Ma 3 Problem Set 2

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1. (10 points) Show that $P(A|B) \leq P(A)$ implies $P(A|B^c) \geq P(A)$, and give an intuitive explanation of why this makes sense.

Solution:

Proof. We know that
$$P(A) = P(A|B) \cdot P(B) + P(A|B^c) \cdot P(B^c).$$
Since $P(A) \geq P(A|B)$ and $P(B^c) = 1 - P(B)$,
$$P(A) = P(A|B) \cdot P(B) + P(A|B^c) \cdot P(B^c)$$
$$P(A) \leq P(A) \cdot P(B) + P(A|B^c) \cdot (1 - P(B))$$
$$P(A)(1 - P(B)) \leq P(A|B^c) \cdot (1 - P(B))$$
$$P(A) \leq P(A|B^c).$$

Intuitively, this makes sense because the probability of A happening is the weighted sum of the probability that it happens given that either B happens or B doesn't happen. Thus, if the probability of A happening given that B happens is lower than the overall weighted average, the probability that A happens given that B doesn't happen needs to compensate by being higher than the weighted average. □

2. (10 points) According to the CDC (Centers for Disease Control and Prevention), men who smoke are 23 times more likely to develop lung cancer than men who don't smoke. Also according to the CDC, 21.6% of men in the U.S. smoke. What is the probability that a man in the U.S. is a smoker, given that he develops lung cancer?

Solution: Let S be the event that a man smokes and L be the event that a man develops lung cancer.

$$P(S|L) = \frac{P(S \cap L)}{P(L)}.$$

Now, we have to find $P(S \cap L)$, the probability that a man smokes and has lung cancer, and $P(L)$, the probability that any man has lung cancer. We are given that $P(S) = 0.216$ and $P(L|S) = 23 \cdot P(L|S^c)$. Thus,

$$P(S \cap L) = P(L|S) \cdot P(S)$$
$$= 23 \cdot P(L|S^c) \cdot 0.216$$
$$P(L) = P(L|S) \cdot P(S) + P(L|S^c) \cdot P(S^c)$$
$$= 23 \cdot P(L|S^c) \cdot P(S) + P(L|S^c) \cdot (1 - P(S))$$
$$= 23 \cdot P(L|S^c) \cdot 0.216 + P(L|S^c) \cdot 0.784.$$

Thus,

$$P(S|L) = \frac{P(S \cap L)}{P(L)}$$
$$= \frac{23 \cdot P(L|S^c) \cdot 0.216}{23 \cdot P(L|S^c) \cdot 0.216 + P(L|S^c) \cdot (0.784)}$$
$$= \frac{0.216 \cdot 23}{0.216 \cdot 23 + 0.784}$$
$$= 0.8636.$$

3. (10 points) A hat contains 100 coins, where 99 are fair but one is double-headed (always landing Heads). A coin is chosen uniformly at random. The chosen coin is flipped 7 times, and it lands Heads all 7 times. Given this information, what is the probability that the chosen coin is double-headed? (Of course, another approach here would be to look at both sides of the coin–but this is a metaphorical coin.)

Solution: Let H be the event that the coin lands heads up all 7 times and D be the even that the random coin is double-headed. We want to find

$$P(D|H) = \frac{P(D \cap H)}{P(H)}.$$

Since D and H are independent,

$$P(D \cap H) = P(D) \cdot P(H|D)$$
$$= \frac{1}{100} \cdot 1^7$$
$$= \frac{1}{100}.$$

We also have

$$P(H) = P(H|D) \cdot P(D) + P(H|D^c) \cdot P(D^c)$$
$$= \frac{1}{100} + \left(\frac{1}{2}\right)^7 \cdot \frac{99}{100}.$$

Thus, our solution is

$$P(D|H) = \frac{P(D \cap H)}{P(H)}$$
$$= 0.5639.$$

4. (10 points) An experiment consists of throwing a fair coin four times. Find the probability mass function

and the cumulative distribution function of the following random variables: (a) the number of heads before the first tail, (b) the number of heads following the first tail, (c) the number of heads minus the number of tails, and (d) the number of tails times the number of heads.

(a) the number of heads before the first tail.

Solution: There are 2^4 possible permutations for 4 coin flips. The number of permutations such that there are x heads before the first tail is $2^{4-(x+1)} = 2^{3-x}$ since the first x flips are heads, the $x+1$-th flip is tails, and every flip afterwards can be whatever. The only exception is when $x = 4$, since we don't put a tail afterwards, and there is only 1 way this can happen. Thus, the PMF is

$$P(X = x) = \begin{cases} \frac{1}{2}, & x = 0 \\ \frac{1}{4}, & x = 1 \\ \frac{1}{8}, & x = 2 \\ \frac{1}{16}, & x = 3 \\ \frac{1}{16}, & x = 4 \end{cases}.$$

Summing the PMF, we have

$$P(X \leq x) = \begin{cases} \frac{1}{2}, & x = 0 \\ \frac{3}{4}, & x = 1 \\ \frac{7}{8}, & x = 2 \\ \frac{15}{16}, & x = 3 \\ 1, & x = 4 \end{cases}.$$

(b) the number of heads following the first tail

Solution: TBH 4 flips is such a small number of flips that I think the cleanest way to do the problem is to just list all possible outcomes:

HHHH HHHT HHTH HHTT HTHH HTHT HTTH HTTT THHH THHT THTH THTT
TTHH TTHT TTTH TTTT

Based on the above outcomes, our PMF is

$$P(X \leq x) = \begin{cases} \frac{5}{16}, & x = 0 \\ \frac{3}{8}, & x = 1 \\ \frac{1}{4}, & x = 2 \\ \frac{1}{16}, & x = 3 \end{cases}.$$

Our CDF is

$$P(X \geq x) = \begin{cases} \frac{5}{16}, & x = 0 \\ \frac{11}{16}, & x = 1 \\ \frac{15}{16}, & x = 2 \\ 1, & x = 3 \end{cases}.$$

(c) the number of heads minus the number of tails

Solution: If the difference between the number of heads and tails is d, then the number of heads is

$$h + (h - d) = 4 \implies h = \frac{4 + d}{2}.$$

Since the number of heads must be an integer, we have $d = -4, -2, 0, 2, 4$. We also have that the number of outcomes with precisely $(4 + d)/2$ heads is

$$\binom{4}{\frac{4+d}{2}}.$$

Thus, our PMF is

$$P(X = d) = \begin{cases} \frac{1}{16}, & d = -4 \\ \frac{1}{4}, & d = -2 \\ \frac{3}{8}, & d = 0 \\ \frac{1}{4}, & d = 2 \\ \frac{1}{16}, & d = 4 \end{cases}.$$

Our CDF is

$$P(X \geq d) = \begin{cases} \frac{1}{16}, & d = -4 \\ \frac{5}{16}, & d = -2 \\ \frac{11}{16}, & d = 0 \\ \frac{15}{16}, & d = 2 \\ 1, & d = 4 \end{cases}.$$

(d) the number of tails times the number of heads

Solution: The support of X is $0, 3, 4$.
Our PMF is

$$P(X = x) = \begin{cases} \frac{1}{8}, & x = 0 \\ \frac{1}{2}, & x = 3 \\ \frac{3}{8}, & x = 4 \end{cases}.$$

Our CDF is

$$P(X = x) = \begin{cases} \frac{1}{8}, & x = 0 \\ \frac{5}{8}, & x = 3 \\ 1, & x = 4 \end{cases}.$$

5. (10 points) Consider the binomial distribution with n trials and probability p of success on each trial. For what value of k is $P(X = k)$ maximized? This value is called the mode of the distribution. (Hint: Consider the ratio of successive terms.)

Solution: The probability of i successful trials is

$$P(X = i) = p^i \cdot (1-p)^{n-i} \cdot \binom{n}{i}$$
$$= p^i \cdot (1-p)^{n-i} \cdot \frac{n!}{i!(n-i)!}.$$

The ratio between successive terms is

$$\frac{P(X = i+1)}{P(X = i)} = \frac{p^{i+1}(1-p)^{n-i-1}\frac{n!}{(i+1)!(n-i-1)!}}{p^i \cdot (1-p)^{n-i} \cdot \frac{n!}{i!(n-i)!}}$$
$$= \frac{p(n-i)}{(1-p)(i+1)}.$$

As long as $p(n-i) \geq (1-p)(i+1)$, the probability of i successful trials is increasing. Solving for the maximum i for which this is true,

$$p(n-i) \geq (1-p)(i+1)$$
$$pn - pi \geq i + 1 - pi - p$$
$$p(n+1) - 1 \geq i.$$

Since k is the maximum $i+1$,

$$k = \max(i) + 1 = \lfloor p(n+1) \rfloor.$$

6. (10 points) Three players play 10 independent rounds of a game, and each player has probability $\frac{1}{3}$ of winning each round. Find the joint distribution of the numbers of games won by each of the three players.

Solution: Let a, b, c be the number of games each player wins, respectively. Then, for $a + b + c = 10$,

$$P(a, b, c) = \binom{10}{a} \cdot \binom{10-a}{b} \cdot \binom{c}{c} \cdot \left(\frac{1}{3}\right)^a \cdot \left(\frac{1}{3}\right)^b \cdot \left(\frac{1}{3}\right)^c = \frac{\binom{10}{a} \cdot \binom{10-a}{b}}{3^{10}}$$

since there are $\binom{10}{a}$ ways to choose the games that the first player wins, $\binom{10-a}{b}$ ways to choose the games that the second player wins, $\binom{c}{c}$ ways to choose the games that the third player wins, and we multiply them together to find the total number of ways for each player to win their number of games. The probability that player 1 wins the games we assign them is $(\frac{1}{3})^a$, the probability that player 2 wins the games we assign them is $(\frac{1}{3})^b$, and same for player 3.

7. (10 points) If X is a discrete uniform random variable, that is, $P(X = k) = \frac{1}{n}$ for $k = 1, 2, ..., n$, find $E(X)$ and $\text{Var}(X)$.

Solution:

$$E(X) = \sum_{k=1}^{n} \frac{1}{n} \cdot k$$

$$= \frac{n+1}{2}$$

$$\text{Var}(X) = E(X^2) - E(X)^2$$

$$= \sum_{k=1}^{n} \frac{1}{n} k^2 - \left(\frac{n+1}{2}\right)^2$$

$$= \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4}$$

$$= \frac{n^2 - 1}{12}.$$

8. (10 points) Let X be a random variable with mean 1 and variance 4. Find the linear transformation $Y = aX + b$ such that Y has mean zero and variance one.

Solution: By the linearity of expectation,

$$E(Y) = E(aX + b)$$
$$= E(aX) + E(b)$$
$$= aE(X) + b$$
$$= a + b$$
$$\text{Var}(Y) = \text{Var}(aX + b)$$
$$= a^2 \text{Var}(X)$$
$$= 4a^2.$$

Thus, for mean zero and variance one,

$$\begin{cases} a + b = 0 \\ 4a^2 = 1 \end{cases} \implies a = \frac{1}{2}, b = -\frac{1}{2} \text{ or } a = -\frac{1}{2}, b = \frac{1}{2}.$$

9. (10 points) Let X be a random variable that takes values $1, 2, 3$ and 4, and we have $P(X \geq 1) = 1$, $P(X \geq 2) = 0.7$, $P(X \geq 3) = 0.2$, $P(X \geq 4) = 0.1$, $P(X > 4) = 0$. Compute $E[X]$.

Solution: From the CDF, we have that the following:

$$\begin{cases} P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) = 1 \\ P(X = 2) + P(X = 3) + P(X = 4) = 0.7 \\ P(X = 3) + P(X = 4) = 0.2 \\ P(X = 4) = 0.1 \end{cases}.$$

This gives us

$$P(X = 1) = 0.3$$
$$P(X = 2) = 0.5$$
$$P(X = 3) = 0.1$$
$$P(X = 4) = 0.1.$$

Thus

$$E(X) = \sum_{i=1}^{4} i \cdot P(X = i) = 0.3 \cdot 1 + 0.5 \cdot 2 + 0.1 \cdot 3 + 0.1 \cdot 4 = 2.$$

10. (10 points) Let X be a discrete random variable with probability mass function $p(x)$ and let $Y = g(X)$. Show that $E(Y) = \sum_x g(x)p(x)$ provided that $|g(x)|p(x) < \infty$.

Solution: Y is a random variable since it is a composition of two functions: X (random variables are just functions from sample space to $\mathbb{R}$), and g. Thus, the expected value of Y is

Proof.

$$E(Y) = \sum_s Y(s)P(\{s\})$$
$$= \sum_s g(X(s))P(\{s\}).$$

Notice that we can group together sets of s by their image $X(s) = x$. Thus,

$$E(Y) = \sum_s g(X(s)P(\{s\})$$
$$= \sum_x g(x) \sum_{s:X(s)=x} P(\{s\})$$
$$= \sum_x g(x)p(x).$$

$\square$

11. (10 points) **Extra credit.** A random square has a side length that is a uniform $[0, 1]$ random variable. Find the expected area of the square.

Solution:
$$\mathbb{E}_{x \in [0,1]}[x^2] = \int_0^1 x^2 \, dx = \frac{1}{3}.$$