Université catholique de Louvain — École de Physique
Faculté des Sciences — 7 November 2024

LPHYS2114 Nonlinear dynamics
Tutorial 5 – Poincaré-Bendixson Theorem

1. Existence of a limit cycle. Given the system

$$\dot{x} = x - y - x^3, \quad \dot{y} = x + y - y^3. \tag{1}$$

Show that for some $0 < r_1 < r_2$ there exists at least one periodic solution in the annulus defined by $r_1 \leqslant \sqrt{x^2 + y^2} \leqslant r_2$.

2. Calculate Perturbation of a 2D limit cycle. The goal of this exercise is to study the system of differential equations :

$$\dot{x} = x(1 - x^2 - y^2) - y + 2\mu x^2, \quad \dot{y} = y(1 - x^2 - y^2) + x + 2\mu xy. \tag{2}$$

Here, μ is a real parameter.

Equilibrium

(a) Show that $(x, y) = (0, 0)$ is a hyperbolic equilibrium.

(b) What kind of hyperbolic equilibrium is it ? Describe qualitatively the dynamics of the system in a neighbourhood around this point.

Existance of a limit cycle.

(c) Show that in polar coordinates r, θ, the system can be written as :

$$\dot{r} = r(1 - r^2) + 2\mu r^2 \cos\theta, \quad \dot{\theta} = 1. \tag{3}$$

(d) Deduce that $(x, y) = (0, 0)$ is the only equilibrium of the system.

(e) Show that for $\mu = 0$, the system has a single limit cycle. Find the limit set (ω limit) of the orbit of the limit cycle for all points $(x, y) \neq (0, 0)$.

(f) Show that for all $\mu \geq 0$, The system has at least one limit cycle. *Hint :* Find an attracting annulus centred on the origin as shown in 1.

Perturbation calculation

We are now interested in the case where $0 < |\mu| \ll 1$. In this case, we can try to find the alteration to the shape of the orbit that was found when $\mu = 0$ by a perturbation calculation. It is convinient to use the parametric form $r = r(\theta)$ and assume the existence of the the series :

$$r(\theta) = \rho_0(\theta) + \rho_1(\theta)\mu + \rho_2(\theta)\mu^2 + \ldots \tag{4}$$

(g) What is the value of $\rho_0(\theta)$?

(h) Find a differential equation for $r(\theta)$. Deduce a differential equation for $\rho_1(\theta)$.

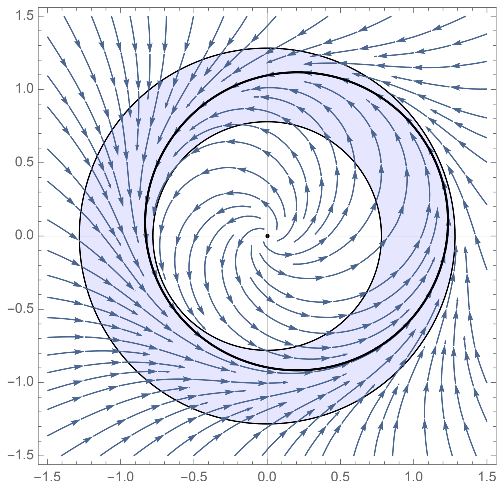

FIGURE 1 – A phase portrait of the system (2), with $\mu = 1/4$, in the neighbourhood of the equilibrium $(x, y) = (0, 0)$. The coloured annulus acts as a trapping or attracting neighbourhood. The solid line in the interior is an approximation of the limit cycle, estimated by a Taylor expansion of order 2, expanded on μ.

(i) Solve and find an approximation of $r(\theta)$ by a polynomial of order 1.

(j) Next, find a differential equation for $\rho_2(\theta)$ and deduce $\rho_2(\theta)$. The approximation of $r(\theta)$ by a polynomial of Taylor order 2 is shown in figure 1.

Hint : For the questions (i) and (j), we could use the indefinite integrals

$$\int \mathrm{e}^{a\theta} \cos b\theta \, \mathrm{d}\theta = \frac{\mathrm{e}^{a\theta}\,(a \cos b\theta + b \sin b\theta)}{a^2 + b^2} + C, \tag{5}$$

$$\int \mathrm{e}^{a\theta} \sin b\theta \, \mathrm{d}\theta = \frac{\mathrm{e}^{a\theta}\,(a \sin b\theta - b \cos b\theta)}{a^2 + b^2} + C, \tag{6}$$

where a, b are the constraints and C is an arbitrary integration constant.

3. The Sel'kov model. Glycolysis is a metabolic pathway for the production of energy in cells and the assimilation of sugar. Adenosine diphosphate (ADP) and fructose-6-phosphate (F6P) are involved in this process. The concentration of ADP in the cell is given by x and the concentration of F6P by y. According to the Sel'kov model, their evolution in time is given by the system of ODEs :

$$\dot{x} = -x + ay + x^2 y, \quad \dot{y} = b - ay - x^2 y, \tag{7}$$

where $a, b > 0$ are constants.

(a) Sketch the x- et la y-nullclines of the system.

(b) Deduce the equilibria of the system. Find a condition where the equilibrium is unstable.

(c) Consider the shape defined between the points $(0, 0)$, $(0, b/a)$, $(b, b/a)$, then a line with slope -1 is drawn from the latter point, to where it intersects the x-nullcline. Lastly a vertical line is drawn from this point to the x-axis. Show that this shape is a trapping region for the system.

(d) Next, deduce that a limit cycle exists in the interior of the shape, so long as the instability of the equilibrium found in part (b) is satisfied.

(e) In the $a - b$ plane, identify the regions where a stable limit cycle exists or the fixed point is stable.