Game Plan for the Next Few Lectures

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The following is a WIP typed adaptation of Dr. Andreas Mihatsch's notes on the general idea of the Igusa construction, available here: https://www.math.uni-bonn.de/people/mihatsch/21u22%20WS/moduli/.

So far, we've shown for $n \geq 3$ that $\mathcal{M}_n[1/6]/\boldsymbol{Z}[1/6n]$ is representable. Ultimately, we'd like to show that $\mathcal{M}_n/\boldsymbol{Z}[1/n]$ is representable. To do this, we will use the **Igusa Construction**:

1. First, we'll show by hand that $\mathcal{M}_3/\boldsymbol{Z}[1/3]$ and $\mathcal{M}_4/\boldsymbol{Z}[1/2]$ are representable (actually, we'll only show that $\mathcal{M}_3/\boldsymbol{Z}[1/3]$ is representable; the $\mathcal{M}_4/\boldsymbol{Z}[1/2]$ construction is similar). To do this, we'll need to develop the **Weil pairing**.

2. Given $n \geq 3$, let G be the kernel of the natural map $\mathrm{GL}_2(\boldsymbol{Z}/(3n)) \to \mathrm{GL}_2(\boldsymbol{Z}/n)$. Then, the group G acts on $\mathcal{M}_{3n}$ and we can consider the quotient $q\colon \mathcal{M}_{3n} \to G\backslash\mathcal{M}_{3n}$. We will prove that $G\backslash\mathcal{M}_{3n}$ and $\mathcal{M}_n[1/3]$ are isomorphic. Similarly, the functor $\mathcal{M}_n[1/2]$ is isomorphic to a quotient of $\mathcal{M}_{4n}$. Then, the schemes $\mathcal{M}_n[1/2]$ and $\mathcal{M}_n[1/3]$ glue to $\mathcal{M}_n$. This step is in principle similar to the quotient construction for $\widetilde{\mathcal{M}_n}[1/6]$. The main difference is that while q is a G-torsor as in the $\widetilde{\mathcal{M}_n}[1/6]$ construction, it is only trivial étale locally. Thus, we'll need to develop some descent statements for elliptic curves.