

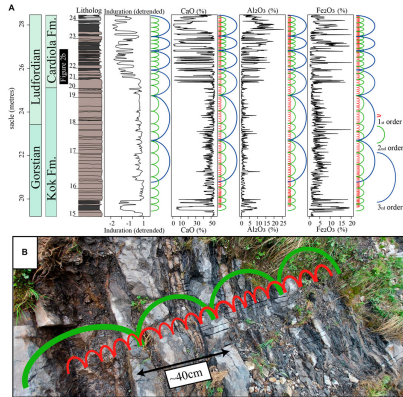


Michel Crucifix

# Plan

- 1 Motivation
- 2 The two-body problem
- 3 The linear perturbation theory
- 4 The non-linear perturbation problem
- 5 The precession mouvement

# Bundles in the rocks



- The whole idea of cyclostratigraphy is to identify bundle structures, which are organised according to a hierarchy of periods.
- Where does this come from ?

M. Arts et al. In: *Frontiers in Earth Science* (2024)

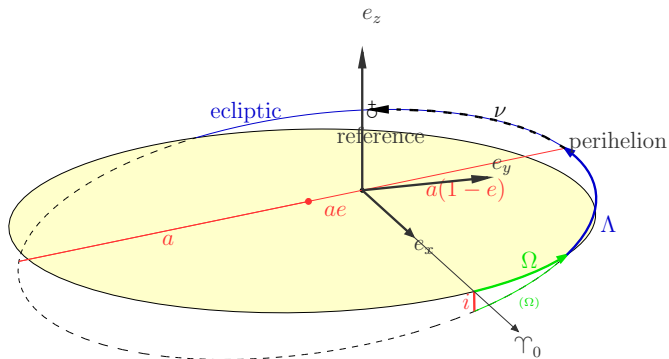
# Plan

- 1 Motivation
- 2 The two-body problem
- 3 The linear perturbation theory
- 4 The non-linear perturbation problem
- 5 The precession mouvement

# Suppose the Sun and the Earth are alone in the Universe

- ▶ You want to describe their motion
- ▶ The Sun is so big that it is practically the center of mass
- ▶ So we take it as the center of our framework. We define a *reference plane* and a *reference direction*.

## The angles and variables describing the orbit (Keplerian elements)



# The beauty of the Keplerian orbit

is that all Keplerian elements are constant, except the “mean longitude”  $\ell$ , which varies in a somehow tricky way

- ▶ Celestial mechanics is an *art* of choosing the right variables to make the solution look as simple as possible. Astronomers look for *constant of motions* : things that are unchanged along the trajectory.
- ▶ The approach used since the XIXth century (angle-action) is a bit more abstract but the idea is there.

the “mean longitude” is linked to angles you can really measure:

$$\ell = E - e \sin E.$$
$$\tan \frac{\nu}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2}$$

# Plan

- 1 Motivation
- 2 The two-body problem
- 3 The linear perturbation theory
- 4 The non-linear perturbation problem
- 5 The precession mouvement



# You terrified me; why do you say all of this?

- ▶ ... because Sun and Earth are not alone !
- ▶ In practice, these equations of  $a$ ,  $i$  etc. will need to be modified to take into account *perturbations* by other planets. But how ?
- ▶ This ambitious program was developed by Lagrange (1736 – 1813) and Laplace (1749 – 1827), and many others.
- ▶ Astronomers have pursued the goal of a *general theory* that describes planetary motion is one that is supposed to be valid over a *very long* time range : the orbital variables do not go to zero or infinity over time.

# Lagrange and Laplace had already a go!

Indeed ! Already Lagrange and Laplace introduced the idea of two abstract vectors (slight re-interpretation):

- ▶ the *eccentricity* vector  $\mathbf{e}$ . It has a size  $e$  and points to the perihelion.
- ▶ the *inclination* vector  $\mathbf{i}$  : It has a size  $\sin i$  and points to the ascending node.

`\end{document}`

# The averaging process

- ▶ To make a long (and complicated) story short: we are concerned here about mutual interactions of orbital planets over many thousands of years. So to first approximation, we care about the mean gravitational field generated by a planet over its whole orbit (Gauss' Keplerian rings).
- ▶ So in a way we are interested about how elements which describe the orbit (the "osculating elements") influence each other.
- ▶ This averaging process is still used today (though not always).
- ▶ Laplace and Lagrange found that to a *very first* approximation, the movements of  $\mathbf{e}$  and  $\mathbf{i}$  can be obtained by solving two sets of equations:

# Linear equations

$$\begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \\ \dot{e}_4 \\ \dot{e}_5 \\ \dot{e}_6 \end{pmatrix} = G \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{pmatrix} \qquad \begin{pmatrix} c\dot{i}_1 \\ \dot{i}_2 \\ \dot{i}_3 \\ \dot{i}_4 \\ \dot{i}_5 \\ \dot{i}_6 \end{pmatrix} = K \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{pmatrix}$$

where the numbers correspond to the different planets. Nowadays, we call this an “eigenvalue” problem: the vector  $e$  and  $i$  will oscillate with angular velocities that are the eigenvalues of the matrices  $G$  and  $K$ .

## In plain terms:

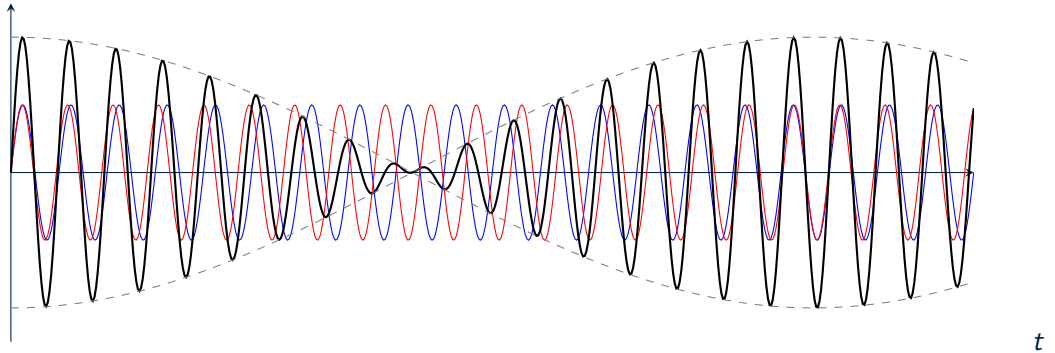
This means that given *initial conditions* (which set the phases), the  $e$  and  $i$  vectors follow quasi periodic movements:

$$\begin{aligned} e \sin \Pi &= \sum a_i \sin g_i t + \phi_t \\ \sin i \sin \Omega &= \sum b_i \sin s_i t + \phi_t \end{aligned}$$

these are the famous  $g$  and  $s$  terms.

# Beating

Amplitude



# Exercise 1 : Beatings with more periods

## Sine Waves and Beating

### Introduction

In this document, we will explore how simple sine waves can interact to produce **beating patterns** and how these can be visualised using a **continuous wavelet transform**. This exercise is designed to show the phenomenon of **amplitude modulation** clearly.

Before starting, make sure you have installed the

# The beating hierarchy of eccentricity



# Exercise 2 : Understanding the hierarchy of eccentricity beatings

## Calculating Eccentricity Periods Introduction

In this exercise, we will use a set of fundamental frequencies, often denoted as  $\{g_i\}$ , to calculate a simplified version of the orbital eccentricity,  $\{e\}$ . These frequencies are derived from celestial mechanics and describe the long-term behavior of planetary orbits.

```
# Load the vector of g_terms from Laskar's
```

# Plan

- 1 Motivation
- 2 The two-body problem
- 3 The linear perturbation theory
- 4 The non-linear perturbation problem
- 5 The precession mouvement

# What's wrong with the linear problem

- ▶ Linear theory: perturbations by others orbits, which are considered to be non-perturbed
- ▶ In reality, they dance together: It implies
  - ▶ that there are many more possible frequencies than the 8 'g' and 's' + background
  - ▶ that 'g' and 's' move
  - ▶ resonances are possible (dancing together). In general: combinations of 'g' and 's' = 0 are dangerous
  - ▶ entering in resonance and resonance escapes are partly unpredictable

# Mathematical approach (work in progress)

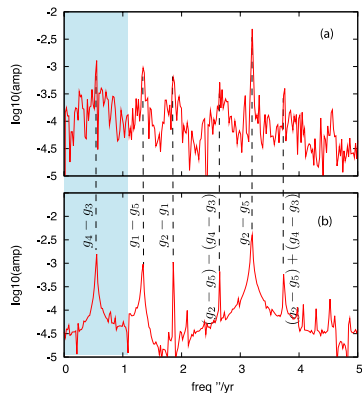
## 1 Analytical approaches :

$$\text{solution} = \sum_i a_i \sin(b_i t + \phi_i)$$

- ▶ Historically : Le Verrier, Hill, Bretagnon, and Laskar's thesis Ch. 1
- ▶ shows of  $g$  and  $s$  combine and contaminate both  $e$  and  $i$
- ▶ Used in the BER78 solution
- ▶ Analytical averaging : Laskar 1988
- ▶ Numerical averaging (Mogaverno and Laskar, using in Hoang et al).
- ▶ First allowed to show resonance, drifts, and chaos
- ▶ by design: respect energy conservation
- ▶ Quinn, Laskar 93 and Laskar 04 (hybrid)
- ▶ Zeebe 2023 (first to be fully open source, based on models available)
- ▶ Best to reassess resonances involving the inner planets

# Effect of non-linearity: broadening and background

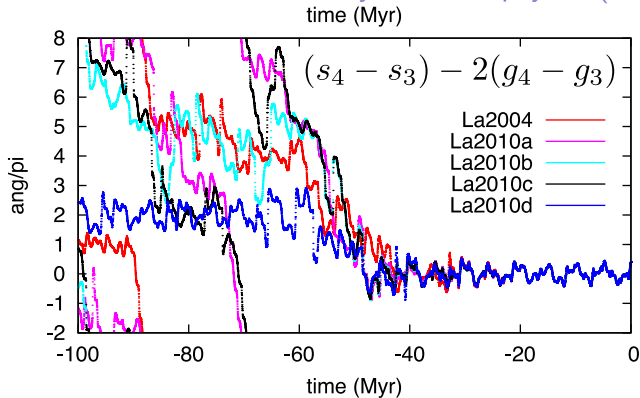
IF I HAVE TIME : SHOW ANALYTICAL VS LA10



{J. Laskar et al. In: *Astronomy and Astrophysics* (2011) see also exercise by ACDS}

# The $(g_3 - g_4) - 2(s_3 - s_4)$ resonance

J. Laskar et al. In: *Astronomy and Astrophysics* (2011)



We introduce here a crucial resonance, which we will further inspect once we have explained what is obliquity and how it is computed.

{J. Laskar et al. In: *Astronomy and Astrophysics* (2011) see also qmd exercise 3}

# The 404-ka bomb

## STILL TO DO

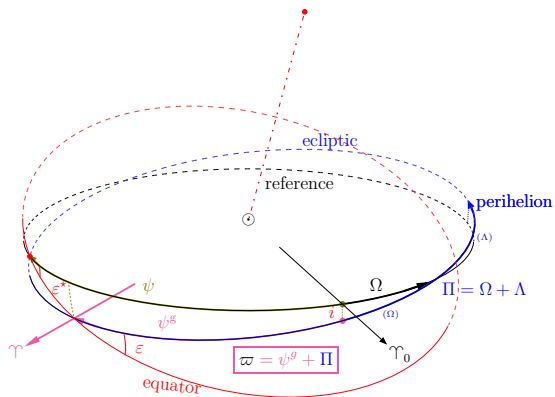
{R. E. Zeebe and M. L. Lantink. In: *The Astronomical Journal* (2024)}

# Plan

- 1 Motivation
- 2 The two-body problem
- 3 The linear perturbation theory
- 4 The non-linear perturbation problem
- 5 The precession mouvement



# Climatic precession



# Climatically relevant

- ▶ What counts, for climate, is insolation that you receive at the different *seasons*.
- ▶ Seasons depict the course of the *altitude* of the Sun.
- ▶ The reference is the point (node) at which the equator and the ecliptic intersect ( $\Upsilon$ ).
- ▶ The relevant angle for precession is thus the *longitude of the perihelion*  $\varpi$
- ▶ The relevant angle for “inclinaison” is the obliquity  $\varepsilon$ .

For climate (and cyclostratigraphy) we care about the following quantities

$\varpi$  : heliocentric longitude of the Sun  $e$  : Eccentricity  $\varepsilon$  : Obliquity

# How does that link to planetary precession ?

- ▶ And to very first order, *general precession* has a regular rate of  $k = 1$  revolution every 23 000 years

What it implies for our frequencies ?

Again, *to very first order* (linear theory)

- ▶ Eccentricity: *differences of  $g$  terms*
- ▶  $\varpi$  :  $g + k$  terms
- ▶ Obliquity :  $s + k$  terms

# Obliquity beats

Obliquity :  $s + k$  terms

Obliquity beats = differences in obliquity terms  $\rightarrow$  differences in  $s$  terms

Eccentricity beats = differences in precession terms  $\rightarrow$  differences in  $g$  terms

## Exercise 3 : Visualise the $g_3 - g_4 - 2(s_3 - s_4)$ resonance

### Resonance between obliquity and eccentricity

```
# Load the package
require(palinsol)
require(gtseries)
```

### Objective of the exercise

In this exercise, we will explore the famous  **$(g_3 - g_4) - 2(s_3 - s_4)$  resonance**, which couples obliquity and

## Exercise 3

Looking further general precession  $\psi$  is caused by - the torque by the Sun and Moon - thus depends (slightly) in  $e$  (for the Earth-Sun distance) and on the angle between equator and moon orbit - both vary (the Moon has a constant tilt with respect to the ecliptic. . . )

- ▶ What this says:
  - ▶ climatic precession more *variable* when  $e$  is small (cf. Huybers 2008)

# Mathematical approaches

- 1 Analytical approach (manipulate all terms; use trigonometric laws etc.)
  - ▶ find the origin of all frequencies
  - ▶ Sharaf - Budnikova ; Berger's thesis ; R code available from my team
- 2 Numerical integration driven by planetary solution
  - ▶ e.g. Laskar and Robutel ; Neron de Surgy (accounts for tidal dissipation)
  - ▶ Fortran code ; Julia code available from my team.
  - ▶ similar approach (based on Quinn) implemented in Zeebe's theme



# Resonances and chaos due to precession (also!)

(still to do)

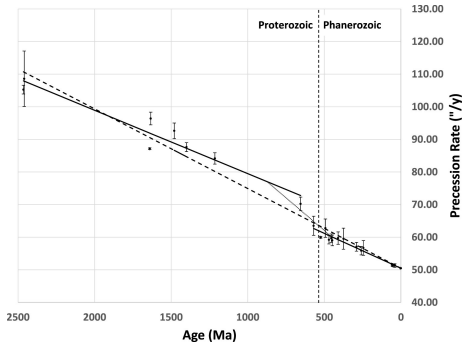
# And multi-million-year trend !

By its action on tides, the Moon acts as if it was trying to 'brake' (slow-down) Earth's rotation

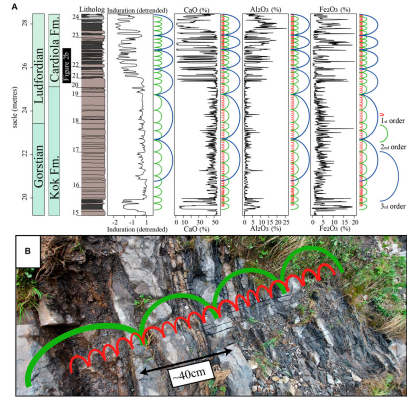
- ▶ Over the years: Earth's slows down and becomes rounder (less torque → less precession)
- ▶ Moon gains speed → accelerates → escapes the Earth

This 'brake' effect is not constant (it is particularly high at this moment; this is not tenable backward in time), prompting evolution models of dissipation. Resonances are possible.

# Two models are currently popular:



D. Waltham and M. Green. In: *Earth and Planetary Science Letters* (2024)



M. Farhat et al. In: *Astronomy and Astrophysics* (2022)

# Understand the consequences on the main precession periods

- ▶ The periods of eccentricity ( $g$ — differences) are unaffected by precession (this is planetary)
- ▶ The periods of precession ( $g - k$  terms) are ( $k$  is the precession rate)

Thus the *ratio* gives you an indication on the precession rate. If you know the LOD, you get the Earth-moon distance.