Selected Exercises of Elements of Statistical Learning

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1 Introduction

This is a collection of certain exercise solutions I did as I was reading through this book. As I am not a mathematician, the ”proofs” listed here are rough and probably not rigorous. They only serve as a means to explain the concepts in a way that aids my own understanding.

2 Overview of Supervised Learning

Exercises

Exercise 2.1. **Suppose each of K-classes has an associated target t_k, which is a vector of all zeros, except a one in the kth position. Show that classifying to the largest element of $\hat{y}$ amounts to choosing the closest target, $min_k||t_k - \hat{y}||$, if the elements of $\hat{y}$ sum to one.**

We are trying to show that

$$arg\,max_k\,\hat{y_k} = arg\,min_k||t_k - \hat{y}||$$

To do this, we can show that for any $k^* \neq arg\,max_k\,\hat{y_k}$ and $k = arg\,max_k\,\hat{y_k}$,

$$||t_{k^*} - \hat{y}|| > ||t_k - \hat{y}||$$

Note that we can equivalently consider the $||\cdot||^2$ instead of $||\cdot||$, because they are both monotonic for ≥ 0.

Using the definition of the Euclidean norm,

$$\begin{aligned} ||t_{k^*} - \hat{y}||^2 &= ||\hat{y}||^2 + ||t_{k^*}||^2 - 2t_{k^*}\hat{y} && (1)\\ &= \hat{y}^2 && (2)\\ & && (3) \end{aligned}$$

Similarly,

$$\begin{aligned} ||t_k - \hat{y}||^2 &= ||\hat{y}||^2 + ||t_k||^2 - 2t_k\hat{y} && (4)\\ &= \hat{y}^2 + 1 - 2\hat{y} && (5) \end{aligned}$$

Therefore

$$\begin{aligned} ||t_{k^*} - \hat{y}||^2 - ||t_k - \hat{y}||^2 &= \hat{y}^2 - (\hat{y}^2 + 1 - 2\hat{y}) && (6)\\ &= -1 + 2\hat{y} && (7)\\ &\geq 0 && (8) \end{aligned}$$

since $\hat{y}$ sums to one. So, $||t_k - \hat{y}||$ is minimized when $k = arg\,max_k\,\hat{y_k}$.

Exercise 2.2. Show how to compute the Bayes decision boundary for the simulation example in Figure 2.5.

Exercise 2.3. **Consider N data points uniformly distributed in a p-dimensional unit ball centered at the origin. Suppose we consider a nearest-neighbor estimate at the origin. Show that the median distance from the origin to the closest data point is given by: $d(p, N) = (1 - \frac{1}{2}^{1/N})^{1/p}$.**

Let m be the median distance from the origin to the closest point. This means that the probability that all data points are further than m is 0.5. "Further" simply means a greater norm. Since samples x_i are i.i.d, we can more formally state this as:

$$\prod_{i=1}^{N} P(||x_i|| > m) = \frac{1}{2}$$

Note that we can flip this around to be $\prod_{i=1}^{N} P(||x_i|| \leq m) = \frac{1}{2}$. Now we can use the cumulative function of the uniform distribution as follows:

$$\prod_{i=1}^{N} P(||x_i|| \leq m) = \prod_{i=1}^{N} 1 - ||m|| \tag{9}$$

$$= \prod_{i=1}^{N} 1 - m^p \tag{10}$$

$$= (1 - m^p)^N = \frac{1}{2} \tag{11}$$

Now we can solve for m:

$$\frac{1}{2} = (1 - m^p)^N \tag{12}$$

$$\frac{1}{2}^{1/N} = 1 - m^p \tag{13}$$

$$m^p = 1 - \frac{1}{2}^{1/N} \tag{14}$$

$$m = (1 - \frac{1}{2}^{1/N})^{1/p} \tag{15}$$

Exercise 2.4. **The edge effect problem discussed on page 23 is not peculiar to uniform sampling from bounded domains. Consider inputs drawn from a spherical multinormal distribution $XN(0, I_p)$. The squared distance from any sample point to the origin has a χ^2_p distribution with mean p. Consider a prediction point x_0 drawn from this distribution, and let $a = x_0/||x_0||$ be an associated unit vector. Let $z_i = a^T x_i$ be the projection of each of the training points on this direction. Show that the z_i are distributed $N(0, 1)$ with expected squared distance from the origin 1, while the target point has expected squared distance p from the origin. Hence for $p = 10$, a randomly drawn test point is about 3.1 standard deviations from the origin, while all the training points are on average one standard deviation along direction a. So most prediction points see themselves as lying on the edge of the training set.**