

# Selected Exercises of Statistical Inference

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# 1 Probability Theory

## Exercises

*Exercise 1.1. For each, describe the sample space.*

**(a) Toss a coin four times.**

A sample in this case is a bit string of length four, where each digit represents a head or tails (0 for head, 1 for tails, for example). Thus the sample space contains  $2^4$  samples (e.g.  $\{0110\}$ ).

**(b) Count the number of insect-damaged leaves on a plant.**

The sample space contains all integers from 0 to  $\infty$ . Practically, this can be reduced, since there are likely not  $\infty$  leaves on a plant.

**(c) Measure the lifetime (in hours) of a particular brand of light bulb.**

Similar to (b), the sample space contains all integers from 0 to  $\infty$ . Again, it is unlikely that the lifetime is  $\infty$ .

**(d) Record the weights of 10-day-old rats.**

The sample space here is all real-valued positive numbers (positive because a 10-day-old rat's weight will be  $\geq 0$ ).

**(e) Observe the population of defectives in a shipment of electronic components.**

Let  $n$  be the number of electronic components in a shipment. The population of defectives can be modeled as the proportion of defectives in the shipment, thereby making the sample space a real-valued number on the range  $[0, 1]$ . Alternatively, we can model the number of defectives directly, thereby making the sample space all integers on the range  $[0, n]$ .

*Exercise 1.2. Verify the following identities.*

**(a)  $A \setminus B = A \setminus (A \cap B) = A \cap B^c$**

We know that  $A \setminus B = A \cap B^c$  from the definition of  $A \setminus B$ . From this, we can see that

$$A \setminus B = A \cap B^c \tag{1}$$

$$= (A \cap A) \cap B^c \tag{2}$$

$$= A \cap (A \cap B^c) \tag{3}$$

$$= A \setminus (A \cap B) \tag{4}$$

**(b)  $B = (B \cap A) \cup (B \cap A^c)$**

Using the distributive laws:

$$(B \cap A) \cup (B \cap A^c) = B \cap (A \cup A^c) \tag{5}$$

$$= B \cap S \tag{6}$$

$$= B \tag{7}$$

**(c)  $B \setminus A = B \cap A^c$**

This is the definition of  $B \setminus A$ .

(d)  $A \cup B = A \cup (B \cap A^c)$

Using the distributive laws:

$$A \cup (B \cap A^c) = (A \cup B) \cap (A \cup A^c) \quad (8)$$

$$= (A \cup B) \cap S \quad (9)$$

$$= A \cup B \quad (10)$$

*Exercise 1.3. Finish the proof of Theorem 1.1.4*

*Exercise 1.4. For events  $A$  and  $B$ , find formulas for the probabilities of the following events in terms of quantities  $P(A)$ ,  $P(B)$ , and  $P(A \cap B)$ .*

(a) **either  $A$  or  $B$  or both**

This is given by

$$P(A \cup B) + P(A \cap B) = P(A) + P(B) - P(A \cap B) + P(A \cap B) \quad (11)$$

$$= P(A) + P(B) \quad (12)$$

(b) **either  $A$  or  $B$  but not both**

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad (13)$$

(c) **at least one of  $A$  or  $B$**

$$P(A \cup B \cup (A \cap B)) = P(A) + P(B) - P(A \cup B) + P(A \cup B) \quad (14)$$

$$= P(A) + P(B) \quad (15)$$

(d) **at most one of  $A$  or  $B$**

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad (16)$$

*Exercise 1.5. Twins.*

**Given the following events:**

$A = \{aUSbirthresultsintwinfemales\}$

$B = \{aUSbirthresultsinidenticaltwins\}$

$C = \{aUSbirthresultsintwins\}$

(a) **State in words, the event  $A \cap B \cap C$ .**

$A \cap B \cap C$  represents the event that a US birth results in identical female twins.

(b) **Find  $P(A \cap B \cap C)$ .**

Among all US births, 1 in 90 are twins. Among all twins, 1 in 3 are identical. Among identical twins, females are as likely as males (1 in 2). So, the probability that a given birth results in identical female twins are

$$\frac{1}{90} \times \frac{1}{3} \times \frac{1}{2} = \frac{1}{540}$$

So, 1 in 540 US births are identical twin females.

**Exercise 1.6. Two pennies, one  $P(head) = u$ , the other  $P(head) = w$ , are tossed. Define**

$$p_0 = P(0heads)$$

$$p_1 = P(1head)$$

$$p_2 = P(2heads)$$

**Can  $u$  and  $w$  be picked such that  $p_0 = p_1 = p_2$ ?**

$$p_0 = (1 - u)(1 - w) = 1 - w - u + wu \quad (17)$$

$$p_1 = u(1 - w) + w(1 - u) = u - wu + w - wu = u + w - 2wu \quad (18)$$

$$p_2 = wu \quad (19)$$

Let's set  $p_0 = p_2$ . We then get

$$1 - w - u + wu = wu \quad (20)$$

$$1 - u = w \quad (21)$$

Plug this into  $p_1 = p_2$ :

$$u + w - 2wu = wu \quad (22)$$

$$u + w = 3wu \quad (23)$$

$$u + (1 - u) = 3(1 - u)u \quad (24)$$

$$1 = 3u - 3u^2 \quad (25)$$

$$\frac{1}{3} = u - u^2 \quad (26)$$

Thus, for  $p_0 = p_1 = p_2$ , we have to find the point at which the parabola  $u - u^2 = \frac{1}{3}$ . However, we note that the maximum of this parabola can be found by taking the derivative and setting equal to 0:

$$\frac{d}{du}(u - u^2) = 1 - 2u \quad (27)$$

$$0 = 1 - 2u \quad (28)$$

$$2u = 1 \quad (29)$$

$$u = \frac{1}{2} \quad (30)$$

So, the parabola has a maximum of  $\frac{1}{2}$ , and therefore is never equal to  $\frac{1}{3}$ . So, there is no  $u$  and  $w$  such that  $p_0 = p_1 = p_2$ .

*Exercise 1.7. Referring to the dart example, suppose the wall is hit with probability 1, and the wall has area  $A$ .*

**(a) Using the fact that the probability of hitting a region is proportional to its area, construct a probability function for  $P(\text{scoringipoints})$ ,  $i = 0, \dots, 5$  (No points are scored if the dart board is not hit.**

The area of the dartboard with radius  $r$  is given by:  $\pi r^2$ . Therefore,

$$P(\text{boardishit}) = \frac{\pi r^2}{A}$$

Note that in order to score a point, you have to hit the board, and then hit the slice of the board that contains that point. From Example 1.2.7 we already know the probability of scoring  $i$  points when the board is definitely hit. This is given by

$$P(\text{scoringipoints}|\text{boardishit}) = \frac{(6-i)^2 - (5-i)^2}{5^2}$$

Since it is only possible to score points if I first hit the board, the new probability of scoring  $i$  points is given by

$$P(\text{scoringipoints}) = P(\text{scoringipoints}|\text{boardishit})P(\text{boardishit}) \quad (31)$$

$$= \frac{\pi r^2((6-i)^2 - (5-i)^2)}{25A} \quad (32)$$

**(b) Show that the conditional probability  $P(\text{scoringipoints}|\text{boardishit})$  is the probability distribution of Example 1.2.7**

Using the definition of conditional probability, we get:

$$P(\text{scoringipoints}|\text{boardishit}) = \frac{P(\text{scoringipoints}, \text{boardishit})}{P(\text{boardishit})} \quad (33)$$

$$= \frac{\pi r^2((6-i)^2 - (5-i)^2)}{5^2 A} \frac{A}{\pi r^2} \quad (34)$$

$$= \frac{(6-i)^2 - (5-i)^2}{5^2} \quad (35)$$

*Exercise 1.8. More darts.*

*Exercise 1.9. Prove the general version of DeMorgan's Laws.*

Using induction, we can see that

$$\left(\bigcup_{i=1}^n A_i\right)^c = A_1^c \cap \left(\bigcup_{i=2}^n A_i\right)^c \quad (36)$$

$$= A_1^c \cap A_2^c \cap \left(\bigcup_{i=3}^n A_i\right)^c \quad (37)$$

$$= \dots \quad (38)$$

$$= A_1^c \cap \dots \cap A_{n-2}^c \cap (A_{n-1} \cup A_n)^c \quad (39)$$

$$= \bigcap_{i=1}^n A_i^c \quad (40)$$

*Exercise 1.10. More DeMorgan's.*

*Exercise 1.11. Let  $S$  be a sample space.*

**(a) Show that the collection  $B = \{\emptyset, S\}$  is a sigma algebra.**

There are three requirements for a sigma algebra  $B$ :

(I)  $\emptyset \in B$  (II) If  $A \in B$ , then  $A^c \in B$  (III) If  $A_1, A_2, \dots \in B$ , then  $\bigcup_{i=1}^{\infty} A_i \in B$

So, to show that  $B = \{\emptyset, S\}$  is a valid sigma algebra, we will show that it satisfies these three requirements.

Requirement I is obvious, as  $\emptyset$  is an element in  $B$ . Since  $\emptyset^c = S$ , and  $S \in B$ , requirement II is satisfied. Finally, we note that  $\emptyset \cup S = S$ , and since  $S \in B$ , requirement III is satisfied.

**(b) Let  $B = \{\text{all subset of } S, \text{ including } S\}$ . Show that  $B$  is a sigma algebra.**

$\emptyset$  is a subset of  $S$ , so requirement I is satisfied.

All subsets include all complements, so requirement II is satisfied.

$\bigcup_{i=1}^{\infty} A_i \in B$ , so requirement III is satisfied.

**(c) Show that the intersection of two sigma algebras is a sigma algebra.**

For requirement I, we note that  $\emptyset \cap A = \emptyset \forall A$ . So, requirement I is satisfied under intersection.

For requirement II, we can use DeMorgan's Law. For any event  $A_1 \in B_1$  and  $A_2 \in B_2$ ,  $(A_1 \cap A_2)^c = A_1^c \cup A_2^c$ . Since  $B_1$  and  $B_2$  are both sigma algebras, they are both closed under complementation and union, and therefore requirement II is satisfied.

For requirement III, we note that  $(A \cap B) \cup (A \cap C) = A \cap (B \cup C) = A^c \cup (B \cup C)^c$ . Since sigma algebras are closed under complementation and union, we know that  $A^c \cup (B \cup C)^c \in B$ . So, requirement III is satisfied.

*Exercise 1.12. Statisticians in the deFinetti school.*

**(a) Show that the Axiom of Countable Additivity implies Finite Additivity.**

Axiom of Countable Additivity states that

If  $A_1, A_2, \dots \in B$  are pairwise disjoint, then  $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$ .

Finite Additivity states that  $P(A \cup B) = P(A) + P(B)$ , if  $A$  and  $B$  are disjoint.

Assume that the Axiom of Countable Additivity is true. Then using induction,

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = P\left(A_1 \cup \bigcup_{i=2}^{\infty} A_i\right) \quad (41)$$

$$= P(A_1) + P\left(\bigcup_{i=2}^{\infty} A_i\right) \quad (42)$$

$$= \dots \quad (43)$$

$$= \sum_{i=1}^{\infty} P(A_i) \quad (44)$$

**Exercise 1.13.** If  $P(A) = \frac{1}{3}$  and  $P(B^c) = \frac{1}{4}$ , can  $A$  and  $B$  be disjoint?

Since  $P(A) = \frac{1}{3}$  and  $P(B) = 1 - P(B^c) = \frac{3}{4}$ , then  $A$  and  $B$  cannot be disjoint, because  $P(A) + P(B) = \frac{13}{12} > 1$ .

**Exercise 1.14. Supersets.**

**Exercise 1.15. Finish proof of Theorem 1.2.14.**

The text proved this for  $k = 2$ . To prove the general, we can use induction. Let  $k = K$ . Each task  $k_i \forall i \in \{1, \dots, K\}$  can be done  $n_i$  ways. Let  $N(i)$  be the number of ways to do  $i$  tasks. Then

$$N(i) = N(i-1) \times n_i N(i-1) = N(i-2) \times n_{i-1} \quad (45)$$

Using the base case of  $k = 2$ , we see that in general, the number of ways to do  $k$  tasks is:  $n_1 n_2 \cdots n_k$ .

**Exercise 1.16. How many sets of initial are there if every person has one surname and**

**(a) exactly two given names?**

Here we have a string of length two, where each character can have 26 values. So,

$$26 \times 26 = 676$$

total initials.

**(b) either one or two given names?**

Here we have some people with initials of length 2, and some of length 3. So,

$$(26^2) + (26^3) = 676 + 17576 = 18252$$

total initials.

**(c) either one or two or three given names?**

Here we have some people with initials of length 2, some of length 3, and others with length 4. So,

$$\sum_{i=2}^4 26^i = 676 + 17576 + 456976 = 475228$$

total initials.

**Exercise 1.17. In dominos, there are two numbers,  $\{1, \dots, n\}$ . Order does not matter. How many possible number pairs are there?**

Since order does not matter, this is  $n \text{ choose } k$ , where  $k = 2$ . So,

$$\binom{n}{2} = \frac{n!}{2!(n-2)!} \quad (46)$$

$$= \frac{n \times (n-1) \times (n-2)!}{2 \times (n-2)!} \quad (47)$$

$$= \frac{n \times (n-1)}{2} \quad (48)$$

**Exercise 1.18. If  $n$  balls are placed at random into  $n$  cells, find the probability that exactly one cell remains empty.**

We first note that there are  $n^n$  ways to place  $n$  balls into  $n$  cells. So, this will likely be the denominator in our answer.

Having one cell remain empty means that  $n-1$  cells contain one ball and one cell contains two balls. This means that each of the  $n$  balls have to go into a specific cell. Once a ball is placed in a cell, there are only  $n-1$  cells left for the next one to go to, etc. So, there are  $n!$  ways to place the  $n$  balls into the correct  $n$  cells. Finally, it doesn't matter which cell is empty, just that there is one and only one. So, there has to be one cell that is empty and one cell that contains two balls. There are  $\binom{n}{2}$  ways to choose these cells. So, the final answer is

$$\binom{n}{2} \frac{n!}{n^n}$$

**Exercise 1.19. Partial derivatives.**

**(a) How many fourth partial derivatives does a function of three variables have?**

Each time you take a derivative, you can take a derivative any of the variables (and you can take a derivative of a particular variable more than once). This is the case of unordered sampling with replacement. We know from the text that in general this is given by

$$\binom{n+r-1}{r}$$

So, there are

$$\binom{3+4-1}{4} = 15$$

fourth partial derivatives on a function of three variables.



*Exercise 1.20. Telephone rings.*

*Exercise 1.21. A closet contains  $n$  pairs of shoes. If  $2r$  shoes are chosen at random ( $2r < n$ ), what is the probability that there will be no matching pair in the sample?*

There are  $n$  pairs of shoes, or  $2n$  single shoes. So, there are  $\binom{2n}{2r}$  total sample points (the denominator). For the numerator, we want to count the number of sample points for which we don't have a matching pair.

If we choose a new pair with each  $2r$ , that would mean that we don't have a matching pair. There are  $\binom{n}{2r}$  ways to select  $2r$  points from  $n$  pairs. But, with each pair that we select, we could either choose the left or the right shoe. So, there are 2 ways to choose each  $2r$  pairs, so  $2^{2r}$  ways to choose within the styles of the  $2r$  points we selected.

Combine these, and we get

$$\binom{n}{2r} \frac{2^{2r}}{\binom{2n}{2r}}$$

*Exercise 1.22. (a) In a draft lottery containing the 366 days of the year, what is the probability that the first 180 days (without replacement) are evenly distributed among the 12 months?*

Having 180 days evenly distributed among the 12 months mean that there should be 15 days picked per month. The probability is then

$$\frac{\binom{31}{15} \binom{29}{15} \cdots \binom{30}{15} \binom{31}{15}}{\binom{366}{180}} \approx 1.67 \times 10^{-8}$$

**(b) What is the probability that the first 30 days contain none from September?**

The no-September probability is just the section of the sample space in which we distribute the 30 days into the days that aren't September. This is given by

$$\frac{\binom{336}{30}}{\binom{366}{30}}$$

*Exercise 1.23.*