Conditional Quantile Regression Article Proposal

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Overview

Let a time series be given by

$$y_t = A(L)y_t + \beta x_t + \varepsilon_t,$$

where the distribution of ε_t is unknown. $(\ldots)$

Nonparametric model

Nonparametric model - Formulation

$$
\begin{aligned}
\min_{q_{\alpha t}, \delta_t^+, \delta_t^-, \xi_t} \quad & \sum\nolimits_{\alpha \in A} \sum\nolimits_{t \in T'} \left(\alpha \delta_{t\alpha}^+ + (1-\alpha) \delta_{t\alpha}^- \right) \\
& + \lambda_1 \sum\nolimits_{t \in T'} \gamma_{t\alpha} + \lambda_2 \sum\nolimits_{t \in T'} \xi_{t\alpha} \\
s.t. \quad & \delta_t^+ - \delta_{t\alpha}^- = y_t - q_{t\alpha}, && \forall t \in T', \forall \alpha \in A, \\
& D_{t\alpha}^1 = \frac{q_{\alpha t+1} - q_{\alpha t}}{x_{t+1} - x_t}, && \forall t \in T', \forall \alpha \in A, \\
& D_{t\alpha}^2 = \frac{\left(\frac{q_{\alpha t+1} - q_{\alpha t}}{x_{t+1} - x_t} \right) - \left(\frac{q_{\alpha t} - q_{\alpha t-1}}{x_t - x_{t-1}} \right)}{x_{t+1} - 2x_t + x_{t-1}}. && \forall t \in T', \forall \alpha \in A, \\
& \gamma_{t\alpha} \geq D_{t\alpha}^1, && \forall t \in T', \forall \alpha \in A, \\
& \gamma_{t\alpha} \geq -D_{t\alpha}^1, && \forall t \in T', \forall \alpha \in A, \\
& \xi_{t\alpha} \geq D_{t\alpha}^2, && \forall t \in T', \forall \alpha \in A, \\
& \xi_{t\alpha} \geq -D_{t\alpha}^2, && \forall t \in T', \forall \alpha \in A, \\
& \delta_{t\alpha}^+, \delta_{t\alpha}^-, \gamma_{t\alpha}, \xi_{t\alpha} \geq 0, && \forall t \in T', \forall \alpha \in A, \\
& q_{t\alpha} \leq q_{t\alpha'}, && \forall t \in T', \forall (\alpha, \alpha') \in A \times A, \alpha <
\end{aligned}
$$

Nonparametric vs. Linear Model

- The nonparametric approach is more flexible to capture heteroscedasticity.

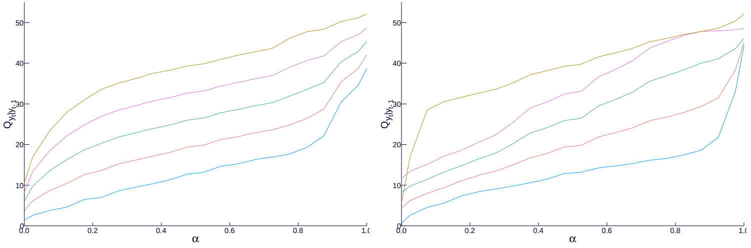


Figure 1: Estimated quantile functions, for different values of y_{t-1}. On the left using a linear model and using a nonparametric approach on the right.

Nonparametric vs. Linear Model

- This flexibility might lead to overfitting, if we don't select a proper penalty, as shown below:

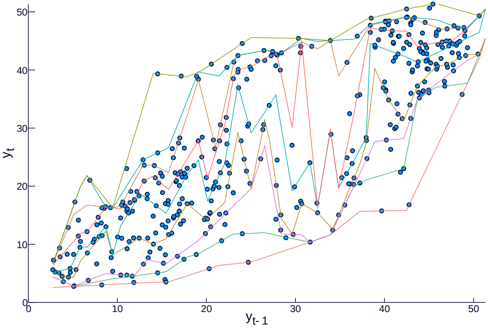


Figure 2: Example of a overfitted quantile function

Linear Models

Formulation

$$
\begin{aligned}
\min_{\beta_{0\alpha},\beta_{\alpha},\varepsilon^{+}_{t\alpha},\varepsilon^{-}_{t\alpha}} \quad & \sum_{\alpha\in A}\sum_{t\in T}\left(\alpha\varepsilon^{+}_{t\alpha} + (1-\alpha)\varepsilon^{-}_{t\alpha}\right) && && (11)\\
\text{s.t.} \quad & \varepsilon^{+}_{t\alpha} - \varepsilon^{-}_{t\alpha} = y_t - \beta_{0\alpha} - \beta^{T}_{\alpha}x_t, && \forall t \in T, \forall \alpha \in A, && (12)\\
& \varepsilon^{+}_{t\alpha}, \varepsilon^{-}_{t\alpha} \geq 0, && \forall t \in T, \forall \alpha \in A, && (13)\\
& \beta_{0\alpha} + \beta^{T}_{\alpha}x_t \leq \beta_{0\alpha'} + \beta^{T}_{\alpha'}x_t, && \forall t \in T, \forall (\alpha,\alpha') \in A \times A, \alpha < \alpha', && (14)
\end{aligned}
$$

One model for each α-quantile - Resume

In this part, we investigate the usage of MILP to select which variables are included in the model, by using a constraint which limits them to a number of K. This means that only K coefficients $\beta_{p\alpha}$ may have nonzero values, for each α-quantile. This assumption is modeled with binary variables $z_{p\alpha}$, which indicates whether $\beta_{p\alpha}$ is included or not.

One model for each α-quantile - Formulation

$$\min_{\beta_{0\alpha},\beta_{\alpha},z_{p\alpha}\varepsilon^{+}_{t\alpha},\varepsilon^{-}_{t\alpha}} \quad \sum_{\alpha\in A}\sum_{t\in T}\left(\alpha\varepsilon^{+}_{t\alpha}+(1-\alpha)\varepsilon^{-}_{t\alpha}\right) \tag{15}$$

$$\text{s.t} \quad \varepsilon^{+}_{t\alpha}-\varepsilon^{-}_{t\alpha}=y_t-\beta_{0\alpha}-\sum_{p=1}^{P}\beta_{p\alpha}x_{t,p}, \qquad \forall t\in T,\forall\alpha\in A, \tag{16}$$

$$\varepsilon^{+}_{t\alpha},\varepsilon^{-}_{t\alpha}\geq 0, \qquad \forall t\in T,\forall\alpha\in A, \tag{17}$$

$$-Mz_{p\alpha}\leq\beta_{p\alpha}\leq Mz_{p\alpha}, \qquad \forall\alpha\in A,\forall p\in P, \tag{18}$$

$$\sum_{p=1}^{P}z_{p\alpha}\leq K, \qquad \forall\alpha\in A, \tag{19}$$

$$z_{p\alpha}\in\{0,1\}, \qquad \forall\alpha\in A,\forall p\in P, \tag{20}$$

$$\beta_{0\alpha}+\beta_{\alpha}^{T}x_t\leq\beta_{0\alpha'}+\beta_{\alpha'}^{T}x_t, \qquad \forall t\in T,\forall(\alpha,\alpha')\in A\times A,\alpha<\alpha', \tag{21}$$

Defining groups for α-quantiles - Resume

- Now, adding groups of quantiles.
- Each probability α belongs to a group g.
- The total number of groups is limited to G.
- Incorporation of new integer variables.
- Total number of valid solution falls.

Defining groups for α-quantiles - Formulation

$$\min_{\beta_{0\alpha}, \beta_{\alpha}, z_{p\alpha} \varepsilon^{+}_{t\alpha}, \varepsilon^{-}_{t\alpha}} \quad \sum_{\alpha \in A} \sum_{t \in T} \left(\alpha \varepsilon^{+}_{t\alpha} + (1-\alpha) \varepsilon^{-}_{t\alpha} \right) \tag{22}$$

$$\text{s.t} \quad \varepsilon^{+}_{t\alpha} - \varepsilon^{-}_{t\alpha} = y_t - \beta_{0\alpha} - \sum_{p=1}^{P} \beta_{p\alpha} x_{t,p}, \qquad \forall t \in T, \forall \alpha \in A, \tag{23}$$

$$\varepsilon^{+}_{t\alpha}, \varepsilon^{-}_{t\alpha} \geq 0, \qquad \forall t \in T, \forall \alpha \in A, \tag{24}$$

$$-Mz_{p\alpha g} \leq \beta_{p\alpha} \leq Mz_{p\alpha g}, \qquad \forall \alpha \in A, \forall p \in P, \forall g \in G \tag{25}$$

$$z_{p\alpha g} := 2 - (1 - z_{pg}) - I_{g\alpha} \tag{26}$$

$$\sum_{p=1}^{P} z_{pg} \leq K, \qquad \forall g \in G, \tag{27}$$

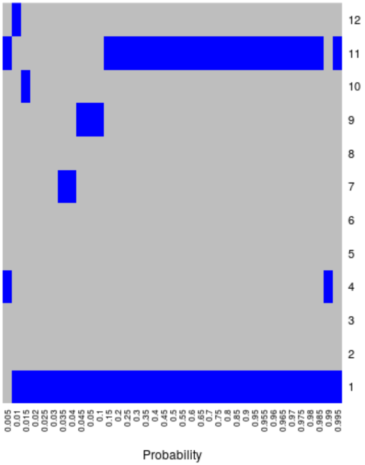
$$\beta_{0\alpha} + \beta^{T}_{\alpha} x_t \leq \beta_{0\alpha'} + \beta^{T}_{\alpha'} x_t, \qquad \forall t \in T, \forall (\alpha, \alpha') \in A \times A, \alpha < \alpha', \tag{28}$$

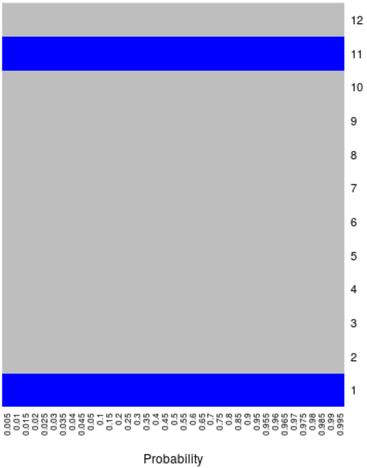
$$\sum_{g \in G} I_{g\alpha} = 1, \qquad \forall \alpha \in A, \tag{29}$$

$$I_{g\alpha}, z_{pg} \in \{0, 1\}, \qquad \forall p \in P, \quad \forall g \in G, \tag{30}$$

Results

- We start to notice, from $K = 2$, that by letting





Defining groups by the introduction of switching variable - Formulation

$$\min_{\beta_{0\alpha},\beta_{\alpha},z_{p\alpha}\varepsilon^{+}_{t\alpha},\varepsilon^{-}_{t\alpha}} \quad \sum_{\alpha\in A}\sum_{t\in T}\left(\alpha\varepsilon^{+}_{t\alpha}+(1-\alpha)\varepsilon^{-}_{t\alpha}\right) \tag{31}$$

$$\text{s.t} \quad \varepsilon^{+}_{t\alpha}-\varepsilon^{-}_{t\alpha}=y_t-\beta_{0\alpha}-\sum_{p=1}^{P}\beta_{p\alpha}x_{t,p}, \qquad \forall t\in T,\forall\alpha\in A, \tag{32}$$

$$\varepsilon^{+}_{t\alpha},\varepsilon^{-}_{t\alpha}\geq 0, \qquad \forall t\in T,\forall\alpha\in A, \tag{33}$$

$$-Mz_{p\alpha}\leq\beta_{p\alpha}\leq Mz_{p\alpha}, \qquad \forall\alpha\in A,\forall p\in\{1,\ldots,P\}, \tag{34}$$

$$\sum_{p=1}^{P}z_{p\alpha}\leq K, \qquad \forall\alpha\in A, \tag{35}$$

$$z_{p\alpha}\in\{0,1\}, \qquad \forall\alpha\in A,\forall p\in\{1,\ldots,P\}, \tag{36}$$

$$\beta_{0\alpha}+\beta^{T}_{\alpha}x_t\leq\beta_{0\alpha'}+\beta^{T}_{\alpha'}x_t, \qquad \forall t\in T,\forall(\alpha,\alpha')\in A\times A,\alpha<\alpha', \tag{37}$$

$$z_{p\alpha}-z_{p\alpha+1}\leq m_{p\alpha}, \qquad \forall\alpha\in A', \quad \forall p\in P \tag{38}$$

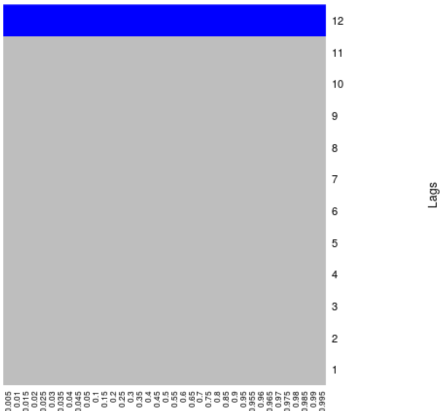
$$\sum_{\alpha\in A'}r_{\alpha}\leq|G|-1 \tag{39}$$

$$\tag{40}$$

where $A'=A\setminus\{|A|\}$

Results

- As there are much less possibilities when $K = 1$, every method gets pretty fast to the optimum result. Selecting the 12^{TH} lag was the best choice.



Results

Results