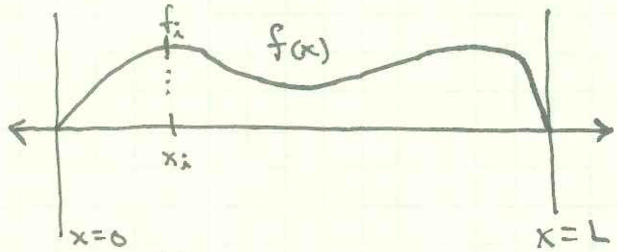
Derivation of finite difference equations to matrix form

- Given a function $f(x)$ which varies in 1-D

- $f_i \simeq f(x_i)$



Given the space from 0 to L is divided into n equidistant points s.t.

① $x_i = x_0 + i \cdot \Delta x$ where $\Delta x = L/n$

- $$\frac{\partial f(x_i)}{\partial x} \simeq \frac{f(x_i + \Delta x) - f(x_i)}{\Delta x} = \frac{f(x_i) - f(x_i - \Delta x)}{\Delta x}$$

$$\shortparallel$$

$$② \quad \frac{f(x_i + \Delta x) - f(x_i - \Delta x)}{2\Delta x}$$

- Similarly, $$\frac{\partial^2 f(x_i)}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f(x_i)}{\partial x}\right) \simeq \frac{\frac{f_{i+1} - f_i}{\Delta x} - \frac{f_i - f_{i-1}}{\Delta x}}{\Delta x}$$

$$\shortparallel$$

$$③ \quad \frac{f_{i+1} - 2f_i + f_{i-1}}{\Delta x^2}$$

- For any point, x_i, $f(x_i)$ depends on f_{i-1}, f_i, and f_{i-1} as well as Δx

- For heat conduction, $\dot{Q} = -k \frac{\partial^2 T}{\partial x^2}$ ... using ③,

  - $$\dot{Q} = -k\left[\frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta x^2}\right] \quad \text{(or)} \quad \frac{\dot{Q}\Delta x^2}{k} = -T_{i+2} + 2T_i - T_{i-1} \quad ④$$

- For all x_is (except x_0 + x_L) ④ can be cast into matrix form as

$$\begin{bmatrix} -1 & 2 & -1 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & -1 & 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} T_i \\ \vdots \\ T_{n-1} \end{bmatrix} = \begin{bmatrix} \frac{\dot{Q}\Delta x^2}{k} \\ \vdots \\ \vdots \end{bmatrix}$$

- Now we must consider boundary conditions for the first and last rows.

- Given the boundry conditions:

$$T_0 + \frac{d}{dx} T_0 = a \quad + \quad T_L + \frac{d}{dx} T_L = b$$

~~the coefficients should~~

$$a = T_0 + \left[\frac{T_1 - T_0}{\Delta x}\right] \quad + \quad b = T_L + \left[\frac{T_L - T_{L-1}}{\Delta x}\right]$$

$$\parallel \qquad\qquad \parallel$$

$$T_0 + \frac{T_1}{\Delta x} - \frac{T_0}{\Delta x} = a \qquad b = T_L + \frac{T_L}{\Delta x} - \frac{T_{L-1}}{\Delta x}$$

$$T_0\left(1 - \frac{1}{\Delta x}\right) + \frac{T_1}{\Delta x} = a \qquad b = T_L\left(1 + \frac{1}{\Delta x}\right) - \frac{T_{L-1}}{\Delta x}$$

- Mult. by Δx

$$T_0(\Delta x - 1) + T_1 = a\Delta x \qquad b\Delta x = T_L(1 + \Delta x) - T_{L-1}$$

- So the first line becomes:

$$[\Delta x - 1, 1, 0, \ldots]$$

- And the last line is:

$$[\ldots, 0, -1, \Delta x + 1]$$

- S.T.

$$\boxed{\begin{bmatrix} \Delta x-1 & 1 & 0 & \cdots & & 0 \\ -1 & 2 & 1 & \cdots & & 0 \\ 0 & -1 & 2 & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & & -1 & 2 & 1 \\ 0 & \cdots & & 0 & -1 & \Delta x+1 \end{bmatrix} \cdot \begin{bmatrix} T_0 \\ T_1 \\ \vdots \\ \vdots \\ T_{L-1} \\ T_L \end{bmatrix} = \begin{bmatrix} a\Delta x \\ \frac{\dot{Q}\Delta x^2}{k} \\ \vdots \\ \vdots \\ \frac{\dot{Q}\Delta x^2}{k} \\ b\Delta x \end{bmatrix}}$$