The Weil Extension Theorem

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The following is a WIP typed adaptation of Dr. Andreas Mihatsch's notes on the Weil Extension Theorem, available here: `https://www.math.uni-bonn.de/people/mihatsch/21u22%20WS/moduli/`.

Statement

Our goal today is to prove the Weil Extension Theorem:

Theorem 1 (Weil Extension Theorem)**.** *Let S be a connected Dedekind scheme with generic point η and let E and E' be elliptic curves over S. Then, the natural map* $\mathrm{Hom}_S(E, E') \to \mathrm{Hom}_S(E_\eta, E'_\eta)$ *is an isomorphism.*

This is in fact only a special case of the full Weil Extension Theorem:

Theorem 2 (Weil Extension Theorem)**.** *Suppose that S is a normal and Noetherian scheme. Let Z be a smooth S-scheme, let G be a separated group scheme, and let $u\colon Z \dashrightarrow G$ be a rational map. If u is defined in codimension* 1*, then it is defined everywhere.*

We'll only prove Theorem 1 today. The proof of Theorem 2 can be found in Section 4.4 of BLR's *Néron Models*.

Important Lemmas

We'll need some important lemmas for the proof of Theorem 1.

Lemma 3. *Suppose that X is a separated S-scheme, and suppose that U is a schematically dense open subscheme of an S-scheme Y. Then, the natural map* $\mathrm{Hom}_S(Y, X) \to \mathrm{Hom}_S(U, S)$ *is an injection.*

To see why separatedness is necessary, consider the case where Y is the affine line and X is the line with two origins, and U is the punctured affine line. Let $\varphi\colon U \to X$ be the natural inclusion of U into X arising from viewing U as X minus both of its two origins. Then, the map φ extends in two ways!

Lemma 4 (see Stacks [01ZC])**.** *Let X be an S-scheme that is locally of finite presentation. Let Y be an S-scheme and let y be a point in Y. The natural map* $\mathrm{colim}_{\textit{opens } U \ni y} \mathrm{Hom}_S(U, X) \to \mathrm{Hom}_S(\mathrm{Spec}\, \mathcal{O}_{Y,y}, X)$ *is an isomorphism.*

An important variation on Lemma 4 is as follows:

Lemma 5. *Let X be an S-scheme that is locally of finite presentation and let Y be an S-scheme with a qcqs structure map. The natural map* $\mathrm{colim}_{\textit{opens } U \ni y} \mathrm{Hom}_S(U \times_S Y, X) \to \mathrm{Hom}_S(\mathrm{Spec}\, \mathcal{O}_{S,s} \times Y, X)$ *is an isomorphism.*

Now, suppose that E and E' are schemes over a connected Dedekind scheme with generic point η, and we are given a map $\varphi_\eta\colon E_\eta \to E'_\eta$. Lemma 4 implies that if φ_η lifts to a map $\varphi\colon E \to E'$, then φ is unique. It also implies that if $\{U_i\}$ is an open cover of E and we can lift φ_η to $\varphi_i \colon U_i \to E'$ for every i, then the various φ_i glue to a lift $E \to E'$.

Moreover, Lemma 5 implies that the existence of a lift of φ_K to a map $E \to E'$ is equivalent to the existence of a lift $\operatorname{Spec} \mathcal{O}_{S,s} \times_S E \to \operatorname{Spec} \mathcal{O}_{S,s} \times_S E'$ for every $s \in S$. In particular, it suffices to prove Theorem 1 assuming that S is the spectrum of a DVR.

The Heart of the Proof

Proof of Theorem 1. **Step 1:** To briefly recall our situation, our goal is to show that if E and E' are two elliptic curves over a connected Dedekind scheme with generic point η, then every map $\varphi_\eta \colon E_\eta \to E'_\eta$ lifts uniquely to a map $\varphi \colon E \to E'$. As explained at the end of the previous section, we can assume that S is the spectrum of a discrete valuation ring R. Let s and η denote the point corresponding to the maximal ideal of R and the generic point, respectively.

Now, let $x \in E_s$ be the generic point of the special fiber E_s. Then, the ring $\mathcal{O}_{E,x}$ is a DVR since x is a height 1 point on a regular scheme. The Valuative Criterion for Properness applied to the structure map $E' \to S$ implies that there exists a map $\operatorname{Spec} \mathcal{O}_{E,x} \to E'$ lifting the natural map $\operatorname{Spec} \mathcal{O}_{E,x} \to S$. By Lemma 4, there exists an open $U \subset E$ such that $\operatorname{codim}_E(E \setminus U)$ equals 2 and there exists an extension $\psi \colon U \to E'$ of $\operatorname{Spec} \mathcal{O}_{E,s} \to S$.

Step 2: Now, suppose the following hold:

1. The set $E \setminus U$ consists of $\kappa(s)$-rational points $\overline{a_1}, \dots, \overline{a_r}$;
2. there exist $a_1, \dots, a_r \in E(R)$ such that for all i, we have $a_i(s) = \overline{a_i}$; and
3. there exists $b \in E(R)$ such that for all i, the point $b + a_i$ lies in $U(R)$.

Then, denoting by t_b translation by b and and $t_{\psi(b)}$ translation by $\psi(b)$, the map $t_{\psi(b)}^{-1} \circ \psi \circ t_b \colon t_b^{-1}(U) \to E'$ agrees with the map $\psi \colon U \to E'$ on $t_b^{-1}(U) \cap U$ since ψ_η is a group homomorphism. Since $t_b^{-1}(U) \cup U$ equals E, so they glue to a map $E \to E'$ extending ψ, which in turn extends φ_η. So we just need to reduce to the situation in which (1), (2), and (3) hold.

To do this, we will show that it suffices to prove our theorem after base changing by a faithfullly flat extension of DVRs; and then we'll show that there exists a DVR R' and a faithfully flat extension of DVRs $R \to R'$ such that (1), (2), and (3) hold after base-changing by R' along $R \to R'$.

Step 3: Suppose that $R \to R'$ is a faithfully flat extension of DVRs. [1] Suppose further that there exists a map $\varphi' \colon R' \times_R E \to R' \times_R E'$ lifting the base change along $R \to R'$ of φ_η. We claim that the map φ' must arise as the base change along $R \to R'$ of some map $\varphi \colon E \to E'$. To see this, we will apply fpqc descent for maps of schemes. Recall that fpqc descent says that if X and Y are schemes and $X' \to X$ is an fpqc morphism of schemes, then, letting p_1 and p_2 denote the projections of $X' \times_X X'$ to X', the set $\operatorname{Hom}(X, Y)$ is in natural bijection with

$$\operatorname{Eq}\left(\operatorname{Hom}(X', Y) \overset{-\circ p_1}{\underset{-\circ p_2}{\rightrightarrows}} \operatorname{Hom}(X' \times_X X', Y) \right).$$

[1] This means that R' is a DVR and the map $R \to R'$ is injective, local, and faithfully flat.

Apply fpqc descent with $X = E$ and $X' = R' \times_R E$ and $Y = E'$, it suffices to check that $\varphi' \circ p_1$ equals $\varphi' \circ p_2$. Since φ' lifts $R' \times_R \varphi_\eta$, we know that $(\varphi' \circ p_1)_\eta$ and $(\varphi' \circ p_2)_\eta$ are both equal to $R' \times_R \varphi_\eta$. From here, the schematic density of $(R' \times_R T' \times_R E)_\eta$ in $R' \times_R R' \times_R E$ yields the claim via Lemma 3.

Step 4: Finally, we claim that there exists a DVR R' and a faithfully flat extension of DVRs $R \to R'$ such that the assumptions (1), (2), and (3) in Step 2 all hold.

To see this, suppose $a \in E_s$ is a closed point. The extension $\kappa(a)/\kappa(s)$ is finite, of the form $\kappa[t_1, \dots, t_m]/(f_1, \dots, f_\ell)$. Denote by $\widetilde{f_1}, \dots, \widetilde{f_\ell}$ respective lifts of the polynomials $f_1, \dots, f_\ell$ to $R[t_1, \dots, t_m]$. Now, let T be the normalization of an irreducible component of $R[t_1, \dots, t_m]/(\widetilde{f_1}, \dots, \widetilde{f_\ell})$. Then, the point a is the image of an R_1-point in the pullback of the faithfully flat $\operatorname{Spec} T \to \operatorname{Spec} R$ along the structure map of E. Iterating this process, we can find a ring R_1 and a faithfully flat map $R \to R_1$ such that $(R_1 \times_R E) \setminus (R_1 \times_R U)$ consists of rational points

Finish the last bit of this proof.

□

The structure map of E_s is a proper (in particular, quasicompact) map into a point, so E_s is quasicompact; thus, the scheme E_s has a closed point.

If $\kappa(a)/\kappa(s)$ happens to be separable, the Primitive Element Theorem yields $m = \ell = 1$.