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A COUNTEREXAMPLE IN STOCHASTIC OPTIMUM CONTROL*

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Abstract. It is sometimes conjectured that nothing is to be gained by using nonlinear controllers when the objective is to minimize the expectation of a quadratic criterion for a linear system subject to Gaussian noise and with unconstrained control variables.

In fact, this statement has only been established for the case where all control variables are generated by a single station which has perfect memory.

Without this qualification the conjecture is false.

1. Introduction. In a stochastic control problem control actions have to be taken at various instants in time as functions of the data then available. One seeks the functions for which the expected value of the cost, under given noise distributions, is minimized. It is usually assumed that all actions to be taken at a given time are based on the same data and that any data available at time t will still be available at any later time $t' > t$. This situation is the *classical information pattern.*

Considering in particular unconstrained control of linear systems with Gaussian noise and quadratic criteria, it is well known that the search for an optimum can safely be confined to the class of affine (linear plus constant) functions [1]. This is the case for both discrete and continuous time systems, with classical information pattern.

In this paper it is shown that the class of affine functions is not always adequate (complete, in decision theory parlance) when the information pattern is not classical.

A counterexample is presented for which it is established that an optimal design exists and that no affine design is optimal. There does not appear to exist any counterexample involving fewer variables than the one presented here.

The practical importance of nonclassical information patterns is discussed.

2. Problem description.

Original Statement. Let x_0 and v be independent random variables with finite second moments. Consider the following 2-stage stochastic control problem. (All variables are real scalars.)

State equations.
$$x_1 = x_0 + u_1,$$
$$x_2 = x_1 - u_2.$$

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Output equations. $y_0 = x_0$,

$y_1 = x_1 + v$.

Cost function. $k^2(u_1)^2 + (x_2)^2, \quad k^2 > 0$.

Admissible controllers. $u_1 = \gamma_1(y_0)$,

$u_2 = \gamma_2(y_1)$,

where (γ_1, γ_2) is any pair of Borel functions. The set of such pairs is designated by Γ.

Objective. For any choice of (γ_1, γ_2) the variables u_1 and x_2 become random variables, and since the cost function is nonnegative it has an expectation that is possibly infinite. The problem is to minimize over Γ the expression $E\{k^2(u_1)^2 + (x_2)^2\}$. The information pattern is nonclassical because the value of y_0 is known at the first control stage but not at the second.

It will be shown that for x_0 and v Gaussian and suitable parameter values the best affine controller is not optimal over Γ.

Restatement. Denoting x_0 by x, γ_2 by g and letting f be defined by $f(x) = x + \gamma_1(x)$, the problem amounts to minimizing, over the set Γ of all pairs of Borel functions (f, g), the expression

$$J(f, g) = E\{k^2(x - f(x))^2 + (f(x) - g(f(x) + v))^2\}, \tag{1}$$

where $k^2 > 0$. Without loss of generality one may assume

$$E\{x\} = E\{v\} = 0, \quad E\{v^2\} = 1. \tag{2}$$

This reduction amounts to ordinate shifts of f and g, abscissa shift of g and rescaling. The case $E\{v^2\} = 0$ is trivial. Problem $\pi(k^2, F)$ is the problem of minimizing (1) with v Gaussian subject to (2) and x having the distribution function F subject to (2) and $0 < E\{x^2\} \equiv \sigma^2 < \infty$. Problem $\pi(k^2, \sigma^2)$ is the special case of problem $\pi(k^2, F)$ with F the Gaussian distribution with zero mean and variance σ^2.

3. Existence of an optimum for problem $\pi(k^2, F)$.

LEMMA 1. (a) $J^* = \inf\{J(f, g) \mid (f, g) \in \Gamma\}$ *satisfies* $0 \leqq J^* \leqq \min(1, k^2\sigma^2)$.

(b) *If* $(f, g) \in \Gamma$, *then there exists* $(f_1, g_1) \in \Gamma$ *such that* $E\{f_1(x)\} = 0$, $E\{(x - f_1(x))^2\} \leqq \sigma^2$, $J(f_1, g_1) \leqq J(f, g)$ *and* $E\{f_1^2(x)\} \leqq 4\sigma^2$.

Proof. (a) For $f \equiv 0$, $g \equiv 0$ one has $J(f, g) = k^2E\{x^2\} = k^2\sigma^2$, while for $f(x) \equiv x$, $g(y) \equiv y$ one has $J(f, g) = E\{v^2\} = 1$.

(b) If $E\{(x - f(x))^2\} > \sigma^2$ so that $J(f, g) \geqq k^2E\{(x - f(x))^2\} > k^2\sigma^2$, then $f_1 \equiv g_1 \equiv 0$ satisfies all requirements. If $E\{(x - f(x))^2\} \leqq \sigma^2$, then $E\{f^2(x)\} \leqq 4\sigma^2$, so that $m = E\{f(x)\}$ exists. Let $f_1(x) \equiv f(x) - m$, $g_1(y) \equiv g(y + m) - m$. Then $E\{f_1(x)\} = 0$, $E\{(x - f_1(x))^2\} = E\{(x - f(x))^2\} - m^2 \leqq \sigma^2$, hence $E\{f_1^2(x)\} \leqq 4\sigma^2$ and $J(f_1, g_1) = J(f, g) - k^2m^2 \leqq J(f, g)$.

Hence one need only consider pairs (f, g) for which $f(x)$ has zero mean and variance not exceeding $4\sigma^2$. For such f we now select $g = g_f^*$ to minimize $J(f, g)$ for fixed f.

With $\varphi(x) \equiv (2\pi e^{x^2})^{-1/2}$ define

$$D_f(y) = \int \varphi(y - f(x))\, dF(x),$$

$$N_f(y) = \int f(x)\varphi(y - f(x))\, dF(x),$$

$$g_f^*(y) = N_f(y)/D_f(y),$$

$$J_2^*(f) = J(f, g_f^*).$$

First we recall a well-known fact.

LEMMA 2. *Let μ be a measure and h a measurable function. Consider the integral*

$$H(s) = \int_{-\infty}^{+\infty} \varphi(s - t)h(t)\, d\mu(t).$$

Then the set of real values of s for which the integral is finite is convex and H is analytic on the interior of this set.

Proof. Since $\varphi(s - t) = \sqrt{2\pi}\,\varphi(s)e^{st}\varphi(t)$, one can interpret H as

$$H(s) = \sqrt{2\pi}\varphi(s) \int_{-\infty}^{+\infty} e^{st}\varphi(t)h(t)\, d\mu(t).$$

The claim then follows from the properties of convergence strips of two-sided Laplace transforms.

LEMMA 3. *Assume* $E\{f^2(x)\} < \infty$. *Then*

(a) N_f, D_f, g_f^* *are analytic with* $D_f > 0$;

(b) D_f *is a density of the random variable* $y = f(x) + v$;

(c) $g_f^*(y) = E\{f(x) \mid y\}$ *a.s.*;

(d) $J_2^*(f) = \min\{J(f, g) \mid g \text{ Borel}\}$;

(e) $dg_f^*(y)/dy = \operatorname{var}\{f(x) \mid y\} \geqq 0$;

(f) $$\begin{aligned} J_2^*(f) - k^2E\{(x - f(x))^2\} &= E\{\operatorname{var}\{f(x) \mid y\}\} \\ &= E\{f^2(x)\} - E\{g_f^{*2}(y)\} \\ &= 1 - I(D_f), \end{aligned}$$

where

$$I(D_f) = \int \left(\frac{d}{dy} D_f(y)\right)^2 \frac{dy}{D_f(y)}$$

$$= \int \frac{dD_f(y)}{dy} \, d\log D_f(y)$$
$$= 4 \int \left(\frac{d}{dy} \sqrt{D_f(y)} \right)^2 dy$$

is the Fisher information of the random variable y;

(g) $\max(0, 1 - E\{f^2(x)\}) \leqq I(D_f) \leqq 1$, *and for* $E\{(x - f(x))^2\} \leqq \sigma^2$ *one has* $J_2^*(f) \leqq k^2\sigma^2 + \min(1, 4\sigma^2)$.

Proof. (a) For each y the integrands $\varphi(y - z)$ and $z\varphi(y - z)$, with $z = f(x)$, are bounded, hence the integrals defining N_f and D_f exist for all y. By Lemma 2, N_f and D_f are analytic. Since φ is strictly positive, so is D_f, hence g_f^* is analytic.

(b) The joint distribution of y and x is defined by

$$\varphi(y - f(x))\, dy\, dF(x)$$

because the measurable transformation $\binom{x}{v} \leftrightarrow \binom{x}{y}$ with $y = f(x) + v$ is measure preserving by Cavalieri's principle (though a Jacobian does not exist for general f). Hence the marginal distribution of y has density D_f.

(c) Since $f(x)$ has finite second moment, its conditional expectation exists. With the joint distribution of x and v as in (b), (c) is immediate.

(d) This states the quadratic minimization property of expectations.

(e) and (f). These follow by simple manipulations. Note that

$$N_f(y) = yD_f(y) + \frac{d}{dy} D_f(y);$$

hence,

$$g_f^*(y) = y + \frac{d}{dy} \log D_f(y).$$

(g) This follows as in Lemma 1.

The problem is thus reduced to the minimization of

$$J_2^*(f) = k^2E\{(x - f(x))^2\} - I(D_f) + 1$$

over all Borel functions (or only those of zero mean and variance $\leqq 4\sigma^2$).

The designer is trying to find a compromise between (i) keeping the cost of the first stage correction small, and (ii) making the Fisher information of the observation available at the second stage large.

The difficulty is that J_2^* is not a convex functional.

Now for f_0 and $g_{f_0}^*$ as in Lemma 3 we attempt to minimize $J(f, g_{f_0}^*)$ over f for fixed $g_{f_0}^*$.

LEMMA 4. *Let P be the distribution of a real random variable. Let α_1 be the set of all points x for which both $(-\infty, x]$ and $[x, +\infty)$ have positive probability. Let α_2 be the set obtained by removing from the convex hull of the support of P those boundary points which are not atoms. Let α_3 be the intersection of all convex sets of probability one. Then $\alpha_1 = \alpha_2 = \alpha_3 \equiv \alpha(P)$, the smallest convex set of probability one.*

Proof. (i) $\alpha_1 \supset \alpha_2$: If x belongs to the interior of α_2, both $[x, \infty)$ and $(-\infty, x]$ have positive probability. If x is a boundary point of α_2, then it is an atom, hence belongs to α_1.

(ii) $\alpha_2 \supset \alpha_3$: By construction α_2 is a convex set of probability one.

(iii) $\alpha_3 \supset \alpha_1$: If E is a convex set and x a point in α_1 but not in E, then E is disjoint from one of the sets $(-\infty, x]$, $[x, +\infty)$, and thus E has probability less than one. Hence all convex sets of probability one contain α_1.

Note that in two (or more) dimensions the intersection of all convex sets of probability one may have probability zero, because the boundary of a nontrivial convex set is uncountable.

LEMMA 5. *For f and g_f^* as in Lemma 3 let P be the distribution of the random variable $f(x)$. Then the range of g_f^* is contained in $\alpha(P)$.*

Proof. By contradiction suppose that for some y the set $[g(y), \infty)$ (or $(-\infty, g(y)]$) has probability zero under P. Then

$$\begin{aligned} g(y)D_f(y) &= N_f(y) = \int dF(x)f(x)\varphi(y - f(x)) \\ &= \int dP(z)z\varphi(y - z) \\ &= \int_{(-\infty, g(y))} dP(z)z\varphi(y - z) \\ &< g(y) \int_{(-\infty, g(y))} dP(z)\varphi(y - z) \\ &= g(y)D_f(y), \end{aligned}$$

which is a contradiction.

LEMMA 6. *For f_0 and $g_{f_0}^*$ as in Lemma 3, fixed, one has*

$$J(f, g_{f_0}^*) = \int dF(x)[k^2(x - f(x))^2 + K(f(x))],$$

where K is a nonnegative analytic function.

Proof. To shorten notation let $g = g_{f_0}^*$ and let P be the distribution of the random variable $f_0(x)$. One has

$$\begin{aligned} J(f, g) &= \int dF(x) \left[k^2(x - f(x))^2 + \int dv\varphi(v)(f(x) - g(f(x) + v))^2 \right] \\ &= \int dF(x)[k^2(x - f(x))^2 + K(f(x))] \end{aligned}$$

with

$$K(z) = \int dv\varphi(v)(z - g(z+v))^2$$

$$= \int dy\varphi(y-z)(z-g(y))^2.$$

Since the integrands are nonnegative, the above formulas are valid whether the integrals are finite or not. Let β be the set $\{z \mid K(z) < \infty\}$. Because of the inequalities

$$g^2(y) \leqq 2z^2 + 2(z - g(y))^2,$$
$$(z - g(y))^2 \leqq 2z^2 + 2g^2(y),$$

the set β coincides with the set of all z for which

$$\int dy\varphi(y-z)g^2(y) < \infty.$$

By Lemma 2 the set β is thus convex. By construction of g,

$$J_2^*(f_0) = J(f_0, g) < \infty,$$

and therefore,

$$\int dF(x)K(f_0(x)) = \int dP(z)K(z) < \infty.$$

Hence the set β has probability one under P. Since it is convex, β contains $\alpha(P)$ defined in Lemma 4 and, by Lemma 5, $\alpha(P)$ contains the range of g. Also by Lemma 3(e) g is monotone nondecreasing.

This author claims that $\beta = (-\infty, +\infty)$; indeed otherwise by convexity of β at least one of the inequalities $-\infty < \inf \beta$, $\sup \beta < \infty$ holds. If both hold, g is bounded which implies $\beta = (-\infty, +\infty)$. If $\inf \beta = -\infty$, $\sup \beta < \infty$, then

$$\int_{-\infty}^{0} dy\varphi(y-z)g^2(y) = \sqrt{2\pi}\varphi(z) \int_{-\infty}^{0} dye^{yz}\varphi(y)g^2(y)$$

converges for $z < \sup \beta$ by the assumption on β and a fortiori for $z \geqq \sup \beta$. But, for all z,

$$\int_0^{\infty} dy\varphi(y-z)g^2(y) < \infty$$

because for $y > 0$, g is bounded, according to Lemmas 3(e) and 5, by

$$g(0) \leqq g(y) \leqq \sup \beta.$$

Hence $\beta = (-\infty, +\infty)$ and a symmetric argument applies for $-\infty < \inf \beta$, $\sup \beta = \infty$.

In conclusion, $\int dy\varphi(y-z)g^2(y)$ is finite for all z, and a fortiori $\int dy\varphi(y-z)g(y)$ is finite for all z. By Lemma 2, both these integrals are analytic in z. Therefore,

$$K(z) = z^2 - 2z\int dy\varphi(y-z)g(y) + \int dy\varphi(y-z)g^2(y)$$

is analytic.

LEMMA 7. *For $E\{(f_0(x) - x)^2\} \leqq \sigma^2$ and $g = g_{f_0}^*$ as in Lemma 3, there exists a function f^*, monotone nondecreasing on $\alpha(F)$, such that*

(a) $J(f^*, g) = \min\{J(f, g) \mid f \text{ Borel}\}$,

(b) $|f^*(x)| < c(x)$ *for x in $\alpha(F)$, where the real-valued function c depends only on F and k^2, not on f_0.*

Proof. For each x the function

$$V_x(z) = k^2(x-z)^2 + K(z)$$

is nonnegative, continuous (by Lemma 6) and radially unbounded (because $K \geqq 0$). Hence it attains its minimum on a nonempty compact set. For each x define $f^*(x)$ as one of the minimizing values of z (e.g., the largest). Then for any x and x',

$$V_x(f^*(x)) \leqq V_x(f^*(x'))$$

and

$$V_{x'}(f^*(x')) \leqq V_{x'}(f^*(x)).$$

Adding these inequalities gives

$$(x - x')(f^*(x) - f^*(x')) \geqq 0.$$

Hence the function f^* is monotone nondecreasing and a fortiori Borel.

$$V_x(f^*(x)) \leqq V_x(f(x))$$

for all x in $\alpha(F)$, which implies

$$\int dF(x)V_x(f^*(x)) \leqq \int dF(x)V_x(f(x))$$

or

$$J(f^*, g) \leqq J(f, g),$$

so that f^* is optimal for fixed g. In particular, $J(f^*, g) \leqq J(f_0, g) = J_2^*(f_0) \leqq k^2\sigma^2 + \min(1, 4\sigma^2)$, a constant independent of f_0.

Hence $E\{f^{*2}(x)\} \leqq a$, where a is a constant. Then for $x \in \alpha(F)$,

$$-\left(\frac{a}{F((-\infty, x])}\right)^{1/2} \leqq f^*(x) \leqq \left(\frac{a}{F([x, +\infty))}\right)^{1/2}.$$

Indeed, if $f^*(x) > (a/F([x, +\infty)))^{1/2}$, then

$$\int dF(\xi) f^{*2}(\xi) \geqq \int_{[x,+\infty)} dF(\xi) f^{*2}(\xi)$$

$$> \frac{a}{F([x, +\infty))} \int_{[x,+\infty)} dF(\xi) = a,$$

and similarly for the lower bound.

One also needs a special form of Helly's selection theorem.

LEMMA 8. *Let S be a convex set of reals and f_n a sequence of monotone nondecreasing functions on S. Assume that, for all n and all x in S, $|f_n(x)| \leqq c(x) < \infty$. Then there exists a subsequence which converges pointwise on S to a monotone nondecreasing function f.*

Proof. Because at each x in S the numerical sequence $f_n(x)$ is bounded, there exists a subsequence converging at that value of x. Given a countable subset of S a subsequence converging on it can be formed by the diagonal process. Let S_0 be the set of rational points in S. It is countable, hence we may assume that f_n is a subsequence converging on S_0, reindexed. Then $\limsup f_n(x)$ is a monotone nondecreasing function to which, by monotony, the sequence f_n converges at all points of continuity interior to S. Since the points of discontinuity of a monotone function are countable and the number of boundary points of S belonging to S is at most two, a second application of the diagonal process yields a subsequence converging on S.

THEOREM 1. *For any $k^2 > 0$ and any distribution F the problem $\pi(k^2, F)$ has an optimal solution.*

Proof. Let $(f_n^{(0)}, g_n^{(0)})$ be a minimizing sequence in Γ, that is,

$$\lim_{n\to\infty} J(f_n^{(0)}, g_n^{(0)}) = J^* = \inf \{J(f, g) \mid f, g \text{ Borel}\}.$$

Observe that $J(f, g)$ depends only upon f through its restriction to $\alpha(F)$. Henceforth we shall only consider functions f so restricted. Observe also that when the construction of Lemma 1(b) is applied to a pair (f, g), where f is monotone on $\alpha(F)$, the resulting function f_1 is monotone on $\alpha(F)$.

For each value of n replace $(f_n^{(0)}, g_n^{(0)})$ by $(f_n^{(1)}, g_n^{(1)})$ according to Lemma 1(b). Then replace by $(f_n^{(1)}, g_n^{(2)})$ with $g_n^{(2)} = g^*_{f_n^{(1)}}$ according to Lemma 3. Then replace by $(f_n^{(2)}, g_n^{(2)})$, where $f_n^{(2)}$ is optimal versus $g_n^{(2)}$ and monotone as by Lemma 7. Then replace by $(f_n, g_n^{(3)})$ according to Lemma 1(b) noting that f_n is still monotone. Then replace by (f_n, g_n), where $g_n = g^*_{f_n}$ according to Lemma 3.

Then

$$J(f_n, g_n) = J_2^*(f_n) \leqq J(f_n, g_n^{(3)}) \leqq J(f_n^{(2)}, g_n^{(2)})$$
$$\leqq J(f_n^{(1)}, g_n^{(2)}) \leqq J(f_n^{(1)}, g_n^{(1)}) \leqq J(f_n^{(0)}, g_n^{(0)}),$$

and therefore the sequence (f_n, g_n) is a fortiori a minimizing sequence, that is,

$$\lim J_2^*(f_n) = J^*.$$

By Lemmas 7 and 8 there exists a subsequence f_{n_k} converging to a limit f pointwise on $\alpha(F)$. Relabel (f_{n_k}, g_{n_k}) as (f_n, g_n). By Fatou's lemma,

$$E\{f^2(x)\} \leqq \liminf E\{f_n^2(x)\} \leqq 4\sigma^2.$$

Let $g = g_f^* = N_f/D_f$. For each y the functions $\varphi(y - z)$ and $z\varphi(y - z)$ are bounded functions of z, and since φ is continuous,

$$\varphi(y - f_n(x)) \to \varphi(y - f(x)),$$

$$f_n(x)\varphi(y - f_n(x)) \to f(x)\varphi(y - f(x)),$$

pointwise in x, for all y.

By the bounded convergence theorem,

$$D_{f_n}(y) = \int_{\alpha(F)} dF(x)\varphi(y - f_n(x)) \to D_f(y) > 0$$

for each y, and similarly,

$$N_{f_n}(y) \to N_f(y).$$

Hence $g_n(y) \to g(y)$ pointwise.

For all x in $\alpha(F)$ and all y the nonnegative expression

$$A_n(x, y) = [k^2(x - f_n(x))^2 + (f_n(x) - g_n(y))^2]\varphi(y - f_n(x))$$

converges to

$$A(x, y) = [k^2(x - f(x))^2 + (f(x) - g(y))^2]\varphi(y - f(x)).$$

By Fatou's lemma,

$$\int_{\alpha(F)} dF(x) \int dy A(x, y) \leqq \liminf_{n\to\infty} \int_{\alpha(F)} dF(x) \int dy A_n(x, y)$$

or

$$J(f, g) \leqq \liminf_{n\to\infty} J(f_n, g_n) = J^*.$$

But, by definition, $J^* \leqq J(f, g)$, hence $J^* = J(f, g)$ and the pair (f, g) is optimal. (Define f as zero outside $\alpha(F)$.)

Note that when $\alpha(F)$ has a (say upper) boundary point b not belonging to $\alpha(F)$ (because b is not an atom), then the function $c(x)$ of Lemma 7 approaches ∞ as $x \to b$ and, in consequence, the function f of Theorem 1 may approach ∞ as $x \to b$. Then a *monotone* real-valued extension of f to $(-\infty, \infty)$ does not exist.

Taking the first variation of J_2^* gives, formally,

$$\delta J_2^*(f) = \int dF(x) G_f(x) \delta f(x),$$

where

$$G_f(x) = 2k^2(f(x) - x)$$
$$+ \int dy \varphi(y - f(x)) \frac{D_f'(y)}{D_f(y)} \left[2(y - f(x))^2 + \frac{D_f'(y)}{D_f(y)} (y - f(x)) - 2 \right]$$

with

$$D_f'(y) = \frac{d}{dy} D_f(y) = N_f(y) - y D_f(y).$$

Hence one has the following necessary condition.

Lemma 9. *If f is optimal, then $E\{f(x)\} = 0$, $E\{f^2(x)\} \leq 4\sigma^2$, and $G_f(x) = 0$ F-almost surely, provided the formal differentiation holds at least in the sense of Gâteaux for δf in $L_\infty[(-\infty, \infty), F]$.*

This condition is of little use because there are in general many local minima of $J_2^*(f)$. Steepest descent in function space can be used to improve a suboptimal solution but not, safely, to find an absolute optimum.

An alternative existence proof can be based on a generalization of Theorem 378 of Hardy, Littlewood and Pólya [2]. All functions f which give the same distribution to $f(x)$ also give the same optimal cost $1 - I(D_f)$ for the second stage. According to the theorem in question, among all these "equimeasurable" functions, the monotone nondecreasing rearrangement maximizes $E\{xf(x)\}$, hence minimizes $E\{(f(x) - x)^2\}$. This establishes the existence of a minimizing sequence $(f_n, g_{f_n}^*)$ with monotone f_n.

4. Optimization of $\pi(k^2, \sigma^2)$ over the affine class. For problem $\pi(k^2, \sigma^2)$ let

$$J_a^* = \inf \{J(f, g) \mid f, g \text{ affine}\}.$$

Observe that the transformation of (f, g) into (f_1, g_1) in Lemma 1(b) maps the class of affine pairs into itself. Hence one need only consider $E\{f(x)\} = 0$ or

$$f(x) = \lambda x.$$

For such f,

$$g_f^*(y) = \mu y$$

with

$$\mu = \frac{\sigma^2\lambda^2}{1 + \sigma^2\lambda^2}$$

and

$$J_2^*(f) = J_{2a}^*(\lambda) = k^2\sigma^2(1 - \lambda)^2 + \frac{\lambda^2\sigma^2}{1 + \lambda^2\sigma^2}.$$

This expression being nonnegative, analytic and radially unbounded, optimal values of λ exist and are stationary points of J_{2a}^*.

Lemma 11. *Optimal affine solutions exist and are of the form $f(x) = \lambda x$, $g(y) = \mu y$, where*

$$\mu = \frac{\sigma^2\lambda^2}{1 + \sigma^2\lambda^2},$$

and $t = \sigma\lambda$ is a real root of the equation

$$(t - \sigma)(1 + t^2)^2 + \frac{1}{k^2} t = 0.$$

Proof. Set $dJ_{2a}^*(\lambda)/d\lambda = 0$.

A great deal of insight is gained by interpreting graphically the condition of Lemma 11. It may be written

$$\frac{t}{(1 + t^2)^2} = k^2(\sigma - t).$$

Hence the stationary points are the abscissas of the points of intersection of the curve

$$s = \frac{t}{(1 + t^2)^2}$$

with the line

$$s = k^2(\sigma - t).$$

The curve is odd and positive for $t > 0$. Since σ and k^2 are positive, all solutions are positive. The curve has a maximum at $t = \sqrt{3}/3$ with value $3\sqrt{3}/16$ and then decays asymptotically to zero with an inflection at $t = 1$, where the value is $\frac{1}{4}$ and the slope $-\frac{1}{4}$.

Hence for $k^2 \geq \frac{1}{4}$ and any σ there is exactly one root which defines a unique optimum.

For $k^2 < \frac{1}{4}$ and σ sufficiently small there is a unique solution with t small. For σ sufficiently large there is a unique solution with t large. For intermediate values of σ there are 3 solutions corresponding to two local minima of J^*_{2a} separated by a local maximum. There is a value σ_c of σ for which the two local minima are equal, hence both optimal. For $\sigma < \sigma_c$ the lowest root is optimal; for $\sigma > \sigma_c$ the largest root is optimal. Hence for fixed $k^2 < \frac{1}{4}$ the plot of the optimal λ versus σ has a jump at σ_c, though J_a^* is continuous in σ. At, and only at, the jump there are two optimal solutions.

LEMMA 12. *For $k^2 < \frac{1}{4}$ the critical value of σ is $\sigma_c = k^{-1}$. At this value the two optimal solutions are $\lambda = \mu = \frac{1}{2}(1 \pm \sqrt{1 - 4k^2})$, both of which yield $J_a^* = 1 - k^2$.*

Proof. Let $k^2\sigma^2 = 1$, $k^2 < \frac{1}{4}$. Then the stationarity condition is $(t - \sigma)(1 + t^2)^2 + \sigma^2 t = 0$ and can be factored into

$$(t^2 - \sigma t + 1)(t^3 + t - \sigma) = 0.$$

Since the two roots $t = \frac{1}{2}(\sigma \pm \sqrt{\sigma^2 - 4})$ give the same value $1 - k^2$ to J^*_{2a}, they are the two local minima, and the real root of the cubic is the intermediate local maximum. Hence $k^2\sigma^2 = 1$ is the critical condition.

Note that for $k^2\sigma^2 = 1$, $k^2 = \frac{1}{4}$, there is a triple root at the inflection point, and for $k^2\sigma^2 = 1$, $k^2 > \frac{1}{4}$, the only real root is that of the cubic and this is then the optimum.

LEMMA 13. *If a design is optimal in the affine class, it is optimal in the class of all pairs of Borel functions (f, g) of which at least one is affine.*

Proof. If either f or g is affine and fixed, the determination of an optimal choice for the other function is a Gaussian-linear-quadratic single-stage problem with classical information pattern, hence it is an affine function.

Clearly this lemma holds in far more general problems with "at least one" replaced by "all but at most one."

LEMMA 14. *If $f(x) = \lambda x$ and $g(y) = \sigma^2\lambda^2 y/(1 + \sigma^2\lambda^2)$ is stationary (in particular, optimal) over the affine class, then it satisfies the formal conditions of Lemma 9.*

Proof. With $f(x) = \lambda x$, substitution yields

$$G_f(x) = 2\left(k^2(\lambda - 1) + \frac{\lambda}{(1 + \lambda^2\sigma^2)^2}\right) x,$$

which vanishes by the stationarity condition of Lemma 11.

Despite the facts stated in Lemmas 13 and 14, we shall find that $J^* < J_a^*$ is possible.

5. Two-point symmetric distributions. Consider problem $\pi(k^2, F)$ for F the two-point symmetric distribution assigning probability $\frac{1}{2}$ to $x = \sigma > 0$ and $x = -\sigma$.

Let $a = f(\sigma)$. For optimization we may assume by Lemma 1(b) that $f(-\sigma) = -a$ and by Lemma 7 that $a \geqq 0$.

The first stage cost is thus $k^2(a - \sigma)^2$. At the second stage,

$$D_f(y) = \tfrac{1}{2}(\varphi(y - a) + \varphi(y + a))$$
$$= \sqrt{2\pi}\,\varphi(a)\varphi(y) \cosh ay.$$

Hence,

$$\frac{D_f'(y)}{D_f(y)} = -y + a \tanh ay$$

and

$$g_f^*(y) = a \tanh ay,$$
$$g_f^{*2}(y) = a^2 - a^2 \operatorname{sech}^2 ay,$$
$$E\{g_f^{*2}(y)\} = a^2 - h(a),$$

where

$$h(a) = \sqrt{2\pi}\, a^2\varphi(a) \int \frac{\varphi(y)}{\cosh ay}\, dy.$$

Thus,

$$J_2^*(f) = k^2E\{(x - f(x))^2\} + E\{f^2(x)\} - E\{g_f^{*2}(y)\}$$
$$= k^2(a - \sigma)^2 + h(a).$$

This is a radially unbounded analytic function of a, and therefore attains a minimum $J^* = V_k(\sigma)$ at one or more optimal values of a. Any optimal value must satisfy the transcendental equation

$$k^2(\sigma - a) = -\tfrac{1}{2}h'(a).$$

A plot of $-\frac{1}{2}h'(\sigma)$ is similar in shape to the plot of $t/(1 + t^2)^2$ which occurred in the optimization of the Gaussian case over the affine class. Hence the qualitative discussion of that case applies also in the present instance. The possible appearance of two local minima has now a simple interpretation. For small k^2 and appropriate σ one policy is to bring a close to zero by means of f so that the second stage will have little work to do; another policy is to make a larger than σ, creating a vast gap between a and $-a$, so that the second stage can almost infallibly separate these two values.

In summary one has the following lemma.

LEMMA 15. *When F is the two-point symmetric distribution with variance σ^2, then the design $f(x) = (a/\sigma)x$, $g(y) = a \tanh ay$ is optimal for an appropriate constant a which gives the minimum in the formula*

$$J^* = V_k(\sigma) \equiv \min_a [k^2(a - \sigma)^2 + h(a)].$$

Note that the functions $h(a)$, $h'(a)$ and $V_k(\sigma)$ can be obtained by computer programs with relative ease. Note also that, for the general problem $\pi(k^2, F)$, whenever $f(x)$ has a two-point symmetric distribution with the values $\pm a$, then the minimum over g of $E\{(f(x) - g(f(x) + v))^2\}$ is $h(a)$.

When $a \gg 1$ (the variance of the noise v), the second stage cost should be close to zero. More precisely one has the following lemma.

LEMMA 16. *The function* $h(a)$ *is bounded by* $\sqrt{2\pi}\, a^2\varphi(a) = a^2 e^{-a^2/2}$.

Proof.

$$\int \frac{\varphi(y)}{\cosh ay}\, dy \leqq \int \varphi(y)\, dy = 1.$$

6. Nonlinear design for the Gaussian case.

THEOREM 2. *There exist values of the parameters* k *and* σ *for problem* $\pi(k^2, \sigma^2)$ *such that* J^*, *the optimal cost, is less than* J_a^*, *the optimal cost achievable in the class of affine designs.*

Proof. Consider the design

$$f(x) = \sigma \operatorname{sgn} x, \quad g(y) = \sigma \tanh \sigma y.$$

For this choice $f(x)$ has a two-point symmetric distribution and $g = g_f^*$. Then

$$J(f, g) = k^2 E\{(x - \sigma \operatorname{sgn} x)^2\} + h(\sigma),$$

where h is the function defined in §5.

The first term is readily evaluated to be

$$2k^2\sigma^2\left(1 - E\left\{\left|\frac{x}{\sigma}\right|\right\}\right) = 2k^2\sigma^2\left(1 - \sqrt{\frac{2}{\pi}}\right).$$

For $k^2\sigma^2 = 1$, by Lemma 16,

$$J(f, g) \leqq 2\left(1 - \sqrt{\frac{2}{\pi}}\right) + \sqrt{2\pi}\,\frac{1}{k^2}\,\varphi\left(\frac{1}{k}\right).$$

As $k \to 0$, the right-hand side approaches $2(1 - \sqrt{2/\pi}) = 0.404230878$, while by Lemma 12, J_a^* approaches 1.

Hence, for small k^2, $J^* \leqq J(f, g) < J_a^*$.

The design of Theorem 2 is far from optimal. Lower values of $J(f, g_f^*)$ for $k^2\sigma^2 = 1$, k^2 small, are obtained by starting with f a $(2n + 1)$-level quantization and then improving this choice by the gradient method in function space.

The optimum, which exists by Theorem 1, is not known.

Computer experimentation suggests that the functional J_2^* has a large (possibly infinite) number of stationary points.

7. A lower bound for the Gaussian case. Since only suboptimal designs for the Gaussian case were found in §6 and these give only upper bounds on J^*, it may be useful to have a loose but positive lower bound on J^*.

Let ξ, u, v be independent random variables: ξ, v Gaussian of zero mean and variances σ^2, 1; u taking the values $+1$ and -1 with probability $\frac{1}{2}$.

Let J_3^* be the infimum, over all pairs (f, g) of Borel functions of two variables, of the expression

$$J_3(f, g) = E\{k^2(u\xi - f(u\xi, \xi))^2 + (f(u\xi, \xi) - g(f(u\xi, \xi) + v, \xi))^2\}.$$

Let $x = u\xi$ and $y = f(u\xi, \xi) + v$; then x is a Gaussian random variable independent of v and distributed like ξ.

Hence for any pair (f_1, g_1) of Borel functions of one variable, the choice

$$f(x, \xi) = f_1(x), \quad g(y, \xi) = g_1(y)$$

is a possible design, for which

$$J_3(f, g) = J(f_1, g_1),$$

where J is the cost functional of problem $\pi(k^2, \sigma^2)$. Hence $J_3^* \leqq J^*$. But

$$J_3(f, g) = E\{E\{\text{expression} \mid \xi\}\},$$

and for fixed ξ the minimization of the conditional expectation is the problem of §5 with the variable σ of that section having the value ξ. Hence for all pairs (f, g) the conditional expectation is, almost surely in ξ, bounded from below by the function $V_k(\xi)$ defined in Lemma 15. This establishes the next theorem.

THEOREM 3. *For problem* $\pi(k^2, \sigma^2)$ *one has*

$$J^* \geqq \frac{1}{\sigma}\int d\xi\varphi(\xi/\sigma)V_k(\xi).$$

Since V_k can be obtained by computer, this bound can be evaluated for any k and σ.

Theorem 3 may be considered a special case of the following observation. Suppose the expected cost, in a stochastic optimization problem with nonnegative cost function, is considered as a function of the design γ and of the distribution F of some of the noise variables. Suppose that the conditional distribution of the other noise variables, given those described by F, is fixed, for instance, because they are independent. Let $K(\gamma, F)$ be this function, with values in $[0, +\infty]$. Then for each γ, K is a linear function of F on the set on which it is finite and is $+\infty$ elsewhere. Therefore $K^*(F) = \inf_\gamma K(\gamma, F)$ satisfies, for all distributions F_1, F_2 and $0 < \theta < 1$, the extended-real number inequality

$$K^*(\theta F_1 + (1 - \theta)F_2) \geqq \theta K^*(F_1) + (1 - \theta)K^*(F_2).$$

In other words, K^* is concave in the extended-real sense. If F is construed as a mixture of distributions F_α under some distribution of α, then by the concavity of K^*, the expectation of $K^*(F_\alpha)$ under α is a lower bound on $K^*(F)$.

In Theorem 3, α is the Gaussian random variable ξ and F_α is the two-point symmetric distribution supported on $\pm\xi$.

8. Physical situations leading to nonclassical information patterns. (a) Nonclassical patterns arise when the controller memory is limited. In particular, one may want to determine an optimal zero-memory controller, that is, one for which each control action depends only upon the most recent output [3].

(b) Whenever the physical system to be controlled is of large size or comprises mobile subsystems, nonclassical patterns appear. Indeed control is then effected from several stations widely separated and in relative motion. Hence the actions applied at a given time-stage by the stations are not based all on the same data, even when each station has perfect memory. Communication links between stations are subject to delay, noise and operating costs. These links should be considered as part of the controlled system and the communication policy as part of the control policy. The nonclassical effects are most likely to be of practical import in such cases, as for control of space missions, air traffic or high-speed ground transportation.

(c) When communications problems are considered as control problems (which they are), the information pattern is never classical since at least two stations, not having access to the same data, are always involved.

If one considers the transmission of Gaussian signals over Gaussian channels with quadratic (power and distortion) criteria, then there is a possibility, in complex cases such as with noisy feedback channels, etc., that the optimum "controller" (i.e., modulator or coder) not be affine.

9. Conclusions. (i) Further study of linear, Gaussian, quadratic control problems with general information patterns appears to be required.

(ii) The existence of an optimum and the question of completeness of the class of affine designs must be examined as a function of the information pattern.

(iii) It would be interesting if a relation could be found between the appearance of several local minima over the affine class and lack of completeness of this class.

(iv) Algorithms for approaching an optimal solution need to be developed. Because of the occurrence of local minima, this appears to be a most difficult task.

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