Critical path estimation in heterogeneous scheduling heuristics

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1 Introduction

The *critical path* of a project is defined as the longest sequence of constituent tasks that must be done in order to complete it [9, 8]. If we express the project in the form of a task graph, then the critical path is simply the longest—i.e., costliest—path through it. This is useful because the time it takes to execute the tasks on the critical path of the graph is therefore a lower bound on the makespan of any possible schedule for the project, no matter how many processors are available. In the context of a listing heuristic for scheduling the task graph on a set of parallel processors, a natural choice then is to prioritize all tasks according to the length of the critical path from that task to the end, the idea being that tasks with the greatest downward path length contribute most heavily to the makespan and should therefore be processed as soon as possible. This approach has a long and successful history in scheduling for homogeneous processors, and is in fact provably optimal for two-processor systems [4].

Now, for homogeneous processors, all tasks have the same cost on all processors, so that there is only one possible cost each task may take. In particular this means that the node weights in the task graph are *fixed* so, disregarding the edge weights for the moment, the longest path through the DAG is also fixed, no matter what schedule we ultimately follow. Therefore, when we refer to the critical path of the task graph it is clear what is meant. Unfortunately, the concept of the critical path is not so clearly defined for heterogeneous processing environments. The problem is, there are now multiple possible costs that each task may take—depending on the schedule—and likewise many different communication costs that may be incurred. In particular this means that the weights of the task DAG are not fixed and we cannot simply compute a longest path. Consider

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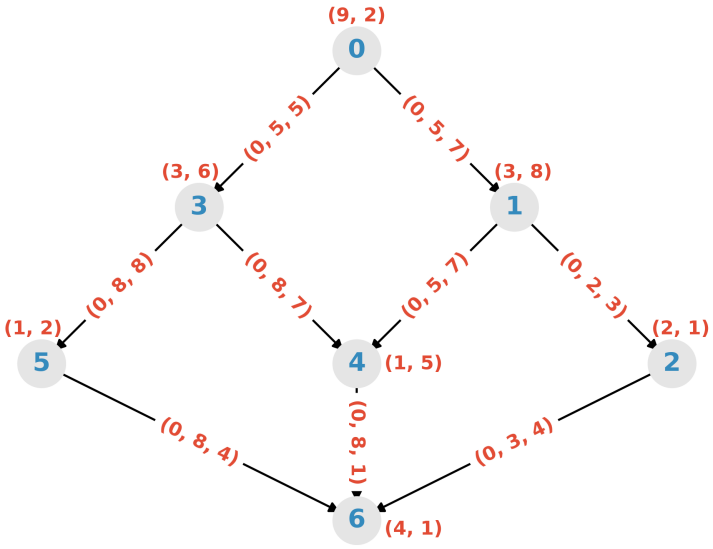


Figure 1: Simple task graph with costs on a two-processor target platform.

for example the simple DAG shown in Figure 1, where the labels represent all the possible weights each task/edge may take on a two-processor heterogeneous target platform; the red labels near the nodes represent the computation costs on processors $P1$ and $P2$ in the form (W_i^1, W_i^2), while the edge labels represent the possible communication costs in the form $(W_{ik}^{11} = W_{ik}^{22} = 0, W_{ik}^{12}, W_{ik}^{21})$. How should the longest path through a graph like that in Figure 1 be defined?

The HEFT approach, as described in previous chapters, is to use *average* values over all sets of possible costs in order to fix the DAG weights and then compute the critical path in a standard dynamic programming manner. In particular, for each $i = 1, \ldots, n$, we compute a number u_i, called the upward rank, which purportedly represents the critical path length from task t_i to the end and is taken to be the task's priority. For example, for the graph in Figure 1, HEFT first (implicitly) converts the DAG to the fixed-cost one shown in Figure 2, and then calculates the task ranks like so:

$$
\begin{aligned}
u_6 &= 2.5, \\
u_5 &= 1.5 + 3 + 2.5 \\
&= 7, \\
u_4 &= 3 + 2.25 + 2.5 \\
&= 7.75, \\
u_2 &= 1.5 + 1.75 + 2.5 \\
&= 5.75, \\
u_3 &= 4.5 + \max\{4 + u_5,\ 3.75 + u_4\}
\end{aligned}
$$

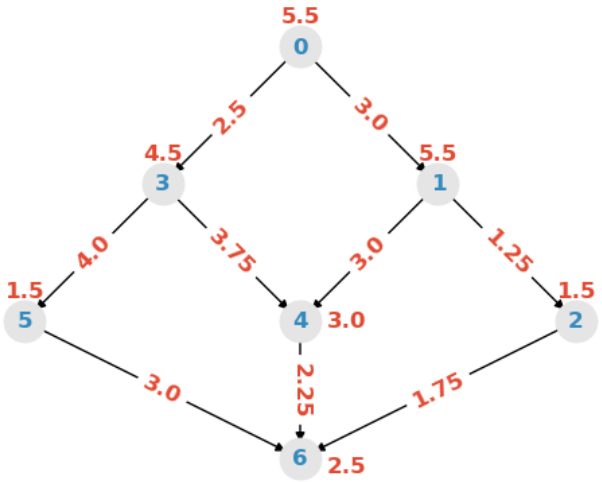


Figure 2: Fixed-cost counterpart of task DAG from Figure 1 which is implicitly used in HEFT.

$$
\begin{aligned}
&= 4.5 + \max\{11,\ 11.5\} \\
&= 16, \\
u_1 &= 5.5 + \max\{3 + u_4,\ 1.25 + u_2\} \\
&= 5.5 + \max\{10.75,\ 7\} \\
&= 16.25, \\
u_0 &= 5.5 + \max\{2.5 + u_3,\ 3 + u_1\} \\
&= 5.5 + \max\{18.25,\ 19.25\} \\
&= 24.75.
\end{aligned}
$$

These ranks give a scheduling priority list of $\{t_0, t_1, t_3, t_4, t_5, t_2, t_6\}$. The resulting schedule length obtained by HEFT with these priorities is 22, as shown in Figure 3. Interestingly, we see that u_0, the rank of the single entry task, is greater than the schedule makespan—and therefore obviously not a lower bound on it. The question then is, what quantity do the u_i values actually represent? Now, because of the averaging of the costs, perhaps the most intuitive interpretation is that the u_i are in turn estimates of the average critical path lengths, taken over the set of all possible critical paths —but there is no robust mathematical justification for believing that this definition of the critical path is more useful than other possibilities.

Given this ambiguity, alternative ways to define the critical path for the ranking phase in HEFT have been considered before, most notably by Zhao and Sakellariou [14], who empirically compared the performance of HEFT when averages other than the mean (e.g., median, maximum, minimum) are used to compute upward (or downward) ranks. Their conclusions were that using the mean is not

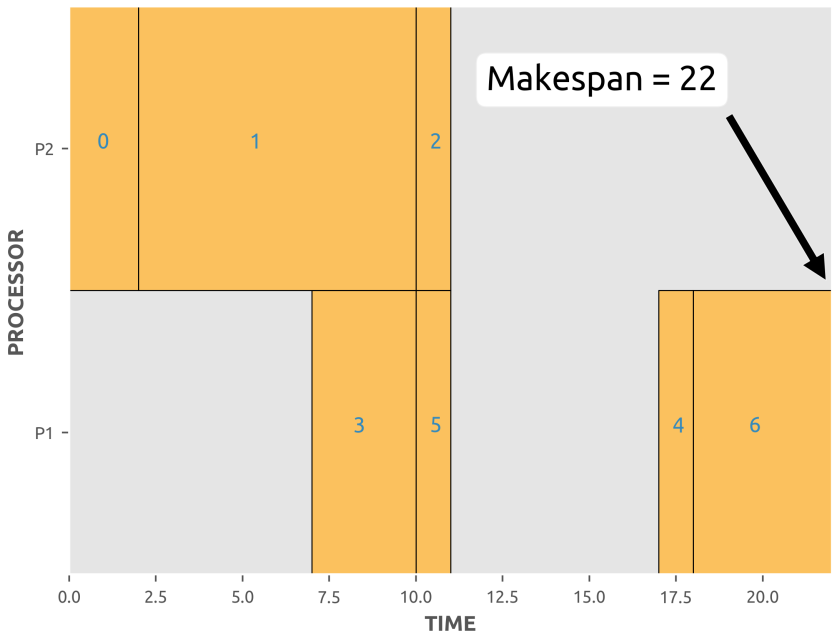


Figure 3: HEFT schedule for DAG in Figure 1.

clearly superior to other averages, although none of the other options they considered were consistently better. Indeed, perhaps the biggest takeaway from their investigation was that HEFT is very sensitive to how priorities are computed, with significant variation being seen for different graphs and target platforms. In this chapter we undertake a similar investigation with the aim of establishing if there are choices which do consistently outperform the standard upward ranking in HEFT. In addition, we consider how critical path estimates can be used to determine processor selection, as well as task prioritization, following the approach of the *Predict Earliest Finish Time* (PEFT) heuristic [1], and attempt to ascertain which definition leads to the best practical performance.

This will be a largely empirical study, as is common in this area, although mathematical results which underlie heuristic methods are also described. To facilitate this investigation we created a software package that simulates heterogeneous scheduling problems, much like that described in the previous chapter, although not restricted to accelerated target platforms. As before, the (`Python`) source code for this simulator can be found on Github[1] and all of the results presented here can be re-run from scripts contained therein.

[1] https://github.com/mcsweeney90/critical-path-estimation

2 A universal lower bound

Functionally, the critical path is used in HEFT as a lower bound on the makespan, so that minimizing the critical path gives us the most scope to minimize the makespan, assuming we make good use of our parallel resources. With this in mind, there are many different ways we can define the critical path so that it gives a lower bound on the makespan of any possible schedule. The most straightforward approach would be to just set all weights to their minimal values but a tighter bound can be computed in the following manner. First, define ℓ_i^a for all tasks t_i and processors p_a to be the critical path length from t_i to the end (inclusive), assuming that it is scheduled on processor p_a. These values can easily be computed recursively by setting $\ell_i^a = W_i^a$, $a = 1, \ldots, q$, for all exit tasks then moving up the DAG and calculating

$$\ell_i^a = W_i^a + \max_{k \in S_i} \left(\min_{b=1,\ldots,q} \left(\ell_k^b + W_{ik}^{ab} \right) \right), \quad a = 1, \ldots, q, \tag{1}$$

for all other tasks. Then, for each $i = 1, \ldots, n$,

$$\ell_i = \min_{a=1,\ldots,q} \ell_i^a \tag{2}$$

gives a true lower bound on the remaining cost of any schedule once the execution of task t_i begins. These ℓ_i values could be useful as alternative task priorities in HEFT, especially since the cost of computing all of the ℓ_i^a in this manner is only $O((m+n)q) \approx O(n^2 q)$ so in particular is the same order as the usual HEFT prioritization phase.

For example, for the simple DAG shown in Figure 1, we find that the ℓ_i values are as given in Table 1 (with the u_i included for comparison). Interestingly, we see that tasks t_1 and t_3 have the same lower bound (8 units) and the performance of the alternative ranks relative to the standard u_i sequence in HEFT depends on which is chosen to be scheduled first: if $t-1$, the priority list does not change so the schedule makespan is 22 units, but if t_3 is selected instead, the final schedule makespan is 20 units—i.e., smaller than the original—as illustrated in Figure 4. Of course, this is only one example: there is no mathematically valid reason to suppose that using the ℓ_i sequence instead of u_i as the task ranks in HEFT will actually lead to superior performance in general. Still, it seems worthwhile to investigate this empirically, which we do in Section 5.

(Note that the lower bound on the critical path as defined here is very similar to the optimistic cost used in PEFT; this will be discussed further in Section 4.)

3 A stochastic interpretation

In this section we propose a family of alternative task ranking phases in HEFT based on the following interpretation of the standard ranking. Given the complex

Table 1: Upward ranks u_i and lower bounds ℓ_i for the DAG in Figure 1.

Task:	0	1	2	3	4	5	6
u_i	24.75	16.25	5.75	16	7.75	7	2.5
ℓ_i	16	8	2	8	5	3	1

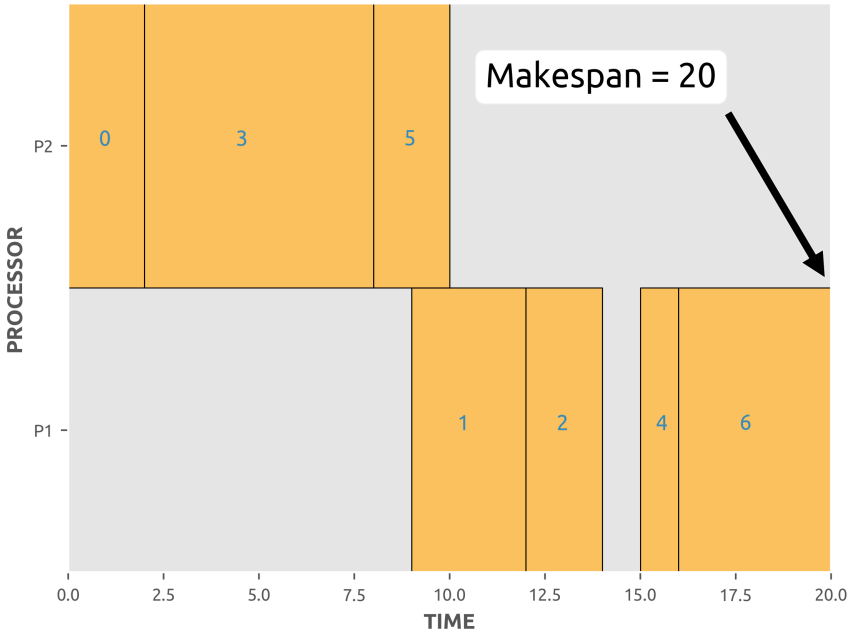


Figure 4: Alternative HEFT schedule for DAG in Figure 1.

interplay between the ranking and processor selection phases, it is impossible to predict exactly which values all DAG weights will take at runtime, at least without modifying the latter. Now, by using average values over all sets of possible costs, HEFT in some sense implicitly assumes that all members of each set are equally likely to be incurred. Conceptually, we can view this as an attempt to *model* the processor selection phase in order to predict which values the weights will assume. This model takes the form of a *stochastic graph*—i.e., all weights are assumed to be independent (discrete) random variables with associated probability mass functions (pmfs) given by the assumption of equal likelihood. More precisely, let m_i be the pmf corresponding to the task weight variable w_i and m_{ik} that for the edge weight w_{ik}, then

$$m_i(W_i^a) \coloneqq \mathbb{P}[w_i = W_i^a] = \frac{1}{q}, \quad a = 1, \ldots, q,$$

and

$$\begin{aligned} m_{ik}(W_{ik}^{ab}) &= m_i(W_i^a) \cdot m_k(W_k^b) \\ &= \frac{1}{q^2}, \quad \forall a, b. \end{aligned}$$

It is important to note here that the model defined by these pmfs is clearly not accurate. In particular, all of the node and edge weight variables are independent of one another. This is induced by the averaging but does not reflect, for example, the fact that edge weights are fully determined once the node weights are realized. Still, we attempt to establish exactly how useful the model is through extensive numerical simulations in Section 5.

Note also that the expected values of the node and edge weight variables are given by

$$\mathbb{E}[w_i] = \sum_{a=1}^{q} W_i^a m_i(W_i^a) = \frac{1}{q} \sum_{a=1}^{q} W_i^a, \tag{3}$$

$$\mathbb{E}[w_{ik}] = \sum_{a=1}^{q} \sum_{b=1}^{q} W_{ik}^{ab} m_{ik}(W_{ik}^{ab}) = \frac{1}{q^2} \sum_{a,b} W_{ik}^{ab}. \tag{4}$$

This means that $\mathbb{E}[w_i] = \overline{w_i}$ and $\mathbb{E}[w_{ik}] = \overline{w_{ik}}$. So the computation of the upward ranks u_i in HEFT can instead be done by setting $u_i = \mathbb{E}[w_i]$ for all exit tasks, then moving up the DAG and recursively computing

$$u_i = \mathbb{E}[w_i] + \max_{k \in S_i} \big(u_k + \mathbb{E}[w_{ik}] \big) \tag{5}$$

for all other tasks.

In summary, since all possible node and edge weights of a task graph G are known but their actual values at runtime aren't, one possible interpretation of the standard HEFT task prioritization phase is that critical path lengths in G are estimated through a two-step process:

1. An associated stochastic graph G_s is implicitly constructed with node and edge pmfs m_i and m_{ik} as defined above.

2. The numbers u_i are recursively computed for all tasks in G_s using (5), and taken as the critical path lengths from the corresponding tasks in G.

In the following two sections, we propose modifications of both steps so as to obtain different critical path estimates that may be used as task ranks in HEFT. The performance of these will then be evaluated through extensive numerical simulations in Section 5.

3.1 The critical path of G_s

Now, since all of its weights are RVs, the critical path of the stochastic graph G_s is clearly itself an RV. But a natural question arises from the interpretation outlined in the previous section: what is the relationship between the sequence of numbers u_i as defined by (5) and the critical path of G_s? In fact, it has long been known in the context of *Program Evaluation and Review Technique* (PERT) network analysis that the numbers u_i are *lower bounds on the expected value* of the critical path lengths of the stochastic DAG. This result dates back at least as far as Fulkerson [6], who referred to it as already being widely-known and gave a simple proof. This prompts another question: does using the actual expected values lead to superior performance in HEFT?

Unfortunately, computing the moments of the critical path length of a graph whose weights are discrete RVs was shown to be a $\#P$-complete problem by Hagstrom [7]. This means that it is generally impractical to compute the true expected values. However, efficient methods which yield better approximations than the u_i numbers are known; we discuss examples in the following two sections.

3.1.1 Monte Carlo sampling

Monte Carlo (MC) methods have a long history as a means of approximating the critical path distribution for PERT networks, dating back to at least the early 1960s [13]. The idea is to simulate the realization of all RVs (according to their pmfs) and then evaluate the critical path of the resulting deterministic graph. This is done repeatedly, giving a set of critical path instances whose empirical distribution function is guaranteed to converge to the true distribution by the Glivenko-Cantelli Theorem [2]. Furthermore, analytical results allow us to quantify the approximation error for any given the number of realizations—and therefore the number of realizations needed to reach a desired accuracy.

The downside of Monte Carlo sampling is its cost. While modern architectures are well-suited to this approach because of their parallelism, it still may be impractical in the context of a scheduling heuristic, especially when the DAG is large; we often found this to be the case for the examples discussed in Section 5.

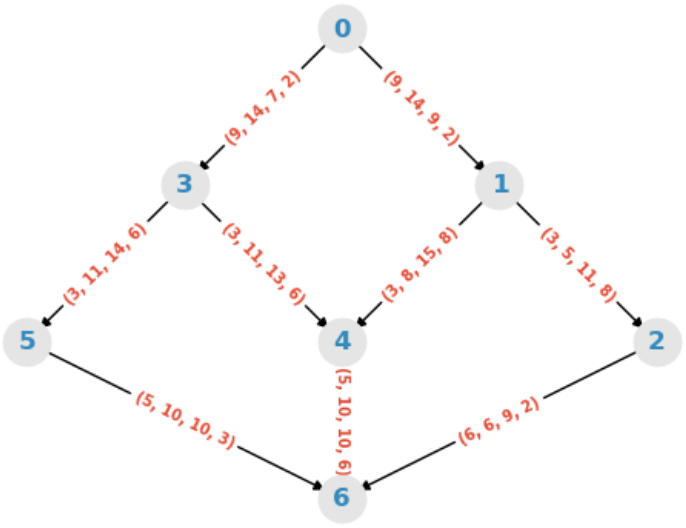


Figure 5: Edge weight-only equivalent of task DAG from Figure 1.

Hence in this report we typically only use the Monte Carlo method as a means of obtaining a reference solution when estimating the critical path of G_s; see, for example, the following section.

3.1.2 Fulkerson's bound

Before introducing the alternative bounds on the critical path lengths proposed by Fulkerson, we first describe how the stochastic graph G_s can be expressed in an equivalent formulation with only edge weights. This step is not mathematically necessary but simply makes the elucidation much cleaner; it should be emphasized that all of the following still holds, with only minor adjustments, if this is not done. The most straightforward approach is to simply redefine the edge weights so that they also include the computation cost of the parent task and, if the child task is an exit, the computation cost of the child as well. More precisely, we define a new set of edge weight variables $\tilde{w}_{ik}$ which take values

$$\tilde{W}_{ik}^{ab} := W_i^a + W_{ik}^{ab} + \delta_k W_k^b, \quad \forall a, b, i, k,$$

where $\delta_k = 1$ if t_k is an exit task and zero otherwise. Figure 5 illustrates how the graph in Figure 1 would be transformed in this manner, where the edge labels are in the form $(\tilde{W}_{ik}^{11}, \tilde{W}_{ik}^{12}, \tilde{W}_{ik}^{22}, \tilde{W}_{ik}^{21})$.

Note that $\mathbb{P}[w_{ik} = W_{ik}^{ab}] = \mathbb{P}[\tilde{w}_{ik} = \tilde{W}_{ik}^{ab}]$ for all a, b, i and k, so that $m_{ik}(\tilde{W}_{ik}^{ab}) \equiv m_{ik}(W_{ik}^{ab})$. However, we do have to make a minor adjustment to how the u_i numbers are computed since the expected value of the node weights no longer has any

meaning. In particular, we set $u_i = 0$ for all exit tasks then recursively compute

$$u_i = \max_{k \in S_i} \left(u_k + \mathbb{E}[\tilde{w}_{ik}] \right) \tag{6}$$

for all others. It can readily be verified that this sequence is identical to that defined by (5) with the exception of the exit tasks, for which the corresponding numbers are now zero. (Technically we should perhaps rename this number sequence but given the fact that it is essentially identical we felt this was unnecessary.)

Now, for all $i = 1, \ldots, n$, let c_i be the critical path length from task t_i to the end and let $e_i = \mathbb{E}[c_i]$ be its expected value. Define Z_i to be the set of all weight RVs corresponding to edges downward of t_i (i.e., the remainder of the graph). Let $R(Z_i)$ be the set of all possible *realizations* of the RVs in Z_i. Given a realization $z_i \in R(Z_i)$, let $\ell(z_i)$ be the critical path length from task t_i to the end (which is a scalar because all weights have been realized). Then by the definition of the expected value we have

$$e_i = \sum_{z_i \in R(Z_i)} \mathbb{P}[Z_i = z_i] \ell(z_i). \tag{7}$$

Let B_i be the set of all the weight RVs corresponding to edges which connect task t_i to its immediate children—i.e., $B_i \coloneqq \{\tilde{w}_{ik}\}_{k \in S_i}$. Further, let $R(B_i)$ be the set of all possible realizations of the RVs in B_i and let $b_i \in R(B_i)$ be any such realization. Suppose we define a sequence of numbers by $f_i = 0$, if t_i is an exit task, and

$$f_i = \sum_{b_i \in R(B_i)} \mathbb{P}[B_i = b_i] \max_{k \in S_i} \{ f_k + b_{ik} \} \tag{8}$$

for all other tasks, where b_{ik} is the realization of the edge weight RV $\tilde{w}_{ik}$ under the set of realizations b_i. Then Fulkerson showed that the following proposition holds.

Proposition 1 *For all $i = 1, \ldots, n$, we have $u_i \leq f_i \leq e_i$, so that in particular the f_i give a tighter bound on the expected values of the critical path lengths.*

Proof. The proof proceeds by induction. Without loss of generality, we assume that the DAG has single entry and exit tasks (artificial zero-cost tasks can be added if necessary) so that in particular n is the index of the exit task. Clearly, since $u_n = f_n = e_n = 0$ by definition, the inequality holds in that case. Now for a generic task t_i we assume that the inequalities $u_k \leq f_k \leq e_k$ hold for all $k \in S_i$ (i.e., all of t_i's child tasks) and show that the inequality $u_i \leq f_i \leq e_i$ therefore holds as well.

First, we prove the left-hand inequality. By bringing the probability into the maximization we can rewrite (8) as

$$f_i = \sum_{b_i \in R(B_i)} \max_{k \in S_i} \{\mathbb{P}[B_i = b_i](f_k + b_{ik})\}.$$

Interchanging the summation and maximization, we get the inequality

$$f_i \geq \max_{k \in S_i} \left\{ \sum_{b_i \in R(B_i)} \mathbb{P}[B_i = b_i](f_k + b_{ik}) \right\}.$$

By expanding out the term in the maximization and making use of the facts that the f_k are independent of b_i and the weight RVs corresponding to distinct edges are also independent, we see that

$$\begin{aligned}
\sum_{b_i \in R(B_i)} \mathbb{P}[B_i = b_i](f_k + b_{ik}) &= \sum_{b_i \in R(B_i)} \mathbb{P}[B_i = b_i] f_k + \sum_{b_i \in R(B_i)} \mathbb{P}[B_i = b_i] b_{ik} \\
&= f_k \sum_{b_i} \mathbb{P}[B_i = b_i] + \sum_{b_i} \mathbb{P}[B_i = b_i] b_{ik} \\
&= f_k + \sum_{b_i} \mathbb{P}[B_i = b_i] b_{ik} \\
&= f_k + \mathbb{E}[\tilde{w}_{ik}].
\end{aligned}$$

By the induction hypothesis, $f_k \geq u_k$ for all $k \in S_i$, so we have

$$\begin{aligned}
f_i &\geq \max_{k \in S_i} \{f_k + \mathbb{E}[\tilde{w}_{ik}]\} \\
&\geq \max_{k \in S_i} \{u_k + \mathbb{E}[\tilde{w}_{ik}]\} = u_i,
\end{aligned}$$

as required to establish the first part of the inequality.

Now we prove the second part. First, note that we can rewrite equation (7) as

$$e_i = \sum_{z_i \in R(Z_i)} \mathbb{P}[Z_i = z_i] \max_{k \in S_i} \{\ell(z_k) + z_{ik}\},$$

where z_{ik} is the realization of the edge weight RV $\tilde{w}_{ik}$ under the set of realizations z_i. The key now is to divide into two disjoint subsets, $Z_i = B_i \cup Y_i$, where the former is defined as before and the latter contains all of the other elements of the set (i.e., RVs corresponding to edges beyond $t_i's$ child tasks). By the independence of the edge weights, we have

$$\mathbb{P}[Z_i = z_i] = \mathbb{P}[B_i = b_i]\mathbb{P}[Y_i = y_i],$$

where y_i runs over all the possible realizations of the set Y_i. This means that

$$\begin{aligned} e_i &= \sum_{z_i \in R(Z_i)} \mathbb{P}[Z_i = z_i] \max_{k \in S_i} \{\ell(z_k) + z_{ik}\} \\ &= \sum_{b_i} \mathbb{P}[B_i = b_i] \sum_{y_i} \mathbb{P}[Y_i = y_i] \max_{k \in S_i} \{(\ell(z_k) + b_{ik})\}, \end{aligned}$$

where we change the z_{ik} to b_{ik} to acknowledge that it is now set according to the realization b_i. Interchanging the rightmost sum and its inner maximization we get the inequality

$$\begin{aligned} e_i &\geq \sum_{b_i} \mathbb{P}[B_i = b_i] \max_{k \in S_i} \left\{ \sum_{y_i} \mathbb{P}[Y_i = y_i](\ell(z_k) + b_{ik}) \right\} \\ &= \sum_{b_i} \mathbb{P}[B_i = b_i] \max_{k \in S_i} \left\{ b_{ik} + \sum_{y_i} \mathbb{P}[Y_i = y_i]\ell(z_k) \right\}, \end{aligned}$$

since the b_{ik} do not depend on y_i. By the independence assumption, for all $k \in S_i$, we have $\mathbb{P}[Y_i = y_i] = \mathbb{P}[Z_k = z_k]$. Therefore

$$\begin{aligned} e_i &\geq \sum_{b_i} \mathbb{P}[B_i = b_i] \max_{k \in S_i} \left\{ b_{ik} + \sum_{z_k} \mathbb{P}[Z_k = z_k]\ell(z_k) \right\} \\ &= \sum_{b_i} \mathbb{P}[B_i = b_i] \max_{k \in S_i} \{b_{ik} + e_k\}. \end{aligned}$$

By the induction assumption, $e_k \geq f_k$ for all $k \in S_i$, so finally we have

$$e_i \geq \sum_{b_i} \mathbb{P}[B_i = b_i] \max_{k \in S_i} \{b_{ik} + f_k\} = f_i,$$

which completes the proof. □

Example As a demonstration of how the Fulkerson numbers f_i are calculated, we consider the small DAG from Figure 5. The procedure is greatly simplified in this case because the edge pmfs m_{ik} are all constant and therefore all realizations any given weight RV are equally likely. Since there are $q^2 = 4$ realizations of each edge weight, we have $|R(B_i)| = 4^{|S_i|}$ for all i. Therefore, for any task t_i and any realization $b_i \in B_i$,

$$\mathbb{P}[B_i = b_i] = \frac{1}{|R(B_i)|} = \frac{1}{4^{|S_i|}}.$$

We begin by setting $f_6 = 0$ and then working our way up the DAG. The Fulkerson numbers corresponding to tasks t_5, t_4 and t_2 are straightforward since in each case

their only child task is t_6:

$$\begin{aligned} f_5 &= \frac{1}{4}(f_6 + 5) + \frac{1}{4}(f_6 + 10) + \frac{1}{4}(f_6 + 10) + \frac{1}{4}(f_6 + 3) \\ &= \frac{5 + 10 + 10 + 3}{4} = 7, \\ f_4 &= \frac{5 + 10 + 10 + 6}{4} = 7.75, \\ f_2 &= \frac{6 + 6 + 9 + 2}{4} = 5.75. \end{aligned}$$

Things are more interesting for f_1, f_3 and f_0 since they each have two children. The calculations are quite long-winded because $|R(B_0)| = |R(B_1)| = |R(B_3)| = 4^2 = 16$, so we only include all of the details for f_1, which is computed as follows,

$$\begin{aligned} f_1 = \frac{1}{16}[&\max(f_2 + 3, f_4 + 3) + \max(f_2 + 3, f_4 + 8) + \max(f_2 + 3, f_4 + 15) + \max(f_2 + 3, f_4 + 8) \\ &+ \max(f_2 + 5, f_4 + 3) + \max(f_2 + 5, f_4 + 8) + \max(f_2 + 5, f_4 + 15) + \max(f_2 + 5, f_4 + 8) \\ &+ \max(f_2 + 11, f_4 + 3) + \max(f_2 + 11, f_4 + 8) + \max(f_2 + 11, f_4 + 15) + \max(f_2 + 11, f_4 + 8) \\ &+ \max(f_2 + 8, f_4 + 3) + \max(f_2 + 8, f_4 + 8) + \max(f_2 + 8, f_4 + 15) + \max(f_2 + 8, f_4 + 8)] \\ = \frac{1}{16}[&10.75 + 15.75 + 22.75 + 15.75 + 10.75 + 15.75 + 22.75 + 15.75 \\ &+ 16.75 + 16.75 + 22.75 + 16.75 + 13.75 + 15.75 + 22.75 + 15.75] \\ = \frac{271}{16} &\approx 16.9. \end{aligned}$$

In a similar manner, we compute $f_3 = \frac{1159}{64} \approx 18.1$ and finally $f_0 = \frac{28899}{1024} \approx 28.2$. Table 2 compares the f_i and u_i values for this example with e_i, the actual expected critical path length, which was estimated using the Monte Carlo method with 1000 realizations (see Section 3.1.1). We see that the f_i do indeed give tighter bounds than the u_i. Furthermore, if we were to rank the tasks according to the former, we would get a task priority list of $\{t_0, t_3, t_1, t_4, t_5, t_2, t_6\}$—i.e., tasks t_1 and t_3 are interchanged compared to the standard HEFT ranking. As seen in Section 2, this leads to a schedule with a smaller makespan (20 units) than the schedule obtained using the u_i numbers as ranks (22 units). Again, this is only one example, but it does support the idea that taking the f_i as task ranks rather than the u_i may be useful, a proposition that we investigate more thoroughly in Section 5.

Complexity To compute each of the f_i using (8) we need to do a lot of work, as suggested by the (deliberately) tedious breakdown of the calculations for f_1 in the example above. The problem is that the computation scales with the size of $R(B_i)$, which is exponential in the number of child tasks $|S_i|$. There are up to q^2 distinct costs for each edge so we may have $|R(B_i)| = q^{2^{|S_i|}}$. In the worst case, a task may have $n-1$ children, which can easily lead to an impractical time complexity for even relatively small values of n and q.

Table 2: Fulkerson numbers f_i for the DAG in Figure 5.

Task:	0	1	2	3	4	5	6
u_i	24.75	16.25	5.75	16	7.75	7	0
f_i	28.2	16.9	5.75	18.1	7.75	7	0
e_i	29.6	17.6	5.9	18.7	7.8	7.3	0

Fortunately, we can compute the f_i numbers more efficiently using an idea first proposed by Clingen [3] in the context of extending Fulkerson's method to the case where edge weights are modeled as continuous random variables. The key is to exploit the structure of the computations as illustrated by the example for f_1 above. In particular, we can see that certain values repeatedly arise from the maximizations—for example, in the computations above, $f_4 + 15$ is larger than the sum of f_2 and any weight realization along the edge (t_1, t_2), so it is the result of all the maximizations in which it occurs. This suggests that perhaps we can improve efficiency by calculating how frequently each term will appear in the summation, rather than working sequentially through the set of realizations $R(B_i)$. This is the intuitive basis for Clingen's method, which is as follows.

For all $i = 1, \ldots, n$ and $k \in S_i$, let $R(\tilde{w}_{ik})$ be the set of all possible realizations of the edge weight RV $\tilde{w}_{ik}$. Further, let V_i be the set of all unique values of $f_k + \tilde{w}_{ik}$ and define

$$\alpha_i = \max_{k \in S_i} (f_k + \min(R(\tilde{w}_{ik}))).$$

Let M_{ik} be the cumulative pmf along edge (t_i, t_k), so that $M_{ik}(x) = \mathbb{P}[\tilde{w}_{ik} \leq x]$, and define the related function $M^*_{ik}(x) = \mathbb{P}[\tilde{w}_{ik} < x]$. Then, if we let v run over the elements of V_i, we can rewrite equation (8) as

$$f_i = \sum_{v \geq \alpha_i} v \left(\prod_{k \in S_i} M_{ik}(v - f_k) - \prod_{k \in S_i} M^*_{ik}(v - f_k) \right). \tag{9}$$

Formally, this approach is based on the well-known fact that the cumulative probability mass function of the maximum of a finite set of (independent) RVs is equal to the product of the individual cumulative pmfs of the RVs. More intuitively, the idea is just that described above, to in some sense "count" the number of times each v is the output of a maximization: all $v < \alpha_i$ are disregarded because they can never be maximal and the modified pmf M^*_{ik} is a kind of correction term to prevent over counting.

Using Clingen's method, we can recalculate f_1 from the example above as

follows,

$$
\begin{aligned}
V_1 &= \{f_4+3, f_4+8, f_4+15, f_2+3, f_2+11, f_2+8\} \\
&= \{10.75, 15.75, 22.75, 8.75, 16.75, 13.75\}, \\
\alpha_1 &= \max\{f_4+\min(3,8,15),\ f_2+\min(3,5,8,11)\} \\
&= \max\{10.75,\ 8.75\} \\
&= 10.75, \\
f_1 &= 10.75[M_{14}(3)M_{12}(5) - M^*_{14}(3)M^*_{12}(5)] \\
&\quad + 15.75[M_{14}(8)M_{12}(10) - M^*_{14}(8)M^*_{12}(10)] \\
&\quad + 22.75[M_{14}(15)M_{12}(17) - M^*_{14}(15)M^*_{12}(17)] \\
&\quad + 16.75[M_{14}(9)M_{12}(11) - M^*_{14}(9)M^*_{12}(11)] \\
&\quad + 13.75[M_{14}(6)M_{12}(8) - M^*_{14}(6)M^*_{12}(8)] \\
&= 10.75\left[\frac{1}{4}\cdot\frac{1}{2} - 0\cdot\frac{1}{4}\right] + 15.75\left[\frac{3}{4}\cdot\frac{3}{4} - \frac{1}{4}\cdot\frac{3}{4}\right] \\
&\quad + 22.75\left[1\cdot 1 - \frac{3}{4}\cdot 1\right] + 16.75\left[\frac{3}{4}\cdot\frac{3}{4} - \frac{3}{4}\cdot\frac{1}{2}\right] \\
&\quad + 13.75\left[\frac{1}{4}\cdot\frac{1}{2} - \frac{1}{4}\cdot\frac{1}{4}\right] \\
&= \frac{271}{16} \approx 16.9.
\end{aligned}
$$

A complete description of a practical procedure for computing the Fulkerson numbers f_i using Clingen's method f_i is given in Algorithm 1.

At first blush this method may not appear to be any more efficient than before but the number of operations required to compute each of the f_i is now $O(q^2|S_i|)$, rather than the first term being exponential in the second. Of course, it should also be noted that this procedure is still more expensive than computing the u_i sequence. Indeed, as we will see in Section 5, we found that it can still be impractical when the sets of realizations for each edge are large. However, this relationship between efficiency and the number of edge weight realizations can also be exploited: for accelerated platforms which follow the model described in the previous chapter, the number of possible realizations remains small, no matter how many processors are available, so Fulkerson's numbers can still be calculated efficiently for large DAGs and target platforms with many processors.

Extensions Elmaghraby [5] proposed two refinements of Fulkerson's method. The first involves computing each of the f_i numbers in the aforementioned manner and then reversing the direction of the remaining subgraph in order to calculate an intermediate result which can be used to improve the quality of the bound. The second is a more general approach based on using two or more *point estimates* of e_i, rather than just f_i, a method that was later generalized by Robillard and

Trahan [10]. In both cases Elmagharaby proved that the new number sequences achieve tighter bounds on e_i than the Fulkerson numbers f_i. However, small-scale experimentation suggested that the improvement of Elmaghraby's new bounds over Fulkerson's were typically minor compared to the improvement of the latter over the u_i sequence so we chose to only evaluate here whether tightening the bounds at all is useful in HEFT.

Algorithm 1: Computing Fulkerson's numbers by Clingen's method.

```
for i = n, ..., 1 do
    f_i = 0, α_i = 0, V_i = {}
    for k ∈ S_i do
        r_m = ∞
        for r ∈ R(w̃_ik) do
            r_m ← min(r_m, r)
            if f_k + r ∉ V_i then
                V_i ← V_i ∪ {f_k + r}
            end
        end
        α_i ← max(α_i, f_k + r_m)
    end
    for v ∈ V_i do
        if v ≥ α_i then
            g = 1, d = 1
            for k ∈ S_i do
                g ← g × M_ik(v − f_k)
                d ← d × M*_ik(v − f_k)
            end
            f_i ← f_i + v × (g − d)
        end
    end
end
```

3.2 Adjusting the pmfs

We previously argued that in some sense the purpose of the node and edge pmfs m_i and m_{ik} is to simulate the dynamics of the processor selection phase of HEFT so that, for example, $m_i(W_i^a)$ should represent the probability that task t_i is scheduled on processor p_a, and so on. In HEFT, tasks are assigned to the processor that is estimated to complete their execution at the earliest time and attempting to model this accurately beforehand can quickly get messy and

expensive—especially given the interaction between the two phases of the algorithm. However, a sensible idea may be to simply *weight* the processor selection probabilities according to their respective computation costs: if, say, a task is 10 times faster on one processor than another then it seems more likely it will be scheduled on the former than the latter, even once the effect of contention is taken into account. With this in mind, for all tasks t_i let

$$s_i = \sum_a \frac{1}{W_i^a}$$

and define a new set of pmfs by

$$\hat{m}_i(W_i^a) = \frac{1}{W_i^a s_i} \quad \forall i, a$$

and

$$\begin{aligned} \hat{m}_{ik}(W_{ik}^{ab}) &= \hat{m}_i(W_i^a) \cdot \hat{m}_k(W_k^b) \\ &= \frac{1}{W_i^a W_k^b s_i s_k} \quad \forall i, k, a, b. \end{aligned}$$

Note that we take the reciprocal of the costs in order to reflect the idea that processors with smaller costs are more likely to be chosen than larger ones.

These modified pmfs can be used in conjunction with either upward ranking, as defined by equation (5), Fulkerson's bound, or even Monte Carlo methods. For example, in the first instance, the expectations simply become

$$\mathbb{E}[w_i] = \sum_{a=1}^{q} W_i^a \hat{m}_i(W_i^a) = \frac{q}{s_i}, \tag{10}$$

$$\mathbb{E}[w_{ik}] = \sum_{a=1}^{q} \sum_{b=1}^{q} W_{ik}^{ab} \hat{m}_{ik}(W_{ik}^{ab}) = \frac{1}{s_i s_k} \sum_{a,b} \frac{W_{ik}^{ab}}{W_i^a W_k^b}, \tag{11}$$

and these can be used with equation (5) to compute an alternative sequence of task ranks $\hat{u}_i$; of course, this is slightly more computationally expensive than computing the standard u_i ranks but only by a constant factor. Similarly, by using the modified pmfs in conjunction with equations (8) or (9) we can define alternative Fulkerson numbers $\hat{f}_i$. Table 3 presents the $\hat{u}_i$ and $\hat{f}_i$ sequences for the small example DAG from Figure 5. While the values no longer have any meaning compared to the e_i from the previous section, we see that the inequality $\hat{u}_i \leq \hat{f}_i$ holds, as we would expect (by sampling realizations according to $\hat{m}$ rather than m we could use the Monte Carlo method to estimate the corresponding number sequence $\hat{e}_i$, although that is not necessary here). The more pertinent takeaway from the table is that in both cases the value for $i = 3$ is greater than for $i = 1$, so task t_3 would be scheduled before t_1 if the sequences were used as ranks in

Table 3: Weighted number sequences $\hat{u}_i$ and $\hat{f}_i$ for the DAG in Figure 5.

Task:	0	1	2	3	4	5	6
$\hat{u}_i$	22.9	15.2	4.3	15.5	8.6	7.5	0
$\hat{f}_i$	25.3	15.5	4.3	17.8	8.6	7.5	0

HEFT. As seen in previous sections, this results in a smaller makespan than the standard HEFT algorithm. Both of these possibilities are therefore considered as alternative task ranks for HEFT in Section 5.

Note that since equations (10) and (11) are simply weighted averages we do not believe that computing $\hat{u}_i$ instead of u_i is a new idea: although we could not find any explicit references in the literature, we suspect this has been done before in practice.

4 Processor selection

Critical path estimates are used in HEFT—and many similar listing heuristics—only at the task prioritization phase. This begs the question, can they also be useful for processor selection? Arabnejad and Barbosa's Predict Earliest Finish Time (PEFT) heuristic [1] represents one sensible way this can be done. Recall from earlier chapters that before scheduling begins PEFT computes a table of *optimistic costs* C_i^a for all task and processor combinations in the following manner. First, set $C_i^a = 0$, $a = 1, \ldots, q$, for all exit tasks, then move up the DAG and recursively compute

$$C_i^a = \max_{k \in S_i} \left(\min_{b=1,\ldots,q} \left(\delta_{ab} \overline{w_{ik}} + W_k^b + C_k^b \right) \right) \tag{12}$$

for all other tasks, where $\delta_{ab} = 1$ if $a = b$ and 0 otherwise. The C_i^a values are referred to in PEFT as optimistic costs but can be interpreted as *conditional critical paths* in that they represent some estimate of future schedule costs given a processor selection. When scheduling, say, task t_i, in PEFT we choose the processor p_{opt} defined by

$$p_{opt} \coloneqq \min_a \left(F_i^a + C_i^a \right),$$

where, as in previous chapters, F_i^a is the estimated schedule makespan when t_i is completed by p_a. This is a nice extension of the dynamic programming principles underlying HEFT: rather than optimizing the schedule makespan up to the current task (i.e., F_i^a), we extend the horizon and optimize an estimate of

the complete schedule makespan. Now, computing the full optimistic cost table is only $O(n^2q)$—i.e., the same as HEFT—but since the values within are similar in nature to the upward ranks u_i it is sensible (and more efficient) to make use of them for prioritizing tasks, rather than going to the effort of computing the upward ranks as well. Hence PEFT defines task priorities C_i through

$$C_i = \frac{1}{q} \sum_a C_i^a. \tag{13}$$

It is important to note here that the task priorities as computed by (13) do not necessarily respect precedence constraints since the equality $C_i^a \leq C_k^a$ for $k \in S_i$ does not hold (because of the internal minimization over all processors). However this is easily remedied by selecting tasks for scheduling from the pool of currently "ready" tasks (i.e., those for which all of their parents have been scheduled) according to their priorities. Note also that although it is arguably more natural that task ranks include the cost of the task itself, which these do not, it is suggested that the savings made through the alternative selection step are more beneficial overall. Certainly, numerical experiments described by the original authors [1] suggest that PEFT is at least competitive with HEFT, especially when there are a large number of processing resources.

The structure of PEFT follows a more general heuristic framework that is defined by the following procedure:

1. Compute a table of conditional critical path estimates C_i^a for all $i = 1, \ldots, n$ and $a = 1, \ldots, q$.

2. Compute all task ranks C_i as some function of the C_i^a.

3. At the processor selection phase, schedule task t_i on the processor which minimizes $F_i^a + C_i^a$.

The standard heuristic is defined by using equations (12) and (13) for the first and second parts of this framework, respectively. A natural question is, are there any better choices?

4.1 Alternative conditional critical paths

All of the methods for estimating critical path lengths at the prioritization phase which were introduced previously can be modified to give conditional critical path estimates (which also disregard the cost of the task itself). No matter which, we first let $C_i^a = 0$ for $a = 1, \ldots, q$ and all exit tasks, then move up the DAG and recursively compute the other values, so from now on we focus only on the latter. The most straightforward method to extend is the optimistic lower bound, for

which we now compute

$$C_i^a = \max_{k \in S_i} \left(\min_{b=1,\ldots,q} \left(C_k^b + W_k^b + W_{ik}^{ab} \right) \right), \quad a = 1, \ldots, q, \tag{14}$$

for all non-exit tasks. Note that these values are extremely similar to the optimistic costs (12) as used in PEFT, with the exception that the specific communication cost W_{ik}^{ab} is used in the minimization rather than the average $\overline{w_{ik}}$. Indeed, this is arguably the most intuitive way to define the conditional critical path since the value C_i^a is a true lower bound on the remaining makespan of any schedule which executes task t_i on processor p_a; locally-optimal processor selections are overruled if the best possible final makespan we can hope to achieve given that selection is inferior to other choices.

Alternatively, we could take a similar tack to the standard HEFT upward ranks and use an estimate of what we expect the conditional critical paths to be. More specifically, we move up the DAG and recursively compute

$$C_i^a = \max_{k \in S_i} \left(\frac{1}{q} \sum_{b=1}^{q} (C_k^b + W_k^b + W_{ik}^{ab}) \right), \quad a = 1, \ldots, q. \tag{15}$$

We can just as easily use a weighted mean inside the maximization. In particular, for all non-exit tasks we recursively compute

$$C_i^a = \max_{k \in S_i} \left(\frac{1}{\hat{s}_k} \sum_{b=1}^{q} \frac{C_k^b + W_k^b + W_{ik}^{ab}}{W_k^b + C_k^b} \right), \quad a = 1, \ldots, q, \tag{16}$$

where

$$\hat{s}_k = \sum_{b=1}^{q} \frac{1}{W_k^b + C_k^b}. \tag{17}$$

This is conceptually similar to the weighted pmf $\hat{m}$ as defined in Section 3.2, except that we also include the conditional critical path lengths in the weighting. The motivation for the change is that the probability a task t_i will ultimately be scheduled on a given processor p_a at the selection phase now also depends on the C_i^a value.

4.2 Computing task priorities

There are many different ways we can use the conditional critical path estimates C_i^a to calculate task priorities C_i. By default, PEFT uses the mean but we could use any other average over the set of processors instead. In light of our comments above, the natural equivalent of the weighted mean ranking for HEFT, as defined

in Section 3.2, would be to compute the task priorities recursively, starting from the leaves, through

$$C_i = \frac{q}{\hat{s}_i} + \max_{k \in S_i} C_k, \tag{18}$$

where the maximization is taken to be zero if S_i is empty (i.e., for exit tasks) and $\hat{s}$ is as defined in (17). By adding the maximization term we ensure that the C_i give a complete valid priority list of tasks, although we could select tasks from the current set of ready tasks according to the first term as in the default PEFT algorithm instead; it makes no difference conceptually, but we use the version given here for the experiments in the following section because it is slightly more efficient in our implementation.

5 Experimental results

In this section we use our software simulator to evaluate the alternative task prioritization phases in HEFT described in Sections 2 and 3, as well as the PEFT variants outlined in Section 4. All of the data from the experiments described here, and the code used to generate it, can be found in the Github repository associated with this chapter in the `scripts`[2] folder. All experiments here were run on a machine...

5.1 Test graphs

In order to compare different heuristics, we need a suitably large and diverse collection of graphs to test them on. As in the previous chapter, we decided to use the Standard Task Graph (STG) [11] set for the topologies of these DAGs, as these are generated using several standard methods. The STG set comprises multiple subsets with differing numbers of tasks n, each of which in turn contains 180 DAGs. Here we use only those subsets with $n \in \{100, 1000\}$ in the hope that the order of magnitude increase will allow us to clearly observe the effect of DAG size. (More accurately, since DAGs from the STG all have single entry and exit tasks, we actually have, e.g., $n = 102$ rather than $n = 100$, but we refer to the rounder value throughout for the sake of clarity.)

For each of these $180 \times 2 = 360$ different DAG topologies, we create multiple copies with different computation and communication costs. These costs are generated according to the following three parameters.

- $q \in \{2, 4, 8, 16\}$, the number of processors in the target platform (and therefore the number of possible computation costs for each task, and so on).

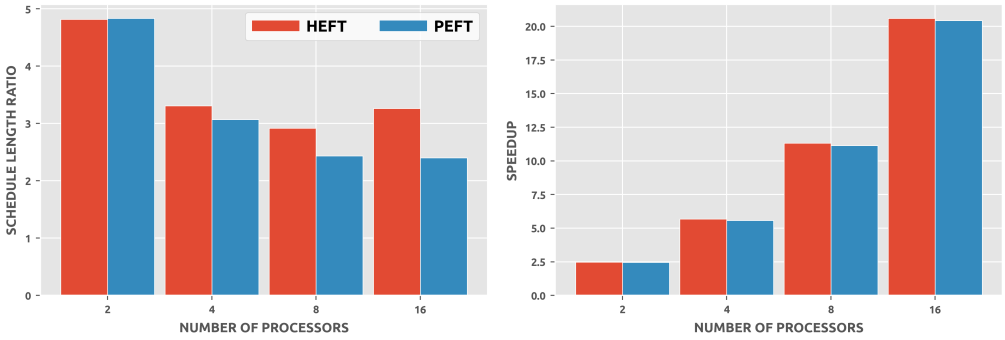
[2] `critical-path-estimation/scripts`

- $\beta \in \{0.1, 1, 10\}$, the computation-to-communication ratio (CCR), defined as in the previous chapter.
- $h \in \{1.0, 2.0\}$, the *heterogeneity factor* of the processors, which is defined below.

Once all of the parameters are specified for a graph, we proceed as follows. First, we select a value $\overline{p}$ uniformly at random from the interval $[1, 100]$ and then for each processor p_a, $a = 1, \ldots, q$, we realize a parameter r_a that we call the *power* from a uniform distribution over the interval $\left[\overline{p}(1 - h/2),\ \overline{p}(1 + h/2)\right]$. Here, the power of a processor roughly corresponds to its processing speed, with the parameter h controlling how similar powers—and therefore task computation costs on different processors—are to one another. By construction, $h \in [0, 2]$, so we consider $h = 1.0$ and $h = 2.0$ in our experiments in order to capture the extremes (disregarding the case $h = 0$). In practice, task computation costs are rarely determined entirely by processor speed, so we actually generate them according to the power as follows. First, an average task cost for the DAG $\overline{t}$ is selected at random from $[1, 100]$ and then for each task t_i, $i = 1, \ldots, n$, we choose a value x_i from $[0, 2\overline{t}]$. Next, for all $a = 1, \ldots, q$, we realize $g_a \sim \Gamma(1, r_a)$ (i.e., choose g_a from a Gamma distribution with mean and variance r_a). A Gamma distribution was chosen because it is always positive and fairly heavy-tailed, but of course other choices are possible. Finally, we let $W_i^a = x_i g_a$ be the computation cost of task t_i on processor p_a.

This approach differs from that used in the original HEFT [12] and PEFT [1] papers, where task computation costs were always generated independently of the processor they represent. That may be unrealistic in many applications since the cost of a task on two or more processors typically depends to some extent on their relative processing speed, although that is not to say that the method used here is necessarily more realistic in general. At any rate, most of the experiments described in this section were also repeated for the alternative method, with substantially similar results.

Communication costs however are generated independently of both processor power and the computation costs. After all of the latter have been determined, we compute an average edge cost $\overline{e}$ such that the specified CCR would be achieved. Then for all edges (t_i, t_k), we realize $\overline{w_{ik}}$ from a $U[0, 2\overline{e}]$ distribution. Specific communication costs between the tasks on all possible pairs of distinct processors are in turn selected uniformly at random from the interval $\left[\overline{w_{ik}}(1 - h/2),\ \overline{w_{ik}}(1 + h/2)\right]$. (Recall that communication costs are assumed to be zero when both tasks are scheduled on the same processor.) Note that the randomness here means that sometimes the target CCR is not precisely achieved, although it is usually acceptably close. We consider the three cases $\beta = 0.1$, $\beta = 1$ and $\beta = 10$ here because these are order of magnitude variations that correspond to graphs for which communication predominates, computation and communication are roughly equal, and communication is low.



(a) SLR. (b) Speedup.

Figure 6: SLR and speedup by processor number q.

For each DAG and each combination of the three parameters q, β and h, we make five copies of the graph and generate a different set of costs for each. There are $4 \times 3 \times 2 = 24$ possible combinations and 360 distinct DAG topologies, so altogether our DAG test set therefore contains $5 \times 24 \times 360 = 43200$ graphs.

5.2 Benchmarking

Before evaluating the variants we have proposed here, we want to establish how effective the existing heuristics HEFT and PEFT actually are for our test set, both in the abstract and relative to one another. As described in the previous chapter, two widely-used metrics in this area are the *speedup* and the *schedule length ratio* (SLR), which measure the quality of a given schedule relative to the best that can be achieved, respectively, on any single processor of the target platform or an idealization of the platform in which all processor contention is ignored. Figure 6 shows how the average SLR and speedup for HEFT and PEFT vary with the number of processors q. (All other parameters are combined so that each pair of blue and red bars represent averages over $43200/4 = 10800$ graphs.)

The figure shows roughly what we would hope: both HEFT and PEFT seem to do very well on average, as far as we can discern from these metrics alone: for example, the average speedup for both in all cases is superlinear[3]. With regards to their comparative performance, we see that the margins are very tight, at least on average, in terms of both metrics. Recalling that, put simply, larger values are good for speedup and bad for SLR, across the entire DAG set, PEFT obtains an average SLR of 3.2, compared to 3.6 for HEFT, whereas HEFT marginally shades PEFT with regard to average speedup at 10 against 9.9. Of course, these average values obscure significant variation: from the figure alone we can see that, for

[3]As explained in the previous chapter, it is entirely possible that speedup exceed the number of processors q in heterogeneous DAG scheduling problems, especially when we are considering the idealized static problem as here.

Table 4: Percentage of DAGs for which HEFT and PEFT failed.

CCR:	0.1	1.0	10.0
HEFT	11.5	1.8	0.2
PEFT	6.7	0.6	0.1

example, the SLR difference between PEFT and HEFT increases with q, which is in line with what has been reported elsewhere [1]. Overall, if we compare the schedules computed by the two heuristics directly, PEFT obtained an average makespan reduction of about 0.7% compared to HEFT, which is fairly small but may be worthwhile in some situations, depending on the relative cost of the two heuristics themselves.

In fact, PEFT was bettered by HEFT 54% of the time, across the set as a whole, and was only likely to be superior for a few parameter choices—in particular, when $\beta = 0.1$, $n = 100$ or $q = 16$. In all such cases, this was largely down to the phenomenon remarked upon in the previous chapter, where HEFT fails altogether—i.e., returns a schedule makespan greater than the minimal serial time—due to difficulty managing communication costs because of its greedy processor selection phase. Table 4 summarizes the failure probabilities for HEFT and PEFT as the CCR varies. The failure rate for PEFT is always smaller than HEFT, as we may well expect given that one interpretation of its processor selection phase is as a kind of correction preventing locally greedy actions which may be detrimental in the long run. However it still may be unacceptably high, so one of the aims of our investigation is to establish whether any of the variants proposed in Section 4 reduce the failure rate further.

While we largely avoid discussion of comparative runtimes here because our code is not optimized, we should note that our implementation of PEFT could be significantly more expensive than HEFT, especially for larger values of n and q.

Given what we have observed in these benchmarking experiments, we decided to investigate the proposed alternative HEFT task ranking phases and PEFT variants separately, which we do in the following two sections, respectively. We then make direct comparisons and draw overall conclusions in Section 5.5.

5.3 Ranking phases in HEFT

We compare the performance of HEFT with all of the following task prioritization phases:

- U, the standard upward ranks u_i,

- LB, the optimistic ranks ℓ_i as defined by (2),
- F, the Fulkerson ranks f_i as defined by (9),
- W, the weighted mean ranks $\hat{u}_i$ as defined in Section 3.2,
- WF, the weighted Fulkerson numbers $\hat{f}_i$ as defined in Section 3.2,
- R, which returns an effectively *random* priority list and is defined more precisely below.

What we really want to know when evaluating a given task ranking scheme in HEFT is how good the task priority list is compared to other *topological sorts*—orderings which respects precedence constraints. Now, the `Networkx` function `topological_sort` returns a topologically sorted list of nodes in an input graph. Furthermore, the algorithm which this implements does not consider any objective other than meeting the precedence constraints, so we can in some sense regard this as a random sample from the set of all possible topological sorts of the tasks. Hence we refer to the ranking phase defined by using the task ordering returned by `topological_sort` as R, and we include it in this comparison as a baseline to gauge the effectiveness of the other task prioritization phases.

5.3.1 Small-scale testing

Our implementations of the F and WF ranking phases are somewhat slow for DAGs with $n = 1000$ and $q = 16$, so before investing considerable computational time we decided to compare the six task prioritization phases on a smaller-scale using only the subset of DAGs with $n = 100$ and $q < 16$, which comprised 16200 DAGs in total. Since we want to directly compare the different rankings, the two main metrics we use here are the *average percentage degradation* (APD), as defined in the previous chapter, and the percentage of instances for which a given ranking achieved the best schedule for the DAGs.

Figure 7 shows the APD for all six of the ranking phases, broken down according to the CCR (so that each of the bars represents an average over $16200/3 = 5400$ DAGs). We have chosen to present the data in this way because it illustrates the fact that none of the new rankings were able to significantly reduce the failure rate compared to the default: when $\beta = 0.1$, the failure rates were all high so the other 5 rankings did not significantly outperform the R ranking phase, whereas they clearly did for the higher CCR values, when their failure rates were much lower. As to the comparative performance of the rankings, W and WF were the two best overall and did almost exactly as well as each other. LB's performance relative to U was mixed, while F was worse than U for all three CCR values.

Of course, the APD is only an average value; we also want to know how likely a given ranking is to achieve the best schedule for a given DAG, and, for the

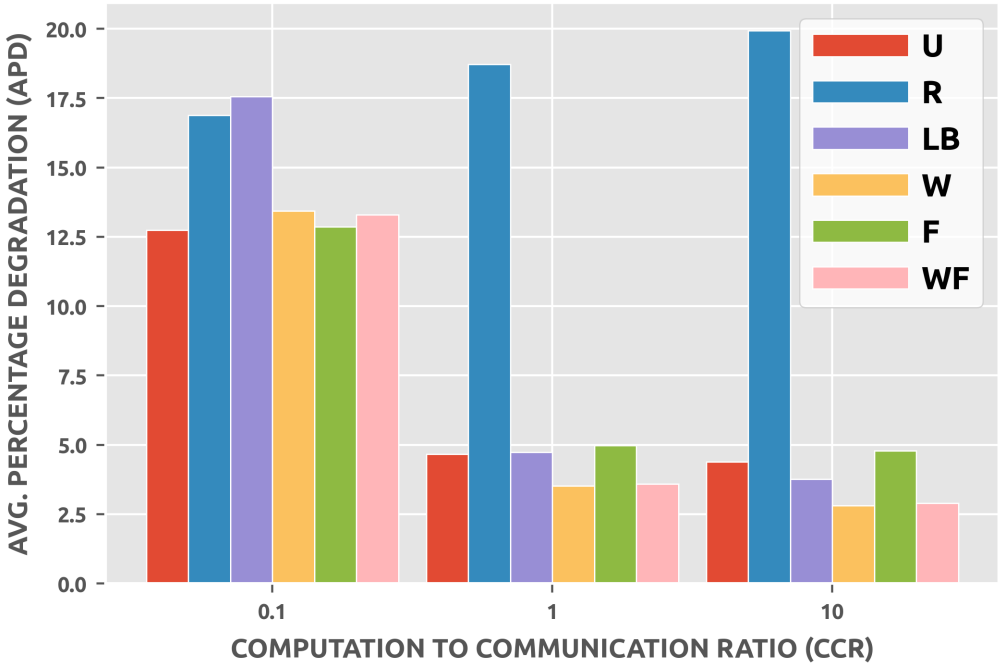


Figure 7: Average percentage degradation for the 6 ranking phases by CCR.

Table 5: Number of best instances and improvements vs default for alternative ranking phases.

RANKING:	U	R	LB	W	F	WF
% BEST	19.5	3.5	22.7	27.6	18.0	27.6
% < U	-	10.2	48.7	56.1	39.8	55.8

alternative rankings, whether they are likely to improve on the standard upward ranking prioritization phase. Hence in Table 5 we present the percentage of the 16200 DAGs for which each of the rankings (including U) achieved the best schedule and the percentage of cases for which each of the alternative rankings produced a schedule better than U. We see again that the W and WF rankings are the best overall, although no single ranking dominated all others.

The results of our small-scale testing indicate that the Fulkerson ranking F in particular does not seem to be an improvement on the standard prioritization phase. Given that it is likely to be much more expensive—considerably so in our implementation—this is a big problem: only large makespan reductions can really justify the extra cost. Indeed, although the weighted Fulkerson ranking WF did much better, both compared to F and the standard U, it was not a significant im-

provement over the much cheaper W ranking. Ultimately we conclude that there is little advantage in obtaining tighter bounds on the critical path lengths of the stochastic graph G_s (see Section 3.1) but simply weighting the averages appears to be beneficial. Therefore, in the following section, we extend the comparison to the full test DAG set but no longer consider the F and WF rankings.

Despite this, it should be noted that there may be circumstances in which these rankings based on the Fulkerson numbers are competitive with the standard ranking in terms of both cost and performance. In particular, for accelerated environments, such as those considered in the previous chapter, the set of possible values each weight may take is much smaller, which significantly improves efficiency. A more detailed investigation of the performance of F and WF for these environments may therefore be useful in the future.

5.3.2 The full set

We repeated the comparison described in the previous section for the U, LB and W rankings on the full set of 43200 DAGs. In addition to F and WF, we also omitted the R ranking, as it is only useful for gauging the effect of failures and we have already established that all of the rankings fail at about the same rate. Figure 8 shows the APD for the three rankings, broken down by the number of processors q. We see that the W ranking dominates the others, being the best for all four values of q, whereas the LB ranking is worse than U for all but the largest processor number. Furthermore, the W ranking achieved the best schedule 47% of the time and was outright superior to U for 60% of DAGs, while the corresponding figures for LB were 32% and 51%, respectively.

Taking the results of this section and the previous one as a whole, it appears that the W ranking is likely to lead to a smaller schedule than the standard ranking U, although there is a lot of variation depending on the parameter values, whereas the LB ranking was typically competitive with U but no more likely to achieve a superior schedule.

5.4 Processor selection

Given the superior performance of the W ranking in the previous section, the three PEFT variants we consider here all use the task prioritization given by (18) and are therefore defined by how they compute the conditional critical paths:

- PEFT-LB, using (14).
- PEFT-M, using (15).
- PEFT-WM, using (16).

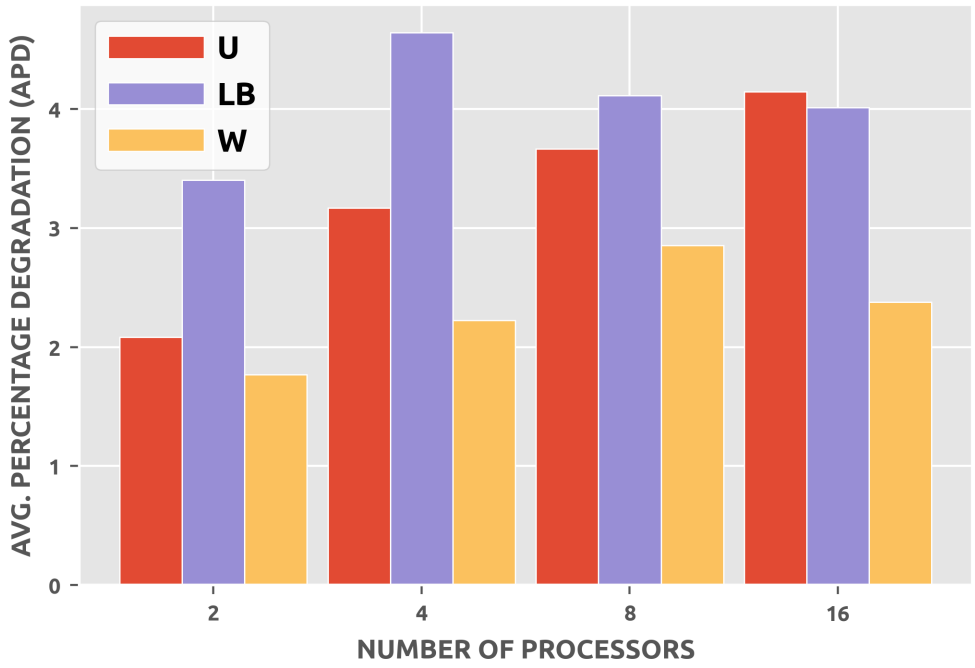


Figure 8: Average percentage degradation for U, LB and W rankings across the entire test DAG set, displayed by number of processors q.

There are really two questions we want to answer here: are any of these alternatives superior to PEFT, and is using the conditional critical path lengths in the processor selection phase actually useful at all?

With regards to the former, Figure X illustrates the percentage of instances across the DAG test set for which each of the three variants achieved a superior schedule to the standard PEFT algorithm, broken down by processor number q. Overall PEFT-LB is most likely to improve on the default, obtaining a better schedule 55% of the time across the entire set, with PEFT-WM second at 53%. One of the questions prompted by the benchmarking experiments in Section 5.2 was whether any of these variants could reduce the number of failures, but in fact all of them had a higher failure rate. PEFT-LB's was only slightly higher but the M and WM variants both recorded almost double the number of failures—this equates to about 5% of the entire DAG set, concentrated in the same parameter regimes identified earlier. Furthermore, when they failed, they could do so almost arbitrarily badly, which is why we chose to display the percentage of better instances in the figure rather than the APD: sometimes the makespan could be literally thousands of times worse, leading to a distorted APD value...

5.5 Conclusions

To summarize, our main conclusions from the experimental comparison of the alternative HEFT task prioritization phases are as follows.

- Largely supporting previous investigations along these lines [14], perhaps our biggest takeaway is that no method appears to be dominant and it is impossible to say, for a given DAG and target platform, which should be used.
- Having said that, the W ranking was consistently more likely to result in a smaller makespan across the set of test DAGs as a whole and many of the subsets. The usual caveats about the limitations of our experimental framework hold of course but overall we would softly recommend it be used instead of the default, particularly since it is conceptually simple (just a weighted mean) and not significantly more expensive.
- There is no clear benefit in obtaining tighter bounds on the associated stochastic graph (i.e., Fulkerson's bound). Given the higher time complexity in the general case, this does not therefore seem to be a worthwhile alternative.
- Similarly, there appears to be little advantage in using the LB ranking, although it also wasn't clearly inferior to the standard ranking in terms of performance or cost.

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