Université catholique de Louvain — École de Physique
Faculté des Sciences — 5 December 2024

LPHYS2114 Non Linear Dynamics
Série 10 – Tent map. Linear 1d maps

Tent map and the Cantor set

We will consider the tent map defined by :

$$f(x) = \begin{cases} 3x, & x \leqslant 1/2, \\ 3(1-x), & x \geqslant 1/2. \end{cases} \tag{1}$$

The graph of f is shown in Figure 1. We are interested in the collection of points x_0 from which orbits originate.

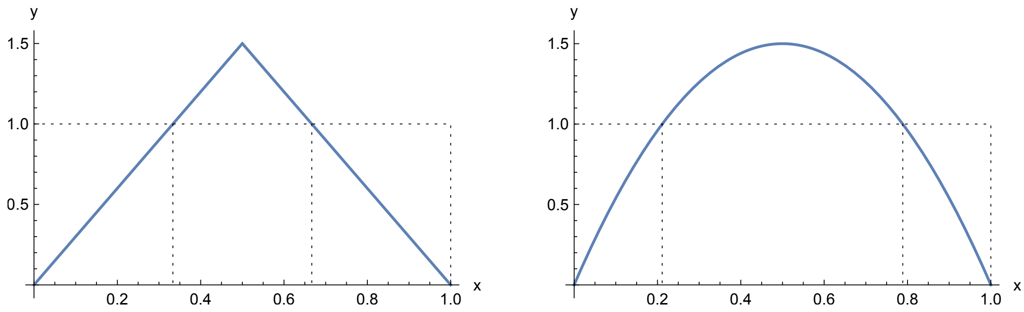

FIGURE 1 – *Left :* Graph of the tent map defined by (1). *Right :* Graph of the logistic map with $a = 6$.

1. Cantor set 1/3. It is convenient to introduce the Cantor set 1/3. To define it we consider the interval $[a, b]$ and a function T defined by

$$T([a,b]) = [a, a + (b-a)/3] \cup [b - (b-a)/3, b]. \tag{2}$$

For the union of disjoint intervals $I = \bigcup_{k=1}^{n} [a_k, b_k]$, we define $T(\bigcup_{k=1}^{n} [a_k, b_k]) = \bigcup_{k=1}^{n} T([a_k, b_k])$, i.e. T agit séparément sur chaque intervalle de l'union.

(a) Given $K_0 = [0, 1]$. We iteratively define $K_{n+1} = T(K_n)$, $n \geqslant 0$. Calculate K_1, K_2, K_3 and the et les tracer.

(b) The Cantor set is $K_\infty = \lim_{n\to\infty} K_n$. Show that this collection is not empty.

(c) Show that the sum $|K_n|$ of the length of the intervals of K_n is given by $|K_n| = (2/3)^n$. Deduce that K_∞ is the null measure.

The ternary representation of a point $x \in K_0$ is given by

$$x = .a_1 a_2 a_3 \cdots = \sum_{n=1}^{\infty} \frac{a_n}{3^n}, \quad a_n \in \{0, 1, 2\}. \tag{3}$$

On admet toutes les suites de chiffres ternaires, ce qui implique que la représentation n'est pas unique. En effet pour $a_n > 0$ on a $.a_1 \ldots a_n \bar{0} = .a_1 \ldots (a_n - 1)\bar{2}$.

(d) Montrer que $K_\infty = \{x = .a_1a_2a_3\ldots | a_n \in \{0,2\},\, n \geqslant 0\}$. En déduire que K_∞ n'est pas dénombrable.

(e) Établir que l'application $x \mapsto 3x$ est une bijection entre $[0,1/3] \cap K_\infty$ et K_∞. Expliquer dans quel sens il est alors autosimilaire.

2. Tent map. We now use the results from the previous question to study the tent map f.

(a) Prove that the orbit of points $x_0 \notin K_0 = [0,1]$ under the map f is not n'est pas bornée.

(b) Show that the points $x_0 \in K_0 \backslash K_1$ leave K_0 after one iteration. Deduce that all the orbits are not En déduire que leur orbite n'est pas bornée.

(c) Find all points which leave the set K_0 after n iterations.

(d) Show taht the set of points Montrer que l'ensemble des points dont l'orbite reste dans K_0 est K_∞. Conclure.

3. L'application logistique avec $a > 4$. We now will look at the logistic map with $a > 4$, shown in Figure (1) and the et de l'étude de l'application en tente, décrire l'ensemble des points x_0 dont l'orbite est bornée.

Linear iterations in the plane

4. A map on the plane. We consider the linear map $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(\boldsymbol{x}) = A\boldsymbol{x}$ with

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}. \tag{4}$$

(a) Calculate the eigan-values and eigan-vectors of A.

(b) Deduce the phase portrait of the map $\boldsymbol{x}_{n+1} = f(\boldsymbol{x}_n)$, $n \geqslant 0$.

5. Conserved Quantities. A function $E : \mathbb{R}^m \to R$ is called quantity conserving for the map $f : \mathbb{R}^m \to \mathbb{R}^m$ if $E \circ f = E$.

(a) Show that E is est constante le long les orbites de l'itération $\boldsymbol{x}_{n+1} = f(\boldsymbol{x}_n)$ définie par f.

(b) On considère l'application linéaire $f(\boldsymbol{x}) = A\boldsymbol{x}$ avec $\boldsymbol{x} \in \mathbb{R}^2$. Show that $E(\boldsymbol{x}) = x_1^2 + x_2^2$ is quantity conserving for

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{5}$$

(c) Sketch the phase portrait of the map defined by f.

(d) Find the non-trivial conserved quantities for

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{et} \quad A = \begin{pmatrix} 2 & 0 \\ 0 & 1/3 \end{pmatrix}. \tag{6}$$