Ma 3 - Problem Set 1

Marco Yang

2237027

1. (20 points) In bridge, there are 4 players (A, B, C, D) and each player receives 13 cards from standard shuffled 52 card deck. What is the probability that

   (a) exactly 1 of 4 players has one ace and one king?

Solution: The probability that exactly 1 of the 4 players has one ace and one king is the ratio of the number of hands such that exactly 1 player has one ace and one king and the rest do not. Let a hand with one ace and one king be denoted as a valid hand.

The number of ways we can choose one player to have a valid hand is

$$A = 4 \cdot 4^2 \cdot \binom{44}{11}$$

since there are 4 players from which we can choose one to have a valid hand, 4 aces and 4 kings we can choose from to form a pair, and there are 44 remaining non-king and non-ace cards from which we can choose 11 cards to create the rest of the hand.

Now, we want the ways we can form the remaining hands such that no player has a king and an ace. This is the same as finding the complement of the number of permutations of hands such that there is at least one valid hand. Notice that to count the complement, we only have to consider two cases:

1. exactly 1 of the remaining 3 players has a valid hand
2. exactly 3 of the remaining 3 players has a valid hand (there is no way for exactly 2 of the remaining 3 players to have a valid hand because the last king and last ace that isn't in the hands of the first three players must go to the last player, in which case the last player also has a valid hand)

For the first case, the number ways we can choose exactly 1 remaining player to have a valid hand is

$$3 \cdot 3^2 \cdot \binom{33}{11} \cdot X$$

since there 3 remaining players from which we can choose one to have a valid hand, 3^2 ways to choose an ace-king pair from the remaining cards, 33 remaining non-ace and non-king cards from which we choose 11 to complete the hand, and X is the number of ways we can choose the remaining two hands such that neither hand is valid.

Notice that X is the complement of the number of ways we can choose the hands such that both of the remaining players have valid hands (we can't have exactly 1 out of the 2 remaining players to have a valid hand since that would cause the other player to also have a valid hand, as explained before). Thus the complement of X, which we denote X^c, is

$$X^c = 2^2 \cdot \binom{22}{11}$$

since there are 2^2 ways to choose an ace-king pair and 22 remaining non-king and non-ace cards from which we choose 11 to complete the players hand. We need not compute the number of ways to form the last players hand since it's already predetermined by the other 3 players' hands.

Thus, for case 1, the number of ways we can choose exactly 1 player out of 3 remaining players to have a valid hand is

$$3 \cdot 3^2 \cdot \binom{33}{11} \cdot (1 - X^c) = 3 \cdot 3^2 \cdot \binom{33}{11} \cdot \left(\binom{26}{13} - 2^2 \cdot \binom{22}{11} \right).$$

For case 2, the number of ways we can we choose all 3 remaining players to have a valid hand is

$$3^2 \cdot \binom{33}{11} \cdot X^c$$

since there are $3^2 \cdot \binom{33}{11}$ ways for the first remaining player to have a valid hand, and X^c ways to have the last 2 remaining players to also have valid hands. Thus, our final number of ways to have at least 1 out of the 3 remaining players to have valid hands is

$$Y = 3 \cdot 3^2 \cdot \binom{33}{11} \cdot \left(\binom{26}{13} - 2^2 \cdot \binom{22}{11} \right) + 3^2 \cdot \binom{33}{11} \cdot 2^2 \cdot \binom{22}{11}.$$

Back to the original problem: we are looking to multiply the number of ways to have one player have a valid hand and the rest to have invalid hands. Thus, we are looking for the complement of the above number, which is

$$B = \binom{39}{13} \cdot \binom{26}{13} \cdot \binom{13}{13} - Y.$$

The total number of permutations of the four player's hands is

$$\binom{52}{13} \cdot \binom{39}{13} \cdot \binom{26}{13} \cdot \binom{13}{13}$$

Putting it all together, the probability that exactly one player has a valid hand is

$$\frac{A \cdot B}{\binom{52}{13} \cdot \binom{39}{13} \cdot \binom{26}{13} \cdot \binom{13}{13}} = 0.3656.$$

(b) player A and B each have exactly one ace?

Solution: There are $4 \cdot 3$ permutations for the ace that A and B receive, and the probability that player A's ace goes to player A is $\frac{13}{52}$, the probablity that player B's ace goes to player B is $\frac{13}{51}$ since the first ace removes a potential spot for player B to go to, and the same approach is used to find the probabilities that the remaining two aces go to either player C or D.

$$4 \cdot 3 \cdot \left(\frac{13}{52}\right) \cdot \left(\frac{13}{51}\right) \cdot \left(\frac{26}{50}\right) \cdot \left(\frac{25}{49}\right) = 0.2029.$$

2. (20 points) A club consists of 10 seniors, 12 juniors, and 15 sophomores. An organizing committee of size 5 is chosen randomly (with all subsets of size 5 equally likely).

(a) Find the probability that there are exactly 3 sophomores in the comittee.

Solution: There are $\binom{15}{3}$ combinations of juniors and $\binom{22}{2}$ combos of seniors/juniors. The total number of combinations is $\binom{37}{5}$. Thus, the probability is

$$P = \frac{\binom{15}{3}\binom{22}{2}}{\binom{37}{5}} = 0.2411.$$

(b) Find the probability that the committee has at least one rep from each of the senior, junior, and sophomore classes.

Solution: I think the complement is easier to calculate. The probability that the the committee is formed by at most 2 classes is the combinations of 5 people chosen from only seniors/juniors, only juniors/sophs, and only seniors/sophs, minus the number of committees only formed by one class since they overlap.

Thus,

$$P = 1 - \left(\frac{\binom{10+12}{5} + \binom{12+15}{5} + \binom{10+15}{5} - (\binom{10}{5} + \binom{12}{5} + \binom{15}{5})}{\binom{37}{5}} \right)$$
$$= 0.6418.$$

3. (15 points) Let A and B be events. The symmetric difference $A \Delta B$ is defined to be the set of all elements that are in A and B but not both. In logic and engineering this event is alos called the XOR of A and B. Show that

$$P(A\Delta B) = P(A) + P(B) - 2P(A \cap B).$$

Solution:

Proof. By PIE, we know that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Since $A \Delta B = (A \cup B) \setminus (A \cap B)$,

$$P(A\Delta B) = P(A \cup B) - P(A \cap B) = P(A) + P(B) - 2P(A \cap B).$$

□

4. (10 points) Suppose A and B are independent events. Show that A^c and B^c are also independent.

Solution:

Proof. From DeMorgan's Law, we know that the intersection of the complements of two sets is the complement of the union. Thus, we have

$$
\begin{aligned}
P(A^c \cap B^c) &= P((A \cup B)^c) \\
&= 1 - P(A \cup B) \\
&= 1 - (P(A) + P(B) - P(A \cap B)) \\
&= 1 - P(A) - P(B) + P(A \cap B).
\end{aligned}
$$

Since A and B are independent, we know that $P(A \cap B) = P(A)P(B)$. Thus,

$$
\begin{aligned}
P(A^c \cap B^c) &= 1 - P(A) - P(B) + P(A)P(B) \\
&= (1 - P(A))(1 - P(B)).
\end{aligned}
$$

By definition, $P(A^c) = 1 - P(A)$ and $P(B^c) = 1 - P(B)$. Thus,

$$P(A^c \cap B^c) = P(A^c)P(B^c).$$

Thus, A^c and B^c fulfill the criteria for independence. $\square$

5. (10 points) Two cards are distributed to each of three players from a standard deck of 52 shuffled cards. What is the probability that at least two of the three players have an ace with either a jack or a king?

Solution: There are two cases: exactly two of the three players have an ace with either a jack or a king or exactly three players have an ace with either a jack or a king.

For the first case, there are $\binom{3}{2} = 3$ to choose the two players who have an ace with either a jack or a king and the player who doesn't have an ace with either a jack or a king. The probability that both selected players satisfy the conditions is

$$\frac{4 \cdot 8}{\binom{52}{2}} \cdot \frac{3 \cdot 7}{\binom{50}{2}}.$$

The probability that the third player does not have an ace with either a king or a jack is the complement of the probability that the third player does have an ace with a king or a jack, which is

$$1 - \frac{2 \cdot 6}{\binom{48}{2}}.$$

Thus, the probability for the first case is

$$P_1 = 3 \cdot \frac{4 \cdot 8}{\binom{52}{2}} \cdot \frac{3 \cdot 7}{\binom{50}{2}} \cdot \left(1 - \frac{2 \cdot 6}{\binom{48}{2}}\right).$$

For the second case, there is $\binom{3}{3} = 1$ way to choose which players have an ace with either a jack or a king. The probability that all three have an ace with either a jack or a king is

$$P_2 = \frac{4 \cdot 8}{\binom{52}{2}} \cdot \frac{3 \cdot 7}{\binom{50}{2}} \cdot \frac{2 \cdot 6}{\binom{48}{2}}.$$

Thus, our answer is

$$P_1 + P_2 = 0.001232.$$

6. (25 points) Alice and Bob play a game. Alice has two blue chips, while Bob has three red chips. They put all five chips into a bag, mix them up, then an impartial referee removces them one at a time and returns them to the players. The winner is the first person to collect all their chips.

   (a) Determine the probability that Alice wins this game, and explain how you calculated your answer.

   Solution: The probability that Alice wins is the probability of a permutation of 5 chips with the last chip being red, which is just

   $$\frac{\binom{4}{2}}{\binom{5}{2}} = \frac{3}{5}.$$

   (b) In the next round, Alice has ony one blue chip, while Bob has six red chips. Now what is the probability that Alice wins, and why?

   Solution: The probability that Alice wins is the probability of a permutation of 5 chips with the last chip being red, which is just the probability that

   $$\frac{\binom{6}{1}}{\binom{7}{1}} = \frac{6}{7}.$$

   (c) Make a conjecture for different numbers of red and blue chips.

   Solution: Notice that our set of all permutations of the chips is the same as the set of all backwards permutations of the chips. Thus, WLOG, we can let the last chip be the first one we "choose" from the set of all chips. Since the probability of Alice winning is the same as the last chip being red, it's just the probability that the first chip we choose from the bag is red, or the ratio of the red chips to all chips:

   $$\frac{\text{number of red chips}}{\text{number of total chips}}.$$