MATH 146 Personal Notes

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Chapter 1: Vector Spaces (S1.1)

Let $\mathbb{F}$ be a field.
Then, we say V is a "vector space" over $\mathbb{F}$ if there exists

① an addition $+: (V \times V) \to V$ by $+(x,y) = x+y$; and

② a scalar multiplication $\times: (\mathbb{F} \times V) \to V$ by $\times(a,x) = ax$;

and the following conditions hold:

① V is an abelian group with respect to addition;

② $1_{\mathbb{F}} x = x \quad \forall x \in V$;

③ multiplication is associative; ie $a(bx) = (ab)x \quad \forall a,b \in \mathbb{F},\ x \in V$;

and

④ the left and right distributive laws hold;
ie $a(x+y) = ax + ay$ and $(a+b)x = ax+bx \quad \forall a,b \in \mathbb{F},\ x \in V$. (D2)

$\mathbb{F}^n$ IS A VECTOR SPACE OVER $\mathbb{F}$ (E2(1))

R_1 We can show that the Cartesian product

$$\mathbb{F}^n = \{(a_1, a_2, \dots, a_n) : a_i \in \mathbb{F} \ \forall i \in \{1,2,\dots,n\}\}$$

is a vector space over $\mathbb{F}$ with respect to the addition operation

$$(a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n) = (a_1+b_1, a_2+b_2, \dots, a_n+b_n)$$

and multiplication operation

$$c\,(a_1, a_2, \dots, a_n) = (ca_1, ca_2, \dots, ca_n).$$

Proof. This follows from verifying each of the conditions above. ☒

R_2 Note that we generally say "the vector space $\mathbb{F}^n$" to refer to the vector space $\mathbb{F}^n$ over $\mathbb{F}$. (R3(4))

COLUMN VECTOR NOTATION (E2(2))

Note that we can also write elements of $\mathbb{F}^n$ as "column vectors"; ie of the form

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \quad \text{where } a_1, a_2, \dots, a_n \in \mathbb{F}.$$

$\mathbb{Q}^n$ IS A VECTOR SPACE OVER $\mathbb{Q}$, $\mathbb{R}^n$ IS A VECTOR SPACE OVER $\mathbb{R}$, & $\mathbb{C}^n$ IS A VECTOR SPACE OVER $\mathbb{C}$ (R3(1))

We can show

① $\mathbb{Q}^n$ is a vector space over $\mathbb{Q}$;

② $\mathbb{R}^n$ is a vector space over $\mathbb{R}$; and

③ $\mathbb{C}^n$ is a vector space over $\mathbb{C}$.

Proof. This directly follows from the fact that $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are fields (MATH 145), and substituting the respective fields into the above lemma. ☒

$\mathbb{R}^n$ IS A VECTOR SPACE OVER $\mathbb{Q}$, & $\mathbb{C}^n$ IS A VECTOR SPACE OVER $\mathbb{R}$ (R3(2))

Moreover, we can also show that

① $\mathbb{R}^n$ is a vector space over $\mathbb{Q}$; and

② $\mathbb{C}^n$ is a vector space over $\mathbb{R}$.

Proof. Essentially, this stems from the fact that we can "multiply" vectors in $\mathbb{R}^n$ by scalars in $\mathbb{Q}$, and vectors in $\mathbb{C}^n$ by scalars in $\mathbb{R}$.
The formal proof is left to the reader. ☒

KEY

- S : section
- D : definition
- R : remark
- E : example
- T : theorem
- L : lemma
- C : corollary

MATRICES (D3(1))

R_1 Let $\mathbb{F}$ be a field, and $m, n \in \mathbb{Z}^+$.
Then, we say A is an "$m \times n$ matrix" with entries from $\mathbb{F}$ if it is of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \\ a_{m1} & a_{m2} & & a_{mn} \end{pmatrix},$$

where $a_{ij} \in \mathbb{F} \quad \forall i \in \{1,2,\dots,m\},\ j \in \{1,2,\dots,n\}$.

R_2 Alternatively, we can represent A via the notation

$$A = (a_{ij}), \quad \text{ie } \{1,2,\dots,m\},\ j \in \{1,2,\dots,n\}.$$

ij-ENTRY OF A MATRIX (D3(2))

Given a $m \times n$ matrix A, the "ij-entry" of A, or "a_{ij}", is defined to be the entry in A at the i^{th} row and j^{th} column.

ZERO MATRIX (D3(3))

The "$m \times n$ zero matrix", or more simply the "zero matrix", denoted as "O", is defined to be

$$O = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & 0 \end{pmatrix} \Bigg\updownarrow m,$$

$$\xleftrightarrow{\quad n \quad}$$

ie the $m \times n$ matrix where which entry equals 0.

MATRIX EQUALITY (D3(4))

We say two matrices A and B are equal if and only if $a_{ij} = b_{ij} \quad \forall i \in \{1,2,\dots,m\},\ j \in \{1,2,\dots,n\}$.

MATRIX ADDITION (D3(5))

Let A and B be $m \times n$ matrices with entries from some field $\mathbb{F}$.
Then, the "addition" of A and B, denoted by "$A+B$", is defined to be the matrix where

$$(A+B)_{ij} = a_{ij} + b_{ij} \quad \forall i \in \{1,2,\dots,m\},\ j \in \{1,2,\dots,n\}.$$

MATRIX SCALAR MULTIPLICATION (D3(6))

Let A be a $m \times n$ matrix with entries from some field $\mathbb{F}$, and $c \in \mathbb{F}$ be arbitrary.
Then the "scalar multiplication" of A by c, denoted by "cA", is defined to be the matrix where

$$(cA)_{ij} = c(a_{ij}) \quad \forall i \in \{1,2,\dots,m\},\ j \in \{1,2,\dots,n\}.$$

SPACE OF $m \times n$ MATRICES (E3)

R_1 Let $\mathbb{F}$ be a field.
Then the "space of all $m \times n$ matrices" with entries from $\mathbb{F}$, denoted by "$M_{m \times n}(\mathbb{F})$", is defined to be the set of all $m \times n$ matrices with entries from $\mathbb{F}$.

R_2 Note that $M_{m \times n}(\mathbb{F})$ is a vector space over $\mathbb{F}$ with respect to the matrix addition and scalar multiplication operations.

Proof. This follows from verifying each of the conditions in D2. ☒

FUNCTION SPACES (E4)

1. Let the set $D \neq \emptyset$ be arbitrary, and let $\mathbb{F}$ be a field.
Then the "space of all functions" from D to $\mathbb{F}$, denoted by "$\mathbb{F}^D$", is defined to be the set of all functions of the form $f : D \to \mathbb{F}$.

2. Similarly, we can show that $\mathbb{F}^D$ is a vector space over $\mathbb{F}$ with respect to the operations of function addition

$$(f+g)(x) := f(x) + g(x) \quad \forall f, g \in \mathbb{F}^D, \; x \in \mathbb{F}$$

and function scalar multiplication

$$(cf)(x) := cf(x) \quad \forall f \in \mathbb{F}^D, \; x, c \in \mathbb{F}.$$

Proof. Similar strategy to E3: verify each condition in D2 holds.

POLYNOMIALS (D4)

SET OF ALL POLYNOMIALS OF DEGREE AT MOST n (D4 (1))

1. Let $\mathbb{F}$ be a field.
Then, we denote $P_n(\mathbb{F})$ to be the set of all polynomials with coefficients from $\mathbb{F}$ and of degree at most n; ie

$$P_n(\mathbb{F}) = \left\{ \sum_{i=0}^{n} a_n x^n \; : \; a_j \in \mathbb{F} \;\; \forall j \in \{0, 1, \dots, n\} \right\}.$$

POLYNOMIAL SPACES (D4 (2))

1. Let $\mathbb{F}$ be a field.
Then, we denote "$\mathbb{F}[x]$" to be the set of all polynomials with coefficients from $\mathbb{F}$; ie

$$\mathbb{F}[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \; : \; a_j \in \mathbb{F} \;\; \forall j \in \mathbb{N} \cup \{0\} \right\}.$$

2. Then, we can show that $\mathbb{F}[x]$ is a vector space over $\mathbb{F}$ with respect to the operations of polynomial addition

$$(f+g)(x) = \sum_{i=0}^{n} (a_n + b_n) x^n \quad \forall f, g \in \mathbb{F}[x]$$

and polynomial scalar multiplication

$$c f(x) = \sum_{i=0}^{n} (c a_n) x^n \quad \forall f \in \mathbb{F}[x], \; c \in \mathbb{F}.$$

Proof. Similar strategy to E4.

BASIC PROPERTIES OF VECTOR SPACES (S1.2)

CANCELLATION PROPERTY FOR VECTOR ADDITION (T1.1)

Let V be a vector space.
Suppose there exists some $x, y, z \in V$ such that $x + z = y + z$.
Then necessarily $x = y$.

Proof. Note that

$$\begin{aligned} x &= x + 0 \\ &= x + (z + (-z)) \\ &= (x+z) + (-z) \\ &= (y+z) + (-z) \\ &= y + (z + (-z)) \end{aligned}$$

and so $x = y$, as required. ∎

UNIQUENESS OF THE ZERO VECTOR IN VECTOR SPACES (C1.1.1 (1))

Let V be a vector space.
Suppose $0_1, 0_2 \in V$ are both zero vectors.
Then necessarily $0_1 = 0_2$.

Proof. This follows from the fact that V is an abelian group under addition. ∎

UNIQUENESS OF ADDITIVE INVERSES IN VECTOR SPACES (C1.1.1 (2))

Let V be a vector space.
Then for any $x \in V$, there exists one and only one vector $y \in V$ that satisfies $x + y = 0$.

Proof. This also follows from the fact that V is an abelian group under addition. ∎

$0x = 0 \;\; \forall x \in V$ (T1.2 (1))

Let V be a vector space over some field $\mathbb{F}$, and let 0 be the additive identity of $\mathbb{F}$.
Then, for any $x \in V$, necessarily $0 \cdot x = 0$.

Proof. This, again, follows from the fact that V is an abelian group under addition. ∎

$a0 = 0 \;\; \forall a \in \mathbb{F}$ (T1.2 (2))

Let V be a vector space over some field $\mathbb{F}$, and let 0 be the zero vector of V.
Then, for any $a \in \mathbb{F}$, necessarily $a\,0 = 0$.

Proof. This, again, follows from the fact that V is an abelian group under addition. ∎

$(-a)x = -(ax) = a(-x) \;\; \forall a \in \mathbb{F}, \; x \in V$ (T1.2 (3))

Let V be a vector space over some field $\mathbb{F}$, and let $a \in \mathbb{F}$, $x \in V$ be arbitrary.
Then necessarily $(-a)x = -(ax) = a(-x)$.

Proof. Proof is similar to the analog of this statement for rings (MATH 145). ∎

SUBSPACES (S1.3)

Let V be a vector space over some field $\mathbb{F}$.
Then we say the subset $W \subseteq V$ is a "subspace" of V if

① $W \neq \emptyset$;
* we usually check whether $0 \in W$ to verify this claim. (R4)

② If $x \in W$ and $y \in W$, then $(x+y) \in W$; and

③ If $c \in \mathbb{F}$ and $x \in W$, then $cx \in W$. (D6)

SUBSPACES ARE VECTOR SPACES OVER $\mathbb{F}$ WITH RESPECT TO THE OPERATIONS OF V (T1.3)

Let W be a subspace of a vector space V over some field $\mathbb{F}$.
Then W is also a vector space over $\mathbb{F}$ under the operations of V restricted to W.

Proof. This follows from verifying the conditions in D2, taking into account the properties of subspaces. ▨

$\{0\}$ AND V ARE SUBSPACES OF V (E8 (1))

Let V be a vector space.
Then $\{0\}$ and V itself are always subspaces of V.

Proof. $\{0\}$ is vacuously a subspace, and V is trivially a subspace. ▨

$P_2(\mathbb{R})$ IS A SUBSPACE OF $\mathbb{R}[x]$ (E8 (2))

We can show that $P_2(\mathbb{R})$ is a subspace of $\mathbb{R}[x]$.

Proof. This stems from the fact that:
- $P_2(\mathbb{R}) \subset \mathbb{R}[x]$ by definition;
- $0 \in P_2(\mathbb{R})$; and
- $P_2(\mathbb{R})$ is closed under the addition & scalar multiplication defined on $\mathbb{R}[x]$. ▨

$\{(a_{ij}) \in M_{n\times n}(\mathbb{F}) \mid \sum_{k=1}^{n} a_{kk} = 0\}$ IS A SUBSPACE OF $M_{n\times n}(\mathbb{F})$ (E8 (3))

We can show that the set $\{(a_{ij}) \in M_{n\times n}(\mathbb{F}) \mid \sum_{k=1}^{n} a_{kk} = 0\}$ is a subspace of $M_{n\times n}(\mathbb{F})$, where $n \in \mathbb{N}$ is arbitrary.

Proof. Similar proof to the above.

$\{(a_{ij}) \in M_{n\times n}(\mathbb{F}) \mid \sum_{k=1}^{n} a_{kk} = 1\}$ IS <u>NOT</u> A SUBSPACE OF $M_{n\times n}(\mathbb{F})$ (E8 (4))

We can show the set $\{(a_{ij}) \in M_{n\times n}(\mathbb{F}) \mid \sum_{k=1}^{n} a_{kk} = 1\}$ is <u>not</u> a subspace of $M_{n\times n}(\mathbb{F})$.

Proof. Let $a, b \in \{(a_{ij}) \in M_{n\times n}(\mathbb{F}) \mid \sum_{k=1}^{n} a_{kk} = 1\}$ be arbitrary.
Then, notice that

$$\begin{aligned}\sum_{k=1}^{n} (a+b)_{kk} &= \sum_{k=1}^{n} (a_{kk} + b_{kk}) \\ &= 1+1 \\ &= 2,\end{aligned}$$

So that $a+b \notin \{(a_{ij}) \in M_{n\times n}(\mathbb{F}) \mid \sum_{k=1}^{n} a_{kk} = 1\}$. ▨