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Yuzhi Cai, *University of Plymouth*

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Autoregression with Non-Gaussian Innovations

Yuzhi Cai

Abstract

Many economics and finance time series are non-Gaussian. In this paper, we propose a Bayesian approach to non-Gaussian autoregressive time series models via quantile functions. This approach is parametric, so we also compare the proposed parametric approach with a semi-parametric approach. Simulation studies and applications to real time series show that this method works very well.

KEYWORDS: Bayesian method, quantile function, non-Gaussian time series, simulation, parametric and semi-parametric approaches

Author Notes: The author would like to thank the referee and Professor W. G. Gilchrist for their very suggestive comments which have greatly enhanced the quality and the presentation of the paper.

1 Introduction

An autoregressive time series model of order k is given by

$$y_t = a_0 + a_1 y_{t-1} + \cdots + a_k y_{t-k} + \varepsilon_t, \tag{1}$$

where ε_t are independently and identically distributed with mean 0 and constant variance σ^2, a_i, $i = 0, \ldots, k$, and σ^2 are model parameters. It is known that model (1) is suitable for Gaussian time series. For non-Gaussian time series, one of the difficulties in using model (1) is to identify a proper distribution for the error term ε_t. Koenker (2005) proposed a semi-parametric approach which can be used to deal with this problem. That is, by estimating a sequence of conditional quantiles of y_t, a conditional distribution function of y_t can be obtained approximately. Specifically, his semi-parametric model says that the τ^{th} quantile of y_t conditional on $\mathbf{y}_{t-1} = (y_1, \ldots, y_{t-1})$ is given by

$$q^{\tau}_{y_t|\mathbf{y}_{t-1}} = a^{\tau}_0 + a^{\tau}_1 y_1 + \cdots + a^{\tau}_k y_k, \quad 0 < \tau < 1 \tag{2}$$

where $a^{\tau}_0, \ldots, a^{\tau}_k$ are model parameters depending on τ.

For a given time series $y_1, y_2, \ldots, y_n$, the parameters can be estimated by minimizing the following cost function

$$\min_{\boldsymbol{\beta}^{\tau}} \sum_{t=k+1}^{n} \rho_{\tau}(u_t),$$

where $\rho_{\tau}(u_t) = u_t(\tau - I_{[u_t<0]})$, and

$$u_t = y_t - a^{\tau}_0 y_{t-1} - \cdots - a^{\tau}_k y_{t-k}, \quad t = k+1, \ldots, n.$$

Different methods have been proposed to solve the above optimization problem, see for example, Koenker and D'Orey (1987, 1994). Yu and Moyeed (2001) also proposed a Bayesian method to estimate the model parameters. Cai (2007) and Cai and Stander (2008) extended the Bayesian approach to deal with quantile self-exciting autoregressive time series models.

The above approach is a semi-parametric approach because the error term of the model is not specified, and hence there is no need to select a distribution for the error term ε_t. Another way to deal with the error term ε_t is to use a parametric approach proposed by Gilchrist (2000). His model says that the conditional quantile function of y_t is given by

$$Q_{y_t}(\tau \mid \mathbf{y}_{t-1}) = a_0 + a_1 y_{t-1} + \cdots + a_k y_{t-k} + \eta Q(\tau, \boldsymbol{\gamma}), \tag{3}$$

where a_i, $i = 0, \ldots, k$, η and $\boldsymbol{\gamma}$ are the model parameters.

It is worth of discussing the differences between models (1), (2) and (3). Model (1) estimates the conditional mean of y_t and residual variance σ^2, and model (2) estimates the specific τ^{th} conditional quantile of y_t for $\tau \in (0, 1)$. So if a sequence of quantiles are estimated, then the whole conditional distribution function of y_t can be estimated by using the estimated conditional quantiles. Model (3) estimates the conditional distribution of y_t directly via the conditional quantile function of y_t. We know that model (1) is not suitable for non-Gaussian time series because it is difficult to identify the distribution of the error term ε_t. Model (2) does not depend on the choice of the distribution of ε_t, but we will see in Sections 3 and 4 that the estimated quantiles may not be consistent. Model (3) depends on the choices of $Q(\tau, \boldsymbol{\gamma})$ which defines the distribution of the error term, but the estimated quantiles are naturally consistent. However, the choice of $Q(\tau, \boldsymbol{\gamma})$ is very flexible compared with model (1) due to the properties of quantile functions. For example, the sum of quantile functions gives a new quantile function; the product of two positive quantile functions is also a quantile function; a proper transformation of a simple quantile function results in another qunatile function etc. Each quantile function defines a distribution, so the properties of quantile functions enable us to construct a proper model (3) for an observed time series and to deal with non-Gaussian time series very flexibly.

Gilchrist (2000) also discussed several methods for estimating the parameters of model (3). However, due to the difficulties involved in the maximum likelihood estimation method, he did not focus on the maximum likelihood estimation method, which motivated our current research.

In this paper, we propose a Bayesian approach to non-Gaussian autoregressive time series models. The general methodology of the Bayesian approach is presented in Section 2. Then in Section 3, we carry out a simulation study and compare our approach with the semi-parametric approach. Applications of the developed methodology to real time series are given in Section 4. Finally, some comments and further discussions are given in Section 5.

2 The MCMC method

Let $y_1, \ldots, y_n$ be an observed time series from model (3), and let $\boldsymbol{\beta} = (a_0, \ldots, a_k, \eta, \boldsymbol{\gamma})$. Then the conditional likelihood of $y_{k+1}, \ldots, y_n$ conditional on $\mathbf{y}_k = (y_1, \ldots, y_k)^\top$ is given by

$$L(y_{k+1}, \ldots, y_n \mid \mathbf{y}_k, \boldsymbol{\beta}) = \prod_{t=k+1}^{n} f(y_t \mid \mathbf{y}_{t-1}, \boldsymbol{\beta}) = \prod_{t=k+1}^{n} \frac{1}{\frac{\partial Q_{y_t}(\tau,\gamma)}{\partial \tau} \big|_{\tau=\tau_t}},$$

where τ_t, $t = k+1, \ldots, n$, satisfy

$$y_t = a_0 + a_1 y_{t-1} + \cdots + a_k y_{t-k} + \eta Q(\tau_t, \boldsymbol{\gamma}). \tag{4}$$

In general cases, equation (4) needs to be solved numerically.

Let $\pi(\boldsymbol{\beta})$ be the prior density function of the parameters. Then the posterior density function of $\boldsymbol{\beta}$ is given by

$$\pi(\boldsymbol{\beta} \mid \mathbf{y}_n) \propto L(y_{k+1}, \ldots, y_n \mid \mathbf{y}_k, \boldsymbol{\beta})\pi(\boldsymbol{\beta}), \quad \boldsymbol{\beta} \in \Omega,$$

where Ω is the parameter space. If $\pi(\boldsymbol{\beta} \mid \mathbf{y}_n)$ is properly defined on Ω, then a MCMC method can be developed to estimate the model parameters. See, for example, Chib and Greenberg (1995). In this paper, a general random walk sampler has been designed which is given below.

Let $\boldsymbol{\beta}$ be the current value of the parameters, τ_t, $t = k+1, \ldots, n$, the corresponding probabilities associated with $\boldsymbol{\beta}$, $\boldsymbol{\beta}'$ the proposed value and τ_t' the associated probabilities. Then a general random walk MCMC sampler for the model can be designed as follows. (a) Obtain a proposed value $\boldsymbol{\beta}'$ from, say, $g(\boldsymbol{\beta}, \boldsymbol{\beta}')$, such that $\boldsymbol{\beta}' \in \Omega$. (b) Solve equation (4) for τ_t'. (c) Accept the proposal with probability $\min\{AB, 1\}$, where

$$A = \frac{\pi(\boldsymbol{\beta}' \mid \mathbf{y}_n)}{\pi(\boldsymbol{\beta} \mid \mathbf{y}_n)}, \quad B = \frac{g(\boldsymbol{\beta}', \boldsymbol{\beta}) / \int_\Omega g(\boldsymbol{\beta}', \boldsymbol{\beta}) d\boldsymbol{\beta}}{g(\boldsymbol{\beta}, \boldsymbol{\beta}') / \int_\Omega g(\boldsymbol{\beta}, \boldsymbol{\beta}') d\boldsymbol{\beta}'}.$$

If the sampler is run long enough, then we would expect that the distribution of the simulated Markov chain will converge to the posterior distribution of the parameters.

Note that if the support of g is also Ω, then the integral involved in B will be 1. Hence it is straight forward to evaluate B. However, this is not true in general, which makes it difficult to evaluate the value of B as the shape of Ω can be very complicated. So in this paper we propose to use a simulation method to estimate any integrals involved in B.

For illustration purposes, we take $Q(\tau, \boldsymbol{\gamma})$ as the exponential quantile function throughout the paper. That is,

$$Q(\tau, \boldsymbol{\gamma}) = -\frac{1}{\lambda} \ln(1-\tau), \quad 0 \le \tau < 1.$$

Therefore, model (3) can be rewritten as

$$Q_{y_t}(\tau \mid \mathbf{y}_{t-1}) = a_0 + a_1 y_{t-1} + \cdots + a_k y_{t-k} - \frac{\ln(1-\tau)}{\gamma}, \tag{5}$$

with $\gamma = \lambda/\eta$ and $\boldsymbol{\beta} = (a_0, \ldots, a_k, \gamma)$. Furthermore, the conditional likelihood of y_t, $t = k+1, \ldots, n$, is given by

$$L(y_{k+1}, \ldots, y_n \mid \mathbf{y}_k, \boldsymbol{\beta}) = \prod_{t=k+1}^{n} f(y_t \mid \mathbf{y}_{t-1}, \boldsymbol{\beta}) = \prod_{t=k+1}^{n} \gamma(1-\tau_t),$$

where τ_t is the solution of

$$y_t = a_0 + a_1 y_{t-1} + \cdots + a_k y_{t-k} - \frac{\ln(1-\tau)}{\gamma}. \tag{6}$$

In this special case, τ_t can be found exactly.

Let the prior density function of the parameters be given by

$$\pi(\boldsymbol{\beta}) = \prod_{i=0}^{k} \pi(a_i)\pi(\gamma),$$

where $\pi(a_i)$ is a normal density function with zero mean and variance $\tilde{\sigma}^2_{a_i}$, and $\pi(\gamma) = \alpha e^{\alpha\gamma}$. Then it can be shown that the posterior distribution is well defined on $\Omega = \Omega_1 \times \Omega_2$, where $\Omega_1 = \{(a_0, \ldots, a_k) \mid a_0 + a_1 y_{t-1} + \cdots + a_k y_{t-k} \leq y_t,\ t = 1, \ldots, n\}$, and $\Omega_2 = (0, \infty)$. Hence, the random walk sampler can be used to estimate the model parameters, where the choice of the proposal distributions is very flexible. In this paper, we take

$$g(\boldsymbol{\beta}, \boldsymbol{\beta}') = \prod_{i=0}^{k} g_i(a_i, a_i') g_{k+1}(\gamma, \gamma'),$$

where $g_i(a_i, a_i') \sim N(a_i, \sigma^2_{a_i})$ and $g_{k+1}(\gamma, \gamma') \sim N(\gamma, \sigma^2_{\gamma})$.

It is worth mentioning that $\pi(a_i)$ can be taken as any proper density functions. This is because the likelihood function is always less than γ^{n-k} for any $0 \leq \tau_t < 1$, $t = k+1, \ldots, n$. While for $\pi(\gamma)$, we need to choose one that makes the posterior distribution proper.

In the following section, we carry out a simulation study to check the performance of the method.

3 Simulation study

If $k = 2$, then model (5) becomes

$$Q_{y_t}(\tau \mid \mathbf{y}_{t-1}) = a_0 + a_1 y_{t-1} + a_2 y_{t-2} - \frac{\ln(1-\tau)}{\gamma}. \quad (7)$$

Let $a_0 = -0.6$, $a_1 = 0.3$, $a_2 = 0.6$ and $\gamma = 1.6$. Starting from $y_1 = y_2 = 0$, we simulated a long time series of length 9200 from the above model. By deleting the first 9000 values we obtained a simulated time series of length 200. The prior information is specified by the values of $\tilde{\sigma}_{a_i}$, $i = 0, 1, 2$, and α. We randomly took $\tilde{\sigma}_{a_i} = 10$ and $\alpha = 0.5$ which are rather large indicating weak prior information. For other values we have very similar results. The starting values were chosen as $(\min_{1 \le t \le n} y_t, 0, 0, \gamma_0)$, where γ_0 is a random sample from the prior distribution of γ. It can be shown that this initial value is in the support Ω of the posterior distribution. For this simulation study, the initial value was $(-2.011,\ 0,\ 0,\ 0.110)$.

The time series plot of the simulated data together with the acf and partial acf plots of the time series are in Figure 1.

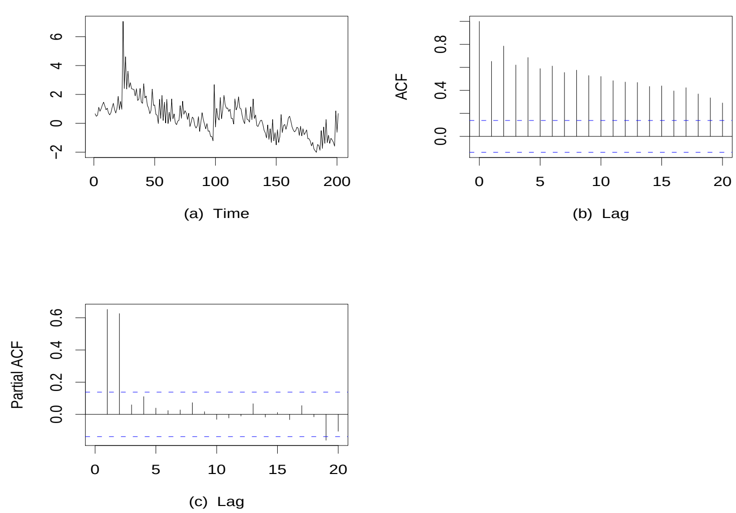

Figure 1: (a) Time series plot of the simulated series. (b) acf plot of the simulated series. (c) Partial acf plot of the simulated series.

We ran the sampler 20,000 steps. After the burn-in period (first 10,000 steps), we save samples every 50 steps. So the collected sample for each parameter is of length 200. Figure 2 shows the histograms of the collected samples, where the solid vertical lines indicate the positions of the true parameter values, and the dashed vertical lines the Bayesian estimates of the parameters. It is clear that the true values are well within the posterior marginals.

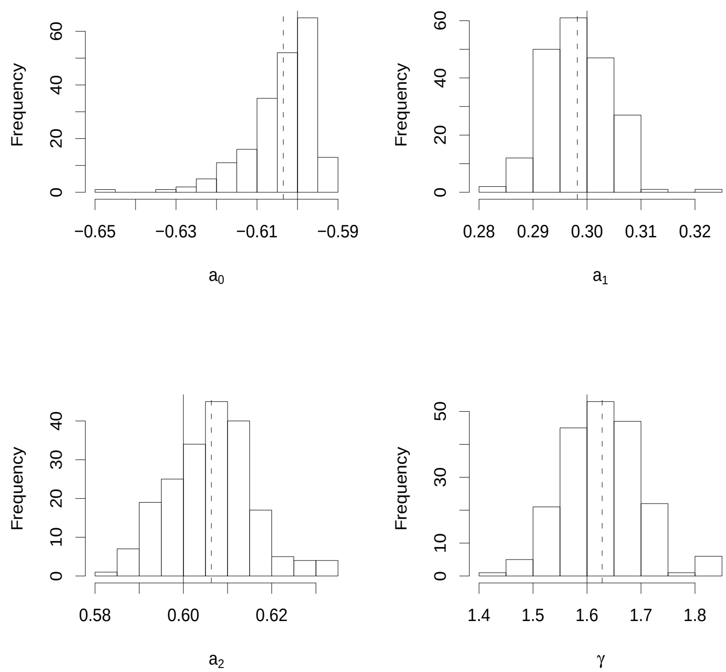

Figure 2: Histograms of the collected samples for each parameter in the Simulation Study.

So the fitted model is given by

$$Q_{y_t}(\tau \mid \mathbf{y}_{t-1}) = -0.6035 + 0.2982 y_{t-1} + 0.6063 y_{t-2} - \frac{\ln(1-\tau)}{1.6280}, \tag{8}$$

which is very similar to the true model. Figure 3 shows the conditional quantiles obtained from the true model (7) and the fitted (8) corresponding to (from the bottom to the top) $\tau = 0.05, 0.5, 0.95$ and 0.995 respectively. It is clear that the conditional distribution of the time series at any time point is skewed to the right, which is what we should have expected. Furthermore, the estimated conditional quantiles can not be distinguished from the true conditional quantiles indicating that they are of very

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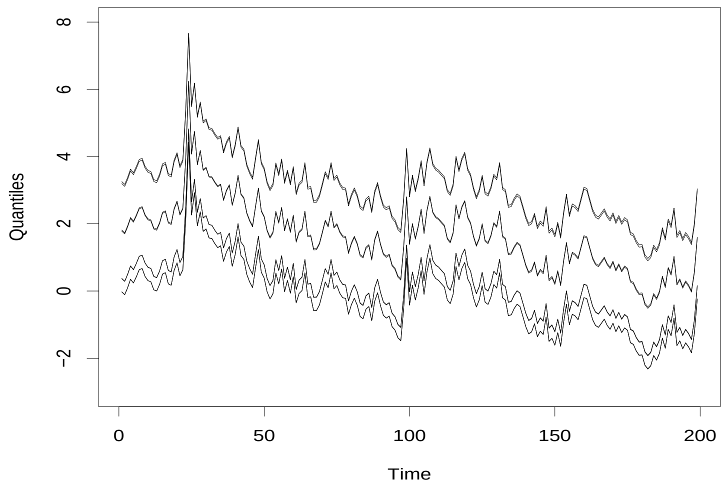

Figure 3: Conditional quantiles from the true model (7) and the fitted model (8) corresponding to (from the bottom to the top) $\tau = 0.05, 0.5, 0.95$ and 0.995 respectively.

good accuracy.

Corresponding to model (7), the semi-parametric model proposed by Koenker (2005) is given by

$$q^{\tau}_{y_t|\mathbf{y}_{t-1}} = a^{\tau}_0 + a^{\tau}_1 y_{t-1} + a^{\tau}_2 y_{t-2}, \tag{9}$$

where the parameters depend on the values of τ. As this model does not depend on the error term, the semi-parametric model (9) can also be applied to the same simulated time series. We used the free statistical software R to fit the τ^{th} conditional quantiles to the simulated time series, where $\tau = 0.05, 0.5, 0.95, 0.995$. Figure 4 shows the conditional quantiles from the true model (7) (darker curves) and the semi-parametric model (9) (lighter curves). It is seen that when τ is small, the conditional quantiles from both models are very similar. However, when τ approaches to extremes, the differences between two conditional quantiles are getting larger. In fact, this is one of the problems with the semi-parametric approach, because when τ approaches to 0 or 1, no data available which makes it impossible to estimate the parameters, leading to undesired estimated quantiles.

Furthermore, although not shown here to save space, the conditional quantiles obtained from the semi-parametric approach cross over, meaning that the estimated quantiles are not consistent. However, the parametric approach does not suffer such

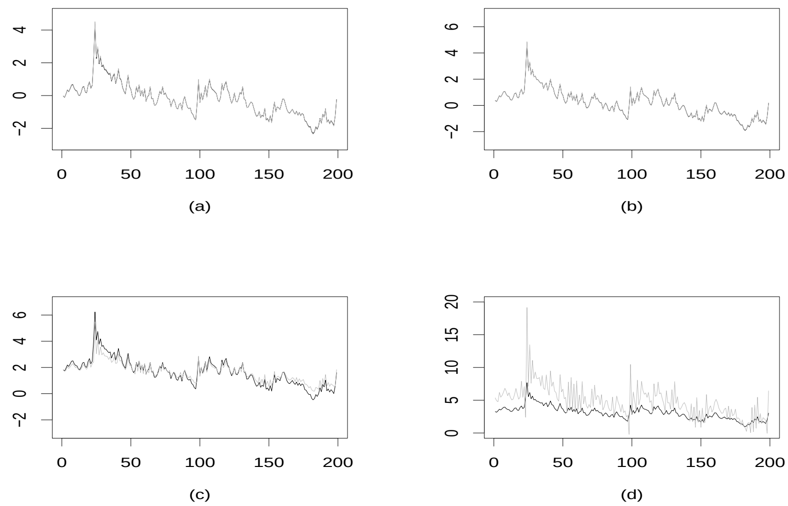

Figure 4: Conditional quantiles from the true model (7) (darker curves) and the fitted model (9) (lighter curves) corresponding to (a) $\tau = 0.05$, (b) $\tau = 0.5$, (c) $\tau = 0.95$ and (d) $\tau = 0.995$.

problems.

We also fitted model (5) with order $k = 0, 1, 2, 3$ and 4 to the same simulated data in order to see whether the methodology enables us to identify the true model. If the fitted model is proper, then we would expect that the standardized residuals

$$e_t = \gamma(y_t - a_0 - a_1 y_{t-1} - \cdots - a_k y_{t-k}), \quad t = k+1, \ldots, n,$$

should follow the unit rate exponential distribution defined by $Q(\tau) = -\ln(1-\tau)$. Therefore, we should expect that the points on the plot of the sample quantiles of the standardized residuals e_t against the quantiles of the unit rate exponential distribution should be approximately along a straight line. Figure 5 shows the QQ-plots for different fitted models. It is seen that for this data set the models of order $k = 0, 1$ perform worse than the models with order $k = 2, 3, 4$. However, for the models with $k \geq 2$, the QQ-plots look very similar. If we check the AIC values calculated by using the Bayesian estimates of the parameters for each fitted model, we see that (Figure 5(f)) the model of order 2 has the minimum value of AIC which is much smaller than any other models and which is in fact the true order of the model. It is worth mentioning that, strictly speaking, the use of the AIC in this context is not appropriate, but it gives us a rough idea on the penalty

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of extra parameters. Combining all the above results we conclude that the method works very well.

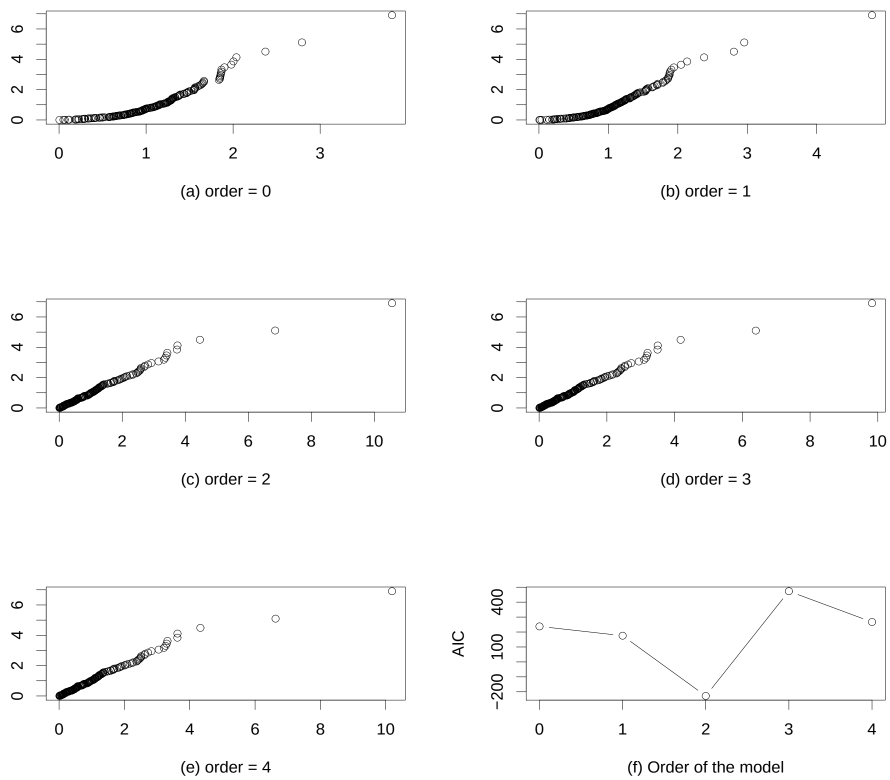

Figure 5: QQ-plots (a)-(e) and the AIC values (f) of the fitted models in the simulation study.

4 Applications to real time series

4.1 Levels of Lake Huron 1875-1972

In this section, we apply the methodology to the Levels of Lake Huron during 1875-1972. The time series plot and the acf, partial acf plots are given in Figure 6. Compared with Figure 1, we see that the Lake Huron time series shows similar autocorrelation structures. So we consider to apply the methodology developed here to this time series.

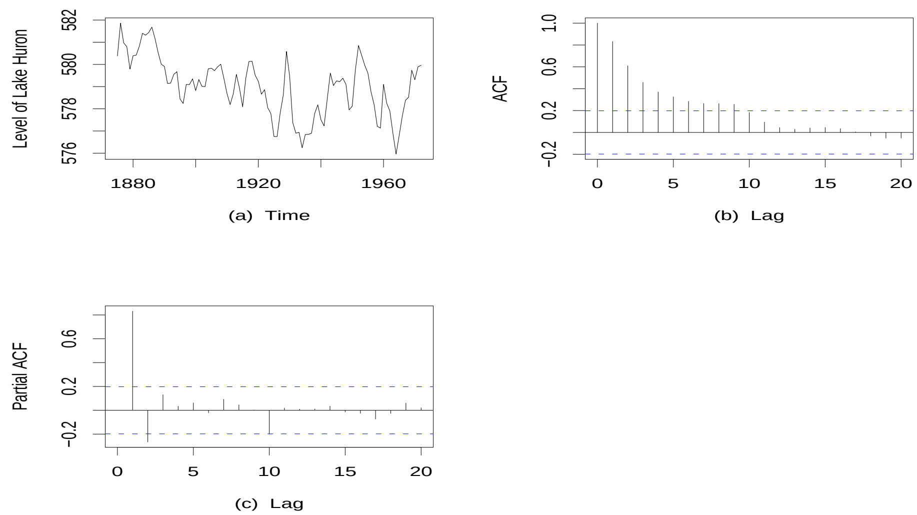

Figure 6: (a) Time series plot of the Lake Huron series. (b) acf plot of the Lake Huron time series. (c) Partial acf plot of the Lake Huron time series.

We fit a sequence of models to the time series with order $k = 1, 2, 3$ and 4. For each fitted model, we run our sampler 550,000 steps. After burn-in period, we save samples every 10 steps. We also checked the AIC values and the QQ-plots of the fitted models. Both AIC values and the QQ-plots suggest that the model with order $k = 3$ is the best. To save space, we only show the histograms of the collected samples (burn-in period: first 150,000 steps) and the QQ-plot of the standardized residuals in Figure 7, where the continuous curves are the density plots, the vertical lines give the locations of the estimated parameter values which are the sample means. So the fitted model is given by

$$Q_{y_t}(\tau \mid \mathbf{y}_{t-1}) = 1.238 + 1.187y_{t-1} - 0.537y_{t-2} + 0.345y_{t-3} - \frac{\log(1-\tau)}{0.767}. \quad (10)$$

The corresponding semi-parametric model is given by

$$q^{\tau}_{y_t|\mathbf{y}_{t-1}} = a^{\tau}_0 + a^{\tau}_1 y_{t-1} + a^{\tau}_2 y_{t-2} + a^{\tau}_3 y_{t-3} \quad (11)$$

The fitted conditional quantiles for $\tau = 0.05$, 0.25, 0.5, 0.75, 0.95 and 0.995 are given in Figure 8. As expected, the inconsistency behavior on the estimated conditional qunatiles from model (11) occurred again.

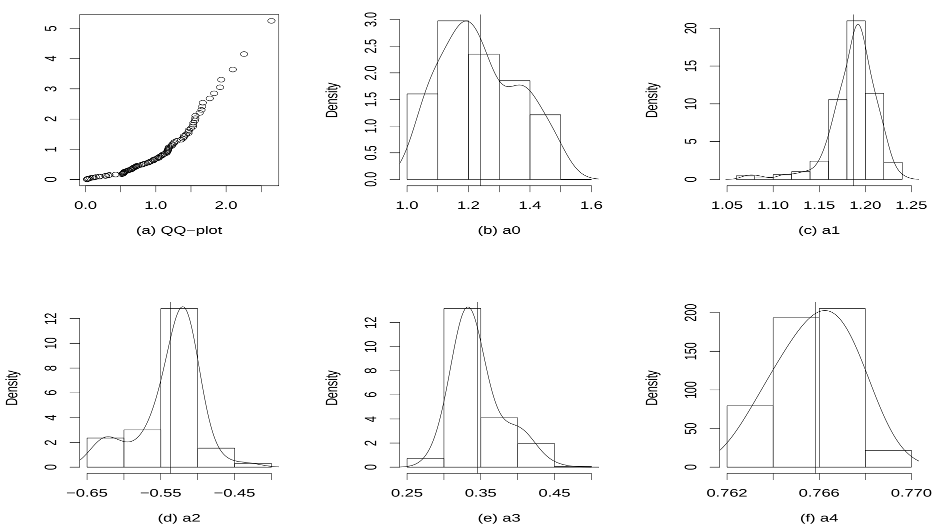

Figure 7: (a) The QQ-plot of the residuals of model (10). (b) $\sim$ (f): Histograms, density plots (continuous curves) and the estimated parameter values (darker vertical lines) of the collected samples.

Note that the QQ-plot in Figure 7 shows some doubts on the quality of the fitted model. However, compared with the semi-parametric approach, model (10) still performs better, which can be further confirmed by the following investigation.

We further estimated the conditional quantile functions of y_7 and y_{33} by using fitted model (10) and the semi-parametric model (11). Note that these two time points were chosen randomly, and we can estimate the conditional quantile functions of y_t for any t similarly. Conditional on the history $\mathbf{y}_6$ and $\mathbf{y}_{32}$, we estimated the quantile functions of y_7 and y_{33} respectively, and the quantile function plots are shown in Figure 9, where thicker curves are the quantile functions obtained by using model (11) and thinner curves are the quantile functions obtained by using model (10). It is clear that the conditional quantile functions estimated by the semi-parametric approach are not monotonic functions of τ at these two time points.

4.2 Major European stock indices

Consider the daily closing prices of major European stock indices in the period of 1995-1996. Specifically, we consider the Germany DAX Index, which is most commonly cited benchmark for measuring the returns posted by stocks on the Frankfurt Stock Exchange. It is a performance-based index, which means that any dividends

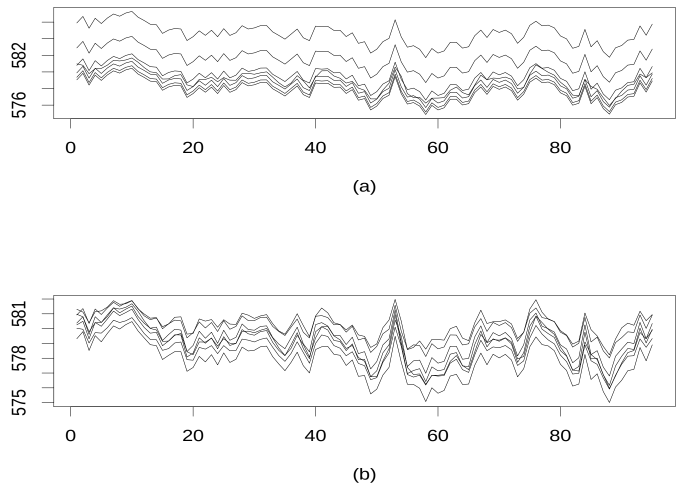

Figure 8: The fitted conditional quantiles for (from the bottom to the top) $\tau = 0.05$, 0.25, 0.5, 0.75, 0.95 and 0.995 from (a) model (10) and (b) model (11), where the darker curves are the Lake Huron time series.

and other events are rolled into the index's final calculation. The time series considered here is of length 400 and is shown in Figure 10. We fitted a sequence of model (5) to the time series and found that the model with order $k = 4$ provides the best fit. Figure 11 shows the QQ-plot of the residuals and the histograms of the collected samples from the MCMC method, where the vertical lines indicate the positions of the estimated parameter values. Therefore, the fitted model is

$$Q_{y_t}(\tau \mid \mathbf{y}_{t-1}) = 0.756 + 0.944 y_{t-1} + 0.343 y_{t-2} + 0.250 y_{t-3} - 0.571 y_{t-4} - \frac{\log(1-\tau)}{0.011}. \tag{12}$$

The fitted conditional quantiles are shown in Figure 12. Based on the fitted model (12), we can further obtain one-step ahead predictive quantile function of y_{401} given $\mathbf{y}_{400}$. Figure 13 shows the conditional density function of y_{401}, where the continuous vertical line gives the position of the true observed value, the dashed and the dotted lines correspond to the predicted median and the predicted mean from the fitted model (12) respectively. It is seen that the observed value is well within the predictive distribution range.

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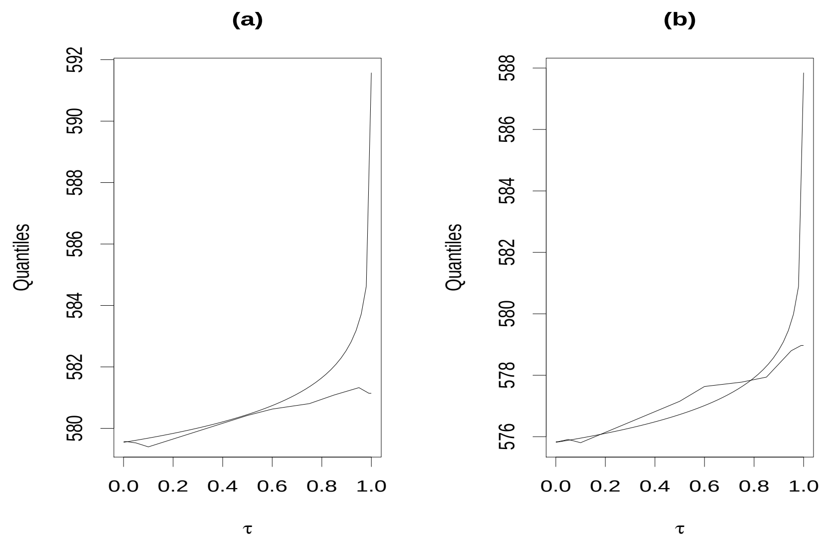

Figure 9: The estimated conditional quantile functions of (a) y_7 and (b) y_{33}, where the thinner curves were obtained from model (10), while the thicker curves were obtained from model (11).

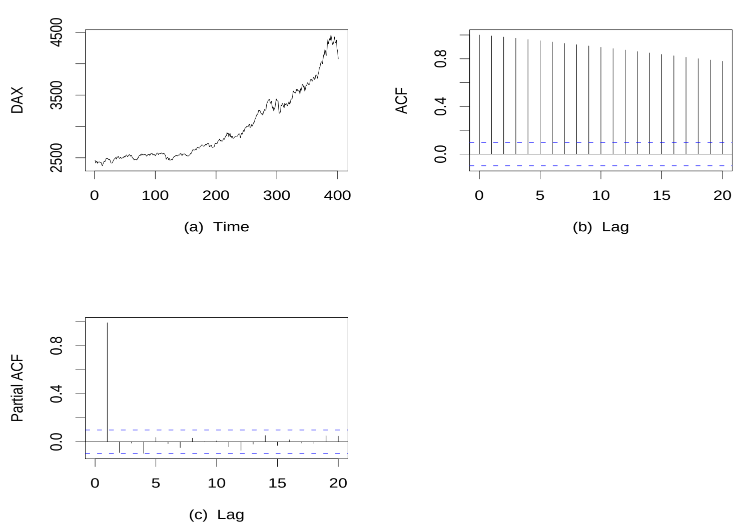

Figure 10: (a) Time series plot of the Germany DAX series. (b) acf plot of the Germany DAX series. (c) Partial acf plot of the Germany DAX series.

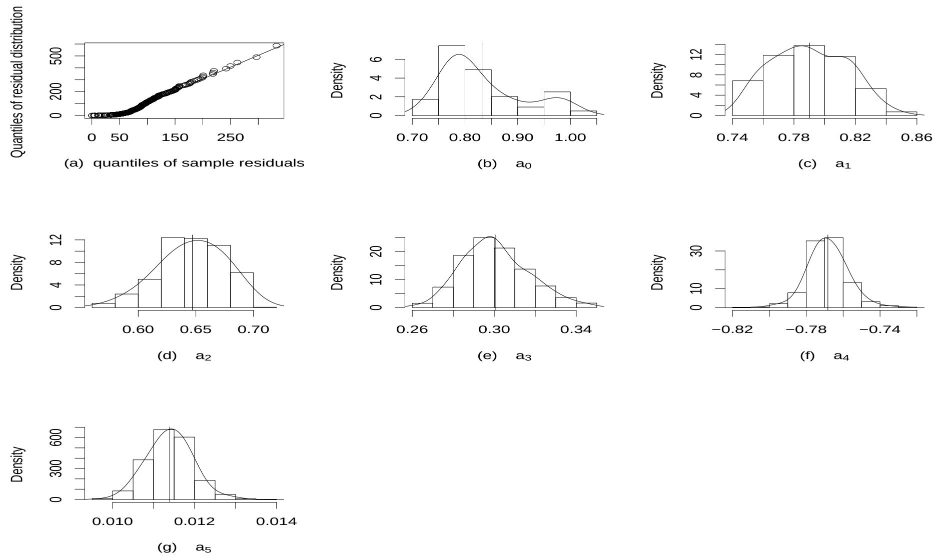

Figure 11: (a) The QQ-plot of the residuals of model (12). (b) $\sim$ (g): Histograms, density plots (continuous curves) and the estimated parameter values (vertical lines) of the collected samples.

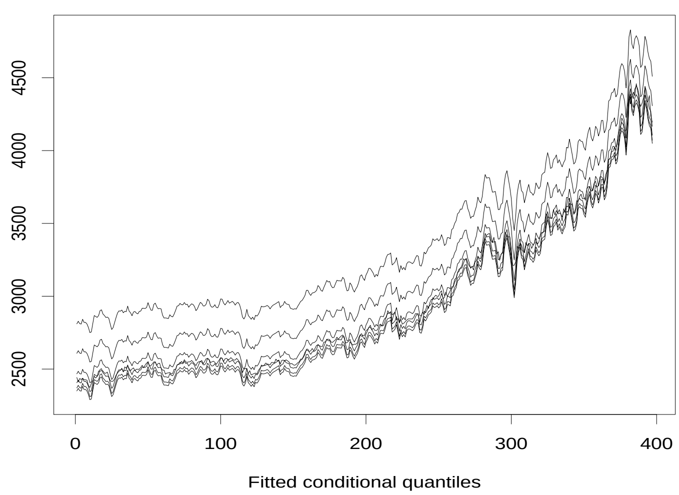

Figure 12: Fitted conditional quantiles (lighter curves), from the bottom to the top, for $\tau = 0.05, 0.25, 0.5, 0.75, 0.95, 0.995$. Darker curve is the true time series.

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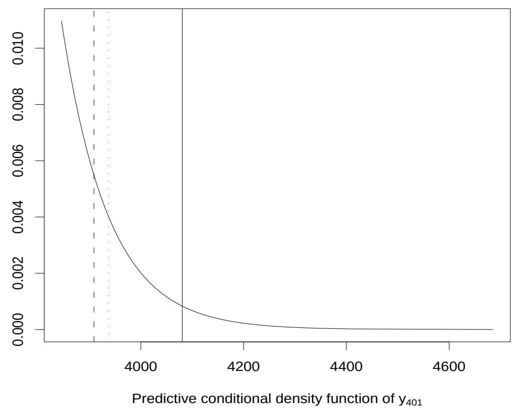

Figure 13: Predictive conditional density function of y_{401}. The continuous vertical line gives the position of the true observed value, the dashed and the dotted lines correspond to the predicted median and the predicted mean from the fitted model (12) respectively.

5 Further comments and conclusions

We have presented a Bayesian approach to non-Gaussian autoregressive time series via quantile functions. Our approach has been compared with the semi-parametric approach through a simulation study and applications to real time series. All the obtained results show that the method works well.

It is worth mentioning that it may be possible to use a tailored proposal along the lines of Chib and Greenberg (1994) to improve the mixing of the Markov chain obtained from the random walk sampler. Further investigation will be carried out in the future.

The methodology can be further developed to deal with nonlinear time series and the case where the scale parameter of the model also depends on the values of the time series. Furthermore, multiple step ahead forecasting methods from model (3) need to be developed in the future, and new methodology also needs to be developed to deal with multivariate time series. The progress on these issues will be reported elsewhere.

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