Contents

1. Motivation and structure

1.1. Foreword

The course LPHYS2114 has now been given for quite a number of years by Prof. Christian Hagendorf, to the great satisfaction of the students. The lecture has been passed on to me. I am a physicist specialised in climate dynamics, and I have been working dynamical systems as models of the climate since 2006. My approach tends to be pragmatic, as I am ultimately interested in the behaviour of the real world system that I am attempting to model with dynamical systems. So will be this lecture. The divide between *continuous* dynamical systems and *discrete* dynamical systems has been conserved, as well as many exercises and the overall evaluation structure. I would like to warmly acknowledge Prof. Hagendorf's for his help and collaboration in preparing this lecture.

As I am giving this course for the first year, adjustments are likely and I at time of writing I have not yet been able to provide a fully detailed weekly program. The basis is 10 2-hour lessons (this including 3 spare weeks in the 13-week official program), plus weekly exercise sessions organised by Victor Couplet.

In preparing this lecture, I have relied on several reference books. Although the current notes are meant to be the formal material upon which evaluation is based, the circumstances are such that writing is work in progress and students are most welcome to check them and provide corrections or suggestions. This text and related material is provided as a free git repository.

- S.H. Strogatz, Nonlinear dynamics and chaos. Westview Press (2015).
- S. Wiggins, Introduction to Applied Nonlinear Dynamical Systems and Chaos, Springer (2003)

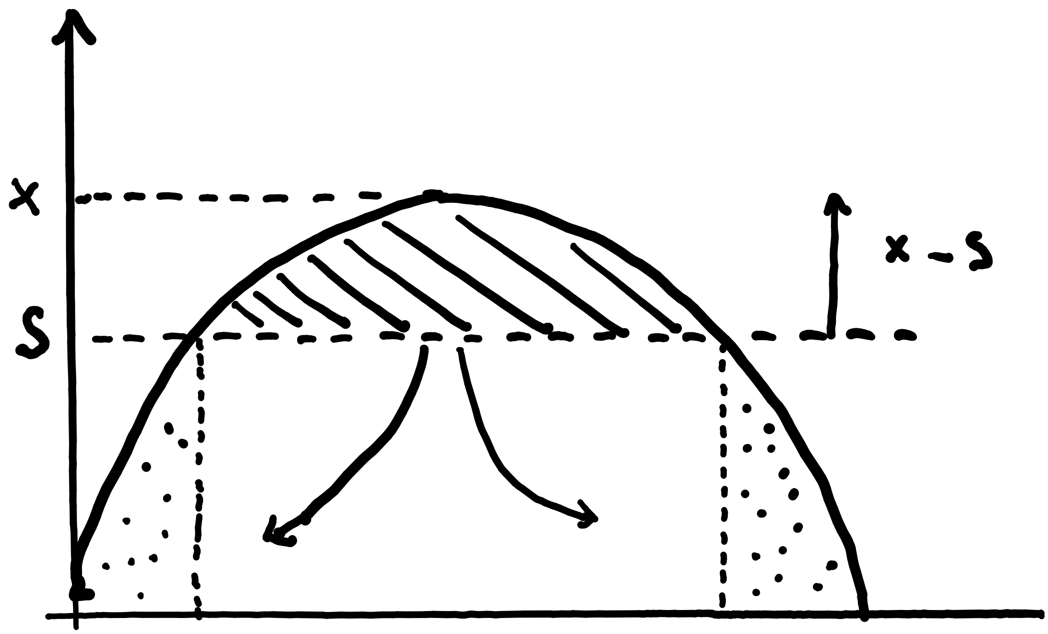
- R. Hilborn, Chaos and Nonlinear Dynamics: An Introduction for Scientists and Engineers (2nd edn) , Oxford University Press (2000)
- H. Dijkstra, Nonlinear Physical Oceanography, A Dynamical Systems Approach to the Large Scale Ocean Circulation and El Niño, Springer Science+Business Media (2000)

1.2. Motivating example: ice sheet

To motivate the course and its objective, consider the following example from glaciology.

An ice sheet is a large accumulation of ice. The largest ice sheets take thousands of years to millennia to form. If the climate is stable enough (say, it is *constant*), the ice sheet may reach a *stable equilibrium*.

This equilibrium results from a balance between *accumulation* of snow above the *snowline* (which is determined by the external climate conditions), and the *ablation* of ice that is being pushed below the snow line.



Intuitively, we may perceive that if climate warms a little, the altitude of the ice sheet will decrease (implying less net accumulation), but the flow of ice towards the ice sheet will be reduced as well, so that a new balance will be reached.

However, we may already anticipate that if the climate warms *too much*, and the top surface of the ice sheet will drown under the snow-line, at which point we expect a catastrophic meltdown of the icesheet.

We witness here a non-linear phenomenon. There is what some might call a *tipping point*, a point of warming above which the qualitative behaviour of the object changes. In this course, we would like to use mathematical modelling and a mathematical language to describe this behaviour, and many others. We will be introduced to the theory and learn the techniques that allow us to describe the expected behaviour of such systems.

The standard theory is the theory of *dynamical systems*. In essence, a dynamical system is the combination of a state space, say Ω, and a *rule* that determines the evolution of every point of the state space.

This is perhaps a bit abstract, so consider again our ice sheet example.

Call the altitude of the summit of the ice sheet x. This is positive real number, so $x \in \mathbb{R}^+$. The altitude of the snowline is S. We consider that it is constant, so in the following we will view it as a *parameter*.

Here, we have *summarised* the state of our real-world system (the ice sheet) with a single variable x. That variable evolves in $\mathbb{R}^+$. So in our case the Ω space is $\mathbb{R}^+$. Now we need a rule for the evolution of an ice sheet that would have an altitude x.

One (standard) approach is to write a *differential equation*. Simply, we equate $\frac{\mathrm{d}x}{\mathrm{d}t}$ with a function of the state x, and the parameter S. At this point, we need a bit of physical intuition. x will evolve as a balance between accumulation and ablation. The difference between accumulation and ablation is called *net accumulation*. One (simplistic) way of putting it, is that net accumulation is proportional to the distance between x and S. It would be an equation of the kind

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x - S \tag{1}$$

Class room discussion: Consider an initial condition $x_0 = x(t_0)$. Finding x_t for $t > t_0$ is an initial value problem, which is a particular case of a Cauchy problem. Find the solutions. Show that their behaviour differ depending on the initial condition. Explain also what is wrong with this model. How can we fix it ?

After discussion, we see that we need a *negative feedback* to prevent ice to grow until infinity when it starts above the snowline. We won't discuss too much the phenomenological details here, but say that the negative feedback is related to the flow of ice towards the ice sheet ablation zone, which is enhanced when the ice sheet grows very big. So our final equation would resemble:

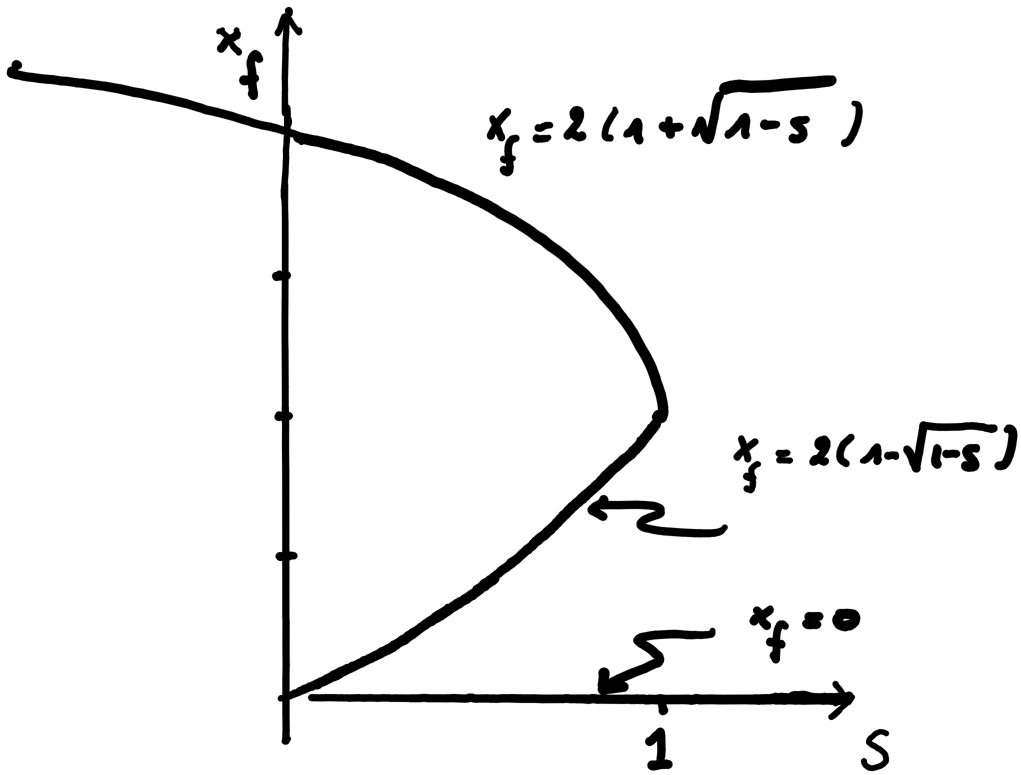
$$\frac{\mathrm{d}x}{\mathrm{d}t} = x - ax^2 - S \text{ if } x > 0, \text{ and } \frac{\mathrm{d}x}{\mathrm{d}t} = \max(0, -S) \text{ if } x = 0, \tag{2}$$

where a is a parameter. Again, we look after qualitative behaviours, and for (mathematical) simplicity we will set $a = 1/4$. In practice we cannot be so relaxed about fixing parameters, but this is for the sake of the demonstration.

What is going on now ?

Class room discussion: A fixed point x_f of the differential equation is an element of the domain Ω such that the system is invariant at that point. That is, if $x(t_0) = x_f$ for a given $t = t_0$, then $x(t) = x_f$ for any time t of the time domain. How do you find the fixed points associated with the dynamical system in (2) ?

After discussion, we find that the number of fixed points depends on the value of the parameter S: one fixed point for $S < 0$ $2(1+\sqrt{(}1-S))$, one fixed point for $S > 1$ (x=0), and three fixed points between these values (which ones ?)



Now we would like to determine the behaviour of the system for initial conditions between these points. One approach would be to resolve the ordinary differential equation (2). For $S = 0$ it is doable (hint: use the separation of variables: put all dx terms on one side; the dt terms on the other side; integrate and solve for x) but it is cumbersome and the strategy will actually not work in most cases. In other words, most non-linear ordinary differential equations have no analytical solutions.

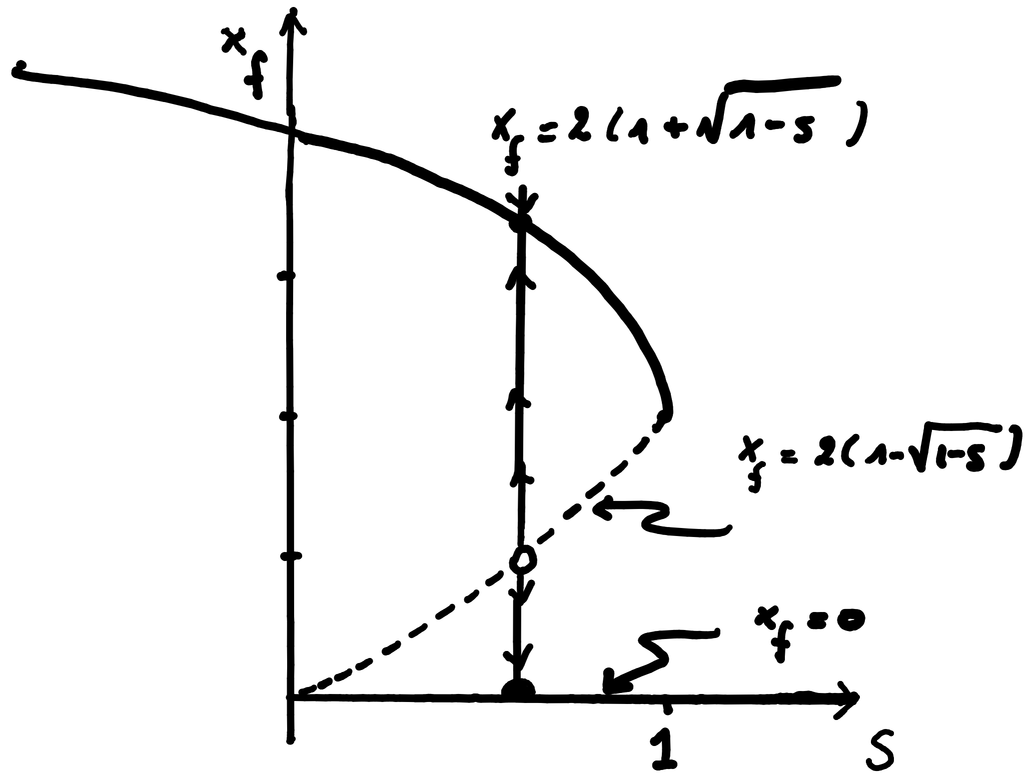
Hence, a more fruitful strategy is to study the behaviour of (t) *near* (in French: *dans le voisinage the*) fixed points, and use theory to connect the flowlines between fixed points.

The model we have been starting with is of the form $\frac{\mathrm{d}x}{\mathrm{d}t} = F(x;\psi)$ with, here, $\psi := \{S\}$. By definition of a fixed point, $F(x_f;\Psi) = 0$ (the dependency on ψ is dropped for clarity). Define $\delta x := x - x_f$.

$$\frac{\mathrm{d}x - x_f}{\mathrm{d}t} \doteq \frac{\mathrm{d}\delta_x}{\mathrm{d}t} = \underbrace{\left.\frac{\partial F(x;\psi)}{\partial x}\right|_{x_f}}_{\lambda} \delta x + \mathcal{O}(x^2) \tag{3}$$

This is a linear differential equation for δx with constant coefficient. That is, near enough to the fixed point, δx decays (if $\lambda < 0$) or grows (if $\lambda > 0$) exponentially with e-folding time $1/\lambda$. This distinguishes a point that is *locally stable* from a point that is *locally unstable*.

Class room discussion: Reconsiders the one to three solutions of the bifurcation diagram. Which ones are stable, and which one unstable ? What we see appearing are *stable* and *unstable* solution *branches*.



Hence, even though we have avoided to resolve the ordinary differential equation; (we have avoided to solve the initial value problem), but provided that we have identified all fixed points, and characterised their local stability, we have gained a good qualitative picture of the system's behavior. For any value of S, we can picture the *flow* associated with the dynamical system, that is, for any point x of the domain Ω, we know the direction taken by x as time progresses.

This is an example of qualitative analysis of a non-linear dynamical system, which, as we see it here, is reasonably simple but not simplistic. We have identified *invariant sets*, that is, sets of points that are left unchanged by the flow. In this example, invariant sets are fixed points. We have identified *bifurcation points*, that is, points of the parameter space S where the number or the stability of the invariant sets (again, here, fixed points).

In this first lecture, meant to motivate the course, we have been informal and did not justify our findings with theorems; nor did we attempt to be too systematic and rigorous with definitions. But we understand our objectives: identify the nature of invariant sets, estimate their stability, and depict the behaviour of the system between these invariant sets.

Specifically, the lecture will be divided into two broad sections:

- continuous dynamical systems (expressed as ordinary differential equations)
- discrete dynamical systems (expressed as iterations)

Dynamical systems will be deterministic (but informally, from time to time I will mention the interest and possibilities brought about by stochastic dynamical systems), and we will not go beyond three dimensions.

1.3. Evaluation

Exercises will be evaluated and provide a *fifth* of the final evaluation that can no longer be modified. There will be a *written* exam (June and September sessions) that is focused on exercises. Exams of the previous years are available on Moodle. They will be focused on applications, knowledge of the theory being instrumental in solving the problems.

2. Continuous Flows: existence and uniqueness

2.1. Definition of trajectories, orbit, and linear stability

(based on [Wig03], section 7)

Consider the following dynamical system:

$$\dot{x} \stackrel{\text{def}}{=} \frac{\mathrm{d}x}{\mathrm{d}t} = f(x,t), \text{ with } x \in \Omega \tag{4}$$

Definition (Autonomous dynamical system) The system is said to be *autonomous* if there is no explicit dependence of f on t, and non-autonomous otherwise.

For simplicity, we will admit that $\Omega = \mathbb{R}^n$, where n is the phase-space dimension.

1. A *trajectory* $x(t, t_0, x_0)$ (or *phase curve*), for $t = I$ is a solution of the differential equation (4), passing through x_0 at time t_0 over an interval of existence I.
2. An orbit through x_0 is the set of points in phase space passing of a trajectory passing through x_0.

2.2. Equilibrium solutions, stability and linearised stability

In this section we briefly formalise and extend notions that have already been covered in the *Motivating example* section.

We consider an *autonomous* vector field

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x), \quad x \in \mathbb{R}^n$$

A fixed point (also said: *equilibrium solution*) is a point $x_f \in \mathbb{R}^n$ such that $f(x_f) = 0$.

We distinguish the Lyapunov stability from the asymptotic stability. Roughly speaking, x_f is *stable* (shorthand for *Lyapunov stable*) if solutions starting close enough to x_f remain near it. It is *asymptotically stable* if these solutions eventually converge to x_f.

Definition (Lyapunov Stability) The fixed point x_f is (Lyapunov) stable if, given $\varepsilon > 0$, there exists a $\delta > 0$ such that, for any other trajectory $y(t)$ satisfying $|x_f - y(t_0)| < \delta$, then $|x_f - y(t_0)| < \varepsilon$ for $t > t_0$, $t_0 \in \mathbb{R}$.

Definition (Asymptotic Stability) A fixed point x_f is asymptotically stable if it is Lyapunov stable *and* for any other solution $y(t)$ there exists a constant $b > 0$ such that, if $|x_f - y(t_0)| < 0$, then $\lim_{t\to\infty} |x_f - y(t)| = 0$.

Class room discussion: Illustrate these notions graphically

We now reconsider the notion of linear stability, based on the developments already started in the Motivation section.

Consider a fixed point x_f, and let $y = x_f + \delta x$.

We find that

$$\frac{\mathrm{d}\delta x}{\mathrm{d}t}_j = \sum \frac{\partial f_j}{\partial x_i} \delta x_i + \mathcal{O}(|\delta x|^2),$$

where the derivatives are evaluated in x_f, which we more simply write

$$\frac{\mathrm{d}\delta x}{\mathrm{d}t}_j = \mathrm{D}f(x_f)\delta x + \mathcal{O}(|\delta x|^2).$$

The notation $\mathrm{D}f(x_f)$ refers to the Jacobian of f evaluated in x_f. If $n = 1$ (one-dimensional system), then $\mathrm{D}f$ is simply $\frac{\mathrm{d}f}{\mathrm{d}x}$, which corresponds to the example in the Motivation section.

Given that for the stability of fixed points we are only concerned with the behaviour of solutions arbitrarily close to x_f, it seems reasonable to inspect the linear system:

$$\frac{\mathrm{d}\delta x}{\mathrm{d}t}_j = \mathrm{D}f(x_f)\delta x,$$

We will inspect more carefully the solutions of this system in the plane in section 4.1 but we may already anticipate that solutions will grow exponentially in the direction of eigenvectors of the Jacobian with positive eigenvalues, and decay exponentially in the direction of eigenvectors with negative eigenvalues.

Definition (Hyperbolic fixed point.) A fixed point is hyperbolic if all the eigenvalues of $\mathrm{D}f(x_f)$ have either a strictly positive or a strictly negative real part.

We will be able to *prove* that a fixed point with all *negative* eigenvalues (said to be linearly stable) is asymptotically stable. But we leave it for later. At this point, we already know enough to understand the idea of sink and source:

1. A hyperbolic fixed point is called a *sink* if all eigenvalues of the Lyapunov spectrum have negative real parts.

2. A hyperbolic fixed point is called a *source* if all eigenvalues of the Lyapunov spectrum have positive real parts.
3. A hyperbolic fixed point is called a *saddle* if some but not all eigenvalues of the Lyapunov spectrum have positive real parts
4. A non-hyperbolic fixed point with purely imaginary eigenvalues, and non-zero is a *center*.

Class room discussion: Consider again the motivating example with the ice sheets. Are the fixed points hyperbolic? Everywhere ? (tip: it is enough to restrict the discussion to the case $x \neq 0$.)

2.3. Properties of vector fields: existence, uniqueness, differentiability and flows

In this section we develop a technical apparatus to deal wit the notions of "long term" and "observable" behaviours of *orbits* of *dynamical systems*. This will later allow us to develop the notions of *attracting sets* and *attractors*. Again, we restrict the discussion to dynamical systems in the form of vector fields, that is:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x,t), \text{ with } x \in \Omega \tag{5}$$

and, again, for simplicity, we will admit that $\Omega = \mathbb{R}^n$. Here, we further suppose that $f(x,t)$ is $\mathbf{C}^r$-differentiable, with $r \geq 1$ on the open set $U \subset \mathbb{R}^n \times \mathbb{R}$.

Theorem (Existence) Let $(x_0, t_0) \in U$. Then, there *exists* a solution of (5) through the point x_0 at $t = t_0$, denoted $x(t, t_0, x_0)$ for $|t - t_0|$ sufficiently small. The solution is unique. Moreover, $x(t, t_0, x_0)$ is a $\mathbb{C}^r$ function of t, t_0, and x_0.

The proof is available in specialised books (e.g. Hirsch and Smale, 1974).

It has also been proved that the unique solution can be (uniquely) *extended* to the boundaries of any closed, compact subset of U. However, this says nothing about what is going on once the solution has reached the boundaries !

Class room discussion: Consider the following example: $\frac{\mathrm{d}x}{\mathrm{d}t} = x^2, \quad x \in \mathbb{R}$. Find the analytical solution, and show that the solution blows up and that the interval of existence of solutions through x_0 at $t = 0$ depends on x_0. Tip: use the methods of variable separation.

There in another handy theorem, that says that if $f(x, t, \psi)$ is differentiable with respect to the parameter ψ, then the solution is also differentiable with respect to that parameter.

2.4. Special properties of autonomous fields

We now consider two important propositions that apply to autonomous dynamical systems, and that we will (at last!) be able to prove. For simplicity, we consider a dynamical system admitting solutions over all times.

Theorem (Time shifted trajectories are trajectories in autonomous fields.) If $x(t)$ is a solution of $\frac{\mathrm{d}x}{\mathrm{d}t} = f(x)$, then so is $x(t+\tau)$ for any τ.

Proof: By definition, $\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f\left(x\left(t\right)\right)$. Hence, we have

$$\left.\frac{\mathrm{d}x(t+\tau)}{\mathrm{d}t}\right|_{t=t_0} = \left.\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right|_{t=t_0+\tau} = f\left(x\left(t+\tau\right)\right) = \left. f\left(x\left(t_0+\tau\right)\right)\right|_{t=t_0}.$$

To pack this up:

$\left.\frac{\mathrm{d}x(t+\tau)}{\mathrm{d}t}\right|_{t=t_0} = f\left(x\left(t+\tau\right)\right)\big|_{t=t_0}$ is true for any $t_0 \in \mathbb{R}$. In other words, $x(t+\tau)$ is a solution of the dynamical system. $\square$

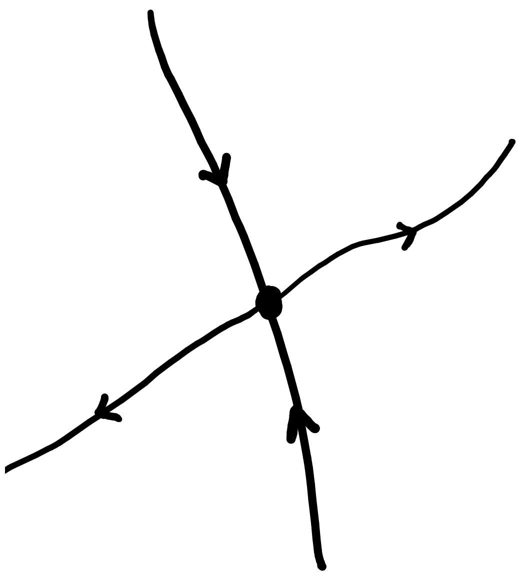
The formal maths make it a bit confusing, but the intuition is reasonably clear: two time-shifted trajectories that pass through the same point *correspond to the same orbit* !

This has an important implication: in an autonomous dynamical system, there is *one single* orbit that passes through any point x_0. There is a more formal proof in [Wig03], but the intuition seems to be reasonably clear from the uniqueness theorem.

In other words,

Theorem (Non-crossing orbits) Orbits *never cross* each other in an autonomous dynamical system.

Class room discussion: Consider the following flow diagram. How would you prove that the intersection is necessarily a fixed point? What does it say about the time a trajectory will need to reach the fixed point?



2.5. Flows

At this point, we have understood (at least intuitively) that the future fate a point in the phase space, in an autonomous system, does not depend on the time at which the snapshot has been taken. This is after all quite straightforward given the definition of an autonomous system (the evolution dictated by f does not depend explicitly on time). In other words, for any point (x_0, t_0), there will be a function $\phi^\tau(x_0)$ that points to the point of the orbit reached at time $t + \tau$, as long as the trajectory has remained within the bounds of the domain. This function ϕ is the *flow function*, sometimes called the *phase flow* function.

Given the uniqueness of solutions, we find that $\phi^{\tau_1+\tau_2} = \phi^{\tau_2} \circ \phi^{\tau_2}$ and $\phi^0 = \phi^{\tau_1-\tau_1} = Id$..

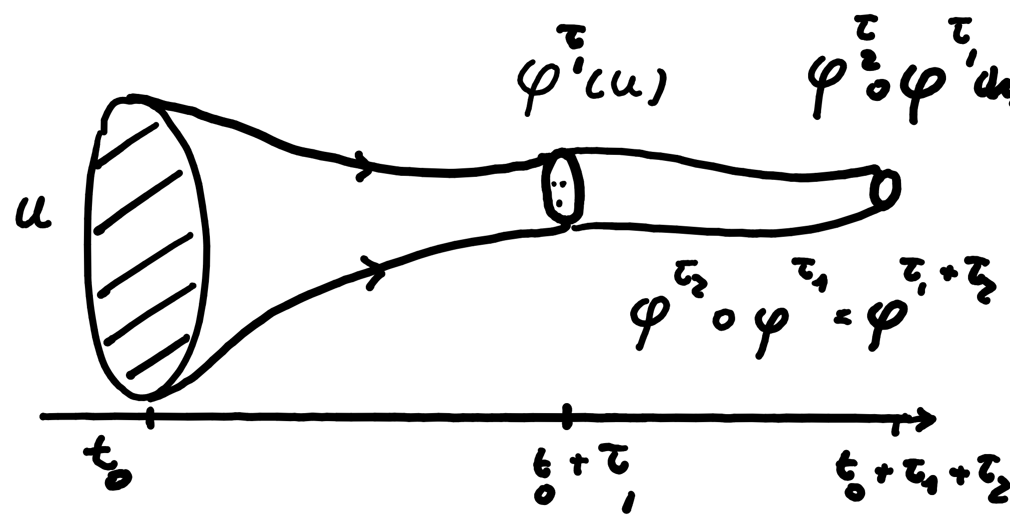
From where it comes that the flow is an invertible function:

$$(\phi^{\tau})^{-1} = \phi^{-\tau}.$$

In our notation we have been a little ambiguous about the domain of ϕ, something programmers will not like (one should always specify the type of inputs of a function). Above we have applied ϕ^{τ} to elements of the phase-space domain. However, we could also map the function on subsets U of this domain, as illustrated on the figure below:

$$\phi^{\tau_1+\tau_2}(U) = \phi^{\tau_2} \circ \phi^{\tau_1}(U).$$

Again, this flow function is invertible.



So the flow function provides an intuitive impression that is reminiscent of a fluid flow. It may be converging; diverging but streamlines never cross each other. At this point, we may already perceive that dynamical systems will fall in different categories.

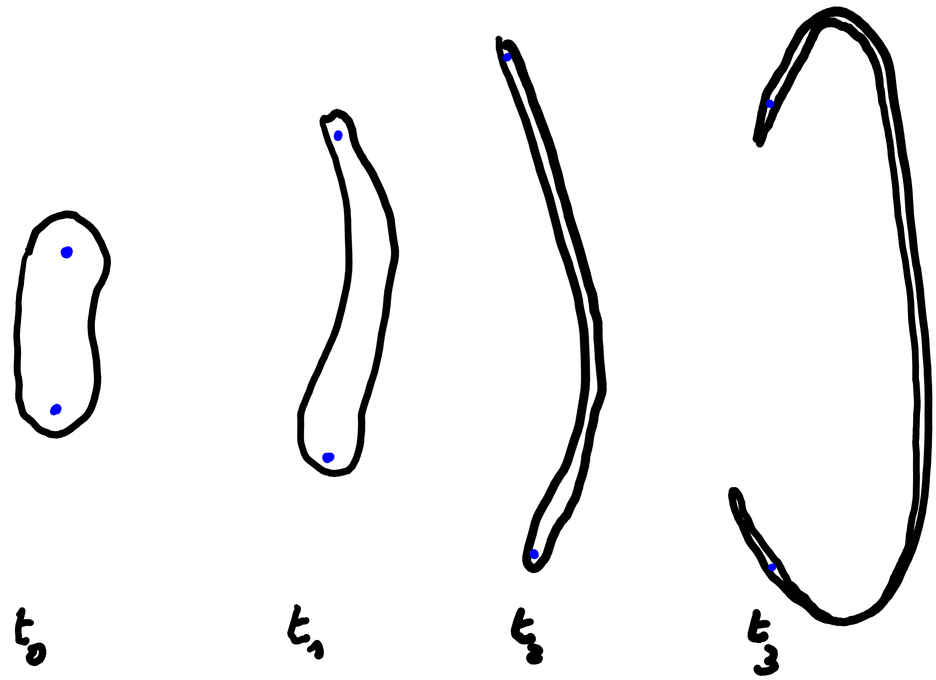
1. Those associated with flow functions that (on average) diverge. Trajectories tend to spread over. Such flows can be considered to be globally unstable because trajectories tend to grow towards the limits of the domain.
2. Those associated with flow functions that (on average) converge. Trajectories tend to cluster towards attracting regions (a notion that still needs to be formalised). These flows are called *dissipative*.
3. Those associated with flow functions that neither diverge nor converge. The phase-space volume of U is conserved through time.

Class room discussion: The latter category represents the class of *Hamiltonian flows*. Can you guess why? Tip: think of Liouville's theorem.

Class room discussion: At this point we may already have some intuitions about how the eigenvectors and eigenvalues of the Jacobian of f governs the properties of the flow. Which conditions do we expect for a converging flow? What happens if at least one eigenvalue is positive? The eigenvalues of the Jacobian are the *Lyapunov spectrum* and the individual values are called the *Lyapunov exponents*. Why this term: "exponent". In non-linear dynamics we tend to pay attention to the *largest Lyapunov exponent*. Can you guess why ?

As we will make it clear in the following, *dissipative flows* are characterised by a *negative Jacobian trace*: the sum of eigenvalues is negative; $\sum \lambda_i < 0$.

The figure below represents a *dissipative* flow that has at least one positive Lyapunov exponent. The phase space volume is shrinking over time (meaning that the *sum* of eigenvalues is negative, even though one is positive). However, because one eigenvalue is positive, the flow gets stretched in one direction, so that to initially close initial conditions get increasingly distant with each other as time grows.



This is the basic explanation to the possibility of sensitive dependence to initial conditions, that is, chaos.

One generally distinguishes dissipative chaos (in dissipative flows) from Hamiltonian chaos (in Hamiltonian flows). Poincaré first discovered sensitive dependence to initial conditions in the 3-body problem, which is characterised by a Hamiltonian flow. The meteorologist Edwar Lorenz is famous for having popularised the notion of dissipative chaos. We will come back to these notions later on.

3. Vector fields on the line: bifurcations and normal forms

We now consider the autonomous dynamical system on the line, with parameter ψ which, for simplicity, we view as one-dimensional.

3.1. Monotonous character of the trajectories

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x;\psi), \text{ with } x \in \mathbb{R},\ \psi \in \mathbb{R}, \tag{6}$$

with f continuous and differentiable over x and ψ.

We start with a remarkable properties of the trajectories:

Theorem (No local extrema.) Trajectories of (6) do not have local maxima and minima.

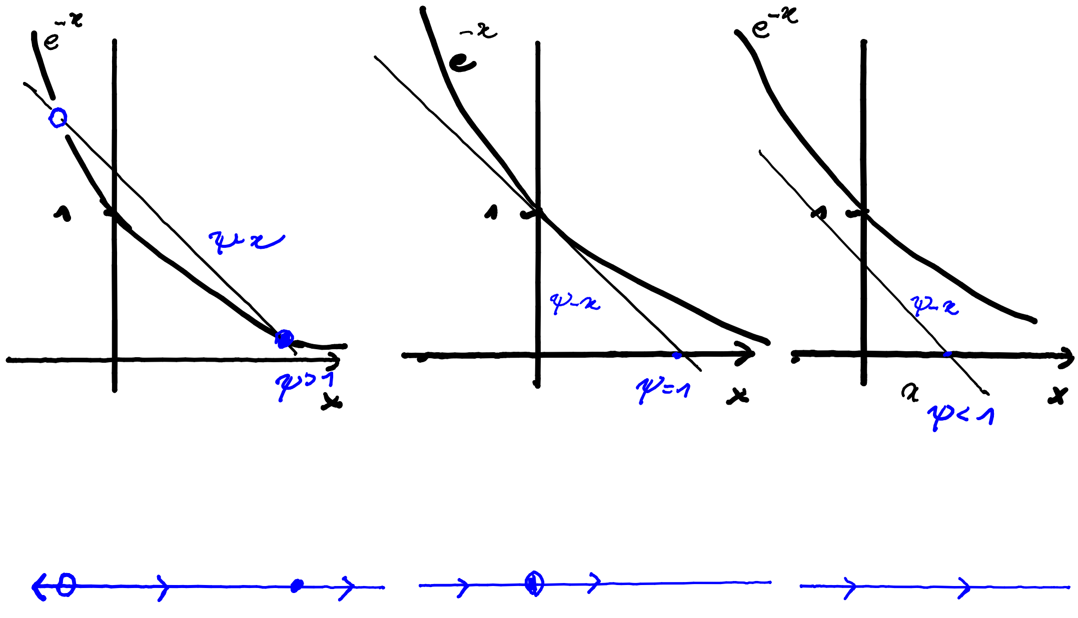
Proof: If there was a local extrema at time t, $x(t)$ at that point would need to have a zero-derivative, it must therefore be a fixed point; thus the trajectory must be constant and it cannot have local extrema. $\square$

It comes up that trajectories on the line must either be constant (fixed point), strictly increasing (towards infinity, or towards a fixed point), or strictly decreasing (ditto).

3.2. The saddle-node bifurcation

We now consider changes in ψ that may affect the existence or the *nature* of the fixed point. To this end, consider the following example: $f(x;\psi) = \psi - x - e^{-x}$.

The fixed point may be visualised graphically, with the equation $e^{-x} = \psi - x$. This is the *phase portrait*.



As is clear from the figure showing the phase portraits, the number of fixed points changes for 2 hyperbolic fixed points for $\psi > 1$, to one non-hyperbolic fixed point at $\psi = 1$, to zero for $\psi < 1$. Hence, the parameter $\psi = 1$ marks a change in the topology of the flow. This is a singularity that is called a *bifurcation point*.

In this section we will learn to *detect* bifurcation points in vector fields on the line and classify them.

To this end, let us continue with the example with have just started. The bifurcation occurs at $\psi = 1$ and we would like to study the behaviour of f near this bifurcation point.

To this end, we develop the function according to Taylor (recall that we have taken care of its differentiability):

$$
\begin{aligned}
f(x;\psi) =& f(x_0;\psi_0) + \left.\frac{\partial f}{\partial \psi}\right|_{x_0,\psi_0}(\psi-\psi_0) + \left.\frac{\partial f}{\partial x}\right|_{x_0,\psi_0}(x-x_0)+ \\
& \frac{1}{2}\left.\frac{\partial^2 f}{\partial x^2}\right|_{x_0,\psi_0}(x-x_0)^2+ \\
& \frac{1}{2}\left.\frac{\partial^2 f}{\partial \psi^2}\right|_{x_0,\psi_0}(\psi-\psi_0)^2+ \\
& \frac{1}{2}\left.\frac{\partial^2 f}{\partial \psi \partial x}\right|_{x_0,\psi_0}(x-x_0)(\psi-\psi_0) \\
& + \dots \\
=& \underbrace{\psi - 1 - \frac{1}{2}x^2}_{\text{normal form}} + \dots
\end{aligned}
$$

Hence, if we develop the vector field around the bifurcation point, it takes a specific form that depends on the first and second-order derivatives on x and ψ. This specific form is called a *normal form*, and the normal form characterises the bifurcation.

Specifically, the normal form $f(x;\psi') = \psi' - \frac{1}{2}x^2$ ($\psi' := \psi - 1$) is characteristic of the *saddle-node* bifurcation (also called a *fold bifurcation*): it is the collision of a stable and an unstable fixed point, with both vanishing beyond the bifurcation point. We have met this exact bifurcation in the motivating example.

Theorem (Sufficient conditions for a saddle-node bifurcation on the line.) A saddle-node bifurcation occurs on x_0, ψ_0 when:

1. $f(x_0;\psi_0) = 0$;
2. $\frac{\partial f}{\partial x}(x_0;\psi_0) = 0$;
3. $\frac{\partial^2 f}{\partial x^2}(x_0;\psi_0) \neq 0$;
4. $\frac{\partial f}{\partial \psi}(x_0;\psi_0) \neq 0$;

The intuition beyond this theorem is reasonably clear. The first condition imposes the existence of the fixed point; the second one is the non-asymptotic character of the fixed point (neither linear stable nor linearly unstable). The third and fourth conditions set the locally parabolic character of the solutions near the bifurcation point.

More formally, from the implicit function theorem, (1) and (4) imply that there must exist a function $\psi(x)$ on an interval $I \subset \mathbb{R}$ such that $f(x, \psi(x)) = 0$ and $\psi(x_0) = \psi_0$.

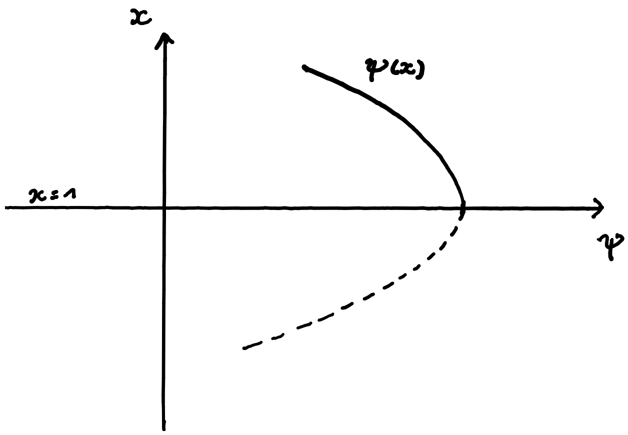
This is the curve of fixed points, as a function of ψ. Furthermore, its derivatives are:

$$\frac{\mathrm{d}\psi}{\mathrm{d}x} = -\frac{\frac{\partial f(x,\psi(x))}{\partial x}}{\frac{\partial f(x,\psi(x))}{\partial \psi}};$$

The assumptions (1) and (4) imply that $\frac{\mathrm{d}\psi}{\mathrm{d}x} = 0$. And

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} = -\frac{\frac{\partial^2 f(x,\psi(x))}{\partial x^2}\frac{\partial f(x;\psi(x))}{\partial \psi} + \frac{\partial f(x,\psi(x))}{\partial x}\frac{\partial f(x;\psi(x))}{\partial \psi \partial x}}{\left(\frac{\partial f(x;\psi(x))}{\partial \psi}\right)^2},$$

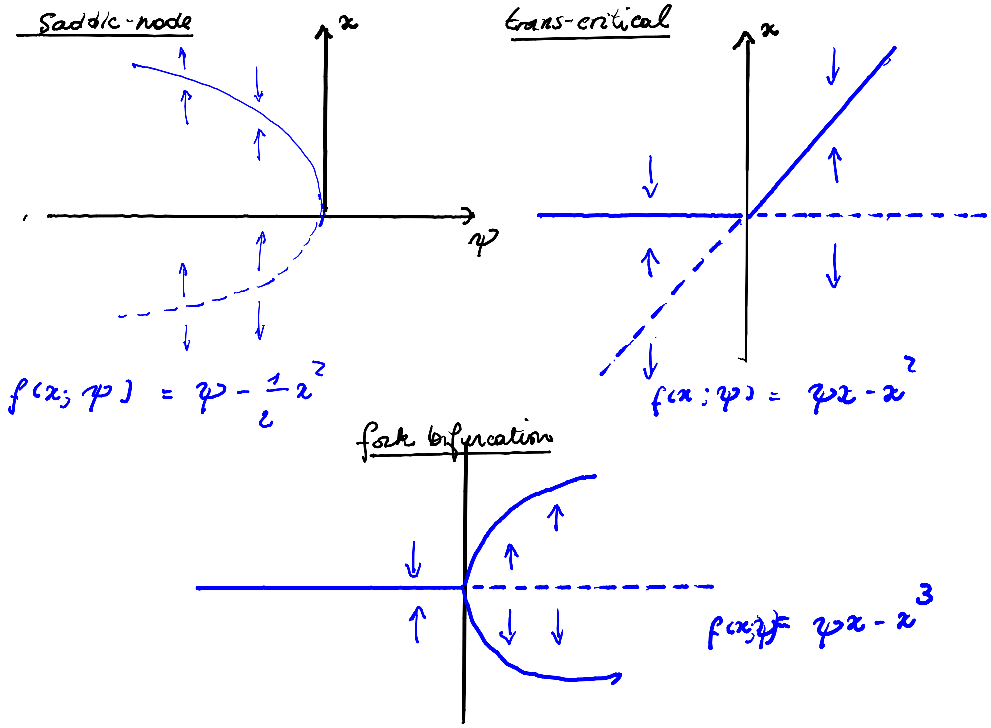
which then must be different from zero from (3) and (4). Both indicate that the solution curve $\psi(x)$ has a minimum or a local maximum in $x = x_0$ and therefore must bear the characteristic of a saddle-node bifurcation, as shown on the following graph:



□

3.3. Summary of the bifurcations on the line

- saddle node: two fixed points collapsing and annihilating each other
- trans-critical: two fixed points exchanging stability
- fork : one fixed point losing stability, sending trajectories towards two adjacent fixed points.



Class room discussion: Think of physical examples corresponding to these bifurcations.

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4. Vector fields on the plane: stable and unstable manifolds

4.1. Linear fields

4.1.1. General solution

We consider the vector field defined in $\mathbb{R}^2 \to \mathbb{R}^2$, with $\boldsymbol{x} \in \mathbb{R}^2$. In the particular case of a linear diffeomorphism, the vector field takes a matrix form, as follows:

$$\dot{\boldsymbol{x}} = A\boldsymbol{x}.$$

The Jordan canonical form theorem tells us that, if A is a real two by two matrix, there is a non-singular real matrix T such that $A = TJT^{-1}$, and T is made of the column vectors $\boldsymbol{u}$ and $\boldsymbol{v}$. Depending on the eigenvalue and eigenvectors of A, J will be of one of three forms:

- *Case A*: Two independent eigenvectors, real eigenvalues : $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$;
- *Case B*: Complex eigenvalues $\alpha \pm i\beta$: $A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$;
- *Case C*: A single independent eigenvector, with one single complex eigenvalue $A = \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix}$.

Geometrically, T is the composition of a rotation, dilation and reflection around the origin.

4.1.2. Case A: Real eigenvalues and linearly independent eigenvectors : invariant manifolds and linear transient growth

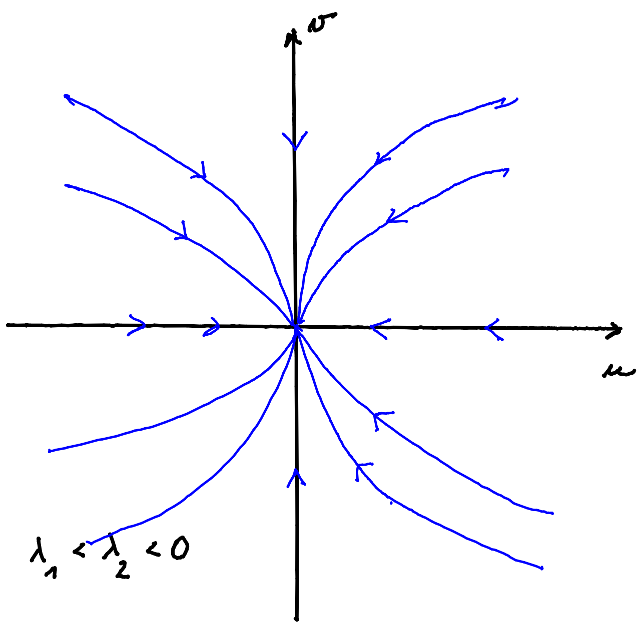
The general solution to the ordinary differential equation is

$$\boldsymbol{x}(t) = C_1 e^{\lambda_1 t}\boldsymbol{u} + C_2 e^{\lambda_2 t}\boldsymbol{v},$$

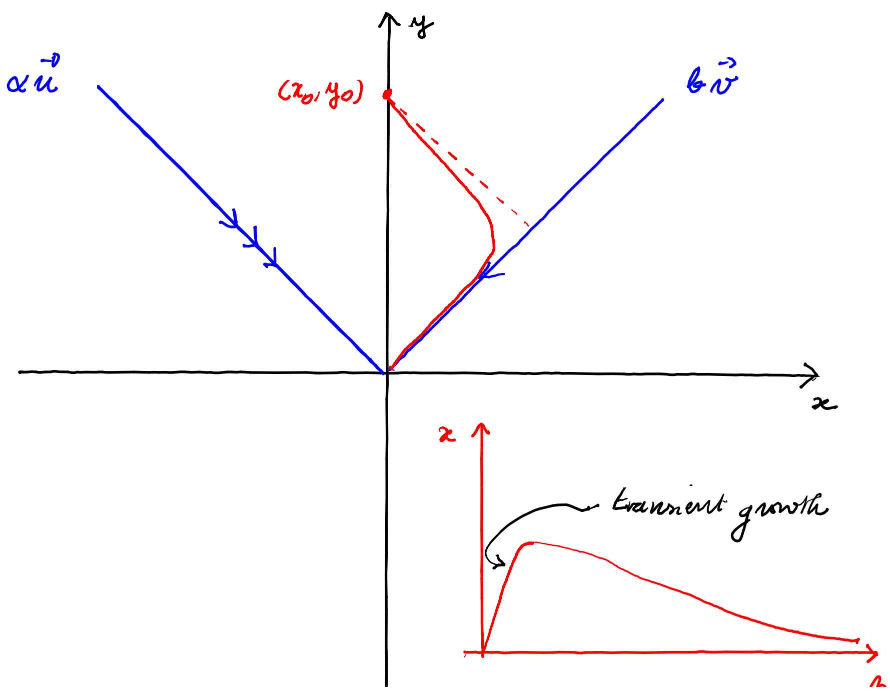
where $\boldsymbol{u}$ and $\boldsymbol{v}$ are the eigenvectors, and λ_1 and λ_2 the respective eigenvalues.

The fixed point $(0, 0)$ is asymptotically attracting if it is a hyperbolic fixed point with $\lambda_{1,2} < 0$. Furthermore, for an initial condition along the eigenvector u, that is, $\boldsymbol{x}_0 = \alpha \boldsymbol{u}$, the particular solution is $\boldsymbol{x} = \alpha e^{\lambda_1 t} \boldsymbol{u}$. In other words, in the particular case of an initial condition along an eigenvector, the orbit is a line.

In other words, the lines defined by $\alpha \boldsymbol{u}$ and $\beta \boldsymbol{v}$, $\alpha, \beta \in \mathbb{R}$ are invariant manifolds: if you are on the line, the flow keeps you on the line. We can already have some intuitive grasp on what is going on, and observe that the manifold might be unstable (the flow propels you to infinity), or stable (the flow asymptotically brings you to the fixed point $(0, 0)$, depending on the eigenvalue.



In general, even if the eigenvalues are real, the orbit is *not* a line. Consider the case of a hyperbolic sink ($\lambda_1, \lambda_2 < 0$). One can view that $-\lambda_1$ and $-\lambda_2$ act as the inverse of attracting time scales. For example, $\lambda_1 \gg \lambda_2$ imply that the trajectory will be quickly attracted towards the $u-$component of the trajectory will vanish quickly, so that it will be quickly attracted towards the manifold associated with the second eigenvalue. The $\beta\boldsymbol{v}$ will act as a *slow manifold*. As it turns out, depending on the initial condition, an observer that would observe one component (say, the first component) may observe *transient growth* before reaching the asymptotic limit $(0, 0)$.



4.1.3.Case B: Complex eigenvalues : spiral and center

The frictionless, linear harmonic oscillator is defined by (after a suitable choice of length and time units):

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}.$$

This is a typical example of linear vector field with purely imaginary eigenvalues $\lambda_{1,2} = \pm i$.

The general solution is $q = C_0 e^i(t + \phi_0)$ and $p = \dot{q}$.

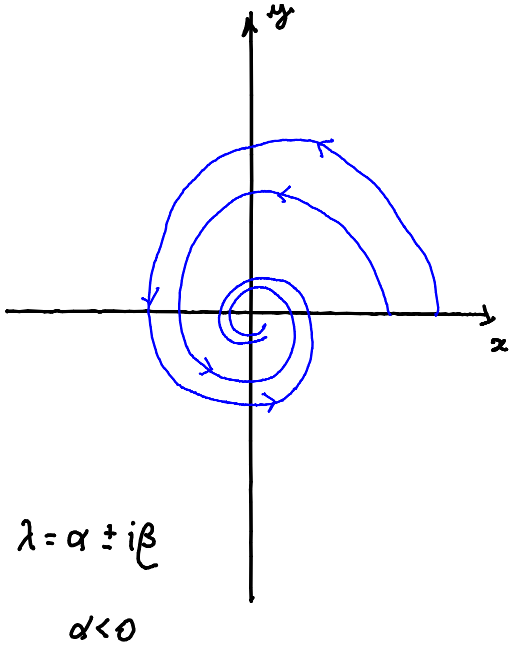
Class room discussion: Can you see that you now have an *infinity* of invariant manifolds? Which are they? This situation is quite typical of Hamiltonian (conservative flows).

Now consider the slightly more realistic case of a harmonic oscillator with friction coefficient ν.

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -\nu \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}$$

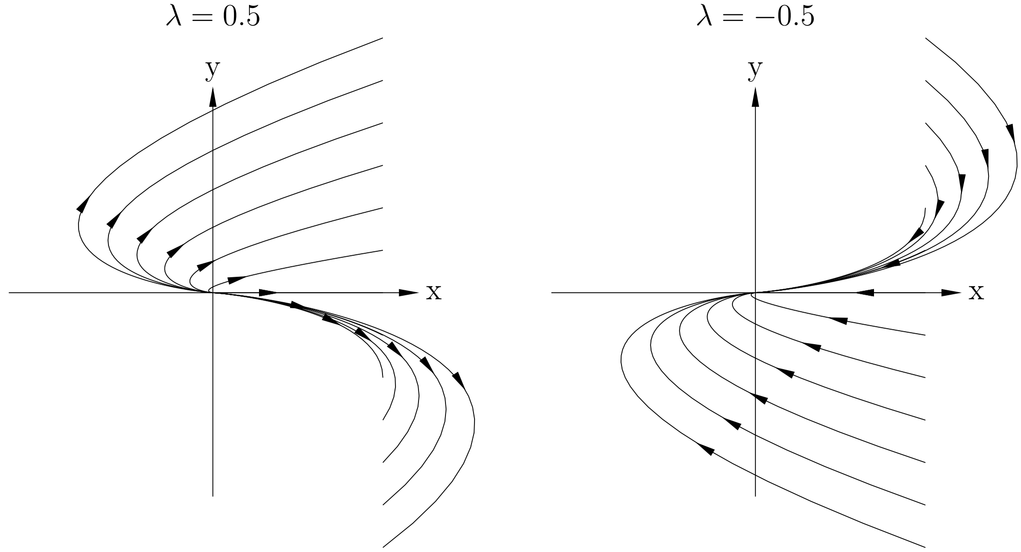
The eigenvalues are found with the characteristic equation $\lambda^2 + \lambda\nu - 1 = 0$. The eigenvalues are $\lambda = \frac{1}{2}(-\nu \pm \sqrt{\nu^2 - 4})$. The weakly damped harmonic oscillator $\nu < 2$ will thus be characterised by a damped oscillation, associated with two eigenvalues that are complex conjugate.

As we see it, the *conservative* harmonic oscillator is associated with a non-hyperbolic fixed point that is a *center*. The more general damped oscillator is associated with a hyperbolic fixed point, asymptotically attracting, that is *spiraling*.



4.1.4. Case C: Jordan form matrix

In this case, the general solution of the trajectories is $\boldsymbol{x}(t) = (C_1 e^{\lambda t} + C_2 t e^{\lambda t})\boldsymbol{u} + C_2 t e^{\lambda t}\boldsymbol{v}$.



Class room discussion: Can you see that for linear fields to generate bounded trajectories within $\mathbb{R}^2$, fixed points must be either hyperbolic sinks, or centers? Sources will generate trajectories flowing to infinity.

4.2. Non-linear vector fields in the plane

4.2.1. Topological equivalence with the linearised system around hyperbolic fixed points

As we see now, non-linear dynamics are needed to combine hyperbolic sources with bounded behaviour, which, in the plane, will typically generate attracting orbits. We will review at a later stage the dynamics of attracting orbits, but for the moment we would like to take advantage of what we have learned about hyperbolic sources and sinks in the linear regime. The theory allows us to show that *near* a hyperbolic fixed point, the non-linear flows behaves *like* a linear system with the same eigenspectrum.

To get there, we start with an example that, unusual fact, can be solved analytically.

$$\dot{x} = x + y^2,$$
$$\dot{y} = -y.$$

The fixed point is $(x_0, y_0) = (0, 0)$. Writing, more generically, the system as:

$$\dot{x} = f(x, y),$$
$$\dot{y} = g(x, y)$$

we find that we can *linearise* the system as follows (writing $x' = x - x_0 = x$):

$$\dot{x} = f(x_0, y_0) + \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} x' + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} y' + N_x(x, y), \tag{7}$$
$$\dot{y} = g(x_0, y_0) + \left.\frac{\partial g}{\partial x}\right|_{(x_0,y_0)} x' + \left.\frac{\partial g}{\partial y}\right|_{(x_0,y_0)} y' + N_y(x, y), \tag{8}$$
$$\tag{9}$$

where $N_{x,y}$ include all the non-linear terms. The *tangent linear system is*, per definition:

$$\dot{x} = f(x_0, y_0) + \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} x' + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} y',$$
$$\dot{y} = g(x_0, y_0) + \left.\frac{\partial g}{\partial x}\right|_{(x_0,y_0)} x' + \left.\frac{\partial g}{\partial y}\right|_{(x_0,y_0)} y',$$

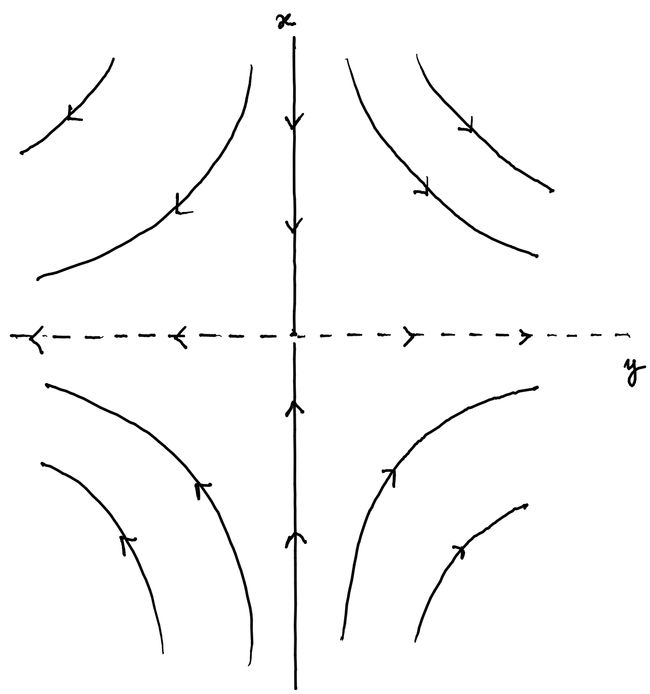
or, written in a more compact form, and considering that, by definition of the fixed point, $f(x_0, y_0)$ and $g(x_0, y_0) = 0$:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathrm{D}(f,g) \begin{pmatrix} x \\ y \end{pmatrix},$$

with $\mathrm{D}(f,g)$ the Jacobian of (f,g) evaluated at the fixed point. In the specific example chosen here,

$$\mathrm{D}(f,g) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It is already diagonal, with eigenvectors along the x and y axes, and eigenvalues $\lambda_1 = 1$ and $\lambda_2 1$. The phase portrait is thus fairly straightforward:



You identify without much difficulty the stable and unstable manifolds.

Let us come back to our full system. The equation for $\dot{y}$ admits, as solution, $y(t) = C_2 e^{-t}$. Plugged into tho $\dot{x}$ equation, we have $\dot{x} = x + C_2^2 e^{-2t}$. The *homogeneous* equation $\dot{x} = x$ admits, as solution, $x^H(t) = C_1 e^t$. One approach for finding a particular solution to the inhomogeneous system is to make the constant variable, that is try $C(t)$ in lieu of C_1:

$$\begin{aligned}
\dot{C}e^t &= C_2^2 e^{-2t}, \\
\Leftrightarrow C(t) &= -\frac{1}{3} C_2^2 e^{-3t}, \\
\Leftrightarrow x^I(t) &= -\frac{1}{3} C_2^2 e^{-2t}.
\end{aligned}$$

The general solution is thus $x(t) = C_1 e^t - \frac{1}{3} C_2^2 e^{-2t}$.

Starting from any point (x_0, y_0) at time $t = 0$, we have $C_2 = y_0$, $C_1 = x_0 - \frac{1}{3} y_0^2$.

In particular, if $y_0 = 0$, then $y_t = 0$ and $x(t) = x_0 e^t$. Hence, along the x axis, the system behaves *exactly* as the linear system (!). The line $y = 0$ is the unstable (invariant) manifold.

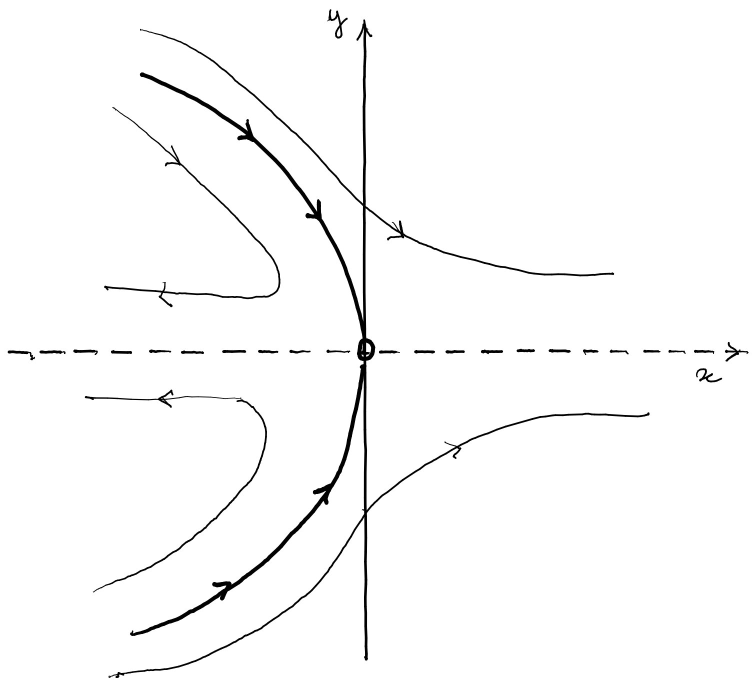
Now, contrary to the linearised system, the line $x_0 = 0$ is *not* an invariant manifold, as, starting from $x_0 = 0$,

$$x(t) = -\frac{1}{3} y_0^2 (e^t + e^{-2t}).$$

However, if we are a tiny bit smarter and arrange for C_1 to be 0, that is, $x_0 + \frac{1}{3} y_0^2 = 0$, then the system evolves as:

$$x(t) = y_0^2 e^{-2t},$$
$$y(t) = y_0 e^{-t},$$

on which the relationship $x_0 + \frac{1}{3}y_0^2 = 0$ is maintained. In other words, the curve defined by $x_0 + \frac{1}{3}y_0^2 = 0$ is the stable invariant manifold.



With this particular non-linear system, about which we could find the analytical solution, we observe that the invariant manifolds of the linear system are *tangent* and have the same stability characteristics as those of the linear tangent system. This is great news, because it suggests that we can learn quite a great deal about the non-linear system from inspection of the much simpler linear system, near the fixed point.

This is not *always* the case, as demonstrated by the following example:

$$\dot{x} = -y + ax(x^2 + y^2), \tag{10}$$

$$\dot{y} = x + ay(x^2 + y^2). \tag{11}$$

Lacking time, full inspection of the system is left as an exercise, but rushing through the lessons to be learned here, one would find that the *non-linear* system spirals downwards or upwards depending on the value of a (negative or positive), while the *linear system* is the Hamiltonian system that we have already studied above and admits circular orbits.

So what is the trouble here? The key for establishing a qualitative equivalence between a linear and a non-linear system is the *hyperbolic* character of the fixed points, as established by the Hartman-Grobman theorem (1959 and 1960):

Theorem (Hartman-Grobman theorem) Consider $p = (0, 0)$ a *hyperbolic* fixed point of the ordinary differential equation $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^2$, and $\boldsymbol{f} \in \mathbf{C}^k(\mathbb{R}^2), k \geq 1$. Then, there is a neighbourhood B around $p = (0, 0)$ and a homeomorphism $h : B \to R^2$ with $h(0, 0) = (0, 0)$ such that, $\forall \boldsymbol{x} \in B$, there is an interval $I \subset \mathbb{R}$ such that:

$$h \circ \phi^t(\boldsymbol{x}) = \bar{\phi}^t \circ h(\boldsymbol{x}), \quad \forall t \in I, \boldsymbol{x} \in B,$$

where ϕ^t is the flow of $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ and $\bar{\phi}^t$ is the flow of the linearised system $\dot{\boldsymbol{x}} = \mathrm{D}(f, g)\boldsymbol{x}$.

It is said that the flows ϕ^t and $\bar{\phi}^t$ are *topologically conjugate.*

The topological equivalence between linearised systems and non-linear systems around the *hyperbolic* fixed points imply that we can predict the existence of stable and unstable invariant manifolds and their tangent directions, from inspection of the eigenvalues and eigenvectors of the Jacobian.

We have a *sink* if all eigenvalues are negative real parts, a *source* if they all have positive real parts, and a *saddle point* if some are negative and negative, provided that none is null in which case the fixed point is no longer hyperbolic. This theory, including the Hartman-Grobman theorem, works also for $n \geq 2$.

Specifically, we define:

Definition (Stable local manifold) The stable local manifold around p in the neighbourhood B is
$S_B = \left\{x \in B : \phi^t(x) \in B \forall t \geq 0\right\}$.

Similarly,

Definition (Unstable local manifold) The unstable local manifold around p in the neighbourhood B is
$U_B = \left\{x \in B : \phi^{-t}(x) \in B \forall t \geq 0\right\}$.

Because of the way the definition is put, fixed points (even if they are saddle points or sources) belong to both the local stable and unstable manifolds. This is a bit counter-intuitive, so pay attention to this.

4.2.2 Stable-unstable manifolds theorem

Let us see one more consequence of the Hartman-Grobman theorem.

If p is a (hyperbolic) sink, then there must be a neighbourhood B around p such that $S_B = B$ and $U_B = \{p\}$. Can you see this? Indeed, in this neighbourhood, the flow is topologically conjugate to that of the linearised system. We now that it is a *sink*, attracting from all directions. The stable manifold *is* the whole neighbourhood. But as we have just seen, $\{p\}$ must also belong to the unstable manifold.

If p is a (hyperbolic) source, then, $U_B = B$ and $S_B = \{p\}$.

We will now formulate an important theorem, not demonstrated here, that will allow us to obtain more information about the shape of the stable and unstable manifolds around the fixed points.

Theorem (Stable and unstable manifolds theorem) Consider $f : \mathbb{R}^2 \to \mathbb{R}^2$, a diffeomorphism $\in \mathbf{C}^k$, $k \geq 1$, and $p = (0, 0)$ a *hyperbolic* equilibrium of the ordinary differential equation $\dot{x} = f(x)$, such that

$$\mathrm{D}f(p) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad \text{with} \lambda_1 < 0 \lambda_2.$$

Then, $\exists \varepsilon > 0$ and $B =]-\varepsilon, \varepsilon[\times]-\varepsilon, \varepsilon[$ such that:

- $S_B = \{(x, s(x)) \text{with} x \in]-\varepsilon, \varepsilon[\}$ where $s(x)$ is $\mathbb{C}^k$ and $s'(0) = 0$;
- $U_B = \{(u(y), y) \text{with} x \in]-\varepsilon, \varepsilon[\}$ where $u(x)$ is $\mathbb{C}^k$ and $u'(0) = 0$;

Let's try to put this in plain text. If we have a hyperbolic saddle, then there is a neighbourhood where the stable manifold through the point has a functional form, tangent to the axis corresponding to the stable direction of the linearised system; and there is a *also* a neighbourhood where the unstable manifold has a functional form, tangent to the axis corresponding to the unstable direction of the linearised system. It results from this theorem that the stable and unstable manifolds are locally orthogonal to each other ! So this is big.

The consequences are pretty wild. Indeed, suppose again that the system has been put such that the eigenvectors correspond to the x

and y axes (as above), with $x = 0$ (the $y-$axis) the unstable manifold through the origin.

Then, we can as we have done already linearise the system:

$$\begin{aligned} \dot{x} &= \lambda_1 x + N_x(x, y), \\ \dot{y} &= \lambda_2 y + N_x(x, y). \end{aligned} \tag{12}$$

The non-linear terms contain second-order terms and more, so by definition $lim_{(x,y)\to(0,0)} \frac{N_{x,y}(x,y)}{x^2+y^2} = 0$.

So, consider now $(x(t), y(t))$ an orbit of the stable manifold that passes through $x, s(x)$. It must remain on the stable manifold, and we find that $y(t) = s(x(t))$. We can re-write system (12) as follows:

$$\begin{aligned} \dot{x} &= \lambda_1 x + N_x(x, s(x)), \\ \dot{x}s'(x) &= \lambda_2 y + N_x(x, s(x)). \end{aligned} \tag{13}$$

Combining both equations to eliminate time (the $\dot{x}$ term), you get

$$s'(x)(\lambda_1 x + N_x(x, s(x))) = \lambda_2 s(x) + N_y(x, s(x)) \tag{14}$$

Similarly, for the unstable form, you would get:

$$(\lambda_1 u(y)x + N_x(u(y), y)) = u'(\lambda_2 u(y) + N_y(u(y), y)) \tag{15}$$

We will now show how we can use these developments. Consider again the system that we have been playing with already, but we just swap the x and y directions to comply with the condition $\lambda_1 < 0 < \lambda_2$:

$$\dot{x} = -x,$$
$$\dot{y} = y + x^2.$$

Thus, $N_x(x, y) = 0$, $N_y(x, y) = x^2$. We had the solution, and we know that $s(x) = -\frac{1}{3}x^2$, but as we said above, it is unusual to be able to find it analytically. So suppose we don't know it. To this end, consider the differential equation for the stable manifold:

$$s'(x)(-x) = s(x) + x^2.$$

If we admit that s is infinitely differentiable, and keeping in mind that its first-order derivative is null, then as a general rule we know that $s(x) = \sum_{k=2}^{\infty} s_k x^k$. Substituting in the differential equation, that makes

$$-x \sum_{k=2}^{\infty} s_k x^{k-1} = x^2 + \sum_{k=2}^{\infty} -k s_k x^k.$$

Identifying terms for the different orders, one by one, we get $-2s_2 = s_2 + 1$ and $-ks_k = s_k$ for $k \geq 3$. The only solution is $s_2 = -\frac{1}{3}$ and $s_k = 0$ for $k \geq 3$.

As an exercise, you could find that, for

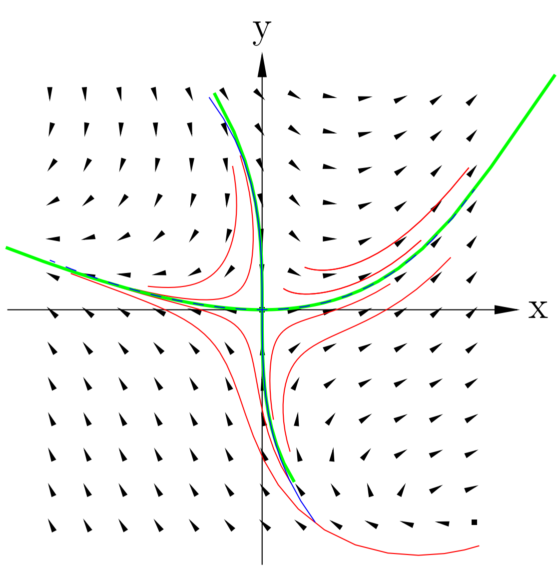
$$\dot{x} = x + y^3$$
$$\dot{y} = -y + 2xy + x^2,$$

we have:

$$
u(x) = \frac{1}{3}x^2 + \frac{1}{6}x^3 + \frac{1}{15}x^4 + \mathcal{O}(x^5)
$$
$$
s(y) = -\frac{1}{4}y^3\mathcal{O}(y^5)
$$

Attention, there is a small difficulty: the roles of x and y are inverted compared to the theory above (in other words: $\lambda_2 < 0 < \lambda_1$. The trick is just to rename $y^\star = x$ and $x^\star = y$ and then rename back when finished.

The graph below represents sample orbits (red) and the approximate stable and unstable manifolds (blue), the computed stable and unstable manifolds (green), and the flow direction (arrows)



(cf. Wiggins chap 2)

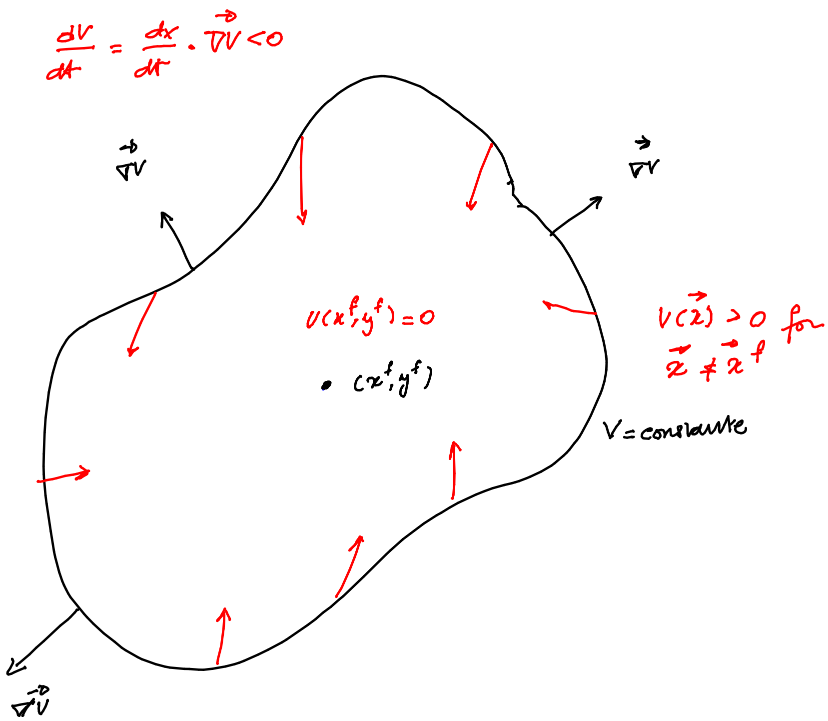
The method of Lyapunov functions can be used to determine the stability of fixed points when the information obtained from the linearisation is inconclusive (e.g. non-hyperbolic fixed points). As noted by Wiggins, the Lyapunov theory is a large area, and we examine here a very small part of it. Furthermore, while it is presented here in a chapter about vector fields in $\mathbb{R}^2$, it is in fact valid in any dimension, but so much easier to visualise on the plane.

The basic idea is to find, in a neighbourhood of the fixed point, a function of the state that is positive, zero at the fixed point, and known to decay strictly monotonously over time. In which case, quite intuitively, the fixed point is asymptotically stable.

More specifically, consider the following vector field:

$$\begin{aligned} \dot{x} &= f(x, y) \\ \dot{y} &= g(x, y), \quad (x, y) \in \mathbb{R}^2, \end{aligned}$$

with fixed point (x_f, y_f). Let $V(x, y)$ a scalar-valued function on $\mathbb{R}^2$ that is at least $\mathbf{C}^1$, such that $V(x_f, y_f) = 0$.



Suppose also that the locus of points satisfying $V(x, y) = C$ (constant) forms closed curves for different values of C encircling (x_f, y_f). Now, recall that the gradient of V, ∇V, is a vector perpendicular to the tangent of the curves, pointing to the direction of increasing V. If this vector is always pointing outwards (decreasing values of C as we get closer to (x_f, y_f), then we get $\nabla V(x, y) \cdot (\dot{x}, \dot{y}) \leq 0$. Now we can state the general theorem

Theorem (Existence of Lyapunov function implies stabilitiy) In the conditions stated above, with $V(x_f, y_f) = 0$ and $V(x_f, y_f) > 0$ if $(x, y) \neq (x_f, y_f)$, if there exists a neighbourhood U around (x_f, y_f) such that:

- $\dot{V}(x) \leq 0$ in $U \backslash \{(x_f, y_f)\}$: (x_f, y_f) is stable
- $\dot{V}(x) < 0$ in $U \backslash \{(x_f, y_f)\}$: (x_f, y_f) is *asymptotically* stable

Although quite intuitive, the proof is not quite straightforward and will only be discussed during the lecture time permitting. It is available in [Wig03], p. 24.

More insightful for the physicist is the physical meaning attached to V. In a conservative (Hamiltonian system), the Hamiltonian would actually satisfy the definition of a Lyapunov function, although this is a limit case, where $\dot{V}(x) = 0$. In a thermodynamic system, then the Free Energy (at constant temperature) or the Gibbs free energy (at constant pressure) work as Lyapunov functions. In the simpler case of a near Hamiltonian system (e.g. pendulum with friction), then the potential + kinetic energy V satisfy the definition of a Lyapunov function near the fixed point.

So, in principle we have a good method for proving the stability of a fixed point; in practice, finding V is not always as straightforward as in the cases above. There is no miracle receipt.

Again, the evolution of the Lyapunov function must be (strictly) monotonous in a neighbourhood of the fixed point. The latter may be arbitrarily small to satisfy the mathematical definition of an asymptotically stable fixed point, though of course, for physical significance, we might want a large neighbourhood !

Indeed, as we see it, a contour C within which $\dot{V}(x) \leq 0$ will effectively act as a *trapping region*, that is defined as a region that contains all its future states.

Definition (Trapping region) A trapping region U relative to a diffeomorphism is a region such that $phi^t(U) \subset U \ \forall t > 0$.

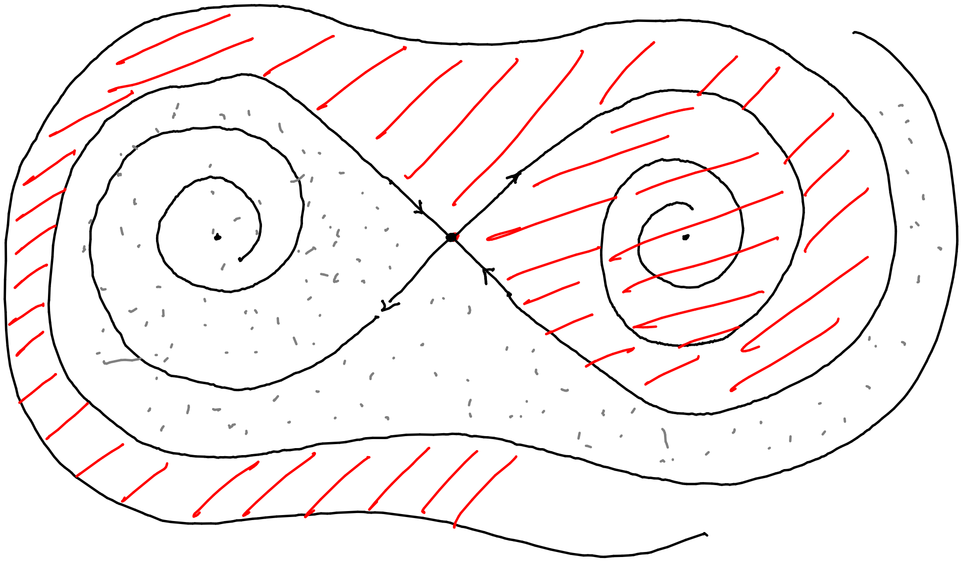
An attracting set is the asymptotic limit of the sequence of contractions of the trapping regions, which is technically defined as follows:

Definition (Attracting set) An attracting set exists if it can be defined as $\bigcap_{t>0}^{\infty} \phi^t(U)$.

Conversely, the *basin* of attraction relative to the attracting set A is the set of initial conditions that eventually falls into it:

Definition (Basin of attraction) The basing of attraction (or *domain*) of an attracting set A is $\bigcup_{t\leq 0}^{\infty} \phi^t(U)$, where U is an open neighbourhood acting as a trapping region leading to A.

Below is an example of basins of attractions of two sinks. The sinks act as two individual *attracting sets*.



References

[Wig03] S. Wiggins. *Introduction to Applied Nonlinear Dynamical Systems and Chaos*. 2003.