MATH 146 Personal Notes

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Chapter 1: Vector Spaces (S1.1)

KEY			
S :	section	P :	proposition
D :	definition	A :	assignment
R :	remark		
E :	example		
T :	theorem		
L :	lemma		
C :	corollary		

Let $\mathbb{F}$ be a field.
Then, we say V is a "vector space" over $\mathbb{F}$ if there exists
① an addition $+: (V \times V) \to V$ by $+(x,y) = x+y$; and
② a scalar multiplication $\times: (\mathbb{F} \times V) \to V$ by $\times(a,x) = ax$;
and the following conditions hold:
① V is an abelian group with respect to addition; (VS 1 = commutativity; 2 = associativity; 3 = identity; 4 = inverse)
② $1_{\mathbb{F}} x = x \quad \forall x \in V$; (VS 5)
③ multiplication is associative; ie $a(bx) = (ab)x \quad \forall a,b \in \mathbb{F},\ x \in V$; and (VS 6)
④ the left and right distributive laws hold;
ie $a(x+y) = ax+ay$ and $(a+b)x = ax+bx \quad \forall a,b \in \mathbb{F},\ x \in V$. (D2)
(VS 7 = former, VS 8 = latter)

$\mathbb{F}^n$ IS A VECTOR SPACE OVER $\mathbb{F}$ (E2(1))

E_1: We can show that the Cartesian product
$$\mathbb{F}^n = \{(a_1, a_2, \dots, a_n) : a_i \in \mathbb{F} \ \ \forall i \in \{1,2,\dots,n\}\}$$
is a vector space over $\mathbb{F}$ with respect to the addition operation
$$(a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n) = (a_1+b_1, a_2+b_2, \dots, a_n+b_n)$$
and multiplication operation
$$c(a_1, a_2, \dots, a_n) = (ca_1, ca_2, \dots, ca_n).$$

Proof. This follows from verifying each of the conditions above. ∎

E_2: Note that we generally say "the vector space $\mathbb{F}^n$" to refer to the vector space $\mathbb{F}^n$ over $\mathbb{F}$. (R3(4))

COLUMN VECTOR NOTATION (E2(2))

Note that we can also write elements of $\mathbb{F}^n$ as "column vectors"; ie of the form
$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \quad \text{where } a_1, a_2, \dots, a_n \in \mathbb{F}.$$

$\mathbb{Q}^n$ IS A VECTOR SPACE OVER $\mathbb{Q}$, $\mathbb{R}^n$ IS A VECTOR SPACE OVER $\mathbb{R}$, & $\mathbb{C}^n$ IS A VECTOR SPACE OVER $\mathbb{C}$ (R3(1))

We can show
① $\mathbb{Q}^n$ is a vector space over $\mathbb{Q}$;
② $\mathbb{R}^n$ is a vector space over $\mathbb{R}$; and
③ $\mathbb{C}^n$ is a vector space over $\mathbb{C}$.

Proof. This directly follows from the fact that $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are fields (MATH 145), and substituting the respective fields into the above lemma. ∎

$\mathbb{R}^n$ IS A VECTOR SPACE OVER $\mathbb{Q}$, & $\mathbb{C}^n$ IS A VECTOR SPACE OVER $\mathbb{R}$ (R3(2))

Moreover, we can also show that
① $\mathbb{R}^n$ is a vector space over $\mathbb{Q}$; and
② $\mathbb{C}^n$ is a vector space over $\mathbb{R}$.

Proof. Essentially, this stems from the fact that we can "multiply" vectors in $\mathbb{R}^n$ by scalars in $\mathbb{Q}$, and vectors in $\mathbb{C}^n$ by scalars in $\mathbb{R}$. The formal proof is left to the reader. ∎

MATRICES (D3(1))

D_1: Let $\mathbb{F}$ be a field, and $m,n \in \mathbb{Z}^+$. Then, we say A is an "$m \times n$ matrix" with entries from $\mathbb{F}$ if it is of the form
$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \\ a_{m1} & a_{m2} & & a_{mn} \end{pmatrix},$$
where $a_{ij} \in \mathbb{F} \quad \forall i \in \{1,2,\dots,m\},\ j \in \{1,2,\dots,n\}$.

D_2: Alternatively, we can represent A via the notation
$$A = (a_{ij}), \quad \text{ie } \{1,2,\dots,m\},\ j \in \{1,2,\dots,n\}.$$

ij-ENTRY OF A MATRIX (D3(2))

Given a $m \times n$ matrix A, the "ij-entry" of A, or "a_{ij}", is defined to be the entry in A at the i^{th} row and j^{th} column.

ZERO MATRIX (D3(3))

The "$m \times n$ zero matrix", or more simply the "zero matrix", denoted as "O", is defined to be
$$O = \underbrace{\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & 0 \end{pmatrix}}_{n} \Big\} m,$$
ie the $m \times n$ matrix where which entry equals 0.

MATRIX EQUALITY (D3(4))

We say two matrices A and B are equal if and only if $a_{ij} = b_{ij} \quad \forall i \in \{1,2,\dots,m\},\ j \in \{1,2,\dots,n\}$.

MATRIX ADDITION (D3(5))

Let A and B be $m \times n$ matrices with entries from some field $\mathbb{F}$.
Then, the "addition" of A and B, denoted by "$A+B$", is defined to be the matrix where
$$(a+b)_{ij} = a_{ij} + b_{ij} \quad \forall i \in \{1,2,\dots,m\},\ j \in \{1,2,\dots,n\}.$$

MATRIX SCALAR MULTIPLICATION (D3(6))

Let A be a $m \times n$ matrix with entries from some field $\mathbb{F}$, and $c \in \mathbb{F}$ be arbitrary.
Then the "scalar multiplication" of A by c, denoted by "cA", is defined to be the matrix where
$$(ca)_{ij} = c(a_{ij}) \quad \forall i \in \{1,2,\dots,m\},\ j \in \{1,2,\dots,n\}.$$

SPACE OF $m \times n$ MATRICES (E3)

E_1: Let $\mathbb{F}$ be a field.
Then the "space of all $m \times n$ matrices" with entries from $\mathbb{F}$, denoted by "$M_{m \times n}(\mathbb{F})$", is defined to be the set of all $m \times n$ matrices with entries from $\mathbb{F}$.

E_2: Note that $M_{m \times n}(\mathbb{F})$ is a vector space over $\mathbb{F}$ with respect to the matrix addition and scalar multiplication operations.

Proof. This follows from verifying each of the conditions in D2. ∎

FUNCTION SPACES (E4)

Let the set $D \neq \emptyset$ be arbitrary, and let $\mathbb{F}$ be a field.

Then the "space of all functions" from D to $\mathbb{F}$, denoted by "$\mathbb{F}^D$", is defined to be the set of all functions of the form $f: D \to \mathbb{F}$.

Similarly, we can show that $\mathbb{F}^D$ is a vector space over $\mathbb{F}$ with respect to the operations of function addition

$$(f+g)(x) := f(x) + g(x) \quad \forall f, g \in \mathbb{F}^D, \ x \in \mathbb{F}$$

and function scalar multiplication

$$(cf)(x) := cf(x) \quad \forall f \in \mathbb{F}^D, \ x, c \in \mathbb{F}.$$

Proof. Similar strategy to E3: verify each condition in D2 holds.

POLYNOMIALS (D4)

SET OF ALL POLYNOMIALS OF DEGREE AT MOST n (D4 (1))

Let $\mathbb{F}$ be a field.

Then, we denote $P_n(\mathbb{F})$ to be the set of all polynomials with coefficients from $\mathbb{F}$ and of degree at most n; ie

$$P_n(\mathbb{F}) = \left\{ \sum_{i=0}^{n} a_n x^n : a_j \in \mathbb{F} \ \forall j \in \{0, 1, \dots, n\} \right\}.$$

POLYNOMIAL SPACES (D4 (2))

Let $\mathbb{F}$ be a field.

Then, we denote "$\mathbb{F}[x]$" to be the set of all polynomials with coefficients from $\mathbb{F}$; ie

$$\mathbb{F}[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i : a_j \in \mathbb{F} \ \forall j \in \mathbb{N} \cup \{0\} \right\}.$$

Then, we can show that $\mathbb{F}[x]$ is a vector space over $\mathbb{F}$ with respect to the operations of polynomial addition

$$(f+g)(x) = \sum_{i=0}^{n} (a_n + b_n) x^n \quad \forall f, g \in \mathbb{F}[x]$$

and polynomial scalar multiplication

$$cf(x) = \sum_{i=0}^{n} (ca_n) x^n \quad \forall f \in \mathbb{F}[x], \ c \in \mathbb{F}.$$

Proof. Similar strategy to E4.

BASIC PROPERTIES OF VECTOR SPACES (S1.2)

CANCELLATION PROPERTY FOR VECTOR ADDITION (T1.1)

Let V be a vector space.

Suppose there exists some $x, y, z \in V$ such that $x + z = y + z$.

Then necessarily $x = y$.

Proof. Note that

$$\begin{aligned} x &= x + 0 \\ &= x + (z + (-z)) \\ &= (x+z) + (-z) \\ &= (y+z) + (-z) \\ &= y + (z + (-z)) \end{aligned}$$

and so $x = y$, as required. ☒

UNIQUENESS OF THE ZERO VECTOR IN VECTOR SPACES (C1.1.1 (1))

Let V be a vector space.

Suppose $0_1, 0_2 \in V$ are both zero vectors.

Then necessarily $0_1 = 0_2$.

Proof. This follows from the fact that V is an abelian group under addition. ☒

UNIQUENESS OF ADDITIVE INVERSES IN VECTOR SPACES (C1.1.1 (2))

Let V be a vector space.

Then for any $x \in V$, there exists one and only one vector $y \in V$ that satisfies $x + y = 0$.

Proof. This also follows from the fact that V is an abelian group under addition. ☒

$0x = 0 \quad \forall x \in V$ (T1.2 (1))

Let V be a vector space over some field $\mathbb{F}$, and let 0 be the additive identity of $\mathbb{F}$.

Then, for any $x \in V$, necessarily $0 \cdot x = 0$.

Proof. This, again, follows from the fact that V is an abelian group under addition. ☒

$a0 = 0 \quad \forall a \in \mathbb{F}$ (T1.2 (2))

Let V be a vector space over some field $\mathbb{F}$, and let 0 be the zero vector of V.

Then, for any $a \in \mathbb{F}$, necessarily $a\,0 = 0$.

Proof. This, again, follows from the fact that V is an abelian group under addition. ☒

$(-a)x = -(ax) = a(-x) \quad \forall a \in \mathbb{F}, \ x \in V$ (T1.2 (3))

Let V be a vector space over some field $\mathbb{F}$, and let $a \in \mathbb{F}$, $x \in V$ be arbitrary.

Then necessarily $(-a)x = -(ax) = a(-x)$.

Proof. Proof is similar to the analog of this statement for rings (MATH 145). ☒

SUBSPACES (S1.3)

Let V be a vector space over some field $\mathbb{F}$.
Then we say the subset $W \subseteq V$ is a "subspace" of V if

1. $W \neq \emptyset$;
   * we usually check whether $0 \in W$ to verify this claim. (R4)
2. If $x \in W$ and $y \in W$, then $(x+y) \in W$; and
3. If $c \in \mathbb{F}$ and $x \in W$, then $cx \in W$. (D6)

SUBSPACES ARE VECTOR SPACES OVER $\mathbb{F}$ WITH RESPECT TO THE OPERATIONS OF V (T1.3)

Let W be a subspace of a vector space V over some field $\mathbb{F}$.
Then W is also a vector space over $\mathbb{F}$ under the operations of V restricted to W.

Proof. This follows from verifying the conditions in D2, taking into account the properties of subspaces. ☒

$\{0\}$ AND V ARE SUBSPACES OF V (E8 (1))

Let V be a vector space.
Then $\{0\}$ and V itself are always subspaces of V.

Proof. $\{0\}$ is vacuously a subspace, and V is trivially a subspace. ☒

$P_2(\mathbb{R})$ IS A SUBSPACE OF $\mathbb{R}[x]$ (E8 (2))

We can show that $P_2(\mathbb{R})$ is a subspace of $\mathbb{R}[x]$.

Proof. This stems from the fact that:

- $P_2(\mathbb{R}) \subset \mathbb{R}[x]$ by definition;
- $0 \in P_2(\mathbb{R})$; and
- $P_2(\mathbb{R})$ is closed under the addition & scalar multiplication defined on $\mathbb{R}[x]$. ☒

$\{(a_{ij}) \in M_{n \times n}(\mathbb{F}) \mid \sum_{k=1}^{n} a_{kk} = 0\}$ IS A SUBSPACE OF $M_{n \times n}(\mathbb{F})$ (E8 (3))

We can show that the set $\{(a_{ij}) \in M_{n \times n}(\mathbb{F}) \mid \sum_{k=1}^{n} a_{kk} = 0\}$ is a subspace of $M_{n \times n}(\mathbb{F})$, where $n \in \mathbb{N}$ is arbitrary.

Proof. Similar proof to the above.

$\{(a_{ij}) \in M_{n \times n}(\mathbb{F}) \mid \sum_{k=1}^{n} a_{kk} = 1\}$ IS NOT A SUBSPACE OF $M_{n \times n}(\mathbb{F})$ (E8 (4))

We can show the set $\{(a_{ij}) \in M_{n \times n}(\mathbb{F}) \mid \sum_{k=1}^{n} a_{kk} = 1\}$ is not a subspace of $M_{n \times n}(\mathbb{F})$.

Proof. Let $a, b \in \{(a_{ij}) \in M_{n \times n}(\mathbb{F}) \mid \sum_{k=1}^{n} a_{kk} = 1\}$ be arbitrary.
Then, notice that

$$\sum_{k=1}^{n} (a+b)_{kk} = \sum_{k=1}^{n} (a_{kk} + b_{kk}) = 1 + 1 = 2,$$

so that $a+b \notin \{(a_{ij}) \in M_{n \times n}(\mathbb{F}) \mid \sum_{k=1}^{n} a_{kk} = 1\}$. ☒

SUBSPACES OF $\mathbb{R}^2$ (E9 (1))

Note that the subspaces of $\mathbb{R}^2$ are

1. $\mathbb{R}^2$ itself;
2. $\{0_{\mathbb{R}^2}\} = \{(0,0)\}$; and
3. all lines in $\mathbb{R}^2$ that pass through $(0,0)$.

SUBSPACES OF $\mathbb{F}^2$ (E9 (4a))

In general, for any field $\mathbb{F}$, the subspaces of $\mathbb{F}^2$ are

1. $\mathbb{F}^2$ itself;
2. $\{0\}$; and
3. all the "lines" in $\mathbb{F}^2$ through 0.
   ie of the form $\{(x,y) \in \mathbb{F}^2 \mid \binom{x}{y} = k\binom{x_1}{y_1},\ \binom{x_1}{y_1} \in \mathbb{F}^2\}$

SUBSPACES OF $\mathbb{R}^3$ (E9 (2))

Similarly, the subspaces of $\mathbb{R}^3$ are

1. $\mathbb{R}^3$ itself;
2. $\{0_{\mathbb{R}^3}\} = \{(0,0,0)\}$;
3. all lines in $\mathbb{R}^3$ that pass through $(0,0,0)$; and
4. all planes in $\mathbb{R}^3$ that pass through $(0,0,0)$.

SUBSPACES OF $\mathbb{F}^3$ (E9 (4b))

Similarly, for any field $\mathbb{F}$, the subspaces of $\mathbb{F}^3$ are

1. $\mathbb{F}^3$ itself;
2. $\{0\}$;
3. all the "lines" in $\mathbb{F}^3$ through 0; and
   ie of the form $\{(x,y,z) \in \mathbb{F}^3 \mid \begin{pmatrix}x\\y\\z\end{pmatrix} = k\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix},\ \begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix} \in \mathbb{F}^3\}$ (E9(3))
4. all the "planes" in $\mathbb{F}^3$ through 0.
   ie of the form $\{(x,y,z) \in \mathbb{F}^3 \mid \exists a,b,c \in \mathbb{F}$ such that $ax+by+cz=0.\}$

LINEAR COMBINATIONS & SYSTEM OF LINEAR EQUATIONS (S1.4)

* knowledge of elimination method is assumed.

LINEAR COMBINATION (D7(1))

Let V be a vector space over a field $\mathbb{F}$, and let the subset $S \subseteq V$ be such that $S \neq \emptyset$.

Then, we say a vector $x \in V$ is a "linear combination" of vectors from S if there exists a finite number of vectors $u_1, u_2, \dots, u_n \in S$ and scalars $a_1, a_2, \dots, a_n \in \mathbb{F}$ such that

$$x = a_1 u_1 + a_2 u_2 + \cdots + a_n u_n,$$

where $n \geq 1$. (D7(1))

In this case, we also say that x is a linear combination of the vectors $u_1, u_2, \dots, u_n$.

COEFFICIENTS OF A LINEAR COMBINATION (D7 (2))

Let V be a vector space over some field $\mathbb{F}$, and let the vector $x \in V$ be a linear combination of the vectors $u_1, u_2, \dots, u_n \in S$, where $S \subseteq V$ and $S \neq \emptyset$.

Assume that $x = a_1 u_1 + a_2 u_2 + \cdots + a_n u_n$, where $a_1, a_2, \dots, a_n \in \mathbb{F}$.

Then we denote the scalars $a_1, a_2, \dots, a_n \in \mathbb{F}$ as the "coefficients" of the linear combination.

SPAN (D7(3))

Let V be a vector space over some field $\mathbb{F}$, and let the subset $S \subseteq V$ be such that $S \neq \emptyset$.

Then, we define the "span" of S, denoted as "$\text{span}(S)$", to be the set of all linear combinations of vectors in S.

Note that, for convenience, we define

$$\text{span}(\emptyset) = \{0\}.$$

EXAMPLE 1: SPAN OF (1,0,0) & (0,1,0) IN $\mathbb{R}^3$ (E10(1))

Observe that in $\mathbb{R}^3$, the span of $(1,0,0)$ & $(0,1,0)$ in $\mathbb{R}^3$ is

$$\{a(1,0,0) + b(0,1,0) : a, b \in \mathbb{R}\}$$

or more simply

$$\{(a, b, 0) : a, b \in \mathbb{R}\}.$$

EXAMPLE 2: SPAN($\{x^n : n \geq 1\}$) IN $\mathbb{Q}[x]$ (E10(2))

We can show that for the vector space $\mathbb{Q}[x]$, the span of $S = \{x, x^2, \dots, x^n, \dots\}$ is the set of all polynomials in $\mathbb{Q}[x]$ whose constant coefficient equals 0.

SPAN OF A FINITE AMOUNT OF VECTORS (E10(3.1))

Suppose V is a vector space over some field $\mathbb{F}$, and let $S \subseteq V$. Further assume that

$$S = \{v_1, v_2, \dots, v_n\},$$

ie the size of S is finite.

Then, it follows that

$$\text{span}(S) = \{a_1 v_1 + a_2 v_2 + \cdots a_n v_n : a_i \in \mathbb{F} \ \forall i \in \{1, 2, \dots, n\}\}.$$

SPAN OF A COUNTABLE AMOUNT OF VECTORS (E10 (3.2))

Suppose V is a vector space over some field $\mathbb{F}$, and let $S \subseteq V$. Further assume that

$$S = \{v_1, v_2, \dots, v_n, \dots\},$$

ie $|S| = |\mathbb{N}|$.

Then, we can show that

$$\text{span}(S) = \text{span}(\{v_1\}) \cup \text{span}(\{v_1, v_2\}) \cup \cdots \cup \text{span}(\{v_1, v_2, \dots, v_n\}) \cup \cdots$$

or in other words, that

$$\text{span}(S) = \bigcup_{n=1}^{\infty} \text{span}\{v_1, v_2, \dots, v_n\}.$$

SPAN OF AN UNCOUNTABLE AMOUNT OF VECTORS (E10 (3.3))

Suppose V is a vector space over some field $\mathbb{F}$, and let $S \subseteq V$. Further assume that $|S| > |\mathbb{N}|$; ie the size of S is uncountable.

Then note that there are no "obvious" simplifications to the formula for $\text{span}(S)$.

SPAN OF A SET IS ALWAYS A SUBSPACE OF THE PARENT VECTOR SPACE (T1.4)

Let V be a vector space over some field $\mathbb{F}$, and let $S \subseteq V$.

Then necessarily $\text{span}(S)$ is a subspace of V.

Proof. This follows from verifying each subspace condition for $\text{span}(S)$. ■

Moreover, $\text{span}(S)$ is the "smallest possible" subspace of V that contains S, in the sense that

1. $S \subseteq \text{span}(S)$; and
2. If W is any other subspace of V containing S, then $\text{span}(S) \subseteq W$.

"GENERATES/SPANS" (D8)

Let V be a vector space, and let $S \subseteq V$.

Then, we say S "generates" V, or S "spans" V, if $\text{span}(S) = V$.

Note to prove $\text{span}(S) = V$, we just need to prove every vector in V can be written as a linear combination of vectors in S, since $\text{span}(S) \subseteq V$ by definition. (This follows from extensionality.) (R6)

LINEAR INDEPENDENCE & DEPENDENCE (S1·5)

LINEARLY DEPENDENT (D9 (1))

Let V be a vector space over some field $\mathbb{F}$, and let $S \subseteq V$.
Then, we say S is "linearly dependent" if there exists a finite number of distinct vectors $u_1, u_2, \dots, u_n \in S$ and scalars $c_1, c_2, \dots, c_n \in \mathbb{F}$, where $c_1, c_2, \dots, c_n$ are all not zero, such that

$$c_1 u_1 + c_2 u_2 + \cdots + c_n u_n = 0.$$

In this case, we also say the vectors of S are linearly dependent.

Note that if S is finite, say $S = \{u_1, u_2, \dots, u_n\}$, then S is linearly dependent if and only if there exists a $(c_1, c_2, \dots, c_n) \in \mathbb{F}^n$, where $(c_1, c_2, \dots, c_n) \neq (0, 0, \dots, 0)$, such that

$$c_1 u_1 + c_2 u_2 + \cdots + c_n u_n = 0. \quad (R7\ (4a))$$

LINEARLY INDEPENDENT (D9 (2))

Let V be a vector space over some field $\mathbb{F}$, and let $S \subseteq V$.
Then, we say S is "linearly independent" if it is not "linearly dependent"; ie for every choice of distinct $u_1, u_2, \dots, u_n \in S$, if $c_1, c_2, \dots, c_n \in \mathbb{F}$ are scalars such that

$$c_1 u_1 + c_2 u_2 + \cdots + c_n u_n = 0,$$

then necessarily $c_1 = c_2 = \cdots = c_n = 0$.

Similarly, if S is finite, say $S = \{u_1, u_2, \dots, u_n\}$, then S is linearly independent if and only if whenever $(c_1, c_2, \dots, c_n) \in \mathbb{F}^n$ are such that

$$c_1 u_1 + c_2 u_2 + \cdots + c_n u_n = 0,$$

then necessarily $c_1 = c_2 = \cdots = c_n = 0$.

TRIVIAL REPRESENTATION OF 0 (R7 (1))

Note that for any vector space V and vectors $u_1, u_2, \dots, u_n \in V$, we denote the "trivial representation of $0 \in V$" as a linear combination of $u_1, u_2, \dots, u_n$ by

$$0u_1 + 0u_2 + \cdots + 0u_n = 0.$$

EMPTY SET IS LINEARLY INDEPENDENT (R7 (2))

Note that the empty set, $\emptyset$, is vacuously linearly independent.

* since linearly dependent sets must be non-empty by definition.

$\{0\}$ IS LINEARLY DEPENDENT (R7 (3))

Note that the set $\{0\}$ is linearly dependent, since $1(0) = 0$ is a non-trivial representation of 0 as a linear combination of finitely many distinct vectors in S.

$0 \in S \Rightarrow S$ IS LINEARLY DEPENDENT (R7 (5))

Note that any subset of a vector space that contains the zero vector is linearly dependent.

EXAMPLE 1: $S = \{(0,1,1), (1,0,1), (1,2,3)\}$ IS LINEARLY DEPENDENT IN $\mathbb{R}^3$ (E(4))

We can show that the set $S = \{(0,1,1), (1,0,1), (1,2,3)\}$ is linearly dependent in $\mathbb{R}^3$.

Proof. We search for scalars $a, b, c \in \mathbb{R}$, not all 0, such that

$$a\begin{pmatrix}0\\1\\1\end{pmatrix} + b\begin{pmatrix}1\\0\\1\end{pmatrix} + c\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

This reduces to solving the system

$$\begin{cases} b + c = 0 \\ a + 2c = 0 \\ a + b + 3c = 0. \end{cases}$$

Simplifying, we get that $a = -2t$, $b = -t$ and $c = t$, where $t \in \mathbb{R}$.
For instance, $(a,b,c) = (-2,-1,1)$ is a solution in which not all of a, b, c are 0.
It follows that S is linearly dependent. ∎

EXAMPLE 2: $S = \{1, x, x^2, x^3\}$ IS LINEARLY INDEPENDENT IN $\mathbb{Z}_5[x]$ (E(5))

We can show that the set $S = \{1, x, x^2, x^3\}$ is linearly independent in $\mathbb{Z}_5[x]$.

Proof. Note that if there exist $a_0, a_1, a_2, a_3 \in \mathbb{Z}_5$ such that

$$a_0(1) + a_1 x + a_2 x^2 + a_3 x^3 = 0,$$

then by definition necessarily $a_0 = a_1 = a_2 = a_3 = 0$, and this is sufficient to prove the claim. ∎

S IS LINEARLY DEPENDENT $\iff$ $S=\{0\}$ OR SOME VECTOR IN S IS A LINEAR COMBINATION OF OTHER VECTORS IN S (T1.5)

Let V be a vector space, and let $S \subseteq V$. Then S is linearly dependent if and only if $S=\{0\}$ or some vector in S is a linear combination of other vectors in S.

Proof. We first prove the backward argument.

First, note we know why $\{0\}$ is linearly dependent from a previous section.

So, suppose there exists a vector $v \in S$ such that

$$v = c_1 u_1 + c_2 u_2 + \cdots + c_n u_n,$$

where $c_i \in \mathbb{F}$ and $u_i \in V$ $\forall i \in \{1,2,\dots,n\}$.

Without loss in generality, assume $u_1, u_2, \dots, u_n$ are distinct.

By assumption, since $v \notin \{u_1, u_2, \dots, u_n\}$, necessarily $u_1, u_2, \dots, u_n, v$ are distinct.

Finally, since

$$0 = (-1)v + c_1 u_1 + c_2 u_2 + \cdots + c_n u_n,$$

and $-1 \neq 0$, it follows S is linearly dependent. ※

Next, we prove the forward argument.

Assume S is linearly dependent, so that there exist distinct $u_1, u_2, \dots, u_n \in S$ and $a_1, a_2, \dots, a_n \in \mathbb{F}$ (not all 0) such that

$$a_1 u_1 + a_2 u_2 + \cdots + a_n u_n = 0.$$

Without loss in generality, assume $a_i \neq 0$ $\forall i \in \{1,2,\dots,n\}$.

Case 1: $n=1$.

Then $a_1 u_1 = 0$, and since $a_1 \neq 0$ it follows that $u_1 = 0$ (since fields are integral domains, so the cancellation property applies.)

Hence $0 \in S$. If $S=\{0\}$ we are done; otherwise, we can pick a $v \in S \setminus \{0\}$, and we can write $0 = 0v$, proving some vector in S, 0, can be written as a linear combination of another vector, v, in S.

Case 2: $n > 1$.

Then since $a_n \neq 0$, we can solve for u_n:

$$u_n = -a_n^{-1}[a_1 u_1 + a_2 u_2 + \cdots + a_{n-1} u_{n-1}],$$

$$\therefore u_n = (-a_n^{-1} a_1) u_1 + (-a_n^{-1} a_2) u_2 + \cdots + (-a_n^{-1} a_{n-1}) u_{n-1},$$

showing u_n can be expressed as a linear combination of other elements in S.

BASES & DIMENSION (S1.6)

BASIS (D10)

Let V be a vector space.
Then, we say a subset $S \subseteq V$ is a "basis" for V if
① S is linearly independent; and
② S spans V.

In this case, we also say that the vectors of S form a basis for V.

STANDARD BASIS (E17)

In $\mathbb{F}^n$, define the "standard basis" for $\mathbb{F}^n$ the subset

$$S = \{e_1, e_2, \dots, e_n\},$$

where $e_j \in \mathbb{F}^n$ is the vector with j^{th} coordinate 1 and other coordinates 0.

(It is easy to prove S is indeed a basis for $\mathbb{F}^n$.)

In $P_n(\mathbb{F})$, define the "standard basis" for $P_n(\mathbb{F})$ as the set

$$S = \{1, x, x^2, \dots, x^n\}.$$

(It is also easy to prove S is indeed a basis for $P_n(\mathbb{F})$).

UNIQUE REPRESENTATION OF ELEMENTS IN VECTOR SPACES UNDER A BASIS (T1.6)

Let $\{v_1, v_2, \dots, v_n\}$ be a basis for a vector space V.
Then for every $x \in V$, x can be uniquely represented as a linear combination of $v_1, v_2, \dots, v_n$;
ie there exists a unique n-tuple $(a_1, a_2, \dots, a_n) \in \mathbb{F}^n$ such that

$$x = a_1 v_1 + a_2 v_2 + \dots + a_n v_n.$$

Proof. Existence: this follows from the fact that $\{v_1, v_2, \dots, v_n\}$ spans V by definition.
Uniqueness: suppose there exists some $b_1, b_2, \dots, b_n \in \mathbb{F}$ such that

$$x = a_1 v_1 + \dots + a_n v_n = b_1 v_1 + \dots + b_n v_n.$$

It follows that

$$0 = (a_1 - b_1) v_1 + \dots + (a_n - b_n) v_n,$$

and since $\{v_1, v_2, \dots, v_n\}$ is linearly independent, necessarily $a_k = b_k$ $\forall k \in \{1, 2, \dots, n\}$. ▩

V IS GENERATED BY S, $|S| = |\mathbb{N}|$ ⇒ $T \subseteq S$ IS ALSO A BASIS FOR V (T1.7)

Let V be a vector space, and assume that V is generated by a countable set S.
Then there exists a subset of S that is a basis for V.

Proof. If $S = \emptyset$ or $S = \{0\}$, then $\emptyset$ is a basis for V trivially.
Otherwise, S contains at least a non-zero vector.
Hence, we can write S as

$$S = \{v_1, v_2, \dots, v_n\} \quad \text{or} \quad S = \{v_1, v_2, \dots\}.$$

By the WOP, we can pick the smallest index $i_1 \geq 1$ such that $v_{i_1} \neq 0$.
Then $\{v_{i_1}\}$ is linearly independent.
Let i_2 be the smallest index such that $v_{i_2} \in S$ and $v_{i_2} \notin \mathrm{span}(\{v_{i_1}\})$.
Continue this "process" until we obtain the set

$$T = \{v_{i_k} \in S \mid v_{i_k} \notin \mathrm{span}(\{v_{i_1}, \dots, v_{i_{k-1}}\}),\ k \geq 1\}.$$

Finally, we can prove T is a basis for V.
① Assume T is linearly dependent.
Then there exists $a_1, a_2, \dots, a_k$, all not 0, such that

$$a_1 v_{i_1} + \dots + a_k v_{i_k} = 0.$$

It follows that

$$v_{i_k} = -a_k^{-1} a_1 v_{i_1} - \dots - a_k^{-1} a_{k-1} v_{i_{k-1}},$$

contradicting the construction of T.
② We can prove by induction that $\mathrm{span}(S_k) = \mathrm{span}(T_k)$ $\forall k \geq 1$, where

$$S_k = \{v_1, v_2, \dots, v_k\} \quad \text{and} \quad T_k = T \cap S_k = \{v_{i_q} \mid i_q \leq k\}.$$

Then, let $x \in V = \mathrm{span}(S)$. Then $x \in \mathrm{span}(S_m)$ for some large m, so that $x \in \mathrm{span}(T_m) \subset \mathrm{span}(T)$.
Hence $V \subseteq \mathrm{span}(T)$, and it follows that $V = \mathrm{span}(T)$. ▩

EVERY VECTOR SPACE HAS A BASIS (T1.8)

We can prove that every vector space has a basis.
(The proof uses Zorn's lemma & maximal linearly independent subsets.)

REPLACEMENT THEOREM (T1.9)

Suppose V is a vector space with a finite spanning set S.
Let T be a linearly independent subset in V. Then

① $|T| \leq |S|$; and
② There exists a set $H \subseteq S$ containing exactly $(|S| - |T|)$ vectors such that $T \cup H$ generates V.

Proof. Let $n = |S|$, and let $m = |T|$.
Then, when $m = 0$, clearly $m = 0 \leq |S|$.
Next, assume the statement is true for some $m \geq 0$.
This implies that if $T_m \subset V$ is any linearly independent subset in V of size m, then $m \leq n$ and there exists a set $H_m \subseteq S$ containing exactly $n-m$ vectors such that $T_m \cup H_m$ generates V.
Let $T_m = \{v_1, v_2, \ldots, v_m\}$ and $T = T_m \cup \{v_{m+1}\}$, such that T is linearly independent and a subset of V.
Note that this implies T_m is also linearly independent.
Now, apply the induction hypothesis on T_m to get that $n \geq m$, and there exist $(n-m)$ vectors $w_{m+1}, \ldots, w_n \in S$ such that $\{v_1, \ldots, v_m, w_{m+1}, \ldots, w_n\}$ generates V.
Then, since $n \geq m$, either $n = m$ or $n > m$.
If $n = m$, $\{v_1, \ldots, v_m, w_{m+1}, \ldots, w_n\} = \{v_1, \ldots, v_m\}$.
Thus, $v_{m+1} \in \text{span}\{v_1, \ldots, v_m\}$, so by Theorem 1.5, the set $\{v_1, \ldots, v_m, v_{m+1}\}$ is linearly dependent.
But this is a contradiction; hence, it follows that $n > m$, so that $n \geq m+1$, proving ①.
Subsequently, write

$$v_{m+1} = a_1 v_1 + \cdots + a_m v_m + a_{m+1} w_{m+1} + \cdots + a_n w_n$$

for some scalars $a_1, \ldots, a_n \in \mathbb{F}$.
Then, if $a_{m+1} = \cdots = a_n = 0$, then we would get that $v_{m+1} = a_1 v_1 + \cdots + a_m v_m$, which is a contradiction; hence, at least one of the scalars $a_{m+1}, \ldots, a_n$ must be non-zero.
Then, without loss in generality, assume $a_{m+1} \neq 0$.
It follows that

$$w_{m+1} = -a_{m+1}^{-1} a_1 v_1 - \cdots - a_{m+1}^{-1} a_m v_m - a_{m+1}^{-1} a_{m+1} v_{m+1}$$
$$- a_{m+1}^{-1} a_{m+2} w_{m+2} - \cdots - a_{m+1}^{-1} a_n w_n.$$

Let $H = \{w_{m+2}, \ldots, w_n\} \subset S$. The above shows that $w_{m+1} \in \text{span}(T \cup H)$.
Moreover, since $v_1, \ldots, v_m \in T \subset T \cup H$ and $w_{m+2}, \ldots, w_n \in H \subset T \cup H$, it follows that

$$V = \text{span}\{w_{m+1}, v_1, \ldots, v_m, w_{m+2}, \ldots, w_n\} \subseteq \text{span}(T \cup H).$$

But since $\text{span}(T \cup H) \subseteq V$, it follows that $V = \text{span}(T \cup H)$, completing the proof. ▨

V IS FINITELY SPANNED ⇒ ALL BASES OF V & HAVE EQUAL CARDINALITIES (C1.9.1)

Suppose V is a finitely spanned vector space.
Then all bases of V are finite and have the same amount of elements.

Proof. Let S be a finite spanning set for V, and let B be an arbitrary basis for V.
Then by definition, B is linearly independent.
By the Replacement Theorem, $|B| \leq |S| < \infty$.
Next, let B_1 and B_2 be two bases of V.
Then, since B_1 is linearly independent and B_2 is a finite spanning set for V, by the Replacement Theorem necessarily $|B_1| \leq |B_2|$.
Similarly, since B_2 is linearly independent and B_1 is a finite spanning set for V, by the Replacement Theorem necessarily $|B_2| \leq |B_1|$.
It follows that $|B_1| = |B_2|$, and we are done.

DIMENSION

FINITE/INFINITE-DIMENSIONAL (D12)

1. We say a vector space V is "finite-dimensional" if it has a basis consisting of a finite number of vectors.
2. Otherwise, we say V is "infinite-dimensional".

DIMENSION (D12)

1. Let V be a finite-dimensional vector space.
Then, the "dimension" of V, denoted as "$\dim V$", is defined to be the unique number of vectors in each basis for V.
2. By convention, we let $\dim\{0\} = 0$.
3. Examples:
① $\dim \mathbb{F}^n = n$;
② $\dim \mathbb{C}^n = 2n$;
③ $\dim M_{m \times n}(\mathbb{F}) = mn$; and
④ $\dim P_n(\mathbb{F}) = n+1$. (E18)

ANY FINITE SPANNING SET FOR V CONTAINS AT LEAST n VECTORS (C1.9.2(1))

Let V be a vector space with $\dim V = n$. Then if S is a finite spanning set for V, necessarily $|S| \geq n$.

Proof. By the Existence Theorem (T1.7), there exists a subset T of S that is a basis for V. Therefore $|T| = \dim V = n$, which implies that $|S| \geq |T| = n$. $\blacksquare$

S GENERATES V, $|V| = n$ $\Rightarrow$ S IS A BASIS FOR V (C1.9.2 (2))

Let V be a vector space with $\dim V = n$, and suppose S generates V, with $|S| = n$. Then S is a basis for V.

Proof. Again, by the Existence Theorem (T1.7), there exists some subset $T \subseteq S$ such that T is a basis for V. By the above corollary, $|T| = n$, so that if $|S| = n$, necessarily $S = T$. It follows that S is a basis for V. $\blacksquare$

S IS LINEARLY INDEPENDENT $\Rightarrow$ S CONTAINS AT MOST n VECTORS (C1.9.2(3))

Let V be a vector space, with $\dim V = n$. Suppose the subset $S \subseteq V$ is linearly independent. Then S contains at most n vectors.

Proof. Applying the Replacement Theorem for the spanning set β, it follows that $|S| \leq |\beta|$, and since $|\beta| = n$, this tells us that $|S| \leq n$, as needed. $\blacksquare$

S IS LINEARLY INDEPENDENT, $|S| = n$ $\Rightarrow$ S IS A BASIS FOR V (C1.9.2 (4))

Let V be a vector space, with $\dim V = n$. Suppose the subset $S \subseteq V$ is linearly independent and $|V| = n$. Then S is a basis for V.

Proof. Applying the Replacement Theorem for the spanning set β and the linearly independent set S, there must exist a subset $H \subseteq \beta$ containing $|\beta| - |S| = n - n = 0$ vectors such that $S \cup H$ generates V. But since $|H| = 0$, hence $H = \emptyset$, so that S generates V (and hence is a basis for V.) $\blacksquare$

EVERY LINEARLY INDEPENDENT SUBSET OF V CAN BE "EXTENDED" TO A BASIS OF V (C1.9.2 (5))

Let V be a vector space, with $\dim V = n$. Suppose $L = \{v_1, \dots, v_k\}$ is a linearly independent subset of V, where $1 \leq k \leq n$. Then there exists a $H \subset V$ such that $L \cup H$ is a basis of V.

Proof. If $k = n$, by C1.9.2(4) L is trivially a basis for V.
If $k < n$, then by the Replacement Theorem for the spanning set β and L, there necessarily exists a subset $H \subset \beta$ containing $|\beta| - |L| = n - k$ vectors such that $L \cup H$ generates V.
By C1.9.2(1), $|L \cup H| \geq n$. But
$$|L \cup H| \leq |L| + |H| = k + (n-k) = n,$$
so that $|L \cup H| = n$.
It follows by C1.9.2(2) that $L \cup H$ is a basis for V. $\blacksquare$

W IS A SUBSPACE OF V $\Rightarrow$ $\dim W \leq \dim V$; $\dim W = \dim V$ $\Leftrightarrow$ $W = V$ (C1.9.2 (6))

Let W be a subspace of the vector space V. Then $\dim W \leq \dim V$, with equality occurring if and only if $V = W$.

Proof. If $W = \{0\}$, then $\dim W = 0 \leq \dim V$.
Otherwise, W contains a non-zero vector w_1. Then $\{w_1\}$ is linearly independent.
Continue to choose the vectors $w_1, \dots, w_n \in W$ such that $\{w_1, \dots, w_k\}$ is linearly independent.
Note that this process cannot go on indefinitely, since $\{w_1, \dots, w_k\}$ is also linearly independent in V.
This implies that $k \leq n$.
Next, by T1.5, $W \subset \operatorname{span}(\{w_1, \dots, w_k\}) = \operatorname{span}(T)$.
Then, since $T \subseteq W$, necessarily $\operatorname{span}(T) \subset \operatorname{span}(W) = W$.
It follows that $W = \operatorname{span}(T)$, so that T is a basis (since it is also linearly independent), and
$$\dim W = |T| = k \leq n = \dim V.$$
Note that if $\dim V = n = \dim W$, then a basis for W is also a linearly independent set containing n elements.
Hence, by C1.9.2 (4), that set is also a basis for V. $\blacksquare$

W IS A SUBSPACE FOR V $\Rightarrow$ ANY BASIS OF W CAN BE "EXTENDED" TO A BASIS IN V (C1.9.2 (7))

Let W be a subspace of the vector space V, and let S be a basis of W. Then we can "extend" S to a basis in V.

Proof. By C1.9.2 (6), $\dim W \leq \dim V$.
Let $T = \{w_1, \dots, w_k\}$ be a basis for W, so that T is linearly independent in W, which in turn implies T is linearly independent in V.
So, by C1.9.2 (5), we can "extend" T to a basis in V. $\blacksquare$

QUOTIENT SPACES (S1.7)

COSET & REPRESENTATIVE (D13)

Let V be a vector space, and W be a subspace of V.
Then, for a given $x \in V$, its corresponding "coset" of W in V, denoted as "$x+W$", is defined to be the set

$$x+W = \{x+w : w \in W\}.$$

* note that $x+W \subseteq V$.

In this case, we call "x" a "representative" of the coset $x+W$.

$x \equiv y \pmod W$ (D13)

Let V be a vector space, and let W be a subspace of V.
Then, we write "$x \equiv y \pmod W$" if and only if $x-y \in W$.

V/W (D13)

Let V be a vector space, and W a subspace of V.
Then, we denote "V/W" (ie "V mod W") as the set

$$V/W = \{x+W : x \in V\};$$

ie let V/W be the collection of cosets of W in V.

$V/\{0\} = V$ (E19 (2))

For any vector space V, necessarily $V/\{0\} = V$.

Proof. $V/\{0\} = \{0+x : x \in V\}$
$= \{x : x \in V\}$
$\therefore V/\{0\} = V$. ☒

COSET TEST (P1)

Let W be a subspace of a vector space V, and let $x, y \in V$ be arbitrary.
Then $x+W = y+W$ if and only if $x-y \in W$.

Proof. Similar to test for cosets in MATH 145.

$\equiv \pmod W$ IS AN EQUIVALENCE RELATION ON V (R8)

Note that the relation "$\equiv \pmod W$" is an equivalence relation on V.

ADDITION & MULTIPLICATION IN V/W (D14)

Let V be a vector space over a field $\mathbb{F}$, and let W be a subspace of V.
Then, we can define an addition on V/W by

$$(x+W) + (y+W) := ((x+y) + W);$$

and a scalar multiplication on V/W by

$$a(x+W) := (ax) + W;$$

for any $a \in \mathbb{F}$ and $x, y \in W$.

Note that these addition and multiplication operations are well-defined. (L1)

Proof. Similar to proof for quotient groups/rings.

V/W IS A VECTOR SPACE (THE QUOTIENT SPACE OF V BY W) (T1.10)

Let V be a vector space, and W a subspace of V.
Then the set V/W is a vector space over $\mathbb{F}$ with the operations of coset addition and scalar multiplication, denoted as "the quotient space of V by W".

Proof. Verify all 8 conditions. (VS 1-8). ☒

BASIS FOR QUOTIENT SPACES (T1.11)

Let V be a vector space with $\dim V = n$, and let W be a subspace of V such that $\dim W = k$.
Let $\{v_1, \dots, v_n\}$ be a basis for V, such that $\{v_1, \dots, v_k\}$ is a basis for W.
Then,

① The set $\{v_{k+1}+W, \dots, v_n+W\}$ is a basis for V/W; and

② $\dim(V/W) = \dim V - \dim W$.

Proof. To prove ①, we show $\{v_{k+1}+W, \dots, v_n+W\}$ is both linearly independent and generates V/W, giving us our basis.
It follows that
$\dim(V/W) = |\{v_{k+1}+W, \dots, v_n+W\}|$
$= n-(k+1)+1$
$= n-k$
$\therefore \dim(V/W) = \dim V - \dim W$. ☒

$\dim V \geq \infty$, $\dim W \geq \infty \not\Rightarrow \dim V/W \geq \infty$ (R9)

Let V be an infinite-dimensional vector space, and let W be an infinite-dimensional subspace of V.
Then, note that it is not necessarily the case that $\dim(V/W) \geq \infty$.

Example: let $V = \mathbb{F}^\infty$ & $W = \{(0, x_2, \dots) : x_k \in \mathbb{F}\}$.
Note that each element of V/W is simply "determined" by the value of the first coordinate x_1, so that $\dim(V/W) = 1$.

SUMS & INTERNAL DIRECT SUMS OF SUBSPACES (SL8)

SUM OF SUBSPACES (DIS)

Let V be a vector space over $\mathbb{F}$, and let W_1, W_2 be subspaces of V. Then, we define the "sum" of W_1 and W_2, denoted as "W_1+W_2", to be the set

$$W_1 + W_2 := \{v_1+v_2 : v_1 \in W_1,\ v_2 \in W_2\}.$$

INDEPENDENT / DISJOINT (DIS)

Let V be a vector space, and let W_1, W_2 be subspaces of V. Then, we say W_1 and W_2 are "independent", or "disjoint", if and only if $W_1 \cap W_2 = \{0\}$.

(INTERNAL) DIRECT SUM (DIS)

Let V be a vector space, and let W_1, W_2 be independent subspaces of V. Then, we define the "(internal) direct sum" of W_1 and W_2", denoted as $W_1 \oplus W_2$, to be the set

$$W_1 \oplus W_2 = W_1 + W_2.$$

* ie "$\oplus$" is the notation for "+" used when W_1 & W_2 are independent.

Note that $W_1 \oplus W_2$ is well-defined as long as $W_1 \cap W_2 = \{0\}$. (R10)

W_1+W_2 IS THE "SMALLEST" SUBSPACE CONTAINING W_1 & W_2 (L2 (2))

Let V be a vector space, and let W_1, W_2 be subspaces of V. Then $W_1 + W_2$ is necessarily the smallest subspace of V containing W_1 and W_2.

Proof. First, we prove W_1+W_2 is a subspace of V.

Let $(v_1+v_2), (u_1+u_2) \in W_1+W_2$ and $a \in F$, where $v_1, u_1 \in W_1$ and $v_2, u_2 \in W_2$.

Then, since W_1 and W_2 are subspaces of W_1+W_2, necessarily $v_1+u_1 \in W_1$ and $v_2+u_2 \in W_2$, so that

$$(v_1+v_2)+(u_1+u_2) = (v_1+u_1)+(v_2+u_2) \in W_1+W_2.$$

Moreover, since $av_1 \in W_1$ and $av_2 \in W_2$, necessarily

$$a(v_1+v_2) = av_1 + av_2 \in W_1+W_2,$$

proving W_1+W_2 is closed under addition and scalar multiplication.

Then, since $v_1 = v_1 + 0 \in W_1+W_2\ \forall v_1 \in W_1$ & $v_2 = 0 + v_2 \in W_1+W_2\ \forall v_2 \in W_2$, it follows that $W_1 \subseteq W_1+W_2$ and $W_2 \subseteq W_1+W_2$.

Finally, let Y be a subspace of V that contains both W_1 & W_2.

Since Y is closed under addition, $v_1+v_2 \in Y$ for every $v_1 \in W_1$ and $v_2 \in W_2$ necessarily.

It follows that $W_1+W_2 \subseteq Y$, completing the proof.

$V = W_1 \oplus W_2 \iff \forall v \in V: \exists$ UNIQUE $w_1 \in W_1$, $w_2 \in W_2 \ni v = w_1 + w_2$ (L2 (3))

Let V be a vector space, and let W_1 and W_2 be subspaces of V. Then $W_1 \oplus W_2 = V$ if and only if for every vector $v \in V$, there exist unique elements $w_1 \in W_1$ and $w_2 \in W_2$ such that $v = w_1 + w_2$.

Proof. ($\Rightarrow$) Since $V = W_1 \oplus W_2$, necessarily $V = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$.

Let $v \in V$, and note that since $V = W_1+W_2$, it implies that $v \in W_1 + W_2$.

So, by definition, there exist some $w_1 \in W_1$, $w_2 \in W_2$ such that $v = w_1 + w_2$.

Next, suppose we have $v = w_1' + w_2'$ for some $w_1' \in W_1$ and $w_2' \in W_2$.

Then

$$0 = (w_1+w_2) - (w_1' + w_2') = (w_1 - w_1') + (w_2 - w_2').$$

Since $w_1, w_1' \in W_1$ & $w_2, w_2' \in W_2$, necessarily $w_1 - w_1' \in W_1$ & $w_2 - w_2' \in W_2$ also, so that

$$(w_1 - w_1') = w_2' - w_2 \in W_1 \cap W_2 = \{0\},$$

Hence $w_1 - w_1' = w_2' - w_2 = 0$, implying that $w_1 = w_1'$ & $w_2 = w_2'$, proving uniqueness. *

($\Leftarrow$) By assumption, every vector $v \in V$ can be written as $v = w_1 + w_2$ for some $w_1 \in W_1$ & $w_2 \in W_2$.

Hence $V \subseteq W_1 + W_2$, and by L2(2) necessarily $W_1 + W_2 \subseteq V$; so $V = W_1 + W_2$.

Next, let $x \in W_1 \cap W_2$. Then $-x \in W_1 \cap W_2$.

Then, note that

$$0 = 0 + 0 = x + (-x) \in W_1 + W_2,$$

and due to the uniqueness assumption, necessarily $x = 0$.

Thus $W_1 \cap W_2 = \{0\}$, so that $V = W_1 \oplus W_2$. ☒

$\dim(W_1), \dim(W_2) < \infty \Rightarrow \dim(W_1+W_2) < \infty$ & $\dim(W_1) + \dim(W_2) = \dim(W_1+W_2) + \dim(W_1 \cap W_2)$ (T1.12 (1))

Let V be a vector space over some field $\mathbb{F}$, and let W_1, W_2 be finite dimensional subspaces of V.
Then necessarily W_1+W_2 is finite dimensional, and
$\dim(W_1) + \dim(W_2) = \dim(W_1+W_2) + \dim(W_1 \cap W_2)$.

Proof. First, note $W_1 \cap W_2$ is a subspace of W_1 (A2), so that
$\dim(W_1 \cap W_2) \le \dim(W_1) < \infty$ (C1.9.2 (6)).
Next, let $\{u_1, u_2, \dots, u_k\}$ be a basis for $W_1 \cap W_2$.
Extend this basis to get the bases
$S_1 = \{u_1, \dots, u_k, v_1, \dots, v_m\}$ of W_1 and
$S_2 = \{u_1, \dots, u_k, z_1, \dots, z_p\}$ of W_2, which we can always do by C1.9.2 (5)).
Let $S = \{u_1, \dots, u_k, v_1, \dots, v_m, z_1, \dots, z_p\}$.
We claim S is a basis for $W_1 + W_2$.
Indeed, consider

$$a_1u_1 + \dots + a_ku_k + b_1v_1 + \dots + b_mv_m + c_1z_1 + \dots + c_pz_p = 0 \quad \text{— ②}$$

for some scalars $a_1, \dots, a_k, b_1, \dots, b_m, c_1, \dots, c_p$.
Then

$$b_1v_1 + \dots + b_mv_m = -a_1u_1 - \dots - a_ku_k - c_1z_1 - \dots - c_pz_p.$$

Since the RHS is a linear combination of vectors in W_2, the RHS $\in W_2$; and since the LHS is a linear combination of vectors in W_1, the LHS $\in W_1$.

Thus $b_1v_1 + \dots + b_mv_m \in W_1 \cap W_2$.
Next, since $\{u_1, \dots, u_k\}$ is a basis for $W_1 \cap W_2$, there exist scalars $d_1, \dots, d_k$ such that

$$b_1v_1 + \dots + b_mv_m = d_1u_1 + \dots + d_ku_k.$$

So

$$b_1v_1 + \dots + b_mv_m - d_1u_1 - \dots - d_ku_k = 0.$$

Since $\{u_1, \dots, u_k, v_1, \dots, v_m\}$ is a basis for W_1, necessarily $b_1 = \dots = b_m = d_1 = \dots = d_k = 0$.
Substitute $b_1 = \dots = b_m$ into ② to get that

$$a_1u_1 + \dots + a_ku_k + c_1z_1 + \dots + c_pz_p = 0.$$

Then, since $\{u_1, \dots, u_k, z_1, \dots, z_p\}$ is a basis for W_2, we have

$$a_1 = \dots = a_k = c_1 = \dots = c_p = 0,$$

proving S is linearly independent.

Subsequently, let $x+y \in W_1 + W_2$ be arbitrary, where $x \in W_1$ and $y \in W_2$.
Then, since S_1 and S_2 are bases for W_1 and W_2 respectively, we can write x and y as linear combinations of vectors in S_1 and S_2, respectively:

$$x = a_1u_1 + \dots + a_ku_k + b_1v_1 + \dots + b_mv_m \; ; \; \&$$
$$y = d_1u_1 + \dots + d_ku_k + c_1z_1 + \dots + c_pz_p,$$

where $a_1, \dots, a_k, b_1, \dots, b_m, d_1, \dots, d_k, c_1, \dots, c_p \in \mathbb{F}$.
Hence

$$x+y = (a_1+d_1)u_1 + \dots + (a_k+d_k)u_k + b_1v_1 + c_1z_1 + \dots + b_mv_m + c_pz_p,$$

which is sufficient to show $x+y \in \mathrm{span}(S)$.
Thus $W_1 + W_2 \subseteq \mathrm{span}(S)$, and since $\mathrm{span}(S) \subseteq W_1 + W_2$ by definition, it follows that $W_1 + W_2 = \mathrm{span}(S)$, verifying that S is indeed a basis for $W_1 + W_2$.
In particular,

$$\begin{aligned}\dim(W_1+W_2) + \dim(W_1 \cap W_2) &= |S| + k \\ &= m+p+k+k \\ &= (m+k) + (p+k) \\ &= \dim W_1 + \dim W_2.\end{aligned}$$

☒

$\dim(V) < \infty$, $W_1 \oplus W_2 = V \Rightarrow \dim W_1 + \dim W_2 = \dim V$ (T1.12 (2))

Let V be a vector space over $\mathbb{F}$, and let W_1, W_2 be finite-dimensional subspaces of V.
Suppose further that V itself is finite dimensional, and $W_1 \oplus W_2 = V$.
Then necessarily $\dim W_1 + \dim W_2 = \dim V$.

Proof. Since $W_1 \oplus W_2 = V$, necessarily $W_1 \cap W_2 = \{0\}$.
So, by T1.12 (1), it follows that

$$\begin{aligned}\dim W_1 + \dim W_2 &= \dim(W_1+W_2) + \dim(W_1 \cap W_2) \\ &= \dim(V) + 0\end{aligned}$$

$\therefore \dim W_1 + \dim W_2 = \dim(V)$. ☒

COMPLEMENTARY SUBSPACES (D15)

Let V be a vector space, and let W be a subspace of V.
Then a subspace W' of V is said to be a "complementary subspace" to W if $W \oplus W' = V$; ie

① $W \cap W' = \{0\}$; and
② $W + W' = V$.

$\dim W + \dim W' = \dim V$

Let V be a vector space, and let W be a subspace of V.
Let W' be a complementary subspace to W.
Then necessarily $\dim W + \dim W' = \dim V$.

Proof. Follows directly from T1.12 (2).

EXISTENCE OF COMPLEMENTARY SUBSPACES (R11 (1))

Let V be a vector space, and let W be a subspace of V.
Then there always exists a complementary subspace W' to W of V such that $W \oplus W' = V$.

Proof. First, note that every finite linearly independent set can be extended to a basis V that has a countable spanning set (A3).
Hence, every linearly independent subset of V can be extended to a basis for V.
It follows that every subspace W of V has a complementary subspace W'. ☒

NON-UNIQUENESS OF COMPLEMENTARY SUBSPACES (R11 (2))

Note that complementary subspaces of a given vector space V are not necessarily unique.

eg $V = \mathbb{R}^3$, $W = \{(1,0,0), (0,1,0)\}$, $W_1' = \{(0,0,1)\}$, $W_2' = \{(0,0,-1)\}$;
observe that both W_1' and W_2' are complementary subspaces to W.

Chapter 2: Linear Transformations and Matrices

LINEAR TRANSFORMATIONS (S2.1)

Let V and W be vector spaces over the same field $\mathbb{F}$.

Then, we say the function $T: V \to W$ is a "linear transformation" from V to W if

(L1) → ① $T(x+y) = T(x) + T(y) \quad \forall x, y \in V$; and

(L2) → ② $T(cx) = cT(x) \quad \forall x \in V,\ c \in \mathbb{F}$. (D16)

In this case, we say the function $T: V \to W$ is "linear".

T IS LINEAR ⟺ $T(cx+y) = cT(x) + T(y)$ (P2)

Let the function $T: V \to W$, where V and W are vector spaces over the same field $\mathbb{F}$.

Then T is linear if and only if $T(cx+y) = cT(x) + T(y)$ for all $x, y \in V$ and $c \in \mathbb{F}$.

ZERO TRANSFORMATION (E23 (1a))

For any vector spaces V and W, the "zero transformation", given by "$T_0: V \to W$", is defined by $T_0(x) = 0 \quad \forall x \in V$.

IDENTITY TRANSFORMATION (E23 (1b))

For any vector space V, the "identity transformation" $I_V: V \to V$ is given by $I_V(x) = x \quad \forall x \in V$.

$T: V \to \mathbb{F}^n$ BY $T(a_1v_1 + \cdots + a_nv_n) := (a_1, \ldots, a_n)$ (E23 (3))

Let V be a finite-dimensional vector space over $\mathbb{F}$, and let $\{v_1, v_2, \ldots, v_n\}$ be a basis for V.

Then, the mapping

$$T: V \to \mathbb{F}^n \text{ by } T(a_1v_1 + \cdots + a_nv_n) := (a_1, \ldots, a_n)$$

is linear.

$T: \mathbb{F}^n \to \mathbb{F}^k$, $T(x_1, \ldots, x_n) := (x_1, \ldots, x_k)$ (E23 (4))

Let $\mathbb{F}$ be a field, and suppose $1 \le k < n$.

Then the projection mapping

$$T: \mathbb{F}^n \to \mathbb{F}^k \text{ by } T(x_1, \ldots, x_n) := (x_1, \ldots, x_k)$$

is linear.

$T(0) = 0$ (P3 (1))

Let $T: V \to W$ be linear.

Then necessarily $T(0) = 0$.

Proof. $T(0) = T(0+0) = T(0) + T(0)$;

thus $0 = T(0) + T(0) - T(0) = T(0)$. ∎

$T(x-y) = T(x) - T(y)$ (P3(2))

Let $T: V \to W$ be linear.

Then necessarily $T(x-y) = T(x) - T(y) \quad \forall x, y \in V$.

Proof. $T(x-y) = T(x) + T(-y)$

$= T(x) + (-1)T(y)$

$\therefore T(x-y) = T(x) - T(y)$. ∎

$T(a_1x_1 + \cdots + a_nx_n) = a_1T(x_1) + \cdots + a_nT(x_n)$ (P3(3))

Let T be linear, and $a_1, \ldots, a_n \in \mathbb{F}$ and $x_1, \ldots, x_n \in V$ be arbitrary.

Then necessarily

$$T(a_1x_1 + \cdots + a_nx_n) = a_1T(x_1) + \cdots + a_nT(x_n).$$

$\{v_1, \ldots, v_n\}$ IS A BASIS FOR V, $\{w_1, \ldots, w_n\}$ ARE ELEMENTS FOR W ⟹ ∃ A UNIQUE LINEAR MAPPING $T: V \to W$ ∋ $T(v_k) = w_k$ (T2.1)

Let $\{v_1, \ldots, v_n\}$ be a basis for a vector space V, and let $\{w_1, \ldots, w_n\}$ be arbitrary elements of another vector space W.

Then there exists a unique linear mapping $T: V \to W$ such that

$$T(v_1) = w_1, \quad \ldots, \quad T(v_n) = w_n.$$

Proof. Let $v \in V$ be arbitrary. Since $\{v_1, v_2, \ldots, v_n\}$ is a basis for V, there must exist $a_1, \ldots, a_n \in \mathbb{F}$ such that

$$v = a_1v_1 + \cdots + a_nv_n.$$

Let $T(v) = T(a_1v_1 + \cdots + a_nv_n) = a_1w_1 + \cdots + a_nw_n$ (by P3(3)).

Then, by construction, for any $1 \le k \le n$, we have

$$T(v_k) = T(0v_1 + \cdots + 0v_{k-1} + 1v_k + 0v_{k+1} + \cdots + 0v_n)$$
$$= 0w_1 + \cdots + 0w_{k-1} + 1w_k + 0w_{k+1} + \cdots + 0w_n$$
$$= w_k,$$

proving uniqueness.

Next, suppose there exists another linear mapping $L: V \to W$ satisfying $L(v_1) = w_1, \ldots, L(v_n) = w_n$.

Let $v = a_1v_1 + \cdots + a_nv_n$, where $v \in V$ and $a_1, \ldots, a_n \in \mathbb{F}$.

Then

$$L(v) = L(a_1v_1 + \cdots + a_nv_n)$$
$$= a_1L(v_1) + \cdots + a_nL(v_n)$$
$$= a_1w_1 + \cdots + a_nw_n$$
$$= a_1T(v_1) + \cdots + a_nT(v_n)$$
$$= T(a_1v_1 + \cdots + a_nv_n)$$
$$\therefore L(v) = T(v).$$

Hence $L(v) = T(v) \quad \forall v \in V$, so that $T = L$, proving uniqueness. ∎

It also follows that we have

$$T(a_1v_1 + \cdots + a_nv_n) = a_1w_1 + \cdots + a_nw_n. \quad \text{(C2.1.1)}$$

NULL SPACE / KERNEL (D17(1))

Let V and W be vector spaces, and let $T: V \to W$ be linear.

Then the "null space" of T, or the "kernel" of T, denoted as "$N(T)$", is defined to be the set

$$N(T) := \{x \in V \mid T(x) = 0\}.$$

RANGE / IMAGE (D17(2))

Let V and W be vector spaces, and let $T: V \to W$ be linear.

Then the "range" of T, or the "image" of T, denoted as "$R(T)$", is defined to be the set

$$R(T) := \{T(x) : x \in V\}.$$

N(T) IS A SUBSPACE OF V (T2.2)

Let $T: V \to W$ be linear.

Then necessarily $N(T)$ is a subspace of V.

R(T) IS A SUBSPACE OF W (T2.2)

Let $T: V \to W$ be linear.

Then necessarily $R(T)$ is a subspace of W.

$\{v_1, \dots, v_n\}$ IS A BASIS FOR $V \Rightarrow$ $\text{span}(\{T(v_1), \dots, T(v_n)\}) = R(T)$ (T2.3)

Let V and W be vector spaces, and let $T: V \to W$ be linear.

Suppose the set $\{v_1, \dots, v_n\}$ is a basis for V.

Then necessarily $\{T(v_1), \dots, T(v_n)\}$ generates $R(T)$.

NULLITY (D18)

Let $T: V \to W$ be linear, and suppose that $\dim(N(T)) < \infty$.

Then, we define the "nullity" of T, denoted by "nullity(T)", to be equal to

$$\text{nullity}(T) = \dim(N(T)).$$

RANK (D18)

Let $T: V \to W$ be linear, and suppose that $\dim(R(T)) < \infty$.

Then, we define the "rank" of T, denoted by "rank(T)", to be equal to

$$\text{rank}(T) = \dim(R(T)).$$

RANK-NULLITY THEOREM (T2.4)

Let V and W be vector spaces, and let $T: V \to W$ be linear with $\dim(V) < \infty$.

Then necessarily

$$\text{rank}(T) + \text{nullity}(T) = \dim(V).$$

Proof. Since $N(T)$ is a subspace of V (T2.2) and $\dim V < \infty$, by C1.9.2 (6) necessarily $\text{nullity}(T) \leq \dim(V) < \infty$.

Then, let $\text{nullity}(T) = k$, and suppose that $\{v_1, \dots, v_k\}$ is a basis for $N(T)$.

We know that we can "extend" $\{v_1, \dots, v_k\}$ to get a basis for V, $\{v_1, \dots, v_n\}$, so let us do so.

Next, we claim $\{T(v_{k+1}), \dots, T(v_n)\}$ is a basis for $R(T)$.

First, we show $\{T(v_{k+1}), \dots, T(v_n)\}$ spans $R(T)$.

By T2.2, $R(T) = \text{span}(\{T(v_1), \dots, T(v_k), T(v_{k+1}), \dots, T(v_n)\})$.

Then, since $\{v_1, \dots, v_k\}$ is a basis for $N(T)$, necessarily

$$T(v_1) = \dots = T(v_k) = 0.$$

Hence,

$$R(T) = \text{span}(\{T(v_{k+1}), \dots, T(v_n)\}),$$

as needed.

Next, we show $\{T(v_{k+1}), \dots, T(v_n)\}$ is linearly independent.

Consider

$$c_{k+1}T(v_{k+1}) + \dots + c_nT(v_n) = 0, \quad \text{where } c_{k+1}, \dots, c_n \in \mathbb{F}$$

$$\Rightarrow T(c_{k+1}v_{k+1} + \dots + c_nv_n) = 0.$$

Hence $c_{k+1}v_{k+1} + \dots + c_nv_n \in N(T)$; then, since $\{v_1, \dots, v_k\}$ is a basis for $N(T)$, there exist $d_1, \dots, d_k \in \mathbb{F}$ such that

$$c_{k+1}v_{k+1} + \dots + c_nv_n = d_1v_1 + \dots d_kv_k.$$

$$\Rightarrow -d_1v_1 - \dots - d_kv_k + c_{k+1}v_{k+1} + \dots + c_nv_n = 0.$$

Since $\{v_1, \dots, v_n\}$ is a basis for V, consequently

$$d_1 = \dots = d_k = c_{k+1} = \dots = c_n = 0,$$

showing $\{T(v_{k+1}), \dots, T(v_n)\}$ is linearly independent.

Consequently,

$$\begin{aligned}\text{rank}(T) + \text{nullity}(T) &= \dim(R(T)) + \dim(N(T)) \\ &= k + (n - (k+1) + 1) \\ &= n\end{aligned}$$

$\therefore \text{rank}(T) + \text{nullity}(T) = \dim(V).$ ∎

ONE-TO-ONE (1-1) (D19)

Let $T: V \to W$ be linear.

Then, we say T is "one-to-one" if, for any $x, y \in V$, $T(x) = T(y)$ implies $x = y$.

ONTO (D19)

Let $T: V \to W$ be linear.

Then, we say T is "onto" if $R(T) = V$.

ISOMORPHISM (D19)

1. Let $T: V \to W$ be linear. Then, we say T is an "isomorphism" if it is both one-to-one and onto.

2. We say V is "isomorphic" to W if an isomorphism $T: V \to W$ exists, (D20) and denote this by the notation "$V \cong W$".

T IS 1-1 $\Leftrightarrow$ $N(T) = \{0\}$ (L3)

Let $T: V \to W$ be linear. Then T is one-to-one if and only if $N(T) = \{0\}$.

Proof. ($\Rightarrow$) Suppose T is one-to-one.
Let $x \in V$ be such that $T(x) = 0$.
Then since $T(0) = 0 = T(x)$, by definition $x = 0$, so that $N(T) = \{0\}$.
($\Leftarrow$) Suppose $N(T) = \{0\}$. Consider $x, y \in V$ such that $T(x) = T(y)$.
Then
$$T(x-y) = T(x) - T(y) = 0,$$
so that $x - y \in N(T)$;
hence $x - y = 0$, so that $x = y$ (and hence T is 1-1). ∎

$\{v_1, \dots, v_n\}$ IS A BASIS FOR V $\Rightarrow$ T IS ISOMORPHIC $\Leftrightarrow$ $\{T(v_1), \dots, T(v_n)\}$ IS A BASIS FOR W (T2.5)

Let V and W be vector spaces over a field $\mathbb{F}$, with $\dim V < \infty$.
Let $\{v_1, \dots, v_n\}$ be a basis for V, and let $T: V \to W$ be linear.
Then T is an isomorphism if and only if $\{T(v_1), \dots, T(v_n)\}$ is a basis for W.

Proof. ($\Rightarrow$) Consider
$$c_1 T(v_1) + \dots + c_n T(v_n) = 0$$
$$\Rightarrow T(c_1 v_1 + \dots + c_n v_n) = 0.$$
Since T is one-to-one by definition, hence
$c_1 v_1 + \dots + c_n v_n = 0$,
and as $\{v_1, \dots, v_n\}$ is a basis for V,
necessarily $c_1 = \dots = c_n = 0$;
hence $\{T(v_1), \dots, T(v_n)\}$ is a basis for W. #

($\Leftarrow$) If $\{T(v_1), \dots, T(v_n)\}$ is a basis for W, by definition $\{T(v_1), \dots, T(v_n)\}$ generates W.
Thus $W = \text{span}(\{T(v_1), \dots, T(v_n)\}) = R(T)$,
where the second equality comes from T2.3.
Then, since $W = R(T)$, T is necessarily onto.
Let $x \in N(T)$. Since $\{v_1, \dots, v_n\}$ is a basis for V,
there must exist some $a_1, \dots, a_n \in \mathbb{F}$ such that
$$x = a_1 v_1 + \dots + a_n v_n.$$
Hence
$$0 = T(x) = a_1 T(v_1) + \dots + a_n T(v_n).$$
Since $\{T(v_1), \dots, T(v_n)\}$ is a basis for W by assumption,
thus $\{T(v_1), \dots, T(v_n)\}$ is linearly independent, so that
$$a_1 = \dots = a_n = 0,$$
and so
$$x = 0v_1 + \dots + 0v_n = 0.$$
Consequently $N(T) = \{0\}$, so that (by L3) T is 1-1. ∎

CONSTRUCTING AN ISOMORPHISM FROM V TO W

Let V and W be vector spaces.
Then, we can construct an isomorphism from V to W as follows:

1. Choose a basis $\{v_1, \dots, v_n\}$ for V, and a basis $\{w_1, \dots, w_n\}$ for W.
2. Let the linear transformation $T: V \to W$ be such that $T(v_k) = w_k$ $\forall k \in \{1, 2, \dots, n\}$.
   (T exists; this follows from T2.1)
3. Then, by T2.5, T is also an isomorphism.

$V \cong W \Leftrightarrow \dim V = \dim W$ (T2.6)

Let V and W be two finite-dimensional vector spaces over a field $\mathbb{F}$.
Then V is isomorphic to W if and only if $\dim V = \dim W$.

$\dim V = \dim W < \infty$; T IS 1-1 $\Leftrightarrow$ T IS ONTO $\Leftrightarrow$ $\text{rank}(T) = \dim(V)$ (T2.7)

Let V and W be two vector spaces over a field $\mathbb{F}$, and assume $\dim V = \dim W < \infty$.
Let $T: V \to W$ be linear.
Then the following are equivalent to one another:

1. T is one-to-one;
2. T is onto; and
3. $\text{rank}(T) = \dim(V)$.