I. Quantile Autoregression

Our main objective in this paper is estimating the p-step ahead α-quantile function $\mathcal{Q}^{\alpha}_{y_t|y_{t-p}}(t)$ for a given set of time series data y_t, as the one on figure 1. So, given a sequence $\{y_t\}$, we can pair an observation y_t with its p-lagged correspondent y_{t-p}. Figure 2 shows this relationship. We will assume that all information regarding the estimated quantile value are past observations, being in accordance with other pure autoregressive models.

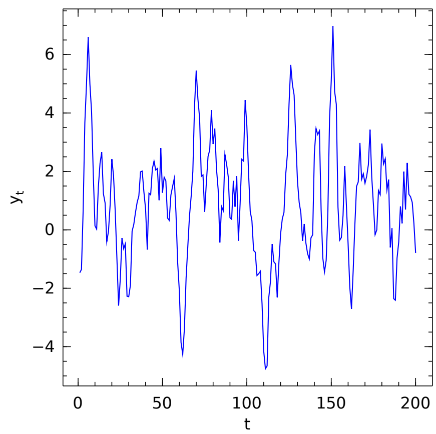

Fig. 1. Time series y_t

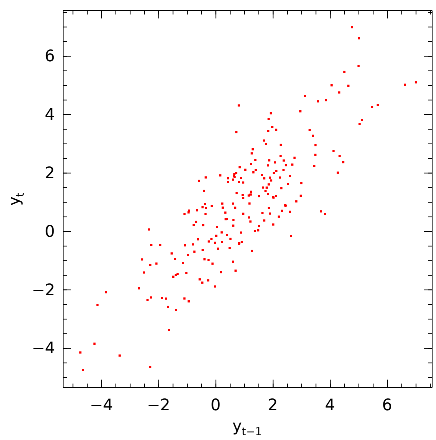

Fig. 2. Relationship between y_t and its first lag y_{t-1}

We will investigate two ways of estimating the quantiles for the aforementioned relationship: we will use a linear and a nonparametric model.

II. Linear-QAR

III. NP-QAR

Fitting a linear estimator for the Quantile Auto Regression isn't appropriate when nonlinearity is present in the data. This nonlinearity may produce a linear estimator that underestimates the quantile for a chunk of data while overestimating for the other chunk (we illustrate this in figure 3). To prevent this issue from occurring we propose a modification which we let the prediction $\mathcal{Q}^{\alpha}_{y_t|y_{t-1}}(t)$ adjust freely to the data and its nonlinearities. To prevent overfitting and smoothen our predictor, we include a penalty on its roughness by including the ℓ_1 norm of its second derivative. For more information on the ℓ_1 norm acting as a filter, one can refer to [1].

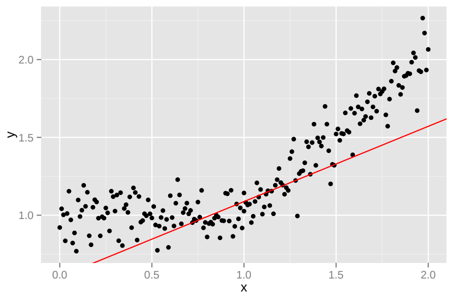

Fig. 3. Example of data where nonlinearity is present and a linear quantile estimator is employed

Let $\{\tilde{y}_t\}_{t=1}^{n}$ be the sequence of observations in time t. Now, let $\tilde{x}_t$ be the $p-$lagged time series of $\tilde{y}_t$, such that $\tilde{x}_t = L^p(\tilde{y}_t)$, where L is the lag operator. Matching each observation $\tilde{y}_t$ with its $p-$lagged correspondent $\tilde{x}_t$ will produce $n-p$ pairs $\{(\tilde{y}_t, \tilde{x}_t)\}_{t=p+1}^{n}$ (note that the first p observations of y_t must be discarded). Consider J to be the set of indexes such that

$$\tilde{x}_{J_1} \leq \tilde{x}_{J_2} \leq \cdots \leq \tilde{x}_{J_{n-P}}.$$

Now, we define $\{x_i\}_{i=1}^{n-p} = \{\tilde{x}_{J_t}\}_{t=p+1}^{n}$ and $\{y_i\}_{i=1}^{n-p} = \{\tilde{y}_{J_t}\}_{t=p+1}^{n}$ and $I = \{2, \ldots, n-p-1\}$. As we need the second difference of q_i, I has to be shortened by two elements.

Our optimization model to estimate the nonparametric quantile is as follows:

$$\mathcal{Q}^{\alpha}_{y_t|y_{t-1}}(i) = \arg\min_{q_i} \sum_{i\in I} \left(|y_i - q_i|^{+}\alpha + |y_i - q_i|^{-}(1-\alpha)\right) + \lambda \sum_{i\in I} |D^2 q_i|, \tag{1}$$

where $D^2 q_t$ is the second derivative of the q_t function, calculated as follows:

$$D^2 q_i = \left(\frac{q_{i+1} - q_i}{x_{i+1} - x_i}\right) - \left(\frac{q_i - q_{i-1}}{x_i - x_{i-1}}\right).$$

The first part on the objective function is the usual quantile regression condition for $\{q_i\}$. The second part is the ℓ_1-filter. The purpose of a filter is to control the amount of variation for our estimator q_i. When no penalty is employed we would always get $q_i = y_i$. On the other hand, when $\lambda \to \infty$, our estimator approaches the linear quantile regression. The penalty for variation on the values of

The output of our optimization problem is a sequence of ordered points $\{(x_i, q_i)\}_{i\in I}$. The next step is to interpolate these points in order to provide an estimation for any other value of x. To address this issue, we propose a B-splines interpolation, that will be discussed in another subsection.

When estimating quantiles for a few different values of α, however, sometimes we find them overlapping each other, which we call crossing quantiles. To prevent this, we include a non-crossing constraint:

$$q_i^{\alpha} \leq q_i^{\alpha'}, \quad \forall i \in I, \alpha < \alpha'. \tag{2}$$

The difference between using or not this constraint can be seen in the two plots below:

REFERENCES

[1] S.-J. Kim, K. Koh, S. Boyd, and D. Gorinevsky, "ℓ_1 trend filtering," *SIAM review*, vol. 51, no. 2, pp. 339–360, 2009.