Lesson 9

Euler-Bernoulli Beam

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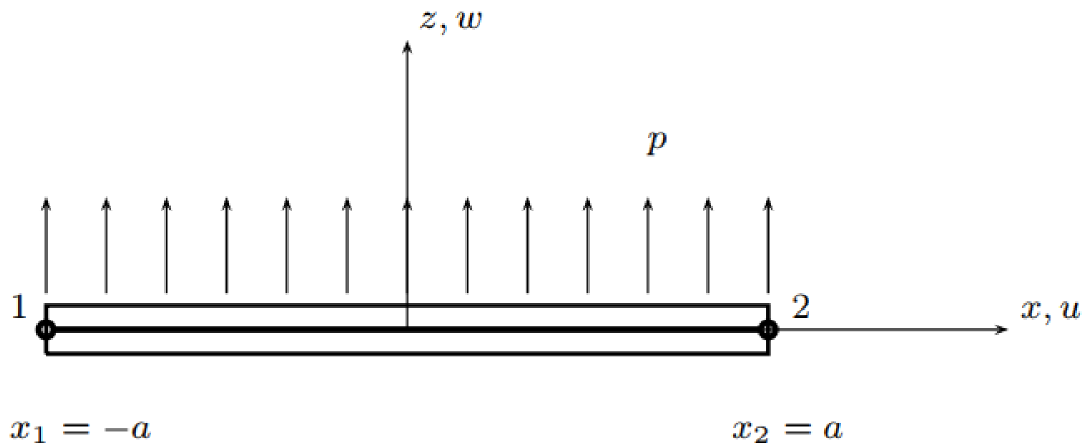
May 31, 2023

Content

1. Bernoulli Beam – Formulation

- A classical beam theory where the transverse shear deformation is neglected.
- The beam is defined in the $x-z$ plane, with constant cross-section area A.
- The transverse deflection w is the only degree of freedom of the model.
- The plane sections remain plane and normal to the axis of the beam after deformation.
- The in-plane rotation given by the derivative of the transverse deflection w.r.t the beam axis.



- The displacement u, at a distance z of the beam middle axis is given by

$$u = -z\frac{\partial w}{\partial x}$$

- Strains are defined as $\varepsilon_x = \dfrac{\partial u}{\partial x} = -z\dfrac{\partial^2 w}{\partial x^2}; \quad \gamma_{xz} = \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x} = 0$

1. Bernoulli Beam – Formulation

- The strain energy is

$$U = \frac{1}{2}\int_V \sigma_x \varepsilon_x dV = \frac{1}{2}\int_V E\varepsilon_x^2 dV = \frac{1}{2}\int_A \int_{-a}^{a} E\left(\frac{\partial^2 w}{\partial x^2}\right)^2 z^2 dA dx = \frac{1}{2}\int_{-a}^{a} EI_y \left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx$$

with I_y *is the second moment of area of the beam cross-section.*

- The kinetic energy is

$$K = \frac{1}{2}\int_V \left(\rho \dot{u}^2 + \rho \dot{w}^2\right) dV = \frac{1}{2}\int_{-a}^{a} \left(\rho I_y \left(\frac{\partial \dot{w}}{\partial x}\right)^2 + \rho A \dot{w}^2\right) dx$$

The first term is the rotary inertia, can be neglected for thin beams.
The second term is the vertical bulk inertia.

- The external work due to the transverse pressure p and the axial load N_0 (that accounts for nonlinear Von Karman strains) is

$$\delta W = \int_{-a}^{a} p\,\delta w\, dx - \int_{-a}^{a} N_0 \frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial x} dx$$

- At each node we consider 2 DOFs: w and $\frac{\partial w}{\partial x}$, respectively are the transverse displacement and rotation of the cross section.

$$\mathbf{w}^{eT} = \begin{bmatrix} w_1 & \frac{\partial w_1}{\partial x} & w_2 & \frac{\partial w_2}{\partial x} \end{bmatrix}$$

1. Bernoulli Beam – Formulation

- The transverse displacement is interpolated using Hermite shape functions as

$$w = \mathbf{N}(\xi)\mathbf{w}^e$$

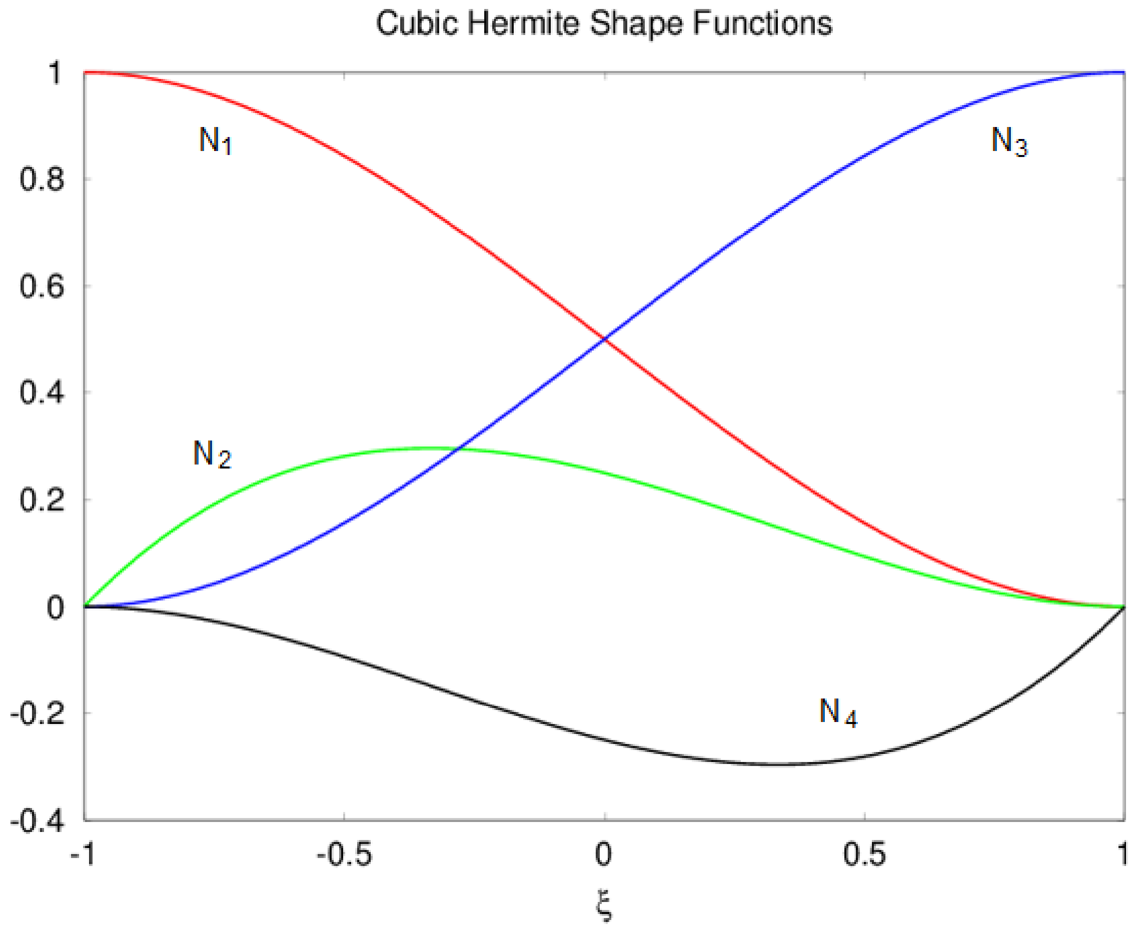
The shape functions are

$$N_1(\xi) = \frac{1}{4}\left(2 - 3\xi + \xi^3\right)$$

$$N_2(\xi) = \frac{a}{4}\left(1 - \xi - \xi^2 + \xi^3\right)$$

$$N_3(\xi) = \frac{1}{4}\left(2 + 3\xi - \xi^3\right)$$

$$N_4(\xi) = \frac{a}{4}\left(-1 - \xi + \xi^2 + \xi^3\right)$$



1. Bernoulli Beam – Formulation

- The transverse displacement is interpolated using Hermite shape functions as

$$w = \mathbf{N}(\xi)\mathbf{w}^e$$

The shape functions are defined as

$$N_1(\xi) = \frac{1}{4}\left(2 - 3\xi + \xi^3\right); \quad N_2(\xi) = \frac{a}{4}\left(1 - \xi - \xi^2 + \xi^3\right)$$

$$N_3(\xi) = \frac{1}{4}\left(2 + 3\xi - \xi^3\right); \quad N_4(\xi) = \frac{a}{4}\left(-1 - \xi + \xi^2 + \xi^3\right)$$

with $\xi = x/a$ *identifies the dimensionless axial coordinate,* $\xi \in [-1, 1]$.

thus $$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x} = \frac{1}{a}\frac{\partial}{\partial \xi}; \quad \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right) = \frac{1}{a}\frac{\partial}{\partial \xi}\left(\frac{1}{a}\frac{\partial}{\partial \xi}\right) = \frac{1}{a^2}\frac{\partial^2}{\partial \xi^2}$$

- The strain energy is obtained as

$$U = \frac{1}{2}\int_{-a}^{a} EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx = \frac{1}{2}\int_{-1}^{1}\frac{EI_y}{a^4}\left(\frac{\partial^2 w}{\partial \xi^2}\right)^2 a\,d\xi = \frac{1}{2}\mathbf{w}^{eT}\frac{EI_y}{a^3}\int_{-1}^{1}\mathbf{N}''^T\mathbf{N}''\,d\xi\,\mathbf{w}^e \quad \textit{with} \quad \mathbf{N}'' = \frac{\partial^2 \mathbf{N}}{\partial \xi^2}$$

- The element stiffness matrix is obtained as

$$\mathbf{K}^e = \frac{EI_y}{a^3}\int_{-1}^{1}\mathbf{N}''^T\mathbf{N}''\,d\xi = \frac{EI_y}{2a^3}\begin{bmatrix} 3 & 3a & -3 & 3a \\ 3a & 4a^2 & -3a & 2a^2 \\ -3 & -3a & 3 & -3a \\ 3a & 2a^2 & -3a & 4a^2 \end{bmatrix}$$

1. Bernoulli Beam – Formulation

- The kinetic energy takes the form

$$K = \frac{1}{2}\int_{-a}^{a} \rho A \dot{w}^2 dx = \frac{1}{2}\int_{-1}^{1} \rho A a \dot{w}^2 d\xi = \frac{1}{2}\dot{\mathbf{w}}^{eT}\int_{-1}^{1} \rho A a \mathbf{N}^T \mathbf{N} d\xi\, \dot{\mathbf{w}}^e$$

The mass matrix is

$$\mathbf{M}^e = \int_{-1}^{1} \rho A a \mathbf{N}^T \mathbf{N} d\xi = \frac{\rho A a}{210}\begin{bmatrix} 156 & 44a & 54 & -26a \\ 44a & 16a^2 & 26a & -12a^2 \\ 54 & 26a & 156 & -44a \\ -26a & -12a^2 & -44a & 16a^2 \end{bmatrix}$$

- The work done by the distributed forces p is

$$\delta W_1^e = \int_{-a}^{a} p\,\delta w\,dx = \int_{-1}^{1} p\,\delta w\; a\,d\xi = \delta \mathbf{w}^{eT} a \int_{-1}^{1} p \mathbf{N}^T d\xi$$

The vector of nodal forces equivalent to distributed uniform forces p is obtained as

$$\mathbf{f}^e = ap\int_{-1}^{1} \mathbf{N}^T d\xi = \frac{ap}{3}\begin{bmatrix} 3 \\ a \\ 3 \\ -a \end{bmatrix}$$

1. Bernoulli Beam – Formulation

❖ The work done by the axial force N^0 is defined as

$$\delta W_2^e = \int_{-a}^{a} N_0 \frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial x} dx = \int_{-1}^{1} \frac{N_0}{a^2}\frac{\partial w}{\partial \xi}\frac{\partial \delta w}{\partial \xi} a d\xi = \delta \mathbf{w}^{eT} \frac{N_0}{a}\int_{-1}^{1} \mathbf{N}'^T \mathbf{N}' d\xi \mathbf{w}^e$$

The stability matrix is defined as

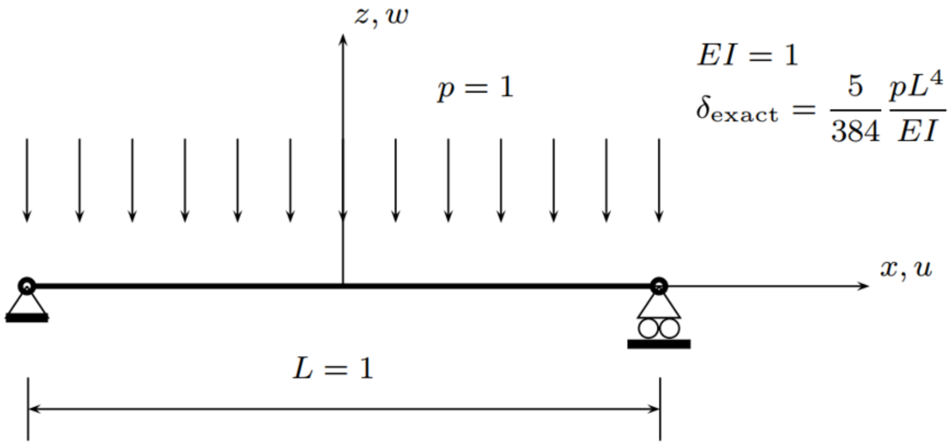
$$\mathbf{G}^e = \frac{1}{a}\int_{-1}^{1} \mathbf{N}'^T \mathbf{N}' d\xi = \frac{1}{60a}\begin{bmatrix} 36 & 6a & -36 & 6a \\ 6a & 16a^2 & -6a & -4a^2 \\ -36 & -6a & 36 & -6a \\ 6a & -4a^2 & -6a & 16a^2 \end{bmatrix}$$

❖ After assembly, the algebraic solving system is

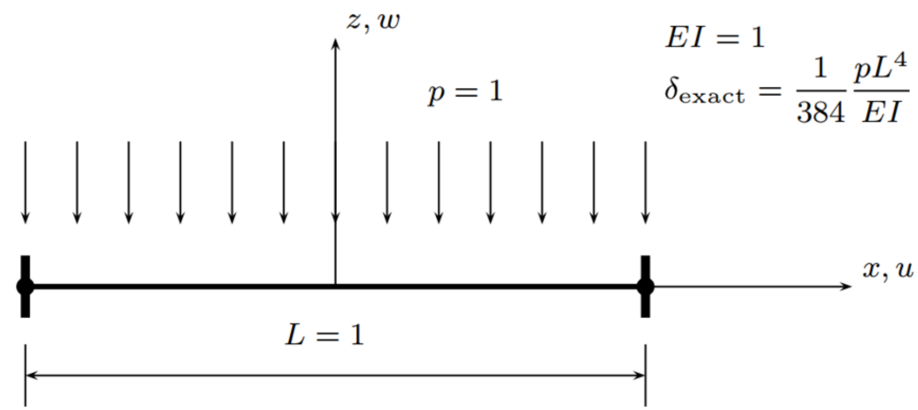
$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} - N_0 \mathbf{G}^e \mathbf{u} = \mathbf{f}$$

From this, the static, buckling, and free vibration problems of Bernoulli beam can be solved.
The buckling and free vibration problems are both solved in the form of an eigenvalue problem.

1. Bernoulli Beam – Bernoulli Beam Problem - Python



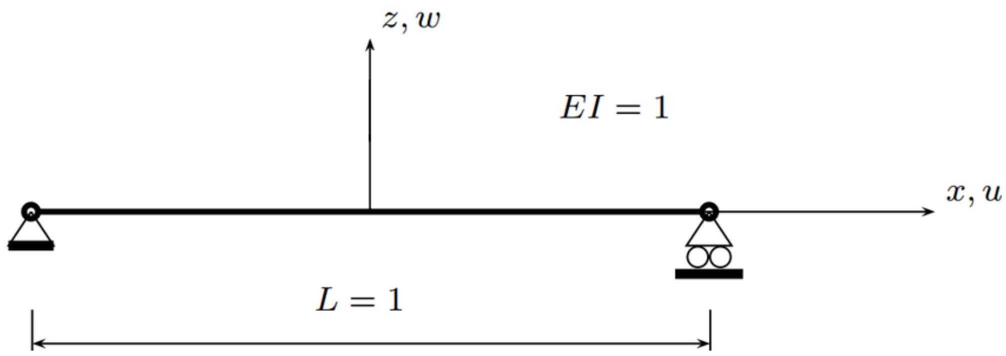
(Simply-supported, uniform load)



(Clamped, uniform load)

❖ In the Python codes, students can define the number of elements.

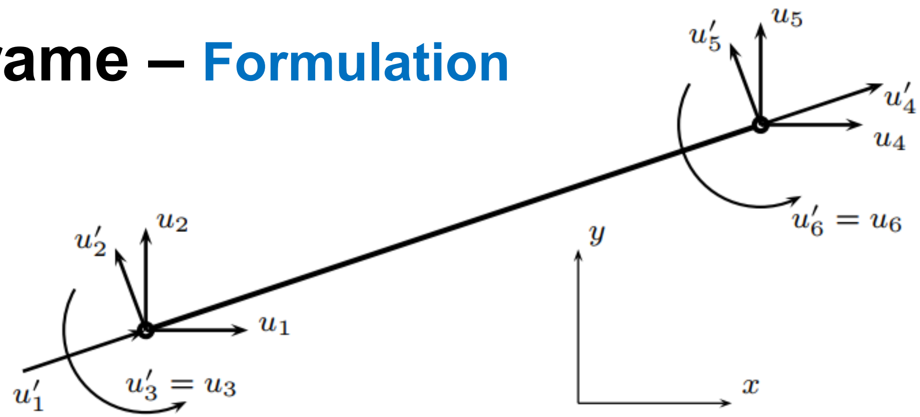
1. Bernoulli Beam – Bernoulli Beam Free Vibrations



(Simply-supported)

❖ In the free vibration, there is no external load.

2. Bernoulli 2D Frame – Formulation



- The previous Bernoulli beam formulations in local coordinates will be generalized.
- The local stiffness and mass matrices are transformed by a matrix of rotation which is a function of the beam slope w.r.t the horizontal axis.
- Each node has 3 DOFs (i.e., 2 displacements and 1 rotation)
- The vector of local displacements is given $\mathbf{u}'^T = \begin{bmatrix} u_1' & u_4' & u_2' & u_5' & u_3' = u_3 & u_6' = u_6 \end{bmatrix}$
- The vector of global displacements is given $\mathbf{u}^T = \begin{bmatrix} u_1 & u_4 & u_2 & u_5 & u_3 = u_3' & u_6 = u_6' \end{bmatrix}$
- The relation between local and global displacement $\mathbf{u}' = \mathbf{L}\mathbf{u}$

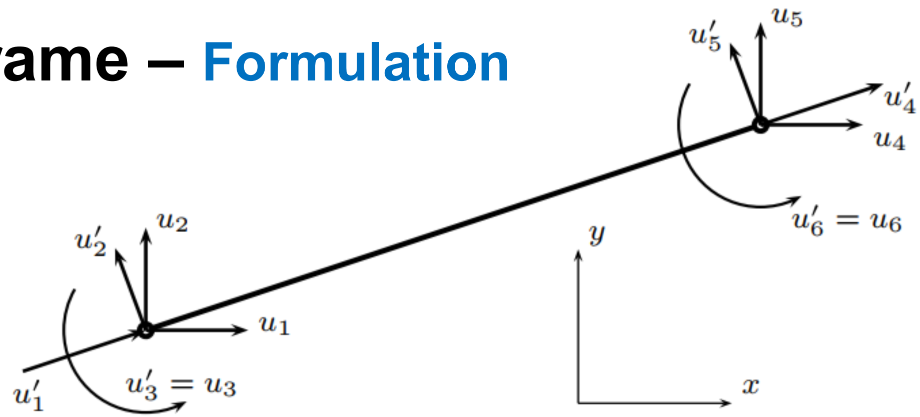
with $\mathbf{L} = \begin{bmatrix} l & 0 & m & 0 & 0 & 0 \\ 0 & l & 0 & m & 0 & 0 \\ -m & 0 & l & 0 & 0 & 0 \\ 0 & -m & 0 & l & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ *and*

$$l = \cos\theta = \frac{x_2 - x_1}{L_e}$$

$$m = \sin\theta = \frac{y_2 - y_1}{L_e}$$

$$L_e = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

2. Bernoulli 2D Frame – Formulation



❖ In local coordinates, the DOFs of the frame element are the combination of the DOFs of a bar element u_1' and the DOFs of a Bernoulli beam element u_2' and u_3'. Thus, the local stiffness matrix of the frame element is the combination of the stiffness of the bar element and the Bernoulli beam element.

$$\mathbf{K}'^e = \frac{E}{L^3}\begin{bmatrix} AL^2 & -AL^2 & 0 & 0 & 0 & 0 \\ & AL^2 & 0 & 0 & 0 & 0 \\ & & 12I & -12I & 6IL & 6IL \\ & & & 12I & -6IL & -6IL \\ & & & & 4IL^2 & 2IL^2 \\ sym & & & & & 4IL^2 \end{bmatrix}$$

Note: the new numbering of global DOFs in the frame element and the Bernoulli beam element when assembling the stiffness matrix.

$$\mathbf{w}^{eT} = \left[u_2' = w_1 \quad u_3' = \frac{\partial w_1}{\partial x} \quad u_5' = w_2 \quad u_6' = \frac{\partial w_2}{\partial x} \right]$$

$$\mathbf{u}^T = \left[u_1 \quad u_4 \quad u_2 \quad u_5 \quad u_3 = u_3' \quad u_6 = u_6' \right]$$

❖ The strain energy is transformed to global coordinates as

$$U^e = \frac{1}{2}\mathbf{u}'^T\mathbf{K}'^e\mathbf{u}' = \frac{1}{2}\mathbf{u}^T\mathbf{L}^T\mathbf{K}'^e\mathbf{L}\mathbf{u} = \frac{1}{2}\mathbf{u}^T\mathbf{K}\mathbf{u}$$ *with the global stiffness of the frame element* $\mathbf{K} = \mathbf{L}^T\mathbf{K}'^e\mathbf{L}$

2. Bernoulli 2D Frame – Formulation

❖ The local mass matrix of the frame element is the combination of the mass of the bar element and the Bernoulli beam element.

$$\mathbf{M}'^e = \frac{\rho AL}{420}\begin{bmatrix} 140 & 70 & 0 & 0 & 0 & 0 \\ & 140 & 0 & 0 & 0 & 0 \\ & & 156 & 54 & 22L & -13L \\ & & & 156 & 13L & -22L \\ & & & & 4L^2 & -3L^2 \\ sym & & & & & 4L^2 \end{bmatrix}$$

Note: the new numbering of global DOFs in the frame element and the Bernoulli beam element when assembling the stiffness matrix.

$$\mathbf{w}^{eT} = \left[u_2' = w_1 \quad u_3' = \frac{\partial w_1}{\partial x} \quad u_5' = w_2 \quad u_6' = \frac{\partial w_2}{\partial x} \right]$$

$$\mathbf{u}^T = \left[u_1 \quad u_4 \quad u_2 \quad u_5 \quad u_3 = u_3' \quad u_6 = u_6' \right]$$

❖ In global coordinates, the kinetic energy is given by

$$M^e = \frac{1}{2}\mathbf{u}'^T\mathbf{M}'^e\mathbf{u}' = \frac{1}{2}\mathbf{u}^T\mathbf{L}^T\mathbf{M}'^e\mathbf{L}\mathbf{u} = \frac{1}{2}\mathbf{u}^T\mathbf{M}\mathbf{u}$$ *with the global mass of the frame element* $\mathbf{M} = \mathbf{L}^T\mathbf{M}'^e\mathbf{L}$

❖ The global load vector is also defined according to the order of the global DOFs

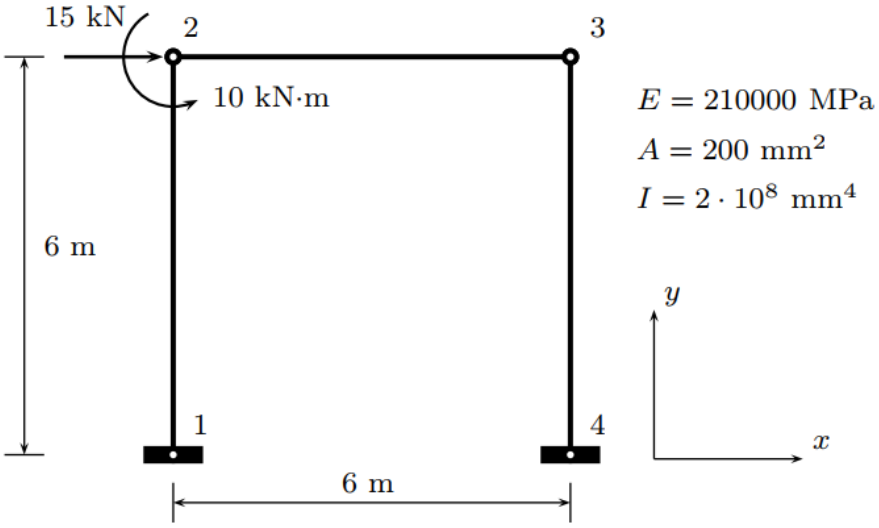
$$\mathbf{F} = \left[F_{x,1} \quad \ldots \quad F_{x,n} \quad F_{y,1} \quad \ldots \quad F_{y,n} \quad M_1 \quad \ldots \quad M_n \right]$$

F_x, F_y, M *are the horizontal and vertical concentrated forces and moments applied at the nodes.*

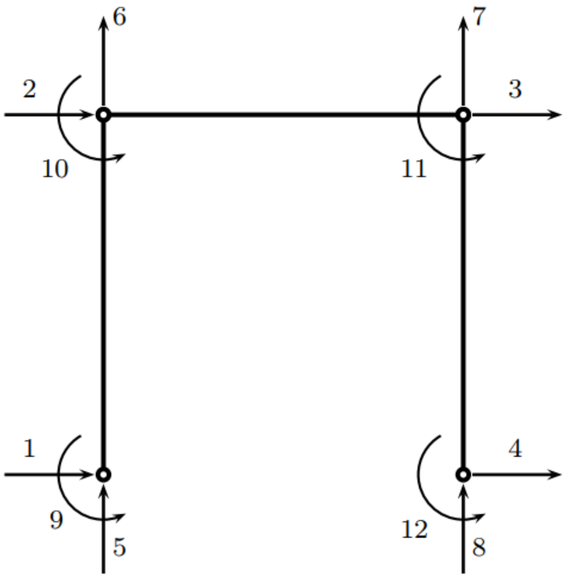
n *is the number of nodes in the mesh*

❖ Static and free vibration problems when $\mathbf{F} = \mathbf{0}$.

2. Bernoulli 2D Frame – 2D Frame Problem - Python

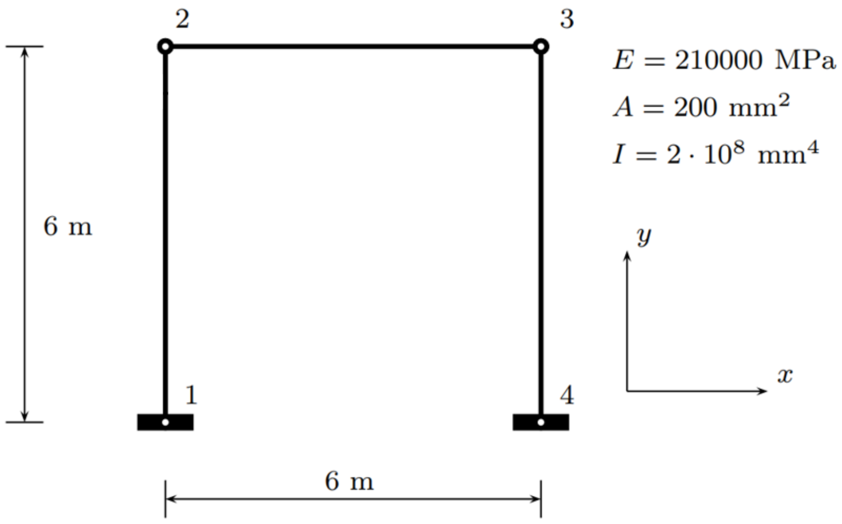


(A 2D frame: geometry, materials, and loads)

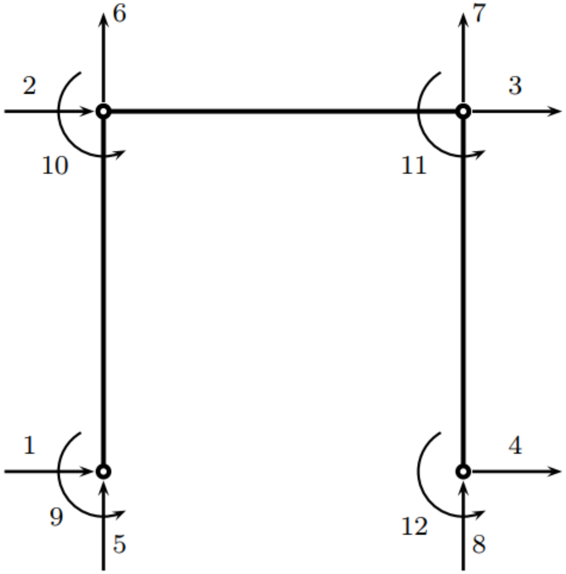


(DOF ordering)

2. Bernoulli 2D Frame – 2D Frame in Free Vibration - Python

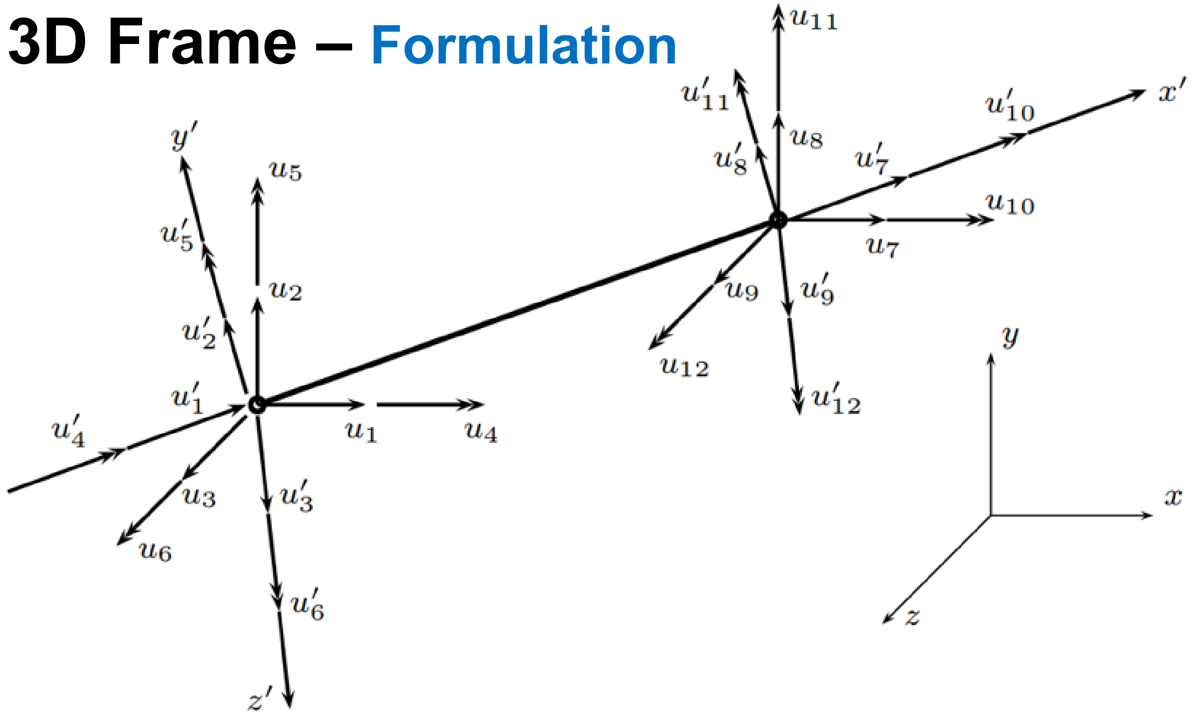


Density: $\rho = 8.05 \cdot 10^{-9}\ ton/mm^3$



(DOF ordering)

3. Bernoulli 3D Frame – Formulation



- In global coordinates, each node has 3 displacement DOFs and 3 rotation DOFs.
- More complex than 2D beam element due to the orientation of the beam in 3D space.
- The local axis x' is aligned with the beam major axis.
- The direction cosines according to axis x' is straightforward and follows that of 2D frame.

$$C_{xx'} = \frac{x_2 - x_1}{L_e}; \quad C_{yx'} = \frac{y_2 - y_1}{L_e}; \quad C_{zx'} = \frac{z_2 - z_1}{L_e}; \quad L_e = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

3. Bernoulli 3D Frame – Formulation

❖ The vector rotation matrix in 3D space

$$\mathbf{r}=\begin{bmatrix} C_{xx'} & C_{yx'} & C_{zx'} \\ C_{xy'} & C_{yy'} & C_{zy'} \\ C_{xz'} & C_{yz'} & C_{zz'} \end{bmatrix}=R_\alpha R_\beta R_\gamma$$

with $R_\alpha, R_\beta, R_\gamma$ *are three rotation matrices.*

α, β, γ *are the rotation angles about* x', y', z' *axes.*

- *Rotation about* z *axis,* R_γ *is given by*

$$R_\gamma=\begin{bmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with $\cos\gamma = C_{xx'}/C_{xy}$, $\sin\gamma = C_{yx'}/C_{xy}$, $C_{xy}=\sqrt{C_{xx'}^2+C_{yx'}^2}$

- *Rotation about* y *axis,* R_β *is given by*

$$R_\beta=\begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$

with $\cos\beta = C_{xy}$, $\sin\beta = C_{yx'}$

- *Combining the rotation about* y *and* z *axis, the vector rotation matrix is*

$$\mathbf{r}=R_\beta R_\gamma=\begin{bmatrix} C_{xx'} & C_{yx'} & C_{zx'} \\ -C_{yx'}/C_{xy} & C_{xx'}/C_{xy} & 0 \\ -C_{xx'}C_{zx'}/C_{xy} & -C_{yx'}C_{zx'}/C_{xy} & C_{xy} \end{bmatrix}$$

3. Bernoulli 3D Frame – Formulation

When $\beta = 90^0$ *or* $\beta = 270^0$*, the 2-node beam element change only along* z *axis, thus the vector rotation matrix takes a special form.*

$$\mathbf{r} = \begin{bmatrix} 0 & 0 & C_{zx'} \\ 0 & 1 & 0 \\ -C_{zx'} & 0 & 0 \end{bmatrix}$$

for $z_2 > z_1$ *or* $\beta = 90^0$, $C_{zx'} = 1$

for $z_1 > z_2$ *or* $\beta = 270^0$, $C_{zx'} = -1$

- *If the extra rotation* α *is included, the rotation matrix is* $R_\alpha = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$
- *Finally, the vector rotation matrix becomes*

$$\mathbf{r} = R_\alpha R_\beta R_\gamma = \begin{bmatrix} C_{xx'} & C_{yx'} & C_{zx'} \\ \left(C_{xx'}C_{zx'}\sin\alpha - C_{yx'}\cos\alpha\right)/C_{xy} & \left(C_{yx'}C_{zx'}\sin\alpha + C_{xx'}\cos\alpha\right)/C_{xy} & -C_{xy}\sin\alpha \\ \left(-C_{xx'}C_{zx'}\cos\alpha + C_{yx'}\sin\alpha\right)/C_{xy} & \left(-C_{yx'}C_{zx'}\cos\alpha + C_{xx'}\sin\alpha\right)/C_{xy} & C_{xy}\cos\alpha \end{bmatrix}$$

- *Special case of vertical members* $\beta = 90^0$ *and* $\beta = 270^0$ *can be derived as*

$$\mathbf{r} = \begin{bmatrix} 0 & 0 & C_{zx'} \\ C_{zx'}\sin\alpha & \cos\alpha & 0 \\ -C_{zx'}\cos\alpha & \sin\alpha & 0 \end{bmatrix}$$

3. Bernoulli 3D Frame – Formulation

❖ In the local coordinate system, the stiffness matrix is given by

$$\mathbf{K}' = \begin{bmatrix} EA/L & 0 & 0 & 0 & 0 & 0 & -EA/L & 0 & 0 & 0 & 0 & 0 \\ & 12EI_z/L^3 & 0 & 0 & 0 & 6EI_z/L^2 & 0 & -12EI_z/L^3 & 0 & 0 & 0 & 6EI_z/L^2 \\ & & 12EI_y/L^3 & 0 & -6EI_y/L^2 & 0 & 0 & 0 & -12EI_y/L^3 & 0 & -6EI_y/L^2 & 0 \\ & & & GJ/L & 0 & 0 & 0 & 0 & 0 & -GJ/L & 0 & 0 \\ & & & & 4EI_y/L & 0 & 0 & 0 & 6EI_y/L^2 & 0 & 2EI_y/L & 0 \\ & & & & & 4EI_z/L & 0 & -6EI_z/L^2 & 0 & 0 & 0 & 2EI_z/L \\ & & & & & & EA/L & 0 & 0 & 0 & 0 & 0 \\ & & & & & & & 12EI_z/L^3 & 0 & 0 & 0 & -6EI_z/L^2 \\ & & & & & & & & 12EI_y/L^3 & 0 & 6EI_y/L^2 & 0 \\ & & & & & & & & & GJ/L & 0 & 0 \\ & & & & & & & & & & 4EI_y/L & 0 \\ sym & & & & & & & & & & & 4EI_z/L \end{bmatrix}$$

❖ The global stiffness matrix is $\mathbf{K} = \mathbf{R}^T\mathbf{K}'\mathbf{R}$ *with* $\mathbf{R} = \begin{bmatrix} \mathbf{r} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{r} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{r} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{r} \end{bmatrix}$ *and* $\mathbf{r}$ *is defined previously.*

❖ After the static problem is solved, the reactions can be computed as $\mathbf{F} = \mathbf{KU}$.

❖ The element nodal forces can be evaluated by axes transformation as $\mathbf{f}_e = \mathbf{K}'\mathbf{R}\mathbf{U}_e$

with $\mathbf{f}_e$ *is the element nodal forces vector,* $\mathbf{U}_e$ *is the global vector of displacement of element e.*

3. Bernoulli 3D Frame – Formulation

❖ In the local coordinate system, the consistent and lumped mass matrices are defined as

$$\mathbf{M}'_C = \frac{\rho AL}{420}\begin{bmatrix} 140 & 0 & 0 & 0 & 0 & 0 & 70 & 0 & 0 & 0 & 0 & 0 \\ & 156 & 0 & 0 & 0 & 22L & 0 & 54 & 0 & 0 & 0 & -13L \\ & & 156 & 0 & -22L & 0 & 0 & 0 & 54 & 0 & 13L & 0 \\ & & & 140r_x^2 & 0 & 0 & 0 & 0 & 0 & 70r_x^2 & 0 & 0 \\ & & & & 4L^2 & 0 & 0 & 0 & -13L & 0 & -3L^2 & 0 \\ & & & & & 4L^2 & 0 & 13L & 0 & 0 & 0 & -3L^2 \\ & & & & & & 140 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & & 156 & 0 & 0 & 0 & -22L \\ & & & & & & & & 156 & 0 & 22L & 0 \\ & & & & & & & & & 140r_x^2 & 0 & 0 \\ & & & & & & & & & & 4L^2 & 0 \\ sym & & & & & & & & & & & 4L^2 \end{bmatrix} \quad \mathbf{M}'_L = \frac{\rho AL}{2}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & r_x^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & 1 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & & 1 & 0 & 0 & 0 & 0 \\ & & & & & & & & 1 & 0 & 0 & 0 \\ & & & & & & & & & r_x^2 & 0 & 0 \\ & & & & & & & & & & 0 & 0 \\ sym & & & & & & & & & & & 0 \end{bmatrix}$$

with $r_x^2 = \left(I'_y + I'_z\right)/A$ *and* I'_y, I'_z *are the second moment of area of the cross-section about the principal* y' *and* z' *axes.*

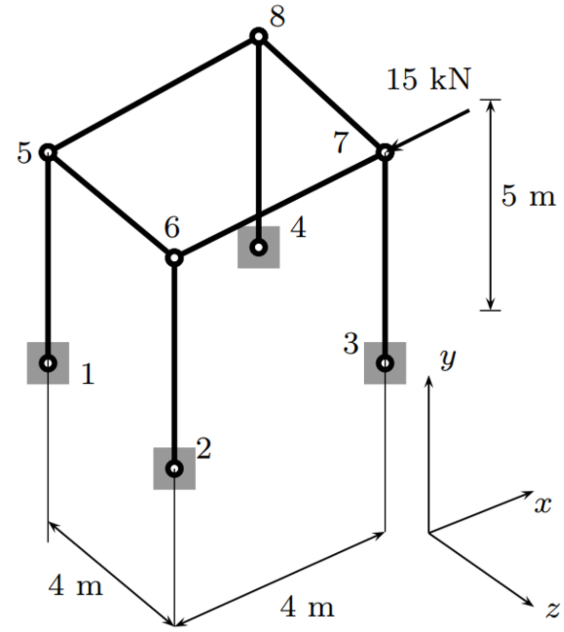
❖ The mass matrix in the global coordinate system takes the form $\mathbf{M} = \mathbf{R}^T\mathbf{M}'\mathbf{R}$

with $\mathbf{R}$ *is the rotation matrix defined previously.*

3. Bernoulli 3D Frame – 3D Frame Problem - Python

$E = 210\,GPa,\quad G = 84\,GPa,\quad A = 2\cdot 10^{-2}\,m^2$

$I_y = 1\cdot 10^{-4}\,m^4,\quad I_z = 2\cdot 10^{-4}\,m^4,\quad J = 0.5\cdot 10^{-4}\,m^4$

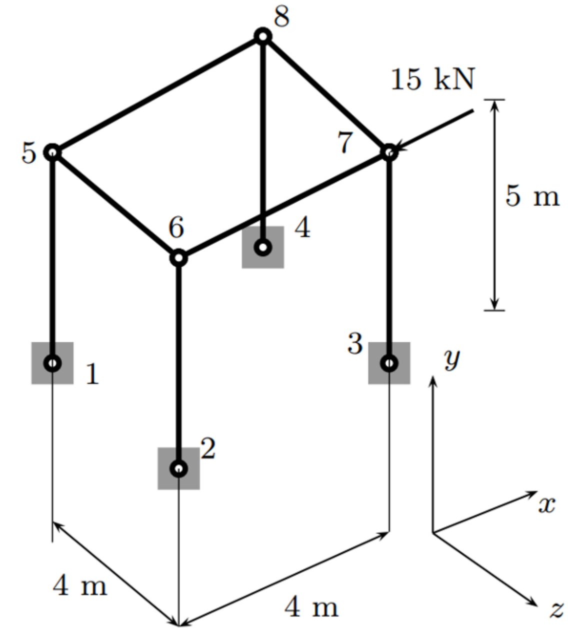


3. Bernoulli 3D Frame – 3D Frame in Free Vibrations - Python

$$E = 210\,GPa, \qquad G = 84\,GPa, \qquad A = 2\cdot 10^{-2}\,m^2$$

$$I_y = 1\cdot 10^{-4}\,m^4, \qquad I_z = 2\cdot 10^{-4}\,m^4, \qquad J = 0.5\cdot 10^{-4}\,m^4$$

$$\rho = 7850\ kg/m^3$$



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thank you