LPHYS2114 Non-linear Dynamics
Série 3 – Non-linear Equilibriums

1. The Lotka-Volterra Model - A Species Competition Model. We will analyse a system the describes the competition between two species. We let x_1 and x_2 gives the population levels of the two species. To simplify this we assume x_1 and x_2 are given by real, positive values. According to the competition model of Lotka-Volterra the growth rates of the two populations are given by :

$$\frac{\dot{x}_1}{x_1} = a(1 - x_1) - bx_2, \quad \frac{\dot{x}_2}{x_2} = c(1 - x_2) - dx_1. \tag{1}$$

where $a, b, c, d > 0$ are parameters.

(a) Interpret the different terms in this system of differential equations.

It is more convenient to write the ODEs in the form $\dot{x}_1 = f_1(\boldsymbol{x})$, $\dot{x}_2 = f_2(\boldsymbol{x})$. For fixed $j = 1, 2$. We call the curve where $\dot{x}_j = 0$ the x_j*-nullcline.*

(b) For a general $f_1(\boldsymbol{x}),\ f_2(\boldsymbol{x})$ characterise the flow crossing the nullclines. Show that the equilibriums are found at the intersections of the x_1-nullcline and the x_2-nullcline.

Extinction of a species

We want to study the competition model of Lotka-Volterra for a particular case :

$$a = 1, \quad b = 2, \quad c = 1, \quad d = 3. \tag{2}$$

(c) Determine the nullclines of the system. Sketch the phase portrait.

(d) Describe the dynamics of the system in the long term, can the two species co-exist ?

Extinction or coexistence ?

We now consider the general case where $a, b, c, d > 0$. We want to determine if there are values of these parameters where the coexistence of the species is possible, or if one species goes extinct over time.

(e) Determine the nullclines. Find that there is one equilibrium with both populations having a non-zero population if *(i)* $a/b > 1$ and $c/d > 1$ where *(ii)* $a/b < 1$ and $c/d < 1$.

(f) Show that the positive equilibrium is not an attractor (sink) if we have the case given in *(i)*.

2. Hopf bifurcations. We consider the system of ODEs :

$$\dot{x}_1 = ax_1 - x_2 - x_1(x_1^2 + x_2^2), \quad \dot{x}_2 = x_1 + ax_2 - x_2(x_1^2 + x_2^2), \tag{3}$$

where $a \in \mathbb{R}$ is a parameter.

(a) Show that for all $a \in \mathbb{R}$, the system has only one equilibrium, which we will determine.

(b) Study the linearised system around this equilibrium for $a < 0, a = 0, a > 0$. Show there is a bifurcation at $a = 0$.

(c) Analyse the bifurcation in the non-linear system, write the equaionts in polar coordinates r, θ.

(d) Show that if we pass from $a < 0$ to $a > 0$ a periodic solution appears. Determine the orbit.

(e) This orbit is an example of a *limit cycle*. Justify this terminology. We call the bifurcation where limit cycles appear *Hopf bifurcations*.

(f) Sketch the phase portrait and describe the dynamics of the solutions in the long term for $a < 0$, $a = 0$ and $a > 0$.

3. Hamiltonian Systems. A Hamiltonian system in 2D is one that can be written in the form

$$\dot{x}_1 = \frac{\partial H(\boldsymbol{x})}{\partial x_2}, \quad \dot{x}_2 = -\frac{\partial H(\boldsymbol{x})}{\partial x_1} \tag{4}$$

where $H : \mathbb{R}^2 \to \mathbb{R}$ is a function of class C^2 called the *Hamiltonian function*.

(a) Find all the matricies $\boldsymbol{A}$ such that the linearised system $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$ is Hamiltonian. Determine a Hamiltonian function for the given $\boldsymbol{A}$.

(b) Show that the eigan-values of $\boldsymbol{A}$ are of the form $\pm\lambda$ (saddle point) where $\pm \mathrm{i}\lambda$ (centre) with $\lambda \in \mathbb{R}$. Find the phase portraits for these two cases.

(c) Let $\boldsymbol{p}$ is a equilibrium of the general Hamiltonian system. Characterise the eigan-values of $\mathrm{d}H(\boldsymbol{p})$. Deduce the dynamics of the system in the neighbourhood of p.

4. Reversible Systems. A 2 dimensional system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^2$, is called *reversible* if it is invariant under the changes $t \to -t$, $x_2 \to -x_2$.

(a) Show the reversibility implies $f_1(x_1, -x_2) = -f_1(x_1, x_2)$ and $f_2(x_1, -x_2) = f_2(x_1, x_2)$.

(b) Find that if $\boldsymbol{p} = (x_1^*, x_2^*)$ is an equilibrium of the system, then $\boldsymbol{p}' = (x_1^*, -x_2^*)$ is also an equilibrium.

(c) Show that if $\boldsymbol{p} = (x_1^*, x_2^*)$ is an equilibrium with $x_2^* = 0$ the eigan values of $\mathrm{d}f(\boldsymbol{p})$ are of the form $\pm\lambda$ (saddle point) or $\pm\mathrm{i}\lambda$ (centre) with $\lambda \in \mathbb{R}$. Can we deduce the dynamics of the system in the neighbourhood of $\boldsymbol{p}$?

We can demonstrait that if $\boldsymbol{p}$ is a centre of a reversible system, there exists a neighbourhood B of $\boldsymbol{p}$ such that all the orbits of the solutions in $B \backslash \{\boldsymbol{p}\}$ are closed curves.

(d) Give an intuitive explanation for establishing this property of reversible systems.