LPHYS2114 Non linear dynamics
Exercise 2 – Bifurcations. Linear systems.

Bifurcations

1. Evolution of fish populations. We will start by looking at a simple model of a fish population. If there is no fishing, the population $P = P(t) \geqslant 0$ follows the logistic model.

Simplistic model

The impacts of fishing is modelled by the term $-H$, added to the logistic equation where $H > 0$ is a constant removal of fish from the population :

$$\dot{P} = aP\left(1 - \frac{P}{N}\right) - H. \tag{1}$$

where $a > 0$ is the growth rate of the population and $N > 0$ is the population saturation.

(a) We will make a change of variables $P(t) = \mu x(\tau = \lambda t)$ to simplify the equation for further analysis, show this results in :

$$\dot{x} = x(1 - x) - h. \tag{2}$$

and find μ, λ et h.

(b) Graphically analyse this ODE as a function of h.

(c) Show there is a bifurcation at a critical value $h = h_c$ which we will determine. What kind of bifurcation is this ?

(d) Discuss the evolution of the fish population over the long term for $h < h_c$ and $h > h_c$.

(e) Discuss the weaknesses of this model.

Improved model

To improve the original model (1), we will consider a model where the decrease of the population through fishing varies with P. We will model this by ensuring the deduction goes to zero as P goes to zero, and reaches a constant $-H$ for large P. A possible model is given by :

$$\dot{P} = aP\left(1 - \frac{P}{N}\right) - \frac{HP}{B + P}. \tag{3}$$

with $B \geqslant 0$.

(f) Discuss the importance of parameter B.

(g) Show that there is a change of variables that allow the equation to be re-written as :

$$\dot{x} = x(1-x) - \frac{hx}{b+x} \tag{4}$$

where b and h are to be determined.

(h) Show that the ODE can have two or three equilibria depending on the values of b and h. Determine their stability. Remember that : $x \geqslant 0$.

(i) Analyse the dynamics in a neighbourhood of $x = 0$. Determine that a bifurcation takes place at $h = b$.

(j) Show that another bifurcation occurs when $h = \frac{1}{4}(1+b)^2$ at a value $b < b_c$ where b_c is to be determined.

(k) Analyse the stability of the system on a $b - h$ diagram.

Linear systems on the plane

2. Solutions of linear systems. For the matrices

$$\textit{(i)}\ A = \begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix} \quad \text{et} \quad \textit{(ii)}\ A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \tag{5}$$

Find the following :

(a) Find the eigan values and eigan vectors of A.

(b) Find the matrix T such that $\bar{A} = TAT^{-1}$ is a normal form.

(c) Find the general solution of $\dot{\bar{\boldsymbol{x}}} = \bar{A}\bar{\boldsymbol{x}}$ and $\dot{\boldsymbol{x}} = A\boldsymbol{x}$.

(d) Sketch the phase portrait of the two systems.

3. Harmonic oscillator with friction. We take the equations of a harmonic oscillator

$$\ddot{x} + 2\gamma\dot{x} + \omega^2 x = 0, \quad \omega > 0, \gamma \geqslant 0. \tag{6}$$

(a) Write the equation in the form a a linear 2-dimensional ODE $\dot{\boldsymbol{x}} = A\boldsymbol{x}$.

(b) For what values of ω and γ, the matrix A has eigan values that are *(i)* complex, *(ii)* real and distinct values, and *(iii)* repeating values ?

(c) Find the general solution of the system in any of the cases *(i)*, *(ii)*, and *(iii)*. Sketch the phase portraits for each case and describe the movement of the oscillator.

4. Classification of 2-dimensional linear systems. We take a generalised 2-dimensional linear system :

$$\dot{\boldsymbol{x}} = A\boldsymbol{x}, \quad \boldsymbol{x} \in \mathbb{R}^2. \tag{7}$$

where, A is a 2×2 real valued matrix. Let $D = \det A$ and $T = \operatorname{tr} A$.

(a) Find the eigan values of A in terms of D and T. Describe the valye of the eigan values according to the sign of T, D and $T^2 - 4D$.

(b) On the T-D plane, sketch a typical phase portrait in each of the regions that are separated by the curves $D = 0$, $T = 0$, and $T^2 - 4D = 0$.