Topics in nonlinear self-dual supersymmetric theories

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Abstract

Theories of self-dual supersymmetric nonlinear electrodynamics are generalized to a curved superspace of 4D $\mathcal{N} = 1$ supergravity, for both the old-minimal and the new-minimal versions of $\mathcal{N} = 1$ supergravity. We derive the self-duality equation, which has to be satisfied by the action functional of any U(1) duality invariant model of a massless vector multiplet, and show that such models are invariant under a superfield Legendre transformation. We construct a family of self-dual nonlinear models, which includes a minimal curved superspace extension of the $\mathcal{N} = 1$ supersymmetric Born-Infeld action. The supercurrent and supertrace of such models are explicitly derived and proved to be duality invariant.

The requirement of nonlinear self-duality turns out to yield nontrivial couplings of the vector multiplet to Kähler sigma models. We explicitly construct such couplings in the case when the matter chiral multiplets are inert under the duality rotations, and more specifically to the dilaton-axion chiral multiplet when the group of duality rotations is enhanced to $\mathrm{SL}(2, \mathbb{R})$.

The component structure of the nonlinear dynamical systems introduced proves to be more complicated, especially in the presence of supergravity, as compared with well-studied effective supersymmetric theories containing at most two derivatives (including nonlinear Kähler sigma-models). As a result, when deriving their canonically normalized component actions, the traditional approach becomes impractical and cumbersome. We find it more efficient to follow the Kugo-Uehara scheme which consists of (i) extending the superfield theory to a super-Weyl invariant system; and then (ii) applying a plain component reduction along with imposing a suitable super-Weyl gauge condition. This scheme is implemented in order to derive the bosonic action of the $\mathrm{SL}(2, \mathbb{R})$ duality invariant coupling to the dilaton-axion chiral multiplet and a Kähler sigma-model.

Another manifestation of the nontrivial component structure of nonlinear self-dual systems (of both the vector and tensor multiplets) is that their fermionic sector turns out to contain higher derivative terms, even in the case of global supersymmetry. However, we demonstrate that these higher derivative terms may be eliminated by a nontrivial field redefinition. Such a field redefinition is explicitly constructed. It brings the fermionic actions to a one-parameter deformation of the Akulov-Volkov action for the goldstino. The Akulov-Volkov form emerges, in particular, in the case of the $\mathcal{N} = 1$ supersymmetric Born-Infeld action and the tensor-Goldstone multiplet action. We also analyze the fermionic sector of the chiral scalar multiplet model obtained by dualizing the tensor-Goldstone model.

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1. S. M. Kuzenko and S. A. McCarthy, “Nonlinear self-duality and supergravity,” JHEP **0302** (2003) 038, hep-th/0212039.
2. S. M. Kuzenko and S. A. McCarthy, “On the component structure of $\mathcal{N} = 1$ supersymmetric nonlinear electrodynamics,” JHEP **0505** (2005) 012, hep-th/0501172.

Permission has been granted to include this work.

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Introduction

Symmetries are of great importance in all areas of physics, but few have captured the imagination and interest of theorists in quite the same way as supersymmetry. Since the pioneering papers of Gol'fand and Likhtman [1], Volkov and Akulov [2, 3], and Wess and Zumino [4, 5], much effort has been put into understanding the consequences of theories possessing this remarkable symmetry. Supersymmetry offers possible solutions to some long standing problems in particle physics, such as the gauge hierarchy problem and force unification at high energies (see, e.g., [6–9] for reviews). It is also widely believed that a solution to the mystery of dark matter may be provided by light supersymmetric particles predicted by supersymmetric extensions of the Standard Model of particle physics (see, e.g., [10] and references therein). Moreover, much impetus for current research into supersymmetry stems from its fundamental relationship with superstring theory (see [11–16] for reviews). Superstring theory is the only known theory with the potential to provide a unified description of the four fundamental interactions, and supersymmetry is one of the underlying principles of its formulation. However, there is currently a lack of direct experimental evidence for supersymmetry, with present particle accelerators unable to reach energies high enough to probe beyond the Standard Model. This may be addressed by the next generation accelerator being built at CERN in Geneva. The Large Hadron Collider is due to begin operating in 2007 and the detection of supersymmetry is one of its primary goals.

The defining feature of supersymmetric theories is a symmetry transforming bosons (the force mediating particles) into fermions (the matter particles) and vice versa. This symmetry, known as supersymmetry, therefore implies a (partial) unification of forces and matter. Unlike ordinary symmetries, the generators of supersymmetry transformations (supercharges) obey anticommutation relations. This implies that the classical concepts of the Lie group and Lie algebra should be properly modified in order to realize supersymmetry. The relevant algebraic structures furnishing such a realization are known as Lie supergroups and Lie superalgebras [17–20]. In four dimensions, the supersymmetric extension of the Poincaré symmetry group ($\mathcal{P}$), is provided by the super-Poincaré ($\mathcal{SP}$) or $\mathcal{N}$-extended super-Poincaré group. These are constructed through the addition of $\mathcal{N}$ conserved Majorana spinor charges to the generators of the Poincaré algebra.

In order to keep supersymmetry manifest while performing calculations, the concepts of *superspace* and *superfields* were introduced [21, 22]. Superspace is an extension of spacetime by anticommuting (Grassmann) coordinates, while superfields are functions of the superspace coordinates. Unitary representations of the $\mathcal{N}=1$ Poincaré superalgebra are realized in terms of superfields on a $\mathcal{N}=1$ superspace. This superspace is constructed[1] as the left coset space $\mathcal{SP}/\mathrm{SO}(3,1)$, in the same way that Minkowski space can be identified as the left coset space $\mathcal{P}/\mathrm{SO}(3,1)$. Superfields are the building blocks of supersymmetric field theories in superspace. Taking into account the anticommuting nature of the fermionic coordinates, a series expansion of a superfield in these coordinates will terminate at some order since, for all i and j, $\theta_i\theta_j = -\theta_j\theta_i \Rightarrow \theta_i^2 = 0$. The coefficients in the expansion are functions of normal spacetime coordinates – the component fields of the superfield. In this way, superfield theories can be reduced to their component fields, resulting in field theories that are supersymmetric by construction.

The introduction of gravity into the supersymmetry framework occurs quite naturally when supersymmetry transformations become local [23, 24]. Local supersymmetry requires the inclusion of the supergravity multiplet containing the graviton and its superpartner, the gravitino. To formulate supergravity theories in superspace one can consider a curved superspace, just as Einstein introduced curved spacetime to formulate gravity. The curved superspace construction gives one a powerful tool with which to study supergravity theories in a way that is manifestly supersymmetric. In this work we will use the curved $\mathcal{N}=1$ superspace formalism in order to study *nonlinear self-dual theories* coupled to supergravity (see the textbooks [25–27] and also [8, 28, 29] for comprehensive reviews of the $\mathcal{N}=1$ superspace technique employed throughout).

Nonlinear self-duality

It is a well known property of Maxwell's electromagnetism that the equations of motion in vacuum[2] are invariant under a continuous U(1) rotation of the electric and magnetic fields, $\vec{E}+\mathrm{i}\vec{B} \to \mathrm{e}^{-\mathrm{i}\tau}(\vec{E}+\mathrm{i}\vec{B})$ (see, e.g., [30, 31]). This is equivalent to a transformation mixing the electromagnetic field strength F_{ab} and its Hodge dual $\tilde{F}_{ab} = \frac{1}{2}\epsilon_{abcd}F^{cd}$. Such transformations are called *electromagnetic duality rotations*. In the case $\tau = \pi/2$, the transformation reads $\vec{E} \to \vec{B}$, $\vec{B} \to -\vec{E}$ (or, in terms of the field strength, $F \to \tilde{F}$, $\tilde{F} \to -F$).

In 1934, Born and Infeld [32], in an effort to solve the problem of the infinite self-

[1] Here SO(3, 1) is the Lorentz group.

[2] Insisting on duality invariance in the presence of matter is equivalent to the existence of particles with magnetic charges or monopoles (see, e.g., [30]).

energy of the electron, introduced a nonlinear extension of electromagnetism. It was soon noticed by Schrödinger [33] that the U(1) duality invariance of the linear theory was also present in the Born-Infeld theory, albeit in a nonlinearly realized form. In more recent times, the Born-Infeld action has been the subject of renewed interest due to its appearance in the low-energy effective actions of string theories [34,35] – a development well beyond the original expectations that lead the authors of [32] to put forward their theory. In conjunction with the appearance of patterns of duality invariance in extended supergravity [36,37] this led to the development by Gaillard and Zumino, Gibbons and others [38–52] of a general theory of nonlinear self-duality in four and higher spacetime dimensions. The condition of self-duality imposes a nontrivial restriction on such theories, with the lagrangian required to satisfy a nonlinear *self-duality equation* (see chapter 2). Some interesting properties arise as a result of this self-duality equation being satisfied, including (i) invariance under Legendre transformation; and (ii) invariance of the energy momentum tensor under duality transformations. In [53,54], the general considerations of [38–50] were extended to 4D $\mathcal{N}=1,2$ globally supersymmetric theories and, in particular, the $\mathcal{N}=1$ supersymmetric Born-Infeld theory [55] was shown to possess U(1) duality invariance.

There exist deep yet mysterious connections between nonlinear self-duality and supersymmetry and here we give three examples. Firstly, in the case of partial spontaneous supersymmetry breakdown $\mathcal{N}=2 \to \mathcal{N}=1$, theories of the Maxwell-Goldstone multiplet [56,57] (coinciding with the $\mathcal{N}=1$ supersymmetric Born-Infeld action [55]) and the tensor-Goldstone multiplet [57,58] were shown in [53,54] to be self-dual, *i.e.* invariant under U(1) duality rotations. Additionally, the supersymmetry breaking may also be described by the chiral-scalar-Goldstone theory, which is obtained by dualizing the tensor-Goldstone model – a procedure consistent with duality transformations. Secondly, self-duality turns out to be quite useful in attempts to construct the correct Maxwell-Goldstone multiplet action for partial supersymmetry breakdown $\mathcal{N}=4 \to \mathcal{N}=2$ – the $\mathcal{N}=2$ supersymmetric Born-Infeld action[3]. It was suggested in [54] to look for an $\mathcal{N}=2$ vector multiplet action which should be (i) self-dual; and (ii) invariant under a nonlinearly realized central charge bosonic symmetry. These requirements turn out to allow one to restore the Goldstone multiplet action uniquely to any fixed order in powers of chiral superfield strength $\mathcal{W}$; this was carried out in [54] up to order $\mathcal{W}^{10}$. Recently, there has been consider-

[3]See [59,60] for earlier attempts to construct an $\mathcal{N}=2$ supersymmetric version of the Born-Infeld action.

able progress in developing the formalism of nonlinear realizations to describe the partial SUSY breaking $\mathcal{N} = 4 \rightarrow \mathcal{N} = 2$ [61–63]. So far, the authors of [61–63] have reproduced the action obtained in [54]. As a final example, we mention that the (Coulomb branch) low energy effective action of the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory is conjectured to be invariant under U(1) duality rotations [53] (a weaker form of self-duality of the effective action was proposed in [64]).

The above features provide enough evidence for considering supersymmetric self-dual systems to be quite interesting and their properties worth studying.

Component reduction

From superspace actions in terms of superfields one can reduce to a component field action. By construction, when reducing from a superfield action to components, the action will remain supersymmetric; however there is no guarantee that the component action will be in a canonically normalized form. For general supergravity-matter systems with *at most two derivatives* at the component level [65–68], the traditional approach (reviewed in [26]) of obtaining canonically normalized component actions consists of two steps: (i) a plain reduction from superfields to components; and (ii) the application of a field-dependent Weyl and local chiral transformation (accompanied by a gravitino shift). When we turn to models of nonlinear supersymmetric electrodynamics, a generic term in the component action may involve *any number of derivatives* – with even the purely electromagnetic part of the theory being a nonlinear function of the field strength. For such supergravity-matter systems, the traditional approach can be argued to become impractical and cumbersome (as regards the component tensor calculus employed in [65–68], it has never been extended, to the best of our knowledge, to the case of the supersymmetric theories we are going to study, and therefore the superspace approach is the only formalism at our disposal). There exist two alternatives [69, 70] to the traditional approach of component reduction [26] that were originally developed for the systems scrutinized in [65–68] or slightly more generals ones, but remain equally powerful in a more general setting. The first approach, described in [70], involves the concept of a Kähler superspace. This is a extension of conventional $\mathcal{N} = 1$ superspace to include a U(1) factor in the structure group. Supersymmetry coupling is then absorbed into the supergeometry by replacing the U(1) gauge potential by the superfield Kähler potential. The second approach by Kugo-Uehara [69], which we will employ in this thesis, conceptually originates in [71–73] and is quite natural in the framework of the Siegel-Gates formulation of superfield supergravity [74, 75]. The scheme consists of (i) extending the superfield theory to a super-Weyl invariant system; and then (ii) applying a plain component reduction along with imposing a suitable super-Weyl gauge condition. We will describe this in more detail in chapter 4.

About this thesis

In this thesis we study various aspects of the supersymmetric nonlinear self-dual theories introduced in [53,54]. In particular we study their properties when coupled to supergravity and supersymmetric matter. We go on to study the component structure of such theories, and in the globally supersymmetric case, investigate their relationship with the standard Akulov-Volkov (AV) action of the goldstino [2,3]. The structure of the thesis is as follows:

In chapter 1 we review relevant points of the old-minimal [76–80] and new-minimal [81,82] formulations of supergravity, and also review the general procedure of reducing locally supersymmetric actions from superfields to components.

Chapter 2 begins with a brief review of nonlinear self-dual electrodynamics in curved spacetime, followed by our generalization to supersymmetric nonlinear self-dual electrodynamics in curved superspace. This involves deriving the self-duality equation as the condition for a $\mathcal{N} = 1$ vector multiplet model to be invariant under U(1) duality rotations. Such models are shown to be invariant under a superfield Legendre transformation. We then introduce a family of self-dual nonlinear models and argue that the supercurrent and supertrace of such models are duality invariant.

To a large extent, the results of chapter 2 are a minimal curved superspace extension of the globally supersymmetric results presented in [54]. However, the considerations become more interesting when we introduce couplings to new minimal supergravity and Kähler sigma models – such couplings are not as trivial as in the globally supersymmetric case. We investigate this in chapter 3, and also consider coupling to the dilaton-axion multiplet. Such models are of interest from the point of view of string theory.

In chapter 4 we commence an examination of the component structure of the models developed in chapters 2 and 3. As will be argued in the chapter, to deal with our supergravity-matter systems, it will be useful to employ the scheme of Kugo and Uehara [69], which eliminates the need for field redefinitions and manipulations at the component level to bring the actions into canonical form. The scheme is first illustrated on the example of a nonlinear Kähler sigma-model coupled to supergravity and then used to derive the component action of new-minimal supergravity. Finally, we apply the method to investigate the component structure of the nonlinear self-dual models of chapter 3.

In chapter 5 we turn our focus to different aspects of the fermionic dynamics of nonlinear self-dual theories, restricting ourselves to the case of global supersymmetry. We will elucidate the relationship between the fermionic sector of nonlinear self-dual models of both the vector and tensor multiplets and the AV action. The relationship is shown to be particularly special for the cases of the Maxwell-Goldstone and tensor-Goldstone models. Additionally, we look at the fermionic sector of the chiral-scalar-Goldstone model dual to the tensor-Goldstone model.

In the appendices we collect conventions (appendix A), as well as give details of calculations not presented in the main body of the thesis, including some original results. In appendix B we present an alternative realization of old-minimal supergravity that has not been discussed in textbooks [25–27]. Appendix C details an explicit derivation of the self-duality equation in curved superspace, while appendix D includes necessary details for the supercurrent and supertrace calculation. Additionally, some nuances of the AV action are presented in appendix E. In particular, we demonstrate that all the terms of eighth order in the AV action completely cancel (a result which, to the best of our knowledge, has not been previously realized in the literature). Finally, details of the dualization of the tensor-Goldstone multiplet are presented in appendix F.

This thesis is based in part upon two publications [83, 84]. The results of [83] are covered in chapters 2 and 3, while those of [84] are covered in chapters 4 and 5. The tensor multiplet and chiral scalar multiplet results presented in sections 5.5 and 5.6 are new results that follow from [84], but are yet to be published.

Superfield supergravity and component reduction

There are three off-shell formulations of $\mathcal{N} = 1$ superfield supergravity: (i) old-minimal supergravity[1] [76–80]; (ii) non-minimal supergravity [75, 85]; and (iii) new-minimal supergravity [81, 82]. Each corresponds to a different way of gauge-fixing 4D, $\mathcal{N} = 1$ conformal supergravity to produce Einstein supergravity. The dynamical objects describing Einstein supergravity are then the gravitational superfield $H^m = \bar{H}^m$ and one of the following compensating superfields: (i) the chiral compensator for old-minimal supergravity; (ii) the complex linear compensator for non-minimal supergravity; and (iii) the real linear compensator for new-minimal supergravity. The supergravity multiplet for each formulation contains the graviton and gravitino as dynamical fields, but each differs in their auxiliary field content. The non-minimal multiplet has $(20 + 20)$ bosonic and fermionic degrees of freedom, while the two minimal formulations, in terms of different sets of auxiliary fields, each have $(12 + 12)$. In the case of pure supergravity, the three off-shell formulations are classically equivalent, *i.e.* they lead to the same dynamics on the mass shell.

Old-minimal supergravity plays an important role in that the other formulations may be treated as a super-Weyl invariant coupling of old-minimal supergravity to some special matter superfields. Even so, in certain circumstances, it is advantageous to formulate some problems in either the new- or non-minimal supergravity settings. In particular, coupling to Kähler sigma models is most natural in the new-minimal supergravity framework; a fact that we will exploit later when we couple Kähler sigma models to nonlinear self-dual electrodynamics.

In this chapter we recall salient points of the old- and new-minimal formulations[2] of $\mathcal{N} = 1$ supergravity (see [25–27] for more details) and also review the general procedure of reducing locally supersymmetric actions from superfields to components.

1.1 Old-minimal supergravity

To formulate gravity, Einstein introduced the concept of a curved spacetime. Likewise, when formulating supergravity theories, it is useful to introduce the concept of a curved superspace. Using the notation[3] and conventions as indicated in appendix

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[1] In appendix B we present an alternative realization of old-minimal supergravity.

[2] Non-minimal supergravity does not lead to interesting matter couplings [27].

[3] In particular, $z^M = (x^m, \theta^\mu, \bar{\theta}_{\dot{\mu}})$ are the coordinates of $\mathcal{N} = 1$ curved superspace, $\mathrm{d}^8 z = \mathrm{d}^4 x\, \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}$ is the full flat superspace measure, and $\mathrm{d}^6 z = \mathrm{d}^4 x\, \mathrm{d}^2\theta$ is the measure in the chiral subspace.

A, the supergeometry of $\mathcal{N} = 1$ curved superspace is described by the covariant derivatives

$$\mathcal{D}_A = (\mathcal{D}_a, \mathcal{D}_\alpha, \bar{\mathcal{D}}^{\dot\alpha}) = E_A + \Omega_A \ , \qquad (1.1.1)$$
$$E_A = E_A{}^M \partial_M \ , \qquad \Omega_A = \frac{1}{2}\,\Omega_A{}^{bc} M_{bc} = \Omega_A{}^{\beta\gamma} M_{\beta\gamma} + \Omega_A{}^{\dot\beta\dot\gamma} \bar{M}_{\dot\beta\dot\gamma} \ ,$$

where $E_A{}^M$ is the supervielbein, Ω_A is the Lorentz superconnection and $M_{bc} \Leftrightarrow (M_{\beta\gamma}, \bar{M}_{\dot\beta\dot\gamma})$ are the Lorentz generators. The supertorsion and supercurvature are defined by the covariant derivative algebra as follows:

$$[\mathcal{D}_A, \mathcal{D}_B\} = T_{AB}{}^C \mathcal{D}_C + R_{AB} \ . \qquad (1.1.2)$$

The supergeometry (1.1.1) proves to be too general to describe Einstein supergravity. To obtain the correct geometry, one must impose the following supertorsion constraints:

$$T_{\alpha\beta}{}^C = 0 \ , \qquad T_{\dot\alpha\dot\beta}{}^C = 0 \ , \qquad T_{\alpha\dot\alpha}{}^B + 2\mathrm{i}\, \delta_c{}^B (\sigma^c)_{\alpha\dot\alpha} = 0 \ ,$$
$$R_{\alpha\dot\alpha}{}^{cd} = 0 \ , \qquad T_{\alpha b}{}^c = 0 \ , \qquad T_{\dot\alpha b}{}^c = 0 \ . \qquad (1.1.3)$$

The covariant derivatives then obey the following algebra (see appendix A for useful identities related to this algebra):

$$\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot\alpha}\} = -2\mathrm{i}\mathcal{D}_{\alpha\dot\alpha} \ , \qquad (1.1.4)$$
$$\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = -4\bar{R} M_{\alpha\beta} \ , \qquad \{\bar{\mathcal{D}}_{\dot\alpha}, \bar{\mathcal{D}}_{\dot\beta}\} = -4R\bar{M}_{\dot\alpha\dot\beta} \ ,$$
$$\left[\bar{\mathcal{D}}_{\dot\alpha}, \mathcal{D}_{\beta\dot\beta}\right] = -\mathrm{i}\varepsilon_{\dot\alpha\dot\beta}\Big(R\,\mathcal{D}_\beta + G_\beta{}^{\dot\gamma}\bar{\mathcal{D}}_{\dot\gamma} - (\bar{\mathcal{D}}^{\dot\gamma} G_\beta{}^{\dot\delta})\bar{M}_{\dot\gamma\dot\delta} + 2W_\beta{}^{\gamma\delta} M_{\gamma\delta}\Big) - \mathrm{i}(\mathcal{D}_\beta R)\bar{M}_{\dot\alpha\dot\beta} \ ,$$
$$\left[\mathcal{D}_\alpha, \mathcal{D}_{\beta\dot\beta}\right] = \mathrm{i}\varepsilon_{\alpha\beta}\Big(\bar{R}\,\bar{\mathcal{D}}_{\dot\beta} + G^\gamma{}_{\dot\beta}\mathcal{D}_\gamma - (\mathcal{D}^\gamma G^\delta{}_{\dot\beta}) M_{\gamma\delta} + 2\bar{W}_{\dot\beta}{}^{\dot\gamma\dot\delta}\bar{M}_{\dot\gamma\dot\delta}\Big) + \mathrm{i}(\bar{\mathcal{D}}_{\dot\beta}\bar{R}) M_{\alpha\beta} \ ,$$

where the tensors R, $G_a = \bar{G}_a$ and $W_{\alpha\beta\gamma} = W_{(\alpha\beta\gamma)}$ satisfy the Bianchi identities

$$\bar{\mathcal{D}}_{\dot\alpha} R = \bar{\mathcal{D}}_{\dot\alpha} W_{\alpha\beta\gamma} = 0 \ , \quad \bar{\mathcal{D}}^{\dot\gamma} G_{\alpha\dot\gamma} = \mathcal{D}_\alpha R \ , \quad \mathcal{D}^\gamma W_{\alpha\beta\gamma} = \mathrm{i}\, \mathcal{D}_{(\alpha}{}^{\dot\gamma} G_{\beta)\dot\gamma} \ . \qquad (1.1.5)$$

Modulo purely gauge degrees of freedom, all geometric objects – the supervielbein and the superconnection – can be expressed in terms of three unconstrained superfields (known as the prepotentials of old-minimal supergravity): gravitational superfield $H^m = \bar{H}^m$, chiral compensator φ ($\bar{E}_{\dot\alpha}\varphi = 0$) and its conjugate $\bar\varphi$.

The action for old-minimal supergravity is given in terms of the super-Poincaré invariant full curved superspace measure[4], $\mathrm{d}^8 z\, E^{-1}$, where $E = \mathrm{Ber}(E_A{}^M)$ is the

[4]We follow the conventions of [27], which slightly differ from those of [26]. To convert to the conventions of [26] one makes the replacements $E \to E^{-1}$, $R \to 2R$, $G_a \to 2G_a$ and $W_{\alpha\beta\gamma} \to 2W_{\alpha\beta\gamma}$.

Berezinian or superdeterminant of the supervielbein. The action takes the form

$$S_{\mathrm{SG,old}} = -3\int \mathrm{d}^8 z\, E^{-1} \ , \tag{1.1.6}$$

where we have set the gravitational coupling constant to one.

In the superspace geometry (1.1.1), integration by parts works as

$$\int \mathrm{d}^8 z\, E^{-1}(-1)^{\epsilon_A}\mathcal{D}_A V^A = \int \mathrm{d}^8 z\, E^{-1}(-1)^{\epsilon_B} V^A T_{AB}{}^B \ , \tag{1.1.7}$$

where ϵ_A is the Grassmann parity of A, *i.e.* $\epsilon_A = 1$, if A is a spinor index, and $\epsilon_A = 0$ otherwise. Consequently, in old-minimal supergravity, the torsion constraints (1.1.3) imply

$$\int \mathrm{d}^8 z\, E^{-1}\mathcal{D}^\alpha \psi_\alpha = \int \mathrm{d}^8 z\, E^{-1}\mathcal{D}_a V^a = 0 \ , \tag{1.1.8}$$

where ψ_α and V^a are, respectively, arbitrary spinor and vector superfields under appropriate boundary conditions.

Integration over full superspace and over chiral superspace can be related by the following *chiral superspace rule*:

$$\begin{aligned}\int \mathrm{d}^8 z\, E^{-1}\, \mathcal{L} &= -\frac{1}{4}\int \mathrm{d}^8 z\, \frac{E^{-1}}{R}\, (\bar{\mathcal{D}}^2 - 4R)\mathcal{L} \\ &= -\frac{1}{4}\int \mathrm{d}^6 z\, \varphi^3\, (\bar{\mathcal{D}}^2 - 4R)\mathcal{L} \ ,\end{aligned} \tag{1.1.9}$$

where the equality in the last line takes place in the so-called chiral representation. This result is especially simple in the chiral case, $\mathcal{L} = \mathcal{L}_{\mathrm{c}}/R$, with $\mathcal{L}_{\mathrm{c}}$ a covariantly chiral scalar, $\bar{\mathcal{D}}_{\dot\alpha}\mathcal{L}_{\mathrm{c}} = 0$.

1.1.1 Supergravity-matter dynamical systems

Systems in which old-minimal supergravity (1.1.6) is coupled to supersymmetric matter χ, have the general structure

$$S = S_{\mathrm{SG,old}} + S_{\mathrm{M}}[\chi\,;\mathcal{D}] \ . \tag{1.1.10}$$

To calculate the dynamical equations of matter superfields in a supergravity background, one varies the matter superfields, whilst keeping the supergravity prepotentials constant. For a real covariantly scalar superfield $V(z)$, the functional variation is given by

$$\delta S[V;\mathcal{D}] = S[V+\delta V;\mathcal{D}] - S[V;\mathcal{D}] = \int \mathrm{d}^8 z\, E^{-1}\, \delta V(z)\frac{\delta S[V;\mathcal{D}]}{\delta V(z)} \ , \tag{1.1.11}$$

where

$$\frac{\delta V(z')}{\delta V(z)} = E\,\delta^4(x-x')\delta^2(\theta-\theta')\delta^2(\bar\theta-\bar\theta') \equiv \delta^8(z-z') \ . \tag{1.1.12}$$

For a superfunctional action $S[\Phi,\bar\Phi]$ of a covariantly chiral scalar superfield $\Phi(z)$, $\bar{\mathcal{D}}_{\dot\alpha}\Phi = 0$, the functional variation is

$$\begin{aligned}\delta S[\Phi,\bar\Phi;\mathcal{D}] &= S[\Phi+\delta\Phi,\bar\Phi+\delta\bar\Phi;\mathcal{D}] - S[\Phi,\bar\Phi;\mathcal{D}] \\ &= \int\mathrm{d}^8 z\,\frac{E^{-1}}{R}\,\delta\Phi(z)\frac{\delta S[\Phi,\bar\Phi;\mathcal{D}]}{\delta\Phi(z)} \ + \ \text{c.c.} \ ,\end{aligned} \tag{1.1.13}$$

where, using the chiral superspace rule (1.1.9),

$$\frac{\delta\Phi(z')}{\delta\Phi(z)} = -\frac{1}{4}(\bar{\mathcal{D}}^2-4R)\delta^8(z-z') \equiv \delta_+(z-z') \ . \tag{1.1.14}$$

To simplify notation in what follows, we introduce

$$A\cdot B = \int\mathrm{d}^8 z\,\frac{E^{-1}}{R}\,A^\alpha B_\alpha \ , \qquad \bar A\cdot\bar B = \int\mathrm{d}^8 z\,\frac{E^{-1}}{\bar R}\,\bar A_{\dot\alpha}\bar B^{\dot\alpha} \ , \tag{1.1.15}$$

with A_α and B_α covariantly chiral spinor superfields, $\bar{\mathcal{D}}_{\dot\alpha}A_\alpha = \bar{\mathcal{D}}_{\dot\alpha}B_\alpha = 0$. We will also use a similar notation for chiral scalars, in which case (1.1.13) can be written as

$$\delta S[\Phi,\bar\Phi] = \delta\Phi\cdot\frac{\delta S[\Phi,\bar\Phi]}{\delta\Phi} + \delta\bar\Phi\cdot\frac{\delta S[\Phi,\bar\Phi]}{\delta\bar\Phi} \ . \tag{1.1.16}$$

If we instead wish to calculate supergravity dynamical equations, then we must vary the supergravity prepotentials, which means a deformation of the supergeometry itself. This technique is outlined in appendix D, where we will use it to calculate the supercurrent and supertrace. The supercurrent, $T_a = \bar T_a$, and supertrace, T, $\bar{\mathcal{D}}_{\dot\alpha}T = 0$, are the superfield generalization of the energy-momentum tensor. They are defined in terms of covariantized variational derivatives with respect to the supergravity prepotentials (see appendix D for more details),

$$T_{\alpha\dot\alpha} = \frac{\Delta S_{\mathrm{M}}}{\Delta H^{\alpha\dot\alpha}} \ , \qquad T = \frac{\Delta S_{\mathrm{M}}}{\Delta\varphi} \ , \tag{1.1.17}$$

and satisfy the conservation equation

$$\bar{\mathcal{D}}^{\dot\alpha}T_{\alpha\dot\alpha} = -\frac{2}{3}\mathcal{D}_\alpha T \ , \tag{1.1.18}$$

when the matter superfields are put on the mass shell.

Using the same technique, one can determine the covariantized variational derivatives, $\Delta S_{\text{SG,old}}/\Delta H^a = G_a$ and $\Delta S_{\text{SG,old}}/\Delta\varphi = -3R$. Thus, the superfield dynamical equations for the supergravity-matter system (1.1.10) are

$$2G_a + T_a = 0 \ , \qquad -3R + T = 0 \ . \tag{1.1.19}$$

In the matter free case, this reduces to the supergravity equations $G_a = R = 0$.

1.2 Super-Weyl transformations

Super-Weyl transformations, originally introduced in [86], are simply local rescalings of the chiral compensator in old-minimal supergravity [74, 75] (see also [25, 27]),

$$\varphi \to \mathrm{e}^{\sigma}\varphi \ , \qquad H^m \to H^m \ , \tag{1.2.1}$$

with $\sigma(z)$ is an arbitrary covariantly chiral scalar parameter, $\bar{\mathcal{D}}_{\dot\alpha}\sigma = 0$. In terms of the covariant derivatives, the super-Weyl transformation is

$$\mathcal{D}_\alpha \ \to \ \mathrm{e}^{\sigma/2-\bar\sigma}\Big(\mathcal{D}_\alpha - (\mathcal{D}^\beta\sigma)\, M_{\alpha\beta}\Big) \ , \qquad \bar{\mathcal{D}}_{\dot\alpha} \ \to \ \mathrm{e}^{\bar\sigma/2-\sigma}\Big(\bar{\mathcal{D}}_{\dot\alpha} - (\bar{\mathcal{D}}^{\dot\beta}\bar\sigma)\bar{M}_{\dot\beta\dot\alpha}\Big) \ . \tag{1.2.2}$$

Additionally, the supergravity tensors R, $G_{\alpha\dot\alpha}$ and $W_{\alpha\beta\gamma}$ transform as

$$\begin{aligned} R \ &\to \ -\frac{1}{4}\mathrm{e}^{-2\sigma}\,(\bar{\mathcal{D}}^2 - 4R)\,\mathrm{e}^{\bar\sigma} \ , \qquad W_{\alpha\beta\gamma} \ \to \ \mathrm{e}^{-3\sigma/2}\, W_{\alpha\beta\gamma} \ , \\ G_{\alpha\dot\alpha} \ &\to \ \mathrm{e}^{-(\sigma+\bar\sigma)/2}\left(G_{\alpha\dot\alpha} + \frac{1}{2}(\mathcal{D}_\alpha\sigma)(\bar{\mathcal{D}}_{\dot\alpha}\sigma) + \mathrm{i}\mathcal{D}_{\alpha\dot\alpha}(\sigma-\bar\sigma)\right) \ . \end{aligned} \tag{1.2.3}$$

It is also useful to note that, under the super-Weyl transformation (1.2.2),

$$(\mathcal{D}^2 - 4\bar{R}) \ \to \ \mathrm{e}^{-2\bar\sigma}\,(\mathcal{D}^2 - 4\bar{R})\,\mathrm{e}^{\sigma} \ , \tag{1.2.4}$$

while the full superspace measure and the chiral superspace measure transform as

$$\mathrm{d}^8 z\, E^{-1} \ \to \ \mathrm{d}^8 z\, E^{-1}\, \mathrm{e}^{\sigma+\bar\sigma} \ , \qquad \mathrm{d}^8 z\, \frac{E^{-1}}{R} \ \to \ \mathrm{d}^8 z\, \frac{E^{-1}}{R}\, \mathrm{e}^{3\sigma} \ , \tag{1.2.5}$$

see eq. (1.1.9).

1.3 New-minimal supergravity

We will also find it useful to deal with the new-minimal formulation of supergravity. New-minimal supergravity and old-minimal supergravity are both 'minimal' in the sense that they are realized in terms of a minimal number of auxiliary field degrees of freedom. The actual field content of each is different, however, as pure supergravity theories, they both result in the same dynamics on the mass shell.

New-minimal supergravity can be treated as a super-Weyl invariant coupling of old-minimal supergravity to the improved tensor multiplet [87], described by a real covariantly linear scalar superfield, $\mathbb{L} = \bar{\mathbb{L}}$,

$$(\bar{\mathcal{D}}^2 - 4R)\,\mathbb{L} = (\mathcal{D}^2 - 4\bar{R})\,\mathbb{L} = 0 \ . \tag{1.3.1}$$

Due to (1.2.4), when acting on a scalar superfield, the super-Weyl transformation of $\mathbb{L}$ is uniquely fixed to be

$$\mathbb{L} \ \to \ \mathrm{e}^{-\sigma-\bar\sigma}\,\mathbb{L} \ . \tag{1.3.2}$$

The new-minimal supergravity action is [88, 89]

$$S_{\rm SG,new} = 3\int {\rm d}^8z\, E^{-1}\, \mathbb{L}\, {\rm ln}\mathbb{L}\ . \tag{1.3.3}$$

The relationship between the old-minimal and new-minimal supergravity actions can be seen by relaxing the constraint (1.3.1) on $\mathbb{L}$ and introducing the following auxiliary action

$$S = 3\int {\rm d}^8z E^{-1}\, (U\, \mathbb{L} - {\rm e}^U)\ , \tag{1.3.4}$$

where the Lagrange multiplier, U, is an unconstrained real scalar superfield. Under a super-Weyl transformation, U must transform as

$$U \ \rightarrow\ U - \sigma - \bar{\sigma}\ . \tag{1.3.5}$$

Eliminating U by its equation of motion returns us to the new-minimal supergravity action (1.3.3) with the linearity constraint (1.3.1) on $\mathbb{L}$. On the other hand, solving the $\mathbb{L}$ equation of motion, we find that U may be written as the sum of covariantly scalar chiral and antichiral superfields

$$U = {\rm ln}\Sigma + {\rm ln}\bar{\Sigma}\ , \qquad \bar{\mathcal{D}}_{\dot\alpha}\Sigma = 0\ . \tag{1.3.6}$$

Substituting into the auxiliary action we obtain

$$\tilde{S}_{\rm SG,old} = -3\int {\rm d}^8z\, E^{-1}\, \bar{\Sigma}\Sigma\ . \tag{1.3.7}$$

There remains a gauge freedom, since under super-Weyl transformations

$$\Sigma \ \rightarrow\ {\rm e}^{-\sigma}\, \Sigma\ . \tag{1.3.8}$$

Thus we are free to make the gauge choice $\Sigma = 1$, by which we regain the old-minimal supergravity action (1.1.6). However, it is not necessary that this particular gauge is chosen. We will find this freedom useful later, when determining the component structure of Kähler sigma models coupled to new-minimal supergravity.

1.4 Components in old-minimal supergravity

The old minimal supergravity multiplet $\{e_a{}^m, \Psi_a{}^\beta, \bar{\Psi}_{a\dot\beta}\,; A_a, B, \bar{B}\}$ comprises the (inverse) vierbein $e_a{}^m$, the gravitino $\boldsymbol{\Psi}_a = (\Psi_a{}^\beta, \bar{\Psi}_{a\dot\beta})$, and the auxiliary fields[5] A_a, B and $\bar{B}$. Within the framework of superfield supergravity [76–78], these component

[5]These auxiliary fields are denoted as $\mathbb{A}_a$, $\mathbb{B}$ and $\bar{\mathbb{B}}$ in [27].

fields naturally appear in a Wess-Zumino gauge [90] (see [25–27] for reviews). Here we use the Wess-Zumino gauge chosen in [27].

We define the component fields of a superfield by space projection and covariant differentiation. For a superfield $V(z)$, the former is the zeroth order term in the power series expansion in θ and $\bar\theta$

$$V| = V(x,\theta=0,\bar\theta=0)\ . \tag{1.4.1}$$

The space projection of the vector covariant derivatives are

$$\mathcal{D}_a| = \nabla_a - \frac{1}{3}\varepsilon_{abcd}\,A^d M^{bc} + \frac{1}{2}\Psi_a{}^\beta\,\mathcal{D}_\beta| + \frac{1}{2}\bar\Psi_{a\dot\beta}\,\bar{\mathcal{D}}^{\dot\beta}|\ , \tag{1.4.2}$$

where we have introduced the spacetime covariant derivatives, $\nabla_a = e_a + \frac{1}{2}\omega_{abc}M^{bc}$, with $\omega_{abc} = \omega_{abc}(e,\Psi)$ the connection and $e_a = e_a{}^m\partial_m$. The explicit expressions for the projections $\mathcal{D}_\alpha|$ and $\bar{\mathcal{D}}^{\dot\alpha}|$ are

$$\mathcal{D}_\alpha| = \partial_\alpha - \mathrm{i}(\sigma^a\bar\Psi^b)_\alpha M_{ab}\ , \qquad \bar{\mathcal{D}}^{\dot\alpha}| = \bar\partial^{\dot\alpha} - \mathrm{i}(\tilde\sigma^a\Psi^b)^{\dot\alpha}M_{ab}\ . \tag{1.4.3}$$

The spacetime covariant derivatives obey the following algebra

$$[\nabla_a,\nabla_b] = \mathcal{T}_{ab}{}^c\,\nabla_c + \frac{1}{2}\,\mathcal{R}_{abcd}M^{cd}\ , \tag{1.4.4}$$

where $\mathcal{R}_{abcd}$ is the curvature tensor and $\mathcal{T}_{abc}$ is the torsion. The torsion is related to the gravitino by

$$\mathcal{T}_{abc} = -\frac{\mathrm{i}}{2}(\Psi_a\sigma_c\bar\Psi_b - \Psi_b\sigma_c\bar\Psi_a)\ . \tag{1.4.5}$$

Additionally, we can write the connection in terms of the supergravity fields as

$$\omega_{abc} = \omega_{abc}(e) - \frac{1}{2}(\mathcal{T}_{bca} + \mathcal{T}_{acb} - \mathcal{T}_{abc})\ , \qquad \omega_{abc}(e) = \frac{1}{2}(\mathcal{C}_{bca} + \mathcal{C}_{acb} - \mathcal{C}_{abc})\ , \tag{1.4.6}$$

where $\mathcal{C}_{abc}$ are the anholonomy coefficients,

$$[e_a,e_b] = \mathcal{C}_{ab}{}^c e_c\ , \qquad \mathcal{C}_{ab}{}^c = \left((e_a e_b{}^m) - (e_b e_a{}^m)\right) e_m{}^c\ . \tag{1.4.7}$$

The supergravity auxiliary fields occur as follows

$$R| = \frac{1}{3}B\ , \qquad G_a| = \frac{4}{3}A_a\ . \tag{1.4.8}$$

One also has

$$\begin{aligned}
\mathcal{D}_\alpha R| &= -\frac{2}{3}(\sigma^{bc}\Psi_{bc})_\alpha - \frac{2\mathrm{i}}{3}A^b\Psi_{b\alpha} + \frac{\mathrm{i}}{3}\bar B(\sigma^b\bar\Psi_b)_\alpha\ ,\\
\bar{\mathcal{D}}_{(\dot\alpha}G^\beta{}_{\dot\beta)}| &= -2\Psi_{\dot\alpha\dot\beta,}{}^\beta + \frac{\mathrm{i}}{3}\bar B\bar\Psi^\beta{}_{(\dot\alpha,\dot\beta)} - 2\mathrm{i}(\tilde\sigma^{ab})_{\dot\alpha\dot\beta}\Psi_a{}^\beta A_b + \frac{2\mathrm{i}}{3}\Psi_{\alpha(\dot\alpha,}{}^\alpha A_{\dot\beta)}{}^\beta\ ,\\
W_{\alpha\beta\gamma}| &= \Psi_{(\alpha\beta,\gamma)} - \mathrm{i}(\sigma_{ab})_{(\alpha\beta}\Psi^a{}_{\gamma)}A^b\ ,
\end{aligned} \tag{1.4.9}$$

and

$$\begin{aligned}\bar{\mathcal{D}}^2\bar{R}| &= \frac{2}{3}\left(\mathcal{R}+\frac{\mathrm{i}}{2}\varepsilon^{abcd}\mathcal{R}_{abcd}\right)+\frac{16}{9}A^aA_a+\frac{4}{9}\epsilon^{abcd}\mathcal{T}_{abc}A_d-\frac{8\mathrm{i}}{3}(\nabla_aA^a)+\frac{8\mathrm{i}}{9}\mathcal{T}_{ab}{}^bA^a\\ &\quad+\frac{8}{9}B\bar{B}+\frac{4}{9}B(\Psi_a\sigma^{ab}\Psi_b)+\mathrm{i}\bar{\mathcal{D}}_{\dot\alpha}\bar{R}|(\tilde\sigma^a\Psi_a)^{\dot\alpha}+\frac{2\mathrm{i}}{3}\Psi^{\alpha\dot\alpha,\beta}\mathcal{D}_{(\alpha}G_{\beta)\dot\alpha}|\ , \qquad (1.4.10)\end{aligned}$$

where

$$\begin{aligned}\Psi_{ab}{}^\gamma &= \nabla_a\Psi_b{}^\gamma-\nabla_b\Psi_a{}^\gamma-\mathcal{T}_{ab}{}^c\Psi_c{}^\gamma\ ,\\ \Psi_{\alpha\beta,}{}^\gamma &= \frac{1}{2}(\sigma^{ab})_{\alpha\beta}\Psi_{ab}{}^\gamma\ , \qquad \Psi_{\dot\alpha\dot\beta,}{}^\gamma=-\frac{1}{2}(\tilde\sigma^{ab})_{\dot\alpha\dot\beta}\Psi_{ab}{}^\gamma\ , \qquad (1.4.11)\end{aligned}$$

is the gravitino field strength and $\mathcal{R}=\eta^{ac}\eta^{bd}\mathcal{R}_{abcd}$ is the scalar curvature.

With these objects and the covariant derivative algebra (1.1.4), the method to obtain the component action is as follows[6].

As a consequence of the chiral rule (1.1.9), modulo a total derivative, it is sufficient to work with chiral actions involving a chiral scalar lagrangian, $\mathcal{L}_\mathrm{c}$, $\bar{\mathcal{D}}_{\dot\alpha}\mathcal{L}_\mathrm{c}=0$. Such a chiral action generates the following component action [27]:

$$\begin{aligned}\int\mathrm{d}^8z\,\frac{E^{-1}}{R}\mathcal{L}_\mathrm{c} &= \int\mathrm{d}^4x\,e^{-1}\Big\{-\frac{1}{4}\mathcal{D}^2\mathcal{L}_\mathrm{c}|-\frac{\mathrm{i}}{2}(\bar\Psi^b\tilde\sigma_b)^\alpha\,\mathcal{D}_\alpha\mathcal{L}_\mathrm{c}|+(B+\bar\Psi^a\tilde\sigma_{ab}\bar\Psi^b)\,\mathcal{L}_\mathrm{c}|\Big\}\ ,\\ e &= \det(e_a{}^m)\ . \qquad (1.4.12)\end{aligned}$$

As an example, applying this procedure to the the old-minimal supergravity action (1.1.6), we obtain the well-known component action

$$\begin{aligned}S_{\mathrm{SG,old}} &= -3\int\mathrm{d}^8z\,E^{-1} = -3\int\mathrm{d}^8z\,\frac{E^{-1}}{R}R \qquad (1.4.13)\\ &= -3\int\mathrm{d}^4x\,e^{-1}\left\{-\frac{1}{4}\mathcal{D}^2R|-\frac{\mathrm{i}}{2}(\bar\Psi^a\tilde\sigma_a)^\alpha\mathcal{D}_\alpha R|+(B+\bar\Psi^a\tilde\sigma_{ab}\bar\Psi^b)R|\right\}\\ &= \int\mathrm{d}^4x\,e^{-1}\left\{\frac{1}{2}\mathcal{R}+\frac{4}{3}A^aA_a-\frac{1}{3}B\bar{B}+\frac{1}{4}\varepsilon^{abcd}(\bar\Psi_a\tilde\sigma_b\Psi_{cd}-\Psi_a\sigma_b\bar\Psi_{cd})\right\}\ .\end{aligned}$$

The auxiliary fields, A^a and B vanish on the mass shell.

1.5 Reduction to flat superspace

In chapter 5, our considerations will turn to flat superspace. Flat $\mathcal{N}=1$ superspace is obtained from curved $\mathcal{N}=1$ superspace by letting $H^a\to0$, $\varphi\to1$. In this limit,

[6] In general one would also require expressions for $\bar{\mathcal{D}}_{(\dot\alpha}\mathcal{D}^{(\gamma}G^{\delta)}{}_{\dot\beta)}|$ and $\mathcal{D}_{(\alpha}W_{\beta\gamma\delta)}|$. However, these were not necessary for our particular calculations, and we refer to [27] for these expressions.

the superfield tensors $R, G_a, W_{\alpha\beta\gamma}$ vanish and covariant derivatives reduce to the flat superspace covariant derivatives

$$D_\alpha = \partial_\alpha + \mathrm{i}\,\bar\theta^{\dot\alpha}\partial_{\alpha\dot\alpha}\ , \qquad \bar D_{\dot\alpha} = -\bar\partial_{\dot\alpha} - \mathrm{i}\,\theta^\alpha\partial_{\alpha\dot\alpha}\ . \tag{1.5.1}$$

For superspace integrals, one should take advantage of the last line in the chiral superspace rule (1.1.9) before taking the flat superspace limit. At the component level, for the supergravity multiplet, the vierbein becomes the Kronecker delta, while the gravitino and auxiliary fields are switched off.

1.6 Matter supermultiplets

Here, we introduce three of the supersymmetry multiplets that we will work with: (i) the vector multiplet; (ii) the chiral scalar multiplet; and (iii) the tensor multiplet.

- *Vector multiplet* $(F_{ab}, \psi^\alpha, \bar\psi_{\dot\alpha}\,; D)$

  The abelian $\mathcal{N} = 1$ vector multiplet is described by the covariantly (anti) chiral superfield stengths $\bar W_{\dot\alpha}$ and W_α,

$$W_\alpha = -\frac{1}{4}(\bar{\mathcal{D}}^2 - 4R)\mathcal{D}_\alpha V\ , \qquad \bar W_{\dot\alpha} = -\frac{1}{4}(\mathcal{D}^2 - 4\bar R)\bar{\mathcal{D}}_{\dot\alpha} V\ , \tag{1.6.1}$$

  where the real scalar superfield V is an unconstrained prepotential. The strengths are constrained superfields satisfying the Bianchi identity

$$\mathcal{D}^\alpha W_\alpha = \bar{\mathcal{D}}_{\dot\alpha}\bar W^{\dot\alpha}\ . \tag{1.6.2}$$

  The component fields in the Wess-Zumino gauge are introduced by

$$W_\alpha| = \psi_\alpha\ , \qquad -\frac{1}{2}\mathcal{D}^\alpha W_\alpha| = D\ , \qquad \mathcal{D}_{(\alpha}W_{\beta)}| = 2\mathrm{i}\hat F_{\alpha\beta} = \mathrm{i}(\sigma^{ab})_{\alpha\beta}\hat F_{ab}\ , \tag{1.6.3}$$

  where

$$\begin{aligned}
\hat F_{ab} &= F_{ab} - \frac{1}{2}(\Psi_a\sigma_b\bar\psi + \psi\sigma_b\bar\Psi_a) + \frac{1}{2}(\Psi_b\sigma_a\bar\psi + \psi\sigma_a\bar\Psi_b)\ , \\
F_{ab} &= \nabla_a V_b - \nabla_b V_a - \mathcal{T}_{ab}{}^c V_c\ ,
\end{aligned} \tag{1.6.4}$$

  with $V_a = e_a{}^m(x)\,V_m(x)$ the gauge one-form and $\mathcal{T}_{abc}$ defined in (1.4.5).

- *Scalar multiplet* $(Y, \chi_\alpha\,; F)$

  The scalar multiplet is described by a chiral scalar superfield ϕ, $\bar{\mathcal{D}}_{\dot\alpha}\phi = 0$. The component fields are defined by[7]

$$\phi| = Y\ , \qquad \mathcal{D}_\alpha\phi| = \chi_\alpha\ , \qquad -\frac{1}{4}\mathcal{D}^2\phi^i| = F\ . \tag{1.6.5}$$

[7] Note that when we introduce the chiral scalar superfields for the Kähler sigma models and the dilaton-axion multiplet in chapter 3, the F-term definitions will be slightly modified so that they transform covariantly under holomorphic reparametrizations of the Kähler manifold.

- *Tensor multiplet* $(\ell, \tilde{\psi}^\alpha, \bar{\tilde{\psi}}_{\dot{\alpha}}, \tilde{V}_a)$

  The tensor multiplet [91] is described by a real linear superfield L, subject to the constraint

$$(\mathcal{D}^2 - 4\bar{R})L = (\bar{\mathcal{D}}^2 - 4R)L = 0 \ , \qquad L = \bar{L} \ . \tag{1.6.6}$$

The constraint may be solved in terms of an unconstrained chiral spinor pre-potential, η_α,

$$L = \frac{1}{2}(\mathcal{D}^\alpha \eta_\alpha + \bar{\mathcal{D}}_{\dot{\alpha}} \bar{\eta}^{\dot{\alpha}}) \ , \qquad \bar{\mathcal{D}}_{\dot{\alpha}} \eta_\alpha = 0 \ . \tag{1.6.7}$$

We introduce the component fields of L in the Wess-Zumino gauge as

$$L| = \ell \ , \qquad \mathcal{D}_\alpha L| = \tilde{\psi}_\alpha \ , \qquad \frac{1}{2}[\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot{\alpha}}]L| = -(\sigma_a)_{\alpha\dot{\alpha}} \hat{V}^a \ , \tag{1.6.8}$$

with

$$\begin{aligned} \hat{V}^a &= \tilde{V}^a - \frac{1}{2}\epsilon^{abcd}\big(\Psi_b\, \sigma_{cd}\, \tilde{\psi} + \bar{\Psi}_b\, \tilde{\sigma}_{cd}\, \bar{\tilde{\psi}} - \mathcal{T}_{bcd}\, \ell\big) - \frac{4}{3} A^a \ell \ , \\ \tilde{V}^a &= \frac{1}{2}\epsilon^{abcd}\big(\nabla_b B_{cd} - \mathcal{T}_{bc}{}^f B_{fd}\big) \ , \end{aligned} \tag{1.6.9}$$

where $\tilde{V}^a$ is the Hodge dual of the field strength of the gauge two-form $B_{ab} = -B_{ba} = e_a{}^m e_b{}^n B_{mn}$.

It is worth pointing out that, for the compensator of new-minimal supergravity, we have used the notation $\mathbb{L}$.

Electromagnetic duality rotations in curved superspace

In 1934 Born and Infeld [32], seeking a solution to the problem of the infinite self-energy of the electron, proposed an alternative formulation of electromagnetism. It involved replacing the linear Maxwell lagrangian by the following lagrangian:

$$-\frac{1}{4}F^{ab}F_{ab} \quad \longrightarrow \quad \frac{1}{\kappa^2}\Big(1-\sqrt{-\det(\eta_{ab}+\kappa\,F_{ab})}\Big)~, \tag{2.1}$$

where κ is a coupling constant. Such a replacement modifies the dynamics at short distances and results in a finite electron self-energy. It was soon realized by Schrödinger [33] that Born-Infeld theory possesses the remarkable property of U(1) duality invariance. The linear theory ($\kappa \to 0$), corresponding to Maxwell's equations in vacuum, is invariant under the continuous U(1) transformation $\vec{E} + \mathrm{i}\vec{B} \to \mathrm{e}^{-\mathrm{i}\tau}(\vec{E}+\mathrm{i}\vec{B})$, and Schrödinger showed that this property extends to the Born-Infeld theory, at the cost of the U(1) duality transformation being realized nonlinearly.

Although the expectations that led the authors of [32] to put forward their theory were never fully realized, the Born-Infeld action has reappeared in the context of the low energy effective actions in string theory [34, 35]. In conjunction with the appearance of patterns of duality invariance in extended supergravity [36, 37], this motivated the development of a general theory of nonlinear self-duality in four and higher spacetime dimensions [38–52], and later an extension to 4D $\mathcal{N} = 1, 2$ globally supersymmetric theories [53, 54]. Indeed, as indicated in the introduction, self-duality is an important feature of a number of supersymmetric theories, particularly ones describing spontaneous partial breaking of supersymmetry. This provides our motivation for the investigation of general nonlinear self-dual supersymmetric theories.

Given that string theory contains gravity as a fundamental excitation, the corresponding low energy effective actions should incorporate gravity. It is therefore important, when studying the nonlinear self-dual electrodynamic models, which are related to such effective actions, to understand how they couple to gravity and, in the extended case, to supergravity. We will begin this chapter with a brief review of self-duality in curved spacetime [40, 41, 92], before going on to develop the curved superspace extension.

2.1 Self-duality in curved spacetime

Consider a nonlinear theory[1] $S[F,g] = \int \mathrm{d}^4x\,\sqrt{-g}\,L(F)$, $g = \det(g_{mn})$ of the electromagnetic field strength[2], $F_{mn} = \partial_m V_n - \partial_n V_m$ in curved spacetime such that $L(F) = (-1/4)F^{mn}F_{mn} + O(F^4)$. We introduce

$$\tilde{G}_{mn}(F) \;=\; \frac{1}{2}\epsilon_{mnpq}G^{pq}(F) \;\equiv\; 2\frac{\partial L(F)}{\partial F^{mn}}\;, \tag{2.1.1}$$

where the totally antisymmetric tensor ϵ_{mnpq} is defined by (A.5). For a theory $S[\lambda]$ in which the parameter λ is varied such that $\lambda \to \lambda + \delta\lambda$, the partial derivative of the lagrangian with respect to λ is defined by

$$S[\lambda + \delta\lambda] - S[\lambda] = \int \mathrm{d}^4x\,\sqrt{-g}\,\delta\lambda\frac{\partial L(F)}{\partial\lambda}\;. \tag{2.1.2}$$

In the linear case, where the lagrangian is the Maxwell lagrangian in vacuum $L(F) = (-1/4)F^{mn}F_{mn}$, we have $G = \tilde{F}$, the Hodge dual of the electromagnetic field strength. The Bianchi identity and equation of motion have the same form

$$\nabla^n\tilde{F}_{mn} = 0\;, \qquad \nabla^n\tilde{G}_{mn} = 0\;, \tag{2.1.3}$$

so we may consider U(1) duality rotations such that

$$\begin{pmatrix} G'(F') \\ F' \end{pmatrix} = \begin{pmatrix} \cos\tau & -\sin\tau \\ \sin\tau & \cos\tau \end{pmatrix}\begin{pmatrix} G(F) \\ F \end{pmatrix}\;, \tag{2.1.4}$$

where

$$\tilde{G}'_{mn}(F') = 2\frac{\partial L'(F')}{\partial F'^{mn}}\;, \tag{2.1.5}$$

and the gravitational field, g_{mn} is inert under duality transformations. If one requires that the theory be self dual, *i.e.* that $L'(F) = L(F)$, then we have the following restriction placed upon the model:

$$G^{mn}\tilde{G}_{mn} + F^{mn}\tilde{F}_{mn} = 0\;. \tag{2.1.6}$$

This is called the self-duality equation. Any solution is a minimal coupling of U(1) duality invariant nonlinear electrodynamics to Einstein gravity,

$$S_{\mathrm{G}} = \frac{1}{2}\int \mathrm{d}^4x\,\sqrt{-g}\,\mathcal{R}\;. \tag{2.1.7}$$

[1]The lagrangian is also a function of the gravitational field g_{mn}, $L(F) = L(F,g)$, however this dependence is not indicated explicitly.

[2]Here, in manifestly a covariant form, we actually have $F_{mn} = \nabla_m V_n - \nabla_n V_m$, where $\nabla_m v_n = \partial_m v_n - \Gamma^p{}_{mn}v_p$, for a covector field v_m, with $\Gamma^p{}_{mn}$ being the Christoffel symbols. However, this reduces to $F_{mn} = \partial_m V_n - \partial_n V_m$, since torsion vanishes in this purely bosonic system.

As an important example, consider the Born-Infeld lagrangian minimally coupled to gravity

$$L_{\mathrm{BI-curved}} = \frac{1}{\kappa^2}\left(1 - \frac{1}{\sqrt{-g}}\sqrt{-\det(g_{mn} + \kappa F_{mn})}\right) . \tag{2.1.8}$$

The model is a solution to the self-duality equation (2.1.6). For later comparison, using (A.2), the Born-Infeld action in the vierbein formalism is

$$S_{\mathrm{BI-curved}} = \frac{1}{\kappa^2}\int \mathrm{d}^4x\, e^{-1}\left(1 - \sqrt{-\det(\eta_{ab} + \kappa\, F_{ab})}\right) , \tag{2.1.9}$$

where F_{ab} is as defined in (1.6.4), but with torsion set to zero.

Such theories have a number of interesting properties, including invariance under Legendre transformation [38, 40] and duality invariance of the energy-momentum tensor [38, 40]. In addition, when coupled to the dilaton and axion, the duality group is enlarged to the non-compact $\mathrm{SL}(2,\mathbb{R})$ group of symmetries [38, 41] (see chapter 3).

2.1.1 Invariance under Legendre transformation

Invariance under Legendre transformation can be shown by simultaneously relaxing the constraint on the field strength so that it is now an unconstrained antisymmetric field, and introducing a Lagrange multiplier field A_{D} in the following manner:

$$L(F, F_{\mathrm{D}}) = L(F) - \frac{1}{2}F_{mn}\tilde{F}_D^{mn} , \qquad F_{\mathrm{D}}^{mn} = \nabla^m V_{\mathrm{D}}^n - \nabla^n V_{\mathrm{D}}^m , \tag{2.1.10}$$

where F_{D} is the dual field strength. Solving the equation of motion for A_{D} requires that F satisfy the Bianchi identity $\nabla^n\tilde{F}_{mn} = 0$, and the model (2.1.10) reduces to the original one. On the other hand, one can also consider the equation of motion for F:

$$G(F) = F_{\mathrm{D}} , \tag{2.1.11}$$

the solution of which is $F = F(F_{\mathrm{D}})$. Substituting back into (2.1.10), one obtains the dual model

$$L_{\mathrm{D}}(F_{\mathrm{D}}) \equiv \left(L(F) - \frac{1}{2}F_{mn}\tilde{F}_D^{mn}\right)\Big|_{F=F(F_{\mathrm{D}})} . \tag{2.1.12}$$

Since the following combination is invariant under arbitrary duality transformations (2.1.4)

$$L(F) - \frac{1}{4}F_{mn}G^{mn}(F) = L(F') - \frac{1}{4}F'_{mn}G'^{mn}(F') , \tag{2.1.13}$$

we find that, for a finite rotation by $\tau = \pi/2$,

$$L(F) - \frac{1}{2}F_{mn}\tilde{F}_{\mathrm{D}}^{mn} = L(F_{\mathrm{D}}) . \tag{2.1.14}$$

Comparing with (2.1.12) we see that $L = L_{\mathrm{D}}$.

2.1.2 Duality invariance of the energy-momentum tensor

An elegant, model independent proof that the energy-momenum tensor is invariant under U(1) duality transformations was given by Gaillard and Zumino [38, 43]. Suppose we have a duality invariant parameter λ, in our nonlinear self-dual theory $S[F,\lambda]$. Then the observable Q, defined by

$$\sqrt{-g}\,Q = \frac{\partial}{\partial\lambda}\Big(\sqrt{-g}\,L(F,\lambda)\Big)\ , \qquad (2.1.15)$$

is itself duality invariant. To see this, consider an infinitesimal duality transformation $\delta F = \tau G$, $\delta G = -\tau F$,

$$\sqrt{-g}\,\delta Q = \frac{\partial}{\partial\lambda}\Big(\sqrt{-g}\,\delta L\Big) = \frac{\tau}{2}\frac{\partial}{\partial\lambda}\Big(\sqrt{-g}\,G_{mn}\tilde{G}^{mn}\Big)\ . \qquad (2.1.16)$$

Using the self-duality equation (2.1.6) we see that

$$\sqrt{-g}\,\delta Q = -\frac{\tau}{2}\frac{\partial}{\partial\lambda}\Big(\sqrt{-g}\,F_{mn}\tilde{F}^{mn}\Big) = -\frac{\tau}{4}\frac{\partial}{\partial\lambda}\Big(\sqrt{-g}\,\epsilon^{mnpq}\,F_{mn}F_{pq}\Big) = 0\ , \qquad (2.1.17)$$

where the totally antisymmetric tensor is given by (A.5). As a consequence, if we take the invariant parameter to be the metric g_{mn}, then the energy-momentum tensor T^{mn}, defined by

$$\sqrt{-g}\,T^{mn} = -2\frac{\partial}{\partial g_{mn}}\Big(\sqrt{-g}\,L(F)\Big)\ , \qquad (2.1.18)$$

is invariant under U(1) duality rotations.

2.2 Self-duality in curved superspace

To extend the concept of nonlinear electromagnetic self-duality to curved superspace we consider models of a single abelian $\mathcal{N}=1$ vector multiplet (see section 1.6) generated by the action $S[W,\bar{W}]$. The multiplet is described by the covariantly (anti) chiral superfield stengths $\bar{W}_{\dot\alpha}$ and W_α, which are defined by (1.6.1) and satisfy the Bianchi identity (1.6.2).

Let us consider general actions that do not involve the combination $\mathcal{D}^\alpha W_\alpha$ as an independent variable. In such cases, $S[W,\bar{W}] \equiv S[v]$ can be unambiguously defined as if it were a functional of *unconstrained* (anti) chiral superfields $\bar{W}_{\dot\alpha}$ and W_α, and one can define (anti) chiral superfields $\bar{M}_{\dot\alpha}$ and M_α as

$$\mathrm{i}\,M_\alpha[v] \equiv 2\,\frac{\delta}{\delta W^\alpha}\,S[v]\ , \qquad -\mathrm{i}\,\bar{M}^{\dot\alpha}[v] \equiv 2\,\frac{\delta}{\delta \bar{W}_{\dot\alpha}}\,S[v]\ . \qquad (2.2.1)$$

Here, using the notation (1.1.15), the functional derivatives are defined by

$$\delta S = S[W+\delta W,\bar{W}+\delta\bar{W}] - S[W,\bar{W}] = \delta W(z)\cdot\frac{\delta S[v]}{\delta W(z)} + \delta\bar{W}(z)\cdot\frac{\delta S[v]}{\delta\bar{W}(z)}\ , \quad (2.2.2)$$

and

$$\frac{\delta}{\delta W^\alpha(z)} W^\beta(z') = \delta_\alpha{}^\beta \delta_+(z-z') \ . \tag{2.2.3}$$

With this definition, the vector multiplet equation of motion following from the action $S[W,\bar{W}]$ is

$$\mathcal{D}^\alpha M_\alpha \ = \ \bar{\mathcal{D}}_{\dot\alpha} \bar{M}^{\dot\alpha} \ . \tag{2.2.4}$$

Now, since the Bianchi identity (1.6.2) and the equation of motion (2.2.4) have the same functional form, one may consider (see appendix C for more details) U(1) duality transformations of the form

$$\left(\begin{array}{c} M'_\alpha[v'] \\ W'_\alpha \end{array} \right) \ = \ \left(\begin{array}{cc} \cos\tau & -\sin\tau \\ \sin\tau & \cos\tau \end{array} \right) \left(\begin{array}{c} M_\alpha[v] \\ W_\alpha \end{array} \right) \ , \tag{2.2.5}$$

where M' should be

$$\mathrm{i}\, M'_\alpha[v'] = 2\,\frac{\delta}{\delta W'^\alpha}\, S[v'] = 2\,\frac{\delta}{\delta W'^\alpha}\, S[v] + 2\,\frac{\delta}{\delta W^\alpha}\, \delta S \ , \tag{2.2.6}$$

with $\delta S = S[v'] - S[v]$.

As detailed in appendix C, the condition of self-duality leads to the following restriction:

$$\mathrm{Im}\Big(W\cdot W \ + \ M\cdot M \Big) \ = \ 0 \ . \tag{2.2.7}$$

We call this the $\mathcal{N}=1$ *self-duality equation.* When combined with the old-minimal supergravity action (1.1.6), any solution $S[W,\bar{W}]$ which satisfies this equation generates a U(1) duality invariant supersymmetric electrodynamics coupled to old-minimal supergravity.

2.3 Invariance under superfield Legendre transformation

As we showed in section 2.1, one of the important properties of all models of self-dual electrodynamics is invariance under Legendre transformation [43]. It was shown in [53] that this property also holds for any globally supersymmetric model of the massless vector multiplet that is invariant under U(1) duality rotations. We will demonstrate that this property also naturally extends to curved superspace.

Consider the auxiliary action

$$S[W,\bar{W},W_{\rm D},\bar{W}_{\rm D}] = S[W,\bar{W}] - \frac{\mathrm{i}}{2}\,(W\cdot W_{\rm D} - \bar{W}\cdot\bar{W}_{\rm D}) \ , \tag{2.3.1}$$

where W_α is now an *unconstrained* covariantly chiral spinor superfield and $W_{\mathrm{D}\,\alpha}$ is the dual field strength defined by

$$W_{\mathrm{D}\,\alpha} = -\frac{1}{4}\,(\bar{\mathcal{D}}^2 - 4R)\mathcal{D}_\alpha\, V_{\rm D} \ , \qquad \bar{W}_{\mathrm{D}\,\dot\alpha} = -\frac{1}{4}\,(\mathcal{D}^2 - 4\bar{R})\bar{\mathcal{D}}_{\dot\alpha}\, V_{\rm D} \ , \tag{2.3.2}$$

with the Lagrange multiplier, $V_{\rm D}$ a real scalar superfield.

Upon elimination of $W_{\rm D}$ by its equation of motion we regain $S[W,\bar W]$ with the condition that W satisfy the Bianchi identity (1.6.2). On the other hand, the equation of motion for W_α is $M[W,\bar W] = W_{\rm D}$. Solving this equation, $W = W[W_{\rm D},\bar W_{\rm D}]$, and substituting into the auxiliary action (2.3.1) we obtain the dual action $S_{\rm D}[W_{\rm D},\bar W_{\rm D}]$,

$$S_{\rm D}[W_{\rm D},\bar W_{\rm D}] = \Big(S[W,\bar W] - \frac{\rm i}{2}\,(W\cdot W_{\rm D} - \bar W\cdot \bar W_{\rm D})\Big)\Big|_{W=W[W_{\rm D},\bar W_{\rm D}]} ~. \qquad (2.3.3)$$

Since the following combination is invariant under arbitrary duality transformations (2.2.5)

$$S[v'] - \frac{\rm i}{4}\Big(W'\cdot M'[v'] - \bar W'\cdot \bar M'[v']\Big) = S[v] - \frac{\rm i}{4}\Big(W\cdot M[v] - \bar W\cdot\bar M[v]\Big) ~, \qquad (2.3.4)$$

we find that, for a finite rotation by $\tau = \pi/2$,

$$S[W,\bar W] - \frac{\rm i}{2}\Big(W\cdot M - \bar W\cdot\bar M\Big) = S[W_{\rm D},\bar W_{\rm D}] ~. \qquad (2.3.5)$$

Comparing with (2.3.3) we see that $S = S_{\rm D}$.

2.4 Family of self-dual models

Extending the globally supersymmetric results of [54], we now present a family of $\mathcal N = 1$ supersymmetric self-dual models with actions of the general form

$$S[W,\bar W] = \frac14\int {\rm d}^8z\,\frac{E^{-1}}{R}\,W^2 + \frac14\int {\rm d}^8z\,\frac{E^{-1}}{\bar R}\,\bar W^2 + \frac14\int {\rm d}^8z\,E^{-1}\,W^2\,\bar W^2\,\Lambda(\omega,\bar\omega) ~, \qquad (2.4.1)$$

where $\Lambda(\omega,\bar\omega)$ is a real analytic function of the variables

$$\omega \equiv \frac18(\mathcal D^2 - 4\bar R)\,W^2 ~, \qquad \bar\omega \equiv \frac18(\bar{\mathcal D}^2 - 4R)\,\bar W^2 ~. \qquad (2.4.2)$$

If this model (2.4.1) is to solve the self-duality equation (2.2.7), it requires that Λ satisfy the following differential equation:

$${\rm Im}\Big\{\Gamma - \bar\omega\,\Gamma^2\Big\} = 0 ~, \qquad \Gamma = \frac{\partial(\omega\,\Lambda)}{\partial\omega} ~. \qquad (2.4.3)$$

We will often refer to this general family of models throughout the thesis.

As an important example, consider a minimal curved superspace extension[3] of the $\mathcal N = 1$ supersymmetric Born-Infeld action. The action can be written in the form

$$S_{\rm SBI} = \frac14\int {\rm d}^8z\,\frac{E^{-1}}{R}\,X ~+~ \frac14\int {\rm d}^8z\,\frac{E^{-1}}{\bar R}\,\bar X ~, \qquad (2.4.4)$$

[3]Such a curved superspace action was discussed in [93].

with the following nonlinear constraint on the chiral scalar superfield X:

$$X + \frac{1}{16}\, X(\bar{\mathcal{D}}^2 - 4R)\, \bar{X} = W^2 \ , \qquad \bar{\mathcal{D}}_{\dot\alpha} X = 0 \ . \tag{2.4.5}$$

Using the fact that $W_\alpha W_\beta W_\gamma = 0$, this can be shown to be equivalent to

$$S_{\rm SBI} = \frac{1}{4}\int {\rm d}^8 z\, \frac{E^{-1}}{R}\, W^2 + \frac{1}{4}\int {\rm d}^8 z\, \frac{E^{-1}}{\bar R}\, \bar W^2 + \frac{1}{4}\int {\rm d}^8 z\, E^{-1} \frac{W^2\, \bar W^2}{1 + \frac{1}{2}\,A + \sqrt{1 + A + \frac{1}{4}\,B^2}} \ , \tag{2.4.6}$$

where

$$A = \omega + \bar\omega \ , \qquad B = \omega - \bar\omega \ . \tag{2.4.7}$$

This action is of the form (2.4.1), and it is easy to check that the differential equation (2.4.3) is satisfied. Thus the minimal curved superspace extension of the $\mathcal{N} = 1$ supersymmetric Born-Infeld action is a self-dual theory.

2.5 Duality invariance of the supercurrent and supertrace

In the bosonic case, self-dual models have the important property that the energy-momentum tensor is invariant under U(1) duality rotations [33, 38, 40, 41, 43]. It is natural to ask whether this property extends to the supersymmetric case, the superfield generalization of the energy-momentum tensor being the supercurrent $T_a = \bar T_a$ and supertrace T, $\bar{\mathcal{D}}_{\dot\alpha} T = 0$ (1.1.17).

Gaillard and Zumino [38, 43] developed an elegant, model-independent proof of the fact that the energy-momentum tensor of any self-dual bosonic system is invariant under U(1) duality rotations (see section 2.1.2). It is not quite trivial however to generalize this proof to the supersymmetric case, and this is why we will follow a brute-force approach, similar to [33, 40, 41], and directly check duality invariance of the supercurrent and supertrace for the family model (2.4.1).

Firstly, using the techniques outlined in appendix D, we find that the supertrace of the model (2.4.1) is

$$T = \frac{1}{8}\, W^2 (\bar{\mathcal{D}}^2 - 4R) \Big[\bar W^2 \Big(\Gamma + \bar\Gamma - \Lambda \Big) \Big] \ , \tag{2.5.1}$$

with Γ defined in (2.4.3). Consider an infinitesimal duality rotation $\delta W_\alpha = \tau M_\alpha$, $\delta M_\alpha = -\tau W_\alpha$, where

$$ {\rm i}\, M_\alpha = W_\alpha \Big\{ 1 - \frac{1}{4} (\bar{\mathcal{D}}^2 - 4R) \Big[\bar W^2 \Big(\Lambda + \frac{1}{8} (\mathcal{D}^2 - 4\bar R) \Big(W^2 \,\frac{\partial \Lambda}{\partial \omega} \Big) \Big) \Big] \Big\} \ . \tag{2.5.2}$$

For such a transformation it can be shown that δT vanishes for $\Lambda \neq 0$ only if the self-duality equation (2.4.3) is taken into account. Now, the conservation equation

(1.1.18) is to be satisfied both before and after applying the duality rotation. Since T is duality invariant, the left hand side of (1.1.18) should also be invariant. This essentially implies duality invariance of the supercurrent.

Turning now to the supercurrent, and again using the techniques described in appendix D, we find

$$
\begin{aligned}
T_{\alpha\dot\alpha} \;=\;& \mathrm{i}\, M_\alpha \bar W_{\dot\alpha} - \mathrm{i}\, W_\alpha \bar M_{\dot\alpha} - \frac{\mathrm{i}}{4}\mathcal{D}_{\alpha\dot\alpha}\big(W^2\bar W^2\,(\Gamma-\bar\Gamma)\big) \\
&- \frac{1}{6}G_{\alpha\dot\alpha}W^2\bar W^2\big(\Gamma+\bar\Gamma-\Lambda\big) - \frac{1}{24}\,[\mathcal{D}_\alpha,\bar{\mathcal{D}}_{\dot\alpha}]\,\big(W^2\bar W^2\big(\Gamma+\bar\Gamma-\Lambda\big)\big) \\
&- \frac{\mathrm{i}}{4}(W^2\overleftrightarrow{\mathcal{D}}_{\alpha\dot\alpha}\bar W^2)\Lambda - \frac{\mathrm{i}}{4}W^2\bar W^2(\mathcal{D}_{\alpha\dot\alpha}\omega)\frac{\partial\Lambda}{\partial\omega} + \frac{\mathrm{i}}{4}W^2\bar W^2(\mathcal{D}_{\alpha\dot\alpha}\bar\omega)\frac{\partial\Lambda}{\partial\bar\omega} \\
&+ \frac{\mathrm{i}}{16}(\mathcal{D}_{\alpha\dot\alpha}W^2)\bar W^2(\mathcal{D}^2-4\bar R)\Big(W^2\frac{\partial\Lambda}{\partial\omega}\Big) - \frac{\mathrm{i}}{16}W^2(\mathcal{D}_{\alpha\dot\alpha}\bar W^2)(\bar{\mathcal{D}}^2-4R)\Big(\bar W^2\frac{\partial\Lambda}{\partial\bar\omega}\Big)\;.
\end{aligned}
\tag{2.5.3}
$$

Off the mass shell, the variational derivative $\Delta S/\Delta H$ can be shown to include the extra (gauge non-invariant) term

$$
\frac{\mathrm{i}}{4}(\mathcal{D}^\beta M_\beta - \bar{\mathcal{D}}_{\dot\beta}\bar M^{\dot\beta})\,[\mathcal{D}_\alpha,\bar{\mathcal{D}}_{\dot\alpha}]\,V\;, \tag{2.5.4}
$$

which involves the naked prepotential V and therefore does not allow a naive generalization of the Gaillard-Zumino proof [38, 43] to superspace. After a tedious calculation one can explicitly show that (i) the conservation equation (1.1.18) is indeed satisfied; and (ii) the supercurrent (2.5.3) is duality invariant.

Self-dual electrodynamics and matter coupling in supergravity

We now turn to systems in which the models for self-dual electrodynamics of chapter 2 are coupled to supersymmetric matter. We will first look at coupling to Kähler sigma models in which the chiral multiplets parametrizing the target space are inert under duality transformations. We will than consider theories in which the coordinates are allowed to change under duality transformations, focussing on the so-called dilaton-axion multiplet.

In phenomenological applications of supergravity-matter theories, chiral matter is described by supersymmetric nonlinear sigma models. As first realized by Zumino [94], target spaces for $\mathcal{N}=1$ supersymmetric sigma models must be complex Kähler manifolds. Chiral scalar superfields correspond to complex coordinates on a Kähler manifold [95–98]. A detailed discussion of supersymmetric sigma models in supergravity is given in [26].

The dilaton and axion are massless scalar fields that appear in the low-energy effective action of string theory [11,12,14,15]. The significance of the dilaton is that its vacuum expectation value determines the string coupling constant, while the axion arises in the solution to the strong CP-problem via the Peccei-Quinn mechanism (see, e.g., [8] for more details). We note that the axion may equivalently be described by a gauge two-form. These fields also turn out to be of importance from the point of view of duality invariance. If one starts with Maxwell electrodynamics and then couples the gauge field to the dilaton and axion, then the lagrangian takes the form

$$-\frac{1}{4}\mathrm{e}^{-\varphi}F^{ab}F_{ab}+\frac{1}{4}\,a\,F^{ab}\tilde{F}_{ab}\ , \tag{3.1}$$

where φ and a are the dilaton and axion respectively. The remarkable feature of this theory is that the original U(1) duality invariance of Maxwell's theory is enhanced to the non-compact group $\mathrm{SL}(2,\mathbb{R})$ [38]. In $\mathcal{N}=1$ supersymmetric theories the dilaton and axion belong to the same multiplet (the dilaton-axion multiplet), which can be realized as either a chiral scalar multiplet or its tensor multiplet dual.

To facilitate our discussion of the coupling of nonlinear electrodynamics to supersymmetric nonlinear sigma models, we begin by formulating nonlinear self-duality in new-minimal supergravity.

3.1 Coupling to new-minimal supergravity

In order to couple our self-dual models to new-minimal supergravity, we note that any system of matter fields Ψ coupled to new-minimal supergravity can be treated as a super-Weyl invariant coupling of old minimal supergravity to the matter superfields Ψ and the real linear scalar superfield $\mathbb{L}$ defined by (1.3.1).

For the massless vector multiplet, the gauge superfield V is inert under super-Weyl transformations, whilst W_α transforms as

$$W_\alpha \;\to\; \mathrm{e}^{-3\sigma/2}\, W_\alpha \;. \tag{3.1.1}$$

Thus the linear part of the family model (2.4.1) is already super-Weyl invariant. To promote the nonlinear part to a super-Weyl invariant form we notice that the following combination is invariant[1]:

$$(\mathcal{D}^2 - 4\bar{R})\Big(\frac{W^2}{\mathbb{L}^2}\Big) \;. \tag{3.1.2}$$

As a result, we can replace the action (2.4.1) by the following functional[2]:

$$\begin{aligned} S[W,\bar{W},\mathbb{L}] = \frac{1}{4}\int \mathrm{d}^8 z\, \frac{E^{-1}}{R}\, W^2 \;+\; \frac{1}{4}\int \mathrm{d}^8 z\, \frac{E^{-1}}{\bar{R}}\, \bar{W}^2 \\ +\; \frac{1}{4}\int \mathrm{d}^8 z\, E^{-1}\, \frac{W^2\,\bar{W}^2}{\mathbb{L}^2}\, \Lambda\Big(\frac{\omega}{\mathbb{L}^2}, \frac{\bar{\omega}}{\mathbb{L}^2}\Big) \;. \end{aligned} \tag{3.1.3}$$

Combined with the new-minimal supergravity action (1.3.3), this gives a description of self-dual electrodynamics in new-minimal supergravity. The action is (i) super-Weyl invariant; and (ii) self-dual, *i.e.* it solves the $\mathcal{N} = 1$ self-duality equation (2.2.7).

3.2 Kähler sigma models in supergravity

Kähler sigma models are most easily described within the framework of new-minimal supergravity [99]. Given a Kähler manifold parameterized by n complex coordinates ϕ^i and their conjugates $\bar{\phi}^{\underline{i}}$, with $K(\phi,\bar{\phi})$ the Kähler potential, the corresponding supergravity-matter action is

$$S = 3\int \mathrm{d}^8 z\, E^{-1}\, \mathbb{L}\, \ln\mathbb{L} + \int \mathrm{d}^8 z\, E^{-1}\, \mathbb{L}\, K(\phi,\bar{\phi}) \;. \tag{3.2.1}$$

[1]Here we actually generalize the construction [53] of $\mathcal{N} = 1$ superconformal $U(1)$ duality invariant systems in flat superspace.

[2]Without spoiling the super-Weyl invariance and self-duality of the action (3.1.3), the 'compensator' $\mathbb{L}$ can be replaced in (3.1.3) by $\mathbb{L}/\kappa$, with κ a coupling constant. We set this constant to be one since it can be absorbed via renormalization of the self-interaction, $\hat{\Lambda}(\omega,\bar{\omega}) = \kappa^2\Lambda(\kappa^2\omega,\kappa^2\bar{\omega})$, see [54].

The dynamical variables ϕ^i are covariantly chiral scalar superfields, $\bar{\mathcal{D}}_{\dot\alpha}\phi^i = 0$, being inert with respect to the super-Weyl transformations. The action is obviously super-Weyl invariant. Moreover, the action is invariant under the Kähler transformations

$$K(\phi,\bar\phi) \to K(\phi,\bar\phi) + \lambda(\phi) + \bar\lambda(\bar\phi) \ , \tag{3.2.2}$$

with $\lambda(\phi)$ an arbitrary holomorphic function.

The relationship with old-minimal supergravity can be shown by using the procedure described in section 1.3. We first relax the constraint (1.3.1) on $\mathbb{L}$ and introduce the auxiliary action

$$S = 3\int \mathrm{d}^8 z\, E^{-1}\, (U\,\mathbb{L} - \Upsilon) \ , \tag{3.2.3}$$

where

$$\Upsilon = \exp\Big(U - \frac{1}{3}K(\phi,\bar\phi)\Big) \ , \tag{3.2.4}$$

and U is a real unconstrained scalar superfield transforming under super-Weyl transformations as in (1.3.5). To maintain invariance under Kähler transformations (3.2.2), U must transform as

$$U \ \to \ U + \frac{1}{3}\left(\lambda(\phi) + \bar\lambda(\bar\phi)\right) \ . \tag{3.2.5}$$

If U is eliminated from the auxiliary action (3.2.3) by its equation of motion we regain the action (3.2.1) with the linearity constraint (1.3.1) on $\mathbb{L}$. On the other hand, substituting in the solution to the $\mathbb{L}$ equation of motion (1.3.6) we obtain the dual model

$$S_{\rm Kahler} = -3\int \mathrm{d}^8 z\, E^{-1}\, \bar\Sigma\Sigma \exp\Big(-\frac{1}{3}K(\phi,\bar\phi)\Big) = -3\int \mathrm{d}^8 z\, E^{-1}\, \tilde\Upsilon \ , \tag{3.2.6}$$

where Σ is a covariantly chiral scalar superfield, $\bar{\mathcal{D}}_{\dot\alpha}\Sigma = 0$, and

$$\tilde\Upsilon = \Sigma\bar\Sigma \exp\Big(-\frac{1}{3}K(\phi,\bar\phi)\Big) \ . \tag{3.2.7}$$

To maintain Kähler invariance, Σ transforms under Kähler transformations (3.2.2) as

$$\Sigma \ \to \ \mathrm{e}^{\lambda(\phi)/3}\,\Sigma \ . \tag{3.2.8}$$

The super-Weyl gauge freedom (1.3.8) allows us some flexibility. If we make the gauge choice $\Sigma = 1$, Kähler transformations (3.2.2) must be accompanied by a special super-Weyl transformation with $\sigma = (1/3)\lambda(\phi)$. In addition to U(1) duality invariance, the model (3.2.6) would then also enjoy the so-called super-Weyl-Kähler invariance.

3.3 Coupling to nonlinear sigma models

For matter superfields, ϕ and $\bar{\phi}$, inert under duality transformations, it is easy to couple the Kähler sigma model (3.2.1) in new-minimal supergravity to self-dual supersymmetric electrodynamics (3.1.3). The supergravity-matter system is described by the action

$$S[W,\bar{W},\phi,\bar{\phi},\mathbb{L}] = 3\int \mathrm{d}^8z\,E^{-1}\,\mathbb{L}\ln\mathbb{L} + \int \mathrm{d}^8z\,E^{-1}\,\mathbb{L}\,K(\phi,\bar{\phi}) + S[W,\bar{W},\mathbb{L}]\ , \quad (3.3.1)$$

and this theory possesses several important symmetries: (i) super-Weyl invariance; (ii) Kähler invariance; and (iii) U(1) duality invariance. To establish the link to this theory's description in the framework of old-minimal supergravity, the procedure described above can be used to obtain the dual model

$$S[W,\bar{W},\phi,\bar{\phi},\Sigma,\bar{\Sigma}] = -3\int \mathrm{d}^8z\,E^{-1}\,\tilde{\Upsilon} + S[W,\bar{W},\tilde{\Upsilon}]\ . \quad (3.3.2)$$

Again, if we choose the super-Weyl gauge such that $\Sigma = 1$, this theory, unlike (3.3.1) enjoys the super-Weyl-Kähler invariance in addition to U(1) duality invariance.

In flat superspace the Kähler sigma model, $K(\phi,\bar{\phi})$ couples only trivially to supersymmetric nonlinear self-dual electrodynamics, through the addition of the kinetic term $\int \mathrm{d}^8z\,K(\phi,\bar{\phi})$ [54]. The result (3.3.2) shows that in curved superspace more care needs to be taken. Even for matter that is inert under duality transformations, the coupling of a Kähler sigma model to supersymmetric nonlinear self-dual electrodynamics is nontrivial.

3.4 Coupling to the dilaton-axion multiplet

In the above analysis of the coupling of self-dual supersymmetric electrodynamics to Kähler sigma models in curved superspace, it was assumed that the matter superfields, ϕ and $\bar{\phi}$, are inert under the electromagnetic duality rotations. Of some interest is a more general situation when, say, a chiral matter superfield Φ and its conjugate $\bar{\Phi}$ *do* transform under duality rotations, which can now span a larger group than the one corresponding to the pure gauge field case. Coupling to the so-called dilaton-axion supermultiplet is an important example.

We start by formulating the conditions of duality invariance for the abelian vector multiplet (W_α, $\bar{W}_{\dot{\alpha}}$) interacting with chiral matter (Φ, $\bar{\Phi}$) in curved superspace. Let $S[v] = S[W,\bar{W},\Phi,\bar{\Phi}]$ be the action functional of the supergravity-matter systems, with the dependence of $S[v]$ on the supergravity prepotentials being implicit. We again introduce covariantly (anti) chiral spinor superfields $\bar{M}^{\dot{\alpha}}$ and M_α defined by the rule (2.2.1). Since the Bianchi identity $\mathcal{D}^\alpha W_\alpha = \bar{\mathcal{D}}_{\dot\alpha}\bar{W}^{\dot\alpha}$ and the gauge field equation of motion $\mathcal{D}^\alpha M_\alpha = \bar{\mathcal{D}}_{\dot\alpha}\bar{M}^{\dot\alpha}$ are of the same functional form, we may consider infinitesimal duality transformations

$$\delta \left(\begin{array}{c} M_\alpha \\ W_\alpha \end{array} \right) = \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \left(\begin{array}{c} M_\alpha \\ W_\alpha \end{array} \right) , \qquad \delta\Phi = \xi(\Phi) , \tag{3.4.1}$$

with $\xi(\Phi)$ a *holomorphic* function and a, b, c and d real numbers.

We understand the conditions for self-duality to be that (i) $M'[v']$ should be defined as in (2.2.6); and (ii) that the equation of motion for Φ,

$$\Pi[v] = \frac{\delta}{\delta\Phi} S[v] , \tag{3.4.2}$$

should transform covariantly under duality transformations

$$\delta\Pi = -\frac{\partial\xi(\Phi)}{\partial\Phi}\Pi[v] , \tag{3.4.3}$$

$$\delta\Pi = \Pi'[v'] - \Pi[v] , \qquad \Pi'[v'] = \frac{\delta}{\delta\Phi'} S[v'] = \frac{\delta}{\delta\Phi'} S[v] + \frac{\delta}{\delta\Phi}\delta S .$$

Following [54] (see appendix C for details), the conditions of duality invariance in the presence of matter can be shown to be

$$\begin{aligned} \delta\Phi \cdot \frac{\delta S}{\delta\Phi} + \delta\bar\Phi \cdot \frac{\delta S}{\delta\bar\Phi} &= \frac{\mathrm{i}}{4}\, b \Big(W \cdot W - \bar W \cdot \bar W \Big) - \frac{\mathrm{i}}{4}\, c \Big(M \cdot M - \bar M \cdot \bar M \Big) \\ &+ \frac{\mathrm{i}}{2}\, a \Big(W \cdot M - \bar W \cdot \bar M \Big) , \end{aligned} \tag{3.4.4}$$

with $d = -a$. We see that the maximal group of duality transformations is $\mathrm{Sp}(2, \mathbb{R}) \cong \mathrm{SL}(2, \mathbb{R})$. The complex variable Φ should then parametrize the homogeneous space $\mathrm{SL}(2, \mathbb{R})/\mathrm{U}(1)$, with the vector field $\xi(\Phi)$ in (3.4.4) generating the action of $\mathrm{SL}(2, \mathbb{R})$ on the coset space. The matter-free case, which was considered before, corresponds to freezing the superfield $\Phi(z)$ to a given point of the space $\mathrm{SL}(2, \mathbb{R})/\mathrm{U}(1)$. In such a case, the duality group, $\mathrm{SL}(2, \mathbb{R})$, reduces to $\mathrm{U}(1)$ – the stabilizer of the point chosen.

To describe the dilaton-axion multiplet, we make use of the lower half-plane realization of the coset space $\mathrm{SL}(2, \mathbb{R})/\mathrm{U}(1)$. Then, the variation $\delta\Phi = \xi(\Phi)$ in (3.4.1) is

$$\delta\Phi = b + 2a\,\Phi - c\,\Phi^2 . \tag{3.4.5}$$

Our solution to the equations (3.4.4) reads

$$
\begin{aligned}
S &= 3\int \mathrm{d}^8 z\, E^{-1}\, \mathbb{L} \ln \mathbb{L} + \int \mathrm{d}^8 z\, E^{-1}\, \mathbb{L}\, \Big(\mathcal{K}(\Phi, \bar{\Phi}) + K(\phi, \bar{\phi}) \Big) \\
&\quad + \frac{\mathrm{i}}{4} \int \mathrm{d}^8 z\, \frac{E^{-1}}{R}\, \Phi\, W^2 - \frac{\mathrm{i}}{4} \int \mathrm{d}^8 z\, \frac{E^{-1}}{\bar{R}}\, \bar{\Phi}\, \bar{W}^2 \\
&\quad - \frac{1}{16} \int \mathrm{d}^8 z\, E^{-1}\, (\Phi - \bar{\Phi})^2\, \frac{W^2\, \bar{W}^2}{\mathbb{L}^2}\, \Lambda\Big(\frac{\mathrm{i}}{2}(\Phi - \bar{\Phi})\, \frac{\omega}{\mathbb{L}^2}\,,\ \frac{\mathrm{i}}{2}(\Phi - \bar{\Phi})\, \frac{\bar{\omega}}{\mathbb{L}^2} \Big)\,,
\end{aligned}
\tag{3.4.6}
$$

where ω is defined in (2.4.2). Here $\mathcal{K}(\Phi, \bar{\Phi})$ is the Kähler potential of the Kähler mainfold $\mathrm{SL}(2, \mathbb{R})/\mathrm{U}(1)$. It takes the form

$$
\mathcal{K}(\Phi, \bar{\Phi}) = -\ln \frac{\mathrm{i}}{2}(\Phi - \bar{\Phi})\,. \tag{3.4.7}
$$

The term $\int \mathrm{d}^8 z\, E^{-1}\, \mathbb{L}\, K(\phi, \bar{\phi})$ in (3.4.6) corresponds to the chiral matter which is inert under the duality rotations. For $\Phi = -\mathrm{i}$, the action (3.4.6) reduces to (3.3.1).

The supergravity-matter system (3.4.6) enjoys the following important properties: (i) super-Weyl invariance; (ii) Kähler invariance; and (iii) $\mathrm{SL}(2, \mathbb{R})$ duality invariance. To re-formulate this theory in the framework of the old-minimal version of $\mathcal{N} = 1$ supergravity, one should eliminate the real linear compensator $\mathbb{L}$ following the procedure described in section 3.2. This will lead to

$$
\begin{aligned}
S &= -3\int \mathrm{d}^8 z\, E^{-1}\, \tilde{\Upsilon} + \frac{\mathrm{i}}{4} \int \mathrm{d}^8 z\, \frac{E^{-1}}{R}\, \Phi\, W^2 - \frac{\mathrm{i}}{4} \int \mathrm{d}^8 z\, \frac{E^{-1}}{\bar{R}}\, \bar{\Phi}\, \bar{W}^2 \\
&\quad - \frac{1}{16} \int \mathrm{d}^8 z\, E^{-1}\, (\Phi - \bar{\Phi})^2\, \frac{W^2\, \bar{W}^2}{\tilde{\Upsilon}^2}\, \Lambda\Big(\frac{\mathrm{i}}{2}(\Phi - \bar{\Phi})\, \frac{\omega}{\tilde{\Upsilon}^2}\,,\ \frac{\mathrm{i}}{2}(\Phi - \bar{\Phi})\, \frac{\bar{\omega}}{\tilde{\Upsilon}^2} \Big)\,,
\end{aligned}
\tag{3.4.8}
$$

where

$$
\tilde{\Upsilon} = \Sigma \bar{\Sigma} \exp\Big(-\frac{1}{3}\mathcal{K}(\Phi, \bar{\Phi}) - \frac{1}{3} K(\phi, \bar{\phi}) \Big)\,. \tag{3.4.9}
$$

The chiral superfield Σ may be gauged away using the super-Weyl gauge freedom (1.3.8). If we make the gauge choice $\Sigma = 1$ then, unlike (3.4.6), this action enjoys the super-Weyl–Kähler invariance.

Nonlinear self-duality in components

While the considerations in chapters 2 and 3 were given mainly in terms of superspace and superfields, here we would now like to subject to scrutiny the component structure of self-dual supersymmetric systems.

As we indicated in the introduction, reducing from a superfield action to components does not guarantee a component action in canonical form. For example, consider the action for a Kähler sigma model coupled to old-minimal supergravity

$$S = -3 \int \mathrm{d}^8 z \, E^{-1} \exp\left(-\frac{1}{3} K(\phi, \bar{\phi})\right) \ , \tag{4.1}$$

(see eq. (3.2.6) in the gauge where $\Sigma = 1$). A plain reduction to components will result in the scalar curvature and Rarita-Schwinger terms having a multiplicative exponential factor $\mathrm{e}^{-K(\phi,\bar{\phi})/3}$. To transform into a canonical form, we need to apply a field-dependent Weyl and local chiral transformation (accompanied by a gravitino shift). This traditional approach (reviewed in [26]) can be applied to general supergravity-matter systems with at most two derivatives at the component level [65–68]. However, the required component level manipulations become extremely impractical and cumbersome, when applied to general theories which may contain any number of derivatives, including the nonlinear supersymmetric electrodynamics that we are investigating. It would make sense to look for an alternative in which such complications are removed through judicious use of superfield techniques. There exist two alternatives [69, 70] that were originally developed for the systems scrutinized in [65–68] or slightly more generals ones, but remain equally powerful in a more general setting.

The first approach, described in [70], involves enlarging the conventional superspace geometry to include a U(1) factor in the structure group. The geometry of the supersymmetry coupling is then obtained by replacing the gauge potential in U(1) superspace by the superfield Kähler potential. Such a superspace is called a Kähler superspace. In the case of the model (4.1), for the suitably modified supergeometry, the exponential factor is absorbed into the E^{-1} term. Accordingly, when reducing to components, the resulting action will be in canonical form.

We prefer to follow the second approach – that of Kugo and Uehara [69]. This approach does not require a modification of the underlying superspace and is quite natural in the framework of the Siegel-Gates formulation of superfield supergravity [74, 75]. We describe this method below.

4.1 Kugo-Uehara method

The idea behind the Kugo-Uehara method [69], which conceptually originates in [71–73], is to follow the pattern of the Weyl invariant extension of Einstein gravity,

$$S[g] = \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\,\mathcal{R} \;\longrightarrow\; S[g,\varphi] = 3\int \mathrm{d}^4x\sqrt{-g}\left\{g^{mn}\,\partial_m\varphi\partial_n\varphi + \frac{1}{6}\mathcal{R}\,\varphi^2\right\}, \tag{4.1.1}$$

and extend any supergravity-matter system to a super-Weyl invariant system (in the Howe-Tucker sense [86]) by introducing a compensating covariantly chiral scalar superfield Σ (in addition to the supergravity chiral compensator [74, 75]). When reducing to components, canonically normalized component actions are obtained simply by imposing a suitable super-Weyl gauge condition to effectively eliminate Σ. Conveniently, we have already obtained such extensions in chapter 3 when relating the old- and new-minimal supergravity formulations of our nonlinear electromagnetic models.

4.1.1 Example: Kähler sigma models

In order to illustrate the Kugo-Uehara method, we obtain the well known result for a Kähler sigma model coupled to supergravity (see, e.g., [26]), *i.e.* we will determine the component structure of the model (4.1). The super-Weyl extension of (4.1) is (3.2.6). We are required to make a particular super-Weyl gauge choice such that the Einstein and Rarita-Schwinger terms in the component action come out in canonical form.

We define[1] the component fields of the chiral scalar superfields ϕ^i by

$$\phi^i| = Y^i\,, \qquad \mathcal{D}_\alpha\phi^i| = \chi^i_\alpha\,, \qquad -\frac{1}{4}\mathcal{D}^2\phi^i| = F^i + \frac{1}{4}\Gamma^i{}_{jk}\,\chi^j\chi^k\,, \tag{4.1.2}$$

where we have introduced the Christoffel symbols $\Gamma^i{}_{jk}$ of the Kähler manifold defined by Kähler potential $K(Y,\bar{Y})$. The metric of the Kähler manifold is

$$g_{i\underline{i}} = g_{\underline{i}i} = \frac{\partial^2 K(Y,\bar{Y})}{\partial Y^i\partial\bar{Y}^{\underline{i}}} \equiv K_{i\underline{i}}\,, \tag{4.1.3}$$

where the subscript i ($\underline{i}$) on K denotes differentiation with respect to Y^i ($\bar{Y}^{\underline{i}}$). Similarly, we can write expressions for the Christoffel symbols and the curvature on the Kähler manifold

$$\Gamma^i{}_{jk} = g^{i\underline{i}}K_{jk\underline{i}}\,, \qquad \Gamma^{\underline{i}}{}_{\underline{j}\underline{k}} = g^{\underline{i}i}K_{i\underline{j}\underline{k}}\,,$$
$$R_{ij\underline{i}\underline{j}} = K_{ij\underline{i}\underline{j}} - g^{k\underline{k}}K_{ij\underline{k}}K_{k\underline{i}\underline{j}}\,, \tag{4.1.4}$$

[1] Note this definition is different from (1.6.5). With this definition, both χ^i and F^i transform as tangent vectors under arbitrary holomorphic reparametrizations, $Y^i \to f^i(Y)$, of the Kähler manifold with Kähler potential $K(Y,\bar{Y})$.

where the matrix elements $g^{i\underline{i}} = g^{\underline{i}i}$ correspond to the inverse Kähler metric, $g^{i\underline{j}}\, g_{\underline{j}k} = \delta^i{}_k$.

To reduce the Kähler sigma model (3.2.6) to components, we apply the reduction formula (1.4.12) to obtain

$$\begin{aligned} S_{\text{Kahler}} = -3\int \mathrm{d}^4x\, e^{-1}\bigg\{&\Big(-\frac{1}{4}\mathcal{D}^2 R| - \frac{\mathrm{i}}{2}(\bar{\Psi}^a\tilde{\sigma}_a)^\alpha \mathcal{D}_\alpha R| + (B + \bar{\Psi}^a\tilde{\sigma}_{ab}\bar{\Psi}^b)R|\Big)\,\tilde{\Upsilon}| \\ &-\frac{1}{2}\mathcal{D}^\alpha R|\mathcal{D}_\alpha\tilde{\Upsilon}| - \frac{1}{4}R|\mathcal{D}^2\tilde{\Upsilon}| - \frac{\mathrm{i}}{2}(\bar{\Psi}^a\tilde{\sigma}_a)^\alpha R|\mathcal{D}_\alpha\tilde{\Upsilon}| \\ &-\frac{1}{16}\mathcal{D}^2\bar{\mathcal{D}}^2\tilde{\Upsilon}| + \frac{\mathrm{i}}{8}(\bar{\Psi}^a\tilde{\sigma}_a)^\alpha\mathcal{D}_\alpha\bar{\mathcal{D}}^2\tilde{\Upsilon}| - \frac{1}{4}(B + \bar{\Psi}^a\tilde{\sigma}_{ab}\bar{\Psi}^b)\bar{\mathcal{D}}^2\tilde{\Upsilon}|\bigg\}\ . \end{aligned} \tag{4.1.5}$$

The first line of (4.1.5) reduces to the supergravity action (1.4.13) if we make a super-Weyl gauge choice such that $\tilde{\Upsilon}| = 1$. This can be done by setting

$$\Sigma| = \mathrm{e}^{K(Y,\bar{Y})/6}\ . \tag{4.1.6}$$

We have now eliminated the need to perform a Weyl rescaling on the component action. Further specification of the components of Σ can be used to remove the need for the chiral rotation (and gravitino shift). We accomplish this with the following choices

$$\begin{aligned} \mathcal{D}_\alpha\Sigma| &= \frac{1}{3}\,\chi^i_\alpha K_i\,\mathrm{e}^{K(Y,\bar{Y})/6}\ , \\ -\frac{1}{4}\mathcal{D}^2\Sigma| &= \Big(\frac{1}{3}F^iK_i - \frac{1}{12}\,\chi^i\chi^j(K_{ij} - \Gamma^k{}_{ij}K_k + \frac{1}{3}K_iK_j)\Big)\ \mathrm{e}^{K(Y,\bar{Y})/6}\ . \end{aligned} \tag{4.1.7}$$

Such a choice implies that $\mathcal{D}_\alpha\tilde{\Upsilon}| = \mathcal{D}^2\tilde{\Upsilon}| = 0$, and thus the second line of (4.1.5) vanishes. The action reduces to

$$S_{\text{Kahler}} = S_{\text{SG,old}} - 3\int \mathrm{d}^4x\, e^{-1}\Big\{ - \frac{1}{16}\mathcal{D}^2\bar{\mathcal{D}}^2\tilde{\Upsilon}| + \frac{\mathrm{i}}{8}(\bar{\Psi}^a\tilde{\sigma}_a)^\alpha\mathcal{D}_\alpha\bar{\mathcal{D}}^2\tilde{\Upsilon}|\Big\}\ . \tag{4.1.8}$$

We are now in a position to write down the component action for supergravity coupled to a Kähler sigma model. In agreement with the well known result (see, e.g., [26]), we obtain

$$\begin{aligned} S_{\text{Kahler}} = \int \mathrm{d}^4x\, e^{-1}\bigg\{&\frac{1}{2}\mathcal{R} \ +\ \frac{4}{3}\mathbb{A}^a\mathbb{A}_a \ -\ \frac{1}{3}\bar{B}B \ +\ \frac{1}{4}\varepsilon^{abcd}(\bar{\Psi}_a\tilde{\sigma}_b\hat{\Psi}_{cd} - \Psi_a\sigma_b\hat{\bar{\Psi}}_{cd}) \\ &-g_{i\underline{i}}\Big(\nabla^aY^i\,\nabla_a\bar{Y}^{\underline{i}} \ +\ \frac{\mathrm{i}}{4}(\chi^i\sigma^a\overset{\leftrightarrow}{\hat{\nabla}}_a\,\bar{\chi}^{\underline{i}}) \ -\ F^i\bar{F}^{\underline{i}} \\ &-\frac{1}{2}(\Psi_a\sigma^b\tilde{\sigma}^a\chi^i)(\nabla_b\bar{Y}^{\underline{i}}) \ -\ \frac{1}{2}(\bar{\Psi}_a\tilde{\sigma}^b\sigma^a\bar{\chi}^{\underline{i}})(\nabla_bY^i) \end{aligned} \tag{4.1.9}$$

$$
\begin{aligned}
&-\frac{1}{8}(\Psi^a\sigma_b\bar{\Psi}_a)(\chi^i\sigma^b\bar{\chi}^{\underline{i}}) \;-\; \frac{\mathrm{i}}{8}\varepsilon^{abcd}(\Psi_a\sigma_b\bar{\Psi}_c)(\chi^i\sigma_d\bar{\chi}^{\underline{i}})\Big) \\
&+\frac{1}{16}\chi^i\chi^j\bar{\chi}^{\underline{i}}\bar{\chi}^{\underline{j}}\,(R_{ij\underline{i}\underline{j}} - \frac{1}{2}g_{i\underline{i}}g_{j\underline{j}})\Bigg\}\ ,
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{\Psi}_{ab}{}^{\gamma} \;&=\; \Psi_{ab}{}^{\gamma} \;+\; \frac{1}{4}\left(K_i\,\nabla_a Y^i - K_{\underline{i}}\,\nabla_a\bar{Y}^{\underline{i}}\right)\Psi_b{}^{\gamma} \;-\; \frac{1}{4}\left(K_i\,\nabla_b Y^i - K_{\underline{i}}\,\nabla_b\bar{Y}^{\underline{i}}\right)\Psi_a{}^{\gamma}\ , \\
\hat{\nabla}_a\chi^i_{\gamma} \;&=\; \nabla_a\chi^i_{\gamma} \;-\; \frac{1}{4}\Big(K_j\,\nabla_a Y^j - K_{\underline{j}}\,\nabla_a\bar{Y}^{\underline{j}}\Big)\,\chi^i_{\gamma} \;+\; \Gamma^i{}_{jk}(\nabla_a Y^j)\chi^k_{\gamma}\ . \qquad (4.1.10)
\end{aligned}
$$

In order to to diagonalize in the auxiliary field sector we have made the redefinition

$$
A_a \;=\; \mathbb{A}_a \;-\; \frac{\mathrm{i}}{4}\left(K_i\,\nabla_a Y^i - K_{\underline{i}}\,\nabla_a\bar{Y}^{\underline{i}}\right) \;-\; \frac{1}{16}\,g_{i\underline{i}}(\chi^i\sigma_a\bar{\chi}^{\underline{i}})\ , \qquad (4.1.11)
$$

so that the auxiliary fields $\mathbb{A}_a$, B and F^i vanish on the mass shell.

4.1.2 Components in new-minimal supergravity

As a further example of the application of the Kugo-Uehara method, we consider the component reduction of the new-minimal supergravity action (1.3.3). In this case, it is not actually necessary to introduce the chiral compensating superfield Σ, since the new-minimal supergravity action is already super-Weyl invariant. As introduced in chapter 1, new-minimal supergravity is a super-Weyl invariant coupling of old-minimal supergravity to the improved tensor multiplet described by the linear multiplet $\mathbb{L}$ (1.3.1). The gauge freedom (1.3.2) will allow us to impose a suitable gauge condition to eliminate $\mathbb{L}$ and result in a canonically normalised component action.

We begin the reduction to components as described in section 1.4. Using the chiral rule (1.1.9) and the constraint (1.3.1), we can rewrite the new-minimal supergravity action (1.3.3) as

$$
S_{\text{SG,new}} = 3\int \mathrm{d}^8 z\, E^{-1}\,\mathbb{L}\ln\mathbb{L} = -3\int \mathrm{d}^8 z\,\frac{E^{-1}}{R}\Big\{R\,\mathbb{L} + \frac{1}{4}(\bar{\mathcal{D}}_{\dot{\alpha}}\,\mathbb{L})(\bar{\mathcal{D}}^{\dot{\alpha}}\mathbb{L})\mathbb{L}^{-1}\Big\}\ . \quad (4.1.12)
$$

Next, applying the the reduction rule (1.4.12), we obtain

$$
\begin{aligned}
S_{\text{SG,new}} \;=\; -3\int \mathrm{d}^4x\, e^{-1}\Bigg\{\Bigg(&-\frac{1}{4}\mathcal{D}^2 R - \frac{\mathrm{i}}{2}(\bar{\Psi}^a\tilde{\sigma}_a)^{\alpha}\mathcal{D}_{\alpha}R + (B + \bar{\Psi}^a\tilde{\sigma}_{ab}\bar{\Psi}^b)R\Bigg)\mathbb{L} \\
&-R\bar{R}\,\mathbb{L} - \frac{1}{8}(\mathcal{D}_{\alpha}\bar{\mathcal{D}}_{\dot{\alpha}}\mathbb{L})(\mathcal{D}^{\alpha}\bar{\mathcal{D}}^{\dot{\alpha}}\mathbb{L})\mathbb{L}^{-1} - \frac{1}{2}(\mathcal{D}^{\alpha}R)(\mathcal{D}_{\alpha}\mathbb{L}) \qquad (4.1.13) \\
&-\frac{\mathrm{i}}{8}(\Psi^a\tilde{\sigma}_a)^{\alpha}R\,\mathcal{D}_{\alpha}\mathbb{L} - \frac{1}{8}(\mathcal{D}^2\bar{\mathcal{D}}_{\dot{\alpha}}\mathbb{L})(\bar{\mathcal{D}}^{\dot{\alpha}}\mathbb{L})\mathbb{L}^{-1} + \frac{1}{4}(\bar{\mathcal{D}}_{\dot{\alpha}}\mathbb{L})(\bar{\mathcal{D}}^{\dot{\alpha}}\mathbb{L})\bar{R}\,\mathbb{L}^{-1}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{4}(\mathcal{D}^\alpha\mathbb{L})(\mathcal{D}_\alpha\bar{\mathcal{D}}_{\dot\alpha}\mathbb{L})(\bar{\mathcal{D}}^{\dot\alpha}\mathbb{L})\mathbb{L}^{-2}+\frac{1}{8}(\bar{\mathcal{D}}_{\dot\alpha}\mathbb{L})(\bar{\mathcal{D}}^{\dot\alpha}\mathbb{L})(\mathcal{D}^\alpha\mathbb{L})(\mathcal{D}_\alpha\mathbb{L})\mathbb{L}^{-3}\\
&-\frac{\mathrm{i}}{4}(\bar\Psi^a\tilde\sigma_a)^\alpha(\mathcal{D}_\alpha\bar{\mathcal{D}}_{\dot\alpha}\mathbb{L})(\bar{\mathcal{D}}^{\dot\alpha}\mathbb{L})\mathbb{L}^{-1}+\frac{\mathrm{i}}{8}(\bar\Psi^a\tilde\sigma_a)^\alpha(\mathcal{D}_\alpha\mathbb{L})(\bar{\mathcal{D}}_{\dot\alpha}\mathbb{L})(\bar{\mathcal{D}}^{\dot\alpha}\mathbb{L})\mathbb{L}^{-2}\\
&+\frac{1}{4}(B+\bar\Psi^a\tilde\sigma_{ab}\bar\Psi^b)(\bar{\mathcal{D}}_{\dot\alpha}\mathbb{L})(\bar{\mathcal{D}}^{\dot\alpha}\mathbb{L})\mathbb{L}^{-1}\Bigg\}\Bigg|\ ,
\end{aligned}
$$

We see that if we now apply the following gauge choice:

$$
\mathbb{L}|=1\ ,\qquad \mathcal{D}_\alpha\mathbb{L}|=0\ , \tag{4.1.14}
$$

then the first line of (4.1.13) reduces to the old-minimal supergravity action (1.4.13), while the rest of the action simplifies greatly,

$$
S_{\text{SG,new}}=S_{\text{SG,old}}-3\int\mathrm{d}^4x\,e^{-1}\Bigg\{-\frac{1}{9}B\bar{B}-\frac{1}{32}[\mathcal{D}_\alpha,\bar{\mathcal{D}}_{\dot\alpha}]\mathbb{L}|[\mathcal{D}^\alpha,\bar{\mathcal{D}}^{\dot\alpha}]\mathbb{L}|\Bigg\}\ , \tag{4.1.15}
$$

For simplicity, we will restrict ourselves to the bosonic sector, setting the gravitino field to zero. We then have have (*c.f.* (1.6.8) and (1.6.9))

$$
\frac{1}{2}[\mathcal{D}_\alpha,\bar{\mathcal{D}}_{\dot\alpha}]\mathbb{L}|=-(\sigma_a)_{\alpha\dot\alpha}\hat{w}^a\ ,\qquad \hat{w}^a=\tilde{w}^a-\frac{4}{3}A^a\ , \tag{4.1.16}
$$

where $\tilde{w}_a$ the Hodge dual of the field strength of the gauge two-form,

$$
\tilde{w}_a=e_a{}^m\tilde{w}_m\ ,\qquad \tilde{w}^m=\frac{1}{2}\epsilon^{mnpq}\partial_n b_{pq}\ ,\quad b_{mn}=-b_{nm}\ . \tag{4.1.17}
$$

From (4.1.15), we find the component action for new-minimal supergravity to be

$$
\begin{aligned}
S_{\text{bosonic}}&=\int\mathrm{d}^4x\,e^{-1}\Bigg\{\frac{1}{2}\mathcal{R}+\frac{4}{3}A^aA_a-\frac{3}{4}\hat{w}^a\hat{w}_a\Bigg\}\\
&=\int\mathrm{d}^4x\,e^{-1}\Bigg\{\frac{1}{2}\mathcal{R}-\frac{3}{4}\tilde{w}^a\tilde{w}_a+2A^a\tilde{w}_a\Bigg\}\ .
\end{aligned} \tag{4.1.18}
$$

Note that the $B\bar{B}$ and A^2 terms from the old-minimal supergravity action (1.4.13) are no longer be present. This can be considered to be as a result of the remaining super-Weyl gauge freedom. The gauge choice (4.1.14) has required that we fix the components $\mathrm{Re}(\sigma)|$ and $\mathcal{D}_\alpha\sigma|$ of the chiral parameter $\sigma(z)$, $\bar{\mathcal{D}}_{\dot\alpha}\sigma=0$ from the super-Weyl transformation (1.2.1). The further freedom derived from $\mathcal{D}^2\sigma|$ allows one to arbitrarily shift B, eliminating it as a physical variable. Meanwhile, the residual freedom from the imaginary part of σ means that the auxiliary field A_a is now a gauge field of the local chiral transformation,

$$
\delta A_m\sim\partial_m\mathrm{Im}(\sigma)|\ . \tag{4.1.19}
$$

Any terms of the form A^2 would then be a mass term for the gauge field and, as such, must not be present. We indeed see that this is the case for the action (4.1.18), which is invariant under the gauge transformation (4.1.19) due to the Bianchi identity $\partial_a \tilde{w}^a = 0$. The two gauge fields b_{mn} and A_a do not propagate on the mass shell, where these auxiliary fields vanish to give the same on-shell dynamics as the component action from old-minimal supergravity (1.4.13).

4.2 Dilaton-axion multiplet model

In chapter 3 we presented a SL(2,$\mathbb{R}$) duality invariant model (3.4.8) for nonlinear electrodynamics coupled to the dilaton-axion multiplet, a duality inert Kähler sigma model and supergravity. We would now like to investigate the component structure of this theory by employing the Kugo-Uehara method.

Similarly to our definition (4.1.2) of the component fields $\{Y^i, \chi^i_\alpha, F^i\}$ of the scalar superfield ϕ^i, we introduce the component fields $\{\mathcal{Y}, \zeta_\alpha, \mathcal{F}\}$ of the dilaton-axion multiplet Φ with Kähler potential (3.4.7). The dilaton φ and axion a fields are related to the superfield Φ by

$$\Phi| = \mathcal{Y} = a - \mathrm{i}\,\mathrm{e}^{-\varphi} \ . \tag{4.2.1}$$

To reduce the dilaton-axion multiplet model (3.4.8) to components, we apply the reduction rule (1.4.12) to obtain

$$\begin{aligned} S = S_V - 3\int \mathrm{d}^4x\, e^{-1}\Big\{\Big(-\frac{1}{4}\mathcal{D}^2 R| - \frac{\mathrm{i}}{2}(\bar{\Psi}^a\bar{\sigma}_a)^\alpha \mathcal{D}_\alpha R| + (B + \bar{\Psi}^a\tilde{\sigma}_{ab}\bar{\Psi}^b)R|\Big)\Omega| \\ -\frac{1}{2}\mathcal{D}^\alpha R|\mathcal{D}_\alpha\Omega| - \frac{1}{4}R|\mathcal{D}^2\Omega| - \frac{\mathrm{i}}{2}(\bar{\Psi}^a\tilde{\sigma}_a)^\alpha\mathcal{D}_\alpha\Omega| \\ +\frac{1}{16}\mathcal{D}^2\bar{\mathcal{D}}^2\Omega| + \frac{\mathrm{i}}{8}(\bar{\Psi}^a\tilde{\sigma}_a)^\alpha\mathcal{D}_\alpha\bar{\mathcal{D}}^2\Omega| - \frac{1}{4}(B + \bar{\Psi}^a\tilde{\sigma}_{ab}\bar{\Psi}^b)\bar{\mathcal{D}}^2\Omega|\Big\} \ , \end{aligned} \tag{4.2.2}$$

where

$$\Omega = \tilde{\boldsymbol{\Upsilon}} + \frac{1}{48}(\Phi - \bar{\Phi})^2\frac{W^2\bar{W}^2}{\tilde{\boldsymbol{\Upsilon}}^2}\,\Lambda\Big(\frac{\mathrm{i}}{2}(\Phi - \bar{\Phi})\,\frac{\omega}{\tilde{\boldsymbol{\Upsilon}}^2}\ ,\ \frac{\mathrm{i}}{2}(\Phi - \bar{\Phi})\,\frac{\bar{\omega}}{\tilde{\boldsymbol{\Upsilon}}^2}\Big) \ , \tag{4.2.3}$$

and we have separated out the following part of the action:

$$S_V = \frac{\mathrm{i}}{4}\int \mathrm{d}^8z\,\frac{E^{-1}}{R}\,\Phi\, W^2 - \frac{\mathrm{i}}{4}\int \mathrm{d}^8z\,\frac{E^{-1}}{\bar{R}}\,\bar{\Phi}\,\bar{W}^2 \ . \tag{4.2.4}$$

Since S_V does not couple to Σ and $\bar{\Sigma}$, its component form is independent of the super-Weyl gauge choice. It is therefore straightforward to evaluate the component structure of this part of the action[2]

$$
\begin{aligned}
S_V = \int \mathrm{d}^4x\, e^{-1}\Bigg\{ & -\frac{1}{4}\,\mathrm{e}^{-\varphi}\,F^{ab}F_{ab} + \frac{1}{4}\,a\,F^{ab}\tilde{F}_{ab} - \frac{1}{2}(\psi\sigma^b\bar{\psi})\nabla_b a - \frac{\mathrm{i}}{2}\,\mathrm{e}^{-\varphi}\,(\psi\sigma^a\nabla_a\bar{\psi}) \\
& + \frac{1}{2}\,\mathrm{e}^{-\varphi}\,F^{ab}(\Psi_a\sigma_b\bar{\psi} + \psi\sigma_b\bar{\Psi}_a) + \frac{\mathrm{i}}{2}\,\mathrm{e}^{-\varphi}\,\tilde{F}^{ab}(\Psi_a\sigma_b\bar{\psi} - \psi\sigma_b\bar{\Psi}_a) \\
& + \frac{1}{4}\,F^{ab}(\zeta\sigma_{ab}\psi + \bar{\zeta}\tilde{\sigma}_{ab}\bar{\psi}) + \frac{1}{4}\left(\mathrm{e}^{-\varphi}(\eta^{ac}\eta^{bd} - \eta^{ad}\eta^{bc}) + a\,\epsilon^{abcd}\right)(\Psi_a\sigma_b\bar{\psi})(\psi\sigma_c\bar{\Psi}_d) \\
& + \frac{1}{16}\,\mathrm{e}^{-\varphi}\Big((3\bar{\Psi}^a\bar{\Psi}_a - 2\bar{\Psi}^a\tilde{\sigma}_{ab}\bar{\Psi}^b)\psi^2 + (3\Psi^a\Psi_a - 2\Psi^a\sigma_{ab}\Psi^b)\bar{\psi}^2\Big) \\
& - \frac{1}{8}(\Psi_a\sigma_b\bar{\psi})(\zeta\sigma^{ab}\psi) - \frac{1}{8}(\bar{\Psi}_a\tilde{\sigma}_b\psi)(\bar{\zeta}\tilde{\sigma}^{ab}\bar{\psi}) - \frac{1}{32}\psi^2(\zeta\sigma^a\bar{\Psi}_a) + \frac{1}{32}\bar{\psi}^2(\Psi_a\sigma^a\bar{\zeta}) \\
& + \frac{1}{16}\,\mathrm{e}^{\varphi}\,(\zeta^2\psi^2 + \bar{\zeta}^2\bar{\psi}^2) + \frac{1}{2}(\psi\sigma^a\bar{\psi})\mathcal{T}_{ab}{}^b - \frac{\mathrm{i}}{4}(\zeta\psi - \bar{\zeta}\bar{\psi})D \\
& + \frac{1}{2}\,\mathrm{e}^{-\varphi}\,D^2 - \mathrm{e}^{-\varphi}(\psi\sigma^a\bar{\psi})A_a + \frac{\mathrm{i}}{4}(\mathcal{F}\psi^2 - \bar{\mathcal{F}}\bar{\psi}^2)\Bigg\}\ ,
\end{aligned}
\tag{4.2.5}
$$

where we have used the explicit form of the Kähler potential (3.4.7).

Looking at the first line of (4.2.2) we notice that if a super-Weyl gauge choice is made such that $\Omega| = 1$ then this will reduce to the supergravity action (1.4.13), and not require a Weyl rescaling. To achieve this, we make the choice

$$
\Sigma| = \exp\left(\frac{1}{6}K(Y,\bar{Y}) + \frac{1}{6}\mathcal{K}(\mathcal{Y},\bar{\mathcal{Y}}) + \frac{1}{24}\mathrm{e}^{-2\varphi}\psi^2\bar{\psi}^2\Lambda\big(\mathrm{e}^{-\varphi}\omega|\ ,\ \mathrm{e}^{-\varphi}\bar{\omega}|\big)\right)\ . \tag{4.2.6}
$$

A number of options are available for the gauge choice for the other components of Σ. If the following gauge choices are made

$$
\begin{aligned}
\mathcal{D}_\alpha\Sigma| &= \frac{1}{3}(\chi^i_\alpha K_i - \frac{\mathrm{i}}{2}\,\mathrm{e}^{\varphi}\zeta_\alpha)\,\Sigma|\ , \\
-\frac{1}{4}\mathcal{D}^2\Sigma| &= \frac{1}{3}\Big(F^iK_i - \frac{1}{4}\,\chi^i\chi^j(K_{ij} - \Gamma^k{}_{ij}K_k + \frac{1}{3}K_iK_j) \\
&\qquad - \frac{\mathrm{i}}{2}\,\mathrm{e}^{\varphi}\mathcal{F} - \frac{1}{24}\,\mathrm{e}^{2\varphi}\,\zeta^2 + \frac{\mathrm{i}}{12}\,\mathrm{e}^{\varphi}\,\zeta\chi^iK_i\Big)\,\Sigma|\ ,
\end{aligned}
\tag{4.2.7}
$$

then $\mathcal{D}_\alpha\tilde{\boldsymbol{\Upsilon}}| = \mathcal{D}^2\tilde{\boldsymbol{\Upsilon}}| = 0$, and the action (4.2.2) simplifies greatly.

The complete component action turns out to be extremely complicated as far as the fermionic sector is concerned. The fermionic sector will be studied in the

[2]This result is in agreement with, for example, [70].

flat-space case in the following chapter. Here we only focus on the bosonic sector.

$$
\begin{aligned}
S_{\text{bosonic}} = \int \mathrm{d}^4x\, e^{-1} \Big\{ & \frac{1}{2}\mathcal{R} - g_{i\underline{i}}\, \nabla^a Y^i\, \nabla_a \bar{Y}^{\underline{i}} - \frac{1}{4}\Big(\mathrm{e}^{2\varphi}\, (\nabla a)^2 + (\nabla\varphi)^2\Big) \\
& -\frac{1}{4}\,\mathrm{e}^{-\varphi} F^{ab} F_{ab} + \frac{1}{4}\, a\, F^{ab} \tilde{F}_{ab} + \mathrm{e}^{-2\varphi}\, \omega|\, \bar{\omega}|\, \Lambda\big(\mathrm{e}^{-\varphi}\omega|\, ,\, \mathrm{e}^{-\varphi}\bar{\omega}|\big) \\
& +\frac{4}{3}\mathbb{A}^a\mathbb{A}_a - \frac{1}{3} B\bar{B} + \frac{1}{2}\mathrm{e}^{-\varphi} D^2 + g_{i\underline{i}}\, F^i \bar{F}^{\underline{i}} + \frac{1}{4}\,\mathrm{e}^{2\varphi}\,\mathcal{F}\bar{\mathcal{F}} \Big\}\ ,
\end{aligned} \tag{4.2.8}
$$

where

$$
\begin{aligned}
\omega| &= F^{\alpha\beta} F_{\alpha\beta} - \frac{1}{2} D^2\ , \qquad\qquad \bar{\omega}| \ = \ \bar{F}^{\dot{\alpha}\dot{\beta}} \bar{F}_{\dot{\alpha}\dot{\beta}} - \frac{1}{2} D^2\ , \\
A_a &= \mathbb{A}_a - \frac{\mathrm{i}}{4}\left(K_i\, \nabla_a Y^i - K_{\underline{i}}\, \nabla_a \bar{Y}^{\underline{i}}\right) - \frac{1}{4}\,\mathrm{e}^{\varphi}\, \nabla_a a\ ,
\end{aligned} \tag{4.2.9}
$$

and $\mathcal{R}$ and F_{ab} are as defined respectively in (1.4.4) and (1.6.4), but with torsion set to zero.

As a special representative in the family of self-dual actions (2.4.1)–(2.4.3), we would like to consider the supersymmetric Born-Infeld action (2.4.6) in curved superpace. In this case the function $\Lambda(\omega,\bar{\omega})$ takes the form

$$
\begin{aligned}
\Lambda(\omega,\bar{\omega}) &= \frac{\kappa^2}{1 + \frac{1}{2}\,A + \sqrt{1 + A + \frac{1}{4}\,B^2}}\ , \\
A = \kappa^2(\omega + \bar{\omega})\ ,& \qquad B = \kappa^2(\omega - \bar{\omega})\ ,
\end{aligned} \tag{4.2.10}
$$

where we have introduced the coupling constant κ. After eliminating the auxiliary fields, the bosonic action (4.2.8) becomes

$$
\begin{aligned}
S = \int \mathrm{d}^4x\, e^{-1} \Big\{ & \frac{1}{2}\mathcal{R} - g_{i\underline{i}}\, \nabla^a Y^i\, \nabla_a \bar{Y}^{\underline{i}} - \frac{1}{4}\Big(\mathrm{e}^{2\varphi}\, (\nabla a)^2 + (\nabla\varphi)^2\Big) \\
& +\frac{1}{\kappa^2}\left(1 - \sqrt{-\det(\eta_{ab} + \kappa\, \mathrm{e}^{-\varphi/2} F_{ab})}\right) + \frac{1}{4}\, a\, F^{ab}\tilde{F}_{ab} \Big\}\ .
\end{aligned} \tag{4.2.11}
$$

Fermionic dynamics in flat spacetime

If supersymmetry were an unbroken symmetry then superpartners, while displaying opposite statistics, would be expected to have the same mass. The fact that superpartners have not yet been observed implies that, if supersymmetry exists, it must be broken at an energy scale beyond the reach of today's accelerators. As indicated in the introduction, self-dual theories have a strong connection to theories in which supersymmetry is partially broken spontaneously. Such a process is always accompanied by a fermionic particle called the goldstino [2, 3].

The $\mathcal{N} = 1$ supersymmetric Born-Infeld (SBI) action [55] is known to describe the Maxwell-Goldstone multiplet for spontaneous partial supersymmetry breaking $\mathcal{N} = 2 \to \mathcal{N} = 1$ [56, 57]. As a consequence, its purely fermionic sector (5.2.14), turns out to describe spontaneous breakdown of $\mathcal{N} = 1$ supersymmetry [100]. However, the standard action describing the breakdown of $\mathcal{N} = 1$ supersymmetry is the Akulov-Volkov (AV) action [2, 3] for the goldstino. Universality of the goldstino dynamics implies that the SBI and AV actions be related. Indeed it was conjectured in [100] that the two actions are related by a nontrivial field redefinition. Moreover, guided by considerations of nonlinearly realized supersymmetry, the authors of [100] proposed a nice scheme for constructing such a field redefinition.

The fermionic sector of the SBI action contains higher derivative terms, and yet, remarkably, the above argument says that there exists a field redefinition that will eliminate these terms bringing the action into the Akulov-Volkov form. Now, it is a general feature of nonlinear self-dual systems that their component structure is highly nontrivial, even in the case of flat global superspace. This occurs not only for the models of the massless vector multiplet discussed in previous chapters, but also for the nonlinear self-dual models of the tensor multiplet introduced in [54]. The fermionic sector of these models proves to contain higher derivative terms. We will show that, as in the specific case of the SBI action, it is possible to remove these terms by implementing a nonlinear field redefinition. The resulting action will be a one-parameter deformation of the AV action. We will also investigate the fermionic sector of the chiral-scalar-Goldstone theory [57, 58], dual to the tensor-Goldstone theory.

5.1 The family model in flat superspace

Our discussion of the fermionic dynamics in $\mathcal{N} = 1$ supersymmetric nonlinear electrodynamics will be restricted to the case of flat global superspace (see section 1.5).

Here, the condition for self-duality becomes

$$\mathrm{Im} \int \mathrm{d}^6 z \left\{ W^2 + M^2 \right\} = 0 \ , \qquad \frac{\mathrm{i}}{2} M_\alpha = \frac{\delta}{\delta W^\alpha} S[W, \bar{W}] \ . \tag{5.1.1}$$

The family action takes the form

$$S[W, \bar{W}] = \frac{1}{4} \int \mathrm{d}^6 z \, W^2 + \frac{1}{4} \int \mathrm{d}^6 \bar{z} \, \bar{W}^2 + \frac{1}{4} \int \mathrm{d}^8 z \, W^2 \, \bar{W}^2 \, \Lambda(\omega, \bar{\omega}) \ , \tag{5.1.2}$$

where now

$$\omega \ \equiv \ \frac{1}{8} D^2 W^2 \ , \qquad \bar{\omega} \ \equiv \ \frac{1}{8} \bar{D}^2 \bar{W}^2 \ , \tag{5.1.3}$$

and the function $\Lambda(\omega, \bar{\omega})$ still satisfies the differential equation (2.4.3).

In the case of the globally supersymmetric version of (2.4.6), we obtain the $\mathcal{N} = 1$ supersymmetric Born-Infeld (SBI) action [55],

$$S_{\mathrm{SBI}} = \frac{1}{4} \int \mathrm{d}^6 z \, W^2 + \frac{1}{4} \int \mathrm{d}^6 \bar{z} \, \bar{W}^2 + \frac{\kappa^2}{4} \int \mathrm{d}^8 z \, \frac{W^2 \, \bar{W}^2}{1 + \frac{1}{2} A + \sqrt{1 + A + \frac{1}{4} B^2}} \ ,$$
$$A = \kappa^2 \left(\omega + \bar{\omega} \right) \ , \qquad B = \kappa^2 \left(\omega - \bar{\omega} \right) \ , \tag{5.1.4}$$

with the coupling constant κ introduced. This action describes the dynamics of the Maxwell-Goldstone multiplet for the spontaneous partial breaking of supersymmetry $\mathcal{N} = 2 \to \mathcal{N} = 1$ [56, 57]. The broken supersymmetry is nonlinearly realized; the SBI action (5.1.4) being invariant under the second supersymmetry transformation

$$\delta X = 2 \, \eta^\alpha W_\alpha \ , \qquad \delta W_\alpha = \eta_\alpha \left(1 + \frac{\kappa^2}{16} \bar{D}^2 \bar{X} \right) + \frac{\mathrm{i}\kappa^2}{4} \partial_{\alpha\dot{\alpha}} X \bar{\eta}^{\dot{\alpha}} \ , \tag{5.1.5}$$

where η_α is a constant parameter and X, $\bar{D}_{\dot{\alpha}} X = 0$, is the constrained chiral scalar superfield of (2.4.4, 2.4.5) in the flat superspace limit,

$$S_{\mathrm{SBI}} = \frac{1}{4} \int \mathrm{d}^6 z \, X + \frac{1}{4} \int \mathrm{d}^6 \bar{z} \, \bar{X} \ , \qquad X + \frac{\kappa^2}{16} X \, \bar{D}^2 \bar{X} = W^2 \ . \tag{5.1.6}$$

5.2 Fermionic dynamics of the family model

Now, to pick out the fermionic sector of the family model (5.1.2), we can safely set the bosonic coordinates to zero. The reason for this is that the action has the following symmetry at the superfield level.

$$W_\alpha(x, \theta) \ \longrightarrow \ W_\alpha(x, -\theta) \ . \tag{5.2.1}$$

At the component level, the spinor field is invariant

$$\psi_\alpha(x) = W_\alpha| \ \longrightarrow \ \psi_\alpha(x) \ , \tag{5.2.2}$$

whilst the bosonic fields transform as

$$F_{\alpha\beta}(x) = \frac{1}{2\mathrm{i}} D_{(\alpha} W_{\beta)}| \longrightarrow -F_{\alpha\beta}(x) , \qquad D(x) = -\frac{1}{2} D^\alpha W_\alpha| \longrightarrow -D(x) . \quad (5.2.3)$$

This symmetry implies that the component action contains only even powers of the bosonic fields. It is therefore consistent, when discussing the component structure, to restrict our attention to the purely fermionic sector specified by[1]

$$D_\alpha W_\beta| = 0 . \quad (5.2.4)$$

Let $S[\psi, \bar\psi]$ be the fermionic action that follows from (5.1.2) upon switching off all the bosonic fields. It turns out that $S[\psi, \bar\psi]$ obeys a functional equation which is induced by the self-duality (5.1.1).

The self-duality equation (5.1.1) must hold for an arbitrary chiral spinor $W_\alpha(z)$ and its conjugate $\bar W_{\dot\alpha}(z)$. This means that the spinors W_α and $\bar W_{\dot\alpha}$ are chosen in (5.1.1) to satisfy only the chirality constraints $\bar D_{\dot\alpha} W_\alpha = 0$ and $D_\alpha \bar W_{\dot\alpha} = 0$, but not the Bianchi identity

$$D^\alpha W_\alpha = \bar D_{\dot\alpha} \bar W^{\dot\alpha} . \quad (5.2.5)$$

Thus W_α now contains two independent fermionic components

$$\psi_\alpha(x) = W_\alpha| , \qquad \rho_\alpha(x) = -\frac{1}{4} D^2 W_\alpha| . \quad (5.2.6)$$

Let $\hat S \equiv S[\psi, \bar\psi, \rho, \bar\rho]$ be the component action that follows from (5.1.2) upon relaxing the Bianchi identity and restricting to the fermionic sector (5.2.4). Then, the self-duality equation (5.1.1) reduces to

$$\mathrm{Im} \int \mathrm{d}^4 x \left\{ \psi^\alpha \rho_\alpha + 4 \frac{\delta \hat S}{\delta \psi^\alpha} \frac{\delta \hat S}{\delta \rho_\alpha} \right\} = 0 . \quad (5.2.7)$$

The genuine fermionic action, $S[\psi, \bar\psi]$, is obtained from the self-dual action $\hat S$ by imposing the "fermionic Bianchi identities" $\rho_\alpha = -\mathrm{i}\, (\sigma^b \partial_b \bar\psi)_\alpha$ and $\bar\rho_{\dot\alpha} = \mathrm{i}\, (\partial_b \psi \sigma^b)_{\dot\alpha}$,

$$S[\psi, \bar\psi] = S[\psi, \bar\psi, \rho, \bar\rho]\Big|_{\rho = -\mathrm{i}\, (\sigma^b \partial_b \bar\psi)} . \quad (5.2.8)$$

[1]As an additional argument, in the case of the SBI action (5.1.4), the restriction (5.2.4) is consistent with the nonlinear second supersymmetry (5.1.5), since it is invariant under this transformation, *i.e.* $\delta(D_\alpha W_\beta)|_{F_{ab} = D = 0} = 0$.

A short calculation leads to the fermionic action

$$
\begin{aligned}
S[\psi,\bar\psi] = \int \mathrm{d}^4x \Big\{ &-\frac{1}{2}\langle u+\bar u\rangle + \Big(\langle u\rangle\langle\bar u\rangle - \frac{1}{4}(\partial^a\psi^2)(\partial_a\bar\psi^2)\Big)\,\Lambda(0,0) \\
&+\Big(\langle u\rangle^2\langle\bar u\rangle + \frac{1}{2}(\bar\psi^2\Box\psi^2)\Big)\,\Lambda_\omega(0,0) + \Big(\langle u\rangle\langle\bar u\rangle^2 + \frac{1}{2}(\psi^2\Box\bar\psi^2)\Big)\,\Lambda_{\bar\omega}(0,0) \\
&+\Big(\langle u\rangle^2\langle\bar u\rangle^2 - \frac{1}{2}(\partial^a\psi^2)(\partial_a\bar\psi^2)\langle u\rangle\langle\bar u\rangle + \frac{1}{16}\psi^2\bar\psi^2(\Box\psi^2)(\Box\bar\psi^2)\Big)\Lambda_{\omega\bar\omega}(0,0) \\
&+\frac{3}{8}(\bar\psi^2\Box\psi^2)\langle u\rangle^2\,\Lambda_{\omega\omega}(0,0) + \frac{3}{8}(\psi^2\Box\bar\psi^2)\langle\bar u\rangle^2\,\Lambda_{\bar\omega\bar\omega}(0,0)\Big\}\ . \qquad (5.2.9)
\end{aligned}
$$

Here we have introduced the following 4×4 matrices:

$$
u_a{}^b = \mathrm{i}\,\psi\sigma^b\partial_a\bar\psi\ , \qquad \bar u_a{}^b = -\mathrm{i}\,(\partial_a\psi)\sigma^b\bar\psi\ , \qquad (5.2.10)
$$

as well as made use of the useful compact notation

$$
\langle F\rangle \equiv \mathrm{tr}\,F = F_a{}^a\ , \qquad (5.2.11)
$$

for an arbitrary 4×4 matrix $F=(F_a{}^b)$.

The fermionic action obtained involves several constant parameters associated with the function $\Lambda(\omega,\bar\omega)$ that enters the original supersymmetric action. However, not all of these parameters are independent since $\Lambda(\omega,\bar\omega)$ must be a solution to the self-duality equation (2.4.3). This restriction proves to imply

$$
\Lambda_\omega(0,0) = \Lambda_{\bar\omega}(0,0) = -\Lambda^2(0,0)\ , \qquad \Lambda_{\omega\omega}(0,0) = \Lambda_{\bar\omega\bar\omega}(0,0) = 2\Lambda^3(0,0)\ . \qquad (5.2.12)
$$

The self-duality equation imposes no restrictions on $\Lambda(0,0)$ and $\Lambda_{\omega\bar\omega}(0,0)$. For later convenience, we represent

$$
\Lambda(0,0) = \frac{\kappa^2}{2}\ , \qquad \Lambda_{\omega\bar\omega}(0,0) = \frac{\kappa^6}{8}(\mu+3)\ . \qquad (5.2.13)
$$

5.2.1 Fermionic sector of the supersymmetric BI action

For the particular case of the SBI action (5.1.4), we have the parameter $\mu=0$ and the fermionic action (5.2.9) taking the form

$$
\begin{aligned}
S_{\rm BG}[\psi,\bar\psi] = \int \mathrm{d}^4x \Big\{ &-\frac{1}{2}\langle u+\bar u\rangle + \frac{\kappa^2}{2}\Big(\langle u\rangle\langle\bar u\rangle - \frac{1}{4}(\partial^a\psi^2)(\partial_a\bar\psi^2)\Big) \qquad (5.2.14) \\
&-\frac{\kappa^4}{4}\Big(\langle u\rangle^2\langle\bar u\rangle + \langle u\rangle\langle\bar u\rangle^2 + \frac{1}{2}(\bar\psi^2\Box\psi^2) + \frac{1}{2}(\psi^2\Box\bar\psi^2)\Big) \\
&+\frac{3\kappa^6}{32}\Big(4\langle u\rangle^2\langle\bar u\rangle^2 - 2(\partial^a\psi^2)(\partial_a\bar\psi^2)\langle u\rangle\langle\bar u\rangle + \frac{1}{4}\psi^2\bar\psi^2(\Box\psi^2)(\Box\bar\psi^2) \\
&+ (\bar\psi^2\Box\psi^2)\langle u\rangle^2 + (\psi^2\Box\bar\psi^2)\langle\bar u\rangle^2\Big)\Big\}\ .
\end{aligned}
$$

5.3 The Akulov-Volkov action

In the pioneering papers [2, 3] on supersymmetry Akulov and Volkov obtained the action for the goldstino associated with the spontaneous breaking of $\mathcal{N}=1$ supersymmetry. The Akulov-Volkov (AV) action takes the form[2]

$$S_{\rm AV}[\lambda,\bar\lambda] = \frac{2}{\kappa^2}\int {\rm d}^4x\,\Big\{1-\det\Xi\Big\}\ , \tag{5.3.1}$$

where

$$\Xi_a{}^b = \delta_a{}^b + \frac{\kappa^2}{4}\Big({\rm i}\,\lambda\sigma^b\partial_a\bar\lambda - {\rm i}\,(\partial_a\lambda)\sigma^b\bar\lambda\Big) \equiv \delta_a{}^b + \frac{\kappa^2}{4}(v+\bar v)_a{}^b\ , \tag{5.3.2}$$

having defined

$$v_a{}^b = {\rm i}\,\lambda\sigma^b\partial_a\bar\lambda\ , \qquad \bar v_a{}^b = -{\rm i}\,(\partial_a\lambda)\sigma^b\bar\lambda\ . \tag{5.3.3}$$

The broken supersymmetry is nonlinearly realized, with the AV action (5.3.1) being invariant under the transformation

$$\delta\lambda_\alpha = \frac{2}{\kappa}\,\eta_\alpha - \frac{{\rm i}\kappa}{2}\big(\lambda\,\sigma^a\bar\eta - \eta\,\sigma^a\bar\lambda\big)\partial_a\lambda_\alpha\ , \tag{5.3.4}$$

where η_α is a constant parameter.

5.4 The family action and the Akulov-Volkov action

Looking at the fermionic action (5.2.9), it is hardly possible to imagine that it is related somehow to the AV action (5.3.1) describing goldstino dynamics. Nevertheless, the two fermionic theories turn out to be closely related in the following sense. There exists a nonlinear field redefinition, $(\psi_\alpha,\bar\psi^{\dot\alpha}) \to (\lambda_\alpha,\bar\lambda^{\dot\alpha})$, that eliminates all the higher derivative terms in (5.2.9) and brings this action to a one-parameter deformation of the AV action. The two theories coincide, modulo such a field redefinition, under the choice

$$\Lambda_{\omega\bar\omega}(0,0) = \frac{3}{8}\kappa^6 = 3\Lambda^3(0,0) \quad \Longleftrightarrow \quad \mu = 0\ , \tag{5.4.1}$$

which occurs, in particular, in the case of the SBI action (5.1.4). The remainder of this section is devoted to the proof of the above statement.

---

[2]Note that the normalization factor used here differs from that of $1/2\kappa^2$ usually found in the literature. This is in order to match up with the coupling constant in the bosonic sector of the SBI action (6.2).

5.4.1 Field redefinition

Before we begin looking for a field redefinition, we first note that the the AV action may be written in the form (5.3.1) (see appendix E for more details)

$$S_{\rm AV}[\lambda,\bar\lambda]=\int{\rm d}^4x\,\Big\{-\frac{1}{2}\langle v+\bar v\rangle-\frac{\kappa^2}{4}\Big(\langle v\rangle\langle\bar v\rangle-\langle v\bar v\rangle\Big) \\ -\frac{\kappa^4}{32}\Big(\langle v^2\bar v\rangle-\langle v\rangle\langle v\bar v\rangle-\frac{1}{2}\langle v^2\rangle\langle\bar v\rangle+\frac{1}{2}\langle v\rangle^2\langle\bar v\rangle\ +\ {\rm c.c.}\Big)\Big\}\ . \tag{5.4.2}$$

We wish to find a field redefinition which will bring the action (5.2.9) in to the form (5.4.2). We begin by first noting that the leading order terms must match, $\psi_\alpha=\lambda_\alpha+O(\kappa^2)$. A general redefinition takes the form

$$\psi_\alpha=\lambda_\alpha+\frac{\kappa^2}{2}\Lambda^{(3)}_\alpha(\alpha_i)+\frac{\kappa^4}{4}\Lambda^{(5)}_\alpha(\beta_i)+\frac{\kappa^6}{8}\Lambda^{(7)}_\alpha(\gamma_i)\ , \tag{5.4.3}$$

where $\Lambda^{(n)}_\alpha$ are terms of n-th order in fields λ_α and $\bar\lambda_{\dot\alpha}$ containing $(n-1)/2$ partial derivatives. They are parametrized by a number of constant coefficients α_i, β_i and γ_i that can be chosen to be real. Higher order terms do not contribute due to the nilpotency of the spinor fields, $\lambda^3=0$.

At third-order in fields we can write $\Lambda^{(3)}_\alpha$ as

$$\Lambda^{(3)}_\alpha(\alpha_i)=\lambda_\alpha\Big\{\alpha_1\langle v\rangle+\alpha_2\langle\bar v\rangle\Big\}+{\rm i}\alpha_3(\sigma^a\bar\lambda)_\alpha(\partial_a\lambda^2)\ . \tag{5.4.4}$$

With this, if we substitute (5.4.3) into (5.2.9) we obtain

$$S[\psi,\bar\psi]=\int{\rm d}^4x\,\Big\{-\frac{1}{2}\langle v+\bar v\rangle-\frac{\kappa^2}{2}\alpha_1\left(\langle v\rangle^2+\langle\bar v\rangle^2\right)-\kappa^2(\alpha_2+\alpha_3-\frac{1}{2})\langle v\rangle\langle\bar v\rangle \\ +\kappa^2\alpha_3\langle v\bar v\rangle+\frac{\kappa^2}{2}(\alpha_3-\frac{1}{4})(\partial^a\lambda^2)(\partial_a\bar\lambda^2)\Big\}+O(\kappa^4)\ . \tag{5.4.5}$$

The requirement that the transformed action match with the AV action (5.4.2), uniquely fixes the coefficients $\alpha_1=0,\alpha_2=1/2,\alpha_3=1/4$.

At fifth-order there exist many more admissible structures that can contribute to the field redefinition under consideration. We can write $\Lambda^{(5)}_\alpha$ as

$$\Lambda^{(5)}_\alpha(\beta_i)=\lambda_\alpha\Big\{\beta_1\langle v\rangle\langle\bar v\rangle+\beta_2\langle\bar v\rangle^2+\beta_3(\partial^a\lambda^2)(\partial_a\bar\lambda^2)+\beta_4\langle v\bar v\rangle+\beta_5\langle\bar v^2\rangle+\beta_6(\bar\lambda^2\Box\lambda^2)\Big\} \\ +{\rm i}(\sigma^a\bar\lambda)_\alpha(\partial_a\lambda^2)\Big\{\beta_7\langle v\rangle+\beta_8\langle\bar v\rangle\Big\}\ . \tag{5.4.6}$$

Substituting into (5.2.9), after a tedious calculation, we are able to match the AV action (5.4.2) if

$$\beta_4=2\beta_3-\frac{1}{4}\ ,\quad\beta_5=-\frac{1}{4}\ ,\quad\beta_6=\frac{1}{16}\ ,\quad\beta_7=\frac{1}{4}-(\beta_1+\beta_2+\beta_3)\ ,\quad\beta_8=\beta_3\ . \tag{5.4.7}$$

Unlike the third-order case, not all coefficients are uniquely fixed – we are left with three free parameters, $\beta_1, \beta_2, \beta_3$.

At the highest-order, we can write $\Lambda^{(7)}_\alpha$ as

$$\Lambda^{(7)}_\alpha(\gamma_i) = \lambda_\alpha\Big\{\gamma_1\langle v\rangle\langle\bar v\rangle^2 + \gamma_2\langle v\bar v^2\rangle + \gamma_3\langle\bar v\rangle(\partial^a\lambda^2)(\partial_a\bar\lambda^2) + \gamma_4\langle v\rangle(\bar\lambda^2\Box\lambda^2)\Big\} \\ + \mathrm{i}\gamma_5(\sigma^a\bar\lambda)_\alpha(\partial_a\lambda^2)\langle v\rangle\langle\bar v\rangle \ . \tag{5.4.8}$$

Again, substituting into (5.2.9), we try to match the AV action (5.4.2), which vanishes at this order. We obtain the following restrictions:

$$\gamma_1 = \frac{1}{8}(3\mu + 1 + 4(\beta_1 - 2\beta_3 - 2\gamma)) \ , \qquad \gamma_2 = \beta_1 + \beta_2 - \beta_3 \ , \tag{5.4.9}$$
$$\gamma_3 = -\frac{1}{4}(\mu - 2\beta_1 - 2\beta_2) \ , \qquad \gamma_4 = \frac{1}{8}(\mu - \frac{1}{4} - 2\beta_1 + 4\beta_3) \ ,$$

where we have gained another free parameter, $\gamma \equiv \gamma_5$. However, at this order, even with this freedom in the redefinition, it is impossible to match the AV action unless a restriction is placed on the type of model we are investigating, *i.e.* we must choose a particular value for μ or, equivalently, $\Lambda_{\omega\bar\omega}(0,0)$.

If we do not restrict the model then, with the following field redefinition

$$\begin{aligned}\psi_\alpha = \lambda_\alpha\Big\{&1 + \frac{\kappa^2}{4}\langle\bar v\rangle + \frac{\kappa^4}{4}\Big(\beta_1\langle v\rangle\langle\bar v\rangle + \beta_2\langle\bar v\rangle^2 + (2\beta_3 - \frac{1}{4})\langle v\bar v\rangle \\ &- \frac{1}{4}\langle\bar v^2\rangle + \beta_3(\partial^a\lambda^2)(\partial_a\bar\lambda^2) + \frac{1}{16}(\bar\lambda^2\Box\lambda^2)\Big) \\ &+ \frac{\kappa^6}{64}\Big((3\mu + 1 + 4(\beta_1 - 2\beta_3 - 2\gamma))\langle v\rangle\langle\bar v\rangle^2 \\ &\quad - 2(\mu - 2\beta_1 - 2\beta_2)\langle\bar v\rangle(\partial^a\lambda^2)(\partial_a\bar\lambda^2) \\ &\quad + (\mu - \frac{1}{4} - 2\beta_1 + 4\beta_3)\langle v\rangle(\bar\lambda^2\Box\lambda^2) + 8(\beta_1 + \beta_2 - \beta_3)\langle v\bar v^2\rangle\Big)\Big\} \\ +\frac{\mathrm{i}}{8}\kappa^2(\sigma^a\bar\lambda)_\alpha(\partial_a\lambda^2)\Big\{&1 + \frac{\kappa^2}{2}(1 - 4(\beta_1 + \beta_2 + \beta_3))\langle v\rangle \\ &+ 2\kappa^2\beta_3\langle\bar v\rangle + \kappa^4\gamma\langle v\rangle\langle\bar v\rangle\Big\} \ ,\end{aligned} \tag{5.4.10}$$

the transformed action is

$$S[\psi,\bar\psi] = S_{\mathrm{AV}}[\lambda,\bar\lambda] + \frac{\kappa^6}{32}\,\mu\int \mathrm{d}^4x\,\langle v^2\bar v^2\rangle \ . \tag{5.4.11}$$

We see that the action (5.2.9), in conjunction with (5.2.12) and (5.2.13), is equivalent to the AV action (5.4.2) if eq. (5.4.1) holds. In particular, the SBI action (5.1.4) has this property.

Our field redefinition (5.4.10) involves four free parameters, $\beta_1, \beta_2, \beta_3$ and γ, which do not show up in the transformed action (5.4.11). This means that these parameters correspond to some symmetries of the original theory (5.2.9). Indeed, if we modify the field redefinition (5.4.10) by varying any of the parameters, the action is not affected.

5.4.2 Discussion

At first glance, the existence of the field redefinition (5.4.10) that turns the action (5.2.9) into (5.4.11), looks absolutely fantastic and unpredictable. However, it has a solid theoretical justification in one special case of self-dual supersymmetric electrodynamics (5.1.2) – the $\mathcal{N}=1$ supersymmetric Born-Infeld action (5.1.4). As we have already mentioned, this action is known to describe the Maxwell-Goldstone multiplet for spontaneous partial supersymmetry breaking $\mathcal{N}=2 \to \mathcal{N}=1$ [56, 57]. As a consequence, its purely fermionic sector (5.2.14), turns out to describe spontaneous breakdown of $\mathcal{N}=1$ supersymmetry [100]. This goldstino action clearly does not coincide with the standard goldstino action (5.3.1) or, equivalently, with (5.4.2). Universality of the goldstino dynamics, on the other hand, implies that the two goldstino actions, (5.3.1) and (5.2.14), should be related to each other. It was therefore conjectured in [100] that the actions (5.3.1) and (5.2.14) are related by a nontrivial field redefinition. Moreover, guided by considerations of nonlinearly realized supersymmetry, the authors of [100] proposed a nice scheme for constructing such a field redefinition and also confirmed it to order κ^2. Pushing their scheme to higher orders seems to give the redefinition (5.4.10) with all the parameters fixed as follows: $\beta_1 = 1/16$, $\beta_2 = 0$, $\beta_3 = 1/32$ and $\gamma = 0$. We have checked the correspondence to order κ^4.

The broken supersymmetry is nonlinearly realized. From (5.1.5) we find that (5.2.14) is invariant under the transformation

$$\delta\psi_\alpha = \delta W_\alpha| = \eta_\alpha \left(1 + \frac{\kappa^2}{16}\bar{D}^2\bar{X}|\right) + \frac{{\rm i}\kappa^2}{4}\partial_{\alpha\dot{\alpha}}X|\,\bar{\eta}^{\dot{\alpha}} \ , \tag{5.4.12}$$

where

$$\begin{aligned} X| \ &= \ \psi^2\Big(1 - \frac{\kappa^2}{2}\langle\bar{u}\rangle + \frac{\kappa^4}{4}\langle\bar{u}\rangle^2 + \frac{\kappa^4}{16}\,\bar{\psi}^2\Box\psi^2\Big) \ , \qquad\qquad (5.4.13)\\ \frac{1}{8}D^2X| \ &= \ \langle u\rangle - \frac{\kappa^2}{2}\langle u\rangle\langle\bar{u}\rangle + \frac{\kappa^4}{4}\Big(\langle u\rangle^2\langle\bar{u}\rangle + \langle u\rangle\langle\bar{u}\rangle^2 - \langle u\rangle(\partial^a\psi^2)(\partial_a\bar{\psi}^2)\\ &\qquad + \frac{1}{4}\langle u\rangle(\bar{\psi}^2\Box\psi^2) + \frac{1}{4}\langle\bar{u}\rangle(\psi^2\Box\bar{\psi}^2) + \frac{1}{4}\psi^2\bar{\psi}^2\Box\langle u\rangle\Big)\\ &\quad - \frac{3\kappa^6}{8}\Big(\langle u\rangle^2\langle\bar{u}\rangle^2 - \frac{1}{2}\langle u\rangle\langle\bar{u}\rangle(\partial^a\psi^2)(\partial_a\bar{\psi}^2) + \frac{1}{4}\langle u\rangle^2(\bar{\psi}^2\Box\psi^2) \end{aligned}$$

$$+\frac{1}{6}\langle\bar{u}\rangle^2(\psi^2\Box\bar{\psi}^2)+\frac{1}{16}\psi^2\bar{\psi}^2(\Box\psi^2)(\Box\bar{\psi}^2)\Big)\ .$$

We would also like to remark that, the field redefinition (5.4.10) corresponds to the purely fermionic sector of the globally supersymmetric theory (5.1.2). In the case when both bosonic and fermionic fields are present, as well as in the presence of supergravity – the case we analyzed in chapter 4, there should exist an extension of (5.4.10) that, at least, eliminates all higher derivative terms from the component action. But here our brute-force approach becomes extremely cumbersome and tedious to follow (even the fermionic case was quite taxing). We believe that there should be a more efficient approach to construct such field redefinitions, but have not yet been able to determine such an approach.

5.5 Self-duality and the tensor multiplet

In [54], the concept of self-duality was extended to general globally supersymmetric models of the tensor multiplet. Motivated by our results from the previous section, we would like to now consider the fermionic sector of such self-dual models of the tensor multiplet, with the expectation that there will again be a close relationship with the AV action. This motivation stems from the fact that the tensor-Goldstone theory describing the spontaneous partial supersymmetry breaking $\mathcal{N}=2\rightarrow\mathcal{N}=1$, belongs to the class of self-dual models introduced in [54], *i.e.* the tensor-Goldstone theory displays U(1) duality invariance.

5.5.1 Self-duality equation for the tensor multiplet

Here, we briefly relate, following [54], the formulation of self-dual models of the tensor multiplet, before going on to investigate the component structure of such models. The tensor multiplet was introduced in section 1.6 in terms of the real linear scalar superfield L, which is subject to the constraint (1.6.6). We consider the antichiral spinor superfield[3] defined by

$$\Psi_\alpha=D_\alpha L\ ,\qquad D_\beta\Psi_\alpha=0\ ,\tag{5.5.1}$$

which, as a consequence of the constraint (1.6.6), must satisfy the Bianchi identity

$$-\frac{1}{4}\bar{D}^2\Psi_\alpha+\mathrm{i}\partial_{\alpha\dot{\alpha}}\bar{\Psi}^{\dot{\alpha}}=0\ .\tag{5.5.2}$$

[3]Our inability to define such a superfield in curved superspace prevents a simple extension of these ideas to the locally supersymmetric case.

Analagous to the electrodynamic case considered in chapter 2, for a theory with action $S[\Psi, \bar{\Psi}]$, we define (anti) chiral superfields Υ_α and $\bar{\Upsilon}_{\dot{\alpha}}$ as

$$
\mathrm{i}\,\Upsilon_\alpha \equiv 2\,\frac{\delta}{\delta\Psi^\alpha}\,S\;, \qquad -\mathrm{i}\,\bar{\Upsilon}^{\dot{\alpha}} \equiv 2\,\frac{\delta}{\delta\bar{\Psi}_{\dot{\alpha}}}\,S\;. \tag{5.5.3}
$$

The equation of motion for such models reads:

$$
-\frac{1}{4}\bar{D}^2\Upsilon_\alpha + \mathrm{i}\partial_{\alpha\dot{\alpha}}\bar{\Upsilon}^{\dot{\alpha}} = 0\;. \tag{5.5.4}
$$

Now, since the Bianchi identity (1.6.2) and the equation of motion (2.2.4) have the same functional form, one may consider U(1) duality transformations of the form

$$
\begin{pmatrix} \Upsilon'_\alpha \\ \Psi'_\alpha \end{pmatrix} = \begin{pmatrix} \cos\tau & -\sin\tau \\ \sin\tau & \cos\tau \end{pmatrix} \begin{pmatrix} \Upsilon_\alpha \\ \Psi_\alpha \end{pmatrix}\;, \tag{5.5.5}
$$

This leads us to the following condition for self-duality

$$
\mathrm{Im}\int \mathrm{d}^6\bar{z}\,\Big(\Psi^\alpha\Psi_\alpha \;+\; \Upsilon^\alpha\Upsilon_\alpha\Big) \;=\; 0\;. \tag{5.5.6}
$$

In [54], the authors showed that such models are invariant under a superfield Legendre transformation, and indicated that many of the results that held for the nonlinear self-dual models of the vector multiplet translated directly to the tensor multiplet case. To go from a vector multiplet theory to a tensor multiplet theory one should (i) switch the chirality of the $\mathcal{N}=1$ superspace by $x^a \to x^a$, $\theta_\alpha \to \bar{\theta}_{\dot{\alpha}}$, $\bar{\theta}_{\dot{\alpha}} \to \theta_\alpha$, mapping the chiral superspace into the anitchiral superspace, and vice versa; and (ii) the chiral spinor superfield W_α becomes the anitchiral spinor superfield Ψ_α, $W_\alpha(x,\theta) \to \Psi_\alpha(x,\bar{\theta})$. In particular, applying this to the vector multiplet family model (2.4.1), we would like to consider the family of actions

$$
S = \frac{1}{4}\int \mathrm{d}^6 z\,\bar{\Psi}^2 + \frac{1}{4}\int \mathrm{d}^6\bar{z}\,\Psi^2 + \frac{1}{4}\int \mathrm{d}^8 z\,\Psi^2\bar{\Psi}^2\Lambda(\omega,\bar{\omega})\;, \tag{5.5.7}
$$

where $\Lambda(\omega,\bar{\omega})$ is now an analytic function of the variables

$$
\omega = \frac{1}{8}\bar{D}^2\Psi^2\;, \qquad \bar{\omega} = \frac{1}{8}D^2\bar{\Psi}^2\;. \tag{5.5.8}
$$

The action (5.5.7) is a solution of the self-duality equation (5.5.6) if $\Lambda(\omega,\bar{\omega})$ satisfies the differential equation (2.4.3).

The action for the tensor-Goldstone multiplet [58] is a particular example of this family of self-dual models (5.5.7). The action is constructed in a similar way to the supersymmetric Born-Infeld action (5.1.4). We consider the following action for an antichiral superfield X

$$S = \frac{1}{4}\int \mathrm{d}^6 z\, \bar{X} \;+\; \frac{1}{4}\int \mathrm{d}^6 \bar{z}\, X \;, \qquad D_\alpha X = 0 \;, \tag{5.5.9}$$

satisfying the nonlinear constraint

$$X + \frac{\kappa^2}{16} X D^2 \bar{X} = \Psi^2 \;. \tag{5.5.10}$$

Solving the constraint, we find that the action (5.5.9) can be written in the form

$$S = \frac{1}{4}\int \mathrm{d}^6 z\, \bar{\Psi}^2 + \frac{1}{4}\int \mathrm{d}^6 \bar{z}\, \Psi^2 + \frac{\kappa^2}{4}\int \mathrm{d}^8 z\, \frac{\Psi^2\,\bar{\Psi}^2}{1+\frac{1}{2}A + \sqrt{1+A+\frac{1}{4}B^2}} \;,$$
$$A = \kappa^2\,(\omega + \bar{\omega}) \;, \qquad B = \kappa^2\,(\omega - \bar{\omega}) \;. \tag{5.5.11}$$

The action describes spontaneous partial supersymmetry breaking $\mathcal{N}=2 \to \mathcal{N}=1$ – the broken supersymmetry being nonlinearly realized in the following way:

$$\delta X = 2\,\eta^\alpha \Psi_\alpha \;, \qquad \delta\Psi_\alpha = \eta_\alpha \left(1 + \frac{\kappa^2}{16} D^2 \bar{X}\right) + \frac{\mathrm{i}\kappa^2}{4}\partial_{\alpha\dot{\alpha}} X\, \bar{\eta}^{\dot{\alpha}} \;, \tag{5.5.12}$$

where η_α is a constant parameter. The action (5.5.11) is of the form (5.5.7), and one can check that the self-duality equation (2.4.3) is satisfied. Thus the dynamics of the tensor-Goldstone multiplet is a U(1) duality invariant theory.

5.5.2 Tensor multiplet family model in components

We would now like to investigate the fermionic structure of the family model (5.5.7). We take the flat superspace limit of the component fields of the tensor multiplet, ℓ, $\tilde{\psi}_\alpha$, $\tilde{V}_a$, defined by (1.6.8) and (1.6.9). The absence of auxiliary fields means we are free to set the bosonic sector to zero, $\ell = \tilde{V}_a = 0$.

Reducing to components in the usual manner, the fermionic action $S[\tilde{\psi}, \bar{\tilde{\psi}}]$ corresponding to the tensor multiplet family model (5.5.7) turns out to have *exactly the same* form as the fermionic action $S[\psi, \bar{\psi}]$ for vector multiplet family model (5.2.9). Therefore, under the trivial field redefinition $\tilde{\psi}_\alpha = \psi_\alpha$, followed by the nonlinear field redefinition (5.4.10), we have

$$S[\tilde{\psi}, \bar{\tilde{\psi}}] = S[\psi, \bar{\psi}] = S_{\mathrm{AV}}[\lambda, \bar{\lambda}] + \frac{\kappa^6}{32}\,\mu \int \mathrm{d}^4 x\, \langle v^2 \bar{v}^2 \rangle \;. \tag{5.5.13}$$

Thus, as for the vector multiplet family action (2.4.1), the fermionic action for the tensor multiplet family of models, (5.5.7), is equivalent to a one-parameter deformation of the AV action (5.3.1). Indeed, under the condition (5.4.1), it reduces to exactly AV action. In particular, this occurs for the tensor-Goldstone action (5.5.11). This is to be expected since, as with the Maxwell-Goldstone case of the previous section, the tensor-Goldstone multiplet describes the spontaneous breakdown of $\mathcal{N}=1$ supersymmetry, and as a result the fermionic action should correspond up to a field redefinition to the standard goldstino action (5.3.1). Moreover, this action will be invariant under the second nonlinear supersymmetry (5.4.12), with $\tilde{\psi}_\alpha = \psi_\alpha$.

We also note that, in the case of the tensor-Goldstone model (5.5.11), it should also be possible to transfer the scheme of [100] to determine this field redefinition in this case.

For completeness, we briefly mention that the bosonic sector of the tensor multiplet family action (5.5.7) is given by

$$S = \int \mathrm{d}^4x \left\{ -\frac{1}{2}(\omega + \bar{\omega})| + \omega|\,\bar{\omega}|\,\Lambda(\omega|, \bar{\omega}|) \right\} , \tag{5.5.14}$$

where

$$\omega| = -\frac{1}{2}(\tilde{V} + \mathrm{i}\partial\ell)^2 , \qquad \bar{\omega}| = -\frac{1}{2}(\tilde{V} - \mathrm{i}\partial\ell)^2 . \tag{5.5.15}$$

In the case of the tensor-Goldstone action (5.5.11), it takes the form

$$S = \frac{1}{\kappa^2}\int \mathrm{d}^4x \left\{ 1 - \sqrt{1 - \kappa^2(\tilde{V}^2 - \partial\ell^2) - \kappa^4(\tilde{V}^a\partial_a\ell)^2} \right\} . \tag{5.5.16}$$

5.6 Chiral-scalar-Goldstone multiplet

There is a third Goldstone multiplet associated with the spontaneous partial breaking of $\mathcal{N}=2 \to \mathcal{N}=1$ supersymmetry. It is given in terms of the chiral scalar multiplet obtained by dualizing the tensor-Goldstone multiplet [58]. We would like to investigate the fermionic action associated with this action. Again, universality of goldstino dynamics implies that there should exist a nonlinear field redefinition taking this action into the Akulov-Volkov form (5.3.1).

5.6.1 Dual of the tensor-Goldstone action

We begin by dualizing the tensor-Goldstone action (5.5.11), as outlined in [58], and detailed in [101] . We start with the the tensor family model (5.5.7), and rewrite the action it in terms of the real linear superfield, L:

$$S[L] = \int \mathrm{d}^8z \left\{ -L^2 + \frac{1}{4}(DL)^2(\bar{D}L)^2\Lambda(\omega, \bar{\omega}) \right\} , \tag{5.6.1}$$

where now

$$\omega = \frac{1}{4}(D^\alpha \bar{D}^{\dot{\alpha}} L)(D_\alpha \bar{D}_{\dot{\alpha}} L) , \qquad \bar{\omega} = \frac{1}{4}(\bar{D}^{\dot{\alpha}} D^\alpha L)(\bar{D}_{\dot{\alpha}} D_\alpha L) . \tag{5.6.2}$$

The constraint (5.5.10) and the Bianchi identity (5.5.2), can be used to determine the nonlinear second supersymmetry in terms of L:

$$\delta L = \theta\eta + \bar{\theta}\bar{\eta} - \frac{\kappa^2}{8} D^\alpha \bar{X}\,\eta_\alpha - \frac{\kappa^2}{8} \bar{D}_{\dot{\alpha}} X\,\bar{\eta}^{\dot{\alpha}} \ . \tag{5.6.3}$$

To dualize $S[L]$, we then relax the constraint (1.6.6) on L while simultaneously adding a Lagrange multiplier to $S[L]$,

$$S = S[L] + 2\int \mathrm{d}^8 z\, L(\phi + \bar{\phi}) \ , \tag{5.6.4}$$

where ϕ is a covariantly chiral scalar superfield, $\bar{\mathcal{D}}_{\dot{\alpha}}\phi = 0$. If we substitute in the solution to the ϕ equation of motion we regain the action $S[L]$ with the linearity contraint (1.6.6) on L. On the other hand, if we solve the L equation of motion (see [101] and appendix F for details) we obtain the dual action. In general, it is difficult to solve the equation of motion for L, however in the case of the tensor-Goldstone action (5.5.11), we are indeed able to find a solution. The dual action of the tensor-Goldstone action is (see appendix F):

$$S[\phi,\bar{\phi}] = \int \mathrm{d}^8 z \left\{ 2\phi\bar{\phi} + \frac{(\kappa^2/4)\,(D\phi)^2(\bar{D}\bar{\phi})^2}{1 + p + \sqrt{(1+p)^2 - q\bar{q}}} \right\} = \int \mathrm{d}^8 z\, \mathcal{L}(\phi,\bar{\phi}) \ , \tag{5.6.5}$$

where

$$p = -\kappa^2(\partial^{\alpha\dot{\alpha}}\phi)(\partial_{\alpha\dot{\alpha}}\bar{\phi}) \ , \quad q = -\kappa^2(\partial^{\alpha\dot{\alpha}}\phi)(\partial_{\alpha\dot{\alpha}}\phi) \ , \quad \bar{q} = -\kappa^2(\partial^{\alpha\dot{\alpha}}\bar{\phi})(\partial_{\alpha\dot{\alpha}}\bar{\phi}) \ . \tag{5.6.6}$$

5.6.2 Chiral-scalar-Goldstone in components

We now turn to the fermionic sector of this action the action (5.6.5). To determine this, we note that it is invariant under the transformation

$$\phi(x,\theta) \ \longrightarrow \ -\phi(x,-\theta) \ . \tag{5.6.7}$$

At the component level, the spinor field is invariant

$$\chi_\alpha(x) = D_\alpha\phi| \to \chi_\alpha(x) \ , \tag{5.6.8}$$

whilst the bosonic fields change as follows

$$Y(x) = \phi| \longrightarrow -Y(x) \ , \qquad F(x) = -\frac{1}{4}D^2\phi| \longrightarrow -F(x) \ . \tag{5.6.9}$$

Thus we are free to set the bosonic fields to zero to give us the purely fermionic sector. A standard reduction to components then reveals the fermionic action to be

$$S[\chi,\bar\chi]=\int\mathrm{d}^4x\,\Big\{-\frac{1}{2}\langle w+\bar w\rangle+\frac{\kappa^2}{2}\Big(\langle w\rangle^2+\langle\bar w\rangle^2+\langle w\bar w\rangle+\frac{1}{4}(\partial^a\chi^2)(\partial_a\bar\chi^2)\Big) \\ -\frac{\kappa^4}{8}\Big(\langle w^2\bar w\rangle+\langle w\rangle\langle w\bar w\rangle-\frac{1}{2}\langle w\rangle(\partial^a\chi^2)(\partial_a\bar\chi^2)\ +\ \text{c.c.}\Big) \\ +\frac{\kappa^6}{32}(\partial^a\chi^2)(\partial_a\bar\chi^2)(\partial^b\chi^2)(\partial_b\bar\chi^2)\Big\}\ , \tag{5.6.10}$$

where

$$w_a{}^b=\mathrm{i}\,\chi\sigma^b\partial_a\bar\chi\ ,\qquad \bar w_a{}^b=-\mathrm{i}\,(\partial_a\chi)\sigma^b\bar\chi\ . \tag{5.6.11}$$

This action is not of the same form, nor does it contain higher-derivative terms as for the fermionic actions of the vector and tensor-Goldstone multiplets (5.2.9). However, it is still the action of the goldstino for spontaneous breaking of $\mathcal{N}=1$ supersymmetry, since it is invariant under the transformation

$$\delta\chi_\alpha=D_\alpha\delta\phi|=\eta_\alpha+\frac{\kappa^2}{16}D_\alpha\big[\eta^\beta\bar D^2D_\beta\mathcal{L}(\phi,\bar\phi)\big]\Big|\ , \tag{5.6.12}$$

where $\mathcal{L}(\phi,\bar\phi)$ is defined by (5.6.5). Therefore, there should exist a field redefinition taking it into the AV action form. To obtain this field redefinition, we use the same brute force technique that we employed in section 5.4 for the vector multiplet model. A general field redefinition takes the form

$$\chi_\alpha=\lambda_\alpha+\frac{\kappa^2}{2}\Lambda^{(3)}_\alpha(\alpha_i)+\frac{\kappa^4}{4}\Lambda^{(5)}_\alpha(\beta_i)+\frac{\kappa^6}{8}\Lambda^{(7)}_\alpha(\gamma_i)\ , \tag{5.6.13}$$

where, again, $\Lambda^{(n)}_\alpha$ are terms of n-th order in fields λ_α and $\bar\lambda_{\dot\alpha}$ containing $(n-1)/2$ partial derivatives, and are parametrized by the real constant coefficients α_i, β_i and γ_i. The general form of the $\Lambda^{(n)}_\alpha$ are given by (5.4.4), (5.4.6) and (5.4.8).

Substituting (5.6.13) into (5.6.10), the requirement that the transformed action match the AV action (5.4.2), means that, to fourth order in κ, the field redefinition takes the form:

$$\chi_\alpha=\lambda_\alpha\Big\{1+\frac{\kappa^2}{2}\big(\langle v\rangle+\frac{1}{2}\langle\bar v\rangle\big)+\frac{\kappa^4}{4}\Big(\beta_1\langle v\rangle\langle\bar v\rangle+\beta_2\langle\bar v\rangle^2+(2\beta_3+\frac{1}{4})\langle v\bar v\rangle \\ -\frac{1}{4}\langle\bar v^2\rangle+\beta_3(\partial^a\lambda^2)(\partial_a\bar\lambda^2)+\frac{3}{4}(\bar\lambda^2\Box\lambda^2)\Big)\Big\} \\ -\frac{\mathrm{i}}{4}\kappa^2(\sigma^a\bar\lambda)_\alpha(\partial_a\lambda^2)\Big\{\frac{1}{2}+\kappa^2(\beta_1+\beta_2+\beta_3-\frac{23}{4})\langle v\rangle-\kappa^2(\beta_3-\frac{7}{4})\langle\bar v\rangle\Big\}+O(\kappa^6)\ , \tag{5.6.14}$$

where $v_a{}^b$ is defined in (5.3.3). Again, at order κ^2, the field redefinition is uniqiuely fixed, while at order κ^4, there remains three free parameters, β_1, β_2 and β_3. We can see that there exists a field redefinition for which $S[\chi,\bar{\chi}] = S_{\rm AV}[\lambda,\bar{\lambda}]$. This result is perhaps not as dramatic as for the vector and tensor multiplet actions (5.2.9), seeing as the fermionic action (5.6.10) does not contain higher derivative terms; however it is a very nice result in that we have now shown that the fermionic sectors of all three actions associated with the spontaneous partial breaking of $\mathcal{N}=2 \to \mathcal{N}=1$ supersymmetry are equivalent, up to a nonlinear field redefinition to the AV action (5.3.1).

For completeness, we briefly mention that the bosonic sector of the chiral-scalar-Goldstone multiplet action (5.6.5) is

$$S_{\rm bosonic} = \frac{1}{\kappa^2}\int {\rm d}^4x \left\{1 - \sqrt{(1+2\kappa^2\,\partial Y\,\partial\bar{Y})^2 - 4\kappa^4(\partial Y)^2(\partial\bar{Y})^2}\right\} . \qquad (5.6.15)$$

Conclusion

We conclude this thesis with a discussion of some of the results obtained, and indicate some future perspectives.

Theories of nonlinear self-dual electrodynamics possess a number of interesting properties, including (i) invariance under Legendre transformation; and (ii) duality invariance the energy-momentum tensor. In chapter 2, we have generalized these models to nonlinear self-dual theories of a massless $\mathcal{N}=1$ vector multiplet in a curved $\mathcal{N}=1$ superspace. We have shown that the properties of the bosonic theories naturally generalize to the supersymmetric case, which possess (i) invariance under superfield Legendre transformation; (ii) duality invariance of the supercurrent and supertrace. These properties arise as a result of the theory satisfying the $\mathcal{N}=1$ self-duality equation (2.2.7), which places a nontrivial restriction on the action functional. The minimal extension of the supersymmetric Born-Infeld action (2.4.6) to curved superspace emerges as a particular example of an action which does indeed satisfy this condition.

In chapter 3 we have investigated the coupling of nonlinear self-dual supersymmetric theories to matter in both the old- and new-minimal formulations of supergravity. The most interesting result concerns coupling to Kähler sigma models. The outcome of our analysis was that this coupling is nontrivial in a curved superspace, even when the chiral superfields of the nonlinear sigma model are inert under duality transformations (3.3.2). In the case where the chiral superfields do transform under duality transformations, we have considered coupling to the dilaton-axion multiplet. For such models, we found that the U(1) group of duality transformation is enlarged to the non-compact group $\mathrm{SL}(2,\mathbb{R})$. We would like to point out that, to describe the dilaton-axion complex, we have used the $\mathcal{N}=1$ chiral multiplet. One may also realize the dilaton-axion complex in terms of the $\mathcal{N}=1$ tensor multiplet – a situation that arises in the context of heterotic string theory. Transition from the chiral to the tensor realization can be implemented as follows. Writing the dilaton-axion Kähler potential in the form (3.4.7), the action (3.4.6) can be brought (at the cost of sacrificing the manifest gauge invariance in the second line of the action) to such a form that Φ and $\bar{\Phi}$ appear only in the real combination $\mathrm{i}(\Phi-\bar{\Phi})/2$. We can then apply a superfield Legendre transformation which turns the description in terms of Φ and $\bar{\Phi}$ into one in terms of a *real* superfield $\mathbb{G}$ under the modified linearity condition

$$(\bar{\mathcal{D}}^2-4R)\mathbb{G}=W^\alpha W_\alpha\ , \qquad (\mathcal{D}^2-4\bar{R})\mathbb{G}=\bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}\ . \tag{6.1}$$

This constraint is known to describe the Chern-Simons coupling of the tensor multiplet to the vector multiplet. It would be interesting to understand what the fate of the $SL(2,\mathbb{R})$ duality symmetry is in this dual version of the theory.

In chapter 4 we have demonstrated the usefulness of the Kugo-Uehara method in the calculation of the component actions of the nonlinear self-dual models developed in previous chapters. The component structure of these models is extremely complicated, containing any number of derivatives. If one applies the traditional approach of component reduction to these models, the resulting action is not necessarily in a canonical form, and one must apply a field dependent Weyl and local chiral transformation (accompanied by a gravitino shift) to obtain a canonically normalized result. The Kugo-Uehara method allows us to avoid this by (i) extending the model to a super-Weyl invariant system; and (ii) when reducing to components, making a gauge choice such that the resulting action is in a canonical form. This method was implemented to derive the bosonic action of the $SL(2,\mathbb{R})$ duality invariant coupling to the dilaton-axion chiral multiplet and a Kähler sigma-model (4.2.8).

With regards the fermionic dynamics investigated in chapter 5, we have presented three fermionic models: (i) $S[\psi,\bar{\psi}]$, the fermionic action (5.2.9) of the vector multiplet family model (2.4.1); (ii) $S[\tilde{\psi},\bar{\tilde{\psi}}]$, the fermionic action (equivalent to (5.2.9) when $\psi_\alpha=\tilde{\psi}_\alpha$) for the tensor multiplet family model (5.5.7); and (iii) $S[\chi,\bar{\chi}]$, the fermionic action (5.6.10) of the chiral-scalar-Goldstone action (5.6.5). In all three cases, the actions may be brought into the Akulov-Volkov form (5.3.1) by a nontrivial field redefinition – for the first two actions, this field redefinition is given by (5.4.10) under the condition (5.4.1), while for the third action it is given by (5.6.14) (correspondence checked to order κ^4). Now, the first two actions are the fermionic sectors of the self-dual family models (2.4.1) and (5.5.7), with $\Lambda(\omega,\bar{\omega})$ a solution of the differential equation (2.4.3). Of these families, the SBI action (5.1.4) and the tensor-Goldstone action (5.5.11) are just special representatives of the infinitely many members which satisfy the condition (5.4.1). It is only these special representatives (5.1.4) and (5.5.11), which describe the spontaneous partial supersymmetry breaking $\mathcal{N}=2\to\mathcal{N}=1$. At the component level, however, the purely fermionic actions of all these members satisfy the condition (5.4.1) and are equivalent, up to a field redefinition (5.4.10), to the AV action (5.3.1), and are invariant under a second nonlinearly realized supersymmetry (5.4.12). In this sense, all such models can be considered to encode information about spontaneous supersymmetry breaking.

In conclusion, we make a final comment regarding the SBI action (5.1.4). In the purely bosonic sector, this theory reduces, upon elimination of the auxiliary field, to the Born-Infeld action

$$S_{\mathrm{BI}} = \frac{1}{\kappa^2}\int \mathrm{d}^4x \,\left(1 - \sqrt{-\det(\eta_{ab} + \kappa\, F_{ab})}\right) \ , \tag{6.2}$$

c.f. (2.1.9). In the purely fermionic sector, it reduces, upon implementing the field redefinition (5.4.10) with $\mu = 0$, to the AV action (5.3.1). In the general case it should describe, upon implementing a nonlinear field redefinition, the spacetime filling D3-brane in a special gauge for kappa-symmetry, see [102, 103] and references therein. Such a gauge would differ from the one chosen in [102]. It would also be of interest to obtain the full component description of the tensor-Goldstone (5.5.11) and chiral-scalar-Goldstone (5.6.5) theories. Such actions may be interpreted in terms of super 3-brane actions in five and six dimensional Minkowski space [57].

Conventions

Throughout this thesis we have used the notation and conventions of [27], and a few of the more important are collected here. As is usual, our conventions for indices have lower case letters from the middle and beginning of the Latin (Greek) alphabets corresponding to curved and flat spacetime (spinor) coordinates respectively. Likewise curved and flat superspace coordinates are denoted by upper case letters from the middle and beginning of the Latin alphabet respectively.

We use the mostly positive Minkowski metric,

$$\eta_{ab} = \mathrm{diag}(-1,\ 1,\ 1,\ 1)\ , \tag{A.1}$$

to raise and lower spacetime indices of tangent space tensors, $V_a = \eta_{ab}V^b$, $V^a = \eta^{ab}V_b$, where the indices $a, b = 0, 1, 2, 3$. Similarly, the curved spacetime metric g_{mn}, can be used to raise and lower indices of curved spacetime tensors. The two metrics are related by introducing the vierbein ${e_m}^a$,

$$g_{mn} = {e_m}^a {e_n}^b \eta_{ab}\ . \tag{A.2}$$

We also introduce the inverse vierbein ${e_a}^m$ by ${e_a}^m {e_m}^b = {\delta_a}^b$ and ${e_m}^a {e_a}^n = {\delta_m}^n$.

Symmetrization and antisymmetrization are denoted by parentheses and square brackets respectively,

$$V_{(A_1 \dots A_n)} = \frac{1}{n!}\sum_{\pi\in S_n} V_{A_{\pi(1)}\dots A_{\pi(n)}}\ , \qquad V_{[A_1 \dots A_n]} = \frac{1}{n!}\sum_{\pi\in S_n} \mathrm{sgn}(\pi) V_{A_{\pi(1)}\dots A_{\pi(n)}}\ . \tag{A.3}$$

The totally antisymmetric tensor ϵ_{abcd} is normalized such that $\epsilon_{0123} = 1$ and $\epsilon^{0123} = -1$. The product of two antisymmetric tensors, with various index contractions is

$$\begin{aligned}
\epsilon^{abcd}\epsilon_{efgh} &= -4!\, \delta^a_{[e}\delta^b_f\delta^c_g\delta^d_{h]}\ ,\\
\epsilon^{abcd}\epsilon_{efgd} &= -3!\, \delta^a_{[e}\delta^b_f\delta^c_{g]}\ ,\\
\epsilon^{abcd}\epsilon_{efcd} &= -4\, \delta^a_{[e}\delta^b_{f]}\ ,\\
\epsilon^{abcd}\epsilon_{ebcd} &= -6\, \delta^a_e\ ,\\
\epsilon^{abcd}\epsilon_{abcd} &= -24\ .
\end{aligned} \tag{A.4}$$

In curved spacetime, the totally antisymmetric tensor ϵ_{mnpq}, is given by

$$\epsilon_{mnpq} = {e_m}^a {e_n}^b {e_p}^c {e_q}^d \epsilon_{abcd}\ , \qquad \epsilon_{0123} = \sqrt{-g} = e^{-1}\ , \tag{A.5}$$

where $g = \det(g_{mn})$ and $e = \det({e_a}^m)$.

Two-component spinors carry undotted and dotted indices, $\alpha = 1, 2$ and $\dot\alpha = \dot 1, \dot 2$. The antisymmetric tensors, $\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}$ and $\epsilon_{\dot\alpha\dot\beta} = -\epsilon_{\dot\beta\dot\alpha}$ are defined by

$$\epsilon^{12} = \epsilon_{21} = 1 \ , \qquad \epsilon^{\dot 1\dot 2} = \epsilon_{\dot 2\dot 1} = 1 \ , \tag{A.6}$$

and $\epsilon^{\alpha\beta}\epsilon_{\beta\gamma} = \delta^{\alpha}{}_{\gamma}$ and $\epsilon^{\dot\alpha\dot\beta}\epsilon_{\dot\beta\dot\gamma} = \delta^{\dot\alpha}{}_{\dot\gamma}$. They can be used to raise and lower spinor indices,

$$\epsilon^{\alpha\beta}\psi_\beta = \psi^\alpha \ , \quad \epsilon_{\alpha\beta}\psi^\beta = \psi_\alpha \ , \qquad \epsilon^{\dot\alpha\dot\beta}\bar\psi_{\dot\beta} = \bar\psi^{\dot\alpha} \ , \quad \epsilon_{\dot\alpha\dot\beta}\bar\psi^{\dot\beta} = \bar\psi_{\dot\alpha} \ , \tag{A.7}$$

The standard summation convention is

$$\psi\chi = \psi^\alpha\chi_\alpha = \chi^\alpha\psi_\alpha = \chi\psi \ , \qquad \bar\psi\bar\chi = \bar\psi_{\dot\alpha}\bar\chi^{\dot\alpha} = \bar\chi_{\dot\alpha}\bar\psi^{\dot\alpha} = \bar\chi\bar\psi \ , \tag{A.8}$$

and $\psi^2 = \psi\psi$, $\bar\psi^2 = \bar\psi\bar\psi$. Spinor conjugation is then understood as Hermitian conjugation, $(\psi\chi)^* = (\psi^\alpha\chi_\alpha)^* = (\chi_\alpha)^*(\psi^\alpha)^* = \bar\chi_{\dot\alpha}\bar\psi^{\dot\alpha}$.

The sigma matrices are defined as

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{A.9}$$

They have the spinor index structure, $(\sigma_a)_{\alpha\dot\alpha}$. Sigma matrices with raised indices are denoted by

$$(\tilde\sigma_a)^{\dot\alpha\alpha} = \epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}(\sigma_a)_{\beta\dot\beta} \ . \tag{A.10}$$

Some useful properties of sigma matrices include

$$\begin{aligned} (\sigma_a\tilde\sigma_b + \sigma_b\tilde\sigma_a)_\alpha{}^\beta &= -2\,\eta_{ab}\,\delta_\alpha{}^\beta \ , & \mathrm{Tr}(\sigma_a\tilde\sigma_b) &= -2\,\eta_{ab} \ , \\ (\tilde\sigma_a\sigma_b + \tilde\sigma_b\sigma_a)^{\dot\alpha}{}_{\dot\beta} &= -2\,\eta_{ab}\,\delta^{\dot\alpha}{}_{\dot\beta} \ , & (\sigma^a)_{\alpha\dot\alpha}(\tilde\sigma_a)^{\dot\beta\beta} &= -2\,\delta_\alpha^\beta\,\delta_{\dot\alpha}^{\dot\beta} \ . \end{aligned} \tag{A.11}$$

and

$$\begin{aligned} \sigma_a\tilde\sigma_b\sigma_c &= \eta_{ac}\sigma_b - \eta_{bc}\sigma_a - \eta_{ab}\sigma_c + \mathrm{i}\epsilon_{abcd}\sigma^d \ , \\ \tilde\sigma_a\sigma_b\tilde\sigma_c &= \eta_{ac}\tilde\sigma_b - \eta_{bc}\tilde\sigma_a - \eta_{ab}\tilde\sigma_c + \mathrm{i}\epsilon_{abcd}\tilde\sigma^d \ . \end{aligned} \tag{A.12}$$

The antisymmetric traceless matrices

$$(\sigma_{ab})_\alpha{}^\beta = -\frac{1}{4}(\sigma_a\tilde\sigma_b - \sigma_b\tilde\sigma_a)_\alpha{}^\beta \ , \qquad (\tilde\sigma_{ab})^{\dot\alpha}{}_{\dot\beta} = -\frac{1}{4}(\tilde\sigma_a\sigma_b - \tilde\sigma_b\sigma_a)^{\dot\alpha}{}_{\dot\beta} \ , \tag{A.13}$$

satisfy the Lorentz algebra.

$$[\sigma_{ab}, \sigma_{cd}] = \eta_{ad}\sigma_{bc} - \eta_{ac}\sigma_{bd} + \eta_{bc}\sigma_{ad} - \eta_{bd}\sigma_{ac} \ , \tag{A.14}$$

and are (anti) self-dual,

$$\frac{1}{2}\,\varepsilon_{abcd}\,\sigma^{cd} = -\mathrm{i}\,\sigma_{ab}\ , \qquad \frac{1}{2}\,\varepsilon_{abcd}\,\tilde{\sigma}^{cd} = \mathrm{i}\,\tilde{\sigma}_{ab}\ . \tag{A.15}$$

Also,

$$\begin{aligned}
(\sigma^{ab}\sigma^{c}) &= -\frac{1}{2}(\eta^{ac}\eta^{bd} - \eta^{bc}\eta^{ad} + \mathrm{i}\epsilon^{abcd})\,\sigma_d\ , \qquad (A.16)\\
(\sigma^{a}\tilde{\sigma}^{bc}) &= -\frac{1}{2}(\eta^{ac}\eta^{bd} - \eta^{ab}\eta^{cd} + \mathrm{i}\epsilon^{abcd})\,\sigma_d\ ,\\
(\tilde{\sigma}^{ab}\tilde{\sigma}^{c}) &= -\frac{1}{2}(\eta^{ac}\eta^{bd} - \eta^{bc}\eta^{ad} - \mathrm{i}\epsilon^{abcd})\,\tilde{\sigma}_d\ ,\\
(\tilde{\sigma}^{a}\sigma^{bc}) &= -\frac{1}{2}(\eta^{ac}\eta^{bd} - \eta^{ab}\eta^{cd} - \mathrm{i}\epsilon^{abcd})\,\tilde{\sigma}_d\ ,
\end{aligned}$$

and

$$\mathrm{tr}(\sigma^{ab}\sigma^{cd}) = -\frac{1}{2}(\eta^{ac}\eta^{bd} - \eta^{ad}\eta^{bc} + \mathrm{i}\epsilon^{abcd})\ , \tag{A.17}$$

$$\mathrm{tr}(\tilde{\sigma}^{ab}\tilde{\sigma}^{cd}) = -\frac{1}{2}(\eta^{ac}\eta^{bd} - \eta^{ad}\eta^{bc} - \mathrm{i}\epsilon^{abcd})\ . \tag{A.18}$$

Products of two spinors may be reduced by

$$\psi_\alpha\chi_\beta = \frac{1}{2}\epsilon_{\alpha\beta}\psi\chi - \frac{1}{2}(\sigma^{ab})_{\alpha\beta}\psi\sigma_{ab}\chi\ , \qquad \bar{\psi}_{\dot\alpha}\bar{\chi}_{\dot\beta} = -\frac{1}{2}\epsilon_{\dot\alpha\dot\beta}\bar{\psi}\bar{\chi} - \frac{1}{2}(\tilde{\sigma}^{ab})_{\dot\alpha\dot\beta}\bar{\psi}\tilde{\sigma}_{ab}\bar{\chi}\ ,$$

$$\psi_\alpha\bar{\chi}_{\dot\alpha} = -\frac{1}{2}(\sigma^a)_{\alpha\dot\alpha}\psi\sigma_a\bar{\chi}\ , \tag{A.19}$$

and the following Fierz identities hold for arbitrary spinors $\psi_1^\alpha, \psi_2^\alpha, \psi_3^\alpha, \psi_4^\alpha$:

$$(\psi_1\psi_2)(\psi_3\psi_4) = -(\psi_1\psi_3)(\psi_2\psi_4) - (\psi_1\psi_4)(\psi_2\psi_3)\ , \tag{A.20}$$

$$(\psi_1\psi_2)(\bar{\psi}_3\bar{\psi}_4) = -\frac{1}{2}(\psi_1\sigma^a\bar{\psi}_4)(\psi_2\sigma_a\bar{\psi}_3)\ .$$

The connection between spacetime and spinor indices is accomplished in terms of the sigma matrices by

$$V_{\alpha\dot\alpha} = (\sigma^a)_{\alpha\dot\alpha}V_a\ , \qquad V_a = -\frac{1}{2}(\tilde{\sigma}_a)^{\dot\alpha\alpha}V_{\alpha\dot\alpha}\ . \tag{A.21}$$

For an antisymmetric tensor, $X_{ab} = -X_{ba}$ we can write

$$X_{\alpha\dot\alpha\,\beta\dot\beta} = (\sigma^a)_{\alpha\dot\alpha}(\sigma^b)_{\beta\dot\beta}X_{ab} = 2\epsilon_{\alpha\beta}X_{\dot\alpha\dot\beta} + 2\epsilon_{\dot\alpha\dot\beta}X_{\alpha\beta}\ , \tag{A.22}$$

$$X_{\alpha\beta} = \frac{1}{2}(\sigma_{ab})_{\alpha\beta}X_{ab}\ , \qquad X_{\dot\alpha\dot\beta} = -\frac{1}{2}(\tilde{\sigma}_{ab})_{\dot\alpha\dot\beta}X_{ab}\ ,$$

$$X_{ab} = (\sigma_{ab})_{\alpha\beta}X^{\alpha\beta} - (\tilde{\sigma}_{ab})_{\dot\alpha\dot\beta}X^{\dot\alpha\dot\beta}\ ,$$

where

$$(\sigma_{ab})_{\alpha\beta} = \epsilon_{\beta\gamma}(\sigma_{ab})_\alpha{}^\gamma = (\sigma_{ab})_{\beta\alpha} \ , \qquad (\tilde{\sigma}_{ab})_{\dot{\alpha}\dot{\beta}} = \epsilon_{\dot{\alpha}\dot{\gamma}}(\tilde{\sigma}_{ab})^{\dot{\gamma}}{}_{\dot{\beta}} = (\tilde{\sigma}_{ab})_{\dot{\beta}\dot{\alpha}} \ . \tag{A.23}$$

The Hodge dual of X is denoted by a tilde,

$$\tilde{X}_{ab} \equiv \frac{1}{2}\epsilon_{abcd}X^{cd} = (\sigma_{ab})_{\alpha\beta}\tilde{X}^{\alpha\beta} - (\tilde{\sigma}_{ab})_{\dot{\alpha}\dot{\beta}}\tilde{X}^{\dot{\alpha}\dot{\beta}} \ , \tag{A.24}$$

where, due to (A.15), $\tilde{X}_{\alpha\beta} = -\mathrm{i}X_{\alpha\beta}$, $\tilde{X}_{\dot{\alpha}\dot{\beta}} = \mathrm{i}X_{\dot{\alpha}\dot{\beta}}$ and $\tilde{\tilde{X}} = -X$. If X_{ab} is a real tensor, then $X_{\alpha\beta}$ and $X_{\dot{\alpha}\dot{\beta}}$ are conjugate to each other, $\bar{X}_{\dot{\alpha}\dot{\beta}} = X_{\dot{\alpha}\dot{\beta}}$. In particular, this applies to the Lorentz generators, $M_{ab} = -M_{ba} \Leftrightarrow (M_{\alpha\beta}, \bar{M}_{\dot{\alpha}\dot{\beta}})$, which satisfy the same algebra (A.14) as the σ_{ab} matrices. They act on arbitrary spinors as follows:

$$\begin{aligned} M_{\alpha\beta}(\psi_\gamma) &= \frac{1}{2}(\epsilon_{\gamma\alpha}\psi_\beta + \epsilon_{\gamma\beta}\psi_\alpha) \ , \qquad & M_{\alpha\beta}(\bar{\psi}_{\dot{\gamma}}) &= 0 \ , \\ \bar{M}_{\dot{\alpha}\dot{\beta}}(\bar{\psi}_{\dot{\gamma}}) &= \frac{1}{2}(\epsilon_{\dot{\gamma}\dot{\alpha}}\bar{\psi}_{\dot{\beta}} + \epsilon_{\dot{\gamma}\dot{\beta}}\bar{\psi}_\alpha) \ , \qquad & \bar{M}_{\dot{\alpha}\dot{\beta}}(\psi_\gamma) &= 0 \ . \end{aligned} \tag{A.25}$$

Also, of some use during calculation are the following identities, which follow from the covariant derivative algebra (1.1.4):

$$\begin{aligned} \mathcal{D}_\alpha\mathcal{D}_\beta &= \frac{1}{2}\epsilon_{\alpha\beta}\mathcal{D}^2 - 2\bar{R}\,M_{\alpha\beta} \ , \qquad \bar{\mathcal{D}}_{\dot{\alpha}}\bar{\mathcal{D}}_{\dot{\beta}} = -\frac{1}{2}\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\mathcal{D}}^2 + 2R\,\bar{M}_{\dot{\alpha}\dot{\beta}} \ , \\ \mathcal{D}_\alpha\mathcal{D}^2 &= 4\bar{R}\,\mathcal{D}^\beta(\epsilon_{\alpha\beta} + M_{\alpha\beta}) \ , \qquad \mathcal{D}^2\mathcal{D}_\alpha = -2\bar{R}\,\mathcal{D}^\beta(\epsilon_{\alpha\beta} + M_{\alpha\beta}) \ , \\ \bar{\mathcal{D}}_{\dot{\alpha}}\bar{\mathcal{D}}^2 &= 4R\,\bar{\mathcal{D}}^{\dot{\beta}}(\epsilon_{\dot{\alpha}\dot{\beta}} + \bar{M}_{\dot{\alpha}\dot{\beta}}) \ , \qquad \bar{\mathcal{D}}^2\bar{\mathcal{D}}_{\dot{\alpha}} = -2R\,\bar{\mathcal{D}}^{\dot{\beta}}(\epsilon_{\dot{\alpha}\dot{\beta}} + \bar{M}_{\dot{\alpha}\dot{\beta}}) \ , \\ \left[\mathcal{D}^2, \bar{\mathcal{D}}_{\dot{\alpha}}\right] &= -4(G_{\alpha\dot{\alpha}} + \mathrm{i}\mathcal{D}_{\alpha\dot{\alpha}})\mathcal{D}^\alpha + 4\bar{R}\,\bar{\mathcal{D}}_{\dot{\alpha}} - 4(\mathcal{D}^\gamma G^\delta{}_{\dot{\alpha}})M_{\gamma\delta} + 8\bar{W}_{\dot{\alpha}}{}^{\dot{\gamma}\dot{\delta}}\bar{M}_{\dot{\gamma}\dot{\delta}} \ , \\ \left[\bar{\mathcal{D}}^2, \mathcal{D}_\alpha\right] &= -4(G_{\alpha\dot{\alpha}} - \mathrm{i}\mathcal{D}_{\alpha\dot{\alpha}})\bar{\mathcal{D}}^{\dot{\alpha}} + 4R\,\mathcal{D}_\alpha - 4(\bar{\mathcal{D}}^{\dot{\gamma}}G_\alpha{}^{\dot{\delta}})\bar{M}_{\dot{\gamma}\dot{\delta}} + 8W_\alpha{}^{\gamma\delta}M_{\gamma\delta} \ , \end{aligned} \tag{A.26}$$

where $\mathcal{D}^2 = \mathcal{D}^\alpha\mathcal{D}_\alpha$, and $\bar{\mathcal{D}}^2 = \bar{\mathcal{D}}_{\dot{\alpha}}\bar{\mathcal{D}}^{\dot{\alpha}}$. Note also that

$$(\mathcal{D}_\alpha V)^* = (-1)^{\epsilon(V)}\bar{\mathcal{D}}_{\dot{\alpha}}V^* \ , \qquad (\mathcal{D}^2 V)^* = \bar{\mathcal{D}}^2 V^* \ , \tag{A.27}$$

where $\epsilon(V)$ is the Grassmann parity of V, *i.e.* $\epsilon(V) = 1$, if V is a fermionic superfield, and $\epsilon(V) = 0$, if it is bosonic.

Old-minimal supergravity: alternative realization

Here we consider an alternative realization of the model for old-minimal supergravity (1.3.7) that is obtained by employing a variant superfield representation [104, 105] of the form

$$\Sigma^3 = -\frac{1}{4}(\bar{\mathcal{D}}^2 - 4R)\, P~, \qquad \bar{P} = P~, \tag{B.1}$$

with P an unconstrained real scalar superfield. It follows from (1.3.8) that the super-Weyl transformation of P should be

$$P ~\rightarrow~ \mathrm{e}^{-\sigma-\bar{\sigma}}\, P~, \tag{B.2}$$

compare with (1.3.2). In this approach, the covariantly chiral scalar Σ occurs not as one of the dynamical fields, but instead as a gauge-invariant field strength associated with the following gauge freedom:

$$\delta P = L~, \qquad (\bar{\mathcal{D}}^2 - 4R)\, L = 0~. \tag{B.3}$$

Unlike the vector multiplet case, the component vector field contained in P should now be interpreted as a gauge three-form, as the action of (B.3) on this field coincides with a three-form gauge transformation [104, 105]. The gauge-invariant four-form field strength, which is associated with the three-form, appears as one of the two auxiliary fields contained in Σ^3. The latter implies, in fact, that a supersymmetric cosmological term [71–75] becomes real,

$$\mathrm{Im} \int \mathrm{d}^8 z\, \frac{E^{-1}}{R}\, \Sigma^3 = 0~. \tag{B.4}$$

An interesting feature of this realization for old minimal supergravity is that it possesses solutions with a non-vanishing cosmological constant, without the need to explicitly add a supersymmetric cosmological term, $\mu \int \mathrm{d}^8 z\, (ER)^{-1}\, \Sigma^3$, to the supergravity action (1.3.7). Another intriguing property of this formulation is that, unlike the standard realization, it allows a simple construction of massive off-shell supergravitons [106, 107].

If one only insists on the super-Weyl invariance (B.2) without requiring the gauge symmetry (B.3), a more general action, involving the naked prepotential P, can be considered

$$\begin{aligned} S &= \int \mathrm{d}^8 z\, P\, \mathcal{F}\Big(\frac{\bar{\Sigma}\,\Sigma}{P}\Big) + \Big\{ g \int \mathrm{d}^8 z\, \frac{E^{-1}}{R}\, W^2 ~+~ \text{c.c.} \Big\}~, \\ W_\alpha &= -\frac{1}{4}(\bar{\mathcal{D}}^2 - 4R)\mathcal{D}_\alpha \ln P~, \end{aligned} \tag{B.5}$$

with a real function $\mathcal{F}(x)$ and a constant parameter g. In general, such an action describes couplings of supergravity to supersymmetric matter, see e.g. [108]. It corresponds to pure supergravity only if $g = 0$ and $\mathcal{F}(x)$ is a linear function, $\mathcal{F}(x) = -3x + \mu$.

Self-duality equation in curved superspace

Here we derive the matter-self-duality equation (3.4.4) and the $\mathcal{N}=1$ self-duality equation (2.2.7).

From section 3.4, we understand self-duality to be the satisfaction of the two conditions (2.2.6) and (3.4.3). The first condition for self-duality is that the transformed superfield M' be given by (2.2.6). Now, since

$$\begin{aligned}
\frac{\delta}{\delta W'^{\alpha}}S[v] &= \left(\frac{\delta W^{\beta}}{\delta W'^{\alpha}}\cdot\frac{\delta}{\delta W^{\beta}}+\frac{\delta \bar{W}_{\dot\beta}}{\delta W'^{\alpha}}\cdot\frac{\delta}{\delta \bar{W}_{\dot\beta}}+\frac{\delta \Phi}{\delta W'^{\alpha}}\cdot\frac{\delta}{\delta \Phi}+\frac{\delta \bar\Phi}{\delta W'^{\alpha}}\cdot\frac{\delta}{\delta \bar\Phi}\right)S[v]\\
&=\left((1-d)\frac{\delta}{\delta W^{\alpha}}-c\left(\frac{\delta}{\delta W^{\alpha}}M^{\beta}[v]\right)\cdot\frac{\delta}{\delta W^{\beta}}-c\left(\frac{\delta}{\delta W^{\alpha}}\bar{M}_{\dot\beta}[v]\right)\cdot\frac{\delta}{\delta \bar{W}_{\dot\beta}}\right)S[v]\\
&=\frac{\mathrm{i}}{2}M_{\alpha}[v]-\frac{\delta}{\delta W^{\alpha}}\left(S[v]+\frac{\mathrm{i}}{4}c\,(M\cdot M-\bar{M}\cdot\bar{M})\right)\ ,
\end{aligned} \tag{C.1}$$

this leads to the following requirement

$$\begin{aligned}
\frac{\delta}{\delta W^{\alpha}}\delta S &= \frac{\mathrm{i}}{2}\delta M_{\alpha}+\frac{\delta}{\delta W^{\alpha}}\left(S[v]+\frac{\mathrm{i}}{4}c\,(M\cdot M+\bar{M}\cdot\bar{M})\right)\\
&= \frac{\delta}{\delta W^{\alpha}}\left((a+d)S[v]+\left(\frac{\mathrm{i}}{4}(c\,M\cdot M+b\,W\cdot W)+\text{c.c}\right)\right)\ ,
\end{aligned} \tag{C.2}$$

where we have used the notation (1.1.15).

The second condition for self-duality is that the equation of motion (3.4.2) for Φ transforms covariantly under duality transformations (3.4.3). Since

$$\begin{aligned}
\frac{\delta}{\delta \Phi'}S[v] &= \left(\frac{\delta \Phi}{\delta \Phi'}\cdot\frac{\delta}{\delta \Phi}+\frac{\delta \bar\Phi}{\delta \Phi'}\cdot\frac{\delta}{\delta \bar\Phi}+\frac{\delta W^{\alpha}}{\delta \Phi'}\cdot\frac{\delta}{\delta W^{\alpha}}+\frac{\delta \bar{W}_{\dot\alpha}}{\delta \Phi'}\cdot\frac{\delta}{\delta \bar{W}_{\dot\alpha}}\right)S[v]\\
&= (1-\frac{\partial\xi(\Phi)}{\partial\Phi})\Pi[v]+\frac{\delta}{\delta\Phi}\left(\frac{\mathrm{i}}{4}c\,(M\cdot M-\bar{M}\cdot\bar{M})\right)\ ,
\end{aligned} \tag{C.3}$$

this leads to the following requirement

$$\frac{\delta}{\delta\Phi}\left(\delta S+\frac{\mathrm{i}}{4}c\,(M\cdot M-\bar{M}\cdot\bar{M})\right)=0\ . \tag{C.4}$$

Comparing with (C.2) we see that, for consistency, we require $a=-d$. So, the action varies under duality transformations as[1]

$$\delta S=\frac{\mathrm{i}}{4}(c\,M\cdot M+b\,W\cdot W)+\text{c.c}\ . \tag{C.5}$$

[1]Here, the constant of integration is set to zero so that the action agrees with the usual supersymmetric Maxwell action in the weak superfield limit of the matter free case, *i.e.* when $\mathrm{i}M_{\alpha}\to W_{\alpha}$.

Now, it is important to note that we may also directly vary the action $S[W, \bar{W}, \Phi, \bar{\Phi}]$ to give

$$\begin{aligned} \delta S &= \left(\delta\Phi \cdot \frac{\delta}{\delta\Phi} + \delta\bar{\Phi} \cdot \frac{\delta}{\delta\bar{\Phi}} + \delta W \cdot \frac{\delta}{\delta W} + \delta\bar{W} \cdot \frac{\delta}{\delta\bar{W}} \right) S[v] \\ &= \delta\Phi \cdot \frac{\delta S}{\delta\Phi} + \delta\bar{\Phi} \cdot \frac{\delta S}{\delta\bar{\Phi}} + \left(\frac{\mathrm{i}}{2}(c\, M \cdot M - a\, W \cdot M) + \text{c.c.} \right) . \end{aligned} \tag{C.6}$$

The two variations (C.5) and (C.6) must coincide. Equating the two leads to the matter-self-duality equation (3.4.4).

In the absence of matter superfields, the $\mathrm{SL}(2, \mathbb{R})$ group of duality transformations reduces to its maximal compact subgroup, $\mathrm{U}(1)$. The parameters are further restricted such that $a = 0$, and $c = -b$ ($= \tau$, the parameter from (2.2.5)). In this event, the matter-self-duality equation (3.4.4) reduces to the $\mathcal{N} = 1$ self-duality equation (2.2.7).

Calculation of the supercurrent and supertrace

The calculation of the supercurrent and supertrace requires a covariant variational technique involving the disturbance of the supergravity prepotentials, *i.e.* a deformation of the supergeometry. We first outline the procedure as detailed in [27], and then go on to give some specific details for the calculation of the supercurrent and supertrace of the family model (2.4.1).

D.1 Covariant variational technique

We consider a disturbance of the supergravity prepotentials H_a and $\varphi \to \mathrm{e}^{\sigma}\varphi$ in terms of the infinitesimal superfields $\mathbf{H}_a(z)$ and $\sigma(z)$, $\bar{\mathcal{D}}_{\dot\alpha}\sigma = 0$,

$$\mathrm{e}^{-2\mathrm{i}\mathbf{H}^a\mathcal{D}_a} \approx 1 - 2\mathrm{i}\mathbf{H}^a\mathcal{D}_a \ , \qquad \boldsymbol{\varphi} = \mathrm{e}^{\sigma} \approx 1 + \sigma \ , \tag{D.1.1}$$

which results in a shift in the covariant derivatives as

$$\begin{aligned}
\mathcal{D}_\alpha &\longrightarrow \nabla_\alpha = \mathrm{e}^{-\mathrm{i}\mathbf{H}^a\mathcal{D}_a}\left(\mathcal{F}\mathcal{D}_\alpha + \frac{1}{2}\Delta\Omega_\alpha{}^{bc}M_{bc}\right)\mathrm{e}^{\mathrm{i}\mathbf{H}^a\mathcal{D}_a} \ , \\
\bar{\mathcal{D}}_{\dot\alpha} &\longrightarrow \bar\nabla_\alpha = \mathrm{e}^{\mathrm{i}\mathbf{H}^a\mathcal{D}_a}\left(\bar{\mathcal{F}}\bar{\mathcal{D}}_{\dot\alpha} + \frac{1}{2}\Delta\bar\Omega_{\dot\alpha}{}^{bc}M_{bc}\right)\mathrm{e}^{-\mathrm{i}\mathbf{H}^a\mathcal{D}_a} \ , \\
\mathcal{D}_{\alpha\dot\alpha} &\longrightarrow \nabla_{\alpha\dot\alpha} = \frac{\mathrm{i}}{2}\{\nabla_\alpha, \bar\nabla_{\dot\alpha}\} \ ,
\end{aligned} \tag{D.1.2}$$

where

$$\mathcal{F} = 1 + \Delta\mathcal{F} \ , \qquad \Delta\mathcal{F} = \frac{1}{2}\sigma - \bar\sigma - \frac{1}{3}G_a\mathbf{H}^a + \frac{2\mathrm{i}}{3}\mathcal{D}_a\mathbf{H}^a - \frac{1}{12}\bar{\mathcal{D}}_{\dot\alpha}\mathcal{D}_\alpha\mathbf{H}^{\alpha\dot\alpha} \ , \tag{D.1.3}$$

and

$$\begin{aligned}
\Delta\bar\Omega_{\dot\alpha\alpha\beta} = \bar{\mathcal{D}}_{\dot\alpha}A_{\alpha\beta} \ , \qquad & \Delta\bar\Omega_{\dot\alpha\dot\beta\dot\gamma} = \epsilon_{\dot\alpha\dot\beta}\bar{\mathcal{D}}_{\dot\gamma}\Delta\bar{\mathcal{F}} + \epsilon_{\dot\alpha\dot\gamma}\bar{\mathcal{D}}_{\dot\beta}\Delta\bar{\mathcal{F}} \ , \\
A_{\alpha\beta} \;\; = \;\; & \frac{1}{2}\bar{\mathcal{D}}^{\dot\beta}\mathcal{D}_{(\alpha}\mathbf{H}_{\beta)\dot\beta} + G_{(\alpha}{}^{\dot\beta}\mathbf{H}_{\beta)\dot\beta} \ .
\end{aligned} \tag{D.1.4}$$

Any variation (D.1.2) of the supergeometry will be accompanied by a variation in the matter superfields

$$\chi \longrightarrow \chi_{(\nabla)} = f(\chi\,; \mathbf{H}, \boldsymbol{\varphi}) \ , \tag{D.1.5}$$

which will depend on the superfield type. For example, a chiral scalar superfield will change as $\phi_{(\nabla)} = \mathrm{e}^{\mathrm{i}\mathbf{H}^a\mathcal{D}_a}\phi$, $\bar\nabla_{\dot\alpha}\phi_{(\nabla)} = 0$. The action transforms as

$$S[\chi\,; \mathcal{D}] \longrightarrow S[\chi_{(\nabla)}\,; \nabla] \equiv S[\chi\,; \mathcal{D}|\mathbf{H}, \boldsymbol{\varphi}] = \int \mathrm{d}^8 z\, \mathcal{E}^{-1}\, \mathcal{L}(\chi_{(\nabla)}\,; \nabla) \ . \tag{D.1.6}$$

In practice, it is helpful to switch to the so-called 'quantum chiral representation',

$$\tilde{\chi}_{(\nabla)} = \mathrm{e}^{-\mathrm{i}\mathbf{H}^a\mathcal{D}_a}\chi_{(\nabla)} \ , \qquad \tilde{\nabla}_A = \mathrm{e}^{-\mathrm{i}\mathbf{H}^a\mathcal{D}_a}\nabla_A \mathrm{e}^{\mathrm{i}\mathbf{H}^a\mathcal{D}_a} \ , \tag{D.1.7}$$
$$S[\chi\,;\mathcal{D}|\mathbf{H},\boldsymbol{\varphi}] = \int \mathrm{d}^8z\, \tilde{\mathcal{E}}^{-1}\, \mathcal{L}(\tilde{\chi}_{(\nabla)}\,;\tilde{\nabla}) \ ,$$

where[1]

$$\tilde{\mathcal{E}}^{-1} = E^{-1}\left(1+\sigma+\bar{\sigma}+\frac{1}{3}G_{\alpha\dot{\alpha}}\mathbf{H}^{\alpha\dot{\alpha}} + \frac{1}{12}[\mathcal{D}_\alpha,\bar{\mathcal{D}}_{\dot{\alpha}}]\mathbf{H}^{\alpha\dot{\alpha}} + \frac{\mathrm{i}}{2}\mathcal{D}_{\alpha\dot{\alpha}}\mathbf{H}^{\alpha\dot{\alpha}}\right) \ ,$$
$$\tilde{\mathbf{R}} = R - \frac{1}{2}(\bar{\mathcal{D}}^2-4R)\Delta\bar{\mathcal{F}} \ . \tag{D.1.8}$$

To determine the supercurrent T_a and supertrace T, $\bar{\mathcal{D}}_{\dot{\alpha}}T = 0$, one calculates how the action has varied under (D.1.2) to first order in $\mathbf{H}_a$ and σ,

$$\Delta S = S[\tilde{\chi}_{(\nabla)}\,;\tilde{\nabla}] - S[\chi\,;\mathcal{D}] = \int \mathrm{d}^8z\, E^{-1}\left\{\mathbf{H}^aT_a + \frac{1}{R}\,\sigma\, T + \frac{1}{\bar{R}}\,\bar{\sigma}\,\bar{T}\right\} \ . \tag{D.1.9}$$

A useful identity, due to (D.1.8) and the chiral superspace rule (1.1.9), is

$$\int \mathrm{d}^8z\, \frac{\tilde{\mathcal{E}}^{-1}}{\tilde{\mathbf{R}}}\,\mathcal{L}_{\mathrm{c}} = \int \mathrm{d}^8z\, \frac{E^{-1}}{R}\,(1+3\sigma)\mathcal{L}_{\mathrm{c}} \ . \tag{D.1.10}$$

We also note that there is a simple prescription for calculating the supertrace, since the local scaling of the chiral prepotential, $\varphi \to \mathrm{e}^{\sigma}\varphi$, with $\sigma(z)$ a chiral scalar superfield, $\bar{\mathcal{D}}_{\dot{\alpha}}\sigma = 0$, is just the super-Weyl transformation (1.2.1) introduced in section 1.2. The supertrace for an action $S[\chi\,;H^a,\varphi]$ can then be calculated by first performing a super-Weyl transformation (1.2.1) and then taking the variational derivative with respect to $\sigma(z)$ and then setting σ to zero,

$$T = \frac{\delta}{\delta\sigma}S[\chi\,;H^a,\mathrm{e}^{\sigma}\varphi]\Big|_{\sigma=0} \ , \tag{D.1.11}$$

where variational differentiation with respect to a chiral scalar superfield is defined by (1.1.13).

D.2 Supercurrent and supertrace of the family model

Now, we would like to use the method described above to determine the supercurrent and supertrace of the family of self-dual models (2.4.1).

[1]For general calculations, expressions for how $G_{\alpha\dot{\alpha}}$ and $W_{\alpha\beta\gamma}$ vary under supergravity prepotential deformations are needed. However, they were unnecessary for our calculation and we refer to [27] for these expressions.

Turning first to the supertrace, as indicated above, we first perform a super-Weyl rescaling (1.2.1) of the family action (2.4.1). The linear part of the family action is super-Weyl invariant, while to determine the variation of the nonlinear part we use (1.2.5), (1.2.4) and (3.1.1). We find that under an infinitesimal super-Weyl rescaling (1.2.1), the superfunctional $S[W,\bar{W}]$ varies as

$$\delta S[W,\bar{W}] = -\frac{1}{2}\int d^8z\, E^{-1}\,\sigma\, W^2\bar{W}^2\left(\Gamma+\bar{\Gamma}-\Lambda\right) \tag{D.2.1}$$

$$= \frac{1}{8}\int d^8z\,\frac{E^{-1}}{R}\,\sigma\, W^2(\bar{\mathcal{D}}^2-4R)\left[\bar{W}^2\left(\Gamma+\bar{\Gamma}-\Lambda\right)\right] . \tag{D.2.2}$$

We see that this gives the supertrace T (2.5.1).

To determine the supercurrent, we first note that since real covariantly scalar prepotential V is invariant under deformations of the supergravity prepotentials, in the quantum chiral representation we have

$$\tilde{V}_{(\nabla)} = e^{-i\mathbf{H}^a\mathcal{D}_a}V . \tag{D.2.3}$$

The superfield strength W_α in the quantum chiral representation is

$$\tilde{W}_{(\nabla)\alpha} = -\frac{1}{4}(\tilde{\bar{\tilde{\nabla}}}^2-4\tilde{\mathbf{R}})\tilde{\nabla}_\alpha\tilde{V}_{(\nabla)} \tag{D.2.4}$$

Using the identity

$$(\tilde{\bar{\tilde{\nabla}}}^2-4\tilde{\mathbf{R}})\lambda_\alpha = (\delta_\alpha{}^\beta-A_\alpha{}^\beta)(\bar{\mathcal{D}}^2-4R)\lambda_\beta+(\bar{\mathcal{D}}^2-4R)[A_\alpha{}^\beta\lambda_\beta+2\Delta\bar{\mathcal{F}}\lambda_\alpha] , \tag{D.2.5}$$

which holds for an arbitrary spinor superfield, λ_α, we can show that

$$\begin{aligned}\tilde{W}^2_{(\nabla)} = W^2-\frac{1}{4}(\bar{\mathcal{D}}^2-4R)\Big[&G_{\alpha\dot{\beta}}\mathbf{H}^{\beta\dot{\beta}}(\mathcal{D}^\alpha V)W_\beta-\frac{1}{2}(\bar{\mathcal{D}}_{\dot{\beta}}\mathcal{D}^\alpha\mathbf{H}^{\beta\dot{\beta}})(\mathcal{D}_\beta V)W_\alpha\\ &+i\,\mathbf{H}^{\beta\dot{\beta}}(\mathcal{D}_{\beta\dot{\beta}}\mathcal{D}^\alpha V)W_\alpha-\frac{i}{2}\mathcal{D}^\alpha(\mathbf{H}^{\beta\dot{\beta}}\mathcal{D}_{\beta\dot{\beta}}V)W_\alpha\Big] .\end{aligned} \tag{D.2.6}$$

Likewise, to determine the transformation of $\omega = (1/8)(\bar{\mathcal{D}}^2-4R)W^2$,

$$\tilde{\bar{\tilde{\omega}}}_{(\nabla)} = \frac{1}{8}(\tilde{\bar{\tilde{\nabla}}}^2-4\tilde{\mathbf{R}})\tilde{\bar{\tilde{W}}}^2_{(\nabla)} , \tag{D.2.7}$$

we use the identity

$$(\tilde{\bar{\tilde{\nabla}}}^2-4\tilde{\mathbf{R}})U = (\bar{\mathcal{D}}^2-4R)U+2(\bar{\mathcal{D}}^2-4R)[\Delta\bar{\mathcal{F}}U] , \tag{D.2.8}$$

which holds for arbitrary scalar superfields U. We also know that since W^2 is a covariantly chiral scalar,

$$\tilde{\bar{\tilde{W}}}^2_{(\nabla)} = e^{-2i\mathbf{H}^a\mathcal{D}_a}(\tilde{W}^2_{(\nabla)})^* . \tag{D.2.9}$$

To calculate the supercurrent $T_{\alpha\dot\alpha}$, we must expand out

$$\Delta S = S[\tilde{W}_{(\nabla)}, \tilde{\bar{W}}_{(\nabla)}\,; \tilde{\nabla}] - S[W, \bar{W}\,; \mathcal{D}] = -\frac{1}{2}\int \mathrm{d}^8 z\, E^{-1}\mathbf{H}^{\alpha\dot\alpha} T_{\alpha\dot\alpha}\ , \tag{D.2.10}$$

to first order in $\mathbf{H}_{\alpha\dot\alpha}$, where

$$\begin{aligned} S[\tilde{W}_{(\nabla)}, \tilde{\bar{W}}_{(\nabla)}\,; \tilde{\nabla}] = \frac{1}{4}\int \mathrm{d}^8 z\, \frac{\tilde{\mathcal{E}}^{-1}}{\tilde{\mathbf{R}}}\, \tilde{W}^2_{(\nabla)} \ + \ \frac{1}{4}\int \mathrm{d}^8 z\, \frac{\tilde{\mathcal{E}}^{-1}}{\tilde{\bar{\mathbf{R}}}}\, \tilde{\bar{W}}^2_{(\nabla)} \\ + \ \frac{1}{4}\int \mathrm{d}^8 z\, \tilde{\mathcal{E}}^{-1}\, \tilde{W}^2_{(\nabla)} \tilde{\bar{W}}^2_{(\nabla)} \Lambda(\tilde{\omega}_{(\nabla)}, \tilde{\bar{\omega}}_{(\nabla)})\ . \end{aligned} \tag{D.2.11}$$

Since ω and R are both chiral scalar superfields, we also have

$$\tilde{\omega}_{(\nabla)} = \mathrm{e}^{-2\mathrm{i}\mathbf{H}^a \mathcal{D}_a} (\tilde{\bar{\omega}}_{(\nabla)})^*\ , \qquad \tilde{\bar{\mathbf{R}}} = \mathrm{e}^{-2\mathrm{i}\mathbf{H}^a \mathcal{D}_a} (\tilde{\mathbf{R}})^*\ . \tag{D.2.12}$$

Putting all these together, after a lengthy calculation we obtain the supercurrent (2.5.3).

D.3 Duality invariance of the supercurrent and supertrace

Duality invariance of the supercurrent and supertrace of the family model (2.4.1) is a consequence of self-duality. To verify that the supertrace is invariant under infinitesimal duality rotations, $\delta W_\alpha = \tau M_\alpha$, $\delta M_\alpha = -\tau W_\alpha$, the following relations which result from the self-duality relation (2.4.3) are useful

$$(1 - 2\,\bar\omega\,\Gamma)\,\frac{\partial \Gamma}{\partial \omega} - (1 - 2\,\omega\,\bar\Gamma)\,\frac{\partial \bar\Gamma}{\partial \omega} = -\bar\Gamma^2\ , \quad (1 - 2\,\bar\omega\,\Gamma)\,\frac{\partial \Gamma}{\partial \bar\omega} - (1 - 2\,\omega\,\bar\Gamma)\,\frac{\partial \bar\Gamma}{\partial \bar\omega} = \Gamma^2\ , \tag{D.3.1}$$

aswell as the reality condition

$$\Gamma - \bar\Gamma + \bar\omega\,\frac{\partial \Gamma}{\partial \bar\omega} - \omega\,\frac{\partial \bar\Gamma}{\partial \omega} = 0\ . \tag{D.3.2}$$

With M_α defined by (2.5.2), we find that

$$\begin{aligned} (\delta W^2)\bar{W}^2 &= -2\mathrm{i}\,\tau\, W^2\bar{W}^2(1 - 2\,\bar\omega\,\Gamma)\ , \\ W^2(\delta \bar{W}^2) &= 2\mathrm{i}\,\tau\, W^2\bar{W}^2(1 - 2\,\omega\,\bar\Gamma)\ , \\ W^2\bar{W}^2\,\delta\omega &= -2\mathrm{i}\,\tau\, W^2\bar{W}^2\,\omega\,(1 - 2\,\bar\omega\,\Gamma)\ , \\ W^2\bar{W}^2\,\delta\bar\omega &= 2\mathrm{i}\,\tau\, W^2\bar{W}^2\,\bar\omega\,(1 - 2\,\omega\,\bar\Gamma)\ . \end{aligned} \tag{D.3.3}$$

With this, if we now vary the supertrace (2.5.1) directly and use (D.3.1) and (D.3.2), we find that $\delta T = 0$.

A similar calculation confirms the duality invariance of the supercurrent. When performing this check, it is easier to work with the expression

$$
\begin{aligned}
T_{\alpha\dot{\alpha}} \quad = \quad & \mathrm{i}M_\alpha \bar{W}_{\dot{\alpha}} - \mathrm{i}W_\alpha \bar{M}_{\dot{\alpha}} \; + \; \frac{\mathrm{i}}{4}\mathcal{D}_{\alpha\dot{\alpha}}\big(W^2\bar{W}^2\big(\Gamma+\bar{\Gamma}-\Lambda\big)\big) \\
- \quad & \frac{1}{6}G_{\alpha\dot{\alpha}}W^2\bar{W}^2\left(\Gamma+\bar{\Gamma}-\Lambda\right) \; - \; \frac{1}{24}\left[\mathcal{D}_\alpha,\bar{\mathcal{D}}_{\dot{\alpha}}\right]\big(W^2\bar{W}^2\big(\Gamma+\bar{\Gamma}-\Lambda\big)\big) \\
- \quad & \frac{\mathrm{i}}{2}W^2\mathcal{D}_{\alpha\dot{\alpha}}\left[\bar{W}^2\left(\Lambda+\frac{1}{8}(\mathcal{D}^2-4\bar{R})\Big(W^2\frac{\partial\Lambda}{\partial\omega}\Big)\right)\right] \\
+ \quad & \frac{\mathrm{i}}{2}W^2\bar{W}^2(\mathcal{D}_{\alpha\dot{\alpha}}\bar{\omega})\frac{\partial\Lambda}{\partial\bar{\omega}} \; - \; \frac{\mathrm{i}}{16}W^2(\mathcal{D}_{\alpha\dot{\alpha}}\bar{W}^2)(\bar{\mathcal{D}}^2-4R)\Big(\bar{W}^2\,\frac{\partial\Lambda}{\partial\bar{\omega}}\Big)\;,
\end{aligned}
\tag{D.3.4}
$$

which is equivalent to (2.5.3). Notice that duality invariance of the supertrace automatically implies duality invariance for the first two lines of (D.3.4).

Akulov-Volkov action

With the notation (5.2.11), the Akulov-Volkov (AV) action (5.3.1) for the goldstino can explicitly be rewritten as a polynomial in v and $\bar{v}$:

$$
\begin{aligned}
S_{\rm AV}[\lambda,\bar\lambda] = \int\! {\rm d}^4x \Big\{ &-\frac{1}{2}\langle v+\bar v\rangle - \frac{\kappa^2}{16}\Big(\langle v+\bar v\rangle^2 - \langle (v+\bar v)^2\rangle\Big) \\
&-\frac{\kappa^4}{32}\Big(\langle v^2\bar v\rangle - \langle v\rangle\langle v\bar v\rangle - \frac{1}{2}\langle v^2\rangle\langle\bar v\rangle + \frac{1}{2}\langle v\rangle^2\langle\bar v\rangle \;+\; {\rm c.c.}\Big) \\
&+\frac{\kappa^6}{128}\Big(\langle v^2\bar v^2\rangle + \frac{1}{2}\langle v\bar v v\bar v\rangle - \Big[\langle v\rangle\langle v\bar v^2\rangle - \frac{1}{4}\langle v\rangle^2\langle\bar v^2\rangle \;+\; {\rm c.c.}\Big] \\
&\qquad +\langle v\rangle\langle\bar v\rangle\langle v\bar v\rangle - \frac{1}{2}\langle v\bar v\rangle^2 - \frac{1}{4}\langle v^2\rangle\langle\bar v^2\rangle - \frac{1}{4}\langle v\rangle^2\langle\bar v\rangle^2\Big)\Big\} \ .
\end{aligned}
\tag{E.1}
$$

The fourth-order terms can be simplified slightly:

$$
\frac{1}{4}\int\! {\rm d}^4x\,\Big(\langle v+\bar v\rangle^2 - \langle (v+\bar v)^2\rangle\Big) = \int\! {\rm d}^4x\,\Big(\langle v\rangle\langle\bar v\rangle - \langle v\bar v\rangle\Big) \ . \tag{E.2}
$$

Regarding the eighth-order terms, the situation is more dramatic. Using the (easily verified) identities

$$
\begin{aligned}
\langle v^2\bar v^2\rangle &= \Big(\langle v\rangle\langle v\bar v^2\rangle - \frac{1}{2}\langle v\rangle^2\langle\bar v^2\rangle + {\rm c.c.}\Big) + \langle v\bar v\rangle\Big(\langle v\bar v\rangle - \langle v\rangle\langle\bar v\rangle\Big) \ , \\
2\langle v\bar v v\bar v\rangle &= \langle v^2\rangle\langle\bar v^2\rangle - 2\langle v\bar v\rangle^2 + \Big(\langle v\rangle^2\langle\bar v^2\rangle + {\rm c.c.}\Big) + \langle v\rangle^2\langle\bar v\rangle^2 \ ,
\end{aligned}
\tag{E.3}
$$

one can check that the eighth-order terms in (E.1) completely cancel out! This result may seem strange, since the eighth-order terms in the AV action are, to the best of our knowledge, explicitly retained in all relevant publications, starting with the classic papers by Akulov and Volkov [2,3,81] and continuing today, e.g. [109] (see, however [110] where it is demonstrated that the energy-momentum tensor for the AV model does not contain any eighth-order terms). Therefore we will give another, purely algebraic and quite elementary, proof.

The whole contribution from the eighth-order terms in the integrand in (E.1) can be shown to be proportional to

$$
\varepsilon_{abcd}\,\varepsilon^{efgh}\,v_e{}^a v_f{}^b \bar v_g{}^c \bar v_h{}^d = \lambda^2\bar\lambda^2\,\varepsilon_{abcd}\,\varepsilon^{efgh}\,\Big(\partial_e\bar\lambda\,\tilde\sigma^{ab}\,\partial_f\bar\lambda\Big)\,\Big(\partial_g\lambda\,\sigma^{cd}\,\partial_h\lambda\Big) \ . \tag{E.4}
$$

Using the well-known property of the sigma-matrices (A.15),

$$
\frac{1}{2}\,\varepsilon_{abcd}\,\sigma^{cd} = -{\rm i}\,\sigma_{ab} \ , \quad \frac{1}{2}\,\varepsilon_{abcd}\,\tilde\sigma^{cd} = {\rm i}\,\tilde\sigma_{ab} \ , \quad \longrightarrow \quad (\sigma^{ab})_\alpha{}^\beta\,(\tilde\sigma_{ab})^{\dot\alpha}{}_{\dot\beta} = 0 \ , \tag{E.5}
$$

we see that the whole contribution under consideration vanishes. As a result, the AV action takes the form (5.4.2).

Dualizing the tensor-Goldstone multiplet

Here we outline the derivation of the action (5.6.5), dual to the tensor-Goldstone multiplet action (5.5.11). Having relaxed the constraint (1.6.6) on the superfield L, we consider the auxiliary action (5.6.4), which we can write as

$$S = \int \mathrm{d}^8 z \left\{ 2\phi\bar{\phi} - (L - \phi - \bar{\phi})^2 + \frac{1}{4}(DL)^2(\bar{D}L)^2\Lambda(\omega,\bar{\omega}) \right\} \ , \tag{F.1}$$

where

$$\omega = \frac{1}{2}Q^2 \ , \qquad \bar{\omega} = \frac{1}{2}\bar{Q}^2 \ , \qquad Q_{\alpha\dot{\alpha}} = \mathrm{i}\bar{D}_{\dot{\alpha}}D_\alpha L \ . \tag{F.2}$$

Solving the equation of motion for L gives

$$L = \phi + \bar{\phi} - \frac{\mathrm{i}}{2}(D^\alpha L)\big(\bar{Q}_{\alpha\dot{\alpha}}\Gamma - Q_{\alpha\dot{\alpha}}\bar{\Gamma}\big)(\bar{D}^{\dot{\alpha}}L) + \dots \ , \tag{F.3}$$

where $\Gamma = \Gamma(\omega,\bar{\omega})$ is defined by (2.4.3). The dots refer to higher order terms in $D_\alpha L$ and $\bar{D}_{\dot{\alpha}}L$ that do not contribute upon substitution back into the action (F.1), and terms involving D^2L and $\bar{D}^2L$. As was explicitly shown in [101] the final solution becomes a power series in these latter terms, with only the lowest order term, independent of D^2L and $\bar{D}^2L$ being physically relevant. The terms can be absorbed into the Lagrange multiplier by a superfield redefinition since

$$2L(\phi+\bar{\phi}) + f(L)D^2L + \bar{f}(L)\bar{D}^2L \ \longrightarrow \ L\big(\phi + \bar{\phi} + D^2 f + \bar{D}^2\bar{f}\big) \ . \tag{F.4}$$

The authors of [101] explicitly determined the required superfield redefinition. For our brief analysis, we allow ourselves to ignore these terms.

Substituting the solution (F.3) in to the auxiliary action (F.1) we obtain

$$S[\phi,\bar{\phi}] = \int \mathrm{d}^8 z \left\{ 2\phi\bar{\phi} + \frac{1}{4}(DL)^2(\bar{D}L)^2\left(\Lambda - \omega\bar{\Gamma}^2 - \bar{\omega}\Gamma^2 + 2v\Gamma\bar{\Gamma}\right) \right\} \ , \tag{F.5}$$

where

$$v = \frac{1}{2}Q^a\bar{Q}_a \ . \tag{F.6}$$

As a consequence of substituting the solution (F.3) into $D_\alpha L$ we find that

$$\begin{aligned} (D\phi)^2 &= (DL)^2\left\{(1-\bar{\omega}\,\Gamma)^2 + 2\,v\,\bar{\Gamma}\,(1-\bar{\omega}\,\Gamma) + \omega\,\bar{\omega}\,\bar{\Gamma}^2\right\} \ , \\ (\bar{D}\bar{\phi})^2 &= (\bar{D}L)^2\left\{(1-\omega\,\bar{\Gamma})^2 + 2\,v\,\Gamma\,(1-\omega\,\bar{\Gamma}) + \omega\,\bar{\omega}\,\Gamma^2\right\} \ . \end{aligned} \tag{F.7}$$

Also, substituting the solution (F.3) into the definition for $Q_{\alpha\dot{\alpha}}$ (F.2) gives

$$\partial_{\alpha\dot{\alpha}}\phi = \frac{1}{2}\,Q_{\alpha\dot{\alpha}}\,(1-\bar{\omega}\,\Gamma) + \frac{1}{2}\,\omega\,\bar{Q}_{\alpha\dot{\alpha}}\,\bar{\Gamma} \,, \qquad \partial_{\alpha\dot{\alpha}}\bar{\phi} = \frac{1}{2}\,\bar{Q}_{\alpha\dot{\alpha}}\,(1-\omega\,\bar{\Gamma}) + \frac{1}{2}\,\bar{\omega}\,Q_{\alpha\dot{\alpha}}\,\Gamma \ . \tag{F.8}$$

Consequently, defining

$$p = -(\partial^{\alpha\dot\alpha}\phi)(\partial_{\alpha\dot\alpha}\bar\phi) \ , \quad q = -(\partial^{\alpha\dot\alpha}\phi)(\partial_{\alpha\dot\alpha}\phi) \ , \quad \bar q = -(\partial^{\alpha\dot\alpha}\bar\phi)(\partial_{\alpha\dot\alpha}\bar\phi) \ , \tag{F.9}$$

we obtain the following coupled equations:

$$\begin{aligned} q &= \omega\left\{(1-\bar\omega\,\Gamma)^2 + 2\,v\,\bar\Gamma\,(1-\bar\omega\,\Gamma) + \omega\,\bar\omega\,\bar\Gamma^2\right\} \ , \\ \bar q &= \bar\omega\left\{(1-\omega\,\bar\Gamma)^2 + 2\,v\,\Gamma\,(1-\omega\,\bar\Gamma) + \omega\,\bar\omega\,\Gamma^2\right\} \ , \\ p &= v - \frac{1}{2}(q+\bar q) + \frac{1}{2}(\omega+\bar\omega) \ . \end{aligned} \tag{F.10}$$

Comparing these equations with (F.7) we see that

$$(D\phi)^2\omega = (DL)^2 q \ , \qquad (\bar D\bar\phi)^2\bar\omega = (\bar DL)^2\bar q \ . \tag{F.11}$$

The equations (F.10) are hard to solve in general, but for the particular case of the tensor-Goldstone multiplet (5.5.11), in which the function $\Lambda(\omega,\bar\omega)$ takes the form (4.2.10), the solution reads

$$\omega+\bar\omega = \frac{(q+\bar q)(1+p)+2q\bar q}{(1+p)^2-q\bar q} \ , \qquad \omega-\bar\omega = \frac{q-\bar q}{\sqrt{(1+p)^2-q\bar q}} \ , \tag{F.12}$$
$$v = p + \frac{1}{2}(q+\bar q) - \frac{1}{2}(\omega+\bar\omega) \ .$$

Substituting (F.12) and (F.11) into (F.5) and reintroducing the coupling constant κ, we obtain the chiral-scalar-Goldstone action (5.6.5).

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