Lesson 04

Hamilton's Principle, Weak Form

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1. Hamilton's Principle – The man

Joseph-Louis Lagrange (25 January 1736 - 10 April 1813) was an Italian Enlightenment Era mathematician and astronomer. He made significant contributions to the fields of analysis, number theory, and both classical and celestial mechanics. In 1766, on the recommendation of Euler and d'Alembert, Lagrange succeeded Euler as the director of mathematics at the Prussian Academy of Sciences in Berlin, Prussia, where he stayed for over twenty years, producing volumes of work and winning several prizes of the French Academy of Sciences.

Sir William Rowan Hamilton MRIA (4 August 1805 - 2 September 1865) was an Irish mathematician. While still an undergraduate he was appointed Andrew's professor of Astronomy and Royal Astronomer of Ireland and lived at Dunsink Observatory. He made important contributions to optics, classical mechanics and algebra.

1. Hamilton's Principle – Equations of Motion

❖ The motion of a particle acted upon by conservative forces between two arbitrary instants of time is such that the line integral over the Lagrangian function *L* is an extremum for the path motion.

$$\delta \int_{t_1}^{t_2} L\,dt = 0 \qquad \text{subjected to} \qquad \delta u(t_1) = \delta u(t_2) = 0$$

with $\quad L = T - \Pi = T - (U + V) \quad \rightarrow \quad \int_{t_1}^{t_2} \left[\delta T - (\delta U + \delta V) \right] dt = 0$

❖ Kinetic Energy $\qquad T = \frac{1}{2} \int_V \rho \dot{u}_i \dot{u}_i dV \quad \rightarrow \quad \delta T = \int_V \rho \dot{u}_i \delta \dot{u}_i dV$

❖ Statement of Hamilton's Principle

$$\int_{t_1}^{t_2} \left[\int_V \rho \dot{u}_i \delta \dot{u}_i dV + \int_V f_i \cdot \delta u_i \, dV + \int_{S_2} \hat{t}_i \cdot \delta u_i \, dS - \int_V \sigma_{ij} \cdot \delta \varepsilon_{ij} \, dV \right] dt = 0$$

❖ Equations of Motion

$$\rho \ddot{u}_i - \sigma_{ij,j} - f_i = 0 \quad \text{in} \quad V$$

$$n_j \sigma_{ij} - \hat{t}_i = 0 \quad \text{in} \quad S_2$$

2. Dynamic Problems – Axially-loaded Bar

❖ Total Potential Energy $\Pi = \int_0^L \frac{EA}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx - Pu(L)$

❖ Kinetic Energy $T = \frac{1}{2}\int_V \rho \dot{u}_i \dot{u}_i dV = \frac{1}{2}\int_0^L \rho A \dot{u}^2 dx$

$$\int_{t_1}^{t_2} \delta T\, dt = \int_{t_1}^{t_2}\int_0^L \rho A \dot{u}\, \delta\dot{u}\, dx\, dt = -\int_{t_1}^{t_2}\int_0^L \rho A \ddot{u}\, \delta u\, dx\, dt \quad \text{as} \quad \delta u(t_1) = \delta u(t_2) = 0$$

❖ Hamilton's Principle

$$0 = -\int_{t_1}^{t_2}\left\{\int_0^L \left[\rho A \ddot{u}\, \delta u + EA\frac{\partial u}{\partial x}\frac{\partial \delta u}{\partial x}\right]dx - P\delta u(L)\right\}dt$$

Integration by parts yield equations of motion and associated boundary conditions

$$0 = \int_0^L \left[\rho A \ddot{u} - \frac{d}{dx}\left(EA\frac{du}{dx}\right)\right]\delta u dx + \left[\left(EA\frac{du}{dx} - P\right)\delta u\right]_{x=L} + \left[\left(EA\frac{du}{dx}\right)\delta u\right]_{x=0}$$

❖ Equations of motion

$$\rho A \ddot{u} - \frac{d}{dx}\left(EA\frac{du}{dx}\right) = 0, \quad 0 < x < L$$

2. Dynamic Problems – Cantilever Beam

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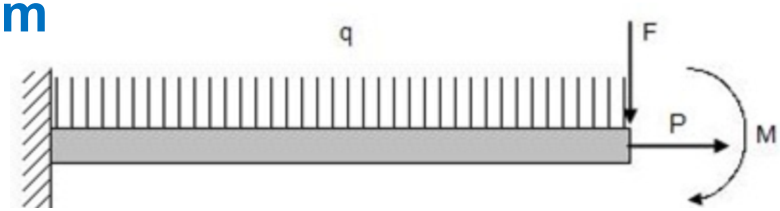


❖ Total virtual work

$$\delta W = \delta W_I + \delta W_E = \delta U + \delta V = \int_0^L \left[N \frac{d\delta u}{dx} + M\left(\frac{-d^2 \delta w}{\partial x^2} \right) - f\delta u - q\delta w \right] dx - P\delta u(L) - F\delta w(L)$$

with $N = EA\dfrac{du}{dx}; \quad M = -EI\dfrac{d^2 w}{dx^2}$

❖ Kinematic energy

$$T = \frac{1}{2}\int_V \rho \dot{u}_i \dot{u}_i dV = \frac{1}{2}\int_0^L \rho A\left(\dot{u}^2 + \dot{w}^2\right) dx \;\rightarrow\; \delta T = \int_0^L \rho A\left(\dot{u}\delta\dot{u} + \dot{w}\delta\dot{w}\right) dx$$

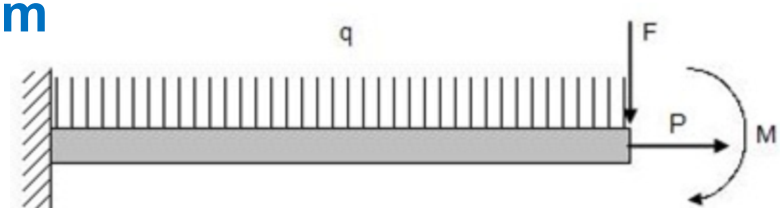
$$\int_{t_1}^{t_2} \delta T\, dt = \int_{t_1}^{t_2}\int_0^L \rho A\left(\dot{u}\delta\dot{u} + \dot{w}\delta\dot{w}\right) dxdt = \int_0^L \rho A\left[\int_{t_1}^{t_2} \left(\dot{u}\delta\dot{u} + \dot{w}\delta\dot{w}\right) dt \right] dx$$

$$= \int_0^L \rho A\left[-\int_{t_1}^{t_2} \left(\ddot{u}\delta u + \ddot{w}\delta w\right) dt + \underbrace{\left[\dot{u}\delta u + \dot{w}\delta w\right]_{t_1}^{t_2}}_{=0} \right] dx = -\int_{t_1}^{t_2}\int_0^L \rho A\left(\ddot{u}\delta u + \ddot{w}\delta w\right) dxdt$$

❖ Hamilton's Principle

$$0 = \int_{t_1}^{t_2} \left[\delta T - \left(\delta U + \delta V\right) \right] dt$$

2. Dynamic Problems – Cantilever Beam



- Hamilton's Principle

$$0=\int_{t_1}^{t_2}\left[\delta T-\left(\delta U+\delta V\right)\right]dt$$

$$0=\int_0^L\left[\frac{d}{dx}\left(EA\frac{du}{dx}\right)\delta u-\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right)\delta w+f\delta u+q\delta w-\rho A\ddot{u}\delta u-\rho A\ddot{w}\delta w\right]dx$$

$$-\left[\left(EA\frac{du}{dx}\right)\delta u+\left(EI\frac{d^2w}{dx^2}\right)\frac{d\delta w}{dx}-\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right)\delta w\right]_0^L+P\delta u(L)+F\delta w(L)$$

- Euler Equations (Equations of Motion)

$$\frac{d}{dx}\left(EA\frac{du}{dx}\right)+f=\rho A\ddot{u}, \qquad -\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right)+q=\rho A\ddot{w}$$

- Boundary Conditions

At x = 0

Either $\left(EA\dfrac{du}{dx}\right)$ or δu

Either $\left(EI\dfrac{d^2w}{dx^2}\right)$ or $\dfrac{d\delta w}{dx}$

Either $\dfrac{d}{dx}\left(EI\dfrac{d^2w}{dx^2}\right)$ or δw

At x = L

Either $\left(EA\dfrac{du}{dx}\right)-P$ or δu

Either $\left(EI\dfrac{d^2w}{dx^2}\right)$ or $\dfrac{d\delta w}{dx}$

Either $\dfrac{d}{dx}\left(EI\dfrac{d^2w}{dx^2}\right)+F$ or δw

3. Weak Form – Linear Functional (Form)

❖ Linear Functional $l(\alpha u+\beta v)=\alpha l(u)+\beta l(v)$

❖ Linear Form (wiki): In linear algebra, a linear functional or linear form (also called a one-form or co-vector) is a linear map from a vector space to its field of scalars. When vectors are represented as column vectors, then linear functionals are represented as row vectors, and their action on vectors is given by the dot product, or the matrix product with the row vector on the left and the column vector on the right. In general, if *V* is a vector space over a field *k*, then a linear functional *f* is a function from *V* to *k* that is linear:

$$f(\vec{v}+\vec{w})=f(\vec{v})+f(\vec{w}) \quad \text{for all} \quad \vec{v},\vec{w}\in V$$

$$f(a\vec{v})=a\,f(\vec{v}) \quad \text{for all} \quad \vec{v}\in V,\, a\in k$$

❖ Linear functionals first appeared in functional analysis, the study of vector spaces of functions. A typical example of a linear functional is integration: the linear transformation defined by the Riemann integral

$$I(f)=\int_a^b f(x)\,dx$$

is a linear functional from the vector space C[a,b] of continuous functions on the interval [a,b] to the real numbers.

3. Weak Form – Bilinear Functional (Form)

- Linear Functional

$$B\left(\alpha u_1 + \beta u_2, v\right) = \alpha B\left(u_1, v\right) + \beta B\left(u_2, v\right)$$
$$B\left(u, \alpha v_1 + \beta v_2\right) = \alpha B\left(u, v_1\right) + \beta B\left(u, v_2\right)$$

- In mathematics, more specifically in abstract algebra and linear algebra, a bilinear form on a vector space V is a bilinear map V ×V → K, where K is the field of scalars. In other words, a bilinear form is a function B : V ×V → K which is linear in each argument separately:

$$\begin{aligned} B(u+v,w) &= B(u,w) + B(v,w) \\ B(u,v+w) &= B(u,v) + B(u,w) \\ B(\lambda u,v) &= B(u,\lambda v) = \lambda B(u,v) \end{aligned}$$

- The definition of a bilinear form can be extended to include modules over a commutative ring, with linear maps replaced by module homomorphisms.

- When K is the field of complex numbers C, one is often more interested in sesquilinear forms, which are similar to bilinear forms but are conjugate linear in one argument.

3. Weak Form – Quadratic Functional (Form)

- Linear Functional $Q(u) = B(u,u)$

- Quadratic forms are homogeneous quadratic polynomials in n variables. In the cases of one, two, and three variables they are called unary, binary, and ternary and have the following explicit form:

$$q(x) = ax^2$$
$$q(x,y) = ax^2 + bxy + cy^2$$
$$q(x,y,z) = ax^2 + by^2 + cz^2 + dxy + exz + fyz$$

where *a,..., f* are the coefficients. Note that quadratic functions, such as $ax^2 + bx + c$ in the one variable case, are not quadratic forms, as they are typically not homogeneous (unless b and c are both 0).

- An *n*-ary quadratic form over a field *K* is a homogeneous polynomial of degree 2 in *n* variables with coefficients in *K*:

$$q\left(x_1, \dots, x_n\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i x_j, \quad a_{ij} \in K$$

- This formula may be rewritten using matrices: let *x* be the column vector with components $x_1, \dots, x_n$ and $A = (a_{ij})$ be the n × n matrix over *K* whose entries are the coefficients of *q*. Then

$$q\left(x\right) = x^T A x$$

3. Weak Form – Need for Weak Form

- 1D Boundary Value Problem (BVP) with Nonhomogeneous Boundary Conditions

$$-\frac{d}{dx}\left[a(x)\frac{du}{dx}\right]+c(x)u=f(x),\quad 0<x<L \qquad u(0)=u_0,\quad a\frac{du}{dx}\bigg|_{x=L}=Q_0$$

- We seek an approximate solution over entire domain Ω = (0,L) in the form

$$U_N=\sum_{j=1}^{N}c_j\phi_j(x)+\phi_0(x)$$

- Let $L=1,\ u_0=1,\ Q_0=0,\ a(x)=x,\ c(x)=1,\ f(x)=0$ the problem becomes

$$-\frac{d}{dx}\left[x\frac{du}{dx}\right]+u=0,\quad 0<x<1 \qquad \textit{subjected to} \qquad u(0)=1,\quad x\frac{du}{dx}\bigg|_{x=1}=0$$

- Choose the approximate solution

$$U_2=c_1\phi_1+c_2\phi_2+\phi_0 \qquad \textit{with} \qquad \phi_0=1,\quad \phi_1=x^2-2x,\quad \phi_2=x^3-3x$$

- To make U_2 satisfy the differential equation, we have

$$c_2x^3+(c_1-9c_2)x^2-(6c_1+3c_2)x+2c_1+3c_2+1=0$$

$$\rightarrow\ c_2=0;\quad c_1-9c_2=0;\quad 6c_1+3c_2=0;\quad 2c_1+3c_2+1=0\ \rightarrow\ \textit{No Solution}$$

3. Weak Form – Need for Weak Form

❖ Need to have Weighted-Integral Form (Weighted-Residual Form)

- To have the means to obtain *N* linearly independent algebraic relations among the coefficients c_j of the approximation.
- Choosing N linearly independent weight functions in the integral statement

$$u \approx U_N = \sum_{j=1}^{N} c_j \phi_j(x) + \phi_0(x)$$

3. Weak Form – Step 1: Weighted Integral Form

- ❖ Move all expressions of the differential equation to one side.
- ❖ Multiply the entire equation with a function w, called the weight function.
- ❖ Integrate over the domain $W = (0, L)$ of the problem

$$0 = \int_0^L w \left[-\frac{d}{dx}\left(a(x)\frac{du}{dx} \right) - q(x) \right] dx$$

- N linearly independent functions for w and obtain N equations for $c_1, c_2, \ldots, c_N$
- Weighted integral does not include any boundary conditions.

Weight function

- Can be any nonzero, integrable function.
- Less continuity requirements than dependent variables.

3. Weak Form – Step 2: Weak Formulation

- ❖ Weaker continuity of dependent variables.
- ❖ Natural boundary conditions are included in the formulation.
- ❖ Equal distribution of differentiation between weight function and dependent variable.

$$0=\int_0^L\left[w\left(-\frac{d}{dx}\left(a\frac{du}{dx}\right)\right)-wq\right]dx=\int_0^L\left(\frac{dw}{dx}a\frac{du}{dx}-wq\right)dx-\left[w\left(a\frac{du}{dx}\right)\right]_0^L$$

Primary Variables

- Dependent variables (and derivatives of dependent variables).
- Essential Boundary Conditions.

Secondary Variables

- Coefficient of weight function (physical meaning).
- Natural Boundary Conditions.

3. Weak Form – Step 3: Imposing Boundary Conditions

- Weight function requires to satisfy the homogeneous form of E.B.C. $\left(u(0)=u_0 \;\rightarrow\; w(0)=0\right)$

$$
\begin{aligned}
0 &= \int_0^L \left(\frac{dw}{dx} a \frac{du}{dx} - wq \right) dx - \left[w \left(a \frac{du}{dx} \right) \right]_0^L \\
&= \int_0^L \left(\frac{dw}{dx} a \frac{du}{dx} - wq \right) dx - w(L) a \frac{du}{dx}\bigg|_{x=L} + \cancel{w(0)}\, a \frac{du}{dx}\bigg|_{x=0} \\
&= \int_0^L \left(\frac{dw}{dx} a \frac{du}{dx} - wq \right) dx - w(L) Q_0
\end{aligned}
$$

- Weight function acts like the 1st variation of dependent variable, i.e., $w = \delta u$

- Functional Expression

$$0 = B(w,u) - l(u)$$

with

$$B(w,u) = \int_0^L a \frac{dw}{dx} \frac{du}{dx} dx$$

$$l(u) = \int_0^L wq\, dx + w(L) Q_0$$

3. Weak Form – Big Picture!

Total Potential Energy

$$\Pi = \int_0^L \frac{EA}{2}\left(\frac{du}{dx}\right)^2 dx - Pu(L)$$

principle of total potential energy

Weak Form

$$0 = \delta\Pi(u) = \int_0^L EA\frac{du}{dx}\frac{d\delta u}{dx}dx - P\delta u(L)$$

Finite Element Model

$$0 = \int_0^L EA\frac{d\phi_i}{dx}\frac{d\phi_j}{dx}dx - p\phi_i$$

Integration by parts

Weighted Integral Form

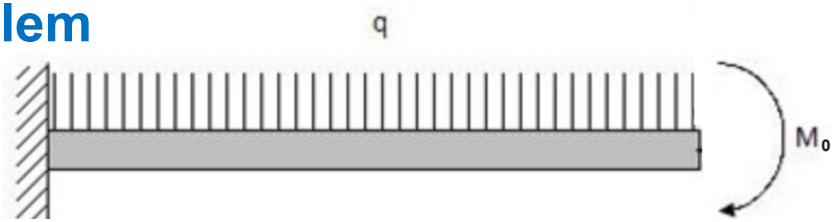
$$0 = -\int_0^L \frac{d}{dx}\left(EA\frac{du}{dx}\right)\delta u dx + \left[\left(EA\frac{du}{dx} - P\right)\delta u\right]_{x=L} - \left[\left(EA\frac{du}{dx}\right)\delta u\right]_{x=0}$$

weak formulation

Strong Form

$$-\frac{d}{dx}\left(EA\frac{du}{dx}\right) = 0; \quad \left(EA\frac{du}{dx} - P\right)\delta u = 0 \quad \text{at} \quad x = L$$

3. Weak Form – Cantilever Beam Problem



❖ Strong Form (Euler Equations + B.C.S)

$$-\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right)+q=0 \quad \text{subjected to} \quad w(0)=\left.\frac{dw}{dx}\right|_{x=0}=0; \quad \left.EI\frac{d^2w}{dx^2}\right|_{x=L}=M_0; \quad \left.\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right)\right|_{x=L}=0$$

❖ Weighted Integral Form

$$0=-\int_0^L\left[-\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right)\delta w+q\delta w\right]dx$$

❖ Weak Form

$$0=\int_0^L\left[-\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right)\frac{d\delta w}{dx}-q\delta w\right]dx+\delta w\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right)\Bigg|_0^L$$

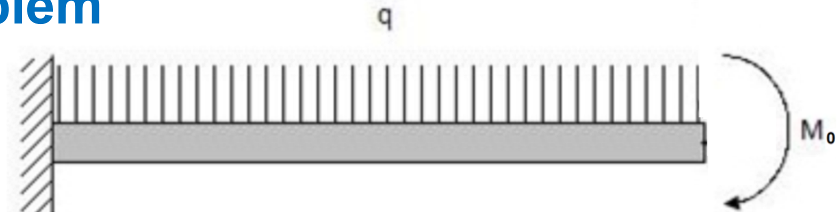
$$=\int_0^L\left[EI\frac{d^2w}{dx^2}\frac{d^2\delta w}{dx^2}-q\delta w\right]dx+\delta w\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right)\Bigg|_0^L-\frac{d\delta w}{dx}\left(EI\frac{d^2w}{dx^2}\right)\Bigg|_o^L$$

❖ Identify the Boundary Conditions

Either $\left(EI\dfrac{d^2w}{dx^2}\right)$ or $\dfrac{d\delta w}{dx}$

Either $\dfrac{d}{dx}\left(EI\dfrac{d^2w}{dx^2}\right)$ or δw

3. Weak Form – Cantilever Beam Problem



- Using Stress Resultants

$$M = EI\frac{d^2w}{dx^2}; \qquad V = \frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right)$$

- Weak Form becomes

$$0 = \int_0^L \left[EI\frac{d^2w}{dx^2}\frac{d^2\delta w}{dx^2} - q\delta w \right] dx + \delta w(L)V(L) - \delta w(0)V(0) - \left.\frac{d\delta w}{dx}\right|_{x=L} M(L) + \left.\frac{d\delta w}{dx}\right|_{x=0} M(0)$$

$$= \int_0^L \left[EI\frac{d^2w}{dx^2}\frac{d^2\delta w}{dx^2} - q\delta w \right] dx - \left.\frac{d\delta w}{dx}\right|_{x=L} M_0$$

- Functional Expression

$$0 = B(w,\delta w) - l(\delta w)$$

with $B(w,\delta w) = \int_0^L EI\frac{d^2w}{dx^2}\frac{d^2\delta w}{dx^2}dx$ *and* $l(\delta w) = \int_0^L q\delta w\,dx + \left.\frac{d\delta w}{dx}\right|_{x=L} M_0$

- Total Potential Energy

$$\Pi(w) = \int_0^L \left[\frac{1}{2}EI\left(\frac{d^2w}{dx^2}\right)^2 - qw \right] dx - \left.\frac{dw}{dx}\right|_{x=L} M_0$$

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thank you