR VECTOR ADDITION (TI.1)

$\exists$ : Let  $V$  be a vector space.

Suppose there exists some  $x, y, z \in V$  such that

$$x+z = y+z$$

Then necessarily  $x=y$ .

Proof. Note that

$$\begin{aligned} x &= x+0 \\ &= x+(z+(-z)) \\ &= (x+z)+(-z) \\ &= (y+z)+(-z) \\ &= y+(z+(-z)) \end{aligned}$$

and so  $x=y$ , as required.  $\blacksquare$

### UNIQUENESS OF THE ZERO VECTOR IN VECTOR SPACES (CI.1.1 (1))

$\exists$ : Let  $V$  be a vector space.

Suppose  $0_1, 0_2 \in V$  are both zero vectors.

Then necessarily  $0_1 = 0_2$ .

Proof. This follows from the fact that  $V$  is an abelian group under addition.  $\blacksquare$

### UNIQUENESS OF ADDITIVE INVERSES IN VECTOR SPACES (CI.1.1 (2))

$\exists$ : Let  $V$  be a vector space.

Then for any  $x \in V$ , there exists one and only one vector  $y \in V$  that satisfies  $x+y=0$ .

Proof. This also follows from the fact that  $V$  is an abelian group under addition.  $\blacksquare$

### $0x=0 \quad \forall x \in V$ (TI.2 (1))

$\exists$ : Let  $V$  be a vector space over some field  $\mathbb{F}$ , and let  $0$  be the additive identity of  $\mathbb{F}$ .

Then, for any  $x \in V$ , necessarily  $0 \cdot x = 0$ .

Proof. This, again, follows from the fact that  $V$  is an abelian group under addition.  $\blacksquare$

### $a0=0 \quad \forall a \in \mathbb{F}$ (TI.2 (2))

$\exists$ : Let  $V$  be a vector space over some field  $\mathbb{F}$ , and let  $0$  be the zero vector of  $V$ .

Then, for any  $a \in \mathbb{F}$ , necessarily  $a \cdot 0 = 0$ .

Proof. This, again, follows from the fact that  $V$  is an abelian group under addition.  $\blacksquare$

### $(-a)x = -(ax) = a(-x) \quad \forall a \in \mathbb{F}, x \in V$ (TI.2 (3))

$\exists$ : Let  $V$  be a vector space over some field  $\mathbb{F}$ , and let  $a \in \mathbb{F}, x \in V$  be arbitrary.

Then necessarily  $(-a)x = -(ax) = a(-x)$ .

Proof. Proof is similar to the analog of this statement for rings (MATH145).  $\blacksquare$

# SUBSPACES (SI.3)

Let  $V$  be a vector space over some field  $\mathbb{F}$ . Then we say the subset  $W \subseteq V$  is a "subspace" of  $V$  if

- ①  $W \neq \emptyset$ ;

\* we usually check whether  $0 \in W$  to verify this claim. (R4)

- ② If  $x \in W$  and  $y \in W$ , then  $(x+y) \in W$ ; and

- ③ If  $c \in \mathbb{F}$  and  $x \in W$ , then  $cx \in W$ . (D6)

## SUBSPACES ARE VECTOR SPACES OVER $\mathbb{F}$ WITH RESPECT TO THE OPERATIONS OF $V$ (TI.3)

Let  $W$  be a subspace of a vector space  $V$  over some field  $\mathbb{F}$ .

Then  $W$  is also a vector space over  $\mathbb{F}$  under the operations of  $V$  restricted to  $W$ .

Proof. This follows from verifying the conditions in D2, taking into account the properties of subspaces.  $\square$

## $\{0\}$ AND $V$ ARE SUBSPACES OF $V$ (E8(1))

Let  $V$  be a vector space.

Then  $\{0\}$  and  $V$  itself are always subspaces of  $V$ .

Proof.  $\{0\}$  is vacuously a subspace, and  $V$  is trivially a subspace.  $\square$

## $P_2(\mathbb{R})$ IS A SUBSPACE OF $\mathbb{R}[x]$ (E8(2))

We can show that  $P_2(\mathbb{R})$  is a subspace of  $\mathbb{R}[x]$ .

Proof. This stems from the fact that:

- $P_2(\mathbb{R}) \subset \mathbb{R}[x]$  by definition;
- $0 \in P_2(\mathbb{R})$ ; and
- $P_2(\mathbb{R})$  is closed under the addition & scalar multiplication defined on  $\mathbb{R}[x]$ .  $\square$

## $\{(a_{ij}) \in M_{n \times n}(\mathbb{F}) \mid \sum_{k=1}^n a_{kk} = 0\}$ IS A SUBSPACE

### OF $M_{n \times n}(\mathbb{F})$ (E8(3))

We can show that the set  $\{(a_{ij}) \in M_{n \times n}(\mathbb{F}) \mid \sum_{k=1}^n a_{kk} = 0\}$  is a subspace of  $M_{n \times n}(\mathbb{F})$ , where  $n \in \mathbb{N}$  is arbitrary.

Proof. Similar proof to the above.

## $\{(a_{ij}) \in M_{n \times n}(\mathbb{F}) \mid \sum_{k=1}^n a_{kk} = 1\}$ IS NOT A

### SUBSPACE OF $M_{n \times n}(\mathbb{F})$ (E8(4))

We can show the set  $\{(a_{ij}) \in M_{n \times n}(\mathbb{F}) \mid \sum_{k=1}^n a_{kk} = 1\}$  is not a subspace of  $M_{n \times n}(\mathbb{F})$ .

Proof. Let  $a, b \in \{(a_{ij}) \in M_{n \times n}(\mathbb{F}) \mid \sum_{k=1}^n a_{kk} = 1\}$  be arbitrary.

Then, notice that

$$\begin{aligned} \sum_{k=1}^n (a+b)_{kk} &= \sum_{k=1}^n (a_{kk} + b_{kk}) \\ &= 1 + 1 \\ &= 2, \end{aligned}$$

so that  $a+b \notin \{(a_{ij}) \in M_{n \times n}(\mathbb{F}) \mid \sum_{k=1}^n a_{kk} = 1\}$ .  $\square$

## SUBSPACES OF $\mathbb{R}^2$ (E9(1))

Note that the subspaces of  $\mathbb{R}^2$  are

- ①  $\mathbb{R}^2$  itself;

- ②  $\{0_{\mathbb{R}^2}\} = \{(0,0)\}$ ; and

- ③ all lines in  $\mathbb{R}^2$  that pass through  $(0,0)$ .

## SUBSPACES OF $\mathbb{F}^2$ (E9(4a))

In general, for any field  $\mathbb{F}$ , the subspaces of

$$\mathbb{F}^2$$
 are

- ①  $\mathbb{F}^2$  itself;

- ②  $\{0\}$ ; and

- ③ all the "lines" in  $\mathbb{F}^2$  through  $0$ .

i.e. of the form  $\{(x,y) \in \mathbb{F}^2 \mid (x) = k(y), (y) \in \mathbb{F}^2\}$

## SUBSPACES OF $\mathbb{R}^3$ (E9(2))

Similarly, the subspaces of  $\mathbb{R}^3$  are

- ①  $\mathbb{R}^3$  itself;

- ②  $\{0_{\mathbb{R}^3}\} = \{(0,0,0)\}$ ;

- ③ all lines in  $\mathbb{R}^3$  that pass through  $(0,0,0)$ ; and

- ④ all planes in  $\mathbb{R}^3$  that pass through  $(0,0,0)$ .

## SUBSPACES OF $\mathbb{F}^3$ (E9(4b))

Similarly, for any field  $\mathbb{F}$ , the subspaces

$$\text{of } \mathbb{F}^3 \text{ are}$$

- ①  $\mathbb{F}^3$  itself;

- ②  $\{0\}$ ;

- ③ all the "lines" in  $\mathbb{F}^3$  through  $0$ ; and

i.e. of the form  $\{(x,y,z) \in \mathbb{F}^3 \mid (x) = k(y), (y) = k(z), (z) \in \mathbb{F}^3\}$  (E9(3))

- ④ all the "planes" in  $\mathbb{F}^3$  through  $0$ .

i.e. of the form  $\{(x,y,z) \in \mathbb{F}^3 \mid \exists a, b, c \in \mathbb{F} \text{ such that } ax+by+cz=0\}$ .

# LINEAR COMBINATIONS & SYSTEM OF LINEAR EQUATIONS (SI.4)

## LINEAR COMBINATION (D7(1))

\* knowledge of elimination method is assumed.

$\exists_1$  Let  $V$  be a vector space over a field  $F$ , and let the subset  $S \subseteq V$  be such that  $S \neq \emptyset$ .

Then, we say a vector  $x \in V$  is a "linear combination" of vectors from  $S$  if there exists a finite number of vectors  $u_1, u_2, \dots, u_n \in S$  and scalars  $a_1, a_2, \dots, a_n \in F$  such that

$$x = a_1 u_1 + a_2 u_2 + \dots + a_n u_n,$$

where  $n \geq 1$ . (D7(1))

$\exists_2$  In this case, we also say that  $x$  is a linear combination of the vectors  $u_1, u_2, \dots, u_n$ .

## COEFFICIENTS OF A LINEAR COMBINATION (D7(2))

$\exists_1$  Let  $V$  be a vector space over some field  $F$ , and let the vector  $x \in V$  be a linear combination of the vectors  $u_1, u_2, \dots, u_n \in S$ , where  $S \subseteq V$  and  $S \neq \emptyset$ . Assume that  $x = a_1 u_1 + a_2 u_2 + \dots + a_n u_n$ , where  $a_1, a_2, \dots, a_n \in F$ .

Then we denote the scalars  $a_1, a_2, \dots, a_n \in F$  as the "coefficients" of the linear combination.

## SPAN (D7(3))

$\exists_1$  Let  $V$  be a vector space over some field  $F$ , and let the subset  $S \subseteq V$  be such that  $S \neq \emptyset$ .

Then, we define the "span" of  $S$ , denoted as "span( $S$ )", to be the set of all linear combinations of vectors in  $S$ .

$\exists_2$  Note that, for convenience, we define

$$\text{span}(\emptyset) = \{\emptyset\}.$$

## EXAMPLE 1: SPAN OF $(1,0,0)$ & $(0,1,0)$ IN $\mathbb{R}^3$ (E10(1))

$\exists_1$  Observe that in  $\mathbb{R}^3$ , the span of  $(1,0,0)$  &  $(0,1,0)$  in  $\mathbb{R}^3$  is

$$\{(a, 0, 0) + b(0, 1, 0) : a, b \in \mathbb{R}\}$$

or more simply

$$\{(a, b, 0) : a, b \in \mathbb{R}\}.$$

## EXAMPLE 2: SPAN( $\{x^n : n \geq 1\}$ ) IN $\mathbb{Q}[x]$ (E10(2))

$\exists_1$  We can show that for the vector space  $\mathbb{Q}[x]$ , the span of  $S = \{x, x^2, \dots, x^n, \dots\}$  is the set of all polynomials in  $\mathbb{Q}[x]$  whose constant coefficient equals 0.

## SPAN OF A FINITE AMOUNT OF VECTORS (E10(3.1))

$\exists_1$  Suppose  $V$  is a vector space over some field  $F$ , and let  $S \subseteq V$ . Further assume that

$$S = \{v_1, v_2, \dots, v_n\},$$

i.e. the size of  $S$  is finite.

Then, it follows that

$$\text{span}(S) = \{a_1 v_1 + a_2 v_2 + \dots + a_n v_n : a_i \in F \text{ for } i=1, 2, \dots, n\}.$$

## SPAN OF A COUNTABLE AMOUNT OF VECTORS (E10(3.2))

$\exists_1$  Suppose  $V$  is a vector space over some field  $F$ , and let  $S \subseteq V$ . Further assume that

$$S = \{v_1, v_2, \dots, v_n, \dots\},$$

i.e.  $|S| = |\mathbb{N}|$ .

Then, we can show that

$$\text{span}(S) = \text{span}(\{v_1\}) \cup \text{span}(\{v_1, v_2\}) \cup \dots \cup \text{span}(\{v_1, v_2, \dots, v_n\}) \cup \dots$$

or in other words, that

$$\text{span}(S) = \bigcup_{n=1}^{\infty} \text{span}(\{v_1, v_2, \dots, v_n\}).$$

## SPAN OF AN UNCOUNTABLE AMOUNT OF VECTORS (E10(3.3))

$\exists_1$  Suppose  $V$  is a vector space over some field  $F$ , and let  $S \subseteq V$ . Further assume that  $|S| > |\mathbb{N}|$ ; i.e. the size of  $S$  is uncountable. Then note that there are no "obvious" simplifications to the formula for  $\text{span}(S)$ .

## SPAN OF A SET IS ALWAYS A SUBSPACE OF THE PARENT VECTOR SPACE (T1.4)

$\exists_1$  Let  $V$  be a vector space over some field  $F$ , and let  $S \subseteq V$ . Then necessarily  $\text{span}(S)$  is a subspace of  $V$ .

Proof: This follows from verifying each subspace condition for  $\text{span}(S)$ .  $\square$

$\exists_2$  Moreover,  $\text{span}(S)$  is the "smallest possible" subspace of  $V$  that contains  $S$ , in the sense that

①  $S \subseteq \text{span}(S)$ ; and

② If  $W$  is any other subspace of  $V$  containing  $S$ , then  $\text{span}(S) \subseteq W$ .

## "GENERATES / SPANS" (D8)

$\exists_1$  Let  $V$  be a vector space, and let  $S \subseteq V$ .

Then, we say  $S$  "generates"  $V$ , or  $S$  "spans"  $V$ , if  $\text{span}(S) = V$ .

$\exists_2$  Note to prove  $\text{span}(S) = V$ , we just need to prove every vector in  $V$  can be written as a linear combination of vectors in  $S$ , since  $\text{span}(S) \subseteq V$  by definition.

(This follows from extensionality.) (R6)

# LINEAR INDEPENDENCE & DEPENDENCE (SI-5)

## LINEARLY DEPENDENT (D9(1))

$\text{💡}$  Let  $V$  be a vector space over some field  $F$ , and let  $S \subseteq V$ .

Then, we say  $S$  is "linearly dependent" if there exists a finite number of distinct vectors  $u_1, u_2, \dots, u_n \in S$  and scalars  $c_1, c_2, \dots, c_n \in F$ , where  $c_1, c_2, \dots, c_n$  are all not zero, such that

$$c_1 u_1 + c_2 u_2 + \dots + c_n u_n = 0.$$

$\text{💡}$  In this case, we also say the vectors of  $S$  are linearly dependent.

$\text{💡}$  Note that if  $S$  is finite, say  $S = \{u_1, u_2, \dots, u_n\}$ , then  $S$  is linearly dependent if and only if there exists a  $(c_1, c_2, \dots, c_n) \in F^n$ , where  $(c_1, c_2, \dots, c_n) \neq (0, 0, \dots, 0)$ , such that

$$c_1 u_1 + c_2 u_2 + \dots + c_n u_n = 0. \quad (\text{R7(4a)})$$

## LINEARLY INDEPENDENT (D9(2))

$\text{💡}$  Let  $V$  be a vector space over some field  $F$ , and let  $S \subseteq V$ .

Then, we say  $S$  is "linearly independent" if it is not "linearly dependent"; ie for every choice of distinct  $u_1, u_2, \dots, u_n \in S$ , if  $c_1, c_2, \dots, c_n \in F$  are scalars such that

$$c_1 u_1 + c_2 u_2 + \dots + c_n u_n = 0,$$

then necessarily  $c_1 = c_2 = \dots = c_n = 0$ .

$\text{💡}$  Similarly, if  $S$  is finite, say  $S = \{u_1, u_2, \dots, u_n\}$ , then  $S$  is linearly independent if and only if whenever  $(c_1, c_2, \dots, c_n) \in F^n$  are such that

$$c_1 u_1 + c_2 u_2 + \dots + c_n u_n = 0,$$

then necessarily  $c_1 = c_2 = \dots = c_n = 0$ .

## TRIVIAL REPRESENTATION OF 0 (R7(1))

$\text{💡}$  Note that for any vector space  $V$  and vectors  $u_1, u_2, \dots, u_n \in V$ , we denote the "trivial representation of  $0 \in V$ " as a linear combination of  $u_1, u_2, \dots, u_n$  by

$$0u_1 + 0u_2 + \dots + 0u_n = 0.$$

## EMPTY SET IS LINEARLY INDEPENDENT (R7(2))

$\text{💡}$  Note that the empty set,  $\emptyset$ , is vacuously linearly independent.

\* since linearly dependent sets must be non-empty by definition.

## $\{0\}$ IS LINEARLY DEPENDENT (R7(3))

$\text{💡}$  Note that the set  $\{0\}$  is linearly dependent, since  $1(0) = 0$  is a non-trivial representation of  $0$  as a linear combination of finitely many distinct vectors in  $S$ .

## $0 \in S \Rightarrow S$ IS LINEARLY DEPENDENT (R7(5))

$\text{💡}$  Note that any subset of a vector space that contains the zero vector is linearly dependent.

**EXAMPLE 1:**  $S = \{(0, 1, 1), (1, 0, 1), (1, 2, 3)\}$  IS LINEARLY DEPENDENT IN  $\mathbb{R}^3$  (E14)

$\text{💡}$  We can show that the set  $S = \{(0, 1, 1), (1, 0, 1), (1, 2, 3)\}$  is linearly dependent in  $\mathbb{R}^3$ .

Proof. We search for scalars  $a, b, c \in \mathbb{R}$ , not all 0, such that

$$a\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + b\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This reduces to solving the system

$$\begin{cases} b+c=0 \\ a+2c=0 \\ a+b+3c=0 \end{cases}$$

Simplifying, we get that

$$a = -2t, b = -t \text{ and } c = t,$$

where  $t \in \mathbb{R}$ .

For instance,  $(a, b, c) = (-2, -1, 1)$  is a solution in which not all of  $a, b, c$  are 0.

It follows that  $S$  is linearly dependent.  $\blacksquare$

**EXAMPLE 2:**  $S = \{1, x, x^2, x^3\}$  IS LINEARLY INDEPENDENT IN  $\mathbb{Z}_5[x]$  (E15)

$\text{💡}$  We can show that the set  $S = \{1, x, x^2, x^3\}$  is linearly independent in  $\mathbb{Z}_5[x]$ .

Proof. Note that if there exist  $a_0, a_1, a_2, a_3 \in \mathbb{Z}_5$  such that

$$a_0(1) + a_1x + a_2x^2 + a_3x^3 = 0,$$

then by definition necessarily  $a_0 = a_1 = a_2 = a_3 = 0$ , and this is sufficient to prove the claim.  $\blacksquare$

$S$  IS LINEARLY DEPENDENT  $\Leftrightarrow$

$S = \{0\}$  OR SOME VECTOR IN  $S$  IS A  
LINEAR COMBINATION OF OTHER VECTORS  
IN  $S$  (TI-S)

Let  $V$  be a vector space, and let  $S \subseteq V$ .  
Then  $S$  is linearly dependent if and only if  
 $S = \{0\}$  or some vector in  $S$  is a linear  
combination of other vectors in  $S$ .

Proof. We first prove the backward argument.

First, note we know why  $\{0\}$  is linearly  
dependent from a previous section.

So, suppose there exists a vector  $v \in S$   
such that

$$v = c_1 u_1 + c_2 u_2 + \dots + c_n u_n,$$

where  $c_i \in \mathbb{F}$  and  $u_i \in V$   $\forall i \in \{1, 2, \dots, n\}$ .

Without loss in generality, assume  $u_1, u_2, \dots, u_n$  are distinct.

By assumption, since  $v \notin \{u_1, u_2, \dots, u_n\}$ , necessarily

$u_1, u_2, \dots, u_n, v$  are distinct.

Finally, since

$$0 = (-1)v + c_1 u_1 + c_2 u_2 + \dots + c_n u_n,$$

and  $-1 \neq 0$ , it follows  $S$  is linearly dependent. \*

Next, we prove the forward argument.

Assume  $S$  is linearly dependent, so that there exist  
distinct  $u_1, u_2, \dots, u_n \in S$  and  $a_1, a_2, \dots, a_n \in \mathbb{F}$  (not all 0)  
such that

$$a_1 u_1 + a_2 u_2 + \dots + a_n u_n = 0.$$

Without loss in generality, assume  $a_1 \neq 0$   $\forall i \in \{1, 2, \dots, n\}$ .

Case 1:  $n=1$ .

Then  $a_1 u_1 = 0$ , and since  $a_1 \neq 0$  it follows that  $u_1 = 0$   
(since fields are integral domains, so the cancellation  
property applies.)

Hence  $0 \in S$ . If  $S = \{0\}$  we are done;  
otherwise, we can pick a  $v \in S \setminus \{0\}$ , and we  
can write  $0 = 0v$ , proving some vector in  $S$ , 0, can  
be written as a linear combination of another  
vector,  $v$ , in  $S$ .

Case 2:  $n > 1$ .

Then since  $a_1 \neq 0$ , we can solve for  $u_1$ :

$$u_1 = -a_1^{-1} [a_1 u_1 + a_2 u_2 + \dots + a_{n-1} u_{n-1}],$$

$$\therefore u_1 = (-a_1^{-1} a_1) u_1 + (-a_1^{-1} a_2) u_2 + \dots + (-a_1^{-1} a_{n-1}) u_{n-1},$$

showing  $u_1$  can be expressed as a linear

combination of other elements in  $S$ .

# BASES & DIMENSION (SI.6)

## BASIS (D10)

Let  $V$  be a vector space.

Then, we say a subset  $S \subseteq V$  is a "basis" for  $V$  if  
①  $S$  is linearly independent; and  
②  $S$  spans  $V$ .

In this case, we also say that the vectors of  $S$  form a basis for  $V$ .

## STANDARD BASIS (C17)

In  $\mathbb{F}^n$ , define the "standard basis" for  $\mathbb{F}^n$  the subset

$$S = \{e_1, e_2, \dots, e_n\},$$

where  $e_j \in \mathbb{F}^n$  is the vector with  $j$ th coordinate 1 and other coordinates 0.

(It is easy to prove  $S$  is indeed a basis for  $\mathbb{F}^n$ )

In  $P_n(\mathbb{F})$ , define the "standard basis" for  $P_n(\mathbb{F})$  as the set

$$S = \{1, x, x^2, \dots, x^n\}.$$

(It is also easy to prove  $S$  is indeed a basis for  $P_n(\mathbb{F})$ ).

## UNIQUE REPRESENTATION OF ELEMENTS IN VECTOR SPACES UNDER A BASIS (T1.6)

Let  $\{v_1, v_2, \dots, v_n\}$  be a basis for a vector space  $V$ .

Then for every  $x \in V$ ,  $x$  can be uniquely represented as a linear combination of  $v_1, v_2, \dots, v_n$ ; ie there exists a unique  $n$ -tuple  $(a_1, a_2, \dots, a_n) \in \mathbb{F}^n$  such that

$$x = a_1 v_1 + a_2 v_2 + \dots + a_n v_n.$$

Proof. Existence: this follows from the fact that  $\{v_1, v_2, \dots, v_n\}$  spans  $V$  by definition.

Uniqueness: suppose there exists some  $b_1, b_2, \dots, b_n \in \mathbb{F}$  such that

$$x = a_1 v_1 + \dots + a_n v_n = b_1 v_1 + \dots + b_n v_n.$$

It follows that

$$0 = (a_1 - b_1)v_1 + \dots + (a_n - b_n)v_n.$$

and since  $\{v_1, v_2, \dots, v_n\}$  is linearly independent, necessarily  $a_i = b_i \quad \forall i \in \{1, 2, \dots, n\}$ .  $\square$

$V$  IS GENERATED BY  $S$ ,  $|S| = |\mathbb{N}|$

$\Rightarrow TCS$  IS ALSO A BASIS FOR  $V$  (T1.7)

Let  $V$  be a vector space, and assume that

$V$  is generated by a countable set  $S$ .

Then there exists a subset of  $S$  that is a basis for  $V$ .

Proof. If  $S = \emptyset$  or  $S = \{0\}$ , then  $\emptyset$  is a basis for  $V$  trivially.

Otherwise,  $S$  contains at least a non-zero vector.

Hence, we can write  $S$  as

$$S = \{v_1, v_2, \dots, v_n\} \text{ or } S = \{v_1, v_2, \dots\}.$$

By the WOP, we can pick the smallest index  $i \geq 1$  such that  $v_i \neq 0$ .

Then  $\{v_i\}$  is linearly independent.

Let  $i_2$  be the smallest index such that  $v_{i_2} \in \text{span}\{v_i\}$ .

Continue this "process" until we obtain the set

$$T = \{v_{i_k} \mid v_{i_k} \notin \text{span}\{v_{i_1}, \dots, v_{i_{k-1}}\}, k \geq 1\}.$$

Finally, we can prove  $T$  is a basis for  $V$ .

① Assume  $T$  is linearly dependent.

Then there exists  $a_1, a_2, \dots, a_k$ , all not 0, such that

$$a_1 v_{i_1} + \dots + a_k v_{i_k} = 0.$$

It follows that

$$v_{i_k} = -a_1^{-1} a_1 v_{i_1} - \dots - a_{k-1}^{-1} a_{k-1} v_{i_{k-1}},$$

contradicting the construction of  $T$ .

② We can prove by induction that  $\text{span}(S_k) = \text{span}(T_k) \quad \forall k \geq 1$ , where

$$S_k = \{v_1, v_2, \dots, v_k\} \text{ and } T_k = T \cap S_k = \{v_{i_q} \mid i_q \leq k\}.$$

Then, let  $x \in \text{span}(S)$ . Then  $x \in \text{span}(S_m)$  for some large  $m$ , so that  $x \in \text{span}(T_m) \subset \text{span}(T)$ .

Hence  $V \subseteq \text{span}(T)$ , and it follows that  $V = \text{span}(T)$ .  $\square$

## EVERY VECTOR SPACE HAS A BASIS

### (T1.8)

We can prove that every vector space has a basis.

(The proof uses Zorn's Lemma & maximal linearly independent subsets.)

## REPLACEMENT THEOREM (TI.9)

Suppose  $V$  is a vector space with a finite spanning set  $S$ . Let  $T$  be a linearly independent subset in  $V$ . Then

- ①  $|T| \leq |S|$ ; and
- ② There exists a set  $H \subseteq S$  containing exactly  $(|S|-|T|)$  vectors such that  $T \cup H$  generates  $V$ .

Proof. Let  $n = |S|$ , and let  $m = |T|$ . Then, when  $m=0$ , clearly  $m=0 \leq |S|$ . Next, assume the statement is true for some  $m \geq 0$ . This implies that if  $T_m \subseteq V$  is any linearly independent subset in  $V$  of size  $m$ , then  $m \leq n$  and there exists a set  $H_m \subseteq S$  containing exactly  $n-m$  vectors such that  $T_m \cup H_m$  generates  $V$ .

Let  $T_m = \{v_1, v_2, \dots, v_m\}$  and  $T = T_m \cup \{v_{m+1}\}$ , such that  $T$  is linearly independent and a subset of  $V$ .

Note that this implies  $T_m$  is also linearly independent.

Now, apply the induction hypothesis on  $T_m$  to get that  $n \geq m$ , and there exist  $(n-m)$  vectors  $w_{m+1}, \dots, w_n \in S$  such that

$\{v_1, \dots, v_m, w_{m+1}, \dots, w_n\}$  generates  $V$ .

Then, since  $n \geq m$ , either  $n=m$  or  $n > m$ .

If  $n=m$ ,  $\{v_1, \dots, v_m, w_{m+1}, \dots, w_n\} = \{v_1, \dots, v_m\}$ .

Thus,  $v_{m+1} \in \text{span}\{v_1, \dots, v_m\}$ , so by Theorem 1.5, the set  $\{v_1, \dots, v_m, v_{m+1}\}$  is linearly dependent.

But this is a contradiction; hence, it follows that  $n > m$ , so that  $n \geq m+1$ , proving ①.

Subsequently, write

$$v_{m+1} = a_1 v_1 + \dots + a_m v_m + a_{m+1} v_{m+1} + \dots + a_n w_n$$

for some scalars  $a_1, \dots, a_n \in \mathbb{F}$ .

Then, if  $a_{m+1} = \dots = a_n = 0$ , then we would get that  $v_{m+1} = a_1 v_1 + \dots + a_m v_m$ , which is a contradiction; hence, at least one of the scalars  $a_{m+1}, \dots, a_n$  must be non-zero.

Then, without loss in generality, assume  $a_{m+1} \neq 0$ .

It follows that

$$\begin{aligned} w_{m+1} &= -a_{m+1}^{-1} a_1 v_1 - \dots - a_{m+1}^{-1} a_m v_m - a_{m+1}^{-1} a_{m+1} v_{m+1} \\ &\quad - a_{m+1}^{-1} a_{m+2} w_{m+2} - \dots - a_{m+1}^{-1} a_n w_n. \end{aligned}$$

Let  $H = \{w_{m+2}, \dots, w_n\} \subset S$ . The above shows that

$w_{m+1} \in \text{span}(T \cup H)$ .

Moreover, since  $v_1, \dots, v_m \in T \subseteq \text{span}(T \cup H)$  and  $w_{m+2}, \dots, w_n \in H \subseteq \text{span}(T \cup H)$ , it follows that

$$V = \text{span}\{w_{m+1}, v_1, \dots, v_m, w_{m+2}, \dots, w_n\} \subseteq \text{span}(T \cup H).$$

But since  $\text{span}(T \cup H) \subseteq V$ , it follows that  $V = \text{span}(T \cup H)$ , completing the proof.  $\square$

## V IS FINITELY SPANNED $\Rightarrow$ ALL BASES OF V & H HAVE EQUAL CARDINALITIES (CI.9.1)

Suppose  $V$  is a finitely spanned vector space.

Then all bases of  $V$  are finite and have the same amount of elements.

Proof. Let  $S$  be a finite spanning set for  $V$ , and let  $B$  be an arbitrary basis for  $V$ . Then by definition,  $B$  is linearly independent.

By the Replacement Theorem,  $|B| \leq |S| < \infty$ .

Next, let  $B_1$  and  $B_2$  be two bases of  $V$ . Then, since  $B_1$  is linearly independent and  $B_2$

is a finite spanning set for  $V$ , by the

Replacement Theorem necessarily  $|B_1| \leq |B_2|$ .

Similarly, since  $B_2$  is linearly independent and  $B_1$  is a finite spanning set for  $V$ , by the

Replacement Theorem necessarily  $|B_2| \leq |B_1|$ .

It follows that  $|B_1| = |B_2|$ , and we are done.

## DIMENSION FINITE/INFINITE-DIMENSIONAL (DI.2)

We say a vector space  $V$  is "finite-dimensional" if it has a basis consisting of a finite number of vectors.

Otherwise, we say  $V$  is "infinite-dimensional".

## DIMENSION (DI.2)

Let  $V$  be a finite-dimensional vector space.

Then, the "dimension" of  $V$ , denoted as "dim  $V$ ", is defined to be the unique number of vectors in each basis for  $V$ .

By convention, we let  $\dim\{0\} = 0$ .

Examples:

- ①  $\dim \mathbb{F}^n = n$ ;
- ②  $\dim \mathbb{C}^n = 2n$ ;
- ③  $\dim M_{m \times n}(\mathbb{F}) = mn$ ; and
- ④  $\dim P_n(\mathbb{F}) = n+1$ . (E18)

## ANY FINITE SPANNING SET FOR $V$ CONTAINS AT LEAST $n$ VECTORS (C1.9.2(1))

Let  $V$  be a vector space with  $\dim V = n$ . Then if  $S$  is a finite spanning set for  $V$ , necessarily  $|S| \geq n$ .

Proof. By the Existence Theorem (T1.7), there exists a subset  $T$  of  $S$  that is a basis for  $V$ . Therefore  $|T| = \dim V = n$ , which implies that  $|S| \geq |T| = n$ .  $\square$

## $S$ GENERATES $V$ , $|V|=n \Rightarrow S$ IS A BASIS FOR $V$ (C1.9.2 (2))

Let  $V$  be a vector space with  $\dim V = n$ , and suppose  $S$  generates  $V$ , with  $|S|=n$ . Then  $S$  is a basis for  $V$ .

Proof. Again, by the Existence Theorem (T1.7), there exists some subset  $T \subseteq S$  such that  $T$  is a basis for  $V$ . By the above corollary,  $|T|=n$ , so that if  $|S|=n$ , necessarily  $S=T$ . It follows that  $S$  is a basis for  $V$ .  $\square$

## $S$ IS LINEARLY INDEPENDENT $\Rightarrow$ $S$ CONTAINS AT MOST $n$ VECTORS (C1.9.2(3))

Let  $V$  be a vector space, with  $\dim V = n$ . Suppose the subset  $S \subseteq V$  is linearly independent. Then  $S$  contains at most  $n$  vectors.

Proof. Applying the Replacement Theorem for the spanning set  $P$ , it follows that  $|S| \leq |P|$ , and since  $|P|=n$ , this tells us that  $|S| \leq n$ , as needed.  $\square$

## $S$ IS LINEARLY INDEPENDENT, $|S|=n$ $\Rightarrow S$ IS A BASIS FOR $V$ (C1.9.2 (4))

Let  $V$  be a vector space, with  $\dim V = n$ . Suppose the subset  $S \subseteq V$  is linearly independent and  $|V|=n$ . Then  $S$  is a basis for  $V$ .

Proof. Applying the Replacement Theorem for the spanning set  $P$  and the linearly independent set  $S$ , there must exist a subset  $H \subseteq P$  containing  $|P|-|S|=n-n=0$  vectors such that  $S \cup H$  generates  $V$ . But since  $|H|=0$ , hence  $H=\emptyset$ , so that  $S$  generates  $V$  (and hence is a basis for  $V$ ).  $\square$

## EVERY LINEARLY INDEPENDENT SUBSET OF $V$ CAN BE "EXTENDED" TO A BASIS OF $V$ (C1.9.2 (5))

Let  $V$  be a vector space, with  $\dim V = n$ . Suppose  $L = \{v_1, \dots, v_k\}$  is a linearly independent subset of  $V$ , where  $1 \leq k \leq n$ . Then there exists a HCV such that  $L \cup H$  is a basis of  $V$ .

Proof. If  $k=n$ , by C1.9.2(4)  $L$  is trivially a basis for  $V$ . If  $k < n$ , then by the Replacement Theorem for the spanning set  $P$  and  $L$ , there necessarily exists a subset HCB containing  $|P|-|L|=n-k$  vectors such that  $L \cup H$  generates  $V$ . By C1.9.2(1),  $|L \cup H| \geq n$ . But  $|L \cup H| \leq |L| + |H| = k + (n-k) = n$ , so that  $|L \cup H| = n$ . It follows by C1.9.2(2) that  $L \cup H$  is a basis for  $V$ .  $\square$

## $W$ IS A SUBSPACE OF $V$

$$\Rightarrow \dim W \leq \dim V ; \quad \dim W = \dim V \\ \Leftrightarrow W = V \quad (\text{C1.9.2 (6)})$$

Let  $W$  be a subspace of the vector space  $V$ . Then  $\dim W \leq \dim V$ , with equality occurring if and only if  $V=W$ .

Proof. If  $W=\{v\}$ , then  $\dim W=0 \leq \dim V$ . Otherwise,  $W$  contains a non-zero vector  $w_1$ . Then  $\{w_1\}$  is linearly independent. Continue to choose the vectors  $w_1, \dots, w_n \in W$  such that  $\{w_1, \dots, w_k\}$  is linearly independent. Note that this process cannot go on indefinitely, since  $\{w_1, \dots, w_k\}$  is also linearly independent in  $V$ . This implies that  $k \leq n$ . Next, by T1.5,  $W \subseteq \text{span}(\{w_1, \dots, w_k\}) = \text{span}(T)$ . Then, since  $T \subseteq W$ , necessarily  $\text{span}(T) \subseteq \text{span}(W) = W$ . It follows that  $W = \text{span}(T)$ , so that  $T$  is a basis (since it is also linearly independent), and  $\dim W = |T| = k \leq n = \dim V$ .

Note that if  $\dim V = n = \dim W$ , then a basis for  $W$  is also a linearly independent set containing  $n$  elements. Hence, by C1.9.2(4), that set is also a basis for  $V$ .  $\square$

## $W$ IS A SUBSPACE FOR $V \Rightarrow$ ANY BASIS OF $W$ CAN BE "EXTENDED" TO A BASIS IN $V$ (C1.9.2 (7))

Let  $W$  be a subspace of the vector space  $V$ , and let  $S$  be a basis of  $W$ . Then we can "extend"  $S$  to a basis in  $V$ .

Proof. By C1.9.2(6),  $\dim W \leq \dim V$ . Let  $T = \{w_1, \dots, w_n\}$  be a basis for  $W$ , so that  $T$  is linearly independent in  $W$ , which in turn implies  $T$  is linearly independent in  $V$ . So, by C1.9.2(5), we can "extend"  $T$  to a basis in  $V$ .  $\square$

# QUOTIENT SPACES (SI.7)

## COSET & REPRESENTATIVE (D13)

Let  $V$  be a vector space, and  $W$  be a subspace of  $V$ . Then, for a given  $x \in V$ , its corresponding "coset" of  $W$  in  $V$ , denoted as " $x+W$ ", is defined to be the set  $x+W = \{x+w : w \in W\}$ .

\* note that  $x+W \subseteq V$ .

In this case, we call " $x$ " a "representative" of the coset  $x+W$ .

## $x \equiv y \pmod{W}$ (D13)

Let  $V$  be a vector space, and let  $W$  be a subspace of  $V$ . Then, we write " $x \equiv y \pmod{W}$ " if and only if  $x-y \in W$ .

## $V/W$ (D13)

Let  $V$  be a vector space, and  $W$  a subspace of  $V$ . Then, we denote " $V/W$ " (ie " $V \pmod{W}$ ") as the set

$V/W = \{x+W : x \in V\}$ ;  
ie let  $V/W$  be the collection of cosets of  $W$  in  $V$ .

## $V/\{0\} = V$ (E19 (2))

For any vector space  $V$ , necessarily  $V/\{0\} = V$ .

Proof:  $V/\{0\} = \{0+x : x \in V\} = \{x : x \in V\} \therefore V/\{0\} = V$ .

## COSET TEST (P1)

Let  $W$  be a subspace of a vector space  $V$ , and let  $x, y \in V$  be arbitrary. Then  $x+W = y+W$  if and only if  $x-y \in W$ .

Proof: Similar to test for cosets in MATH 145.

## $\equiv \pmod{W}$ IS AN EQUIVALENCE RELATION ON $V$ (R8)

Note that the relation " $\equiv \pmod{W}$ " is an equivalence relation on  $V$ .

## ADDITION & MULTIPLICATION IN $V/W$ (D14)

Let  $V$  be a vector space over a field  $F$ , and let  $W$  be a subspace of  $V$ . Then, we can define an addition on  $V/W$  by

$(x+W) + (y+W) := ((x+y)+W)$ ;  
and a scalar multiplication on  $V/W$  by

$a(x+W) := (ax)+W$ ;

for any  $a \in F$  and  $x, y \in W$ .

Note that these addition and multiplication operations are well-defined. (L1)

Proof: Similar to proof for quotient groups/rings.

## $V/W$ IS A VECTOR SPACE

### (THE QUOTIENT SPACE OF $V$ BY $W$ ) (TI.10)

Let  $V$  be a vector space, and  $W$  a subspace of  $V$ . Then the set  $V/W$  is a vector space over  $F$  with the operations of coset addition and scalar multiplication, denoted as "the quotient space of  $V$  by  $W$ ".

Proof: Verify all 8 conditions. (VS 1-8).

## BASIS FOR QUOTIENT SPACES (TI.11)

Let  $V$  be a vector space with  $\dim V = n$ , and let  $W$  be a subspace of  $V$  such that  $\dim W = k$ . Let  $\{v_1, \dots, v_n\}$  be a basis for  $V$ , such that  $\{v_1, \dots, v_k\}$  is a basis for  $W$ .

Then,

- ① The set  $\{v_{k+1}+W, \dots, v_n+W\}$  is a basis for  $V/W$ ; and
- ②  $\dim(V/W) = \dim V - \dim W$ .

Proof: To prove ①, we show  $\{v_{k+1}+W, \dots, v_n+W\}$  is both linearly independent and generates  $V/W$ , giving us our basis.

It follows that

$$\begin{aligned}\dim(V/W) &= |\{v_{k+1}+W, \dots, v_n+W\}| \\ &= n - (k+1) \\ &= n - k \\ \therefore \dim(V/W) &= \dim V - \dim W.\end{aligned}$$

$$\dim V \geq \infty, \dim W \geq \infty \Rightarrow \dim V/W \geq \infty \quad (\text{R9})$$

Let  $V$  be an infinite-dimensional vector space, and let  $W$  be an infinite-dimensional subspace of  $V$ .

Then, note that it is not necessarily the case that  $\dim(V/W) \geq \infty$ .

Example: let  $V = \mathbb{F}^\infty$  &  $W = \{(0, x_2, \dots) : x_2 \in \mathbb{F}\}$ . Note that each element of  $V/W$  is simply "determined" by the value of the first coordinate  $x_1$ , so that  $\dim(V/W) = 1$ .

# SUMS & INTERNAL DIRECT SUMS OF SUBSPACES (SL8)

## SUM OF SUBSPACES (DIS)

Let  $V$  be a vector space over  $\mathbb{F}$ , and let  $W_1, W_2$  be subspaces of  $V$ . Then, we define the "sum" of  $W_1$  and  $W_2$ , denoted as  $W_1 + W_2$ , to be the set

$$W_1 + W_2 := \{v_1 + v_2 : v_1 \in W_1, v_2 \in W_2\}.$$

## INDEPENDENT/DISJOINT (DIS)

Let  $V$  be a vector space, and let  $W_1, W_2$  be subspaces of  $V$ . Then, we say  $W_1$  and  $W_2$  are "independent", or "disjoint", if and only if  $W_1 \cap W_2 = \{0\}$ .

## (INTERNAL) DIRECT SUM (DIS)

Let  $V$  be a vector space, and let  $W_1, W_2$  be independent subspaces of  $V$ .

Then, we define the "(internal) direct sum" of  $W_1$  and  $W_2$ , denoted as  $W_1 \oplus W_2$ , to be the set

$$W_1 \oplus W_2 = W_1 + W_2.$$

\* ie " $\oplus$ " is the notation for "+" used when  $W_1$  &  $W_2$  are independent.

Note that  $W_1 \oplus W_2$  is well-defined, as long as  $W_1 \cap W_2 = \{0\}$ . (R10)

## $W_1 + W_2$ IS THE "SMALLEST" SUBSPACE CONTAINING $W_1$ & $W_2$ (L2 (2))

Let  $V$  be a vector space, and let  $W_1, W_2$  be subspaces of  $V$ .

Then  $W_1 + W_2$  is necessarily the smallest subspace of  $V$  containing  $W_1$  and  $W_2$ .

Proof. First, we prove  $W_1 + W_2$  is a subspace of  $V$ .

Let  $(v_1 + v_2), (u_1 + u_2) \in W_1 + W_2$  and  $a \in \mathbb{F}$ , where  $v_1, v_2 \in W_1$  and  $u_1, u_2 \in W_2$ .

Then, since  $W_1$  and  $W_2$  are subspaces of  $W_1 + W_2$ , necessarily  $v_1 + u_1 \in W_1 + W_2$  and  $v_2 + u_2 \in W_1 + W_2$ .

so that

$$(v_1 + v_2) + (u_1 + u_2) = (v_1 + u_1) + (v_2 + u_2) \in W_1 + W_2.$$

Moreover, since  $av_1 \in W_1$  and  $av_2 \in W_2$ , necessarily

$$a(v_1 + v_2) = av_1 + av_2 \in W_1 + W_2.$$

proving  $W_1 + W_2$  is closed under addition and scalar multiplication.

Then, since  $v_1 = v_1 + 0 \in W_1 + W_2 \quad \forall v_1 \in W_1$  &  $v_2 = 0 + v_2 \in W_1 + W_2 \quad \forall v_2 \in W_2$ , it follows that

$$W_1 \subseteq W_1 + W_2 \text{ and } W_2 \subseteq W_1 + W_2.$$

Finally, let  $Y$  be a subspace of  $V$  that contains both  $W_1$  &  $W_2$ .

Since  $Y$  is closed under addition,  $v_1 + v_2 \in Y$

for every  $v_1 \in W_1$  and  $v_2 \in W_2$  necessarily.

It follows that  $W_1 + W_2 \subseteq Y$ , completing the proof.

$$V = W_1 \oplus W_2 \iff \forall v \in V : \exists \text{ unique } w_1 \in W_1,$$

$$w_2 \in W_2 \ni v = w_1 + w_2 \quad (\text{L2 (3)})$$

Let  $V$  be a vector space, and let  $W_1$  and  $W_2$  be subspaces of  $V$ .

Then  $W_1 \oplus W_2 = V$  if and only if for every vector  $v \in V$ , there exist unique elements  $w_1 \in W_1$  and  $w_2 \in W_2$  such that  $v = w_1 + w_2$ .

Proof. ( $\Rightarrow$ ) Since  $V = W_1 \oplus W_2$ , necessarily  $V = W_1 + W_2$ , and  $W_1 \cap W_2 = \{0\}$ .

let  $v \in V$ , and note that since  $V = W_1 + W_2$ , it implies that  $v \in W_1 + W_2$ .

So, by definition, there exist some  $w_1 \in W_1, w_2 \in W_2$  such that  $v = w_1 + w_2$ .

Next, suppose we have  $v = w'_1 + w'_2$  for some  $w'_1 \in W_1$  and  $w'_2 \in W_2$ . Then

$$0 = (w_1 + w_2) - (w'_1 + w'_2) = (w_1 - w'_1) + (w_2 - w'_2).$$

Since  $w_1, w'_1 \in W_1$  &  $w_2, w'_2 \in W_2$ , necessarily  $w_1 - w'_1 \in W_1$  &  $w_2 - w'_2 \in W_2$  also, so that

$$(w_1 - w'_1) = w'_2 - w_2 \in W_1 \cap W_2 = \{0\},$$

Hence  $w_1 - w'_1 = w'_2 - w_2 = 0$ , implying that  $w_1 = w'_1$  &  $w_2 = w'_2$ , proving uniqueness. \*

( $\Leftarrow$ ) By assumption, every vector  $v \in V$  can be written as  $v = w_1 + w_2$  for some  $w_1 \in W_1$  &  $w_2 \in W_2$ . Hence  $V \subseteq W_1 + W_2$ , and by L2(2) necessarily  $W_1 + W_2 \subseteq V$ ; so  $V = W_1 + W_2$ .

Next, let  $x \in W_1 \cap W_2$ . Then  $-x \in W_1 \cap W_2$ .

Then, note that

$$0 = 0 + 0 = x + (-x) \in W_1 + W_2,$$

and due to the uniqueness assumption, necessarily  $x = 0$ .

Thus  $W_1 \cap W_2 = \{0\}$ , so that  $V = W_1 \oplus W_2$ .  $\blacksquare$

$$\dim(W_1), \dim(W_2) < \infty \Rightarrow \dim(W_1 + W_2) < \infty \text{ &} \\ \dim(W_1) + \dim(W_2) = \dim(W_1 + W_2) + \dim(W_1 \cap W_2)$$

(T1.12 (1))

Let  $V$  be a vector space over some field  $\mathbb{F}$ , and let  $W_1, W_2$  be finite dimensional subspaces of  $V$ . Then necessarily  $W_1 + W_2$  is finite dimensional, and  $\dim(W_1) + \dim(W_2) = \dim(W_1 + W_2) + \dim(W_1 \cap W_2)$ .

Proof. First, note  $W_1 \cap W_2$  is a subspace of  $W_1$  (A2), so that  $\dim(W_1 \cap W_2) \leq \dim(W_1) < \infty$  (C1.9.2(6)).

Next, let  $\{u_1, u_2, \dots, u_k\}$  be a basis for  $W_1 \cap W_2$ .

Extend this basis to get the bases

$S_1 = \{u_1, \dots, u_k, v_1, \dots, v_m\}$  of  $W_1$  and  $S_2 = \{u_1, \dots, u_k, z_1, \dots, z_p\}$  of  $W_2$ , which we can always do by C1.9.2(5)).

Let  $S = \{u_1, \dots, u_k, v_1, \dots, v_m, z_1, \dots, z_p\}$ .

We claim  $S$  is a basis for  $W_1 + W_2$ .

Indeed, consider

$$a_1 u_1 + \dots + a_k u_k + b_1 v_1 + \dots + b_m v_m + c_1 z_1 + \dots + c_p z_p = 0 \quad \text{--- (2)}$$

for some scalars  $a_1, \dots, a_k, b_1, \dots, b_m, c_1, \dots, c_p$ .

Then

$$b_1 v_1 + \dots + b_m v_m = -a_1 u_1 - \dots - a_k u_k - c_1 z_1 - \dots - c_p z_p.$$

Since the RHS is a linear combination of vectors in  $W_2$ , the RHS  $\in W_2$ ; and since the LHS is a linear combination of vectors in  $W_1$ , the LHS  $\in W_1$ .

Thus  $b_1 v_1 + \dots + b_m v_m \in W_1 \cap W_2$ .

Next, since  $\{u_1, \dots, u_k\}$  is a basis for  $W_1 \cap W_2$ , there exist scalars  $d_1, \dots, d_k$  such that

$$b_1 v_1 + \dots + b_m v_m = d_1 u_1 + \dots + d_k u_k.$$

So

$$b_1 v_1 + \dots + b_m v_m - d_1 u_1 - \dots - d_k u_k = 0.$$

Since  $\{u_1, \dots, u_k, v_1, \dots, v_m\}$  is a basis for  $W_1$ , necessarily  $b_1 = \dots = b_m = d_1 = \dots = d_k = 0$ .

Substitute  $b_1 = \dots = b_m$  into (2) to get that

$$a_1 u_1 + \dots + a_k u_k + c_1 z_1 + \dots + c_p z_p = 0.$$

Then, since  $\{u_1, \dots, u_k, z_1, \dots, z_p\}$  is a basis for  $W_2$ , we have

$$a_1 = \dots = a_k = c_1 = \dots = c_p = 0,$$

proving  $S$  is linearly independent.

Subsequently, let  $x+y \in W_1 + W_2$  be arbitrary, where  $x \in W_1$  and  $y \in W_2$ .

Then, since  $S_1$  and  $S_2$  are bases for  $W_1$  and  $W_2$  respectively, we can write  $x$  and  $y$  as linear combinations of vectors in  $S_1$  and  $S_2$ , respectively:

$$x = a_1 u_1 + \dots + a_k u_k + b_1 v_1 + \dots + b_m v_m; \quad \text{--- (3)}$$

$$y = d_1 u_1 + \dots + d_k u_k + c_1 z_1 + \dots + c_p z_p;$$

where  $a_1, \dots, a_k, b_1, \dots, b_m, d_1, \dots, d_k, c_1, \dots, c_p \in \mathbb{F}$ .

Hence

$$x+y = (a_1+d_1)u_1 + \dots + (a_k+d_k)u_k + (b_1+c_1)z_1 + \dots + (b_m+c_p)z_p,$$

which is sufficient to show  $x+y \in \text{span}(S)$ .

Thus  $W_1 + W_2 \subseteq \text{span}(S)$ , and since  $\text{span}(S) \subseteq W_1 + W_2$ ,

by definition, it follows that  $W_1 + W_2 = \text{span}(S)$ ,

verifying that  $S$  is indeed a basis for

$W_1 + W_2$ .

In particular,

$$\begin{aligned} \dim(W_1 + W_2) + \dim(W_1 \cap W_2) &= |S| + k \\ &= m + p + k + k \\ &= (m+k) + (p+k) \\ &= \dim W_1 + \dim W_2. \end{aligned}$$

$$\dim(V) < \infty, \quad W_1 \oplus W_2 = V \Rightarrow \dim W_1 + \dim W_2 = \dim V$$

(T1.12 (2))

Let  $V$  be a vector space over  $\mathbb{F}$ , and let  $W_1, W_2$  be finite-dimensional subspaces of  $V$ .

Suppose further that  $V$  itself is finite-dimensional, and  $W_1 \oplus W_2 = V$ .

Then necessarily  $\dim W_1 + \dim W_2 = \dim V$ .

Proof. Since  $W_1 \oplus W_2 = V$ , necessarily  $W_1 \cap W_2 = \{0\}$ .

So, by T1.12(1), it follows that

$$\begin{aligned} \dim W_1 + \dim W_2 &= \dim(W_1 + W_2) + \dim(W_1 \cap W_2) \\ &= \dim(V) + 0 \end{aligned}$$

$$\therefore \dim W_1 + \dim W_2 = \dim(V). \quad \blacksquare$$

## COMPLEMENTARY SUBSPACES

Let  $V$  be a vector space, and let  $W$  be a subspace of  $V$ .

Then a subspace  $W'$  of  $V$  is said to be a "complementary subspace" to  $W$  if  $W \oplus W' = V$ ; ie

$$\textcircled{1} \quad W \cap W' = \{0\}; \quad \text{and}$$

$$\textcircled{2} \quad W + W' = V.$$

$$\dim W + \dim W' = \dim V$$

Let  $V$  be a vector space, and let  $W$  be a subspace of  $V$ .

$W'$  be a complementary subspace to  $W$ .

Then necessarily  $\dim W + \dim W' = \dim V$ .

Proof. Follows directly from T1.12(2).

## EXISTENCE OF COMPLEMENTARY SUBSPACES (R11(1))

Let  $V$  be a vector space, and let  $W$

be a subspace of  $V$ .

Then there always exists a complementary subspace  $W'$  to  $W$  of  $V$  such that  $W \oplus W' = V$ .

Proof. First, note that every linearly independent set can be extended to a basis  $V$  that has a countable spanning set (A3).

Hence, every linearly independent subset of  $V$  can be extended to a basis for  $V$ .

It follows that every subspace  $W$  of  $V$  has a complementary subspace  $W'$ .  $\blacksquare$

## NON-UNIQUENESS OF COMPLEMENTARY SUBSPACES

(R11(2))

Note that complementary subspaces of a given vector space  $V$  are not necessarily unique.

eg  $V = \mathbb{R}^3$ ,  $W = \{(1,0,0), (0,1,0)\}$ ,  $W'_1 = \{(0,0,1)\}$ ,  $W'_2 = \{(0,0,-1)\}$ ;

observe that both  $W'_1$  and  $W'_2$  are complementary subspaces to  $W$ .