Information Structures, Stability, and Optimality

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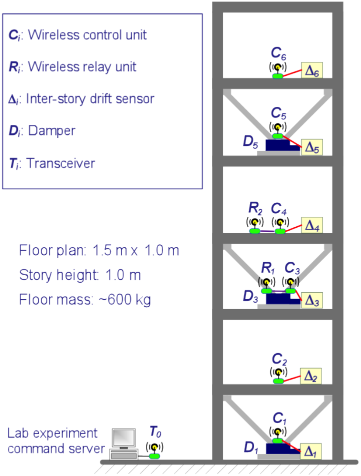
Outline

Outline

Aircraft Formation Flight

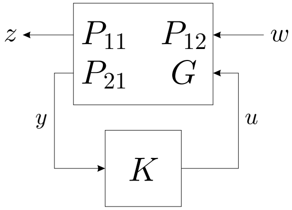
- Want tight formation for drag reduction.
- Dynamics are coupled.
- Each aircraft has its own sensor measurements.
- Possibly able to share information with other aircraft.

Structural Dampening



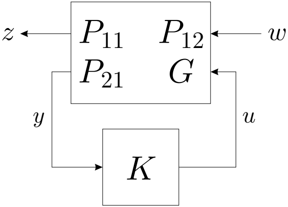
- Sensors measure displacement between floors
- Controllers between each floor can actuate active dampers
- Relays may send measurements to other floors' controllers as well
- Trade-offs between delays and amount of information used
- Ack: Yang Wang, GaTech-CE

Standard Formulation



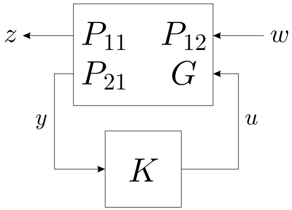
- y: measurements
- u: control inputs
- w: exogenous inputs
- z: regulated outputs

Standard Formulation



$$
\begin{array}{rl}
\text{minimize} & \|P_{11} + P_{12}K(I - GK)^{-1}P_{21}\| \\
\text{subject to} & K \text{ stabilizes } P
\end{array}
$$

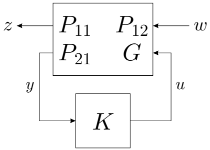
Standard Formulation

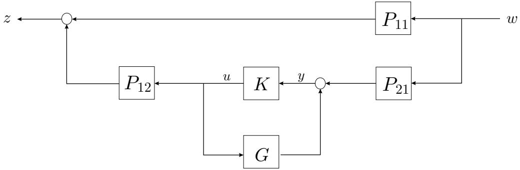


$$\begin{array}{ll} \text{minimize} & \|f(P,K)\| \\ \text{subject to} & K \text{ stabilizes } P \end{array}$$

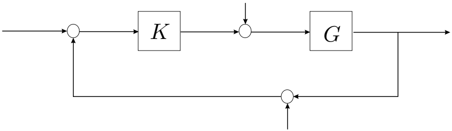
Block Diagrams

Two-Input Two-Output

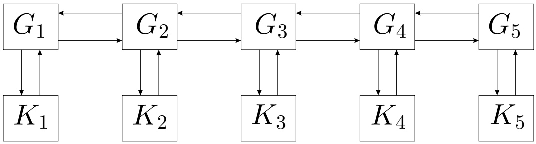




Classical



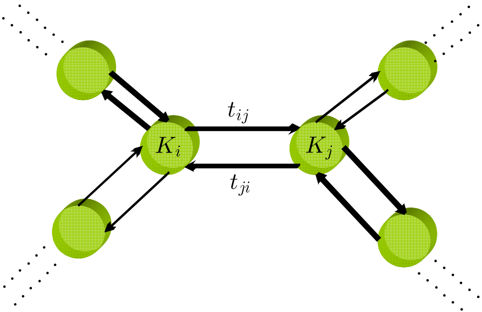
Structural Constraints



Control design problem is to find K which is block diagonal, whose diagonal blocks are 5 separate controllers.

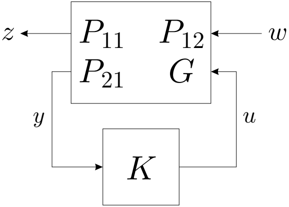
$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} K_1 & 0 & 0 & 0 & 0 \\ 0 & K_2 & 0 & 0 & 0 \\ 0 & 0 & K_3 & 0 & 0 \\ 0 & 0 & 0 & K_4 & 0 \\ 0 & 0 & 0 & 0 & K_5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}$$

Delay Constraints



$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \underbrace{\begin{bmatrix} D_{t_{11}}\tilde{K}_{11} & D_{t_{12}}\tilde{K}_{12} & D_{t_{13}}\tilde{K}_{13} \\ D_{t_{21}}\tilde{K}_{21} & D_{t_{22}}\tilde{K}_{22} & D_{t_{23}}\tilde{K}_{23} \\ D_{t_{31}}\tilde{K}_{31} & D_{t_{32}}\tilde{K}_{32} & D_{t_{33}}\tilde{K}_{33} \end{bmatrix}}_{K} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Standard Decentralized Formulation



$$
\begin{array}{ll}
\text{minimize} & \|f(P,K)\| \\
\text{subject to} & K \text{ stabilizes } P \\
 & K \in S
\end{array}
$$

Outline

Fixed Modes

Wang and Davison, TAC 1973.

- Defined as modes which cannot be altered by any (possibly dynamic) LTI decentralized controller.
- Shown to be the same as the modes which cannot be altered by any static gain controller.

Taken together, fixed modes are

$$\Lambda \quad = \quad \cap_{K \in S \cap \mathbb{R}^{n_u \times n_y}} \quad \sigma(A + BKC)$$

Time-Varying Control of Fixed Modes

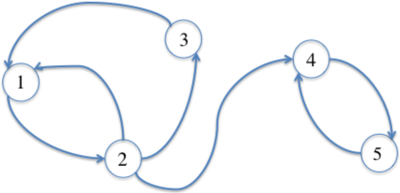
Even though the plant is LTI, modes which are fixed w.r.t. LTI control may be moved with LTV control.

H. Kobayashi, H. Hanafusa, and T. Yoshikawa.
Controllability under decentralized information structure.
IEEE Transactions on Automatic Control, 1978.

B. Anderson and J. Moore.
Time-Varying feedback laws for decentralized control.
IEEE Transactions on Automatic Control, 1981.

S.-H. Wang.
Stabilization of decentralized control systems via time-varying controllers.
IEEE Transactions on Automatic Control, 1982.

Quotient Fixed Modes (Gong and Aldeen, 1997)



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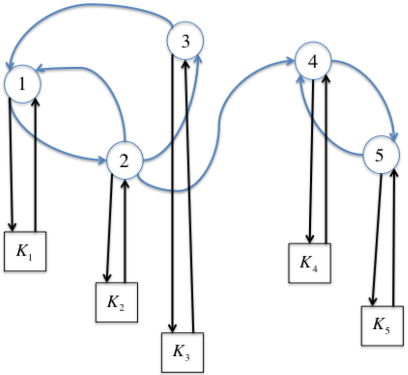

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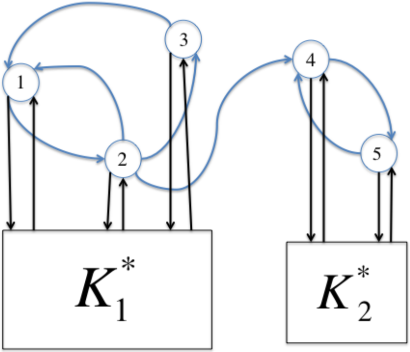

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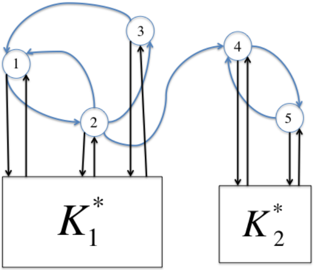
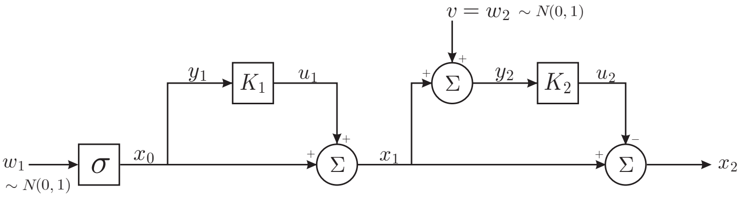


$$\Lambda^* \quad = \quad \bigcap_{K^* \in S^* \cap \mathbb{R}^{n_u \times n_y}} \quad \sigma(A + B^* K^* C^*)$$

Outline

Witsenhausen Counterexample (1968)

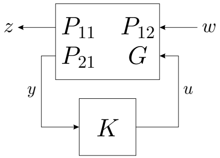


- Objective: $\min \ \mathbb{E}(k^2 u_1^2 + x_2^2)$
- Showed that optimal controller is not necessarily linear.
- We also see that optimal control problem is not convex.

Outline

General Formulation

The set of K with a given decentralization constraint is a subspace S, called the *information constraint*.

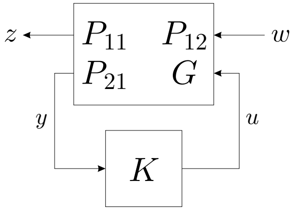


We would like to solve

$$\begin{array}{rl} \text{minimize} & \|f(P,K)\| \\ \text{subject to} & K \text{ stabilizes } P \\ & K \in S \end{array}$$

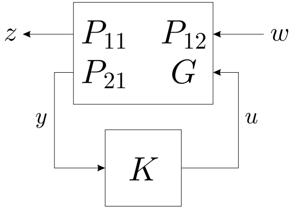
- For general P and S, there is no known tractable solution.
- A few cases were shown to be tractable. (Voulgaris, Bamieh)
- Certain classes are known to be $\mathcal{NP}$-hard.

Change of Variables



minimize	$\|f(P,K)\|$		minimize	$\|T_1 - T_2QT_3\|$
subject to	K stabilizes P	$\iff$	subject to	Q stable
	$K \in S$			$Q \in$??

Change of Variables



$$\begin{array}{ll} \text{minimize} & \|f(P,K)\| \\ \text{subject to} & K \text{ stabilizes } P \\ & K \in S \end{array} \iff \begin{array}{ll} \text{minimize} & \|T_1 - T_2 Q T_3\| \\ \text{subject to} & Q \text{ stable} \\ & Q \in \ ?? \end{array}$$

Quadratic Invariance

Definition

The set S is called quadratically invariant with respect to G if

$$KGK \in S \qquad \text{for all} \quad K \in S$$

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M. Rotkowitz and S. Lall
A Characterization of Convex Problems in Decentralized Control, 2006

Optimal Stabilizing Controller

We would like to solve

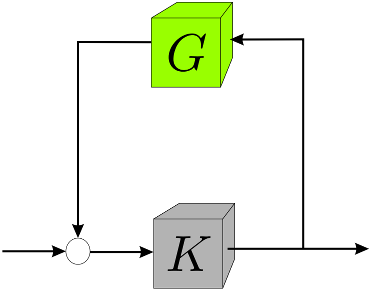
$$\begin{aligned} \text{minimize} \quad & \|f(P,K)\| \\ \text{subject to} \quad & K \text{ stabilizes } P \\ & K \in S \end{aligned}$$

If S is quadratically invariant with respect to G, we may solve

$$\begin{aligned} \text{minimize} \quad & \|T_1 - T_2 Q T_3\| \\ \text{subject to} \quad & Q \text{ stable} \\ & Q \in S \end{aligned}$$

which is convex.

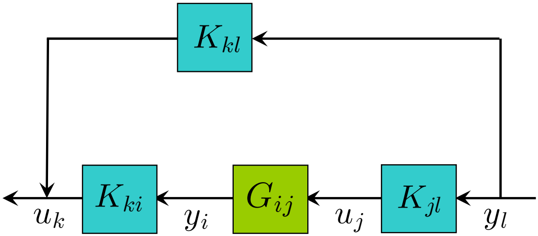
Block Diagram Interpretation



$$\begin{aligned} K(I-GK)^{-1} &= K + KGK + K(GK)^2 + \dots \\ &\in S \end{aligned}$$

Outline

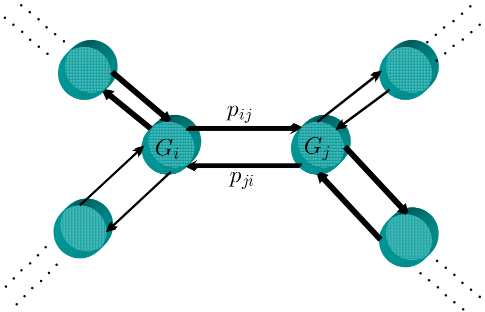
QI - Sparsity



S is quadratically invariant with respect to G if and only if

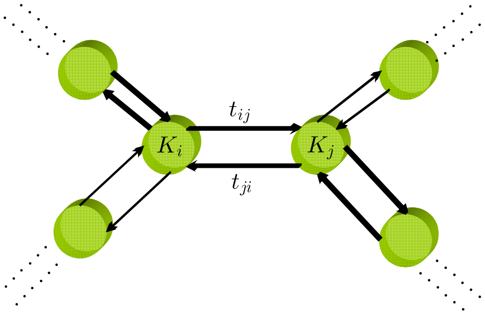
$$K_{ki},\ G_{ij},\ K_{jl} \quad \text{non-zero} \quad \Longrightarrow \quad K_{kl} \quad \text{non-zero}$$

Arbitrary Network with Propagation Delays



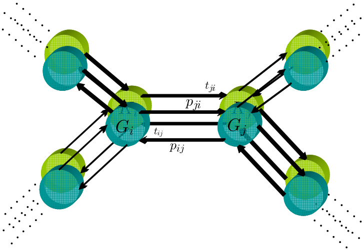
p_{ij} : propagation / dynamics delay from node i to node j

Arbitrary Network with Transmission Delays



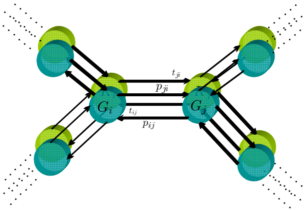
t_{ij} : transmission / communication delay from node i to node j

Control over Network with Delays



Objective: find all such controllers which stabilize the given plant and/or find optimal stabilizing controller.

Control over Network with Delays

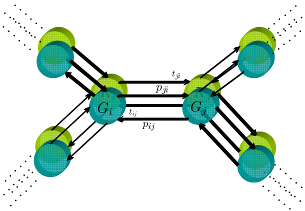


Assuming transmission delays satisfy the triangle inequality (i.e. transmissions take the quickest path)

Optimal control problem may be cast as a convex optimization problem if

$$p_{ij} \geq t_{ij} \qquad \text{for all } i,j$$

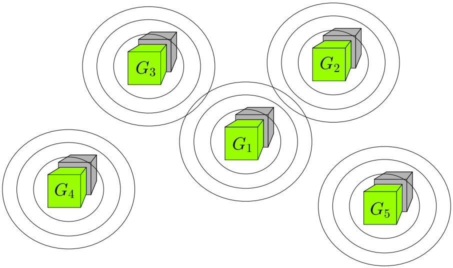
Control over Network with Delays



Assuming transmission delays satisfy the triangle inequality (i.e. transmissions take the quickest path)

Ack: Randy Cogill, University of Virginia

Arbitrary Positions



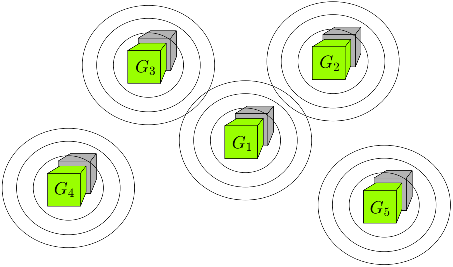
- Subsystems have arbitrary positions in any number of dimensions.
- Propagation delays and transmission delays scale with distance.

$$p_{ij} = \gamma_p \left\|x_i - x_j\right\| \qquad\qquad t_{ij} = \gamma_t \left\|x_i - x_j\right\|$$

- Condition for quadratic invariance is

$$\gamma_p \geq \gamma_t$$

Arbitrary Positions



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$$p_{ij} = \gamma_p \|x_i - x_j\| \qquad c_i = C \qquad t_{ij} = \gamma_t \|x_i - x_j\|$$

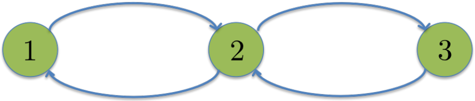
- Condition for quadratic invariance is

$$\gamma_p + (C/R) \geq \gamma_t$$

Spatio-Temporal Systems

- Convex spatio-temporal problems similarly classified.
- Simplifies and generalizes the convex spatially invariant problems.
  - Subadditive support functions generalize notion of funnel causality (Bamieh, Voulgaris).
  - Allows for arbitrary spatial dimension.
  - Allows for computational delay.

Same Structure Synthesis



Suppose

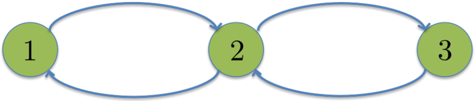
$$G \sim \begin{bmatrix} \bullet & \bullet & \circ \\ \bullet & \bullet & \bullet \\ \circ & \bullet & \bullet \end{bmatrix} \qquad S = \left\{ K \;\middle|\; K \sim \begin{bmatrix} \bullet & \bullet & \circ \\ \bullet & \bullet & \bullet \\ \circ & \bullet & \bullet \end{bmatrix} \right\}$$

Then

$$KGK \sim \begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix}$$

so S is ***not*** quadratically invariant with respect to G.

Same Structure Synthesis



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Outline

1. Examples / Framework
2. Stabilizability (Fixed Modes)
3. Optimal Decentralized Control
   - Quadratic Invariance
   - Convex (QI) Examples
     - Information Structures (Sparsity)
     - Control over Networks (with Delays)
     - Spatio-Temporal Systems
   - Non-Convex / Non-QI
     - Perfectly Decentralized
4. Stabilization
5. Nonlinear Decentralized Control

What if it's not?

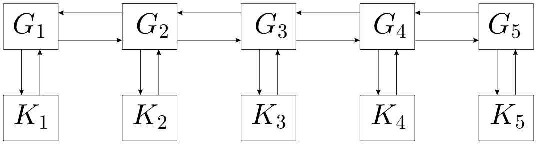
Many heuristics long before QI

- e.g. One controller at a time

Several methods since QI:

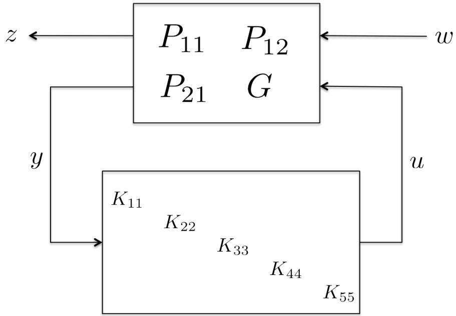
- e.g. Work with Nuno Martins on finding the closest quadratically invariant constraint

Perfectly Decentralized Control



- Each controller can only see information from its local subsystem.
- Never quadratically invariant, unless G is block diagonal as well.

Perfectly Decentralized Control



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Key Trick

Generalization of trick from:

V. Manousiouthakis.
On the parametrization of all stabilizing decentralized controllers.
System and Control Letters, 1993.

D. Sourlas and V. Manousiouthakis.
Best achievable decentralized performance.
IEEE Transactions on Automatic Control, 1995.

Key Trick

$$L_l = \begin{bmatrix} I & & & & \\ & 2I & & & \\ & & 3I & & \\ & & & 4I & \\ & & & & 5I \end{bmatrix} \qquad L_r = \begin{bmatrix} I & & & & \\ & 2I & & & \\ & & 3I & & \\ & & & 4I & \\ & & & & 5I \end{bmatrix}$$

$$L_l K = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} \\ 2K_{21} & 2K_{22} & 2K_{23} & 2K_{24} & 2K_{25} \\ 3K_{31} & 3K_{32} & 3K_{33} & 3K_{34} & 3K_{35} \\ 4K_{41} & 4K_{42} & 4K_{43} & 4K_{44} & 4K_{45} \\ 5K_{51} & 5K_{52} & 5K_{53} & 5K_{54} & 5K_{55} \end{bmatrix}, \qquad K L_r = \begin{bmatrix} K_{11} & 2K_{12} & 3K_{13} & 4K_{14} & 5K_{15} \\ K_{21} & 2K_{22} & 3K_{23} & 4K_{24} & 5K_{25} \\ K_{31} & 2K_{32} & 3K_{33} & 4K_{34} & 5K_{35} \\ K_{41} & 2K_{42} & 3K_{43} & 4K_{44} & 5K_{45} \\ K_{51} & 2K_{52} & 3K_{53} & 4K_{54} & 5K_{55} \end{bmatrix}$$

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$$L_l = \begin{bmatrix} I & & & & \\ & 2I & & & \\ & & 3I & & \\ & & & \ddots & \\ & & & & aI \end{bmatrix} \qquad L_r = \begin{bmatrix} I & & & & \\ & 2I & & & \\ & & 3I & & \\ & & & \ddots & \\ & & & & aI \end{bmatrix}$$

$$\begin{aligned} & L_l K = K L_r \\ \Leftrightarrow \quad & iK_{ij} = jK_{ij} \quad \forall\, i,j \\ \Leftrightarrow \quad & K_{ij} = 0 \quad \forall\, i \neq j \\ \Leftrightarrow \quad & K \in S. \end{aligned}$$

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Constraint

$$
\begin{aligned}
& K \in S \\
\Leftrightarrow \quad & L_l K = K L_r \\
\Leftrightarrow \quad & W_1 + W_2 Q + Q W_3 + Q W_4 Q = 0 \\
\Leftrightarrow \quad & q(Q) = 0
\end{aligned}
$$

Constraint

$$
\begin{aligned}
& K \in S \\
\Leftrightarrow \quad & L_l K = K L_r \\
\Leftrightarrow \quad & W_1 + W_2 Q + Q W_3 + Q W_4 Q = 0 \\
\Leftrightarrow \quad & q(Q) = 0
\end{aligned}
$$

Constraint

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M. Rotkowitz
Parametrization of All Stabilizing Controllers Subject to Any Structural Constraint, 2010

Optimization

The optimal (perfectly) decentralized control problem:

$$\begin{aligned} \text{minimize} \quad & \|f(P,K)\| \\ \text{subject to} \quad & K \text{ stabilizes } P \\ & K \in S \end{aligned}$$

is then equivalent to:

$$\begin{aligned} \text{minimize} \quad & \|T_1 - T_2 Q T_3\| \\ \text{subject to} \quad & Q \text{ stable} \\ & q(Q) = 0 \end{aligned}$$

Optimal Control Summary

- Quadratically invariant constraint $K \in S$ may be passed on to parameter as $Q \in S$.
- Block diagonal constraint may be rewritten as $L_l K = K L_r$.
- Leads to nearly complete story for optimal linear (decentralized) control:

Optimal (centralized) control:

$$\begin{aligned} \text{minimize} \quad & \|T_1 - T_2 Q T_3\| \\ \text{subject to} \quad & Q \text{ stable} \end{aligned}$$

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Optimal control subject to a quadratically invariant contraint:

$$\begin{aligned} \text{minimize} \quad & \|T_1 - T_2 Q T_3\| \\ \text{subject to} \quad & Q \text{ stable} \\ & a(Q) = 0 \end{aligned}$$

Optimal Control Summary

- Quadratically invariant constraint $K \in S$ may be passed on to parameter as $Q \in S$.
- Block diagonal constraint may be rewritten as $L_l K = K L_r$.
- Leads to nearly complete story for optimal linear (decentralized) control:

Optimal control subject to an arbitrary structural constraint:

$$\begin{aligned} \text{minimize} \quad & \|T_1 - T_2 Q T_3\| \\ \text{subject to} \quad & Q \text{ stable} \\ & q(Q) = 0 \end{aligned}$$

Outline

Fine print

- Main QI results assume the existence of an admissible, stable, stabilizing controller, $K_{\text{nom}} \in S$.
- Similar work using Mori parameterization allows removal of 'stable' for most cases of interest. (Sabau, Martins, Rotkowitz)
- Still need stabilizing decentralized controller, itself a difficult problem.

QI Stabilization (Sabau, Martins)

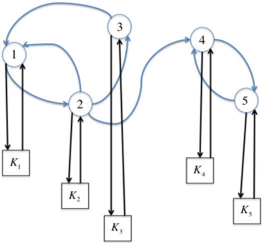
Theorem

Given a D.C.P.F. of G, if S is QI w.r.t. G,

$$K \in S \quad \iff \quad M_r Q M_l - M_r Y_l \in S$$

- For at least structural constraints, can solve for Q (or find that it doesn't exist), and thus find stabilizing $K_{\text{nom}} \in S$.
- Combines with main QI results for complete stabilization and optimization algorithm.
- Or, provides different convex problem to achieve both.

Possible connection with QFM



- Finding closest QI superset (connecting indirect connections) gives quotient system, plus some extra links which don't affect modes.
- This is a system for which we can now compute stabilizing (when they exist) and optimal controllers.

Outline

NLTV Invariance Condition

$$K_1(I \pm GK_2) \in S \qquad \text{for all} \quad K_1, K_2 \in S$$

- Similarly allows for a convex parameterization of all stabilizing controllers and all achievable closed-loop maps, without assumptions of linearity or time-invariance on plant or controller.

M. Rotkowitz
Information Structures Preserved under Nonlinear, Time-Varying Feedback, 2006

NLTV Invariance Condition

$$K_1(I \pm GK_2) \in S \qquad \text{for all } \; K_1, K_2 \in S$$

- Objective: develop a condition that similarly allows for a convex parameterization of all stabilizing controllers and all achievable closed-loop maps, without assumptions of linearity or time-invariance on plant or controller.
- Bonus: Information constraint S no longer required to be a subspace.
  - Could allow for other types of constrained control.
  - Will be crucial for integration of other types of networking constraints, such as packet drops.

Witsenhausen Counterexample, 1968

SIAM J. CONTROL
Vol. 6, No. 1, 1968
Printed in U.S.A.

A COUNTEREXAMPLE IN STOCHASTIC OPTIMUM CONTROL*

H. S. WITSENHAUSEN†

9. Conclusions. (i) Further study of linear, Gaussian, quadratic control problems with general information patterns appears to be required.

(ii) The existence of an optimum and the question of completeness of the class of affine designs must be examined as a function of the information pattern.

(iii) It would be interesting if a relation could be found between the appearance of several local minima over the affine class and lack of completeness of this class.

(iv) Algorithms for approaching an optimal solution need to be developed. Because of the occurrence of local minima, this appears to be a most difficult task.

Partially Nested (Ho and Chu, 1971)

- N team members, each making decision $\gamma_i : \tilde{y}_i \to u_i$.
- Gaussian noise, expected quadratic cost (LQG).

An information structure is *partially nested* if whenever the jth control input u_j can affect the ith measurement $\tilde{y}_i$, then the jth measurement $\tilde{y}_j$ can be deduced from $\tilde{y}_i$.

In others words, whenever someone can affect what you see, you can see everything that they can.

Main PN Result

If the information structure is partially nested, then the optimal control exists, is unique, and is linear.

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PN/QI

For the class of problems where both are well-defined

$$\text{PN} \quad \Longleftrightarrow \quad \text{QI}$$

M. Rotkowitz
On Information Structures, Convexity, and Linear Optimality, 2008

PN/QI

For the class of problems where both are well-defined

$$\text{PN} \iff \text{QI}$$

Remaining questions:

- LQG
  - other vector spaces
  - other types of constraints
  - removal of implicit constraints for PN framework
- Other cost functions

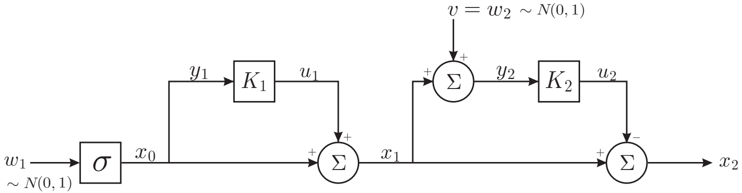
A. Stoorvogel, 1995

Find K to minimize

$$J(K) = \sup_{w \neq 0} \frac{\|z\|_\infty}{\|w\|_\infty}$$

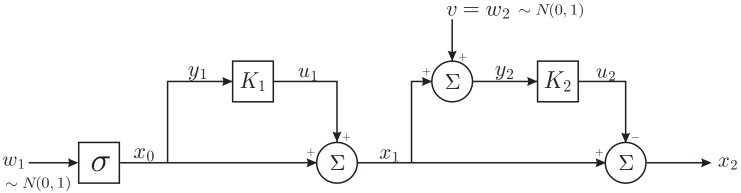
- Nonlinear outperforms linear - for a centralized problem!
- Trivially PN/QI, but linear controllers suboptimal.

Witsenhausen Counterexample (1968)



- $z = \begin{bmatrix} ku_1 & x_2 \end{bmatrix}^T$
- Original Objective: $\min \ \mathbb{E}\|z\|_2^2$
  - Optimal controller is not linear.
- Consider $\min \ \sup_{w \neq 0} \frac{\|z\|_2}{\|w\|_2}$

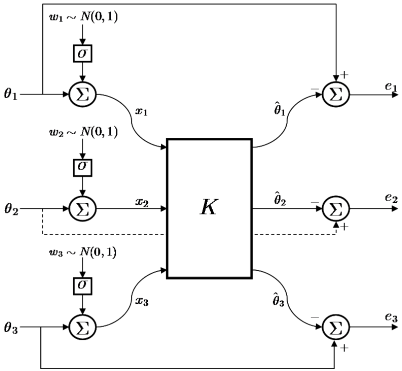
Witsenhausen Counterexample (1968)



- Consider $\min \ \sup_{w \neq 0} \frac{\|z\|_2}{\|w\|_2}$
  - Optimal controller is linear.

M. Rotkowitz
Linear Controllers are Uniformly Optimal for the Witsenhausen Counterexample, 2006

Stein's Paradox



- Estimate unknown θ
- No prior
- Quadratic cost: $\mathbb{E}(\sum e_i^2)$
- Linear suboptimal

Summary

- Discussed stabilizability, stabilization, convexity, and linear optimality of decentralized control problems, and links between these questions.

"Information Structures in Optimal Decentralized Control", tutorial at CDC'12 in Maui.
N. Martins, S. Yuksel, A. Mahajan, M. Rotkowitz.