Tutorial 5 – Systems in three dimensions. Poincaré L'application de Poincaré

Three dimensional systems

1. Rikitake model. The Rikitake equations are given by

$$\dot{x} = -\nu x + yz, \quad \dot{y} = -\nu y + (z-a)x, \quad \dot{z} = 1 - xy, \tag{1}$$

where $a, \nu > 0$ are parameters. These equations were proposed in 1958 as a model for the generation of the earth's magnetic field by the currents in the mantle.

(a) Show that the system is dissipative, meaning that volumes degrease over time.

(b) Show that the equilibriums are $(\pm k, \pm k^{-1}, \nu k^2)$ where k is a solution of $\nu(k^2 - k^{-2}) = a$. Classify these equilibria.

Numerical simulations show that the model exhibits chaotic behaviour for certain parameter values. This behaviour is similar to the irregular reversals in the earth's magnetic field. A solution of the model is shown in Figure 1.

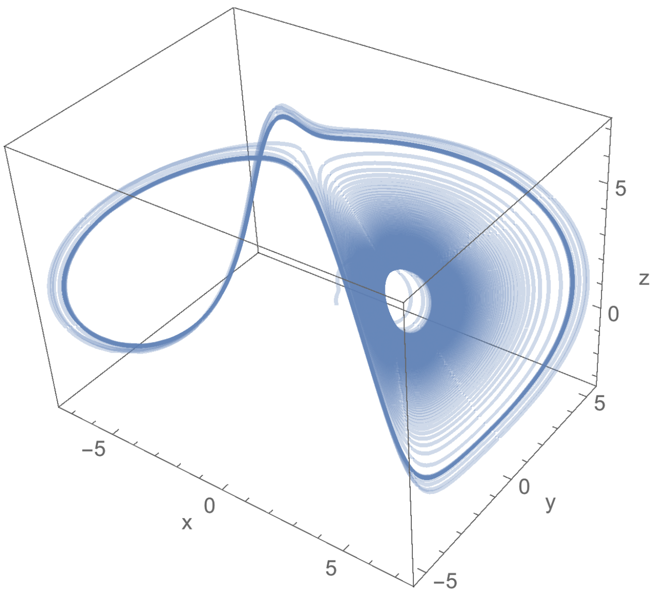

FIGURE 1 – Orbit of a solution of the Rikitake model with $a = 2, \nu = 1/6$ and and initial condition $(x_0, y_0, z_0) = (0, 1, 0)$.

2. Lorenz System for Large Reynolds Numbers. We will investigate the Lorenz system in the limit $r \to \infty$. By taking this limit in a particular way we can make the dissipative terms disappear.

(a) We will let $\epsilon = r^{-1/2}$. In the limit $r \to \infty$ we have $\epsilon \to 0$. Find a change of variables $x, y, z, t \to X, Y, Z, \tau$ dependant on ϵ as $\epsilon \to 0$ the lorenz system becomes :

$$\dot{X} = Y, \quad \dot{Y} = -XZ, \quad \dot{Z} = XY. \tag{2}$$

(b) Show that the new system is not dissipative : That it conserves volumes over time.

(c) Find two conserved quantities for the new system (these are $E = E(X, Y, Z)$ where $\dot{E} = 0$.) Discuss the consequences of the existence of the existence of these conserved quantities.

Poincaré Sections

3. Logistic equation with a periodic forcing. In this exercise we consider a model for the evolution of a population. The equation is given by $\dot{x} = f(t, x)$ with

$$f(t, x) = ax(1 - x) - h(1 + \sin 2\pi t) \tag{3}$$

where $a, h > 0$ are constants. The first term corresponds to growth rate given by the logistic equation. The second term represents a diminishing term of the population which varies with time t.

(a) Show that if $x(t)$ is a solution of the differential equation then $x(t + 1)$ is also a solution. Show that it is sufficient to constrain the solutions between $0 \leqslant t \leqslant 1$.

(b) Given $\phi(t, x_0) = \phi^t(x_0)$ is a unique solution with the initial condition x_0 at $t = 0$. We define a *Poincaré section* by $p(x_0) = \phi(t = 1, x_0)$. Show that if x_0 is a fixed point, i.e. $p(x_0) = x_0$ then the solution is a periodic solution of period $T = 1$.

(c) Show that

$$\frac{\partial \phi(t, x_0)}{\partial x_0} = \exp\left(\int_0^t f_x(s, \phi(s, x_0))\mathrm{d}s\right) \tag{4}$$

where $f_x(t, x) = \partial f(t, x)/\partial x$ is a partial derivative of f with respect to x. Show that $p'(x_0) > 0$.

(d) By differentiating (4) with respect to x_0, and show that $p''(x_0) < 0$. Show that the Poincaré section has at least two fixed points. *Hint :* It is useful to make $p(x_0)$ as a function of x_0.

(e) Show that $p(x_0)$ is dependant on the value of h. Deduce that there exists a unique value $h = h_c$ where this Poincaré section that has exactly one fixed point. For $h > h_c$, no fixed point exists. What is the significance for the evolution of the population ?

The behaviour of the solutions for $h < h_c$ is shown in Figure **??**.

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