Fine Moduli (away from 2 and 3)

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The following is a WIP typed adaptation of Dr. Andreas Mihatsch's notes on the fine moduli, available here: https://www.math.uni-bonn.de/people/mihatsch/21u22%20WS/moduli/.

Torsors

Let G/S be a group scheme and let X and Y be schemes over S. Suppose that G acts on X and suppose we have a G-invariant map $\varphi\colon X \to Y$.

Definition 1. *The G-invariant map φ is called a **G-torsor** if the following two conditions are satisfied:*

1. *the map $G \times_S X \to X \times_Y X\colon (g, x) \mapsto (gx, x)$ is an isomorphism; and*
2. *there exists a covering $Y' \to Y$, usually fppf, and a section $Y' \to Y' \times_Y X$ to the natural map $Y' \times_Y X \to Y'$.*

Sometimes we want the covering to be fpqc, étale, or something else. If we don't specify, we mean fppf.

The idea is as follows. Given $y\colon T \to Y$, the first axiom says that any two lifts $x_1, x_2\colon T \rightrightarrows X$ differ by a *unique* $g \in G(T)$, i.e., there holds $x_2 = g \circ x_1$. In other words, the set $\{x \in X(T)$ lifting $y\}$ is either empty or isomorphic to $G(T)$. The second axiom says that there exists a covering $T' \to T$ and $x \in X(T')$ lifting Y.

This excludes, for example, the case $X = \emptyset$.

Example 2. Suppose that L/K is a finite Galois extension with Galois group Γ. Letting X and Y denote $\operatorname{Spec} L$ and $\operatorname{Spec} K$, respectively, the group Γ acts on X and the map $X \to Y$ is Γ-invariant. Then, the map $X \to Y$ is a torsor for the étale topology, where the cover of Y is X itself.

Example 3. If $X \to Y$ is an isogeny of abelian varieties over a field with kernel K, then the map $X \to Y$ is K-invariant and a torsor for the fppf topology, where the cover of Y is again X itself.

Definition 4. *A group action $G \times_S X \to X$ is called **free** if the map $G \times_S X \to X \times_S X$ is a closed immersion.*

Theorem 5. *Suppose that G/S is finite and locally free, that X/S is separable, that G acts on X freely, and that there exists a cover of X by G-stable affine sets. Then, the quotient $Y = G\backslash X$ exists, the map $X \to Y$ is finite and locally free, and the map $G \times_S X \to X \times_Y X$ is an isomorphism. In particular, the map $X \to Y$ is a G-torsor for the fppf topology.*

Proof. Abelian Varieties, Lecture 14. □

$\boldsymbol{G}_m$-torsors

Lemma 6. *Suppose that $X \to Y$ is a GL_n-torsor for the étale or fppf or fpqc topology. Then, there exists a Zariski covering $\{U_i\}$ of Y such that $X(U_i)$ is nonempty.*

Proof. This is a consequence of fpqc descent for vector bundles. The full proof is omitted. □

The above lemma means that torsors for the different topologies coincide.

Lemma 7. *The $\boldsymbol{G}_m$-torsors $\pi\colon X \to Y$ are in bijection with elements of $\mathrm{Pic}(Y)$.*

Sketch. One can check that the maps

$$\{\boldsymbol{G}_m\text{-torsors } X \xrightarrow{\pi} Y\} \to \mathrm{Pic}(Y)\colon X \mapsto \{\ell \in \pi_* \mathcal{O}_X : \mu^*(\ell) = t \otimes \ell\} \text{ and}$$
$$\mathrm{Pic}(Y) \to \{\boldsymbol{G}_m\text{-torsors } X \xrightarrow{\pi} Y\}\colon \mathcal{L} \mapsto \mathcal{S}pec \bigoplus_{i \in \boldsymbol{Z}} \mathcal{L}^{\otimes i}$$

are mutually inverse. □

Now, suppose that $X = \operatorname{Spec} A$ is an affine scheme over an affine scheme $S = \operatorname{Spec} R$. Suppose that $\boldsymbol{G}_m$ acts on X with an action corresponding to a grading $A = \bigoplus_{i \in \boldsymbol{Z}} A_i$.

Proposition 8. *If the action of $\boldsymbol{G}_m$ on X is free, then the quotient map $q\colon X \to Y = \operatorname{Spec} A_0$ is a $\boldsymbol{G}_m$-torsor. More precisely, the sheaf $\widetilde{A_1}$ is a line bundle over Y and A equals $\bigoplus_{i \in \boldsymbol{Z}} A_1^{\otimes i}$.*

Proof. Since the action of $\boldsymbol{G}_m$ on X is free, the corresponding ring map $A \otimes_R A \to R[t, t^{-1}] \otimes_R A\colon a \otimes b \mapsto \sum_{i \in \boldsymbol{Z}} t^i \otimes a_i b$ is surjective. In particular, the element $t \otimes 1$ is in the image. This implies that there exist elements $e_1, \dots, e_r \in A_1$ and $f_1, \dots, f_r \in A_{-1}$. such that $\sum_{i=1}^r e_i f_i$ equals 1. For each i, let u_i denote $e_i f_i$. Since the degree of each u_i is zero, the scheme $\operatorname{Spec} A_0$ is equal to the union $\bigcup_{i=1}^r D(u_i)$.

Now, on $D(u_i)$, the element e_i is invertible (as is f_i). Given an element $a \in A_1[u_i^{-1}]$, we may write

$$a = a \sum_{j=1}^r e_j f_j = \underbrace{\left(a \sum_{j=1}^r \frac{e_j f_j}{e_i} \right)}_{\in A_0} e_i.$$

Since e_i is invertible in $A[u_i^{-1}]$, multiplication by e_i yields an isomorphism between $A_0[u_i^{-1}]$ and $A_1[u_i^{-1}]$. Thus, the sheaf $\widetilde{A_1}$ is a line bundle over Y.

Now, given $a \in A_d[u_i^{-1}]$, we can write $a = (a e_i^{-d}) e_i^d$, so each element of $A_d[u_i^{-1}]$ is an element of $A_0[u_i^{-1}] e_i^d$. This shows that the natural map $\bigoplus_{i \in \boldsymbol{Z}} A_1^{\oplus i} \to A$ is an isomorphism. □

An important consequence of this is that $X \to Y$ has sections Zariski locally and these are unique up to the $\boldsymbol{G}_m$-action.

Write up the example and proof that there is no elliptic curve over $\boldsymbol{Q}[j]_{(j)}$ of j-invariant j, as well as the remark.

Level Structure

Let $n \geq 1$ be an integer and suppose that S is a scheme such that n is invertible in $\mathcal{O}_S(S)$.

Let $E \to S$ be an elliptic curve. Then, the map $E[n] \to S$ is finite étale of order n^2.

First, let's recall two important principles:

1. If $f: X \to Y$ is finite étale, then f is closed since it is finite and open since flat maps of locally finite presentation are open (see Stacks [01UA]).

2. If the diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "u"'] & & Y \arrow[dl, "v"] \\ & S & \end{tikzcd}$$

commutes and both u and v are finite étale, then f is finite étale as well.

Proposition 9. *Suppose that $X \to S$ is finite étale of degree d. Then, there exists a scheme S' and a finite étale map $S' \to S$ such that $S' \times_S X$ is isomorphic as an S'-scheme to a disjoint union of d copies of S'.*

Proof. We proceed by induction on d.

Base case: If d equals 1, then X is isomorphic to S and the claim is obvious.

Inductive step: By Principle 2, the diagonal map $X \to X \times_S X$ is finite étale. Principle 1 implies that $X \times_S X$ is the disjoint union of the diagonal and everything else. Thus, we can pull back $X \to S$ along itself and $X \times_S X$. Applying the inductive hypothesis to the diagonal component of $X \times_S X$ and the complement thereof (and taking the fiber product of the respective schemes resulting from these applications) yields the claim. □

Proposition 10. *Let E/S be an elliptic curve. Maintaining the assumption that n is invertible in $\mathcal{O}_S(S)$, there exists a scheme S' and a finite étale map $S' \to S$ such that $S' \times_S E[n]$ is isomorphic to the constant group scheme $\underline{(\mathbf{Z}/n)^{\otimes 2}}_{S'}$*

Proof. By Proposition 9, there exists a finite étale covering $S' \to S$ such that $S' \times_S E[n]$ is isomorphic to the disjoint union of n^2 copies of S'.

Type up the rest of this proof.

□

Proposition 11. *Let E/S be an elliptic curve. Maintaining the assumption that n is invertible in $\mathcal{O}_S(S)$, the contravariant functor*

$$\begin{aligned} L_{E,n} &: \mathbf{Sch}/S \to \mathbf{Set} \\ &: T/S \to \{\alpha : \underline{(\mathbf{Z}/n)^{\oplus 2}}_T \xrightarrow{\sim} T \times_S E[n]\} \end{aligned}$$

is representable by a finite étale S-scheme. Moreover, the map $L_{E,n} \to S$ is a $\mathrm{GL}_2(\mathbf{Z}/n)$-torsor for the étale topology (where $\mathrm{GL}_2(\mathbf{Z}/n)$ acts on $L_{E,n}$ via $g \cdot \alpha := \alpha \circ g$).

Note that a group homomorphism $\underline{(\mathbf{Z}/n)^{\oplus 2}}_T \to T \times_S E[n]$ is the same as two elements in $E[n](T)$.

Proof. Consider the scheme $X := E[n] \times_S E[n]$. A group homomorphism $\underline{(\boldsymbol{Z}/n)^{\oplus 2}}_T \to T \times_S E[n]$ is the same as two elements in $E[n](T)$, so $X(T)$ is the same as $\mathrm{Hom}(\underline{(\boldsymbol{Z}/n)^{\oplus}}_T, T \times_S E[n])$. Let a and b be two elements of $(\boldsymbol{Z}/n)^{\oplus 2}$ that are not both zero. Let $m_{a,b} \colon X \to E[n]$ be the map $(\alpha_1, \alpha_2) \mapsto (a\alpha_1, b\alpha_2)$. This map is finite étale, so $B_{a,b} \colon X \times_{m_{a,b}, E[n], e} S \to X$, which is informally the locus where a and b give non-trivial relations of α_1 and α_2, is both open and closed. Then, the space $X \setminus \bigcup_{(a,b)\neq 0} B_{a,b}$ represents $L_{E,n}$.

To see that $L_{E,n}$ is a GL_2-torsor, first note that Proposition 10 implies that there exists a finite étale covering $T \to S$ such that $T \times_S E[n]$ is isomorphic to $\underline{(\boldsymbol{Z}/n)^{\oplus 2}}_T$. So there holds

$$T \times_S \underline{\mathrm{Iso}}(\underline{(\boldsymbol{Z}/n)^{\oplus 2}}_S, E[n]) \cong \underline{\mathrm{Iso}}(\underline{(\boldsymbol{Z}/n)^{\oplus 2}}_T, \underline{(\boldsymbol{Z}/n)^{\oplus 2}}_T) \cong \underline{\mathrm{GL}_2(\boldsymbol{Z}/n)}_T,$$

which verifies the torsor property. □

Definition 12. *A **level-n structure** on an elliptic curve E/S is an isomorphism from $E[n]$ to $\underline{(\boldsymbol{Z}/n)^{\oplus 2}}_S$. An **isomorphism of elliptic curves with level-n structure** $(E, \alpha) \to (E', \alpha')$ is an isomorphism $\varphi \colon E_1 \xrightarrow{\sim} E_2$ such that α_2 equals $\varphi \circ \alpha_1$.*

Let $\mathcal{M}_n$ denote the contravariant functor

$$\begin{aligned} \mathbf{Sch}/\boldsymbol{Z}\left[\frac{1}{n}\right] &\to \mathbf{Set} \\ S &\mapsto \{(E, \alpha)/S\}/\cong \end{aligned}$$

and let $\widetilde{\mathcal{M}_n}$ denote the functor

$$\begin{aligned} \mathbf{Sch}/\boldsymbol{Z}\left[\frac{1}{6n}\right] &\to \mathbf{Set} \\ S &\mapsto \{(E, \alpha, \pi)/S : (E, \alpha) \in \widetilde{\mathcal{M}}\left[\frac{1}{6}\right](S), \alpha \in L_{E,n}(S)\}/\cong. \end{aligned}$$

Proposition 13. *The scheme $\widetilde{\mathcal{M}_n}$ is representable by an affine scheme.*

Proof. The functor $\widetilde{\mathcal{M}_n}$ is naturally isomorphic to $L_{\mathscr{E},n}$. □

Now, we have a natural "forget" map $\pi \colon \widetilde{\mathcal{M}_n} \to \mathcal{M}_n[1/6]$. We also have a quotient map $q \colon \widetilde{\mathcal{M}_n} \to \boldsymbol{G}_m \backslash \widetilde{\mathcal{M}_n}$.

Theorem 14. *Assume that n is not less than 3. Then, the schemes $\boldsymbol{G}_m \backslash \widetilde{\mathcal{M}_n}$ and $\mathcal{M}_n[1/6]$ are isomorphic. In particular, the functor $\mathcal{M}_n$ is representable by an affine scheme.*

Before we prove this, we state and prove the following proposition, which explains why this is plausible.

Proposition 15. *If n is not less than 3, then an elliptic curve (E, α) with level-n structure has no nontrivial automorphisms.*

Type up the first proof.

Proof. Let φ be an element of $\operatorname{Aut}(E)$ and suppose that $\varphi|_{E[n]}$ is the identity, i.e., that $E[n]$ is contained in $\ker(\varphi - 1)$. Last term, it was proved that the following sequence is exact:

$$0 \to E[n] \to E \xrightarrow{\cdot n} E \to 0.$$

By the quotient property, there exists a map ψ making the following diagram commute:

$$\begin{tikzcd} E \arrow[r, "\cdot n"] \arrow[d, "\varphi-1"'] & E = E/E[n] \arrow[dl, dashed, "\exists\psi"] \\ E. & \end{tikzcd}$$

The map $\varphi - 1$ equals $n\psi$, so we have

$$\begin{aligned} n^2 \deg \psi &= (\varphi - 1)(\varphi^* - 1) \\ &= \deg \varphi - (\varphi + \varphi^*) + 1 \\ &= 2 - (\varphi + \varphi^*) \end{aligned}$$

By our classification of elliptic curve endomorphism rings, we must have $|\varphi + \varphi^*| \leq 2$, so $n^2 \deg \psi$ is not greater than 4. If n is at least 3, this forces ψ to be 0. $\square$

Proof of Theorem 14

Recall that n is assumed to be not less than 3 and that we're working over $\boldsymbol{Z}[1/(6n)]$.

First, *assume* we know that $\boldsymbol{G}_m$ acts on $\widetilde{\mathscr{M}_n}$ freely. Then, the quotient map $q\colon \widetilde{\mathscr{M}_n} \to \boldsymbol{G}_m \backslash \widetilde{\mathscr{M}_n}$ is a $\boldsymbol{G}_m$-torsor and, as such, has local sections, unique up to the $\boldsymbol{G}_m$-action.

We will construct mutually inverse maps

$$\boldsymbol{G}_m \backslash \widetilde{\mathscr{M}_n} \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \mathscr{M}_n.$$

For $y \in (\boldsymbol{G}_m \backslash \widetilde{\mathscr{M}_n})(s)$, define $\Phi(y)$ as follows. We can cover S by a set of Zariski opens $\{S_i\}_{i\in I}$ such that on each S_i, there exists (E_i, α_i, π_i) lifting $y|_{S_i}$. Letting S_{ij} denote $S_i \cap S_j$ for each $i, j \in I$, the torsor property implies that $(E_i, \alpha_i \pi_i)|_{vS_i}$ equals $(E_j, \alpha_j, \lambda_{ij}\pi_j)_{S_{ij}}$ for a unique $\lambda_{ij} \in \mathscr{O}_S(S_{ij})^\times$, so we get a unique isomorphism $\varphi_{ij}\colon (E_i, \alpha_i)|_{S_{ij}} \xrightarrow{\sim} (E_j, \alpha_j)|_{S_{ij}}$. By Proposition 15, the various φ_{ij} satisfy the cocycle condition, and thus we can glue the various (E_i, α_i) into an element $(E, \alpha) \in \mathscr{M}_n(S)$. Set $\Phi(y)$ to (E, α).

For $(E, \alpha) \in \mathscr{M}_n(S)$, define $\Psi((E, \alpha))$ as follows. We can cover S by a set of Zariski opens $\{S_i\}_{i\in I}$ such that $\omega_E|_{S_i}$ equals $\mathscr{O}_{S_i}$ for every $i \in I$. On each of the S_i, pick lifts $(E, \alpha, \pi_i) \in \widetilde{\mathscr{M}_n}(S_i)$. Since q is $\boldsymbol{G}_m$-invariant, the various $q((E, \alpha, \pi_i))$ glue and define an element $y \in \boldsymbol{G}_m \backslash \widetilde{\mathscr{M}_n}$. Set $\Psi((E, \alpha))$ to y.

One readily verifies that Φ and Ψ are mutually inverse, so as long as we accept that $\boldsymbol{G}_m$ acts on $\widetilde{\mathscr{M}_n}$ freely, the schemes $\boldsymbol{G}_m \backslash \widetilde{\mathscr{M}_n}$ and $\mathscr{M}_n$ are isomorphic as claimed.

So it remains to show that the action of $\boldsymbol{G}_m$ on $\widetilde{\mathscr{M}_n}$ is free.

The Weil Extension Theorem

We want to show that $\boldsymbol{G}_m \times_{\boldsymbol{Z}[1/(6n)]} \widetilde{\mathcal{M}_n} \to \widetilde{\mathcal{M}_n} \times_{\boldsymbol{Z}[1/(6n)]} \widetilde{\mathcal{M}_n}$ is a closed immersion. Equivalently, we want to show that this map is a proper monomorphism (see Stacks [04XV]).

To see that $\boldsymbol{G}_m \times_{\boldsymbol{Z}[1/(6n)]} \widetilde{\mathcal{M}_n} \to \widetilde{\mathcal{M}_n} \times_{\boldsymbol{Z}[1/(6n)]} \widetilde{\mathcal{M}_n}$ is a monomorphism, consider λ and λ' in $\boldsymbol{G}_m(S)$ and two elements (E, α, π) and (E', α', π') in $\widetilde{\mathcal{M}_n}(S)$. We claim that if there exist isomorphisms $\varphi\colon (E, \alpha, \pi) \xrightarrow{\sim} (E', \alpha', \pi')$ and $\psi\colon (E, \alpha, \lambda\pi) \xrightarrow{\sim} (E', \alpha', \lambda'\pi')$, then λ equals λ'. Since n is assumed to be not less than 3, there is at most one isomorphism $(E, \alpha) \xrightarrow{\sim} (E', \alpha')$, so φ equals ψ. In particular, we have $\varphi^*(\pi') = \pi$ and $\lambda'\varphi^*(\pi') = \lambda\pi$. Thus, we have $\lambda = \lambda'$.

It remains to check that $\boldsymbol{G}_m \times_{\boldsymbol{Z}[1/(6n)]} \widetilde{\mathcal{M}_n} \to \widetilde{\mathcal{M}_n} \times_{\boldsymbol{Z}[1/(6n)]} \widetilde{\mathcal{M}_n}$ is proper. We would like to use the valuative criterion for properness; that is, we would like to show that if R is a DVR with fraction field K, and (E, α, π) and (E', α', π') are elements of $\widetilde{M}_n(R)$ such that there exists an isomorphism $\varphi_K\colon (E, \alpha)_K \xrightarrow{\sim} (E', \alpha')_K$ with $\varphi_K(\pi') = \lambda\pi$ for $\lambda \in K^\times$, then λ is in fact an element of $R^\times$ and φ_K lifts uniquely to a map $(E, \alpha) \to (E', \alpha')$. We will make use of the Weil Extension Theorem:

The valuative criterion for properness states that a finite type, quasiseparated map $X \to Y$ is proper if and only if for every Dedekind ring R and every commutative square

$$\begin{tikzcd} \operatorname{Spec}(\operatorname{Frac} R) \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} R \arrow[r] & Y \end{tikzcd}$$

there exists a unique map $\operatorname{Spec} R \to X$ such that the diagram

$$\begin{tikzcd} \operatorname{Spec}(\operatorname{Frac} R) \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed, "\exists!"] & Y \end{tikzcd}$$

commutes. Equivalently, the natural map $\operatorname{Hom}_Y(\operatorname{Spec} R, X) \to \operatorname{Hom}_Y(\operatorname{Spec}(\operatorname{Frac} R), X)$ is an isomorphism (this is the version we're using in the text).

Theorem 16 (Weil Extension Theorem)**.** *If S is a connected Dedekind scheme with generic point η, and E and E' are elliptic curves over S, then the natural map* $\operatorname{Hom}(E, E') \to \operatorname{Hom}(E_\eta, E'_\eta)$ *is an isomorphism.* [1]

[1] Since E_η and E'_η are schematically dense and E' is separated, injectivity is immediate; the nontrivial part is surjectivity.

We'll prove the Weil Extension Theorem soon, but for now, let's see why it implies our desired properties. The Weil Extension Theorem tells us that there is a unique lift of φ_K to a map $\varphi\colon E \to E'$, which is necessarily an isomorphism since φ_K^{-1} also lifts. Since $E[n]_K$, $E'[n]_K$, and $\underline{(\boldsymbol{Z}/n)^{\oplus 2}}_K$ are schematically dense and $\varphi_K \circ \alpha_K$ equals α'_K, we must have $\varphi \circ \alpha = \alpha'$. We also have $\varphi^*(\pi') = \mu\pi$ for some $\mu \in R^\times$, and thus we have $\mu = \lambda$ since $\Gamma(E, \Omega_{E/R})$ injects into $\Gamma(E_K, \Omega_{E/K})$.