Frontiers in Networked Control Lecture 1
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A Simple But Intractable Example

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Review of 2 Input-2 Output Framework

Consider Figure 1. w are the exogenous inputs, z are the exogenous outputs, y are the controller inputs and u are the control signals. We typically want to minimize the map from w (e.g. wind gust) to z, "error signals".

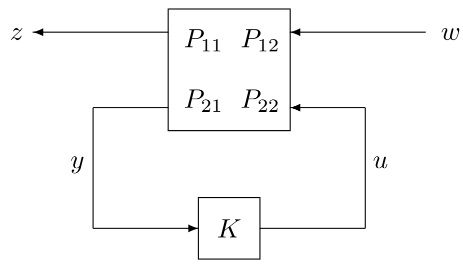

Figure 1: Classical 2 Input-2 Output Framework

$$\begin{aligned} z &= P_{11}w + P_{12}u \\ y &= P_{21}w + P_{22}u \end{aligned} \tag{1}$$

$u = Ky$ gives

$y = P_{21}w + P_{22}Ky$
$(I - P_{22}K)y = P_{21}w$
$y = (I - P_{22}K)^{-1}P_{21}w$
$u = K(I - P_{22}K)^{-1}P_{21}w$
$z = [P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}]w$
$z = f_l(P, K)w$

E.g. we want to keep $\|f_l(P, K)\|$ small

Considering 3-dimensional vectors this gives

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \underbrace{\begin{bmatrix} * & & \\ * & * & \\ * & * & * \end{bmatrix}}_{G} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \qquad \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \underbrace{\begin{bmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{bmatrix}}_{K} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

This holds when we have

*) Classical Information Pattern
*) Linear Dynamics
*) Quadratic Cost
*) Gaussian Noise

$\Rightarrow$

-Separation of estimation and control.

-Optimal controller is linear.

Witsenhausen showed that without a Classical Information Pattern the rest of the theory brakes down.

Witsenhausen Counterexample (1968)

Consider Figure 2

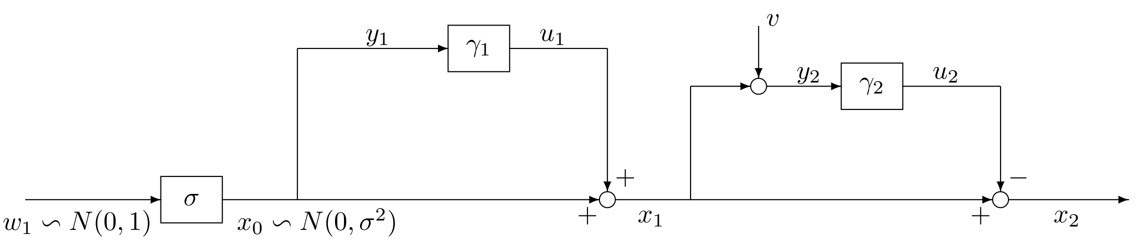

Figure 2: Block diagram to show Witsenhausen's Counterexample

$x_0 = \sigma w_1$
$y_1 = x_0$
$u_1 = \gamma_1(y_1)$
$x_1 = x_0 + u_1$
$w_2 = v$
$y_2 = x_1 + v$
$u_2 = \gamma_2(y_2)$
$x_2 = x_1 - u_2$

$J = E(ku_1^2 + x_2^2)$

In the 2nd time step:
have only access to y_2

Now

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \underbrace{\begin{bmatrix} * & \\ * & * \end{bmatrix}}_{G} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \qquad \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \underbrace{\begin{bmatrix} * & \\ & * \end{bmatrix}}_{K} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Where K is not triangular!

The result of some "usual" control strategies are summarized in Table 1;

(0) -classical case, can choose $u_2 = *\, y_1$

(1) -no input cost

(2) -no output cost

Consider now $u_1 = -y_1 + \sigma sign(y_1)$

$x_1 = \sigma sign(y_1)$

$u_2 = \sigma sign(y_2)$

$\Rightarrow$

$u_1 = -y_1 + \sigma sign(y_1) = -\sigma w_1 + \sigma sign(w_1)$

$u_1^2 = \sigma^2 w_1^2 + \sigma^2 - 2\sigma^2 w_1 sign(w_1) =$

$\sigma^2(w_1^2 + 1) - 2\sigma^2 |w_1|$

$E(u_1^2) = 2\sigma^2 - 2\sigma^2 E(|w_1|)$

We have

$$E(|w_1|) = 2 \cdot \frac{1}{\sqrt{2\pi}} \int_0^\infty x e^{-\frac{x^2}{2}} dx \quad = \quad \sqrt{\frac{2}{\pi}} \Big[-e^{-\frac{x^2}{2}} \Big]_0^\infty \quad = \quad \sqrt{\frac{2}{\pi}}$$

$\Rightarrow$

$E(u_1^2) = 2\sigma^2 \Big(1 - \sqrt{\frac{2}{\pi}}\Big)$

$\backsim 0.4\sigma^2$ (MATLAB)

$$x_2 = \begin{cases} -2\sigma & w_1 < 0, \quad v > 0 \\ 2\sigma & w_1 > 0, \quad v < 0 \\ 0 & else \end{cases}$$

This controller, denoted (3), is compared with some the other control strategies in Table 1, using

$$\begin{cases} k = 0.01 \\ \sigma = 10 \end{cases} \Rightarrow \qquad k\sigma^2 = 1$$

An optimal linear controller arrives at

$$J \backsim 0.9$$

	γ_1	γ_2	$E(u_1^2)$	x_2	$E(x_2^2)$	J
(0) full info	0	y_1	0	0	0	0
(1) No Input Cost	0	y_2	0	v	1	1
(2) No Output Cost	$-y_1$	0	σ^2	0	0	$k\sigma^2 = 1$
(3) M&S	$-y_1 + \sigma sign(y_1)$	$\sigma sign(y_2)$	$\backsim 0.4\sigma^2$	$\{0, \pm 2\sigma\}$	$\backsim 0$	$\backsim 0.4k\sigma^2 \backsim 0.4$

Table 1: Result for different strategies