Frontiers in Networked Control	Lecture 6
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Spatially distributed systems

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This lecture reviews the paper "Distributed Control Design for Spatially Interconnected Systems" by Rafaello D'Andrea and Geir E. Dullerud.

1 Motivation

An example of a spatially distributed system is a vibrating flexible rope, where the separate parts of the rope are discretizised as illustrated in Figure 1.

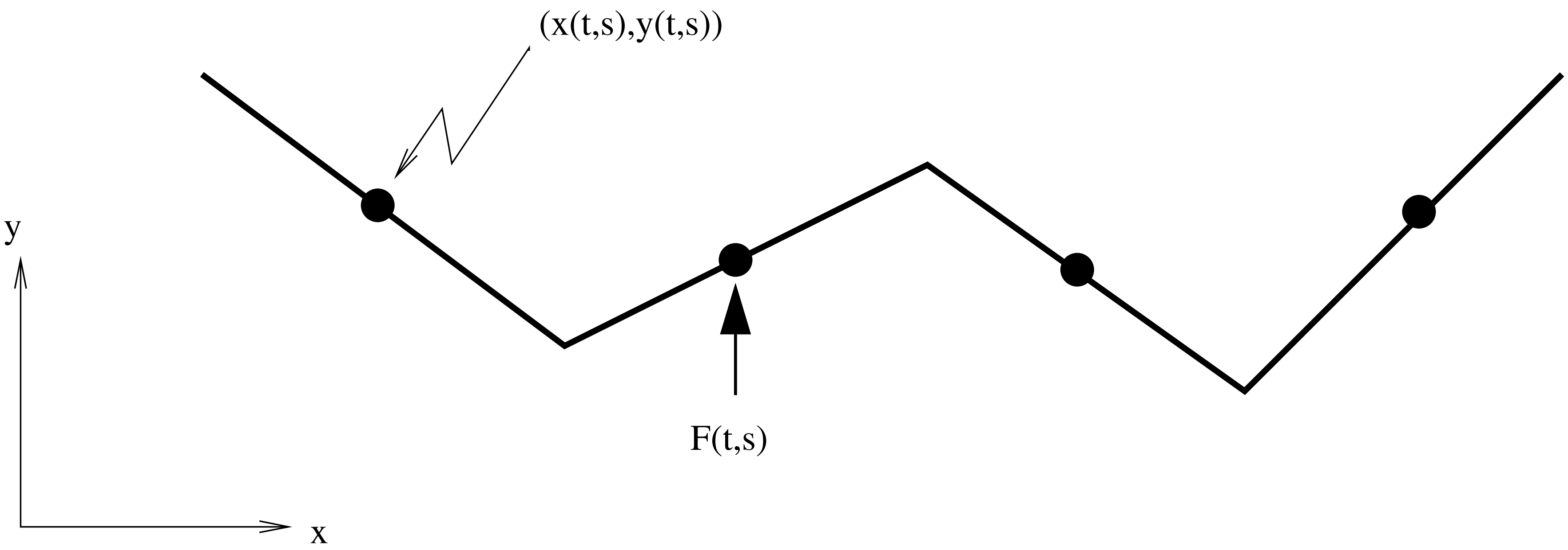

Figure 1: A discretizised rope as an example of a spatially distributed system.

Here $t \in \mathbb{R}^+$ is the time and

$$s_i \in \mathbb{D}_i \begin{cases} \mathbb{Z} \\ \mathbb{Z}_{N_i} = \{1, \ldots, N_i\} \end{cases}$$

is the spatial variable(s). In the rope case, s_1 tells what segment we are looking at. The movement of the rope depends on the movement of all other segments:

$$\begin{aligned} M\ddot{y}(s) &= y(s-1) + y(s+1) - 2y(s) + v(s-1) - v(s+1) + F(s) \\ M\ddot{x}(s) &= \ldots \\ J\nu(s) &= \ldots \end{aligned}$$

2 Introduction

The paper discusses three aspects of spatially distributed systems:

- Well-posedness

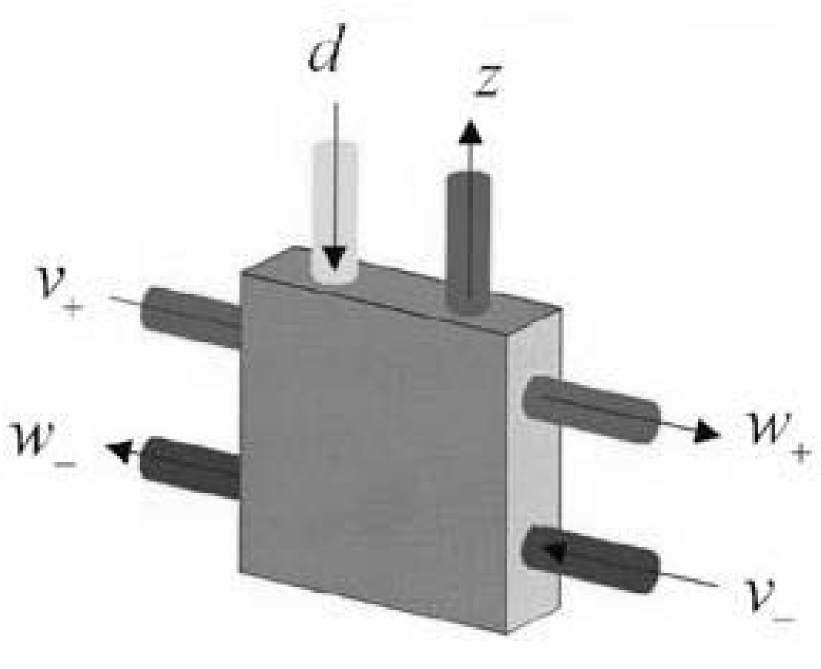

Figure 2: System building block, with input and output signals. Figure copied from the reviewed paper.

- Stability
- Performance

It also considers how to design controllers for such systems. The system is modelled as consisting of blocks, as depicted in Figure 2. The variables are:

$$\begin{aligned} d &= \text{input (disturbance, reference, ...)} \\ z &= \text{output (errors, control effort, ...)} \\ w, v &= \text{interconnection variables} \end{aligned}$$

A distributed system can then be assembled by such blocks, each of which is governed by the following equations:

$$\begin{aligned} \dot{x}(t,s) &= A_{TT}x(t,s) + A_{TS}v(t,s) + B_T d(t,s) \\ w(t,s) &= A_{ST}x(t,s) + A_{SS}v(t,s) + B_S d(t,s) \\ z(t,s) &= C_T x(t,s) + C_S v(t,s) + Dd(t,s), \end{aligned}$$

where

$$w = \begin{pmatrix} w^+ \\ w^- \end{pmatrix};\ v = \begin{pmatrix} v^+ \\ v^- \end{pmatrix}.$$

3 Preliminaries

The blocks can be assembled in one- or two dimensional structures (although Figure 2 illustrates a block for one-dimensional structures) and have finite or infinite lengths. Finally, they can also be connected in periodic configurations, as illustrated in Figures 2-4 in the paper. All of the above holds if S lies in the finite commutative group over $\mathbb{Z}_{N_i}$, where the plus operator should be taken modulo N_i, so that

$$N_i + 1 = 0.$$

We can also let $S \in \mathbb{Z}_{N_i} \times \mathbb{Z}_{M_i} \times \ldots$ to get higher-dimensional networks.

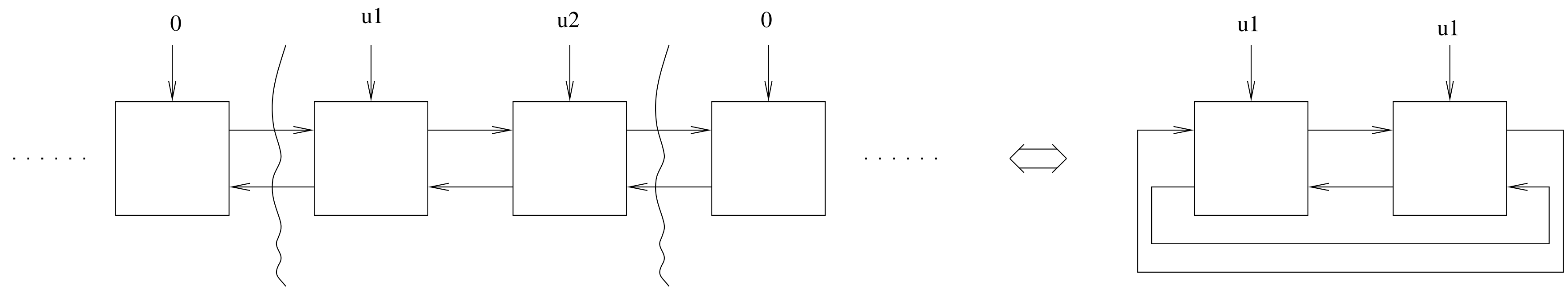

Figure 3: Infinitely long chains of equal blocks are equivalent to periodic chains if the elements outside an interval have input signal zero. The converse transformation can also be done.

Infinitely long chains of equal elements can be replaced by periodic chains if the elements outside an interval have input signal zero. This is illustrated in Figure 3. Conversely, a periodic chain can be transformed to an infinite chain, where the added elements have input signal zero.

The authors introduce a spatial shift operator that acts on s:

$$\begin{array}{rcl} (Sv)(t,s) & := & v(t,s+1) \\ (S^{-1}v)(t,s) & := & v(t,s-1) \end{array}$$

This operator then allows them to express the operator Δ:

$$\Delta = \begin{pmatrix} SI_{m+} & 0 \\ 0 & S^{-1}I_{m-} \end{pmatrix},$$

where $m+$ is the dimension of the variables v^- and $m-$ is the dimension of the v^+ variables. Now $w^+(t,s) = v^+(t,s+1)$ and the system can be written as

$$\left\{ \begin{array}{rcl} \dot{x} & = & Ax + Bd \\ z & = & Cx + Dd \end{array} \right. ,$$

with

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A_{TT} & B_T \\ C_T & D \end{pmatrix} + \begin{pmatrix} A_{TS} \\ C_S \end{pmatrix} (\Delta - A_{SS})^{-1}(A_{ST} \; B_S).$$

It is assumed here that $(\Delta - A_{SS})$ is invertible.

4 Well-posedness, stability and performance

4.1 Well-posedness

The authors first give two examples of systems that are not well-posed. They both refer to Figure 4. If both P_1 and P_2 have unitary gain, we have $w_1(t) = v_1(t)$ and $w_2(t) = v_2(t)$. This yields an algebraic loop, and there does not exist solutions for all input signals. Another example is if we choose

$$\left. \begin{array}{r} P_1 \text{ has unitary gain} \\ P_2(\zeta) = 1 - \frac{1}{\zeta} \end{array} \right\} \Rightarrow \frac{v_1}{n_1}(\zeta) = \zeta.$$

This transfer function is not proper, so the system is not physically realizable. The paper gives the following result on well-posedness:

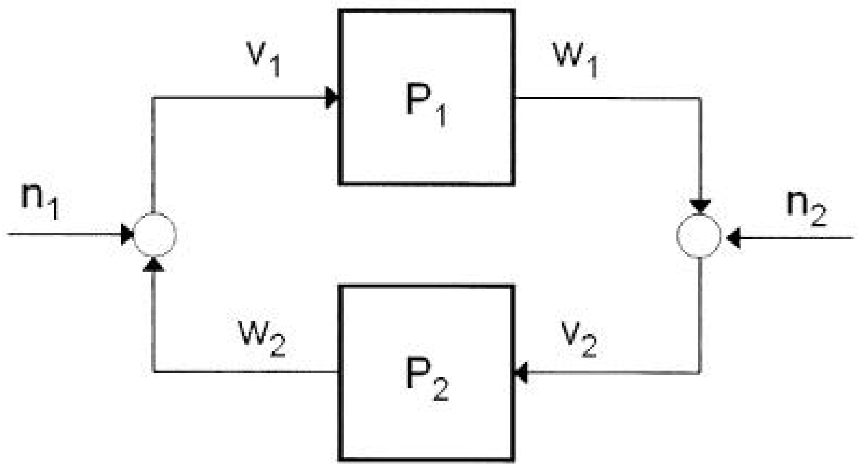

Figure 4: Example of a system that may not be well-posed. Figure copied from the reviewed paper.

Theorem 1 (Well-posedness) *Let*

$$A_{SS} = \begin{pmatrix} A_{SS}^{++} & A_{SS}^{+-} \\ A_{SS}^{-+} & A_{SS}^{--} \end{pmatrix}.$$

Then the system is well-posed iff $(\Delta - A_{SS})^{-1}$ *exists and is bounded. This is equivalent to*

$$\exists\ x = x^* \ \ s.t. \ \ {A_{SS}^{+}}^{*} x A_{SS}^{+} - {A_{SS}^{-}}^{*} x A_{SS}^{-} < 0,$$

where

$$A_{SS}^{+} = \begin{pmatrix} A_{SS}^{++} & A_{SS}^{+-} \\ 0 & I \end{pmatrix}; \ A_{SS}^{-} = \begin{pmatrix} I & 0 \\ A_{SS}^{-+} & A_{SS}^{--} \end{pmatrix}$$

4.2 Stability

Theorem 2 (Stability) *The system* $\dot{x} = Ax$, $x(t=0)$ *is stable, i.e.*

$$\lim_{t \to \infty} ||x(t)|| = 0$$

if

$$\exists\ X_S = X_S^*,\ X_T > 0$$

such that $J_{UL} < 0$*. Here,* $J_{UL} < 0$ *is the 2-by-2 upper left part of* J*, as defined in (the very long) Equation 42 in the paper.*

The condition for well-posedness is included in this, so if the system is stable, it is also well-posed.

4.3 Performance

We want to achieve performance in the sense of disturbance attenuation, i.e. $||z|| < ||d||$. This is called that the system is *contractive*.

Theorem 3 (Performance) *The performance goal is achieved, i.e.* $||z|| < ||d||$ *if*

$$\exists\ X_S = X_S^*,\ X_T > 0$$

such that $J < 0$*.* J *is defined in Equation 42 in the paper.*

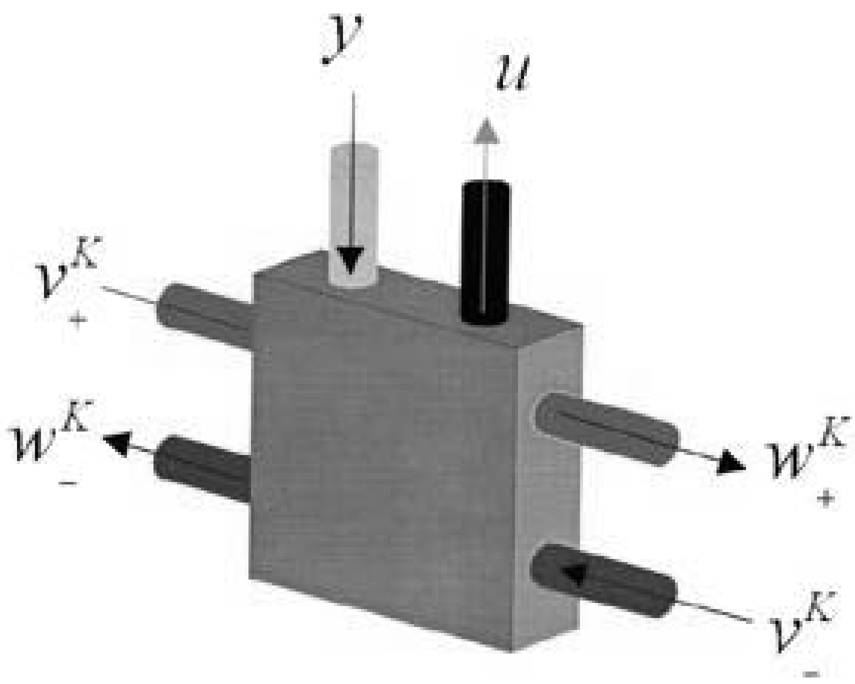

Figure 5: The structure of each controller. Figure copied from the reviewed paper.

Since the stability condition considered a submatrix of J, the performance condition implies both stability and well-posedness of the system. A big advantage is that with this formulation, the matrices only grow with the number of neighbors to each subsystem, not with the size of the whole system.

We conclude this section with a theorem on equivalence betweem different kinds of systems:

Theorem 4 *If an infinitely extended system is well-posed, stable and contractive, then both the finite and periodic versions are well-posed, stable and contractive.*

5 Control design

After having investigated the properties of the system, how do we design controllers for it? The authors assume that the controllers have a similar structure as the subplants, as illustrated in Figure 5. One controller and one subplant together form a new closed-loop subsystem, with output signals

$$\tilde{w}_+ = \begin{pmatrix} w_+ \\ w_+^K \end{pmatrix}.$$

The same LMI:s as before hold, but now we get a number of unknown controller parameters, that need to be selected. We get a "Bilinear LMI" (BLMI). Thanks to the similarity in structure of the controller and plant, the BLMI can be made convex and solvable.

5.1 Example

This example could be a particle repelled by two other particles. The dynamic equations are expressed on operator form:

$$
\begin{aligned}
\ddot{p} &= \frac{1}{8}(S_1 + S_1^{-1} + S_2 + S_2^{-1} + G)p + \\
&\quad + \frac{1}{16}(S_1 + S_1^{-1} - 2)(S_2 + S_2^{-1} - 2)d_1 + u \\
z_1 &= \frac{1}{16}(S_1 + S_1^{-1} - 2)(S_2 + S_2^{-1} - 2)p \\
z_2 &= u \\
y &= p + d_2
\end{aligned}
$$

The spatial variables now take values in a two-dimensional grid,

$$(s_1, s_2) \in \mathbb{Z} \times \mathbb{Z}.$$

The particles are affected by particles in neighboring grids, where the spatial variables are shifted one step up or down. The system has two states:

$$
\begin{aligned}
x_1(t, s_1, s_2) &= p(t, s_1, s_2) \text{ (particle positions)} \\
x_2(t, s_1, s_2) &= \dot{p}(t, s_1, s_2) \text{ (particle velocities)}
\end{aligned}
$$

The authors then use different controller design techniques to compare the performance (i.e. the L_2 gain) and the computational time for an example as above, with a 10-by-10 grid of particles.