Math 3 - Problem Set 3

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1. (20 points) If X, Y, Z are r.v.s s.t. X and Y are independent and Y and Z are indepdent, does it follow that X and Z are independent?

Solution: No. For example, we can let X be the event that a fair coin, call it coin 1, lands heads, let Y be the event that a second fair coin, call it coin 2, lands tials, and Z be the event that coin 1 lands tails. Even though the outcomes of the first and second coins are independent, meaning X and Y are independent and Y and Z are independent, the outcomes of the same coin (coin 1) are not, meaning X and Z are dependent.

2. (20 points) (a) A fair die is rolled. Find the expected value of the roll.

Solution:
$$E(X) = \frac{1}{6}(1+2+3+4+5+6) = 3.5.$$

(b) Four fair dice are rolled. Find the expected total of the rolls.

Solution: By the linearity of expectation, the expected value of four rolls is
$$4 \cdot E(X) = 4 \cdot 3.5 = 14.$$

3. (20 points) Two teams are going to play a best-of-7 match (the match will end as soon as either team has won 4 games). Each game ends in a win for one team and a loss for the other team. Assume that each team is equally likely to win each game, and that the games played are independent. Find the mean and variance of the number of games played.

Solution: The total number of ways the best-of-7 can end is $2 \cdot \binom{7}{4}$ since there are 2 ways to choose the team that wins and $\binom{7}{4}$ ways to choose which games the winning team wins.
The probability that exactly i games are played is
$$2 \cdot \binom{i-1}{3}$$
since the last game must be won by the winning team (otherwise the best-of-7 terminates earlier), and we from the remaining $i-1$ games we choose 3 games for the winning team to win.
Thus, the expected value (mean) is
$$E(X) = \sum_{i=4}^{7} \frac{2 \cdot \binom{i-1}{3}}{2 \cdot \binom{7}{4}} \cdot i = 6.4.$$
The variance is the expected value of the squared difference between each outcome and the mean, which can be expressed as

$$\mathrm{Var}(X) = E(X^2) + E(X)^2 = \sum_{i=4}^{7} \frac{2 \cdot \binom{i-1}{3}}{2 \cdot \binom{7}{4}} \cdot i^2 - 6.4^2 = 0.64.$$

4. (20 points) Let X be a discrete r.v. with support $-n, -n+1, \ldots, 0, \ldots, n-1, n$ for some positive integer n. Suppose that the PMF of X satisfies the symmetry propert for all integers k. Find $E(X)$.

Solution:

$$E(X) = \sum_{k=-n}^{n} p(k) \cdot k = 0 \cdot p(0) + \sum_{k=1}^{n} (p(k) \cdot k + p(-k) \cdot (-k)) = \sum_{k=1}^{n} (p(k) \cdot k - p(k) \cdot k) = \sum_{k=1}^{n} 0 = 0.$$

5. (20 points) An airline overbooks a flight, selling more tickets for the flight than there are seats on the plane (figuring that it's likely that some people won't show up). The plane has 100 seats, and 110 people have booked the flight. Each person will show up for the flight with probability 0.9, independently. Find the probability that there will be enough seats for everyone who shows up for the flight.

Solution: I think it's easier to find the complement of the probability that there are enough seats. This happens when there are at most 9 be people who don't show up. The probability that this happens is

$$\sum_{i=0}^{9} \binom{110}{i} \cdot (0.1)^i \cdot (0.9)^{110-i}$$

since there are $\binom{110}{i}$ ways to choose the i people that don't show up, the probability that none of them show up is $(0.1)^i$, and the probability that everyone else does show up is $(0.9)^{110-i}$.

Thus, the probability that there are enough seats is

$$1 - \sum_{i=0}^{9} \binom{110}{i} \cdot (0.1)^i \cdot (0.9)^{110-i} = 0.6710.$$