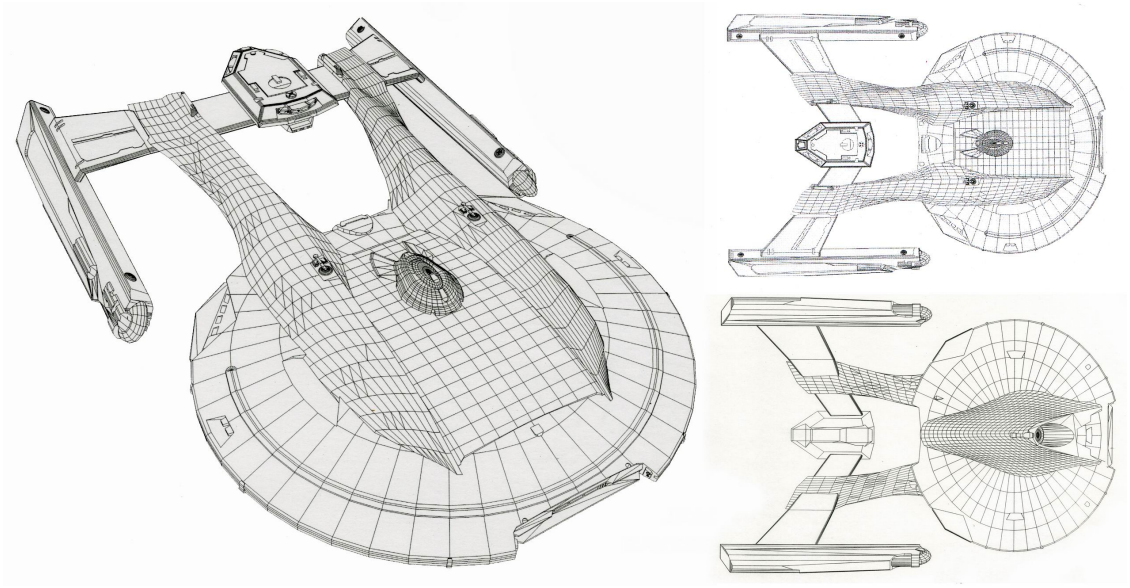
# Computer Graphics

Today we'll study the mathematics used to create and manipulate computer graphics.

The computer graphics seen in movies and videogames works in three stages:

1. A 3D model of the scene objects is created;
2. The model is converted into (many small) polygons in 3D that approximate the surfaces of the model; and
3. The polygons are transformed via a linear transformation to yield a 2D representation that can be shown on a flat screen.

There is interesting mathematics in each stage, but the transformations that take place in the third stage are **linear**, and that's what we'll study today.



Initially, object models may be expressed in terms of smooth functions like polynomials.

However the first step is to convert those smooth functions into a set of discrete pieces – coordinates and line segments.

All subsequent processing is done in terms of the discrete coordinates that approximate the shape of the original model.

The reason for this conversion is that most transformations needed in graphics are *linear.*

Expressing the scene in terms of coordinates is equivalent to expressing it in terms of vectors, eg, in $\mathbb{R}^3$.

And linear transformations on vectors are always matrix multiplications, so implementation is simple and uniform.

The resulting representation consists of lists of 3D coordinates called *faces.* Each face is a polygon.

The lines drawn between coordinates are implied by the way that coordinates are grouped into faces.

**Example.**

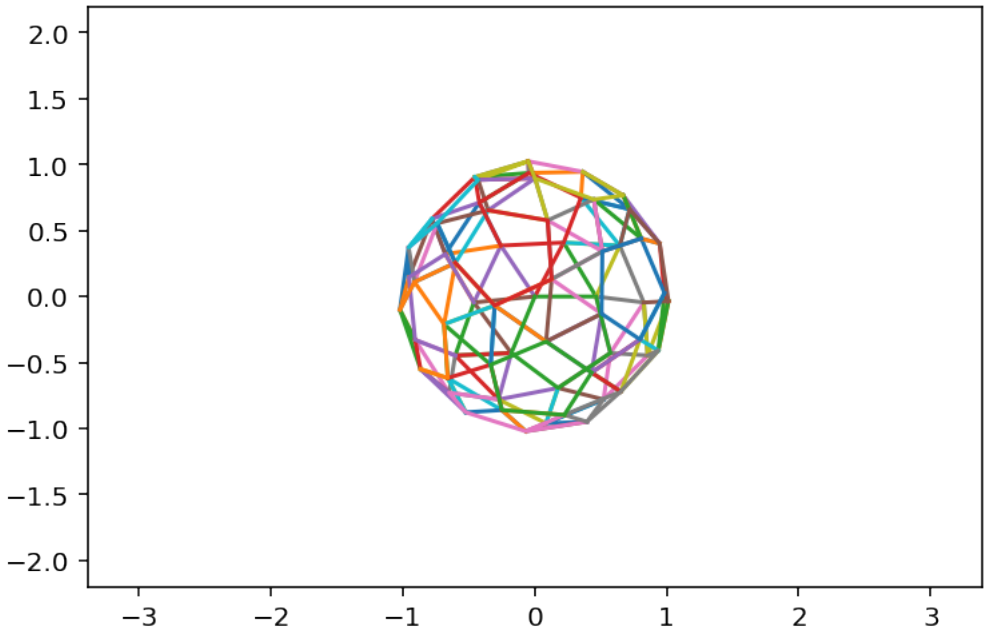
Here is a view of a ball-like object. It is centered at the origin.

The ball is represented in terms of 3D coordinates, but we are only plotting the $x$ and $y$ coordinates here.

Colors correspond to faces.

```
[3]: #
     # load ball wireframe
     import obj2clist as obj
     with open('snub_icosidodecahedron.wrl','r') as fp:
```

```
    ball = obj.wrl2flist(fp)
#
# set up view
fig = plt.figure()
ax = plt.axes(xlim=(-5,5),ylim=(-5,5))
plt.plot(-2,-2,'')
plt.plot(2,2,'')
plt.axis('equal')
#
# plot the ball
for b in ball:
    ax.plot(b[0],b[1])
```




Imagine that we want to circle the camera around the ball. This is equivalent to rotating the ball around the $y$ axis.

This is a linear transformation. We can implement it by multiplying the coordinates of the ball by a *rotation matrix*. To define the rotation matrix, we need to think about what happens to each of the columns of $I$ : $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$.

To rotate through an angle of $\alpha$ radians around the $y$ axis, the vector $\mathbf{e_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ goes to $\begin{bmatrix} \cos\alpha \\ 0 \\ \sin\alpha \end{bmatrix}$.

Of course, $\mathbf{e_2}$ is unchanged.

And $\mathbf{e_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ goes to $\begin{bmatrix} -\sin\alpha \\ 0 \\ \cos\alpha \end{bmatrix}$.

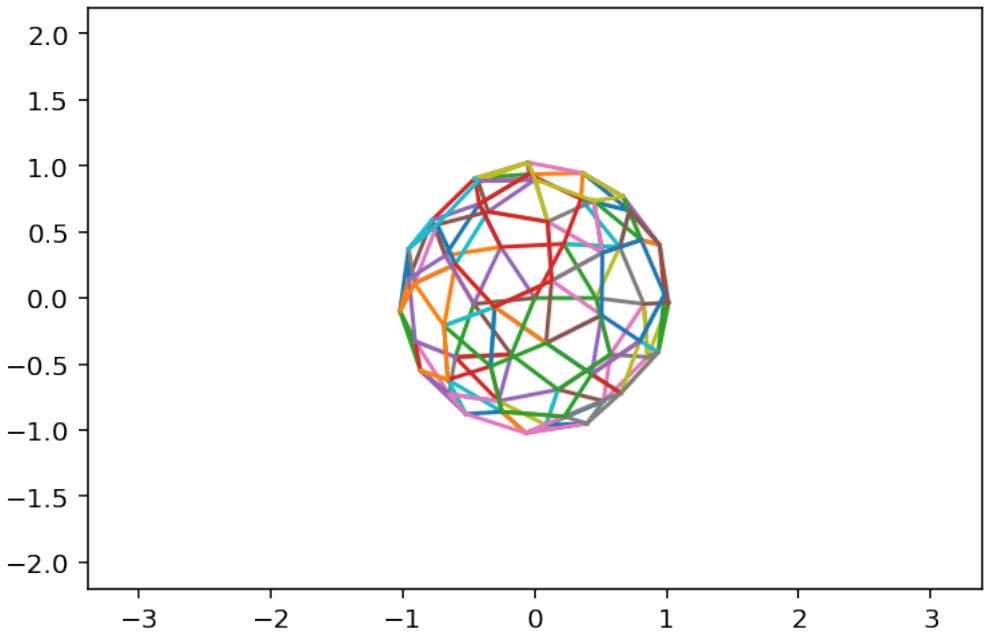
So the entire rotation matrix is:

$$\begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix}.$$

[4]:
```
# set up view
import matplotlib.animation as animation
mp.rcParams['animation.html'] = 'jshtml'

fig = plt.figure()
ax = plt.axes(xlim=(-5,5),ylim=(-5,5))
plt.plot(-2,-2,'')
plt.plot(2,2,'')
plt.axis('equal')
ballLines = []
for b in ball:
    ballLines += ax.plot([],[])
#
#to get additional args to animate:
#def animate(angle, *fargs):
#    fargs[0].view_init(azim=angle)
def animate(frame):
    angle = 2.0 * np.pi * (frame/100.0)
    rotationMatrix = np.array([[np.cos(angle), 0, -np.sin(angle)],
                               [0,             1,              0],
                               [np.sin(angle), 0, np.cos(angle)]])
    for b,l in zip(ball,ballLines):
        rb = rotationMatrix @ b
        l.set_data(rb[0],rb[1])
    fig.canvas.draw()
#
# create the animation
animation.FuncAnimation(fig, animate,
                        frames=np.arange(0,100,1),
                        fargs=None,
                        interval=100,
                        repeat=False)
```

```
<matplotlib.animation.FuncAnimation at 0x7fccd8bd5350>
```




## Translation

Manipulating graphics objects using matrix multiplication is very convenient.

However, there is a common operation that is **not** a linear transformation: *translation*, aka motion.

The standard way to avoid this difficulty is to use what are called **homogeneous coordinates.**

We identify each point $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$ with a corresponding point $\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \in \mathbb{R}^4$.

The coordinates of the point in $\mathbb{R}^4$ are the homogeneous coordinates for the point in $\mathbb{R}^3$.

The extra component gives us a constant that we can scale and add to the other coordinates, as needed, via matrix multiplication.

This means for 3D graphics, all transformation matrices are $4 \times 4$.

**Example.**

Let's say we want to move a point $(x, y, z)$ to location $(x + h, y + k, z + m)$.

We represent the point in homogeneous coordinates as $\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$.

The transformation corresponding to this 'translation' is:

$$\begin{bmatrix} 1 & 0 & 0 & h \\ 0 & 1 & 0 & k \\ 0 & 0 & 1 & m \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x+h \\ y+k \\ z+m \\ 1 \end{bmatrix}.$$

If we only consider $x$, $y$, and $z$ this is not a linear transformation. But of course, in $\mathbb{R}^4$ this most definitely **is** a linear transformation.

We have 'sheared' the fourth dimension, which affects the other three. A very useful trick!

**Constructing Matrices for Homogeneous Coordinates**

For any transformation $A$ that is linear in $\mathbb{R}^3$ (such as scaling, rotation, reflection, shearing, etc.), we can construct the corresponding matrix for homogeneous coordinates quite simply:

If

$$A = \begin{bmatrix} \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare \end{bmatrix}$$

Then the corresponding transformation for homogeneous coordinates is:

$$\begin{bmatrix} \blacksquare & \blacksquare & \blacksquare & 0 \\ \blacksquare & \blacksquare & \blacksquare & 0 \\ \blacksquare & \blacksquare & \blacksquare & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In other words, when performing a linear transformation on $x, y$, and $z$, one simply 'carries along' the extra coordinate without modifying it.

## Perspective Projections

There is another nonlinear transformation that is important in computer graphics: perspective.

Happily, we will see that homogeneous coordinates allow us to capture this too as a linear transformation in $\mathbb{R}^4$.

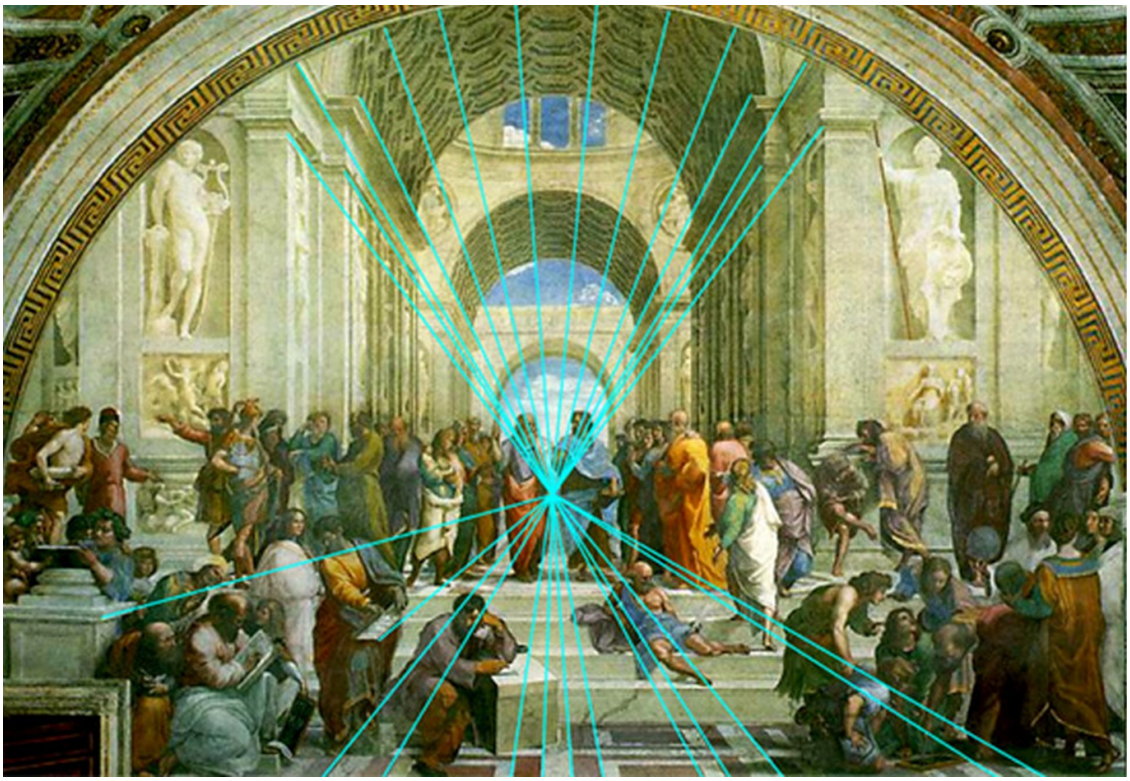
The eye, or a camera, captures light (essentially) in a single location, and hence gathers light rays that are converging.

So, to portray a scene with realistic appearance, it is necessary to reproduce this effect.

The effect can be thought of as "nearer objects are larger than further objects."

This was (re)discovered by Renaissance artists (supposedly first by Filippo Brunelleschi, around 1405).

Here is a famous example: Raphael's *School of Athens* (1510).



We now understand that this effect interacts in a powerful way with neural circuitry in our brains.

The mind reconstructs a sense of three dimensions from the two dimensional information presented to it by the retina.

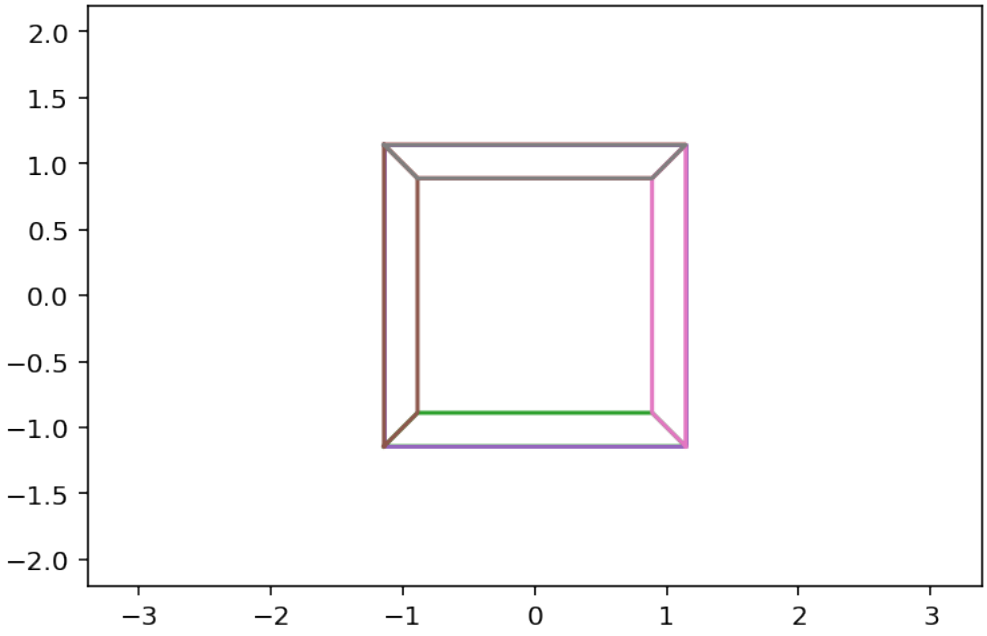
This is done by sophisticated processing in the visual cortex, which is a fairly large portion of the brain.

[7]:
```
# set up view
import matplotlib.animation as animation
mp.rcParams['animation.html'] = 'jshtml'

fig = plt.figure()
ax = plt.axes(xlim=(-5,5),ylim=(-5,5))
plt.plot(-2,-2,'')
plt.plot(2,2,'')
plt.axis('equal')
with open('cube.obj','r') as fp:
    cube = obj.obj2flist(fp)
cube = obj.homogenize(cube)
cubeLines = []
for c in cube:
    cubeLines += ax.plot([],[])
#
def animate(i):
    angle = 2.0 * np.pi * (i/100.0)
    P = np.array([[1.,0,0,0],[0,1.,0,0],[0,0,0,0],[0,0,-1./8,1]]).dot(np.array([[np.cos(angle),0,-n
    for b,l in zip(cube,cubeLines):
        rb = P.dot(b)
        l.set_data(rb[0]/rb[3],rb[1]/rb[3])
```

```
        fig.canvas.draw()
    #
    # create the animation
    animation.FuncAnimation(fig, animate,
                            frames=np.arange(0,100,1),
                            fargs=None,
                            interval=100,
                            repeat=False)
```

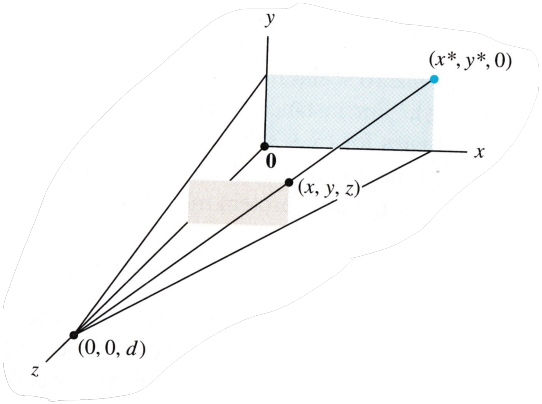
```
<matplotlib.animation.FuncAnimation at 0x7fccc8c76c50>
```




Notice that when the image is stationary, it appears flat (like a picture frame). As soon as it starts to move, it springs into 3D in perception.

**Computing Perspective.**

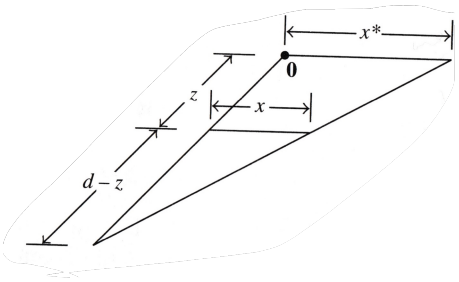
A standard setup for computing a perspective transformation is as shown in this figure:



For simplicity, we will let the $xy$-plane represent the screen. This is called the *viewing plane.*

The eye/camera is located on the $z$ axis, at the point $(0, 0, d)$. This is call the *center of projection.*

A *perspective projection* maps each point $(x, y, z)$ onto an image point $(x^*, y^*, 0)$ so that the center of projection and the two points are all on the same line.




The way to compute the projection is using similar triangles.

The triangle in the $xz$-plane shows the lengths of corresponding line segments.

Similar triangles show that

$$\frac{x^*}{d} = \frac{x}{d-z}$$

and so

$$x^* = \frac{dx}{d-z} \;=\; \frac{x}{1-z/d}.$$

Notice that the function $T : (x, z) \mapsto \frac{x}{1-z/d}$ is **not** linear.

Using homogeneous coordinates, we can construct a linear version of $T$ in $\mathbb{R}^4$.

To do so, we establish the following convention: we will allow the fourth coordinate to vary away from 1.

However, when we plot, for a point $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ we will plot the point $\begin{bmatrix} \frac{x}{h} \\ \frac{y}{h} \\ \frac{z}{h} \end{bmatrix}$.

In this way, by dividing the $x, y, z$ coordinates by the $h$ coordinate, we can implement a **nonlinear** transform in $\mathbb{R}^3$.

So, to implement the perspective transform, we want $\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$ to map to $\begin{bmatrix} \frac{x}{1-z/d} \\ \frac{y}{1-z/d} \\ 0 \\ 1 \end{bmatrix}$.

The way we will implement this is to actually cause it to map to $\begin{bmatrix} x \\ y \\ 0 \\ 1-z/d \end{bmatrix}$.

Then, when we plot (dividing $x$ and $y$ by the $h$ value) we will get the proper transform.

The matrix that implements this transformation is quite simple:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1/d & 1 \end{bmatrix}.$$

So:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1/d & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \\ 1 - z/d \end{bmatrix}.$$

## Composing Transformations

One big payoff for casting all graphics operations as linear transformations comes in the **composition** of transformations.

Consider two linear transformations $T$ and $S$. For example, $T$ could be a scaling and $S$ could be a rotation. Assume $S$ is implemented by a matrix $A$ and $T$ is implemented by a matrix $B$.

To first scale and then rotate a vector $\mathbf{x}$ we would compute $S(T(\mathbf{x}))$.

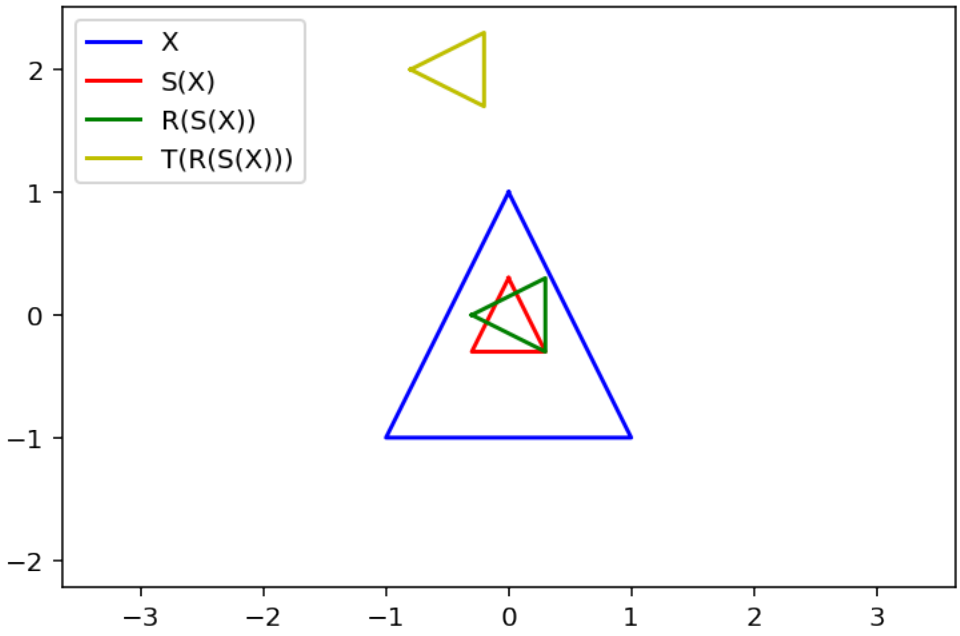
Of course this is implemented as $A(B\mathbf{x})$.

But note that this is the same as $(AB)\mathbf{x}$. In other words, $AB$ is a *single matrix that both scales and rotates* $\mathbf{x}$.

By extension, we can combine any arbitrary sequence of linear transformations into a single matrix. This greatly simplifies high-speed graphics.

Note though that if $C = AB$, then $C$ is the transformation that *first* applies $B$, *then* applies $A$.

**Example.** Let's work in homogenous coordinates for points in $\mathbb{R}^2$. Find the $3 \times 3$ matrix that corresponds to the composite transformation of first scaling by 0.3, then rotation of 90° about the origin, and finally a translation of $\begin{bmatrix} -0.5 \\ 2 \end{bmatrix}$ to each point of a figure.




The scaling matrix is:

$$S = \begin{bmatrix} 0.3 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The rotation matrix is:

$$R = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(Note that $\sin 90^\circ = 1$ and $\cos 90^\circ = 0$.)
The translation matrix is:

$$T = \begin{bmatrix} 1 & 0 & -0.5 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

So the matrix for the composite transformation is:

$$TRS = \begin{bmatrix} 1 & 0 & -0.5 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.3 & 0.3 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -0.3 & -0.5 \\ 0.3 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

## Homework 7

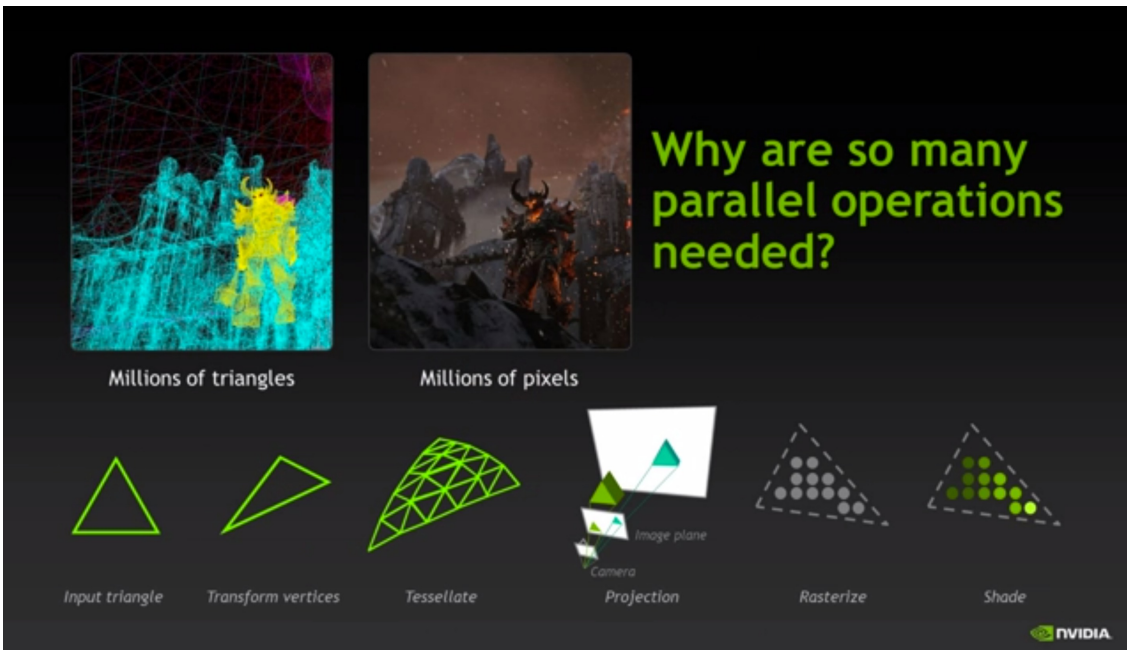
Homework 7 will allow you to explore these ideas by computing various transformation matrices. I will give you some 3D wireframe objects and you will compute the necessary matrices to create a simple animation.

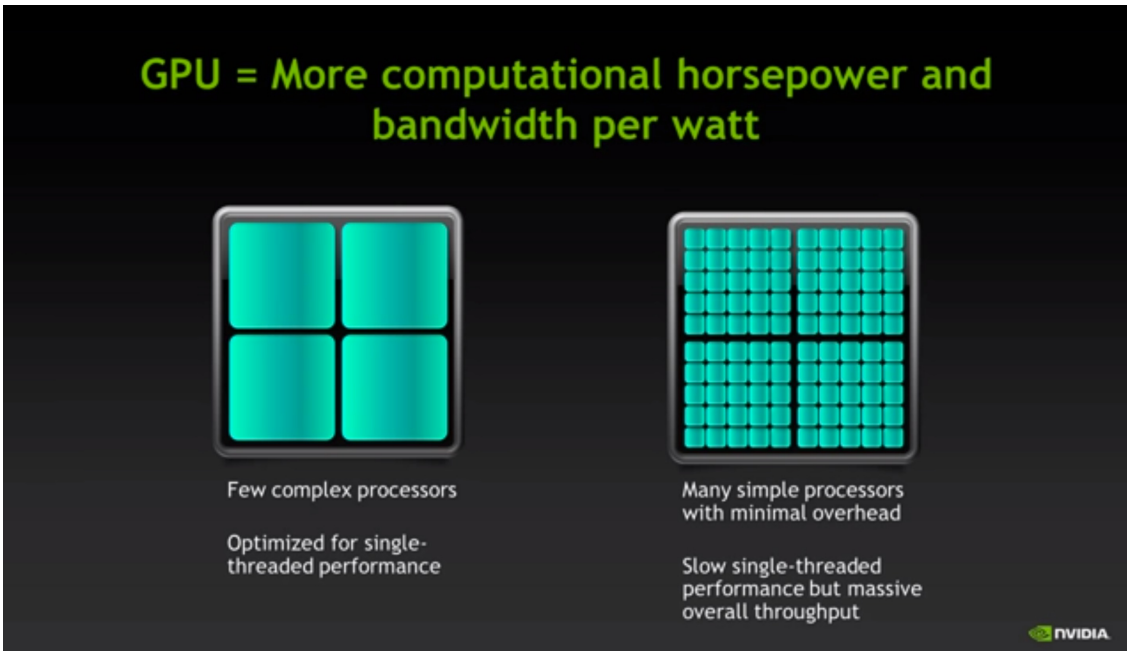
Here is what your finished product will look like:

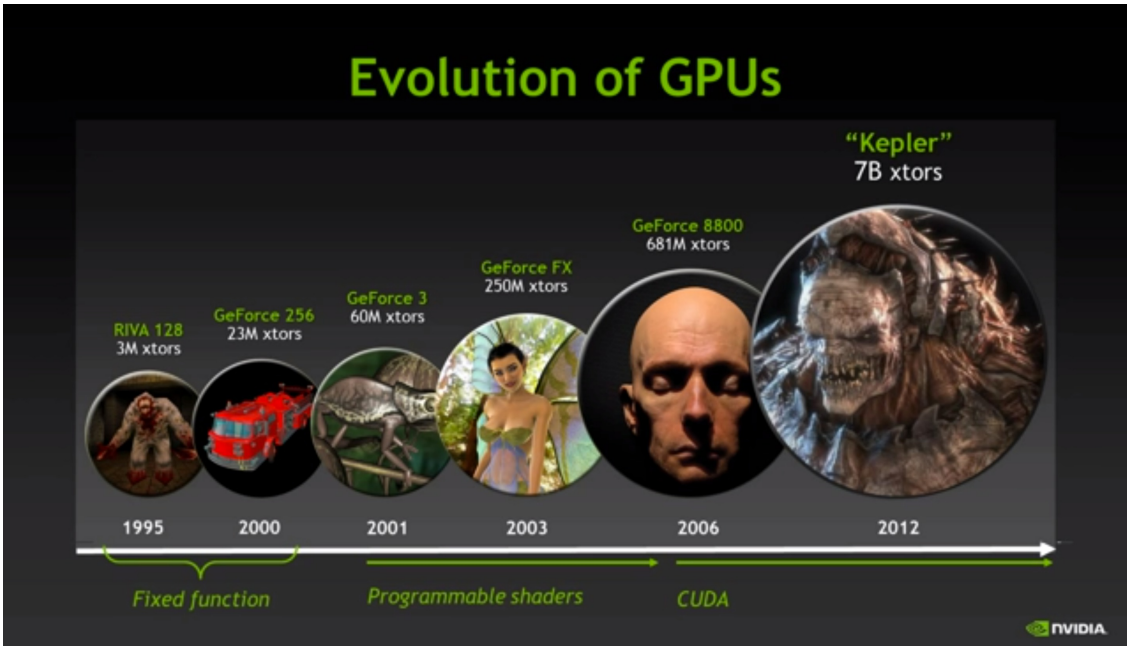
```
<IPython.core.display.HTML object>
```

## Fast Computation of Linear Systems using Graphics Processing Units (GPUs)

```
[12]: # Note: Another good source is http://pixeljetstream.blogspot.de/2015/02/life-of-triangle-nvidias-
```



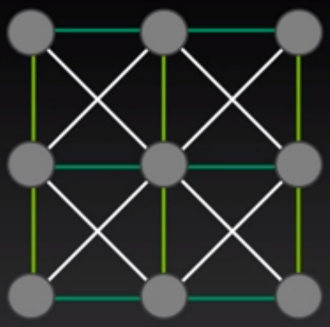



**Examples of Other Linear Systems Used in Computer Graphics**

## PARTICLE SIMULATION — GAMING

*Hair simulation* • *Cloth simulation*

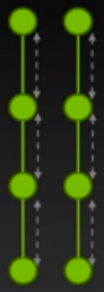


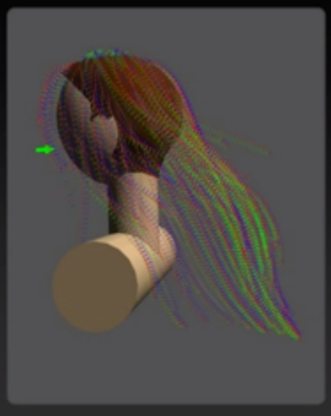

Samaritan demo© Epic Games

NVIDIA

## PARTICLE SIMULATION — GAMING

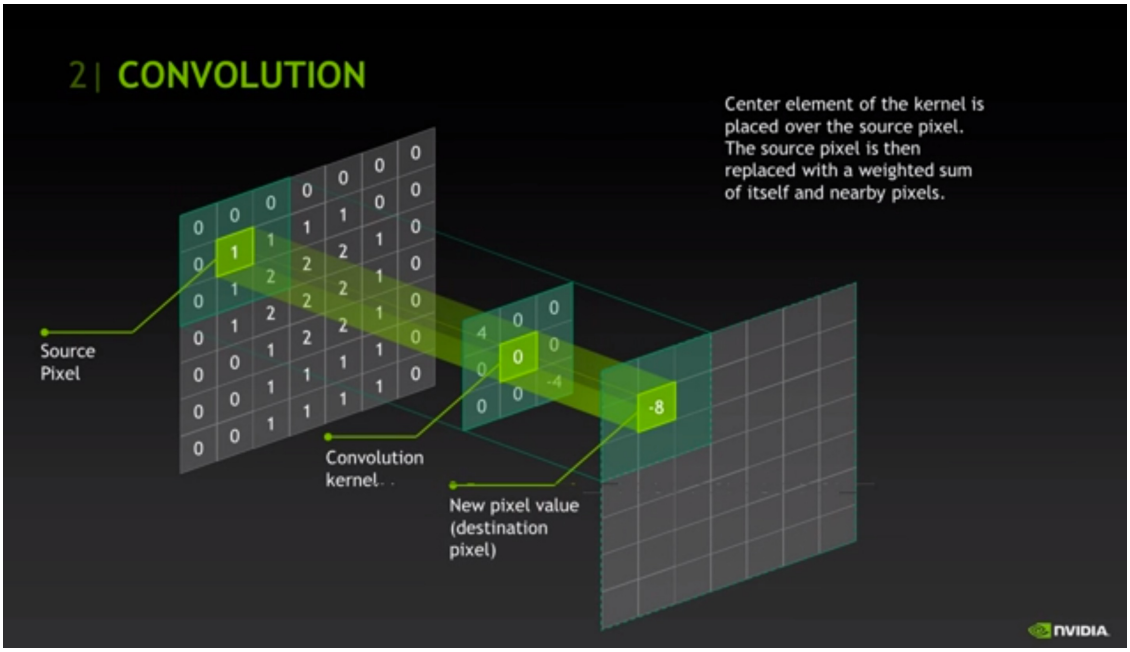
*Hair simulation*

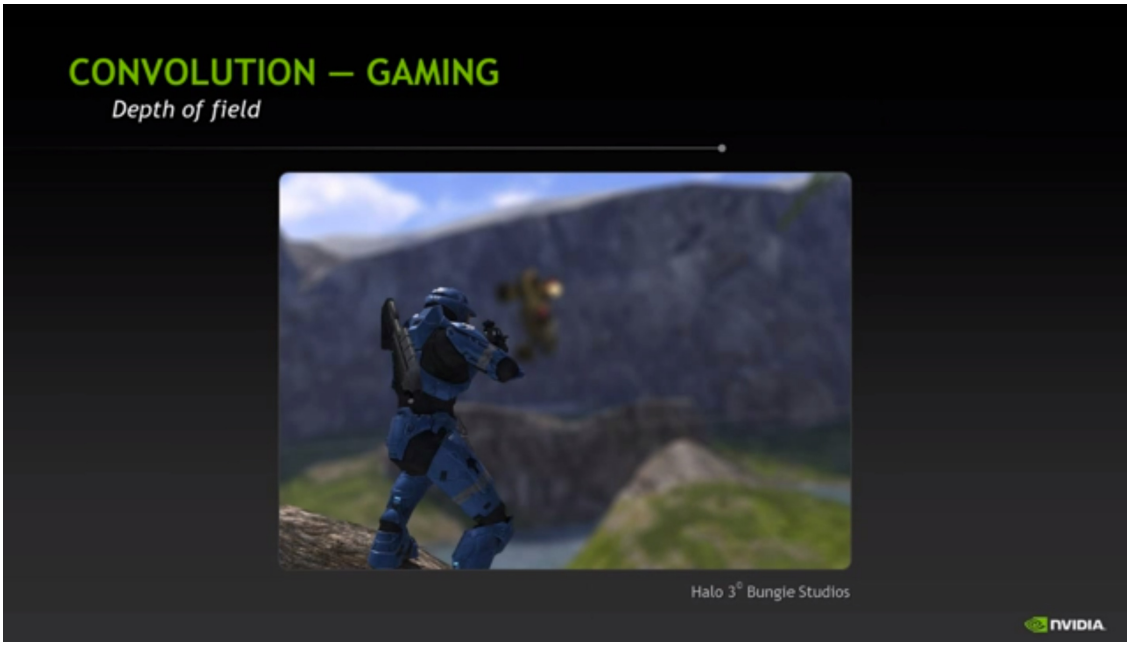




NVIDIA Hair Demo

NVIDIA


2| CONVOLUTION
Center element of the kernel is placed over the source pixel. The source pixel is then replaced with a weighted sum of itself and nearby pixels.
Source Pixel
Convolution kernel
New pixel value (destination pixel)
-8
NVIDIA


CONVOLUTION — GAMING
Depth of field
Halo 3© Bungie Studios
NVIDIA