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LPHYS2114 Non-linear Dynamics
Série 8 – Superstable fixed points, basins of attraction and conjugation

1. Newton-Raphson Method. Show that the fixed points of the map of the Newton-Rhapson method are superstable.

2. Basins of attraction. Given a function f on $\mathbb{R}$. The *basin of attraction*, of a fixed point p of the map, is defined by the collection of points x which converge towards p. The following theorem can help with finding a basin of attraction of p :

Theorem 1. *Given a continuous f on $\mathbb{R}$ and $a < b < c$.*

(i) If $f(b) = b$ and $x < f(x) < b$ for all $x \in [a, b[$ then $\lim_{n\to\infty} f^n(a) = b$.

(ii) If $f(b) = b$ and $b < f(x) < x$ for all $x \in]b, c]$ then $\lim_{n\to\infty} f^n(c) = b$.

The first part of this exercise consists in demonstrating this theorem. The second part uses the theorem for finding the basins of attraction of attracting fixed points.

Demonstrating the theorem

We will only demonstrate section *(i)*. (The demonstration of part *(ii)* follows in a similar manner.)

(a) Given $x_0 = a$ and $x_{n+1} = f(x_n)$, $n \geqslant 0$. Show that $(x_n)_{n=0}^{\infty}$ is a strictly increasing series. Show that it converges.

(b) Given that $p = \lim_{n\to\infty} x_n$. Justify that $a \leqslant p \leqslant b$. Deduce that $p = b$.

Applications

(c) Given $f(x) = \frac{4}{\pi} \arctan x$. Show that f has fixed points $p = 0, \pm 1$. Show that $p = 0$ is a source and $p = \pm 1$ are the attractors. Show that the basin of attraction of $p = \pm 1$ are $\{x \in \mathbb{R} : \pm x > 0\}$.

(d) Given $f(x) = ax(1 - x)$. Show that for $1 < a < 2$ the basin of attraction of the fixed point $p = 1 - 1/a$ is the open interval $]0, 1[$.

3. Conjugation. In this exercise we consider the maps that can be linked by conjugation.

Example

(a) Show that the logistic map $x_{n+1} = ax_n(1 - x_n)$ can be transformed into the map $y_{n+1} = y_n^2 + c$ by a linear change of variables $y_n = \alpha x_n + \beta$ with α, β which are to be determined.

(b) Show that we can write the relation between the maps in the form

$$C \circ f = g \circ C \tag{1}$$

where $C(x) = \alpha x + \beta$, $g(x) = x^2 + c$ et $f(x) = ax(1 - x)$.

Properties of corresponding maps

More generally, two functions f, g linked by a homeomorphism C according to (1) are called (topological) conjugates. The properties of the dynamics of the maps can be linked. Here we consider the relations between their periodic points.

(c) Show that f^k and g^k are conjugates by C for all integers $k \geqslant 1$.

(d) Show that p is a k-periodic point of f iff $C(p)$ is a k-periodic point of g.

(e) Show that if f, g, C are differentiable and if $C'(p) \neq 0$ then

$$(g^k)'(C(p)) = (f^k)'(p). \tag{2}$$

Application : Logistic map with $a = 4$

In this part we consider the logistic map $g(x) = ax(1-x)$ with $a = 4$ and *The tent map*

$$f(x) = \begin{cases} 2x, & 0 \leqslant x \leqslant 1/2, \\ 2(1-x) & 1/2 \leqslant x \leqslant 1, \end{cases} \tag{3}$$

on the interval $[0, 1]$. The graphs of the functions f, g are shown in Figure 1.

(f) Show that f and g are conjugates by $C(x) = \frac{1}{2}(1 - \cos \pi x)$.

(g) Show that the periodic orbits of g are repulsive. *Hint :* Start with $k = 1$ and $k = 2$, and then generalise.

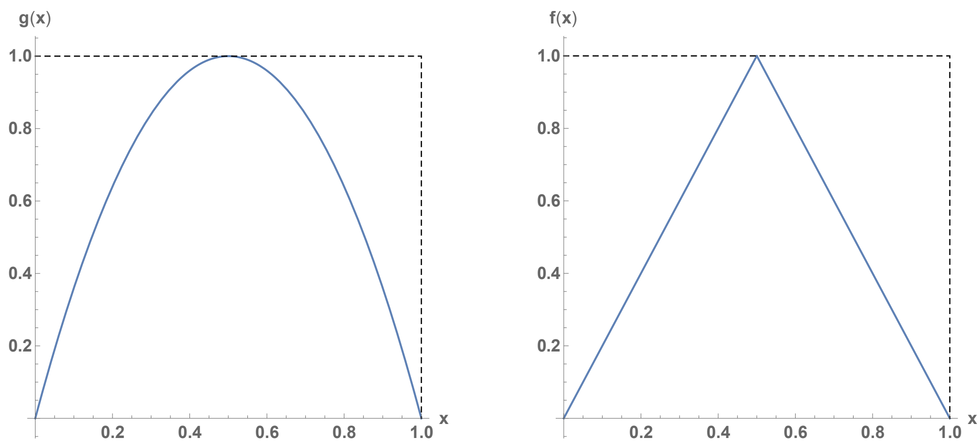

Figure 1 – Logistic map $a = 4$ (left) and the tent map (right).