Computing Casimir Energies

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1. The Casimir Effect

The Casimir Effect is an attractive force between two uncharged conductors in a vacuum due to quantum fluctuations of the electromagnetic field. Due to quantized energies, only the photons which have whole number multiples of their wavelengths that fit into the gap between the conductors contribute to the total energy of the configuration. Hence, the total energy decreases as we move the conductors closer together. This indicates the existence of an attractive force – the Casimir force.

Scattering theory methods allow utilizing scattering amplitudes (T-matrices) to express the Casimir interaction energy for different shapes and boundary conditions [3]. For small separations, calculating these matrices becomes more computationally intensive, therefore a different approach is required. In this project, we consider short-separation configurations by utilizing "proximity force approximation" method that allows us to express energy as an inverse polynomial expansion of separation.

2. Cases Considered

The first configuration we consider is a conducting strip parallel to a conducting plane, as described in [1]. Such a strip can be approximated as an elliptic cylinder in the limit of zero radius. For this reason we use elliptic cylinder coordinates, (μ, θ, z) where $x = d\cosh\mu\cos\theta$, $y = d\sinh\mu\cos\theta$, and $z = z$.

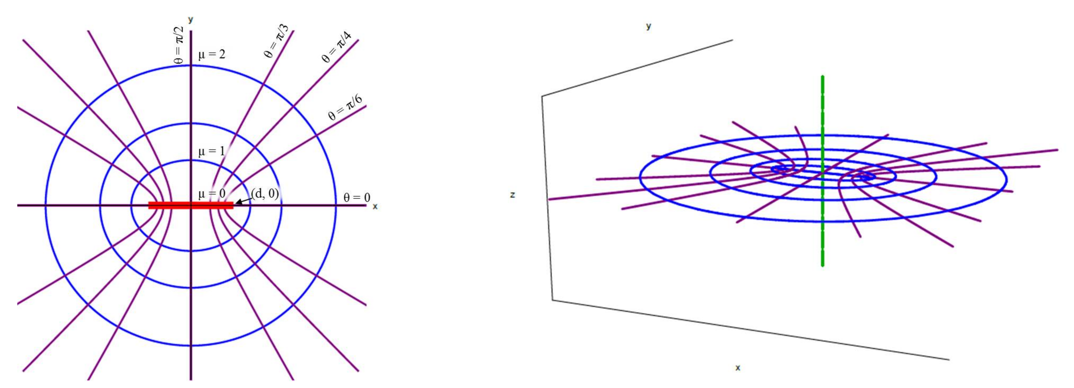


Figure 1: *Elliptic cylinder coordinate system.*

Obtaining the Casimir energies requires solving the Helmholtz differential equation in elliptical coordinates. The angular and radial solutions are known as Mathieu functions. We then consider the expansion of the Casimir interaction energy per unit length:

$$\frac{\mathcal{E}}{\hbar c L} = -\frac{\pi^2}{720}\frac{2d}{H^3} + \frac{2\beta}{H^2} + \frac{\gamma}{2dH} + \dots \tag{1}$$

as described in [1].

The leading term in this expansion is the proximity force approximation, the second term represents the interaction between the two edges and the plane, and the third term represents the interaction between the edges. The graph below represents the ratio of the exact Casimir interaction energy to the proximity force approximation for a strip parallel to a plane as a function of separation, and the best fit line, leading to $\beta = 0.00092$ and $\gamma = -0.0040$. [1]

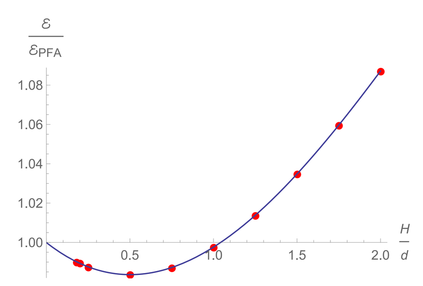


Figure 2: *Ratio of the exact Casimir interaction energy to the proximity force approximation as a function of separation for a perfectly conducting strip parallel to a plane. [1]*

The second configuration we consider is the case of a flat disk parallel to a plane. This is the limiting case of an oblate spheroid. Similarly to the previous case, we obtain solutions to the Helmholtz equation in the spheroidal coordinate system which are spheroidal functions. At short distances, we expand the Casimir interaction energy as:

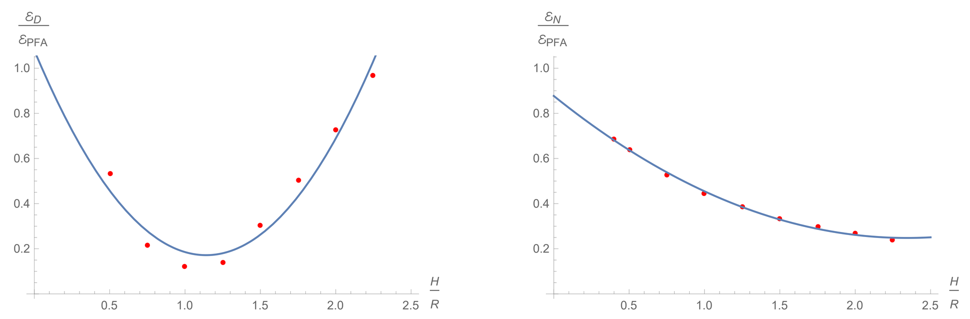
$$\frac{\mathcal{E}_D}{\hbar c} = -\frac{\pi^3}{1440}\frac{R^2}{H^3} + \beta_D\frac{2R\pi}{H^2} + \dots \tag{2}$$

for the Dirichlet's boundary conditions, and

$$\frac{\mathcal{E}_N}{\hbar c} = -\frac{\pi^3}{1440}\frac{R^2}{H^3} + \beta_N\frac{2R\pi}{H^2} + \dots \tag{3}$$

for Neumman's.

Analogously to the previous case, we find the correction coefficients, β_D and β_N. The graphs below represent the ratio of the exact Casimir interaction energy to the proximity force approximation, as a function of separation. From the polynomial fit, we extract values for $\beta_D = -0.00343$ and $\beta_N = 0.00171$.



(a) Dirichlet Boundary Condition (b) Neumann Boundary Condition

Figure 3: *Ratio of the exact Casimir interaction energy to the proximity force approximation as a function of separation for a perfectly conducting disk parallel to a plane.*

Comparing these two configurations will potentially give us more insight about the contribution from the edges of a disk and its curvature to the total Casimir energies for the considered configuration. We assume that the net β coefficients in the case of a strip and the disk should be approximately the same for the short distances. This requires further examination for different separations in the case of a disk, that in turn requires improving the existing code.

3. Taylor Expansions of Mathieu Functions

The Helmholtz equation in elliptic cylinder coordinates is as follows:

$$\frac{1}{d^2(\cosh^2\mu - \cos^2\theta)}\frac{\partial^2\Psi}{\partial\mu^2} + \frac{\partial^2\Psi}{\partial\theta^2} + \frac{\partial^2\Psi}{\partial z^2} + k^2\Psi = 0 \tag{4}$$

This partial differential equation is separable and the angular and radial parts become

$$\frac{d^2\Theta}{d\theta^2} + (a - 2qcos2\theta)\Theta(\theta) = 0 \tag{5}$$

$$\frac{d^2M}{d\mu^2} - (a - 2qcosh2\mu)M(\mu) = 0 \tag{6}$$

The solutions to these two differential equations are the Mathieu functions. The separation constant, a, is called the characteristic value. The parameter q is proportional to (d), squared. Beyond angular and radial, the solutions can be characterized as ordinary or modified, first kind or second kind, and even or odd. Therefore, there is a total of sixteen Mathieu functions.Typically the characteristic values are denoted with an a for even solutions and with a b for odd solutions.

We are interested in periodic solutions. Characteristic values which yield periodic solutions are indexed by the integer r, which runs from 0 to ∞ for even solutions and 1 to ∞ for odd solutions. Our goal was to take an existing Mathematica package developed by [1] which computes Mathieu functions numerically and make it capable of computing them symbolically. To achieve this, we expand the Mathieu functions as Taylor series about $q = 0$. The angular Mathieu functions can be represented by series of sines and cosines and the radial functions by series of Bessel functions, in a manner similar to Fourier series. The coefficients are specified by the given characteristic value. For example, the angular, ordinary, first kind, even Mathieu function for odd indices is

$$ce_r(q,\theta) = \frac{1}{\sum_{m=0}^{\infty} A_{2m+1}(r,q)^2}\sum_{m=0}^{\infty} A_{2m+1}(r,q)\cos[(2m+1)\theta] \tag{7}$$

Mathematica is capable of computing series expansion for trigonometric and Bessel functions. In theory, the expressions for the Fourier series could be run through the Mathematica Series function in order to produce Taylor series. However, in order to speed up calculations, the existing package used a built in Mathematica function which returns the characteristic value for a given index r and parameter q. Because this function requires a numerical input for q, the existing code did not allow for symbolic calculations. To solve this problem, we use a method developed by [2], which allows us to represent the even and odd characteristic values symbolically for a given r in the following way:

$$a_r = r^2 + \sum_{k=1}^{\infty}\alpha_k^{(r)}q^k \tag{8}$$

$$b_r = r^2 + \sum_{k=1}^{\infty}\beta_k^{(r)}q^k \tag{9}$$

The coefficients α and β are given by recursion relations. Replacing the numerical version of the characteristic value in the code with the new symbolic version, we can run the formulas given by [1] through Mathematicas series functions and obtain Taylor series about $q = 0$.

The series of the angular, ordinary, first kind, even Mathieu function of index 1, called with an order 2, is:

$$ce_1(q,\theta) = \cos\theta - \frac{1}{8}\cos 3\theta q + \frac{1}{384}(-3\cos\theta - 6\cos 3\theta + 2\cos 5\theta)q^2 + O[q]^3 \tag{10}$$

4. Optimizing the Taylor Series Expansion of the Mathieu Equations

The original Mathematica package built the Mathieu functions out of series of trigonometric or Bessel functions. While these series need to be infinite to exactly represent Mathieu functions, when doing numerical calculations it is sufficient to set the upper limit high enough that they are almost exact. However, running these trigonometric and Bessel series with large upper limits through the Mathematica Series function to create Taylor series is time consuming. Luckily, for creating Taylor series, not all the terms are needed. Only terms which contain powers of the index q equal to or lower than the order of the series being called are required. In order to optimize the run time and assure accuracy of our package, it was necessary to find formulas for the minimum number terms needed. These formulas were functions of both the order of the series being called and the index r. To obtain such equations, we gathered many different data points. Using Mathematicas built in Exponent functions, we were able to determine when terms in the Fourier series contained powers of q greater than the order of the Taylor series being asked for which can be omitted. Once we considered enough points, trends appeared which we were able to implement into the package.

References

[1] Blose, E.N.; Ghimire, B.; Graham, N.; and Stratton-Smith, J. 2014
[2] Frenkel, D.; Portugal, R. 2001
[3] Lambrecth, A; Maia Neto, P.A.; Raynaud S. 2006
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[5] Maia Neto, P.A.; Lambrecth, A; Raynaud S. 2008