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1 Motivation and structure

1.1 foreword

The course LPHYS2114 has now been given for quite a number of years by Prof. Christian Hagendorf, to the great satisfaction of the students. The lecture has been passed on to me. I am a physicist specialised in climate dynamics, and I have been working dynamical systems as models of the climate since 2006. My approach tends to be pragmatic, as I am ultimately interested in the behaviour of the real world system that I am attempting to model with dynamical systems. So will be this lecture. The divide between *continuous* dynamical systems and *discrete* dynamical systems has been conserved, as well as many exercises and the overall evaluation structure. I would like to warmly acknowledge Prof. Hagendorf's for his help and collaboration in preparing this lecture.

As I am giving this course for the first year, adjustments are likely and I at time of writing I have not yet been able to provide a fully detailed weekly program. The basis is 10 2-hour lessons (this including 3 spare weeks in the 13-week official program), plus weekly exercise sessions organised by Victor Couplet.

In preparing this lecture, I have relied on several reference books. Although the current notes are meant to be the formal material upon which evaluation is based, the circumstances are such that writing is work in progress and students are most welcome to check them and provide corrections or suggestions. This text and related material is provided as a free git repository.

- S.H. Strogatz, Nonlinear dynamics and chaos. Westview Press (2015).
- S. Wiggins, Introduction to Applied Nonlinear Dynamical Systems and Chaos, Springer (2003)
- R. Hilborn, Chaos and Nonlinear Dynamics: An Introduction for Scientists and Engineers (2nd edn) , Oxford University

Press (2000)

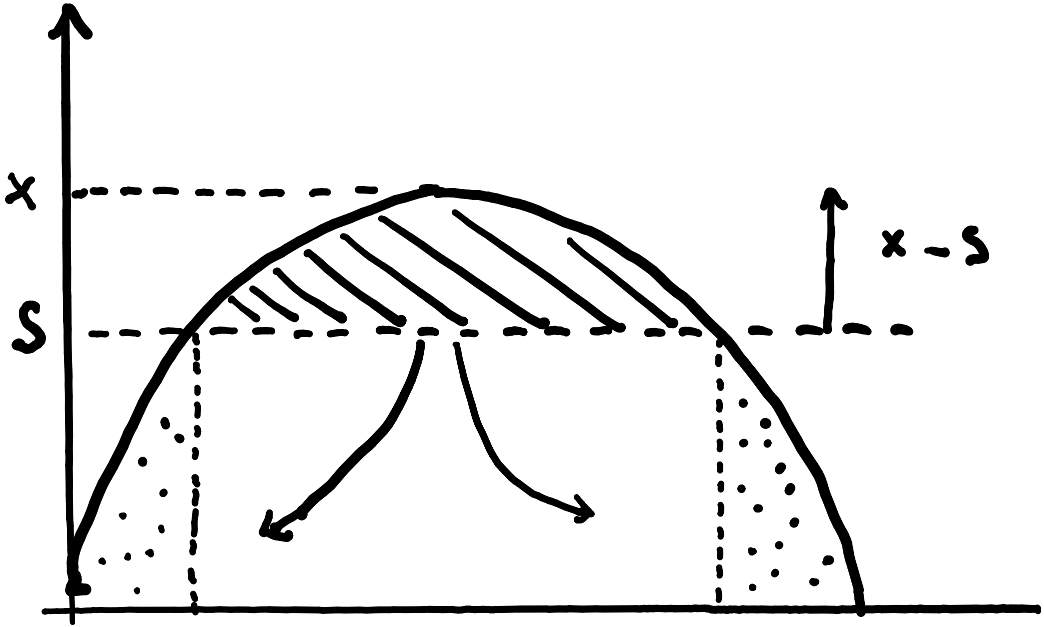
- H. Dijkstra, Nonlinear Physical Oceanography, A Dynamical Systems Approach to the Large Scale Ocean Circulation and El Nio, Springer Science+Business Media (2000)

1.2 Motivating example: ice sheet

To motivate the course and its objective, consider the following example from glaciology.

An ice sheet is a large accumulation of ice. The largest ice sheets take thousands of years to millennia to form. If the climate is stable enough (say, it is *constant*), the ice sheet may reach a *stable equilibrium.*

This equilibrium results from a balance between *accumulation* of snow above the *snowline* (which is determined by the external climate conditions), and the *ablation* of ice that is being pushed below the snow line.



x
S
x - S

Intuitively, we may perceive that if climate warms a little, the altitude of the ice sheet will decrease (implying less net accumulation), but the flow of ice towards the ice sheet will be reduced as well, so that a new balance will be reached. However, we may already anticipate that if the climate warms *too much*, and the top surface of the ice sheet will drown under the snowline, at which point we expect a catastrophic meltdown of the icesheet.

We witness here a non-linear phenomenon. There is what some might call a *tipping point*, a point of warming above which the qualitative behaviour of the object changes. In this course, we would like to use mathematical modelling and a mathematical language to describe this behaviour, and many others. We will be introduced to the theory and learn the techniques that allow us to describe the expected behaviour of such systems.

The standard theory is the theory of *dynamical systems*. In essence, a dynamical system is the combination of a state space, say Ω, and a *rule* that determines the evolution of every point of the state space.

This is perhaps a bit abstract, so consider again our ice sheet example.

Call the altitude of the summit of the ice sheet x. This is positive real number, so $x \in \mathbb{R}^+$. The altitude of the snowline is S. We consider that it is constant, so in the following we will view it as a *parameter*.

Here, we have *summarised* the state of our real-world system (the ice sheet) with a single variable x. That variable evolves in $\mathbb{R}^+$. So in our case the Ω space is $\mathbb{R}^+$. Now we need a rule for the evolution of an ice sheet that would have an altitude x.

One (standard) approach is to write a *differential equation*. Simply, we equate $\frac{\mathrm{d}x}{\mathrm{d}t}$ with a function of the state x, and the parameter S. At this point, we need a bit of physical intuition. x will evolve as a balance between accumulation and ablation. The difference between accumulation and ablation is called *net accumulation*. One (simplistic) way of putting it, is that net accumulation is proportional to the distance between x and S. It would be an equation of the kind

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x - S \tag{1}$$

Class room discussion: Consider an initial condition $x_0 = x(t_0)$. Finding x_t for $t > t_0$ is an initial value problem, which is a particular case of a Cauchy problem. Find the solutions. Show that their behaviour differ depending on the initial condition. Explain also what is wrong with this model. How can we fix it ?

After discussion, we see that we need a *negative feedback* to prevent ice to grow until infinity when it starts above the snowline. We won't discuss too much the phenomenological details here, but say that the negative feedback is related to the flow of ice towards the ice sheet ablation zone, which is enhanced when the ice sheet grows very big. So our final equation would resemble:

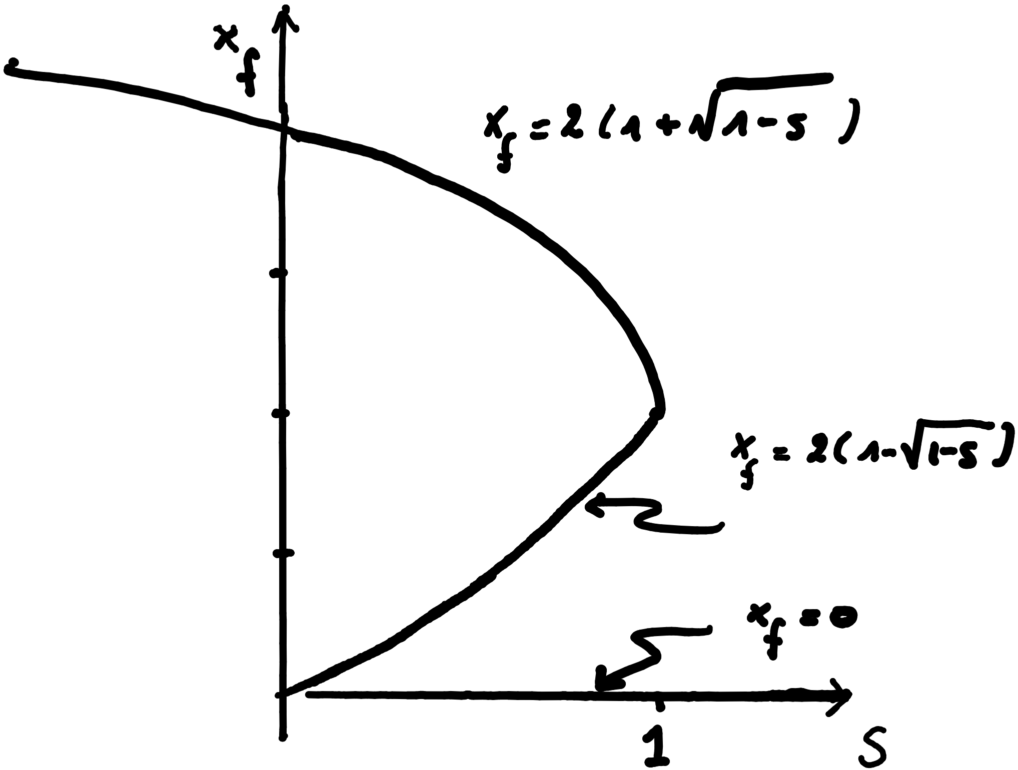
$$\frac{\mathrm{d}x}{\mathrm{d}t} = x - ax^2 - S \text{ if } x > 0, \text{ and } \frac{\mathrm{d}x}{\mathrm{d}t} = \max(0, -S) \text{ if } x = 0, \tag{2}$$

where a is a parameter. Again, we look after qualitative behaviours, and for (mathematical) simplicity we will set $a = 1/4$. In practice we cannot be so relaxed about fixing parameters, but this is for the sake of the demonstration.

What is going on now ?

Class room discussion: A fixed point x_f of the differential equation is an element of the domain Ω such that the system is invariant at that point. That is, if $x(t_0) = x_f$ for a given $t = t_0$, then $x(t) = x_f$ for any time t of the time domain. How do you find the fixed points associated with the dynamical system in (2) ?

After discussion, we find that the number of fixed points depends on the value of the parameter S: one fixed point for $S < 0$ $2(1 + \sqrt{(}1 - S))$, one fixed point for $S > 1$ (x=0), and three fixed points between these values (which ones ?)



x_f
$x_f = 2(1+\sqrt{1-s})$
$x_f = 2(1-\sqrt{1-s})$
$x_f = 0$
1
s

Now we would like to determine the behaviour of the system for initial conditions between these points. One approach would be to resolve the ordinary differential equation (2). For $S = 0$ it is doable (hint: use the separation of variables: put all $\mathrm{d}x$ terms on one side; the $\mathrm{d}t$ terms on the other side; integrate and solve for x) but it is cumbersome and the strategy will actually not work in most cases. In other words, most non-linear ordinary differential equations have no analytical solutions.

Hence, a more fruitful strategy is to study the behaviour of (t) *near* (in French: *dans le voisinage the*) fixed points, and use theory to connect the flowlines between fixed points.

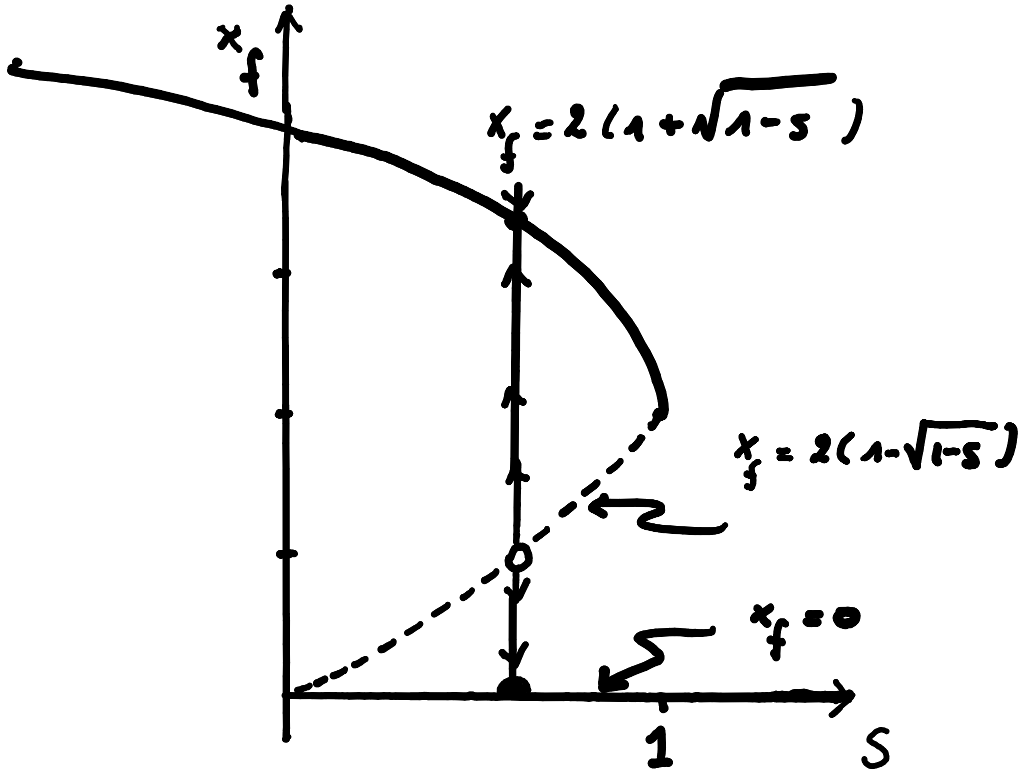
The model we have been starting with is of the form $\frac{\mathrm{d}x}{\mathrm{d}t} = F(x;\psi)$ with, here, $\psi := \{S\}$. By definition of a fixed point, $F(x_f;\Psi) = 0$

(the dependency on ψ is dropped for clarity). Define $\delta x := x - x_f$.

$$\frac{\mathrm{d}x - x_f}{\mathrm{d}t} \doteq \frac{\mathrm{d}\delta_x}{\mathrm{d}t} = \underbrace{\left.\frac{\partial F(x;\psi)}{\partial x}\right|_{x_f}}_{\lambda} \delta x + \mathcal{O}(x^2) \tag{3}$$

This is a linear differential equation for δx with constant coefficient. That is, near enough to the fixed point, δx decays (if $\lambda < 0$) or grows (if $\lambda > 0$) exponentially with e-folding time $1/\lambda$. This distinguishes a point that is *locally stable* from a point that is *locally unstable*.

Class room discussion: Reconsiders the one to three solutions of the bifurcation diagram. Which ones are stable, and which one unstable ? What we see appearing are *stable* and *unstable* solution *branches*.



x_f
$x_f = 2(1+\sqrt{1-s})$
$x_f = 2(1-\sqrt{1-s})$
$x_f = 0$
1
s

Hence, even though we have avoided to resolve the ordinary differential equation; (we have avoided to solve the initial value problem), but provided that we have identified all fixed points, and characterised their local stability, we have gained a good qualitative picture of the system's behavior. For any value of S, we can picture the *flow* associated with the dynamical system, that is, for any point x of the domain Ω, we know the direction taken by x as time progresses.

This is an example of qualitative analysis of a non-linear dynamical system, which, as we see it here, is reasonably simple but not simplistic. We have identified *invariant sets*, that is, sets of points that are left unchanged by the flow. In this example, invariant sets are fixed points. We have identified *bifurcation points*, that is, points of the parameter space S where the number or the stability of the invariant sets (again, here, fixed points).

In this first lecture, meant to motivate the course, we have been informal and did not justify our findings with theorems; nor did we attempt to be too systematic and rigorous with definitions. But we understand our objectives: identify the nature of invariant sets, estimate their stability, and depict the behaviour of the system between these invariant sets.

Specifically, the lecture will be divided into two broad sections:

- continuous dynamical systems (expressed as ordinary differential equations)
- discrete dynamical systems (expressed as iterations)

Dynamical systems will be deterministic (but informally, from time to time I will mention the interest and possibilities brought about by stochastic dynamical systems), and we will not go beyond three dimensions.

1.3 Evaluation

Exercises will be evaluated and provide a third of the final evaluation that can no longer be modified. There will be a *written* exam (June and September sessions) that is focused on exercises. Exams of the previous years are available on Moodle. They will be focused on applications, knowledge of the theory being instrumental in solving the problems.