LPHYS2114 Non-linear Dynamics
Série 9 – Chaotic Maps

One dimensional Maps

1. px mod 1. Given $\Omega = [0, 1[$ and $f : \Omega \to \Omega$ defined by $f(x) = px \mod 1$ where $p \geqslant 2$ is an integer.

(a) Study the map map by using the p-adique representation for $x \in \Omega$:

$$x = .b_1 b_2 b_3 \cdots = \sum_{n=1}^{\infty} \frac{b_n}{p^n}, \quad b_n \in \{0, 1, \ldots, p-1\}. \tag{1}$$

(b) Show that the map f is chaotic.

2. $(x + \alpha)$ mod 1. We will look at the interval $\Omega = [0, 1[$. To measure the distance between two points $x, y \in \Omega$, we use the metric $d(x, y) = \min(|x - y|, 1 - |x - y|)$. We can visualise Ω as a circle and d as the length of an arc (à un facteur près).

Given $f : \Omega \to \Omega$ the map defined by $f(x) = (x + \alpha) \mod 1$ where $0 \leqslant \alpha < 1$ is a real number.

(a) Show that if α is rational then the points in Ω are periodic. Deduce that there are no dense orbits in Ω.

(b) Show that if α is irrational then all the points have a dense orbit. Show that f does not have periodic points.

(c) Show that for all α, the map f does not have the property of sensitivity to initial conditions and conclude.

Maps in two dimensions

3. Bakers Map. Given $\Omega = [0, 1[\times[0, 1[$. We denote $\boldsymbol{x} = (x_1, x_2) \in \Omega$ are points. The map $f : \Omega \to \Omega$ of the Baker Map is given by $\boldsymbol{x}' = f(\boldsymbol{x})$ with

$$x_1' = 2x_1 \mod 1, \quad x_2' = \frac{1}{2}(x_2 + [2x_1]) \tag{2}$$

where $[\xi]$ is a est la partie entière de $\xi \in \mathbb{R}$.

Geometrically, the map can be visualised in two steps : *(i)* We transform Ω a rectangle of length 2 and height 1/2 by the transformation $(x_1, x_2) \to (2x_1, x_2/2)$. *(ii)* We cut the rectangle vertically into two rectangles of length 1 and height 1/2 and combine the two halves to regather the original square Ω. An illustration is given in Figure 1.

(a) Why is f called the “bakers map” ?

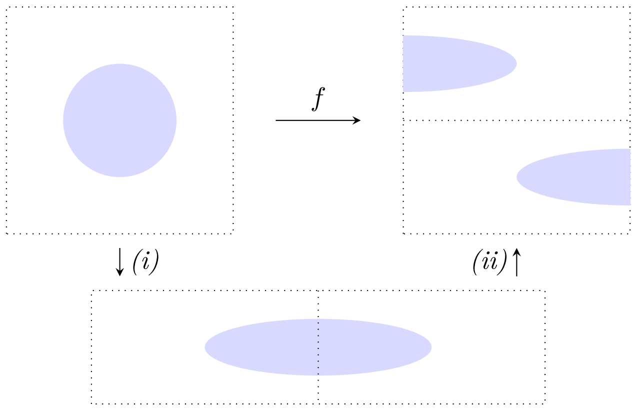

FIGURE 1 – An image of a disk of radius $r = 1/4$ centred on the square Ω under one transformation of the bakers map.

Dyadic Representation

We are interested in the map defined by this method. As the map $2x \mod 1$ on the interval, it is convenient to focus on f and use a dyadic representation. For $x_1 = .b_0 b_1 b_2 \cdots$ and $x_2 = .b_{-1} b_{-2} b_{-3} \cdots$ we write :

$$\boldsymbol{x} = (x_1, x_2) = \cdots b_{-3} b_{-2} b_{-1} . b_0 b_1 b_2 \cdots \tag{3}$$

In the series Dans la suite on écrira également $b_n = b_n(\boldsymbol{x})$ for the binary digits of $\boldsymbol{x}$.

(a) Find (x_1', x_2') and deduce that the action of f in this representation.

(b) Show that f is invertible on Ω and show that the action of f^{-1} in the representation (3). And deduce f^{-1} in cartesian coordinates.

Norm

To show that f is chaotic we must introduce a norm on the interval Ω which allows us to measure distances. We can recall that all on the plane all norms are equivalent. It is convenient to choose the Manhattan norm :

$$||\boldsymbol{x}|| = |x_1| + |x_2|, \quad \boldsymbol{x} = (x_1, x_2) \in \Omega. \tag{4}$$

(c) Given $\boldsymbol{x}, \boldsymbol{x}' \in \Omega$ and $N > 0$ an integer. Show that if $b_n(\boldsymbol{x}) = b_n(\boldsymbol{x}')$ for all $-N \leqslant n \leqslant N-1$ then $||\boldsymbol{x} - \boldsymbol{x}'|| \leqslant 2^{-(N-1)}$.

(d) Given E a sub-set of Ω. Show that E is dense in Ω if for all $\boldsymbol{x} \in \Omega$, $N > 0$ an integer, there exists $\boldsymbol{x}' \in E$ such that $b_n(\boldsymbol{x}) = b_n(\boldsymbol{x}')$ for all $-N \leqslant n \leqslant N-1$.

Chaos

(e) Show that the bakers map is chaotic.