UNIVERSITÉ CATHOLIQUE DE LOUVAIN
Faculté des Sciences

École de Physique
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LPHYS2114 Non-linear dynamics
Série 4 – Variété stable-unstable and Lyapunov Functions

1. Phase portrait of two dimensional systems. Given the two dimensional system

$$\dot{x} = -x(1+y), \quad \dot{y} = y + x^2. \tag{1}$$

(a) Show that the system is invariant under the transformation $(x, y) \mapsto (-x, y)$. What does this tell us about the phase space of the system ?

(b) Find and classify all the equilibria of the system. Qualitatively describe the dynamics in the neighbourhood of these equilibria.

(c) Approximately calculate the stable and unstable local manifolds of the saddle point equilibria by developing the Taylor series up to order 4. *Hint :* Use the symmetry of the system found in part (a) to simplify the calculation.

(d) Sketch the dynamics of the system.

2. Stable and unstable manifold. Given the system of ODEs

$$\dot{x} = x(4 - x - y), \quad \dot{y} = y(x - 2). \tag{2}$$

(a) Show that the system has saddle point two equilibria at $\boldsymbol{p} = (0, 0)$ and $\boldsymbol{p} = (4, 0)$, and a single attractor at $\boldsymbol{p} = (2, 2)$.

(b) Determine the unstable and stable manifolds in the neighbourhood of the saddle points by developing the Taylor series up to 3.

(c) Sketch the phase portrait of the system.

3. A Lyapunov function for the Lorenz 63 system. The famous Lorenz 1963 system of ODEs is given by :

$$\dot{x} = \sigma(y - x), \quad \dot{y} = rx - y - xz, \quad \dot{z} = xy - bz. \tag{3}$$

Where, $\sigma, r, b > 0$ are parameters. In this exercise we are interested in one of the equilibria :

(a) Show that $\boldsymbol{p} = (0, 0, 0)$ is an equilibria of the system for all σ, r, b.

(b) Show that $E(x, y, z) = \sigma^{-1}x^2 + y^2 + z^2$ is a strict Lyapunov function for this equilibria for $0 < r < 1$.

(c) Describe the basin of attraction of this equilibria. (The basin of attraction of a equilibria is a collection of points that converge to the equilibria under the flow.)

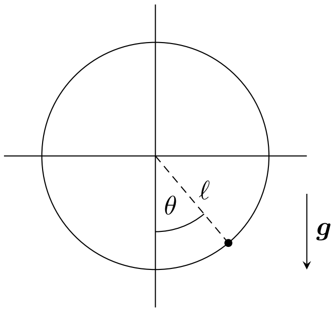

FIGURE 1 – Sketch of the idealised pendulum.

4. Ideal pendulum with friction. THe ideal pendulum, is a rigid pendulum of length ℓ under the influence of gravity and a friction force such as air pressure. The equation of motion is given by :

$$\ddot{\theta} + 2\gamma\dot{\theta} + \omega^2 \sin\theta = 0. \tag{4}$$

Here, $\gamma > 0$ is the coefficient of friction and $\omega = \sqrt{g/\ell}$.

(a) Write the equations of movement as a two dimensional system where $x_1 = \theta$ and $x_2 = \dot{\theta}$.

(b) Show that the equilibria of the system are of the form $\boldsymbol{p} = (n\pi, 0)$, $n \in \mathbb{Z}$. Why is it sufficient to consider only the points described by $n = 0$ et $n = 1$. Distinguish the physical meaning of these two equilibria.

(c) Show that the energy function $E = \frac{1}{2}x_2^2 + \omega^2(1 - \cos x_1)$ is a Lyapunov function of the equilibria $\boldsymbol{p} = (0, 0)$. Deduce that this equilibria is stable.

(d) Can we also deduce that $\boldsymbol{p} = (0, 0)$ is asymptotically stable ? Conclude about the dynamics of the system.

5. Bendixson Criteria. Show that the ODE $\ddot{x} = x - x^3 + (b - x^2)\dot{x}$ does not have periodic solutions for $b < 0$.