MATH 247
Personal Notes

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Module 1.1: Normed Vector Spaces

1. A "normed vector space", or "NVS", is a vector space V over $\mathbb{R}$ equipped with a function

$$\|\cdot\| : V \to [0,\infty),$$

and the function satisfies the following:

① $\|v\| = 0 \iff v = 0$;

② $\|\alpha v\| = |\alpha| \cdot \|v\| \quad \forall \alpha \in \mathbb{R}$; and

③ $\|u+v\| \leq \|u\| + \|v\|$. (Triangle Inequality).

2. Such a function $\|\cdot\|$ is called a "norm".

3. We generally use the notation "$(V, \|\cdot\|)$" to indicate a normed vector space.

4. Geometrically,

① $\|v\|$ refers to the "length" of v, or the distance between v & 0; and

② $\|v-w\|$ refers to the "distance" between v & w.

5. Normed vector spaces are useful in real analysis because the "notion" of "distance" in NVS helps us talk about "approximating" real numbers with more well-behaved ones (eg $\mathbb{Q}$).

p-NORMS : $\|v\|_p$

1. Let $p \geq 1$, and let $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n$.

Then, the "p-norm" of v, denoted as "$\|v\|_p$", is equal to

$$\|v\|_p = \left(\sum_{i=1}^{n} |v_i|^p \right)^{\frac{1}{p}}.$$

2. We can show that the p-norm is indeed a norm of $\mathbb{R}^n$.

(see A1.)

3. In particular, the 2-norm,

$$\|v\|_2 = \left(\sum_{i=1}^{n} |v_i|^2 \right)^{\frac{1}{2}},$$

is called the "Euclidean norm" on $\mathbb{R}^n$;

In this course, we equip $\mathbb{R}^n$ with $\|\cdot\|_2$, unless stated otherwise.

INFINITY NORM : $\|v\|_\infty$

Let $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n$.

Then, the "infinity norm" of v, denoted as "$\|v\|_\infty$", is defined to be

$$\|v\|_\infty = \max\{|v_1|, |v_2|, \ldots, |v_n|\}.$$

$(\ell^p, \|\cdot\|_p)$ & $(\ell^\infty, \|\cdot\|_\infty)$ ARE NVS OF $\mathbb{R}^N$

1. Let $v = (v_1, v_2, \ldots) \in \mathbb{R}^N$ (ie let v be a sequence of reals).

Then, we let ℓ^p be defined by

$$\ell^p = \{ v \in \mathbb{R}^N : \|v\|_p < \infty \},$$

where

$$\|v\|_p = \left(\sum_{i=1}^{\infty} |v_i|^p \right)^{\frac{1}{p}}.$$

2. Similarly, we let ℓ^∞ be defined by

$$\ell^\infty = \{ v \in \mathbb{R}^N : \|v\|_\infty < \infty \},$$

where

$$\|v\|_\infty = \max\{|v_1|, |v_2|, \ldots\}.$$

3. Then, $(\ell^p, \|\cdot\|_p)$ & $(\ell^\infty, \|\cdot\|_\infty)$ are NVS.

UNIFORM NORM ON $C([a,b])$: $\|f\|_\infty$

1. Let $V = C([a,b])$, ie the set of all continuous functions $f : [a,b] \to \mathbb{R}$.

Then, the "uniform norm" of a $f \in V$, denoted as "$\|f\|_\infty$", is defined to be

$$\|f\|_\infty = \sup\{ |f(x)| : x \in [a,b] \}$$
$$= \max\{ |f(x)| : x \in [a,b] \} \quad \text{(by EVT)}$$

Unless otherwise stated, we assume the uniform norm is used if working with $\mathbb{R}^N$ as a NVS.

2. An alternative norm to $\mathbb{R}^N$ is the "integration-based" norm $\|f\|_p$, where $p \geq 1$ and $f \in V$, which is defined as

$$\|f\|_p = \left(\int_a^b |f(x)|^p \, dx \right)^{\frac{1}{p}}.$$

Module 1.2: Convergence

CONVERGENCE / DIVERGENCE

Let V be a NVS, and let the sequence $(a_n) \subseteq V$.
Then, we say (a_n) "converges" to some $a \in V$, denoted "$a_n \to a$", if for all $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ such that $\|a_n - a\| < \varepsilon$ $\forall n \geq N$.

Otherwise, we say that (a_n) diverges.

eg[1] $V = \ell^\infty$, $(a_n) \subseteq V$

$a_n = (1, \frac{1}{2}, \dots, \frac{1}{n}, 0, 0, \dots)$

$a = (1, \frac{1}{2}, \frac{1}{3}, \dots)$

Claim: $a_n \to a$

Proof. Let $\varepsilon > 0$, and choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$.
Then, for $n \geq N$, note that

$$\|a_n - a\|_\infty = \|(\underbrace{0, 0, \dots, 0}_{n}, \tfrac{1}{n+1}, -\tfrac{1}{n+2}, \dots)\|_\infty$$
$$= \sup\{0, \tfrac{1}{n+1}, \tfrac{1}{n+2}, \dots\}$$
$$= \tfrac{1}{n+1} < \tfrac{1}{n} < \tfrac{1}{N} < \varepsilon,$$

showing that the sequence converges. ∎

eg[2] $V = \ell^\infty$, $(a_n) \subseteq V$

$a_n = (\underbrace{1, \dots, 1}_{n}, 0, 0, \dots)$

$a = (1, 1, \dots, 1, \dots)$

Claim: $a_n \not\to a$.

Proof. $\forall n \in \mathbb{N}$: $\|a_n - a\|_\infty = 1$
(since always 1 present in the sequence $(a_n - a)$).

BOUNDED (SUBSET)

Let $A \subseteq V$, where V is a NVS.
Then, we say A is "bounded" if there exists a $M > 0$ such that $\|a\| \leq M$ for all $a \in A$.

BOUNDED (SEQUENCES)

Let $(a_n) \subseteq V$, where V is a NVS.
Then, we say (a_n) is "bounded" if $\{a_1, \dots, a_n, \dots\}$ is itself bounded.

(a_n) IS CONVERGENT $\Rightarrow$ (a_n) IS BOUNDED

Let $(a_n) \subseteq V$ be so that (a_n) is convergent.
Then necessarily (a_n) is bounded.

Proof. Suppose $a_n \to a \in V$.
$\Rightarrow \exists N \in \mathbb{N}$ such that if $n \geq N$, then $\|a_n - a\| < 1$.
Then, notice that for $n \geq N$,

$$\|a_n\| = \|a_n - a + a\|$$
$$\leq \|a_n - a\| + \|a\|$$
$$\leq 1 + \|a\|.$$

Let $M = \max\{\|a_1\|, \dots, \|a_{N-1}\|, 1 + \|a\|\}$,
we have that $\|a_n\| \leq M$ $\forall n \in \mathbb{N}$, as needed. ∎

Note the converse is not necessarily true!
eg $(a_n) = (1, -1, 1, -1, \dots)$

$a_n \to a$, $b_n \to b$ $\Rightarrow$ $a_n + b_n \to a + b$

Let $(a_n), (b_n) \subseteq V$, and let $a_n \to a$ & $b_n \to b$.
Then necessarily $a_n + b_n \to a + b$.

$a_n \to a$ $\Rightarrow$ $\alpha a_n \to \alpha a$

Let $(a_n) \subseteq V$, and let $a_n \to a$.
Then necessarily $\alpha a_n \to \alpha a$.

Module 1.3: Completeness

CAUCHY SEQUENCE

Let $(a_n) \subseteq V$, where V is a NVS.
Then, we say (a_n) is a "Cauchy sequence" if for all $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ such that $\|a_n - a_m\| < \varepsilon$ for all $n, m \geq N$.

(a_n) IS CONVERGENT $\Rightarrow$ (a_n) IS CAUCHY

Let $(a_n) \subseteq V$ be convergent.
Then necessarily (a_n) is also Cauchy.

Proof. Let $\varepsilon > 0$.
We know $N \in \mathbb{N}$, $a \in V$ such that $\|a_n - a\| < \frac{\varepsilon}{2}$ $\forall n \geq N$.
Also, for $n, m \geq N$, we know

$$\begin{aligned} \|a_n - a_m\| &\leq \|a_n - a\| + \|a_m - a\| \\ &< \varepsilon, \end{aligned}$$

as needed. $\blacksquare$

(a_n) IS CAUCHY $\not\Rightarrow$ (a_n) IS CONVERGENT

Note that $(a_n) \subseteq V$ is Cauchy does <u>not necessarily imply</u> it is also convergent.

For example, take the NVS $(C_{00}, \|\cdot\|_\infty)$, where

$$C_{00} = \{(x_n) \in \ell^\infty : \exists N \ni x_n = 0 \ \forall n \geq N\},$$

and let $(a_n) \subseteq C_{00}$ be defined by

$$a_n = (1, \tfrac{1}{2}, \ldots, \tfrac{1}{n}, 0, 0, \ldots),$$

and let a be equal to

$$a = (1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots) \notin C_{00}.$$

We know $a_n \to a$ in ℓ^∞, so $(a_n) \subseteq \ell^\infty$ is Cauchy.
$\Rightarrow (a_n) \subseteq C_{00}$ is still Cauchy.
However, since $a_n \to a \notin C_{00}$, and limits are unique, it follows that $(a_n) \subseteq C_{00}$ diverges.

COMPLETE

Let $A \subseteq V$, where V is a NVS.
Then, we say A is "complete" if whenever $(a_n) \subseteq A$ is Cauchy, it follows that there exists an $a \in A$ such that $a_n \to a$.

BANACH SPACE

Let V be a NVS.
Then, we say V is a "Banach space" if V is complete.

Module 1.4: Banach Spaces

$\mathbb{R}^n$ IS A BANACH SPACE WRT $\|\cdot\|_p$ & $\|\cdot\|_\infty$

First, let $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n$ & $1 \le p \le \infty$.

Then, note the following:

① $\|v\|_p^p = |v_1|^p + \dots + |v_n|^p$

$\le n \max\{|v_1|, \dots, |v_n|\}^p$

$\therefore \|v\|_p^p = n\|v\|_\infty^p$;

② $\|v\|_\infty^p \le |v_1|^p + \dots + |v_n|^p = \|v\|_p^p$.

③ So, $\|v\|_p \le \sqrt[p]{n}\, \|v\|_\infty$, and

$\|v\|_\infty \le \|v\|_p$.

We can also show $(\mathbb{R}^n, \|\cdot\|_\infty)$ is a Banach space.

Proof. Suppose $(a_k) \subseteq \mathbb{R}^n$ is Cauchy, say

$a_k = (a_k^{(1)}, \dots, a_k^{(n)})$,

for each $k \in \mathbb{N}$, where $a_k^{(j)} \in \mathbb{R}$.

Let $\varepsilon > 0$. We know $\exists N \in \mathbb{N}$ such that $\|a_k - a_\ell\|_\infty < \varepsilon$ for all $k, \ell \ge N$.

Then, for all $k, \ell \ge N$ and $1 \le i \le n$, we have that

$|a_k^{(i)} - a_\ell^{(i)}| \le \|a_k - a_\ell\|_\infty < \varepsilon$,

and so it follows that $(a_k^{(i)})_{k=1}^\infty \subseteq \mathbb{R}$ is Cauchy.

Finally, since $\mathbb{R}$ is complete, it follows that

$a_k^{(i)} \to b_i \in \mathbb{R} \quad \forall 1 \le i \le n$.

To finish, we want to prove $a_k \to (b_1, \dots, b_n)$, which is sufficient to prove the statement in question.

Let $\varepsilon > 0$. Fix $1 \le i \le n$.

We know there exists a $N_i \in \mathbb{N}$ such that $|a_k^{(i)} - b_i| < \varepsilon \ \forall k \ge N_i$.

Let $N = \max\{N_1, N_2, \dots, N_n\}$, so that for $k \ge N$, we have that

$\left\|a_k - \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}\right\|_\infty = \max\{|a_k^{(i)} - b_i| : 1 \le i \le n\}$

$\le \varepsilon$,

completing the proof (as $a_k \to \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$). ▨

This is sufficient to prove $(\mathbb{R}^n, \|\cdot\|_p)$ is a Banach space.

Proof. Assume $(\mathbb{R}^n, \|\cdot\|_\infty)$ is a Banach space.

Let $1 \le p < \infty$, and let $(a_k) \subseteq \mathbb{R}^n$ be Cauchy wrt $\|\cdot\|_p$.

Thus, (a_k) is Cauchy wrt $\|\cdot\|_\infty$, and so

$a_k \to a \in \mathbb{R}^n$ wrt $\|\cdot\|_\infty$.

Hence $a_k \to a$ wrt $\|\cdot\|_p$, and so $(\mathbb{R}^n, \|\cdot\|_p)$ is also a Banach space. ▨

ℓ^∞ IS A BANACH SPACE

We can prove ℓ^∞ is a Banach space.

Proof. Let $(a_n) \subseteq \ell^\infty$ be Cauchy.

For all $n \in \mathbb{N}$, we can write

$a_n = (a_n^{(1)}, a_n^{(2)}, \dots)$,

where $a_n^{(i)} \in \mathbb{R}$.

We claim for an $i \in \mathbb{N}$, that $(a_n^{(i)})$ is Cauchy.

Proof. Let $\varepsilon > 0$ be given.

Then, there exists a $N \in \mathbb{N}$ such that $\|a_n - a_m\|_\infty < \varepsilon$ $\forall n, m \ge N$.

Next, fix $i \in \mathbb{N}$. For $n, m \ge N$, note that

$|a_n^{(i)} - a_m^{(i)}| \le \sup\{|a_n^{(i)} - a_m^{(i)}| : i \in \mathbb{N}\}$

$= \|a_n - a_m\|_\infty$

$< \varepsilon$,

proving the claim. ⋇

So, by the completeness of $\mathbb{R}$, we have that

$a_n^{(i)} \to b_i$ (as $n \to \infty$)

for all $i \in \mathbb{N}$.

Next, we claim that $a_n \to b$, where $b = (b_1, b_2, \dots)$.

Proof. Let $\varepsilon > 0$ be given.

Then, we know $\exists N \in \mathbb{N}$ such that $\|a_n - a_m\|_\infty < \varepsilon$ $\forall n, m \ge N$.

We also have that

$|a_n^{(i)} - a_m^{(i)}| < \varepsilon \quad \forall n, m \ge N$ and $i \in \mathbb{N}$

by definition.

Next, taking $m \to \infty$, we note for $n \ge N$ that

$|a_n^{(i)} - b_i| \le \varepsilon \quad \forall i \in \mathbb{N}$.

Thus $\|a_n - b\|_\infty < \varepsilon$ for all $n \ge N$, and so indeed

$a_n \to b$. #

It follows that ℓ^∞ is a Banach space, as needed. ▨

Module 2.1: Closed and Open Sets

CLOSED SET

Let V be a NVS, and let $C \subseteq V$.
Then, we say C is "closed" if whenever $(a_n) \subseteq C$ with $a_n \to a \in V$, then $a \in C$.

OPEN SET

Let V be a NVS, and let $U \subseteq V$.
Then, we say U is "open" if $V \setminus U$ is closed.

TOPOLOGY ON A SPACE

Let V be a NVS.
Then, the "topology" on V is defined to be the set
$$\mathcal{T} = \{ U \subseteq V \mid U \text{ is open} \}.$$

$\emptyset, V$ ARE OPEN & CLOSED IN V

Note $\emptyset$ and V are always open and closed in V.

CLOSED BALL: $\overline{B_r(a)}$

1. Let $r > 0$ and $a \in V$, where V is a NVS.
Then, the "closed ball" of radius r centred at a, denoted as "$\overline{B_r(a)}$", is defined to be the set
$$\overline{B_r(a)} = \{ x \in V : \|x - a\| \le r \}.$$

2. We can prove $\overline{B_r(a)}$ is closed.

Proof. Let $(a_n) \subseteq \overline{B_r(a)} \ni a_n \to b \in V$.
By def^n, $\|a_n - a\| \le r \quad \forall n \in \mathbb{N}$.
But since $a_n \to a$, it follows that
$$\|a_n - a\| \to \|b - a\|,$$
and so $\|b - a\| \le \max\{\|a_n - a\| : n \in \mathbb{N}\} \le r$,
and so $b \in \overline{B_r(a)}$, which is sufficient to prove the statement. ▨

OPEN BALL: $B_r(a)$

1. Let $r > 0$ and $a \in V$, where V is a NVS.
Then, the "open ball" of radius r centred at a, denoted as "$B_r(a)$", is defined to be the set
$$B_r(a) = \{ x \in V : \|x - a\| < r \}.$$

2. We can show $B_r(a)$ is open.

Proof. By a similar proof to the above, we can show $\{x \in V : \|x - a\| \ge r\}$ is closed.
Hence
$$B_r(a) = V \setminus \{x \in V : \|x - a\| \ge r\}$$
is open. ▨

$V = \ell^\infty$, $c_0 = \{(x_n) \in \ell^\infty \mid x_n \to 0\}$ IS CLOSED

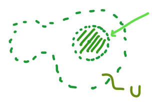
We can show $c_0 = \{(x_n) \in \ell^\infty \mid x_n \to 0\}$ is closed in $V = \ell^\infty$.

Proof. Let $(a_n) \subseteq c_0 \ni a_n \to a \in \ell^\infty$.
Let $a_n = (a_n^{(1)}, a_n^{(2)}, \dots) \quad \forall n \in \mathbb{N}$.
Hence, we know
$$\lim_{k \to \infty} a_n^{(k)} = 0 \quad \forall n \in \mathbb{N}.$$
Then, say $a = (b_1, b_2, \dots)$, and let $\varepsilon > 0$.
We know there exists $N_1, N_2 \in \mathbb{N} \ni$
① $\|a_n - a\|_\infty < \frac{\varepsilon}{2} \quad \forall n \ge N_1$; and
② $|a_{N_1}^{(k)}| < \frac{\varepsilon}{2} \quad \forall k \ge N_2$.
Finally, for $k \ge N_2$, note that
$$\begin{aligned} |b_k| &= |a_{N_1}^{(k)} - b_k - a_{N_1}^{(k)}| \\ &\le |a_{N_1}^{(k)} - b_k| + |a_{N_1}^{(k)}| \\ &\le \|a_{N_1} - a\|_\infty + |a_{N_1}^{(k)}| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$
showing $b_k \to 0$ and so $a = (b_1, b_2, \dots) \in c_0$, and so c_0 is closed. ▨

U IS OPEN $\iff \forall a \in U : \exists r > 0 \ni B_r(a) \subseteq U$

1. Let V be a NVS, and let $U \subseteq V$.
Then, U is open if and only if for any $a \in U$, there exists a $r > 0$ such that $B_r(a) \subseteq U$.

eg in $\mathbb{R}^2$



For any arbitrary point, you can draw a "circle" that is contained in U.

Proof. ($\Rightarrow$) Assume U is open, so $V \setminus U$ is closed.
Suppose, for a contradiction, that (negation of $\forall a \in U : \exists r > 0 \ni B_r(a) \subseteq U$).
$\exists a \in U \ni (\nexists r > 0 \ni B_r(a) \subseteq U)$.
In particular, $\forall n \ge \mathbb{N}$, there exists some $a_n \in B_{\frac{1}{n}}(a)$ such that $a_n \notin U$.
Note that
① $\|a_n - a\| < \frac{1}{n} \Rightarrow a_n \to a$ (since $\frac{1}{n} \to 0$); and
② $(a_n) \subseteq V \setminus U$, and $V \setminus U$ is closed
$\Rightarrow a \in V \setminus U$.
But $a \in U$ by assumption, a contradiction.
Thus $\forall a \in U : \exists r > 0 \ni B_r(a) \subseteq U$, as needed. #

($\Leftarrow$) Assume $\forall a \in U : \exists r > 0 \ni B_r(a) \subseteq U$.
We claim $V \setminus U$ is closed. Indeed, let
$$(a_n) \subseteq V \setminus U \ni a_n \to a \in V.$$
Suppose $a \in U$, so in particular $\exists r > 0 \ni B_r(a) \subseteq U$.
But since $a_n \to a$, $\exists N \in \mathbb{N} \ni \|a_N - a\| < r$,
implying $a_N \in B_r(a) \subseteq U$, and so $a_N \in U$.
However we assumed $a_N \in V \setminus U$, so this is a contradiction!
Hence $V \setminus U$ is closed, and so U is open, as needed.

Module 2.2-2.4: Closure and Interior

UNION OF OPEN SETS IS OPEN

Let V be a NVS, and let $\{U_\alpha\}_{\alpha \in I}$ be a collection of open sets in V.
Then necessarily $U = \bigcup_{\alpha \in I} U_\alpha$ is open.

INTERSECTION OF CLOSED SETS IS CLOSED

Let V be a NVS, and let $\{C_\alpha\}_{\alpha \in I}$ be a collection of closed sets in V.
Then necessarily $C = \bigcap_{\alpha \in I} C_\alpha$ is closed.

FINITE INTERSECTION OF OPEN SETS IS OPEN

Let $U_1, \dots, U_n \subseteq V$ be open.
Then necessarily $U = U_1 \cap \dots \cap U_n$ is open.

FINITE UNION OF CLOSED SETS IS CLOSED

Let $C_1, \dots, C_n \subseteq V$ be closed.
Then necessarily $C = C_1 \cup \dots \cup C_n$ is closed.

CLOSURE OF A SUBSET: $\overline{A}$

Let $A \subseteq V$.
Then, we define the "closure" of A, denoted as "$\overline{A}$", to be the set

$$\overline{A} = \bigcap_{\substack{A \subseteq C, \\ C \text{ closed}}} C.$$

* $\overline{A}$ is the smallest closed set containing A.

INTERIOR OF A SUBSET: Int(A)

Let $A \subseteq V$.
Then, we define the "interior" of A, denoted as "$\text{int}(A)$", to be the set

$$\text{Int}(A) = \bigcup_{\substack{U \subseteq A, \\ U \text{ open}}} U.$$

* $\text{Int}(A)$ is the largest open set contained in A.

LIMIT POINT

Let V be a NVS, and let $A \subseteq V$.
Then, we say $a \in V$ is a "limit point" of A if there exists a $(a_n) \subseteq V$ with $a_n \to a$.

INTERIOR POINT

Let V be a NVS, and let $A \subseteq V$.
Then, we say $a \in V$ is an "interior point" of A if there exists a $r > 0$ such that $B_r(a) \subseteq A$.

$\overline{A} = \{\text{limit points of } A\}$

We can show that $\overline{A}$ is the set of limit points of A for any $A \subseteq V$.

Proof. Let $X = \{\text{limit points of } A\}$.
Claim: X is closed.
↳ Proof: Let $(a_n) \subseteq X \ni a_n \to a \in V$.
In particular, we know
$\forall n \in \mathbb{N},\ \exists b_n \in A \ni |a_n - b_n| < \frac{1}{n}$.
But we also know
$b_n = b_n - a_n + a_n \to 0 + a = a$,
and so $a \in X$, showing X is closed. #

By definition, we know $\overline{A} \subseteq X$, since $\overline{A}$ is the smallest closed set that contains A.
Now, let $x \in X$, so that $\exists (a_n) \subseteq A \ni a_n \to x$.
Let $C \subseteq V$ be closed, so that $A \subseteq C$.
Then, note $(a_n) \subseteq C$, and so $x \in C$ (since C is closed).
Thus x is in any closed set containing A, and so $X \subseteq \overline{A}$ (and thus $X = \overline{A}$, as needed). ▨

$\text{Int}(A) = \{\text{interior points of } A\}$

Similarly, we can show $\text{int}(A)$ is the set of interior points of A for any $A \subseteq V$.

Proof. Similar to above.

$A = \{(a_n) \in \ell^1 : a_n \in \mathbb{Q}\}$: $\overline{A} = \ell^1$

We claim $\overline{A} = \ell^1$.

Note: for $x \in \ell^1$, assume $\forall \varepsilon > 0, \exists a \in A \ni \|x - a\|_1 < \varepsilon$.
Then, $x \in \overline{A}$.
Why? $\because \forall n \in \mathbb{N} : \exists a_n \in A \ni \|x - a_n\|_1 < \frac{1}{n}$.
$\Rightarrow (a_n) \subseteq A$ and so $a_n \to x$.

Proof. Let $x = (x_1, x_2, \dots) \in \ell^1$ and let $\varepsilon > 0$.
By the density of $\mathbb{Q}$,
$\forall n \in \mathbb{N} : \exists y_n \in \mathbb{Q} \ni |x_n - y_n| < \frac{\varepsilon}{2^n}$.
Consider $y = (y_1, y_2, \dots)$.
Then, note that

$$\|x - y\|_1 = \sum_{n=1}^{\infty} |x_n - y_n| < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon,$$

which is sufficient to show $x \in \overline{A}$, and so $\ell^1 \subseteq \overline{A}$.
As $\overline{A} \subseteq \ell^1$ by definition, it follows that $\overline{A} = \ell^1$, as needed. ▨

$V = \ell^\infty$; $\overline{C_{00}} = C_0$

We can show $\overline{C_{00}} = C_0$ in $V = \ell^\infty$.

Proof. We know $C_{00} \subseteq C_0$, and since C_0 is closed, it follows that $\overline{C_{00}} \subseteq C_0$.
Now, let $x = (x_1, x_2, \dots) \in C_0$, and let $\varepsilon > 0$.
Since $x_n \to 0$, it follows that
$\exists N \in \mathbb{N} \ni |x_n| < \frac{\varepsilon}{2}\ \forall n \geq N$.
Let $y = (x_1, \dots, x_{N-1}, 0, 0, \dots) \in C_{00}$.
It follows that

$$\|x - y\|_\infty = \|(0, \dots, 0, x_N, x_{N+1}, \dots)\|_\infty = \sup\{|x_k| : k \geq N\} \leq \frac{\varepsilon}{2} < \varepsilon,$$

and so necessarily $x \in \overline{C_{00}}$, and so $C_0 \subseteq \overline{C_{00}}$.
Thus $C_0 = \overline{C_{00}}$, as needed. ▨

$\overline{A \cup B} = \overline{A} \cup \overline{B}$

Let $A, B \subseteq V$.
Then necessarily $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Proof. Since $\overline{A} \cup \overline{B}$ is closed & $A \cup B \subseteq \overline{A} \cup \overline{B}$, it follows $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$.
Then, since $A, B \subseteq A \cup B$, it follows that $\overline{A}, \overline{B} \subseteq \overline{A \cup B}$.
Thus $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$, and so $\overline{A} \cup \overline{B} = \overline{A \cup B}$. ∎

$\mathrm{Int}(A \cap B) = \mathrm{Int}(A) \cap \mathrm{Int}(B)$

Let $A, B \subseteq V$.
Then necessarily $\mathrm{Int}(A \cap B) = \mathrm{Int}(A) \cap \mathrm{Int}(B)$.

$\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$

Let $A, B \subseteq V$.
Then necessarily $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$.

$\mathrm{Int}(A \cup B) \supseteq \mathrm{Int}(A) \cup \mathrm{Int}(B)$

Let $A, B \subseteq V$.
Then necessarily $\mathrm{Int}(A \cup B) \supseteq \mathrm{Int}(A) \cup \mathrm{Int}(B)$.

$A = (0,1)$, $B = (1,2)$, $V = \mathbb{R}$: $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$

Let $V = \mathbb{R}$, and let $A = (0,1)$, $B = (1,2)$.
Then note that $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$.

Proof. See that
$\overline{A \cap B} = \overline{\emptyset} = \emptyset$ but $\overline{A} \cap \overline{B} = [0,1] \cap [1,2] = \{1\}$. ∎

$A = [0,1]$, $B = [1,2]$, $V = \mathbb{R}$: $\mathrm{Int}(A \cup B) \neq \mathrm{Int}(A) \cup \mathrm{Int}(B)$

Let $V = \mathbb{R}$, and let $A = [0,1]$, $B = [1,2]$.
Then note that $\mathrm{Int}(A \cup B) \neq \mathrm{Int}(A) \cup \mathrm{Int}(B)$.

Proof. See that
$\mathrm{Int}(A \cup B) = (0,2)$ but $\mathrm{Int}(A) \cup \mathrm{Int}(B) = (0,1) \cup (1,2) = (0,2) \setminus \{1\}$.

$\mathrm{Int}(V \setminus A) = V \setminus \overline{A}$

Let $A \subseteq V$.
Then necessarily $\mathrm{Int}(V \setminus A) = V \setminus \overline{A}$.

Proof. Since $V \setminus \overline{A} \subseteq V \setminus A$ and $V \setminus \overline{A}$ is open, it follows (by the largeness of the interior) that $V \setminus \overline{A} \subseteq \mathrm{Int}(V \setminus A)$.
Then, observe that
$V \setminus \mathrm{Int}(V \setminus A) \supseteq V \setminus (V \setminus A) = A$,
and since $\mathrm{Int}(V \setminus A)$ is open, hence $V \setminus \mathrm{Int}(V \setminus A)$ is closed; thus A is closed, and so
$\overline{A} \subseteq V \setminus \mathrm{Int}(V \setminus A)$.
Hence
$V \setminus \overline{A} \supseteq V \setminus (V \setminus \mathrm{Int}(V \setminus A)) = \mathrm{Int}(V \setminus A)$,
and so necessarily $V \setminus \overline{A} = \mathrm{Int}(V \setminus A)$, as needed. ∎

$\overline{V \setminus A} = V \setminus \mathrm{Int}(A)$

Let $A \subseteq V$.
Then necessarily $\overline{V \setminus A} = V \setminus \mathrm{Int}(A)$.

Boundary of a subset: $\partial(A)$

Let $A \subseteq V$.
Then, the "boundary" of A, denoted as "$\partial(A)$", is defined to be the set
$\partial(A) = \overline{A} \setminus \mathrm{Int}(A)$.

$\partial(A)$ is closed

Let $A \subseteq V$.
Then necessarily $\partial(A)$ is closed.

Proof. $\partial(A) = \overline{A} \setminus \mathrm{Int}(A)$
$= \underbrace{\overline{A}}_{\text{closed}} \cap \underbrace{(V \setminus \mathrm{Int}(A))}_{\text{closed}}$,
and so $\partial(A)$ is closed. ∎

A is closed $\Leftrightarrow$ $\partial(A) \subseteq A$

Let $A \subseteq V$.
Then necessarily A is closed if and only if $\partial(A) \subseteq A$.

Proof. ($\Rightarrow$) A is closed $\Rightarrow$ $\partial(A) \subseteq \overline{A} = A$.
($\Leftarrow$) Suppose $\partial(A) \subseteq A$.
Recall $\partial(A) = \overline{A} \setminus \mathrm{Int}(A)$; in particular, we can write
$\overline{A} = \partial(A) \cup \mathrm{Int}(A)$,
and since $\partial(A), \mathrm{Int}(A) \subseteq A$ it follows that
$\overline{A} \subseteq A$.
As $\overline{A} \subseteq A$ it follows that $A = \overline{A}$, and so A is closed. ∎

Module 3: Compactness & Open Covers

BOUNDED (SETS)

Let $A \subseteq V$, where V is a NVS.
Then, we say that A is "bounded" if there exists a $M \in \mathbb{N}$ such that
$\|a\| < M \quad \forall a \in A$.

COMPACT (SUBSETS)

Let $C \subseteq V$, where V is a NVS.
Then, we say C is "compact" if every $(c_n) \subseteq C$ has a subsequence $(c_{n_k}) \subseteq (c_n)$ with $c_{n_k} \to a \in C$.

$A \subseteq \mathbb{R}^n$ IS CLOSED & BOUNDED $\Rightarrow$ A IS COMPACT

Let $A \subseteq \mathbb{R}^n$ be closed and bounded.
Then necessarily A is compact.

Proof. Let $(a_k) \subseteq A$.
Since A is bounded, $\Rightarrow (a_k)$ is bounded.
By A2, we know $\exists (a_{k_l}) \subseteq (a_k) \ni a_{k_l} \to a \in \mathbb{R}^n$.
Since A is closed, it follows that $a \in A$,
and so A is compact. ☒

$A = \{e_1 = (1,0,\dots,0),\ e_2 = (0,1,0,\dots 0),\ \dots\} \subseteq \ell^\infty$: A IS NOT COMPACT

Let A be the above set. We can show A is not compact.

Proof. $\forall n \neq m$, $\|e_n - e_m\|_\infty = 1$
$\Rightarrow (e_n)$ is not Cauchy
$\Rightarrow (e_n)$ is not convergent
$\Rightarrow A$ is not compact. ☒

Then, notice that
$A = (e_1, e_2, \dots) \subseteq \overline{B_1(0)}$.
The RHS is trivially closed and bounded, but since A is not compact, necessarily $\overline{B_1(0)}$ cannot be compact!

Therefore, closed & bounded does not automatically imply compactness.

C IS COMPACT $\Rightarrow$ C IS CLOSED & BOUNDED

Let $C \subseteq V$ be compact.
Then necessarily C is closed and bounded.

Proof. Let $C \subseteq V$ be compact.
① We claim C is closed.
Proof. Let $(a_n) \subseteq C \ni a_n \to a \in V$.
$\Rightarrow \exists (a_{n_k}) \subseteq (a_n) \ni a_{n_k} \to b \in C$.
However, we must have $a = b \in C$,
and so C is closed. #
② We claim C is bounded.
Proof. Suppose this was not the case.
$\Rightarrow \exists n \in \mathbb{N}, \quad \exists a_n \in C \ni \|a_n\| \geq n$.
Then consider $(a_n) \subseteq C$.
$\Rightarrow \exists (a_{n_k}) \subseteq (a_n) \ni a_{n_k} \to a \in C$.
But $\|a_{n_k}\| \geq n_k$! So it is unbounded,
and so has to be divergent.
This is a contradiction!
Hence C is bounded, as needed. ☒

HEINE-BOREL THEOREM: $C \in \mathbb{R}^n$ IS COMPACT $\Leftrightarrow$ C IS CLOSED & BOUNDED

Let $C \in \mathbb{R}^n$.
Then C is compact if and only if it is also closed & bounded.
(This has been proved by previous observations!)

$C \subseteq V$ IS COMPACT, $A \subseteq C$ IS CLOSED $\Rightarrow$ A IS COMPACT

Let $C \subseteq V$ be compact, and let $A \subseteq C$ be closed.
Then necessarily A is compact.

Proof. Let $(a_n) \subseteq A$. Since $A \subseteq C$,
$\exists (a_{n_k}) \subseteq (a_n) \ni a_{n_k} \to a \in C$.
But as A is closed $\Rightarrow a \in A$, and
so A is compact. ☒

OPEN COVERS (OF SUBSETS)

Let $A \subseteq V$, where V is a NVS.
Then, an "open cover" of A is a collection of open sets $\{U_\alpha : \alpha \in I\}$ such that
$A \subseteq \bigcup_{\alpha \in I} U_\alpha$.

We say the open cover is "finite" if $|I| < |\mathbb{N}|$.

Examples:
① $V = \mathbb{R}$, $A = [0,1]$, $A \subseteq (-\frac{1}{4}, \frac{1}{4}) \cup (0, \frac{1}{2}) \cup \dots \cup (\frac{3}{4}, \frac{5}{4})$
② $V = \mathbb{R}^2$, $A = \mathbb{Z} \times \mathbb{Z}$, $A \subseteq \bigcup_{\alpha \in \mathbb{Z} \times \mathbb{Z}} B_{\frac{1}{2}}(\alpha)$
③ $V = \mathbb{R}$, $A = (0,1]$, $A \subseteq \bigcup_{n \in \mathbb{N}} (\frac{1}{n}, 2)$.

SUBCOVERS

Let $\{U_\alpha : \alpha \in I\}$ be an open cover of $A \subseteq V$.
Then, we say a subset of $\{U_\alpha : \alpha \in I\}$ is a "subcover" of $\{U_\alpha : \alpha \in I\}$.

Note that subcovers are also open covers of A.

$A \subseteq V$ compact, $A \subseteq \bigcup_{\alpha \in I} U_\alpha$ is an open cover $\Rightarrow \exists R > 0 \ni \forall a \in A: B_R(a) \subseteq U_\alpha$ for some $\alpha \in I$

Let $A \subseteq V$ be compact, and let $A \subseteq \bigcup_{\alpha \in I} U_\alpha$ be an open cover of A.
Then, there necessarily exists some $R > 0$ such that for each $a \in A$, we have
$$B_R(a) \subseteq U_\alpha$$
for some $\alpha \in I$.

Proof. Suppose no such $R>0$ exists.
In particular, $\forall n \in \mathbb{N}: \exists a_n \in A \ni B_{\frac{1}{n}}(a_n) \not\subseteq U_\alpha$ $\forall \alpha \in I$.
Since $(a_n) \subseteq A$ and A is compact,
$\Rightarrow \exists (a_{n_k}) \subseteq (a_n) \ni a_{n_k} \to a \in A$.
Say $a \in U_{\alpha_0}$, where $\alpha_0 \in I$ (since the union is an open cover).
Pick $m \in \mathbb{N} \ni B_{\frac{2}{m}}(a) \subseteq U_{\alpha_0}$
Moreover, since $a_{n_k} \to a$, we may find $N \in \mathbb{N}$
$\ni a_{n_k} \in B_{\frac{1}{m}}(a)$ $\forall k \geq N$.
Then, for $k \geq N$ such that $n_k > m$, take
$x \in B_{\frac{1}{m}}(a_{n_k})$.
But
$$\Rightarrow \|x-a\| = \|x - a_{n_k} + a_{n_k} - a\| \leq \|x - a_{n_k}\| + \|a_{n_k} - a\| < \frac{1}{m} + \frac{1}{m} = \frac{2}{m},$$
and so $x \in B_{\frac{2}{m}}(a)$.
$\therefore B_{\frac{1}{m}}(a_{n_k}) \subseteq B_{\frac{2}{m}}(a) \subseteq U_{\alpha_0}$.
But since $n_k > m$, it follows that
$B_{\frac{1}{n_k}}(a_{n_k}) \subseteq B_{\frac{1}{m}}(a_{n_k}) \subseteq U_{\alpha_0}$,
which is a contradiction to our earlier assumption that $B_{\frac{1}{n}}(a_n) \not\subseteq U_\alpha$ $\forall \alpha \in I$. ∎

$A \subseteq V$ is compact $\Rightarrow$ every open cover of A has a finite subcover

Let $A \subseteq V$ be compact.
Then necessarily every open cover of A has a finite subcover.

Proof. Suppose $A \subseteq V$ is compact.
Let $A \subseteq \bigcup_{\alpha \in I} U_\alpha$ be an open cover of A.
Since A is compact, by the above lemma,
$\exists R > 0 \ni \forall a \in A: B_R(a) \subseteq U_\alpha$ for some $\alpha \in I$.
If $\exists a_1, \dots, a_n \in A \ni A \subseteq B_R(a_1) \cup \dots \cup B_R(a_n)$, by the lemma we are done.
So, suppose no such covering existed.
Then, we can find $a_1 \in A$, $a_2 \in A \ni a_2 \notin B_R(a_1)$,
$a_3 \in A \ni a_3 \notin B_R(a_1) \cup B_R(a_2)$, ...
Since $(a_n) \subseteq A$ & A is compact,
$\exists (a_{n_k}) \subseteq (a_n) \ni a_{n_k} \to a$.
However, for $n < m$, we have that
$$a_m \notin B_R(a_n),$$
or in other words,
$$\|a_m - a_n\| \geq R.$$
$\Rightarrow (a_n)$ has no Cauchy subsequences,
$\Rightarrow (a_n)$ has no convergent subsequences,
giving us our contradiction. ∎

Every open cover of $A \subseteq V$ has a finite subcover, & $A \subseteq \bigcup_{\alpha \in I} U_\alpha$ where each U_α is relatively open in A $\Rightarrow \exists \alpha_1, \dots, \alpha_n \in I$ $\ni A \subseteq U_{\alpha_1} \cup \dots \cup U_{\alpha_n}$

Why is the above lemma true?
Proof. $A \subseteq \bigcup_{\alpha \in I} U_\alpha$, where $U_\alpha = A \cap O_\alpha$, $O_\alpha \subseteq V$ open
$$\Rightarrow A \subseteq \bigcup_\alpha (A \cap O_\alpha) = A \cap \left(\bigcup_\alpha O_\alpha\right) \subseteq \bigcup_\alpha O_\alpha,$$
$\Rightarrow A \subseteq O_{\alpha_1} \cup \dots \cup O_{\alpha_n}$
$\Rightarrow A \subseteq \underbrace{U_{\alpha_1} \cup \dots \cup U_{\alpha_n}}_{\text{the relatively open cover.}}$ ∎

Every open cover of A has a finite subcover $\Rightarrow A \subseteq V$ is compact

Let $A \subseteq V$, and suppose every open cover of A has a finite subcover.
Then necessarily $A \subseteq V$ is compact.

Proof. Let $(a_n) \subseteq A$.
For $k \in \mathbb{N}$, consider $C_k = \overline{(a_n)_{n=k}^\infty} \cap A$.
We want to show $\bigcap_{k=1}^\infty C_k \neq \emptyset$.
Each C_k is relatively closed in A. Hence every $U_k = A \setminus C_k$ is relatively open.
For contradiction, assume $\bigcap_{k=1}^\infty C_k = \emptyset$.
$$\Rightarrow A = A \setminus \emptyset = A \setminus \left(\bigcap_{k=1}^\infty C_k\right) = \bigcup (A \setminus C_k) = \bigcup_{k=1}^\infty U_k \leftarrow \text{the relatively open subcover.}$$
By the lemma above,
$\exists i_1, \dots, i_\ell \ni A \subseteq U_{i_1} \cup \dots \cup U_{i_\ell}$.
Since $C_1 \supseteq C_2 \supseteq \cdots$, we have that $U_1 \subseteq U_2 \subseteq \cdots$;
$\therefore A \subseteq U_{i_\ell} \subseteq A$,
and so $A = U_{i_\ell}$.
$\Rightarrow C_{i_\ell} = A \setminus U_{i_\ell} = A \setminus A = \emptyset$.
But since $a_{i_\ell} \in C_{i_\ell} = \emptyset$, this is a contradiction!
Hence, we may find some $a \in \bigcap_{k=1}^\infty C_k$.
$\therefore \exists n_1 < n_2 < \dots \ni \|a_{n_k} - a\| < \frac{1}{k}$ $\forall k \in \mathbb{N}$,
and so $(a_{n_k}) \subseteq A \ni a_{n_k} \to a \in A$, showing A is compact, as needed. ∎

Module 4.1: Limits

LIMIT

1. Let $f: A \to W$, where $A \subseteq V$, and let $a \in V$.
Then, the "limit" of $f(x)$ as x approaches a is $w \in W$ if
   ① $a \in \overline{A \setminus \{a\}}$; and
   ② $\forall \varepsilon > 0 : \exists \delta > 0$ such that if $x \in A$ with $0 < \|x - a\| < \delta$, then $\|f(x) - w\| < \varepsilon$.

2. In this case, we write
$$\lim_{x \to a} f(x) = w.$$
* note that w is unique.

ISOLATED POINT

1. Let $a \in A$, where $A \subseteq V$.
Then, we call a an "isolated point with respect to A" if $a \notin \overline{A \setminus \{a\}}$.

2. If $a \notin \overline{A \setminus \{a\}}$, then there exists $r > 0$ such that $B_r(a) \cap A = \{a\}$ or $\emptyset$.
In other words, there does not exist $x \in A$ with $0 < \|x - a\| < r$.

LIMITS PRESERVE ORDER

Let $A \subseteq V$, and let $f, g, h : A \to \mathbb{R}$ and $a \in \overline{A \setminus \{a\}}$.
Suppose $\lim_{x \to a} f(x)$ & $\lim_{x \to a} g(x)$ exist and $f(x) \leq g(x) \ \forall x \in A$.
Then necessarily $\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$.

SQUEEZE THEOREM

Let $A \subseteq V$, and let $f, g, h : A \to \mathbb{R}$ and $a \in \overline{A \setminus \{a\}}$.
Suppose $f(x) \leq g(x) \leq h(x) \ \forall x \in A$ and
$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$.
Then necessarily $\lim_{x \to a} g(x) = L$ as well.

LIMITS OF MULTIVARIABLE FUNCTIONS

eg[1] Evaluate
$$\lim_{(x,y,z) \to (0,0,0)} \frac{xy^2 + x^2 z + xyz}{\sqrt{x^2 + y^2 + z^2}}.$$

Sol$\underline{n}$. If $x \neq 0$, then observe that
$$0 \leq \left| \frac{xy^2 + x^2 z + xyz}{\sqrt{x^2 + y^2 + z^2}} \right| \leq \frac{|xy^2 + x^2 z + xyz|}{\sqrt{x^2}}$$
$$= \frac{|xy^2 + x^2 z + xyz|}{|x|}$$
$$\leq \frac{|x||y|^2 + x^2 |z| + |x||y||z|}{|x|}$$
$$= y^2 + |x||z| + |y||z|.$$

If $x = 0$, then $f(x, y, z) = 0$.
Since
$$\lim_{(x,y,z) \to (0,0,0)} y^2 + |x||z| + |y||z| = 0,$$
by S.T it follows that $\lim_{(x,y,z) \to (0,0,0)} f(x, y, z) = 0$. #

eg[2] Evaluate
$$\lim_{(x,y) \to (0,0)} \frac{xy^2}{x^2 + y^4}. \ \} f(x,y)$$

Sol$\underline{n}$. As $(\frac{1}{n}, 0) \to (0,0)$, we see that
$$f(\tfrac{1}{n}, 0) = 0 \to 0.$$
As $(\frac{1}{n^2}, \frac{1}{n}) \to (0,0)$, we see that
$$f(\tfrac{1}{n^2}, \tfrac{1}{n}) = \frac{\frac{1}{n^4}}{\frac{1}{n^4} + \frac{1}{n^4}} = \frac{1}{2} \to \frac{1}{2}.$$
Since $0 \neq \frac{1}{2}$, the limit does not exist. #

Module 4.2: Continuity

CONTINUOUS (FUNCTIONS)

Let $f: A \to W$, where $A \subseteq V$.
Then, we say f is "continous" at $a \in A$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $x \in A$ with $\|x - a\| < \delta$, then $\|f(x) - f(a)\| < \varepsilon$.

Note that f is continuous at $a \in \overline{A \setminus \{a\}}$ if and only if $\lim_{x \to a} f(x) = f(a)$.

However, if $a \notin \overline{A \setminus \{a\}}$, then f is automatically continuous at a.

Why? $\Rightarrow$ $\exists\, r > 0$, $B_r(a) \cap A = \{a\}$
Let $\varepsilon > 0$, & chose $\delta = r$.
If $x \in A$ & $\|x - a\| < \delta$, then $x = a$.
$\therefore \|f(x) - f(a)\| = \|f(a) - f(a)\| = 0 < \varepsilon$

We say f is continuous if f is continuous at all $a \in A$.

f IS CTS $\Leftrightarrow$ f PRESERVES CONVERGENCE $\Leftrightarrow$ $\forall$ OPEN $U \subseteq W$, $f^{-1}(U)$ IS RELATIVELY OPEN IN A

Let $f: A \to W$, where $A \subseteq V$. Then, the following are equivalent:
1. f is continuous;
2. f preserves convergence; and
3. $\forall$ open $U \subseteq W$, $f^{-1}(U)$ is relatively open in A.

Proof. ③ $\Leftrightarrow$ ① from Assignment 2.
①$\Rightarrow$②: Suppose f is cts, and let $(a_n) \subseteq A \ni a_n \to a \in A$.
Let $\varepsilon > 0$. There exists $\delta > 0 \ni x \in A$ & $\|x - a\| < \delta$,
then $\|f(x) - f(a)\| < \varepsilon$.
Take $N \in \mathbb{N} \ni \|a_n - a\| < \delta \;\; \forall n \geq N$.
But then, for $n \geq N$, we see that $\|f(a_n) - f(a)\| < \varepsilon$,
showing that $f(a_n) \to f(a)$. #

②$\Rightarrow$①: Assume f preserves convergence, and suppose f is discontinuous at a.
$\Rightarrow \exists\, \varepsilon > 0$ & $(a_n) \subseteq A \ni \|a_n - a\| < \frac{1}{n}$ but $\|f(a_n) - f(a)\| \geq \varepsilon$.
Then, $a_n \to a$ but $f(a_n) \not\to f(a)$ — contradiction! ☒

PROJECTION MAP IS CTS

The "i^{th} projection map" $P_i: \mathbb{R}^n \to \mathbb{R}$, where $1 \leq i \leq n$, is defined by
$P_i(x_1, \dots, x_n) = x_i$.

We can prove the projection map is continuous for any $1 \leq i \leq n$.

Proof. Let $(a_k) \subseteq \mathbb{R}^n \ni a_k \to a \in \mathbb{R}^n$, say $a_k = (a_k^{(1)}, \dots, a_k^{(n)})$ and $a = (b_1, \dots, b_n)$.
We know $a_k^{(i)} \to b_i \;\; \forall 1 \leq i \leq n$ as $k \to \infty$;
$\Rightarrow P_i(a_i) \to P_i(a)$
$\therefore P_i$ is cts. ☒

$f, g: A \to W$ ARE CTS $\Rightarrow$ $f+g$, αf $(\alpha \in \mathbb{R})$ ARE CTS

Let $f, g: A \to W$ be continuous, and let $\alpha \in \mathbb{R}$.
Then $f+g$ and αf are necessarily also continuous.

Proof. Let $(a_n) \subseteq A \ni a_n \to x$.
$\Rightarrow$ (since f, g are cts) $\Rightarrow f(a_n) \to f(a)$ & $g(a_n) \to g(a)$.
$\Rightarrow f(a_n) + g(a_n) \to f(a) + g(a)$ & $\alpha f(a_n) \to \alpha f(a)$. ☒

$f: A \to W_1$, $g: B \to W_2$, $B \subseteq W_1$ ARE CTS $\Rightarrow$ $g \circ f$ IS CTS

Let $f: A \to W_1$ and $g: B \to W_2$ be continuous, where $B \subseteq W_1$.
Then necessarily $(g \circ f)$ is continuous.

Proof. Let $(a_n) \subseteq A \ni a_n \to a$.
Since f is cts, $\Rightarrow f(a_n) \to f(a)$.
Since g is cts, $\Rightarrow g(f(a_n)) \to g(f(a))$.
$\Rightarrow g \circ f$ preserves convergence
$\Rightarrow g \circ f$ is continuous. ☒

Module 4.3: Uniform Continuity

UNIFORM CONTINUITY

Let $f: A \to V$, where $f: A \to W$.
Then, we say f is "uniformly continuous" if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $x, a \in A$ with $\|x - a\| < \delta$, then $\|f(x) - f(a)\| < \varepsilon$.

Note that uniform continuity implies continuity.

LIPSCHITZ (FUNCTIONS)

Let $f: A \to W$.
We say f is "Lipschitz" if there exists a $M > 0$ such that

$$\|f(a) - f(b)\| \leq M\|a - b\| \quad \forall a, b \in A.$$

LIPSCHITZ $\Rightarrow$ UNIFORM CONTINUITY

Let $f: A \to W$ be Lipschitz.
Then necessarily f is uniformly continuous.

Proof. Let $\varepsilon > 0$. Choose $\delta = \frac{\varepsilon}{M}$.
If $a, b \in A$ with $\|a - b\| < \delta$, then

$$\|f(a) - f(b)\| \leq M\|a - b\|$$
$$< M\delta = M\left(\frac{\varepsilon}{M}\right) = \varepsilon,$$

showing f is uniformly cts. ∎

C IS COMPACT, $f: C \to W$ IS CTS $\Rightarrow$ f IS UNIF CTS

Let $C \subseteq V$ be compact & $f: C \to W$ be continuous.
Then f is uniformly continuous.

Why? $\to$ Suppose for contradiction that f is not unif. cts.

$\Rightarrow \exists (a_n), (b_n) \subseteq C \ni \|a_n - b_n\| < \frac{1}{n}, \|f(a_n) - f(b_n)\| \geq \varepsilon$.

By compactness,
$\exists (a_{n_k}) \subseteq (a_n) \ni a_{n_k} \to a \in C$.

$$\Rightarrow b_{n_k} = \underbrace{b_{n_k} - a_{n_k}}_{\to 0} + \underbrace{a_{n_k}}_{\to a}$$

$\Rightarrow b_{n_k} \to a$.

By continuity, $f(a_{n_k}) \to f(a)$, $f(b_{n_k}) \to f(b)$.

$\Rightarrow \|f(a_{n_k}) - f(b_{n_k})\| \to 0$.

But this is a contradiction $\because \|f(a_n) - f(b_n)\| \geq \varepsilon$ by earlier assumption! ∎

Module 4.4: Extreme Value Theorem

$C \subseteq W$ IS COMPACT, $f: C \to W$ IS CTS $\Rightarrow$ $f(C)$ IS COMPACT

Let $C \subseteq W$ be compact, and let $f: C \to W$ be continuous.
Then necessarily $f(C)$ is compact.

Why? Take $(f(a_n)) \subseteq f(C)$, $a_n \in C$.

$\Rightarrow (a_n) \subseteq C$.

(compact) $\exists (a_{n_k}) \subseteq (a_n) \ni a_{n_k} \to a$.

(continuity) $\Rightarrow f(a_{n_k}) \to f(a) \in f(C)$

$\Rightarrow f(C)$ is compact. ☒

$\phi \neq A \subseteq \mathbb{R}$ IS BOUNDED $\Rightarrow$ $\inf A$, $\sup A \in \overline{A}$

Let $\phi \neq A \subseteq \mathbb{R}$ be bounded.
Then necessarily $\inf A$, $\sup A \in \overline{A}$.

Proof. We prove the claim for $\sup A$; $\inf A$ is similar.

$\forall n \in \mathbb{N}$, we know

$\sup A - \frac{1}{n} < \underset{A}{\overset{\in}{a_n}} \leq \sup A$.

$\Rightarrow$ (By S.T) $a_n \to \sup A$.

$\therefore \sup A \in \overline{A}$.

$\phi \neq C \subseteq V$ IS COMPACT, $f: C \to \mathbb{R}$ IS CTS $\Rightarrow$ $\exists a, b \in C \ni f(a) = \min f(C)$ & $f(b) = \max f(C)$ (EXTREME VALUE THEOREM (EVT))

Let $\phi \neq C \subseteq V$ be compact, and let $f: C \to \mathbb{R}$ be continuous.
Then, there must exist some $a, b \in C$ such that $f(a) = \min f(C)$ & $f(b) = \max f(C)$.

Proof. $f(C)$ is compact.

$f(C) \subseteq \mathbb{R} \Rightarrow f(C)$ is closed & bounded.

$\Rightarrow$ (by bounded) $\sup f(C)$, $\inf f(C) \in \overline{f(C)}$.

(since $f(C)$ is compact) $\therefore \overline{f(C)} = f(C)$.

$\therefore \exists a, b \in C \ni f(a) = \inf f(C) = \min f(C)$

& $f(b) = \sup f(C) = \max f(C)$. ☒

UNIFORM NORM (FOR $C(K, W)$)

Let $K \subseteq V$ be compact, and let W be a NVS.
Then, $C(K, W) = \{f: K \to W \text{ cts}\}$ is a NVS when equipped with the uniform norm

$\|f\|_\infty = \max\{\|f(x)\| : x \in K\}$.

Module 5: Sequences of Functions

POINTWISE CONVERGENCE [OF FUNCTIONS]

Let $A \subseteq V$, $f_n : A \to W$ and $f : A \to W$.
Then, we say f_n converges to f "pointwise" if

$$f_n(x) \to f(x) \quad \forall x \in A.$$

UNIFORM CONVERGENCE [OF FUNCTIONS]

(1) Let $A \subseteq V$, $f_n : A \to W$ and $f : A \to W$.
Then, we say f_n converges to f "uniformly" if for any $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ such that

$$\| f_n(x) - f(x) \| < \varepsilon \quad \forall n \geq N,\ x \in A.$$

(2) In this case, note that the same N works uniformly for all $x \in A$.

$\| f_n - f \|_\infty := \sup \{ \| f_n(x) - f(x) \| : x \in A \}$

(1) Let $f_n, f : A \to W$, where $A \subseteq V$.
Then, we define

$$\| f_n - f \|_\infty := \sup \{ \| f_n(x) - f(x) \| : x \in A \}.$$

(2) Note that

$f_n \to f$ uniformly $\iff$ $\| f_n - f \|_\infty < \infty$ eventually & $\| f_n - f \|_\infty \to 0$.

*note that since A may not be compact & f may not be cts, $\| f_n - f \|_\infty$ could be infinite.

(3) For example, take $f_n : \mathbb{R} \to \mathbb{R}$, with $f_1(x) = x$ & $f_n(x) = 0$ $\forall n > 1$.
Then $f_n \to 0$ uniformly, even though $\| f_1 - 0 \|_\infty = \infty$.

EXAMPLES

For each sequence of functions, find the pointwise limit and determine whether the convergence is uniform.

eg 1 $f_n : (0,1) \to \mathbb{R}$, $f_n(x) = \frac{nx}{1+nx}$

Pointwise limit: for $x \in (0,1)$,
$f_n(x) = \frac{nx}{1+nx} \to 1$.
$\therefore f_n \to 1$ pointwise.
For $n > 1$, we have
$| f_n(\frac{1}{n}) - 1 | = \frac{1}{2}$.
$\therefore \| f_n - f \|_\infty \not\to 0$
$\Rightarrow$ convergence is not uniform.

eg 2 $f_n : c_0 \to \mathbb{R}$, $f_n((a_k)) \to a_n$.

Pointwise: For $(a_k) \in c_0$, see that
$f_n((a_k)) = a_n \to 0$.
$\therefore f_n \to 0$ pointwise.
Uniform? For $n \in \mathbb{N}$, see that

$$| f_n((\underbrace{1, \dots, 1}_{n}, 0, 0, \dots)) - 0 | = |1 - 0| = 1.$$

$\therefore \| f_n - 0 \| \geq 1 \Rightarrow \| f_n - 0 \|_\infty \not\to 0$.
$\Rightarrow$ convergence is not uniform.

eg 3 $f_n : [0,1] \times [0,1] \to \mathbb{R}$, $f_n(a,b) = \frac{a^n}{n} + \frac{1}{b+n}$

Pointwise: For $(a,b) \in [0,1] \times [0,1]$,
$f_n(a,b) = \frac{a^n}{n} + \frac{1}{b+n} \in [0, \frac{1}{n}]$
$\therefore$ (by ST) $f_n(a,b) \to 0$.
$\Rightarrow f_n(a,b) \to 0$ pointwise.
Uniform? Note that

$$| f_n(a,b) - 0 | = \frac{a^n}{n} + \frac{1}{b+n} \leq \frac{1}{n} + \frac{1}{n} = \frac{2}{n}$$

$\therefore \| f_n - 0 \|_\infty \leq \frac{2}{n} \to 0$,
$\therefore f_n \to 0$ uniformly.

$f_n : A \to W$, $A \subseteq V$: f_n IS CTS $\forall n \in \mathbb{N}$, $f_n \to f$ UNIFORMLY $\Rightarrow$ f IS CTS

Let $f_n : A \to W$, where $A \subseteq V$.
Suppose each f_n is continuous, and $f_n \to f$ uniformly.
Then necessarily f is continuous.

Proof. Let $(a_n) \subseteq A$ $\ni$ $a_n \to a$ & let $\varepsilon > 0$.
We know we may find $N \in \mathbb{N}$ $\ni$
$\| f_N - f \|_\infty < \boxed{\varepsilon/3}$.
Since f_N is cts, we know $\exists M \in \mathbb{N}$ $\ni$
$| f_N(a_n) - f_N(a) | < \boxed{\varepsilon/3}$ $\forall n \geq M$.
Then, for $n \geq M$, see that

$$\begin{aligned} | f(a_n) - f(a) | &= | f(a_n) - f_N(a_n) + f_N(a_n) - f_N(a) + f_N(a) - f(a) | \\ &\leq | f(a_n) - f_N(a_n) | + | f_N(a_n) - f_N(a) | + | f_N(a) - f(a) | \\ &\leq \| f_n - f \|_\infty + | f_n(a_n) - f_N(a) | + \| f_n - f \|_\infty \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

and so $f(a_n) \to f(a)$, and so f is cts. ∎

$A \subseteq V$ IS COMPACT, W IS A BANACH SPACE $\Rightarrow$ $(C(A,W), \| \cdot \|_\infty)$ IS A BANACH SPACE

Let $A \subseteq V$ be compact, and let W be a Banach space.
Then necessarily $(C(A,W), \| \cdot \|_\infty)$ is a Banach space.

Proof. Let $(f_n) \subseteq C(A,W)$ be Cauchy, and let $\varepsilon > 0$.
We know $\exists N \in \mathbb{N}$ $\ni$
$\| f_n - f_m \|_\infty < \varepsilon$ $\forall n, m \geq N$.
For $x \in A$ & $n, m \geq N$, see that
$\| f_n(x) - f_m(x) \| \leq \| f_n - f_m \|_\infty < \varepsilon$,
and so $(f_n(x)) \subseteq W$ is Cauchy.
Since W is a Banach space, it is complete,
and so $f_n(x) \to f(x) \in W$ for some $f(x) \in W$.
Thus, we have constructed a fn $f : A \to W$ $\ni$
$f_n \to f$ pointwise.
For $x \in A$ and $n \geq N$, we have that
$\lim_{m \to \infty} \| f_n(x) - f_m(x) \| \leq \varepsilon$,
and so
$\| f_n(x) - f(x) \| \leq \varepsilon$ (limits preserve order);
$\Rightarrow \| f_n - f \|_\infty \leq \varepsilon$ since $x \in A$ was arbitrary.
$\Rightarrow f_n \to f$ uniformly.
So, by the previous theorem, it follows that $f \in C(A,W)$,
and so $f_n \to f$ in $C(A,W)$.
$\Rightarrow C(A,W)$ is a Banach space (since (f_n) was an arbitrary Cauchy sequence). ∎

Module 6.1: Partial Derivatives

SCALAR FUNCTION

1. A "scalar function" is any function of the form
$$f: A \to \mathbb{R}, \quad A \subseteq \mathbb{R}^n.$$

2. Note for any $f: A \to \mathbb{R}^m$, $A \subseteq \mathbb{R}^n$, there exist scalar functions $f_1, \dots, f_m : A \to \mathbb{R}$ such that
$$f = (f_1, f_2, \dots, f_m).$$
eg $f: \mathbb{R}^3 \to \mathbb{R}^2$ by $f(x,y,z) = (xze^y, x^2+z^2)$.
Then $f_1(x,y,z) = xze^y$ & $f_2(x,y,z) = x^2+z^2$.
Then $f = (f_1, f_2)$.

i^{th} PARTIAL DERIVATIVE [OF SCALAR FUNCTIONS]:
$\frac{\partial f}{\partial x_i}(a) = f_{x_i}(a)$

1. Let $f: A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^n$. Let $\{e_1, \dots, e_n\}$ be the standard basis for $\mathbb{R}^n$. Then, for $1 \le i \le n$, we define the "i^{th} partial derivative" of f at $a = (a_1, \dots, a_n) \in A$, denoted as $\frac{\partial f}{\partial x_i}(a)$ or $f_{x_i}(a)$, to be equal to
$$f_{x_i}(a) = \frac{\partial f}{\partial x_i}(a) := \lim_{h \to 0} \frac{f(a+he_i) - f(a)}{h},$$
provided the limit exists.

2. We use the notation "$f(x_1, \dots, x_n)$" when talking about functions $f: A \to \mathbb{R}$, $A \subseteq \mathbb{R}^n$.

3. Note that $f_{x_i}(a)$ is the derivative of f at a wrt x_i, treating the other x_j, $j \ne i$ as constants.

4. Moreover, $f_{x_i}(a)$ is the slope of the tangent line to the surface $y = f(x_1, x_2, \dots, x_n)$ which is parallel to e_i.

5. For example, for $f: \mathbb{R}^2 \to \mathbb{R}$, find $\frac{\partial f}{\partial x}(a)$ & $\frac{\partial f}{\partial y}(a)$:
$$f_x(a) = \lim_{h \to 0} \frac{f(a+he_1) - f(a)}{h}$$
$$= \lim_{h \to 0} \frac{f(a_1+h, a_2) - f(a_1, a_2)}{h},$$
and similarly
$$f_y(a) = \lim_{h \to 0} \frac{f(a_1, a_2+h) - f(a_1, a_2)}{h}.$$

6. We also treat $\frac{\partial f}{\partial x_i}$ as a function, and write
$$f_{x_i}(x_1, \dots, x_n) \quad \text{or} \quad \frac{\partial}{\partial x_i} f(x_1, \dots, x_n).$$

EXAMPLE 1: $f: \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = xy^2z + e^{xy}$, FIND PARTIAL DERIVATIVES

Solⁿ. $f_x(x,y,z) = y^2z + ye^{xy}$
$f_y(x,y,z) = 2xyz + xe^{xy}$
$f_z(x,y,z) = xy^2$.

i^{th} PARTIAL DERIVATIVE [OF FUNCTIONS]:
$\frac{\partial f}{\partial x_i}(a) = f_{x_i}(a)$

1. Let $A \subseteq \mathbb{R}^n$ and $f: A \to \mathbb{R}^m$, where $f = (f_1, \dots, f_m)$. For $a \in A$, we define the "i^{th} partial derivative" of f at a, denoted as $\frac{\partial f}{\partial x_i}(a)$ or $f_{x_i}(a)$, to be equal to
$$f_{x_i}(a) = \frac{\partial f}{\partial x_i}(a) := \left(\frac{\partial f_1}{\partial x_i}(a), \dots, \frac{\partial f_m}{\partial x_i}(a)\right) \in \mathbb{R}^m.$$
provided it exists.

EXAMPLE 2: $f: \mathbb{R}^2 \to \mathbb{R}^3$, $f(x,y) = (2x^2y, 4x, e^{xy})$, FIND $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$

Solⁿ. $f_x(x,y) = (4xy, 4, ye^{xy})$; &
$f_y(x,y) = (2x^2, 0, xe^{xy})$.

Module 6.2: Differentiability

DIFFERENTIABLE [FUNCTIONS AT $a \in A$]

1. Let $a \in A \subseteq \mathbb{R}^n$, and let $f : A \to \mathbb{R}^m$.
Then, we say f is "differentiable" at $a \in A$ if

① $a \in \mathrm{Int}(A)$; and

② There exists a $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ such that

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - T(h)}{\|h\|} = 0.$$

(recall $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) = \{T : \mathbb{R}^n \to \mathbb{R}^m \mid T \text{ is linear}\}$)

2. Note that by ①, $f(a+h)$ is defined for small enough h.

OPERATOR NORM [ON $M_{m \times n}(\mathbb{R})$]: $\|A\|_{op}$

Let $A \in M_{m \times n}(\mathbb{R})$.
Then, the "operator norm" of A, denoted as "$\|A\|_{op}$", is defined to be equal to

$$\|A\|_{op} = \sup\{\|Ax\| : x \in \mathbb{R}^n, \|x\| = 1\}.$$

$\|Ax\| \leq \|A\|_{op} \|x\|$

Note that for any $A \in M_{m \times n}(\mathbb{R})$ and $x \in \mathbb{R}^n$, we have

$$\|Ax\| \leq \|A\|_{op} \|x\|.$$

Proof. Clear if $x = 0$.
otherwise, see that

$$\left\| \frac{x}{\|x\|} \right\| = 1.$$

$$\Rightarrow \|A\|_{op} \geq \left\|A \frac{x}{\|x\|}\right\| = \frac{\|Ax\|}{\|x\|}.$$

Proof follows. ∎

DIFFERENTIABILITY ⇒ CONTINUITY

Let $a \in A \subseteq \mathbb{R}^n$, and $f : A \to \mathbb{R}^m$.
Then, if f is diff at a, then necessarily f is cts at a.

Proof. f is diff $\Rightarrow \exists T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) \ni$

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - T(h)}{\|h\|} = 0$$

$$\Rightarrow \lim_{h \to 0} \frac{f(a+h) - f(a) - Bh}{\|h\|} = 0 \quad (\text{where } B \in M_{m \times n}(\mathbb{R}) \ni T(x) = Bx)$$

$\Rightarrow$ we can find $\delta > 0 \ni$ if $0 < \|h\| < \delta$, then

$$\left\| \frac{f(a+h) - f(a) - Bh}{\|h\|} \right\| < 1$$

$$\Rightarrow \|f(a+h) - f(a) - Bh\| < \|h\|$$

$$\Rightarrow \|f(a+h) - f(a)\| - \|Bh\| < \|h\|$$

$$\Rightarrow \|f(a+h) - f(a)\| < \|Bh\| + \|h\| \leq \|B\|_{op}\|h\| + \|h\|$$

As $h \to 0$, $\|B\|_{op}\|h\| + \|h\| \to 0$.

$\Rightarrow$ (by ST) $\lim_{h \to 0} f(a+h) = f(a)$.

Letting $x = a+h$,

$\Rightarrow \lim_{x \to a} f(x) = f(a)$. ∎

DIFFERENTIABLE [FUNCTIONS ON OPEN $U \subseteq \mathbb{R}^n$]

Let $U \subseteq \mathbb{R}^n$ be open, and let $f : U \to \mathbb{R}^m$.
Then, we say f is "differentiable" on U if f is differentiable at every point in U.

Module 6.3: Total Derivatives

TOTAL DERIVATIVE [OF f AT a]

Let $a \in A \subseteq \mathbb{R}^n$, and $f: A \to \mathbb{R}^m$.
Then, the "total derivative" of f at a, denoted as "$D_f(a)$", is defined to be the matrix

$$D_f(a) = \left(\frac{\partial f_i}{\partial x_j}(a) \right) \in M_{m \times n}(\mathbb{R}),$$

provided it exists.

f IS DIFFERENTIABLE $\Rightarrow$ "B" $= D_f(a)$

Let $a \in A \subseteq \mathbb{R}^n$, $f: A \to \mathbb{R}^m$.
Suppose f is differentiable at a, so that there exists a $B \in M_{m \times n}(\mathbb{R})$ such that

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - Bh}{\|h\|} = 0.$$

Then necessarily $B = D_f(a)$.

Proof. It suffices to show that

$$b_j = \frac{\partial f}{\partial x_j} = \left(\frac{\partial f_1}{\partial x_j}, \dots, \frac{\partial f_m}{\partial x_j} \right),$$

So let's do so.
Observe that as $t \in \mathbb{R}$, $t \to 0$, hence $te_j \to 0$, where $\{e_1, \dots, e_n\}$ is the std basis for $\mathbb{R}^n$.
Then,

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - Bh}{\|h\|} = 0 \iff \lim_{t \to 0} \frac{f(a+te_j) - f(a) - B(te_j)}{|t|} = 0$$

$$\iff \lim_{t \to 0^+} \frac{f(a+te_j) - f(a)}{t} = Be_j \quad \& \quad \lim_{t \to 0^-} \frac{f(a+te_j) - f(a)}{-t} = -Be_j$$

$$\iff \lim_{t \to 0} \frac{f(a+te_j) - f(a)}{t} = Be_j$$

$$\iff \frac{\partial f}{\partial x_j}(a) = Be_j = b_j,$$

as needed. ∎

In particular, if f is diff at a, then

1. $\frac{\partial f}{\partial x_i}$ exists $\forall 1 \leq i \leq n$; and
2. $\lim_{h \to 0} \frac{f(a+h) - f(a) - D_f(a)h}{\|h\|} = 0$.

GRADIENT [OF $f: A \to \mathbb{R}$ AT a]: $\nabla f(a)$

Let $a \in A \subseteq \mathbb{R}^n$, and $f: A \to \mathbb{R}$.
Then, the "gradient" of f at a, denoted as $\nabla f(a)$, is defined to be equal to

$$\nabla f(a) = D_f(a) = \left(\frac{\partial f}{\partial x_1}(a), \dots, \frac{\partial f}{\partial x_n}(a) \right).$$

Module 6.4: Continuous Partials

$U \subseteq \mathbb{R}^n$ OPEN, $f: U \to \mathbb{R}$; $\frac{\partial f}{\partial x_j}$ EXISTS $\forall 1 \leq j \leq n$ AND IS CTS AT $a \in U$ $\Rightarrow$ f IS DIFF AT a

1 Let $U \subseteq \mathbb{R}^n$ be open, and let $f: U \to \mathbb{R}$.
Suppose, for some $a \in U$, that $\frac{\partial f}{\partial x_j}$ exists on U
and is cts at a for each $1 \leq j \leq n$.
Then necessarily f is diff at a.

Proof. Let $a = (a_1, \dots, a_n)$. Since U is open, $\exists r > 0 \ni B_r(a) \subseteq U$.
Then, for any $h = (h_1, \dots, h_n) \neq 0 \ni a+h \in B_r(a)$, we have that

$$\begin{aligned} f(a+h) - f(a) &= f(a_1+h_1, \dots, a_n+h_n) - f(a_1, \dots, a_n) \\ &= f(a_1+h_1, \dots, a_n+h_n) - f(a_1, a_2+h_2, \dots, a_n+h_n) \\ &\quad + f(a_1, a_2+h_2, \dots, a_n+h_n) - f(a_1, a_2, a_3+h_3, \dots, a_n+h_n) \\ &\quad + \dots \\ &\quad + f(a_1, \dots, a_{n-1}, a_n+h_n) - f(a_1, \dots, a_n). \end{aligned}$$

By the single variable MVT on x_i, $\forall 1 \leq i \leq j$, $\exists c_j$ b/w $a_j, a_j+h_j \ni$

$$\frac{f(a_1, \dots, a_{j-1}, a_j+h_j, \dots, a_n+h_n) - f(a_1, \dots, a_j, a_{j+1}+h_{j+1}, \dots, a_n+h_n)}{a_j + h_j - a_j}$$

$$= \frac{\partial f}{\partial x_j}(a_1, \dots, a_{j-1}, c_j, a_{j+1}+h_{j+1}, \dots, a_n+h_n).$$

Thus

$$f(a+h) - f(a) = \sum_{j=1}^{n} h_j \frac{\partial f}{\partial x_j}(a_1, \dots, a_{j-1}, c_j, a_{j+1}+h_{j+1}, \dots, a_n+h_n).$$

Next, for $1 \leq j \leq n$, let

$$\delta_j = \frac{\partial f}{\partial x_j}(a_1, \dots, a_{j-1}, c_j, a_{j+1}+h_{j+1}, \dots, a_n+h_n),$$

and $\delta = (\delta_1, \dots, \delta_n)$, so that

$$f(a+h) - f(a) - \nabla f(a) \cdot h = h\delta.$$

Since each partial is cts at a, as $h \to 0$, each $\delta_j \to 0$, and so $\delta \to 0$ in $\mathbb{R}^n$. Thus

$$\begin{aligned} 0 &\leq \lim_{h \to 0} \frac{|f(a+h) - f(a) - \nabla f(a) \cdot h|}{\|h\|} \\ &= \lim_{h \to 0} \frac{|\delta \cdot h|}{\|h\|} \quad (\cdot \text{ is the dot product}) \\ &\leq \lim_{h \to 0} \frac{\|\delta\| \|h\|}{\|h\|} \quad (\text{by Cauchy-Schwartz}) \\ &= 0. \end{aligned}$$

Hence

$$\lim_{h \to 0} \frac{|f(a+h) - f(a) - \nabla f(a) \cdot h|}{\|h\|} = 0,$$

and so

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - \nabla f(a) \cdot h}{\|h\|} = 0,$$

showing f is diff at a. ∎

2 Note that the **converse** is **not necessarily true**!

eg Let $f: \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right), & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

We know f is diff at $(0,0)$ (using the Theorem).
But

$$f_x(x,y) = 2x\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) - \cos\left(\frac{1}{\sqrt{x^2+y^2}}\right)\frac{x}{\sqrt{x^2+y^2}} \quad \forall (x,y) \neq (0,0).$$

See that $(\frac{1}{n}, 0) \to (0,0)$, but

$$f_x\left(\frac{1}{n}, 0\right) = \frac{2}{n}\sin(n) - \cos(n)$$

diverges, so f_x is not cts at $(0,0)$. #