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Polynomial Optimization Methods for Determining Lower Bounds on Decentralized Assignability

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Abstract— **This work considers the determination of non-binary measures of controllability or robustness with respect to decentralized controllers. A measure developed by Vaz and Davison in 1988 nicely captures the distance from a plant to the closest one with a fixed mode, and ties it to eigenvalue assignability; that is, how much effort is at most required to move the modes a given amount with the prescribed information structure. This metric is intractable to compute, but recent work has been very successful in finding close upper bounds. Finding lower bounds is not only important for providing guarantees on where the true metric lies, but is typically more important since determining whether the metric is bounded away from zero corresponds to whether the system can be controlled at all. This paper will address these lower bounds, in particular by using the Courant-Fischer formulation of singular values, we will formulate our problem as a polynomial optimization problem, for which we can then use Sum-of-Squares (SOS) techniques to find a lower bound.**

I. Introduction

A seminal result in decentralized control is the development of fixed modes by [1] - that plant modes which cannot be moved with a static decentralized controller cannot be moved by a dynamic one either, and that the other modes which can be moved can be shifted to any chosen locations with arbitrary precision.

In many cases one needs to know more than just whether or not a fixed mode is present. It could be the case that although the plant is theoretically controllable (i.e., there exists no fixed modes), that a large control effort is required to move the states, and/or that a small perturbation to the plant would result in a fixed mode. These questions have been well answered for the centralized case through controllability, observability, and Hankel operators. In particular, Hankel singular values of a stable plant provide a non-binary measure of how controllable and observable that plant is, and are easy to compute.

In the decentralized case, Vaz & Davison have defined the decentralized assignability measure based on the distance of the plant from the set of plants that have a fixed mode [2]. They characterized and connected the mobility of an eigenvalue of the plant, which is the change in its location when a decentralized controller of bounded magnitude is applied, to the aforementioned measure. They have also proven that this measure would be non-zero if and only if there exist no fixed modes. However, this metric is hard to compute for all but the smallest problems. As an alternative strategy, the approach taken in [3] has explored the use of the Hankel operator to develop an easily computable metric which could provide information regarding proximity to a fixed mode for decentralized control. The developed metric in [3] combines the controllability gramian, observability gramian, and a cross-gramian that incorporates the information structure. That metric closely tracks the one of the Vaz & Davison near presence of a fixed mode, for some but not all the considered classes of fixed modes.

The developed metric by Vaz & Davison in [2] corresponds to the minimization of the n-th singular value over a power set. Recent approaches in [4], and [5] have been successful in obtaining an upper bound for this metric. Namely, [4] has first relaxed the optimization problem in Vaz & Davison into a form that would no longer involve minimization over the power set. Then, three methods for computational of the relaxed version has been proposed. In all of these three methods, the convex heuristic of nuclear norm [6] for rank minimization has been used instead of minimizing over the n-th singular value. In [5], the most promising method among those has been taken, and a subgradient-based method has been applied to directly minimize the n-th singular value over the continuous variable.

In this paper, we aim to provide guaranteed certificates of how far a plant is from having a fixed mode. We want to derive polynomially solvable programs that give lower bounds on the decentralized assignability measure. This can then be interpreted separately, or in conjunction with the upper bounds, to provide certificates and estimates on how nonassignable the modes of the plant are.

To this end, we will first consider the problem of finding a lower bound on any arbitrary selected singular value of a polynomial matrix, we will reformulate this as a polynomial optimization problem, for which we can use sum-of-squares techniques [7] to derive a sequence of convex programs that would approach that singular value from below. We will then present an equivalent form for the decentralized assignability measure, for which we can take the same approach, leading to polynomial time methods for lower bounds on that measure.

The organization of the paper is as follows. We state the preliminary notations in Section II, and review the fixed modes in Section III-A, the decentralized assignability measure of [2] in Section III-B, and related concepts in the polynomial optimization in Section III-C. We will derive

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polynomially solvable methods for lower bounds on any arbitrary singular value of a matrix in Section IV, and then adopt it to our main interest, which is a lower bound on the decentralized assignability measure in Section V. Finally, a numerical examples is provided in Section VI to illustrate, and investigate the performance of the proposed method.

II. Preliminaries

We assume that we have a causal, linear time-invariant, and strictly proper plant G, with n_u inputs and n_y outputs. A state-space representation for G is given by $(A, B, C, 0)$, and thus we have that $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n_u}$, and $C \in \mathbb{R}^{n_y\times n}$. We next introduce some notation that will help encapsulate the main type of decentralization we consider in this paper.

We are not only interested in the input-output characteristics of the plant G, but also in imposing structure on the feedback controller. We will denote the feedback controller by K, such that we have $u(s) = K(s)y(s)$, and suppose that the controller is finite dimensional, causal, linear time-invariant, and proper. We are interested in confining the space of admissible controllers by imposing an information constraint on K. We will primarily focus on sparsity constraints, and denote the set of controllers that satisfy the sparsity constraint by $\mathcal{S}$.

We can associate the set of controllers that satisfy some sparsity constraint (i.e., are in $\mathcal{S}$) to a binary matrix $K^{\text{bin}} \in \mathbb{B}^{n_u\times n_y}$, with $\mathbb{B} = \{0, 1\}$. We have that $K^{\text{bin}}_{ij} = 1$, if and only if the i-th control input may access the j-th measurement output, and 0 otherwise. We also denote the diagonal information structures by $\mathcal{S}_{\text{d}}$, for which only the diagonal elements of the controller are allowed to be non-zero, and thus we have $K^{\text{bin}}_{ij} = 1$ if and only if $i = j$ and 0 otherwise. Also, for a given $i \in \{1, \cdots, n_u\}$, define $\mathtt{J}_i \triangleq \{j \in \{1, \cdots, n_y\} \mid K^{\text{bin}}_{ij} = 1\}$, which are the set of sensor measurements that control action u_i is allowed to access. For a subset $\mathtt{I} \subseteq \{1, \cdots, n_u\}$, denote its complement by $\bar{\mathtt{I}} \triangleq \{1, \cdots, n_u\} - \mathtt{I}$. Similarly define $\mathtt{J}_{\mathtt{I}} \triangleq \bigcup_{i\in\mathtt{I}} \mathtt{J}_i$. Let B_i and C_j be the i-th column of B and j-th row of C, and for any subset $\mathtt{I} = \{i_1, \cdots, i_{|\mathtt{I}|}\}$, define $B_{\mathtt{I}} \triangleq \begin{bmatrix} B_{i_1} & \cdots B_{i_{|\mathtt{I}|}} \end{bmatrix}$. Likewise, for any subset $\mathtt{J} = \{j_1, \cdots, j_{|\mathtt{J}|}\}$, define $C_{\mathtt{J}} \triangleq \begin{bmatrix} C^T_{j_1} & \cdots & C^T_{j_{|\mathtt{J}|}} \end{bmatrix}^T$. It is noteworthy to mention that for a diagonal information structure $\mathcal{S}_{\text{d}}$, we have that $\mathtt{J}_{\bar{\mathtt{I}}} = \bar{\mathtt{I}}$.

III. Review

In this section, we will review the related materials for our problem formulation. We will first state the definition of the fixed modes, and review an algebraic test for determining them in Section III-A. We will then review the decentralized assignability measure in Section III-B, which describes how far a plant is from having a fixed mode. Lastly, we will review polynomial optimization and some related results in Section III-C.

A. Fixed Modes

We will first define the decentralized fixed modes and then state a rank condition to check for their existence.

Definition 1: The set of fixed-modes of a plant G with respect to a sparsity pattern $\mathcal{S}$ and a type $\mathcal{T}$, is defined to be:

$$\begin{aligned}\Lambda(G, \mathcal{S}, \mathcal{T}) &\triangleq \\ &\{\lambda \in \mathbb{C} \mid \lambda \in \operatorname{eig}(A_{\text{CL}}(G, K)),\ \forall\ K \in \mathcal{S} \cap \mathcal{T}\} \\ &= \bigcap_{K\in\mathcal{S}\cap\mathcal{T}} A_{\text{CL}}(G, K),\end{aligned}$$

where $A_{\text{CL}}(G, K)$ gives the resulting closed-loop A matrix when controller K is closed around plant G.

Remark 2: This reduces to the well-known definition of fixed modes in [1] if $\mathcal{S}$ is block-diagonal, and $\mathcal{T} = \mathcal{T}^{\text{s}}$, where $\mathcal{T}^{\text{s}}$ is the set of static controllers.

An algebraic test to check for the existence of a fixed mode (similar to the PBH rank test for controllability or observability) was given in [8, Theorem 4.1]. The generalized version of this test is given in the following theorem:

Theorem 3 ([9, Theorem 2]): Given a strictly proper plant G, represented in the state-space by $(A, B, C, 0)$, and an information structure $\mathcal{S}$, we have that $\lambda \in \mathbb{C}$ is a fixed-mode of G, i.e., $\lambda \in \Lambda(G, \mathcal{S}, \mathcal{T}^{\text{s}})$, if and only if there exists a subset $\mathtt{I} \subseteq \{1, \cdots, n_u\}$, such that:

$$\operatorname{rank}\begin{bmatrix} A - \lambda I & B_{\mathtt{I}} \\ C_{\mathtt{J}_{\bar{\mathtt{I}}}} & 0 \end{bmatrix} < n, \tag{1}$$

where n is the dimension of the state, i.e., $A \in \mathbb{R}^{n\times n}$, and $\mathtt{J}_{\bar{\mathtt{I}}}$ are the sensors that can be seen from inputs other than those in $\mathtt{I}$.

B. Decentralized Assignability Measure

We will first state an existing metric on how far a system is from having decentralized fixed modes, and then review some of its properties. The materials in this section are from [2], and adopted to the notation used in this paper.

We first define the set of plants that have the same dimension as G, and have a fixed mode with respect to $\mathcal{S}$.

Definition 4: Given dimension of state space matrices by $\dim(G)$, and an information structure $\mathcal{S}$, define the set of *unassignable* systems as:

$$\begin{aligned}&\textbf{UNA}(\dim(G), \mathcal{S}) \triangleq \\ &\quad\{\tilde{G} \mid \tilde{G} = \left(\tilde{A}, \tilde{B}, \tilde{C}, 0\right),\ \text{where } \tilde{A} \in \mathbb{R}^{n\times n}, \tilde{B} \in \mathbb{R}^{n\times n_u}, \\ &\qquad\tilde{C} \in \mathbb{R}^{n_y\times n},\ \text{such that } \Lambda\left(\tilde{G}, \mathcal{S}, \mathcal{T}^{\text{s}}\right) \neq \varnothing\},\end{aligned} \tag{2}$$

where dependence on G is implicitly through dimension of its state-space matrices.

We are interested in the minimum distance between G, and the set of plants that have fixed-mode(s) with respect to the information structure $\mathcal{S}$, i.e., we are interested in the distance of G from $\textbf{UNA}(\dim(G), \mathcal{S})$. To this end, define the following notion of distance:

$$\begin{aligned}&d(G, \textbf{UNA}(\dim(G), \mathcal{S})) \triangleq \\ &\qquad\inf_{\tilde{G}\in\textbf{UNA}(\dim(G),\mathcal{S})} \left\| \begin{bmatrix} A - \tilde{A} & B - \tilde{B} \\ C - \tilde{C} & 0 \end{bmatrix} \right\|_2,\end{aligned} \tag{3}$$

where $(\tilde{A}, \tilde{B}, \tilde{C}, 0)$ is a state-space representation for $\tilde{G}$.

Vaz & Davison [2] have defined the *decentralized assignability measure* as the above distance, and have shown that it can equivalently be written as an another optimization problem:

Theorem 5 ([2, Theorem 3]): Given an LTI plant G, and an information structure $\mathcal{S}$, the *decentralized assignability measure* is given by:

$$\sigma_{\mathrm{VD}}(G,\mathcal{S}) \triangleq d\left(G, \mathbf{UNA}\left(\dim(G), \mathcal{S}\right)\right) = \min_{\substack{\lambda\in\mathbb{C},\\ \mathtt{I}\subseteq\{1,\cdots,n_u\}}} \sigma_n\left(\begin{bmatrix} A-\lambda I & B_{\mathtt{I}} \\ C_{\mathtt{J}_{\mathtt{I}}} & 0 \end{bmatrix}\right), \qquad (4)$$

where $\mathtt{I}$ can be any non-empty proper subset.

Remark 6: This metric is zero if and only if (1) has rank less than n, which in turn is a necessary and sufficient condition for having a fixed mode.

Remark 7: This metric possess interesting properties, but it is hard to compute due to two reasons. Firstly, minimizing the n-th singular value is non-convex, and secondly, minimizing over the partitions $\mathtt{I} \subseteq \{1,\cdots,n_u\}$ would involve integer programming ($2^{n_u}-2$ cases). This is the main motivation for approximation algorithms in [4], [5].

C. Polynomial Optimization

We will first give a brief overview of polynomial optimization problems, and then review some related results that we will use later in this paper. The materials in this section are mostly adopted from [7].

Definition 8 (Polynomial Optimization Problem): Given real-valued polynomials $p(x)$, $g_1(x),\cdots,g_r(x)$ all from $\mathbb{R}^n$ to $\mathbb{R}$, the following optimization problem is called a Polynomial Optimization (P.O.) problem:

$$p_K^* \triangleq \min_{x\in K} p(x), \qquad (5)$$

with variable $x\in\mathbb{R}^n$, and where the set $K\subseteq\mathbb{R}^n$ is defined by polynomial inequalities as $K \triangleq \{x\in\mathbb{R}^n | g_i(x)\geq 0,$ for $i=1,\cdots,r\}$.

Remark 9: The equality constraints $h(x)=0$ can be expressed by two inequality constraints $h(x)\geq 0$, and $(-h(x))\geq 0$.

Furthermore, define the sum of squares polynomials as:

Definition 10 (Sum-of-Squares): A real-valued polynomial $p(x):\mathbb{R}^n\to\mathbb{R}$ is called ***Sum-of-Squares*** (SOS), if it can be written as:

$$p(x) = \sum_{i=1}^{\bar{i}} \left(p_i(x)\right)^2,$$

for some $\bar{i}\in\mathbb{N}$, and where $p_i(x):\mathbb{R}^n\to\mathbb{R}$ are real polynomials in x for $i=1,\cdots,\bar{i}$.

We would like to derive lower bounds on one particular instance of such P.O. problems. This could be achieved if the set K satisfies the following assumption:

Assumption 11 ([7, Assumption 4.1]): The set K is compact and there exists a real-valued polynomial $u(x):\mathbb{R}^n\to\mathbb{R}$ such that the set $\{x\in\mathbb{R}^n | u(x)\geq 0\}$ is compact, and:

$$u(x) = u_0(x) + \sum_{k=1}^{r} g_i(x)u_i(x), \qquad \text{for all} \quad x\in\mathbb{R}^n,$$

where the polynomials $u_i(x)$ are all SOS for $i=0,\cdots,r$.

Remark 12: One way to ensure that this assumption holds is that the variable x would be bounded, i.e., it would be known that the solution of (5) would lie in some bounded region $\|x\|_2^2\leq a$. In this case, one can add an inequality constraint $g_{r+1}(x)\geq 0$, with $g_{r+1}(x)=a-\|x\|_2^2$, to the set K, and take $u_i(x)=1$, if $i=r+1$, and 0 otherwise.

It can then be proved that with this assumption one could obtain a sequence of finite dimensional Semidefinite Programs (SDP) that would converge to the optimum value p_K^* from below. This is stated in the following theorem:

Theorem 13 ([7, Theorem 4.2]): Let $p(x)$, p_K^*, and the set K be given as in Definition 8. Assume that K is a compact set that satisfies Assumption 11, then there exists a sequence of finite dimensional SDP of order N, denoted by $\mathbb{Q}_K^N$, that converges to the optimal value p_K^* from below, i.e.:

$$\inf \mathbb{Q}_K^N \uparrow p_K^*, \qquad \text{as} \quad N\to\infty.$$

IV. A Lower Bound on the k-th Singular Value by Polynomial Programming

In this section, we will provide a polynomial optimization problem for a lower bound on the k-th singular value of a polynomial matrix. This would then be used to derive a lower bound on the Vaz & Davison decentralized assignability measure in the next section.

We will first review the Courant-Fischer formulation of the singular values , and will transform this formulation into a polynomial program by rewriting the constraints. To this end, given a Hermitian matrix M, and a non-zero complex vector w with compatible dimension, define the ***Rayleigh quotient***, denoted by $R(M,w)$, as:

$$R(M,w) \triangleq \frac{w^*Mw}{w^*w}.$$

The next theorem reviews the Courant-Fischer formulation of the singular values, and is adopted to the notation used in this paper.

Theorem 14 (Courant-Fischer): Given a complex matrix $M\in\mathbb{C}^{p\times q}$ that has the ordered singular values $\sigma_1(M)\geq\sigma_2(M)\geq\cdots\geq\sigma_{\min(p,q)}(M)$, we have the following relations for the non-trivial cases of $1\leq k\leq\min(p,q)$:

$$\sigma_k^2(M) = \min_{w_1,w_2,\cdots,w_{k-1}\in\mathbb{C}^q} \max_{\substack{w\neq 0, w\in\mathbb{C}^q,\\ w\perp w_1,\cdots,w_{k-1}}} R(M^*M,w) \qquad (6)$$

$$\sigma_k^2(M) = \max_{w_1,w_2,\cdots,w_{q-k}\in\mathbb{C}^q} \min_{\substack{w\neq 0, w\in\mathbb{C}^q,\\ w\perp w_1,\cdots,w_{q-k}}} R(M^*M,w), \qquad (7)$$

where M^* denotes the conjugate transpose of M.

Proof: The proof is a direct consequence of [10, Theorem, 4.2.11, p. 179] when considering that M^*M is a $q\times q$ matrix, for which we have that $\lambda_k(M^*M)=\sigma_k^2(M)$. ■

Remark 15 (Rayleigh-Ritz): This theorem extends the Rayleigh-Ritz theorem on the largest and smallest eigenvalues of a Hermitian matrix [10, Theorem 4.2.2, p. 176], which states that for a Hermitian matrix $M \in \mathbb{C}^{m\times m}$, with eigenvalues $\lambda_1(M) \geq \cdots \geq \lambda_m(M)$, we have that:

$$\lambda_{\max}(M) = \lambda_1(M) = \max_{w\in\mathbb{C}^m} R(M, w),$$

and

$$\lambda_{\min}(M) = \lambda_m(M) = \min_{w\in\mathbb{C}^m} R(M, w).$$

We can equivalently write (6) and (7) using rank constraints on the considered subspaces:

Corollary 16: Given a complex matrix $M \in \mathbb{C}^{p\times q}$ with singular values $\sigma_1(M) \geq \sigma_2(M) \geq \cdots \geq \sigma_{\min(p,q)}(M)$, we have the following relations for the non-trivial values of $1 \leq k \leq \min(p,q)$:

$$\sigma_k^2(M) = \min_{\substack{W\in\mathbb{C}^{(k-1)\times q},\\ \mathrm{rank}(W)=k-1}} \max_{\substack{w\in\mathbb{C}^q,\\ Ww=0\\ w^*w=1}} w^*M^*Mw \quad (8)$$

$$\sigma_k^2(M) = \max_{\substack{W\in\mathbb{C}^{(q-k)\times q},\\ \mathrm{rank}(W)=q-k}} \min_{\substack{w\in\mathbb{C}^q,\\ Ww=0\\ w^*w=1}} w^*M^*Mw. \quad (9)$$

Proof: It is straightforward to replace the Rayleigh quotient with only its numerator, while enforcing the denominator to be equal to one, as any non-zero w can be scaled to have unit norm. For the min-max formulation of (6), we can gather $w_1, \cdots, w_{k-1}$ in a $(k-1)\times q$ matrix as $W = \begin{bmatrix} w_1 & \cdots & w_{k-1}\end{bmatrix}^*$. Then, the minimum would be achieved when all the $w_1, \cdots, w_{k-1}$ are independent of each other so as to make the feasible set for the w in the max part as small as possible, which is equivalent to have $\mathrm{rank}(W) = k-1$. Similar reasoning applies for the max-min formulation. ■

We would further alter (8) and (9) by considering finite samples of the rank-constrained subspaces in those equations. This would lead to upper and lower bounds on the singular values:

$$\sigma_k^2(M) \leq \min_{i\in\{1,\cdots,m\}} \left\{ \max_{\substack{w_i\in\mathbb{C}^q,\\ \bar{W}_i w_i=0\\ w_i^*w_i=1}} w_i^*M^*Mw_i \right\}, \quad (10)$$

where $\bar{W}_i \in \mathbb{C}^{(k-1)\times q}$ are all fixed sampled from the rank constrained subspace in (8) and thus all have rank $k-1$, for $i = 1, \cdots, m$. The upper-bound is achieved due to the fact that by finite sampling we are minimizing over a smaller set rather than the rank constrained subspace in (8), and hence the minimum value over these finite samples would increase compared to the original one. Similarly we have:

$$\sigma_k^2(M) \geq \max_{i\in\{1,\cdots,m\}} \left\{ \min_{\substack{w_i\in\mathbb{C}^q,\\ \underline{W}_i w_i=0\\ w_i^*w_i=1}} w_i^*M^*Mw_i \right\}, \quad (11)$$

where $\underline{W}_i \in \mathbb{C}^{(q-k)\times q}$ are all fixed sampled from the rank constrained subspace in (9) and thus all have rank $q-k$, for $i = 1, \cdots, m$. Similar to the min-max form, the lower bound is achieved as we are considering only finite numbers of $\underline{W}_1, \cdots, \underline{W}_m$, rather than the original subspace specified by the rank constraint in (9), and hence the maximum value would decrease, resulting in a lower bound for $\sigma_k(M)$.

We will use an alternative form for (10) and (11) to form optimization problem that only involve maximization or minimization.

Corollary 17: Suppose that a complex matrix $M \in \mathbb{C}^{p\times q}$ is given, which has singular values $\sigma_1(M) \geq \sigma_2(M) \geq \cdots \geq \sigma_{\min(p,q)}(M)$. Let $m \in \mathbb{N}$ be the desired number of the samples from the rank constrained subspaces, and for $1 \leq k \leq \min(p,q)$, let $\bar{W}_1, \cdots, \bar{W}_m$ be matrices that are sampled from the subspace specified by the rank constraint in (8), i.e., they are all in $\mathbb{C}^{(k-1)\times q}$ and have rank $k-1$. Similarly let $\underline{W}_1, \cdots, \underline{W}_m$ be matrices that are sampled from the rank constraint in (9), that are all in $\mathbb{C}^{(q-k)\times q}$ and have rank $q-k$. Then we have:

$$\begin{aligned} \sigma_k^2(M) \leq \quad & \max \quad \gamma \\ & \text{s.t.} \quad \gamma \leq \bar{w}_i^* M^* M \bar{w}_i \quad && i = 1, \cdots, m \\ & \qquad \bar{W}_i \bar{w}_i = 0 && i = 1, \cdots, m \\ & \qquad \bar{w}_i^* \bar{w}_i = 1 && i = 1, \cdots, m, \end{aligned} \quad (12)$$

with variables $\gamma \in \mathbb{R}$, and $\bar{w}_1, \cdots, \bar{w}_m \in \mathbb{C}^q$. Similarly, we have the following lower bound

$$\begin{aligned} \sigma_k^2(M) \geq \quad & \min \quad \gamma \\ & \text{s.t.} \quad \gamma \geq \underline{w}_i^* M^* M \underline{w}_i \quad && i = 1, \cdots, m \\ & \qquad \underline{W}_i \underline{w}_i = 0 && i = 1, \cdots, m \\ & \qquad \underline{w}_i^* \underline{w}_i = 1 && i = 1, \cdots, m, \end{aligned} \quad (13)$$

with variables $\gamma \in \mathbb{R}$, and $\underline{w}_1, \cdots, \underline{w}_m \in \mathbb{C}^q$.

Proof: We are minimizing over a finite number of real numbers in the minimization problem (10), for which we can equivalently maximize γ such that γ would be less than each of those numbers, which would give (12). Similarly, in (11) we are maximizing over a finite set; we can equivalently minimize γ such that γ is greater than each of the elements of that set, which would give (13). ■

We would want to have a non-trivial lower-bound on the k-th singular values of a variable matrix. The following theorem states that such an algorithm is possible to obtain, if we consider polynomial matrices subject to real-valued polynomial constraints.

Theorem 18: A non-trivial lower bound for the k-th singular value of a complex polynomial matrix $M(x) \in \mathbb{C}^{p\times q}$ subject to real-valued polynomial constraints can be obtained in polynomial time by a convex program, assuming that x, and all the variables in (13) lie in a compact set.

Proof: Optimization problem (13) would still be a polynomial optimization problem when M itself is a complex polynomial matrix which can be further subjected to real-valued polynomial constraints. To see this, observe that further than the objective γ which is real, all the constraints in (13) are real-valued (even for complex M), as $\underline{w}_i^* M^* M \underline{w}_i = \|M\underline{w}_i\|_2^2 \in \mathbb{R}$, $\underline{w}_i^* \underline{w}_i = \|\underline{w}_i\|_2^2 \in \mathbb{R}$, and the equality constraint $\underline{W}_i \underline{w}_i = 0$ could be separately set to zero for its real and imaginary parts. This problem satisfies the P.O. Definition 8, and since all the variables are assumed to be bounded, we can apply Remark 12 to ensure that Assumption 11 holds. Then, the SOS-based SDP which provide lower bounds can be derived using Theorem 13. ■

Remark 19: The compactness condition in Theorem 18 is satisfied for $\underline{w}_i$ by the unit norm constraint. Also, γ is lower bounded by zero by construction. Hence, imposing bounds on x, and γ are sufficient to guarantee that Remark 12 could be applied.

V. A Lower Bound for the Decentralized Assignability Measure

In this section, we are going to address the lower bound for the main measure that we are interested in this paper, namely, we will use the formulation developed in Section IV and show how to apply it on the decentralized assignability measure of Vaz & Davison by giving an equivalent form for this measure. We will first present this equivalent form, and then connect it to the polynomial optimization problem in the previous section. For the the ease of notation define:

$$\underline{F}(\lambda, \alpha) \;\triangleq\; \begin{bmatrix} A - \lambda I & B \operatorname{diag}(\alpha) \\ L(\alpha)\, C & 0 \end{bmatrix}, \tag{14}$$

where $\lambda \in \mathbb{C}$, $\alpha \in \mathbb{B}^{n_u}$, and $L(\alpha) \in \mathbb{B}^{n_y \times n_y}$ is a diagonal matrix that is zero everywhere, except for the diagonals that are given by:

$$[L(\alpha)]_{jj} \;=\; 1 - \prod_{\{i | K^{\text{bin}}_{ij} = 1\}} \alpha_i,$$

for $j = 1, \cdots, n_y$. We can then form another optimization problem based on this binary formulation:

$$\sigma_{\mathbb{B}}(G, \mathcal{S}) \;\triangleq\; \min_{\substack{\lambda \in \mathbb{C} \\ \alpha \in \mathbb{B}^{n_u}}} \sigma_n\left(\underline{F}(\lambda, \alpha)\right). \tag{15}$$

The next theorem will prove that this formulation is equal to the original form of Vaz & Davison metric.

Theorem 20: Given a plant G, and an arbitrary information structure $\mathcal{S}$, we have that:

$$\sigma_{\text{VD}}(G, \mathcal{S}) \;=\; \sigma_{\mathbb{B}}(G, \mathcal{S}). \tag{16}$$

Proof: Given any λ and $\mathtt{I}$ in (4), for each $i \in \{1, \cdots, n_u\}$ take $\alpha_i = 1$ if $i \in \mathtt{I}$, and 0 otherwise. Then, the matrix in (4) would be equal to the $\underline{F}(\lambda, \alpha)$ in (14) except for possibly extra zero rows or columns. These extra zero rows or columns does not affect the the n-th singular value of the $(n+n_y) \times (n+n_u)$ dimensional matrix $\underline{F}(\lambda, \alpha)$, which in turn render the equality. ■

We can now form a polynomial optimization problem based on Corollary 17 and this equivalent formulation:

Corollary 21: Assume that a plant G, an information structure $\mathcal{S}$, and an $m \in \mathbb{N}$ are given. Then, the following optimization problem gives a non-trivial lower bound for the (squared of) the decentralized assignability measure, $\left(\sigma_{\text{VD}}(G, \mathcal{S})\right)^2$:

$$\begin{array}{lll} \min & \gamma & \\ \text{s.t.} & \gamma \geq \underline{w}_i^* \left(\underline{F}(\lambda, \alpha)\right)^* \underline{F}(\lambda, \alpha) \underline{w}_i & \text{for} \quad i = 1, \cdots, m \\ & \underline{W}_i \underline{w}_i = 0 & \text{for} \quad i = 1, \cdots, m \\ & \underline{w}_i^* \underline{w}_i = 1 & \text{for} \quad i = 1, \cdots, m \\ & \alpha_i (1 - \alpha_i) = 0 & \text{for} \quad i = 1, \cdots, n_u, \end{array} \tag{17}$$

with variables $\gamma \in \mathbb{R}$, $\lambda \in \mathbb{C}$, $\alpha \in \mathbb{R}^{n_u}$, $\underline{w}_1, \cdots, \underline{w}_m \in \mathbb{C}^{n+n_u}$, and where $\underline{W}_1, \cdots, \underline{W}_m$ are fixed matrices in $\mathbb{C}^{n_u \times (n+n_u)}$ that all have rank n_u.

Proof: This can be seen by using the equality in Theorem 20, and then applying (13) in Corollary 17 with M replaced by $\underline{F}(\lambda, \alpha)$, and then enforcing $\alpha \in \mathbb{B}^{n_u}$ by adding the last equality constraint in (17). ■

The following theorem establishes that under some mild conditions a lower bound the Vaz & Davison metric can be found by a convex program.

Theorem 22: Given a plant G, and any information structure $\mathcal{S}$, assume that some bounds on the optimal λ, and γ in (17) are known (i.e., $\|\lambda\| \leq \bar{\lambda}$, and $\|\gamma\| \leq \bar{\gamma}$), then a non-trivial lower-bound for $\left(\sigma_{\text{VD}}(G, \mathcal{S})\right)^2$ can be obtained by a convex program.

Proof: The proof is a direct consequence of applying Theorem 18, and Remark 19 on problem (17). ■

VI. Numerical Example

In this section, we will give a numerical example to illustrate the methods and algorithms that are developed in this paper. The considered plant is strictly proper, stable, LTI, and is further centrally controllable and observable.

Example 23: Consider the following plant, with parameter $\beta \in \mathbb{R}$:

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 0 & \beta \\ 1 & 1 \end{bmatrix}, K^{\text{bin}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This plant has a fixed mode only at $\beta = 0$. We vary β and plot the lower bound obtained from sampling two or three $\underline{W}$ in (17) (i.e., $m = 2$, or 3) in Figure 1. We have observed that when $m = 1$, the lower bound is zero in the considered range. Also, it is noteworthy that the lower bound is not only dependent on m, but also on the values of $\underline{W}$ in (17), i.e., different choices for $\underline{W}_1, \cdots, \underline{W}_m$ may result in different lower bounds. We have randomly generated these $\underline{W}_i$ in this example. We have used `gloptipoly` [11] to form the SDP relaxation corresponding to the lower bound on the polynomial optimization problem (17) . The Vaz & Davison metric ($\sigma_{\text{VD}}(G, \mathcal{S})$ in (4)) is computed for the numerical example by evaluating the singular values over a discrete grid in the complex plane for each of the $2^{n_u} - 2$ possible subsets $\mathtt{I}$, which is clearly only an option for very small problems. The upper bound on this metric is obtained by the ADMM method followed by a subgradient-based tuning step as in [5]. As illustrated in this figure, as we increase the number of samples (m) from the rank constrained subspace in (9), the lower bound becomes more accurate.

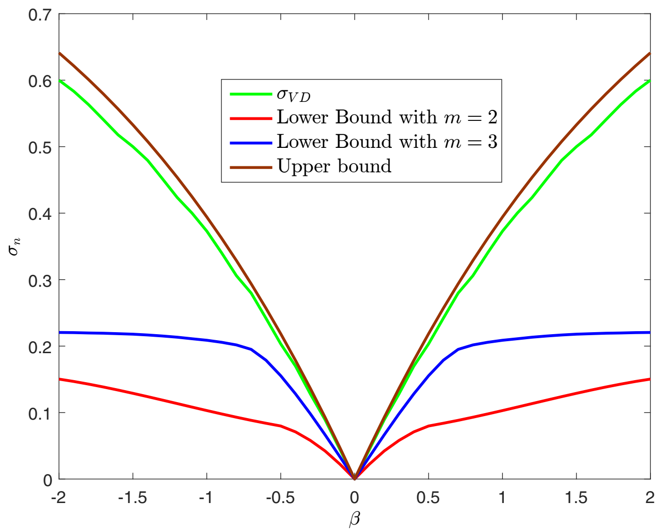

Fig. 1: Upper bound and lower bounds for the Vaz & Davison metric in Example 23

VII. Conclusions

We have studied the problem of how nonassignable the modes of a plant can be. This problem was first addressed by the decentralized assignability measure of Vaz & Davison, which can be connected to the mobility of the modes of the plant. However, this measure is hard to compute exactly. We have formulated bounds on any arbitrary singular value of a polynomial matrix as a polynomial optimization problem, which can be easily extended to such bounds on any eigenvalue of Hermitian matrices as well. We then have obtained polynomial time lower bounds for this problem by applying sum of squares procedures. This has been extended to obtain a lower bound on the decentralized assignability measure which can be solved in polynomial time. Finally, this method has been investigated by a numerical example.

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