Frontiers in Networked Control	Lecture 4
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Tools from graph theory

Lecturer: Magnus Lindhé
Instructor: Michael Rotkowitz Scribe: Alberto Speranzon

1 Introduction

In this lecture a distributed control law for a group of agents is designed and analyzed. The control law is such that the four following global behaviors are obtained: (i) the velocity vectors are aligned, (ii) the speed converges to a common value, (iii) agents avoid collisions and (iv) agents minimizes an artificial potential function. This lecture is based on the following paper: "H. Tanner, A. Jadbabaie and G. J. Pappas, *Flocking in Fixed and Switching Networks*, submitted to Transaction on Automatic and Control, 2005".

2 Preliminaries

We recall here some important fact about graphs that will be instrumental for what follows.

Let us consider a graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ with vertex (or node) set $\mathcal{V}$ and edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. An example of a graph with four nodes $\mathcal{V} = \{1, 2, 3, 4\}$ and edges

$$\mathcal{E} = \{(1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2), (3, 4), (4, 3)\}$$

is shown in Figure 1.

From now on, we will always consider undirected graph, so that if $(i, j) \in \mathcal{E}$ then $(j, i) \in \mathcal{E}$ if $i \neq j$. To any given graph is always possible to associate the *adjacency matrix* A, namely the matrix

$$[A]_{ij} = \begin{cases} 1 & \text{if } (i, j) \in \mathcal{E} \text{ or } i = j \\ 0 & \text{otherwise} \end{cases}$$

The matrix A univocally describes a graph, up to a relabeling.

Consider the graph in Figure 1. It is easy to see that its adjacency matrix is

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} .$$

If we consider a vertex i of an undirected graph $\mathcal{G}$ we denote with *degree* of the vertex i the number of edges entering (or leaving) the vertex plus one. The *degree matrix* of a graph is the diagonal matrix with the degree of each vertex as entry. Considering again the graph in Figure 1, we have that

$$D = \text{diag}(3, 3, 4, 2) .$$

A graph is called *connected* if and only if there is a sequence of edges, *path*, connecting any pair of vertexes of the graph. Given a graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ a subgraph is the graph $\mathcal{S}(\mathcal{Q}, \mathcal{W})$ with $\mathcal{Q} \subset \mathcal{V}$ and $\mathcal{W} \subset \mathcal{E}$ such that if $i, j \in \mathcal{Q}$ then $(i, j) \in \mathcal{W}$ iff $(i, j) \in \mathcal{E}$. A *connected component* is a maximal connected subgraph. That is, two vertices are in the same connected component if and only if there exists a path between them.

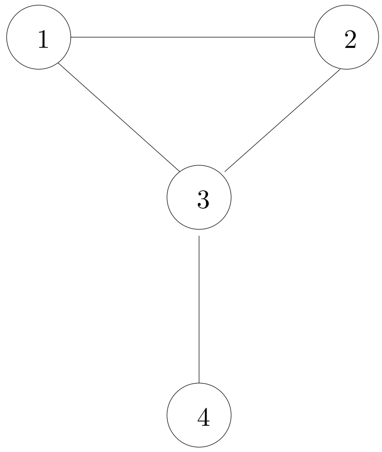

Figure 1: Example of a graph with four vertexes.

From the degree matrix and the adjacency matrix it is possible to define the *Laplacian matrix* of the graph $\mathcal{G}$ as follows

$$L = D - A\,.$$

For the graph of Figure 1 we have

$$L = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

The Laplacian matrix of an undirected graph has the following properties:

(i) It is a semi-definite matrix, $L \geq 0$, and symmetric.

(ii) The smallest eigenvalue is $\lambda_1 = 0$ with associated eigenvector $\mathbb{1} = (1, \ldots, 1)^T$.

(iii) The multiplicity of the eigenvalue $\lambda_1 = 0$ corresponds to the number of connected components

(vi) The first non-zero eigenvalue, λ_2, is related to the connectivity of a graph.

(iv) All other eigenvalue are such that $\lambda_i > 0$.

3 Problem formulation

Consider a group of N agents moving on the plane. Each agent is modelled as a double integrator

$$\begin{aligned} \dot{r}_i &= v_i \\ \dot{v}_i &= u_i \end{aligned}$$

for $i = 1, \ldots, N$. To a group of agents we associate an undirected graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ such that $|\mathcal{V}| = N$ and $(i,j) \in \mathcal{E})$ if there is interaction between agent i and agent j.

The control action for each agent consists of two contributions

$$u_i = \underbrace{-\sum_{j \in \mathcal{N}_i} (v_i - v_j)}_{\alpha_i} \underbrace{- \sum_{j \in \mathcal{N}_i} \nabla_{r_i} V_{ij}}_{a_i}$$

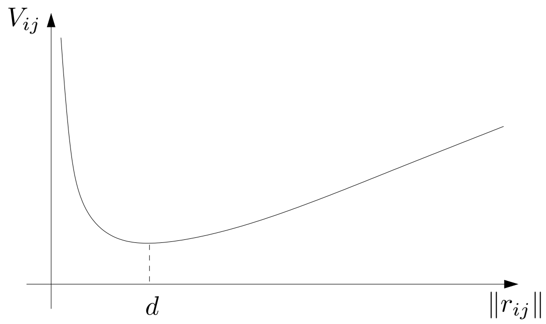

Figure 2: Example of a graph with four vertexes.

were $\mathcal{N}_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$. The term α_i is responsible of making the velocity of the agents equal, whereas the second term a_i is responsible of maintaining a fixed distance between interacting agents. The main assumption here is that the neighbor $\mathcal{N}_i$ of an agent i does not change with time. The function V_{ij} is an artificial potential function which is differentiable, nonnegative, radially unbounded function of the distance $\|r_{ij}\|$ between agents i and j, and such that

(1) $V_{ij}(\|r_{ij}\|) \to +\infty$ as $\|r_{ij}\| \to 0$

(2) V_{ij} has a unique minimum, exactly at the value corresponding to the desired interagent distance.

An example of such function with minimum at d is shown in Figure 2.

The main result is the following.

Theorem 1 *If the agents have a connected neighboring graph, then interconnected agents will not collide. Moreover they will approach a configuration that minimizes the potential energy and they will reach equal velocities.*

Proof.

Let us consider the following Lyapunov function:

$$W(\bar{r}, v) = \frac{1}{2} \sum_{i=1}^{N} (V_i + v_i^T v_i)$$

with $v_i = (v_{i_x}, v_{i_y})^T$ where v_{i_x} is the $x-$component of the velocity and similarly v_{i_y}. We also defined

$$V_i = \sum_{j \in \mathcal{N}_i} V_{ij} \, .$$

The vector $\bar{r}$ denotes relative distances. It is possible to show [1] that the set

$$\Omega = \{(r_{ij}, v_i) : W \leq c\}$$

with $c > 0$ is compact. We also have that

$$\dot{W} = [...] = -\sum_{i=1}^{N} V_i^T \sum_{j \in \mathcal{N}_i} (v_i - v_j) = -v_x^T L v_x - v_y^T - v_y \leq 0$$

since L is semi-definite. Notice that v_x and v_y denotes the stack vectors of the components of the agent velocities projected to the x and y axis. Since the eigenvector of L associated with the single zero eigenvalue

[1]See original paper.

is $\mathbb{1}$, this means that $\dot{W}$ will only be zero whenever both v_x and v_y belong to $\mathrm{span}(\mathbb{1})$, implying that all agent velocities have the same components and are therefore equal. It follows immediately that $\dot{\bar{r}} = 0$. Applying LaSalles invariant principle, it follows that if the initial conditions of the system lie in Ω, its trajectories will converge to the largest invariant set inside the region $S = \{v | \dot{W} = 0\}$. Using the incident matrix of the graph, it is possible to show that

$$\dot{v}_x, \dot{v}_y \in \mathrm{span}(\mathbb{1})^{\perp} .$$

Since S is invariant

$$v_x, v_y \in \mathrm{span}(\mathbb{1}) \Rightarrow \dot{v}_x, \dot{v}_y \in \mathrm{span}(\mathbb{1}) .$$

This yield a contradiction unless

$$\dot{v}_x, \dot{v}_y \in \mathrm{span}(\mathbb{1})^{\perp} \cap \mathrm{span}(\mathbb{1}) \equiv 0 .$$

This means that the agents velocities do not change in steady state and that the potential of each agent is minimized. $\square$

Remark

- If the graph is a tree, then interagent distances are exactly d.
- The speed of convergence is bounded by λ_2.
- The feedback has the same structure of the adjacency matrix.