Inference in Functional Linear Quantile Regression

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Abstract

In this paper we study statistical inference in functional quantile regression for scalar response and a functional covariate. Specifically, we consider linear functional quantile model where the effect of the covariate on the quantile of the response is modeled through the inner product between the functional covariate and an unknown smooth regression parameter function that varies with the level of quantile. The objective is to test that the regression parameter is constant across several quantile levels of interest. The parameter function is estimated by combining ideas from functional principal component analysis and quantile regression. We establish asymptotic properties

of the parameter function estimator, for a single quantile level as well as for a set of quantile levels. An adjusted Wald testing procedure is proposed for this hypothesis of interest and its chi-square asymptotic null distribution is derived. The testing procedure is investigated numerically in simulations involving sparsely and noisy functional covariates and in the capital bike share study application. The proposed approach is easy to implement and the R code is published online.

Keywords: Functional quantile regression; Functional principal component analysis; Measurement error; Wald test; Composite quantile regression.

1 Introduction

The advance in computation and technology generated an explosion of data that have functional characteristics. The need to analyze these type of data triggered a rapid growth of the functional data analysis (FDA) field; see Ramsay and Silverman (2005); Ferraty and Vieu (2006) for two comprehensive treatments. The current research in functional data analysis has been primarily focused on mean regression and only limited interest has been on quantile regression (Koenker, 2005). Quantile regression is appealing in many applications by allowing us to describe the entire conditional distribution of the response at various quantile levels. In our capital bike share data application, it is of interest to study the effect of the bikes rental behavior of casual users in the previous day on the total bike rentals in the current day quantile regression allows us to form a better description of this relationship. Previous papers examining this problem assumed a mean regression functional model, which implicitly assumes that the effect of the previous day casual bike rentals is the same for current busy days (hight total number of casual rentals) and for current non-busy days. This is a rather restrictive assumption for our data application.

Most of the functional regression research considers modeling the mean response (see for example, Yao et al. (2005); Jiang and Wang (2010); Gertheiss et al. (2013); Ivanescu et al. (2015); Usset et al. (2016)); only few works are accommodating higher order moment effects (Staicu et al., 2012; Li et al., 2015). Quantile regression models for scalar responses and functional covariates have been introduced in Cardot et al. (2005). Functional quantile

regression (fQR) models essentially extend the standard quantile regression framework to account for functional covariates: the effect of the covariate on a particular quantile of the response is modeled through the inner product between the functional covariate and an unknown smooth regression parameter function that varies with the level of quantile. Cardot et al. (2005) considers a smoothing splines-based approach to represent the functional covariates and establishes a convergence rate result; Kato (2012) studied principal component analysis (PCA)-based estimation and establishes a sharp convergence rate. Ferraty et al. (2005) and Chen and Müller (2012) estimated the conditional quantile function by inverting the corresponding conditional distribution function; they too studied consistency properties of the regression estimator. Nevertheless, hitherto there is no available work on statistical inference of the quantile regression estimator. Additionally all the functional quantile regression research assumes that the functional covariate is observed either completely on the domain or at very dense grids of points and typically with little or no error contamination.

In this paper we are interested in formally assessing whether the effect of the true smooth signal, as measured by the covariate, varies across few quantile levels of the response, when the smooth signal is observed at finite grids and possibly perturbed with error. For example, if the covariate has no effect on the response, then the above hypothesis holds true; however our hypothesis covers a variety of other situations. The functional linear regression model (Cardot et al., 2005; Chen and Müller, 2012; Kato, 2012), which is very popular in functional data analysis, essentially assumes that this null hypothesis is valid. The hypothesis of interest is important in its own right, yielding a more comprehensive description of the relationship between the covariate and the conditional distribution of the response. Additionally, formally assessing such hypothesis is critical when one wishes to improve the estimation accuracy of the conditional quantile at a single level; if several quantiles of the response are equal, then the estimation accuracy can be improved by borrowing information across these quantile levels. For example, in the case of standard quantile regression where the predictor is a scalar or a vector, there are several approaches to aggregate information across quantile levels in order to improve quantile estimation. This approach is known in the literature by the name composite quantile regression (Koenker, 1984; Zou and Yuan, 2008; Zhao and Xiao, 2014; Jiang et al., 2014). Composite quantile regression improves the efficiency of the quantile estimators at single quantile level, which is important for extreme quantiles (Wang and Wang, 2016). We consider these ideas in our real application analysis, though a more formal investigation to explore this direction is left for future research.

To approach this problem, we assume a linear functional quantile regression (fQR) model that relates the τ th quantile level of the response to the covariate through the inner product between a bivariate regression coefficient function and the true covariate signal. In the case when the true signal is measured at same time points across the study, one naïve way to test the null hypothesis, that the effect of the true covariate signal is constant across several quantile level of interest, is to treat the discretely observed functional covariates as high dimensional covariate and apply standard testing procedures (Wald test) in linear quantile regression for vector covariates (Koenker, 2005). As expected, such approach results in inflated type I errors rates due to the high correlation between the repeated measurements corresponding to the same subject; the situation gets progressively worse when the covariate includes noise. Another alternative is to consider a single number summary of the covariate, such as average or median, and carry out this hypothesis testing by employing standard testing methods in quantile regression. Our numerical investigation of this direction shows that while the Type I error rates are preserved well, the power is substantially affected.

We propose to represent the smooth signal covariate and the quantile regression parameter function using the same orthogonal basis system; this reduces the inner product part of the linear fQR model to an infinite sum of products of basis coefficients of the smooth covariate and parameter function. There are many options in terms of orthogonal bases: we consider the data-driven basis that is formed by the leading eigenbasis functions of the covariance of the true covariate signal and use the percentage of variance explained (PVE) criterion to determine a finite truncation for this basis. While using a finite basis system reduces the dimensionality of the problem, an important challenge is handling the variability of the basis coefficients of the smooth covariate, called functional principal component (fPC) scores. We develop the asymptotic distributions of the quantile estimators based on the estimated fPC scores, when functional covariate is sampled at fine grid of points (dense design). Finally, we introduce an adjusted Wald test statistic and develop its asymptotic null distribution. The introduced testing procedure shows excellent numerical results even

in the situations when the functional covariate is sampled at few and irregular time points across the study (sparse design) and the measurements are contaminated by error.

The development of asymptotic distributions of the quantile estimators based on the estimated fPC scores has important differences from the standard linear quantile regression with vector covariates for a couple of reasons. First, the predictors, fPC scores, are unknown and require estimation; this induces increased uncertainty in the model. We show that asymptotically the quantile estimators are still unbiased, but the variance is inflated. This implies that, in this reduced framework, a direct application of the Wald testing procedure for null hypotheses involving regression parameters is not appropriate. Second, dealing with estimated fPC scores in this situation is different from the measurement errors in predictors setting. For the latter, it is typically assumed that the measurement error and the true predictors are mutually independent or that the errors are independent across subjects (Wei and Carroll, 2009; Wang et al., 2012; Wu et al., 2015). However, in the functional data setting the resulting errors, due to the difference between the estimated fPC scores and the true scores, are dependent with the true predictors and are also dependent across subjects. As a result, the theoretical development requires more careful quantification in terms of the estimated scores and the usage of quantile loss.

This article makes three main contributions. First, we establish the asymptotic distribution of the coefficient estimator for both one single quantile level and multiple quantile levels for dense sampled functional covariates. The results show that, while the estimators are asymptotically unbiased, their variance is inflated. As far as we know, this is the first work to develop the theoretical properties of the quantile estimators allowing inference; previous works have mainly discussed consistency and minimax rates (see Chen and Müller (2012); Kato (2012)). Second, we propose an adjusted Wald test statistic for formally assessing that the quantile regression parameter is constant across specified quantile levels and show that its asymptotic null distribution is chi-square. Third, we consider cases where the functional covariate is observed sparsely and contaminated with noise and illustrate through detailed numerical investigation that the testing procedure continues to have excellent performance. Furthermore, we demonstrates the usage of the composite quantile regression and the corresponding advantage in terms of estimation and prediction accuracy, using a capital bike

rental data set.

The rest of the paper is organized as follows. Section 2 introduces the statistical framework, describes the null and alternative hypotheses, discusses a simpler approximation of the testing procedure and presents the estimation approach. Section 3 develops the asymptotic normality of the proposed estimators, introduces the adjusted Wald test and derives its null asymptotic distribution. Section 4 presents extensive simulation studies confirming excellent performance of the proposed test procedure in various scenarios for both dense and sparse designs. Section 5 applies the proposed test to a bike rental data and illustrates the improvement of combined quantile regressions compared to a single level quantile regression after the proposed tests being used. Proofs of Theorem 3.1 and Thereon 3.2 are given in Section 6. Some additional useful results and technical proofs are provided in the Appendix.

2 Methodology

2.1 Statistical framework

Suppose we observe data $\{Y_i; (t_{ij}, W_{ij}) : j = 1, ..., m_i, t_{ij} \in \mathcal{T}\}_{i=1}^n$, where Y_i is a scalar response variable, $\{W_{i1}, ..., W_{im_i}\}$ is the evaluation of a latent and smooth process $X_i(\cdot)$ at the finite grid of points $\{t_{i1}, ..., t_{im_i}\}$ which is measured with noise, and \mathcal{T} is a bounded closed interval. It is assumed that the observed functional covariate is perturbed by white noise, i.e. $W_{ij} = X_i(t_{ij}) + e_{ij}$, where e_{ij} has mean 0 and variance σ^2 . Furthermore, we assume that the true functional signal $X_i(\cdot) \in L^2(\mathcal{T})$ and $X_i(\cdot)$ are independent and identically distributed. Without loss of generality, we use $\mathcal{T} = [0,1]$ throughout the paper. Our objective is to formally assess whether the smooth covariate signal $X_i(\cdot)$ has constant effect at specified quantile levels of the response.

Let $Q_{Y_i|X_i}(\tau)$ be the conditional τ th quantile function of the response Y_i given the true covariate signal $X_i(\cdot)$ where $\tau \in (0,1)$. We assume the following linear functional quantile regression (fQR) model:

$$Q_{Y_i|X_i}(\tau) = \beta_0(\tau) + \int_0^1 \beta(t, \tau) X_i^c(t) dt,$$
 (1)

where $\beta_0(\tau)$ is the quantle-level varying intercept function, $\beta_0(\tau) \in \mathbb{R}$ and $\beta(t,\tau)$ is the bivariate regression coefficient function and the main object of interest; it is assumed that $\beta(\tau,\tau) \in L^2[0,1]$. Here $X_i^c(t)$ is the de-meaned smooth covariate signal, defined as $X_i^c(t) = X_i(t) - EX_i(t)$. Model (1) is an extension of the standard linear quantile regression model (Koenker, 2005) to functional covariates. It was first introduced by Cardot et al. (2005) and later considered by (Chen and Müller, 2012; Kato, 2012). For simplicity, in the following it is assumed that the smooth covariate signal has zero mean function, $E[X_i(t)] = 0$, for all $t \in [0, 1]$.

Let $\mathcal{U} = \{\tau_1, \dots, \tau_L\}$ be a set with quantile levels of interest where $\tau_1 < \dots < \tau_L$. Our goal is to test the null hypothesis:

$$H_0: \beta(\cdot, \tau_1) = \dots = \beta(\cdot, \tau_L),$$
 (2)

against the alternative hypothesis $H_a: \beta(\cdot, \tau_l) \neq \beta(\cdot, \tau_{l'})$, for some $l \neq l' \in \{1, \ldots, L\}$. This null hypothesis involves infinite dimensional objects, which is very different from the common null hypotheses considered in quantile regression.

One approach to simplify the null hypothesis is by using basis functions expansion. Specifically, let $\{\phi_k(\cdot)\}_{k\geq 1}$ be an orthogonal basis in $L^2[0,1]$ such that $\int_0^1 \phi_k(t)\phi_{k'}(t)dt = 0$ if $k \neq k'$ and 0 if k = k'. We represent the unknown parameter function $\beta(\cdot,\tau)$ using this orthogobal basis $\beta(t,\tau) = \sum_{k\geq 1} \beta_k(\tau)\phi_k(t)$ where $\beta_k(\tau) = \int \beta(t,\tau)\phi_k(t)dt$ are unknown parameter loadings that are varying with the quantile level τ . It follows that the equality $\beta(\cdot,\tau_l) = \beta(\cdot,\tau_{l'})$ is equivalent to $\beta_k(\tau_l) = \beta_k(\tau_{l'})$, $k \geq 1$. Thus, the null hypothesis (2) can be written as $H_0: \beta_k(\tau_1) = \beta_k(\tau_2) = \cdots = \beta_k(\tau_L)$ for $k \geq 1$. Furthermore, we represent the smooth covariate using the same basis function as $X_i(t) = \sum_{k\geq 1} \xi_{ik}\phi_k(t)$ where $\xi_{ik} = \int X_i(t)\phi_k(t) dt$ are smooth covariate loadings. Then, the linear fQR model (1) can be equivalently represented as $Q_{Y_i|X_i}(\tau) = \beta_0(\tau) + \sum_{k=1}^{\infty} \beta_k(\tau)\xi_{ik}$. In practice the infinite summation is typically truncated to some finite truncation K. As a result the fQR model can be approximated by

$$Q_{Y_i|X_i}^K(\tau) = \beta_0(\tau) + \sum_{k=1}^K \beta_k(\tau)\xi_{ik},$$

and the null hypothesis to be tested can be approximated by a reduced version

$$H_0^K : \begin{pmatrix} \beta_1(\tau_1) \\ \beta_2(\tau_1) \\ \vdots \\ \beta_K(\tau_1) \end{pmatrix} = \begin{pmatrix} \beta_1(\tau_2) \\ \beta_2(\tau_2) \\ \vdots \\ \beta_K(\tau_2) \end{pmatrix} = \cdots = \begin{pmatrix} \beta_1(\tau_L) \\ \beta_2(\tau_L) \\ \vdots \\ \beta_K(\tau_L) \end{pmatrix}. \tag{3}$$

If the value of K and the loadings ξ_{ik} 's were known then the above model is exactly the conventional quantile regression model (Koenker, 2005). In such case, the standard approach is to represent the reduced null hypothesis (3) in a more convenient way and then use a Wald testing procedure. More specifically denote by $\theta_{\tau} := (\beta_0(\tau), \beta_1(\tau), \dots, \beta_K(\tau))^T$ the (K+1)-dimensional parameter vector and define $\zeta := (\theta_{\tau_1}^T | \dots | \theta_{\tau_L}^T)^T$ the full quantile regression parameter vector of dimension L(K+1). The reduced null hypothesis can be equivalently re-written as $H_0^K : R \zeta = 0$, where $R = R_1 \otimes R_2$ and

$$R_1 = \begin{bmatrix} 1 & -1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -1 & \dots & 0 & 0 \\ & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & -1 \end{bmatrix} \qquad R_2 = [\mathbf{0}_K | I_K].$$

Here $\mathbf{0}_K$ denotes the K-dimensional vector of zeros and I_K is the $K \times K$ dimensional identity matrix. Wald test is typically formulated as $T_W = (R\widehat{\zeta})^T \ (R\widehat{\Gamma}_{\widehat{\zeta}}R^T)^{-1} \ R\widehat{\zeta}$, where $\widehat{\zeta}$ is the quantile regression estimator of ζ and $\widehat{\Gamma}_{\widehat{\zeta}}$ is a consistent estimator of $\Gamma_{\widehat{\zeta}}$ - the covariance of $\widehat{\zeta}$ - conditional on the true loadings ξ_{ik} 's. However, the truncation K is unknown and so are the loadings of the smooth covariate signal, ξ_{ik} , and they both represent important sources of uncertainty that a valid approach has to account for. In particular the uncertainty in estimating K and ξ_{ik} will affect the covariance $\Gamma_{\widehat{\zeta}}$ and furthermore may afect the asymptotic null distribution of the standard Wald.

Depending on the choice of the orthogonal basis, the approaches used to select the finite truncation K and to develop the theoretical properties for the quantile regression estimators differ. Several choices have been commonly used in functional data analysis literature: Fourier basis functions (Staicu et al., 2015), Wavelet basis (Morris and Carroll, 2006) or orthogonal B-spline (Zhou et al., 2008; Redd, 2012). One important aspect to keep in mind when selecting the basis functions is how to handle the finite truncation K. In this paper we consider the orthogonal basis given by the eigenfunctions of the covariance of the smooth covariate signal $X_i(\cdot)$. Let $G(s,t) := \text{Cov}\{X_i(s), X_i(t)\}$ be the covariance of $X_i(\cdot)$; Mercer's theorem gives the following spectral decomposition of the covariance $G(s,t) = \sum_{k=1}^{\infty} \lambda_k \phi_k(s) \phi_k(t)$, where $\{\phi_k(\cdot), \lambda_k\}_k$ are the pairs of eigenfunctions and corresponding eigenvalues. The eigenvalues λ_k 's are nondecreasing and nonnegative and the eigenfunctions $\phi_k(\cdot)$'s are mutually orthogonal functions in $L^2[0,1]$. Using the Karhunen-Loève expansion, the zero-mean smooth covariate $X_i(\cdot)$ can be represented as $X_i(t) = \sum_{k=1}^{\infty} \xi_{ik} \phi_k(t)$, where $\xi_{ik} = \int_0^1 X_i(t) \phi_k(t) dt$ are commonly known as functional principal component (fPC) scores of $X_i(\cdot)$, satisfying that $E(\xi_{ik}) = 0$, $Var(\xi_{ik}) = \lambda_k$ and uncorrelated over k. A popular way to select the finite truncation, or equivalently the number of leading eigenfunctions, is the percentage of variance explained; alternative options for selecting the finite truncation K are considered in Li et al. (2013).

2.2 Estimation procedure

We discuss estimation for the case when the functional covariate is observed on a fine grid of points - a setting known in the literature by the name of dense sampling design. Nevertheless our procedure can be successfully applied to the case when the covariate is observed on an irregular sampling design with few points (sparse sampling design) and contaminated with noise, as illustrated later in the numerical investigation. When the sampling design is dense, and thus m_i is very large for each i, a common approach in functional data analysis is to reconstruct the underlying trajectories $\widehat{X}_i(\cdot)$ by smoothing the data $\{(t_{ij}, W_{ij}) : j = 1, \ldots, m_i\}$ using a working independence assumption; this approach has been theoretically studied by Zhang and Chen (2007). Let $\overline{X}(\cdot)$ be the sample mean of these reconstructed trajectories and denote by $\widehat{X}_i^c(t) := \widehat{X}_i(t) - \overline{X}(t)$ the centered covariate. Furthermore, let $S(\cdot,\cdot)$ be the sample covariance of $\widehat{X}_i(t)$; the spectral decomposition of $S(\cdot,\cdot)$ yields the pairs of estimated eigenfunctions and eigenvalues $\{\widehat{\phi}_k(\cdot), \widehat{\lambda}_k\}_k$. The theoretical properties of the estimated eigenfunctions $\widehat{\phi}_k(\cdot)$ have been discussed in Hall and Hosseini-Nasab (2006);

Hall et al. (2006); Zhang and Chen (2007). Finally the fPC scores ξ_{ik} are estimated as $\widehat{\xi}_{ik} = \int \widehat{X}_i^c(t) \widehat{\phi}_k(t) dt$; in practice numerical integration is used to approximate the integral; see also Li et al. (2010).

Using the estimated fPC scores $\hat{\xi}_{ik}$'s, the quantile regression parameter of the approximated linear fQR mdoel, θ_{τ} , is estimated by

$$\widehat{\theta}_{\tau} = \underset{(b_0, b_1, \dots, b_K)^T \in \mathbb{R}^{K+1}}{\operatorname{arg \, min}} \sum_{i=1}^n \rho_{\tau} \left(y_i - b_0 - \sum_{k=1}^K \widehat{\xi}_{ik} b_k \right). \tag{4}$$

where $\rho_{\tau}(x) = x\{\tau - I(x < 0)\}$ is the quantile loss function and I(x < 0) is the indicator function that equals 1 if x < 0 and 0 otherwise. In the next section we study the theoretical properties of the quantile regression estimator obtained in this way. In contrast to the quantile regression, the estimated fPC scores $\hat{\xi}_{ik}$'s are dependent across subjects and this makes the corresponding theoretical developments much more challenging.

3 Theoretical properties

3.1 Assumptions

Let $F_i(y) = P(Y_i < y | X_i(\cdot))$, and $f_i(\cdot)$ be the corresponding density function. We make the following assumptions:

- A1. $\{Y_i, X_i(\cdot), e_i(\cdot)\}_{i=1}^n$ are independent and identically distributed (i.i.d.) as $\{Y, X(\cdot), e(\cdot)\}$, and $X(\cdot)$ and $e(\cdot)$ are independent where $E\{e(t)\} = 0$ and $Cov\{e(t), e(t')\} = \sigma^2 I(t = t')$ for any t, t'.
- A2. The conditional distribution $F_i(\cdot)$ is twice continuously differentiable and the corresponding density function $f_i(\cdot)$ is uniformly bounded away from 0 and ∞ at points $Q_{Y_i|X_i}(\tau)$.
- A3. The functional covariates $X(\cdot)$ satisfy that $\mathrm{E}\{X(t_1)X(t_2)X(t_3)X(t_4)\}<\infty$ uniformly for $(t_1,t_2,t_3,t_4)\in[0,1]^4$.

A4. There exists a finite number p_0 such that $\lambda_k = 0$ if $k \geq p_0$; where recall that λ_k 's are the eigenvalues of the covariance of $X_i(\cdot)$'s.

The i.i.d. assumption in A1 and the assumption on the conditional distribution in A2 are standard in quantile regression when the covariates are vectors; see Koenker (2005), Ch. 4. A1 assumes that the functional covariates $X_i(\cdot)$'s are observed with independent white noise $e_i(\cdot)$, making the model more realistic compared to error free assumptions made by Kato (2012). The assumption A3 holds for Gaussian processes and is common in FDA literature; for example, see Hall et al. (2006); Li et al. (2010). Finally A4 requires that the functional covariates $X_i(\cdot)$ have a finite number of fPC's.

The following assumptions are commonly when describing a dense sampling design (Zhang and Chen, 2007; Li et al., 2010). For convenient mathematical derivations, we assume that there are the same number of observations per subject, i.e. $m_i = m$ for all i.

B1. The time points $t_{ij} \stackrel{i.i.d.}{\sim} g(\cdot)$ for i = 1, ..., n; j = 1, ..., where the density $g(\cdot)$ has bounded support [0,1] and is continuously differentiable.

B2. $m \ge Cn^{c_m}$ where $c_m > 5/4$ and C is some constant.

For our theoretical development, we require the following condition for the kernel bandwidth h_X that is used in smoothing the functional covariates.

C1.
$$h_X = O(n^{-c_m/5})$$
.

3.2 Asymptotic distribution

The following theorem gives the asymptotic distribution of the quantile estimator. Kato (2012) gave the minimax rate of the coefficient function estimation when there is no measurement error on the discrete functional covariates. The author assumed that the number of eigenvalues is infinite instead of finite as in our assumption A4. Assuming finite number of non-zero eigenvalues is critical since otherwise the quantile regression problem is ill-posed and the estimation of the coefficient function cannot achieve the root-n rate as argued by Kato (2012). We denote D_0 as the diagonal matrix whose diagonal entries are $(1, \lambda_1, \dots, \lambda_{p_0})$

and $D_1(\tau) = \mathbb{E}[f_i\{Q_{Y_i|X_i}(\tau)\}\xi_i\xi_i^T]$ which is positive definite, where $\xi_i = (1, \xi_{i1}, \dots, \xi_{ip_0})^T$. Similarly, we denote $\hat{\xi}_i = (1, \hat{\xi}_{i1}, \dots, \hat{\xi}_{ip_0})^T$. For now, assume that p_0 is known; the discussion of estimating p_0 is deferred to Section 3.4.

Theorem 3.1 Denote by $\widehat{\theta}_{\tau}$ the quantile regression estimator defined by (4) for $K = p_0$, where $\tau \in (0,1)$. Under conditions A1-A4, B1-B2, C1, we have

$$\sqrt{n}(\widehat{\theta}_{\tau} - \theta_{\tau}) \stackrel{d}{\to} N\left\{0, \tau(1 - \tau)D_1^{-1}(\tau)D_0D_1^{-1}(\tau) + \Theta_{\tau}\Sigma_0\Theta_{\tau}\right\}$$
 (5)

where $\Theta_{\tau} = 1_{(p_0+1)\times(p_0+1)} \otimes \theta_{\tau}^T$ and the matrix Σ_0 is defined in Section 6 which does not depend on τ .

If $\tau_l \neq \tau_{l'} \in (0,1)$ then the asymptotic covariance matrix for $\widehat{\theta}_{\tau_l}$, $\widehat{\theta}_{\tau_{l'}}$ for $1 \leq l, l' \leq L$ is

$$Acov\left\{\sqrt{n}(\widehat{\theta}_{\tau_{l}} - \theta_{\tau_{l}}), \sqrt{n}(\widehat{\theta}_{\tau_{l'}} - \theta_{\tau_{l'}})\right\}$$

$$= \left\{\min(\tau_{l}, \tau_{l'}) - \tau_{l}\tau_{l'}\right\} D_{1}^{-1}(\tau_{l}) D_{0} D_{1}^{-1}(\tau_{l'}) + \Sigma(\tau_{l}, \tau_{l'}), \tag{6}$$

where $\Sigma(\tau_l, \tau_{l'}) = \Theta_{\tau_l} \Sigma_0 \Theta_{\tau_{l'}}$.

Remark that the asymptotic covariances in both (5) and (6) contain two components: a Huber (Huber, 1967) sandwich term that typical in quantile regression theory and one that we call it "variance inflation". Specifically, if the true scores ξ_i 's were observed, then the asymptotic variance of $\hat{\theta}_{\tau}$ would be $\tau(1-\tau)D_1^{-1}(\tau)D_0D_1^{-1}(\tau)$, and the asymptotic covariance matrix for $\hat{\theta}_{\tau_l}$, $\hat{\theta}_{\tau_{l'}}$ would be $\{\min(\tau_l,\tau_{l'})-\tau_l\tau_{l'}\}D_1^{-1}(\tau_l)D_0D_1^{-1}(\tau_{l'})$; see Pollard (1991); Koenker (2005). The variance inflation terms, $\Theta_{\tau}\Sigma_0\Theta_{\tau}$ in (5) and $\Sigma(\tau_l,\tau_{l'})$ in (6), quantify the effect of uncertainty in estimation the fPC scores on the quantile regression estimators. Thus, when the covariates are functional data, the asymptotic distribution of $\hat{\theta}_{\tau}$ is unbiased but the variance is inflated and furthermore the variance inflation terms depend on the true parameter value θ_{τ} .

The proof of Theorem 3.1 is detailed in Section 6. The reasoning follows two main steps: 1) approximate the estimated fPC scores $\hat{\xi}_i$'s by linear combinations of random vectors of the true fPC scores ξ_i ; and 2) show that the approximation error in the predictors is negligible to the quantile loss function. As a direct application of the Theorem 3.1, we can obtain the asymptotic distribution of the coefficient function according to Slutsky's theorem. Under the same conditions of Theorem 3.1, for $\tau \in (0,1)$ and $t \in \mathcal{T}$, we have

$$\sqrt{n}\{\widehat{\beta}(t,\tau) - \beta(t,\tau)\} \xrightarrow{d} N\left\{0, \tau(1-\tau)a^{T}(t)D_{1}^{-1}(\tau)D_{0}D_{1}^{-1}(\tau)a(t) + a^{T}(t)\Theta(\tau)\Sigma_{0}\Theta(\tau)a(t)\right\},$$
where $a^{T}(t) = (1, \phi_{1}(t), \dots, \phi_{p_{0}}(t)).$

3.3 Adjusted Wald test

Using the asymptotic properties of the quantile regression estimators, we are now ready to develop a Wald type testing procedure for assessing the general null hypothesis (2) or its finite reduced version (3) represented in vector form by $H_0^K: R \zeta = 0$. Recall that $\zeta = (\theta_{\tau_1}^T | \dots | \theta_{\tau_L}^T)^T$ denotes the full quantile regression parameter.

We define a modified version of Wald test, called the *adjusted Wald test*, by ignoring the variance inflation terms in the above asymptotic covariances. Specifically let $\widetilde{\Sigma}_{\widehat{\zeta}}$ be $L(K+1) \times L(K+1)$ matrix that has $\tau_l(1-\tau_l)D_1^{-1}(\tau_l)D_0D_1^{-1}(\tau_l)$ as the *l*th block matrix on the diagonal and $\{\min(\tau_l,\tau_{l'})-\tau_l\tau_{l'}\}D_1^{-1}(\tau_l)D_0D_1^{-1}(\tau_{l'})$ as the off-diagonal (l,l') block matrices. Furthermore denote by $\widehat{\Sigma}_{\widehat{\zeta}}$ a consistent estimator of $\widetilde{\Sigma}_{\widehat{\zeta}}$. Define the adjusted Wlad test by

$$T_n = n(R\widehat{\zeta})^T (R\widehat{\Sigma}_{\widehat{\zeta}}R^T)^{-1} R\widehat{\zeta}, \tag{7}$$

where $\hat{\zeta} = (\hat{\theta}_{\tau_1}^T | \dots | \hat{\theta}_{\tau_L}^T)^T$. This test is not a proper Wald test as the covariance matrix used is not the valid covariance of ζ . The following result studies the asymptotic null distribution of the proposed test.

Theorem 3.2 Assume the regularity conditions A1-A4, B1-B2 and C1 hold. If the null hypothesis is true, $R\zeta = 0$, then the asymptotic distribution of T_n is χ_K^2 .

The proof of this result relies on the observation that if $\Sigma_{\widehat{\zeta}}$ is the proper covariance of $\widehat{\zeta}$ as described by Theorem 3.1, then $R(\widetilde{\Sigma}_{\widehat{\zeta}} - \Sigma_{\widehat{\zeta}})R^T = 0$. Thus, although the estimation of

the fPC scores yields inflated covariance of the regression estimator, its effect on testing the null hypothesis (2) is negligible. Nevertheless, if one is interested in testing a different type of null hypothesis for ζ , then this variance inflated term has to be taken into account for a proper testing procedure.

A consistent estimator of $\widetilde{\Sigma}_{\widehat{\zeta}}$ is a plug-in estimator using $\widehat{D}_0 = \sum_{i=1}^n \widehat{\xi}_i \widehat{\xi}_i^T/n$ as an estimator for D_0 and $\widehat{D}_1(\tau) = \sum_{i=1}^n \widehat{f}_i(Q\{\tau|\widehat{\xi}_i\})\widehat{\xi}_i\widehat{\xi}_i^T/n$ as an estimator for $D_1(\tau)$. Both estimators are consistent estimators of D_0 and $D_1(\tau)$ respectively, using law of large numbers-based arguments. Lemma 6.1 discusses the closeness between $\widehat{\xi}_i$ and ξ_i . It implies that, for testing the null hypothesis of equal functional covariate effect across various quantile levels, the common Wald test based on the estimated fPC scores provides a valid testing procedure. The adjusted Wald test, that disregards the variance component due to the estimation uncertainty of the fPC scores, has a chi-square asymptotic null distribution.

3.4 Estimation of the finite truncation

Our results rely on the assumption that there is a finite number of positive eigenvalues of the covariance function of $X_i(\cdot)$'s. The number of principal components p_0 is unknown and should be selected in practice. The selection of p_0 has been studied intensively in the literature such as the criterion of percentage of variance explained (PVE). Recently Li et al. (2013) proposed a Bayesian information criterion (BIC) based on marginal modeling that can consistently select the number of principal components for both sparse and dense functional data, i.e. the resulting estimator $K = K_n$ converges to p_0 in probability. Using this result, we can show that the test statistic T_n with p_0 replaced by the consistent estimator K_n has an approximately null distribution given by $\chi^2_{K_n}$ asymptotically under H_0 : $R\zeta = 0$. In the remaining studies and data analysis section, we use the PVE criterion.

4 Simulation

4.1 Settings

The simulated data is of the form $[Y_i, \{(t_{ij}, W_{ij}) : j = 1, ..., m_i\}]_{i=1}^n$, where Y_i is the scalar response and $W_{ij} = X_i(t_{ij}) + e_{ij}$ is the functional covariate contaminated with measurement error e_{ij} , $t_{ij} \in [0, 1]$, and $X_i(\cdot)$ is the true functional covariate. We generate the data from the following heteroscedastic model:

$$Y_i = \int X_i(t)tdt + \{1 + \gamma \int X_i(t)t^2dt\}\epsilon, \ \epsilon \sim N(0, \sigma_{\epsilon}^2 = 1),$$

which leads to a quantile regression model of the form (1) with $\beta_0(\tau) = \Phi^{-1}(\tau)$, and $\beta(t,\tau) = t + \gamma t^2 \Phi^{-1}(\tau)$. Here the scalar γ controls the heteroscedasticity and determines how the coefficient function $\beta(\cdot,\tau)(\tau \in \mathcal{U})$ varies across τ . Specifically, if $\gamma = 0$ then the effect of $X_i(\cdot)$ is constant across different quantile levels of $Y_i|X_i(\cdot)$, while if $\gamma \neq 0$ then the effect of $X_I(\cdot)$ varies across different quantile levels of $Y_i|X_i(\cdot)$.

The true functional covariate $X_i(\cdot)$ is generated from a Gaussian process with zero mean and covariance function $\operatorname{cov}\{X_i(s), X_i(t)\} = \sum_{k\geq 1} \lambda_k \phi_k(s) \phi_k(t)$, where $\lambda_k = (1/2)^{k-1}$ for k=1,2,3 and $\lambda_k=0$ for $k\geq 4$, and $\{\phi_k(\cdot)\}_k$ are the orthonormal Legendre polynomials on [0,1]: $\phi_1(t)=\sqrt{3}(2t^2-1), \phi_2(t)=\sqrt{5}(6t^2-6t+1), \phi_3(t)=\sqrt{7}(20t^3-30t^2+12t-1)$. It is assumed that the measurement error $e_{ij}\sim N(0,\sigma^2)$. Figure 1 shows an example of simulated data when n=200 and $\gamma=1$: observed functional covariates in the left panel and the histogram of the responses in the right panel.

The objective is to test the null hypothesis $H_0: \beta(\cdot, \tau_l) = \beta(\cdot, \tau_{l'})$ for $\tau_l, \tau_{l'} \in \mathcal{U}$, that the effect of the true functional covariate on the conditional distribution of the response is the same for all the quantile levels in a given set \mathcal{U} . When $\gamma = 0$, the coefficient function $\beta(\cdot, \tau)$ is independent of τ , which means that null hypothesis is true; when $\gamma \neq 0$ then $\beta(\cdot, \tau)$ is varies with τ and thus the null hypothesis is false.

We use the proposed adjusted Wald test using a number of fPC selected via the PVE criterion and using PVE=95%. The R package refund (Huang et al., 2015) is used to es-

timate the fPC scores. We investigate the performance of the test for low and high level of measurement error in the functional covariate ($\sigma = 0.05$ and $\sigma = 1$ respectively), for two sets of quantile levels - $\mathcal{U}_1 = \{0.1, 0.2, 0.3, 0.4\}$, containing one-sided quantile levels, and $\mathcal{U}_2 = \{0.1, 0.2, 0.6, 0.7\}$ with two-sided quantile levels - and for varying sample sizes n from 100 to 5000. We compare the proposed testing procedure with existing alternatives in terms of the Type I error and power performance when the sampling design of the functional covariate $\{t_{ij}: j = 1..., m_i\}$ is regular across i's and dense, as well as when the sampling design irregular and sparse.

4.2 Dense design

We first consider a dense design for the functional covariates: the grid of points for each i is an equispaced grid of $m_i = 100$ timepoints in [0,1]. We are not aware of any testing procedures for testing the null hypothesis of constant effect at various quantile levels, when the covariate is functional; however we can exploit this particular setting and pretend the covariates are vectors and thus use or directly extend existing testing procedures from quantile regression. In particular, consider the following three alternative approaches: (1) treat the observed functional covariate as vector and use the common Wald testing procedure for vector covariates in quantile regression (NaïveQR); (2) summarize observed functional covariate via a single number summary of the functional covariate in conjunction with Wald procedure (SSQR); and treat the observed functional covariate as a vector, reduce the dimensionality using principal component analysis and then apply Wald testing procedure using the vector of principal component scores (pcaQR). For the pcaQR approach, the number of principal components are selected via PVE and using a level PVE=95%. Wald-type test for vector covariates (Koenker, 2005, Chapter 3.2.3) is used to carry out both NaïveQR and pcaQR.

Table 1 summarizes the empirical Type I error rates of the adjusted Wald test when testing H_0 at one-sided quantile levels (\mathcal{U}_1) as well as two-sided quantile levels (\mathcal{U}_2), and furthermore when the functional covariate is observed with low ($\sigma = 0.05$) and large ($\sigma = 1$) measurement error. The results are presented for three significance levels $\alpha = 0.01$, $\alpha = 0.05$ and $\alpha = 0.10$; they indicate irrespective of quantile levels set or magnitude of the measurement error the Type I error rates are slightly inflated for moderate sample sizes.

Nevertheless the empirical Type I error rates converge to the nominal level. The empirical Type I error rates for the alternative approaches are presented in Table 2. As expected the NaïveQR approach has very poor performance. The NaïveQR approach does hypothesis testing when the covariates are highly correlated; this leads to numerical instability due to singularity of the design matrix. Therefore the NaïveQR method produces many missing values (reported as "—") in the table, and yields to very inflated empirical Type I error rates for any significance level.

The third alternative approach, pcaQR, performs relatively good when the magnitude of the error is small ($\sigma=0.05$): the empirical Type I error is close to the nominal level. Nevertheless, as the error variance increases ($\sigma=1$), the empirical rejection probabilities are excessively inflated; additionally the approach yields to many missing values (reported as "-"); see Table 1, the last three columns and the rows corresponding to $\sigma=1$ and sample sizes n=100 and n=500. The results are not surprising, because in the case of large error variance, a direct application of principal component analysis yields a large number of principal components. As a consequence, the application of the classical Wald test for vector covariate leads to numerical instability due to singularity of the design matrix, in a similar way to the NaïveQR approach. The performance of SSQR approach is very good for all the scenarios considered and across various sample sizes: the empirical Type I error rates are close to the nominal levels. This is expected, as in the case when H_0 , the functional covariate effect is through its mean, and this effect is invariant over quantile levels.

[Table 1 about here.]

[Table 2 about here.]

Next we evaluate the performance in terms of empirical rejection probabilities when the null hypothesis is not true. We only focus on the proposed adjusted Wald testing and SSQR procedures, as they have the correct size. Figure 2 illustrates the power curves of the two approaches based on 2000 simulation, when the magnitude of the noise is large, $\sigma = 1$; the results are similar in the case of low noise ($\sigma = 0.05$) and for brevity are not included in the manuscript. The adjusted Wald procedure is much more powerful than the SSQR approach irrespective of the departure from the null hypothesis as reflected by the coefficient γ : for

example, when $\gamma = 1$ the probability to correctly reject H_0 using the adjusted Wald is about 100 % when the sample size is 500 or more, where as the counterpart obtained with SSQR is less than 70 % for sample sizes smaller than n = 5000. The results are not surprising, as SSQR summarizes the entire functional covariate through a single scalar, whereas the proposed adjusted Wald uses the full functional covariate.

[Figure 2 about here.]

4.3 Sparse design

Next, we study the performance of the adjusted Wald testing procedure when the functional covariate is observed sparsely and with measurement error. In this regard we set an overall grid of 101 equispaced points in [0,1] and consider two settings. First, for each i we randomly generate $m_i = 50$ time points to form t_{i1}, \ldots, t_{im_i} - we call this setting 'moderately sparse' sampling design. Second, for each i we randomly generate $m_i = 10$ time points to form t_{i1}, \ldots, t_{im_i} - we call this setting 'sparse' design. The generation of the true functional covariate and the observed covariate, as well as of the observed response follows as described in the previous section. We use the adjusted Wald test which relies on sparse fPCA techniques, that estimate the fPC scores ξ_{ik} 's using conditional expectation proposed by Yao et al. (2005). When the sampling design of the functional covariate is sparse, there are no obvious reasonable alternative approaches to compare. Thus in this section we only discuss the performance of the proposed Wald-type procedure.

Table 3 shows the empirical Type I error when the noise level equals to $\sigma = 1$. They show excellent performance of the adjusted Wald test in maintaining the nominal levels when the sample size is moderately large (n = 1000 or larger) for both moderately sparse and sparse sampling design of the functional covariate. Figure 3 shows the power of the adjusted Wald test for moderately sparse and highly sparse designs for $\sigma = 1$. It indicates that the sparsity of the functional covariates slightly affects the proposed functional Wald-type procedure, as expected. Nevertheless the adjusted Wald test continues to display excellent performance. The results are similar for low level of measurement error and for brevity are not included here.

[Table 3 about here.]

[Figure 3 about here.]

5 Application

In this section we consider the capital bike sharing study and discuss the application of the proposed testing procedure to formally assess whether the effect of the previous day casual bike rentals on the current day total bike rentals varies across several quantile levels. The bike data (Fanaee-T and Gama, 2014) is recorded by the Capital Bikeshare System (CBS), Washington .C., USA, which is available at http://capitalbikeshare.com/system-data. As the new generation of bike rentals, bike sharing systems possess membership, rental and return automatically. With currently over 500 bike-share programs around the world (Larsen, 2013) and the fast growing trend, data analysis on these systems regarding the effects to public traffic and the environment has become popular. The bike data includes hourly rented bikes for casual users that are collected during January 1st 2011 to December 31st 2012, for a total of 731 days.

Our objective is to formally assess how the previous day casual bike rentals, $X_i(\cdot)$, affects the distribution of the current day total bike rentals counts, Y_i . A subsequent interest is to predict the 90% quantile of the total casual bike rentals. Figure 4 plots the hourly profiles of casual bike rentals in the left panel and the histogram of the total casual bike rentals in the right panel.

We assume the functional quantile regression model (1), $Q_{Y_i|X_i}(\tau) = \beta_0(\tau) + \int \beta(t,\tau) X_i^c(t) dt$, where Y_i is the total bike casual bike rentals for the current day and $X_i(\cdot)$ is the true profile of the casual bike rentals recorded in the previous day. As described earlier $\beta_0(\cdot)$ is the quantile varying intercept function and $\beta(\cdot,\tau)$ is the slope parameter and quantifies the effect of the functional covariate at the τ quantile level of the distribution of the response.

To address the first objective we consider a set of quantile levels and use the proposed testing procedure to test the null hypothesis

$$H_0: \beta(\cdot, 0.20) = \beta(\cdot, 0.40) = \beta(\cdot, 0.60) = \beta(\cdot, 0.80).$$

The number of fPC is selected using PVE = 99%; this choice selects three fPC. We use the adjusted Wald test T_n and its asymptotic null distribution; the resulting p-value is close to zero indicating overwhelming evidence that low and large number of bike rentals are affected differently by the hourly rentals on the previous day.

Next we turn to the problem of predicting the 90% quantile of the total bike rentals for the current day. When some quantile coefficients in a region of quantile levels are constant, we may improve the estimator's efficiency by borrowing information from neighboring quantiles to estimate the common coefficients, especially when the quantile level of interest is high. Here consider the quantile level set $\mathcal{U} = (0.8, 0.825, 0.85, 0.875, 0.9)$ around the 90%th quantile. We apply the proposed method to estimate the coefficient functions at various quantile levels \mathcal{U} as shown in Figure 5. The corresponding adjusted Wald test leads to a p-value = 0.466, which suggests that the quantile coefficients are not significantly different across the quantile levels. We consider combined quantile regression at \mathcal{U} by using the methods of quantile average estimator (QAE) and composite regression of quantiles (CRQ) with equal weights; see Koenker (1984); Wang and Wang (2016) for more technical details. We denote the single quantile regression estimation at the 90th quantile by RQ.

[Figure 4 about here.]

[Figure 5 about here.]

We use 1000 bootstrap samples to study the efficiency of the three estimators. Figure 6 plots the bootstrap means and standard errors of the estimates of $\beta(t)$ when using QAE, CRQ and RQ. The QAE and CRQ estimators have smaller standard errors uniformly for all t, indicating the efficiency improvement. This suggests the efficiency gain by combining information across quantile levels. We also observe that the number of fPC's is either 3 or 4 in all bootstrap samples, suggesting that the assumption A4 makes sense to this data set.

[Figure 6 about here.]

Furthermore, we conduct a cross-validation by randomly selecting 50% of the data as the training data set and using the other half as the testing data set. We use 1000 replications

and calculate the prediction error for each replication and each $\tau \in \mathcal{U}$ as follows:

$$PE = \sum_{i \in \text{test sample}} \rho_{\tau} (y_i - \hat{\xi}_i^T \hat{\theta}_{\tau}),$$

where the estimated coefficients $\hat{\theta}_{\tau}$ are based on the training data and the summation is over the test data. We obtain the RQ estimators separately at each of \mathcal{U} while the QAE and CRQ estimators are shared across \mathcal{U} . The mean prediction errors based on 1000 replications are reported in Table 4. We can see that the application of QAE and CRQ improves the prediction significantly for the 87.5%th and 90%th quantiles; differences among the three methods are not significant at the lower quantiles. This makes sense since the data sparsity becomes more severe for more extreme quantile levels, therefore to incorporate lower quantile levels improves efficiency. In contrast, it may be not able to benefit the prediction performance at lower quantile levels by considering the higher levels.

[Table 4 about here.]

6 Proofs

In this section, we sketch the proof for the main Theorem 3.1 in Section 6.1 and the proof of Theorem 3.2 in Section 6.2. Proofs for all lemmas used in this section are deferred to the Appendix. We use $\|\cdot\|_{L^2}$ as the L^2 -norm for a function and $\|\cdot\|$ as the Euclidean norm for a vector.

6.1 Proof of Theorem 3.1

The proof of Theorem 3.1 is composed of three steps. In step 1, we approximate the estimated scores $\hat{\xi}_i$'s by linear combinations of ξ_i 's. In step 2, we obtain the asymptotic distribution of $\hat{\theta}_{\tau}$ at a single quantile level. In step 3, we extend the results in step 2 to multiple quantile levels.

Step 1. Approximation of the estimated scores. Most of the existing literature has been focused on establishing error bounds for the estimation of eigenvalues and eigenfunctions; see for example Hall and Hosseini-Nasab (2006, 2009) and the discussion therein. In this paper we concentrate on the accuracy in predicting the fPC scores.

Lemma 6.1 Under assumptions B1, B2 and C1, we have

$$E\|\widehat{\xi}_i - \xi_i\|^2 = o(n^{-1/2}). \tag{8}$$

In addition,

$$\widehat{\xi}_i - \xi_i = n^{-1/2} B \xi_i + O_p(n^{-1}), \tag{9}$$

where B is a $(p_0+1) \times (p_0+1)$ dimensional matrix with the bottom right $p_0 \times p_0$ block matrix equal to B^+ described next and the rest of the elements equal to zero. Here $B^+ = (b_{kk'})$ is a $p_0 \times p_0$ random matrix such that $b_{kk} = 0$ for $k = 1, \ldots, p_0$ and $b_{kk'} = n^{-1/2}(\lambda_k - \lambda_{k'})^{-1} \left(\sum_{i=1}^n \xi_{ik} \xi_{ik'}\right)$ if $k \neq k'$.

The result given by (9) indicates that the leading term of $\hat{\xi}_i - \xi_i$ is $n^{-1/2}B\xi_i$, which is a linear combination of ξ_i and the random matrix B does not depend on i. The proof of this lemma is in Section 8.

Step 2. Quantile regression on estimated scores. We focus on a single quantile level τ in this step. For any $\delta \in \mathbb{R}^{p_0+1}$, let

$$Z_n(\delta) = \sum_{i=1}^n \{ \rho_{\tau}(\widehat{u}_i - \widehat{\xi}_i^T \delta / \sqrt{n}) - \rho_{\tau}(\widehat{u}_i) \},$$

where $\widehat{u}_i = y_i - \widehat{\xi}_i^T \theta_{\tau}$. Then $Z_n(\delta)$ is a convex function which is minimized at $\widehat{\delta}_n = \sqrt{n}(\widehat{\theta}_{\tau} - \theta_{\tau})$. Therefore, the asymptotic distribution of $\widehat{\delta}_n$ is determined by the limiting behavior of $Z_n(\delta)$. Let $\psi_{\tau}(t) = \tau - I(t < 0)$, then according to the Knight's identity (Knight, 1998), we can decompose $Z_n(\delta)$ into two parts: $Z_n(\delta) = Z_{1n}(\delta) + Z_{2n}(\delta)$, where

$$Z_{1n}(\delta) = -\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \widehat{\xi}_{i}^{T} \delta \psi_{\tau}(\widehat{u}_{i})$$

$$Z_{2n}(\delta) = \sum_{i=1}^{n} \int_{0}^{\widehat{\xi}_{i}^{T} \delta / \sqrt{n}} \{ I(\widehat{u}_{i} \leq s) - I(\widehat{u}_{i} \leq 0) \} ds = \sum_{i=1}^{n} Z_{2ni}(\delta).$$

$$(10)$$

In order to show (5), it is sufficient to prove that

$$Z_n(\delta) \xrightarrow{d} -\delta^T W(\tau) + \frac{1}{2} \delta^T D_1(\tau) \delta,$$
 (11)

where $W(\tau) \sim N\{0, \tau(1-\tau)D_0 + D_1(\tau)\Sigma_0(\tau)D_1(\tau)\}$, since one can apply the convexity lemma (Pollard, 1991) to the quadratic form of δ in (11).

We shall derive the limiting distributions of $Z_{1n}(\delta)$ and $Z_{2n}(\delta)$ separately. Similarly to the definitions in (10), we define $Z_{1n}^*(\delta)$ based on the true scores ξ_i :

$$Z_{1n}^*(\delta) = -\frac{1}{\sqrt{n}} \sum \xi_i^T \delta \psi_\tau(u_i),$$

where $u_i = y_i - Q_{Y_i|X_i}(\tau) = y_i - \xi_i^T \theta_{\tau} = y_i - \sum_{k=0}^{p_0} \xi_{ik} \beta_k(\tau)$. By a direct application of the central limit theorem (CLT), we obtain that the asymptotic distribution of $Z_{1n}^*(\delta)$ is $N(0, \tau(1-\tau)\delta^T D_0\delta)$. However, when the predictors are estimated with errors, the difference $Z_{1n}(\delta) - Z_{1n}^*(\delta)$ is not negligible. Lemma 6.2 provides the following representation of $Z_{1n}(\delta)$.

Lemma 6.2 Under assumptions B1, B2 and C1,

$$Z_{1n}(\delta) = \delta^T \left[-\frac{1}{\sqrt{n}} \sum_{i=1}^n \{ \xi_i \psi_\tau(u_i) - D_1(\tau) d_i \} \right] + o_p(1),$$

where $d_i = (0, d_{i1}, \dots, d_{ip_0})^T$ and $d_{ik} = \sum_{r=1, r \neq k}^{p_0} (\lambda_k - \lambda_r)^{-1} \xi_{ik} \xi_{ir} \beta_r(\tau), k \ge 1$.

Since $\xi_i \psi_{\tau}(u_i) - D_1(\tau) d_i$ are i.i.d., Lemma 6.2 allows us to directly apply Linderberg's CLT to obtain the asymptotic distribution of $Z_{1n}(\delta)$. Note that $\mathrm{E}\{\xi_i \psi_{\tau}(u_i)\} = 0$ and $\mathrm{Var}\{\xi_i \psi_{\tau}(u_i)\} = \tau(1-\tau)D_0$. In addition, $\mathrm{E}d_i = 0$ because ξ_{ik} and ξ_{ir} are uncorrelated and have mean 0 (when $r \neq k$). Let the matrix $\Sigma(\tau)$ be the covariance matrix of d_i whose first row and first column is all 0 and the (k+1,k'+1)th element $(k,k'=1,\ldots,p_0)$ is given by $\mathrm{Cov}(d_{ik},d_{ik'}) = \theta_{\tau}^T A^{k,k'}\theta_{\tau}$ for some (p_0+1) by (p_0+1) matrix $A^{k,k'}$. While the first row and first column of $A^{k,k'}$ is all 0, simple calculation leads to that its bottom right block $A^{k,k',+} = (\sigma_{j,j'})$ is given by

$$\sigma_{j,j'} = \begin{cases} 0 & \text{if } j = k \text{ or } j' = k' \\ (\lambda_k - \lambda_j)^{-1} (\lambda_{k'} - \lambda_{j'})^{-1} \mathcal{E}(\xi_{1k} \xi_{1j} \xi_{1k'} \xi_{1j'}) & \text{Otherwise.} \end{cases}$$
(12)

Let $\Theta_{\tau} = 1_{(p_0+1)\times(p_0+1)} \otimes \theta^T$, and Σ_0 be a $(p_0+1)^2$ by $(p_0+1)^2$ matrix whose (k+1,k'+1)th block is $A^{k,k'}$ $(k,k'=1,\ldots,p_0)$ and (k+1,k'+1)th block is $0_{(p_0+1)\times(p_0+1)}$ for k=0 or k'=0. Then $\Sigma(\tau)$ can be rewritten as $\Sigma(\tau) = \Theta_{\tau}\Sigma_0\Theta_{\tau}^T$. Furthermore, we have

$$\operatorname{Cov}\{\xi_i\psi_\tau(u_i), d_i\} = \operatorname{E}\{\psi_\tau(u_i)\xi_i^T d_i\} = \operatorname{E}\{\xi_i^T d_i \operatorname{E}\psi_\tau(u_i)|\xi_i\} = 0,$$

which leads to

$$-\frac{1}{\sqrt{n}} \sum_{i} \{ \xi_{i} \psi_{\tau}(u_{i}) - D_{1}(\tau) d_{i} \} \stackrel{d}{\to} N(0, \tau(1-\tau)D_{0} + D_{1}(\tau)\Sigma(\tau)D_{1}(\tau)).$$

Equivalently, we have $Z_{1n}(\delta) \stackrel{d}{\to} -\delta^T W(\tau)$ where $W(\tau) \sim N\left(0, \tau(1-\tau)D_0 + D_1(\tau)\Sigma(\tau)D_1(\tau)\right)$. Consequently, the following result for $Z_{2n}(\delta)$ concludes the asymptotic distribution in (11). **Lemma 6.3** Under assumptions B1, B2 and C1, we have

$$Z_{2n}(\delta) = \frac{1}{2}\delta^T D_1(\tau)\delta + o_p(1).$$

Step 3. Asymptotic distributions across quantile levels. When considering various quantile levels, the same arguments can be made via a convex optimization and the limiting distribution of the objective function. The asymptotic covariance in (6) is obtained by the covariance between $\xi_i \psi_{\tau_l}(u_i) + D_1(\tau_l) d_i(\tau_l)$ and $\xi_i \psi_{\tau_{l'}}(u_i) + D_1(\tau_{l'}) d_i(\tau_{l'})$, following similar calculation as in (12).

6.2 Proof of Theorem 3.2

We just need to show that $R(\widetilde{\Sigma}_{\widehat{\zeta}} - \Sigma_{\widehat{\zeta}})R^T = 0$. The (l, l')th block of the matrix $\widetilde{\Sigma}_{\widehat{\zeta}} - \Sigma_{\widehat{\zeta}}$ is $\Theta_{\tau_l} \Sigma_0 \Theta_{\tau_{l'}}$, where $1 \leq l, l' \leq L$. Therefore, we have $\widetilde{\Sigma}_{\widehat{\zeta}} - \Sigma_{\widehat{\zeta}} = A\Sigma_0 A^T$ where $A = (\Theta_{\tau_1}, \dots, \Theta_{\tau_L})^T$ is a $(p_0 + 1)L \times (p_0 + 1)$ matrix. Noting that $\Theta_{\tau_l} = 1_{(p_0+1)\times(p_0+1)} \otimes \theta_l^T$ $(l = 1, \dots, L)$, we have $A^T = 1_{(p_0+1)\times(p_0+1)} \otimes \zeta^T$ and thus $A = 1_{(p_0+1)\times(p_0+1)} \otimes \zeta$. Therefore, when $R\zeta = 0$, it follows that $RA = 1_{(p_0+1)\times(p_0+1)} \otimes (R\zeta) = 0$ which concludes the proof.

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8 Appendix

The Appendix includes the proofs for all lemmas that are needed for showing Theorem 3.1. **Proof of Lemma 6.1.** The bound in (8) is a result following standard bounds for the estimations of the eigenfunctions and covariance kernel in FDA literature. According to Theorem 1 in Hall and Hosseini-Nasab (2006), we have

$$\|\widehat{\phi}_k - \phi_k\|_{L^2} \le 8^{1/2} s_k^{-1} \|\widehat{G} - G\|,$$

where $s_k = \min_{r \leq k} (\lambda_r - \lambda_{r+1})$ and $|||\widehat{G} - G||| = [\int_0^1 \int_0^1 \{\widehat{G}(u, v) - G(u, v)\}^2 du dv]^{1/2}$. Therefore,

$$|\widehat{z}_{ik} - \xi_{ik}| = \left| \int_0^1 X_i(t) \{ \widehat{\phi}_k(t) - \phi_k(t) \} dt \right| \le ||X_i||_{L^2} \cdot ||\widehat{\phi}_k - \phi_k||_{L^2} \le \operatorname{constant} \cdot ||X_i||_{L^2} s_k^{-1} \cdot |||\widehat{G} - G|||,$$

which leads to $\|\widehat{\xi}_i - \xi_i\| \le \text{constant} \cdot \|X_i\|_{L^2} s_{p_0}^{-1} |||\widehat{G} - G|||$. Therefore, for any c > 0,

$$\mathbb{E}\|\widehat{\xi}_i - \xi_i\|^c \le \text{constant} \cdot s_{p_0}^{-c}(\mathbb{E}\|\widehat{G} - G\|^{2c})^{1/2} \le \text{constant} \cdot s_{p_0}^{-c} n^{-c/2},$$

by noting that $\mathrm{E}|||\widehat{G} - G|||^c \leq \mathrm{constant} \cdot n^{-c/2}$ for any c > 0 (Hall and Hosseini-Nasab, 2009, Lemma 3.3). Thus for finite p_0 , we have $\mathrm{E}||\widehat{\xi}_i - \xi_i||^c = o(n^{-c/4})$; in particular, $\sqrt{n}\mathrm{E}||\widehat{\xi}_i^T - \xi_i^T||^2 = o(1)$.

Next we prove the representation in (9). Let \widetilde{G} be the estimator of the kernel G based on the fully observed covariate $X_i(\cdot)$, and recall that \widehat{G} is the estimate based on the discretized W_{ij} with measurement error. Denote $\widetilde{\mathcal{Z}} = \sqrt{n}(\widetilde{G} - G)$ and $\widehat{\mathcal{Z}} = \sqrt{n}(\widehat{G} - G)$. We use the notation $\int \widehat{\mathcal{Z}} \phi_k \phi_{k'}$ to denote $\int_0^1 \int_0^1 \widehat{\mathcal{Z}}(u, v) \phi_k(u) \phi_{k'}(v) du dv$.

Since $\{\phi_k : k \geq 1\}$ forms a basis of the L^2 space on [0, 1], we have $\widehat{\phi}_k = \sum_{k'=1}^{\infty} a_{kk'} \phi'_k$, where $k = 1, \ldots, p_0$ and the generalized Fourier coefficients $a_{kk'} = \int_0^1 \widehat{\phi}_k(t) \phi_{k'}(t) dt$. Furthermore, we have the following expansion for $a_{kk'}$'s:

$$a_{kk} = 1 + O_p(n^{-1}); \quad a_{kk'} = n^{-1/2}(\lambda_k - \lambda_{k'})^{-1} \int \widehat{\mathcal{Z}} \phi_k \phi_{k'} + O_p(n^{-1}) \text{ if } k \neq k',$$

according to (2.6) and (2.7) in Hall and Hosseini-Nasab (2006). Therefore, for $k = 1, \ldots, p_0$, we have

$$\int_{0}^{1} X_{i}(t) \{\widehat{\phi}_{k}(t) - \phi_{k}(t)\} dt = \sum_{k'=1}^{p_{0}} \{a_{kk'} - I(k'=k)\} \xi_{ik'}$$

$$= \sum_{k'=1, k' \neq k}^{p_{0}} n^{-1/2} (\lambda_{k} - \lambda_{k'})^{-1} \int \widehat{\mathcal{Z}} \phi_{k} \phi_{k'} \xi_{ik'} + O_{p}(n^{-1}).$$

A direct calculation gives that

$$\int \widetilde{\mathcal{Z}} \phi_k \phi_{k'} = n^{-1/2} \sum_{i=1}^n \xi_{ik} \xi_{ik'} - n^{1/2} \bar{\xi}_k \bar{\xi}_{k'}$$

for $k, k' = 1, ..., p_0$ and $k \neq k'$, where $\bar{\xi}_k = n^{-1} \sum_{i=1}^n \xi_{ik}$. Since $n^{1/2} \bar{\xi}_k \bar{\xi}_{k'} = n^{-1/2} \cdot (n^{1/2} \bar{\xi}_k) \cdot (n^{1/2} \bar{\xi}_{k'}) = n^{-1/2} \cdot O_p(1) \cdot O_p(1) = O_p(n^{-1/2})$, we have $\int \widetilde{\mathcal{Z}} \phi_k \phi_{k'} = n^{-1/2} \sum_i \xi_{ik} \xi_{ik'} + O_p(n^{-1/2})$. The same approximation holds when using $\widehat{\mathcal{Z}}$ since $\widehat{\mathcal{Z}} - \widetilde{\mathcal{Z}}$ is uniformly $o_p(n^{-1/2})$ as shown by Zhang and Chen (2007). Consequently,

$$\int_0^1 X_i(t) \{ \widehat{\phi}_k(t) - \phi_k(t) \} dt = \sum_{k'=1, k' \neq k}^{p_0} n^{-1} (\lambda_k - \lambda_{k'})^{-1} \left(\sum_{i=1}^n \xi_{ik} \xi_{ik'} \right) \xi_{ik'} + O_p(n^{-1}).$$
 (13)

This approximation will not be affected if we use $\widehat{X}_i(\cdot)$ instead of the true curve $X_i(\cdot)$ because the difference $\widehat{X}_i(\cdot) - X_i(\cdot)$ is negligible uniformly for all i(e.g. see Theorem 2 in Zhang and Chen (2007) or Lemma 1 in Zhu et al. (2014)). Let a p_0 -dimension random matrix $B^+ = (b_{kk'})$ where $b_{kk'} = 0$ if k = k' and $b_{kk'} = n^{-1/2}(\lambda_k - \lambda_{k'})^{-1} (\sum_{i=1}^n \xi_{ik} \xi_{ik'})$ if $k \neq k'$. Let B be a $(p_0 + 1) \times (p_0 + 1)$ zero matrix but the bottom right block is replaced by B^+ , then the right hand side in (13) becomes $n^{-1/2}B\xi_i + O_p(n^{-1})$. Consequently, we have

$$\widehat{\xi}_i - \xi_i = n^{-1/2} B \xi_i + O_p(n^{-1}).$$

Proof of Lemma 6.2. We first decompose the difference between $Z_{1n}(\delta)$ and $Z_{1n}^*(\delta)$ into three parts S_1, S_2 and S_3 as follows.

$$Z_{1n}(\delta) - Z_{1n}^*(\delta) = -\frac{1}{\sqrt{n}} \sum_{i} \widehat{\xi}_{i}^T \delta \psi_{\tau}(\widehat{u}_i) + \frac{1}{\sqrt{n}} \sum_{i} z_{i}^T \delta \psi_{\tau}(u_i)$$

$$= -\frac{1}{\sqrt{n}} \sum_{i} (\widehat{\xi}_{i}^T - \xi_{i}^T) \delta \{ \psi_{\tau}(\widehat{u}_i) - \psi_{\tau}(u_i) \}$$

$$-\frac{1}{\sqrt{n}} \sum_{i} (\widehat{\xi}_{i}^T - \xi_{i}^T) \delta \psi_{\tau}(u_i)$$

$$-\frac{1}{\sqrt{n}} \sum_{i} \xi_{i}^T \delta \{ \psi_{\tau}(\widehat{u}_i) - \psi_{\tau}(u_i) \}$$

$$=: S_1 + S_2 + S_3.$$

We next prove Lemma 6.2 by the following three steps, namely $S_2 = o_p(1)$ (Step i), $S_1 = o_p(1)$ (Step ii) and $S_3 = n^{-1/2} \delta^T D_1(\tau) \sum_{i=1}^n d_i + o_p(1)$ (Step iii). Step i and Step ii indicate that the first two terms S_1 and S_2 are negligible, and it is sufficient to show that

 $\mathrm{E}(S_2^2) = o(1)$ and $\mathrm{E}|S_1| = o(1)$ according to Chebyshev's inequality. The third term S_3 is challenging to analyze since the function of $\psi_{\tau}(\cdot)$ is not differentiable. In **Step iii**, we approximate the term S_3 mainly using the uniform approximation on $\psi_{\tau}(\cdot)$.

Step i. First notice that $E\{\psi_{\tau}(u_i)|\xi_i,\widehat{\xi}_i\}=0$ and $E\{\psi_{\tau}(u_i)^2|\xi_i,\widehat{\xi}_i\}=\tau-\tau^2$. Therefore, we have $E(S_2)=0$, and further

$$E(S_2^2) = E\left\{\frac{1}{\sqrt{n}}\sum_{i=1}^n (\widehat{\xi}_i^T \delta - \xi_i^T \delta)\psi_\tau(u_i)\right\}^2$$
$$= \frac{1}{n}\sum_{i=1}^n \sum_{i'=1}^n E\left\{(\widehat{\xi}_i^T - \xi_i^T)\delta\psi_\tau(u_i) \cdot (\widehat{\xi}_{i'}^T - \xi_{i'}^T)\delta\psi_\tau(u_{i'})\right\}.$$

For i = i',

$$\mathbb{E}\left\{ (\widehat{\xi}_i^T - \xi_i^T) \delta \psi_{\tau}(u_i) \cdot (\widehat{\xi}_{i'}^T - \xi_{i'}^T) \delta \psi_{\tau}(u_{i'}) \right\} = \mathbb{E}\left\{ (\widehat{\xi}_i^T \delta - \xi_i^T \delta) \psi_{\tau}(u_i) \right\}^2 \\
= \mathbb{E}\left\{ (\widehat{\xi}_i^T \delta - \xi_i^T \delta)^2 \mathbb{E}\{\psi_{\tau}^2(u_i) | \xi_i, \widehat{\xi}_i\} \right\} = \tau (1 - \tau) \mathbb{E}\left\{ (\widehat{\xi}_i^T \delta - \xi_i^T \delta)^2 \right\}.$$

Since $\widehat{\xi}_i$ is identically distributed for all i, we have $E\left\{(\widehat{\xi}_i^T\delta - \xi_i^T\delta)^2\right\} = E\left\{(\widehat{\xi}_1^T\delta - \xi_1^T\delta)^2\right\}$. For $i \neq i'$, we have

$$E\left\{ (\widehat{\xi}_i^T - \xi_i^T) \delta \psi_\tau(u_i) \cdot (\widehat{\xi}_{i'}^T - \xi_{i'}^T) \delta \psi_\tau(u_{i'}) \right\} = 0$$

since

$$\mathrm{E}\{\psi_{\tau}(u_i)\psi_{\tau}(u_{i'})|\xi_i,\widehat{\xi}_i,\xi_{i'},\widehat{\xi}_{i'}\} = \mathrm{E}\{\psi_{\tau}(u_i)|\xi_i,\widehat{\xi}_i,\xi_{i'},\widehat{\xi}_{i'}\} \cdot \mathrm{E}\{\psi_{\tau}(u_{i'})|\xi_i,\widehat{\xi}_i,\xi_{i'},\widehat{\xi}_{i'}\} = 0.$$

Therefore $E(S_2^2) = \tau(1-\tau)E\left\{(\widehat{\xi}_1^T\delta - \xi_1^T\delta)^2\right\} = O(E\|\widehat{\xi}_i - \xi_i\|^2) = o(1).$

Step ii. For S_1 , we first introduce the notation

$$\Delta_i = \mathrm{E}(\psi_\tau(\widehat{u}_i)|\xi_i,\widehat{\xi}_i) = \tau - F_i(\widehat{\xi}_i^T \theta_\tau) = F_i(\xi_i^T \theta_\tau) - F_i(\widehat{\xi}_i^T \theta_\tau).$$

In addition, the random variable Δ_i satisfy that

$$\Delta_i = \mathcal{E}(\psi_\tau(\widehat{u}_i) - \psi_\tau(u_i)|\xi_i, \widehat{\xi}_i), \tag{14}$$

$$|\Delta_i| = \mathrm{E}(|\psi_{\tau}(\widehat{u}_i) - \psi_{\tau}(u_i)||\xi_i, \widehat{\xi}_i). \tag{15}$$

The result given in (14) is obtained by noting that $\psi_{\tau}(u_i)$ has mean 0 conditional on ξ_i ; while (15) holds because of that $|\psi_{\tau}(\widehat{u}_i) - \psi_{\tau}(u_i)| = I\{\min(\widehat{\xi}_i\theta_{\tau}, \xi_i\theta_{\tau}) < y_i < \max(\widehat{\xi}_i\theta_{\tau}, \xi_i\theta_{\tau})\}.$

By Taylor's theorem, for any $a, b \in \mathbb{R}$, we have

$$F(a+b) - F(a) = f(a)b + b^{2} \int_{0}^{1} f'(a+tb)(1-t)dt =: f(a)b + \frac{b^{2}}{2}R(a,b),$$

where $|R(a,b)| \leq C_0$. Therefore,

$$\Delta_i = -(\widehat{\xi}_i^T \theta_\tau - \xi_i^T \theta_\tau) f_i(\xi_i^T \theta_\tau) + (\widehat{\xi}_i^T \theta_\tau - \xi_i^T \theta_\tau)^2 R(\widehat{\xi}_i^T, \xi_i^T),$$

where $|R(\widehat{\xi}_i^T, \xi_i^T)| \leq 2C_0$. We also have the bound

$$\mathrm{E}\Delta_i^2 \leq \mathrm{constant} \cdot \mathrm{E}\|\widehat{\xi_i} - \xi_i\|^2 = o(n^{-1/2}).$$

Therefore, $|S_1| \leq \frac{1}{\sqrt{n}} \sum_i |(\widehat{\xi}_i^T \delta - \xi_i^T \delta)| \cdot |\psi_\tau(\widehat{u}_i) - \psi_\tau(u_i)|$ and consequently

$$\begin{aligned} \mathbf{E}|S_1| &\leq \frac{1}{\sqrt{n}} \mathbf{E}\{\sum_i |(\widehat{\xi}_i^T \delta - \xi_i^T \delta)||\Delta_i|\} \\ &= \sqrt{n} \mathbf{E}|(\widehat{\xi}_1^T \delta - \xi_1^T \delta)\Delta_1| \leq \sqrt{n} \mathbf{E}||\widehat{\xi}_i - \xi_i||^2 \mathbf{E}\Delta_1^2 = o(1). \end{aligned}$$

Step iii. Define

$$R_n(t) = \sum_{i=1}^{n} \xi_i \{ \psi_{\tau}(u_i - \xi_i^T t) - \psi_{\tau}(u_i) \},$$

for any vector such that $||t|| \leq C$ for some constant C. Then the uniform approximation (He and Shao, 1996) indicates that

$$\sup ||R_n(t) - \mathbb{E}\{R_n(t)\}|| = O_p(n^{1/2}(\log n)||t||^{1/2}).$$

On the other hand,

$$\begin{aligned} \mathbf{E}\{R_n(t)\} &= \sum_{i} \mathbf{E}[\xi_i \{F_i(\xi_i^T \theta_\tau) - F_i(\xi_i^T \theta_\tau - \xi_i^T t)\}] = n \mathbf{E}[\xi_1 \{F_1(\xi_1^T \theta_\tau) - F_1(\xi_1^T \theta_\tau - \xi_1^T t)\}] \\ &= -n \mathbf{E}\xi_1 \xi_1^T f_1(\xi_1^T \theta_\tau) t + O(n \mathbf{E} \|\xi_1\|^3 \|t\|^2) \\ &= -n D_1(\tau) t + O(n \|t\|^2). \end{aligned}$$

Therefore,

$$R_n(t) = -nD_1(\tau)t + O(n||t||^2) + O_p\left(n^{1/2}(\log n)||t||^{1/2}\right). \tag{16}$$

Note that $\widehat{u}_i = u_i + \xi_i^T \theta_\tau - \widehat{\xi}_i^T \theta_\tau$ and $\widehat{\xi}_i - \xi_i = B \xi_i$ up to a negligible term $O_p(n^{-1})$, let $t = n^{-1/2} B \theta_\tau$, then

$$-\frac{1}{\sqrt{n}} \sum_{i} \xi_{i}^{T} \delta\{\psi_{\tau}(\widehat{u}_{i}) - \psi_{\tau}(u_{i})\} = -n^{-1/2} R_{n}(n^{-1/2}B\theta_{\tau}) + o_{p}(1),$$

where the term $o_p(1)$ is caused by the remainder term $\hat{\xi_i} - \xi_i - B\xi_i = O_p(n^{-1})$ and obtained by using the same technique in Step ii via conditional expectation and Taylor theorem. Combining the result in (16) and the fact that $||n^{-1/2}B\theta_\tau|| = O_p(n^{-1/2})$, we obtain that $R_n(n^{-1/2}B\theta_\tau) = -n^{1/2}D_1(\tau)B\theta_\tau + O(1) + O_p(n^{1/4}\log n)$, leading to that

$$S_3 = -\frac{1}{\sqrt{n}} \sum_{i} \xi_i^T \delta \{ \psi_{\tau}(\widehat{u}_i) - \psi_{\tau}(u_i) \} = \delta^T D_1(\tau) B \theta_{\tau} + o_p(1).$$

According to the definition of B in (9), it is easy to verify that $B\theta_{\tau} = n^{-1/2} \sum_{i=1}^{n} d_i$, where $d_i = (0, d_{i1}, \dots, d_{ip_0})$ and $d_{ik} = \sum_{r=1, r \neq k}^{p_0} (\lambda_k - \lambda_r)^{-1} \xi_{ik} \xi_{ir} \beta_r(\tau), k \geq 1$. Therefore, it follows that $S_3 = n^{-1/2} \delta^T D_1(\tau) \sum_{i=1}^n d_i + o_p(1)$, which concludes Step iii and thus Lemma 6.2.

Proof of Lemma 6.3. Recall that $Z_{2n} = \sum_{i=1}^{n} Z_{2ni}$, where

$$Z_{2ni}(\delta) = \int_0^{\widehat{\xi}_i^T \delta/\sqrt{n}} \{ I(\widehat{u}_i \le s) - I(\widehat{u}_i \le 0) \} ds.$$

First, we have

$$\mathrm{E}[Z_{2ni}(\delta)|\xi_i,\widehat{\xi_i}] = \int_0^{\widehat{\xi_i^T}\delta/\sqrt{n}} F_i(\widehat{\xi_i^T}\theta_\tau + s) - F_i(\widehat{\xi_i^T}\theta_\tau) ds = \frac{1}{\sqrt{n}} \int_0^{\widehat{\xi_i^T}\delta} F_i(\widehat{\xi_i^T}\theta_\tau + \frac{t}{\sqrt{n}}) - F_i(\widehat{\xi_i^T}\theta_\tau) dt.$$

Therefore, by Taylor's theorem, we have

$$E[Z_{2ni}(\delta)|\xi_i,\widehat{\xi_i}] = \frac{1}{\sqrt{n}} \int_0^{\widehat{\xi_i}^T \delta} f_i(\widehat{\xi_i}^T \theta_\tau) \frac{t}{\sqrt{n}} + \frac{t^2}{2n} R(\widehat{\xi_i}^T \delta, \frac{t}{\sqrt{n}}) dt = \frac{1}{2n} \delta^T \widehat{\xi_i} f_i(\widehat{\xi_i}^T \theta_\tau) \widehat{\xi_i}^T \delta + R_{n,i},$$

where R_{ni} is the remainder satisfying that $|R_{ni}| \leq cn^{-3/2} |\hat{\xi}_i^T \delta|^3$. Consequently,

$$E[Z_{2ni}(\delta)|\xi_i,\widehat{\xi}_i] = \frac{1}{2} \cdot \delta^T \frac{1}{n} \widehat{\xi}_i f_i(\widehat{\xi}_i^T \theta_\tau) \widehat{\xi}_i^T \delta + R_{ni}.$$

Therefore, the unconditional expectation of $Z_{2ni}(\delta)$ is

$$EZ_{2ni}(\delta) = E\{E[Z_{2ni}(\delta)|\xi_i, \widehat{\xi}_i]\} = \frac{1}{2} \cdot \delta^T E\left(\frac{1}{n} \widehat{\xi}_i f_i(\widehat{\xi}_i^T \theta_\tau) \widehat{\xi}_i^T\right) \delta + ER_n = \frac{1}{2} \cdot \frac{1}{n} \delta^T D_1(\tau) \delta + E(R_{ni}),$$

leading to

$$EZ_{2n} = \frac{1}{2}\delta^T D_1(\tau)\delta + \sum_{i=1}^n E(R_{ni}).$$

The second term $\sum_{i=1}^{n} E(R_{ni})$ is negligible because

$$\left| \sum_{i=1}^{n} \mathrm{E}(R_{ni}) \right| \leq \sum_{i=1}^{n} \mathrm{E}|R_{ni}| \leq cn^{-3/2} \sum_{i=1}^{n} \mathrm{E}|\hat{\xi}_{i}^{T}\delta|^{3} = cn^{-1/2} \mathrm{E}|\hat{\xi}_{1}^{T}\delta|^{3}$$
(17)

$$\leq O(n^{-1/2}) \cdot \left(\mathbb{E} \| \widehat{\xi}_1 \|_2^3 \right) \| \delta \|_2^3 = O(n^{-1/2}) \cdot O(1) = o(1), \tag{18}$$

where the last step is due to the fact that $\|\widehat{\xi}_1\|_2 = \|\widehat{X}_1\|_2 \le \|\widehat{X}_1 - X_1\|_2 + \|X_1\|_2$.

We next will show that $\max_{i=1,\dots,n} \|\xi_i\|/\sqrt{n} \stackrel{p}{\to} 0$. Note that $\|\xi_i\|^2 = 1 + \xi_{i1}^2 + \dots + \xi_{ip_0}^2$, $i = 1,\dots,n$, and $\|\xi_i\|^2$'s are i.i.d. with a finite second moment $\mathbf{E}\|\xi_i\|^2 = 1 + \lambda_1 + \dots + \lambda_{p_0} < \infty$.

For any $\epsilon > 0$, we have

$$P\left(\max_{i=1,\dots,n} \frac{\|\xi_{i}\|}{\sqrt{n}} > \epsilon\right) \leq \sum_{i=1}^{n} P(\|\xi_{i}\| > \sqrt{n}\epsilon)$$

$$\leq \frac{1}{n\epsilon^{2}} \sum_{i=1}^{n} E\{\|\xi_{i}\|^{2} I(\|\xi_{i}\| > \sqrt{n}\epsilon)\}$$

$$= \frac{1}{\epsilon^{2}} E\{\|\xi_{1}\|^{2} I(\|\xi_{1}\| > \sqrt{n}\epsilon)\} \to 0,$$

according to the dominated convergence theorem. It implies that $\max_{i=1,\dots,n} \|\widehat{\xi_i}\|/\sqrt{n} = o_p(1)$ since $\|\widehat{\xi_i} - \xi_i\| = o_p(1)$ uniformly for all i's. Consequently, $\operatorname{Var}(Z_{2n}|\xi_i$'s, $\widehat{\xi_i}$'s) $\leq \max_{i=1,\dots,n} \|\widehat{\xi_i}^T \delta\|/\sqrt{n} \cdot \operatorname{E}(Z_{2ni}|\xi_i$'s, $\widehat{\xi_i}$'s) $= o_p(1)$, i.e., the conditional variance converges to 0 in probability. Therefore, following the martingale argument in the proof of Theorem 2 in Pollard (1991), we have $Z_{2n} - \operatorname{E}(Z_{2n}) = o_p(1)$, which completes the proof.

References

Cardot, H., Crambes, C., and Sarda, P. (2005). Quantile regression when the covariates are functions. *Nonparametric Statistics*, 17(7):841–856.

Chen, K. and Müller, H.-G. (2012). Conditional quantile analysis when covariates are functions, with application to growth data. *Journal of the Royal Statistical Society: Series B* (Statistical Methodology), 74(1):67–89.

Fanaee-T, H. and Gama, J. (2014). Event labeling combining ensemble detectors and background knowledge. *Progress in Artificial Intelligence*, 2(2-3):113–127.

Ferraty, F., Rabhi, A., and Vieu, P. (2005). Conditional quantiles for dependent functional data with application to the climatic *El Niño* phenomenon. *Sankhyā: The Indian Journal of Statistics*, 67(2):378–398.

Ferraty, F. and Vieu, P. (2006). *Nonparametric functional data analysis*. Springer, New York.

- Gertheiss, J., Maity, A., and Staicu, A.-M. (2013). Variable selection in generalized functional linear models. *Stat*, 2(1):86–101.
- Hall, P. and Hosseini-Nasab, M. (2006). On properties of functional principal components analysis. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 68(1):109–126.
- Hall, P. and Hosseini-Nasab, M. (2009). Theory for high-order bounds in functional principal components analysis. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 146, pages 225–256. Cambridge Univ Press.
- Hall, P., Müller, H.-G., and Wang, J.-L. (2006). Properties of principal component methods for functional and longitudinal data analysis. *The Annals of Statistics*, 34(3):1493–1517.
- He, X. and Shao, Q.-M. (1996). A general Bahadur representation of *M*-estimators and its application to linear regression with nonstochastic designs. *The Annals of Statistics*, 24(6):2608–2630.
- Huang, L., Scheipl, F., Goldsmith, J., Gellar, J., Harezlak, J., McLean, M. W., Swihart, B.,
 Xiao, L., Crainiceanu, C., and Reiss, P. (2015). refund: Regression with Functional Data.
 R package version 0.1-13.
- Huber, P. J. (1967). The behavior of maximum likelihood estimates under nonstandard conditions. In *Proceedings of the fifth Berkeley symposium on mathematical statistics and* probability. Berkeley: University of California Press.
- Ivanescu, A. E., Staicu, A.-M., Scheipl, F., and Greven, S. (2015). Penalized function-on-function regression. *Computational Statistics*, 30(2):539–568.
- Jiang, C.-R. and Wang, J.-L. (2010). Covariate adjusted functional principal components analysis for longitudinal data. *The Annals of Statistics*, 38(2):1194–1226.
- Jiang, L., Bondell, H. D., and Wang, H. (2014). Interquantile shrinkage and variable selection in quantile regression. *Computational Statistics & Data Analysis*, 69:208–219.

- Kato, K. (2012). Estimation in functional linear quantile regression. *The Annals of Statistics*, 40(6):3108–3136.
- Knight, K. (1998). Limiting distributions for L_1 regression estimators under general conditions. The Annals of Statistics, 26(2):755–770.
- Koenker, R. (1984). A note on L-estimates for linear models. Statistics & Probability Letters, 2(6):323–325.
- Koenker, R. (2005). Quantile regression, volume 38. Cambridge university press.
- Larsen, J. (2013). Policy institute: Bike-sharing programs hit the streets in over 500 cities worldwide. http://www.earth-policy.org/plan_b_updates/2013/update112.
- Li, M., Staicu, A.-M., and Bondell, H. D. (2015). Incorporating covariates in skewed functional data models. *Biostatistics*, 16(3):413–426.
- Li, Y., Wang, N., and Carroll, R. J. (2010). Generalized functional linear models with semiparametric single-index interactions. *Journal of the American Statistical Association*, 105(490):621–633.
- Li, Y., Wang, N., and Carroll, R. J. (2013). Selecting the number of principal components in functional data. *Journal of the American Statistical Association*, 108(504):1284–1294.
- Morris, J. S. and Carroll, R. J. (2006). Wavelet-based functional mixed models. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 68(2):179–199.
- Pollard, D. (1991). Asymptotics for least absolute deviation regression estimators. *Econometric Theory*, 7:186–199.
- Ramsay, J. and Silverman, B. (2005). Functional Data Analysis. Springer Series in Statistics. Springer.
- Redd, A. (2012). A comment on the orthogonalization of b-spline basis functions and their derivatives. *Statistics and Computing*, 22(1):251–257.

- Staicu, A.-M., Crainiceanu, C. M., Reich, D. S., and Ruppert, D. (2012). Modeling functional data with spatially heterogeneous shape characteristics. *Biometrics*, 68(2):331–343.
- Staicu, A.-M., Lahiri, S. N., and Carroll, R. J. (2015). Significance tests for functional data with complex dependence structure. *Journal of Statistical Planning and Inference*, 156:1–13.
- Usset, J., Staicu, A.-M., and Maity, A. (2016). Interaction models for functional regression.

 Computational Statistics & Data Analysis, 94:317–330.
- Wang, H. J., Stefanski, L. A., and Zhu, Z. (2012). Corrected-loss estimation for quantile regression with covariate measurement errors. *Biometrika*, 99(2):405–421.
- Wang, K. and Wang, H. J. (2016). Optimally combined estimation for tail quantile regression. Statistica Sinica, 26(1):295–311.
- Wei, Y. and Carroll, R. J. (2009). Quantile regression with measurement error. *Journal of the American Statistical Association*, 104(487):1129–1143.
- Wu, Y., Ma, Y., and Yin, G. (2015). Smoothed and corrected score approach to censored quantile regression with measurement errors. *Journal of the American Statistical Association*, in press.
- Yao, F., Müller, H.-G., and Wang, J.-L. (2005). Functional data analysis for sparse longitudinal data. *Journal of the American Statistical Association*, 100(470):577–590.
- Zhang, J.-T. and Chen, J. (2007). Statistical inferences for functional data. *The Annals of Statistics*, 35(3):1052–1079.
- Zhao, Z. and Xiao, Z. (2014). Efficient regressions via optimally combining quantile information. *Econometric Theory*, 30(06):1272–1314.
- Zhou, L., Huang, J. Z., and Carroll, R. J. (2008). Joint modelling of paired sparse functional data using principal components. *Biometrika*, 95(3):601–619.

- Zhu, H., Yao, F., and Zhang, H. H. (2014). Structured functional additive regression in reproducing kernel hilbert spaces. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 76(3):581–603.
- Zou, H. and Yuan, M. (2008). Composite quantile regression and the oracle model selection theory. *The Annals of Statistics*, 36(3):1108–1126.

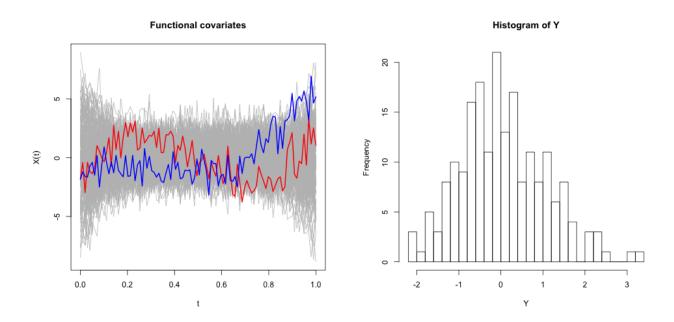


Figure 1: Simulated data when n=200 and $\gamma=1$. The left panel plots the functional covariates and two randomly selected curves are highlighted in blue and red; the right panel is the histogram of the response.

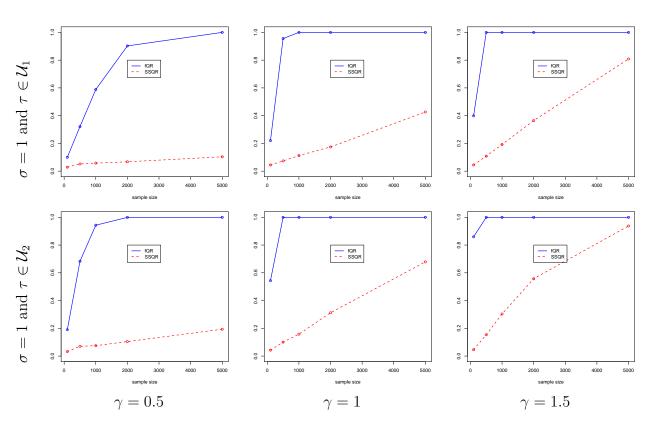


Figure 2: Power curves of the adjusted Wald test and SSQR in various scenarios.

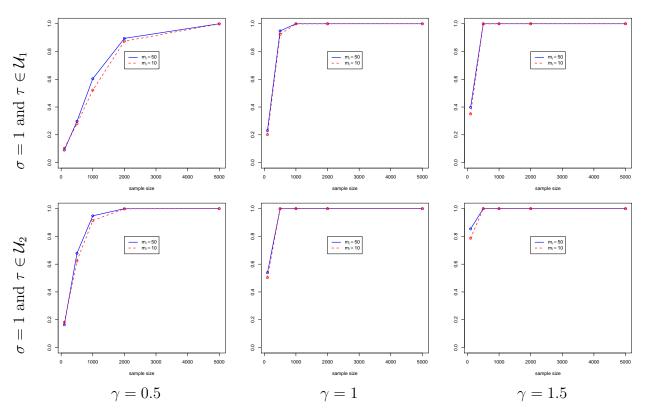


Figure 3: Power curves of the functional Wald test for moderately sparse and highly sparse designs in various scenarios.

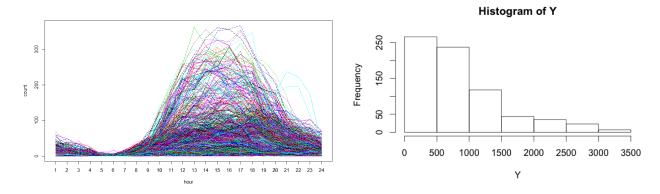


Figure 4: Bike rental data (casual users). The left panel plots hourly bike rentals for casual users on the previous day, and the right panel plots the histogram of the total casual bike rentals on the current day.

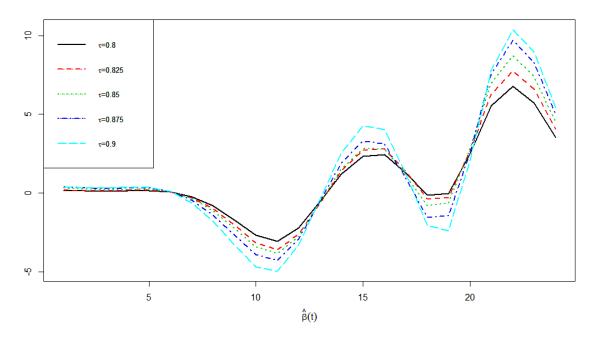


Figure 5: $\widehat{\beta}(t)$ at various quantile levels.

Figure 6: Bootstrap means (the left panel) and standard errors (the right panel) of the estimates of $\beta(t)$ from the QAE, CRQ and the local quantile regression estimation at the 0.9-th quantile (RQ).

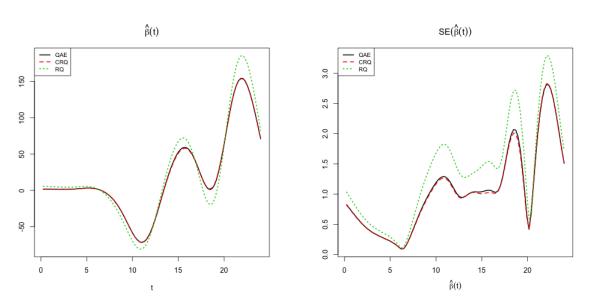


Table 1: Type I error of the adjusted Wald-type test for various levels of significance α , quantile level sets \mathcal{U} and sample sizes n. Recall the quantile levels are $\mathcal{U}_1 = \{0.1, 0.2, 0.3, 0.4\}$ and $\mathcal{U}_2 = \{0.1, 0.2, 0.6, 0.7\}$. Results are based on 5000 simulations.

Scenario	n		α		Scenario	n		α	
		0.01	0.05	0.10			0.01	0.05	0.10
	100	0.021	0.060	0.104		100	0.030	0.076	0.123
$\sigma = 1$	500	0.014	0.057	0.107	$\sigma = 1$	500	0.015	0.062	0.116
$u \in \mathcal{U}_1$	1000	0.017	0.052	0.106	$u \in \mathcal{U}_2$	1000	0.015	0.059	0.112
	2000	0.011	0.051	0.101		2000	0.010	0.053	0.103
	5000	0.010	0.054	0.105		5000	0.012	0.056	0.103

Table 2: Type I error of the alternative approaches for various levels of significance α , quantile level sets \mathcal{U} and sample sizes n. Recall the quantile levels are $\mathcal{U}_1 = \{0.1, 0.2, 0.3, 0.4\}$ and $\mathcal{U}_2 = \{0.1, 0.2, 0.6, 0.7\}$. Results are based on 5000 simulations. When one method returns error (due to singularity of the design matrix) in more than 20% replications, we report it as "–".

Scenario	n	NaïveQR			SSQR			pcaQR		
		0.01	0.05	0.10	0.01	0.05	0.10	0.01	0.05	0.10
	100	_	_	_	0.008	0.033	0.071	_	_	_
$\sigma = 1$	500	_	_	_	0.008	0.036	0.080	_	_	_
$u \in \mathcal{U}_1$	1000	_	_	_	0.010	0.049	0.092	0.996	0.999	1.000
	2000	1.000	1.000	1.000	0.009	0.048	0.097	1.000	1.000	1.000
	5000	1.000	1.000	1.000	0.008	0.053	0.099	0.999	1.000	1.000
	100	_	=	=	0.009	0.040	0.077	=	=	
$\sigma = 1$	500	_	_	_	0.009	0.050	0.096	_	_	_
$u \in \mathcal{U}_2$	1000	1.000	1.000	1.000	0.009	0.046	0.095	1.000	1.000	1.000
	2000	1.000	1.000	1.000	0.010	0.048	0.099	1.000	1.000	1.000
	5000	1.000	1.000	1.000	0.011	0.051	0.100	1.000	1.000	1.000

Table 3: Type I error of the functional Wald test at given significance level α for sparse designs in various scenarios. The two sets of quantile levels are defined as $\mathcal{U}_1 = \{0.1, 0.2, 0.3, 0.4\}$ and $\mathcal{U}_2 = \{0.1, 0.2, 0.6, 0.7\}$. The missing rate is 50% for moderate sparsity and 90% for high sparsity.

Scenario	n	$\alpha \text{ (missing rate} = 50\%)$			α (mis	α (missing rate = 90%)		
		0.01	0.05	0.10	0.01	0.05	0.10	
	100	0.021	0.063	0.104	0.024	0.075	0.119	
$\sigma = 1$	500	0.014	0.055	0.104	0.011	0.058	0.110	
$u \in \mathcal{U}_1$	1000	0.014	0.055	0.106	0.013	0.052	0.101	
	2000	0.011	0.055	0.106	0.013	0.053	0.103	
	5000	0.011	0.052	0.100	0.010	0.048	0.100	
	100	0.026	0.075	0.120	0.034	0.092	0.143	
$\sigma = 1$	500	0.016	0.058	0.110	0.021	0.069	0.119	
$u \in \mathcal{U}_2$	1000	0.013	0.057	0.106	0.014	0.063	0.114	
	2000	0.011	0.053	0.100	0.011	0.056	0.108	
	5000	0.010	0.048	0.103	0.011	0.049	0.100	

Table 4: Mean prediction errors from different methods over 1000 cross-validations. The standard errors are reported in parentheses.

au	QAE	CRQ	RQ
0.8	154.163	153.073	152.396
	(0.277)	(0.260)	(0.272)
0.825	146.163	145.598	145.504
	(0.268)	(0.260)	(0.268)
0.85	137.028	136.758	137.071
	(0.256)	(0.259)	(0.258)
0.875	126.138	125.949	126.819
	(0.239)	(0.252)	(0.240)
0.9	112.774	112.842	113.823
	(0.219)	(0.238)	(0.223)