Conditional Quantile Regression Article Proposal

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Overview

- Let a time series be given by

$$y_t = A(L)y_t + \beta x_t + \varepsilon_t,$$

where the distribution of ε_t is unknown. In this case, the usage of a parametric model is hardly atrrative.

- Quantile regression is one technique available to model this time series dynamics, by estimating a phin grid of α-quantiles at once and forming a data-driven conditional distribution.
- We explore different strategies of estimating the Conditional Quantile Regression focused on approaching the conditional distribution or $y_{t+h|t}$, for a given horizon h.

Linear Models

Linear Models - Formulation

$$\min_{\beta_{0\alpha},\beta_{\alpha},\varepsilon^+_{t\alpha},\varepsilon^-_{t\alpha}} \sum_{\alpha\in A}\sum_{t\in T}\left(\alpha\varepsilon^+_{t\alpha} + (1-\alpha)\varepsilon^-_{t\alpha}\right) \tag{1}$$

subject to

$$\varepsilon^+_{t\alpha} - \varepsilon^-_{t\alpha} = y_t - \beta_{0\alpha} - \beta_{\alpha}^T x_t, \forall t \in T, \qquad \forall \alpha \in A, \tag{2}$$

$$\varepsilon^+_{t\alpha}, \varepsilon^-_{t\alpha} \geq 0, \qquad \forall t \in T, \forall \alpha \in A, \tag{3}$$

$$\beta_{0\alpha} + \beta_{\alpha}^T x_t \leq \beta_{0\alpha'} + \beta_{\alpha'}^T x_t, \qquad \forall t \in T, \forall(\alpha,\alpha') \in A\times A, \alpha < \alpha', \tag{4}$$

Linear Models - Resume

- As there are many explanatory variables for y_t, it is interesting to do a regularization process in order to select only a subset to compose the model.
- The next slides are going to cover a few different ways to achieve this using Mixed Integer Linear Programming.

LM

Regularization by MILP - Resume

- MILP models allow only K variables to be used for each α-quantile. This means that only K coefficients $\beta_{p\alpha}$ may have nonzero values, for each α-quantile. It must be guaranteed by constraints on the optimization problem.
- We present three forms of grouping probabilities while selecting variables

One model for each α-quantile - Formulation

$$\min_{\beta_{0\alpha},\beta_{\alpha},z_{p\alpha},\varepsilon^{+}_{t\alpha},\varepsilon^{-}_{t\alpha}} \sum_{\alpha\in A}\sum_{t\in T}\left(\alpha\varepsilon^{+}_{t\alpha} + (1-\alpha)\varepsilon^{-}_{t\alpha}\right) \tag{5}$$

subject to

$$\varepsilon^{+}_{t\alpha} - \varepsilon^{-}_{t\alpha} = y_t - \beta_{0\alpha} - \sum_{p=1}^{P}\beta_{p\alpha}x_{t,p}, \qquad \forall t \in T, \forall\alpha \in A, \tag{6}$$

$$\varepsilon^{+}_{t\alpha}, \varepsilon^{-}_{t\alpha} \geq 0, \qquad \forall t \in T, \forall\alpha \in A, \tag{7}$$

$$-Mz_{p\alpha} \leq \beta_{p\alpha} \leq Mz_{p\alpha}, \qquad \forall\alpha \in A, \forall p \in P, \tag{8}$$

$$\sum_{p=1}^{P} z_{p\alpha} \leq K, \qquad \forall\alpha \in A, \tag{9}$$

$$z_{p\alpha} \in \{0,1\}, \qquad \forall\alpha \in A, \forall p \in P, \tag{10}$$

$$\beta_{0\alpha} + \beta^{T}_{\alpha}x_t \leq \beta_{0\alpha'} + \beta^{T}_{\alpha'}x_t, \qquad \forall t \in T, \forall(\alpha,\alpha') \in A\times A, \alpha < \alpha', \tag{11}$$

Defining groups for α-quantiles - Formulation

$$\min_{\beta_{0\alpha},\beta_{\alpha},z_{p\alpha}\varepsilon^{+}_{t\alpha},\varepsilon^{-}_{t\alpha}} \quad \sum_{\alpha\in A}\sum_{t\in T}\left(\alpha\varepsilon^{+}_{t\alpha}+(1-\alpha)\varepsilon^{-}_{t\alpha}\right) \tag{12}$$

$$\text{s.t} \quad \varepsilon^{+}_{t\alpha}-\varepsilon^{-}_{t\alpha}=y_t-\beta_{0\alpha}-\sum_{p=1}^{P}\beta_{p\alpha}x_{t,p}, \qquad \forall t\in T, \forall\alpha\in A, \tag{13}$$

$$\varepsilon^{+}_{t\alpha},\varepsilon^{-}_{t\alpha}\geq 0, \qquad \forall t\in T, \forall\alpha\in A, \tag{14}$$

$$-Mz_{p\alpha g}\leq\beta_{p\alpha}\leq Mz_{p\alpha g}, \qquad \forall\alpha\in A, \forall p\in P, \forall g\in G \tag{15}$$

$$z_{p\alpha g}:=2-(1-z_{pg})-I_{g\alpha} \tag{16}$$

$$\sum_{p=1}^{P}z_{pg}\leq K, \qquad \forall g\in G, \tag{17}$$

$$\beta_{0\alpha}+\beta_{\alpha}^{T}x_t\leq\beta_{0\alpha'}+\beta_{\alpha'}^{T}x_t, \qquad \forall t\in T, \forall(\alpha,\alpha')\in A\times A, \alpha<\alpha', \tag{18}$$

$$\sum_{g\in G}I_{g\alpha}=1, \qquad \forall\alpha\in A, \tag{19}$$

$$I_{g\alpha},z_{pg}\in\{0,1\}, \qquad \forall p\in P, \quad \forall g\in G, \tag{20}$$

Defining groups by the introduction of switching variable - Formulation

$$\min_{\beta_{0\alpha}, \beta_{\alpha}, z_{p\alpha} \varepsilon^{+}_{t\alpha}, \varepsilon^{-}_{t\alpha}} \quad \sum_{\alpha \in A} \sum_{t \in T} \left(\alpha \varepsilon^{+}_{t\alpha} + (1-\alpha) \varepsilon^{-}_{t\alpha} \right) \tag{21}$$

$$\text{s.t} \quad \varepsilon^{+}_{t\alpha} - \varepsilon^{-}_{t\alpha} = y_t - \beta_{0\alpha} - \sum_{p=1}^{P} \beta_{p\alpha} x_{t,p}, \qquad \forall t \in T, \forall \alpha \in A, \tag{22}$$

$$\varepsilon^{+}_{t\alpha}, \varepsilon^{-}_{t\alpha} \geq 0, \qquad \forall t \in T, \forall \alpha \in A, \tag{23}$$

$$-Mz_{p\alpha} \leq \beta_{p\alpha} \leq Mz_{p\alpha}, \qquad \forall \alpha \in A, \forall p \in \{1, \ldots, P\}, \tag{24}$$

$$\sum_{p=1}^{P} z_{p\alpha} \leq K, \qquad \forall \alpha \in A, \tag{25}$$

$$z_{p\alpha} \in \{0, 1\}, \qquad \forall \alpha \in A, \forall p \in \{1, \ldots, P\}, \tag{26}$$

$$\beta_{0\alpha} + \beta^{T}_{\alpha} x_t \leq \beta_{0\alpha'} + \beta^{T}_{\alpha'} x_t, \qquad \forall t \in T, \forall (\alpha, \alpha') \in A \times A, \alpha < \alpha', \tag{27}$$

$$z_{p\alpha} - z_{p\alpha+1} \leq m_{p\alpha}, \qquad \forall \alpha \in A', \quad \forall p \in P \tag{28}$$

$$\sum_{\alpha \in A'} r_{\alpha} \leq |G| - 1 \tag{29}$$

$$\tag{30}$$

where $A' = A \setminus \{|A|\}$

Nonparametric model

Nonparametric model - Formulation

$$
\begin{aligned}
\min_{q_{\alpha t},\delta_t^+,\delta_t^-,\xi_t} \quad & \sum_{\alpha\in A}\sum_{t\in T'}\left(\alpha\delta_{t\alpha}^+ + (1-\alpha)\delta_{t\alpha}^-\right) \\
& +\lambda_1\sum_{t\in T'}\gamma_{t\alpha} + \lambda_2\sum_{t\in T'}\xi_{t\alpha} \\
s.t. \quad & \delta_t^+ - \delta_{t\alpha}^- = y_t - q_{t\alpha}, && \forall t\in T', \forall\alpha\in A, \\
& D_{t\alpha}^1 = \frac{q_{\alpha t+1}-q_{\alpha t}}{x_{t+1}-x_t}, && \forall t\in T', \forall\alpha\in A, \\
& D_{t\alpha}^2 = \frac{\left(\frac{q_{\alpha t+1}-q_{\alpha t}}{x_{t+1}-x_t}\right)-\left(\frac{q_{\alpha t}-q_{\alpha t-1}}{x_t-x_{t-1}}\right)}{x_{t+1}-2x_t+x_{t-1}}. && \forall t\in T', \forall\alpha\in A, \\
& \gamma_{t\alpha} \geq D_{t\alpha}^1, && \forall t\in T', \forall\alpha\in A, \\
& \gamma_{t\alpha} \geq -D_{t\alpha}^1, && \forall t\in T', \forall\alpha\in A, \\
& \xi_{t\alpha} \geq D_{t\alpha}^2, && \forall t\in T', \forall\alpha\in A, \\
& \xi_{t\alpha} \geq -D_{t\alpha}^2, && \forall t\in T', \forall\alpha\in A, \\
& \delta_{t\alpha}^+, \delta_{t\alpha}^-, \gamma_{t\alpha}, \xi_{t\alpha} \geq 0, && \forall t\in T', \forall\alpha\in A, \\
& q_{t\alpha} \leq q_{t\alpha'}, && \forall t\in T', \forall(\alpha,\alpha')\in A\times A, \alpha <
\end{aligned}
$$

Nonparametric vs. Linear Model

- The nonparametric approach is more flexible to capture heteroscedasticity.

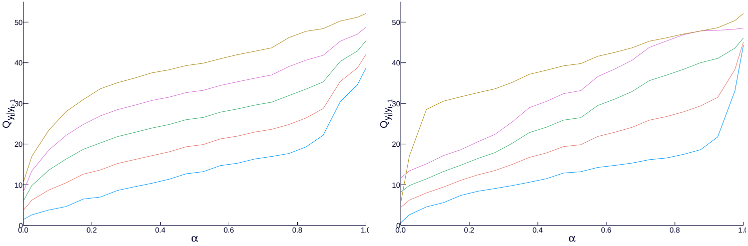

Figure 1: Estimated quantile functions, for different values of y_{t-1}. On the left using a linear model and using a nonparametric approach on the right.

Nonparametric vs. Linear Model

- This flexibility might lead to overfitting, if we don't select a proper penalty, as shown below:

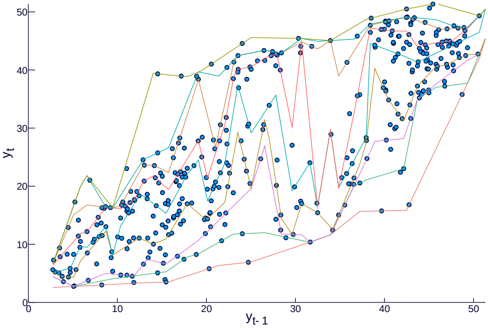

Figure 2: Example of a overfitted quantile function

Goals

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Goals of this work are:

- Using multiple quantiles to estimate the empirical conditional distribution of variables in a time series
- Producing a model identification methodology
- Natural Ressources and Financial applications