Frontiers in Networked Control	Lecture 8
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Consensus Problems

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This talk is based on the paper *Consensus and cooperation in Networked MAS* by Olifati-Saber et al, Feb 2006

Background

Objective of consensus problems:

Reach an agreement regarding a certain quantity (consensus state) that depends on the state of the agents.

Applications:

- Synchronization of coupled oscillators (consensus state: frequency, amplitude)
- Flocking, formation control (consensus state: velocity, inter vehicle distance)
- Rendevouz problems (consensus state: target point, time of arrival)
- Distributed sensor fusion (consensus state: estimate $\hat{x}$)
- Networked processes (consensus state: load balancing between CPU:s, event ordering)

Outline

1. Consensus Problems on Graphs
2. Laplacian Protocol
3. Quadratic Invariance
4. Physical Analogies (electrical networks, discrete heat equation)

1. Consensus Problems on Graphs

Assume we have n agents: $\dot{x}_i = u_i$. The interaction topology ("who talks to who") is modeled by an undirected and connected graph $G(v, \varepsilon, A)$ where v = vertices, ε = edges and $A = [a_{ij}]$ weights. $e_{ij} \in \varepsilon$ is equivalent with $a_{ij} > 0$.

Consensus for a graph means that

$$x(t) = (x_1 \quad x_2 \quad \dots \quad x_n)^T \to \alpha \mathbb{1} \quad \text{as} \quad t \to \infty, \quad \alpha \in \mathbb{R}, \quad \mathbb{1} = (1 \quad 1 \quad \dots \quad 1)^T$$

That is, all x_i are equal in steady state. For instance, in $\mathbb{R}^2$ with $x_1 = x_2$ steady state would be a straight line.

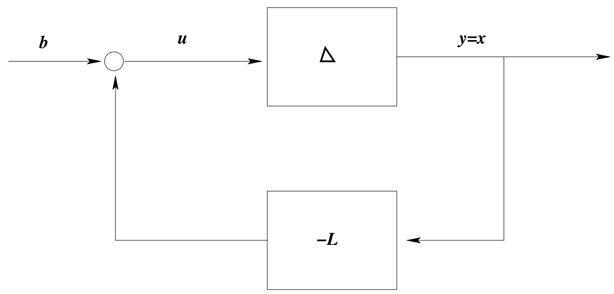

Figure 1: Block diagram for the Laplacian system.

2. Laplacian Protocol

For a group of agents, choosing

$$u_i = \sum_{j \in N_i} (x_j - x_i) + b_i(t), \quad b_i = \sum_{j \in N_i} r_{ij}$$

where r_{ij} are the inter-agent distances, the control u_i drives x_i to the average position of its neighbours. This yields

$$\dot{x} = -Lx$$

where $L = D - A$ is the so called *Laplacian*, with properties

$$L \geq 0, \quad \lambda_1 = 0, \quad L\mathbb{1} = 0.$$

We can show that $x^* = \alpha\mathbb{1}$ is a GAS with consensus value

$$\alpha = \frac{1}{h} \sum_{i=1}^{n} x_i(0).$$

The system is illustrated in a block diagram in Figure 1, where

$$\Delta = \begin{pmatrix} \frac{1}{s} & & \\ & \ddots & \\ & & \frac{1}{s} \end{pmatrix}.$$

3. Quadratic Invariance

Rewrite the consensus problem as

$$\min \|f(P, K)\| \quad \text{s.t. } K \text{ stabilizes } P, \quad K \in S.$$

The system is illustrated in Figure 2 and the transfer function from w to z is

$$z = (p_{11} + p_{12}L(I - p_{22}K)^{-1}p_{21})w$$

How do we solve this consensus problem? Three steps:

1. Objective function?

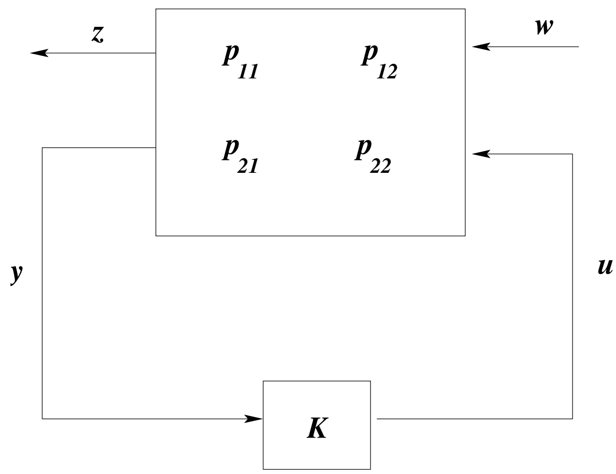

Figure 2: Block diagram for the quadratic invariance problem.

2. Constraint subspace S?
3. Plant P?

1. The standard consensus problem is not an optimal synthesis problem. Hence, different objective functions could be considered:

   (a)
$$y = x, \quad z = My$$
   with
$$M = \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{pmatrix}.$$

   (b)
$$z = (y - \alpha \mathbb{1})$$
$$\min_{u,\alpha} \int_0^\infty (z^T(t)Qz(t) + u^T Ru)dt$$

2. Note: there is a trivial solution to the consensus problem: $u_i = -x_i$ yields meeting at the origin. To rule this out, require $K\mathbb{1} = 0$. To incorporate local interaction we require $K \lesssim L$ (K *dominated* by the Laplacian L).

$$\text{Def: } A \lesssim B \Leftrightarrow B_{ij} = 0 \Rightarrow A_{ij} = 0$$

   In this context, $K \lesssim L$ means that the controller cannot use information that is not present in the Laplacian. This is a pre order relation: reflective and transitive but not symmetric or antisymmetric.

   Example:
$$A = \begin{pmatrix} -1 & 0 \\ 10 & 0 \end{pmatrix} \lesssim \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix} = B.$$

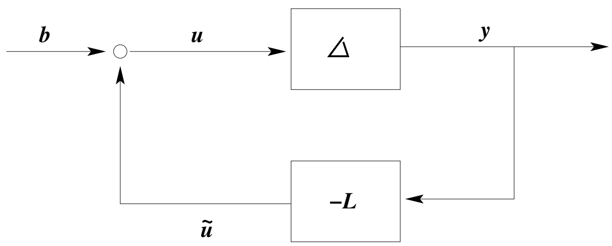

Figure 3: Block diagram.

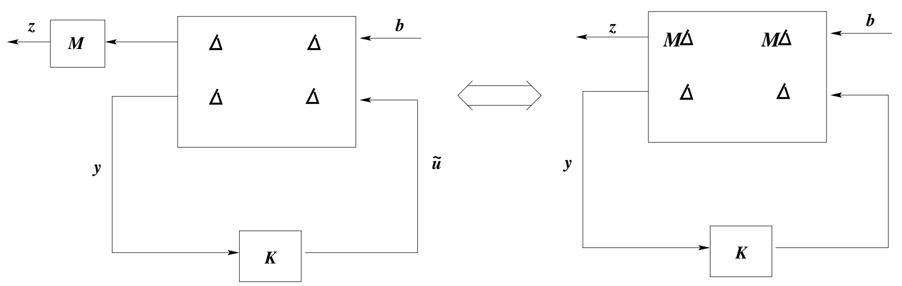

Figure 4: Equivalence of block diagrams.

3. Derivation of the plant P can be done using the block diagrams in Figures 3 and 4. We start at the diagram of Figure 3 and want to rewrite it as LFT.

$$y = \Delta u, \qquad u = b + Ky \qquad \Rightarrow$$

$$y = (I - \Delta K)^{-1}\Delta b$$

$$y = (\Delta + \Delta K(I - \Delta K)^{-1})\Delta)b$$

$$z = (p_{11} + p_{12}K(I - p_{22}K)^{-1}p_{12})w$$

Recap:

$$K\Delta K \in S \quad \forall K \in S \quad \Rightarrow \quad S \quad \text{quadraticly invariant.}$$

This does not hold for our S.
Counterexample:

Consider the graph in Figure 5. The Laplacian for this graph is

$$L = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

If choosing $K = -L$ we have

$$K\Delta K = \frac{1}{s}\begin{pmatrix} 6 & -3 & -3 \\ -3 & 2 & -1 \\ -3 & -1 & 2 \end{pmatrix} \notin S.$$

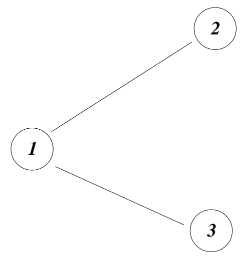

Figure 5: Counterexample.

Conclusion

- Optimal synthesis may be non-convex.
- A stabilizing controller is given by $K = -L$.

4. Physical Analogies

Questions

1. Can $K = -L$ be obtained from a convex problem?
2. Why is L called the Laplacian matrix?
3. Other names include Kirchoff's and admittance matrix, why?

Example A - Electrical Networks (E.N.)

There is a useful interplay between resistive E.N. and weighted graphs. (See i.e. Béla Bollobás, Modern Graph Theory). The network in Figure 6 models a nerve fibre. Find potentials and currents satisfying Ohm's Law (OL), Kirchoff's Current Law (KCL) and Kirchoff's Potentail Law (KPL).

To solve this problem, use some of these conditions and instead minimize the energy function that attains its minima at points where the remaining equations are satisfied. (ex: $Ax = b$ is solved by min $\frac{1}{2}x^T Ax - bx$.) We choose to fulfill OL and KPL explicitly:

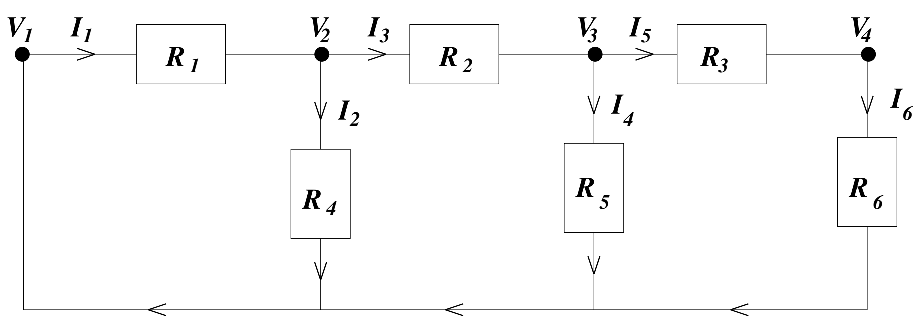

Figure 6: Nerve fibre model.

$$
\begin{aligned}
I_1 &= \tfrac{V_1 - V_2}{R_1}, \quad & I_3 &= \tfrac{V_2 - V_3}{R_2}, \quad & I_5 &= \tfrac{V_3 - V_4}{R_3}, \\
I_2 &= \tfrac{V_2 - V_1}{R_4}, \quad & I_4 &= \tfrac{V_3 - V_1}{R_5}, \quad & I_6 &= \tfrac{V_4 - V_1}{R_1}
\end{aligned}
$$

Define

$$V = (V_1 \quad \dots \quad V_4), \quad I = (I_1 \quad \dots \quad I_6), \quad G = \begin{pmatrix} \frac{1}{R_1} & & \\ & \ddots & \\ & & \frac{1}{R_6} \end{pmatrix}, \quad A = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix}.$$

$$\Rightarrow \quad I = -GAV \tag{1}$$

KCL:

$$\begin{array}{ll} \text{node 1:} & I_5 = I_6 \\ \text{node 2:} & I_3 - I_4 - I_5 = 0 \\ \text{node 3:} & I_1 - I_2 - I_3 = 0 \\ \text{node 4:} & -I_1 + I_2 + I_4 + I_6 = 0 \end{array}$$

$$\Rightarrow \quad A^T I = 0 \tag{2}$$

Combining (1) with (2) yields

$$-A^T GAV = 0 \quad \Rightarrow LV = 0.$$

$\Rightarrow$ a solution is found by solving

$$\min_v V^T LV = \phi(V)$$

which is a convex function. Apparently Laplacian protocol is the same as $\dot{V} = -\nabla\phi(V) = -LV$ (the steepest descent algorithm).

Example B - Spatially discrete heat equation

- Spatial domain: nodes
- Temporal domain: dynamics of the nodes

Partial difference equations (PdE): mimic PDEs on discrete graphs.
Let x_i denote the state (temperature) of node i. The partial derivative of x_i is defined as

$$\partial_j x_i = x_j - x_i = \begin{cases} \partial_j x_i &= \partial_i x_j \\ \partial_i x_i &= 0 \\ \partial_j^2 x_i &= \partial_j x_j - \partial_i x_j = \partial_i x_j = x_i - x_j \end{cases}$$

The discrete Laplacian of x_i is given by

$$\Delta x_i = \sum_{j \in N_i} \partial_j^2 x_i \sum_{j \in N_i} (x_i - x_j).$$

Laplacian protocol/control is simply $\dot{x} = \Delta x$. Compare with the heat equation:

$$\frac{\partial T}{\partial t} = \Delta T.$$

We might expect

1. Convergence to an average of the initial states. Since

$$\sum_{i=1}^{n} u_i(t) = 0, \quad \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i(t),$$

   then

$$\dot{\bar{x}} = \sum_{i=1}^{n} u_i(t) = 0$$

   $\Rightarrow$ $\bar{x}$ is a time-invariant quantity.

2. Constantly adding energy in one node $\not\Rightarrow$ consensus.