Aggregate Uncertainty, HANK, and the ZLB*

Alessandro Lin†
Bank of Italy

Marcel Peruffo‡
University of Sydney

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Abstract

We propose a novel methodology for solving Heterogeneous Agents New Keynesian (HANK) models with aggregate uncertainty and the Zero Lower Bound (ZLB) on nominal interest rates. Our solution strategy combines the sequence-state Jacobian methodology in Auclert et al. (2021) with a tractable structure for aggregate uncertainty by means of a two-regimes shock structure. Using our methodology, we show that: 1) in the presence of the ZLB, a dichotomy emerges between the economy's impulse responses under aggregate uncertainty against the deterministic case; 2) aggregate uncertainty amplifies downturns at the ZLB, and household heterogeneity increases the strength of this amplification; 3) the impact of forward guidance is stronger when there is aggregate uncertainty.

Keywords: Monetary Policy, New Keynesian Model, HANK, Liquidity Traps, Zero Lower Bound, Computational Methods.

JEL Classification: D14, E44, E52, E58

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†Bank of Italy, Directorate General for Economics, Statistics and Research. E-mail: alessandro.lin@bancaditalia.it. Website: https://sites.google.com/view/alessandro-lin

‡University of Sydney, Faculty of Arts and Social Sciences. E-mail: marcel.peruffo@sydney.edu.au. Website: https://sites.google.com/view/marcelperuffo.

1 Introduction

The two most recent recessions in the United States were characterized by: 1) a dramatic spike in aggregate uncertainty; 2) a sharp increase in the unemployment rate above its natural level; and 3) the collapse of monetary policy rates to the Zero Lower Bound (ZLB). The literature has well documented that measures of aggregate uncertainty (Bloom et al., 2018; Bloom, 2014) and of idiosyncratic income risk increase during recessions (Guvenen, Ozkan and Song, 2014; Shimer, 2005). At the same time, it has been shown, both theoretically and empirically, that there are strong interactions between aggregate uncertainty and the ZLB (Basu and Bundick, 2016, 2017; Caggiano, Castelnuovo and Pellegrino, 2017). Yet there is little work in understanding the interactions between the ZLB and uncertainty both at macro and micro levels. This literature is still at its dawn, particularly because solving models that display those features is challenging.[1] Our work fills this gap.

In this paper, we investigate the macroeconomic interactions between aggregate uncertainty, heterogeneity at the micro level, and the ZLB. We first propose a novel solution strategy for Heterogeneous Agents New Keynesian models (HANK) with aggregate uncertainty and occasionally binding constraints at the aggregate level.[2] We then employ our methodology to quantify how the effects of a negative demand shock are amplified due to the combination of uncertainty and monetary policy inaction. We find that such amplification is much stronger in heterogeneous-agents economies, relative to a representative-agent setting.

We extend the method by Eggertsson et al. (2021) to accommodate heterogeneity at the micro level and, at the same time, deal with the aggregate curse of dimensionality problems arising due to aggregate uncertainty in a tractable way. In this setup we assume

[1] Recent developments in this direction are Fernández-Villaverde et al. (2023) and Kase, Melosi and Rottner (2022), who use neural networks techniques, and Schaab (2020), who develops an adaptive grid methodology.

[2] Although our applications concerns the ZLB, the methodology can be easily used in the presence of aggregate nonlinearities other than the ZLB. For instance, nonlinear Phillips curves in the presence of inflationary shocks (Benigno and Eggertsson, 2023; Gitti, 2023), downward wage rigidities (Eggertsson, Mehrotra and Robbins, 2019), and leverage constraints among others.

that the economy is subject to a shock in the form a two-states Markov process with one absorbing state. In particular, the economy begins in the steady state and is hit by a shock which pushes it into a "bad" state. From then on it can either remain in there, or enter a (perfect-foresight) path back to the steady state. The proposed structure allows us to represent the equilibrium as a *finite* number of sequences, which we group under two regimes. In the first, labeled *TS*, the shock has not yet subsided and, thus, there is aggregate uncertainty. The second regime, labeled *PF*, consists of a series of possible paths, distinct from each other, the economy can follow after the shock subsides.

We then develop a method that iteratively solves the equilibrium in one regime at the time, taking the other as given. First, we guess the aggregate state variables (e.g.: the outstanding amount of public debt) and the sequence of distributions of households over individual states in the *TS* regime. These serve as *initial conditions* for each perfect-foresight branch. To solve for the equilibrium in each branch, we introduce a novel way to deal with the perturbations caused by changes in the initial conditions, essentially treating them as exogenous. Once those are accounted for, we solve for the equilibrium in each branch computing heterogeneous-agents Jacobians as in Auclert et al. (2021) and dealing with occasionally binding constraints as in Guerrieri and Iacoviello (2015).

The second step takes the values of all *forward-looking* equilibrium variables in the *PF* regime, including households' (expected) value functions, and solves for the equilibrium in the *TS* regime. Here, we again devise a novel way to deal with perturbations caused by changes in forward-looking variables, treating those as exogenous. We then solve for the equilibrium adapting the heterogeneous-agent Jacobians in Auclert et al. (2021) to account for aggregate uncertainty, and again dealing with occasionally binding constraints as in Guerrieri and Iacoviello (2015). The resulting set of state variables is then fed into the first step, until a fixed point is achieved.

The key advantage of our methodology is that, unlike existing projection methods (Schaab, 2020; Fernández-Villaverde et al., 2023), we solve the model in the space of sequences. Thus, problems that would arise due to the curse of dimensionality are muted. As a result, our method can be applied in models with a rich set of state variables, including multidimensional distributions of households over idiosyncratic states, such as Bayer et al. (2019) (two-asset HANK), Birinci et al. (2022) (job ladder), and Kekre (2022) (rich description of unemployment insurance policies), to name a few.

One potential caveat is that the two-states nature of our shock could be restrictive. We believe that this concern is mitigated by two facts. First, it can in fact accommodate arbitrary paths for exogenous variables, with the only restriction being that the probability of exiting the "bad" state needs to be constant.[3] Second, we argue that this framework can represent how economic agents interpret certain macroeconomic shocks: the economy undergoes a recession, yet there is an anticipation of eventual recovery. For instance, during Covid in 2020, uncertainty surrounded vaccine availability, yet there was a consensus that once they were accessible, economic recovery would ensue.

Our second contribution is to investigate the interaction between aggregate uncertainty and the ZLB in heterogeneous-agents economies. To do so, we consider a standard "one-asset HANK" model calibrated to the US and study the effects of a discount factor shock that depresses aggregate demand and that follows our proposed two-states Markov structure. As its duration is uncertain, we refer to it as the stochastic shock. The shock leads the central bank to reduce nominal rates down to the ZLB. We then compare it with its "deterministic counterpart", i.e. a shock whose magnitude matches the average of the stochastic one, but whose path is certain and known by economic agents. Finally, we quantify the *amplification due to aggregate uncertainty*, which we measure as the difference between the average impact of the stochastic shock and that of its deterministic counterpart (measured in present-discounted terms).

We show that if the central bank were unconstrained in its capacity to determine short-term nominal rates, the impact of aggregate uncertainty is nearly trivial: the average impact of the stochastic shock is essentially identical to the deterministic case, a result known as certainty equivalence (Boppart, Krusell and Mitman, 2018). However, when the ZLB can be binding, the average decline in output caused by the stochastic shock is more than twice as large as in the deterministic case. This result indicates that the

[3]We may however – as usual in models of this class – run into stability problems when choosing those values. See Ascari and Mavroeidis (2022) and Holden (2023) for a discussion on existence and uniqueness of equilibrium at the ZLB under perfect foresight.

interactions between aggregate uncertainty and nonlinearities such as the ZLB might be sizable in amplifying and prolonging recessions.

What is the role of household heterogeneity in determining this amplification? To answer this question, we repeat the experiment in a comparable Representative Agent (RANK) economy. When we introduce the ZLB, we find that the output loss is also significantly larger with the stochastic shock, compared to the deterministic scenario. However, the amplification in HANK is about twice as large as in RANK. Thus, we conclude that household heterogeneity amplifies the effect of aggregate uncertainty at the ZLB.

Our results can be explained by two facts. First, they are linked to the presence of both high-MPC and forward-looking (locally unconstrained) individuals in heterogeneous-agents economies. As argued by Kaplan, Moll and Violante (2018), income effects are paramount in HANK: for a given aggregate income fall, the overall consumption response is larger in HANK, as opposed to RANK. Second, the nonlinearity introduced by the ZLB interacts with the aggregate uncertainty. That is, the *average effect of the stochastic shock* differs significantly from the *effect of the average shock*, implying a larger aggregate income decline. Taken together, these two facts explain the amplifications we documented.

Finally, to illustrate how our solution method can be used to study several policy alternatives, we consider the impact of forward guidance. We find that it is more effective under the stochastic shock against the deterministic scenario, suggesting that this policy can be particularly strong in uncertain environments. The reason is that forward guidance keeps interests rate at the lower bound regardless of the shock realization, essentially removing the regime uncertainty and, thus, its consequences, in the first place.

Related Literature Our paper is most closely connected to a set of works developing solution methods for HANK economies in the presence of both nonlinearities at the aggregate levels and aggregate uncertainty. This literature attempts to depart from first-order perturbation solutions (e.g. Reiter, 2009; Boppart, Krusell and Mitman, 2018; Winberry, 2018; Bayer and Luetticke, 2020) to be able to capture higher-order nonlinear effects, such as those arising from aggregate uncertainty.

The closest papers to ours are Fernández-Villaverde, Hurtado and Nuño (2023), Kase, Melosi and Rottner (2022), and Fernández-Villaverde et al. (2023), which use different machine learning techniques to solve heterogeneous-agents models with aggregate non-linearities and uncertainty. Our work is also related to Schaab (2020), who uses an adaptive sparse grid approach. Relative to those, our key advantage is that we solve the model in the space of sequences. This, in turn, allows us to address issues concerning the curse of dimensionality. In addition, in our main application, as opposed to Schaab (2020), we show that the strong interaction between aggregate uncertainty and the ZLB is independent of cyclical earnings risk.

Finally, our solution methodology is also related to the Extended Path Algorithm (Adjemian and Juillard, 2013), where, contrary to our setup, the uncertainty only lasts for a few periods before the economy reverts to a perfect-foresight path towards the steady state. We conjecture that some of the techniques we introduce in this work, particularly those related to heterogeneous-agents Jacobians, can be adapted to the Extended Path Algorithm, and leave that for future work.[4]

One last word of caution: we refrain from asserting superiority over any of the aforementioned methods. Determining the most suitable approach for a specific question has to be done on a case-by-case basis. Our method has the advantages of being simple, relatively tractable, and sufficiently flexible to address a specific type of uncertainty frequently used in the literature.

2 A Simple Model

This section illustrates the key interaction between a tractable stochastic shock and the ZLB in the context of the textbook Representative Agent New Keynesian model. Aggregate uncertainty displays a simple a simple two-states structure.

Environment. Consider the textbook New Keynesian model (Galí, 2015; Woodford, 2003). The economy is populated by a representative agent who makes standard intertemporal consumption-savings decisions to maximize her expected lifetime log-utility, with discount factor β_t. She can take a non-negative position in the liquid bond that pays a risk-

[4]Our method is also closely related to that of Bigio, Nuño and Passadore (2019), who study debt maturity management. Their shock structure is similar to ours, with the difference that the economy begins at a "risky" steady state. It is worth mentioning that our method can be easily adapted to cases where the economy initially features initial conditions for state variables that differ from their the steady state values.

less interest rate and is in zero supply. Prices are fully rigid. There is a central bank that chooses the gross nominal interest rate R_t following a simple interest rate rule that reacts to output Y_t and is subject to a lower bound $\underline{R}$.

Equilibrium. The equilibrium at any point in time is characterized by an aggregate Euler equation and the interest rate rule:

$$Y_t^{-1} = \beta_t R_t \mathbb{E}_t Y_{t+1}^{-1} = \frac{\beta_t R_t}{\beta R_{ss}} \mathbb{E}_t Y_{t+1}^{-1}, \tag{1}$$

$$R_t = \max\left\{\underline{R}, R_{ss} Y_t^{\phi}\right\}, \tag{2}$$

where $\mathbb{E}_t$ is the expectation operator, ϕ governs the reactivity of the central bank and is assumed to satisfy the Taylor principle, β is the steady-state value of the discount factor, and $R_{ss} = \frac{1}{\beta}$ is the steady-state gross interest rate.[5]

For a given value of expected future marginal utility $\mathbb{E}_t Y_{t+1}^{-1}$, the solution is as follows:

$$Y_t = \begin{cases} \left(\frac{\beta_t}{\beta} \mathbb{E}_t Y_{t+1}^{-1}\right)^{-\frac{1}{1+\phi}} & \text{if } \beta_t \leq \beta \left(\frac{R_{ss}}{\underline{R}}\right)^{\frac{1+\phi}{\phi}} \left(\mathbb{E}_t Y_{t+1}^{-1}\right)^{-1} \\ \left(\frac{\beta_t}{\beta} \frac{\underline{R}}{R_{ss}} \mathbb{E}_t Y_{t+1}^{-1}\right)^{-1} & \text{otherwise} \end{cases}, \tag{3}$$

$$R_t = \begin{cases} R_{ss} \left(\frac{\beta_t}{\beta} \mathbb{E}_t Y_{t+1}^{-1}\right)^{-\frac{\phi}{1+\phi}} & \text{if } \beta_t \leq \beta \left(\frac{R_{ss}}{\underline{R}}\right)^{\frac{1+\phi}{\phi}} \left(\mathbb{E}_t Y_{t+1}^{-1}\right)^{-1} \\ \underline{R} & \text{otherwise} \end{cases}. \tag{4}$$

Note that the higher the expected future marginal utility, the larger the current recession. We will therefore focus on the effects of aggregate uncertainty on the expected future marginal utility and then move to the actual effects on current output.

A Stochastic Shock. We consider the following chain of events. The economy begins at $t = 0$ and households enter with no wealth. Households know their discount factor $\beta_0 \geq \beta$ and $\beta_t = \beta$ for any $t > 1$. They also know that there is a probability μ that the discount factor at $t = 1$ will be $\beta_1 = \beta_{1L} > \beta$, $\beta_1 = \beta$ otherwise.

The assumption on the stochastic structure allows us to divide into only two possible paths that the economy can follow in the aggregate: 1) the history in which at $t = 1$ the discount factor is back to its stationary level, $\beta_1 = \beta$, or 2) the one in which agents are more patient with $\beta_1 = \beta_{1L}$. We will refer to the first case as *contingency 1*, indicating the

[5] The notation $\underline{R}$ generalizes the possibility of an effective lower bound (ELB) as opposed to the ZLB.

time at which the shock dissipates. Similarly, the second history is denoted as *contingency 2* because the discount factor goes back to its steady-state level at $t = 2$.

Solution - Without the ZLB. The model is purely forward looking and simple enough that the solution at $t = 2$ is the steady state, meaning that, in both contingencies $Y_2 = Y_{ss} = 1$ and $R_2 = R_{ss}$.

Now consider the solution at a generic $t = 1$ under the assumption that the lower bound on interest rates does not exist (i.e. $\underline{R} = -\infty$). It is represented by function of the discount factor at $t = 1$ only. We write it in terms of marginal utility:

$$Y_1^{-1} = \left(\frac{\beta_1}{\beta}\right)^{\frac{1}{1+\phi}}.$$

Output at $t = 0$ is in turn given by:

$$Y_0^{-1} = \left(\frac{\beta_0}{\beta}\right)^{\frac{1}{1+\phi}} \left[\mu Y_{1L}^{-1} + (1-\mu)\, Y_{ss}^{-1}\right] = \left(\frac{\beta_0}{\beta}\right)^{\frac{1}{1+\phi}} \underbrace{\left[\mu \left(\frac{\beta_{L1}}{\beta}\right)^{\frac{1}{1+\phi}} + (1-\mu)\right]}_{\mathbb{E}_0 Y_1^{-1}}, \tag{5}$$

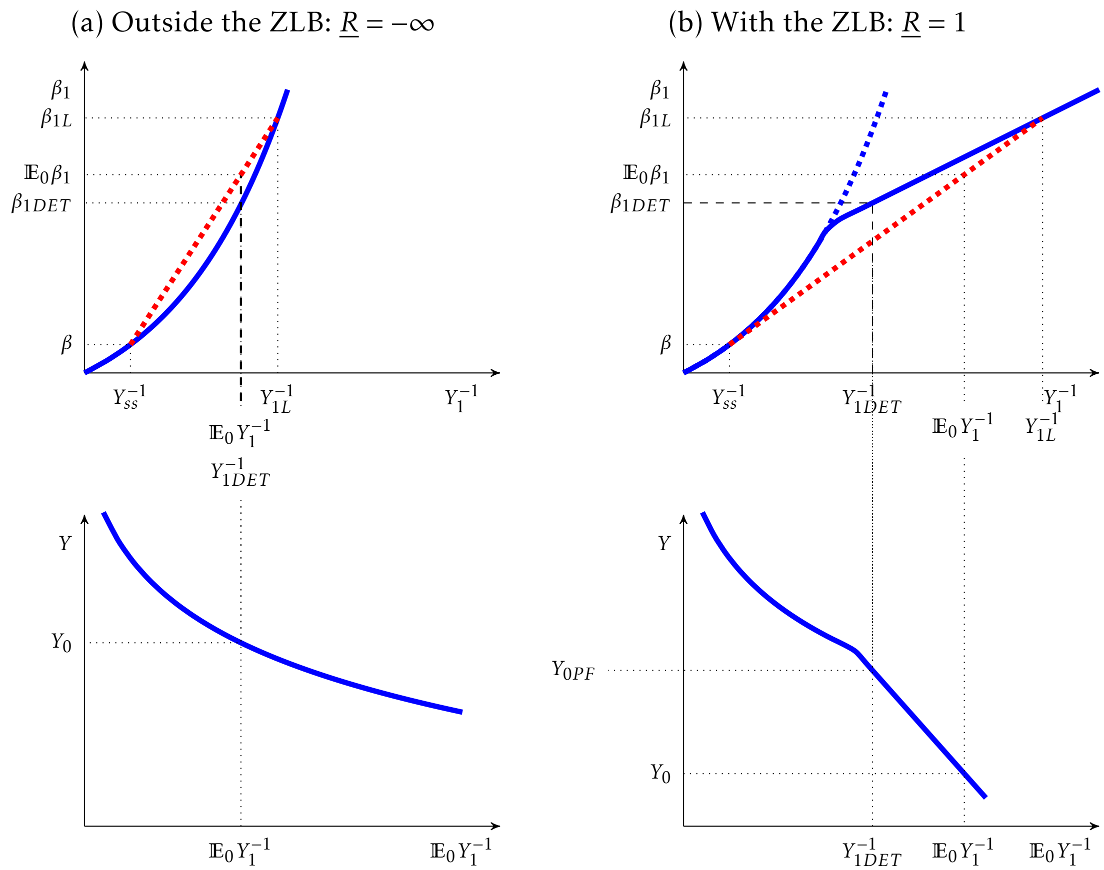
where Y_{1L} denotes the output at $t = 1$ in *contingency 2*. The top-left panel in Figure 1 shows the equilibrium marginal utilities at $t = 1$ as a function of the discount factor β_1. A larger the discount factor leads to a larger marginal utility, in turn implying a larger output loss. The plot also reports the expected future marginal utility on the red dotted line, corresponding to a linear combination between the marginal utilities in the two contingencies. Finally, expected marginal utility at $t = 1$ on the top left is mapped into output at $t = 0$, shown in the bottom-left, via the Euler Equation. Higher expected marginal utilities at $t = 1$ imply lower output at $t = 0$.

The Deterministic Counterpart. Our goal is to study the effects of a demand shock with uncertainty, relative to the deterministic case. To that end, we consider a similar economy whose only difference is the discount factor at $t = 1$ will be β_{1DET} with probability 1, such that the effect at $t = 0$ is the same *in the absence of the ZLB*, meaning that the expected future marginal utilities are equalized:

$$\mu Y_{1L}^{-1} + (1-\mu)\, Y_{ss}^{-1} = Y_{1DET}^{-1} \qquad \Longrightarrow \qquad \mu \left(\frac{\beta_{L1}}{\beta}\right)^{\frac{1}{1+\phi}} + (1-\mu) = \left(\frac{\beta_{1DET}}{\beta}\right)^{\frac{1}{1+\phi}}. \tag{6}$$

The deterministic case is illustrated by the dashed line in the top-left panel of Figure 1. Note that if, when we introduce the ZLB, the impact of the shock is larger in the uncertain environment than in the deterministic case, we can say that uncertainty is amplifying the recession. We are now ready to perform this comparison.

Figure 1: Equilibrium in the Simple Model



Notes: The figure shows the equilibrium of the simple model without the ZLB (left column, $\underline{R} = -\infty$) and with the ZLB (right column, $\underline{R} = 1$). The top panels report the equilibria at $t = 1$. The blue solid lines show the relationship between the discount factor β_1 on the y-axis and the corresponding marginal utility Y_1^{-1} on the x-axis. They also report the corresponding expected value $\mathbb{E}_0 Y_1^{-1}$, obtained with a linear combination along the red dotted line. The blue dotted line on the top-right panel is reported for comparison. The bottom panels report the equilibria at $t = 0$. The blue solid lines show the relationship between output Y_0 on the y-axis and expected future marginal utility $\mathbb{E}_0 Y_1^{-1}$ on the x-axis.

Introducing the ZLB. We now show that the introduction of the ZLB generates different responses in the two economies. To this end, we use the same shocks as in the previous exercise but with $\underline{R} = 1$. Once monetary policy becomes inactive, the slope of equation

(3) becomes steeper (in the graph, the slope is flatter, as the axis are inverted), as shown in the top-right panel of Figure 1. Intuitively, the central bank is unable to counteract the initial decline in aggregate demand induced by the shock by cutting interest rates, so the total impact of the shock is larger.

Consider the case in which the ZLB binds at $t = 1$, both under aggregate uncertainty (contingency 2) and in the deterministic case. That leads to increases in both the expected future marginal utility ($\mathbb{E}_0 Y_1^{-1}$) for the stochastic shock and for the future marginal utility Y_{1DET}^{-1} in the deterministic case. This is depicted in Figure 1 on the top and bottom right panels. On the top, the expected marginal utility is larger under uncertainty.[6] Intuitively, households foresee a chance that the economy ends up in a very deep recession (contingency 2), with ensuing large impact on expected marginal utility, and react by reducing consumption today. Alternatively, in the deterministic counterpart, the incoming recession is not as severe, so the reduction in consumption at $t = 0$ is not as strong.

The resulting amplification due to the interaction between the ZLB and aggregate uncertainty can be seen on the bottom-right plot. In the stochastic economic, output Y_0 is lower than its deterministic counterpart. Note that the ZLB also binds at $t = 0$, as illustrated by the kink. This amplification, however, is also present even if monetary policy were active at $t = 0$ (see Figure D.5 in the Appendix).

In conclusion, we have shown that aggregate uncertainty interacts with the ZLB to amplify recessions. The current section laid the basics of the mechanisms at play, but to understand how household heterogeneity determines the magnitude of that amplification, we now turn our attention to a quantitative HANK model.

3 A HANK Model with Aggregate Uncertainty

This section describes a richer HANK model that we use to illustrate our solution strategy and, later, will serve as our benchmark model in investigating the interaction between the ZLB and aggregate uncertainty. In addition, we explain the notion of aggregate uncertainty that we use, and introduce the notational convention employed in later sections.

[6] We provide the analytical proof, under a general CRRA utility, for $\mathbb{E}_0 Y_1^{-\sigma} > Y_{1DET}^{-\sigma}$ in Appendix D.

3.1 Model

The model economy is populated by households, intermediate producers, a final goods aggregator, and fiscal and monetary authorities.

Households There is a unit measure of infinitely lived households i who maximize their discounted lifetime utility from consumption, $\mathbb{E}_t \sum_{s=t}^{\infty} \left(\prod_{j=t}^{s} \beta_j \right) \frac{c_{it}^{1-\sigma}}{1-\sigma}$. Households inelastically supply labor n_t demanded by firms and receive labor income $z_{it} w_t n_t$, where w_t is the real wage rate per efficient hour and z_{it} is the household idiosyncratic productivity. The matrix $Q_t(\cdot)$ disciplines the transition between productivity states. We assume a time-invariant transition matrix ($Q_t = Q$), implying that individual risk is acyclical. This is a rather conservative assumption, taken to avoid further amplifications due to the (empirically observed) counter-cyclicality of earnings risk (Guvenen et al., 2017). This assumption is by no means required by our solution strategy.

The productivity level also determines dividend payments $z_{it} d_t$ and taxation $z_{it} t_t$, where d_t and t_t are aggregate profits from the firm sector and aggregate taxation from the fiscal authority.[7] Those assumptions imply that the income flow is proportional to aggregate output Y_t net of taxation, $z_{it}(Y_t - t_t)$. Households can save in nominal riskless bonds a_{it} (in real terms), whose price is the inverse of risk-free gross nominal interest rate R_t. Finally, households can only borrow up to a limit $\underline{a}$.

Consider a household with idiosyncratic state z_{it} and initial savings a_{it-1}, whose real value is depreciated by (gross) current inflation Π_t. Its problem is given by:

$$
\begin{aligned}
V_t(z_{it}, a_{it-1}) = \max_{c_{it}, a_{it}} \frac{c_{it}^{1-\sigma}}{1-\sigma} + \beta_t \mathbb{E}_t V_{t+1}(z_{it+1}, a_t), \qquad (7) \\
\text{s.t. } c_{it} + \frac{a_{it}}{R_t} = \frac{a_{it-1}}{\Pi_t} + z_{it}(Y_t - t_t), \\
a_{it} \geq \underline{a}.
\end{aligned}
$$

Note that the expectation operator $\mathbb{E}_t$ embeds both the aggregate and idiosyncratic uncertainty.[8] The optimization problem yields the standard Euler equation optimality con-

[7] This is a practical and conservative assumption. Similarly to our assumption about the idiosyncratic productivity shocks, it deals with the problems resulting from countercyclical dividend dynamics that has been shown to be counterfactual.

[8] The time dependence of the value function captures all the variations in aggregate variables (Y_t, Π_t,

dition and individual asset demand, which we write in the individual state space.

$$\frac{c_t\left(z_{it},a_{it-1}\right)^{-\sigma}}{\Pi_t} \geq \beta_t \frac{R_t}{\Pi_t}\mathbb{E}_t\left[c_{t+1}\left(z_{it},a_{it-1}\right)^{-\sigma}\frac{1}{\Pi_{t+1}}\right] \tag{8}$$

$$a_t\left(z_{it},a_{it-1}\right) = R_t\left[\frac{a_{it-1}}{\Pi_t} + z_{it}\left(Y_t - t_t\right) - c_t\left(z_{it},a_{it-1}\right)\right] \tag{9}$$

We denote the beginning-of-period distribution over individual states by $D_t\left(z_{it},a_{it-1}\right)$. The solution to the household problem yields two aggregates, consumption and asset demand, determined as follows:

$$C_t \equiv \int c_t\left(z_{it},a_{it-1}\right) dD_t,$$

$$A_t \equiv \int a_t\left(z_{it},a_{it-1}\right) dD_t.$$

Supply Side. The supply side follows the New Keynesian tradition with a continuum of intermediate producers with monopolistic power and quadratic price adjustment costs, a competitive final goods producer, and labor supply union that allocates hours equally across households.[9] Intermediate producers are subject to sales subsidy to eliminate monopolistic distortions, with proceeds rebated lump-sum to firms. The final goods producer combines intermediate varieties into the final goods in a competitive market. We relegate details to Appendix E.

The supply side is ultimately summarized by a New Keynesian Phillips curve, with the Frisch elasticity of labor supply represented by ω and the slope parameter κ being inversely related to the degree of price adjustment.

$$\left(\Pi_t - \overline{\Pi}\right)\Pi_t = \beta_t \mathbb{E}_t\left[\left(\frac{Y_{t+1}}{Y_t}\right)^{1-\sigma}\left(\Pi_{t+1} - \overline{\Pi}\right)\Pi_{t+1}\right] + \kappa\left[Y_t^{\omega+\sigma} - 1\right] \tag{10}$$

Fiscal Policy The government imposes lump-sum taxes to ensure a balanced budget, given by:

$$t_t + \frac{b_t}{R_t} = \frac{b_{t-1}}{\Pi_t}. \tag{11}$$

R_t, and t_t).

[9] The resulting aggregate labor supply curve is given by $w_t = N_t^{\omega} Y_t^{\sigma}$, where ω is the Frisch elasticity of labor supply. This formulation is equivalent to, for instance, Auclert, Bardóczy and Rognlie (2023) with flexible wages and unit markdown.

In addition, the government sets the supply of bonds according to a fiscal rule:

$$b_t = b(Y_t, b_{t-1}). \tag{12}$$

Monetary Policy The central bank follows a standard Taylor rule, reacting to deviations of inflation and output from their respective steady state values, $\overline{\Pi}$ and $\overline{Y}$, and is subject to the ZLB. It is given by:

$$R_t = \max\left\{\underline{R}, \overline{R}\left(\frac{\Pi_t}{\overline{\Pi}}\right)^{\phi_\pi}\left(\frac{Y_t}{\overline{Y}}\right)^{\phi_y}\right\} \tag{13}$$

Market Clearing Conditions Assets and goods markets clearing conditions are given by:

$$b_t = A_t \tag{14}$$

$$C_t = Y_t \tag{15}$$

General Description of an Equilibrium. An equilibrium in this economy is represented by a series of stochastic processes for the aggregate variables $X_t = \{Y_t, \Pi_t, b_t, t_t, R_t\}$, a series of stochastic functions for the individual choices $g_t(z_{it}, a_{it-1}) = \{c_t(z_{it}, a_{it-1}), a_t(z_{it}, a_{it-1})\}$, given an initial distribution D_0, government debt b_0, and a stochastic process for the discount factor β_t, such that (i) individual policy functions solve the household maximization problem (7); (ii) the distribution law of motion is consistent with individual policy functions (8-9); and (iii) equations (10-15) hold at all times.

3.2 Aggregate Uncertainty

Solution methodologies for models similar to ours are potentially burdensome for several reasons. First, one needs to keep track of the evolution of distribution over individual states, which is itself a state variable.[10] Second, the possibility of *many* different possible trajectories for the economy increases the complexity in the aggregate variables (since one needs to consider all possible future realizations for a variable at a certain period)

[10] There are a few exceptions to this problem. Acharya and Dogra (2020) and Acharya, Challe and Dogra (2023) work around it by assuming CARA utility functions, which allows for linear aggregation of individual policies. There is also a strand of the literature that uses continuous time techniques to work around some of the computational hurdles. See for instance Achdou et al. (2022), Ahn et al. (2018), and Kaplan, Moll and Violante (2018) among others.

and exacerbates the first problem. Third, the presence of aggregate nonlinearities such as the ZLB renders standard perturbation techniques inadequate.

In this subsection we introduce our notion of aggregate uncertainty, which allows us to significantly reduce the severity of the computational burdens described above, even in heterogeneous-agents models with a rich micro structure. To that end, our key assumption is that the shock follows a two-states Markov process with an absorbing state. Even though, at first, this assumption can sound restrictive, we believe it consists of a fairly accurate representation of the events following macroeconomic disruptions.

Most recessions can be viewed as a consequence of an unexpected event whose precise duration is unknown from an ex-ante perspective. Once such shock dissipates, then the economy moves "back on track". One prominent example is COVID-19. On the onset of the pandemic, the timing frame for the availability of vaccines was far from clear. However, the consensus was that, once a considerable portion of the population would be vaccinated, normalcy would be restored and the economy would recover to pre-pandemic levels (see for instance Pinsker, 2020). A second example is the Great Recession, in which uncertainty about the speed of recovery was also high, but there was little disagreement about the fact that the economy would eventually be back on track. Finally, during the post-pandemic bout of inflation, even though long-term inflation expectations remained fairly stable, there was substantial uncertainty with regards to when inflation would return to near-target levels. In all cases, the uncertainty regarding future economy dynamics was large and we believe that our assumptions and solution strategy are able to adequately capture their economic effects.

Assumptions Our uncertainty structure follows that of Eggertsson and Woodford (2003) and Eggertsson et al. (2021), among others. We assume that the economy begins at its stationary equilibrium and at time $t = 0$ the discount factor unexpectedly becomes $\beta_L > \beta$, i.e. there is an increased desire to save. From then on, in every period there is a fixed probability μ that it remains in this "bad regime". Alternatively, with probability $1 - \mu$, it reverts back to its steady-state value and the economy moves towards the stationary equilibrium.[11] For simplicity, in this exposition, we assume that once the economy departs the bad regime, the value of β_t reverts immediately to its steady-state value.[12]

Formally we have the following expression for the discount factor at $t > 0$:

$$\beta_t = \begin{cases} \beta & \text{if } \beta_{t-1} = \beta \\ \beta & \text{w.p. } (1-\mu) \text{ if } \beta_{t-1} = \beta_L \\ \beta_L & \text{w.p. } \mu \text{ if } \beta_{t-1} = \beta_L \end{cases} \tag{16}$$

Figure 2 represents the possible paths β_t can follow (blue lines) and the thickness corresponds to their unconditional probability. The red line represents the unconditional expectation.[13] The figure also reports a black dashed line, which represents a deterministic counterpart to the whole stochastic structure. In the case of the shock, the deterministic counterpart coincides with the unconditional expectation. However, this is not necessarily true for endogenous variables.[14]

Notation and Terminology We refer to a *contingency* as the time τ when the shock switches back to its steady state value as well as the equilibrium dynamics of the economy following such event. As an example, if the discount factor in our model switches back to its steady-state value β at time 8, the economy's trajectory following this event is what we refer to as contingency 8, i.e. $\tau = 8$. Note that this convention implies that there is no such thing as contingency 0.

We use the notation x_t^τ to indicate the value of variable x (say, inflation), at time t,

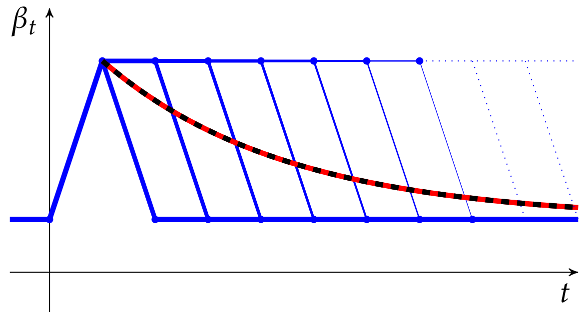
[11] A more structured shock process (a convolution of an AR(1) process together with a two-states Markov process) can be found in Lin (2020).

[12] This assumption, however, is not required for our solution method to be implemented. A necessary condition is that the discount factor eventually returns to its steady-state value, but it can assume arbitrary values after the economy departs the bad regime as long as the system does not violate the solution uniqueness conditions. See Ascari and Mavroeidis (2022) and Holden (2023) for a discussion on existence and uniqueness of equilibrium at the ZLB under perfect foresight.

[13] To construct the unconditional expectation we weight each possible profile by its corresponding probability. This is also what applies to households, that is, they have rational expectations. In principle, our solution strategy allows us to depart from rational expectations but only in the particular way in which agents just apply a different probability than μ to the aggregate process.

[14] It is worth mentioning that the stochastic structure in our setup is fundamentally different from the standard shock structure present in most DSGE models, where every period there is a random disturbance drawn from a normal distribution.

Figure 2: Graphical Representation of Shock



Notes: The figure represents the shock structure and its average (red line). The black dashed line being the deterministic counterpart.

under contingency τ, and x_t to indicate the value of variable x, at time t, when the shock has *not yet* reverted. Note that, in case the shock has reverted, it must be that $t \geq \tau$. As an example of what this notation aims to represent, consider inflation at time $t = 2$. In our setup, there are three distinct "inflation-at-time-2" economic objects that are relevant: time 2 in contingency 1, Π_2^1; time 2 in contingency 2, Π_2^2; and time 2 in any contingency larger than 2, Π_2.

We also define the *collection* of aggregate variables at time t before the shock regime reverts as $X_t \equiv \{Y_t, \Pi_t, R_t, t_t, b_t\}$ and the corresponding set of variables in contingency τ at time t as $X_t^\tau \equiv \{Y_t^\tau, \Pi_t^\tau, R_t^\tau, t_t^\tau, b_t^\tau\}$. Similarly, we denote the distribution over individual states and the value functions at the beginning of the period t in contingency τ respectively by D_t^τ and V_t^τ, and at the beginning of period t, when the shock has not yet reverted, by D_t and V_t.

Implications of the Two-States Shock Structure The shock structure assumption has three main implications in our model. First, the assumption of the two-states structure significantly reduces the multiplicity of the economic objects that can affect agents' decisions at any point in time. Specifically, it implies that there are only $t+1$ possible values for a certain aggregate variable at time t. Back to our example of inflation, note that at $t = 2$ it can assume $t + 1 = 3$ different values, depending on the realizations of the shock.

The second implication is that, once in a contingency (i.e. the bad regime reverts), predetermined variables at the moment of reversion such as the initial distribution D_τ^τ, or past aggregate variables $X_{\tau-1}$, can be taken as given and, from that point on, aggregate variables follow a deterministic path. In fact, we will heavily exploit this fact to make our solution strategy more efficient. Additionally, note that, conditional on those pre-determined variables, contingencies are independent of each other. Thus, their equilibrium solution can be parallelized, yielding further computational efficiency.

The third implication relates to forward-looking variables during the periods in which the shock has not yet reverted, in which case we need to explicitly take the uncertainty into account. The simple structure allows us to write expectations in a compact way. Consider inflation at $t = 1$, from the perspective of $t = 0$. The expectation of one period ahead inflation can be compactly written as $\mu\Pi_1 + (1-\mu)\Pi_1^1$. Now consider the consumption Euler equation (8) at time 0, when households are aware of the uncertainty:

$$\frac{c_0(z,a_{-1})^{-\sigma}}{\Pi_0} \geq \beta_0 \frac{R_0}{\Pi_0} \left\{ (1-\mu) \left[\sum_{z'} Q_{z,z'} c_1^1 (z',a)^{-\sigma} \frac{1}{\Pi_1^1} \right] + \mu \left[\sum_{z'} Q_{z,z'} c_1 (z',a)^{-\sigma} \frac{1}{\Pi_1} \right] \right\},$$

We will use this fact when adapting the Algorithm in Auclert et al. (2021) to compute heterogeneous-agents Jacobians, as we explain in Section 4.

3.3 Calibration

The calibration we use is summarized in Table 1. In the steady state, quarterly output is normalized to 1, the annualized inflation rate is set to 2%, and the supply of liquid bonds equals 25% of yearly GDP, yielding a quarterly average marginal propensity to consume of 0.44. The discount factor is set to clear the asset market at an annualized nominal rate of 2.5%, thus the real rate is 0.5% per anuum.

The CRRA utility parameter is set to 1.5 as in Smets and Wouters (2007). The Frisch elasticity is set to $\omega = 1$. We set the monetary policy parameters to standard values, $\phi_\pi = 1.5$ and $\phi_y = 0.125$. The idiosyncratic risk process is taken from McKay, Nakamura and Steinsson (2016). Further, as a benchmark we set our fiscal rule to a constant amount of debt, i.e. $b_t = \bar{b}$, where $\bar{b}$ represents the steady-state level of bond supply.

We calibrate the shock size $\beta_L = 0.993$ and the slope of the New Keynesian Phillips curve $(\kappa(\omega+\sigma))$ to 0.01 to obtain initial output and inflation, in the HANK model with the ZLB, that match those of the Great Recession. The shock reversal parameter μ is taken from Eggertsson et al. (2021).

Table 1: Calibration

Parameter	Value	Source	Note
σ	1.5	Smets and Wouters (2007)	EIS Parameter
β	0.9805	Calibrated	Discount Factor
ω	1	Standard	Frisch elasticity
$\kappa \times (\omega + \sigma)$	0.01	Calibrated	Slope of Phillips Curve
$\overline{\Pi}$	$1.02^{0.25}$	Standard	Inflation target
ϕ_π	1.5	Standard	Monetary Policy
ϕ_y	0.125	Standard	Monetary Policy
z		McKay, Nakamura and Steinsson (2016)	Idiosyncratic Shocks
Q		McKay, Nakamura and Steinsson (2016)	Transition Matrix
μ	0.9	Eggertsson et al. (2021)	Switching Probability
β_L	0.993	Calibrated	Shock Size

Notes: the table reports the calibration used in the paper. See text for more details.

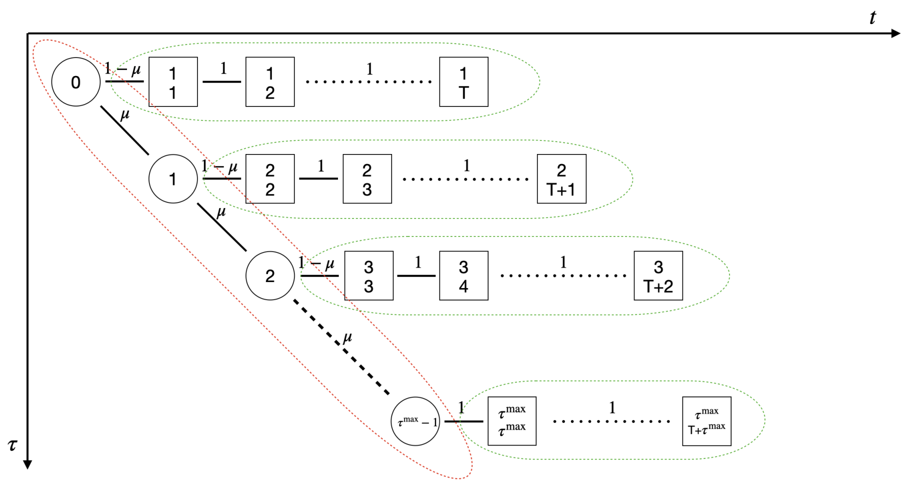
The calibration above is used to obtain our benchmark results in Section 5. At the end of that section, however, we consider several robustness exercises in which we use the earnings risk process by Krueger and Perri (2005) and different assumptions regarding the risk-aversion, the credit limit, and the total bond supply. Further, we truncate the uncertainty by setting $\tau^{\max} = 100$, and report nearly unchanged results for $\tau^{\max} = 200$ in Appendix G.

4 Solution Approach

Our solution methodology can also accommodate the presence of nonlinearities on the behavior of the (macro) economy. In our applications, the central bank is constrained by the ZLB on nominal interest rates. Other applications of our solution algorithm include, but are not limited to, aggregate financial constraints, downward wage rigitidies (see Eggertsson, Mehrotra and Robbins, 2019), or nonlinear Phillips curves (see Benigno and Eggertsson, 2023; Comin, Johnson and Jones, 2023; Gitti, 2023).

We distinguish between the types of paths the economy can take into two: the (*two-states* "bad regime") (henceforth **TS**), where uncertainty has yet to be resolved, and the set of *perfect-foresight* paths (henceforth **PF**), where no uncertainty remains. Figure 3 illustrates the classification. The TS path is represented by the diagonal line, highlighted in red, whereas the PF paths are highlighted in green. In the diagram, the vertical axis represents contingencies τ, whereas the horizontal axis indicates actual time t.

Figure 3: Representation of the Economy under the Two-States Shock Structure



Notes: Time, measured by t, is represented on the horizontal axis. Different contingencies, represented by the vertical axis, are enumerated by τ.

We impose two technical assumptions to implement our methodology. First, there is a period $\tau^{\max}$ at which the shock reverts with probability one. And second, we assume that the economy returns to its steady state within T periods after the resolution of the aggregate uncertainty.

Before proceeding to the solution methodology, we need to define further notation. Along the TS path, we define $\mathbb{X}^{TS}$ as the (stacked) vector of equilibrium aggregate variables $\{X_t\}_{t=0}^{\tau^{\max}-1}$, $\mathbb{D}^{TS}$ as a matrix made of $\tau^{\max}$ distributions over individual states $\left(\{D_t\}_{t=0}^{\tau^{\max}-1}\right)$, and, similarly, $\mathbb{V}^{TS}$ as a matrix of $\tau^{\max}$ value functions $\left(\{V_t\}_{t=0}^{\tau^{\max}-1}\right)$. For each PF contingency τ, we let $\mathbb{X}^{\tau}$, $\mathbb{D}^{\tau}$, and $\mathbb{V}^{\tau}$ respectively represent the set of equilibrium aggregate variables $\left(\{X_t^{\tau}\}_{t=\tau}^{T+\tau-1}\right)$, distributions $\left(\{D_t^{\tau}\}_{t=\tau}^{T+\tau-1}\right)$, and value functions $\left(\{V_t^{\tau}\}_{t=\tau}^{T+\tau-1}\right)$ along contingency τ. Furthermore, we denote $\mathbb{X}^{PF}$, $\mathbb{D}^{PF}$, and $\mathbb{V}^{PF}$ as respectively the complete set of the equilibrium aggregate variables $\left(\{\mathbb{X}^{\tau}\}_{\tau=1}^{\tau^{\max}}\right)$, distributions $\left(\{\mathbb{D}^{\tau}\}_{\tau=1}^{\tau^{\max}}\right)$, and value functions $\left(\{\mathbb{V}^{\tau}\}_{\tau=1}^{\tau^{\max}}\right)$ along the entire set of PF branches. Lastly, let $\mathbb{X}_{ss}^{TS}$ and $\mathbb{X}_{ss}^{PF}$ be the (stacked) vectors of steady-state values with respective dimension $n_x \times \tau^{\max} \times 1$ and $n_x \times T \times 1$, where n_x represents the number of endogenous variables in our equilibrium system. In our benchmark case, $n_x = 5$. Refer to Appendix A for a more detailed overview on our notation.

Our solution methodology iterates between solutions of the TS and the PF paths until a fixed point of all state variables - including the entire distribution of households over individual states - along the diagonal path is achieved. A brief description is provided in Algorithm 1 below:

Algorithm 1 *Broad Overview of the Solution Methodology*

1. *Provide a guess for the economy's state variables at the initial period in each PF path,* $\left\{\{D_\tau^\tau\}_{\tau=1}^{\tau^{\max}}, \{X_\tau\}_{\tau=0}^{\tau^{\max}-1}\right\}^n$, *with* $n = 0$.

2. ***Equilibrium in PF Contingencies.*** *The guess consists of a set of initial conditions in each PF path. Conditional on those, solve for the equilibrium in each of the contingencies. Collect the forward-looking variables that are relevant for each node along the diagonal path,* $\left\{\{V_\tau^\tau\}_{\tau=1}^{\tau^{\max}}, \{X_\tau^\tau\}_{\tau=1}^{\tau^{\max}}\right\}$.

3. ***Equilibrium in TS.*** *Keeping the forward looking variables fixed, solve for the equilibrium along the TS path. Obtain a new set of initial conditions for each PF path,* $\left\{\{D_\tau^\tau\}_{\tau=1}^{\tau^{\max}}, \{X_\tau\}_{\tau=0}^{\tau^{\max}-1}\right\}^{n+1}$.

4. *If the newly obtained state variables are sufficiently close to the guess, an equilibrium for the economy is found. Otherwise, return to step 2.*

General Equilibrium Representation. Let $\mathbb{Z}^{\text{TS}} = \{Z_t\}_{t=0}^{\tau^{\max}-1}$ represent the dynamics of exogenous disturbances along the diagonal path. In our baseline model, this is given by $\beta_t = \beta_L$ for $t \in \{0, 1..., \tau^{\max-1}\}$. For simplicity, we have assumed that once the shock reverts the shock is immediately back to its steady-state value.[15] Following Auclert et al. (2021), the general equilibrium in our model can be expressed by the system of equations:

$$\mathbf{F}\left(\mathbb{X}^{\text{TS}}, \mathbb{X}^{\text{PF}}, \mathbb{Z}^{\text{TS}}\right) = 0, \tag{17}$$

In the model laid out in Section 3, $\mathbf{F}(\cdot)$ consists of equations (10-14) at *each* period of both the TS and the PF paths.

[15] As pointed out earlier, this is not necessary for implementation of our solution strategy. In particular, if there are still shocks once the uncertainty has been resolved, equation (17) would instead read $\mathbf{F}\left(\mathbb{X}^{\text{TS}}, \mathbb{X}^{\text{PF}}, \mathbb{Z}^{\text{TS}}, \mathbb{Z}^{\text{PF}}\right) = 0$.

The system of equations (17) is of high dimension, due to all the possible combinations of time and contingencies. In particular, its dimensions are $n_x \times (T+1) \times \tau^{\max}$. We set $\tau^{\max} = 100$ and $T = 300$, meaning that in our baseline implementation $\mathbf{F}(\cdot)$ has around 150 thousand rows. Our methodology involves computing the Jacobian of the equilibrium system (17). Even though we are dealing with a relatively simple HANK model, this matrix would contain more than 90 million entries. Dividing the equilibrium conditions in two groups, one corresponding to the TS branch, and one corresponding to the entire set of PF branches, allows us to circumvent that problem.

From inputs to outputs. Similar to Auclert et al. (2021), we recast the representation of a heterogeneous-agent model with aggregate uncertainty of the type proposed in Section 3 as a mapping from aggregate variables (inputs) X_t and X_t^τ into outputs $\mathcal{Y}_t$ and $\mathcal{Y}_t^\tau$. Each component of X_t, as well as X_t^τ, has n_x inputs, while each component of $\mathcal{Y}_t$, as well as $\mathcal{Y}_t^\tau$, displays n_y outputs.[16] We assume the existence of functions $y(\cdot)$ and $y^{TS}(\cdot)$, functions $v(\cdot)$ and $v^{TS}(\cdot)$, and transition matrices $\Lambda(\cdot)$ and $\Lambda^{TS}(\cdot)$ such that, conditional on the initial distribution D_0, the set of outcomes $\mathcal{Y}_t$ and $\mathcal{Y}_t^\tau$ solve the following system of equations:[17]

$$V_t^\tau = v\left(V_{t+1}^\tau, X_t^\tau\right) \tag{18}$$

$$D_{t+1}^\tau = \Lambda\left(V_{t+1}^\tau, X_t^\tau\right)' D_t^\tau \tag{19}$$

$$\mathcal{Y}_t^\tau = y\left(V_{t+1}^\tau, X_t^\tau\right)' D_t^\tau \tag{20}$$

$$V_t = v^{TS}\left(V_{t+1}, V_{t+1}^{t+1}, X_t\right) \tag{21}$$

$$D_{t+1}^{t+1} = D_{t+1} = \Lambda^{TS}\left(V_{t+1}, V_{t+1}^{t+1}, X_t\right)' D_t \tag{22}$$

$$\mathcal{Y}_t = y^{TS}\left(V_{t+1}, V_{t+1}^{t+1}, X_t\right)' D_t \tag{23}$$

[16] The number of inputs can be reduced with the help of a Directed Acyclic graph (DAG), as suggested in Auclert et al. (2021).

[17] With some abuse of notation, the distributions and the value functions are henceforth defined over a discretized grid of assets and idiosyncratic productivity.

Equations (18-20) are analogue to equations (10-12) in Auclert et al. (2021).[18] However, in our case, because there is uncertainty regarding when the regime will revert the economy can follow $\tau^{\max}$ possible perfect-foresight paths, indexed by τ. Equation (18) translates future value functions and current inputs into current value functions; equation (19) in turn provides a (linear) mapping between today's and tomorrow's distributions, through the matrix $\Lambda\left(V_{t+1}^{\tau} X_t^{\tau}\right)$; and equation (20) computes (aggregate) outcomes $\mathcal{Y}_t^{\tau}$ based on individual decisions $y\left(V_{t+1}^{\tau}, X_t^{\tau}\right)$, aggregated using the distribution over individual states.

Equations (21-23), on the other hand, are unique to our setup. They explicitly take uncertainty into account: the first two arguments of the functions $v^{TS}(\cdot)$, $\Lambda^{TS}(\cdot)$, and $y^{TS}(\cdot)$ correspond to the two distinct future value functions, on the TS path and on the "$t+1$" PF path respectively. In addition, note that the future distribution determined by equation (22) will be the same if the economy continues in the TS branch (D_{t+1}) or if the shock reverts (D_{t+1}^{t+1}).

In the model presented in Section 3, the five inputs are $X_t = \{Y_t, \Pi_t, b_t, t_t, R_t\}$ (and similarly for X_t^{τ}). The only output is aggregate savings, the relevant variable for market clearing. Thus $\mathcal{Y}_t = y^a(V_{t+1}, V_{t+1}^{t+1}, X_t)' D_t$, with y^a representing the asset policy function (and similarly for $\mathcal{Y}_t^{\tau}$). Finally, the asset market clearing is given by $b_t = \mathcal{Y}_t$ (and $b_t^{\tau} = \mathcal{Y}_t^{\tau}$).

4.1 Solving for the Equilibrium in Perfect Foresight

Consider the situation in which the shock has just reverted at time τ (i.e. $t = \tau$ and agents know the economy is under contingency τ). The state of the economy in the first period of contingency τ is characterized by the vector $X_{\tau-1}$ and the distribution D_{τ}^{τ}. The equilibrium in a generic contingency τ can be summarized by the following system of equations:

$$\mathbf{F}^{\mathrm{PF}}\left(\mathbb{X}^{\tau} | D_{\tau}^{\tau}, X_{\tau-1}\right) = 0. \tag{24}$$

As each contingency features distinct initial conditions, the set of values $\mathbb{X}^{\tau}$ that solves (24) in each of them is distinct. Thus, our task is to solve $\tau^{\max}$ *different* perfect-foresight

[18] Our method also accommodates the case when the entire distribution D_t affects optimal policies, such as in Marbet (2023) and Birinci et al. (2022). The latter shows how to adapt the methodology by Auclert et al. (2021) for those cases.

equilibria, which is done in parallel using a quasi-Newton method similarly to Auclert et al. (2021), Section H. For further speed gains, we treat the initial state of the economy similarly to how we treat shocks, as exogenous arguments of F^{PF}. This is a key contribution of our methodology.

We begin by solving (24) via first-order perturbation. We later return to explain how we obtain the *exact* (nonlinear) solution. The first-order approximation of (24) around the steady state is given by:

$$\mathbf{F}_{\mathbb{X}}^{PF} d\mathbb{X}^{\tau} + \mathbf{F}_{D}^{PF} dD_{\tau}^{\tau} + \mathbf{F}_{X_{-1}}^{PF} dX_{\tau-1} = 0. \tag{25}$$

In the equation above, differentials (d) are taken relative to the steady state, i.e. $d\mathbb{X}^{\tau} = \mathbb{X}^{\tau} - \mathbb{X}_{ss}$, $dD_{\tau}^{\tau} = D_{\tau}^{\tau} - D_{ss}$, and $dX_{\tau-1} = X_{\tau-1} - X_{ss}$. $\mathbf{F}_{\mathbb{X}}^{PF}$ represents the steady-state Jacobian of equilibrium conditions with respect to aggregates in contingency τ, whose dimensions are $n_x \times T$. Finally, the term $\mathbf{F}_{D}^{PF} dD_{\tau}^{\tau}$ evaluates how equilibrium conditions at each period of the contingency are impacted by changes in the distribution D_{τ}^{τ} only, while the term $\mathbf{F}_{X_{-1}}^{PF} dX_{\tau-1}$ evaluates the impact of pre-determined variables $X_{\tau-1}$.

Rearranging equation (25), we obtain:

$$d\mathbb{X}^{\tau} = \left(\mathbf{F}_{\mathbb{X}}^{PF}\right)^{-1} \left(\mathbf{F}_{D}^{PF} dD_{\tau}^{\tau} + \mathbf{F}_{X_{-1}}^{PF} dX_{\tau-1}\right) \tag{26}$$

We directly employ the methodology in Auclert et al. (2021) to compute the heterogeneous-agents part (corresponding to equation 14) of the Jacobian $\mathbf{F}_{\mathbb{X}}^{PF}$, and compute the Jacobians of equations (10-13) analytically. The novelty of our method involves the terms in parenthesis, $\mathbf{F}_{D}^{PF} dD_{\tau}^{\tau}$ and $\mathbf{F}_{X_{-1}}^{PF} dX_{\tau-1}$.[19] We compute the first as the right-hand-side of:

$$\mathbf{F}_{D}^{PF} dD_{\tau}^{\tau} \approx \mathbf{F}^{PF}\left(\mathbb{X}_{ss}^{PF} \middle| D_{\tau}^{\tau}, X_{ss}\right) - \mathbf{F}^{PF}\left(\mathbb{X}_{ss}^{PF} \middle| D_{ss}, X_{ss}\right) \tag{27}$$

An analogue expression applies to $\mathbf{F}_{X_{-1}}^{PF} dX_{\tau-1}$.[20] To obtain (27), we again follow a key insight from Auclert et al. (2021): as we treat changes in initial conditions as a one-time *shock*, and simulate the effect of that shock conditional on other inputs (and policies) remaining at their steady-state values.

[19] If shocks also occur in the perfect-foresight paths, equation (25) is substituted by: $d\mathbb{X}^{\tau} = (\mathbf{F}_{\mathbb{X}}^{PF})^{-1}\left(\mathbf{F}_{\mathbb{Z}} d\mathbb{Z}^{\tau} + \mathbf{F}_{D}^{PF} dD_{\tau}^{\tau} + \mathbf{F}_{X_{-1}}^{PF} dX_{\tau-1}\right)$, where $\mathbb{Z}^{\tau}$ would represent the entire sequence of shocks along contingency τ.

[20] Given our assumed fiscal rule, our baseline model does not feature pre-determined aggregate variables.

Changes in the initial distribution of households will directly affect only the total supply of savings at all periods in a given contingency τ, as this is the only endogenous household decision the model features. Let F_t^τ be the one entry of $\mathbf{F}^{PF}$ representing the asset market clearing condition at time t, contingency τ:

$$F_t^\tau(\mathbb{X}^\tau, D_\tau^\tau) = y^a(V_{t+1}^\tau, X_t^\tau)' D_\tau^\tau - b_t^\tau \equiv \mathcal{Y}_t^\tau - b_t^\tau$$

The term b_t^τ equation above is independent of the heterogeneous-agents block. The challenge is computing derivatives of the first term. For that, we use:

$$d\mathcal{Y}_t^\tau = y_{ss}^{a}{}'(\Lambda_{ss}')^{t-\tau} dD_\tau^\tau \equiv \mathcal{E}_{t-\tau}' dD_\tau^\tau, \tag{28}$$

where $\mathcal{E}_j \equiv (\Lambda_{ss})^j y_{ss}^a$ is the expectation vector defined by Auclert et al. (2021). In our example, it represents the asset demand for when households display their steady-state policies y_{ss}^a for $j+1$ periods, but the initial distribution has changed to D_τ^τ.[21] The key advantage of exploiting equation (28) is that the linear transformation $\mathcal{E}_{t-\tau}'$ can be precomputed and therefore recycled at each iteration of step 1 in algorithm 1, thus yielding speed gains.

With all terms of the right-hand-side of equation (26) at hand, we can easily compute $d\mathbb{X}^\tau$ for each τ. We turn our attention to dealing with the Zero Lower Bound.

Occasionally Binding Constraints - The Zero Lower Bound. We follow the approach of Guerrieri and Iacoviello (2015): in each branch, we first compute $d\mathbb{X}^\tau$ without imposing the bound. This gives us the shadow rates $SR^\tau \equiv \{SR_t^\tau\}_{t=\tau}^{T+\tau-1}$ - the nominal rates the central bank would select if it were unconstrained. Finally, we then reset $R_t^\tau = \underline{R}$ at each period in which $SR_t \leq \underline{R}$ and adjust the Jacobian $\mathbf{F}_{\mathbb{X}}^{PF}$ to account for the fact that the central bank is constrained in those periods. We repeat the approach until the set of periods in which the ZLB binds is stable. The procedure, which consists of step 2 of Algorithm 1, is described below:

[21] The intuition for (28) is the following: even though at date zero, the distribution over individual states is different than the steady-state one, moving forward households maintain policies unchanged (as the Jacobian is evaluated at the steady state), and thus the distribution D converges back to the D_{ss} over time, the convergence being dictated by the transition matrix Λ_{ss}.

Algorithm 2 ***(Step 2 of Algorithm 1) Equilibrium PF Contingencies with Occasionally Binding ZLB.*** *Given $\{D_\tau^\tau\}_{\tau=1}^{\tau^{\max}}$ and $\{\mathbb{X}^\tau\}_{\tau=0}^{\tau^{\max}-1}$, initialize the sets of periods in which the ZLB binds $o_\tau^n = \emptyset$, $n = 0$. For each contingency τ:*

1. *Compute $d\mathbb{X}^\tau$ following (26).*

2. *Compute the shadow rates SR_t^τ for each t.*

3. *Compute $o_\tau^{n+1} = \{t\}$ such that $SR_t^\tau \leq \underline{R}$. In the set of model equilibrium equations, substitute the Taylor Rule for all $\{t\} \in o_\tau^{n+1}$ for $R_t^\tau = \underline{R}$. Modify the Jacobian $\mathbf{F}_{\mathbb{X}}^{PF}$ accordingly.*

4. *If $o_\tau^n \neq o_\tau^{n+1}$, return to 1.*

5. *Proceed to step 3 in Algorithm 1.*

Fully Nonlinear Solution PF Contigencies. Equation (26) computes the equilibrium variables in each contingency using a first-order perturbation, which approximates the solution to the system of equations (24). As the asset demand function is nonlinear, there might be an approximation inaccuracy. This is likely to happen if the initial distribution D_τ^τ is too distant from steady state, or with strong nonlinearities in the economy.

It is straightforward to test if our approximated solution, $\mathbb{X}^\tau \equiv \mathbb{X}_{ss}^{PF} + d\mathbb{X}^\tau$, is accurate, i.e. approximates well the solution to (24). We can forward-simulate the economy along each contingency τ and evaluate the whole set of equilibrium conditions. This, however, can be a burdensome step, as it involves the computation of several transition matrices Λ along each of the $\tau^{\max}$ PF paths. Yet the procedure can be parallelized.

Finally, we can use the Jacobians $\mathbf{F}_{\mathbb{X}}^{PF}$ in a quasi-Newton method to solve for the fully nonlinear equilibrium along each perfect-foresight branch. In fact, the first step of Algorithm 2 consists of the first iteration of such method, which finds the nonlinear solution to the system of equations (24). In Appendix F, we describe it in detail.

4.2 Solving for the Equilibrium in the TS Path

Due to the recursive nature of consumer's problem, solving the equilibrium in the TS path only requires knowledge of the value functions and the set of aggregate variables in the initial period of each perfect-foresight contingency, which are taken as given in step 3 of Algorithm 1. We denote the (stacked) vector of value functions ($\{V_\tau^\tau\}_{\tau=1}^{\tau^{\max}}$) in the *initial* period of each contingency by $\mathbb{V}_1^{PF}$ and the analogue stacked vector of aggregates ($\{X_\tau^\tau\}_{\tau=1}^{\tau^{\max}}$) by $\mathbb{X}_1^{PF}$. Given those, the equilibrium in the TS branch is characterized by:

$$\mathbf{F}^{\text{TS}}\left(\mathbb{X}^{TS}, \mathbb{Z}^{TS} \middle| \mathbb{X}_1^{PF}, \mathbb{V}_1^{PF}\right) = 0, \tag{29}$$

given the initial conditions $D_0 = D_{ss}$ and $X_{-1} = X_{ss}$.

As before, we first describe how we solve find the equilibrium in TS by first-order approximation. In this case:

$$\mathbf{F}_{\mathbb{X}}^{TS} d\mathbb{X}^{TS} + \mathbf{F}_{\mathbb{Z}}^{TS} d\mathbb{Z}^{TS} + \mathbf{F}_{\mathbb{X}_1}^{TS} d\mathbb{X}_1^{PF} + \mathbf{F}_{\mathbb{V}}^{TS} d\mathbb{V}_1^{PF} = 0$$

And rearranging:

$$d\mathbb{X}^{TS} = (\mathbf{F}_{\mathbb{X}}^{TS})^{-1}\left(\mathbf{F}_{\mathbb{Z}}^{TS} d\mathbb{Z}^{TS} + \mathbf{F}_{\mathbb{X}_1}^{TS} d\mathbb{X}_1^{PF} + \mathbf{F}_{\mathbb{V}}^{TS} d\mathbb{V}_1^{PF}\right) \tag{30}$$

Note that we treat *future* conditions the same way we treat *initial* conditions in equation (25), as exogenous shocks.

The last term in equation (30) represents the impact of changes in households' future value functions - at the initial period of the PF branches - on equilibrium conditions along the TS branch. We compute it making use of the following:

$$\mathbf{F}_{\mathbb{V}}^{TS} d\mathbb{V}_1^{PF} \approx \mathbf{F}^{TS}\left(\mathbb{X}_{ss}^{TS}, \mathbb{Z}_{ss} \middle| \mathbb{X}_1^{PF} = \mathbb{X}_{ss}^{TS}, \mathbb{V}_1^{PF}\right) - \mathbf{F}^{TS}\left(\mathbb{X}_{ss}^{TS}, \mathbb{Z}_{ss} \middle| \mathbb{X}_1^{PF} = \mathbb{X}_{ss}^{TS}, \mathbb{V}_{ss}\right),$$

where $\mathbb{Z}_{ss}$ represent a stacked vector of shocks at their steady-state values.

The computation of the expression above is done by solving the households' problem and forward-simulating the economy along the two-states branch in response *only* to the changes in $\mathbb{V}_1^{TS}$, with all other inputs at their steady-state values. At each iteration of Algorithm 1, it has to be done once, but this is not a costly step, since there is only a single TS path.

As changes in *future* inputs (X_τ^τ) on current household policies can only have an effect through changes in future value functions, the term $\mathbf{F}_{\mathbb{X}_1}^{TS} d\mathbb{X}_1^{PF}$ only refers to aggregate equations and, thus, is computed analytically. In the case of our baseline model, the only analytical equilibrium equation with a forward-looking term is the New Keynesian

Phillips Curve (10). Solving the expectations term in this expression along the TS branch yields:

$$\left(\Pi_t - \overline{\Pi}\right)\Pi_t = \beta_t \mu \left(\frac{Y_{t+1}}{Y_t}\right)^{1-\sigma} \left(\Pi_{t+1} - \overline{\Pi}\right)\Pi_{t+1} +$$
$$+ \beta_t (1-\mu) \left(\frac{Y_{t+1}^{t+1}}{Y_t}\right)^{1-\sigma} \left(\Pi_{t+1}^{t+1} - \overline{\Pi}\right)\Pi_{t+1}^{t+1} + \kappa \left[Y_t^{\omega+\sigma} - 1\right]$$

We use the expression above to compute how changes in the initial period of each PF branch (in this case Π_{t+1}^{t+1} and Y_{t+1}^{t+1}) impact equilibrium conditions on the TS branch.

Another novelty of our methodology is in computing the heterogeneous-agent part of the term $\mathbf{F}_{\mathbb{X}}^{TS}$.[22] For that, we adapt the "Fake News Algorithm" by Auclert et al. (2021) to account for aggregate uncertainty. First, we simulate the response of households to an announced shock s periods ahead. The difference to Auclert et al. (2021) is that, in our case, agents take the aggregate uncertainty into account. In particular, there is a probability $1-\mu$ that the economy will leave the TS branch at any point in time, and they react accordingly (see equation 8). The remaining steps of the algorithm are unchanged. In the interest of space, we relegate the discussion about how aggregate uncertainty affects heterogeneous-agents Jacobians to Appendix F.3.

Zero Lower Bound, Fully Nonlinear Equilibria, and Implementation. Along the TS path, we deal with occasionally binding ZLB in the same way as in the PF branches, by following Guerrieri and Iacoviello (2015) (see Algorithm 2). In addition, to compute the exact equilibrium along the TS branch, the steps are analogue to Algorithm 3 in Appendix F.1. Finally, in Appendix F.2, we provide additional implementation details, together with running times for our benchmark model and calibration, along with more sophisticated model versions such as a two-asset HANK model as in Bayer et al. (2019).

5 Aggregate Uncertainty at the ZLB

In this section, we show that aggregate uncertainty interacts with the ZLB, and this interaction is amplified with household heterogeneity. To do so, we perform three ex-

[22] The remaining term of equation (30), $\mathbf{F}_{\mathbb{Z}}^{TS} d\mathbb{Z}^{TS}$, corresponds to the partial equilibrium impact of shocks. Obtaining $\mathbf{F}_{\mathbb{Z}}^{TS}$ is similar to computing the Jacobian of equilibrium conditions with respect to inputs, $\mathbf{F}_{\mathbb{X}}^{TS}$.

periments. First, we use the model described in Section 3 and compare the effects of a stochastic shock (blue lines in Figure 2) and its deterministic counterpart (black line in Figure 2) in the absence of the ZLB. Second, we repeat the experiment but imposing the lower bound, i.e. $\underline{R} = 1$. This experiment quantifies the interaction between aggregate uncertainty and the ZLB. Third, after constructing a RANK economy whose demand shocks imply observational equivalence with the HANK model *without* lower bound, we repeat the experiment of introducing the ZLB. This exercise quantifies the role of household heterogeneity in determining the interaction between aggregate uncertainty and the ZLB.

The magnitude of the shock is set to 0.0125, generating output and inflation responses of 8.1% and -0.6% respectively, roughly matching the case in the US during the financial crisis. Our qualitative results are robust to the magnitude of the shock. Later, we explain how we construct this shock counterpart in the representative-agent economy.

Impulse Responses and Amplification Measure. In what follows, we display graphs containing (i) all the possible paths the economy can follow under the stochastic shock, (ii) the impulse-response-function of the deterministic shock, and (iii) the unconditional average of the stochastic paths. We henceforth refer to the latter as the IRF of the stochastic shock.

Broadly speaking, we consider aggregate uncertainty to amplify a shock when *the average impact of the stochastic shock is larger than the impact of its deterministic counterpart*. Specifically, we measure amplification by comparing the present-discount value of the impulse responses, i.e. comparing $\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t (X_t - X_{ss})$ for a certain variable X_t, under each type of shock.

5.1 IRFs when the Central Bank is not constrained

We consider our HANK economy subject to the stochastic shock, as described by equation (16), and compare it with the same economy subject to a deterministic shock. We set the sequence of demand shocks in the deterministic case to be equal to the unconditional expectation of the stochastic case (see Section 3.2 for details). Formally, $\beta_t^{DET} = \mu^t \beta_L + (1 - \mu^t)\beta$.

The left column panels of Figure 4 show the effects on output, inflation, and nominal rates, of the stochastic and deterministic shocks, plotting contingencies (blue solid lines), the impulse-response-function of the deterministic shock (black dotted line), and the unconditional average of the stochastic paths, represented by the red solid line. The impulse responses under the two shocks are identical, with a 3.75% recession on impact and inflation at 0.8% (below the 2% target). Also, note that nominal rates become negative (bottom-left panel). Since the average responses are nearly same, their discounted sums are also identical, as we report in the first column of Figure 7, which plots the discounted IRFs for output (left) and inflation (right) relative to the corresponding ones under a deterministic shock (blue bar, first column).

This result confirms that certainty equivalence holds in our model, despite the non-linearities at the individual level. Certainty equivalence states that "the average impact of stochastic shock on the economy is the same as the impact of its deterministic counterpart". This is not a novel result, since Achdou et al. (2022) have shown, under more classical shock structures (e.g. AR(1) processes), that certainty equivalence survives the introduction of rich household heterogeneity. The equivalence partly relies on the fact that the response of aggregate variables in the model are approximately linear, a fact noted by Boppart, Krusell and Mitman (2018).

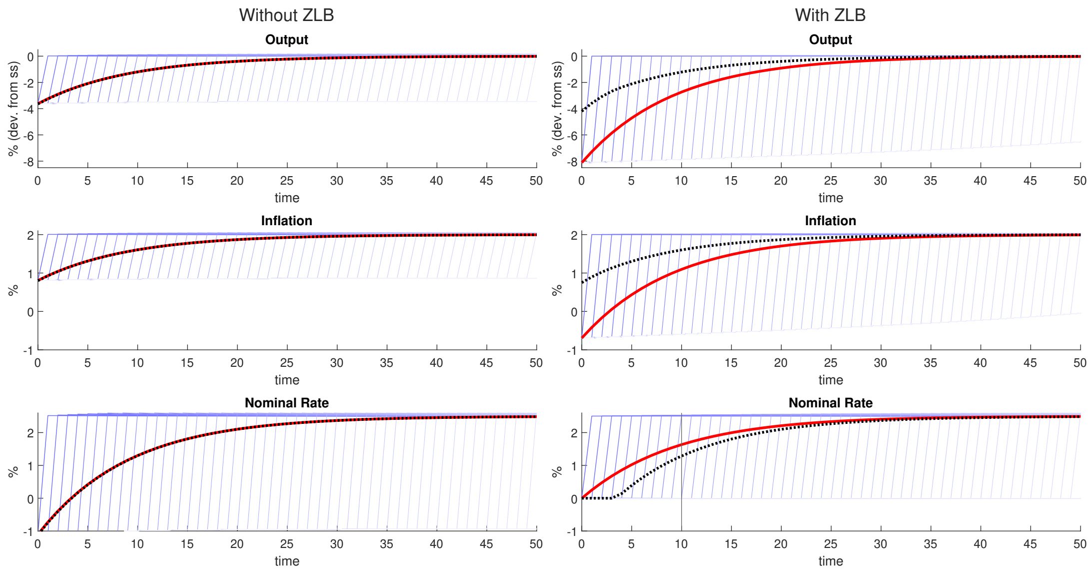
5.2 Introducing the ZLB

To assess the interaction between aggregate uncertainty and the ZLB, we repeat the previous experiment with the only difference that the lower bound to nominal rates is now set to 0 ($\underline{R} = 1$). The right column panels of Figure 4 display the results.

By preventing nominal rates from entering negative territory, the ZLB generates an unwarranted tightening and a dichotomy between the impulse responses under the two shocks. The recession on impact under the stochastic shock (8%) is about double the size of that under the deterministic shock (4%). Similarly, while the price dynamics implies an inflation level of 0.7% under the deterministic shock, the model predicts a mild deflation on impact under the stochastic shock.

The introduction of the ZLB increases the present-discounted expected output losses by 4.5% in the deterministic case and by 125% in the stochastic case. For inflation the

Figure 4: IRF and Contingencies



Notes: The figure reports the effect of a stochastic demand shock as defined in equation (16) and its deterministic counterpart (black dotted line), in our baseline HANK model without the ZLB (left panels) and with the ZLB (right panels), i.e. $\underline{R} = 1$. Each blue line corresponds to one individual contingency, with thickness proportional to its unconditional probability, and red solid line is the impulse response function obtained as a weighted average across all contingencies. The first row reports the effects on output, in deviation from steady state. The second and third rows correspond to annualized inflation and nominal interest rate levels. The black vertical solid line reports the expected duration of the ZLB under the stochastic shock. The x-axis in all panels measures time in quarters.

corresponding numbers are 1.9% and 127%. Thus, the impacts, as well as the duration of the liquidity trap, are (greatly) amplified under aggregate uncertainty.[23]

We then conduct the same exercise but with varying shock sizes. Figure 5 summarizes the amplification by reporting output (left) and inflation (right) on impact as a function of the shock size ($\beta_L - \beta$). We compare the responses of the economies with and without imposing the ZLB, both for the deterministic and stochastic shocks.

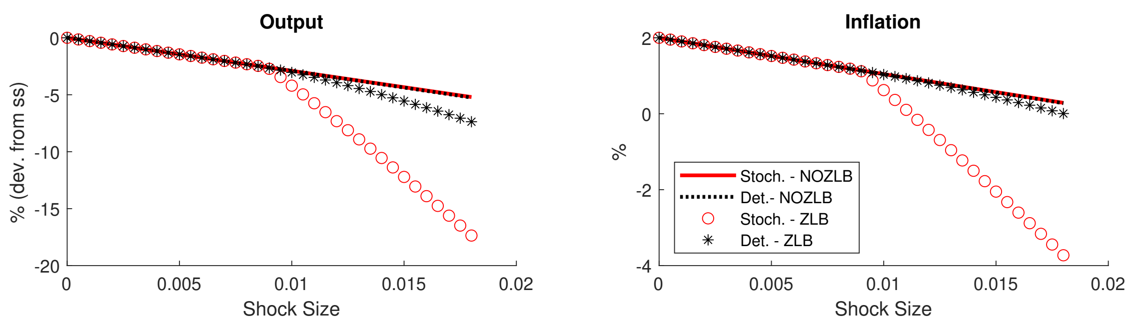
Larger shock sizes generate a larger recession and stronger downward pressure on inflation, in the four cases considered. When we ignore the ZLB (or when the shock is sufficiently small), the effect is linear in the shock size, but most importantly identical

[23] By merely looking at the IRF for nominal rates, one could be misled to believe that, with the stochastic shock, the expected duration of the liquidity trap would be 0. Instead, the actual expected duration is at least twice as large when compared to the deterministic case. This can be seen in the bottom panel of Figure 4, where the vertical line corresponds to the expected time of lift-off for policy rates under the stochastic case and is about twice as large when compared to the deterministic case.

in the stochastic (red solid line) and deterministic case (black dotted line). On the other hand, once the ZLB is taken into account, the linearity breaks once the economy enters the liquidity trap, i.e. the shock is sufficiently large.

Furthermore, there are shock values that trigger the liquidity trap under the stochastic shock (red circles) but not under the deterministic shock (black stars).[24] In other words, the threshold shock value such that the central bank becomes constrained with positive probability is lower with aggregate uncertainty. This is at the heart of the mechanism behind why the aggregate uncertainty interacts with the ZLB to amplify the shock. Finally, note that the marginal effects are much larger under the stochastic shock, as can be seen by the steeper slope of the red circles when compared to the black stars. Thus, the amplification (here measured as the difference between the red circles and the black stars) is not only robust to shock size, but also grows bigger with the size of the shock. This suggests that the mechanisms we investigate in this paper are particularly important when recessions are deep.

Figure 5: Effects on Output and Inflation on Impact



Notes: The figure plots the effects on output (in deviation from steady state) and inflation (in annualized levels) of a demand shock as described in equation (16). The shock size, $\beta_L - \beta$, varies on the x-axis. The red solid (black dotted) line corresponds to the HANK model, with the stochastic (deterministic) shock and without the ZLB. The red circles (black stars) correspond to the HANK model, with the stochastic (deterministic) shock and with the ZLB.

5.3 The Role of Heterogeneous Agents

What is the role of household heterogeneity in the results above? We now quantify the role of heterogeneity by studying a RANK model, replicating the experiments done in the

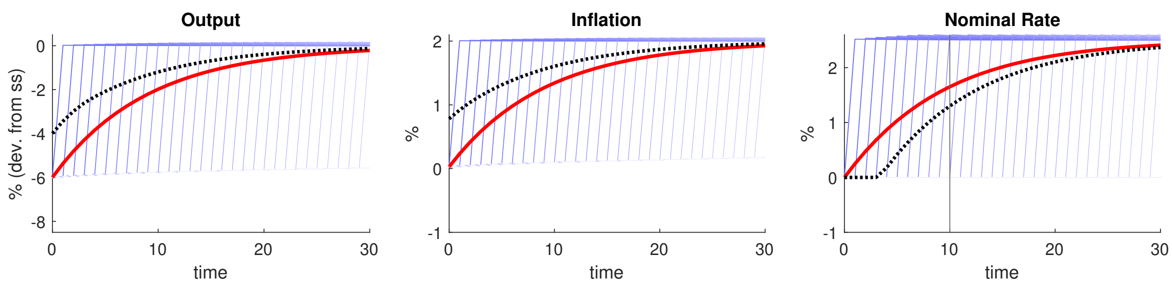
[24]See Figure D.4 for such example in light of the simple model presented in Section 2.

previous two subsections, and comparing the results with those of the HANK model. We substitute the heterogeneous agents by a single representative consumer, whose intertemporal optimization condition, i.e. the consumption Euler equation is:

$$C_t^{-\sigma} = \beta_t^{RANK} R_t \mathbb{E}_t \frac{C_{t+1}^{-\sigma}}{\Pi_{t+1}}. \tag{31}$$

To set our benchmark comparison, we calibrate the discount factors β_t^{RANK} under the stochastic case so that they exactly replicate the effects on output and inflation in all contingencies observed in the HANK model. As a deterministic shock, we consider the unconditional expectations of the stochastic shocks, as done in the previous exercise.[25]

Figure 6: IRF and Contingencies - With the ZLB



Notes: The figure reports the effect of a stochastic demand shock calibrated as described in the text (each blue line corresponds to one individual contingency and its thickness is proportional to its unconditional probability, the red solid line is the impulse response function obtained as a weighted average across all contingencies) and its deterministic counterpart (black dotted line), in a standard RANK model with the ZLB (i.e. imposing $\underline{R} = 1$). The first panel reports the effects on output, in deviation from steady state. The second and third panels correspond to annualized inflation and nominal interest rate levels. The black vertical solid line reports the expected duration of the ZLB under the stochastic shock. The x-axis in all panels measures time in quarters.

When we ignore the ZLB, the resulting simulations in the RANK model are, by construction, identical to the corresponding ones in the HANK model, both under the stochastic shock (by construction) and the deterministic shock (due to certainty equivalence). Once we introduce the ZLB, as in the HANK case, a dichotomy emerges between the impulse responses under the two shocks. Quantitatively, the recession on impact under the

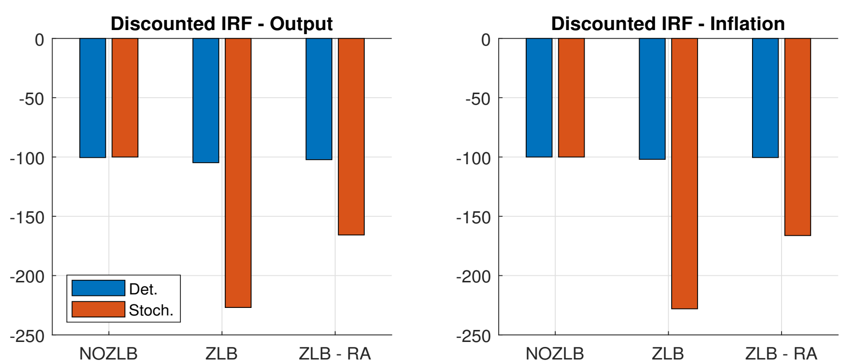
[25]The resulting shock process retains the uncertainty structure with the same fixed probability of reversal μ. It is, however, slightly different from a purely two-states Markov process, in that the discount factor will also vary along the PF paths. As previously mentioned, our solution method can also accommodate this situation.

stochastic shock (6%) is larger than that under the deterministic shock (4%), while inflation declines to 0.7% under the deterministic shock and to near 0% on impact under the stochastic shock.

Figure 7 compares the magnitudes of amplification due to aggregate uncertainty with and without the ZLB in different models. Blue bars correspond to the deterministic case, and red bars corresponds to the stochastic shock. In each panel, the first set of columns plot the discounted IRFs absent the ZLB (which are the same in the HANK and RANK models), the second set plots the (discounted) IRFs with the ZLB in the HANK model, and the the third plots discounted IRFs of the RANK model with ZLB. The left panel displays the IRFs for output, and the right panel displays those for inflation. All bars are normalized with respect to the deterministic case absent the ZLB.

While the introduction of the ZLB in the RANK model increases the expected loss in terms of output by 2.3% in the deterministic case and by 66.7% in the stochastic case, the corresponding numbers for the HANK model are 4.5% and by 125% Thus, this amplification, resulting from the interaction between aggregate uncertainty and the ZLB, is much stronger in the presence of heterogeneous agents.

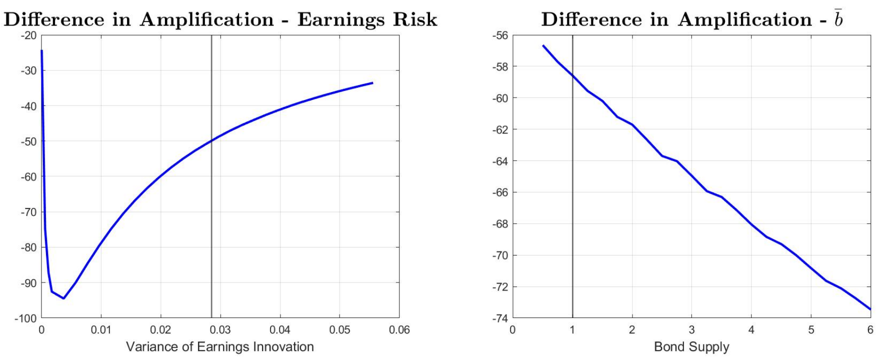
Figure 7: Discounted IRF - Output and Inflation



Notes: The figure reports the implied discounted impulse response functions for output (left panel) and inflation (right panel) under the HANK model without the ZLB (first column), the HANK model with the ZLB (second column), and the RANK model with the ZLB (third column). Within each column, the blue (red) bar corresponds to the deterministic (stochastic) case. All bars are relative to the one in the HANK model without the ZLB, under the deterministic shock (left most blue bar).

Robustness. Figure 8 shows that the result shown in Figure 7, i.e. that the interaction between aggregate uncertainty and the ZLB is larger in HANK compared to RANK, is robust to different specifications of the heterogeneous-agents economy. In particular, Figure 8 reports the difference between the gaps among the “ZLB” and “ZLB - RA” red and blue bars in Figure 7, but for different values of the earnings process variance (left panel) and the total supply of bonds $\bar{b}$ (right panel). On the former, we re-calibrate the model using the log-normal earnings process of Krueger and Perri (2005), using 41 grid points, and repeat our main exercise in this section for different values of the variance of the shock. In addition, Appendix G shows that our results are also robust to different values of the debt limit $\underline{a}$, risk aversion coefficient, and to setting $\tau_{\max} = 200$.

Figure 8: Amplification due to Aggregate Uncertainty and ZLB - HANK vs. RANK - Earnings Risk and Bond Supply



Notes: The figure plots the difference between the gaps among the “ZLB” and “ZLB - RA” red and blue bars in Figure 7, but for different degrees of earnings risk (left) and bond supply (right). For each different value of the parameters considered, β is recalibrated so that the steady-state nominal interest rate is unchanged. Except for the earnings process on the left panel, the remaining parameters are the same as in Table 1. Vertical lines denote the baseline calibration values.

The left panel of Figure 8 shows that, as the variance of the earnings shock approaches zero, i.e. as the model approaches the representative-agent one, the amplification due to aggregate uncertainty and the ZLB is reduced in HANK, relative to RANK. Note that it is not monotone: it reaches its maximum at a point below the baseline earnings risk calibration of Krueger and Perri (2005). Yet it remains substantial at all empirically plausible values.

5.4 Intuition

Why is the amplification of the demand shock due to uncertainty at the ZLB larger in HANK relative to RANK? To answer this question, we impose additional assumptions that enable us to derive analytical expressions for the response of both model economies to the type of demand shock considered in Section 5.

Consider our baseline model as presented in Section 3. We begin by proposing the following simplifying assumptions:

Assumption 1 *The net supply of assets $\bar{b}$ is null.*

Assumption 2 *The asset distribution has a negligible impact on the dynamics of the economy.*

Assumption 3 $\phi_y = 0$*, the monetary policy rule is given by* $R_t = \max\{\overline{R}, \overline{R} + \phi_\pi(\Pi_t - \bar{\Pi})\}$

Assumption 1 simplifies the analytical expressions, but is not necessary for our results, as we show in Appendix H. Assumption 2 is at the core of the method proposed Krusell and Smith (1998). In our main exercises in Section 5, the distribution evolves over time and can certainly have an impact on the economy. In the current section, however, we will consider shocks of short duration (one period), and thus the impact of the distribution on aggregates is unlikely to be relevant. This allows us to obtain analytical expressions for the effect of the demand shock. Finally, we assume that the central bank follows a Taylor rule that only responds to inflation deviations and is constrained by an effective lower bound set to equal the steady-state nominal rate $\overline{R}$.[26]

We begin by investigating the impact of a one-time contemporaneous shock to β_0, in an economy without aggregate uncertainty.

Proposition 1 ***(Response of the Economy Under Perfect Foresight.)*** *Let the economy be in its steady state and consider an unexpected infinitesimal shock to the discount factor at the initial period, $d\beta_0$. Under Assumptions 1-3, to a first order, the general-equilibrium impact of the shock on output in the initial period, Y_0, is given by:*

$$\frac{dY_0}{d\beta_0} = \left(\frac{1}{1 - \overline{MPC}_z + \kappa \overline{MPC}_a}\right) \int \frac{\partial c_{i,0}}{\partial \beta_0} di \equiv \mathcal{M} \int \frac{\partial c_{i,0}}{\partial \beta_0} di, \tag{32}$$

with $\overline{MPC}_z \equiv \int z_i MPC_i di$ and $\overline{MPC}_a \equiv \int a_i MPC_i di$ respectively representing the (steady-state) labor-productivity- and the asset-weighted averages of individual marginal propensities to consume.

[26] The analogue to (32) when the central bank is not constrained by the lower bound is shown and explained in Appendix H.

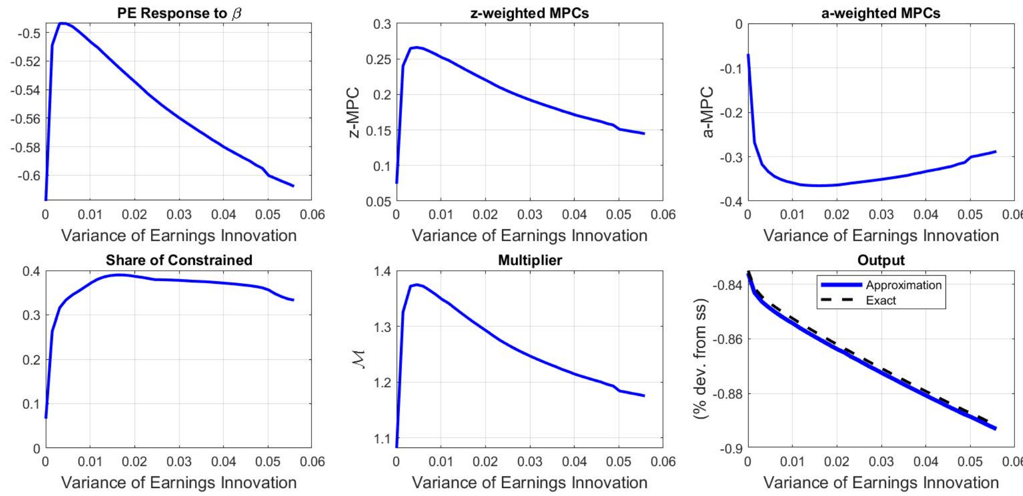
The right hand-side of (32) can be divided into two terms. The latter, $\int \frac{\partial c_{i,0}}{\partial \beta_0}$, corresponds to the partial equilibrium consumption response to the shock. The former, in parenthesis, is a general equilibrium multiplier, which we denote by $\mathcal{M}$. The key to understand the differences in equilibrium responses is to compare these terms across economies. In Figure 9, we consider how they vary in economies with different degrees of earnings risk, as in the left panel of Figure 8.

First, we focus on the partial equilibrium term. In the heterogeneous-agents economy, only a portion of households reacts to changes in the discount factor, precisely those who, in equilibrium, are *unconstrained* in their asset choice. In the representative-agent model, instead, all households are unconstrained and thus react to the changes in β. Figure 9 shows that the term $\int \frac{\partial c_{i,0}}{\partial \beta_0}$ is linked to the share of constrained individuals in the economy (top- and bottom-left panels). With a low degree of idiosyncratic risk, only a few individuals are constrained, and the response is relatively strong. On the other hand, very high risk levels also induce households to save away from the constraint, resulting in an inverse-U-shaped pattern of the share of constrained individuals. Consequently, the partial equilibrium response exhibits a similar behavior. In all though, over a wide range around our preferred calibration, the partial equilibrium response remains well below that of the representative agent.

The multiplier $\mathcal{M}$ features two distinct channels. The first is the classic consumption multiplier of changes in income, represented by $\overline{MPC}_z$. A decline in output will induce a further decline in consumption, which feeds back into output. Because changes in individual income resulting from changes in output are proportional to labor productivity, the magnitude of this mechanism depends on the $\overline{MPC}_z$, or the *aggregate* MPC of this economy. The second term, $\kappa\overline{MPC}_a$, represents the impact of changes in the real value of assets due to inflation, i.e. the Fisher channel. An unexpected fall in inflation revalues household assets, redistributing wealth in the economy, thus affecting aggregate consumption.

The size of $\mathcal{M}$ depends on how reactive consumption is to changes in income and wealth. Figure 9 shows that, as we depart from the RANK case (from the left), households' sensitivity to consumption rises quickly, with $\overline{MPC}_z$ rising fast and $\overline{MPC}_a$'s de-

Figure 9: Decomposition of Equation (32)



Notes: The figures show the different terms of (32), together with the share of constrained households (bottom left), and the output response at $t = 0$. See text for explanations for each term. The x-axis displays different values for the earnings variance.

clining fast. Note that the latter is negative due to the fact that high-MPC households are debtors in this economy, which further contributes to amplifying the shock. Finally, if risk becomes sufficiently high, the sensitivity of consumption begins to decline, as, in equilibrium, low-income individuals hold relatively more savings compared to the case with low risk. In all, though, the multiplier $\mathcal{M}$ follows a similar pattern as the $\overline{MPC}_z$'s, as the Fisher channel has limited impact.[27]

In sum, the multiplier is more important in HANK relative to RANK.[28] Yet the overall response also depends on the partial equilibrium effect of the shock. To contrast those offsetting forces, the bottom right panel of Figure 9 displays the impact of a one-period shock to β_0 under Assumptions 1-3. Note that the response of the economy is well approximated by (32), suggesting that, for a one-period shock, Assumption 2 is a satisfactory. In this case, as earnings dispersion increases, so does the overall impact of the

[27] In the classic version of the representative-agent model, where the net supply of assets is null, the MPC equal $1-\beta$, thus the multiplier equals $\frac{1}{\beta} \approx 1$. Instead, in our baseline calibration of Section 3, $\overline{MPC}_z = 0.15$, $\overline{MPC}_a = -0.20$, yielding a multiplier of 1.18.

[28] This fact is qualitatively unchanged when we consider the role of active monetary policy, as we show in Appendix H. However, in that case, the quantitative importance of MPCs in determining the response of the economy is much smaller than in the case when the lower bound is active.

shock. In other words, although the change in the PE response partially compensates for the change in the multiplier, the net effect is still a relatively stronger impact of the shock in the heterogeneous-agent economy.[29] In all, the movement in MPCs and the ensuing indirect effects dominate the partial equilibrium direct effect of the shock, which explains the larger impact of the shock in the heterogeneous-agent model.

Aggregate Uncertainty. We now evaluate the implications of Proposition 1 in light of the uncertainty structure proposed in Section 2. Consider an economy similar to that proposed in Proposition 1, but it can instead follow two possible branches ($\tau^{max} = 2$). In the first, entered with a probability $1 - \mu$, no shocks are expected, nor will ever realize. In the second branch, $\tau = 2$, a shock to β_1^2, $d\beta_1^2$ is expected at time $t = 1$.[30]

Thus, the impact of a shock to β_1^2 on output at period 1, branch $\tau = 2$ is given by the expression in Proposition 1, evaluated at $\tau = 2$, $t = 1$. Thus, if the economy reaches the branch $\tau = 2$, it will enter a recession. Naturally, the recession will be deeper if the central bank is constrained by the lower bound. What is more, this amplification is stronger in HANK, as shown in Proposition 1. The recession, in turn, feeds back into period one, as individuals increase their savings and consequently aggregate demand is reduced.

The strength of the feedback mechanism depends on how strongly forward-looking households in the economy are. This, in turn, is related to the share of constrained individuals, which is higher in HANK (relative to RANK). In the interest of space, we relegate this discussion to Appendix H. In all, though, as we have seen in Figure 8, the mechanisms associated to Proposition 1 prevail, and the amplification caused by aggregate uncertainty is larger in HANK.

Finally, note that the effects described above occur at every node along the diagonal TS path (Figure 3). That is, whenever the ZLB binds at a node τ, forward-looking indi-

[29]Consider a Two-Asset New-Keynesian model (Bilbiie, 2008), where there are two types of agents: savers (low MPC) and hand-to-mouth (unit MPC) agents, and assets are in zero net supply. In this case, a higher proportion of hand-to-mouth agents increases the aggregate MPC of the economy, but also reduces the partial equilibrium term $\int \frac{\partial c_{i,0}}{\partial \beta_0}$, as these agents are credit-constrained. It turns out that, in this model, these effects *exactly offset* each other (Kaplan, Moll and Violante, 2018). As a result, the amplification at the lower bound is similar to that in RANK.

[30]In this explanation, similarly to Section 2, we do not consider changes to β_0, the arguments are completely unchanged in that case.

viduals at note $\tau - 1$ increase their savings, leading to a recession. This process proceeds backwards, amplifying the recession at every node.

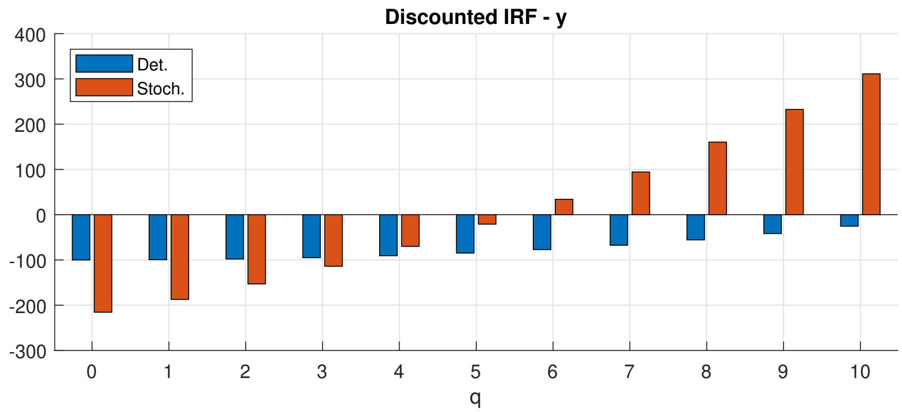
6 Application - Forward Guidance

In this section, we study the effect of forward guidance in our HANK model and the differential effects between the stochastic and deterministic environment. This analysis, as the one highlighted in Figure 5 requires multiple simulations of our model, which is rendered feasible by our solution methodology.

We consider the following forward guidance policy: the central bank credibly announces that it will set the nominal interest rate to 0 for q additional quarters, relative to what would be implied by the Taylor rule. The extra stimulus q is unconditional on the specific contingency realization, implying that this policy increases the expected duration of the of the ZLB periods exactly by q quarters. This analysis is similar to what McKay, Nakamura and Steinsson (2016) label as extended policy (where they choose q to minimize output loss on impact in a RANK economy) and to some extent goes in the direction of the state-contingency mentioned by Woodford (2012). Under the deterministic shock, such policy also corresponds to the "fixed length forward guidance" policy in Eggertsson et al. (2021). However, the equivalence does not hold with the stochastic shock as the actual duration of the aggregate shock is unknown until the reversion is realized.

Figure 10 shows the effects, on the discounted impulse response of output, of forward guidance under the deterministic (blue bars) and stochastic (red bars) shocks as a function of the q quarters of extra stimulus. All bars are relative to the no forward guidance policy under the deterministic case. The plot reveals that forward guidance is more effective under the stochastic case: with the calibrated shock, it takes 6 quarters of extra stimulus to actually flip the output loss to an output gain. The same does not happen under the deterministic shock, despite the fact that the output loss is smaller under it than under the stochastic shock to begin with. The reason is that forward guidance keeps interests rate at the lower bound regardless of the shock realization, essentially removing the nonlinearity and, thus, its interactions with the aggregate uncertainty, in the first place.

Figure 10: Discounted IRF and Forward Guidance - Output



Notes: The figure reports the implied discounted impulse response functions for output under the HANK model with the ZLB, in the forward guidance experiment. The order of the columns corresponds to the quarters of extra stimulus under the forward guidance policy. Within each column, the blue (red) bar corresponds to the deterministic (stochastic) case. All bars are relative to the one in the HANK model with the ZLB, under the deterministic shock and with no extra stimulus (left most blue bar).

7 Conclusions

We develop a novel methodology to solve heterogeneous agents models with aggregate uncertainty and a Zero Lower Bound on nominal interest rates. By considering a two-states Markov shock structure as in Eggertsson et al. (2021), our methodology exploits and expands the techniques proposed by Auclert et al. (2021) and Guerrieri and Iacoviello (2015). Its efficiency and flexibility let us consider several counterfactual policies and robustness scenarios.

We show that, when the Zero Lower Bound binds, aggregate uncertainty amplifies a demand shock, and this amplification is much stronger if we consider a HANK economy. In our benchmark calibration, household heterogeneity nearly doubles the amplification that takes place due to aggregate uncertainty at the ZLB. However, if the monetary authority is unconstrained, no amplification takes place, either in RANK or HANK economies. The increased amplification in HANK is robust to several calibrations and choice of parameters. We show that this result is linked to the presence of both high-MPC and forward-looking (locally unconstrained) individuals in heterogeneous-agents economies.

Finally, we also use our solution methodology to study the impact of forward guidance. The model simulations indicate that the marginal effects of a promise to keep interest rates at the lower bound for an extra quarter are larger when there is aggregate uncertainty.

We hope that our methodology allows future researchers to better understand the role of uncertainty both in the micro and macro level, and to study other types of policy such as government transfers and the impact of other nonlinearities at the macro level such as occasionally binding constraints in the financial sector. Our results concerning amplification in heterogeneous-agents economies are certainly a step in that direction.

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Appendix

A Notation List

- X_t and X_t^τ are vectors of 5 entries (output, inflation, nominal rates, taxes, government debt)

- D_t and D_t^τ are vectors of $n_a \times n_z$ entries, representing the distribution over individual states at the beginning of period t.

- V_t and V_t^τ are vectors of $n_a \times n_z$ entries, representing the value function at period t.

- $\mathbb{X}^{TS}$ is a stacked vector made of $\tau^{\max}$ vectors of 5 entries representing inputs along the TS branch. $\mathbb{X}^{TS} = \{X_t\}_{t=0}^{\tau^{\max}-1}$

- $\mathbb{D}^{TS}$ is a stacked vector made of $\tau^{\max}$ vectors of $n_a \times n_z$ entries representing the distribution at the beginning of each period along the TS branch. $\mathbb{D}^{TS} = \{X_t\}_{t=0}^{\tau^{\max}-1}$

- $\mathbb{V}^{TS}$ is a stacked vector made of $\tau^{\max}$ vectors of $n_a \times n_z$ entries representing the value functions the TS branch. $\mathbb{V}^{TS} = \{V_t\}_{t=0}^{\tau^{\max}-1}$.

- $\mathbb{X}^\tau$ is a stacked vector made of T vectors of 5 entries representing inputs along one of the PF branches. $\mathbb{X}^\tau = \{X_t^\tau\}_{t=\tau}^{T+\tau-1}$.

- $\mathbb{D}^\tau$ is a stacked vector made of T vectors of $n_a \times n_z$ entries representing the distribution at the beginning of each period of the perfect foresight branch. $\mathbb{D}^\tau = \{D_t^\tau\}_{t=\tau}^{T+\tau-1}$.

- $\mathbb{V}^\tau$ is a stacked vector made of $\tau^{\max}$ vectors of $n_a \times n_z$ entries representing the value functions in one of the PF branches. In loose sense $\mathbb{V}^\tau = \{V_t^\tau\}_{t=\tau}^{T+\tau-1}$

- $\mathbb{X}^{PF} = \{\mathbb{X}^\tau\}_{\tau=1}^{\tau^{\max}}$, $\mathbb{D}^{PF} = \{\mathbb{D}^\tau\}_{\tau=1}^{\tau^{\max}}$, $\mathbb{V}^{PF} = \{\mathbb{V}^\tau\}_{\tau=1}^{\tau^{\max}}$.

- $\mathbb{X}_1^{PF} = \{X_\tau^\tau\}_{\tau=1}^{\tau^{\max}}$ is the collection of $5x1$ vectors of inputs in the first period of *each* PF path.

- $\mathbb{V}_1^{PF} = \{V_\tau^\tau\}_{\tau=1}^{\tau^{\max}}$.

B Technical Details

B.1 Aggregate State Variables

One of the arguments in equation (26) is the vector of aggregate variables $X_{\tau-1}$, during the period right before the contingency is revealed. Those are initial conditions for the τ-th PF branch under consideration. Given the dynamic programming structure for the households' problem, those initial conditions do not enter the heterogeneous-agent block. However, they might enter some aggregate equilibrium conditions. One example is the stock of public debt $b_{\tau-1}$. This is an initial condition that should be taken into account under a more general fiscal policy rule. We account for the effects of aggregate state variables in equation (26) by deriving the (analytical) Jacobian of aggregate equilibrium conditions with respect to these variables.

B.2 Lags

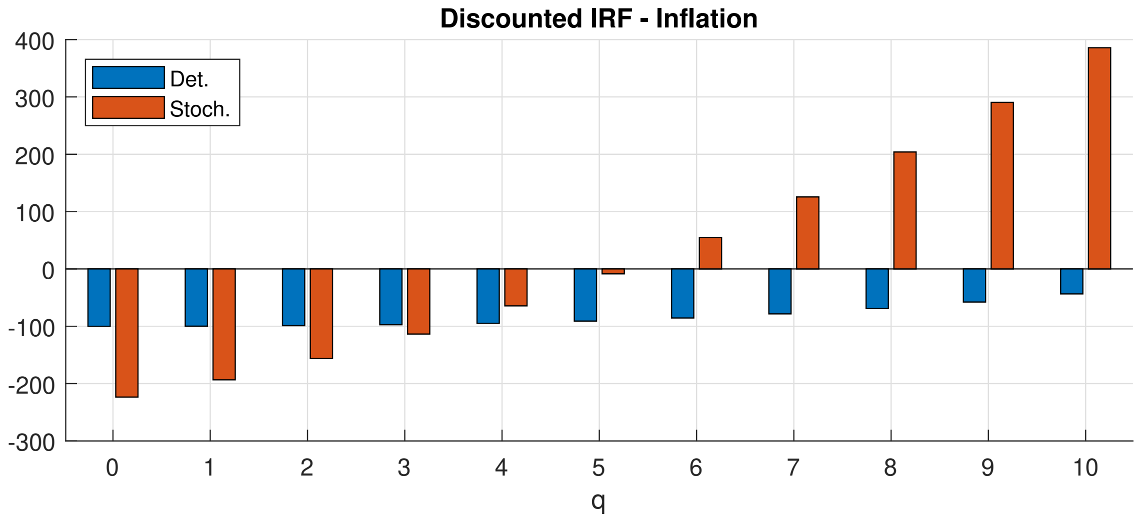
In our model economy, there were no significant variables that entered with a lag larger than 1. This might not be true for more complex models. For instance, if one was to study the new Average Inflation Targeting framework (AIT) of the Federal Reserve, we need to keep track of many past levels of inflation. In particular, once entering a contingency τ, it would not be sufficient to carry over the information in $X_{\tau-1}$ as it is currently defined. The solution is to define an aggregate variable which at time t takes the value of the lag variable of interest. To give a practical example, if the model requires to keep track of inflation 2 periods in the past, define $\Pi_{Lag2,t} = \Pi_{Lag1,t-1}$ and $\Pi_{Lag1,t} = \Pi_{t-1}$, and use them as other structural equation. The variable $\Pi_{Lag2,t}$ corresponds to the inflation with two lags.

B.3 Leads

Some models might require to form expectations of future variables with lead larger than 1. As an example, suppose that we are interested in considering in the equilibrium conditions the expectations for a variable x in l quarters in the future. The solution is to define l auxiliary variables as follows. $\Pi_{Lead1,t} = \mathbb{E}_t\Pi_{t+1}$, $\Pi_{Lead2,t} = \mathbb{E}_t\Pi_{Lead1,t+1}$, $\Pi_{Leadl,t} = \mathbb{E}_t\Pi_{Leadl-1,t+1}$.

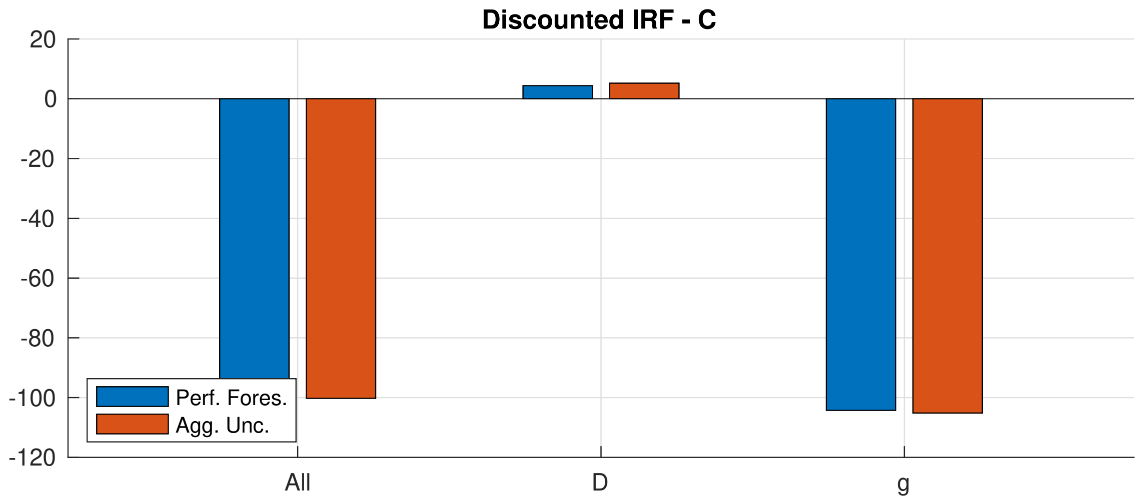
C Other Figures

Figure C.1: Discounted IRF and Forward Guidance - Inflation



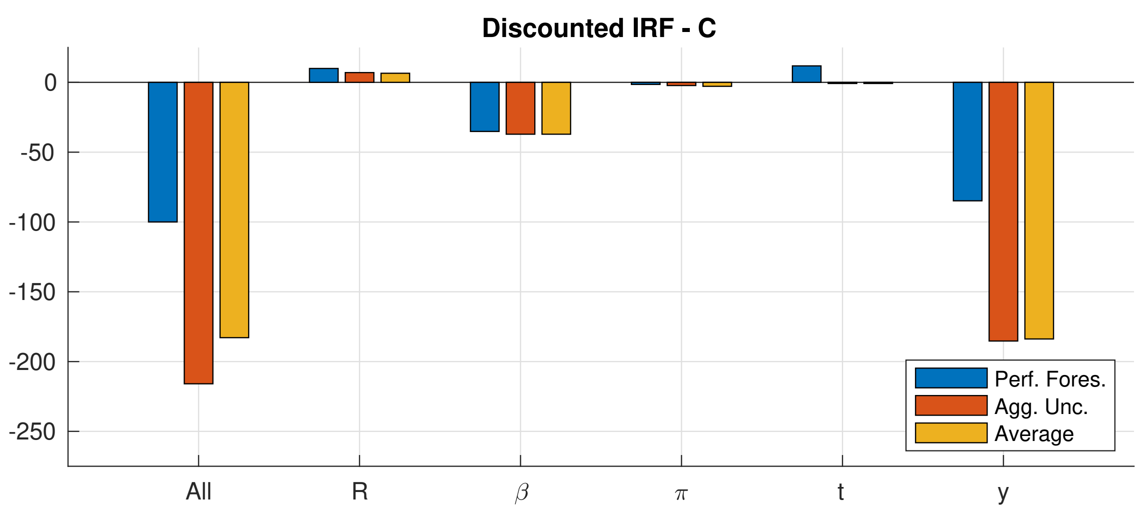
Notes: The figure reports the implied discounted impulse response functions for inflation under the HANK model with the ZLB, in the forward guidance experiment. The order of the columns corresponds to the quarters of extra stimulus under the forward guidance policy. Within each column, the blue (red) bar corresponds to the deterministic (stochastic) case. All bars are relative to the one in the HANK model with the ZLB, under the deterministic shock and with no extra stimulus (left most blue bar).

Figure C.2: Discounted IRF - Decomposition - D and g



Notes: The figure reports the implied discounted impulse response functions for consumption under the HANK model with the ZLB. The columns correspond to the full effects, the effects of the distribution, and the effects of the individual policies. Within each column, the blue (red) bar corresponds to the deterministic (stochastic) case. All bars are relative to the one in the HANK model with the ZLB, under the deterministic shock and with no extra stimulus (left most blue bar).

Figure C.3: Discounted IRF - Decomposition - Prices



Notes: The figure reports the implied discounted impulse response functions for consumption under the HANK model with the ZLB. The columns correspond to the full effects, the effects of nominal rate, discount factor, inflation, taxes, incomes. Within each column, the blue (red) bar corresponds to the deterministic (stochastic) case. The yellow bar corresponds to a deterministic counterfactual where agents are given the average of the prices in the stochastic case. All bars are relative to the one in the HANK model with the ZLB, under the deterministic shock and with no extra stimulus (left most blue bar).

D Simple Model - Additional Proofs and Results

The proofs and generalizations below refer to the simple model presented in Section 2. In addition, in what follows we generalize the utility function to a CRRA form with coefficient σ, of which log utility is a special case.

D.1 Amplification at $t = 0$

Proposition D.2 *If the ZLB binds, the marginal utility at $t = 1$ in the stochastic economy is always larger than its deterministic counterpart, i.e. $\mathbb{E}_0 Y_1^{-\sigma} > Y_{1DET}^{-\sigma}$, where the demand shock for the deterministic economy is constructed according to (6):*

Proof.

$$\mathbb{E}_0 Y_1^{-\sigma} > Y_{1DET}^{-\sigma} \iff$$

$$(1-\mu) + \mu \frac{\beta_{1L}}{\beta} > \frac{\beta_{1DET}}{\beta} = \left[\mu \left(\frac{\beta_{1L}}{\beta} \right)^{\frac{\sigma}{\sigma+\phi}} + (1-\mu) \right]^{\frac{\sigma+\phi}{\sigma}} \iff$$

$$\left((1-\mu) + \mu \frac{\beta_{1L}}{\beta} \right)^{\frac{\sigma}{\sigma+\phi}} > \mu \left(\frac{\beta_{1L}}{\beta} \right)^{\frac{\sigma}{\sigma+\phi}} + (1-\mu)$$

The last inequality is simply Jensen's inequality, given that $\frac{\sigma}{\sigma+\phi} \in (0,1)$. Note that $\mathbb{E}_0 Y_1^{-\sigma} > Y_{1DET}^{-\sigma}$ implies that, at $t = 0$, output in the stochastic economy is lower than its deterministic counterpart (see equation 5).

D.2 The case where the ZLB does not bind at $t = 1$ in the Deterministic Economy

In Figure 1 in the main text, we considered a shock β_{1L} that was sufficiently large so that, in the deterministic case, the ecomomy subject to β_{1DET} reached the zero lower bound at $t = 1$. That does not need to be the case if β is sufficiently low. Figure D.4, analogue to Figure 1, shows a situation in which β_{1L} is sufficiently large so that $R_{1L} = \underline{R}$, but the deterministic economy does not reach the ZLB at $t = 1$ (i.e. $R_{1DET} > \underline{R}$). We lay out the formal conditions over β_{1L} below. Note that, in this case, there is amplification due to the ZLB in the economy with aggregate uncertainty, but there is no such amplification in the deterministic economy, as it does not reach the ZLB.

D.3 The Case when the ZLB does not Bind at $t = 0$

Figure D.5 below shows the case in which the ZLB does not bind at $t = 0$. Note that the curve on the bottom-right panel exhibits no kink, yet output is still lower in the stochastic economy. This happens because the expected marginal utility is higher under the stochastic shock.

D.4 Conditions over β_{1L} for ZLB binding at $t = 1$ in the Deterministic Economy

Consider the deterministic economy at $t = 1$, with discount factor β_{1DET}. In all $t > 1$, the economy returns to the steady state. Thus, following equations (3) and (4), we have that the economy will reach the ZLB if at $t = 1$ if

$$\beta_{1DET} \geq \beta R_{ss}^{\frac{\sigma+\phi}{\phi}}.$$

In the equation above, we used $\underline{R} = 1$. Using the definition of β_{1DET}, we obtain:

$$\begin{aligned} \beta\left[\mu\left(\frac{\beta_{L1}}{\beta}\right)^{\frac{\sigma}{\sigma+\phi}} + (1-\mu)\right]^{\frac{\sigma+\phi}{\sigma}} &\geq \beta R_{ss}^{\frac{\sigma+\phi}{\phi}}, \\ \left[\mu\left(\frac{\beta_{L1}}{\beta}\right)^{\frac{\sigma}{\sigma+\phi}} + (1-\mu)\right] &\geq R_{ss}, \\ \beta_{L1} \geq \beta\left(\frac{R_{ss}-(1-\mu)}{\mu}\right)^{\frac{\sigma+\phi}{\sigma}} &\equiv \hat{\beta}_{L1}. \end{aligned} \tag{D.1}$$

Conditions over β_{1L} for ZLB binding at $t = 1$, Contingency 2 For the ZLB to bind under contingency 2 in the stochastic economy, we have a condition similar to the one above, but over β_{1L}

$$\beta_{1L} \geq \beta R_{ss}^{\frac{\sigma+\phi}{\phi}} \equiv \beta_{1L}^{*}.$$

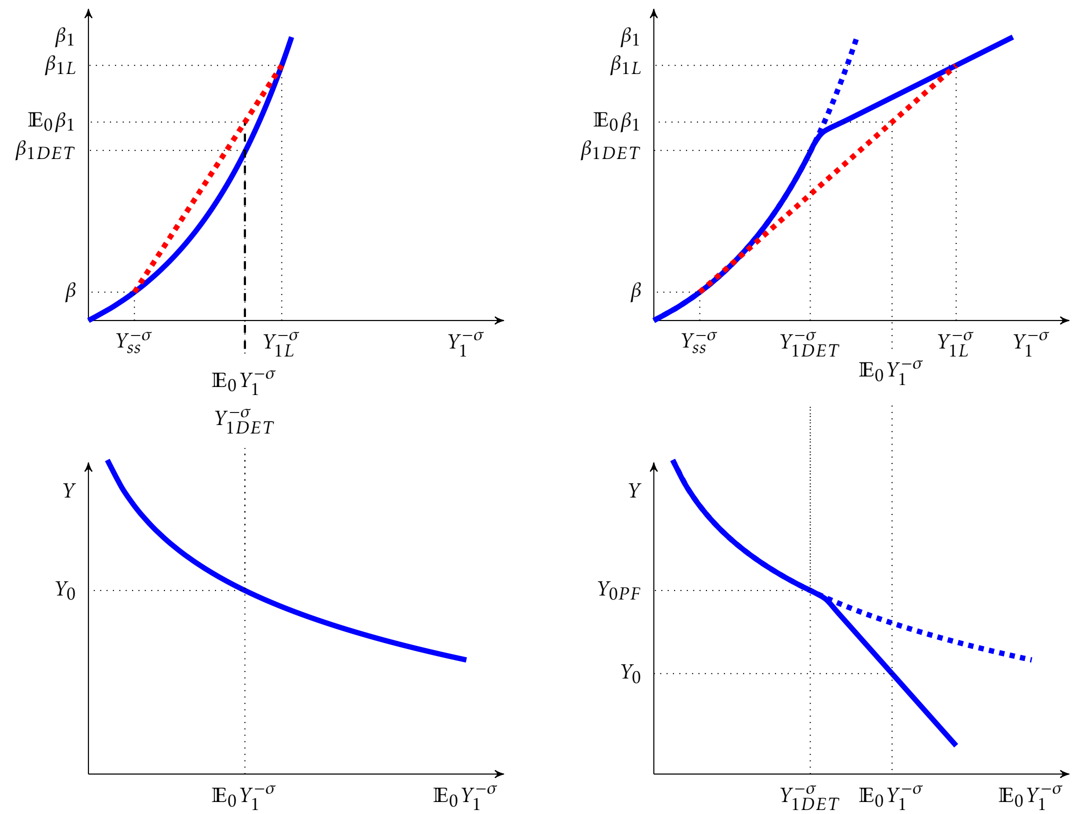
We can now compare β^*_{1L} and $\hat{\beta}_{1L}$

$$\begin{aligned}
\beta^*_{1L} < \hat{\beta}_{1L} &\iff \\
\beta R_{ss}^{\frac{\sigma+\phi}{\phi}} < \beta\left(\frac{R_{ss}-(1-\mu)}{\mu}\right)^{\frac{\sigma+\phi}{\sigma}} &\iff \\
R_{ss} < \left(\frac{R_{ss}-(1-\mu)}{\mu}\right) &\iff \\
\mu R_{ss} < R_{ss}-(1-\mu) &\iff \\
(1-\mu) < R_{ss}(1-\mu) &\iff \\
1 < R_{ss},
\end{aligned}$$

which is always true. Thus, it is always the case that there are values of β_1 such that, if $\beta_1 \in (\beta^*_{1L}, \hat{\beta}_{1L})$, the stochastic economy at $t = 1$ and contingency 2 will hit the ZLB, but its deterministic counterpart will not. This is the case depicted in Figure D.4. Further, if $\beta_1 > \hat{\beta}_{1L}$, then the ZLB will be reached in both cases, which is the scenario depicted in Figure 1.

Figure D.4: Equilibrium in the Simple Model

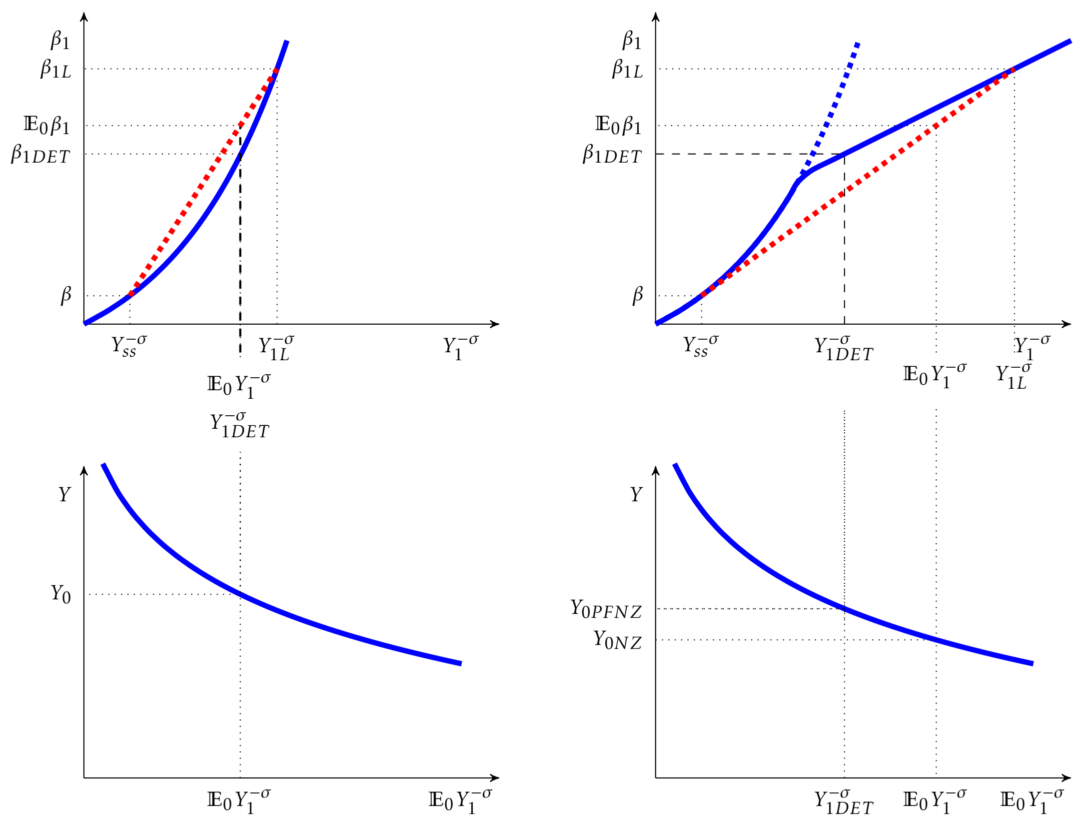
(a) Outside the ZLB: $\underline{R} = -\infty$ (b) With the ZLB: $\underline{R} = 1$



Notes: The figure shows the equilibrium of the simple model without the ZLB (left column, $\underline{R} = -\infty$) and with the ZLB (right column, $\underline{R} = 1$). The top panels report the equilibria at $t = 1$. The blue solid lines show the relationship between the discount factor β_1 on the y-axis and the corresponding marginal utility $Y_1^{-\sigma}$ on the x-axis. They also report the corresponding expected value $\mathbb{E}_0 Y_1^{-\sigma}$, obtained with a linear combination along the red dotted line. The blue dotted line on the top-right panel is reported for comparison. The bottom panels report the equilibria at $t = 0$. The blue solid lines show the relationship between output Y_0 on the y-axis and expected future marginal utility $\mathbb{E}_0 Y_1^{-\sigma}$ on the x-axis. The kink on the bottom right panel corresponds to the level value of future expected marginal utility above which the ZLB binds at $t = 0$. Y_{0PF} denotes output at $t = 0$ without uncertainty, whereas Y_0 refers to that in the economy with uncertainty.

Figure D.5: Equilibrium in the Simple Model

(a) Outside the ZLB: $\underline{R} = -\infty$ (b) With the ZLB: $\underline{R} = 1$



Notes: The figure shows the equilibrium of the simple model without the ZLB (left column, $\underline{R} = -\infty$) and with the ZLB (right column, $\underline{R} = 1$). The top panels report the equilibria at $t = 1$. The blue solid lines show the relationship between the discount factor β_1 on the y-axis and the corresponding marginal utility $Y_1^{-\sigma}$ on the x-axis. They also report the corresponding expected value $\mathbb{E}_0 Y_1^{-\sigma}$, obtained with a linear combination along the red dotted line. The blue dotted line on the top-right panel is reported for comparison. The bottom panels report the equilibria at $t = 0$. where we assume that the ZLB does not bind. The blue solid lines show the relationship between output Y_0 on the y-axis and expected future marginal utility $\mathbb{E}_0 Y_1^{-\sigma}$ on the x-axis. Y_{0PFNZ} corresponds to output at $t = 0$ without uncertainty, whereas Y_{0NZ} corresponds to the economy with uncertainty.

E Additional Model Derivations - Algorithms

E.1 Final Good Producer

A competitive representative final good producer transforms intermediate goods from a continuum of firms indexed by $j \in [0,1]$, using the following technology:

$$Y_t = \left[\int_0^1 y_{jt}^{\frac{\theta-1}{\theta}} dj\right]^{\frac{\theta}{\theta-1}},$$

where $\theta > 1$ is a parameter that governs the substitutability across different types of intermediate goods. The final good producer sells the homogeneous good Y_t to the consumers for a price P_t and pays inputs with their individual price P_{jt}. The price index P_t is defined as:

$$P_t \equiv \left(\int_0^1 P_{jt}^{1-\theta} dj\right)^{\frac{1}{1-\theta}}.$$

The Lagrangian of the problem faced by the final good producer follows.

$$\mathcal{L} = \int_0^1 P_{jt} y_{jt} dj - \lambda\left(\left[\int_0^1 y_{jt}^{\frac{\theta-1}{\theta}} dj\right]^{\frac{\theta}{\theta-1}} - Y_t\right).$$

Taking the first-order condition for an arbitrary j':

$$P_{j't} = \lambda\left[\int_0^1 Y_{jt}^{\frac{\theta-1}{\theta}} dj\right]^{\frac{1}{\theta-1}} y_{j't}^{-\frac{1}{\theta}}.$$

The above condition implies

$$y_{jt} = P_{j't}^{\theta} y_{j't} P_{jt}^{-\theta}$$

for any j and j'. By substituting the above condition into the production function, one obtains the following.

$$\left[\int_0^1 \left[P_{j't}^{\theta} y_{j't} P_{jt}^{-\theta}\right]^{\frac{\theta-1}{\theta}} dj\right]^{\frac{\theta}{\theta-1}} = Y_t$$

$$P_{j't}^{\theta} y_{j't}\left[\int_0^1 P_{jt}^{1-\theta} dj\right]^{\frac{-\theta}{1-\theta}} = Y_t$$

$$Y_{j't} = \left(\frac{P_{j't}}{P_t}\right)^{-\theta} Y_t.$$

E.2 Intermediate Good Producer

The profits maximization problem of an intermediate producer j can be simplified to the following:

$$\begin{aligned}
\max \quad & \left(1+\tau^d\right) P_{jt} y_{jt} - W_t N_{jt} - \frac{\psi}{2}\left(\frac{P_{jt}}{P_{jt-1}} - \overline{\Pi}\right)^2 P_t Y_t + \tau^d P_t Y_t + \\
& + \beta_t \mathbb{E}_t \left(\frac{Y_{t+1}}{Y_t}\right)^{-\sigma} \frac{P_t}{P_{t+1}} \left[\left(1+\tau^d\right) P_{jt+1} y_{jt+1} - W_{t+1} N_{jt+1} - \frac{\psi}{2}\left(\frac{P_{jt+1}}{P_{jt}} - \overline{\Pi}\right)^2 P_{t+1} Y_{t+1} + \tau^d P_{t+1} Y_{t+1}\right] \\
\text{s.t.} \quad & y_{jt} = N_{jt} \\
& y_{jt} = \left(\frac{P_{jt}}{P_t}\right)^{-\theta} Y_t,
\end{aligned}$$

where $\psi > 0$ governs the strength of the quadratic adjustment costs, W_t is the nominal wage rate, and τ^d is a standard subsidy that corrects the steady-state markup distortion. We substitute the production function and the demand function.

$$\begin{aligned}
& \left(1+\tau^d\right) P_{jt} \left(\frac{P_{jt}}{P_t}\right)^{-\theta} Y_t - W_t \left[\left(\frac{P_{jt}}{P_t}\right)^{-\theta} Y_t\right] - \frac{\psi}{2}\left(\frac{P_{jt}}{P_{jt-1}} - \overline{\Pi}\right)^2 P_t Y_t + \tau^d P_t Y_t + \\
& + \beta_t \mathbb{E}_t \left(\frac{Y_{t+1}}{Y_t}\right)^{-\sigma} \frac{P_t}{P_{t+1}} \left\{ \left(1+\tau^d\right) P_{jt+1} \left(\frac{P_{jt+1}}{P_{t+1}}\right)^{-\theta} Y_{t+1} \right. \\
& \left. - W_{t+1} \left[\left(\frac{P_{jt+1}}{P_{t+1}}\right)^{-\theta} Y_{t+1}\right] - \frac{\psi}{2}\left(\frac{P_{jt+1}}{P_{jt}} - \overline{\Pi}\right)^2 P_{t+1} Y_{t+1} + \tau^d P_{t+1} Y_{t+1} \right\}.
\end{aligned}$$

Taking the FOC one obtains the following condition.

$$\begin{aligned}
& \left(1+\tau^d\right)(1-\theta) \left(\frac{P_{jt}}{P_t}\right)^{-\theta} Y_t + \theta W_t \left[\left(\frac{P_{jt}}{P_t}\right)^{-\theta} Y_t\right] P_{jt}^{-1} - \psi \left(\frac{P_{jt}}{P_{jt-1}} - \overline{\Pi}\right) P_t Y_t \frac{1}{P_{jt-1}} + \\
& + \beta_t \mathbb{E}_t \left(\frac{Y_{t+1}}{Y_t}\right)^{-\sigma} \frac{P_t}{P_{t+1}} \psi \left(\frac{P_{jt+1}}{P_{jt}} - \overline{\Pi}\right) P_{t+1} Y_{t+1} \frac{P_{jt+1}}{P_{jt}^2} = 0
\end{aligned}$$

Assuming symmetry (i.e. $P_{jt} = P_{j't} = P_t$), the above simplifies to:

$$\left(1+\tau^d\right)(1-\theta) Y_t + \theta W_t Y_t P_t^{-1} - \psi \left(\Pi_t - \overline{\Pi}\right) \Pi_t Y_t + \beta_t \mathbb{E}_t \left(\frac{Y_{t+1}}{Y_t}\right)^{-\sigma} \psi \left(\Pi_{t+1} - \overline{\Pi}\right) Y_{t+1} \Pi_{t+1} = 0.$$

Rearranging the above expression one obtains:.

$$\left(\Pi_t - \overline{\Pi}\right) \Pi_t = \beta_t \mathbb{E}_t \left(\frac{Y_{t+1}}{Y_t}\right)^{-\sigma} \left(\Pi_{t+1} - \overline{\Pi}\right) \frac{Y_{t+1}}{Y_t} \Pi_{t+1} + \frac{\theta}{\psi} \left[w_t - \left(1+\tau^d\right) \frac{\theta - 1}{\theta}\right] \qquad \text{(E.2)}$$

Substituting the markup $\left(\tau^d = \frac{\theta}{\theta-1}\right)$ and the aggregate labor supply condition yields:

$$\left(\Pi_t - \overline{\Pi}\right)\Pi_t = \beta_t \mathbb{E}_t\left[\left(\frac{Y_{t+1}}{Y_t}\right)^{1-\sigma}\left(\Pi_{t+1} - \overline{\Pi}\right)\Pi_{t+1}\right] + \kappa\left[Y_t^{\omega+\sigma} - 1\right]$$

Finally, aggregate dividends and the production function are given by:

$$d_t = Y_t - w_t N_t \tag{E.3}$$

$$Y_t = N_t. \tag{E.4}$$

F Additional Information - Solution Method

F.1 Computation of Exact Equilibrium

Below we describe how to compute the *exact equilibrium* in the perfect-foresight contingencies of our economy, with the possibility of a binding Zero Lower Bound.

Algorithm 3 ***Perfect-Foresight Contingencies - Exact Equilibrium with ZLB.*** *Given $\{D_\tau^\tau\}_{\tau=1}^{\tau^{\max}}$ and $\{\mathbb{X}^\tau\}_{\tau=0}^{\tau^{\max}-1}$, initialize the set of periods in which the ZLB binds $o^n = \emptyset$, $n = 0$.*

1. *Perform step 1 in Algorithm 2, obtaining $\mathbb{X}^{\tau,0} = \mathbb{X}_{ss} + d\mathbb{X}^\tau$.*

2. *Compute $\mathbf{F}^{PF}(\mathbb{X}^\tau | D_\tau^\tau, X_{\tau-1})$ by forward-simulating the economy along all contingencies.*

3. *If $\|\mathbf{F}^{PF}(\mathbb{X}^\tau | D_\tau^\tau, X_{\tau-1})\| \le \epsilon$ for a given $\epsilon > 0$, conditional on o^n, the exact equilibrium is found (up to the tolerance ϵ). If not, update the endogenous variables in each contingency according to the formula:*

$$\mathbb{X}^{\tau,m+1} = \mathbb{X}^{\tau,m} - \left(\mathbf{F}_{\mathbb{X}}^{PF}\right)^{-1}\mathbf{F}^{PF}(\mathbb{X}^\tau | D_\tau^\tau, X_{\tau-1})$$

   and return to step 2.

4. *Using the resulting $\mathbb{X}^\tau$, perform steps 2-3 in Algorithm 2.*

5. *If $o^n \neq o^{n+1}$, return to step 1. Else, $\mathbb{X}^\tau$ represents the exact equilibrium inputs for contingency τ, given on pre-set initial conditions.*

F.2 Running Times and Additional Implementation Details

To solve the household problem in the model described in Section 3, we discretize the asset grid in n_a points and the income grid in n_z points, and employ the endogenous grid method proposed by Carroll (2006).

In Table F.1 we show the running times for each distinct specification, together with the maximum deviation of equilibrium conditions (17). The benchmark model features $n_z = 3$ and $n_a = 100$. In addition, we include a case with $n_z = 15$ where earnings follow an AR(1) process whose innovations are drawn from a mix of normal distributions, and calibrate the parameters as in Mendicino, Nord and Peruffo (2021), matching high-order moments of the distribution of earnings changes.[31] Finally, the last two columns present the algorithm performance in a basic two-asset model, whose details are relegated to Appendix I. In this case, grids feature $n_z = 7, n_a = 31$, and 30 points for the illiquid asset. We keep the aggregate shock structure the same as in the benchmark. Codes are written in Matlab and were ran on an Aurora Desktop with 3.00Ghz processor and 32GB RAM.[32]

Table F.1: Running Times

Specification	**Benchmark**		**MNP**		**Two Asset**	
Step	Time	Max. Err.	Time	Max. Err.	Time	Max. Err.
Steady State	1.22	-	0.58	-	82	-
All Jacobians	1.25	-	13.7	-	21.4	-
Algo. 1 (Step) - First-Order	2.6	0.5%	3.3	0.5%	-	-
Algo. 1 (Step) - Exact only on TS	2.8	0.004%	6.1	0.0006%	40.0	1.2%
Algo. 1 (Step) - Exact Equilibrium	4.6	0.000006%	39.9	0.000001%	163	0.06%

Notes: Times are given in seconds. "Benchmark" refers to the model calibrated as in Section 3, while "MNP" stands for the model calibrated as in Mendicino, Nord and Peruffo (2021), and "Two Asset" refers to the two-asset HANK model presented in Appendix I. The row "Algo. 1 - First-Order" refers to the solution of both PF and TS branches via first-order perturbation. The row "Algo. 1 - Exact only on TS" refers to the solution of PF paths via perturbation but the exact equilibrium computed in the TS branch. The row "Algo. 1 - Exact Equilibrium" solves for the exact equilibrium in the economy. For the latter three rows, the time is given in seconds per iteration of Algorithm 1. Max errors correspond the maximum absolute value of the asset market clearing equilibrium condition, given as a percentage of steady-state total asset holdings.

The runs in Table F.1 correspond to the equilibrium of the economy in response to a shock to the discount factor β that introduces uncertainty in the economy. For the two versions of the one-asset economy, the shock is the same as in Section 5 ($\mu = 0.9$, $d\beta = 0.0125$), whereas for the two-asset economy $\mu = 0.75$ and $d\beta = 0.0250$, ensuring that

[31] Specifically, their estimation targets the cross-sectional variance of log annual earnings, the standard deviation, the skewness and kurtosis of log annual earnings changes, and the ratio of the 90th to the 10th percentile of log changes.

[32] Parallelization with 12 cores is used in Step 2 of Algorithm 1.

the ZLB also binds.

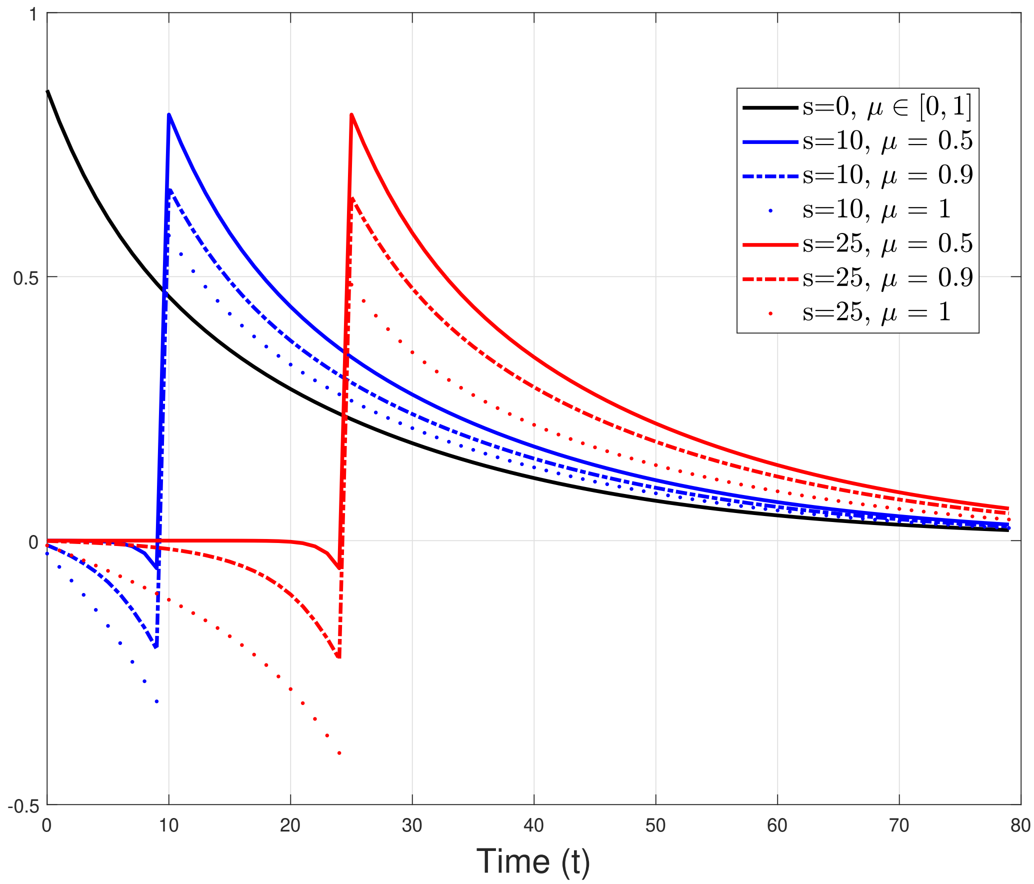
For the two versions of the one-asset model, we see that under all scenarios the maximum errors are small, even with the first-order solution. The main results we present in Section 5 are in practice unchanged quantitatively for any of the setup choices. The initial impact of the shock on output equals -8.02% in the first-order approximation and -8.10% in the other two cases. On the other hand, precision is somewhat reduced for the case two-asset model when we do not enforce the exact equilibrium in the two types of branches.[33]

F.3 Heterogeneous-Agents Jacobian with Aggregate Uncertainty - An Example

How does aggregate uncertainty affect the heterogeneous-agents Jacobians? Intuitively, it affects the reaction to news regarding changes in future inputs. In particular, because households attribute a probability $\mu^s < 1$ that a node in the TS branch s periods ahead will be reached, they under-react to future news, relative to the case in which $\mu = 1$. Figure F.6 plots the response of aggregate savings to changes in output, i.e. a partial equilibrium change in Y_s, at different horizons s, for different degrees of uncertainty μ, i.e. $\left\{\frac{d\mathcal{Y}_t}{dY_s}\right\}_{t=0}^{\tau^{\max}-1}$, where $\mathcal{Y}_t$ representing aggregate savings. Recall that changes in output Y_s have a direct impact on individual labor income, as household i's gross earnings is given by $z_i Y_s$.

[33]The two-asset model requires the computation of the exact equilibrium in the TS branch to ensure convergence, hence the row for "First-Order" is unfilled.

Figure F.6: Asset Market Clearing Jacobian $\left(\frac{d\mathcal{Y}_t}{dY_s}\right)_{t=0}^{\tau^{\max}-1}$ for Distinct μ's



Notes: The figure displays the columns of the heterogeneous-agent Jacobians for changes in *aggregate savings* (A) in response to news about changes in income (Y_s) at different times (s) under different degrees of uncertainty (as governed by μ). The black-solid lines represent the case of contemporaneous income changes, thus the degree of uncertainty is immaterial.

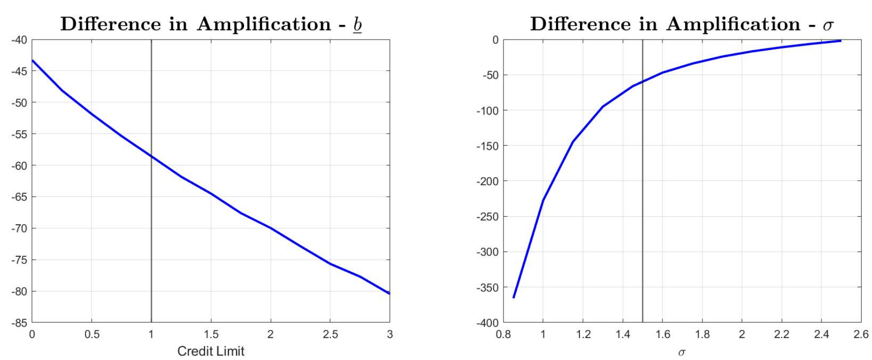
Note that, because contemporaneous changes in inputs are *certain*, different values for the uncertainty parameter μ have no impact on the change in households savings (black line) when $s = 0$. At horizons $s > 0$, though, uncertainty matters. Recall that agents are informed of a state of the world in which income is higher at some future point s, so in the times leading to such period, they start consuming part of this future income by tapping on their savings stock. The lower μ is, the weaker is the reduction in savings in anticipation of changes in output, as households attribute low probabilities to that event. The anticipation is particularly muted for distant horizons. This can be seen, for instance, in the solid red line: there is essentially no reaction to news of a potential change in output happening 25 periods ahead, because the perceived probability of this event actually taking place is negligible. On the other hand, in the case of $\mu = 1$ (dotted line), households immediately react to the certain expectation of a change in output happening even 25 periods ahead.

When the shock materializes ($t \geq s$) the impact on savings is stronger with aggregate uncertainty, relative to the case when $\mu = 1$. This can be seen by comparing the solid and dashed blue and red lines with their dotted counterparts. The intuition is that, when $\mu = 1$, households front-load the consumption a relatively large portion of the expected income windfall. Instead, with uncertainty, consumption front-loading is relatively muted, and a relatively larger portion of the windfall is consumed after it materializes.

G Robustness Exercises

Figure G.7 below examines the robustness of our results in Section 5 with respect to different calibrations of the debt limit (left panel) and the parameter σ, measuring the curvature of the utility function.

Figure G.7: Amplification due to Aggregate Uncertainty and ZLB - HANK vs. RANK - Debt Limit and σ



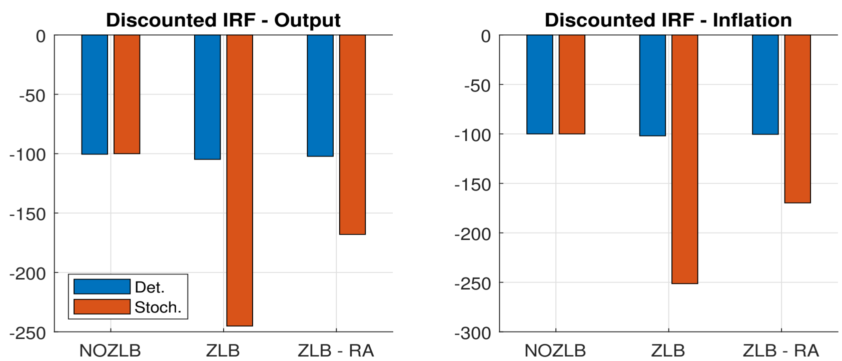
Notes: The figure plots the difference between the gaps among the "ZLB" and "ZLB - RA" red and blue bars in Figure 7, but for different credit limits (left) and parameter σ (right). For each different value of the parameters considered, β is recalibrated so that the steady-state nominal interest rate is unchanged. The remaining parameters are the same as in Table 1. Vertical lines denote the baseline calibration values.

In all cases considered, the differences in amplification remain positive. Note that, as σ increases, the difference is smaller. This is because, for high values of σ,the partial equilibrium impact of the shock gets muted, as the elasticity of intertemporal substitution is low. As a result, the recession is less deep and, for sufficiently high values, the Zero Lower Bound does not bind and no amplification is observed whatsoever.

G.1 Sensitivity - $\tau^{\max}$

To ensure that we selected a sufficiently large horizon τ^{max}, we re-run our main exercise with $\tau^{\max} = 200$. Results are nearly unchanged (quantitatively), with the expected loss in terms of output at 2.2% in the deterministic case and by 68.0% in the stochastic case, while the corresponding numbers for the HANK model at 4.8% and 145%.

Figure G.8: Discounted IRF - Output and Inflation ($\tau^{\max} = 200$)



Notes: The figure reports the implied discounted impulse response functions for output (left panel) and inflation (right panel) under the HANK model without the ZLB (first column), the HANK model with the ZLB (second column), and the RANK model with the ZLB (third column). Within each column, the blue (red) bar corresponds to the deterministic (stochastic) case. All bars are relative to the one in the HANK model without the ZLB, under the deterministic shock (left most blue bar). $\tau^{\max}$ is set to 200.

H Proofs and Derivations - Section 5.4

H.1 Replication of Proposition 1 without a Lower Bound to Nominal Rates

Proposition H.3 *Consider the shock described in Proposition 1, but now assume that $\overline{R}$ is larger than $\underline{R}$, thus the lower bound is not achieved for infinitesimal shocks. Then, the response* *of the economy is given by:*

$$dY_0 = \left(\frac{1}{1-(1-\kappa(\phi-1)+\bar{\tau})\overline{MPC}_z + \kappa\overline{MPC}_a - \kappa\phi_\pi \int \frac{\partial c_{i,0}}{\partial R_0} di} \int \frac{\partial c_{i,0}}{\partial \beta_0} di \right) d\beta_0. \tag{H.5}$$

Note that the conditions outlined above are analogue to those in which the economy's steady-state interest rate is the lower bound itself. We thus interpret the difference between (32) and (H.5) as the impact of the lower bound to nominal rates. Further, note that Proposition is a generalization of Proposition 1 for when (i) $\phi_\pi \neq 0$ (i.e. monetary policy is active) and $\bar{b} \neq 0$. Accordingly, we offer a proof for the latter.

Given the simplifications, household i consumption function at the initial period can be written as:

$$c_{i0} \equiv c(\{\beta_0, x_i, Y_0, \pi_0, R_0, \tau_0\}, \{X_{ss}, \beta_{ss}\}), \tag{H.6}$$

where x_i is the household individual state variable (z_0, a_{-1}) and $\pi_0 = \Pi_0 - 1$. In what follows, we also define $r_t \equiv R_t - 1$ as the net nominal rate. In addition, variables with *bars* refer to their steady-state values.

Take the total derivative with respect to variables at $t = 0$, as those at $t = 1$ onward are unchanged.

$$dc_{i0} = \left(\frac{\partial c_{i0}}{\partial \beta} + \frac{\partial c_{i0}}{\partial Y_0}\frac{dY_0}{d\beta_0} + \frac{\partial c_{i0}}{\partial \pi_0}\frac{d\pi_0}{d\beta_0} + \frac{\partial c_{i0}}{\partial R_0}\frac{dR_0}{d\beta} + \frac{\partial c_{i0}}{\partial \tau_0}\frac{d\tau_0}{d\beta_0} \right) d\beta_0.$$

Aggregate and use market clearing ($C_0 = Y_0$):

$$\int dc_{i0} di = dY_0 = \left(\int \frac{\partial c_{i0}}{\partial \beta} di + \int \frac{\partial c_{i0}}{\partial Y_0}\frac{dY_0}{d\beta} di + \int \frac{\partial c_{i0}}{\partial \pi_0}\frac{d\pi_0}{d\beta} di + \int \frac{\partial c_{i0}}{\partial \tau_0}\frac{d\tau_0}{d\beta} + \frac{\partial c_{i0}}{\partial R_0}\frac{dR_0}{d\beta} di \right) d\beta. \tag{H.7}$$

Our goal is to simplify some elements on the right-hand-side. First, the term $\frac{\partial c_{i0}}{\partial Y_0}$ is related to the individual marginal propensity to consume. Recall that individual income $y_{it} = z_{it}(Y_t - \tau_t)$. Let MPC_i be the change in consumption for individual i with respect to an infinitesimal increase in the right-hand-side of the budget constraint in the steady state. We have:

$$\frac{\partial c_{i0}}{\partial Y_0} = \frac{\partial c_{i0}}{\partial y_{i0}}\frac{\partial y_{i0}}{\partial Y_0} = z_i MPc_i.$$

The term $\frac{\partial c_{i0}}{\partial \pi_0}$ represents the Fisher channel, i.e. the change in consumption due to a revaluation of nominal assets via inflation. It is also related to the marginal propensity to consume of agent i:

$$\frac{\partial c_{i0}}{\partial \pi_0}\frac{d\pi_0}{d\beta_0} = \frac{\partial c_{i0}}{\partial \pi_0}\frac{d\pi_0}{dY_0}\ \frac{dY_0}{d\beta} = \frac{\partial c_{i0}}{\partial \pi_0}\kappa\ \frac{dY_0}{d\beta_0}.$$

To understand how individual consumption changes with *current* inflation (in a partial sense), return to the budget constraint:

$$c_t + \frac{a_t}{R_t} = z_i(Y_t - \tau_t) + \frac{a_{t-1}}{1+\pi_t}.$$

We can also use the definition of MPC_i:

$$\begin{aligned} MPC_{i0} &\equiv \frac{\partial c_{i0}}{\partial \frac{a}{1+\pi_0}} = \frac{\partial c_{i0}}{\partial \pi_0}\frac{\partial \pi_0}{\partial \frac{a}{1+\pi_0}} \\ &\approx \frac{\partial c_{i0}}{\partial \pi_0}\frac{\partial \pi_0}{\partial (1-\pi_0)a}. \end{aligned}$$

where the approximation is valid for small values of π_0. Inverting the equation above, we obtain:

$$\frac{\partial c_{i0}}{\partial \pi_0} \approx MPC_{i0}\frac{\partial(1-\pi_0)a_i}{\partial \pi_0} = -a_i MPC_{i0}. \tag{H.8}$$

Government Budget and Taxes. The government budget constraint, given the fiscal rule, is:

$$\frac{\bar{b}}{R_0} + \tau_0 Y_0 = \frac{\bar{b}}{1+\pi_0}.$$

Using a similar approximation to before:

$$-\bar{b}dr_0 + d\tau_0\bar{Y} + dY_0\bar{\tau} \approx -\bar{b}d\pi_0,$$

where $r_0 = R_0 - 1$ is (net) nominal rate. Note that $dr_0 = dR_0$. Isolating $d\tau_0$ (ignoring the approximation):

$$\begin{aligned} d\tau_0 &= \frac{-\bar{b}d\pi_0 + \bar{b}dr_0 - \bar{\tau}dY_0}{\bar{Y}} \\ &= \frac{\bar{b}(dr_0 - d\pi_0) - \bar{\tau}dY_0}{\bar{Y}}. \end{aligned}$$

That is, taxes need to increase if r increases, they are reduced if inflation rises (due to the Fisher channel), and they are also fall if income rises. Using (i) $dr_0 = \phi\kappa dY_0$ and (ii) $d\pi_0 = \kappa dY_0$:

$$d\tau_0 = \left(\bar{b}\kappa(\phi_\pi - 1) - \bar{\tau}\right)\frac{dY_0}{\bar{Y}} \tag{H.9}$$

Thus, taxes increase to the extent that output falls (so that debt-to-GDP rises) and to compensate the Fisher Channel, as a higher-than-expected inflation reduces the real value of government debt if $\bar{b} > 0$, requiring lower taxes to service it.[34]

Plugging equations (H.9) and (H.8) and using the definition of MPC in (H.10), we obtain:

$$dY_0 = \left(\int \frac{\partial c_{i0}}{\partial \beta_0} di + \int \frac{\partial c_{i0}}{\partial Y_0}\frac{dY_0}{d\beta_0} di + \int \frac{\partial c_{i0}}{\partial \pi_0}\frac{d\pi_0}{d\beta_0} di + \int \frac{\partial c_{i0}}{\partial \tau_0}\frac{d\tau_0}{d\beta_0} + \frac{\partial c_{i0}}{\partial R_0}\frac{dR_0}{d\beta_0} di\right) d\beta_0 \tag{H.10}$$

Recasting the equation above in terms of dY and using the definitions of MPC's and the chain rule, we obtain:

$$dY_0 = \left(\int \frac{\partial c_{i0}}{\partial \beta_0} di + \overline{MPC}_z \frac{dY_0}{d\beta} - \kappa \overline{MPC}_a \frac{dY_0}{d\beta_0} - \overline{MPC}_z(\kappa(\phi - 1) - \bar{\tau})\frac{dY_0}{d\beta_0} + \int \frac{\partial c_{i0}}{\partial R_0} di \kappa\phi \frac{dY_0}{d\beta}\right) d\beta_0,$$

where $\overline{MPC}_z \equiv \int_i z_i MPc_{i0} di$ and $\overline{MPC}_a \equiv \int_i a_i MPc_{i0} di$ are respectively the $z-$ and asset-weighted ecomomy-wide marginal propensity to consume. Reorganizing:

$$\frac{dY}{d\beta} = \frac{1}{1 - (1 - \kappa(\phi - 1) + \tau)\overline{MPC}_z + \kappa\overline{MPC}_a - \kappa\phi \int \frac{\partial c_{i0}}{\partial R_0} di} \int \frac{\partial c_{i0}}{\partial \beta_0} di \tag{H.11}$$

Relative to (32) in Proposition 1, there are two new terms:

1. The impact of taxes through interest rates and the fiscal rule given by the term ϕ that multiplies $\overline{MPC}_z$.

2. The **direct impact of interest rates on consumption.** This is the impact of reactive monetary policy, given by the term $\kappa\phi \int \frac{\partial c_{i0}}{\partial R_0} di$, where $\int \frac{\partial c_{i0}}{\partial R_0} di < 0$.

Note that, if $\phi_\pi = 0$ and $\bar{b} = 0$, so that $\bar{\tau} = 0$, we are back to (32) in Proposition 1, completing the proof.

[34]The effect due to the decline in output would be absent if debt to GDP were constant.

H.2 Impact of Declines in Future Aggregate Demand

The impact of aggregate uncertainty operates through changes in the expected value of the future marginal utility, as shown in Section 2. In this section, we show how that effect propagates in a more complicated setup, allowing for consumer heterogeneity. To be clear, this section's goal is not to study how aggregate uncertainty in itself amplifies recessions. This, we have explored in Section 2. Instead, what we seek to understand here is how the strength of backward propagation of future shocks depends on features of the economy linked to the presence of heterogeneous agents. To that end, we consider a setup with aggregate uncertainty but, as will become clear, the insights obtained are independent of that particular feature.

We employ a similar model to the previous section to study the impact of aggregate uncertainty. To simplify the exposition, we begin by studying assume a "partial-general" equilibrium setup, in which changes in future output affect current outcomes. This change can be certain or uncertain, but there is no need to take a stand on it. We then study how this change affects current output via endogenous transmission to current ($t = 0$) general equilibrium variables, but not future variables (which, by assumption, are fixed). This analysis is intended to be a middle step – once completed, we move to a full general equilibrium analysis.

The setup is the same as the one laid out in Section 2, except that now we consider a richer economy populated by heterogeneous agents. It begins at its steady-state equilibrium. Households then become aware of a possible change in output one period ahead ($t = 1$). If this shock materializes, which happens with probability μ, the economy enters contingency 2, and output is Y_1^2. With probability $1 - \mu$, instead, all the exogenous variables remain at their state-state values forever. If this is the case, the economy enters (or remains at) branch 1.

In branch 1, $t = 1$, the economy will start with a potential new state D_1^1 (and implied b_1^1). As before, we assume that the distribution of households is irrelevant for the dynamics of the economy. The absence of any state variables and shocks, in turn, means that, if the economy enters branch 1, it will be back at its stationary equilibrium for $t \geq 1$.

In branch 2, we assume that at $t = 1$ the endogenous variables are fixed, except for Y_{1L}.

At $t = 2$, branch 2, using a similar argument as before, and given the absence of further shocks, the economy is back at the steady state.

Given these simplifying assumptions, the individual consumption function in period $t = 0$ can be written as:

$$c_{i0} \equiv c(\{x_i, Y_0, \pi_0, R_0, \tau_0, Y_2^1\}, \{X_{ss}, \beta_{ss}\}).$$

Note that, for now, we are assuming that variables at $t = 1$ are exogenous, but variables at $t = 0$ are endogenous.

The total derivative, when aggregated, is similar to equation (H.10):

$$\int dc_{i0} di = dY_1^2 = \left(\int \frac{\partial c_{i0}}{\partial Y_1^2} di + \int \frac{\partial c_{i0}}{\partial Y_0} \frac{dY_0}{dY_1^2} di + \int \frac{\partial c_{i0}}{\partial \pi_0} \frac{d\pi_0}{dY_1^2} di + \int \frac{\partial c_{i0}}{\partial \tau_0} \frac{d\tau_0}{dY_1^2} di \right) dY_1^2. \tag{H.12}$$

The first term in the parenthesis represents the partial equilibrium response of households to declines in future income. This is determined by the Euler Equation and by the probability attributed to the economy entering branch 2, μ. As stated before, we do not explicitly need to account for μ in expression (H.12), but it implicitly determines the impact of the derivatives presented. In our model presented in Section 3, the impact of μ is shown in Figure F.6.

The other three terms correspond to general equilibrium responses, or multipliers. We can treat them in a completely analogue way to before. Assuming the ZLB binds at $t = 0$, we obtain:

$$dY_0 = \left(\int \frac{\partial c_{i0}}{\partial Y_1^2} di + \int z_i MPC_i \frac{dY_0}{dY_1^2} di - \int a_i MPC_i \kappa \frac{dY_0}{dY_1^2} di + \int_i \bar{\tau} z_i MPC_i (1+\kappa) \frac{dY_0}{dY_1^2} \right) dY_1^2. \tag{H.13}$$

We can then rearrange it to obtain:

$$\frac{dY_0}{dY_1^2} = \left(\frac{1}{1 - (1 + (1+\kappa)\bar{\tau})\overline{MPC}_z + \kappa \overline{MPC}_a} \right) \int \frac{\partial c_{i0}}{\partial Y_1^2} di. \tag{H.14}$$

This is analogue to equation (H.10), only that the "shock" is to Y_1^2 and the ZLB binds at $t = 0$. We now proceed to a full general equilibrium analysis, where (H.14) will make a reappearance.

General Equilibrium. We now allow for changes in all endogenous variables in $t = 1$, branch 2. The rest of the setup is unchanged. In particular, $t = 1$, branch 1, we assume that the economy is back at the steady-state equilibrium. We now consider a shock to β_1^2 only.[35]

In this case, focusing on period 1, the consumption function is:

$$c_{i0} \equiv c_{i0} = c(\{x_i, Y_0, \pi_0, R_0, \tau_0, Y_1^2, \pi_1^2, R_1^2, \tau_1^2, \beta_1^2\}, \{X_{ss}, \beta_{ss}\}),$$

where all variables with superscript 2 refer to branch 2 (under which the shock materializes). The total change with respect to β_2 is:

$$\begin{aligned} dc_{i0} =& \bigg(\frac{\partial c_{i0}}{\partial \beta_2} + \frac{\partial c_{i0}}{\partial Y_0} \frac{dY_0}{d\beta_1^2} + \frac{\partial c_{i0}}{\partial \pi_0} \frac{d\pi_0}{d\beta_1^2} + \frac{\partial c_{i0}}{\partial \tau_0} \frac{d\tau_0}{d\beta_1^2} \\ & + \frac{\partial c_{i0}}{\partial Y_1^2} \frac{dY_1^2}{d\beta_1^2} + \frac{\partial c_{i0}}{\partial \pi_1^2} \frac{d\pi_1^2}{d\beta_1^2} + \frac{\partial c_{i0}}{\partial \tau_2} \frac{d\tau_2}{d\beta_1^2} \bigg) d\beta_1^2. \end{aligned}$$

Rearrange it slightly:

$$\begin{aligned} dc_{i0} =& \left(\frac{\partial c_{i0}}{\partial \beta_2} + \frac{\partial c_{i0}}{\partial Y_0} \frac{dY_0}{d\beta_1^2} + \frac{\partial c_{i0}}{\partial \pi_0} \frac{d\pi_0}{d\beta_1^2} + \frac{\partial c_{i0}}{\partial \tau_0} \frac{d\tau_0}{d\beta_1^2} \right) d\beta_1^2 \\ & + \left(\frac{\partial c_{i0}}{\partial Y_1^2} \frac{dY_1^2}{d\beta_1^2} + \frac{\partial c_{i0}}{\partial \pi_1^2} \frac{d\pi_1^2}{d\beta_1^2} + \frac{\partial c_{i0}}{\partial \tau_2} \frac{d\tau_2}{d\beta_1^2} \right) d\beta_1^2. \end{aligned}$$

Note the similarity between the first line of equation above and (H.12). Both refer to changes in consumption at $t = 0$ due to the shock, all else equal (although, in equation H.12, we assumed the shock was to dY_1^2). In addition, note that (i) π_0 is immediately determined by Y_0 via the Phillips Curve and (ii) τ_0 is determined by π_0 and Y_0. As a result, analogue derivations to those in Section H.1 can be applied to the first line of the equation above.

Integrating, we obtain:

$$\begin{aligned} dY_0 =& \left(\int \frac{\partial c_{i0}}{\partial \beta_1^2} di + \int z_i MPc_{i0} \frac{dY_0}{d\beta_1^2} di - \int a_i MPC_{i0} \kappa \frac{dY_0}{d\beta_1^2} di + \int_i \bar{\tau} z_i MPC_i (1+\kappa) \frac{dY_0}{d\beta_1^2} \right) d\beta_1^2 \\ & + \int_i \left(\frac{\partial c_{i0}}{\partial Y_1^2} \frac{dY_1^2}{d\beta_1^2} + \frac{\partial c_{i0}}{\partial \pi_1^2} \frac{d\pi_1^2}{d\beta_1^2} + \frac{\partial c_{i0}}{\partial \tau_2} \frac{d\tau_2}{d\beta_1^2} \right) d\beta_1^2. \end{aligned} \tag{H.15}$$

[35] Including changes in β_1^1 would add no further intuition to our results, and would make the algebraic expressions longer and less tractable. We thus abstract from it.

We now proceed to simplify the second line. Note that, due to our assumptions, there is no dependence of any variable at time $t = 2$, branch 2, on outcomes in period 1.

It will be useful, at this stage, to recast the second line in terms of changes in Y_1^2. The goal is to obtain an equation similar to (H.13), and this is the reason we, at first, assumed Y_1^2 as a shock. First, note that:

$$\begin{aligned}\frac{\partial c_{i0}}{\partial \pi_1^2}\frac{d\pi_1^2}{d\beta_1^2} &= \frac{\partial c_{i0}}{\partial \pi_1^2}\frac{d\pi_1^2}{dY_1^2}\frac{dY_1^2}{d\beta_1^2} \\ &= \frac{\partial c_{i0}}{\partial \pi_1^2}\kappa\frac{dY_1^2}{d\beta_1^2}.\end{aligned}$$

The term $\frac{\partial c_{i0}}{\partial \pi_1^2}$ cannot be recast in terms of consumer reaction to changes in the budget constraint. In our framework, it would be determined by the Euler Equation, with an influence of the intertemporal elasticity of substitution.[36] Recall that π_1^2 directly affects the real interest rate in period 1. We leave this term as is, as it is not our particular goal to investigate it.[37]

Moving to the last term, recall that $\frac{d\tau_1^2}{dY_1^2} = -1 - \frac{d\pi_1^2}{dY_1^2} = -1 - \kappa$. We can use this below:

$$\begin{aligned}\frac{\partial c_{i0}}{\partial \tau_1^2}\frac{d\tau_1^2}{d\beta_1^2} &= \frac{\partial c_{i0}}{\partial \tau_1^2}\frac{d\tau_1^2}{dY_1^2}\frac{dY_1^2}{d\beta_1^2} \\ &= (-1-\kappa)\frac{\partial c_{i0}}{\partial \tau_1^2}\frac{dY_1^2}{d\beta_1^2}.\end{aligned}$$

One additional step is to note that τ_1^2 and Y_1^2 affect household consumption via the budget constraint in period 2:

$$c_2 + \frac{a_3}{R_2} = z_i(Y_1^2 - \tau_1^2) + \frac{a_2}{1+\pi_1^2}.$$

Thus, a unit change in Y_1^2 has (exactly) the opposite effect on the RHS of the budget constraint than a unit change in τ_2^2. Assuming that the only **partial** impact of those variables on consumption in **period 1** is through its impact on the budget constraint at

[36] In addition, π_1^2 also affects wealth in period 2 via the Fisher channel, which also appears in the Euler Equation.

[37] Werning (2015) explores how consumption in heterogeneous- versus representative-agents models reacts to changes in the real interest rate.

$t = 2$ (which in our model, with the Euler Equation, is true), we have:

$$\frac{\partial c_{i0}}{\partial \tau_1^2} = -\frac{\partial c_{i0}}{\partial Y_1^2}.$$

The second line of (H.15) then becomes:

$$\begin{aligned}
\int_i \left(\frac{\partial c_{i0}}{\partial Y_1^2} \frac{dY_1^2}{d\beta_1^2} + \frac{\partial c_{i0}}{\partial \pi_1^2} \frac{d\pi_1^2}{d\beta_1^2} + \frac{\partial c_{i0}}{\partial \tau_2} \frac{d\tau_2}{d\beta_1^2} \right) d\beta_1^2 &= \int_i \left(\frac{\partial c_{i0}}{\partial Y_1^2} \frac{dY_1^2}{d\beta_1^2} + \frac{\partial c_{i0}}{\partial \pi_1^2} \kappa \frac{dY_1^2}{d\beta_1^2} + (1+\kappa) \frac{\partial c_{i0}}{\partial Y_1^2} \frac{dY_1^2}{d\beta_1^2} \right) d\beta_1^2 \\
&= \int_i \left(\frac{\partial c_{i0}}{\partial \pi_1^2} \kappa \frac{dY_1^2}{d\beta_1^2} + (2+\kappa) \frac{\partial c_{i0}}{\partial Y_1^2} \frac{dY_1^2}{d\beta_1^2} \right) d\beta_1^2 \\
&= \int_i \left[\left(\frac{\partial c_{i0}}{\partial \pi_1^2} \kappa + (2+\kappa) \frac{\partial c_{i0}}{\partial Y_1^2} \right) \frac{dY_1^2}{d\beta_1^2} \right] d\beta_1^2.
\end{aligned}$$

Each term of this expression has a very intuitive interpretation. The impact of a shock to β_1^2 on consumption in the initial period *through changes in output in period 2* occur via:

1. Uncertain net-of-taxes income:

   - Gross income in period 2 affects consumption in period 1 via precautionary savings. This is part of the "2" (one of the 1+1 multiplying $\frac{\partial c_{i0}}{\partial Y_1^2}$) in the expression above).

   - The tax affects net income one to one (another of the 1's) and indirectly via changes in inflation affecting the burden of debt, which in turn impacts households budgets. This is the κ, in $2+\kappa$.

2. Inflation: changes in expected future output are linked to future inflation via κ. This has two effects: intertemporal substitution, via the real interest rate, and an indirect impact through the Fisher channel in $t = 1$ affecting households' budgets. Both are contained within $\frac{\partial c_{i0}}{\partial \pi_1^2}$, happening through the Euler Equation.

Equation (H.15) becomes:

$$\begin{aligned}
dY_0 = &\left(\int \frac{\partial c_{i0}}{\partial \beta_2} di + \left[(1 + (1+\kappa)\bar{\tau}) \overline{MPC}_z - \kappa \overline{MPC}_a \right] \frac{dY_0}{d\beta_1^2} \right) d\beta_1^2 \\
&+ \int_i \left[\left(\frac{\partial c_{i0}}{\partial \pi_1^2} \kappa + (2+\kappa) \frac{\partial c_{i0}}{\partial Y_1^2} \right) \frac{dY_1^2}{d\beta_1^2} \right] d\beta_1^2.
\end{aligned}$$

And reorganizing:

$$\frac{dY_0}{d\beta_1^2} = \frac{1}{1 - \left[(1+(1+\kappa)\bar{\tau})\overline{MPC}_z - \kappa\overline{MPC}_a\right]} \int \left\{ \frac{\partial c_{i0}}{\partial \beta_2} + \left(\frac{\partial c_{i0}}{\partial \pi_1^2}\kappa + (2+\kappa)\frac{\partial c_{i0}}{\partial Y_1^2} \right) \frac{dY_1^2}{d\beta_1^2} \right\} di. \tag{H.16}$$

Finally, recall that $\frac{dY_1^2}{d\beta_1^2}$ is given by (H.12) (evaluated at $t = 1$). In itself, this derivative depends on multipliers related to the heterogeneity structure of the economy.

Equation (H.16) features many components on which the micro structure of the economy plays a role. The first is the multiplier $\mathcal{M} = \frac{1}{1-\left[(1+(1+\kappa)\bar{\tau})\overline{MPC}_z+\kappa\overline{MPC}_a\right]}$, discussed at length in Section 5.4. The total impact also depends on a series of forward-looking variables: the impacts of future inflation and output, as well as the impact of the partial equilibrium shock. All of those depend on the underlying heterogeneity structure, for instance via the share of constrained individuals (Figure 9) and wealth effects (Werning, 2015). Finally, there is also the general equilibrium impact of the shock on output at $t = 1$, given by $\frac{dY_1^2}{d\beta_1^2}$, which in itself depends on the multiplier (H.14).

We can re-write (H.16) and a quadratic term of $\mathcal{M}$ will appear:

$$\frac{dY_0}{d\beta_1^2} = \int \left\{ \mathcal{M}\frac{\partial c_{i0}}{\partial \beta_2} + \left(\frac{\partial c_{i0}}{\partial \pi_1^2}\kappa + (2+\kappa)\frac{\partial c_{i0}}{\partial Y_1^2} \right) \mathcal{M}^2 \int_i \frac{\partial dc_{i1}^2}{d\beta_1^2} \right\} di$$

The expression above can indicate why the backward-propagation of uncertainty at the ZLB is stronger in heterogeneous agents economies, as the multiplier is larger due to high MPCs.[38]

In conclusion, Section 2 in the main text show why aggregate uncertainty at the ZLB leads to a deeper recession, while the current exposition highlights the mechanisms that determine the strength of that mechanism, and why it is stronger in heterogeneous-agents economies.

I Two-Asset Model

This section describes the two-asset model used in our simulations in Appendix F. We select a standard version in which households have access to two assets, capital and

[38] A similar result also holds for perfect-foresight economies.

government bonds. The latter is liquid and its holdings can be adjusted costlessly, while households face a probability θ of being able to adjust their capital holdings. The model also features a production sector that combines capital and labor to produce an intermediate good, retailers that produce a final good, a labor union that faces nominal wage adjustment costs, and a capital producer.

I.1 Households

Households are ex-ante identical but ex-post heterogeneous due to idiosyncratic shocks to their labor productivity.

Earnings. Households supply labor n (decided by the union) and receive wz per unit of labor supplied, depending on the market wage w, idiosyncratic productivity z. Dividends are paid proportionately to labor income.

Savings. Households can freely adjust their liquid assets a, which are supplied by the government. In addition, households can invest directly in capital k, but they at any period there is only a probability θ that they can adjust their capital holdings.

Non-adjusting. A non-adjusting household keeps capital holdings constant at $k_t = k_{t-1}$. It solves the dynamic optimization problem given by

$$V_t^n(a_{t-1}, k_{t-1}, z_t) = \max_{c_t \geq 0, a_t \geq \underline{a}} \left\{ u(c_t, n_t) + \beta \mathbb{E}_t V_{t+1}(a_t, k_{t-1}, z_{t+1}) \right\} \tag{I.17}$$

$$\text{s.t. } c_t + \frac{a_t}{R_t} \leq \frac{a_{t-1}}{\Pi_t} + (r_t^K - \delta)k_{t-1} + w_t z_t n_t + d_t z_t - T_t z_t,$$

with $\underline{a}$ as the (exogenous) borrowing limit and β as the discount factor. $r_t^K - \delta$ is the return to capital holdings.

Adjusting. The problem of households that can adjust their capital is:

$$V_t^a(a_{t-1}, k_{t-1}, z_t) = \max_{c_t \geq 0, a_t \geq \underline{a}, k_t \geq 0} \left\{ u(c_t, n_t) + \beta \mathbb{E}_t V_{t+1}(a_t, k_t, z_{t+1}) \right\} \tag{I.18}$$

$$\text{s.t. } c_t + \frac{a_t}{R_t} + q_t k_t \leq \frac{a_{t-1}}{\Pi_t} + (r_t^K + q_t - \delta)k_{t-1} + w_t z_t n_t + d_t z_t - T_t z_t. \tag{I.19}$$

Finally, the value function V_t is given by:

$$V_t(a_{t-1}, k_{t-1}, z_t) = \theta V_t^a(a_{t-1}, k_{t-1}, z_t) + (1-\theta) V_t^n(a_{t-1}, k_{t-1}, z_t). \tag{I.20}$$

I.2 Production

Intermediate Goods Producers. A continuum of identical production firms combine K efficiency units of capital and labor input N to produce intermediate goods using production technology

$$Y_t = A_t K_{t-1}^{\alpha} N_t^{1-\alpha}, \tag{I.21}$$

where A_t represents total factor productivity.

Denote the rental rate per efficiency unit of capital as r_t^K and the wage per unit of labor as w_t. Production firms sell the intermediate consumption good at price p_t^I to retailers. Assuming competitive input and output markets, profit maximization of production firms yields factor prices as

$$w_t = p_t^I (1-\alpha) A_t K_{t-1}^{\alpha} N_t^{-\alpha} \tag{I.22}$$

$$r_t^K = p_t^I \alpha A_t K_{t-1}^{\alpha-1} N_t^{1-\alpha}. \tag{I.23}$$

Final Good Production. We keep the production structure unchanged from that described in Appendix E.1.

Capital Producers. Capital producers transform the final consumption good into the next period's capital, sold at price q_t. They are subject to adjustment costs on to the net-of-depreciation investment. At each period, they select net investment to maximize:

$$\max_{I_{nt}} \mathbb{E}_0 \sum_{t=0} \beta^t \left\{ (q_t - 1) I_{nt} - \frac{\phi_K}{2} \left(\frac{I_{nt} + I^{ss}}{I_{n,t-1} + I^{ss}} \right)^2 (I_{nt} + I^{ss}) \right\}, \tag{I.24}$$

where $I_{nt} \equiv I_t - \delta K_{t-1}$ and investment is defined is $I_t = K_t - (1-\delta) K_{t-1}$. Note that net investment is nil in steady state, while gross steady-state I^{ss} refurbishes existing capital, thus $I^{ss} = \delta K^{ss}$. The resulting optimality condition yields the price of capital as

$$q_t = 1 + \phi^k \left(\frac{I_{nt} + I_{ss}}{I_{n,t-1} + I^{ss}} - 1 \right)^2 + \frac{\phi^k}{2} \cdot \left(\frac{I_{nt} + I^{ss}}{I_{n,t-1} + I^{ss}} - 1 \right)^2 \tag{I.25}$$
$$- \beta \phi^k \left(\frac{(I_{n,t+1} + I_{ss})}{(I_{n,t} + I_{ss})} - 1 \right) \left(\frac{I_{n,t+1} + I^{ss}}{I_{n,t} + I^{ss}} \right)^2.$$

The profits from capital production given by (I.24) are distributed to households as dividends div_t^I.

I.3 Union

We take an off-the-shelf setup from Auclert, Bardóczy and Rognlie (2023), consisting of a union that sets wages with potential adjustments costs. Importantly, it sets wages on behalf of and allocates labor hours equally across households. Households are assumed to supply a continuum of differentiated labor services, indexed by k, aggregated with a CES function and supplied to the intermediate producer. The union for labor type k maximizes the following problem:

$$\max_{W_t} \mathbb{E}_0 \sum_{t=0}^{\infty} \int \beta^t \left[(\mathcal{U}_c(c_{it}, n_{it}) w_{kt} N_{kt} z_{it} + \mathcal{U}_n(c_{it}, n_{it}) N_{kt}) \, di \right].$$

In the expression above, $\mathcal{U}_c$ and $\mathcal{U}_n$ represent respectively the marginal utilities of consumption and labor, and W_{kt} and w_{kt} are respectively the nominal and real wages for type k. The demand curve is:

$$N_{kt} = \left(\frac{w_{kt}}{w_t} \right)^{-\varepsilon_w} N_t,$$

where w_t is the aggregate wage index consistent with CES demand, which is the real wage paid to households.

The first-order conditions for the union yields (see Appendix F.2 in Auclert, Bardóczy and Rognlie, 2023):

$$0 = \left[-N_t \int \mathcal{U}_n(c_{it}, n_{it}) di - w_t N_t \frac{\varepsilon_w - 1}{\varepsilon_w} \int z_{it} \mathcal{U}_c(c_{it}, n_{it}) \right]. \tag{I.26}$$

I.4 Central Bank.

We assume the same targeting rule as in in our benchmark model of Section 3.

I.5 Government

We assume the presence of a fiscal entity that supplies bonds at a constant level $B_t = 1$. Taxes are given by:

$$T_t = \frac{B_{t-1}}{\Pi_t} - \frac{B_t}{R_t}.$$

I.6 Market Clearing

Define $\lambda(a,k,z)$ as the beginning of period distribution of households over the state space. For bonds, we have:

$$B_t = \int_{(a,k,z)} a_t(a,k,z)\lambda_t(a,k,z). \tag{I.27}$$

In addition, aggregate capital holdings are given by

$$K_t = \int_{(a,k,z)} k_t(a,k,z)\lambda_t(a,k,z). \tag{I.28}$$

Total dividends are the sum of dividends from retailers and capital producers, distributed among all households proportionately to z_t:

$$div_t = \frac{div_t^Y + div_t^I}{\int_{(a,k,z)} z\lambda_t(a,k,z)}. \tag{I.29}$$

Market clearing in the goods market requires

$$C_t + I_t + \Xi_t = Y_t, \tag{I.30}$$

where Ξ_t consists of deadweight losses from the cost of capital adjustment, given by:

$$\Xi_t = \frac{\phi^k}{2}\left(\frac{I_{nt} + I^{ss}}{I_{n,t-1} + I^{ss}}\right)^2 (I_{nt} + I^{ss}). \tag{I.31}$$

Finally, labor market clearing is given by

$$N_t = \int_{(a,k,z)} zn_t(a,k,z)\lambda_t(a,k,z).$$

I.7 Calibration

Our exercise is meant to be illustrative of our method, so we take most of our parameters from the existing literature and only calibrate β and θ internally to match a liquid-asset-to-GDP ratio of 1 (given the same nominal rate and inflation target as in our benchmark model) and a capital-to-output ratio of 3.

Table I.2 summarizes the parameters that do not overlap with those of our benchmark model in Section 3:

Table I.2: Externally Calibrated Parameters - Two-Asset Model

Parameter	Value
α	$\frac{1}{3}$
Earnings Process	Mendicino, Nord and Peruffo (2021)
Preferences	Separable, Inv Frisch Elasticity = 2
ϕ^k	11.4
μ	1.1
A_{ss}	1
ε_w	$\to \infty$ (no wage markdown)