Distributed Control of Spatially Invariant Systems - Lecture Notes

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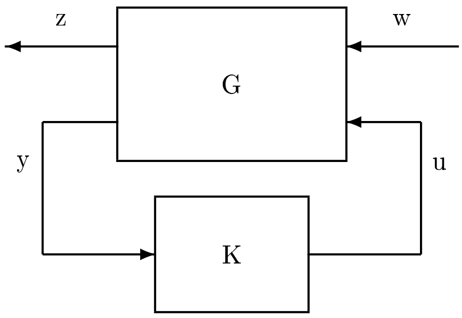
1 Introduction

Consider the general H_∞, LQ or H_2 control of spatial invariant systems:



where G is a spatial invariant system with invariant properties. The controller K is such that it

- stabilizes G globally
- has certain distributed properties
- is design using spatial truncation

The system G is described by:

$$G : \left\{ \begin{array}{rcl} \frac{\partial}{\partial t}\psi(.,t) & = & A\psi(.,t) + B_1 w(.,t) + B_2 u(.,t) \\ z(.,t) & = & C_1\psi(.,t) + D_{12}u(.,t) \\ y(.,t) & = & C_2\psi(.,t) + D_{21}w(.,t) \end{array} \right. \tag{1}$$

where $A, B_1, B_2, C_1, C_2, D_{12}$ and D_{21} are translation invariant operators (see below) and the controller $K(s)$:

$$K : \left\{ \begin{array}{rcl} \frac{\partial}{\partial t}\psi_K(.,t) & = & A_K\psi_K(.,t) + B_K y(.,t) \\ u(.,t) & = & C_K\psi_K(.,t) + D_K y(.,t) \end{array} \right. \tag{2}$$

Definition We define the *Translator operator:* T_{x_0} by:

$$T_{x_0} f(x) = f(x - x_0)$$

and we say that the operator A is *translation invariant* if

$$T_x A = A T_x$$

for every translation T_{x_0}.

For example:

$$\begin{aligned}
\frac{\partial^2}{\partial^2 x} T_{x_0} e^{-x} &= e^{-x-x_0} \\
&= T_{x_0}(e^{-x}) \\
&= T_{x_0} \frac{\partial^2}{\partial^2 x} e^{-x}
\end{aligned}$$

Consider now a linear (continous-time) dynamic system where the dynamics, defined by :

$$\begin{aligned}
\frac{\partial}{\partial t}\psi(x,t) &= [A\psi](x,t) + [Bu](x,t) &(3)\\
y(x,t) &= [Cy]((x,t) + [Du](x,t) &(4)
\end{aligned}$$

This system is said to be spatially invariant if the operators A, B, C, and D are translation invariant.If we take the Fourier transform of 4 and 4 we have that, based on Assumption 1 and Definition 1 from [1], we get a digonalized system in the decoupled form:

$$\begin{aligned}
\frac{\partial}{\partial t}\psi(\hat{\lambda},t) &= \hat{A}(\lambda)\psi(\lambda,t) + \hat{B}(\lambda)\hat{u}(\lambda,t) &(5)\\
\hat{y}(\lambda,t) &= \hat{C}(\lambda)y(\lambda),t) + \hat{D}(\lambda)u(\lambda,t) &(6)
\end{aligned}$$

Example:

$$\begin{aligned}
\frac{\partial}{\partial t}\psi(x,t) &= \frac{\partial^2}{\partial^2}\psi(x,t) + u(x,t) &(7)\\
&\Downarrow \\
\frac{\partial}{\partial t}\psi(\hat{\lambda},t) &= -c\lambda^2\hat{\psi}(\lambda,t) + \hat{u}(\lambda,t), \lambda \in \Re &(8)
\end{aligned}$$

2 LQR Controller design

Now consider the distributed LQR problem, where the problem is to minimize the functional

$$\min_{u(t)\in L_2} J = \int_0^\infty (\langle Q\psi, \psi\rangle + \langle Ru, u\rangle)dt$$

s.t.

$$\begin{aligned}
\frac{\partial}{\partial t}\psi(x,t) &= [A\psi](x,t) + [Bu](x,t) &(9)\\
\psi(x,0) &= \psi_0(x) &(10)
\end{aligned}$$

By taking spatial transforms, this problem can be re-written as

$$J = \int_{\hat{\mathbb{G}}} \int_0^\infty (\hat{\psi}_\lambda^*(t)\hat{Q}_\lambda\hat{\psi}_\lambda(t) + \hat{u}_\lambda^*(t)\hat{R}_\lambda\hat{u}_\lambda(t))dtd\lambda$$

s.t.

$$\frac{\partial}{\partial t}\psi(\hat{\lambda},t) = \hat{A}(\lambda)\psi(\lambda,t)+\hat{B}(\lambda)\hat{u}(\lambda,t) \tag{11}$$
$$\hat{y}(\lambda,0) = \hat{y}_0(\lambda) \tag{12}$$

The optimal feed-back controller for this system is found from the (translation invariant) feedback

$$u = R^{-1}B^*Px \tag{13}$$

where P is a translation invariant operator whose Fourier symbol $\hat{P}_(\lambda)$ is the positive definite solution to the matrix algebraic Riccati equation:

$$\hat{A}^*_\lambda\hat{P}_\lambda+\hat{P}_\lambda\hat{A}_\lambda-\hat{P}_\lambda\hat{B}_\lambda\hat{R}^{-1}_\lambda\hat{B}^*_\lambda\hat{P}_\lambda+\hat{Q}_\lambda=0 \tag{14}$$

for all $\lambda\in\hat{\mathbb{G}}$. In the original spatial domain the controller is as follows

$$u(x,t)=\int_{\mathbb{R}}K(x-\zeta)\psi_K(\zeta)d\zeta \tag{15}$$

where ψ_K is the estimated or measured distributed state and K is the convolution kernel of the controller, i.e. $K:=R^{-1}B^*P$. Thus, the feedback at position x is calculated by a convolving the neighboring state estimates with the kernel of K (the size of this neighborhood is determined by the spread of K).

Example Consider again:

$$\frac{\partial}{\partial t}\psi(x,t)=\frac{\partial^2}{\partial^2}\psi(x,t)+u(x,t) \tag{16}$$

which yields the Fourier transformed system:

$$\frac{\partial}{\partial t}\psi(\hat{\lambda},t)=-c\lambda^2\hat{\psi}(\lambda,t)+\hat{u}(\lambda,t) \tag{17}$$

With LQR optimization and $Q=qI$ and $R=I$, the Riccati equation becomes

$$-2c\lambda^2\hat{p}(\lambda)-\hat{p}^2(\lambda)+q=0 \tag{18}$$

which has the positive solution

$$\hat{p}(\lambda)=-c\lambda^2+\sqrt{c^2\lambda^4+q} \tag{19}$$

In the transform domain, our optimal controller will be of form $\hat{u}(\lambda,t)=\hat{k}(\lambda)\hat{\psi}(\lambda,t)$, with $\hat{k}(\lambda)=-\hat{p}(\lambda)$. The control law is

$$u(x,t)=\int_{\mathbb{R}}k(x-\zeta)\psi(\zeta)d\zeta \tag{20}$$

where $k(x-\zeta)=-p(x)$ and $p(x)$ is the inverse Fourier transform of $\hat{p}(\lambda)$. To determine the spread of $p(x)$ one can show (see [1]) that $p(x)$ decays exponentially, for certain values of q and c. More precisely,

$$\lim_{x\to 0}|p(x)|e^{\eta|x|}\to 0,\ \text{for } 0<\eta<\frac{\sqrt{2}}{2}\left(\frac{q}{c^2}\right)^{1/4} \tag{21}$$

Since $\{k(x)\}$ decays exponentially with $|x|$, it is possible to truncate it to form a localized feedback convolution operator whose closed loop performance is close to the optimal. E.g:

$$K_T(x) \begin{cases} K(x), & |x| \leq T \\ 0, & |x| > T. \end{cases}$$

Notice that the cheaper the control is, i.e. $q \to \infty$ the larger the region of stability is. Thus cheaper control enables a more decentralized controller configuration but also a potential for the controllers to "fight" each other.

3 Some remarks

In [1] it is noted that this spatial truncation, i.e. truncation done in the spatial domain, is advantageous to more standard procedures of truncating in the Fourier domain ("picking a number of modes"). The major reason for this is said to be that the Fourier truncation ignores the location of the controller in the spatial coordinates. However, from practical experience with spatially distributed systems (e.g. paper machine control) it is well known that the sensitivity to uncertainty in the actuator/sensor position is much higher for spatially based control systems compared to modal based control systems.

References

[1] Bassam Bamieh, Fernando Paganini, and Munther A. Dahleh. Distributed control of spatially invariant systems. *IEEE Transactions on Automatic Control*, 47(7), 2002.