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Quantile Regression

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List of variables

$Q_Y(\cdot)$	Quantile function of real random variable Y	
$F_Y(\cdot)$	Distribution function of real random variable Y	
A	A set of probabilities. $A = \{\alpha_1, \ldots, \alpha_{\lvert A\rvert}$	$q_\alpha(x_t)$ α-quantile, given x_t
T_n	Set of size n of observation indexes, such that $T_n = \{1, 2, \ldots, n\}$	
$\{y_t\}_{t\in T_n}$	Sample of time series y_t	
$\{x_t\}_{t\in T_n}$	Sample of d-dimensional time series x_t	
S	Number of different paths in the simulation	
K	Size of each path in the simulation	

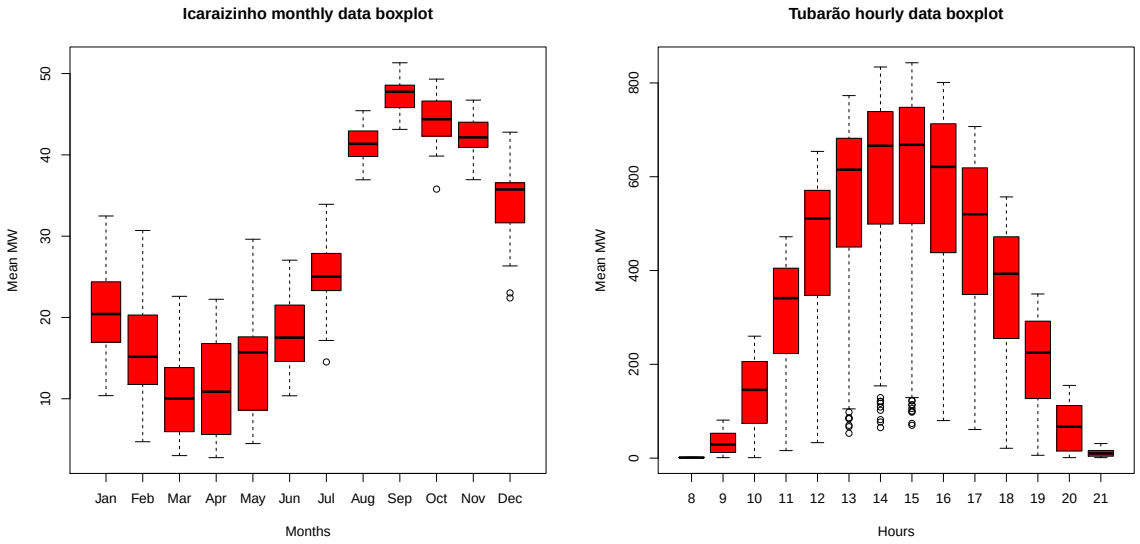


Figure 1.1: Boxplots showing seasonality for monthly and hourly data.

1 Introduction

Quantile Regression is a powerful tool for measuring quantiles others than the median or predicting the mean. A quantile of a random variable is important in risk measuring, as we can measure the probability of occurrence of extreme events, and in many other fields. While working with energy forecasts, quantile regression can produce interesting results when working with both short term (hourly) or long term (monthly) data. As an example, we present a solar time series for the short term and a wind time series for long term. The first set of data is measured at the location of Tubarao (Brazil) on the year of 2014, while the latter is a dataset of mean power monthly observations from Icaraizinho (Brazil) between 1981 to 2011 of measured in Megawatts. Figure 1.1 illustrate the seasonality present in these datasets.

In this work, we apply a few different techniques to forecast the quantile function a few steps ahead. The main frameworks we investigate are parametric linear models and a non-parametric regression. We also investigate how to apply quantile estimations to produce an empirical distribution for the k-step ahead forecasting by using a nonparametric approach.

To make good predictions of random variables, one must find good explanatory variables: it can be either autoregressive, exogenous terms or even a deterministic function that repeats itself. Figure 1.2 shows scatter plots relating y_t with its first lag for both short and long term. We can see that in both of them past values are good explanatory variables to use for forecasting.

In contrast to the linear regression model through ordinary least squares (OLS), which provides only an estimation of the dependent variable conditional mean, quantile regression model yields a much more detailed information concerning the complex relationship about the dependent variable and its covariates. Here we denote as parametric linear model the well-known quantile regression model [4].

Let Y be a real valued random variable. The quantile function $Q_Y : [0, 1] \to \mathbb{R}$ is defined pointwise by its α-quantile, which is given by

$$Q_Y(\alpha) = F_Y^{-1}(\alpha) = \inf\{y : F_Y(y) \geq \alpha\}, \tag{1.1}$$

where F_Y is the distribution function of random variable Y and $\alpha \in [0, 1]$. Equation 1.1 defines what we call from now on as the quantile function $Q_Y(\cdot)$, in relation to random variable Y. In this article, we are interested in a conditional quantile function $Q_{Y|X=x}(\alpha, x)$ (in short, from now on, $Q_{Y|X}(\alpha, x)$), where X can be a vector.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. The conditional quantile function can be found as the result

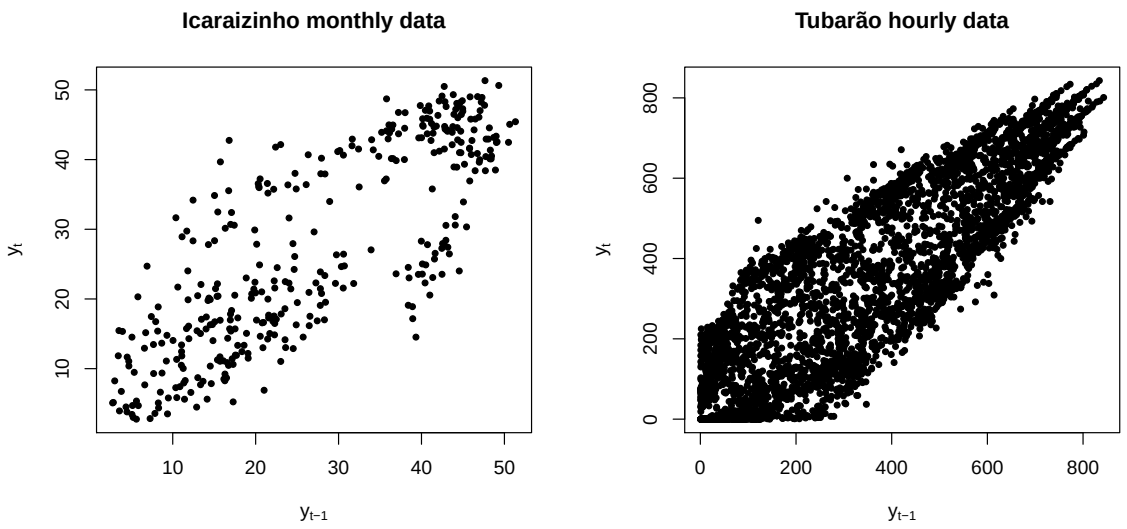


Figure 1.2: Relationship between y_t and its first lags for two selected series.

of the following optimization problem:

$$Q_{Y|X}(\alpha,\cdot) \quad \in \quad \underset{q_\alpha(\cdot)}{\arg\min} \quad (1-\alpha)\int_{\omega\in\Omega}|Y(w)-q_\alpha(X(w))|^- P(dw) \\ +(\alpha)\int_{\omega\in\Omega}|Y(w)-q_\alpha(X(w))|^+ P(dw) \tag{1.2}$$

$$q_\alpha \in \mathcal{Q}, \tag{1.3}$$

The argument of optimization problem described on equation 1.2 is the function q_α, which belongs to a function space $\mathcal{Q}$. We might have different assumptions for the space $\mathcal{Q}$, depending on the type of function we want to find for q_α. A few properties, however, must be achieved by our choice. The conditional quantile function $Q_{Y|X}(\alpha)$ must be monotone on α, and its first derivative must be limited.

In this work, we use the sample quantile, where we calculate the optimization based on a finite number of observations, instead of integrating over all domain of random variable Y. For the specific case where the random variable is a time series y_t, quantiles are estimated from a n size sample of observations of y_t and a explanatory variable x_t for each t, such that our random sample is formed by the sequence $\{y_t, x_t\}_{t=1}^n$. To estimate the α-quantile from a sample, we change 1.2 for the following optimization problem:

$$\hat{Q}_{Y|X}(\alpha,\cdot) \quad \in \quad \underset{q_\alpha(\cdot)}{\arg\min} \sum_{t\in T}\alpha|y_t-q_\alpha(\cdot)|^+ + \sum_{t\in T}(1-\alpha)|y_t-q_\alpha(\cdot)|^-, \tag{1.4}$$

$$q_\alpha \in \mathcal{Q}, \tag{1.5}$$

where $T=\{0,\ldots,n\}$, $|x|^+=\max\{0,x\}$ and $|x|^-=-\min\{0,x\}$. The solution from the above problem is an estimator $\hat{Q}_{Y|X}$ for the quantile function $Q_{Y|X}$.

To model this problem as a Linear Programming problem, thus being able to use a modern solver to fit our model, we create variables ε_t^+ e ε_t^- to represent $|y-q(\cdot)|^+$ and $|y-q(\cdot)|^-$, respectively. The optimal argument $q_\alpha^*(\cdot)$ on the Linear Programming problem 1.6 is the estimated α-quantile for the given random sample.

$$\begin{aligned} \min_{q_\alpha(\cdot),\varepsilon_t^+,\varepsilon_t^-} & \sum_{t\in T}\left(\alpha\varepsilon_t^+ + (1-\alpha)\varepsilon_t^-\right) \\ \text{s.t. } & \varepsilon_t^+ - \varepsilon_t^- = y_t - q_\alpha(x_t), \qquad \forall t\in T, \\ & \varepsilon_t^+, \varepsilon_t^- \geq 0, \qquad \forall t\in T. \end{aligned} \tag{1.6}$$

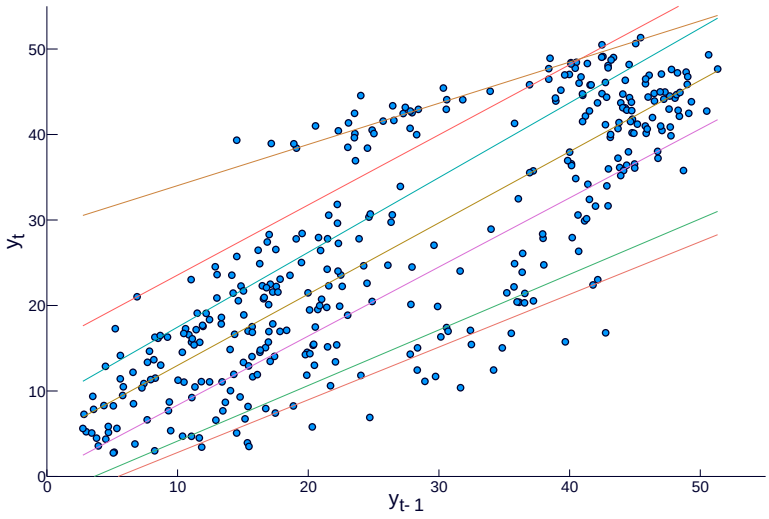


Figure 1.3: Linear quantile estimator with crossing quantiles for $\alpha = 0.95$ and $\alpha = 0.9$

Let A be a set containing a sequence of probabilities α_i such that $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_Q < 1$. This set represents a finite discretization of the interval $[0, 1]$. These values must be close enough so that $Q_{Y|X}$ has a precise representation.

One of our goals with quantile regression is to estimate a quantile function $\hat{Q}_{Y|X}$ of a given real valued random variable X from a sequence of quantiles $q_{\alpha_1}(x_t) \leq q_{\alpha_2}(x_t) \leq \cdots \leq q_{\alpha_{|A|}}(x_t)$, with $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_{|A|} < 1$, for any given t. The process of fitting $\hat{Q}_{Y|X}$ is by mapping every α_i with its estimated quantile $\hat{q}_{\alpha_i}(x_t)$. When this sequence of chosen α_i is thin enough, we have a good approximation of $Q_{Y|X}$. Thus, the distribution found for Y is nonparametric, as no previous assumptions are made about its shape, and its form is fully recovered by the data we have.

A typical problem, however, arises when working with quantile regression. When quantiles are estimated independently, it is possible to find $q_\alpha(x_t) > q_{\alpha'}(x_t)$, for a given t, when $\alpha_1 < \alpha_2$. An example can be seen on Figure 1.3, where quantiles $\alpha = 0.95$ and $\alpha = 0.9$ cross. This problem, called *crossing quantiles*, can be prevented by estimating all quantiles with a single minimization problem.

In order to estimate all quantiles simultaneously, the new objective function will be the sum of all individual objective functions, as well as include all constraints from all individual problems. The only difference is the inclusion of an equation to guarantee that quantiles won't cross. When modifying problem 1.6 to account for all quantiles, we have the following new problem:

$$\{q^*_\alpha(\cdot)\}_{\alpha\in A} \in \underset{q_\alpha(\cdot),\varepsilon^+_{t\alpha},\varepsilon^-_{t\alpha}}{\arg\min} \quad \sum_{\alpha\in A}\sum_{t\in T}\left(\alpha\varepsilon^+_{t\alpha} + (1-\alpha)\varepsilon^-_{t\alpha}\right) \tag{1.7}$$

$$\text{s.t.} \quad \varepsilon^+_{t\alpha} - \varepsilon^-_{t\alpha} = y_t - q_\alpha(x_t), \qquad \forall t \in T, \forall\alpha \in A, \tag{1.8}$$

$$\varepsilon^+_{t\alpha}, \varepsilon^-_{t\alpha} \geq 0, \qquad \forall t \in T, \forall\alpha \in A, \tag{1.9}$$

$$q_\alpha(x_t) \leq q_{\alpha'}(x_t), \qquad \forall t \in T, \forall(\alpha, \alpha') \in A \times A, \alpha' \geq \alpha, \tag{1.10}$$

where constraint 1.10 assures that no lower quantile will have a bigger value than a higher quantile.

The next section discusses with bigger details how to fit a distribution function $Q_{Y|X}(\alpha, x)$ from a sequence of estimated quantiles, as well as showing two different strategies to estimate them: linear models and nonparametric models. In the former, q_α is a linear function of an explanatory variable x_t. In the latter, we let $q_\alpha(x_t)$ assume any functional form. To prevent overfitting, however, we penalize the function's roughness by incorporating a penalty on the second derivative.

In section 4 we investigate how to simulate S scenarios of y_t, considering a linear model and errors ε_t for which the distribution is unknown. To address this issue, we use quantile linear regression to calculate a thin grid of quantiles and fit a distribution function $\hat{F}_{y_t}$. This function will be used to simulate the innovations on the model.

2 Estimating distribution function from quantile regressions

In many applications where a time series model is employed, we often consider the innovations' distribution as known. Take, for example, the AR(p) model:

$$Y = c + \sum_{i=1}^{p} \phi_i X_i + \varepsilon_t,$$

where X_i is a past value of random variable Y. In this model, errors ε_t are assumed to have normal distribution with zero mean.

When we are dealing with natural resources data, however, we can't always assume normality. In these cases, one can either find a distribution that has a better fit to the data or have a nonparametric method to estimate the distribution directly from the available data.

In a time series framework, where a time series y_t is given by a linear model of its regressors x_t

$$Y_t = \beta^T X_t + \varepsilon_t,$$

we propose to estimate the k-step ahead distribution of Y_t with a nonparametric approach. Let an empirical α-quantile $\hat{q}_\alpha \in \mathcal{Q}$ be a functional belonging to a functional space. In any given t, we can estimate the sequence of quantiles $\{q_\alpha(x_t)\}_{\alpha \in A}$ by solving the problem defined on equations (1.7)-(1.10). After evaluating this sequence, by making equal

$$\hat{Q}_{y_t|X=x_t}(\alpha) = \hat{q}_\alpha(x_t), \qquad \forall \alpha \in A, \tag{2.1}$$

we have a set of size $|A|$ of values to define the discrete function over the first argument $\hat{Q}_{y_t|x_t}(\alpha, X = x_t) : A \times \mathbb{R}^d \to \mathbb{R}$. The goal of having function $\hat{Q}$ is to use it as base to construct the estimated quantile function $\hat{Q}'_{y_t|X=x_t}(\alpha, x_t) : [0,1] \times \mathbb{R}^d \to \mathbb{R}$.

A problem arises for the distribution extremities, because when $\alpha = 0$ or $\alpha = 1$, the optimization problem becomes unbounded. In order to find values for $\hat{Q}(\alpha, x_t)$ when $\alpha \in \{0, 1\}$, we chose to linearly extrapolate its values. Note that as $A \subset [0, 1]$, the domain of $\hat{Q}$ is also a subset of the domain of $\hat{Q}'$. The estimative of $\hat{Q}'$ is done by interpolating points of $\hat{Q}$ over the interval $[0, 1]$. Thus, the distribution found for $\hat{y}_\tau$ is nonparametric, as no previous assumptions are made about its shape, and its form is fully recovered by the data we have.

We investigate two different approaches for Q_{y_t} by the functional structure of each individual $q_\alpha(x_t)$. In section 2.1, we explore the case where the individual quantiles $q_\alpha(x_t)$ are a linear function of its arguments:

$$\hat{q}_\alpha(x_t) = \beta_{0,\alpha} + \beta_\alpha^T x_t, \tag{2.2}$$

where β^α is a vector of coefficients for the explanatory variables.

In section 2.2 we introduce a Nonparametric Quantile Autoregressive model with a ℓ_1-penalty term, in order to properly simulate densities for several α-quantiles. In this nonparametric approach we don't assume any form for $q_\alpha(x_t)$, but rather let the function adjust to the data. To prevent overfitting, the ℓ_1 penalty for the second derivative (approximated by the second difference of the ordered observations) is included in the objective function. The result of this optimization problem is that each $q_\alpha(x_t)$ will be a function with finite second derivative.

In order to find good estimates for $Q_{y_t}(\alpha)$ when α approaches 0 or 1, as well as performing interpolation on the values that were not directly estimated, we can either use a kernel smoothing function, splines, linear approximation, or any other method.

2.1 Linear Models for the Quantile Autoregression

Given a time series $\{y_t\}$, we investigate how to select which lags will be included in the Quantile Autoregression. We won't be choosing the full model because this normally leads to a bigger variance in our estimators, which is often linked with bad performance in forecasting applications. So our strategy will be to use some sort of regularization method in order to improve performance. We investigate two ways of accomplishing this goal. The first of them consists of selecting the best subset

of variables through Mixed Integer Programming, given that K variables are included in the model. Using MIP to select the best subset of variables is investigated in [2]. The second way is including a ℓ_1 penalty on the linear quantile regression, as in [3], and let the model select which and how many variables will have nonzero coefficients. Both of them will be built over the standard Quantile Linear Regression model. In the end of the section, we discuss a information criteria to be used for quantile regression and verify how close are the solutions in the eyes of this criteria.

When we choose $q_\alpha(x_t)$ to be a linear function

$$\hat{q}_\alpha(x_t) = \beta_{0\alpha} + \beta_\alpha^T x_t \tag{2.3}$$

we can substitute it on problem 1.6, getting the following LP problem:

$$\begin{aligned} \min_{\beta_0,\beta,\varepsilon_t^+,\varepsilon_t^-} & \sum_{t\in T} \left(\alpha\varepsilon_t^+ + (1-\alpha)\varepsilon_t^-\right) \\ \text{s.t. } & \varepsilon_t^+ - \varepsilon_t^- = y_t - \beta_0 - \beta^T x_t, \qquad \forall t \in \{1,\dots,n\}, \\ & \varepsilon_t^+, \varepsilon_t^- \geq 0, \qquad \forall t \in \{1,\dots,n\}. \end{aligned} \tag{2.4}$$

To solve the crossing quantile issue discussed earlier, we employ a minimization problem with all quantiles at the same time and adding the non crossing constraints. The new optimization problem is shown below:

$$\min_{\beta_{0\alpha},\beta_\alpha,\varepsilon_{t\alpha}^+,\varepsilon_{t\alpha}^-} \quad \textstyle\sum_{\alpha\in A}\sum_{t\in T}\left(\alpha\varepsilon_{t\alpha}^+ + (1-\alpha)\varepsilon_{t\alpha}^-\right) \tag{2.5}$$

$$\text{s.t.} \quad \varepsilon_{t\alpha}^+ - \varepsilon_{t\alpha}^- = y_t - \beta_{0\alpha} - \beta_\alpha^T x_t, \qquad \forall t \in T, \forall \alpha \in A, \tag{2.6}$$

$$\varepsilon_{t\alpha}^+, \varepsilon_{t\alpha}^- \geq 0, \qquad \forall t \in T, \forall \alpha \in A, \tag{2.7}$$

$$\beta_{0\alpha} + \beta_\alpha^T x_t \leq \beta_{0\alpha'} + \beta_{\alpha'}^T x_t, \qquad \forall t \in T, \forall (\alpha,\alpha') \in A \times A, \alpha < \alpha', \tag{2.8}$$

When solving problem (2.5)-(1.10), the sequence $\{q_\alpha\}_{\alpha\in A}$ is fully defined by the values of $\beta_{0\alpha}^*$ and β_α^*, for every α.

2.2 Quantile Autoregression with a nonparametric approach

Fitting a linear estimator for the Quantile Auto Regression isn't appropriate when nonlinearity is present in the data. This nonlinearity may produce a linear estimator that underestimates the quantile for a chunk of data while overestimating for the other chunk. To prevent this issue from occurring we propose a modification which we let the prediction $q_\alpha(x_t)$ adjust freely to the data and its nonlinearities. To prevent overfitting and smoothen our predictor, we include a penalty on its roughness by including the ℓ_1 norm of its second derivative. For more information on the ℓ_1 norm acting as a filter, one can refer to [3].

This time, as opposed to when employing linear models, we don't suppose any functional form for $q_\alpha(x_t)$. This forces us to build each q_α differently: instead of finding a set of parameters that fully defines the function, we find a value for $q_\alpha(x_t)$ at each instant t. On the optimization problem, we will find optimal values for a variable $q_{\alpha t} \in \mathbb{R}$, each consisting of a single point. The sequence of $\{q_{\alpha t}^*\}$ will provide a discretization for the full function $\hat{q}_\alpha(x_t)$, which can be found by interpolating these points.

Let $\{\tilde{y}_t\}_{t=1}^n$ be the sequence of observations in time t. Now, let $\tilde{x}_t$ be the $p-$lagged time series of $\tilde{y}_t$, such that $\tilde{x}_t = L^p(\tilde{y}_t)$, where L is the lag operator. Matching each observation $\tilde{y}_t$ with its $p-$lagged correspondent $\tilde{x}_t$ will produce $n-p$ pairs $\{(\tilde{y}_t, \tilde{x}_t)\}_{t=p+1}^n$ (note that the first p observations of y_t must be discarded). When we order the observation of x in such way that they are in growing order

$$\tilde{x}^{(p+1)} \leq \tilde{x}^{(p+2)} \leq \cdots \leq \tilde{x}^{(n)},$$

we can then define $\{x_i\}_{i=1}^{n-p} = \{\tilde{x}^{(t)}\}_{t=p+1}^n$ and $\{y_i\}_{i=1}^{n-p} = \{\tilde{y}^{(t)}\}_{t=p+1}^n$ and $T' = \{2,\dots,n-p-1\}$. As we need the second difference of q_i, I has to be shortened by two elements.

Our optimization model to estimate the nonparametric quantile is as follows:

$$\begin{aligned}\hat{q}_\alpha(x_t) = \arg\min_{q_{\alpha t}} \sum_{t\in T'} \left(\alpha|y_t - q_{\alpha t}|^+ + (1-\alpha)|y_t - q_{\alpha t}|^-\right) \\ +\lambda_1 \sum_{t\in T'} |D^1_{x_t} q_{\alpha t}| + \lambda_2 \sum_{t\in T'} |D^2_{x_t} q_{\alpha t}|,\end{aligned} \tag{2.9}$$

where $D^1 q_t$ and $D^2 q_t$ are the first and second derivatives of the $q_\alpha(x_t)$ function, calculated as follows:

$$D^2_{x_t} q_{\alpha t} = \frac{\left(\frac{q_{\alpha t+1} - q_{\alpha t}}{x_{t+1} - x_t}\right) - \left(\frac{q_{\alpha t} - q_{\alpha t-1}}{x_t - x_{t-1}}\right)}{x_{t+1} - 2x_t + x_{t-1}},$$

$$D^1_{t\alpha} = \frac{q_{\alpha t+1} - q_{\alpha t}}{x_{t+1} - x_t}.$$

The first part on the objective function is the usual quantile regression condition for $\{q_{t\alpha}\}_{\alpha\in A}$. The second part is the ℓ_1-filter. The purpose of a filter is to control the amount of variation for our estimator $q_\alpha(x_t)$. When no penalty is employed we would always get $q_{\alpha t} = y_t$, for any given α. On the other hand, when $\lambda \to \infty$, our estimator approaches the linear quantile regression.

The full model can be rewritten as a LP problem as bellow:

$$\min_{q_{\alpha t}, \delta_t^+, \delta_t^-, \xi_t} \quad \sum_{\alpha\in A}\sum_{t\in T'} \left(\alpha\delta^+_{t\alpha} + (1-\alpha)\delta^-_{t\alpha}\right) \tag{2.10}$$

$$+\lambda_1 \sum_{t\in T'} \gamma_{t\alpha} + \lambda_2 \sum_{t\in T'} \xi_{t\alpha}$$

$$s.t. \quad \delta^+_t - \delta^-_{t\alpha} = y_t - q_{t\alpha}, \qquad \forall t \in T', \forall \alpha \in A, \tag{2.11}$$

$$D^1_{t\alpha} = \tfrac{q_{\alpha t+1} - q_{\alpha t}}{x_{t+1} - x_t}, \qquad \forall t \in T', \forall \alpha \in A, \tag{2.12}$$

$$D^2_{t\alpha} = \tfrac{\left(\frac{q_{\alpha t+1} - q_{\alpha t}}{x_{t+1} - x_t}\right) - \left(\frac{q_{\alpha t} - q_{\alpha t-1}}{x_t - x_{t-1}}\right)}{x_{t+1} - 2x_t + x_{t-1}}. \qquad \forall t \in T', \forall \alpha \in A, \tag{2.13}$$

$$\gamma_{t\alpha} \geq D^1_{t\alpha}, \qquad \forall t \in T', \forall \alpha \in A, \tag{2.14}$$

$$\gamma_{t\alpha} \geq -D^1_{t\alpha}, \qquad \forall t \in T', \forall \alpha \in A, \tag{2.15}$$

$$\xi_{t\alpha} \geq D^2_{t\alpha}, \qquad \forall t \in T', \forall \alpha \in A, \tag{2.16}$$

$$\xi_{t\alpha} \geq -D^2_{t\alpha}, \qquad \forall t \in T', \forall \alpha \in A, \tag{2.17}$$

$$\delta^+_{t\alpha}, \delta^-_{t\alpha}, \gamma_{t\alpha}, \xi_{t\alpha} \geq 0, \qquad \forall t \in T', \forall \alpha \in A, \tag{2.18}$$

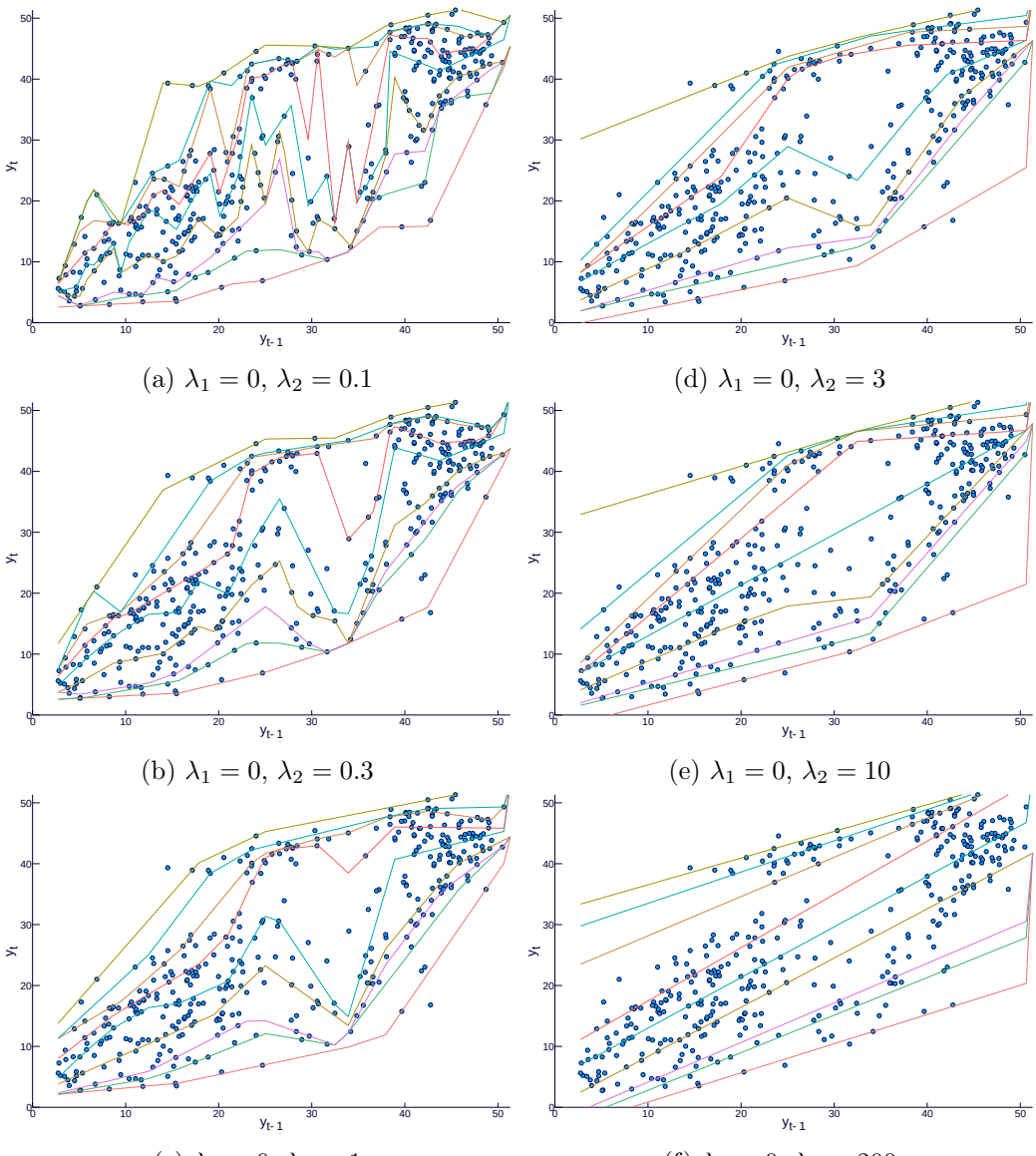
$$q_{t\alpha} \leq q_{t\alpha'}, \qquad \forall t \in T', \forall (\alpha, \alpha') \in A \times A, \alpha < \alpha', \tag{2.19}$$

The output of our optimization problem is a sequence of ordered points $\{(x_t, q_{t\alpha})\}_{t\in T}$, for all $\alpha \in A$. The next step is to interpolate these points in order to provide an estimation for any other value of x_t. To address this issue, we propose using a linear interpolation, that will be developed in another study. Note that $q_{t\alpha}$ is a variable that represents only one point of the α-quantile function $q_\alpha(x_t)$.

The quantile estimation is done for different values of λ. By using different levels of penalization on the second difference, the estimation can be more or less adaptive to the fluctuation. It is important to notice that the usage of the ℓ_1-norm as penalty leads to a piecewise linear solution $q_{t\alpha}$. Figure 3.1 shows the quantile estimation for a few different values of λ.

The first issue is how to select an appropriate value for λ. A simple way is to do it by inspection, which means to test many different values and pick the one that suits best our needs by looking at them. The other alternative is to use a metric to which we can select the best tune. We can achieve this by using a cross-validation method, for example.

The other issue occurs when we try to add more than one lag to the analysis at the same time. This happens because the problem solution is a set of points that we need to interpolate. This multivariate interpolation, however, is not easily solved, in the sense that we can either choose using a very naive estimator such as the K-nearest neighbors or just find another method that is not yet adopted for a wide range of applications.



(a) $\lambda_1 = 0$, $\lambda_2 = 0.1$

(d) $\lambda_1 = 0$, $\lambda_2 = 3$

(b) $\lambda_1 = 0$, $\lambda_2 = 0.3$

(e) $\lambda_1 = 0$, $\lambda_2 = 10$

(c) $\lambda_1 = 0$, $\lambda_2 = 1$

(f) $\lambda_1 = 0$, $\lambda_2 = 200$

Figure 2.1: Quantile estimations for a few different values of λ. The quantiles represented here are $\alpha = (5\%, 10\%, 25\%, 50\%, 75\%, 90\%, 95\%)$. When $\lambda = 0.1$, on the upper left, we clarly see a overfitting on the estimations. The other extreme case is also shown, when $\lambda = 200$ the nonparametric estimator converges to the linear model.

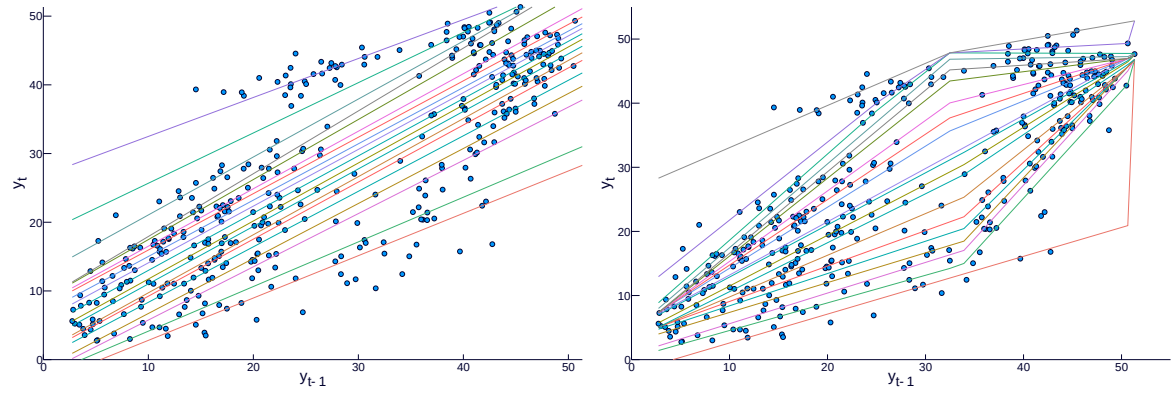


Figure 2.2: Estimated α-quantiles. On the left using a linear model and using a nonparametric approach (using $\lambda = 100$) on the right.

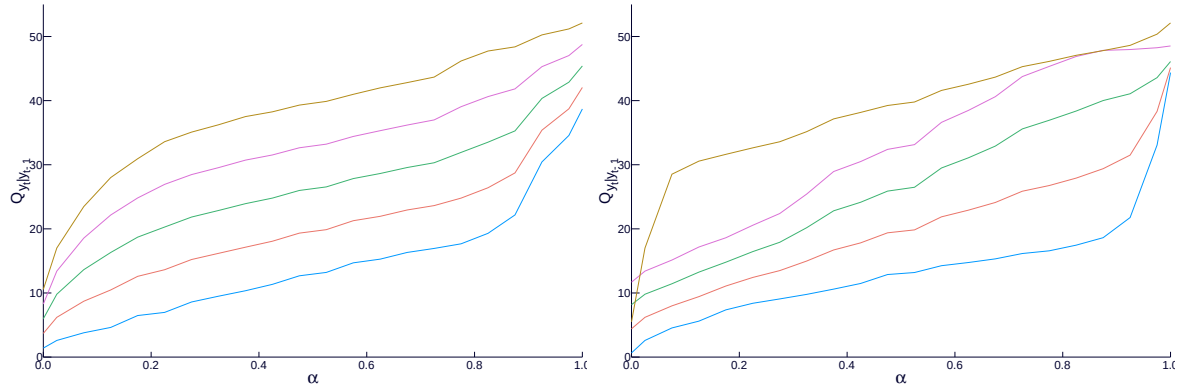


Figure 2.3: Estimated quantile functions, for different values of y_{t-1}. On the left using a linear model and using a nonparametric approach on the right.

2.3 A comparison between both approaches

The last two sections introduced two different strategies to arrive in a Quantile Function $Q_{y_t|X}$. But what are the differences between using one method or the other?

To provide a comparison between both approaches, we estimate a quantile function to predict the one-step ahead quantile function. We use as explanatory variable only the last observation y_{t-1} - so $x_t = y_{t-1}$ - and estimate $\hat{q}_\alpha(y_{t-1})$, for every $\alpha \in \{0.05, 0.1, \ldots, 0.9, 0.95\}$. The result of both methods is shown on Figure 2.2.

While the linear model produces α-quantile functions which are linear by imposition, on the nonparametric model the α-quantiles are flexible enough to form a hull on the data and adapt to its nonlinearities. The difference between the estimated quantile functions $\hat{Q}_{y_t|y_{t-1}}$ on both methods are shown on Figure 2.2.

It is also important to test how the choice of the set A affects the estimated quantile function. We experimented with two different sizes of A. In one of them, a dense grid of probabilities is used: $A = \{0.005, 0.01, \ldots, 0.99, 0.995\}$, consisting of 199 elements. On the other only 19 elements are used to produce the quantile function ($A = \{0.05, 0.1, \ldots, 0.9, 0.95\}$).

2.4 Testing convergence

In this computational exercise, we simulated the following stochastic process:

$$Y_t = \rho Y_{t-1} + \varepsilon_t, \qquad \varepsilon_t \sim LogNormal(\mu, \sigma^2), \tag{2.20}$$

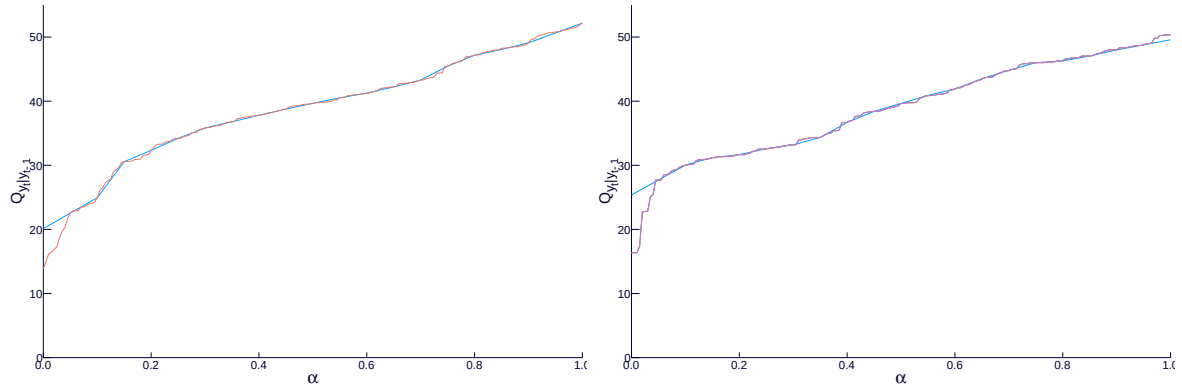


Figure 2.4: Sensitivity to different choices of set A. On the left, we have the estimated quantiles for the linear model, while on the right for the nonparametric model. On both, the red line shows the quantile function estimated with $A = \{0.005, 0.01, \ldots, 0.99, 0.995\}$, consisting of 199 elements. The blue line is the estimated quantile function when $A = \{0.05, 0.1, \ldots, 0.9, 0.95\}$, consisting of only 19 elements.

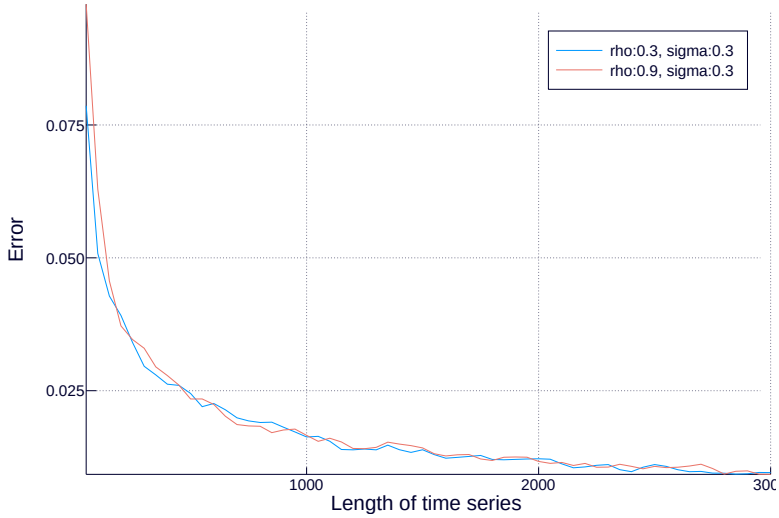


Figure 2.5: Converge of estimated quantile function to true quantile, for a LogNormal distribution

to test how fast the estimated quantile function $\hat{Q}'_{Y|X}$ converges to the real $Q_{Y|X}$, using the parametric approach. An error metric defined as

$$\sum_{\alpha \in A} |\hat{Q}'_{Y|X}(\alpha) - Q_{Y|X}(\alpha)| \tag{2.21}$$

is measured for different values of size of data. The result is shown on figure 2.5.

3 Model Selection

In the following sections we present two ways of doing regularization and selecting the best model for estimating the quantile function. In the first, we use a Mixed Integer optimization problem (MIP) to find the best subset among all choices of covariates. The second is by using the a LASSO-type technique, by penalizing the ℓ_1-norm of the regressors, thus shrinking the size of coefficients towards zero.

3.1 Best subset selection with Mixed Integer Programming

In this part, we investigate the usage of Mixed Integer Programming to select which variables are included in the model, up to a limit of inclusions imposed *a priori*. The optimization problem is described below:

$$\min_{\beta_{0\alpha},\beta_\alpha,z_{p\alpha}\varepsilon^+_{t\alpha},\varepsilon^-_{t\alpha}} \quad \sum_{\alpha\in A}\sum_{t\in T}\left(\alpha\varepsilon_t^+ + (1-\alpha)\varepsilon_t^-\right) \tag{3.1}$$

$$\text{s.t} \quad \varepsilon_t^+ - \varepsilon_{t\alpha}^- = y_t - \beta_{0\alpha} - \textstyle\sum_{p=1}^P \beta_{p\alpha}x_{t,p}, \qquad \forall t\in T, \forall\alpha\in A, \tag{3.2}$$

$$\varepsilon_{t\alpha}^+, \varepsilon_{t\alpha}^- \geq 0, \qquad \forall t\in T, \forall\alpha\in A, \tag{3.3}$$

$$-M_U z_{p\alpha} \leq \beta_{p\alpha} \leq M_U z_{p\alpha}, \qquad \forall p\in\{1,\dots,P\}, \tag{3.4}$$

$$\textstyle\sum_{p=1}^P z_{p\alpha} \leq K, \qquad \forall\alpha\in A, \tag{3.5}$$

$$z_{p\alpha}\in\{0,1\}, \qquad \forall p\in\{1,\dots,P\}, \forall\alpha\in A. \tag{3.6}$$

$$\beta_{0\alpha} + \beta_\alpha^T x_t \leq \beta_{0\alpha'} + \beta_{\alpha'}^T x_t, \qquad \forall t\in T, \forall(\alpha,\alpha')\in A\times A, \alpha<\alpha', \tag{3.7}$$

The objective function and constraints (3.3) and (3.4) are those from the standard linear quantile regression. The other constraints implement the process of regularization, forcing a maximum of K variables to be included. By (3.4), variable $z_{p\alpha}$ is a binary that assumes 1 when the coefficient β_p is included. M_U is chosen in order to guarantee that $M_U \geq \|\hat{\beta}\|_\infty$. The solution given by β_0 and β will be the best linear quantile regression with K nonzero coefficients.

We ran this optimization for each value of $K\in\{1,\dots,12\}$ and quantiles $\alpha\in\{0.05, 0.1, 0.5, 0.9, 0.95\}$. We could see that for all quantiles the 12^{th} lag was the one included when $K=1$. When $K=2$, the 1^{st} lag was always included, sometimes with β_{12}, some others with β_4 and once with β_{11}. These 4 lags that were present until now are the only ones selected when $K=3$. For $K=4$, those same four lags were selected for three quantiles (0.05, 0.1 and 0.5), but for the others (0.9 and 0.95) we have β_6, β_7 and β_9 also as selected. From now on, the inclusion of more lags represent a lower increase in the fit of the quantile regression. The estimated coefficient values for all K's are available in the appendices section.

3.2 Best subset selection with a ℓ_1 penalty

Another way of doing regularization is including the ℓ_1-norm of the coefficients on the objective function. The advantage of this method is that coefficients are shrunk towards zero, and only some of them will have nonzero coefficients. By lowering the penalty we impose on the ℓ_1-norm, more variables are being added to the model. This is the same strategy of the LASSO, and its usage for the quantile regression is discussed in [5]. The proposed optimization problem to be solved is:

$$\min_{\beta_{0\alpha},\beta_\alpha} \sum_{t\in T}\alpha|y_t - q_\alpha(x_t)|^+ + \sum_{t\in T}(1-\alpha)|y_t - q_\alpha(x_t)|^- + \lambda\|\beta\|_1 \tag{3.8}$$

$$q_\alpha(x_t) = \beta_0 - \sum_{p=1}^P \beta_p x_{t,p},$$

where the regressors $x_{t,p}$ used are its lags. In order to represent the above problem to be solved with linear programming solver, we restructure the problem as below:

$$\beta_\lambda^{*LASSO} = \underset{\beta_0,\beta,\varepsilon_{t\alpha}^+,\varepsilon_{t\alpha}^-}{\arg\min} \quad \sum_{\alpha\in A}\sum_{t\in T}\left(\alpha\varepsilon_{t\alpha}^+ + (1-\alpha)\varepsilon_{t\alpha}^-\right) + \lambda\sum_{p=1}^{P}\xi_p \tag{3.9}$$

$$\text{s.t.} \qquad \varepsilon_{t\alpha}^+ - \varepsilon_{t\alpha}^- = y_t - \beta_0 - \sum_{p=1}^{P}\beta_p x_{t,p}, \qquad \forall t \in T, \tag{3.10}$$

$$\varepsilon_{t\alpha}^+, \varepsilon_{t\alpha}^- \geq 0, \qquad \forall t \in T, \forall \alpha \in A \tag{3.11}$$

$$\xi_p \geq \beta_{p\alpha}, \qquad \forall p \in \{1,\dots,P\}, \forall \alpha \in A \tag{3.12}$$

$$\xi_p \geq -\beta_{p\alpha}, \qquad \forall p \in \{1,\dots,P\}, \forall \alpha \in A, \tag{3.13}$$

Once again, this model is built upon the standard linear programming model for the quantile regression (equation 2.4). On the above formulation, the ℓ_1 norm of equation (3.8) is substituted by the sum of ξ_p, which represents the absolute value of β_p. The link between variables ξ_p and β_p is made by constraints (3.12) and (3.13). Note that the linear coefficient β_0 is not included in the penalization, as the sum of penalties on the objective function 3.9.

For such estimation to produce good results, however, each variable must have the same relative weight in comparison with one another. So, before solving the optimization problem, we normalize all variables to have mean $\mu = 0$ and variance $\sigma^2 = 1$. For the vector of observations for each covariate (that in our problem represents is a vector of observations of lags y_{t-p}), we apply the transformation $\tilde{y}_{t-p,i} = (y_{t-p,i} - \bar{y}_{t-p})/\sigma_{t-p}$, where $\bar{y}_{t-p}$ is the p-lag mean and σ_{t-p} the p-lag standard deviation. We use the $\tilde{y}_{t-p,i}$ series to estimate the coefficients. Once done that, we multiply each coefficient for its standard deviation to get the correct coefficient: $\beta_i = \tilde{\beta}_i\dot{\sigma}_{t-p}$.

For low values of λ, the penalty is small and thus we have a model where all coefficients have a nonzero value. On the other hand, while λ is increased the coefficients shrink towards zero; in the limit we have a constant model. For instance, we don't penalize the linear coefficient β_0. For the same quantiles values α we experimented on section 3.1 ($\alpha \in \{0.05, 0.1, 0.5, 0.9, 0.95\}$).

It is important to mention that even though we have coefficients that are estimated by this method, we don't use them directly. Instead, the nonzero coefficients will be the only covariates used as explanatory variables of a regular quantile autoregression, solved by the linear programming problem 2.4. In summary, the optimization in equation 3.8 acts as a variable selection for the subsequent estimation, which is normally called the post-lasso estimation [1].

We are interested, finally, in finding the post-lasso coefficients β_λ^*, which is the solution of the optimization problem given below:

$$\begin{aligned}\beta_\lambda^* = \underset{\beta_0,\beta,\varepsilon_t^+,\varepsilon_t^-}{\arg\min} & \sum_{t=1}^{n}\left(\alpha\varepsilon_t^+ + (1-\alpha)\varepsilon_t^-\right) \\ \text{s.t. } & \varepsilon_t^+ - \varepsilon_t^- = y_t - \beta_0 - \sum_{p\in L_\lambda}\beta_p x_{t,p}, \qquad \forall t \in \{1,\dots,n\}, \\ & \varepsilon_t^+, \varepsilon_t^- \geq 0, \qquad \forall t \in \{1,\dots,n\}.\end{aligned} \tag{3.14}$$

Note that only a subset of the P covariates will have nonzero values, which are given by the set

$$L_\lambda = \{p \mid p \in \{1,\dots,P\},\ |\beta_{\lambda,p}^{*LASSO}| \neq 0\}.$$

Hence, we have that

$$\beta_{\lambda,p}^{*LASSO} = 0 \iff \beta_{\lambda,p}^* = 0.$$

3.3 Simulation Study

If we knew an autoregressive process true model

$$y_t = \phi_0 + \sum_{p=1}^{P}\phi_p y_{t-p} + \varepsilon_t,$$

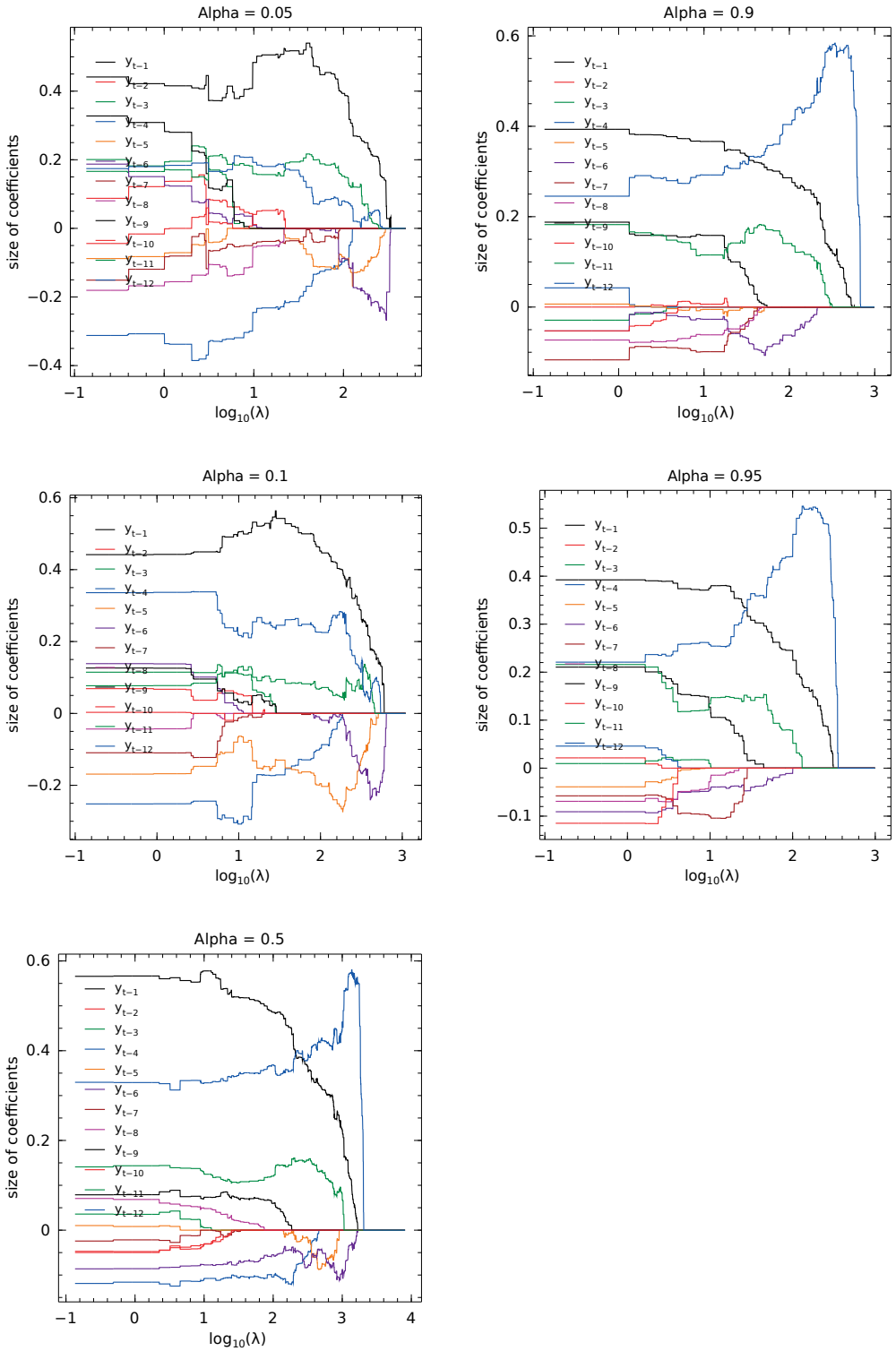


Figure 3.1: Coefficients path for a few different values of α-quantiles. λ is presented in a $\log_{10}$ scale, to make visualization easier.

and knew the error distribution, we could estimate the one-step ahead quantile as the sum $\hat{y}_{t+1} + t_\alpha$, where t_α is the α-quantile for the error distribution. This means that we would be able to use all developments made on conditional mean estimation and simply add an error to estimate quantiles.

However, that is not the case. When working with quantile estimation for real data we don't know the generating process exactly. Using a quantile regression model provides us a good solution even without making assumptions about the error distribution.

We propose simulating an AR(1) model

$$y_t = \phi_0 + \phi y_{t-1} + \varepsilon_t, \qquad \varepsilon_t \sim N(0, \sigma_\varepsilon^2), \tag{3.15}$$

and test two approaches to predict the one-step ahead quantile. On the first one, we consider known the process true model, given on equation 3.15. Thus, our task is to estimate values for $\hat{\phi}_0$, $\hat{\phi}$ and $\hat{\sigma}_\epsilon^2$. In order to calculate the one-step ahead α-quantile, we need to compute

$$\hat{q}_\alpha^{AR}(x_t) = \hat{y}_{t+1|t} + z_\alpha \hat{\sigma}_\varepsilon, \tag{3.16}$$

where $\hat{y}_{t+1|t} = \hat{\phi}_0 + \hat{\phi} y_t$ stands for the one-step ahead conditional mean and $z_\alpha = F^{-1}(\alpha)$, where F is the gaussian distribution function.

On the second approach, we fit a quantile regression by solving problem 2.4. The solution of this optimization problem are coefficients $\hat{\beta}_0$ and $\hat{\beta}$. In order to find the one-step ahead α-quantile, we use the following expression:

$$\hat{q}_\alpha^{QR}(x_t) = \hat{\beta}_0 + \hat{\beta} y_t. \tag{3.17}$$

Note that in both approaches we have one intercept term ($z_\alpha \sigma_\varepsilon + \phi_0$ and β_0) and a coefficient for the first lag (ϕ and β).

We generate data according to equation 3.15, with different values for ϕ (0.25, 0.5, 0.7 and 0.9) and different signal to noise ratios (0.01, 0.05, 0.1, 0.5, 1). We use the signal to noise ratio (RSN) to form the error variance such that $\sigma_e^2 = \dfrac{\phi_0}{1-\phi} \cdot RSN$. This experiment was run with samples of size $n = 20000$. The first half was used to fit coefficients and the second half was used as testing set, on which forecasting was done.

Our first goal is then to evaluate the ability of these two approaches to predict the one-step ahead α-quantile for a few selected α's. We define the model m forecasting error ε_t^m as the quantity

$$\varepsilon_{t\alpha}^m = \hat{q}_\alpha^m(x_t) - q_\alpha(x_t) = \hat{q}_\alpha^m(x_t) - \phi_0 - \phi y_{t-1} - z_\alpha \sigma_\epsilon. \tag{3.18}$$

We use the Root Mean Squared Error (RMSE) to evaluate forecasting performance, which is defined for the model m as follows:

$$RMSE^m = \sqrt{\frac{1}{n-1} \sum_{i=2}^{n} (\varepsilon_{t\alpha}^m)^2} \tag{3.19}$$

On section 5.3, we show a comparison of RMSE for both approaches. To compare them, we compute their RMSE ratio

$$R^{Q/A} = \frac{RMSE^{QR}}{RMSE^{AR}}. \tag{3.20}$$

If $R^{Q/A}$ is smaller than 1, it means that the quantile regression approach performed better in terms of RMSE than the conditional mean based on autoregressive models approach.

Once simulations are executed, we notice forecasting errors are similar for both approaches. The exception being when ϕ is large (0.9) and the noise is small ($RSN = 0.01$). In those cases, quantile regression performed on average 10% better than when using the conditional mean approach. We also provide tables showing differences for the estimated autoregressive and intercept coefficients on section 5.3.

3.4 Model selection

On sections 3.1 and 3.2, we presented ways of doing regularization. But regularization can be done with different levels of parsimony. For example, we can select a different number K of variables to be included in the best subset selection via MIP or choose different values of λ for the ℓ_1 penalty. Each of these choices leads to a different model, so one needs to know how to select the best one among the options we have. One way of achieving this is by using an information criteria to guide our decision.

An information criteria summarizes two aspects. One of them refers to how well the model fits the in-sample observations. The other part penalizes the quantity of covariates used in the model. By penalizing how big our model is, we prevent overfitting from happening. So, in order to enter the model, the covariate must supply enough goodness of fit. In [6], it is presented a variation of the Schwarz criteria for M-estimators that includes quantile regression. The Schwarz Information Criteria (SIC) adapted to the quantile autoregression case is presented below:

$$SIC(j) = n \log(\hat{\sigma}_j) + \frac{1}{2} p_j \log n, \tag{3.21}$$

where $\hat{\sigma}_j = \frac{1}{n} \sum_{t=1}^{n} \rho_\alpha(y_t - x_t^T \hat{\beta}_n(\alpha))$, $\rho_\alpha(\cdot)$ is the penalization function and p_j the j^{th} model's dimension. This procedure leads to a consistent model selection if the model is well specified.

Optimizing a LP problem is many times faster than a similar-sized MIP problem. One of our goals is to test whether a solution of a model with a ℓ_1-norm can approximate well a solution given by the MIP problem. We propose an experiment that is described as follows. First, we calculate the quantity $k(\lambda)$ of nonzero coefficients, for each given lambda:

$$k_\lambda = \|\beta^*_\lambda\|_0. \tag{3.22}$$

Then, for each number K of total nonzero coefficients (from 1 until 13, where 1 means that only the intercept is included), there will be a penalty λ^*_K which minimizes the SIC:

$$\lambda^*_K = \arg\min_\lambda \left\{ SIC\left(k_\lambda\right) \,|\, k_\lambda = K \right\}. \tag{3.23}$$

Thus, we can compare the SIC of the best lasso fit where exactly K variables are selected with the SIC selected by the MIP problem, also with K variables selected.

To help us view the difference of results between both methods, we define a distance metric d between the subset of coefficients chosen by each one of them. Let

$$d(\beta^*_{MIP(K)}, \beta^*_{\lambda^*_K}) = \frac{1}{2K} \sum_{p=1}^{P} \left| I(\beta^*_{MIP(K),p}) - I(\beta^*_{\lambda^*_K,p}) \right|, \tag{3.24}$$

where I is an indicator function such that $I(x) = 0$ if $x = 0$ and $I(x) = 1$ otherwise.

Figure 3.2 shows the results of these experiments for quantiles $\alpha \in \{0.05, 0.1, 0.5, 0.9, 0.95\}$. The results point us that for small values of K the distance between coefficients is bigger and where we observe the biggest differences between the SIC values. The minimum SIC value for the MIP problem is usually found between 4 and 6 variables in the model.

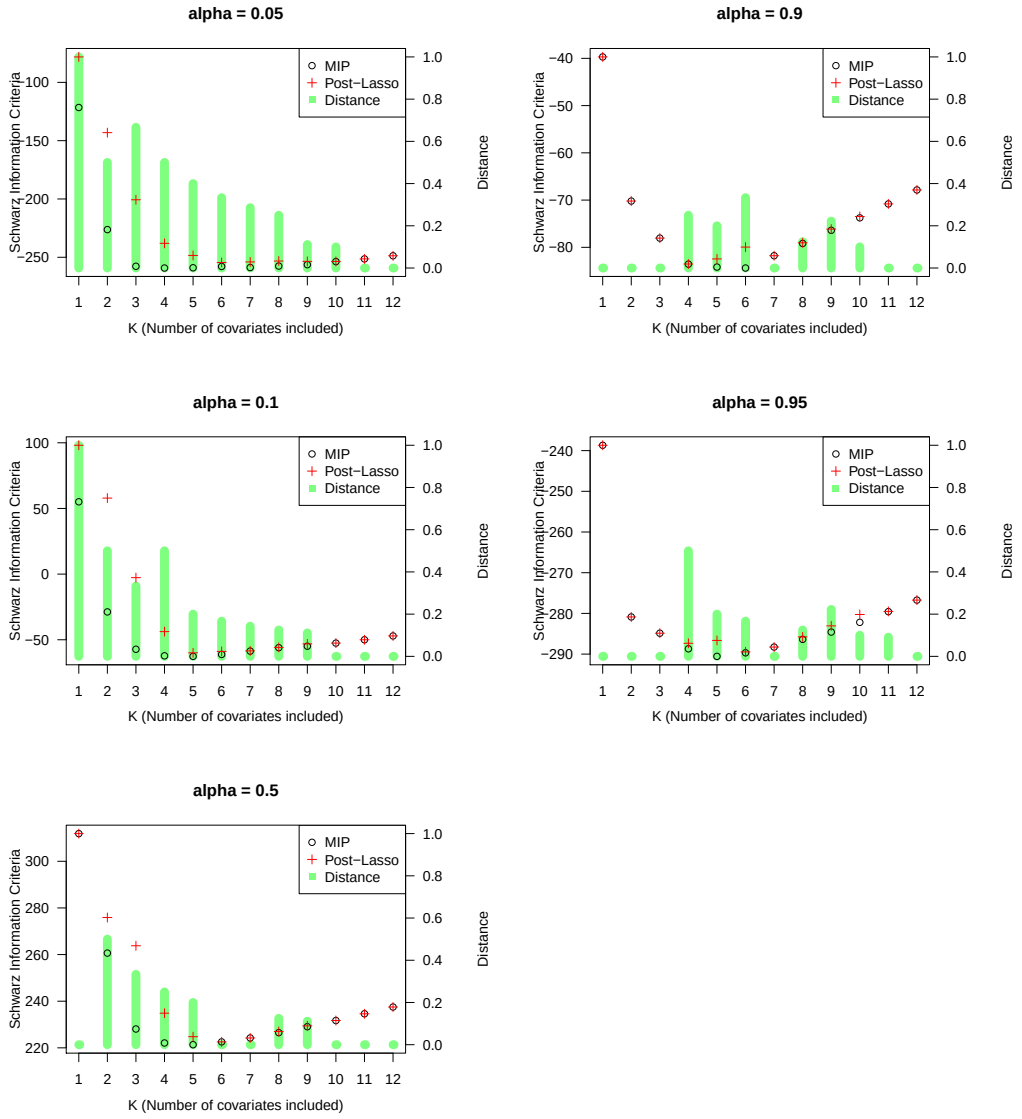


Figure 3.2: Comparison of SIC between a solution with Lasso as a variable selector and the best subset selection with MIP. The bars represent the distance d as defined by equation 3.24.
(*) When the distance is zero, it means that the same variables are selected from both methods for a given k. Thus, in these cases we have the same SIC for both of them.

4 Simulation

In this section, we investigate how to simulate future paths of the time series y_t. Let n be the total number of observations of y_t. We produce S different paths with size K for each. We have n observations of y_t and we want to produce . Given a vector of explanatory variables x_t, let $q_\alpha(x_t)$ be given by the α-quantile estimated as described on section 2.

The variables chosen to compose x_t can be either exogenous variables, autoregressive components of y_t or both. As the distribution of ε_t is unknown, we have to use a nonparametric approach in order to estimate its one-step ahead density.

The coefficients $\beta_{0\alpha}$ and β_α are the solution of the minimization problem given in the problem defined in (1.7)-(1.10), reproduced here for convenience:

$$\min_{q_\alpha, \varepsilon^+_{t,\alpha}, \varepsilon^-_{t,\alpha}} \quad \sum_{\alpha \in A} \sum_{t \in T} \left(\alpha \varepsilon^+_{t,\alpha} + (1-\alpha)\varepsilon^-_{t,\alpha} \right) \tag{4.1}$$

$$\text{s.t.} \quad \varepsilon^+_{t,\alpha} - \varepsilon^-_{t,\alpha} = y_t - q_\alpha(x_t), \qquad \forall t \in T_\tau, \forall \alpha \in A, \tag{4.2}$$

$$\varepsilon^+_{t,\alpha}, \varepsilon^-_{t,\alpha} \geq 0, \qquad \forall t \in T_\tau, \forall \alpha \in A, \tag{4.3}$$

$$q_\alpha(x_t) \leq q_{\alpha'}(x_t), \qquad \forall t \in T_\tau, \forall (\alpha, \alpha') \in A \times A, \alpha < \alpha', \tag{4.4}$$

To produce S different paths of $\{\hat{y}_t\}_{t=n+1}^{n+K}$, we use the following procedure:

Procedure for simulating S scenarios of y_t

---

1. At first, let $\tau = n + 1$.

2. In any given period τ, for every $\alpha \in A$, we use the problem defined on (1.7)-(1.10) to estimate quantiles $q_\alpha(x_\tau)$. Note that x_τ is supposed to be known at time τ. In the presence of exogenous variables that are unknown, it is advisable to incorporate its uncertainty by considering different scenarios. In each scenario, though, x_τ must be considered fully known.

3. Let $\hat{Q}_{y_\tau | X = x_\tau}(\alpha, x_\tau)$ be the estimated quantile function of y_τ. To estimate $\hat{Q}_{y_\tau}$, we first define a discrete quantile function $\tilde{Q}_{y_\tau}$. By mapping every $\alpha \in A$ with its estimated quantile $\hat{q}_\alpha$, we define function $\tilde{Q}_{y_\tau}$. When we interpolate

   This process is described in more details on section 2.

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4. Once we have a distribution for y_{n+1}, we can generate S different simulated values, drawn from the distribution function $\hat{F}_{y_{n+1}} = \hat{Q}^{-1}_{y_\tau}$, derived from the quantile function found by doing steps 2 and 3. Let X be a random variable with uniform distribution over the interval $[0, 1]$. By using results from the Probability Integral Transform, we know that the random variable $F^{-1}_{y_{n+1}}(X)$ has the same distribution as y_{n+1}. So, by drawing a sample of size S from X and applying the quantile function $Q_{y_{n+1}}(\alpha)$, we have our sample of size K for y_{n+1}.

5. Each one of the S different values for y_{n+1} will be the starting point of a different path. Now, for each $\tau \in [n+2, n+K]$ and $s \in S$, we have to estimate quantiles $q_{\alpha\tau,s}$ and find a quantile function for $\hat{Q}_{y_{\tau,s}}$ just like it was done on steps 2 and 3. Note that when $\tau > n + 2$, every estimate will be scenario dependent, hence there will be S distribution functions estimated for each period τ. From now on, in each path just one new value will be drawn randomly from the one-step ahead distribution function - as opposed to what was carried on step 3, when S values were simulated. As there will be S distribution functions - one for each path, in each period τ it will be produced exact S values for y_τ, one for its own path. Repeating this step until all values of τ and s are simulated will give us the full simulations that we are looking for.

We applied this procedure on the ENA data for the brazilian southeast region. The quantiles of scenarios are shown on figure

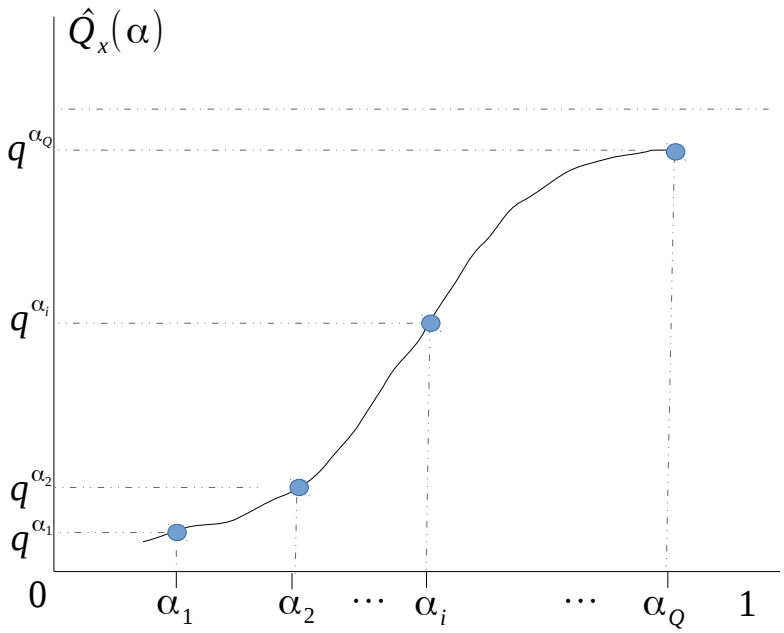

Figure 4.1: Fitting a distribution function from quantile estimations

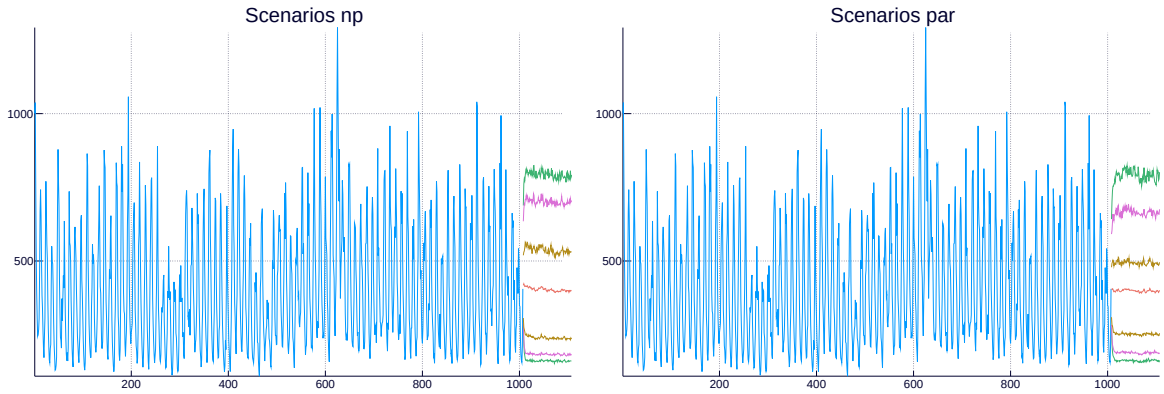

Figure 4.2: Scenario simulation for the nonparametric (on the left) and parametric (on the right) for the ENA Southeast dataset

5 Appendices

5.1 Proof of quantiles as an optimization problem

Let $Z^\alpha = \arg\min_Q E[\alpha \max\{0, X-Q\} + (1-\alpha)\max\{0, Q-X\}]$. We can rewrite the function as

$$\begin{aligned}
Y &= \alpha \int_Q^\infty (X-Q) dF_x + (1-\alpha) \int_{-\infty}^Q (Q-X) dF_X \\
&= \alpha \int_Q^\infty X dF_x - \alpha Q \int_Q^\infty Q dF_x + Q \int_{-\infty}^Q dF_x - \int_{-\infty}^Q X dF_x - \alpha Q \int_{-\infty}^Q dF_x + \alpha \int_{-\infty}^Q X dF_x \\
&= \alpha \int_Q^\infty X dF_x - \alpha Q + Q F_X(Q) - \int_{-\infty}^Q X dF_x - \alpha Q F_X(Q) + \alpha \int_{-\infty}^Q X dF_x \\
&= \alpha \int_Q^\infty X dF_x - \alpha Q + Q F_X(Q) - \int_{-\infty}^Q X dF_x + \alpha \int_{-\infty}^Q X dF_x
\end{aligned}$$

By the first order condition for optimality, we need that $\frac{dZ(Q^*)}{dQ} = 0$. So, we have:

$$-\alpha Q^* f(Q^*) - \alpha + F_X(Q^*) + Q^* f(Q^*) - Q^* f(Q^*) + \alpha Q^* f(Q^*) = 0$$
$$F_X(Q^*) = \alpha.$$

Thus, we have that Z^α is the $\alpha - quantile$ of random variable X.

5.2 MIP coefficients tables

The following tables inform the size of Coefficients when using the regularization method based on MIP described on session 3.1. When using this method, we choose a parameter K which defines the total number of nonzero coefficients (without accounting the intercept β_0, which is always included). In each column we find the estimated values of coefficients for each different choice of K. As coefficients are quantile dependent, we provide tables for $\alpha \in (0.05, 0.1, 0.25, 0.5, 0.75, 0.9, 0.95)$.

	K=1	K=2	K=3	K=4	K=5	K=6	K=7	K=8	K=9	K=10	K=11	K=12
β_0	-15.33	9.38	1.48	1.34	8.72	-1.68	4.94	0.65	-0.27	-0.16	-3.96	-2.55
β_1	-0.00	0.79	0.66	0.58	0.46	0.40	0.48	0.46	0.46	0.47	0.42	0.44
β_2	-0.00	-0.00	-0.00	-0.00	-0.00	0.33	-0.00	-0.00	-0.00	-0.00	0.14	0.09
β_3	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	0.20	0.20	0.19	0.20	0.17
β_4	-0.00	-0.47	-0.28	-0.27	-0.29	-0.35	-0.31	-0.40	-0.35	-0.35	-0.34	-0.31
β_5	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.05	-0.07	-0.09
β_6	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	0.11	0.08	0.11	0.17	0.12	0.19
β_7	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.16	-0.15	-0.08	-0.15
β_8	-0.00	-0.00	-0.00	-0.00	-0.15	-0.00	-0.31	-0.26	-0.17	-0.17	-0.16	-0.18
β_9	-0.00	-0.00	-0.00	-0.00	-0.00	0.14	0.16	0.20	0.26	0.23	0.28	0.33
β_{10}	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.04
β_{11}	-0.00	-0.00	0.26	0.17	0.21	0.08	0.16	0.19	0.17	0.18	0.17	0.20
β_{12}	1.17	-0.00	-0.00	0.18	0.15	0.19	0.22	0.20	0.20	0.18	0.18	0.17

Table 5.1: Coefficients for quantile $\alpha = 0.05$

5.3 Simulation study tables

On section 3.3 we explain a simulation study to try evaluating differences between estimating a quantile with a quantile regression model or using the conditional mean when knowing the true generating model. From this experiment, we present below table of results for three different aspects, for five different quantiles. Tables from 5.6 to 5.10 shows the difference between the root mean square

	K=1	K=2	K=3	K=4	K=5	K=6	K=7	K=8	K=9	K=10	K=11	K=12
β_0	-10.68	10.07	3.56	1.24	0.76	3.01	3.33	3.02	1.05	2.26	1.55	1.57
β_1	-0.00	0.81	0.63	0.61	0.55	0.49	0.49	0.50	0.48	0.44	0.44	0.44
β_2	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	0.04	-0.00	-0.00	0.04	0.07	0.07
β_3	-0.00	-0.00	-0.00	-0.00	0.15	0.20	0.16	0.15	0.13	0.11	0.12	0.12
β_4	-0.00	-0.43	-0.33	-0.28	-0.37	-0.33	-0.34	-0.30	-0.24	-0.24	-0.26	-0.25
β_5	-0.00	-0.00	-0.00	-0.00	-0.00	-0.08	-0.07	-0.12	-0.14	-0.15	-0.17	-0.17
β_6	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	0.11	0.10	0.10	0.14	0.14
β_7	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.07	-0.11	-0.13	-0.11	-0.11
β_8	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.04	-0.04
β_9	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	0.09	0.10	0.13	0.13
β_{10}	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	0.00
β_{11}	-0.00	-0.00	-0.00	0.14	0.17	0.17	0.16	0.15	0.11	0.09	0.08	0.08
β_{12}	1.09	-0.00	0.35	0.27	0.25	0.22	0.22	0.26	0.33	0.34	0.33	0.33

Table 5.2: Coefficients for quantile $\alpha = 0.1$

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	K=1	K=2	K=3	K=4	K=5	K=6	K=7	K=8	K=9	K=10	K=11	K=12
β_0	2.72	-3.38	8.64	4.88	0.62	2.98	2.70	2.62	2.27	1.87	2.43	2.53
β_1	-0.00	0.59	0.52	0.51	0.57	0.54	0.56	0.56	0.58	0.58	0.57	0.57
β_2	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.03	-0.06	-0.05	-0.05
β_3	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	0.04	0.03	0.04
β_4	-0.00	-0.00	-0.25	-0.18	-0.14	-0.11	-0.11	-0.12	-0.11	-0.11	-0.11	-0.12
β_5	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	0.01
β_6	-0.00	-0.00	-0.00	-0.00	-0.00	-0.06	-0.09	-0.08	-0.08	-0.08	-0.09	-0.09
β_7	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.02	-0.02
β_8	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	0.06	0.06	0.05	0.06	0.08	0.07
β_9	-0.00	-0.00	-0.00	-0.00	0.08	0.09	0.06	0.09	0.07	0.07	0.08	0.08
β_{10}	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.05	-0.04	-0.05	-0.05	-0.05
β_{11}	-0.00	0.54	-0.00	0.15	0.14	0.11	0.10	0.11	0.14	0.14	0.15	0.14
β_{12}	0.92	-0.00	0.42	0.34	0.32	0.33	0.32	0.34	0.33	0.34	0.32	0.33

Table 5.3: Coefficients for quantile $\alpha = 0.5$

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errors between both methods of predicting the one-step ahead quantile for a few different values of α. Tables 5.11-5.15 are the ones that shows the nominal difference between the autoregressive coefficients $\hat{\phi}$ and $\hat{\beta}$. The nominal difference between the intercept terms $\hat{\phi}_0 + z_\alpha \hat{\sigma}_\varepsilon^2$) and $\hat{\beta}_0$) is

	K=1	K=2	K=3	K=4	K=5	K=6	K=7	K=8	K=9	K=10	K=11	K=12
β_0	12.14	10.06	6.60	11.05	13.22	12.04	13.34	13.28	12.58	13.69	13.47	13.71
β_1	-0.00	0.24	0.39	0.39	0.40	0.38	0.38	0.38	0.38	0.40	0.40	0.40
β_2	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.02
β_3	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.01	-0.04	-0.03	-0.02
β_4	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	0.03	-0.00	0.05	0.05	0.04
β_5	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	0.00	0.01
β_6	-0.00	-0.00	-0.00	-0.14	-0.00	-0.00	-0.03	-0.05	-0.01	-0.07	-0.07	-0.07
β_7	-0.00	-0.00	-0.00	-0.00	-0.19	-0.10	-0.10	-0.11	-0.09	-0.11	-0.11	-0.10
β_8	-0.00	-0.00	-0.00	-0.00	-0.00	-0.08	-0.07	-0.08	-0.08	-0.07	-0.07	-0.08
β_9	-0.00	-0.00	-0.00	0.14	0.16	0.15	0.16	0.18	0.16	0.19	0.19	0.19
β_{10}	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.04	-0.06	-0.06	-0.06
β_{11}	-0.00	-0.00	0.20	-0.00	0.11	0.15	0.12	0.16	0.16	0.18	0.18	0.19
β_{12}	0.80	0.63	0.39	0.42	0.26	0.29	0.28	0.23	0.29	0.24	0.24	0.25

Table 5.4: Coefficients for quantile $\alpha = 0.9$

.

	K=1	K=2	K=3	K=4	K=5	K=6	K=7	K=8	K=9	K=10	K=11	K=12
β_0	16.73	11.74	11.51	13.77	13.45	13.48	14.36	14.84	12.36	14.04	13.09	14.00
β_1	-0.00	0.26	0.32	0.35	0.38	0.38	0.40	0.43	0.40	0.40	0.39	0.39
β_2	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	0.02	0.02
β_3	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	0.01
β_4	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	0.04	0.06	0.06	0.05
β_5	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.04	-0.03	-0.04
β_6	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.05	-0.10	-0.07	-0.09	-0.08	-0.09
β_7	-0.00	-0.00	-0.00	-0.15	-0.14	-0.12	-0.09	-0.05	-0.06	-0.06	-0.06	-0.06
β_8	-0.00	-0.00	-0.00	-0.00	-0.00	-0.04	-0.05	-0.07	-0.05	-0.08	-0.07	-0.07
β_9	-0.00	-0.00	-0.00	0.16	0.11	0.14	0.16	0.19	0.19	0.22	0.22	0.21
β_{10}	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.15	-0.14	-0.11	-0.12	-0.11
β_{11}	-0.00	-0.00	0.17	-0.00	0.14	0.13	0.12	0.25	0.23	0.18	0.21	0.22
β_{12}	0.71	0.59	0.37	0.41	0.28	0.28	0.25	0.21	0.27	0.25	0.24	0.22

Table 5.5: Coefficients for quantile $\alpha = 0.95$

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	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
1981	23.36	28.34	12.44	18.35	17.10	22.49	23.57	40.10	48.40	42.13	43.70	37.23
1982	20.54	17.48	7.42	10.87	16.57	20.79	27.95	42.55	49.12	42.48	44.78	40.20
1983	27.94	24.50	22.60	22.24	29.62	27.05	33.92	45.06	50.64	49.32	43.83	36.14
1984	20.37	15.35	3.94	3.57	7.85	14.65	20.56	41.01	44.58	44.31	42.94	31.65
1985	10.38	4.71	5.15	2.84	7.27	10.36	14.53	39.33	45.18	41.21	42.15	23.02
1986	18.86	8.25	3.00	5.23	17.29	17.85	23.08	41.36	48.30	42.83	44.36	36.41
1987	26.09	24.71	6.90	21.02	20.73	19.53	28.42	42.94	48.06	44.26	43.11	39.67
1988	15.75	11.66	4.51	4.36	8.29	11.50	19.10	38.40	46.47	44.80	41.79	22.40
1989	19.92	14.52	5.08	2.75	5.62	11.42	17.17	38.94	43.92	43.70	40.69	26.34
1990	29.74	11.70	15.69	14.02	14.85	22.28	24.02	44.55	48.18	44.66	41.51	32.41
1991	17.09	13.46	7.68	6.63	8.51	16.17	26.46	43.36	49.00	45.86	40.14	36.57
1992	21.41	19.78	14.25	21.45	24.24	24.64	30.34	45.43	51.33	47.66	44.50	37.97
1993	27.86	20.13	14.36	16.63	20.94	26.43	30.60	44.07	44.73	43.78	41.40	34.18
1994	12.45	11.06	4.70	5.85	10.49	11.04	23.03	38.50	48.92	47.30	44.97	36.55
1995	20.31	5.80	9.47	5.36	5.62	14.15	23.54	42.48	50.49	42.74	41.15	29.90
1996	19.89	11.85	3.43	5.08	8.26	16.29	24.89	40.52	48.44	44.92	40.15	36.37
1997	23.89	27.80	14.30	11.95	17.55	22.22	31.82	44.07	43.14	40.00	37.94	28.36
1998	15.04	21.70	10.61	17.28	21.57	22.31	27.26	42.45	49.04	46.76	37.22	35.74
1999	22.18	15.39	8.18	13.66	8.67	16.49	22.30	40.43	47.75	39.85	36.95	35.54
2000	16.75	7.95	11.33	10.47	16.73	15.07	18.90	38.91	44.26	46.34	41.98	31.62
2001	24.03	11.82	11.09	9.23	16.30	14.53	25.73	41.57	45.79	40.99	41.52	42.76
2002	16.81	22.08	13.40	11.07	15.71	17.52	26.55	41.64	45.80	45.94	40.64	30.58
2003	17.42	14.05	10.03	11.26	15.39	17.01	28.29	39.98	47.02	47.07	40.47	34.85
2004	15.04	13.34	17.84	16.97	20.10	19.48	25.03	40.11	48.25	47.21	44.13	35.79
2005	24.89	20.47	13.01	20.88	19.98	21.48	27.81	42.74	46.09	46.93	44.98	36.08
2006	32.48	15.44	12.93	6.59	12.19	19.08	27.79	40.72	46.01	44.38	42.85	33.99
2007	28.93	11.13	16.10	11.91	17.68	21.57	30.56	42.95	47.80	47.61	42.97	35.98
2008	20.42	15.46	3.51	9.37	8.71	13.02	23.61	36.93	45.82	46.49	43.91	35.19
2009	21.48	15.16	6.74	3.80	4.48	12.88	24.53	38.40	47.70	40.87	46.73	38.03
2010	24.75	30.70	16.99	16.95	15.72	16.86	27.43	43.18	48.71	35.79	41.30	30.15
2011	16.33	14.79	9.30	7.70	13.35	18.60	23.53	39.62	46.97	40.99	44.75	42.79

$\phi \backslash RSN$	0.01	0.05	0.1	0.5	1
0.25	0.99454499	0.99987753	0.99726238	0.99996553	0.99997820
0.50	0.99928262	1.00002733	0.99902661	1.00017501	1.00072601
0.70	0.97952103	0.99154652	1.00049301	0.99991315	1.00014586
0.90	0.89928729	0.99117750	0.99730087	1.00106737	0.99970135

Table 5.6: RMSE ratio ($RMSE^{QR}/RMSE^{AR}$) for estimating quantile $\alpha = 0.05$. ϕ stands for the autoregressive coefficient and RSN is the signal to noise ratio. Details for these experiments can be found on section 3.3

$\phi \backslash RSN$	0.01	0.05	0.1	0.5	1
0.25	0.99404378	1.00071379	0.99865043	1.00009650	1.00031150
0.50	0.99302385	1.00012893	1.00008177	1.00020172	0.99997679
0.70	0.96163465	0.99948879	1.00007350	1.00076714	1.00002219
0.90	0.82867689	0.99989835	1.00007245	0.99997754	1.00016876

Table 5.7: RMSE ratio ($RMSE^{QR}/RMSE^{AR}$) for estimating quantile $\alpha = 0.1$. ϕ stands for the autoregressive coefficient and RSN is the signal to noise ratio. Details for these experiments can be found on section 3.3

$\phi \backslash RSN$	0.01	0.05	0.1	0.5	1
0.25	0.98066380	0.99981708	0.99972368	1.00003875	1.00027661
0.50	0.99671712	0.99884992	0.99718531	0.99996169	0.99939155
0.70	0.96160330	0.99971110	1.00006113	1.00001116	1.00024519
0.90	0.93616426	0.99985323	0.99934405	0.99991411	0.99994730

Table 5.8: RMSE ratio ($RMSE^{QR}/RMSE^{AR}$) for estimating quantile $\alpha = 0.5$. ϕ stands for the autoregressive coefficient and RSN is the signal to noise ratio. Details for these experiments can be found on section 3.3

$\phi \backslash RSN$	0.01	0.05	0.1	0.5	1
0.25	0.99662610	0.99945043	0.99538478	0.99966478	1.00025526
0.50	1.00028533	0.99613341	0.99981279	0.99996292	1.00108295
0.70	0.91878126	0.99889801	0.99885007	1.00023457	1.00067966
0.90	0.80536841	0.99826964	0.99806239	1.00003275	0.99993464

Table 5.9: RMSE ratio ($RMSE^{QR}/RMSE^{AR}$) for estimating quantile $\alpha = 0.9$. ϕ stands for the autoregressive coefficient and RSN is the signal to noise ratio. Details for these experiments can be found on section 3.3

$\phi \backslash RSN$	0.01	0.05	0.1	0.5	1
0.25	1.00009586	0.99962843	1.00162321	1.00002799	1.00023607
0.50	0.99977885	0.99742853	0.99415437	1.00058173	0.99971703
0.70	0.93295124	0.99948015	0.99936096	1.00048504	0.99969376
0.90	0.97068193	1.00001858	1.00007662	1.00020540	0.99928392

Table 5.10: RMSE ratio ($RMSE^{QR}/RMSE^{AR}$) for estimating quantile $\alpha = 0.95$. ϕ stands for the autoregressive coefficient and RSN is the signal to noise ratio. Details for these experiments can be found on section 3.3

$\phi \backslash RSN$	0.01	0.05	0.1	0.5	1
0.25	-0.02501316	0.01504215	-0.00565925	-0.01735387	-0.00060544
0.50	-0.01319163	0.00222293	0.00477565	0.01005289	-0.01981673
0.70	0.02775786	-0.01203545	0.01320180	-0.00721887	-0.00665778
0.90	0.01743705	0.00303553	0.01319473	-0.01785909	0.00046374

Table 5.11: Diffeference between the autoregressive coefficients ($\hat{\phi} - \hat{\beta}$) for estimating quantile $\alpha = 0.05$. ϕ stands for the autoregressive coefficient and RSN is the signal to noise ratio. Details for these experiments can be found on section 3.3

$\phi \backslash RSN$	0.01	0.05	0.1	0.5	1
0.25	-0.00866846	0.02731984	0.01494079	-0.01572119	0.01000855
0.50	0.02547593	0.00642516	0.00247766	-0.00871352	0.00277974
0.70	0.00503748	0.00751987	-0.00107390	0.00327506	-0.00044987
0.90	0.00709175	0.00692367	-0.00624776	0.00298916	-0.00493012

Table 5.12: Diffeference between the autoregressive coefficients ($\hat{\phi} - \hat{\beta}$) for estimating quantile $\alpha = 0.1$. ϕ stands for the autoregressive coefficient and RSN is the signal to noise ratio. Details for these experiments can be found on section 3.3

$\phi \backslash RSN$	0.01	0.05	0.1	0.5	1
0.25	0.00504450	-0.00552938	0.00198665	0.00331038	0.00493278
0.50	0.00816919	-0.00518034	-0.00514326	0.00056696	0.00347201
0.70	0.00817503	0.00319246	-0.00703127	-0.00478434	0.00539972
0.90	0.01257091	0.00502689	-0.00469782	-0.00543626	0.00172323

Table 5.13: Diffeference between the autoregressive coefficients $(\hat{\phi} - \hat{\beta})$ for estimating quantile $\alpha = 0.5$. ϕ stands for the autoregressive coefficient and RSN is the signal to noise ratio. Details for these experiments can be found on section 3.3

$\phi \backslash RSN$	0.01	0.05	0.1	0.5	1
0.25	0.00063949	-0.00587333	-0.00343132	0.00057888	-0.01576153
0.50	-0.01953741	-0.00099696	-0.01220643	-0.00459181	-0.02725897
0.70	0.00879188	-0.00578564	-0.01365016	-0.01735324	-0.01595786
0.90	-0.00432531	-0.00674863	0.00059043	0.00040195	0.00452383

Table 5.14: Diffeference between the autoregressive coefficients $(\hat{\phi} - \hat{\beta})$ for estimating quantile $\alpha = 0.9$. ϕ stands for the autoregressive coefficient and RSN is the signal to noise ratio. Details for these experiments can be found on section 3.3

$\phi \backslash RSN$	0.01	0.05	0.1	0.5	1
0.25	-0.00217703	0.01773060	-0.01892853	0.01070602	-0.01105531
0.50	-0.01530692	-0.01035399	-0.02514421	0.01278911	-0.00835432
0.70	0.04323376	-0.00354721	0.03589676	-0.00176410	-0.00545403
0.90	0.04279246	0.00727074	-0.00552405	0.00634520	-0.00118189

Table 5.15: Diffeference between the autoregressive coefficients $(\hat{\phi} - \hat{\beta})$ for estimating quantile $\alpha = 0.95$. ϕ stands for the autoregressive coefficient and RSN is the signal to noise ratio. Details for these experiments can be found on section 3.3

$\phi \backslash RSN$	0.01	0.05	0.1	0.5	1
0.25	0.04603811	-0.01775659	-0.03214647	0.02145871	-0.00765273
0.50	0.03352012	-0.00448700	-0.02823629	-0.04445694	0.03234653
0.70	-0.04953532	0.10768253	-0.03871770	0.02006049	0.01848257
0.90	-0.01908312	0.05744539	-0.17970986	0.12894696	0.02136838

Table 5.16: Coefficient difference between the non-autoregressive part $((\hat{\phi}_0 + z_\alpha \hat{\sigma}_\varepsilon^2) - \hat{\beta}_0)$ for estimating quantile $\alpha = 0.05$. ϕ stands for the autoregressive coefficient and RSN is the signal to noise ratio. Details for these experiments can be found on section 3.3

$\phi \backslash RSN$	0.01	0.05	0.1	0.5	1
0.25	0.02260184	-0.04514987	-0.04187708	0.02309700	-0.02037122
0.50	-0.03481513	-0.01041481	-0.00038617	-0.00296439	-0.01172149
0.70	0.03459960	-0.03797767	0.01480016	0.04972062	0.01002299
0.90	0.14704364	-0.06159980	0.08091946	-0.03306211	0.01302636

Table 5.17: Coefficient difference between the non-autoregressive part $((\hat{\phi}_0 + z_\alpha \hat{\sigma}_\varepsilon^2) - \hat{\beta}_0)$ for estimating quantile $\alpha = 0.1$. ϕ stands for the autoregressive coefficient and RSN is the signal to noise ratio. Details for these experiments can be found on section 3.3

$\phi \backslash RSN$	0.01	0.05	0.1	0.5	1
0.25	0.01297018	-0.00048766	-0.01041199	-0.01243439	0.02395973
0.50	-0.00774322	0.03340235	0.04581712	-0.00941144	0.02834337
0.70	0.02631287	-0.00313672	0.05261023	-0.00929484	-0.05534836
0.90	-0.00600514	-0.01798551	0.01424034	0.07490435	-0.10554228

Table 5.18: Coefficient difference between the non-autoregressive part $((\hat{\phi}_0 + z_\alpha \hat{\sigma}_\varepsilon^2) - \hat{\beta}_0)$ for estimating quantile $\alpha = 0.5$. ϕ stands for the autoregressive coefficient and RSN is the signal to noise ratio. Details for these experiments can be found on section 3.3

$\phi \backslash RSN$	0.01	0.05	0.1	0.5	1
0.25	-0.01239889	0.02293524	-0.02274380	-0.01803445	0.00913629
0.50	0.03837217	0.01562634	0.02285269	0.00939266	0.08607257
0.70	0.05059941	0.03251627	0.07964660	0.06332138	0.06101961
0.90	0.27272743	0.10535168	0.04777041	0.02227744	-0.04024329

Table 5.19: Coefficient difference between the non-autoregressive part $((\hat{\phi}_0 + z_\alpha \hat{\sigma}_\varepsilon^2) - \hat{\beta}_0)$ for estimating quantile $\alpha = 0.9$. ϕ stands for the autoregressive coefficient and RSN is the signal to noise ratio. Details for these experiments can be found on section 3.3

$\phi \backslash RSN$	0.01	0.05	0.1	0.5	1
0.25	0.00335900	-0.03337376	0.03659738	-0.02854912	0.01837794
0.50	0.03376431	-0.00203817	-0.00638700	-0.06833761	-0.00379743
0.70	-0.07861339	0.02610872	-0.11362453	0.05524907	-0.00006457
0.90	-0.33397830	-0.06626543	0.05843421	-0.05809618	-0.04813865

Table 5.20: Coefficient difference between the non-autoregressive part $((\hat{\phi}_0 + z_\alpha \hat{\sigma}_\varepsilon^2) - \hat{\beta}_0)$ for estimating quantile $\alpha = 0.95$. ϕ stands for the autoregressive coefficient and RSN is the signal to noise ratio. Details for these experiments can be found on section 3.3

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