A convex characterization of classes of problems in control with specific interaction and communication structures *

Petros G. Voulgaris

Coordinated Science Laboratory

University of Illinois at Urbana-Champaign

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Abstract

We present a list of optimal disturbance rejection problems in systems in which the overall control scheme is required to have a certain structure. These structures correspond to various classes of controlled systems which include what we refer to as nested, chained, hierarchical, delayed interaction and communication, and, symmetric systems. The common thread in all of these classes is that by taking an input-output point of view we can characterize all stabilizing controllers in terms of convex constraints in the Youla-Kucera parameter. The disturbance rejection problem can therefore be casted as a convex, yet nonstandard, model matching problem. Approaches that solve this problem are presented for various optimality criteria.

1 Introduction

In large, complex and distributed systems there is often the need of considering a specific structure on their overall control scheme (e.g., [15].) There are a number of practical reasons, among which cost and reliability, that result to constraints on how an individual local control station interacts with the overall system, what part of information it has access to and what communication mechanism is in place. Hence it is important to have analysis and design techniques when interaction and communication constraints are imposed on the global controller structure.

In this paper we consider the general framework of Figure 1 where G may represent a complex system consisting of subsystems interacting with each other. The overall controller for G is K. Both G and K are assumed to be linear, discrete-time systems. The controller K has to respect a specific structure that is imposed by interaction and communication constraints. A typical example of structure that has been studied extensively in the literature is when K is totally decentralized i.e., diagonal. However, the optimal performance problem when structural constraints are present still remains a challenge to the control community due to each complexity, notably the lack of a

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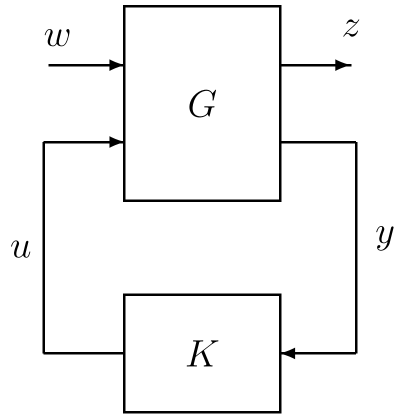

Figure 1: Standard Framework

convex characterization of the problem. To mention only a few samples of related work we refer to [14, 16, 9] and references therein. Taking an input-output point of view and parametrizing all K via the Youla-Kucera [22] parameter Q one can see as a major source of difficulty the fact that structural constraints in K may lead to non-convex constraints in Q. Despite this discouraging point, a main theme in the paper is identifying specific classes of problems for which the constraints in Q are convex with the appropriate choice of the coprime factors of G. These classes can be associated with several practical applications such as integrated flight propulsion systems, platoons of vehicles, networked control, production lines, chemical processes, etc. The various classes of systems identified include what we refer to as nested, chained, hierarchical, delayed interaction and communication, and, symmetric systems. A key feature in all the structures considered is that G_{22}, the part of G that relates the controls u to measurements y, has the same structure as the one imposed on the controller K. It is the structure of G_{22} that matters for convexity; the remaining part of G can be totally unstructured. In the majority of the situations presented here there is an algebraic property between the K's and G_{22}'s under consideration. That is, they form a ring as the structure is preserved in products, additions and in $(I + G_{22}K)^{-1}$ whenever the inverse exists, as it should, for well-posedness.

In the sections to follow we first expose the various structures of interest, continue on to controller parametrization and finally describe the solution procedures to the optimal performance problems in the $\mathcal{H}^2$, ℓ^1 and $\mathcal{H}^\infty$ sense.

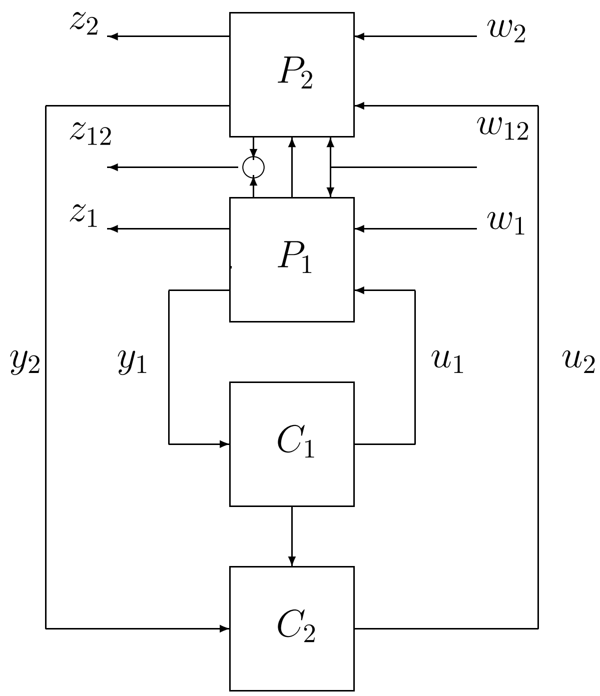

Figure 2: Nested Structure

2 Specific Structures

Throughout the paper we assume, unless stated otherwise, all systems to be finite dimensional linear, time-invariant and described in discrete-time.

2.1 Triangular Structures

In this class of systems G_{22} and K are triangular transfer function matrices. This class includes what we term as nested structures, chains (or strings) with leader and followers, and hierarchical type of schemes. In the sequel we elaborate more on these. Some parts of this exposition can also be found in [19, 20].

Nested systems

This is the case where a subsystem is inside another and there is only one-way interaction, from inside to outside, or, the reverse. A practical application that associates with this set-up is the Integrated Flight Propulsion Control (IFPC) e.g., [5]. To illustrate the nested problem in simple terms we consider only two nests. The generalization to n nests is straightforward. Thus we consider the case of Figure 2 where there is a system comprised of two nests (subsystems.) The internal subsystem consists of a plant P_1 together with its controller C_1 whereas the external consists of the plant P_2 with the controller C_2. The internal and external subsystems have control inputs u_1, u_2 and measured outputs y_1 and y_2 respectively. Due to the nested structure depicted in the figure, the control input u_1 depends only on the measurement y_1 whereas u_2 can depend on both y_1 and y_2. Moreover, y_1 is affected only by u_2 while y_2 is affected by both u_1 and u_2. The overall system is subjected to exogenous inputs (e.g., commands, disturbances, sensor noise) and there are also outputs to be regulated. In particular, we allow for inputs w_1 affecting directly the internal subsystem, inputs w_2 that affect the external subsystem only, and, inputs w_{12} that affect both subsystems. Similarly, the outputs of interest z_1, z_2 and z_{12} are related respectively directly to the internal, directly to the external and to combination of both subsystems. A necessary assumption for the existence of a stabilizing overall controller K is that each subsystem P_i is stabilizable by each subcontroller C_i. Thus, if the exogenously unforced (with no external disturbances) state space description for P_1 is

$$x_1(k+1) = A_1x_1(k) + B_1u_1(k), \quad y_1(k) = C_1x_1(k) + D_1u_1(k)$$

and for P_2

$$x_2(k+1) = A_2x_2(k) + B_{21}u_1(k) + B_2u_2(k), \quad y_2(k) = C_2x_2(k) + D_{21}u_1(k) + D_2u_2(k)$$

we have that there exist feedback and observer gains F_i and L_i respectively such that $A_i + B_iF_i$ and $A_i + L_iC_i$ are Hurwitz (i.e., eigenvalues in the unit disk) for $i = 1, 2$. Bringing the system of Figure 2 to the the standard $G - K$ control design framework of Figure 1 we have the following signal identifications

$$y := \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \; u := \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \; z := \begin{pmatrix} z_1 \\ z_{12} \\ z_2 \end{pmatrix}, \; w := \begin{pmatrix} w_1 \\ w_{12} \\ w_2 \end{pmatrix}.$$

The structure of G_{22} is of the form

$$G_{22} = \begin{pmatrix} g_1 & 0 \\ g_{12} & g_2 \end{pmatrix}$$

i.e., G_{22} has a lower (block) triangular (l.b.t.) structure. Moreover, for the controller K to correspond to the nested structure of Figure 2 it should be of the form

$$K = \begin{pmatrix} k_1 & 0 \\ k_{12} & k_2 \end{pmatrix}$$

i.e., it should be a lower (block) triangular system.

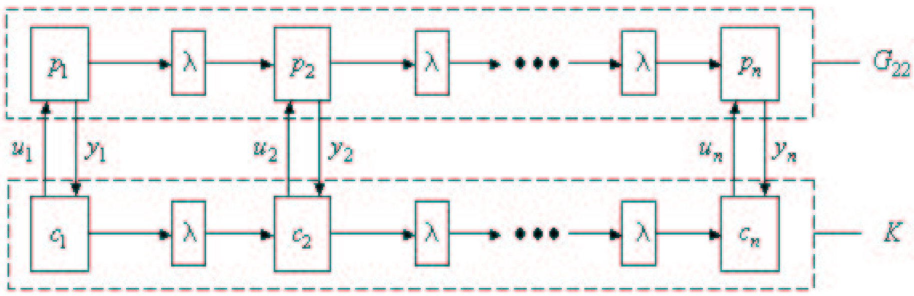

Figure 3: Chain Structure

Chains

In the chain (or string) system of Figure 3 there are n subsystems P_i with their corresponding subcontrollers C_i. Platoons of vehicles where there is a leader and followers that are obtaining information form their leading vehicles is a good example to associate with this structure. The control action u_i in the subsystem P_i affects its follower P_{i+1} by a 1-step delay while the control action u_{i+1} in P_{i+1} does not affect its leader P_i. Also, subcontroller C_i passes information to its follower C_{i+1} with a 1-step delay while C_{i+1} does not transmit any information to C_i. Exogenous inputs w and regulated outputs z are admitted that may couple the dynamics but are not shown in the picture for clarity. Bringing in the general $G - K$ framework we have that in the case of the chain the structure of G_{22} is

$$G_{22} \;=\; \begin{pmatrix} g_1 & & & & \\ \lambda g_{21} & g_2 & & & \\ \lambda^2 g_{31} & \lambda g_{32} & g_3 & & \\ \vdots & & \ddots & \ddots & \\ \lambda^n g_{n1} & \cdots & \cdots & \lambda g_{nn-1} & g_n \end{pmatrix}$$

and similarly the structure of K is

$$K \;=\; \begin{pmatrix} k_1 & & & & \\ \lambda k_{21} & k_2 & & & \\ \lambda^2 k_{31} & \lambda k_{32} & k_3 & & \\ \vdots & & \ddots & \ddots & \\ \lambda^n k_{n1} & \cdots & \cdots & \lambda k_{nn-1} & k_n \end{pmatrix}$$

Noting that

$$G_{22} = \begin{pmatrix} 1 & & & \\ & \lambda & & \\ & & \ddots & \\ & & & \lambda^{n-1} \end{pmatrix} \begin{pmatrix} g_1 & & & \\ g_{21} & g_2 & & \\ \vdots & & \ddots & \\ g_{n1} & \cdots & \cdots & g_n \end{pmatrix} \begin{pmatrix} 1 & & & \\ & \lambda^{-1} & & \\ & & \ddots & \\ & & & \lambda^{-n+1} \end{pmatrix}$$

and

$$K = \begin{pmatrix} 1 & & & \\ & \lambda & & \\ & & \ddots & \\ & & & \lambda^{n-1} \end{pmatrix} \begin{pmatrix} k_1 & & & \\ k_{21} & k_2 & & \\ \vdots & & \ddots & \\ k_{n1} & \cdots & \cdots & k_n \end{pmatrix} \begin{pmatrix} 1 & & & \\ & \lambda^{-1} & & \\ & & \ddots & \\ & & & \lambda^{-n+1} \end{pmatrix}$$

it follows that the chain problem is a special case of a the nested problem. The necessary stabilizability assumption for this problem requires that g_{ij} are stable for $i \neq j$.

An additional structure which can be imposed that requires fewer building blocks is that of Toeplitz, i.e.,

$$G_{22} \;=\; \begin{pmatrix} g_1 & & & \\ \lambda g_2 & g_1 & & \\ \vdots & \ddots & \ddots & \\ \lambda^{n-1} g_n & \dots & \lambda g_2 & g_1 \end{pmatrix}$$

with K as in G_{22} by replacing g's with k's.

Hierarchical

Yet another structure in this category is that of open hierarchies as depicted in Figure 4. This is a multi-input (or, multi-agent) system that needs to be regulated to follow certain external commands

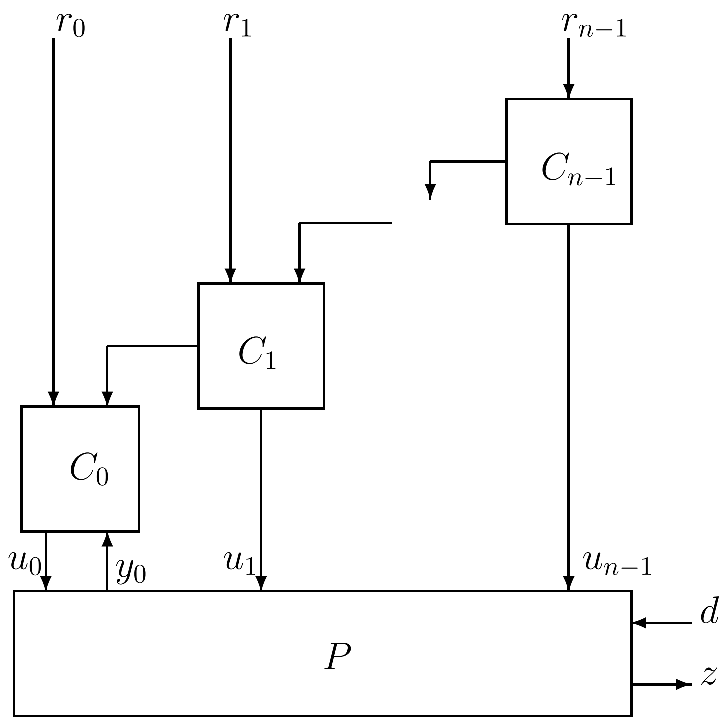

Figure 4: Hierarchical Structure

as well as to reject disturbances. Each control input is authorized by a single decision maker at a specified level in a decision making hierarchy. The decision maker receives signals from upper levels, possibly direct external commands, and, is allowed to pass information only to lower levels. The lowest level is the only level that receives feedback from the system. In particular, P is a LTI, discrete-time, system, the plant, that needs to be regulated, z represents the variables to be controlled, d some direct external disturbances and y_0 variables that are measured and can be used for feedback. The control inputs to P can be grouped to n (possibly vector) variables $u_0, \dots, u_{n-1}$. Each variable u_i is authorized by a corresponding decision maker C_i at the ith level of a n-level hierarchical structure. In this structure there is only one-way flow of signals, from upper to lower levels, and not vice-versa. Decision maker C_0 is the only one that processes the measurements y_0 from the plant P. There are also external direct commands to any level i denoted by r_i. For the setup to make sense C_0 should be able to stabilize P. The above defined system can be put in a standard $G - K$ control design framework of Figure 1 with the following signal identifications:

$$y := \begin{pmatrix} \psi_0 \\ r_1 \\ \vdots \\ r_{n-1} \end{pmatrix}, \; u := \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{n-1} \end{pmatrix}, \; w := \begin{pmatrix} d \\ r_0 \\ \vdots \\ r_{n-1} \end{pmatrix}$$

where $\psi_0 := \begin{pmatrix} y_0 \\ r_0 \end{pmatrix}$. For the controller K to be generated by the hierarchical structure described above and vice-versa the constraint that K is of the form

$$K = \begin{pmatrix} * & * & \dots & * \\ 0 & * & \dots & * \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & * \end{pmatrix}$$

i.e., it should be an upper block triangular (u.b.t.) system. Considering the structure of G_{22} we have that it is of the form

$$G_{22} = \begin{pmatrix} * & * & \dots & * \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

i.e., the only non-zero row is the first.

An additional feature in the overall scheme of Figure 4 is that there can be different, yet commensurate, communication rates from layer to layer. For example, if C_0 samples y_0 and produces u_0 every T time units, C_1 can be sending signals to C_0 every n_1T time units, C_2 can be sending signals to C_1 every n_2n_1T time units, etc. where the n_i's are integers.

2.2 Delayed Interaction and Communication Networks

The network in this case is as in Figure 5. In this figure subsystem P_i and its subcontroller C_i interact with their respective neighbors with a 1-step delay in the transmition and reception of signals. Exogenous inputs w and regulated outputs z are admitted that may couple the dynamics but are not shown in the figure for clarity. A good relevant example to associate in this case is the control of a large networked system over a network where, as an aggregate model, the neighbor-to neighbor interaction and communication is subject to a unit delay. In the $G-K$ framework the structure is reflected in G_{22} and K as

$$G_{22} = \begin{pmatrix} g_{11} & \lambda g_{12} & \cdots & \lambda^{n-1} g_{1n} \\ \lambda g_{11} & g_{22} & \cdots & \lambda^{n-2} g_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda^{n-1} g_{n1} & \cdots & \cdots & g_{nn} \end{pmatrix}$$

and similarly for K

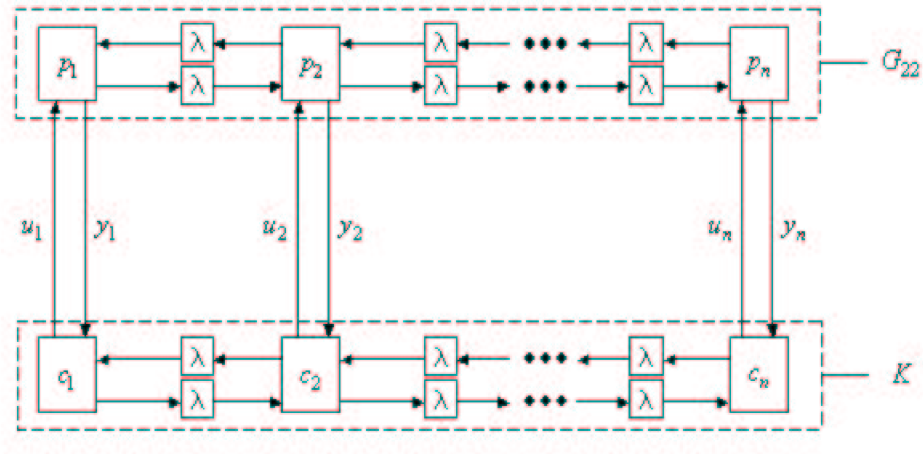

Figure 5: Delayed Interaction and Communication Structure

$$K = \begin{pmatrix} k_{11} & \lambda k_{12} & \cdots & \lambda^{n-1}k_{1n} \\ \lambda k_{11} & k_{22} & \cdots & \lambda^{n-2}k_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda^{n-1}k_{n1} & \cdots & \cdots & k_{nn} \end{pmatrix}$$

For a distributed control system of a similar type in the case of spatially invariant systems we refer to [21].

2.3 Other Structures

Delayed Observation Sharing

Two examples of communication patterns are given. Both are shown in the Figures 6, 7. In Figure 6 the measurement information from a local control station C_i is passed to the other with the delay

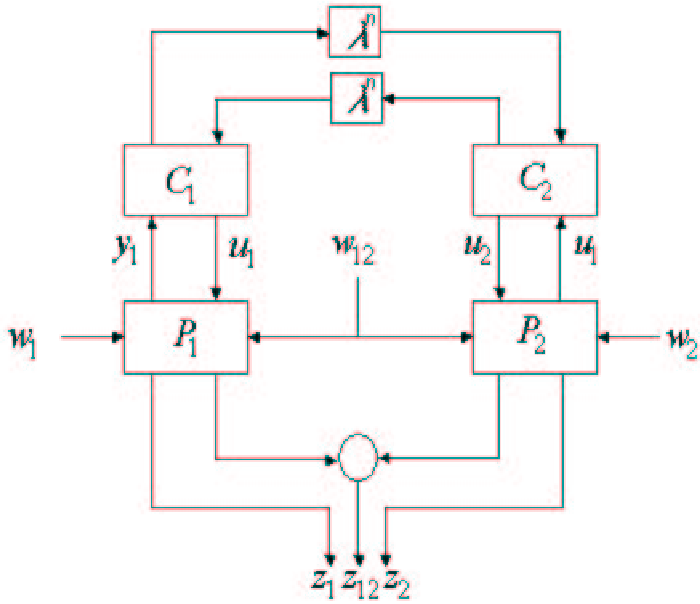

Figure 6: n-Step Delayed Information Exchange

of n time-steps. In Figure 7 information exchanges between stations C_i are performed every n time-steps through a data recording and supervising unit S. In both of these scenarios there is no interaction between the local plants P_i. There is however a coupling through the disturbances w and the variables z to be regulated. In the $G - K$ frame the structure of the required controller K for the first case is

$$K = \begin{pmatrix} k_1 & \lambda^n k_{12} \\ \lambda^n k_{21} & k_2 \end{pmatrix}$$

For the second case, one can lift the system stacking the inputs and outputs to integer multiples of

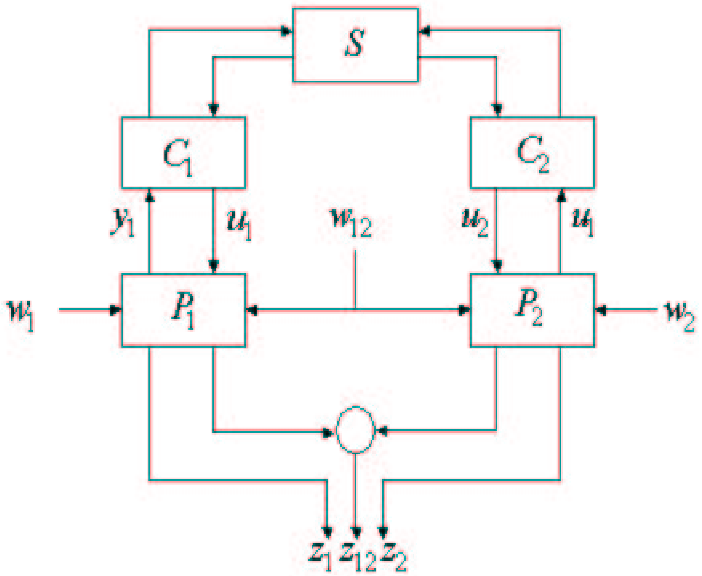

Figure 7: n-Step Information Sharing

n time steps to realize that the lifted controller K should have a feedthrough D-term of the form

$$D_K = \begin{pmatrix} f_1 & & & \\ f_2 & d_1 & & \\ \vdots & \vdots & \ddots & \\ f_n & d_{n-1} & \cdots & d_1 \end{pmatrix}$$

where f_i are full 2×2 matrices and d_i are diagonal. This is a case where n-time periodic C_i's are considered. Generalizations to tree-clusters of this structure can also be considered as in Figure 8 where S_1 and S_2 communicate with a higher level unit Σ every $m \times n$ time-steps.

Symmetric Structures

This is the case where $G_{22} = G_{22}^T$ and $K = K^T$. Figure 9 shows the case of a two-control input and two-measured output symmetric feedback system with g and k representing the coupling dynamics

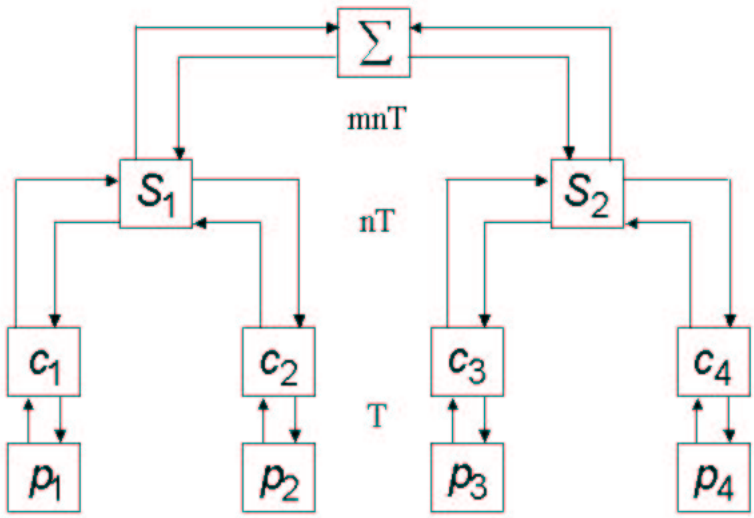

Figure 8: Tree Structure

in the plant and controller respectively. If $g_1 = g_2$ then we have a circulant symmetry in G_{22} [3].

3 Controller Parametrization

Employing the Youla-Kucera parametrization, all stabilizing K, not necessarily with the structure required, are given by the parametrization [22]

$$K = (Y_l - D_l Q)(X_l - N_l Q)^{-1} = (X_r - QN_r)^{-1}(Y_r - QD_r)$$

where Q is a stable free parameter and $Y_l,\ D_l,\ X_l,\ N_l,\ X_r,\ N_r,\ Y_r,\ D_r$ can be obtained from a coprime factorization (e.g., [7, 17]) of G_{22}. The structural constraints on K transform to constraints in Youla parameter Q. With a particular choice of the coprime factors the constraints in Q are convex. In fact, these constraints are the same as in the required structure for K as indicated below.

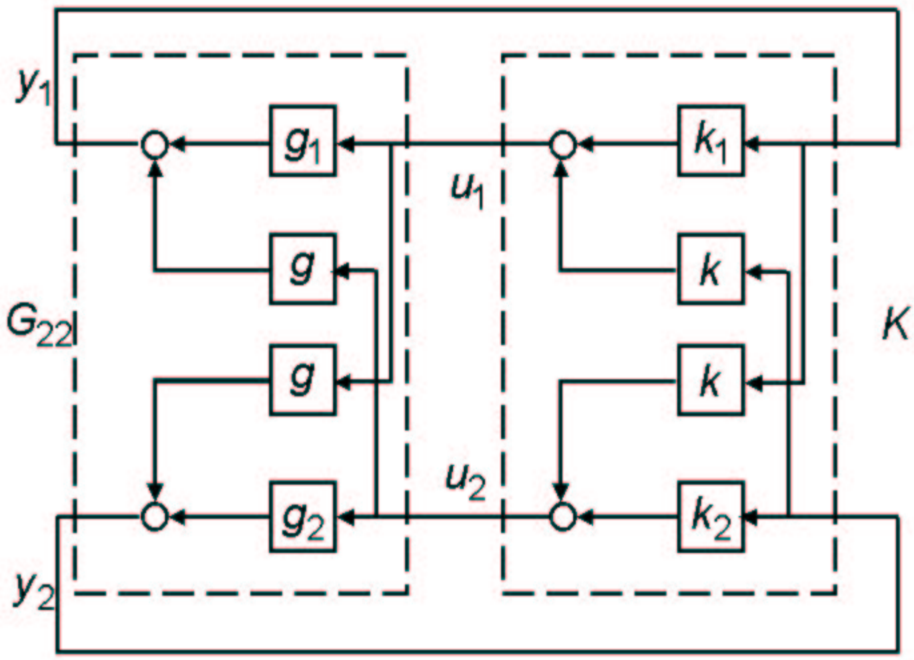

Figure 9: Symmetric Structure

3.1 Triangular Structures

For simplicity we will treat only the case of 2-nested systems as generalizations are straightforward. In this case G_{22} has the state-space description

$$G_{22} \sim (\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \begin{pmatrix} B_1 & 0 \\ B_{21} & B_2 \end{pmatrix}, \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix}, \begin{pmatrix} D_1 & 0 \\ D_{21} & D_2 \end{pmatrix}).$$

By the necessary assumption on the problem formulation we have that there exist feedback and observer gains F_i and L_i respectively such that $A_i+B_iF_i$ and $A_i+L_iC_i$ are Hurwitz (i.e., eigenvalues in the unit disk) for $i = 1, 2$. Hence one can choose a state feedback and an observer gain for G_{22} respectively as

$$F = \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix}, \quad L = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}.$$

With this choice the standard set of doubly coprime factors in [7] have the required triangular structure and hence the structural constraints on K transform to the same constraints on Q. I.e., Q is required to be lower (block) triangular.

For Toeplitz triangular structures similar type of arguments can be used to show that Q is constrained to be Toeplitz. This is simple to establish when G_{22} is stable since all K's are represented as $K = -Q(I - G_{22}Q)^{-1}$. For the unstable case it is necessary for the existence of a stabilizing K with Toeplitz structure to have stable g_i for $i = 2, 3, \ldots$. But then an argument as in the general triangular case above can be used to establish existence of suitable coprime factors that force Q to be Toeplitz.

3.2 Delayed Interaction and Communication Structures

For this types of structures G_{22} has a pulse response $\{G_{22}(i)\}_{i=0}^{\infty}$ with the following band-structure: $G_{22}(0)$ is diagonal, $G_{22}(1)$ is 3-diagonal, $G_{22}(2)$ is 5-diagonal, etc.; that is $G_{22}(i)$ is a $2i+1$-diagonal matrix for $i = 0, 1, \ldots, n-2$. Similar is the imposed structure on K. Representing G_{22} as $G_{22} = (G_1 \; G_2 \; \ldots \; G_n)$ and obtaining a controllable state-space description for each G_i as $G_i \sim (A_i, B_i, C_i, D_i)$, a state space description of $G_{22} \sim (A, B, C, D)$ can be obtained [4] with $A = \mathrm{diag}(A_i)$, $B = \mathrm{diag}(B_i)$, $C = (C_1 \; \ldots \; C_n)$, $D = (D_1 \; \ldots \; D_n)$. This is a controllable realization. If F_i are such that $A_i + B_iF_i$ are Hurwitz one can use a state feedback gain $F = \mathrm{diag}(F_i)$ and the standard formulas in [7] to obtain the coprime factors N_l and D_l. It is straightforward to check that FA^jB is $2j+1$-diagonal for $j = 0, 1, \ldots$ and hence N_l and D_l posses the same structure as G_{22}. A similar argument for K shows that a K with the band-structure can be factored as $K = Y_lX_l^{-1}$ with Y_l, X_l coprime, possessing the band-structure. The existence of such a stabilizing K with band-structure follows from the fact that $\lambda^n G_{22}$ can be stabilized by a controller (assuming G_{22} can). Hence as $K = (Y_l - D_lQ)(X_l - N_lQ)^{-1}$ it follows that K has the band-structure iff Q does.

3.3 Other Structures

The case of delayed observation sharing structures is similar to the previous subsection and thus it will not be discussed further. The result is that Q has to have the same structure as K. For the symmetric structures mentioned we consider the case where G_{22} is stable and symmetric, i.e., $G_{22}^T = G_{22}$. Then all stabilizing K, possibly non-symmetric, are given as $K = (I + QG_{22})^{-1}Q$. If K is to be symmetric, then

$$Q \;=\; K(I - G_{22}K)^{-1} = K^T(I - G_{22}^TK^T)^{-1} = (I - K^TG_{22}^T)^{-1}K^T = ((I - G_{22}K)^T)^{-1}K^T = Q^T$$

i.e., Q is symmetric. Similar argument shows that if Q is symmetric then $K = (I + QG_{22})^{-1}Q$ is symmetric

4 Optimal Performance

All the classes of discussed in section 2 require when viewed in the G-K framework that K is stabilizing and has a specific structure (triangular, banded, etc). From the discussion in the previous section, this is equivalent to requiring that Q has the same structure as K. The structures considered correspond to subspace type of restrictions on Q. We denote by $\mathcal{S}$ the subspace of stable systems Q that have the required structure.

The problem of interest is as follows:

Problem: *Find K with the appropriate structure such that, subject to internal stability, the norm $\|\Phi\|$ is minimized.*

The norm $\|\Phi\|$ may refer to any norm, e.g., $\mathcal{H}^2$, $\mathcal{H}^\infty$ or ℓ^1. By internal stability here we mean the usual stability requirement in the G-K framework.

Based on the parametrization in the previous section the problem of minimizing $\|\Phi\|$ can be casted as

$$\mu := \inf_{Q \in \mathcal{S}} \|H - UQV\|$$

where H, U, V are stable systems. Therefore, the resulting problem is convex, yet infinite dimensional (the pulse response coefficients of Q.)

4.1 Approaches for solving the equivalent problem

In principle, one can solve the problem by considering truncations of the Q parameter [2] and thus approximating the problem with a finite dimensional (the pulse response coefficients of the truncated Q) convex programming problem

$$\mu_N := \inf_{Q_N} \|H - UQ_N V\|$$

where Q_N is a Finite Impulse Response (FIR) of length N, system in $\mathcal{S}$. It can be checked that $\mu_N \to \mu$ monotonically from above as $N \to \infty$. The main shortcoming of this method is that it cannot indicate how close to the optimal solution is the converging upper bound μ_N. In the sequel we discuss approaches to completely solve the problem.

4.1.1 $\mathcal{H}^2$-norm minimization

In this case we can invoke the projection theorem along the lines in [18] to obtain the solution directly. To this end let $U = U_i U_o$, $V = V_o V_i$ be an inner-outer factorization of U and V respectively. Define the subspace $\mathcal{M} := \{Z : Z = U_o Q V_o, Q \in \mathcal{S}\}$. Then the following can be shown:

Theorem 4.1 *The optimal solution Z_o for the problem*

$$\mu = \inf_{Z \in \mathcal{M}} \|H - U_i Z V_i\|$$

is given by the projection onto $\mathcal{M}$

$$Z_o = \Pi_{\mathcal{M}} U_i^* H V_i^*.$$

Once Z_o is found an optimal Q can be found as $Q = U_o^{-r} Z_o V_o^{-l}$ where U_o^{-r} is a right inverse of U_o and V_o^{-l} is a left inverse of U_o. Characterizing $\Pi_{\mathcal{M}}$ in a simple enough manner to allow computations is possible in many of the systems presented. For example, for triangular problems a specialized inner-outer factorization (see [6] chapt. 14) leads to triangular U_o, V_o so that $\mathcal{M}$ coincides with $\mathcal{S}$ which makes the projection $\Pi_{\mathcal{M}} U_i^* H V_i^*$ trivial. An alternative however that avoids solving the non-standard problem can be found in [8] where the original problem is transformed to a standard model matching by "vectorizing" Q. The equivalent problem is of the form

$$\inf_{q \in \mathcal{H}^2} \|h - \bar{U} q\|$$

where q is a vector unconstrained with one-to-one correspondence with $Q \in \mathcal{S}$. The solution for q is then the same as in Theorem 4.1 where $\mathcal{M}$ is simply the space $\mathcal{H}^2$ i.e.,

$$q_{\text{opt}} = \bar{U}_o^{-r} \Pi_{\mathcal{H}^2} \bar{U}_i^* h$$

with $\bar{U} = \bar{U}_i \bar{U}_o$ is an inner-outer factorization of $\bar{U}$ and $\bar{U}_o^{-r}$ is a right inverse of $\bar{U}_o$.

4.1.2 ℓ^1-norm minimization

In this case one can use an extension of the scaled-Q method in [10] to provide converging lower and lower bounds to μ. In particular, for the problem at hand let P_N denote the Nth truncation operator and define the two finite dimensional linear programs:

$$\nu_N(\alpha) := \min \max\{\|H - R\|, \alpha \|Q\|\}$$

subject to

$$P_N(R) = P_N(HQV),\ Q \in \mathcal{S}$$

and

$$\mu_N(\alpha) := \min \max\{\|H - R\|, \alpha \|Q\|\}$$

subject to

$$R = UP_N(Q)V,\ Q \in \mathcal{S}$$

where α is a scalar positive parameter. Then, using elements of duality theory the following can be shown

Theorem 4.2 *There exists an a priori computable α_0 such that for all α with $0 < \alpha \leq \alpha_0$, $\mu_N(\alpha) \to \mu$ monotonically from above and $\nu_N(\alpha) \to \mu$ monotonically from below as $N \to \infty$.*

Hence, with the above theorem one obtains close to optimal solutions to any any prespecified accuracy.

4.2 $\mathcal{H}^\infty$ control

For the $\mathcal{H}^\infty$ problem a Nehari-based approach [18] to get a sequence of converging lower bounds is possible. However, more efficient computations are needed. Recent work in [12, 13] on multi-objective $\mathcal{H}^\infty/\mathcal{H}^2$ control provides a Q-based design technique that gives converging lower as well as upper bounds. This technique readily applies to the "constrained in Q" problems described in this paper when $\mathcal{H}^\infty$ optimization is of interest. It should be noted that the main approach in this method is not conceptually different from that in [10].

Remark 1

For certain problems with specific symmetries constraining the controller to be symmetric is redundant as the optimal K will posses the required structure regardless. Such is the case for example when circulant symmetry is present [3, 1]. In general however this may not hold. Certainly if in the map of interest $\Phi = H - UQV$, H or U or V are not symmetric there is no guarantee that the optimal Q optimal is symmetric. But even if H and U and V are symmetric, the following (static) example shows that the optimal Q may not be symmetric.

Consider $\Phi = H - UQ$ with

$$
\begin{aligned}
H &= \begin{pmatrix} 1 & 0 & \gamma \\ 0 & 1 & \alpha \\ \gamma & \alpha & 1 \end{pmatrix} \qquad\qquad \alpha \neq 0,\ \gamma \neq 0 \\
U &= \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \beta \end{pmatrix}, \qquad\qquad \beta \neq 0,\ \beta \neq 2 \\
Q &= \left(\begin{array}{c|c} \overline{Q} & \begin{pmatrix} q_{13} \\ q_{23} \end{pmatrix} \\ \hline (q_{31} q_{32}) & q_{33} \end{array} \right)
\end{aligned}
$$

then

$$\Phi = \left(\begin{array}{c|c} I - \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \overline{Q} & \begin{pmatrix} \gamma - \tilde{q} \\ \alpha - \tilde{q} \end{pmatrix} \\ \hline (\gamma - \beta q_{31} \ \ \alpha - \beta q_{32}) & 1 - \beta q_{33} \end{array} \right)$$

with $\tilde{q} = q_{13} + q_{23}$. Φ can be though of as the map from w to z in the Figure 10 which is a model following set-up with $G_{22} = U$.

For $\|\Phi\|_{H_2}$ to be minimal $(\gamma - \tilde{q})^2 + (\alpha - \tilde{q})^2$ should be minimal i.e.,

$$q_{13} + q_{23} = \tilde{q} = \frac{\alpha + \gamma}{2}$$

and also

$$(\gamma - \beta q_{31})^2, (\alpha - \beta q_{32})^2$$

should be minimal i.e.,

$$q_{31} = \frac{\gamma}{\beta}, \ q_{32} = \frac{\alpha}{\beta},$$

but

$$q_{31} + q_{32} = \frac{\alpha + \gamma}{\beta} \neq \frac{\alpha + \gamma}{2} = q_{13} + q_{32}$$

for $\beta \neq 2$. Hence optimal Q is not symmetric.

The same example above can be used to show that ℓ_1 optimization as well may lead to a non-symmetric optimal Q. An easy way to construct more examples of this form is when U (and V) are symmetric and stably invertible; then the unconstrained optimal in any norm cost is zero and the optimizing Q is $Q = U^{-1}HV^{-1}$ which is in general non-symmetric.

Remark 2

A number of multi-objective problems can be considered for the types of structures listed in this paper. The key approaches to use are the Q based designs as in [12, 13, 11] where converging upper and lower bounds for the optimal cost are obtained and, arbitrarily close to optimal, suboptimal solutions are furnished.

5 Conclusions and Discussion

In this paper we presented a list of optimal disturbance rejection problems in systems in which the overall control scheme is required to have a certain structure. These structures correspond to

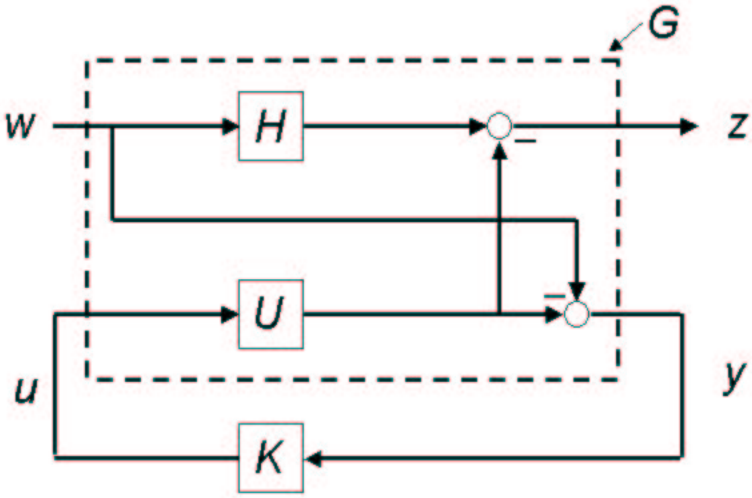

Figure 10: A command following example

various classes of controlled systems such as nested, chained, hierarchical, delayed interaction and communication, and, symmetric systems. These classes can be associated with several practical applications in integrated flight propulsion systems, platoons of vehicles, networked control, production lines, chemical processes, etc. The common thread in all of these classes is that by taking an input-output point of view we can characterize all stabilizing controllers in terms of convex constraints in the Youla-Kucera parameter. The disturbance rejection problem can therefore be casted as a convex, yet nonstandard, model matching problem. Approaches that solve this problem were presented for $\mathcal{H}^2$, ℓ^1 and $\mathcal{H}^\infty$ optimality.

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