# Lesson 6

# Finite Element Method for 2nd-order Boundary Value Problem

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# 1. Finite Element Method – Why FEM?

❖ Shortcomings of Variational Methods

- Difficulty in constructing the approximate functions.
- Even more difficult for complex domain.
- No systematic procedure to construct approximate functions

❖ Requirements of Effective Computational Method

- Mathematical and physical basis (convergent solutions).
- No limitation to geometry or loading.
- Formulation should be independent of the shape of the domain and boundary conditions.
- Allow different degrees of approximation without reformulation.
- Systematic procedure for use on computer.

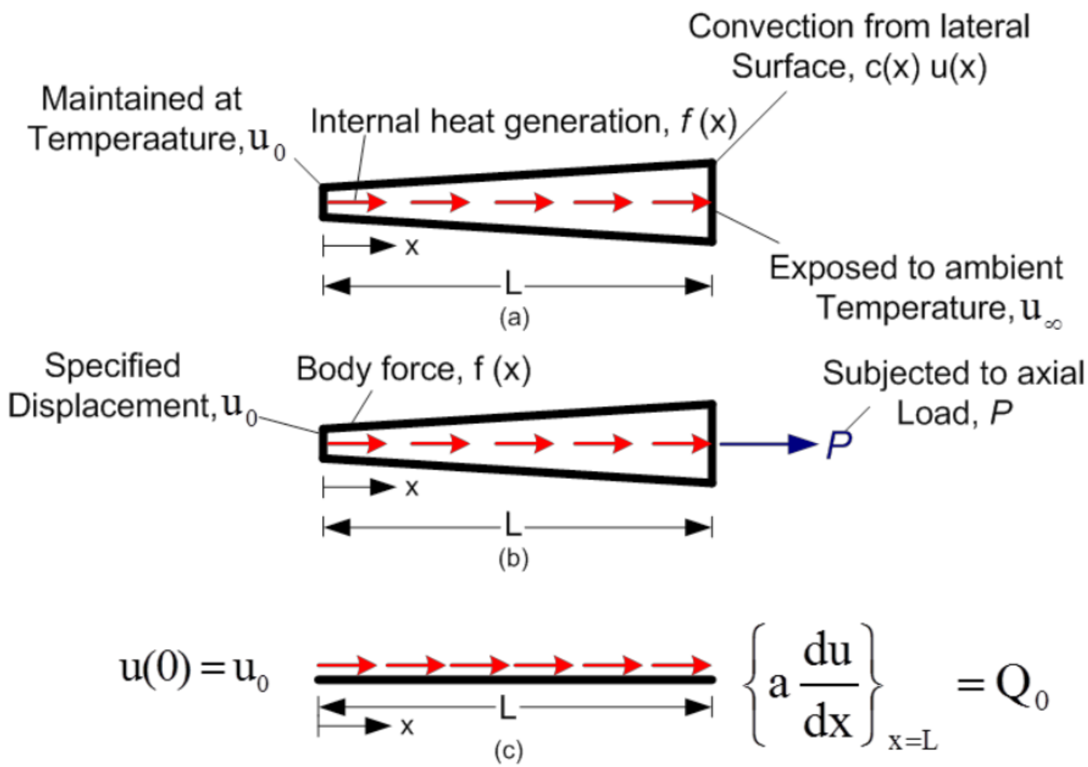
❖ The answer is Finite Element Method with these basic features

- Division of whole into parts, allows representation of geometrically complex domains as collections of geometrically simple domains.
- Derivation of approximate functions over each element: derived using interpolation theory.
- Assembly of elements: continuity of solution and balance of the internal forces.

# 1. Finite Element Method – Model Boundary Value Problem

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right)+cu-f=0, \quad 0<x<L \quad \text{subjected to} \quad u(0)=u_0, \quad a\frac{du}{dx}\Big|_{x=L}=Q_0$$

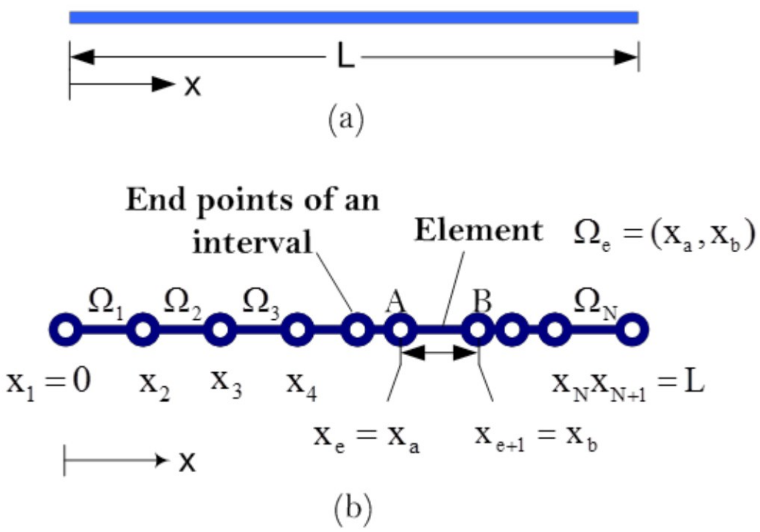
with $a=a(x)$, $c=c(x)$, $f=f(x)$. $u_0$ and $Q_0$ are known.



# 1. Finite Element Method – Model Boundary Value Problem

## Discretization of the Domain

- ❖ The domain is divided into a set of subintervals, called finite elements.
- ❖ The collection of finite element in a domain is called finite element mesh.
- ❖ Why dividing domain?
  - *Domains are composite of geometrically and materially different parts, The solution on these subdomains is represented by different functions that are continuous at the interfaces of these subdomains. Therefore, to seek approximate solution over each subdomain.*
  - *Approximation of the solution over each element of the mesh is simpler than its approximation over the entire domain.*

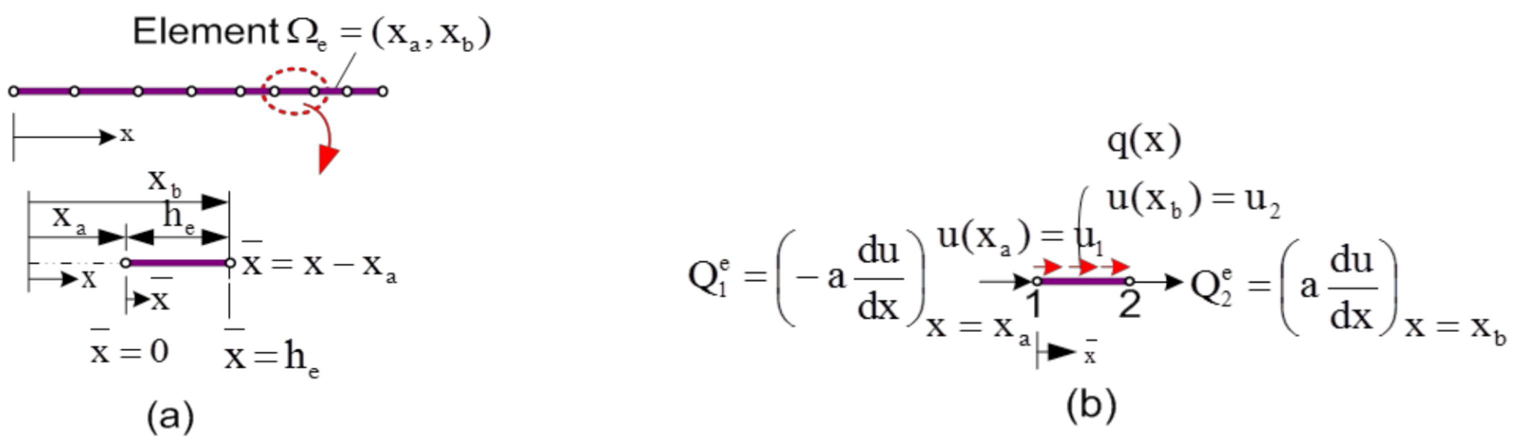


# 1. Finite Element Method – Model Boundary Value Problem

## Derivation of Element Equations

❖ Three steps in derivation of finite element equations

- *Construct the weak form of the differential equations.*
- *Assume the form of the approximate solution over a typical element.*
- *Derive the finite element solution by substituting the approximate solution into the weak form.*



# 1. Finite Element Method – Model Boundary Value Problem

## Weak Formulation

❖ Approximate Solution over each finite element.

$$u_h^e = \sum_{j=1}^{N} u_j^e \psi_j^e$$

*with* $u_j^e$ *are nodal solutions,* $\psi_j^e$ *are the approximation functions over the element.*

❖ Weighted-Integral Form

$$0 = \int_{x_a}^{x_b} w\left[-\frac{d}{dx}\left(a\frac{du}{dx}\right) + cu - f\right]dx$$

*Approximation functions must be twice-differentiable.*

❖ Weak Form

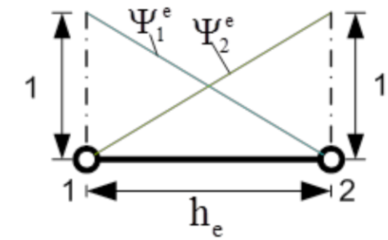
$$0 = \int_{x_a}^{x_b}\left[a\frac{dw}{dx}\frac{du}{dx} + cwu - wf\right]dx - \left[wa\frac{du}{dx}\right]_{x_a}^{x_b}$$

$$= \int_{x_a}^{x_b}\left[a\frac{dw}{dx}\frac{du}{dx} + cwu - wf\right]dx - w(x_a)Q_a - w(x_b)Q_b = B(w,u) - l(w)$$

*with* $\quad Q_a = -a\left.\frac{du}{dx}\right|_{x_a}, \quad Q_b = -a\left.\frac{du}{dx}\right|_{x_b}$

# 1. Finite Element Method – Model Boundary Value Problem

## Approximation of the Solution



(a)

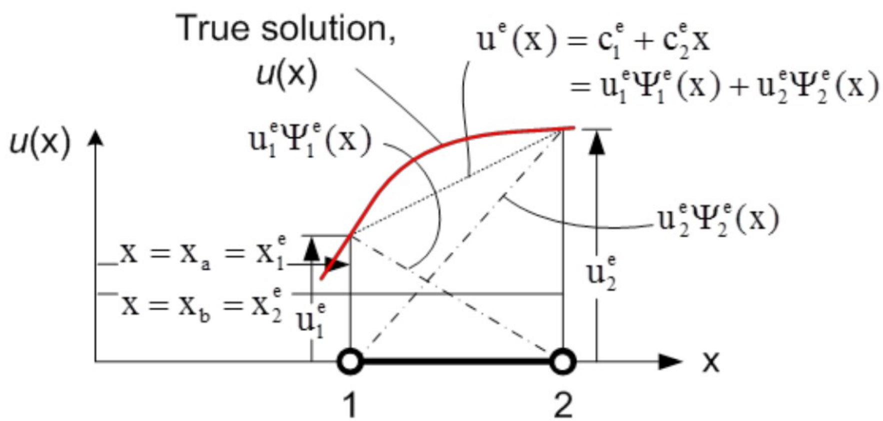
- ❖ Natural boundary conditions are included in the weak form.
- ❖ Essential boundary conditions are not included in the weak form.
- ❖ Approximation solution must fulfill certain requirements
  - Should be continuous over the element, and differentiable (for nonzero coefficients).
  - Should be complete polynomial (to capture all possible states).
  - Should be an interpolant of the primary variables at the finite element nodes. ( to satisfy the essential boundary conditions).
- ❖ Try Linear polynomial

$$u^e = a + bx \quad \textit{subj. to} \quad u^e(x_a) = u_1^e, \quad u^e(x_b) = u_2^e$$

$$\begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix} = \begin{bmatrix} 1 & x_a \\ 1 & x_b \end{bmatrix} \begin{Bmatrix} a \\ b \end{Bmatrix}$$

$$u^e = \psi_1^e u_1^e + \psi_2^e u_2^e = \sum_{j=1}^{2} \psi_j^e u_j^e$$

$$\textit{with} \quad \psi_1^e = \frac{x_b - x}{x_b - x_a} = 1 - \frac{\bar{x}}{h_e}, \qquad \psi_2^e = \frac{x - x_a}{x_b - x_a} = \frac{\bar{x}}{h_e}$$



(b)

# 1. Finite Element Method – Model Boundary Value Problem

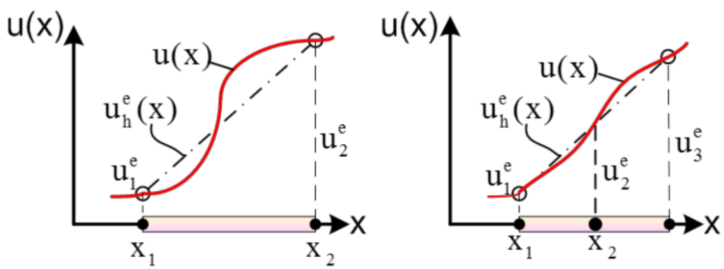
## Approximation of the Solution

❖ Try Quadratic polynomial $u^e = a + bx + cx^2$ *subj. to* $u^e\left(x_1^e\right) = u_1^e,\ \ u^e\left(x_2^e\right) = u_2^e,\ \ u^e\left(x_3^e\right) = u_3^e$

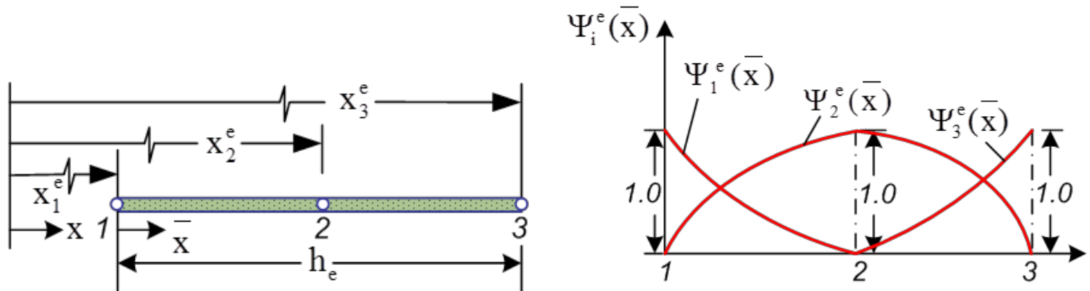
$$\begin{Bmatrix} u_1^e \\ u_2^e \\ u_3^e \end{Bmatrix} = \begin{bmatrix} 1 & x_1^e & \left(x_1^e\right)^2 \\ 1 & x_2^e & \left(x_2^e\right)^2 \\ 1 & x_3^e & \left(x_3^e\right)^2 \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \end{Bmatrix},$$

$$u^e = \psi_1^e u_1^e + \psi_2^e u_2^e + \psi_3^e u_3^e = \sum_{j=1}^{3} \psi_j^e u_j^e$$

*with* $\psi_1^e = \left(1 - \dfrac{\bar{x}}{h_e}\right)\left(1 - \dfrac{2\bar{x}}{h_e}\right),\ \ \psi_2^e = 4\dfrac{\bar{x}}{h_e}\left(1 - \dfrac{\bar{x}}{h_e}\right),\ \ \psi_3^e = -\dfrac{\bar{x}}{h_e}\left(1 - \dfrac{2\bar{x}}{h_e}\right)$




(a) (b)



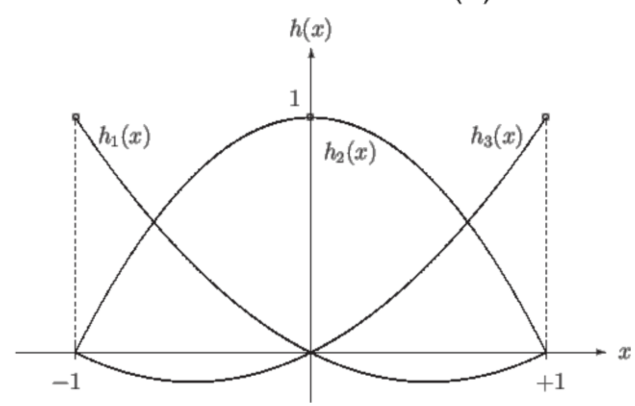

(a) (b)

❖ Quadratic Interpolation

$$\psi_1^e = \left(1 - \frac{x}{h}\right)\left(1 - \frac{2x}{h}\right), \qquad \psi_2^e = 4\frac{x}{h}\left(1 - \frac{x}{h}\right)$$

$$\psi_3^e = -\frac{x}{h}\left(1 - \frac{2x}{h}\right)$$

*(Different coordinate system)*



# 1. Finite Element Method – Model Boundary Value Problem

## Lagrangian Interpolation Function

❖ Suppose that the interpolation polynomial is in the form.

$$u(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

❖ The statement that the interpolation polynomial interpolates the data points means that

$$u(x_i) = u_i \quad \textit{for all} \quad i \in \{0, 1, \ldots, n\}$$

❖ Substituting equation here, we get a system of linear equations in the coefficients $a_k$ . The system in matrix-vector form reads

$$\begin{bmatrix} x_0^n & x_0^{n-1} & x_0^{n-2} & \ldots & x_0 & 1 \\ x_1^n & x_1^{n-1} & x_1^{n-2} & \ldots & x_1 & 1 \\ \ldots & \ldots & \ldots & & \ldots & \ldots \\ x_n^n & x_n^{n-1} & x_n^{n-2} & \ldots & x_n & 1 \end{bmatrix} \begin{bmatrix} a_n \\ a_{n-1} \\ \ldots \\ a_0 \end{bmatrix} = \begin{bmatrix} u_0 \\ u_1 \\ \ldots \\ u_n \end{bmatrix}$$

❖ We have to solve this system for $a_k$ to construct the interpolant $u(x)$. The matrix on the left is commonly referred to as a Vandermonde matrix.

## Lagrangian Interpolation Function

❖ Alternatively, we may write the polynomial immediately in terms of Lagrange polynomials:

$$u(x)=\frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}u_0+\frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)}u_1+\ldots$$

$$\ldots+\frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})}u_n=\sum_{i=0}^{n}\psi_i u_i=\sum_{i=0}^{n}\left(\prod_{\substack{0\le j\le n\\ j\ne i}}\frac{x-x_j}{x_i-x_j}\right)u_i$$

❖ Lagrangian Interpolation Function

$$\psi_i^e\left(x_j^e\right)=\delta_{ij}$$

$$\sum_{i=0}^{n}\psi_i^e(x)=1$$

# 1. Finite Element Method – Finite Element Model

❖ Interpolate the dependent variables

$$u_h^e = \sum_{j=1}^{n} u_j^e \psi_j^e$$

❖ Weak Form

$$0 = \int_{x_a}^{x_b} \left[ a \frac{dw}{dx}\frac{du}{dx} + cwu - wf \right] dx - w(x_a)Q_a - w(x_b)Q_b = \int_{x_a}^{x_b} \left[ a \frac{dw}{dx}\frac{du}{dx} + cwu - wf \right] dx - \sum_{i=1}^{n} w(x_i^e)Q_i^e$$

❖ Rayleigh-Ritz Procedure

$$0 = \int_{x_a}^{x_b} \left[ a \frac{d\psi_1^e}{dx} \left( \sum_{j=1}^{n} u_j^e \frac{d\psi_j^e}{dx} \right) + c\psi_1^e \left( \sum_{j=1}^{n} u_j^e \psi_j^e \right) - \psi_1^e f \right] dx - \sum_{j=1}^{n} \psi_1^e (x_j^e) Q_j^e$$

$$0 = \int_{x_a}^{x_b} \left[ a \frac{d\psi_2^e}{dx} \left( \sum_{j=1}^{n} u_j^e \frac{d\psi_j^e}{dx} \right) + c\psi_2^e \left( \sum_{j=1}^{n} u_j^e \psi_j^e \right) - \psi_2^e f \right] dx - \sum_{j=1}^{n} \psi_2^e (x_j^e) Q_j^e$$

$$\cdots$$

$$0 = \int_{x_a}^{x_b} \left[ a \frac{d\psi_n^e}{dx} \left( \sum_{j=1}^{n} u_j^e \frac{d\psi_j^e}{dx} \right) + c\psi_n^e \left( \sum_{j=1}^{n} u_j^e \psi_j^e \right) - \psi_n^e f \right] dx - \sum_{j=1}^{n} \psi_n^e (x_j^e) Q_j^e$$

# 1. Finite Element Method – Finite Element Model

- The $i^{th}$ algebraic equation can be written as $0=\sum_{j=1}^{n} K_{ij}^e u_j^e - f_i^e - Q_i^e \quad (i=1,2,\ldots,n)$
- Coefficient matrix and source vector in the global coordinate system

$$K_{ij}^e=\int_{x_e}^{x_{e+1}}\left(a\frac{d\psi_i^e}{dx}\frac{d\psi_j^e}{dx}+c\psi_i^e\psi_j^e\right)dx,\qquad f_i^e=\int_{x_e}^{x_{e+1}} f\psi_i^e dx$$

- Coefficient matrix and source vector in the local coordinate system

$$K_{ij}^e=\int_{0}^{h_e}\left(a\frac{d\psi_i^e}{d\bar{x}}\frac{d\psi_j^e}{d\bar{x}}+c\psi_i^e\psi_j^e\right)d\bar{x},\qquad f_i^e=\int_{0}^{h_e} f\psi_i^e d\bar{x}$$

- Matrix form

$$\begin{bmatrix} K_{11} & K_{12} & \cdots & K_{1n} \\ K_{21} & K_{22} & \cdots & K_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ K_{n1} & K_{n2} & \cdots & K_{nn} \end{bmatrix}\begin{Bmatrix} u_1^e \\ u_2^e \\ \cdots \\ u_n^e \end{Bmatrix}=\begin{Bmatrix} f_1^e \\ f_2^e \\ \cdots \\ f_n^e \end{Bmatrix}+\begin{Bmatrix} Q_1^e \\ Q_2^e \\ \cdots \\ Q_n^e \end{Bmatrix}$$

- Linear Element

$$\left[K^e\right]=\frac{a}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}+\frac{ch_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},\qquad \left\{f^e\right\}=\frac{qh_e}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

- Quadratic Element

$$\left[K^e\right]=\frac{a}{3h_e}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}+\frac{ch_e}{30}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix},\qquad \left\{f^e\right\}=\frac{qh_e}{6}\begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix}$$

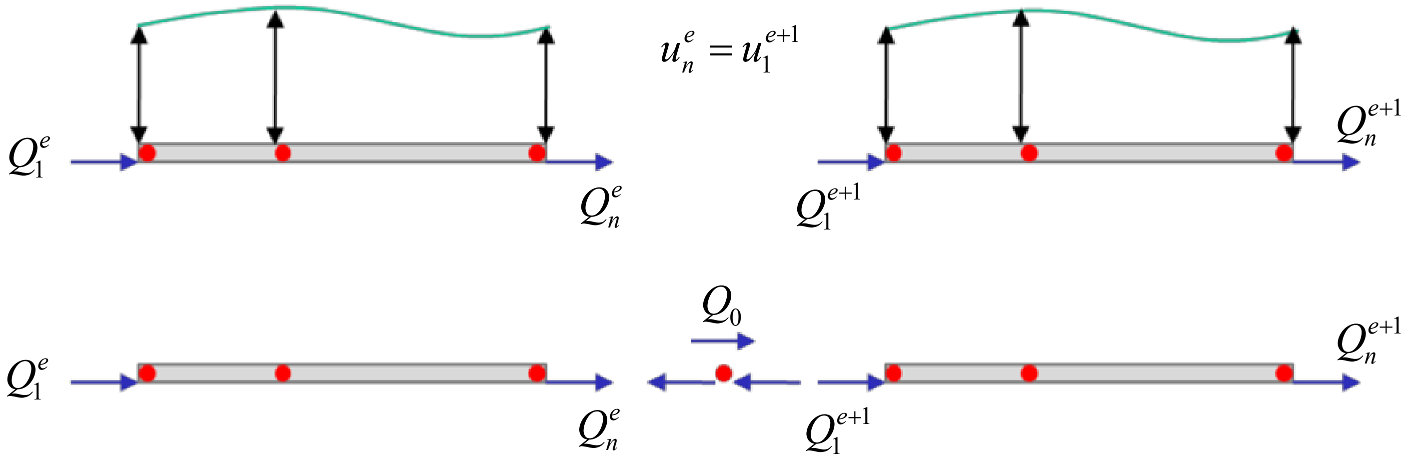
# 1. Finite Element Method – Principle of Connectivity

- Continuity of primary variables at connecting nodes (compatibility)

$$u_n^e = u_1^{e+1}$$

- Balance of secondary variables at connecting nodes (equilibrium)

$$Q_n^e + Q_1^{e+1} = \begin{cases} 0 & \textit{if no external point source is applied} \\ Q_0 & \textit{if an external point source is applied} \end{cases}$$



# 1. Finite Element Method – Connectivity for N linear elements

- Continuity of primary variables at connecting nodes (compatibility)

$$
\begin{aligned}
&u_1^1 = U_1 \\
&u_2^1 = u_1^2 = U_2 \\
&u_2^2 = u_1^3 = U_3 \\
&\ldots \\
&u_2^{N-1} = u_1^N = U_N \\
&u_2^N = u_1^{N+1} = U_{N+1}
\end{aligned}
$$

- 1st Element

$$
\begin{aligned}
K_{11}^1 U_1 + K_{12}^1 U_2 &= f_1^1 + Q_1^1 \\
K_{21}^1 U_1 + K_{22}^1 U_2 &= f_2^1 + Q_2^1
\end{aligned}
$$

- 2nd Element

$$
\begin{aligned}
K_{11}^2 U_2 + K_{12}^2 U_3 &= f_1^2 + Q_1^2 \\
K_{21}^2 U_2 + K_{22}^2 U_3 &= f_2^2 + Q_2^2
\end{aligned}
$$

- Nth Element

$$
\begin{aligned}
K_{11}^N U_N + K_{12}^N U_{N+1} &= f_1^N + Q_1^N \\
K_{21}^N U_N + K_{22}^N U_{N+1} &= f_2^N + Q_2^N
\end{aligned}
$$

# 1. Finite Element Method – Connectivity for N linear elements

❖ Resulting Equations

$$K_{11}^1 U_1 + K_{12}^1 U_2 = f_1^1 + Q_1^1 \qquad \textit{(unchanged)}$$

$$K_{21}^1 U_1 + \left(K_{22}^1 + K_{11}^2\right) U_2 + K_{12}^2 U_3 = f_2^1 + f_1^2 + Q_2^1 + Q_1^2$$

$$K_{21}^2 U_2 + \left(K_{22}^2 + K_{11}^3\right) U_3 + K_{12}^3 U_4 = f_2^2 + f_1^3 + Q_2^2 + Q_1^3$$

$$\dots$$

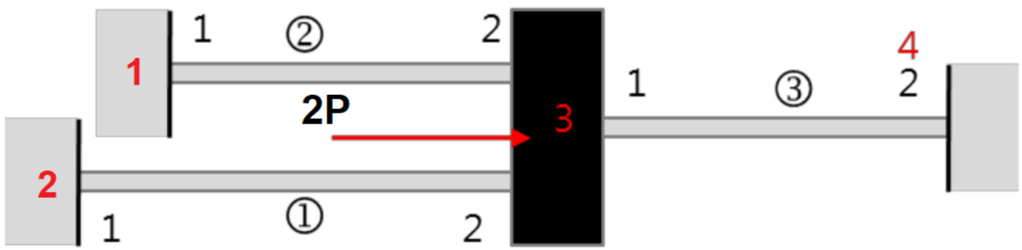
$$K_{21}^{N-1} U_{N-1} + \left(K_{22}^{N-1} + K_{11}^N\right) U_N + K_{12}^N U_{N+1} = f_2^{N-1} + f_1^N + Q_2^{N-1} + Q_1^N$$

$$K_{21}^N U_N + K_{22}^N U_{N+1} = f_2^N + Q_2^N \qquad \textit{(unchanged)}$$

❖ Matrix Form

$$\begin{bmatrix} K_{11}^1 & K_{12}^1 & & & \\ K_{21}^1 & K_{22}^1 + K_{11}^2 & K_{12}^2 & & \\ & K_{21}^2 & K_{22}^2 + K_{11}^3 & \dots & \\ & & \dots & \dots & K_{12}^N \\ & & & K_{21}^N & K_{22}^N \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ \dots \\ U_N \\ U_{N+1} \end{Bmatrix} = \begin{Bmatrix} f_1^1 \\ f_2^1 + f_1^2 \\ f_2^2 + f_1^3 \\ \dots \\ f_2^{N-1} + f_1^N \\ f_2^N \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 + Q_1^2 \\ Q_2^2 + Q_1^3 \\ \dots \\ Q_2^{N-1} + Q_1^N \\ Q_2^N \end{Bmatrix}$$

# 1. Finite Element Method – Three-bar Structure




❖ Connectivity of primary variables at connecting nodes (compatibility)

$$u_1^1 = U_1$$
$$u_1^2 = U_2$$
$$u_2^1 = u_2^2 = u_1^3 = U_3$$
$$u_2^3 = U_4$$

❖ Balance of secondary variables at connecting nodes (equilibrium)

$$Q_2^1 + Q_2^2 + Q_1^3 = 2P$$

$$\begin{bmatrix} K_{11}^1 & 0 & K_{12}^1 & 0 \\ 0 & K_{11}^2 & K_{12}^2 & 0 \\ K_{21}^1 & K_{21}^2 & K_{22}^1 + K_{22}^2 + K_{11}^3 & K_{12}^3 \\ 0 & 0 & K_{21}^3 & K_{22}^3 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} f_1^1 \\ f_1^2 \\ f_2^1 + f_2^2 + f_1^3 \\ f_2^3 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_1^2 \\ Q_2^1 + Q_2^2 + Q_1^3 \\ Q_2^3 \end{Bmatrix}$$

# 1. Finite Element Method – Three-bar Structure

❖ Global finite element equations after applying boundary conditions

$$\begin{bmatrix} K_{11}^1 & 0 & K_{12}^1 & 0 \\ 0 & K_{11}^2 & K_{12}^2 & 0 \\ K_{21}^1 & K_{21}^2 & K_{22}^1 + K_{22}^2 + K_{11}^3 & K_{12}^3 \\ 0 & 0 & K_{21}^3 & K_{22}^3 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ U_3 \\ 0 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 \\ Q_1^2 \\ 2P \\ Q_2^3 \end{Bmatrix}$$

*or*

$$\begin{bmatrix} \mathbf{K}^{11} & \mathbf{K}^{12} \\ \mathbf{K}^{21} & \mathbf{K}^{22} \end{bmatrix} \begin{Bmatrix} \mathbf{U}^* \\ \mathbf{U} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F} \\ \mathbf{F}^* \end{Bmatrix}$$

❖ Condensed Equation for the unknown

$$\left(K_{22}^1 + K_{22}^2 + K_{11}^3\right)U_3 = 2P \quad \rightarrow \quad U_3 = \frac{2P}{K_{22}^1 + K_{22}^2 + K_{11}^3}$$

*or*

$$\mathbf{U} = \left(\mathbf{K}^{22}\right)^{-1}\left(\mathbf{F}^* - \mathbf{K}^{21}\mathbf{U}^*\right)$$

❖ Computation of Secondary Variables from Equilibrium

$$\begin{Bmatrix} Q_1^1 \\ Q_1^2 \\ Q_2^3 \end{Bmatrix} = \begin{bmatrix} K_{11}^1 & 0 & K_{12}^1 & 0 \\ 0 & K_{11}^2 & K_{12}^2 & 0 \\ 0 & 0 & K_{21}^3 & K_{22}^3 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ U_3 \\ 0 \end{Bmatrix} = \begin{Bmatrix} K_{12}^1 U_3 \\ K_{12}^2 U_3 \\ K_{21}^3 U_3 \end{Bmatrix}$$

*or*

$$\begin{aligned} \mathbf{F} &= \mathbf{K}^{11}\mathbf{U}^* + \mathbf{K}^{12}\mathbf{U} \\ &= \mathbf{K}^{11}\mathbf{U}^* + \mathbf{K}^{12}\left(\mathbf{K}^{22}\right)^{-1}\left(\mathbf{F}^* - \mathbf{K}^{21}\mathbf{U}^*\right) \end{aligned}$$

# 1. Finite Element Method – Three-bar Structure

## Postprocessing

- Computation of the Primary Variables at points of interest

$$u_h^e = \sum_{j=1}^{n} u_j^e \psi_j^e \quad \rightarrow \quad \frac{du_h^e}{dx} = \sum_{j=1}^{n} u_j^e \frac{d\psi_j^e}{dx}$$

- Interpolation of the results to check whether the solution makes sense.
- Tabular or graphical presentation of the results.
- Computation of the Secondary Variables from Definition

$$Q_a = -a\left.\frac{du}{dx}\right|_{x_a}, \quad Q_b = a\left.\frac{du}{dx}\right|_{x_b}$$

$$Q_1^1 = -EA\left.\frac{du^1}{dx}\right|_{x=0} = -EA\frac{U_3 - U_1}{h_1} = -\frac{EA}{h_1}U_3 = K_{12}^1 U_3$$

$$Q_1^2 = -EA\left.\frac{du^2}{dx}\right|_{x=0} = -EA\frac{U_3 - U_2}{h_2} = -\frac{EA}{h_2}U_3 = K_{12}^2 U_3$$

$$Q_1^3 = -EA\left.\frac{du^3}{dx}\right|_{x=h_1+h_3} = EA\frac{U_4 - U_3}{h_1} = -\frac{EA}{h_3}U_3 = K_{21}^3 U_3$$

# 1. Finite Element Method – Example Problem (4 Linear Elements)

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right)+cu-f=0,\quad 0<x<1 \quad \text{subjected to} \quad u(0)=0,\ u(1)=0 \quad \text{with} \quad a=1,\ c=-1,\ f=-x^2$$

❖ The equation becomes $-\dfrac{d^2u}{dx^2}-u+x^2=0,\quad 0<x<1$

❖ Element Stiffness Matrix and Force Vector $K_{ij}^e=\int_{x_e}^{x_{e+1}}\left(\dfrac{d\psi_i^e}{dx}\dfrac{d\psi_j^e}{dx}-\psi_i^e\psi_j^e\right)dx,\quad f_i^e=\int_{x_e}^{x_{e+1}}\left(-x^2\right)\psi_i^e dx$

❖ For 04 Linear Elements

- *Element 1:* $\left(h_1=1/4,\ x_a=0,\ x_b=1/4\right)$
- *Element 2:* $\left(h_2=1/4,\ x_a=1/4,\ x_b=1/2\right)$
- *Element 3:* $\left(h_3=1/4,\ x_a=2,\ x_b=3/4\right)$
- *Element 4:* $\left(h_4=1/4,\ x_a=3/4,\ x_b=1\right)$

$$\left[K^e\right]=\frac{1}{24}\begin{bmatrix}94 & -97\\ -97 & 94\end{bmatrix},\qquad \left\{f^e\right\}=-\frac{1}{h_e}\begin{Bmatrix}\dfrac{x_b}{3}\left(x_b^3-x_a^3\right)-\dfrac{1}{4}\left(x_b^4-x_a^4\right)\\ -\dfrac{x_a}{3}\left(x_b^3-x_a^3\right)+\dfrac{1}{4}\left(x_b^4-x_a^4\right)\end{Bmatrix}$$

*with typically,*

$$f_1^1=\int_{x_a}^{x_b}\left(-x^2\right)\left(1-x/h\right)dx=\frac{1}{4h}\left(x_b^4-x_a^4\right)-\frac{1}{3}\left(x_b^3-x_a^3\right)=\left(1/4\right)^4-\frac{1}{3}\left(1/4\right)^3=-0.001302$$

# 1. Finite Element Method – Example Problem (4 Linear Elements)

- The equation becomes

$$\begin{bmatrix} 3.9167 & -4.0417 & 0 & 0 & 0 \\ -4.0417 & 7.8333 & -4.0417 & 0 & 0 \\ 0 & -4.0417 & 7.8333 & -4.0417 & 0 \\ 0 & 0 & -4.0417 & 7.8333 & -4.0417 \\ 0 & 0 & 0 & -4.0417 & 3.9167 \end{bmatrix} \begin{Bmatrix} \cancel{U_1} \\ U_2 \\ U_3 \\ U_4 \\ \cancel{U_5} \end{Bmatrix} = -\begin{Bmatrix} 0.00130 \\ 0.01823 \\ 0.06510 \\ 0.14323 \\ 0.10547 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 + Q_1^2 \\ Q_2^2 + Q_1^3 \\ Q_2^3 + Q_1^4 \\ Q_2^4 \end{Bmatrix}$$

- After Boundary Conditions

$$\begin{bmatrix} 7.8333 & -4.0417 & 0 \\ -4.0417 & 7.8333 & -4.0417 \\ 0 & -4.0417 & 7.8333 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} = -\begin{Bmatrix} 0.01823 \\ 0.06510 \\ 0.14323 \end{Bmatrix}$$

- Solution $U_1 = 0, \quad U_2 = -0.02323, \quad U_3 = -0.04052, \quad U_4 = -0.03919, \quad U_5 = 0$

- Secondary Variables
  - *By Equilibrium* $Q_1^1 = K_{11}^1 U_1 + K_{12}^1 U_2 - f_1^1 = 0.09520, \quad Q_2^4 = K_{21}^4 U_4 + K_{22}^4 U_5 - f_2^4 = 0.26386$
  - *By Definition*

$$Q_1^1 = -\left.\frac{du_h^1}{dx}\right|_{x=0} = -\sum_{j=1}^{n} u_j^1 \frac{d\psi_j^1}{dx} = -u_1^1 \frac{d\psi_1^1}{dx} - u_2^1 \frac{d\psi_2^1}{dx} = -U_1\left(-1/h_e\right) - U_2\left(1/h_e\right) = \frac{U_1 - U_2}{h_e} = 0.09293$$

$$Q_2^4 = \left.\frac{du_h^4}{dx}\right|_{x=1} = \sum_{j=1}^{n} u_j^4 \frac{d\psi_j^4}{dx} = u_1^4 \frac{d\psi_1^4}{dx} + u_2^4 \frac{d\psi_2^4}{dx} = U_4\left(-1/h_e\right) + U_5\left(1/h_e\right) = \frac{U_5 - U_4}{h_e} = 0.15676$$

# 1. Finite Element Method – Example Problem (2 Quadratic Elements)

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right)+cu-f=0,\quad 0<x<1 \quad \text{subjected to} \quad u(0)=0,\ u(1)=0 \quad \text{with} \quad a=1,\ c=-1,\ f=-x^2$$

- ❖ The equation becomes $-\dfrac{d^2u}{dx^2}-u+x^2=0,\quad 0<x<1$
- ❖ Element Stiffness Matrix and Force Vector $K_{ij}^e=\int_{x_e}^{x_{e+1}}\left(\dfrac{d\psi_i^e}{dx}\dfrac{d\psi_j^e}{dx}-\psi_i^e\psi_j^e\right)dx,\quad f_i^e=\int_{x_e}^{x_{e+1}}\left(-x^2\right)\psi_i^e\,dx$
- ❖ For 02 Quadratic Elements

$$\left[K^e\right]=\frac{1}{60}\begin{bmatrix}276 & -322 & 41\\ -322 & 624 & -322\\ 41 & -322 & 276\end{bmatrix},\qquad \left\{f^e\right\}=-\frac{h_e}{60}\begin{Bmatrix}-h_e^2+10x_a^2\\ 12h_e^2+40x_a^2+40x_a^2h_e\\ 9h_e^2+20x_a^2+20x_ah_e\end{Bmatrix}$$

- ❖ Assembled Equations

$$\begin{bmatrix}4.6000 & -5.3667 & 0.6833 & 0 & 0\\ -5.3667 & 10.4000 & -5.3667 & 0 & 0\\ 0.6833 & -5.3667 & 9.2000 & -5.3667 & 0.6833\\ 0 & 0 & -5.3667 & 10.4000 & -5.3667\\ 0 & 0 & 0.6833 & -5.3667 & 4.6000\end{bmatrix}\begin{Bmatrix}\cancel{U_1}\\ U_2\\ U_3\\ U_4\\ \cancel{U_5}\end{Bmatrix}=-\begin{Bmatrix}-0.00208\\ 0.02500\\ 0.03750\\ 0.19167\\ 0.08125\end{Bmatrix}+\begin{Bmatrix}Q_1^1\\ Q_2^1\\ Q_3^1+Q_1^2\\ Q_2^2\\ Q_3^2\end{Bmatrix}$$

# 1. Finite Element Method – Example Problem (2 Quadratic Elements)

❖ Condensed Equations

$$\begin{bmatrix} 10.4000 & -5.3667 & 0 \\ -5.3667 & 9.2000 & -5.3667 \\ 0 & -5.3667 & 10.4000 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} = -\begin{Bmatrix} 0.02500 \\ 0.03750 \\ 0.19167 \end{Bmatrix}$$

❖ Solution

$$U_1 = 0, \quad U_2 = -0.02345, \quad U_3 = -0.04078, \quad U_4 = -0.03947, \quad U_5 = 0$$

❖ Secondary Variables

- *By Equilibrium*

$$Q_1^1 = K_{11}^1 U_1 + K_{12}^1 U_2 + K_{13}^1 U_3 - f_1^1 = 0.10006$$

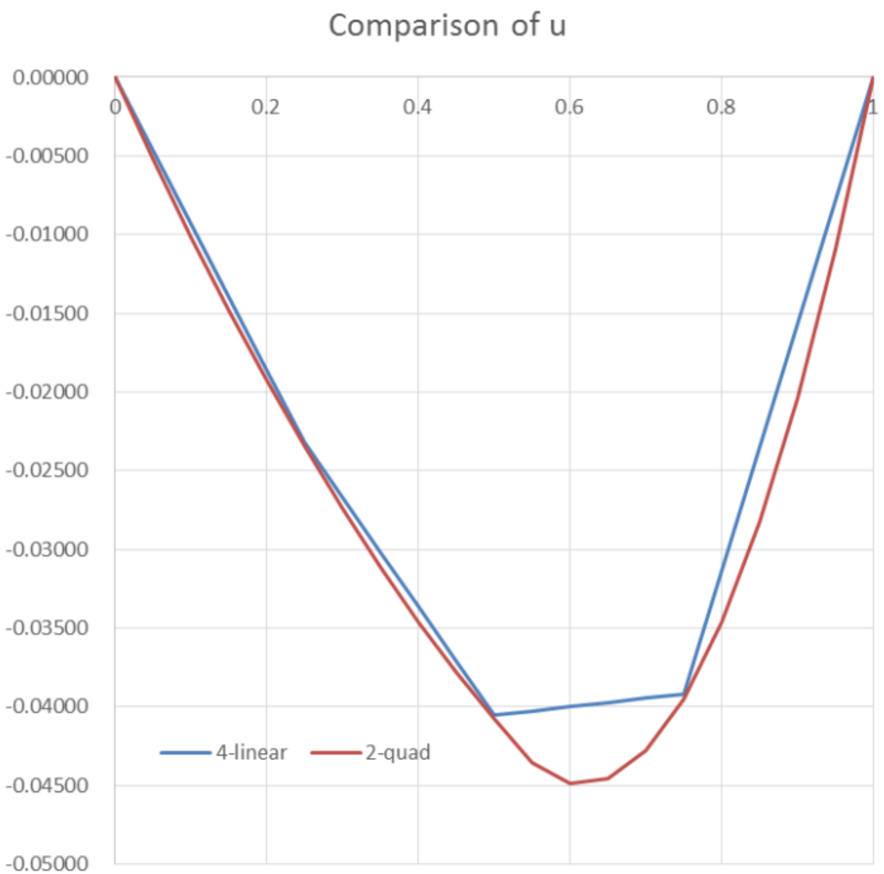
$$Q_3^2 = K_{13}^2 U_3 + K_{23}^2 U_4 + K_{33}^2 U_5 - f_3^2 = 0.26521$$

- *By Definition*

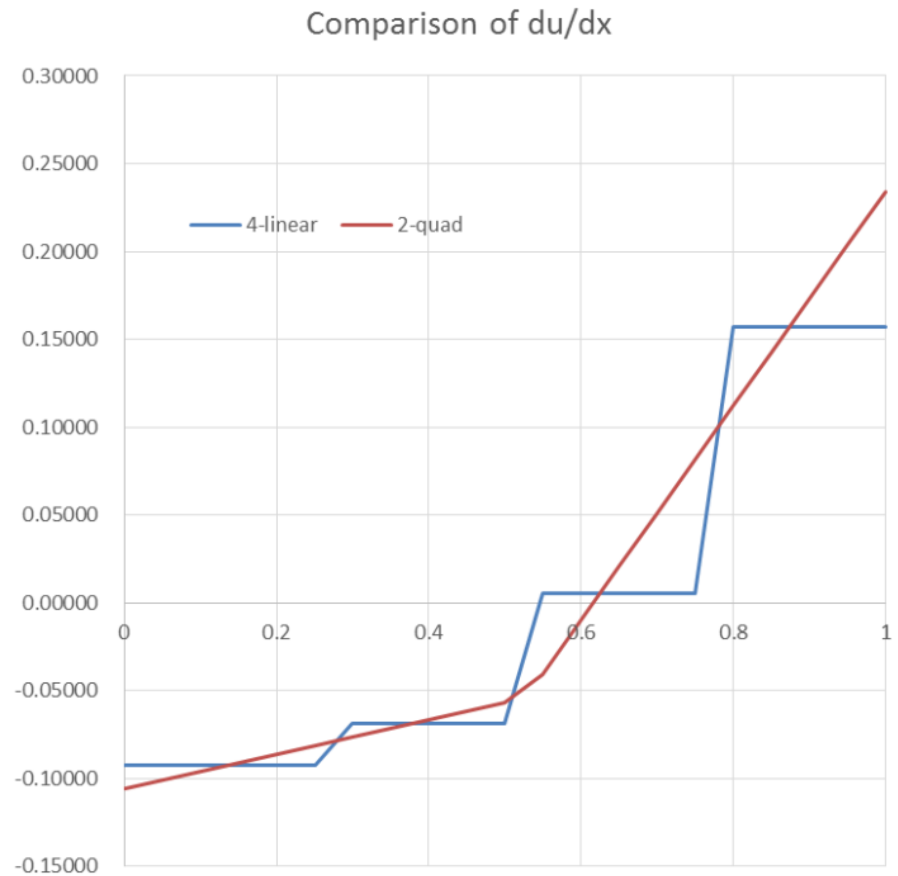
$$Q_1^1 = -\left.\frac{du_h^1}{dx}\right|_{x=0} = -\sum_{j=1}^{n} u_j^1 \frac{d\psi_j^1}{dx} = \frac{1}{h_e}\left(U_1 - 4U_2 + U_3\right) = 0.10602$$

$$Q_3^2 = \left.\frac{du_h^2}{dx}\right|_{x=1} = \sum_{j=1}^{n} u_j^2 \frac{d\psi_j^2}{dx} = \frac{1}{h_e}\left(U_3 - 4U_4 + 3U_5\right) = 0.23442$$

# 1. Finite Element Method – Example Problem (Comparison)

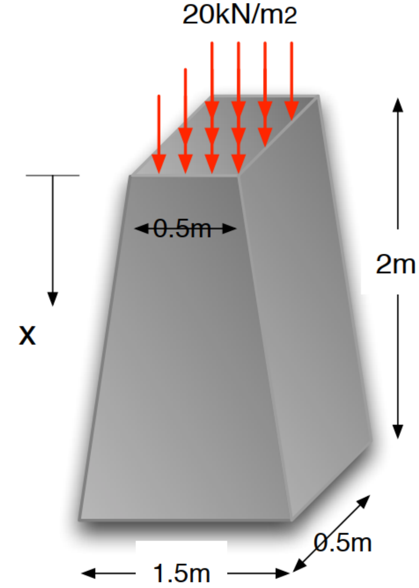



Comparison of Primary Variables




Comparison of Derivatives of Primary Variables

# 2. More Example Problem – Concrete Pier




- Modulus of Elasticity $E = 28\times 10^6\ kN/m^2$

  Boundary conditions $EA\dfrac{du}{dx}\bigg|_{x=0} = 20\times 0.5\times 0.5 = 5kN$

  Body force for unit length $f(x) = \dfrac{dW}{dx} = 6.25(1+x)$

  Cross-section Area $A(x) = 0.25(1+x)$

- Governing Equations

$$-\frac{d}{dx}\left(EA\frac{du}{dx}\right) - f = 0 \quad \rightarrow \quad -\frac{d}{dx}\left[0.25E(1+x)\frac{du}{dx}\right] = 6.25(1+x)$$

- Boundary conditions $\left[0.25E(1+x)\dfrac{du}{dx}\right]_{x=0} = 5, \quad u(2) = 0$

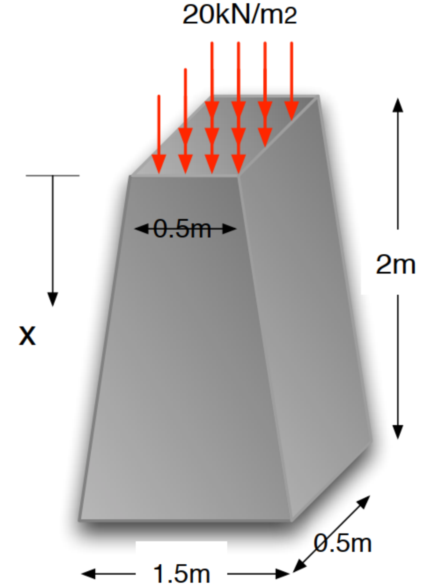
- For a linear element

$$K_{ij}^e = \int_{x_e}^{x_{e+1}} \left[0.25E(1+x)\frac{d\psi_i^e}{dx}\frac{d\psi_j^e}{dx}\right]dx, \qquad f_i^e = \int_{x_e}^{x_{e+1}} 6.25(1+x)\psi_i^e dx$$

- Explicitly

$$\left[K^e\right] = \frac{E}{4h_e}\left[1+0.5(x_e + x_{e+1})\right]\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \qquad \left\{f^e\right\} = 6.25\frac{h_e}{2}\left[\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \frac{1}{3}\begin{Bmatrix} x_{e+1} + 2x_e \\ 2x_{e+1} + x_e \end{Bmatrix}\right]$$

# 2. More Example Problem – Concrete Pier




- Two linear elements

$$\left[K^1\right]=\frac{E}{4}\begin{bmatrix}1.5 & -1.5\\ -1.5 & 1.5\end{bmatrix},\qquad \left\{f^1\right\}=\frac{6.25}{6}\begin{Bmatrix}3+1\\ 3+2\end{Bmatrix}$$

$$\left[K^2\right]=\frac{E}{4}\begin{bmatrix}2.5 & -2.5\\ -2.5 & 2.5\end{bmatrix},\qquad \left\{f^2\right\}=\frac{6.25}{6}\begin{Bmatrix}3+4\\ 3+5\end{Bmatrix}$$

- Assembled Equation

$$E\begin{bmatrix}0.375 & -0.375 & 0\\ -0.375 & 1 & -0.625\\ 0 & -0.625 & 0.625\end{bmatrix}\begin{Bmatrix}U_1\\ U_2\\ \cancel{U_3}\end{Bmatrix}=\begin{Bmatrix}4.167\\ 12.500\\ 8.333\end{Bmatrix}+\begin{Bmatrix}Q_1^1=5\\ \cancel{Q_2^1+Q_1^2}\\ Q_2^2\end{Bmatrix}$$

- Solution $\quad U_1=2.111\times10^{-6}m;\quad U_2=1.238\times10^{-6}m;\quad Q_2^2=-30kN$

- Secondary Variable by definition $\quad Q_2^2=EA\left.\frac{du}{dx}\right|_{x=2}=EAU_2\left(-1/h\right)\Big|_{x=2}=-25.998kN$

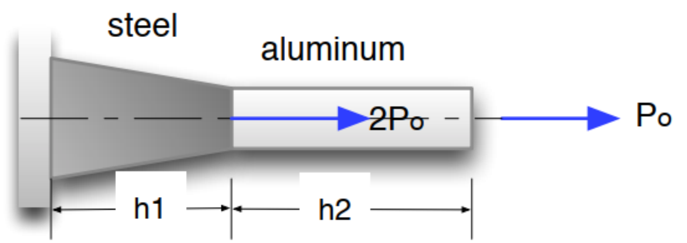
- Four linear elements $\quad U_1=2.008\times10^{-6}m;\quad U_2=1.228\times10^{-6}m$

- Exact Solution

$$u(x)=\frac{1}{E}\left[56.25-6.25(1+x)^2-7.5\ln\left(\frac{1+x}{3}\right)\right]$$

$$u(0)=2.008\times10^{-6}m;\quad u(1)=1.225\times10^{-6}m$$

# 2. More Example Problem – Composite Axial Bar




- Material Properties and data

$$E_s = 30\times10^6\ psi;\quad A_s = (c_1 + c_2 x)^2\ in^2;\quad E_a = 10^7\ psi$$

$$A_a = 1\ in^2;\quad h_1 = 96\ in;\quad h_2 = 120\ in;\quad P_0 = 10{,}000\ lb$$

- Governing Equations

$$-\frac{d}{dx}\left(E_s A_s \frac{du}{dx}\right) = 0\ ,\quad 0 < x < h_1$$

$$-\frac{d}{dx}\left(E_a A_a \frac{du}{dx}\right) = 0\ ,\quad h_1 < x < L$$

- Finite Element Model

$$K_{ij}^e = \int_{x_e}^{x_{e+1}} E_e (c_1 + c_2 x)^2 \frac{d\psi_i^e}{dx}\frac{d\psi_j^e}{dx}dx;\qquad f_i^e = 0 \quad \textit{with} \quad c_1^1 = 1.5,\quad c_2^1 = -0.5/96$$

$$Q_1^e = \left(-EA\frac{du}{dx}\right)_{x_e}\ ,\qquad Q_2^e = \left(EA\frac{du}{dx}\right)_{x_{e+1}}$$

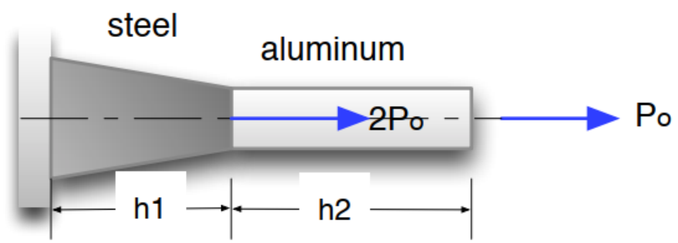
- Assembled Equation for two linear elements

$$10^4\begin{bmatrix} 49.479 & -49.479 & 0 \\ -49.479 & 57.812 & -8.333 \\ 0 & -8.333 & 8.333 \end{bmatrix}\begin{Bmatrix} U_1 = 0 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 \\ Q_2^1 + Q_1^2 = 2P_0 \\ Q_2^2 = P_0 \end{Bmatrix}$$

- Solution

$$U_2 = 0.06063;\quad U_3 = 0.18063;\quad Q_1^1 = -30{,}000$$

# 2. More Example Problem – Composite Axial Bar




❖ Secondary Variables by definition

$$Q_1^1 = -EA\frac{du_h^1}{dx}\bigg|_{x=0} = -EA\left(u_1^1\frac{d\psi_1^1}{dx} + u_2^1\frac{d\psi_2^1}{dx}\right)\bigg|_{x=0}$$

$$= -30\times10^6\times1.5^2\left(\underbrace{U_1}_{=0}\left(-1/h_e\right) + \underbrace{U_2}_{=0.06063}\left(1/h_e\right)\right) = -42{,}630$$

❖ Displacements $u(x) = \begin{cases} u_1^1\psi_1^1 + u_2^1\psi_2^1 = 0.06063x/96\,, & 0 \le x \le 96 \\ u_1^2\psi_1^2 + u_2^2\psi_2^2 = -0.03537 + 0.001x\,, & 96 \le x \le 216 \end{cases}$

❖ Derivative of the dependent variables

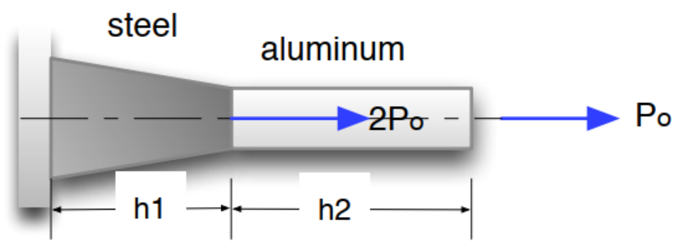
$$\frac{du}{dx} = \begin{cases} u_1^1\dfrac{d\psi_1^1}{dx} + u_2^1\dfrac{d\psi_2^1}{dx} = 0.06063/96\,, & 0 \le x \le 96 \\ u_1^2\dfrac{d\psi_1^2}{dx} + u_2^2\dfrac{d\psi_2^2}{dx} = 0.001\,, & 96 \le x \le 216 \end{cases}$$

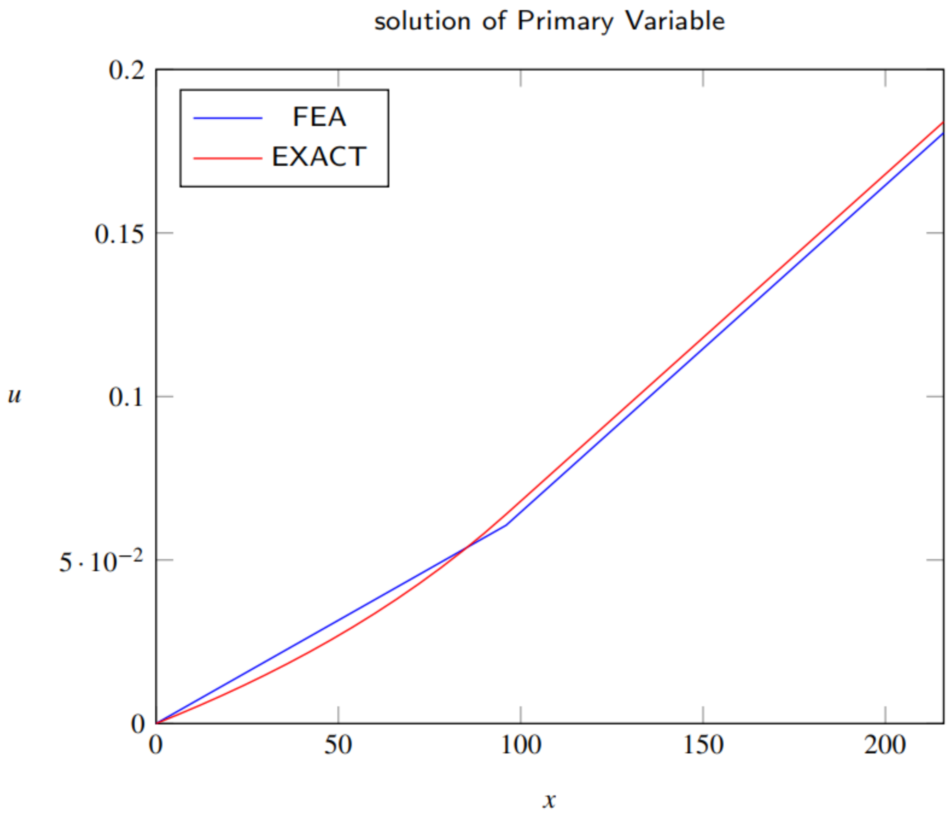
❖ Exact Solution $u(x) = \begin{cases} 0.128\left[x/(288-x)\right], & 0 \le x \le 96 \\ 0.001(x-32)\,, & 96 \le x \le 216 \end{cases} \rightarrow \frac{du}{dx} = \begin{cases} 36.864/(288-x)^2, & 0 \le x \le 96 \\ 0.001\,, & 96 \le x \le 216 \end{cases}$

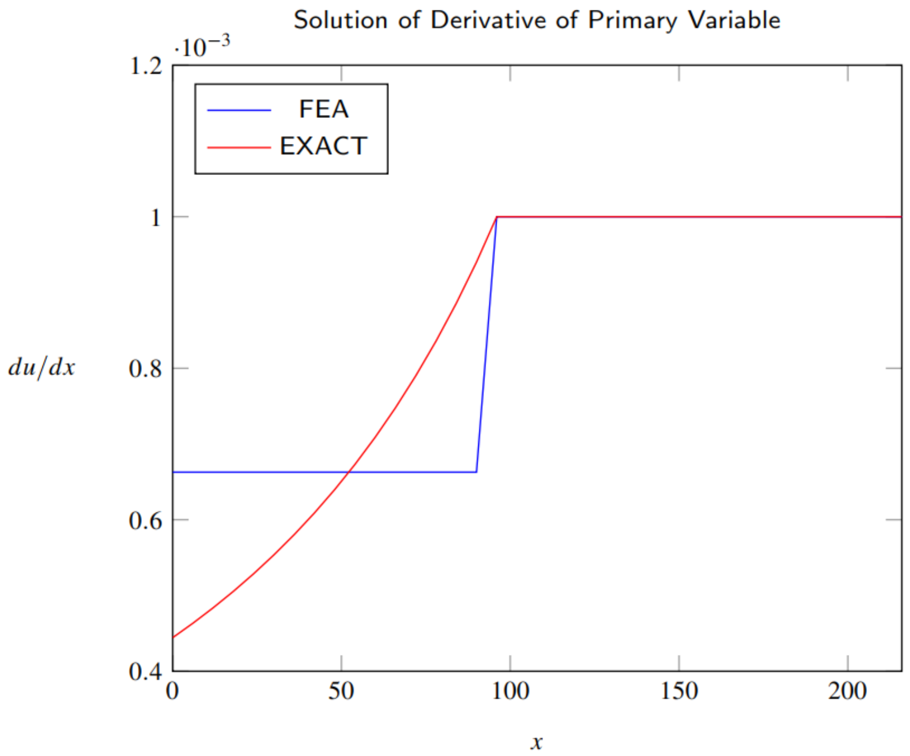
❖ Solution with Two Quadratic Elements

$$U_2 = 0.02572;\quad U_3 = 0.06392;\quad U_4 = 0.12392;\quad U_5 = 0.18392$$

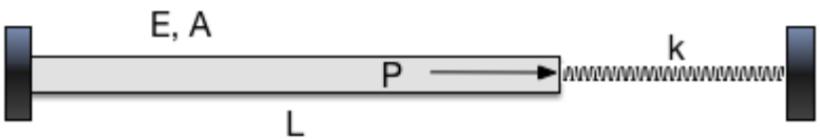
# 2. More Example Problem – Composite Axial Bar (Comparison)



# 2. More Example Problem – Axial Bar with Spring




## Two Linear Elements

- For element 1 $$\frac{EA}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_1^1\\ u_2^1\end{Bmatrix}=\begin{Bmatrix}Q_1^1\\ Q_2^1\end{Bmatrix}$$
- For element 2 $$k\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_1^2\\ u_2^2\end{Bmatrix}=\begin{Bmatrix}Q_1^2\\ Q_2^2\end{Bmatrix}$$
- Assembled Equation $$\begin{bmatrix}EA/L & -EA/L & 0\\ -EA/L & EA/L+k & -k\\ 0 & -k & k\end{bmatrix}\begin{Bmatrix}U_1=0\\ U_2\\ U_3=0\end{Bmatrix}=\begin{Bmatrix}Q_1\\ Q_2=P\\ Q_3\end{Bmatrix}$$
- Solution $$U_2=\frac{P}{EA/L+k};\quad Q_1=-\frac{EA}{L}U_2;\quad Q_3=-kU_2$$
- Consider
  - For $k=0$ (Free Edge) $$U_2=\frac{PL}{EA};\quad Q_1=-P;\quad Q_3=0$$
  - For $k=EA/L$ (Loading at the center) $$U_2=\frac{PL}{2EA};\quad Q_1=Q_2=-P/2$$
  - For $k\to\infty$ (Clamped) $$U_2=Q_1=Q_3=0$$

# 2. More Example Problem – Axial Bar with Spring


### One Linear Element

❖ We already know the boundary conditions

$$U\big|_{x=0} = 0\,; \quad EA\frac{du}{dx} + k\,u\big|_{x=L} = P$$

❖ Use one linear element

$$\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} U_1 = 0 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_2 = P - kU_2 \end{Bmatrix}$$

❖ Same results.

### More Problem!

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right) + cu - f = 0, \quad 0 < x < 1$$ subjected to $u(0) = 0,\ u(1) = 0$ with $a = 1,\ c = 1,\ f = x$

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thank you