# *Weierstraß Moduli*

*Scribe: Matthew Stevens*

*January 27, 2022*

The following is a typed adaptation of Dr. Andreas Mihatsch's notes on Weierstraß moduli, available here: https://www.math.uni-bonn.de/people/mihatsch/21u22%20WS/moduli/.

Some of the statements below are missing the assumption that the base scheme is locally Noetherian.

## Goal: Classify Elliptic Curves

We want to find a space "parameter space" $\mathscr{M}$ and an relative elliptic curve $\mathscr{E}/\mathscr{M}$ that is universal in the following sense: Given a relative elliptic curve $E/S$, there should be a unique map $u\colon S \to \mathscr{M}$ such that $E$ is isomorphic to $u^*\mathscr{E} := S \times_{\mathscr{M}} \mathscr{E}$.

Unfortunately, this isn't possible. One way to see that this can't be done: Let $D$ be a rational number that is not a square. Then, the elliptic curves $y^2 = f(x)$ and $Dy^2 = f(x)$ are isomorphic over $\mathbf{Q}(\sqrt{D})$ but not over $\mathbf{Q}$. If $\mathscr{M}$ were a scheme, there would be unique maps $u, u' \in \mathscr{M}(\mathbf{Q})$ such that $u^*\mathscr{E}$ and $(u')^*\mathscr{E}$ are isomorphic to $E$ and $E'$ respectively. But $u$ and $u'$ are equal in $\mathscr{M}(\mathbf{Q}(\sqrt{D}))$. This is a problem, since $\mathscr{M}(\mathbf{Q})$ should inject into $\mathscr{M}(\mathbf{Q}(\sqrt{D}))$.

Now, there does exist a *stack* $\mathscr{M}$ satisfying the properties above. But we'd really like to have something like this in scheme-land. It turns out that we can find variants of $\mathscr{M}$ for schemes. Broadly, there are three ways to proceed:

1. with equations (Igusa);
2. with geometric invariant theory (Mumford); and
3. functorially (Grothendieck, Artin).

## The Main Result

Today, we'll look at a variant of $\mathscr{M}$ in scheme-land (at least away from 2 and 3).

**Theorem 1.** *The contravariant functor*

$$\begin{aligned} \widetilde{M}\left[\frac{1}{6}\right] &: \mathbf{Sch}/\mathbf{Z}\left[\frac{1}{6}\right] \to \mathbf{Set} \\ &: S \mapsto \left\{(E, \pi) \mid E/S \textit{ is an elliptic curve}, \pi \in \Gamma(E, \Omega^1_{E/C})\right\} / \cong \end{aligned}$$

*is representable by the affine scheme*

$$\operatorname{Spec} \overbrace{\mathbf{Z}\left[\frac{1}{6}\right][a,b]\,[\Delta^{-1}]}^{=:R}, \qquad \Delta = 4a^3 + 27b^2$$

*with universal pair*

$$\begin{aligned}
\mathscr{E} &= V_+(y^2z - x^3 - axz^2 - bz^3) \subset \boldsymbol{P}^2_R \\
\pi &= \textit{the unique section of } \Omega^1_{\mathscr{E}/R} \textit{ such that} \\
&\quad \pi|_{D^+(z)\cap\mathscr{E}} = \frac{-dx}{2y} \textit{ on } D(y) = \frac{-dy}{3x^2+a} \textit{ on } D(3x^2-a).
\end{aligned}$$

The set $D(3x^2 - a)$ contains $V(y) \cap \mathscr{E}$ because $x^3 - ax + b$ is separable and thus its derivative isn't zero.

*The section $\pi$ is well-defined because there holds*

$$0 = d(y^2 - x^3 - ax - b) = 2ydy - (3x^2 + a)dx.$$

Let's clarify a few things.

1. The relation $\cong$ is defined as follows: Given $(E, \pi)$ and $(E', \pi')$ satisfy $(E, \pi) \cong (E', \pi')$ when there exists an isomorphism $f\colon E \xrightarrow{\sim} E'$ such that $f^*\pi'$ equals $\pi$.

2. The section $\pi$ is a global generator if and only if $\pi(s) \in \Omega^1_{E(s)/\kappa(s)}$ is a global generator for all $s \in S$.

3. Implicit in the theorem is that a pair $(E, \pi)$ has no automorphisms if 6 is in $\mathscr{O}_S^\times$. If 2 equals 0 in $\mathscr{O}_S$, then any pair has the automorphism $[-1]$ since there holds $[-1]^*\pi = -\pi = \pi$. This shows that this method can't extent to characteristic 2.

4. For any $(E, \pi) \in \widetilde{M}(S)$, we have $\Omega^1_{E/S} \cong \mathscr{O}_E$, which gives an obstruction for some $E/S$ to occur in $\mathscr{M}$. Since $\Omega^1_{E/S}$ equals $p^*e^*\Omega^1_{E/S}$, any family $E$ occurs Zariski locally in $\widetilde{\mathscr{M}}$.

## Proof of the Main Result

Consider an elliptic curve $E/S$ with structure map $p\colon E \to S$ and identity section $S \to E$.

By Lecture 8 of the Abelian Varieties course, the section $e$ is a closed immersion and $e(S) =: V(\mathscr{I})$ is a Cartier divisor.

Meaning $e(S)$ is locally cut out by a single equation that isn't a zero-divisor.

We know that $\mathscr{O}(3e)$ is relatively very ample since it is fiberwise relatively very ample. We also know that $p_*\,\mathscr{O}(3a)$ is a vector bundle of rank 3 on $S$. Thus, we get a closed embedding $E \hookrightarrow \boldsymbol{P}((p_*\,\mathscr{O}(3e))^\vee)$ into a twist of $\boldsymbol{P}^2_S$.

How do we make this more explicit? Consider $\mathscr{O}_E \subset \mathscr{O}_E(e) \subset \mathscr{O}_E(2e) \subset \cdots$.

**Lemma 2.** *The sheaf $\omega_E := e^*\Omega^1_{E/S}$ is isomorphic to $\mathscr{I}/\mathscr{I}^2$. Moreover, for all $n \in \boldsymbol{Z}$, we have $\mathscr{O}_E(ne)/\,\mathscr{O}_E((n-1)e) \cong e_*\omega_E^{\oplus(-n)}$.*

*Proof.* We have

$$\begin{tikzcd}
S \arrow[r, hook, "e"] \arrow[dr, equal] & E \arrow[d, two heads, "p"] \\
& S.
\end{tikzcd}$$

Since $S/S$ is smooth, the conormal exact sequence on $S$ is left-exact:

$$0 \to \mathscr{I}/\mathscr{I}^2 \to e^*\Omega^1_{E/S} \to \overbrace{\Omega^1_{S/S}}^{=0} \to 0.$$

Thus, there holds $\mathcal{I}/\mathcal{I}^2 \cong e^*\Omega^1_{E/S}$.

Now, we have the exact sequence $0 \to \mathcal{I} \to \mathcal{O}_E \to \mathcal{O}_{e(S)} \to 0$ on $E$. Tensoring by $\mathcal{I}^n$, we obtain the exact sequence

$$0 \to \mathcal{O}_E((n-1)e) \to \mathcal{O}_E(ne) \to \mathcal{I}^{-n} \otimes \mathcal{O}_{e(S)} \to 0.$$

Since we have $\mathcal{I}^{-n} \otimes \mathcal{O}_{e(S)} = (\mathcal{I}/\mathcal{I}^2)^{\otimes(-n)} = e_*\omega_E^{\otimes(-n)}$, the claim follows. □

Here, we are using the fact that if $i\colon Z \to T$ is any closed immersion, then for any quasicoherent sheaf $\mathcal{F}$ on $T$, the natural map $\mathcal{F} \to i_*i^*\mathcal{F}$ is an isomorphism. We'll continue using this fact. See Stacks 01QX for more information.

Now, recall that for any $n \geq 1$, the higher pushforward $R^1p_*\,\mathcal{O}(ne)$ equals 0. For $n \geq 2$, this implies that there is an exact sequence

$$0 \to p_*\,\mathcal{O}_E((n-1)e) \to p_*\,\mathcal{O}_e(ne) \to p_*\mathcal{I}/\mathcal{I}^2 \to 0.$$

Since $\mathcal{I}/\mathcal{I}^2$ is isomorphic to $e_*e^*\mathcal{I}/\mathcal{I}^2$ and $p_*e_* = (p \circ e)_* = \mathrm{id}_*$, this means that there is an exact sequence

$$0 \to p_*\,\mathcal{O}_E((n-1)e) \to p_*\,\mathcal{O}_e(ne) \to \omega_E^{\otimes(-n)} \to 0$$

for all $n \geq 2$.

Further, recall that for any $n \geq 1$, the sheaf $p_*\,\mathcal{O}(ne)$ is a rank $n$ vector bundle on $S$ and $R^1p_*\,\mathcal{O}_E$ is a line bundle. For $n = 1$, we have an exact sequence

$$0 \to p_*O_E = \mathcal{O}_S \to p_*O_E(e) \to \omega_E^{\otimes(-1)} \to R^1p_*\,\mathcal{O}_E \to R^1p_*\,\mathcal{O}_E(e) = 0.$$

Since $\omega_E^{\otimes(-1)}$ and $R^1p_*\,\mathcal{O}_E$ are both rank 1 line bundles, this implies that $\mathcal{O}_S$ is isomorphic to $p_*\,\mathcal{O}_E(e)$.

The above discussion yields a sequence of vector bundles

$$0 \subset \mathcal{O}_S \subset p_*\,\mathcal{O}_E(2e) \subset p_*\,\mathcal{O}(3e) \subset \cdots \subset p_*\,\mathcal{O}(ne) \subset \cdots$$

with factor sheaves $\mathcal{O}_S, \omega_E^{\otimes(-2)}, \omega_E^{\otimes(-3)}$.

If $\omega_E$ is trivial and equal to $\mathcal{O}_S \cdot \pi$, then $\omega_E^{\otimes i}$ equals $\mathcal{O}_S \cdot \pi^i$.

Note that such a $\pi$ yields a generator of $\Gamma(E, \Omega^1_{E/S})$, since $\mathcal{O}_E$ is isomorphic to $\Omega^1_{E/S}$.

Of course, the sheaf $\omega_E$ might not be trivial globally, but if we look at a small enough open $U \subset S$, we can pick

1. an element $x \in p_*\,\mathcal{O}(2e)$ such that $x \bmod \mathcal{O}_S$ equals $\pi^{-2}$; and
2. an element $y \in p_*\,\mathcal{O}(3e)$ such that $y \bmod p_*\,\mathcal{O}_E(2e)$ equals $\pi^{-3}$.

Note that $(1, x)$ and $(1, x, y)$ must be bases of $p_*\,\mathcal{O}_E(2e)$ and $p_*\,\mathcal{O}(3e)$, respectively.

The sections $1, x, y$ give an isomorphism between $\boldsymbol{P}(p_*\,\mathcal{O}(3e)^\vee)$ and $\boldsymbol{P}^2_U$ and realize $E|_U$ in Weierstraß form:

In math.colostate.edu/~miranda/BTES-Miranda.pdf, the same idea is applied in a more classical setting.

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

This is **the** Weierstraß form for the choice $1, x, y$, with the $a_i$ determined uniquely from the linear dependence of $1, x, y, x^2, xy, y^2, x^3$ in $p_*\,\mathcal{O}(6e)$.

Conversely, giving $E$ in such a Weierstraß form, coordinates $x, y$ provide sections of $\mathcal{O}_E(2e)$ and $\mathcal{O}_E(3e)$ of the considered type.

Let $(\pi', x', y')$ be another choice of generator and sections. Then, there exist unique elements $u \in \mathcal{O}_S^\times$ and $r, s, t \in \mathcal{O}_S$ such that there hold

$$\begin{aligned} \pi &= u^{-1}\pi', \\ x &= u^2 x' + r, \text{ and} \\ y &= u^3 y' + su^2 x' + t. \end{aligned}$$

See Deligne's "Courbes Elliptiques: Formulaire d'apres J. Tate" for a reference.

Then, the coefficients $a_i'$ in the Weierstraß form associated to $(\pi', x', y')$ satisfy

$$\begin{aligned} ua_1' &= a_1 + 2s, \\ u^2 a_2' &= a_2 - sa_1 + 3r - s^2, \\ u^3 a_3' &= a_3 + ra_1 + 2t, \text{ and so on.} \end{aligned}$$

Fix $\pi$ as above and assume that 6 is in $\mathcal{O}_S^\times$. From the above discussion, we see that there exist unique sections $x$ and $y$ such that the $a_1, a_2$, and $a_3$ in the corresponding Weierstraß form are all zero. Namely, if we start with $x' \in p_* \mathcal{O}(2e)$ and $y' \in p_* \mathcal{O}(3e)$ with $x \bmod \mathcal{O}_S = \pi^{-2}$ and $y' \bmod p_* \mathcal{O}_E(2e) = \pi^{-3}$, we can replace $x'$ and $y'$ them with the $x$ and $y$ arising from $s = -a_1/2$ and $r = (-a_2 + sa_1 + s^2)/3$ and $t = (-a_3 - ra_1)/2$.

**Corollary 3.** *Suppose that 6 is in $\mathcal{O}_S^\times$. For a pair $(E, \pi)/S$ of an elliptic curve over $S$ and a generator $\pi$ of $\Omega_{E/S}$, there holds* $\mathrm{Aut}(E, \pi) = \{\mathrm{id}\}$.

*Proof.* Let $x$ and $y$ be the unique sections yielding a Weierstrass equation with $a_1 = a_2 = a_3 = 0$. Then, the sections $\varphi^* x$ and $\varphi^* y$ satisfy the same Weierstraß equation, so $\varphi^* x$ and $\varphi^* y$ equal $x$ and $y$, respectively. So $\varphi$ extends to $\mathrm{id}_{\boldsymbol{P}^2}$. □

This shows in particular that $\mathrm{Aut}(E) \to \mathcal{O}_S^\times : \varphi \mapsto [\varphi^* : \omega_E \xrightarrow{\sim} \omega_E]$ is injective. In characteristic 0, it can be shown that $\mathrm{End}(E) \to \mathrm{End}(\omega_E)$ is injective.

Now, let's prove that $\widetilde{\mathscr{M}}[1/6]$ is representable.

We've already see that given $\pi$, there is a unique way to write $E$ in Weierstraß form

$$E : y^2 = x^3 + ax + b$$

with $x \equiv \pi^{-2}$ and $y \equiv \pi^{-3}$.

Conversely, if $E$ and $x$ and $y$ are as above, then $\pi := (x \bmod \mathcal{O}_S)/(y \bmod p_* \mathcal{O}_E(2e))$ is the unique generator of $\omega$ that yields $x$ and $y$ as the sections yielding a Weierstraß form with $a_1 = a_2 = a_3 = 0$ via the process above.

It remains to show that $p^*\pi$ is given by

$$\pi' := \frac{-dx}{2y} = \frac{-dy}{3x^2 + a} \text{ on } D_+(z).$$

Both $p^*\pi$ and $\pi'$ define generators of $\Omega'_{E/S}$ and hence differ by an element of $\mathcal{O}_S(S)^\times$.

We can show this after restriction to $e(S)$. Working locally near $e(S)$, assume $\mathscr{I}$ equals $(t)$. Write $x = f/t^2$ and $y = g/t^3$. Then, we have

$$\begin{aligned}\pi &= \frac{x}{y} \bmod \mathscr{I}^2 \\ &= \frac{f}{g} \cdot t \bmod \mathscr{I}^2 \\ &= \left(\frac{f}{g}\right)(e) \cdot (t \bmod \mathscr{I}^2).\end{aligned}$$

The map $\mathscr{I}/\mathscr{I}^2 \xrightarrow{\sim} e^*\Omega'_{E/S}$ is $\varphi \mapsto d\varphi$, so this is $(f/g)(e)dt \in e^*\Omega'_{E/S}$. Now, we have

$$-\frac{dx}{2y} = -\frac{t^2df - 2tf\,dt}{t^4} \cdot \frac{t^3}{2g} = \frac{f}{g}dt - \frac{t}{2g}df.$$

Since $e^*((f/g)dt - t/(2g)df)$ equals $(f/g)(e)dt$, the claim follows.

## Coarse Moduli and the Other Main Result

Let $F\colon \mathbf{Sch}/S \to \mathbf{Set}$ be a contravariant functor.

**Definition 4.** *A scheme $M$ is a **fine moduli space** for $F$ if it represents $F$, in the sense that $F$ is naturally isomorphic to* $\mathrm{Hom}(-, M)$.

We've already seen that the functor $\mathscr{M}$ doesn't have a fine moduli space. But we can consider a weaker notion.

**Definition 5.** *A **coarse moduli space** for $F$ is a scheme $M$ together with a natural transformation $j\colon F \to M$ such that the following conditions hold:*

1. *the natural transformation $j$ is universal in the sense that if $Y$ is a scheme and $i\colon F \to Y$ is a natural transformation, there exists a unique map $j_i\colon X \to Y$ such that the following diagram commutes:*

$$\begin{tikzcd} F \arrow[r, "j"] \arrow[dr, "i"'] & X \arrow[d, "j_i"] \\ & Y; \end{tikzcd}$$

Even if a course moduli space for $F$ has a section, the functor $F$ might not be representable. For example, the map $[\mathrm{Pic}\colon S \mapsto \mathrm{Pic}(S)] \to \operatorname{Spec} \boldsymbol{Z}$ is a coarse moduli space and there exists a line bundle on $\operatorname{Spec} \boldsymbol{Z}$, but the Picard functor is not representable.

*and*

2. *if $\bar{k}$ is an algebraically closed field, then $j(\bar{k})\colon F(\bar{k}) \to M(\bar{k})$ is an isomorphism.*

A fine moduli space for $F$ is also a coarse moduli space for $F$. The functor $F$ may have a coarse moduli space but not a fine moduli space, and it may be that $F$ has neither fine moduli nor coarse moduli.

Our goal is to construct a coarse moduli space for the functor $\mathscr{M}[1/6]$.

### Game Plan

Let $\boldsymbol{G}_m$ denote the multiplicative group scheme over $\boldsymbol{Z}[1/6]$. We have a $\boldsymbol{G}_m$-action

$$\begin{aligned}\mu\colon \boldsymbol{G}_m \times_{\boldsymbol{Z}[1/6]} \widetilde{\mathscr{M}}[1/6] &\to \widetilde{\mathscr{M}}[1/6] \\ : (\lambda, (E, \pi)) &\mapsto (E, \lambda \cdot \pi)\end{aligned}$$

Let $f\colon \widetilde{\mathscr{M}}[1/6] \to \mathscr{M}[1/6]$ denote the "forget" map sending $(E, \pi)$ to $E$. The forget map $f$ is $\boldsymbol{G}_m$-invariant in the sense that if $p\colon \boldsymbol{G}_m \times_{\boldsymbol{Z}[1/6]} \widetilde{\mathscr{M}}[1/6] \to \widetilde{\mathscr{M}}[1/6]$ is the projection map, then $f \circ \mu$ equals $f \circ p$. Thus, for all maps $\nu\colon \mathscr{M}[1/6] \to Y$, we get a $\boldsymbol{G}_m$-invariant map $\nu \circ f\colon \widetilde{\mathscr{M}}[1/6] \to Y$.

To construct a coarse moduli space for $\mathscr{M}[1/6]$, we'll start by constructing categorical quotients by $\boldsymbol{G}_m$. We'll get a quotient map $q\colon \widetilde{\mathscr{M}}[1/6] \to \boldsymbol{G}_m \backslash \widetilde{\mathscr{M}}[1/6]$ and a unique map $g\colon \boldsymbol{G}_m \backslash \widetilde{\mathscr{M}}[1/6]$ induced by $\nu$ such that $\nu \circ f$ equals $g \circ q$. Finally, we'll get a unique map $h$ that is *independent* of $\nu$ such that the following diagram commutes:

$$\begin{tikzcd}
\widetilde{\mathscr{M}}[\tfrac{1}{6}] \arrow[r, "f"] \arrow[d, "q"'] & \mathscr{M}[\tfrac{1}{6}] \arrow[d, "\nu"] \arrow[dl, dashed, "\exists! h"'] \\
\boldsymbol{G}_m \backslash \widetilde{\mathscr{M}}[\tfrac{1}{6}] \arrow[r, dashed, "\exists! g"'] & Y.
\end{tikzcd}$$

This will verify that the first axiom of coarse moduli spaces holds. Then, we'll conclude the proof by checking that for every algebraically closed field $\overline{k}$, the map $h(\overline{k})\colon \mathscr{M}[1/6](\overline{k}) \to \boldsymbol{G}_m \backslash \widetilde{\mathscr{M}}[1/6](\overline{k})$ is an isomorphism, which will imply that $h\colon \mathscr{M}[1/6] \to \boldsymbol{G}_m \backslash \widetilde{\mathscr{M}}[1/6]$ is a coarse moduli space.

## Quotients by $\boldsymbol{G}_m$

Let $X = \operatorname{Spec} A$ and $S = \operatorname{Spec} R$ be two integral affine schemes. Then, there is a bijective correspondence between actions $\boldsymbol{G}_m \times_S X \to X$ and $R$-algebra gradings $A = \bigoplus_{i \in \boldsymbol{Z}} A_i$.

How do we construct this grading? Recall that an action $\mu\colon \boldsymbol{G}_m \times_S X \to X$ satisfies two axioms:

1. the composition $X \xrightarrow{(1, \mathrm{id}_X)} \boldsymbol{G}_m \times_S X \xrightarrow{\mu} X$ must be the identity, and
2. the map $\mu$ must be associative in the sense that the following diagram commutes, where $m$ denotes the multiplication map on $\boldsymbol{G}_m$:

$$\begin{tikzcd}
\boldsymbol{G}_m \times_S \boldsymbol{G}_m \times_S X \arrow[r, "{(m, \mathrm{id}_X)}"] \arrow[d, "{(\mathrm{id}_{\boldsymbol{G}_m}, \mu)}"'] & \boldsymbol{G}_m \times_S X \arrow[d, "\mu"] \\
\boldsymbol{G}_m \times X \arrow[r, "\mu"'] & X.
\end{tikzcd}$$

**Proposition 6.** *Let $a$ be an element of $A$ and write $\mu(a) = \sum_i t^i \otimes a_i$. For every $i$, there holds $\mu^*(a_i) = t^i \otimes a_i$.*

*Proof.* The associativity of the $\boldsymbol{G}_m$-action implies that there holds

$$\sum_i t^i \otimes t^i \otimes a_i = \sum_i \sum_j t^i \otimes t^j \otimes (a_i)_j.$$

Since elements of the form $t^i \otimes t^j$ form an $R$-basis for $R[t \otimes 1, t^{-1} \otimes 1, 1 \otimes t, 1 \otimes t^{-1}]$, there holds

$$(a_i)_j = \begin{cases} a_i & i = j \\ 0 & i \neq j. \end{cases}$$

□

Now, define $A_i$ to be $\{a \in A : \mu^*(a) = t^i \otimes a\}$.

**Proposition 7.** *The ring $A$ is equal to $\bigoplus_{i \in \boldsymbol{Z}} A_i$.*

*Proof.* The first group-action axiom $[X \xrightarrow{(1,\mathrm{id})} \boldsymbol{G}_m \times_S X \xrightarrow{\mu} X] = \mathrm{id}_X$ translates to

$$a \mapsto \sum_i t^i \otimes a_i \mapsto \sum_i a_i = a.$$

□

For any $i$ and $j$, the set $A_i \cdot A_j$ is a subset of $A_{i+j}$ since $\mu^*$ is a ring map. Thus, we have an $R$-algebra grading $A = \bigoplus_{i \in \boldsymbol{Z}} A_i$.

Note that one can recover the group action from the grading by writing $a = \sum_i a_i$ and setting $\mu^*(a) = \sum_i t^i a_i$. One readily verifies that defining a grading from an action and defining an action from a grading via the methods described are mutually inverse operations.

Now, let $T = \operatorname{Spec} B$ be an affine scheme. Let $p \colon \boldsymbol{G}_m \times_S X \to X$ denote the projection map. To say that a map $\nu \colon X \to T$ is an affine scheme is to say that $\nu \circ \mu$ equals $\nu \circ p$, which translates to $p^* \circ \nu^* = \mu^* \circ \nu^*$. For any $b \in B$, the element $p^* \circ \nu^*(b)$ equals $1 \otimes \nu(b)$, while $\mu^* \circ \nu^*(b)$ equals $\sum_i t^i \otimes \nu(b)_i$, so to say that $\nu$ is $\boldsymbol{G}_m$-invariant is to say that the image of $\nu^*$ is contained in the degree 0 piece of $A$ with respect to the grading induced by the $\boldsymbol{G}_m$-action.

In schematic language, letting $Y$ denote $\operatorname{Spec} A_0$, a map $\nu \colon X \to T$ is $\boldsymbol{G}_m$-invariant if and only if it factors uniquely through $Y$ via the map $q \colon X \to Y$ induced by the inclusion $A_0 \subset A$. That is, the map $q$ is the categorical quotient in the category of affine $S$-schemes.

We are going to prove that the following more general result holds:

**Proposition 8.** *If $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} A_0$ are Noetherian, then the map $q \colon X \to Y$ is the categorical quotient in the category of all affine $S$-schemes.*

*Proof.* **Step 1:** For all open subsets $V \subset Y$, the set $U = q^{-1}(X) \subset X$ is open and $\boldsymbol{G}_m$-stable. Thus, the group action $\mu$ restricts to a group action $\boldsymbol{G}_m \times U \to U$.

This proof generalizes to reductive groups acting on affine schemes, at least when $S$ is the spectrum of a field; see Mumford's Geometric Invariant Theory for more information.

Now, define a presheaf $(q_* \mathcal{O}_X)^{\boldsymbol{G}_m}$ by

$$(q_* \mathcal{O}_X)^{\boldsymbol{G}_m}(V) := \{f \in (q_* \mathcal{O}_X)(V) \mid \mu^* f = p^* f\}.$$

If $f$ is a section in $q_* \mathcal{O}_X(V)$, we can check if there holds $\mu^* f = p^* f$ locally, so $(q_* \mathcal{O}_X)^{\boldsymbol{G}_m}$ is in fact a sub*sheaf* of $q_* \mathcal{O}_X$. Since $q$ is $\boldsymbol{G}_m$-invariant, the map $q^* \colon \mathcal{O}_Y \to q_* \mathcal{O}_X$ factors through $(q_* \mathcal{O}_X)^{\boldsymbol{G}_m}$. In fact, we claim that the map $\mathcal{O}_Y \to (q_* \mathcal{O}_X)^{\boldsymbol{G}_m}$ is an isomorphism. To see this, let $(A_0)_g$ be a distinguished open in $Y$. We have

$$\begin{aligned}(q_* \mathcal{O}_X)^{\boldsymbol{G}_m}(\operatorname{Spec}(A_0)_f) &= \{f \in q_* \mathcal{O}_X(\operatorname{Spec}(A_0)_g) : \mu^* f = p^* f\} \\ &= \{f \in q_* \mathcal{O}_X(\operatorname{Spec}(A_0))_g : \mu^* f = p^* f\} \\ &= \{f \in A_f : \mu^* f = p^* f\} \\ &= (A_0)_f \\ &= \mathcal{O}_Y(\operatorname{Spec}(A_0)_f),\end{aligned}$$

where in the second step we used that $X$ is Noetherian.

**Step 2:** Let $Z = V(I) \subset X$ be a closed subset of $X$ cut out by an ideal $I$. We say that $Z$ is $\boldsymbol{G_m}$**-stable** if there exists a map $\mu_Z \colon \boldsymbol{G}_m \times_S Z \to Z$ making the following diagram commute:

$$\begin{tikzcd} \boldsymbol{G}_m \times_S Z \arrow[r, dashed, "\exists \mu_Z"] \arrow[d, hook] & Z \arrow[d, hook] \\ \boldsymbol{G}_m \times_S X \arrow[r, "\mu"'] & X; \end{tikzcd}$$

in ring-theoretic language, this means that $Z$ is $\boldsymbol{G}_m$-stable when there exists a ring map $\mu_Z^* \colon A/I \to R[t, t^{-1}] \otimes_S A/I$ making the following diagram commute:

$$\begin{tikzcd} R[t,t^{-1}] \otimes_R A/I & A/I \arrow[l, dashed, "\exists \mu_Z^*"'] \\ R[t,t^{-1}] \otimes_R A \arrow[u] & A. \arrow[l, "\mu^*"] \arrow[u] \end{tikzcd}$$

Thus, to say that $Z$ is $\boldsymbol{G}_m$-stable is to say that every $a \in I$ maps to maps to zero under the map $A \to R[t,t^{-1}] \otimes_R A \to R[t,t^{-1}] \otimes_R A/I$, i.e., the element $\sum_i t^i \otimes (a_i \bmod I)$ is zero. In other words, to say that $Z$ is $\boldsymbol{G}_m$-stable is to say that $I$ is homogeneous with respect to the grading on $A$ induced by the $\boldsymbol{G}_m$-action.

A consequence of this is that for a family $\{I_i\}_{i \in J}$ of ideals, with $J$ an index set, such that $I_i$ cuts out a $\boldsymbol{G}_m$-stable closed $Z_i$ for every $i \in J$, there holds $(\sum_{i \in J} I_i) \cap A_0 = \sum_{i \in J}(I_i \cap A_0)$. Thus, there holds $\overline{q(\bigcap_{i \in J} Z_i)} = \overline{\bigcap_{i \in J} q(Z_i)} = \bigcap_{i \in J} \overline{q(Z_i)}$.

It's not difficult to see that we can pull out the intersection in the second step in this particular instance, but just remember that intersection doesn't in general commute with closure!

**Step 3:** We claim that the map $q \colon X \to Y$ is surjective. Indeed, one readily verifies for any $\mathfrak{p} \in \operatorname{Spec} A_0$ that $\mathfrak{p}$ equals $A_0 \cap \mathfrak{p}A$. Since $A$ is Noetherian, the ring $A/\mathfrak{p}A$ has finitely many minimal primes $\mathfrak{q}_1, \cdots, \mathfrak{q}_r$. Now, for $n$ sufficiently large, there holds

$$\left(\bigcap_{i=1}^r \mathfrak{q}_i\right)^n \subset \mathfrak{p}A \subset \bigcap_{i=1}^r \mathfrak{q}_i.$$

Thus, we have

Should the first "⊂" be "=" instead?

$$\left(\bigcap_{i=1}^r \mathfrak{q}_i \cap A_0\right)^n \subset \left(\bigcap_{i=1}^r \mathfrak{q}_i\right)^n \cap A_0 \subset \mathfrak{p}A \cap A_0 = \mathfrak{p},$$

so for some $i$, the ideal $\mathfrak{p}$ must equal $\mathfrak{q}_i \cap A$.

Now, we claim that if $Z$ is a $\boldsymbol{G}_m$-stable closed subset, then $q(Z)$ is closed. To see this, suppose there exists a point $y \in \overline{q(Z)} \setminus q(Z)$ and let $Y$ denote $\overline{\{y\}}$. Then, the closed set $q^{-1}(Y)$ with the reduced scheme structure is $\boldsymbol{G}_m$-stable. We have

$$\begin{aligned} Y &= \overline{q(Z)} \cap Y \\ &= \overline{q(Z)} \cap q(q^{-1}(Y)) \\ &= q(Z \cap q^{-1}(Y)), \end{aligned}$$

where the second step used surjectivity and the third step used Step 2. This is a contradiction, since $q(Z \cap q^{-1}(Y))$ doesn't contain $Y$, so $q(Z)$ must be closed.

**Step 4 (Grand Finale):** Let $\nu \colon X \to T$ be a $\boldsymbol{G}_m$-invariant map. Let $W \subset T$ be an open affine. Then, the open subscheme $\nu^{-1}(W) \subset X$ is open and $\boldsymbol{G}_m$-stable and $Z := X \setminus \nu^{-1}(W)$ with the reduced scheme structure is closed and $\boldsymbol{G}_m$-stable. By Step 3, the set $q(Z)$ is closed, so $V_W := Y \setminus q(Z)$ is open with $q^{-1}(V) \subset \nu^{-1}(W)$.

Covering $V$ by affines and using Step 1, we get a unique map $V_W \to W$ making the following diagram commute:

$$\begin{tikzcd} \nu^{-1}(W) & \supset & q^{-1}(V_W) \arrow[r, "\nu"] \arrow[d, "q"'] & W \\ & & V_W. \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

Now, varying $W$ across all affine opens and gluing using Step 1, we get a unique map

$$\bigcup_{\text{open affines } W \subset T} V_W \to T$$

making the following diagram commute:

$$\begin{tikzcd} \bigcup_{\text{open affines } W \subset T} q^{-1}(V_W) \arrow[r, "\nu"] \arrow[d, "q"'] & W \\ \bigcup_{\text{open affines } W \subset T} V_W. \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

So it all comes down to showing that the union of all the various $V_W$ cover $Y$. We have

$$\begin{aligned} \bigcap_{\text{open affines } W \subset T} (Y \setminus V) &= \bigcap_{\text{open affines } W \subset T} q(X \setminus \nu^{-1}(W)) \\ &= q\left( \bigcap_{\text{open affines } W \subset T} X \setminus \nu^{-1}(W) \right) \\ &= q(\emptyset) = \emptyset, \end{aligned}$$

where in the second step we used Steps 2 and 3. □

**Example 9.** Suppose $S$ is Spec $k$ for $k$ a field. The group scheme $\boldsymbol{G}_m$ acts on $\boldsymbol{A}^n$ via $\lambda \cdot (x_1, \cdots, x_n) = (\lambda x_1, \cdots, \lambda x_n)$, which corresponds to the standard grading on $k[T_1, \cdots, T_n]$ given by $\deg(T_i) = 1$ for all $i$. The quotient $\boldsymbol{G}_m \backslash \boldsymbol{A}^n$ equals Spec $k$, for all orbits have $\{0\}$ in the closure and thus any $\boldsymbol{G}_m$-invariant function on $\boldsymbol{A}^n$ is constant.

**Example 10.** Suppose $S$ is Spec $k$ for $k$ a field. The group scheme $\boldsymbol{G}_m$ acts on $\boldsymbol{A}^2$ via $\lambda \cdot (x, y) = (\lambda x, \lambda^{-1} y)$. The quotient space $\boldsymbol{G}_m \backslash \boldsymbol{A}^2$ is Spec $k[xy]$. The orbit $q^{-1}(0)$ is $V(xy)$, while for nonzero $a \in \boldsymbol{G}_m \backslash \boldsymbol{A}^2 \cong \boldsymbol{A}^1$, the orbit $q^{-1}(a)$ is the hyperbola $xy = a$.

## Application to $\widetilde{\mathscr{M}}[1/6]$

Recall that $\widetilde{\mathscr{M}}[1/6]$ is represented by the spectrum of the ring $R = \boldsymbol{Z}[1/6][a_4, a_6][\Delta^{-1}]$, where $\Delta$ is given by $4a_4^3 + 27a_6^2$, with universal object given by the curve $\mathscr{E} \colon y^2 = x^3 + a_4 x + a_6$ together with differential $\pi = -dx/2y$. We saw last time that replacing $\pi$ with $u\pi$, then we get a different Weierstraß form for $\mathscr{E}$ by replacing $x$ with $u^2 x$ and $y$ with $u^3 y$. In other words, replacing $\pi$ with $u\pi$ replaces $a_i$ with $u^{-i} a_i$. In this way, the group $\boldsymbol{G}_m$ acts on $\widetilde{\mathscr{M}}[1/6]$ such that $\deg(a_4)$ and $\deg(a_6)$ are 4 and 6, respectively, and $\deg(\Delta)$ is 12. One can then check that the degree 0 piece $R_0$ corresponding to this grading is

$$R_0 = \boldsymbol{Z}\left[\frac{1}{6}\right]\left[\frac{a_4^3}{-16\Delta}\right] = \boldsymbol{Z}\left[\frac{1}{6}\right][j], \qquad j = -1728 \cdot \frac{4a_4^3}{-16\Delta},$$

so we have an isomorphism $\boldsymbol{G}_m \backslash \widetilde{\mathscr{M}}[1/6] \xrightarrow{\sim} \boldsymbol{A}^1_{\boldsymbol{Z}[1/6]}$. This gives us an intrinsic definition of the $j$-invariant (up to constants in $\boldsymbol{Z}[1/6]^\times$)!

## Proof that $\widetilde{\mathscr{M}}[1/6]$ is a coarse moduli space for $\mathscr{M}[1/6]$

As before, let $q \colon \widetilde{\mathscr{M}}[1/6] \to \boldsymbol{G}_m \backslash \widetilde{\mathscr{M}}[1/6] \cong \boldsymbol{A}^1_{\boldsymbol{Z}[1/6]}$ denote the quotient map and let $f \colon \widetilde{\mathscr{M}}[1/6] \to \mathscr{M}[1/6]$ denote the forget map. Now, define a map $\mathscr{M}[1/6] \to \boldsymbol{A}^1_{\boldsymbol{Z}[1/6]}$ via the $j$-invariant as follows: Given $E \in \mathscr{M}[1/6](S)$, trivialize $\omega_{E/S}$ locally on $S$ in a sufficiently small neighborhood $U$. Pick a Weierstraß equation, take the $j$-invariant $j \in \mathscr{O}_S(U)$. Since the $j$-invariant is independent of Weierstraß equation, we can glue the local $j$-invariants to a section $j(E) \in \mathscr{O}_S$. Now, since the base change of an elliptic curve cut out by a given Weierstraß equation is the curve given by the Weierstraß equation with coefficients pulled back from the original Weierstraß equation. Thus, if $u \colon T \to S$ is a map, then $j(T \times_S E)$ equals $u^* j(E) \in \mathscr{O}_T(T)$, so $E \mapsto j(E)$ is a natural transformation.

Now, let's prove the categorical property of coarse moduli spaces. Given a scheme $Y$ and a map $\nu \colon \mathscr{M}[1/6] \to Y$, the map $\nu \circ f \colon \widetilde{\mathscr{M}}[1/6] \to Y$ is $\boldsymbol{G}_m$-invariant, so we get a map $u \colon \boldsymbol{G}_m \backslash \widetilde{\mathscr{M}}[1/6] \to Y$ such that $\nu \circ f$ equals $u \circ q$. We have the diagram

$$\begin{tikzcd}
\widetilde{\mathscr{M}}[\frac{1}{6}] \arrow[r, "f"] \arrow[d, "q"'] & \mathscr{M}[\frac{1}{6}] \arrow[dl, "j"'] \arrow[d, "\nu"] \\
\boldsymbol{G}_m \backslash \widetilde{\mathscr{M}}[\frac{1}{6}] \cong \boldsymbol{A}^1_{\boldsymbol{Z}[1/6]} \arrow[r, "u"'] & Y,
\end{tikzcd}$$

the upper left triangle of which commutes by construction. To finish proving the categorical property, we need to check that the lower right triangle commutes. To do this, it suffices to check "pointwise", for it suffices to check for every scheme

$S$, there holds $\nu(S) = u(S) \circ j(S)$. To see this, note that if $E/S$ is a point in $\mathscr{M}[1/6](S)$, we can locally trivialize $\omega_{E/S}$ on a sufficiently small neighborhood $U \subset S$, at which point we can lift $E|_U$ through the map $f$. Thus, we have

$$\nu(U)(E|_U) = u(U) \circ q(U) \circ f(U)(E|_U) = j(U) \circ f(U)(E|_U),$$

so $\nu(S)$ equals $u(S) \circ j(S)$.

In order to finish the proof that $\boldsymbol{G}_m \backslash \widetilde{\mathscr{M}}[1/6]$ is a coarse moduli space for $\mathscr{M}[1/6]$, we need to show that for all algebraically closed fields $k$, the map $j(k)$ is an isomorphism. We omit the proof of this here; one can be found, however, in Silverman's book.

## A Fun Observation

There is no elliptic curve $\mathscr{E} \to \operatorname{Spec} \boldsymbol{Z}[1/6][j]$ with $j$-invariant $j$, as we now explain. If there did exist such an elliptic curve, the sheaf $\omega_{\mathscr{E}|_{\boldsymbol{A}^1_{\boldsymbol{Q}}}}$ would be trivial since $\boldsymbol{Q}[j]$ is a PID, and $\Delta(\mathscr{E})$ would be an element of $\boldsymbol{Q}[j]^\times = \boldsymbol{Q}$, and we would have a solution $\frac{a_4[j]^3}{\Delta} = j$. But $\frac{a_4[j]^3}{\Delta} = j$ has no solution in $\boldsymbol{Q}[j]$. This again shows that $\mathscr{M}[1/6]$ has no fine moduli space.