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# On the Closest Quadratically Invariant Constraint

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***Abstract*— Quadratic invariance is a condition which has been shown to allow for optimal decentralized control problems to be cast as convex optimization problems. The condition relates the constraints that the decentralization imposes on the controller to the structure of the plant. In this paper, we consider the problem of finding the closest subset and superset of the decentralization constraint which are quadratically invariant when the original problem is not. We show that this can itself be cast as a convex problem for the case where the controller is subject to delay constraints between subsystems, but that this fails when we only consider sparsity constraints on the controller. For that case, we develop an algorithm that finds the closest superset in a fixed number of steps, and discuss methods of finding a close subset.**

## I. Introduction

The design of decentralized controllers has been of interest for a long time, as evidenced in the survey [1], and continues to this day with the advent of complex interconnected systems. The counterexample constructed by Hans Witsenhausen in 1968 [2] clearly illustrates the fundamental reasons why problems in decentralized control are difficult.

Among the recent results in decentralized control, new approaches have been introduced that are based on algebraic principles, such as the work in [3]–[5]. Very relevant to this paper is the work in [3], [4], which classified the problems for which optimal decentralized synthesis could be cast as a convex optimization problem. Here, the plant is linear, time-invariant and it is partitioned into dynamically coupled subsystems, while the controller is also partitioned into subcontrollers. In this framework, the decentralization being imposed manifests itself as constraints on the controller to be designed, often called the *information constraint*.

The information constraint on the overall controller specifies what information is available to which controller. For instance, if information is passed between subsystems, such that each controller can access the outputs from other subsystems after different amounts of transmission time, then the information constraints are delay constraints, and may be represented by a matrix of these transmission delays. If instead, we consider each controller to be able to access the outputs from some subsystems but not from others, then the information constraint is a sparsity constraint, and may be represented by a binary matrix.

Given such pre-selected information constraints, the existence of a convex parameterization for all stabilizing controllers that satisfy the constraint can be determined via the algebraic test introduced in [3], [4], which is denoted as *quadratic invariance*. In contrast with prior work, where the information constraint on the controller is fixed beforehand, this paper addresses the design of the information constraint itself. More specifically, given a plant and a pre-selected information constraint that is not quadratically invariant, we give explicit algorithms to compute the quadratically invariant information constraint that is closest to the pre-selected one. We consider finding the closest quadratically invariant superset, which corresponds to relaxing the pre-selected constraints as little as possible to get a tractable decentralized control problem, which may then be used to obtain a lower bound on the original problem, as well as finding the closest quadratically invariant subset, which corresponds to tightening the pre-selected constraints as little as possible to get a tractable decentralized control problem, which may then be used to obtain upper bounds on the original problem.

We consider the two particular cases of information constraint outlined above. In the first case, we consider constraints as transmission delays between the output of each subsystem and the subcontrollers that are connected to it. The distance between any two information constraints is quantified via a norm of the difference between the delay matrices, and we show that we can find the closest quadratically invariant set, superset, or subset as a convex optimization problem.

In the second case, we consider sparsity constraints that represent which controllers can access which subsystem outputs, and represent such constraints with binary matrices. The distance between information constraints is then given by the hamming distance, applied to the binary sparsity matrices. We provide an algorithm that gives the closest superset; that is, the quadratically invariant constraint that can be obtained by way of *allowing* the least number of additional links, and show that it terminates in a fixed number of iterations. For the problem of finding a close subset, we propose two heuristic-based solutions.

**Paper organization:** Besides the introduction, this paper has six sections. Section II presents the notation and the basic concepts used throughout the paper. The delay and sparsity constraints adopted in our work are described in detail in Section III, while their characterization using quadratic invariance is given in Section IV. The main problems addressed in this paper are formulated and solved in Section V. Section VI briefly notes how this work also applies when assumptions of linear time-invariance are dropped, while conclusions are given in Section VII.

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## II. Preliminaries

We suppose that we have a generalized plant $P$ partitioned as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & G \end{bmatrix}$$

We define the ***closed-loop map*** by

$$f(P, K) = P_{11} + P_{12}K(I - GK)^{-1}P_{21}$$

The map $f(P, K)$ is also called the (lower) ***linear fractional transformation*** (LFT) of $P$ and $K$. Note that we abbreviate $G = P_{22}$, since we will refer to that block frequently, and so that we may refer to its subdivisions without ambiguity. This interconnection is shown in Figure 1.

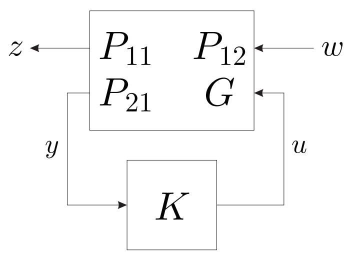


Fig. 1. Linear fractional interconnection of $P$ and $K$

We suppose that there are $n_y$ sensor measurements and $n_u$ control actions, and thus partition the sensor measurements and control actions as

$$y = \begin{bmatrix} y_1^T & \dots & y_{n_y}^T \end{bmatrix}^T \qquad u = \begin{bmatrix} u_1^T & \dots & u_{n_u}^T \end{bmatrix}^T$$

and then further partition $G$ and $K$ as

$$G = \begin{bmatrix} G_{11} & \dots & G_{1n_u} \\ \vdots & & \vdots \\ G_{n_y1} & \dots & G_{n_yn_u} \end{bmatrix} \qquad K = \begin{bmatrix} K_{11} & \dots & K_{1n_y} \\ \vdots & & \vdots \\ K_{n_u1} & \dots & K_{n_un_y} \end{bmatrix}$$

This will typically represent $n$ subsystems, each with its own controller, in which case we will have $n = n_y = n_u$, but this does not have to be the case.

Given $A \in \mathbb{R}^{m\times n}$, we may write $A$ in term of its columns as

$$A = \begin{bmatrix} a_1 & \dots & a_n \end{bmatrix}$$

and then associate a vector $\mathrm{vec}(A) \in \mathbb{R}^{mn}$ defined by

$$\mathrm{vec}(A) = \begin{bmatrix} a_1^T & \cdots & a_n^T \end{bmatrix}^T$$

### A. Delays

We define $\mathrm{Delay}(\cdot)$ for a causal operator as the smallest amount of time in which an input can affect its output. For any causal $H : \mathcal{L}_e^m \to \mathcal{L}_e^n$,

$$\begin{aligned} \mathrm{Delay}(H) \;=\; \inf\{\tau \geq 0 \mid\ & z_1(T+\tau) \neq z_2(T+\tau), \\ & z_1 = H(w_1), z_2 = H(w_2), \\ & w_1, w_2 \in \mathcal{L}_e^m, \\ & w_1(t) = w_2(t)\ \forall\ t \leq T\} \end{aligned}$$

and if $H = 0$, we consider its delay to be infinite.

When $H$ is time-invariant, we may choose $T = 0$, and when $H$ is linear, we may choose $w_1 = 0$, so for a causal linear time-invariant (LTI) $H : \mathcal{L}_e^m \to \mathcal{L}_e^n$,

$$\begin{aligned} \mathrm{Delay}(H) \;=\; \inf\{\tau \geq 0 \mid\ & z(\tau) \neq 0,\ z = H(w), \\ & w \in \mathcal{L}_e^m,\ w(t) = 0\ \forall\ t \leq 0\} \end{aligned}$$

Given an impulse response $h$ which characterizes the map $H$, we can then also give the delay as

$$\mathrm{Delay}(H) \;=\; \inf\{\tau \geq 0 \mid h(\tau) \neq 0\}$$

### B. Sparsity

We introduce some notation which will be useful when we consider sparsity patterns and sparsity constraints.

*1) Binary algebra:* Let $\mathbb{B} = \{0, 1\}$ represent the set of binary numbers. Given $x, y \in \mathbb{B}$,

$$x + y \;=\; \begin{cases} 0, & \text{if } x = y = 0 \\ 1, & \text{otherwise} \end{cases}$$

and

$$xy \;=\; \begin{cases} 1, & \text{if } x = y = 1 \\ 0, & \text{otherwise} \end{cases}$$

Given $X, Y \in \mathbb{B}^{m\times n}$, we say that $X \leq Y$ iff $X_{ij} \leq Y_{ij}$ for all $i, j$.

Given $X, Y, Z \in \mathbb{B}^{m\times n}$, these definitions lead to a few immediate consequences:

$$Z = X + Y \;\Rightarrow\; Z \geq X \tag{1}$$

$$X + Y = X \;\Leftrightarrow\; Y \leq X \tag{2}$$

$$X \leq Y,\ Y \leq X \;\Leftrightarrow\; X = Y \tag{3}$$

Given $X \in \mathbb{B}^{m\times n}$, let $\mathcal{N}(X) = \sum\sum X$, with the sum taken in the usual way, and thus giving the total number of nonzero indices in the binary matrix.

*2) Sparsity patterns:* Suppose $A^{\text{bin}} \in \mathbb{B}^{m\times n}$ is a binary matrix. We define the subspace

$$\begin{aligned} \mathrm{Sparse}(A^{\text{bin}}) = \Big\{ & B \in \mathcal{R}_p^{m\times n} \mid B_{ij}(j\omega) = 0 \text{ for all } i, j \\ & \text{such that } A^{\text{bin}}_{ij} = 0 \text{ for almost all } \omega \in \mathbb{R} \Big\} \end{aligned}$$

giving all of the proper transfer function matrices which satisfy the given sparisty constraint. When appropriate, we could instead define this more simply for all such $B \in \mathbb{R}^{m\times n}$ satisfying the sparsity constraint. Conversely, if $B \in \mathcal{R}_{sp}^{m\times n}$ (or $\mathbb{R}^{m\times n}$), let $A^{\text{bin}} = \mathrm{Pattern}(B)$ be the binary matrix given by

$$A^{\text{bin}}_{ij} = \begin{cases} 0, & \text{if } B_{ij}(j\omega) = 0 \text{ for almost all } \omega \in \mathbb{R} \\ 1, & \text{otherwise} \end{cases}$$

which gives the corresponding sparsity pattern.

## III. Problem Formulation

We now introduce the two main types of problems we will consider in this paper, delay constraints and sparsity constraints, and then formulate our optimization problem.

### A. Delays

We now consider the problem where we have multiple subsystems which may affect one another with some propagation delays, and which may communicate with one another with some transmission delays.

*1) Propagation Delays:* For any pair of subsystems $i$ and $j$ we define the propagation delay $p_{ij}$ as the amount of time before a controller action at subsystem $j$ can affect an output at subsystem $i$ as such

$$p_{ij} \;=\; \text{Delay}(G_{ij}) \qquad \forall\; i \in 1,\ldots,n_y,\; j \in 1,\ldots,n_u$$

*2) Transmission Delays:* For any pair of subsystems $k$ and $l$ we define the (total) transmission delay $t_{kl}$ as the minimum amount of time before the controller of subsystem $k$ may use outputs from subsystem $l$. Given these constraints, we can define the overall subspace of admissible controllers $S$ such that $K \in S$ if and only if

$$\text{Delay}(K_{kl}) \;\geq\; t_{kl} \qquad \forall\; k \in 1,\ldots,n_u,\; l \in 1,\ldots,n_y$$

### B. Sparsity Constraints

We now introduce the other main class of problems we will consider in this paper, where each control input may access certain sensor measurements, but not others.

We then represent the constraints on the overall controller with a binary matrix $K^{\text{bin}} \in \mathbb{B}^{n_u \times n_y}$ where

$$K^{\text{bin}}_{kl} \;=\; \begin{cases} 1, & \text{if control input } k \\ & \text{may access sensor measurement } l \\ 0, & \text{if not.} \end{cases}$$

The subspace of admissible controllers is then given as

$$S \;=\; \text{Sparse}(K^{\text{bin}})$$

We find that the relevant information about the plant is then its sparsity pattern, given as

$$G^{\text{bin}} \;=\; \text{Pattern}(G)$$

where we then have

$$G^{\text{bin}}_{ij} \;=\; \begin{cases} 1, & \text{if control input } j \\ & \text{affects sensor measurement } i \\ 0, & \text{if not.} \end{cases}$$

### C. Problem Setup

Given a generalized plant $P$ and a subspace of admissible controllers $S$, we would then like to solve the following problem:

$$\begin{aligned} \text{minimize} \quad & \|f(P,K)\| \\ \text{subject to} \quad & K \text{ stabilizes } P \\ & K \in S \end{aligned} \tag{4}$$

Here $\|\cdot\|$ is any norm on the closed-loop map chosen to encapsulate the control performance objectives. The delays associated with dynamics propagating from one subsystem to another, or the sparsity associated with them not propagating at all, are embedded in $P$. The subspace of admissible controllers, $S$, has been defined to encapsulate the constraints on how quickly information may be passed from one subsystem to another (delay constraints) or whether it can be passed at all (sparsity constraints). We call the subspace $S$ the ***information constraint***.

Many decentralized control problems may be expressed in the form of problem (4). In this paper, we focus on the case where $S$ is defined by delay constraints or sparsity constraints as discussed above.

This problem is made substantially more difficult in general by the constraint that $K$ lie in the subspace $S$. Without this constraint, the problem may be solved with many standard techniques. Note that the cost function $\|f(P,K)\|$ is in general a non-convex function of $K$. No computationally tractable approach is known for solving this problem for arbitrary $P$ and $S$.

## IV. Quadratic Invariance

In this subsection we define quadratic invariance, and give a brief overview of results regarding this condition, in particular, that it allows convex synthesis of optimal linear decentralized controllers.

*Definition 1:* The set $S$ is called ***quadratically invariant*** under $G$ if

$$KGK \in S \qquad \text{for all } K \in S$$

Note that, given $G$, we can define a quadratic map by $\Psi(K) = KGK$. Then a set $S$ is quadratically invariant if and only if $S$ is an invariant set of $\Psi$; that is $\Psi(S) \subseteq S$.

It was shown in [3] that if $S$ is a closed subspace and $S$ is quadratically invariant under $G$, then with a change of variables, problem (4) is equivalent to the following optimization problem

$$\begin{aligned} \text{minimize} \quad & \|T_1 - T_2 Q T_3\| \\ \text{subject to} \quad & Q \in \mathcal{RH}_\infty \\ & Q \in S \end{aligned} \tag{5}$$

where $T_1, T_2, T_3 \in \mathcal{RH}_\infty$.

This is a convex optimization problem. We may solve it to find the optimal $Q$, and then recover the optimal $K$ for our original problem.

If the norm of interest is the $\mathcal{H}_2$-norm, it was shown in [3] that the problem can be further reduced to an unconstrained optimal control problem and then solved with standard software.

We have assumed for this overview that these operators are all real-rational proper and thus acting on $L_{2e}$ or $\ell_e$. Similar results have been achieved [4] for other function spaces as well, also showing that quadratic invariance allows optimal linear decentralized control problems to be recast as convex optimization problems.

The main focus of this paper is thus characterizing constraints which are as close as possible to our original constraints, and for which the information constraint $S$ is quadratically invariant under the plant $G$.

### A. QI - Delay Constraints

For the case of delay constraints, it was shown in [6] that a necessary and sufficient condition for quadratic invariance is

$$t_{ki} + p_{ij} + t_{jl} \ \geq \ t_{kl} \qquad \forall\ i, j, k, l \tag{6}$$

Note that it was further shown in [6] that as long as the transmission delays satisfy the triangle inequality, then the above condition for convexity can be further reduced to

$$p_{ij} \ \geq \ t_{ij} \qquad \forall\ i, j \tag{7}$$

that is, the communication between any two nodes needs to be as fast as the propagation between the same pair of nodes.

### B. QI - Sparsity Constraints

For the case of sparsity constraints, it was shown in [3] that a necessary and sufficient condition for quadratic invariance is

$$K^{\text{bin}}_{ki}\ G^{\text{bin}}_{ij}\ K^{\text{bin}}_{jl}\ (1 - K^{\text{bin}}_{kl}) = 0 \quad \forall\ i, j, k, l \tag{8}$$

It can be shown that this is equivalent to Condition (6) if we let

$$t_{kl} \ = \ \begin{cases} R, & \text{if } K^{\text{bin}}_{kl} = 0 \\ 0, & \text{if } K^{\text{bin}}_{kl} = 1 \end{cases} \tag{9}$$

and let

$$p_{ij} \ = \ \begin{cases} R, & \text{if } G^{\text{bin}}_{ij} = 0 \\ 0, & \text{if } G^{\text{bin}}_{ij} = 1 \end{cases} \tag{10}$$

for any $R > 0$, the interpretation being that a sparsity constraint can be thought of as a large delay, and a lack thereof can be thought of as no delay.

## V. Closest QI Constraint

We now address the main question of this paper, which is finding the closest constrints when the above conditions fail; that is, when the original problem is not quadratically invariant.

### A. Closest - Delays

Suppose that we have propagation delays $\tilde{p}$ and transmission delays $\tilde{t}$, and that they do not satisfy Condition (6). The problem of finding the closest constraint set, that is, the closest set of transmission delays, such that the set is quadratically invariant, can be set up as

$$\begin{array}{ll} \text{minimize} & \|\text{vec}(t - \tilde{t})\| \\ \text{subject to} & t_{ki} + \tilde{p}_{ij} + t_{jl} \ \geq \ t_{kl} \qquad \forall\ i, j, k, l \\ & t_{kl} \ \geq \ 0 \qquad \forall\ k, l \end{array} \tag{11}$$

This is a convex optimization problem in the new transmission delays $t$. The norm is arbitrary, and may be chosen to encapsulate whatever notion of closeness is most appropriate. If the 1-norm is chosen, corresponding to minimizing the sum of the differences in transmission delays, or the $\infty$-norm is chosen, corresponding to minimizing the largest difference, then the problem may be cast as a linear program (LP).

If we want to find the closest quadratically invariant set which is a superset of the original set, so that we may obtain a lower bound to the solution of the main problem (4), then we simply add the constraint $t_{kl} \leq \tilde{t}_{kl}$ for all $k, l$ (resulting in the final constraint $0 \ \leq \ t_{kl} \ \leq \ \tilde{t}_{kl}$), and the problem remains convex (or remains an LP). Note that if we choose this and choose the 1-norm, then the objective is equivalent to $\ \max \sum \sum t_{kl}$, maximizing the total delay sum.

Similarly, if we want to find the closest quadratically invariant set which is a subset of the original set, so that we may obtain an upper bound to the solution of the main problem (4), then we simply add the constraint $t_{kl} \ \geq \ \tilde{t}_{kl}$ for all $k, l$ (which then replaces the final nonnegativity constraint), and the problem remains convex (or remains an LP). Note that if we choose this and choose the 1-norm, then the objective is equivalent to $\ \min \sum \sum t_{kl}$, minimizing the total delay sum.

### B. Closest - Sparsity

Now suppose that we similarly want to construct the closest quadratically invariant set, superset, or subset, defined by sparsity constraints. Given the sparsity pattern of the plant, $G^{\text{bin}}$, and the original sparsity pattern imposed on the controller, $K^{\text{bin}}$, we can convert these to delays as in (9),(10), and then set up problem (11). The only problem is that for the resulting solution to correspond to a sparsity constraint, we need to add the binary constraints $t_{kl} \in \{0, R\}$ for all $k, l$, and this destroys the convexity of the problem.

*1) Sparsity Superset:* Consider first finding the closest quadratically invariant superset of the original constaint set; that is, the sparsest quadratically invariant set for which all of the original connections $y_l \rightarrow u_k$ are still in place.

This is equivalent to solving the above problem (11) with $t_{kl} \leq \tilde{t}_{kl}$ for all $k, l$, and with the binary constraints, an intractable combinatorial problem, but we present an algorithm which solves it and terminates in a fixed number of steps.

We can write the problem as

$$\begin{array}{ll} \text{minimize} & \mathcal{N}(Z) \\ \text{subject to} & ZG^{\text{bin}}Z \ \leq \ Z \\ & K^{\text{bin}} \ \leq \ Z \end{array} \tag{12}$$

for the variable $Z \in \mathbb{B}^{n_u \times n_y}$, where additions and multiplications are as defined for the binary algebra in the preliminaries, and where we will then wish to use the information constraint $S = \text{Sparse}(Z)$. The objective is defined to give us the sparsest possible solution, the first constraint ensures that the constraint set associated with the solution is quadratically invariant with respect to the plant, and the last constraint requires the resulting set of controllers to be able to access any information that could be accessed with the original constraints. Let the optimal solution to this optimization problem be denoted as $Z^* \in \mathbb{B}^{n_u \times n_y}$.

Define a sequence of sparsity constraints $\{Z_m \ \in$

$\mathbb{B}^{n_u \times n_y},\ m \in \mathbb{N}\}$ given by

$$Z_0 \;=\; K^{\text{bin}} \tag{13}$$

$$Z_{m+1} \;=\; Z_m + Z_m G^{\text{bin}} Z_m, \quad \forall\ m \geq 0 \tag{14}$$

again using the binary algebra.

Our main result will be that this sequence converges to $Z^*$, and that it does so within $n_u n_y$ iterations. We first prove a preliminary lemma showing that the optimal solution can be no more sparse than any element of the sequence.

*Lemma 2:* For $Z^* \in \mathbb{B}^{n_u \times n_y}$ and the sequence $\{Z_m \in \mathbb{B}^{n_u \times n_y},\ m \in \mathbb{N}\}$ defined as above,

$$Z^* \;\geq\; Z_m \quad \forall\ m \in \mathbb{N} \tag{15}$$

*Proof:* First, $Z^* \geq Z_0 = K^{\text{bin}}$ is given by the satisfaction of the last constraint of (12), and it just remains to show the inductive step.

Suppose that $Z^* \geq Z_m$ for some $m \in \mathbb{N}$. It then follows that

$$Z^* + Z^* G^{\text{bin}} Z^* \;\geq\; Z_m + Z_m G^{\text{bin}} Z_m$$

From the first constraint of (12) and (2) we know that the left hand-side is just $Z^*$, and then using the definition of our sequence, we get

$$Z^* \;\geq\; Z_{m+1}$$

and this completes the proof. ■

We now give the main result, that the sequence converges, that it does so within $n_u n_y$ steps, and that it achieves the optimal solution to our problem.

*Theorem 3:*

$$Z_{n_u n_y} \;=\; Z^* \tag{16}$$

*Proof:* We first note from (1) that $\mathcal{N}(Z_{m+1}) \geq \mathcal{N}(Z_m)$ for all $m \in \mathbb{N}$. Since clearly $0 \leq \mathcal{N}(Z_m) \leq n_u n_y$ for all $m \in \mathbb{N}$, this tells us that there exists some $m^* \in 1, \ldots, n_u n_y$ such that $\mathcal{N}(Z_{m+1}) = \mathcal{N}(Z_m)$. Since $Z_{m+1} \geq Z_m$ for all $m \in \mathbb{N}$, it follows that $Z_{m^*+1} = Z_{m^*}$, and then, that $Z_{m+1} = Z_m$ for all $m \geq m^*$.

Writing this as $Z_m + Z_m G^{\text{bin}} Z_m = Z_m$ for all $m \geq m^*$, it then follows from (2) that $Z_m G^{\text{bin}} Z_m \leq Z_m$ for all $m \geq m^*$. Since $Z_{m+1} \geq Z_m$ for all $m \in \mathbb{N}$, it also follows that $Z_m \geq Z_0 = K^{\text{bin}}$ for all $m \in \mathbb{N}$. Thus the two constrains of (12) are satisfied for $Z_m$ for all $m \geq m^*$.

Since $Z^*$ is the sparsest binary matrix satisfying these constraints, it follows that $Z^* \leq Z_m$ for all $m \geq m^*$. Together with Lemma 2 and equation (3), it follows that $Z_m = Z^*$ for all $m \geq m^*$. ■

*Remark 4:* For the case typically of interest, where we are representing multiple $(n)$ subsystems, each with its own controller, and thus $n = n_y = n_u$ , we find that the algorithm terminates in at most $n^2$ iterations.

*2) Sparsity Subset:* We now notice an interesting asymmetry. For the case of delay constraints, if we were interested in finding the most restrictive superset (for a lower bound), or the least restrictive subset (for an upper bound), we simply flipped the sign of our final constraint, and the problem was convex either way. When we instead consider sparsity constraints, the binary constraint ruins the convexity, but we see that in the former (superset) case we can still find the closest constraint in a fixed number of iterations in polynomial time; however, for the latter (subset) case, there is no clear way to "flip" the algorithm.

This can be understood as follows. If there exists indices $i, j, k, l$ such that $K^{\text{bin}}_{ki} = G^{\text{bin}}_{ij} = K^{\text{bin}}_{jl} = 1$, but $K^{\text{bin}}_{kl} = 0$; that is, indices for which condition (8) fails, then the above algorithm resets $K^{\text{bin}}_{kl} = 1$. In other words, if there is an indirect connection from $y_l \rightarrow u_k$, but not a direct connection, it hooks up the direct connection.

But now consider what happens if we try to develop an algorithm that goes in the other direction, that finds the least sparse constraint set which is more sparse than the original. If we again have indices for which condition (8) fails, then we need to disconnect the indirect connection, but it's not clear if we should set $K^{\text{bin}}_{ki}$ or $K^{\text{bin}}_{jl}$ to zero, since we could do either. The goal is generally to disconnect the one that will ultimately lead to having to make the fewest subsequent disconnections, so that we end up with the closest possible constraint set to the original.

We suggest two heuristics for dealing with this problem. It is likely that they can be greatly improved upon, but are meant as a first cut at a reasonable polynomial time algorithm to find a close sparse subset.

For the first heuristic, we set up transmission delays and propagation delays as in (9) and (10), and then instead of adding the binary constraint and making the problem nonconvex, add the constraint $0 \leq t_{kl} \leq R$ for all $k, l$, and solve the resulting convex problem.

Then, for a set of indices violating condition (8), set $K^{\text{bin}}_{ki}$ to zero if $t^*_{ki} \geq t^*_{jl}$, and set $K^{\text{bin}}_{jl}$ to zero otherwise. The motivation is that we disconnect the one that has a larger delay, that is, which is more constrained, in the case where we allowed varying degrees of constraint.

For the second heuristic, we more directly keep track of how many indirect connections are associated with a direct connection. Define this weight as $w_{kl} = \sum \sum K^{\text{bin}}_{ki} G^{\text{bin}}_{ij} K^{\text{bin}}_{jl}$ thus giving the amount of 3-hop connections from $y_l \rightarrow u_k$. This is a crude measure of how many subsequent disconnections we'll have to make to obtain quadratic invariance if we were to disconnect a direct path from $y_l \rightarrow u_k$. Then, given indices for which condition (8) is violated, we set $K^{\text{bin}}_{ki}$ to zero if $w_{ki} \leq w_{jl}$, and set $K^{\text{bin}}_{jl}$ to zero otherwise.

Note that for either heuristic, we have many options for how often to reset the guiding variables, that is, to re-solve the convex program or recalculate the weights, such as after each disconnection, or after each pass through all $n_u n_y$ indices.

## VI. Nonlinear Time-Varying Control

It was shown in [7] that if we consider the design of possibly nonlinear, possibly time-varying (but still causal) controllers to stabilize possibly nonlinear, possibly time-varying (but still causal) plants, then while the quadratic invariance results no longer hold, the following condition

$$K_1(I \pm GK_2) \in S \qquad \text{for all } K_1, K_2 \in S$$

similarly allows for a convex parameterization of all stabilizing controllers subject to the given constraint.

This condition is equivalent to quadratic invariance when $S$ is defined by delay constraints or by sparsity constraints, and so the algorithms in this paper may also be used to find the closest constraint for which this is achieved.

## Acknowledgment

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## VII. Conclusions

The overarching goal of this paper is the design of linear time-invariant, decentralized controllers for plants comprising dynamically coupled subsystems. Given pre-selected constraints on the controller which capture the decentralization being imposed, we addressed the question of finding the closest constraint which is quadratically invariant under the plant. Problems subject to such constraints are amenable to convex synthesis, so this is important for bounding the optimal solution to the original problem.

We focused on two particular classes of this problem. The first is where the decentralization imposed on the controller is specified by delay constraints; that is, information is passed between subsystems with some given delays, represented by a matrix of transmission delays. The second is where the decentralization imposed on the controller is specified by sparsity constraints; that is, each controller can access information from some subsystems but not others, and this is represented by a binary matrix.

For the delay constraints, we showed that finding the closest quadratically invariant constraint can be set up as a convex optimization problem. We further showed that finding the closest superset; that is, the closest set that is less restrictive than the pre-selected one, to get lower bounds on the original problem, is also a convex problem, as is finding the closest subset.

For the sparsity constraints, the convexity is lost, but we provided an algorithm which is guaranteed to give the closest quadratically invariant superset in at most $n^2$ iterations, where $n$ is the number of subsystems. We also provided two heuristics to give close quadratically invariant subsets.

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