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Quantile Regression

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1 Introduction

Wind Firm Energy Certificate (FEC) [7] estimation imposes several challenges. First, it is a quantile function of an aleatory quantity, named here on wind capacity factor (WP). Due to its non-dispachable profile, accurate scenario generation models could reproduce a fairly dependence structure in order to the estimation of FEC. Second, as it is a quantile functions, the more close to the extremes of the interval, the more sensitive to sampling error.

In this work, we apply a few different techniques to forecast the quantile function a few steps ahead. The main frameworks we investigate are parametric linear models and a non-parametric regression. In all approaches we use the time series lags as the regression covariates. To study our methods performance, we use the mean power monthly data of Icaraizinho, a wind farm located in the Brazilian northeast.

The Icaraizinho dataset consists of monthly observations from 1981 to 2011 of mean power measured in Megawatts. The full Icaraizinho serie can be found on the appendices from this article. As is common in renewable energy generation, there is a strong seasonality component. Figures 1.1 and 1.2 illustrate this seasonality, where we can observe low amounts of power generation for the time span between February and May, and a yearly peek between August and November. Figure 1.3 shows four scatter plots relating y_t with some of its lags. We choose to present here the four lags that were selected for the quantile regression in the experiment of section **??**, which are the 1^{st}, 4^{th}, 11^{th} and 12^{th}. They are most likely the four main lags to use for these analysis.

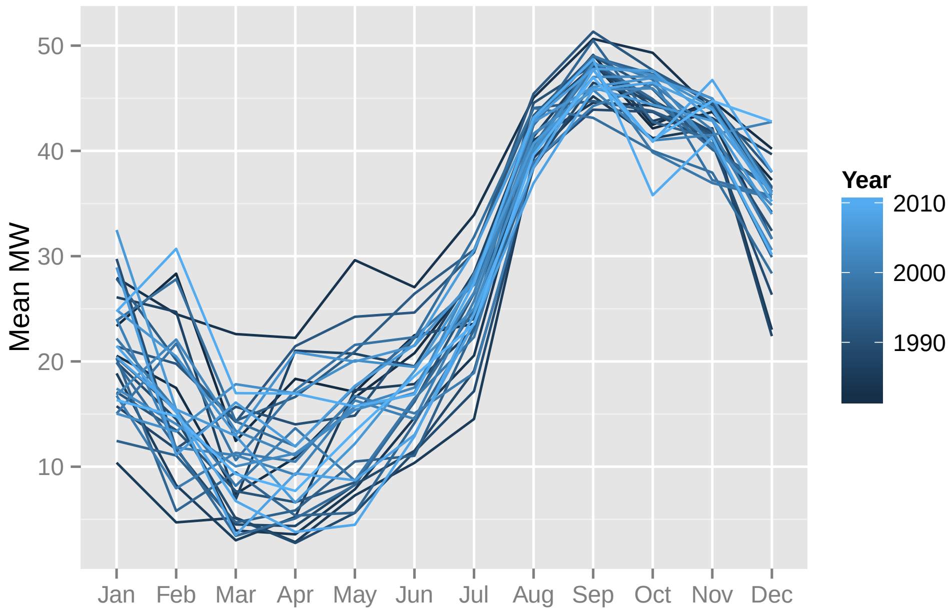


Figure 1.1: Icaraizinho yearly data. Each serie consists of monthly observations for each year.

Here we denote as parametric linear model the well-known quantile regression model [4]. In contrast to the linear regression model through ordinary least squares (OLS), which provides only an estimation of the dependent variable conditional mean, quantile regression model yields a much more detailed information concerning the complex relationship about the dependent variable and its covariates. A Quantile Regression for the α-quantile is the solution of the following optimization problem:

$$\min_{q} \sum_{t=1}^{n} \alpha |y_t - q(x_t)|^+ + (1-\alpha)|y_t - q(x_t)|^-, \tag{1.1}$$

where $q(x_t)$ is the estimated quantile value at a given time t and $|x|^+ = \max\{0, x\}$ and $|x|^- = -\min\{0, x\}$. To model this problem as a Linear Programming problem, thus being able to use a

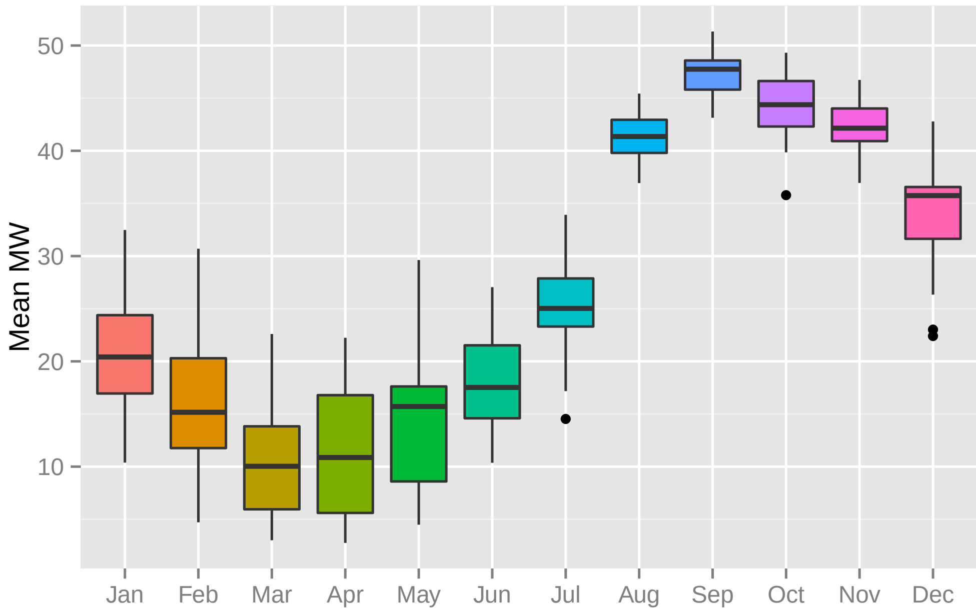


Figure 1.2: Boxplot for each month for the Icaraizinho dataset

modern solver to fit our model, we can create variables ε_t^+ e ε_t^- to represent $|y - q(x_t)|^+$ and $|y - q(x_t)|^-$, respectively. So we have:

$$\begin{aligned} \min_{q, \varepsilon_t^+, \varepsilon_t^-} & \sum_{t=1}^{n} \left(\alpha \varepsilon_t^+ + (1-\alpha) \varepsilon_t^- \right) \\ \text{s.t. } & \varepsilon_t^+ - \varepsilon_t^- = y_t - q(x_t), && \forall t \in \{1, \ldots, n\}, \\ & \varepsilon_t^+, \varepsilon_t^- \geq 0, && \forall t \in \{1, \ldots, n\}. \end{aligned} \tag{1.2}$$

Section 2 is about linear models, so we investigate the quantile estimation when q is a linear function of the series past values, up to a maximum number of lags p:

$$q(y_t, \alpha; \beta) = \beta_0(\alpha) + \beta_1(\alpha) y_{t-1} + \beta_2(\alpha) y_{t-2} + \cdots + \beta_p(\alpha) y_{t-p}. \tag{1.3}$$

In that section we investigate two ways of estimating coeficients, one based on Mixed Integer Programming ideas and the other based on the LASSO [8] penalty. Both of them are strategies to make regularization.

In section 3 we introduce a Nonparametric Quantile Autoregressive model with a ℓ_1-penalty term, in order to properly simulate FEC densities for several α-quantiles. In this nonparametric approach we don't assume any form for $q(x_t)$, but rather let the function adjust to the data. To prevent overfitting, the ℓ_1 penalty for the second derivative (approximated by the second difference of the ordered observations) is included in the objective function.

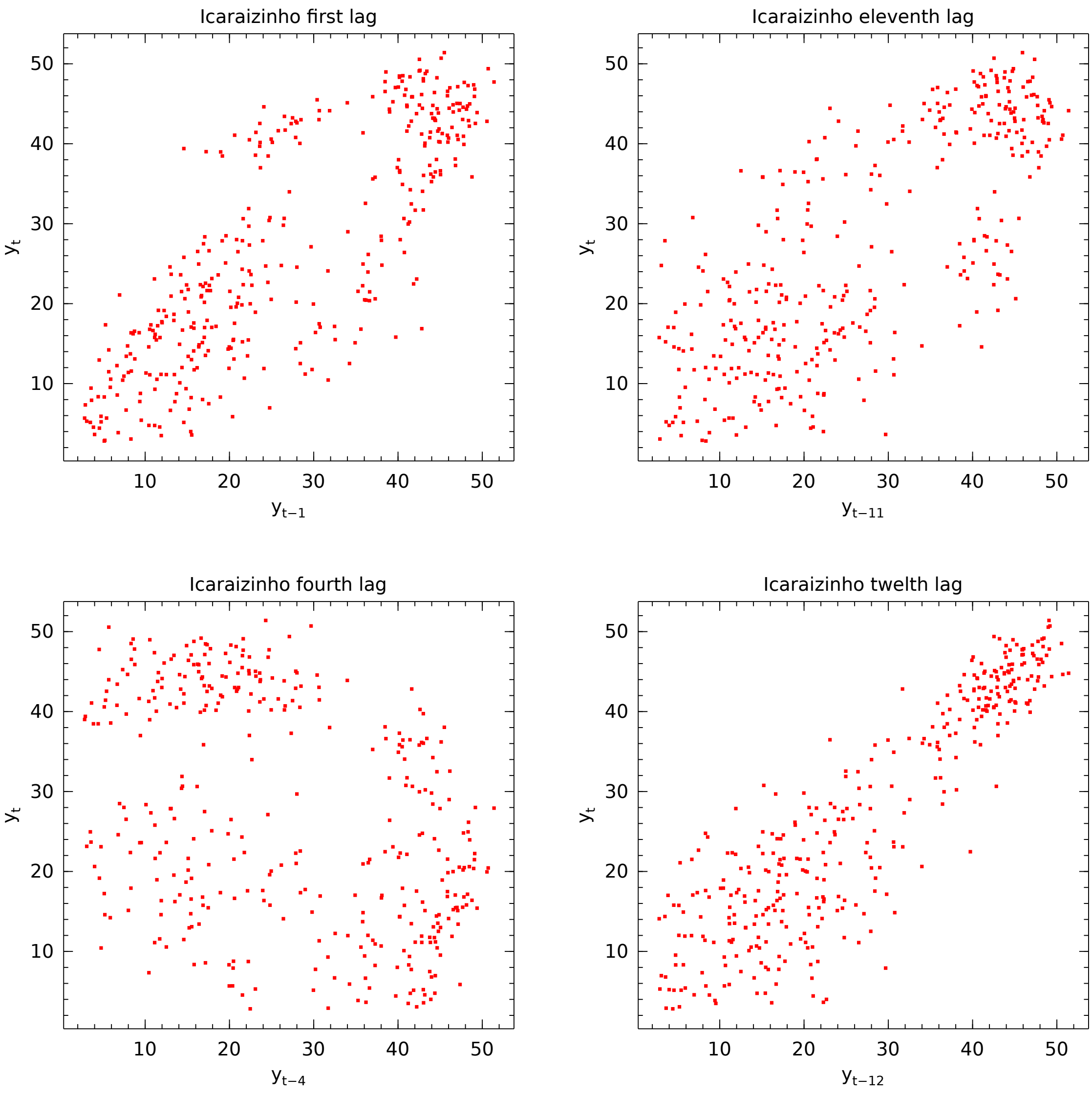


Figure 1.3: Relationship between y_t and some chosen lags.

2 Linear Models for the Quantile Autoregression

Given a time series $\{y_t\}$, we investigate how to select which lags will be included in the Quantile Autoregression. We won't be choosing the full model because this normally leads to a bigger variance in our estimators, which is often linked with bad performance in forecasting applications. So our strategy will be to use some sort of regularization method in order to improve performance. We investigate two ways of accomplishing this goal. The first of them consists of selecting the best subset of variables through Mixed Integer Programming, given that K variables are included in the model. Using MIP to select the best subset of variables is investigated in [2]. The second way is including a ℓ_1 penalty on the linear quantile regression, as in [3], and let the model select which and how many variables will have nonzero coefficients. Both of them will be built over the standard Quantile Linear Regression model. In the end of the section, we discuss a information criteria to be used for quantile regression and verify how close are the solutions in the eyes of this criteria.

When we choose $q(x_t)$ to be a linear function, as on equation 1.1 (that we reproduce below for convenience):

$$\min_{f} \sum_{t=1}^{n} \alpha |y_t - q(x_t)|^{+} + (1-\alpha)|y_t - q(x_t)|^{-}, \tag{2.1}$$

we can substitute it on problem 1.2, getting the following LP problem:

$$\begin{aligned} \min_{\beta_0,\beta,\varepsilon_t^+,\varepsilon_t^-} \quad & \sum_{t=1}^{n} \left(\alpha\varepsilon_t^+ + (1-\alpha)\varepsilon_t^-\right) \\ \text{s.t.} \quad & \varepsilon_t^+ - \varepsilon_t^- = y_t - \beta_0 - \beta^T x_t, \qquad \forall t \in \{1,\dots,n\}, \\ & \varepsilon_t^+, \varepsilon_t^- \geq 0, \qquad \forall t \in \{1,\dots,n\}. \end{aligned} \tag{2.2}$$

In this work, we didn't explore the addition of terms other than the terms y_t past lags. For example, we could include functions of y_{t-p}, such as $log(y_{t-p})$ or $exp(y_{t-p})$. We leave such inclusion for further works.

3 Quantile Autoregression with a nonparametric approach

Fitting a linear estimator for the Quantile Auto Regression isn't appropriate when nonlinearity is present in the data. This nonlinearity may produce a linear estimator that underestimates the quantile for a chunk of data while overestimating for the other chunk (for example, scatter plot of y_t versus y_{t-1} that is seen on the upper left of figure 1.3). To prevent this issue from occurring we propose a modification which we let the prediction $\mathcal{Q}_{y_t|y_{t-1}}(\alpha)$ adjust freely to the data and its nonlinearities. To prevent overfitting and smoothen our predictor, we include a penalty on its roughness by including the ℓ_1 norm of its second derivative. For more information on the ℓ_1 norm acting as a filter, one can refer to [3].

Let $\{\tilde{y}_t\}_{t=1}^n$ be the sequence of observations in time t. Now, let $\tilde{x}_t$ be the $p-$lagged time series of $\tilde{y}_t$, such that $\tilde{x}_t = L^p(\tilde{y}_t)$, where L is the lag operator. Matching each observation $\tilde{y}_t$ with its $p-$lagged correspondent $\tilde{x}_t$ will produce $n-p$ pairs $\{(\tilde{y}_t, \tilde{x}_t)\}_{t=p+1}^n$ (note that the first p observations of y_t must be discarded). When we order the observation of x in such way that they are in growing order

$$\tilde{x}^{(p+1)} \leq \tilde{x}^{(p+2)} \leq \cdots \leq \tilde{x}^{(n)},$$

we can then define $\{x_i\}_{i=1}^{n-p} = \{\tilde{x}^{(t)}\}_{t=p+1}^n$ and $\{y_i\}_{i=1}^{n-p} = \{\tilde{y}^{(t)}\}_{t=p+1}^n$ and $T = \{2, \dots, n-p-1\}$. As we need the second difference of q_i, I has to be shortened by two elements.

Our optimization model to estimate the nonparametric quantile is as follows:

$$\begin{aligned}\mathcal{Q}^{\alpha}_{y_t|y_{t-1}}(t) = \arg\min_{q_t} \sum_{t\in T} \left(|y_t - q_t|^+\alpha + |y_t - q_t|^-(1-\alpha)\right) \\ +\lambda \sum_{t\in T} |D^2_{x_t} q_t|,\end{aligned} \tag{3.1}$$

where $D^2 q_t$ is the second derivative of the q_t function, calculated as follows:

$$D^2_{x_t} q_t = \frac{\left(\frac{q_{t+1}-q_t}{x_{t+1}-x_t}\right) - \left(\frac{q_t - q_{t-1}}{x_t - x_{t-1}}\right)}{x_{t+1} - 2x_t + x_{t-1}}.$$

The first part on the objective function is the usual quantile regression condition for $\{q_t\}$. The second part is the ℓ_1-filter. The purpose of a filter is to control the amount of variation for our estimator q_t. When no penalty is employed we would always get $q_t = y_t$. On the other hand, when $\lambda \to \infty$, our estimator approaches the linear quantile regression.

The full model can be rewritten as a LP problem as bellow:

$$\min_{q_t} \quad \textstyle\sum_{t=1}^n \left(\alpha\delta_t^+ + (1-\alpha)\delta_t^-\right) + \lambda \sum_{t=1}^n \xi_t \tag{3.2}$$

$$s.t. \quad \delta_t^+ - \delta_t^- = y_t - q_t, \qquad \forall t \in \{3, \dots, n-1\}, \tag{3.3}$$

$$D_t = \left(\tfrac{q_{t+1}-q_t}{x_{t+1}-x_t}\right) - \left(\tfrac{q_t - q_{t-1}}{x_t - x_{t-1}}\right) \qquad \forall t \in \{3, \dots, n-1\}, \tag{3.4}$$

$$\xi_t \geq D_t, \qquad \forall t \in \{3, \dots, n-1\}, \tag{3.5}$$

$$\xi_t \geq -D_t, \qquad \forall t \in \{3, \dots, n-1\}, \tag{3.6}$$

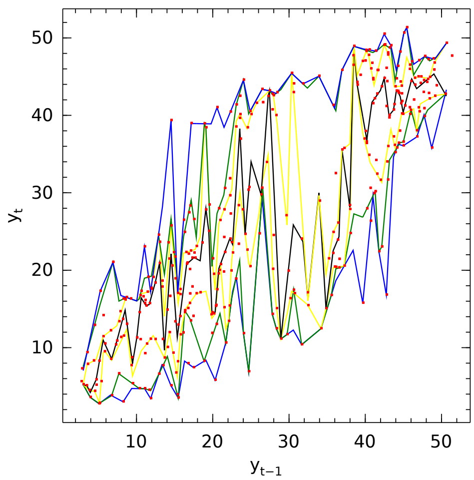
$$\delta_t^+, \delta_t^-, \xi_t \geq 0, \qquad \forall t \in \{3, \dots, n-1\}. \tag{3.7}$$

The output of our optimization problem is a sequence of ordered points $\{(x_t, q_t)\}_{t\in T}$. The next step is to interpolate these points in order to provide an estimation for any other value of x. To address this issue, we propose using a B-splines interpolation, that will be developed in another study.

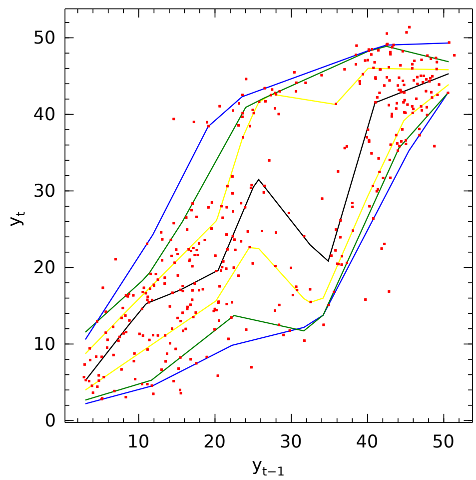
The quantile estimation is done for different values of λ. By using different levels of penalization on the second difference, the estimation can be more or less adaptive to the fluctuation. It is important to notice that the usage of the ℓ_1-norm as penalty leads to a piecewise linear solution q_t. Figure 3.1 shows the quantile estimation for a few different values of λ.

When estimating quantiles for a few different values of α, however, sometimes we find them overlapping each other, which we call crossing quantiles. This effect can be seen in figure 3.1f, where the 95%-quantile crosses over the 90%-quantile. To prevent this, we can include a non-crossing constraint:

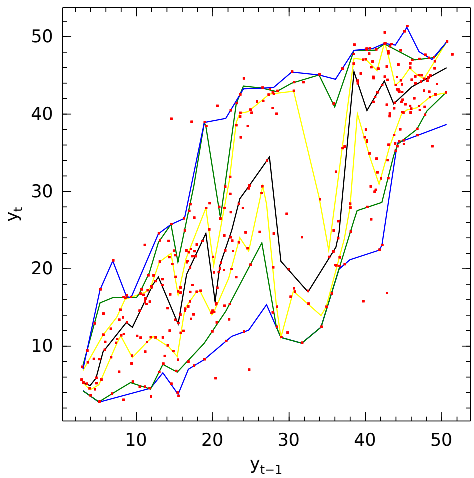
$$q_i^{\alpha} \leq q_i^{\alpha'}, \quad \forall i \in I, \alpha < \alpha'. \tag{3.8}$$



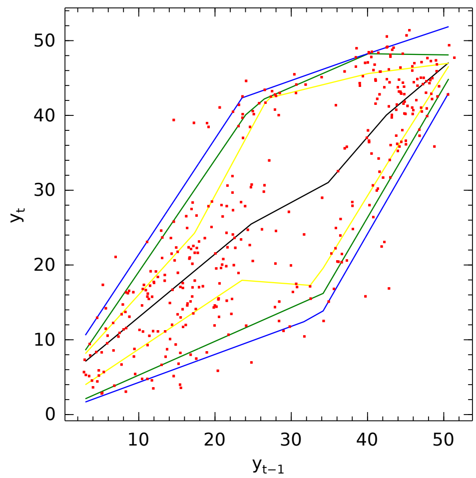
(a) $\lambda = 0.1$



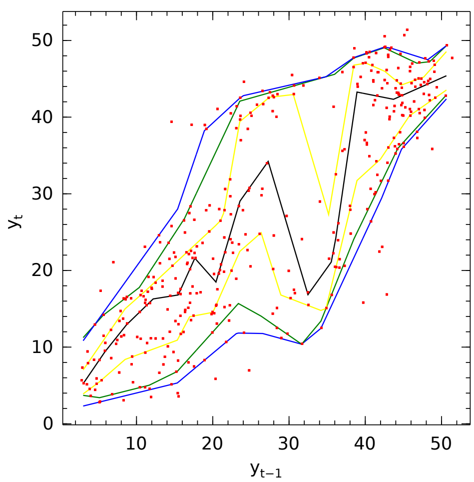
(d) $\lambda = 3$



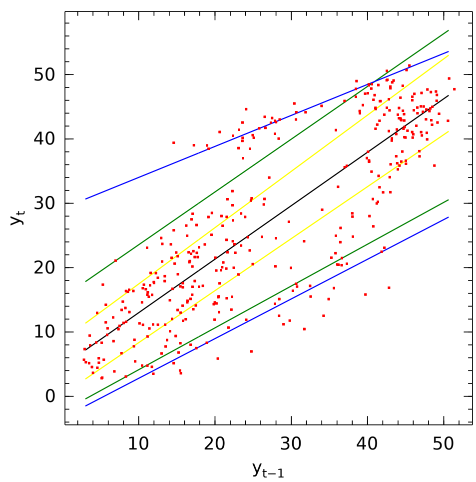
(b) $\lambda = 0.3$



(e) $\lambda = 10$



(c) $\lambda = 1$



(f) $\lambda = 150$

Figure 3.1: Quantile estimations for a few different values of λ. The quantiles represented here are $\alpha = (5\%, 10\%, 25\%, 50\%, 75\%, 90\%, 95\%)$. When $\lambda = 0.1$, on the upper left, we clarly see a overfitting on the estimations. The other extreme case is also shown, when $\lambda = 200$ the nonparametric estimator converges to the linear model.

This means that when α' is a higher quantile than α, then the values from the α'-quantile must be bigger than those of the α-quantile for each and every point.

As a result of this nonparametric estimation, we are able to establish a relation between y_t and y_{t-p} in a way that the model adjusts itself automatically to the present nonlinearities. For this, we only have to supply a numeric value for λ. This approach, however, have yet some issues do be discussed.

The first issue is how to select an appropriate value for λ. A simple way is to do it by inspection, which means to test many different values and pick the one that suits best our needs by looking at them. The other alternative is to use a metric to which we can select the best tune. We can achieve this by using a cross-validation method, for example.

The other issue occurs when we try to add more than one lag to the analysis at the same time. This happens because the problem solution is a set of points that we need to interpolate. This multivariate interpolation, however, is not easily solved, in the sense that we can either choose using a very naive estimator such as the K-nearest neighbors or just find another method that is not yet adopted for a wide range of applications.

3.1 Solar power data

While the Icaraizinho dataset has monthly observations for a wide range of time, we also test the same model for hourly data. In this case, we use the NP-QAR for solar power data. This dataset was retrieved from *https://www.renewables.ninja/* and includes predicted hourly power data for the city of Tubarão (Brazil). This location was chosen because it is the spot of the biggest solar power plant in Brazil.

As solar power production is irradiation dependent, the best single predictor we may have is the hour of the day, as can be seen in the boxplot shown in Figure 3.2

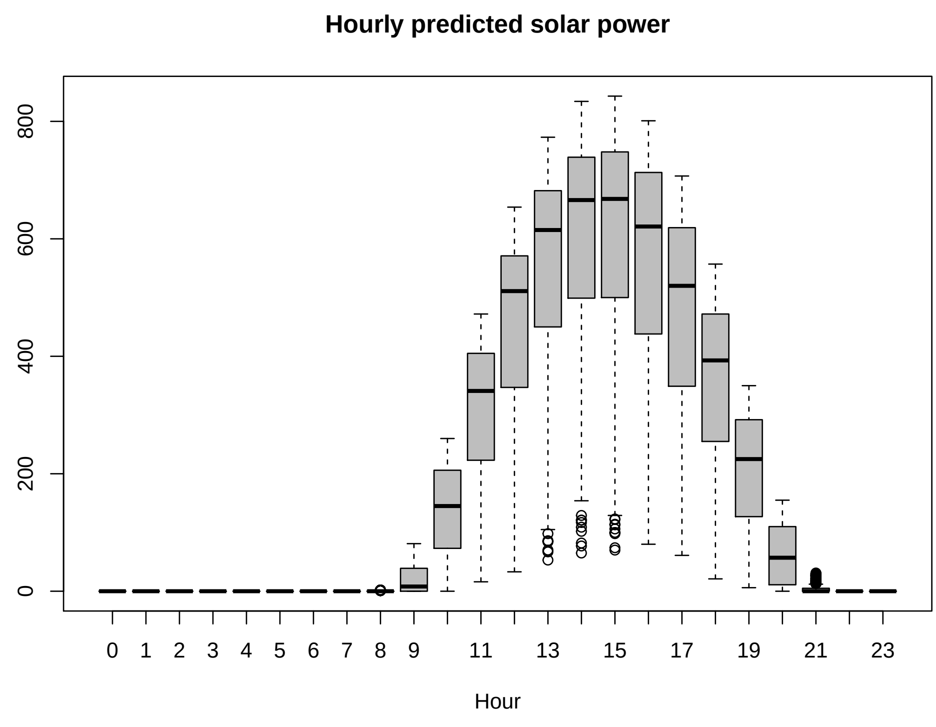


Figure 3.2: Icaraizinho yearly data. Each serie consists of monthly observations for each year.

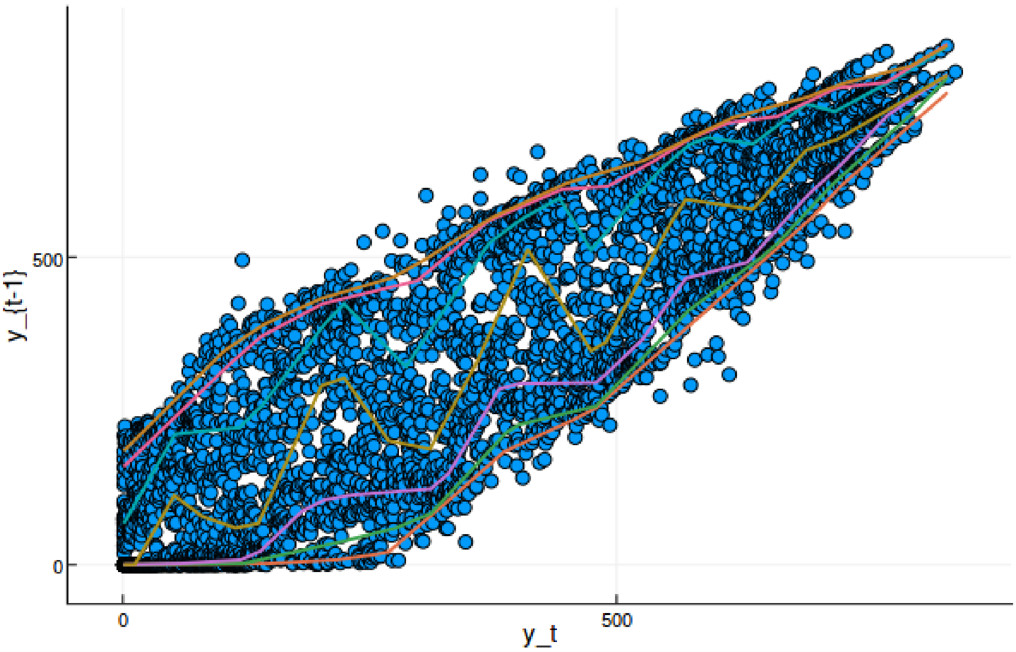


Figure 3.3: Icaraizinho yearly data. Each serie consists of monthly observations for each year.

References

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4 Appendices

4.1 Proof of quantiles as an optimization problem

Let $Z^\alpha = \arg\min_Q E[\alpha \max\{0, X-Q\} + (1-\alpha)\max\{0, Q-X\}]$. We can rewrite the function as

$$
\begin{aligned}
Y &= \alpha \int_Q^\infty (X-Q)dF_x + (1-\alpha)\int_{-\infty}^Q (Q-X)dF_X \\
&= \alpha \int_Q^\infty XdF_x - \alpha Q \int_Q^\infty QdF_x + Q\int_{-\infty}^Q dF_x - \int_{-\infty}^Q XdF_x - \alpha Q \int_{-\infty}^Q dF_x + \alpha \int_{-\infty}^Q XdF_x \\
&= \alpha \int_Q^\infty XdF_x - \alpha Q + QF_X(Q) - \int_{-\infty}^Q XdF_x - \alpha Q F_X(Q) + \alpha \int_{-\infty}^Q XdF_x \\
&= \alpha \int_Q^\infty XdF_x - \alpha Q + QF_X(Q) - \int_{-\infty}^Q XdF_x + \alpha \int_{-\infty}^Q XdF_x
\end{aligned}
$$

By the first order condition for optimality, we need that $\frac{dZ(Q^*)}{dQ} = 0$. So, we have:

$$
\begin{gathered}
-\alpha Q^* f(Q^*) - \alpha + F_X(Q^*) + Q^* f(Q^*) - Q^* f(Q^*) + \alpha Q^* f(Q^*) = 0 \\
F_X(Q^*) = \alpha.
\end{gathered}
$$

Thus, we have that Z^α is the $\alpha - quantile$ of random variable X.

4.2 MIP coefficients tables

The following tables inform the size of Coefficients when using the regularization method based on MIP described on session **??**. When using this method, we choose a parameter K which defines the total number of nonzero coefficients (without accounting the intercept β_0, which is always included). In each column we find the estimated values of coefficients for each different choice of K. As coefficients are quantile dependent, we provide tables for $\alpha \in (0.05, 0.1, 0.25, 0.5, 0.75, 0.9, 0.95)$.

	K=1	K=2	K=3	K=4	K=5	K=6	K=7	K=8	K=9	K=10	K=11	K=12
β_0	-15.33	9.38	1.48	1.34	8.72	-1.68	4.94	0.65	-0.27	-0.16	-3.96	-2.55
β_1	-0.00	0.79	0.66	0.58	0.46	0.40	0.48	0.46	0.46	0.47	0.42	0.44
β_2	-0.00	-0.00	-0.00	-0.00	-0.00	0.33	-0.00	-0.00	-0.00	-0.00	0.14	0.09
β_3	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	0.20	0.20	0.19	0.20	0.17
β_4	-0.00	-0.47	-0.28	-0.27	-0.29	-0.35	-0.31	-0.40	-0.35	-0.35	-0.34	-0.31
β_5	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.05	-0.07	-0.09
β_6	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	0.11	0.08	0.11	0.17	0.12	0.19
β_7	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.16	-0.15	-0.08	-0.15
β_8	-0.00	-0.00	-0.00	-0.00	-0.15	-0.00	-0.31	-0.26	-0.17	-0.17	-0.16	-0.18
β_9	-0.00	-0.00	-0.00	-0.00	-0.00	0.14	0.16	0.20	0.26	0.23	0.28	0.33
β_{10}	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.04
β_{11}	-0.00	-0.00	0.26	0.17	0.21	0.08	0.16	0.19	0.17	0.18	0.17	0.20
β_{12}	1.17	-0.00	-0.00	0.18	0.15	0.19	0.22	0.20	0.20	0.18	0.18	0.17

Table 4.1: Coefficients for quantile $\alpha = 0.05$

	K=1	K=2	K=3	K=4	K=5	K=6	K=7	K=8	K=9	K=10	K=11	K=12
β_0	-10.68	10.07	3.56	1.24	0.76	3.01	3.33	3.02	1.05	2.26	1.55	1.57
β_1	-0.00	0.81	0.63	0.61	0.55	0.49	0.49	0.50	0.48	0.44	0.44	0.44
β_2	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	0.04	-0.00	-0.00	0.04	0.07	0.07
β_3	-0.00	-0.00	-0.00	-0.00	0.15	0.20	0.16	0.15	0.13	0.11	0.12	0.12
β_4	-0.00	-0.43	-0.33	-0.28	-0.37	-0.33	-0.34	-0.30	-0.24	-0.24	-0.26	-0.25
β_5	-0.00	-0.00	-0.00	-0.00	-0.00	-0.08	-0.07	-0.12	-0.14	-0.15	-0.17	-0.17
β_6	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	0.11	0.10	0.10	0.14	0.14
β_7	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.07	-0.11	-0.13	-0.11	-0.11
β_8	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.04	-0.04
β_9	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	0.09	0.10	0.13	0.13
β_{10}	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	0.00
β_{11}	-0.00	-0.00	-0.00	0.14	0.17	0.17	0.16	0.15	0.11	0.09	0.08	0.08
β_{12}	1.09	-0.00	0.35	0.27	0.25	0.22	0.22	0.26	0.33	0.34	0.33	0.33

Table 4.2: Coefficients for quantile $\alpha = 0.1$

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4.3 Simulation study tables

On section **??** we explain a simulation study to try evaluating differences between estimating a quantile with a quantile regression model or using the conditional mean when knowing the true generating model. From this experiment, we present below table of results for three different aspects, for five different quantiles. Tables from 4.7 to 4.11 shows the difference between the root mean square errors between both methods of predicting the one-step ahead quantile for a few different values of α. Tables 4.12-4.16 are the ones that shows the nominal difference between the autoregressive coefficients $\hat{\phi}$ and $\hat{\beta}$. The nominal difference between the intercept terms $\hat{\phi}_0 + z_\alpha \hat{\sigma}^2_\varepsilon)$ and $\hat{\beta}_0)$ is

	K=1	K=2	K=3	K=4	K=5	K=6	K=7	K=8	K=9	K=10	K=11	K=12
β_0	2.72	-3.38	8.64	4.88	0.62	2.98	2.70	2.62	2.27	1.87	2.43	2.53
β_1	-0.00	0.59	0.52	0.51	0.57	0.54	0.56	0.56	0.58	0.58	0.57	0.57
β_2	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.03	-0.06	-0.05	-0.05
β_3	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	0.04	0.03	0.04
β_4	-0.00	-0.00	-0.25	-0.18	-0.14	-0.11	-0.11	-0.12	-0.11	-0.11	-0.11	-0.12
β_5	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	0.01
β_6	-0.00	-0.00	-0.00	-0.00	-0.00	-0.06	-0.09	-0.08	-0.08	-0.08	-0.09	-0.09
β_7	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.02	-0.02
β_8	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	0.06	0.06	0.05	0.06	0.08	0.07
β_9	-0.00	-0.00	-0.00	-0.00	0.08	0.09	0.06	0.09	0.07	0.07	0.08	0.08
β_{10}	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.05	-0.04	-0.05	-0.05	-0.05
β_{11}	-0.00	0.54	-0.00	0.15	0.14	0.11	0.10	0.11	0.14	0.14	0.15	0.14
β_{12}	0.92	-0.00	0.42	0.34	0.32	0.33	0.32	0.34	0.33	0.34	0.32	0.33

Table 4.3: Coefficients for quantile $\alpha = 0.5$

.

	K=1	K=2	K=3	K=4	K=5	K=6	K=7	K=8	K=9	K=10	K=11	K=12
β_0	12.14	10.06	6.60	11.05	13.22	12.04	13.34	13.28	12.58	13.69	13.47	13.71
β_1	-0.00	0.24	0.39	0.39	0.40	0.38	0.38	0.38	0.38	0.40	0.40	0.40
β_2	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.02
β_3	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.01	-0.04	-0.03	-0.02
β_4	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	0.03	-0.00	0.05	0.05	0.04
β_5	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	0.00	0.01
β_6	-0.00	-0.00	-0.00	-0.14	-0.00	-0.00	-0.03	-0.05	-0.01	-0.07	-0.07	-0.07
β_7	-0.00	-0.00	-0.00	-0.00	-0.19	-0.10	-0.10	-0.11	-0.09	-0.11	-0.11	-0.10
β_8	-0.00	-0.00	-0.00	-0.00	-0.00	-0.08	-0.07	-0.08	-0.08	-0.07	-0.07	-0.08
β_9	-0.00	-0.00	-0.00	0.14	0.16	0.15	0.16	0.18	0.16	0.19	0.19	0.19
β_{10}	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.04	-0.06	-0.06	-0.06
β_{11}	-0.00	-0.00	0.20	-0.00	0.11	0.15	0.12	0.16	0.16	0.18	0.18	0.19
β_{12}	0.80	0.63	0.39	0.42	0.26	0.29	0.28	0.23	0.29	0.24	0.24	0.25

Table 4.4: Coefficients for quantile $\alpha = 0.9$

.

	K=1	K=2	K=3	K=4	K=5	K=6	K=7	K=8	K=9	K=10	K=11	K=12
β_0	16.73	11.74	11.51	13.77	13.45	13.48	14.36	14.84	12.36	14.04	13.09	14.00
β_1	-0.00	0.26	0.32	0.35	0.38	0.38	0.40	0.43	0.40	0.40	0.39	0.39
β_2	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	0.02	0.02
β_3	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	0.01
β_4	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	0.04	0.06	0.06	0.05
β_5	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.04	-0.03	-0.04
β_6	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.05	-0.10	-0.07	-0.09	-0.08	-0.09
β_7	-0.00	-0.00	-0.00	-0.15	-0.14	-0.12	-0.09	-0.05	-0.06	-0.06	-0.06	-0.06
β_8	-0.00	-0.00	-0.00	-0.00	-0.00	-0.04	-0.05	-0.07	-0.05	-0.08	-0.07	-0.07
β_9	-0.00	-0.00	-0.00	0.16	0.11	0.14	0.16	0.19	0.19	0.22	0.22	0.21
β_{10}	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.15	-0.14	-0.11	-0.12	-0.11
β_{11}	-0.00	-0.00	0.17	-0.00	0.14	0.13	0.12	0.25	0.23	0.18	0.21	0.22
β_{12}	0.71	0.59	0.37	0.41	0.28	0.28	0.25	0.21	0.27	0.25	0.24	0.22

Table 4.5: Coefficients for quantile $\alpha = 0.95$

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	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
1981	23.36	28.34	12.44	18.35	17.10	22.49	23.57	40.10	48.40	42.13	43.70	37.23
1982	20.54	17.48	7.42	10.87	16.57	20.79	27.95	42.55	49.12	42.48	44.78	40.20
1983	27.94	24.50	22.60	22.24	29.62	27.05	33.92	45.06	50.64	49.32	43.83	36.14
1984	20.37	15.35	3.94	3.57	7.85	14.65	20.56	41.01	44.58	44.31	42.94	31.65
1985	10.38	4.71	5.15	2.84	7.27	10.36	14.53	39.33	45.18	41.21	42.15	23.02
1986	18.86	8.25	3.00	5.23	17.29	17.85	23.08	41.36	48.30	42.83	44.36	36.41
1987	26.09	24.71	6.90	21.02	20.73	19.53	28.42	42.94	48.06	44.26	43.11	39.67
1988	15.75	11.66	4.51	4.36	8.29	11.50	19.10	38.40	46.47	44.80	41.79	22.40
1989	19.92	14.52	5.08	2.75	5.62	11.42	17.17	38.94	43.92	43.70	40.69	26.34
1990	29.74	11.70	15.69	14.02	14.85	22.28	24.02	44.55	48.18	44.66	41.51	32.41
1991	17.09	13.46	7.68	6.63	8.51	16.17	26.46	43.36	49.00	45.86	40.14	36.57
1992	21.41	19.78	14.25	21.45	24.24	24.64	30.34	45.43	51.33	47.66	44.50	37.97
1993	27.86	20.13	14.36	16.63	20.94	26.43	30.60	44.07	44.73	43.78	41.40	34.18
1994	12.45	11.06	4.70	5.85	10.49	11.04	23.03	38.50	48.92	47.30	44.97	36.55
1995	20.31	5.80	9.47	5.36	5.62	14.15	23.54	42.48	50.49	42.74	41.15	29.90
1996	19.89	11.85	3.43	5.08	8.26	16.29	24.89	40.52	48.44	44.92	40.15	36.37
1997	23.89	27.80	14.30	11.95	17.55	22.22	31.82	44.07	43.14	40.00	37.94	28.36
1998	15.04	21.70	10.61	17.28	21.57	22.31	27.26	42.45	49.04	46.76	37.22	35.74
1999	22.18	15.39	8.18	13.66	8.67	16.49	22.30	40.43	47.75	39.85	36.95	35.54
2000	16.75	7.95	11.33	10.47	16.73	15.07	18.90	38.91	44.26	46.34	41.98	31.62
2001	24.03	11.82	11.09	9.23	16.30	14.53	25.73	41.57	45.79	40.99	41.52	42.76
2002	16.81	22.08	13.40	11.07	15.71	17.52	26.55	41.64	45.80	45.94	40.64	30.58
2003	17.42	14.05	10.03	11.26	15.39	17.01	28.29	39.98	47.02	47.07	40.47	34.85
2004	15.04	13.34	17.84	16.97	20.10	19.48	25.03	40.11	48.25	47.21	44.13	35.79
2005	24.89	20.47	13.01	20.88	19.98	21.48	27.81	42.74	46.09	46.93	44.98	36.08
2006	32.48	15.44	12.93	6.59	12.19	19.08	27.79	40.72	46.01	44.38	42.85	33.99
2007	28.93	11.13	16.10	11.91	17.68	21.57	30.56	42.95	47.80	47.61	42.97	35.98
2008	20.42	15.46	3.51	9.37	8.71	13.02	23.61	36.93	45.82	46.49	43.91	35.19
2009	21.48	15.16	6.74	3.80	4.48	12.88	24.53	38.40	47.70	40.87	46.73	38.03
2010	24.75	30.70	16.99	16.95	15.72	16.86	27.43	43.18	48.71	35.79	41.30	30.15
2011	16.33	14.79	9.30	7.70	13.35	18.60	23.53	39.62	46.97	40.99	44.75	42.79

Table 4.6: Monthly Mean Power data of Icaraizinho wind farm, located on the Brazilian northeast. It is available for download here: https://raw.githubusercontent.com/mcruas/data/master/icaraizinho.csv.

$\phi \backslash RSN$	0.01	0.05	0.1	0.5	1
0.25	0.99454499	0.99987753	0.99726238	0.99996553	0.99997820
0.50	0.99928262	1.00002733	0.99902661	1.00017501	1.00072601
0.70	0.97952103	0.99154652	1.00049301	0.99991315	1.00014586
0.90	0.89928729	0.99117750	0.99730087	1.00106737	0.99970135

Table 4.7: RMSE ratio ($RMSE^{QR}/RMSE^{AR}$) for estimating quantile $\alpha = 0.05$. ϕ stands for the autoregressive coefficient and RSN is the signal to noise ratio. Details for these experiments can be found on section **??**

$\phi \backslash RSN$	0.01	0.05	0.1	0.5	1
0.25	0.99404378	1.00071379	0.99865043	1.00009650	1.00031150
0.50	0.99302385	1.00012893	1.00008177	1.00020172	0.99997679
0.70	0.96163465	0.99948879	1.00007350	1.00076714	1.00002219
0.90	0.82867689	0.99989835	1.00007245	0.99997754	1.00016876

Table 4.8: RMSE ratio ($RMSE^{QR}/RMSE^{AR}$) for estimating quantile $\alpha = 0.1$. ϕ stands for the autoregressive coefficient and RSN is the signal to noise ratio. Details for these experiments can be found on section **??**

$\phi \backslash RSN$	0.01	0.05	0.1	0.5	1
0.25	0.98066380	0.99981708	0.99972368	1.00003875	1.00027661
0.50	0.99671712	0.99884992	0.99718531	0.99996169	0.99939155
0.70	0.96160330	0.99971110	1.00006113	1.00001116	1.00024519
0.90	0.93616426	0.99985323	0.99934405	0.99991411	0.99994730

Table 4.9: RMSE ratio ($RMSE^{QR}/RMSE^{AR}$) for estimating quantile $\alpha = 0.5$. ϕ stands for the autoregressive coefficient and RSN is the signal to noise ratio. Details for these experiments can be found on section **??**

$\phi \backslash RSN$	0.01	0.05	0.1	0.5	1
0.25	0.99662610	0.99945043	0.99538478	0.99966478	1.00025526
0.50	1.00028533	0.99613341	0.99981279	0.99996292	1.00108295
0.70	0.91878126	0.99889801	0.99885007	1.00023457	1.00067966
0.90	0.80536841	0.99826964	0.99806239	1.00003275	0.99993464

Table 4.10: RMSE ratio ($RMSE^{QR}/RMSE^{AR}$) for estimating quantile $\alpha = 0.9$. ϕ stands for the autoregressive coefficient and RSN is the signal to noise ratio. Details for these experiments can be found on section **??**

$\phi \backslash RSN$	0.01	0.05	0.1	0.5	1
0.25	1.00009586	0.99962843	1.00162321	1.00002799	1.00023607
0.50	0.99977885	0.99742853	0.99415437	1.00058173	0.99971703
0.70	0.93295124	0.99948015	0.99936096	1.00048504	0.99969376
0.90	0.97068193	1.00001858	1.00007662	1.00020540	0.99928392

Table 4.11: RMSE ratio ($RMSE^{QR}/RMSE^{AR}$) for estimating quantile $\alpha = 0.95$. ϕ stands for the autoregressive coefficient and RSN is the signal to noise ratio. Details for these experiments can be found on section **??**

$\phi \backslash RSN$	0.01	0.05	0.1	0.5	1
0.25	-0.02501316	0.01504215	-0.00565925	-0.01735387	-0.00060544
0.50	-0.01319163	0.00222293	0.00477565	0.01005289	-0.01981673
0.70	0.02775786	-0.01203545	0.01320180	-0.00721887	-0.00665778
0.90	0.01743705	0.00303553	0.01319473	-0.01785909	0.00046374

Table 4.12: Diffeference between the autoregressive coefficients ($\hat{\phi} - \hat{\beta}$) for estimating quantile $\alpha = 0.05$. ϕ stands for the autoregressive coefficient and RSN is the signal to noise ratio. Details for these experiments can be found on section **??**

$\phi \backslash RSN$	0.01	0.05	0.1	0.5	1
0.25	-0.00866846	0.02731984	0.01494079	-0.01572119	0.01000855
0.50	0.02547593	0.00642516	0.00247766	-0.00871352	0.00277974
0.70	0.00503748	0.00751987	-0.00107390	0.00327506	-0.00044987
0.90	0.00709175	0.00692367	-0.00624776	0.00298916	-0.00493012

Table 4.13: Diffeference between the autoregressive coefficients $(\hat{\phi}-\hat{\beta})$ for estimating quantile $\alpha = 0.1$. ϕ stands for the autoregressive coefficient and RSN is the signal to noise ratio. Details for these experiments can be found on section **??**

$\phi \backslash RSN$	0.01	0.05	0.1	0.5	1
0.25	0.00504450	-0.00552938	0.00198665	0.00331038	0.00493278
0.50	0.00816919	-0.00518034	-0.00514326	0.00056696	0.00347201
0.70	0.00817503	0.00319246	-0.00703127	-0.00478434	0.00539972
0.90	0.01257091	0.00502689	-0.00469782	-0.00543626	0.00172323

Table 4.14: Diffeference between the autoregressive coefficients $(\hat{\phi}-\hat{\beta})$ for estimating quantile $\alpha = 0.5$. ϕ stands for the autoregressive coefficient and RSN is the signal to noise ratio. Details for these experiments can be found on section **??**

$\phi \backslash RSN$	0.01	0.05	0.1	0.5	1
0.25	0.00063949	-0.00587333	-0.00343132	0.00057888	-0.01576153
0.50	-0.01953741	-0.00099696	-0.01220643	-0.00459181	-0.02725897
0.70	0.00879188	-0.00578564	-0.01365016	-0.01735324	-0.01595786
0.90	-0.00432531	-0.00674863	0.00059043	0.00040195	0.00452383

Table 4.15: Diffeference between the autoregressive coefficients $(\hat{\phi}-\hat{\beta})$ for estimating quantile $\alpha = 0.9$. ϕ stands for the autoregressive coefficient and RSN is the signal to noise ratio. Details for these experiments can be found on section **??**

$\phi \backslash RSN$	0.01	0.05	0.1	0.5	1
0.25	-0.00217703	0.01773060	-0.01892853	0.01070602	-0.01105531
0.50	-0.01530692	-0.01035399	-0.02514421	0.01278911	-0.00835432
0.70	0.04323376	-0.00354721	0.03589676	-0.00176410	-0.00545403
0.90	0.04279246	0.00727074	-0.00552405	0.00634520	-0.00118189

Table 4.16: Diffeference between the autoregressive coefficients $(\hat{\phi}-\hat{\beta})$ for estimating quantile $\alpha = 0.95$. ϕ stands for the autoregressive coefficient and RSN is the signal to noise ratio. Details for these experiments can be found on section **??**

$\phi \backslash RSN$	0.01	0.05	0.1	0.5	1
0.25	0.04603811	-0.01775659	-0.03214647	0.02145871	-0.00765273
0.50	0.03352012	-0.00448700	-0.02823629	-0.04445694	0.03234653
0.70	-0.04953532	0.10768253	-0.03871770	0.02006049	0.01848257
0.90	-0.01908312	0.05744539	-0.17970986	0.12894696	0.02136838

Table 4.17: Coefficient difference between the non-autoregressive part $((\hat{\phi}_0+z_\alpha\hat{\sigma}_\varepsilon^2)-\hat{\beta}_0)$ for estimating quantile $\alpha = 0.05$. ϕ stands for the autoregressive coefficient and RSN is the signal to noise ratio. Details for these experiments can be found on section **??**

$\phi \backslash RSN$	0.01	0.05	0.1	0.5	1
0.25	0.02260184	-0.04514987	-0.04187708	0.02309700	-0.02037122
0.50	-0.03481513	-0.01041481	-0.00038617	-0.00296439	-0.01172149
0.70	0.03459960	-0.03797767	0.01480016	0.04972062	0.01002299
0.90	0.14704364	-0.06159980	0.08091946	-0.03306211	0.01302636

Table 4.18: Coefficient difference between the non-autoregressive part $((\hat{\phi}_0 + z_\alpha \hat{\sigma}_\varepsilon^2) - \hat{\beta}_0)$ for estimating quantile $\alpha = 0.1$. ϕ stands for the autoregressive coefficient and RSN is the signal to noise ratio. Details for these experiments can be found on section **??**

$\phi \backslash RSN$	0.01	0.05	0.1	0.5	1
0.25	0.01297018	-0.00048766	-0.01041199	-0.01243439	0.02395973
0.50	-0.00774322	0.03340235	0.04581712	-0.00941144	0.02834337
0.70	0.02631287	-0.00313672	0.05261023	-0.00929484	-0.05534836
0.90	-0.00600514	-0.01798551	0.01424034	0.07490435	-0.10554228

Table 4.19: Coefficient difference between the non-autoregressive part $((\hat{\phi}_0 + z_\alpha \hat{\sigma}_\varepsilon^2) - \hat{\beta}_0)$ for estimating quantile $\alpha = 0.5$. ϕ stands for the autoregressive coefficient and RSN is the signal to noise ratio. Details for these experiments can be found on section **??**

$\phi \backslash RSN$	0.01	0.05	0.1	0.5	1
0.25	-0.01239889	0.02293524	-0.02274380	-0.01803445	0.00913629
0.50	0.03837217	0.01562634	0.02285269	0.00939266	0.08607257
0.70	0.05059941	0.03251627	0.07964660	0.06332138	0.06101961
0.90	0.27272743	0.10535168	0.04777041	0.02227744	-0.04024329

Table 4.20: Coefficient difference between the non-autoregressive part $((\hat{\phi}_0 + z_\alpha \hat{\sigma}_\varepsilon^2) - \hat{\beta}_0)$ for estimating quantile $\alpha = 0.9$. ϕ stands for the autoregressive coefficient and RSN is the signal to noise ratio. Details for these experiments can be found on section **??**

$\phi \backslash RSN$	0.01	0.05	0.1	0.5	1
0.25	0.00335900	-0.03337376	0.03659738	-0.02854912	0.01837794
0.50	0.03376431	-0.00203817	-0.00638700	-0.06833761	-0.00379743
0.70	-0.07861339	0.02610872	-0.11362453	0.05524907	-0.00006457
0.90	-0.33397830	-0.06626543	0.05843421	-0.05809618	-0.04813865

Table 4.21: Coefficient difference between the non-autoregressive part $((\hat{\phi}_0 + z_\alpha \hat{\sigma}_\varepsilon^2) - \hat{\beta}_0)$ for estimating quantile $\alpha = 0.95$. ϕ stands for the autoregressive coefficient and RSN is the signal to noise ratio. Details for these experiments can be found on section **??**