20 $P_3(\mathbb{R})$

$B = \{x^3, x^2, x^1, 1\}$. Dados los subespacios:

$$U = \mathcal{L}(x^2+2x, -x^2+x, x^2+x)$$

$$W \equiv \begin{cases} x_2 + x_3 = 0 \\ 2x_2 - x_3 = 0 \end{cases}$$

$$V \equiv \begin{cases} x_1 = 0 \\ x_2 = -\beta \\ x_3 = 0 \\ x_4 = \alpha + \beta \end{cases}$$

Calcular:

1. $U \cap W$
2. $U + W$
3. ¿Son U y W suplementarios?
4. Una base de $W \cap V$ — $p(x) = a + bx + cx^2 + dx^3$
5. Unas ecs. cartesianas de $U+V$.

$$= \mathcal{L}(\{(0,1,2,0), (0,-1,1,0), (0,1,1,0)\})$$

① $U = \mathcal{L}(x^2+2x, -x^2+x, x^2+x)$

$$W \equiv \begin{cases} x_2 + x_3 = 0 \\ 2x_2 - x_3 = 0 \end{cases}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 1 \\ 2 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$U = \mathcal{L}\{(0,1,2,0), (0,0,2,0)\}$$

$$U = \mathcal{L}(\{(0,1,0,0), (0,0,1,0)\})$$

$$U \equiv \begin{cases} x_1 = 0 \\ x_2 = \lambda \\ x_3 = \mu \\ x_4 = 0 \end{cases}$$

Implícitas de U: $\Rightarrow \dim = 2$

$$\begin{cases} x_1 = 0 \\ x_4 = 0 \end{cases}$$

Paramétricas de U

$$U \cap W \equiv \begin{cases} x_1 = 0 \\ x_4 = 0 \\ x_2 + x_3 = 0;\ x_2 = -x_3 = 0; \\ 2x_2 - x_3 = 0;\ -2x_3 - x_3 = 0;\ x_3 = 0; \end{cases}$$

$\boxed{U \cap W} = \{0\} \Rightarrow \dim(U \cap W) = 0$

$\boxed{U+W}$ $\dim(U+W) = P_3(\mathbb{R})$

③ Sí p9 $U \oplus W = P_3(\mathbb{R})$

④ $W = \begin{cases} x_2 + x_3 = 0 \\ 2x_2 - x_3 = 0 \end{cases}$ $\quad V = \begin{cases} x_1 = 0 \\ x_2 = -\beta \\ x_3 = 0 \\ x_4 = \alpha + \beta \end{cases} \Rightarrow V = \mathcal{L}(\{(0,-1,0,1),(0,0,0,1)\})$

$\Downarrow$

$\begin{cases} x_1 = 0 \\ x_3 = 0 \end{cases}$

$W \cap V \equiv \begin{cases} x_2 + x_3 = 0 \ ; \ x_2 = 0 \\ x_1 = 0 \\ x_3 = 0 \\ 2x_2 - x_3 = 0 \ ; \ -x_3 = 0 \ ; \ x_3 = 0 \end{cases}$; $\quad W \cap V \equiv \begin{cases} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{cases}$ $\quad \dim(W \cap V) = 1$,

$W \cap V = \mathcal{L}(\{(0,0,0,1)\})$.

⑤ Ecuaciones cartesianas de $U+V$

$V = \mathcal{L}(\{(0,-1,0,1),(0,0,0,1)\})$

$U = \mathcal{L}(\{(0,1,0,0),(0,0,1,0)\})$

$U+V = \mathcal{L}(\{(0,-1,0,1),(0,0,0,1),(0,1,0,0),(0,0,1,0)\})$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

$(0,1,0,0)$

$\begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 1 \neq 0$; $\quad U+V = \mathcal{L}(\{(0,0,0,1),(0,1,0,1),(0,0,1,0)\})$

$U+V \equiv \begin{cases} x_1 = 0 \\ x_2 = \lambda \\ x_3 = \mu \\ x_4 = \beta \end{cases} \rightarrow x_1 = 0 \Rightarrow \dim(U+V) = 3$

22 $P_3(\mathbb{R})$

1. Demostrar que el polinomio x^3 y sus 3 primeras derivadas forman una base de $P_3(\mathbb{R})$.

$$p(x) = x^3; \quad p'(x) = 3x^2; \quad p''(x) = 6x; \quad p'''(x) = 6;$$

Si $B_{P_3(x)} = \{1, x, x^2, x^3\}$

$B' = \mathcal{L}(\{(0,0,0,1), (0,0,3,0), (0,6,0,0), (6,0,0,0)\})$

↳ Al ser escalonado son todos LI ⇒ es una base de $P_3(\mathbb{R})$ pq $\dim = 4$

2. Estudiar si
$$\left.\begin{aligned} p_1(x) &= 1+3x+5x^2 \\ p_2(x) &= -1+2x^2 \\ p_3(x) &= 3+3x+x^2 \end{aligned}\right\} \text{ ¿son LD / LI?}$$

$$\begin{pmatrix} 1 & -1 & 3 \\ 3 & 0 & 3 \\ 5 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}; \qquad \begin{vmatrix} 1 & -1 & 3 \\ 3 & 0 & 3 \\ 5 & 2 & 1 \end{vmatrix} = -15+18-6+3 = 0 \Rightarrow \text{son LD.}$$

3. $U = \begin{cases} 1+x^2 = q_1(x) \\ 1-x^2 = q_2(x) \end{cases}$ ¿Pertenecen $r(x) = 1+x$ y $p(x) = 1+5x^2$ a U?

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad \begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{vmatrix} = 1+1 \neq 0 \to r(x) \text{ no pertenece}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & -1 & 5 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & -1 & 5 \end{vmatrix} = 0 \to p(x) \text{ es comb. lineal} \Rightarrow p(x) \text{ sí pertenece a } U.$$

4. Sea $W = \mathcal{L}(p_1(x), p_2(x), p_3(x))$. Calcular $W+U$ y $W \cap U$.

$W = \mathcal{L}(\{(1,3,5,0), (-1,0,2,0)\})$. ($p_3(x)$ es LD)

$U = \mathcal{L}(\{(1,0,1,0), (1,0,-1,0)\})$

$U+W = \mathcal{L}(\{(1,3,5,0), (-1,0,2,0), (1,0,1,0), (1,0,-1,0)\})$.

$$\begin{vmatrix} 1 & -1 & 1 & 1 \\ 3 & 0 & 0 & 0 \\ 5 & 2 & 1 & -1 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{vmatrix} = 0; \qquad \begin{vmatrix} 1 & -1 & 1 \\ 3 & 0 & 0 \\ 5 & 2 & 1 \end{vmatrix} = 6+3 \neq 0$$

$U+W = L(\{(1,3,5,0), (-1,0,2,0), (1,0,1,0)\})$.

Calculemos ahora la intersección:

$W = L(\{(1,3,5,0), (-1,0,2,0)\})$.

$U = L(\{(1,0,1,0), (1,0,-1,0)\})$

Calculamos las paramétricas de W:

$$W \equiv \begin{cases} x_1 = \lambda - \mu \\ x_2 = 3\lambda \\ x_3 = 5\lambda + 2\mu \\ x_4 = 0 \end{cases} \qquad \lambda = \frac{x_2}{3}; \qquad \mu = \frac{x_2}{3} - x_1 \qquad W = \{ x_4 = 0$$

$$x_3 = \frac{5x_2}{3} + \frac{2x_2}{3} - 2x_1;$$

No da pereza.

(23) $\mathbb{R}^3$. Subespacios:

$U = L((2,0,-1), (1,2,0), (0,4,1))$

$$W \equiv \begin{cases} x = 0 \\ y + z = 0 \end{cases}$$

$$\begin{vmatrix} 2 & 1 & 0 \\ 0 & 2 & 4 \\ -1 & 0 & 1 \end{vmatrix} = 4 - 4 = 0;$$

$$\begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} = 4 \neq 0;$$

Calcular bases de los subespacios:

- $\mathbb{R}^3/U$:

$$\left.\begin{array}{l} \dim(\mathbb{R}^3) = 3 \\ \dim(U) = 2 \end{array}\right\} \dim(\mathbb{R}^3/U) = 3 - 2 = 1$$

$U = L((2,0,-1), (1,2,0));$

Cojo un complementario de U:

$W = L((0,0,1));$

$$\begin{vmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{vmatrix} = 4 \neq 0$$

$\mathbb{R}^3/U = \{(0,0,1) + U\}$

$(-1,2,1) + U = a(0,0,1) + U;$

$(-1,2,1) - (0,0,a) \in U;$

$1 - a \in U;$

sustituimos en las implícitas de $U \Rightarrow \boxed{a = 0}$

• $\mathbb{R}^3/W$: $\dim(\mathbb{R}^3)=3$; $\dim(W)=1 \Rightarrow \dim(\mathbb{R}^3/W) = 3-1=2$;

$W \equiv \begin{cases} x=0 \\ y+z=0 \end{cases}$; $W = L(\{(0,1,-1)\})$

$\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \end{vmatrix} = -1 \neq 0$;

Complementario a W:

$W^c = L(\{(1,0,0),(0,0,1)\})$

$\mathbb{R}^3/W = \{(1,0,0)+W, (0,0,1)+W\}$

24 $M_2(\mathbb{R})$

$A = \begin{pmatrix} 1 & 2 \\ 3 & m \end{pmatrix}$ $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

1. Demostrar que $F = \{B \in M_2(\mathbb{R}) \mid AB = 0\}$ es un subespacio vectorial.

$A \cdot B_1 = \begin{pmatrix} a_1+2c_1 & b_1+2d_1 \\ 3a_1+mc_1 & 3b_1+md_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$A \cdot B_2 = \begin{pmatrix} a_2+2c_2 & b_2+2d_2 \\ 3a_2+mc_2 & 3b_2+md_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$AB_1 + AB_2 = \begin{pmatrix} a_1+a_2+2c_1+2c_2 & b_1+b_2+2d_1+2d_2 \\ 3a_1+3a_2+mc_1+mc_2 & 3b_1+3b_2+md_1+md_2 \end{pmatrix}$

2. Calcular en función de m la dimensión de F y una base.

$A \cdot B = \begin{pmatrix} a+2c & b+2d \\ 3a+mc & 3b+md \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$;

$\begin{vmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & m & 0 \\ 0 & 3 & 0 & m \end{vmatrix} = C$

$\Updownarrow$

$m^2-12m+36=0 \Leftrightarrow m=6$

$\begin{cases} a+2c=0; \ a=-2c=0; \\ b+2d=0; \ b=-2d; \\ 3a+mc=0; \ -6c+mc=0; \ c\underbrace{(-6+m)}_{\neq 0}=0; \Rightarrow c=0 \\ 3b+md=0; \ -6d+md=0; \ d(-6+m)=0; \end{cases}$

$-6+m=0$;

$\boxed{m=6}$

• Si $m=6 \Rightarrow$ Rango $C=2 \Leftrightarrow \dim F = 4-2=2$

• Si $m \neq 6 \Rightarrow$ Rango $C=4 \Leftrightarrow \dim F = 4-4=2$