Lesson 02

Supplementary Materials

1. Preliminary Algebra

1.1. Scalar, Vector, Matrix, Tensor

✓ **Scalar** (α): A real number to describe a physical quantity.

i.e., length, temperature, height, …

✓ **Vector** ($\vec{a}$, $\mathbf{a}$, $\underline{a}$): An array of scalars to describe a physical quantity.

i.e., force, velocity, moment, …

✓ **Matrix** ($\mathbf{A}$, $\vec{\vec{A}}$, $\underline{\underline{A}}$): An array of vector to describe a physical quantity.

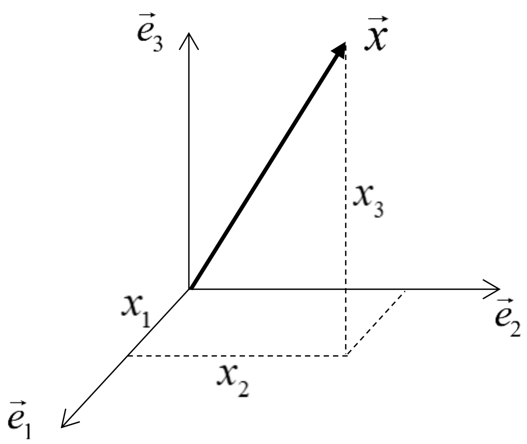
i.e., stress, strain, stiffness, curvature, …

✓ **Tensor**: “a generalized quantity”. (# order = # of indices)

Tensor = Scalar (0^{th} order) $\cup$ Vector (1^{st} order) $\cup$ Matrix (2^{nd} order) $\cup$ 3^{rd} order $\cup$ 4^{th} order $\cup$ …

1.2. Vector Algebra

✓ Position vector in 3D: $\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3$ in which $\vec{e}_1, \vec{e}_2, \vec{e}_3$ are base vectors of the Cartesian coordinate system.



(Cartesian coordinate system)

- ✓ The geometric representation of $\vec{x}$ depends completely on the coordinate system chosen.

- ✓ In general, the geometric vector can be represented by components with their base vectors.

$$\vec{x} = \sum_{i=1}^{3} x_i \vec{v}_i \quad \rightarrow \quad \vec{x} = x_1 \vec{v}_1 + x_2 \vec{v}_2 + x_3 \vec{v}_3$$

- ✓ Summation convention (Einstein convention)

$\vec{x} = \sum_{i=1}^{3} x_i \vec{v}_i = x_i \vec{v}_i$ is component notation, i is dummy index or free index.

- ✓ Tensorial representation of a vector

$$\mathbf{x} = x_i \mathbf{v}_i$$

In the Cartesian coordinate system given,

$$\mathbf{x} = x_i \mathbf{e}_i \quad \rightarrow \quad x_i$$

$$\mathbf{u} = u_i \mathbf{e}_i \quad \rightarrow \quad u_i$$

- ✓ Vector sum

$$\vec{c} = \vec{a} + \vec{b} \quad \rightarrow \quad \mathbf{c} = \mathbf{a} + \mathbf{b} \quad \rightarrow \quad c_i = a_i + b_i$$

- ✓ Scalar multiplication

$$\vec{b} = \alpha \vec{a} \quad \rightarrow \quad \mathbf{b} = \alpha \mathbf{a} \quad \rightarrow \quad b_i = \alpha a_i$$

- ✓ Dot product, or scalar product, or inner product:

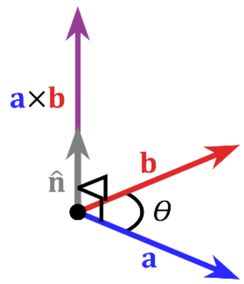
In the Cartesian coordinate system,

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \quad \text{with} \quad \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \quad \text{(Kronecker delta)}$$

Question 1: Compute δ_{ii} ($\delta_{11} + \delta_{22} + \delta_{33} = 3$)

Dot product between 02 vectors $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$:

$$\alpha = \vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \quad \rightarrow \quad \alpha = a_i b_i$$

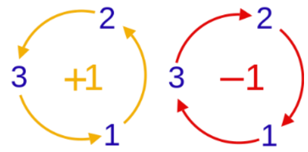


✓ Cross product

$$\vec{c} = \vec{a} \times \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \sin\theta \cdot \vec{n}$$

ε_{ijk} is “permutation symbol” (also Levi-Civita symbol):

$$\varepsilon_{ijk} = \begin{cases} 0 & \text{for } i = j,\ j = k \text{ or } k = i \\ 1 & \text{for } i,j,k \in \{(1,2,3),(2,3,1),(3,1,2)\} \\ -1 & \text{for } i,j,k \in \{(1,3,2),(3,2,1),(2,1,3)\} \end{cases}$$



In the Cartesian coordinate system,

$$\vec{e}_i \times \vec{e}_j = \varepsilon_{ijk} \vec{e}_k$$

Cross product between 02 vectors,

$$\vec{c} = \vec{a} \times \vec{b} = a_j b_k \left(\vec{e}_j \times \vec{e}_k \right) = \varepsilon_{ijk} a_j b_k \vec{e}_i \quad \rightarrow \quad c_i = \varepsilon_{ijk} a_j b_k$$

$$\vec{c} = (a_2 b_3 - a_3 b_2)\vec{e}_1 + (a_3 b_1 - a_1 b_3)\vec{e}_2 + (a_1 b_2 - a_2 b_1)\vec{e}_3$$

✓ $\varepsilon - \delta$ identity (relation)

$$\varepsilon_{ijk}\varepsilon_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix} = \delta_{il} \begin{vmatrix} \delta_{jm} & \delta_{jn} \\ \delta_{km} & \delta_{kn} \end{vmatrix} - \delta_{im} \begin{vmatrix} \delta_{jl} & \delta_{jn} \\ \delta_{kl} & \delta_{kn} \end{vmatrix} + \delta_{in} \begin{vmatrix} \delta_{jl} & \delta_{jm} \\ \delta_{kl} & \delta_{km} \end{vmatrix}$$

$$= \delta_{il}\left(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}\right) - \delta_{im}\left(\delta_{jl}\delta_{kn} - \delta_{jn}\delta_{kl}\right) + \delta_{in}\left(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}\right)$$

A special case with summation convention,

$$\sum_{k=1}^{3} \varepsilon_{ijk}\varepsilon_{mnk} = \varepsilon_{ijk}\varepsilon_{mnk} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$$

1.3. Matrix Algebra

✓ A set of linear equations

$$\begin{aligned} 3x_1 + 3x_2 - \ \ x_3 &= \ \ 1 \\ 2x_1 + 7x_2 + 3x_3 &= -3 \\ x_1 - \ \ x_2 - \ \ x_3 &= \ \ 4 \end{aligned}$$

In matrix form,

$$\begin{bmatrix} 3 & 3 & -1 \\ 2 & 7 & 3 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix} \quad \rightarrow \quad \mathbf{A}\vec{x} = \mathbf{b} \quad \rightarrow \quad \mathbf{Ax} = \mathbf{b}$$

- ✓ In 3D geometry,

$$\mathbf{A} = \sum_{i=1}^{3} \sum_{j=1}^{3} a_{ij} (\vec{e}_i \otimes \vec{e}_j) \text{ in which } a_{ij} \text{ are components of matrix } \mathbf{A}$$

- ✓ Note: $\otimes$ is the tensor product.

$$\vec{e}_1 \otimes \vec{e}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ is a base matrix.}$$

$$\mathbf{A} = a_{11}(\vec{e}_1 \otimes \vec{e}_1) \ + \ a_{12}(\vec{e}_1 \otimes \vec{e}_2) \ + \ \cdots \ + \ a_{33}(\vec{e}_3 \otimes \vec{e}_3)$$

$$= a_{11} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_{12} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \cdots + a_{33} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- ✓ Matrix sum

$$\mathbf{C} = \mathbf{A} + \mathbf{B} \quad \rightarrow \quad c_{ij} = a_{ij} + b_{ij}$$

- ✓ Scalar multiplication

$$\mathbf{B} = \alpha \mathbf{A} \quad \rightarrow \quad b_{ij} = \alpha a_{ij}$$

- ✓ Dot product

$$\mathbf{A} = \mathbf{BC} \quad \rightarrow \quad a_{ij} = b_{ik} c_{kj} = b_{i1} c_{1j} + b_{i2} c_{2j} + b_{i3} c_{3j}$$

✓ Identity matrix

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \quad \text{(Kronecker delta)}$$

✓ A set of linear equations

$$\mathbf{A}\vec{x} = \mathbf{b} \rightarrow \mathbf{A}\mathbf{x} = \mathbf{b} \rightarrow A_{ij}x_j = b_i \quad \text{(03 equations)}$$

✓ Summary

Direct tensor notation	Tensor component notation	Matrix notation
$\alpha = \mathbf{a} \cdot \mathbf{b}$	$\alpha = a_i b_i$	$\alpha = \mathbf{a}^T \mathbf{b}$
$\mathbf{A} = \mathbf{a} \otimes \mathbf{b}$	$A_{ij} = a_i b_j$	$\mathbf{A} = \mathbf{a}\mathbf{b}^T$
$\mathbf{b} = \mathbf{A} \cdot \mathbf{a}$	$b_i = A_{ij} a_j$	$\mathbf{b} = \mathbf{A}\mathbf{a}$
$\mathbf{b} = \mathbf{a} \cdot \mathbf{A}$	$b_j = a_i A_{ij}$	$\mathbf{b}^T = \mathbf{a}^T \mathbf{A}$

1.4. Tensor Calculus

✓ A vector is a rank-1 tensor (i.e., a single index $v_i \vec{e}_i$).

✓ A matrix is a rank-2 tensor (i.e., Two indices $a_{ij} \vec{e}_i \otimes \vec{e}_j$, $\mathbf{1} = [\delta_{ij}]$ is a 3x3 matrix).

✓ Any tensor can be indicated as $\mathbf{T} = T_{ijk...lm} \vec{e}_i \otimes \vec{e}_j \otimes \vec{e}_k \otimes \cdots \otimes \vec{e}_l \otimes \vec{e}_m$.

✓ Tensor product of two vectors

$$\mathbf{u} \otimes \mathbf{v} \neq \mathbf{v} \otimes \mathbf{u}$$

$$(\alpha \mathbf{u}) \otimes \mathbf{v} = \alpha (\mathbf{u} \otimes \mathbf{v})$$

$$\mathbf{u} \otimes (\mathbf{v} + \mathbf{w}) = \mathbf{u} \otimes \mathbf{v} + \mathbf{u} \otimes \mathbf{w}$$

$$(\mathbf{u}\otimes\mathbf{v})\cdot\mathbf{w} = (\mathbf{v}\cdot\mathbf{w})\mathbf{u}$$

$$\mathbf{u}\cdot(\mathbf{v}\otimes\mathbf{w}) = (\mathbf{u}\cdot\mathbf{v})\mathbf{w}$$

✓ Inner product of two tensors

$$\mathbf{A} = A_{ij}\vec{e}_i\otimes\vec{e}_j \text{ and } \mathbf{B} = B_{kl}\vec{e}_k\otimes\vec{e}_l$$

$$\mathbf{C} = \mathbf{A}\cdot\mathbf{B} = \left(A_{ij}\vec{e}_i\otimes\vec{e}_j\right)\cdot\left(B_{kl}\vec{e}_k\otimes\vec{e}_l\right) = A_{ij}B_{kl}\delta_{jk}\vec{e}_i\otimes\vec{e}_l = A_{ik}B_{kl}\vec{e}_i\otimes\vec{e}_l$$

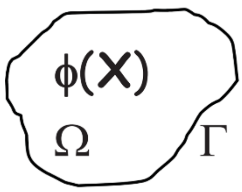
$C_{ij} = A_{ik}B_{kl}$ same to matrix multiplication between components of $\mathbf{A}$ and $\mathbf{B}$.

1.5. Gradient, Divergence, Laplace

✓ $\vec{X}$ is a position vector (spatial coordinates, field variables).

✓ Gradient is considered a vector,

$$\nabla = \frac{\partial}{\partial\vec{X}} = \vec{e}_i\frac{\partial}{\partial X_i}$$



✓ Gradient of a scalar field $\phi\left(\vec{X}\right)$ is a vector,

$$\nabla\phi = \mathrm{grad}\phi = \vec{e}_i\frac{\partial\phi}{\partial X_i} = \phi_{,i}\vec{e}_i$$

✓ Gradient of a vector field $\mathbf{u}\left(\vec{X}\right)$ (rank-1 tensor) is a rank-2 tensor,

$$\nabla\mathbf{u} = \left(\vec{e}_i\frac{\partial}{\partial X_i}\right)\otimes u_j\vec{e}_j = \frac{\partial u_j}{\partial X_i}\vec{e}_i\otimes\vec{e}_j = u_{j,i}\vec{e}_i\otimes\vec{e}_j$$

✓ The divergence of a vector field $\mathbf{u}\left(\vec{X}\right)$ (rank-1 tensor) is a scalar,

$$\nabla\cdot\mathbf{u} = \left(\vec{e}_i\frac{\partial}{\partial X_i}\right)\cdot u_j\vec{e}_j = \frac{\partial u_j}{\partial X_i}\delta_{ij} = \frac{\partial u_i}{\partial X_i} = \frac{\partial u_1}{\partial X_1} + \frac{\partial u_2}{\partial X_2} + \frac{\partial u_3}{\partial X_3}$$

✓ The divergence of a rank-2 tensor $\boldsymbol{\sigma} = \sigma_{ij}\vec{e}_i \otimes \vec{e}_j$ is a rank-1 tensor,

$$\nabla \cdot \boldsymbol{\sigma} = \left(\vec{e}_i \frac{\partial}{\partial X_i} \right) \cdot \sigma_{jk}\vec{e}_j \otimes \vec{e}_k = \frac{\partial \sigma_{jk}}{\partial X_i} \delta_{ij}\vec{e}_k = \frac{\partial \sigma_{jk}}{\partial X_j} \vec{e}_k$$

✓ The Laplace operator is the inner product of two gradient operators,

$$\nabla^2 = \nabla \cdot \nabla = \left(\vec{e}_i \frac{\partial}{\partial X_i} \right) \cdot \left(\vec{e}_i \frac{\partial}{\partial X_i} \right) = \frac{\partial}{\partial X_i} \frac{\partial}{\partial X_j} \delta_{ij} = \frac{\partial}{\partial X_j} \frac{\partial}{\partial X_j}$$
$$= \frac{\partial^2}{\partial X_1^2} + \frac{\partial^2}{\partial X_2^2} + \frac{\partial^2}{\partial X_3^2}$$

✓ Curl of a vector,

$$\nabla \times \mathbf{v} = \left(\vec{e}_i \frac{\partial}{\partial X_i} \right) \times v_j \vec{e}_j = \frac{\partial v_j}{\partial X_i} \varepsilon_{ijk} \vec{e}_k$$

1.6. Integral Theorems

✓ Divergence Theorem

$\iint_\Omega \nabla \cdot \mathbf{A}\, \mathrm{d}\Omega = \int_\Gamma \mathbf{n} \cdot \mathbf{A}\, \mathrm{d}\Gamma$ ($\mathbf{n}$ is a unit outward normal vector, $\mathbf{A}$ is a tensor).

✓ Gradient Theorem

$$\iint_\Omega \nabla \mathbf{A}\, \mathrm{d}\Omega = \int_\Gamma \mathbf{n} \otimes \mathbf{A}\, \mathrm{d}\Gamma$$

✓ Stokes Theorem

$$\int_\Gamma \mathbf{n} \cdot (\nabla \times \mathbf{v})\, \mathrm{d}\Gamma = \oint_c \mathbf{r} \cdot \mathbf{v}\, \mathrm{d}c$$

✓ Reynolds Transport Theorem

$$\frac{\mathrm{d}}{\mathrm{dt}} \iint_\Omega \mathbf{A}\, \mathrm{d}\Omega = \iint_\Omega \frac{\mathrm{d}\mathbf{A}}{\mathrm{dt}} \mathrm{d}\Omega + \int_\Gamma (\mathbf{n} \cdot \mathbf{v}) \mathbf{A}\, \mathrm{d}\Gamma$$

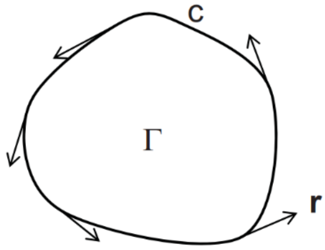
1.7. Integration-by-Parts

- ✓ $u(x)$ and $v(x)$ are continuously differentiable functions.
- ✓ 1D

$$\int_a^b u(x)v'(x)\,dx = \left[u(x)v(x)\right]_a^b - \int_a^b u'(x)v(x)\,dx$$

- ✓ 2D, 3D

$$\int_\Omega \frac{\partial u}{\partial x_i} v\,\mathrm{d}\Omega = \int_\Gamma u\,v\,n_i\,\mathrm{d}\Gamma - \int_\Omega u\frac{\partial v}{\partial x_i}\,\mathrm{d}\Omega$$



- ✓ For a vector field $\mathbf{v}(x)$

$$\int_\Omega \nabla u\cdot\mathbf{v}\,\mathrm{d}\Omega = \int_\Gamma u(\mathbf{v}\cdot\mathbf{n})\,\mathrm{d}\Gamma - \int_\Omega u\nabla\cdot\mathbf{v}\,\mathrm{d}\Omega$$

- ✓ Green's identity

$$\int_\Omega \nabla u\cdot\nabla v\,\mathrm{d}\Omega = \int_\Gamma u\nabla v\cdot\mathbf{n}\,\mathrm{d}\Gamma - \int_\Omega u\nabla^2 v\,\mathrm{d}\Omega$$

1.8. Example: Divergence Theorem

- ✓ S: unit sphere $\left(x^2+y^2+z^2=1\right)$, $\mathbf{F}=2x\mathbf{i}+y^2\mathbf{j}+z^2\mathbf{k}$
- ✓ Integrate $\int_S \mathbf{F}\cdot\mathbf{n}\,dS$

$$\begin{aligned}\int_S \mathbf{F}\cdot\mathbf{n}\,dS &= \iint_\Omega \nabla\cdot\mathbf{F}\,d\Omega\\ &= 2\iint_\Omega (1+y+z)\,d\Omega\\ &= 2\iint_\Omega d\Omega + 2\iint_\Omega y\,d\Omega + 2\iint_\Omega z\,d\Omega\\ &= 2\iint_\Omega d\Omega\\ &= \frac{8\pi}{3}\end{aligned}$$