Lesson 7

Shape Functions
Bars or Trusses

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   - Requirements
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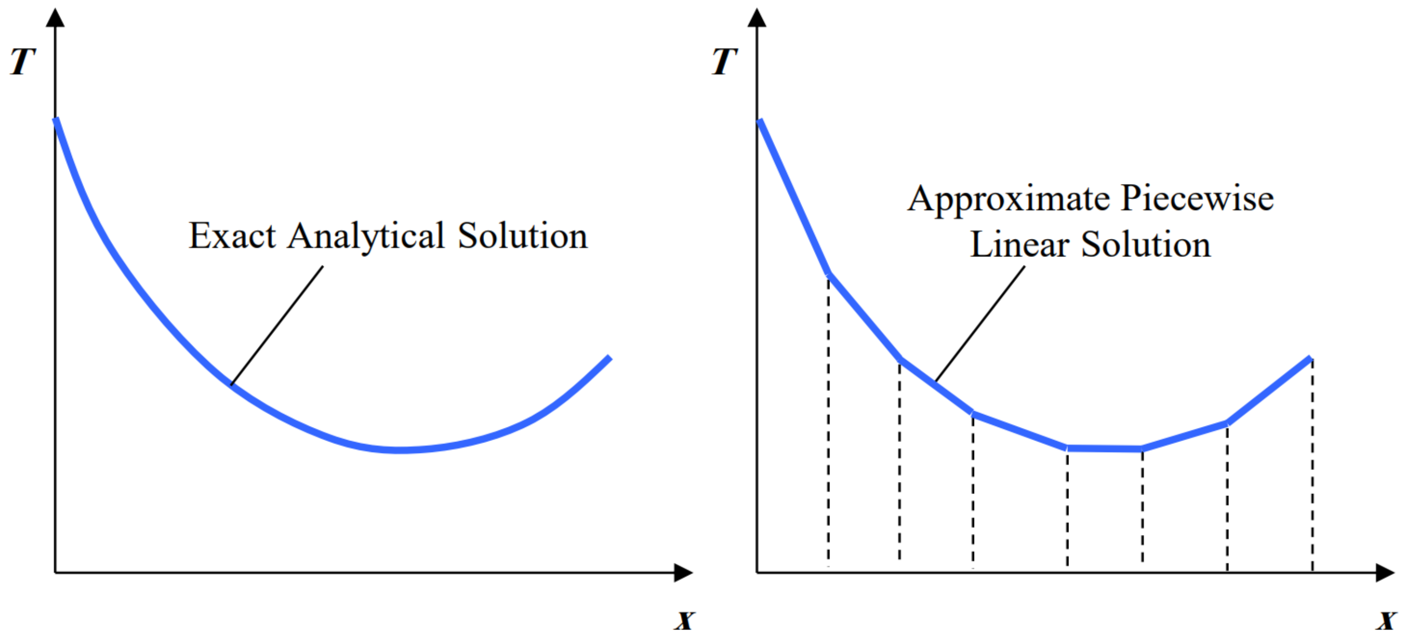
2. Bars or Trusses
   - A bar element
     - ✓ Stiffness Matrix
     - ✓ Mass Matrix
     - ✓ Stress
   - Numerical Integration

1. Shape Functions – Basic Concepts

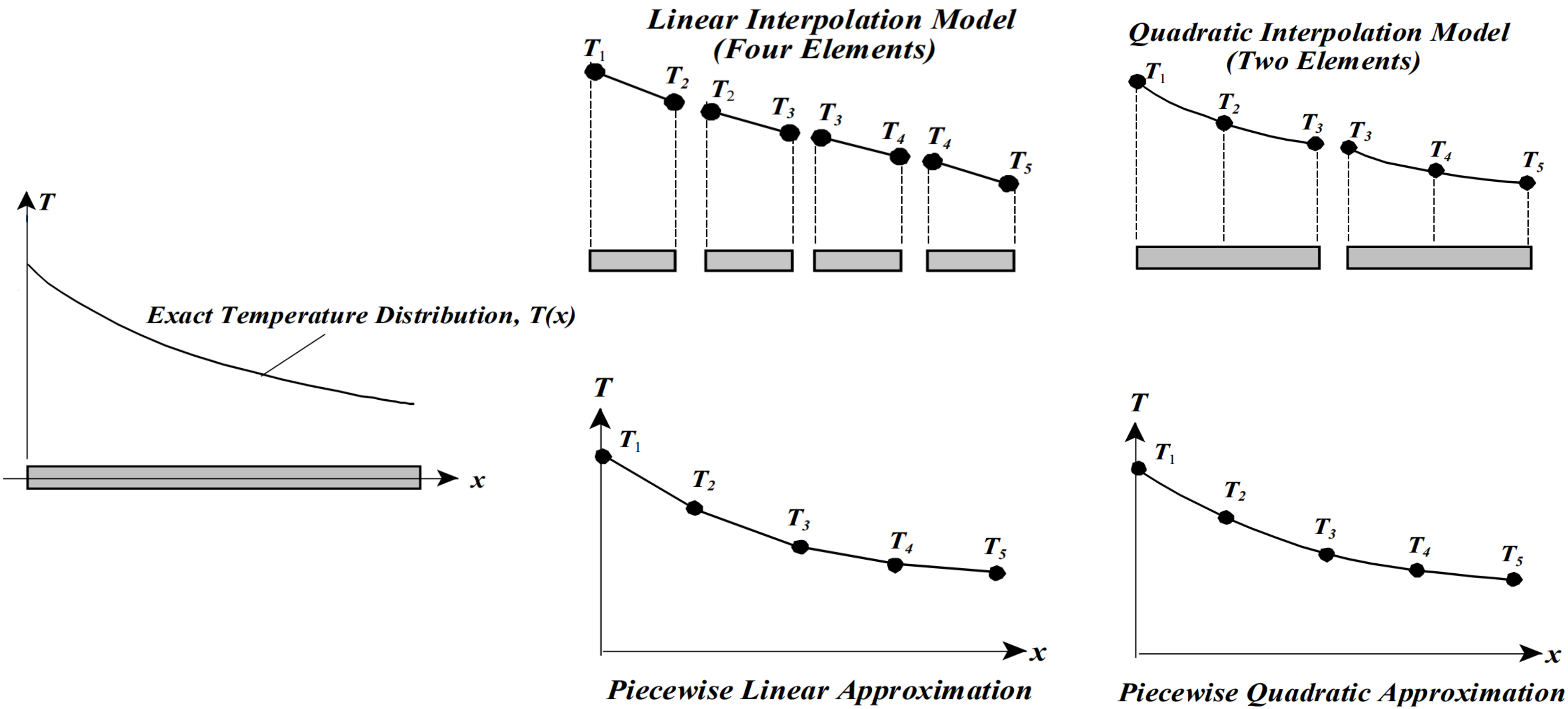
- In the FEM, any continuous solution field such as stress, displacement, temperature, pressure, etc. can be approximated by a discrete model composed of a set of piecewise continuous functions defined over a finite number of subdomains.



One-Dimensional Temperature Distribution

1. Shape Functions – Basic Concepts

❖ Finite Element Discretization Concepts



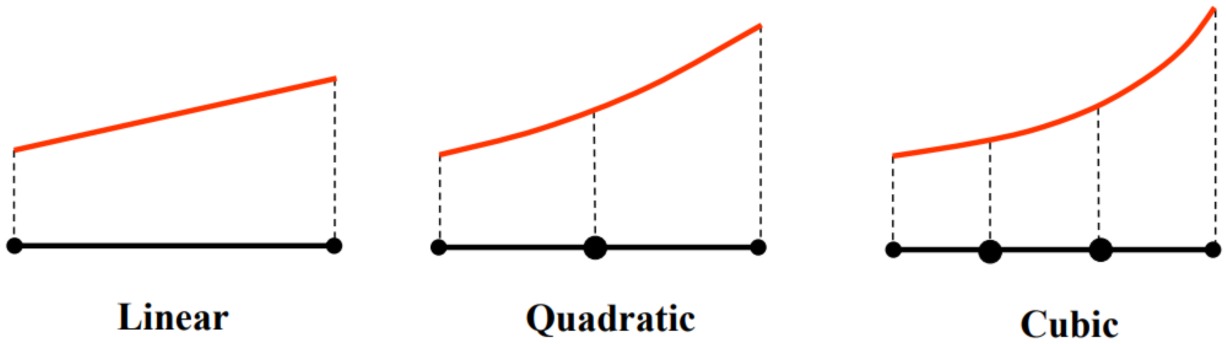
Temperature Continuous but with Discontinuous Temperature Gradients

Temperature and Temperature Gradients Continuous

1. Shape Functions – Basic Concepts

❖ Common Approximation Schemes: One-Dimensional Examples

- Polynomial Approximation: Most often polynomials are used to construct approximation functions for each element. Depending on the order of approximation, different numbers of element parameters are needed to construct the appropriate function.



- Special Approximation: For some cases (e.g., infinite elements, crack or other singular elements) the approximation function is chosen to have special properties as determined from theoretical considerations.

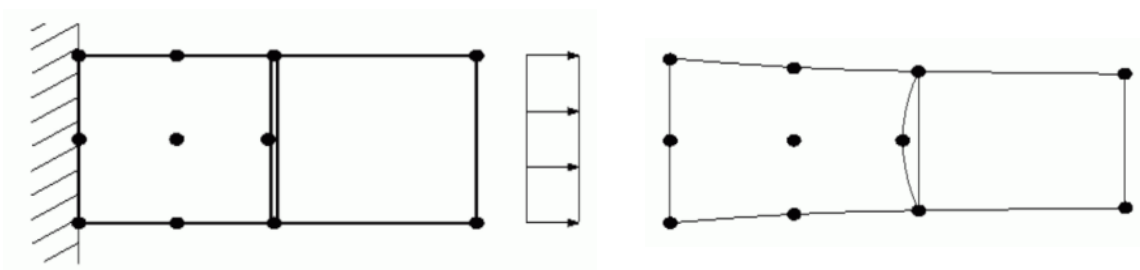
1. Shape Functions – Requirements

❖ Requirements for shape functions are motivated by convergence: as the mesh is refined the FEM solution should approach the analytical solution of the mathematical model.

The requirement for compatibility

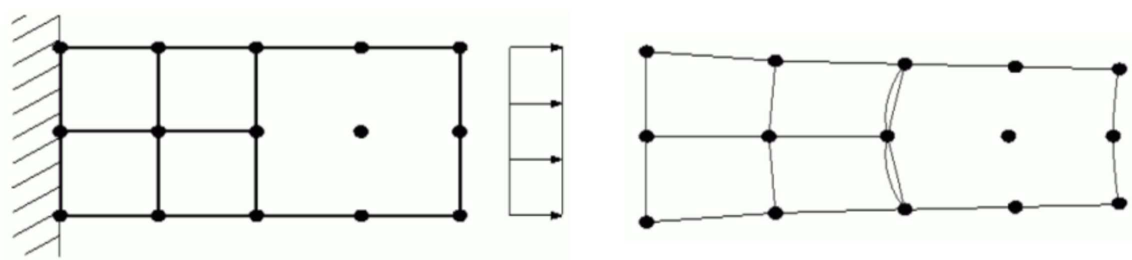
❖ The shape functions should provide displacement continuity inside and between elements. This physically means that there is no material gaps appear when the elements deforms. When the mesh is deformed, such gaps may absorb or release spurious energy.

❖ The finite elements that satisfy this property are called conforming, or compatible.



Compatibility violation:

Gaps in deformed shape



Compatibility violation:

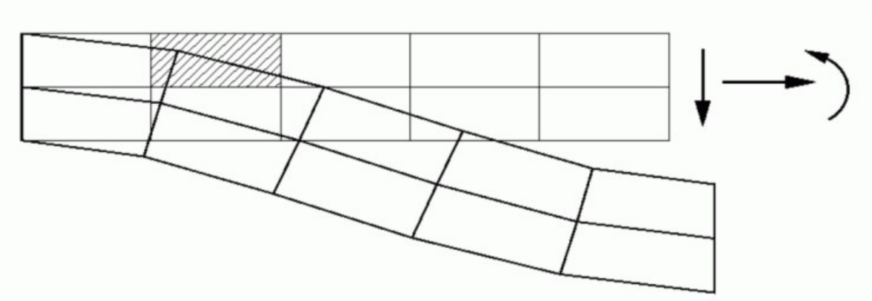
Overlapping in deformed shape

1. Shape Functions – Requirements

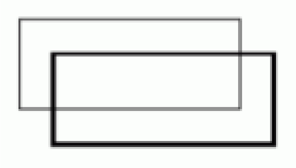
- ❖ Requirements for shape functions are motivated by convergence: as the mesh is refined the FEM solution should approach the analytical solution of the mathematical model.

The requirement for completeness

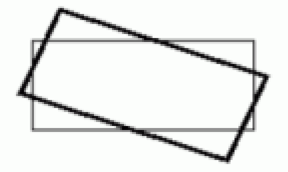
- ❖ The shape functions should be able to represent the rigid body displacement.
- ❖ The shape functions should be able to represent constant strain state.



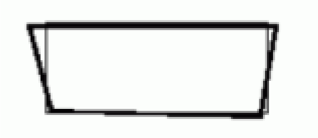
Deformation of Cantilever Beam



Rigid body translation



Rigid body rotation



Deformation

1. Shape Functions – Requirements

❖ If the stiffness integrands involve derivatives of order *m*, then requirements for shape functions can be formulated as follows:

The requirement for compatibility

❖ The shape functions must be C^{m-1} continuous between elements, and C^{m} piecewise differentiable inside each element.

The requirement for completeness

❖ The shape functions must represent exactly all polynomial terms of order ≤ *m* in the Cartesian coordinates. A set of shape functions that satisfies this condition is called *m*-complete.

Compatibility + Completeness = Convergence

"As more elements are introduced, the solution gets better and approaches the exact solution."

The properties of the shape functions

❖ Kronecker delta property: The shape function at any node has a value of 1 at that node and a value of 0 at all other nodes. $N_i\left(\xi_j\right)=\begin{cases}1, & i=j\\ 0, & i\neq j\end{cases}$

1. Shape Functions – Lagrange Interpolation Functions

- Lagrange Interpolation Functions can be constructed for any number of points at any designated locations.
- Lagrange Interpolation Polynomial of order *m* at node *k* (ξ is the local/natural coordinate)

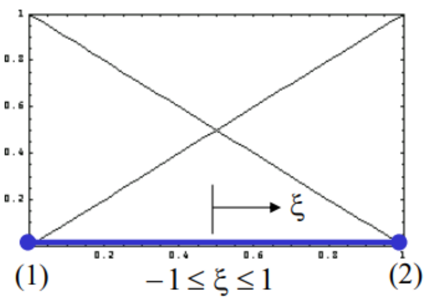
$$L_k^{(m)}(\xi)=\frac{(\xi-\xi_0)(\xi-\xi_1)\cdots(\xi-\xi_{k-1})(\xi-\xi_{k+1})\cdots(\xi-\xi_m)}{(\xi_k-\xi_0)(\xi_k-\xi_1)\cdots(\xi_k-\xi_{k-1})(\xi_k-\xi_{k+1})\cdots(\xi_k-\xi_m)}=\prod_{\substack{i=0\\ i\neq k}}^{m}\frac{(\xi-\xi_i)}{(\xi_k-\xi_i)}$$

No term $(\xi-\xi_k)$

Linear

$$N_1=\frac{1}{2}(1-\xi)$$

$$N_2=\frac{1}{2}(1+\xi)$$



Cubic

$$N_1=-\frac{9}{16}(1-\xi)\left(\frac{1}{3}+\xi\right)\left(\frac{1}{3}-\xi\right),\ N_2=\frac{27}{16}(1-\xi)(1+\xi)\left(\frac{1}{3}-\xi\right)$$

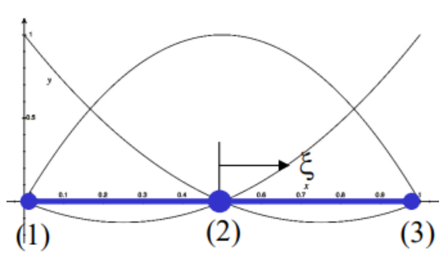
$$N_3=\frac{27}{16}(1-\xi)(1+\xi)\left(\frac{1}{3}+\xi\right),\ N_4=-\frac{9}{16}(1+\xi)\left(\frac{1}{3}+\xi\right)\left(\frac{1}{3}-\xi\right)$$

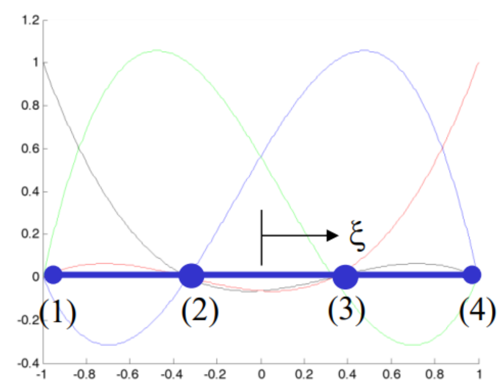
Quadratic

$$N_1=-\frac{1}{2}\xi(1-\xi)$$

$$N_2=(1-\xi)(1+\xi)$$

$$N_3=\frac{1}{2}\xi(1+\xi)$$





1. Shape Functions – Lagrange Interpolation Functions

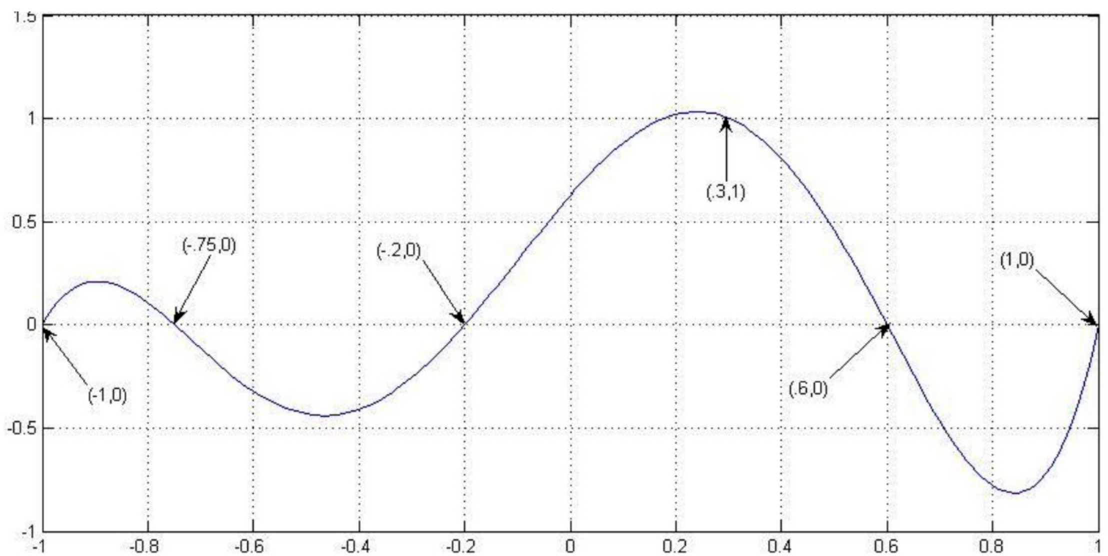
- Lagrange Interpolation Functions use a procedure that automatically satisfies the Kronecker delta property for shape functions.

Demonstration

- Consider 1D example of 6 points: $\xi_0 = -1.0,\ \xi_1 = -0.75,\ \xi_2 = -0.2,\ \xi_3 = 0.3,\ \xi_4 = 0.6,\ \xi_5 = 1.0$

  Construct the Lagrange Interpolation Polynomial (shape function) at point $\xi_3 = 0.3$, the function receives value 1 at $\xi_3 = 0.3$, and receives value 0 at other designated points.

$$L_3^{(5)}(\xi) = \frac{(\xi-\xi_0)(\xi-\xi_1)(\xi-\xi_2)(\xi-\xi_4)(\xi-\xi_5)}{(\xi_3-\xi_0)(\xi_3-\xi_1)(\xi_3-\xi_2)(\xi_3-\xi_4)(\xi_3-\xi_5)}$$



1. Shape Functions – Shape Functions of Plane Elements

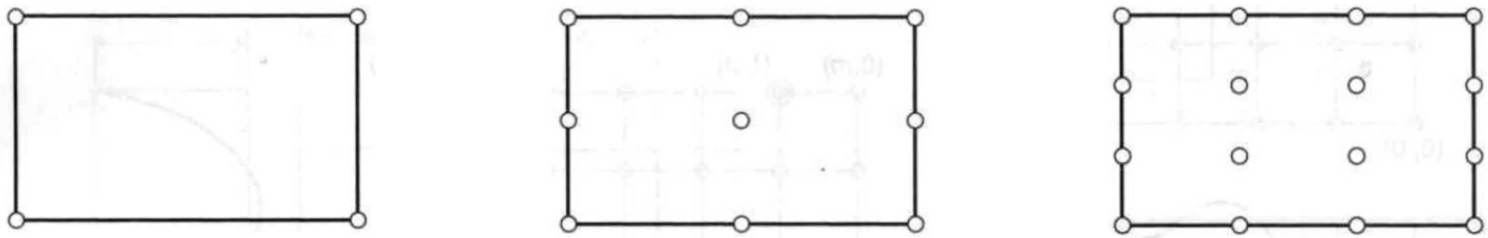
Classification according to

- ❖ The element form
  - Triangular elements
  - Rectangular elements
- ❖ Polynomial degree of the shape functions
  - Linear
  - Quadratic
  - Cubic
  - …
- ❖ Type of the shape functions
  - Lagrange shape functions
  - Serendipity shape functions

1. Shape Functions – Rectangular elements – Lagrange family

Lagrangian Elements

❖ Order n element has $(n+1)^2$ nodes arranged in square-symmetric pattern, might have internal nodes.



$n=1$ $n=2$ $n=3$

❖ Shape functions are products of nth order polynomials in each direction. ("biquadratic", "bicubic", …)

❖ Bilinear quad is a Lagrangian element of order $n=1$.

❖ Lagrange interpolation polynomial in one direction: $l_k^n(\xi_1)=\prod_{\substack{i=0\\ i\neq k}}^{n}\frac{\xi_1-\xi_1^i}{\xi_1^k-\xi_1^i}$

❖ An easy and systematic method of generating shape functions of any order now can be achieved by simple products of Lagrange polynomials in the two coordinates :

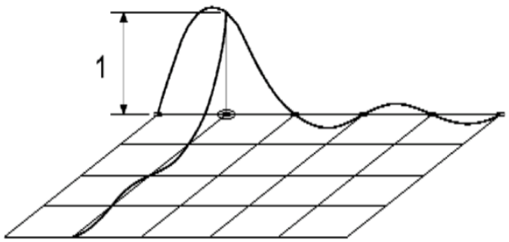
$$N_a=N_{IJ}=l_I^n(\xi_1)l_J^m(\xi_2) \qquad \textit{with} \qquad \xi_1=\frac{2(x-x_e)}{a_e},\quad \xi_2=\frac{2(y-y_e)}{b_e}$$

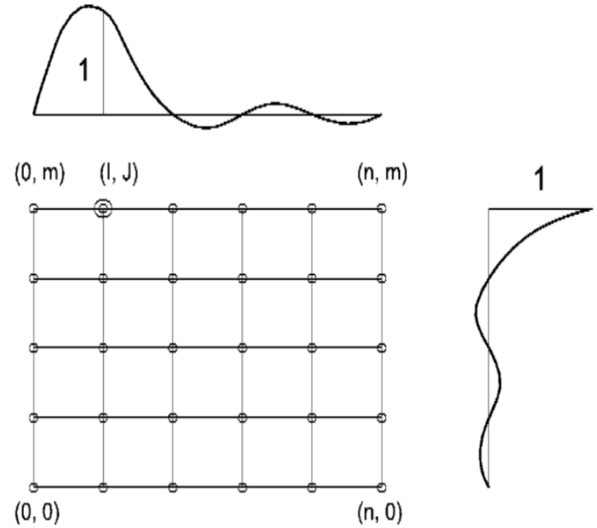
x_e, y_e *are coordinates of the center of the element*

a_e, b_e *are dimensions of the element*

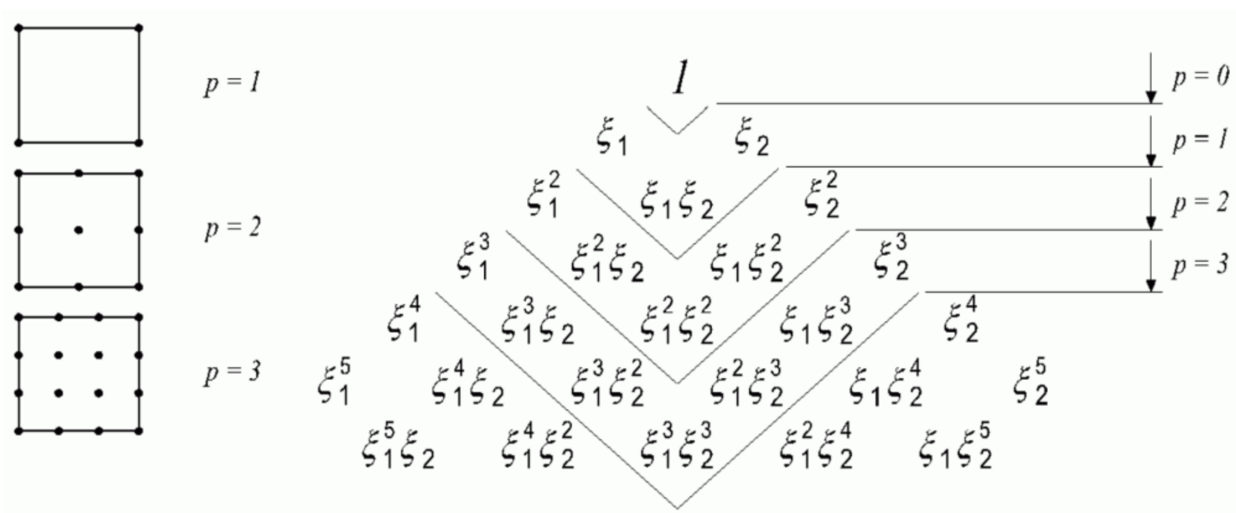
1. Shape Functions – Rectangular elements – Lagrange family

Lagrangian Elements





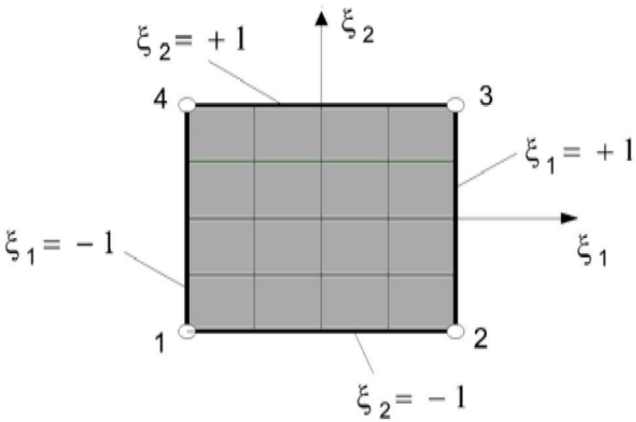
❖ Complete two-dimensional Lagrange polynomials in the Pascal triangle



1. Shape Functions – Rectangular elements – Lagrange family

Quadrilateral Lagrangian elements

(4-node bilinear)

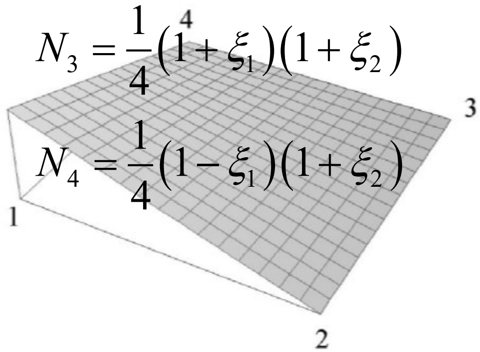


$$N_1 = \frac{1}{4}(1-\xi_1)(1-\xi_2)$$

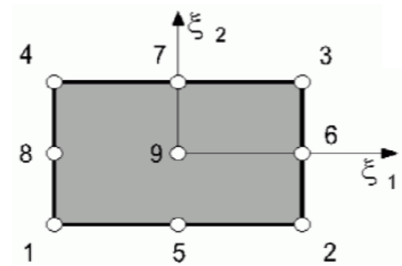
$$N_2 = \frac{1}{4}(1+\xi_1)(1-\xi_2)$$

$$N_3 = \frac{1}{4}(1+\xi_1)(1+\xi_2)$$

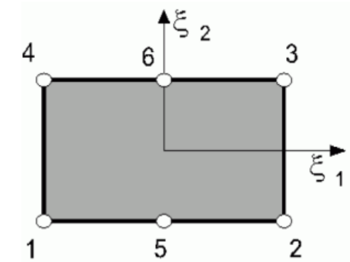
$$N_4 = \frac{1}{4}(1-\xi_1)(1+\xi_2)$$



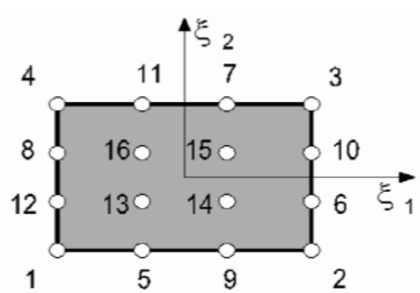
(9-node biquadratic)



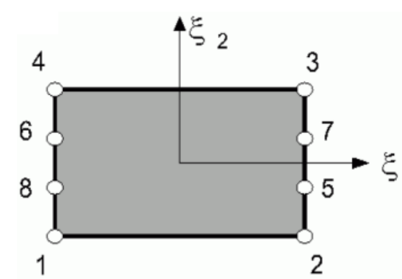
(6-node quadratic-linear)



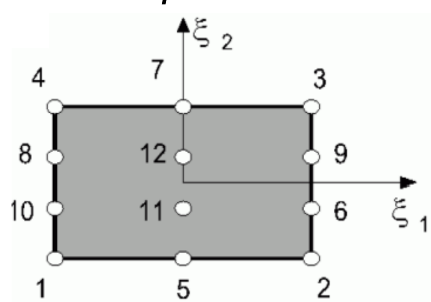
(16-node bicubic)



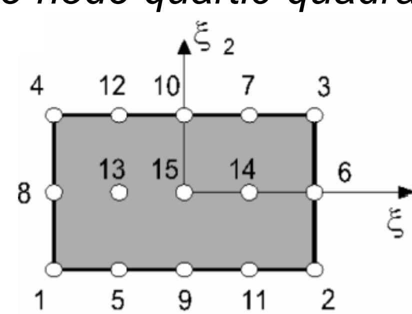
(8-node linear-cubic)



(12-node quadratic-cubic)

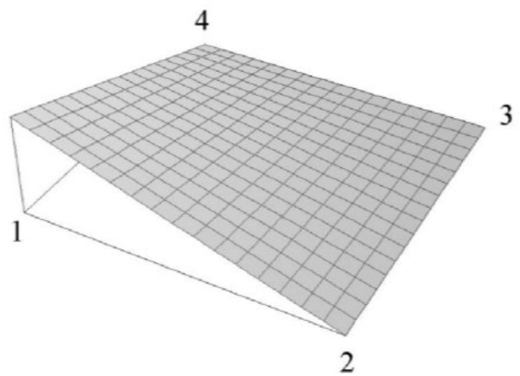


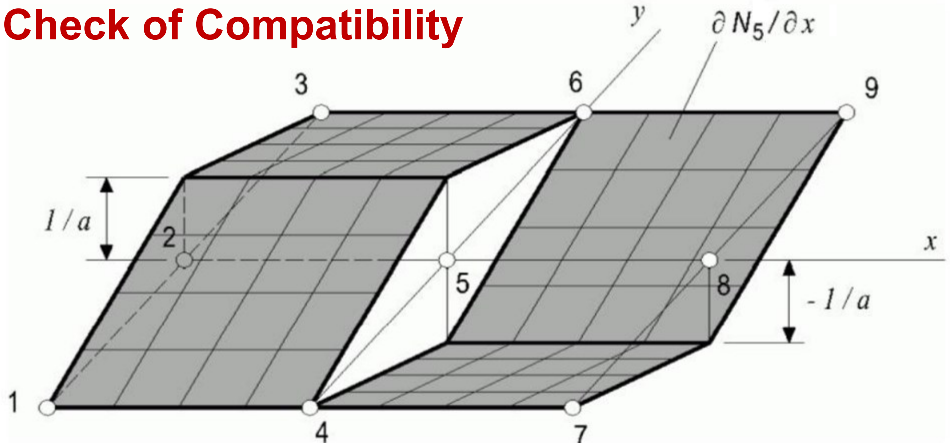
(15-node quartic-quadratic)



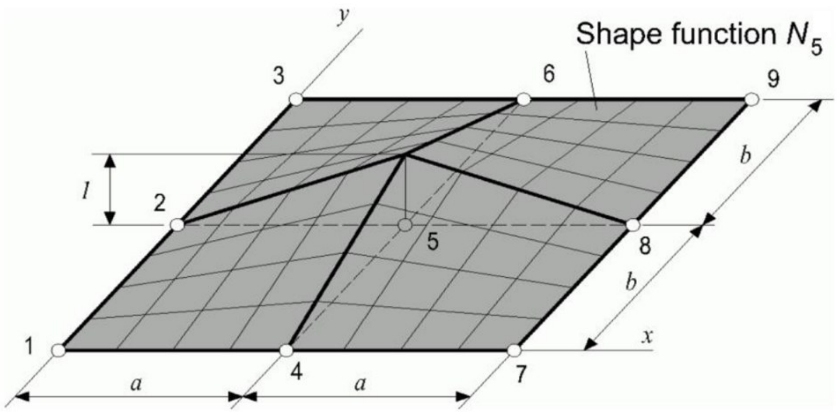
1. Shape Functions – Rectangular elements – Lagrange family

4-node Bilinear Quadrilateral Element – Check of Compatibility

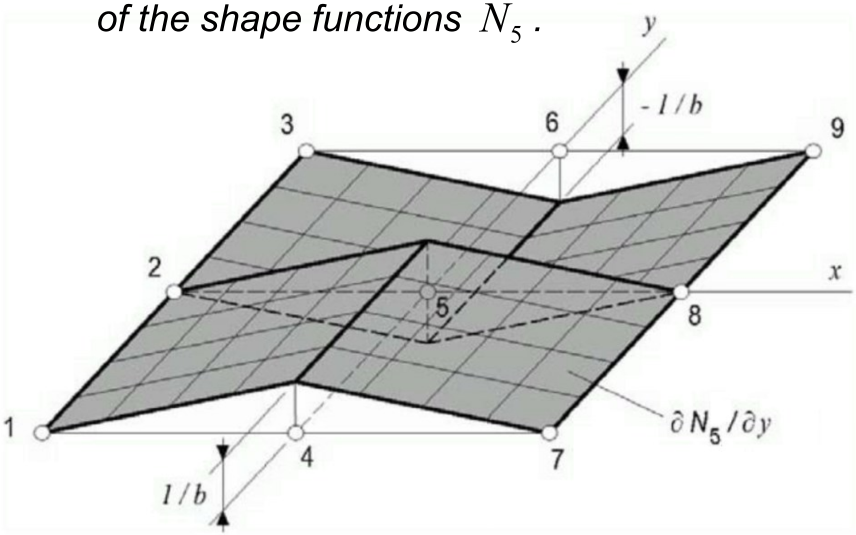




Partial derivatives with respect to x and y of the shape functions N_5 .



Assemblage of four bilinear quadrilateral elements. Change of N_5 along the edge is linear and it is uniquely defined by two nodes.



Derivative inside element exist, and on the boundary has finite discontinuity

1. Shape Functions – Rectangular elements – Lagrange family

4-node Bilinear Quadrilateral Element – Check of Completeness

❖ A set of shape functions is complete for a continuum element if they can represent exactly any linear displacement motions such as :

$$u_x = \alpha_0 + \alpha_1 x + \alpha_2 y, \qquad u_y = \beta_0 + \beta_1 x + \beta_2 y \tag{1}$$

❖ The nodal point displacements corresponding to this displacement field are:

$$u_{xi} = \alpha_0 + \alpha_1 x_i + \alpha_2 y_i, \qquad u_{yi} = \beta_0 + \beta_1 x_i + \beta_2 y_i \tag{2}$$

❖ The displacements (1) can be obtained within the element when the element nodal point displacements are given by (2).

❖ In the iso-parametric formulation, we have the displacement interpolation:

$$u_x = \sum_{i=1}^{n} u_{xi} N_i^e, \quad u_y = \sum_{i=1}^{n} u_{yi} N_i^e$$

❖ Computation for the displacement $u_x = \sum_{i=1}^{n} \left(\alpha_0 + \alpha_1 x_i + \alpha_2 y_i \right) N_i^e = \alpha_0 \sum_{i=1}^{n} N_i^e + \alpha_1 \sum_{i=1}^{n} x_i N_i^e + \alpha_2 \sum_{i=1}^{n} y_i N_i^e$

❖ In the iso-parametric formulation, the coordinates are interpolated in the same way as the displacements, we can use:

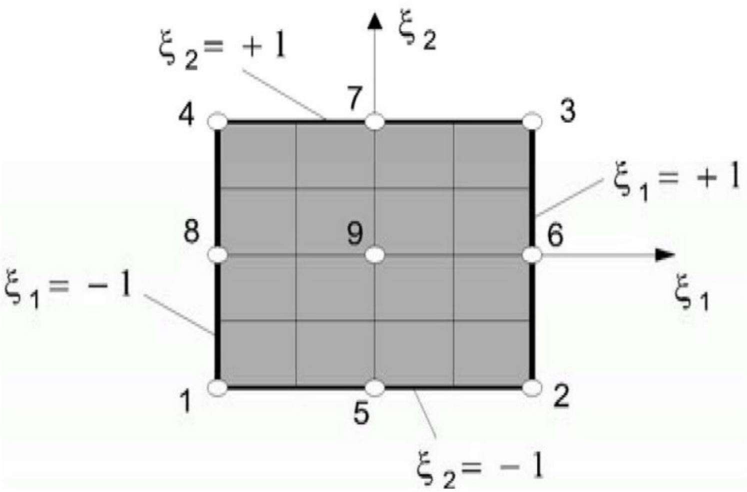
$$x = \sum_{i=1}^{n} x_i N_i^e, \quad y = \sum_{i=1}^{n} y_i N_i^e \qquad \textit{to obtain} \qquad u_x = \alpha_0 \sum_{i=1}^{n} N_i^e + \alpha_1 x + \alpha_2 y \tag{3}$$

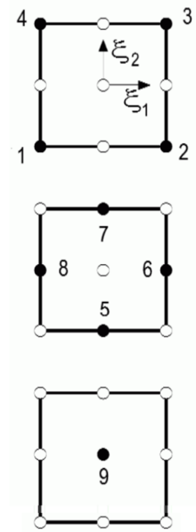
❖ The displacements defined in (3) are the same as those given in (1), thus at any point

$\sum_{i=1}^{n} N_i^e = 1$ *is the condition for the completeness requirements to be satisfied.*

1. Shape Functions – Rectangular elements – Lagrange family

9-node Biquadratic Quadrilateral



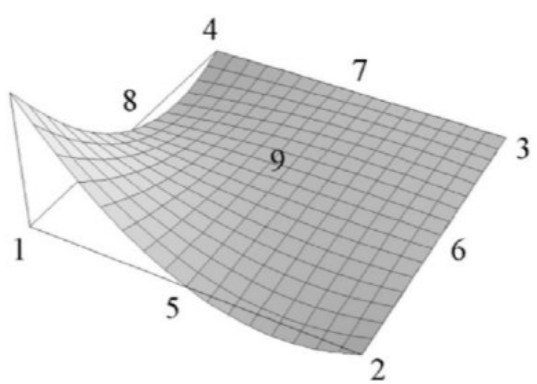


$$N_i = \frac{1}{4}\left(1+\xi_1^{(i)}\xi_1\right)\xi_1^{(i)}\xi_1\left(1+\xi_2^{(i)}\xi_2\right)\xi_2^{(i)}\xi_2$$

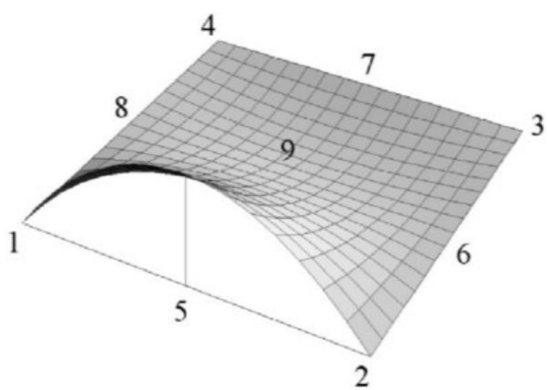
$$N_i = \frac{1}{2}\left(1-\xi_2^2\right)\left(1+\xi_1^{(i)}\xi_1\right)\xi_1^{(i)}\xi_1, \quad \xi_2^{(i)} = 0$$

$$N_i = \frac{1}{2}\left(1-\xi_1^2\right)\left(1+\xi_2^{(i)}\xi_2\right)\xi_2^{(i)}\xi_2, \quad \xi_1^{(i)} = 0$$

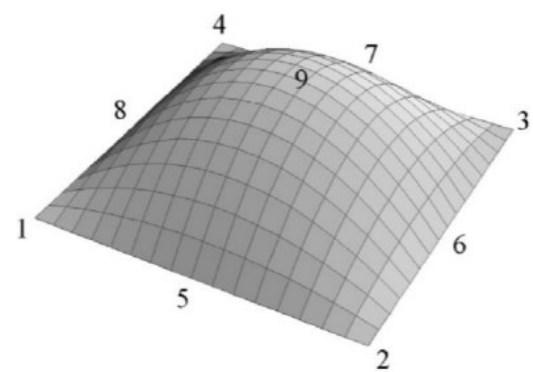
$$N_9 = \left(1-\xi_1^2\right)\left(1-\xi_2^2\right)$$



$$N_1(\xi) = \frac{1}{4}(1-\xi_1)(1-\xi_2)\xi_1\xi_2$$



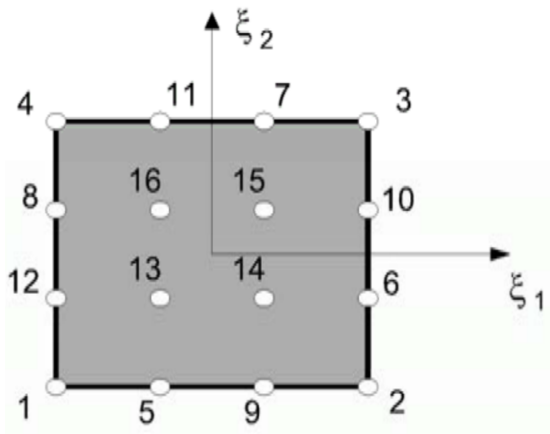
$$N_5(\xi) = \frac{1}{2}\left(1-\xi_1^2\right)(\xi_2-1)\xi_2$$

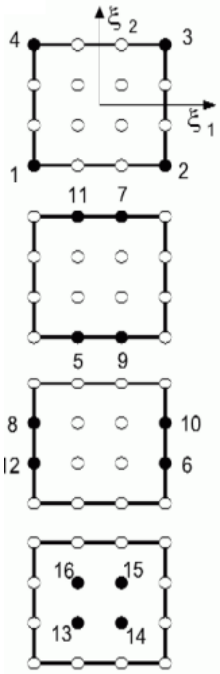


$$N_9(\xi) = \left(1-\xi_1^2\right)\left(1-\xi_2^2\right)$$

1. Shape Functions – Rectangular elements – Lagrange family

16-node Bicubic Quadrilateral



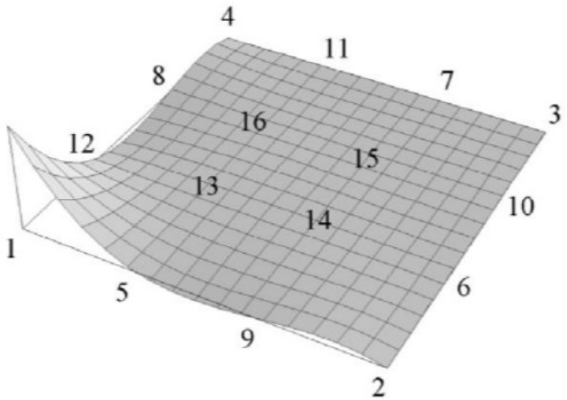


$$N_i = \frac{81}{256}\left(1+\xi_1^{(i)}\xi_1\right)\left(1+\xi_2^{(i)}\xi_2\right)\left(\frac{1}{9}-\xi_1^2\right)\left(\frac{1}{9}-\xi_2^2\right)$$

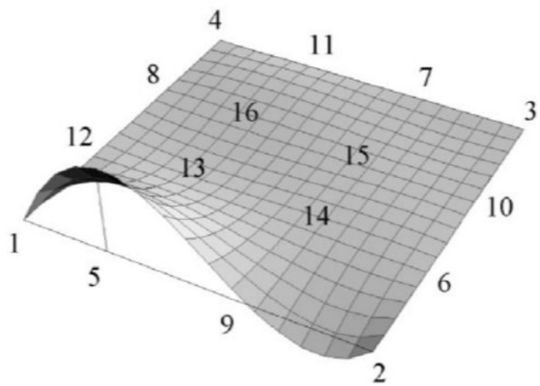
$$N_i = \frac{243}{256}\left(1-\xi_1^2\right)\left(\xi_2^2-\frac{1}{9}\right)\left(\frac{1}{3}+3\xi_1^{(i)}\xi_1\right)\left(1+\xi_2^{(i)}\xi_2\right)$$

$$N_i = \frac{243}{256}\left(1-\xi_2^2\right)\left(\xi_1^2-\frac{1}{9}\right)\left(\frac{1}{3}+3\xi_2^{(i)}\xi_2\right)\left(1+\xi_1^{(i)}\xi_1\right)$$

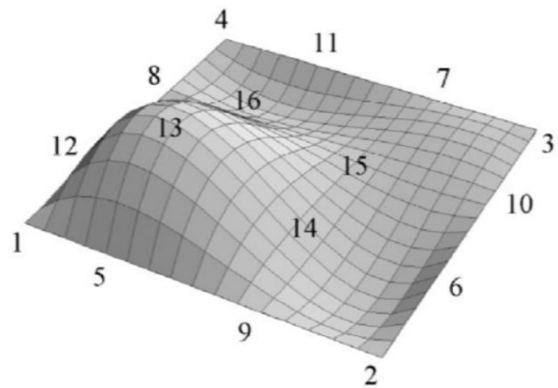
$$N_i = \frac{729}{256}\left(1-\xi_1^2\right)\left(1-\xi_2^2\right)\left(\frac{1}{3}+3\xi_1^{(i)}\xi_1\right)\left(\frac{1}{3}+3\xi_2^{(i)}\xi_2\right)$$



$N_1(\xi)$



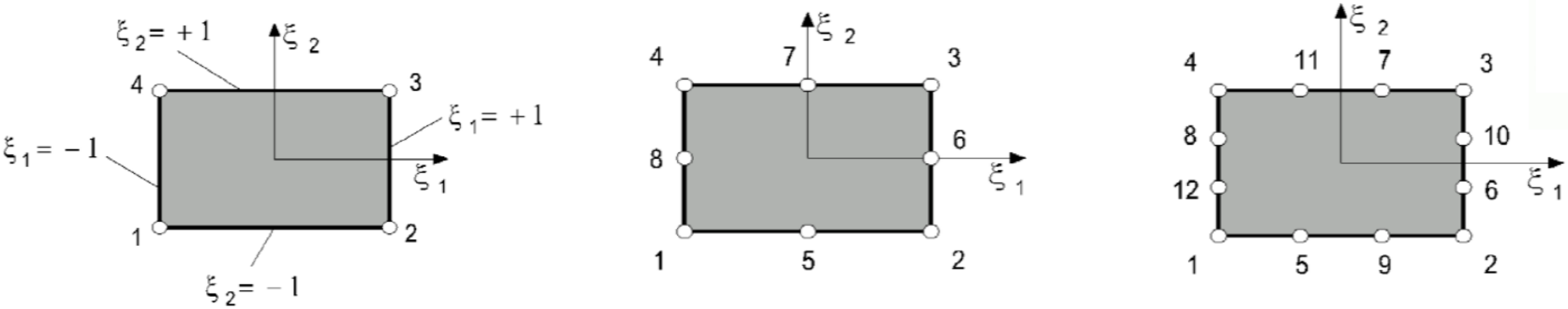
$N_5(\xi)$



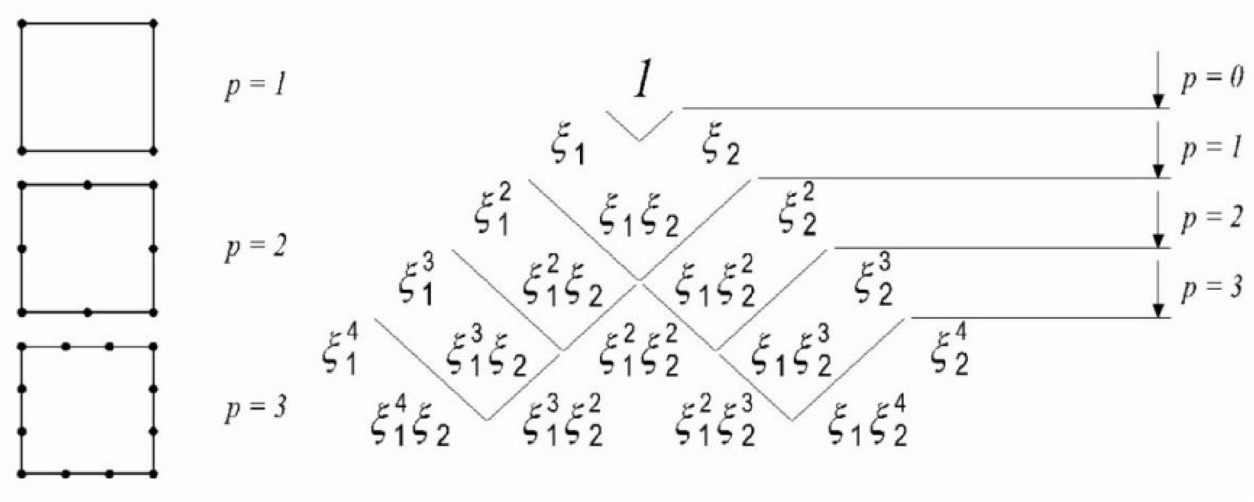
$N_{13}(\xi)$

1. Shape Functions – Rectangular elements – Serendipity elements

❖ Serendipity elements are constructed with nodes only on the element boundary



(4-node bilinear) *(8-node biquadratic)* *(12-node bicubic)*

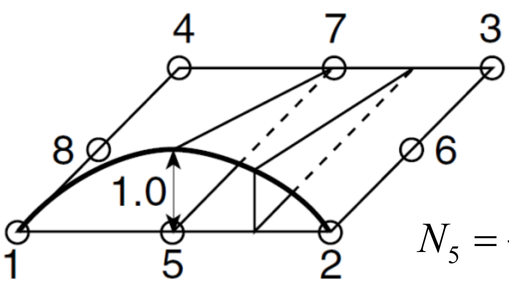


Pascal triangle of 2D serendipity polynomials of quadrilateral elements

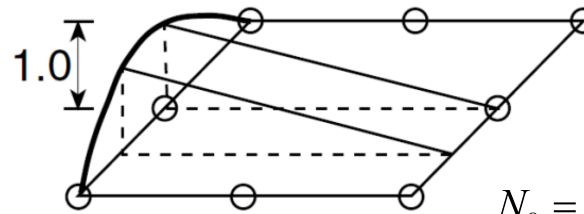
1. Shape Functions – Rectangular elements – Serendipity elements

Serendipity Biquadratic Shape Functions

❖ For mid-side nodes, a Lagrangian interpolation of quadratic x linear type suffices to determine N_i at nodes 5 to 8.

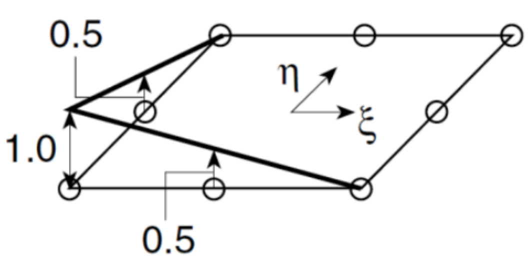


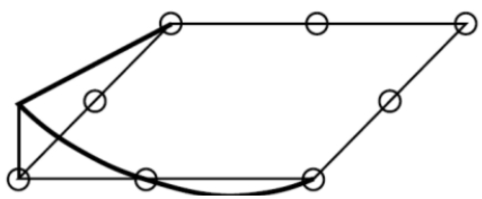
$$N_5 = \frac{1}{2}\left(1-\xi_1^2\right)\left(1-\xi_2\right)$$

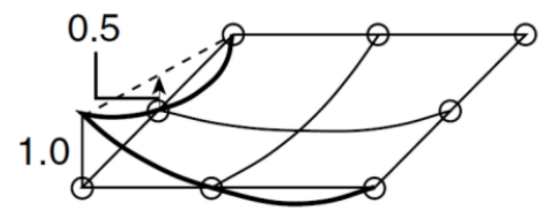


$$N_8 = \frac{1}{2}\left(1-\xi_1\right)\left(1-\xi_2^2\right)$$

❖ For corner nodes start with bilinear Lagrangian family (step 1), and successive subtraction (step 2, step 3) ensures zero value at nodes 5, 8.





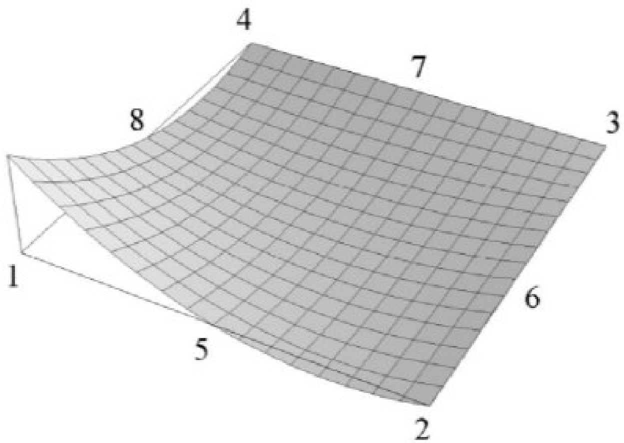


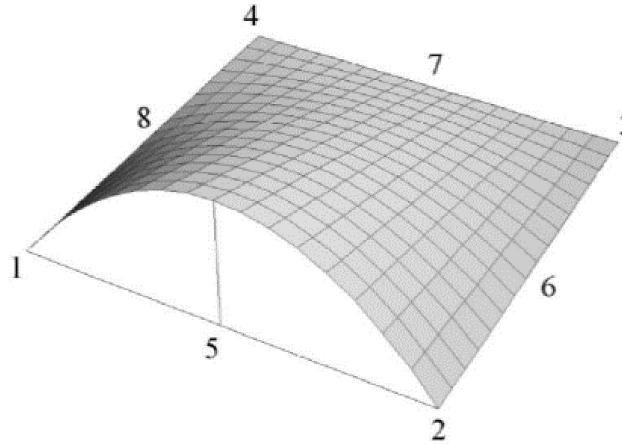
(Step 1) $\hat{N}_1 = \frac{1}{4}\left(1-\xi_1\right)\left(1-\xi_2\right)$ *(Step 2)* $\hat{N}_1 - \frac{1}{2}N_5$ *(Step 3)* $N_1 = \hat{N}_1 - \frac{1}{2}N_5 - \frac{1}{2}N_8$

$$N_1 = \frac{1}{4}\left(1-\xi_1\right)\left(1-\xi_2\right) - \frac{1}{2}\cdot\frac{1}{2}\left(1-\xi_1^2\right)\left(1-\xi_2\right) - \frac{1}{2}\cdot\frac{1}{2}\left(1-\xi_1\right)\left(1-\xi_2^2\right) = \frac{1}{4}\left(1-\xi_1\right)\left(1-\xi_2\right)\left(-1-\xi_1-\xi_2\right)$$

1. Shape Functions – Rectangular elements – Serendipity elements

Serendipity Biquadratic Shape Functions





$$N_1(\xi) = \frac{1}{4}(1-\xi_1)(1-\xi_2)(-1-\xi_1-\xi_2)$$

$$N_i(\xi) = \frac{1}{4}\left(1+\xi_1^{(i)}\xi_1\right)\left(1+\xi_2^{(i)}\xi_2\right)\left(-1+\xi_1^{(i)}\xi_1+\xi_2^{(i)}\xi_2\right)$$

$$N_5(\xi) = \frac{1}{2}\left(1-\xi_1^2\right)(1-\xi_2)$$

$$N_i(\xi) = \frac{1}{2}\left(1-\xi_1^2\right)\left(1+\xi_2^{(i)}\xi_2\right), \quad \xi_1^{(i)} = 0$$

$$N_i(\xi) = \frac{1}{2}\left(1-\xi_2^2\right)\left(1+\xi_1^{(i)}\xi_1\right), \quad \xi_2^{(i)} = 0$$

1. Shape Functions – Rectangular elements – Serendipity elements

Serendipity Shape Functions

❖ In general, serendipity shape functions can be obtained with the following expression:

$$
\begin{aligned}
N_i(\xi_1,\xi_2) = &\frac{1}{2}(1-\xi_2)N_i(\xi_1,-1)+\frac{1}{2}(1+\xi_1)N_i(1,\xi_2)+\frac{1}{2}(1+\xi_2)N_i(\xi_1,1)+\frac{1}{2}(1-\xi_1)N_i(-1,\xi_2) \\
&-\frac{1}{4}(1-\xi_1)(1-\xi_2)N_i(-1,-1)-\frac{1}{4}(1+\xi_1)(1-\xi_2)N_i(1,-1) \\
&-\frac{1}{4}(1+\xi_1)(1+\xi_2)N_i(1,1)-\frac{1}{4}(1-\xi_1)(1+\xi_2)N_i(-1,1)
\end{aligned}
$$

with

$N_i(\xi_1,-1),\ N_i(1,\xi_2),\ N_i(\xi_1,1),\ N_i(-1,\xi_2)$ *are Lagrangian interpolations along the corresponding boundary.*

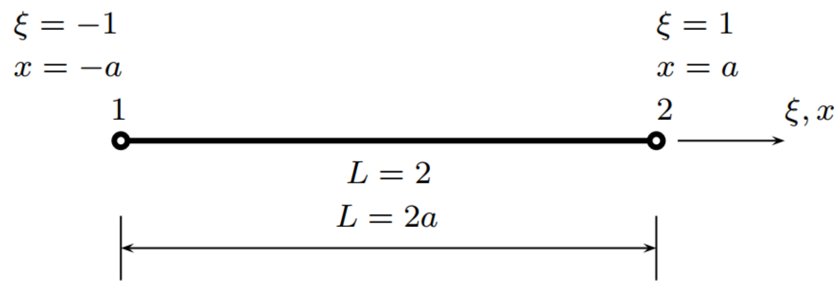
$N_i(-1,-1),\ N_i(1,-1),\ N_i(1,1),\ N_i(-1,1)$ *have values 0 or 1, and represent values of interpolation on corners.*

2. Bars or Trusses – A bar element

- Bars or trusses are axially loaded structural elements.
- A truss connects to other elements only through pins, the connection is free of rotation.
- A truss can be modeled as a discrete element (i.e., spring) as only axial force and elongation are evaluated.
- Depends on the degree of approximation of both geometry and dependent variables (i.e., displacement), finite element formulation are classified as
  - *Superparametric*: the geometry approximation is higher order than the dependent variable approximation.
  - *Isoparametric*: the geometry approximation and the dependent variable approximation have equal degree.
  - *Subparametric*: the geometry approximation is lower order than the dependent variable approximation.
- An isoparametric finite element formulation is considered in this class by then.

2. Bars or Trusses – A bar element

- Constant cross-section area A, length $L = 2a$
- Axial stresses uniform over cross-section σ_x
- Hamilton's Principle



(A 2-node bar element)

$$\int_{t_1}^{t_2} \delta K - \left(\delta U - \delta W\right) dt = 0$$

K *is the kinetic energy*

U *is the internal, strain energy*

W *is the external work due to applied loads*

$\Pi = U - W$ *is the external work due to applied loads*

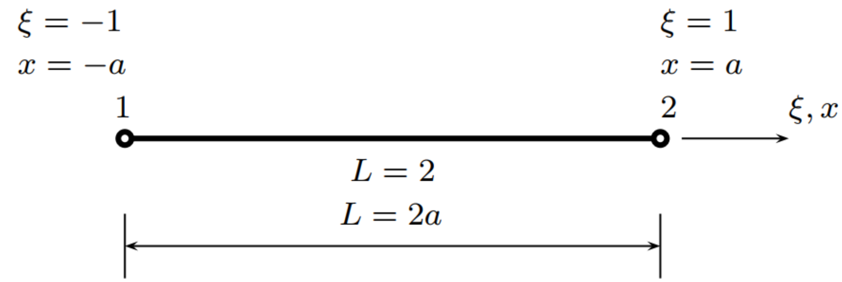
- The kinetic energy $K = \frac{1}{2}\int_V \rho \left(\frac{\partial u}{\partial t}\right)^2 dV = \frac{\rho A}{2}\int_{-a}^{a}\left(\frac{\partial u}{\partial t}\right)^2 dx \rightarrow \int_{t_1}^{t_2} \delta K dt = -\int_{t_1}^{t_2}\int_{-a}^{a} \rho A \frac{\partial^2 u}{\partial t^2}\delta u \, dx \, dt$
- Strain-displacement relation $\varepsilon_x = \frac{\partial u}{\partial x}$
- Assume linear elastic material $\sigma_x = E\varepsilon_x = E\frac{\partial u}{\partial x}$
- The internal strain energy $U = \frac{EA}{2}\int_{-a}^{a}\left(\frac{\partial u}{\partial x}\right)^2 dx \rightarrow \delta U = EA\int_{-a}^{a}\frac{\partial u}{\partial x}\frac{\partial \delta u}{\partial x} dx$
- Virtual external work caused by the applied force by unit length p

$$\delta W = \int_{-a}^{a} p \, \delta u \, dx$$

2. Bars or Trusses – A bar element

- The equilibrium of the bar (also called weak or variational form)

$$\rho A\int_{-a}^{a}\frac{\partial^2 u}{\partial t^2}\delta u\,dx + EA\int_{-a}^{a}\frac{\partial u}{\partial x}\frac{\partial \delta u}{\partial x}dx - \int_{-a}^{a} p\,\delta u\,dx = 0$$



(A 2-node bar element)

- The axial displacements of a 2-noded finite element can be interpolated as

$$u = N_1(\xi)u_1 + N_2(\xi)u_2 = \begin{bmatrix} N_1 & N_2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \mathbf{N}\mathbf{u}^e \;\rightarrow\; \delta u = \mathbf{N}\delta\mathbf{u}^e$$

with $N_1(\xi) = \frac{1}{2}(1-\xi),\; N_2(\xi) = \frac{1}{2}(1+\xi)$ *defined in the natural coordinate system* $\xi \in [-1,+1]$

Coordinate transformation $x = a\xi \rightarrow dx = a\,d\xi \rightarrow \frac{d\mathbf{N}}{dx} = \frac{d\mathbf{N}}{d\xi}\frac{d\xi}{dx} = \frac{1}{a}\frac{d\mathbf{N}}{d\xi} = \frac{1}{a}\mathbf{N}' = \frac{1}{a}\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$

- The equilibrium becomes

$$\rho A\int_{-a}^{a}\left(\mathbf{N}\delta\mathbf{u}^e\right)^T\left(\mathbf{N}\ddot{\mathbf{u}}^e\right)dx + EA\int_{-a}^{a}\left(\frac{d\mathbf{N}}{dx}\delta\mathbf{u}^e\right)^T\left(\frac{d\mathbf{N}}{dx}\mathbf{u}^e\right)dx - \int_{-a}^{a} p\left(\mathbf{N}\delta\mathbf{u}^e\right)^T dx = 0$$

$$\leftrightarrow\; \delta\mathbf{u}^{eT}\underbrace{\left(\rho A a\int_{-1}^{1}\mathbf{N}^T\mathbf{N}\,d\xi\right)}_{\mathbf{M}^e}\ddot{\mathbf{u}}^e + \delta\mathbf{u}^{eT}\underbrace{\left(\frac{EA}{a}\int_{-1}^{1}\mathbf{N}'^T\,\mathbf{N}'\,d\xi\right)}_{\mathbf{K}^e}\mathbf{u}^e - \delta\mathbf{u}^{eT}\underbrace{\left(a\int_{-1}^{1}p\mathbf{N}^T d\xi\right)}_{\mathbf{f}^e} = 0$$

$$\leftrightarrow\; \delta\mathbf{u}^{eT}\mathbf{M}^e\ddot{\mathbf{u}}^e + \delta\mathbf{u}^{eT}\mathbf{K}^e\mathbf{u}^e - \delta\mathbf{u}^{eT}\mathbf{f}^e = 0$$

2. Bars or Trusses – A bar element

- Evaluate the element matrices

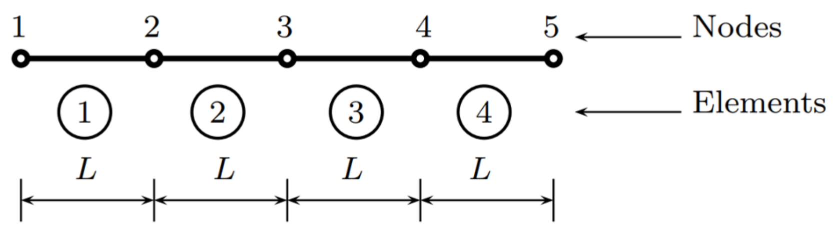
$$\mathbf{K}^e = \frac{EA}{a}\int_{-1}^{1}\mathbf{N}'^T\mathbf{N}'d\xi = \frac{EA}{a}\int_{-1}^{1}\begin{bmatrix}-1/2\\1/2\end{bmatrix}\begin{bmatrix}-1/2 & 1/2\end{bmatrix}d\xi = \frac{EA}{2a}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}$$

$$\mathbf{M}^e = \rho Aa\int_{-1}^{1}\mathbf{N}^T\mathbf{N}d\xi = \frac{\rho Aa}{4}\int_{-1}^{1}\begin{bmatrix}1-\xi\\1+\xi\end{bmatrix}\begin{bmatrix}1-\xi & 1+\xi\end{bmatrix}d\xi = \frac{\rho Aa}{3}\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}$$

$$\mathbf{f}^e = a\int_{-1}^{1}p\mathbf{N}^T d\xi = \frac{ap}{2}\int_{-1}^{1}\begin{bmatrix}1-\xi\\1+\xi\end{bmatrix}d\xi = \frac{ap}{2}\begin{bmatrix}1\\1\end{bmatrix}$$

Use the lumped mass matrix to avoid the integration for the mass matrix $\mathbf{M}^e = \rho Aa\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

- Example: A bar with 5 nodes, 4 elements



The structure vector of displacements:

$$\mathbf{u}^T = \begin{bmatrix}u_1 & u_2 & u_3 & u_4 & u_5\end{bmatrix}$$

Summing the contribution of all elements, the equilibrium becomes:

$$\delta\mathbf{u}^T\sum_{e=1}^{4}\mathbf{M}^e\ddot{\mathbf{u}} + \delta\mathbf{u}^T\sum_{e=1}^{4}\mathbf{K}^e\mathbf{u} - \delta\mathbf{u}^T\sum_{e=1}^{4}\mathbf{f}^e = 0 \quad\leftrightarrow\quad \delta\mathbf{u}^T\mathbf{M}\ddot{\mathbf{u}} + \delta\mathbf{u}^T\mathbf{K}\mathbf{u} - \delta\mathbf{u}^T\mathbf{f} = 0$$

$\mathbf{M}, \mathbf{K}, \mathbf{f}$ *are the structure stiffness matrix, mass matrix, and the force vector, respectively*

2. Bars or Trusses – A bar element

❖ Example (cont.)

Evaluate the structure stiffness matrix

$$\mathbf{K}=\frac{EA}{L}\left\{\underbrace{\begin{bmatrix}1&-1&0&0&0\\-1&1&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}}_{Element\,1}+\underbrace{\begin{bmatrix}0&0&0&0&0\\0&1&-1&0&0\\0&-1&1&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}}_{Element\,2}+\dots\right\}=\frac{EA}{L}\begin{bmatrix}1&-1&0&0&0\\-1&2&-1&0&0\\0&-1&2&-1&0\\0&0&-1&2&-1\\0&0&0&-1&1\end{bmatrix}$$

Evaluate the structure mass matrix and the vector of equivalent forces

$$\mathbf{M}=\frac{\rho AL}{6}\begin{bmatrix}2&1&0&0&0\\1&4&1&0&0\\0&1&4&1&0\\0&0&1&4&1\\0&0&0&1&2\end{bmatrix}\qquad\qquad \mathbf{f}=ap\begin{bmatrix}1\\2\\2\\2\\1\end{bmatrix}$$

A global system of equations is obtained and can be solved after applying the BCs.

$$\mathbf{M}\ddot{\mathbf{u}}+\mathbf{K}\mathbf{u}=\mathbf{f}$$

2. Bars or Trusses – A bar element

❖ Example (cont.)

- When $\mathbf{M} = 0$, it becomes a static problem $\mathbf{Ku} = \mathbf{f}$
- When $\mathbf{f} = 0$, it becomes a free vibrations problem $\mathbf{M\ddot{u}} + \mathbf{Ku} = \mathbf{0}$

Solution form: $\mathbf{u} = \hat{\mathbf{u}} e^{i\omega t}$

The final algebraic problem becomes $\left(\mathbf{K} - \omega^2 \mathbf{M}\right)\hat{\mathbf{u}} = \mathbf{0}$

$\hat{\mathbf{u}}$ represents the eigenvector and ω^2 represent the eigenvalue.

❖ Post-computation of Stress

- Using the finite element approximation and the coordinate transformation, the stress in the generic element is

$$\sigma_x = E\varepsilon_x = E\frac{\partial u}{\partial x} = E\frac{\partial\left(\mathbf{N}\mathbf{u}^e\right)}{\partial \xi}\frac{\partial \xi}{\partial x} = \frac{E}{a}\frac{d\mathbf{N}}{d\xi}\mathbf{u}^e = \frac{E}{a}\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} u_1^e \\ u_2^e \end{bmatrix} = \frac{E}{2a}\left(u_2^e - u_1^e\right)$$

- u_1 and u_2 are the nodal displacements of the generic element.
- The stress at the element is constant due to using the linear interpolation.

2. Bars or Trusses – Numerical Integration

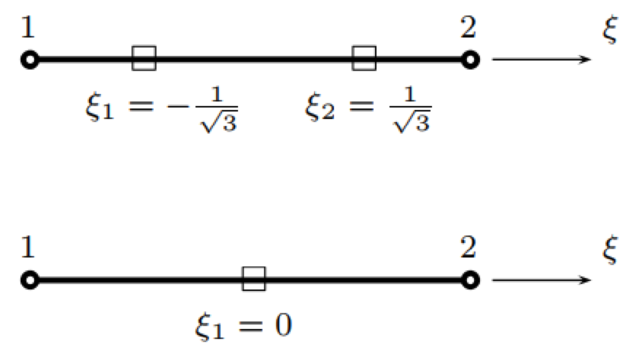
- Integrals in the variational method can be evaluated using numerical integration.
- Consider a function $f(x),\ x \in [-1,1]$, its integral can be approximated by a sum of p Gauss points, in which the function at those points is multiplied by some weights.

$$I = \int_{-1}^{1} f(x)\,dx = \sum_{i=1}^{p} f(x_i) W_i$$

Coordinates and weights for Gauss integration

n	x_i	W_i
1	0.0	2.0
2	±0.5773502692	1.0
3	±0.7745966692 0.0	0.5555555556 0.8888888889
4	±0.8611363116 ±0.3399810436	0.3478548451 0.6521451549

1D Gauss quadrature for 2 and 1 integration points



- If $f(x)$ is a polynomial of degree p, the Gauss integration method yields exact values when $\frac{1}{2}(p+1)$ Gauss integration points are used. Nearest larger integer is selected.

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thank you