Lesson 03

Virtual Work, Calculus of Variation Hamilton's Principle

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1. Virtual Work – Work and Energy

- Work is the product of force and displacement in the direction of the force.

- The work done by the force in moving the particle from A to B is given by

$$W = \int_A^B F \cdot du$$

- Total work done on the body is the sum of work done on all particles of the body.

$$W = \int_V F \cdot u \, dv$$

- Energy is the capacity to do work. Measure of the capacity of all forces.

$$E = \int_t \int_V F \cdot u \, dv \, dt$$

- Both work and energy are scalars (independent of the coordinate system).

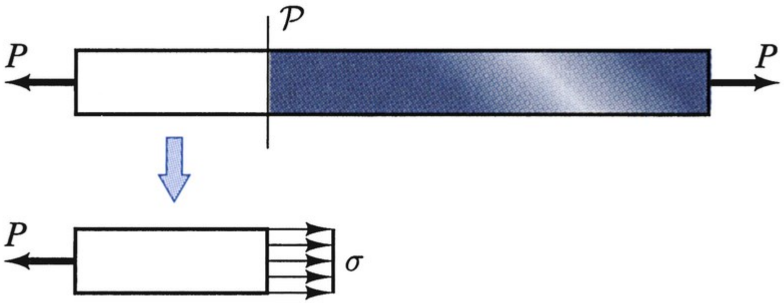
1. Virtual Work – Strain Energy

❖ Strain energy density. $$U_0 = \int_{\varepsilon} \sigma_{ij} d\varepsilon_{ij}$$

❖ Complementary strain energy density. $$U_0^* = \int_{\sigma} \varepsilon_{ij} d\sigma_{ij}$$

❖ For linear elastic materials: $$U_0 = U_0^*$$

Axially Loaded Bar Problem



❖ Strain energy density.

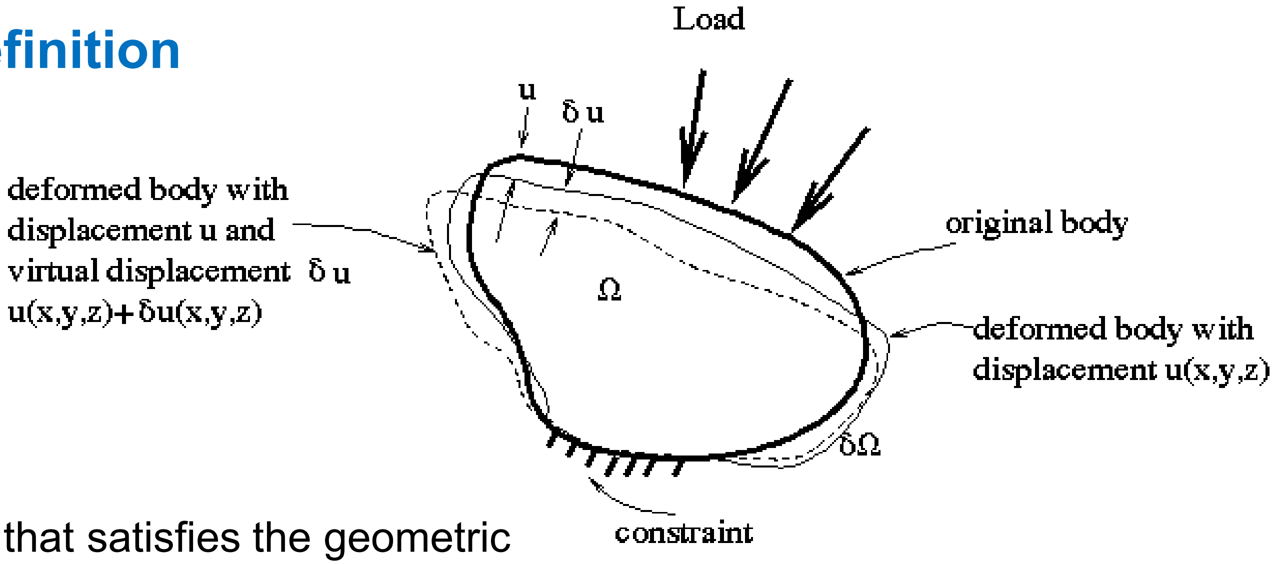
$$U_0 = \frac{1}{2}\sigma_x \varepsilon_x = \frac{1}{2}E\varepsilon^2 = \frac{1}{2}E\left(\frac{du}{dx}\right)^2$$

❖ Strain energy. $$U = \int_V \frac{1}{2}E\left(\frac{du}{dx}\right)^2 dV = \int_0^L \int_A \frac{1}{2}E\left(\frac{du}{dx}\right)^2 dA\, dx = \int_0^L \frac{EA}{2}\left(\frac{du}{dx}\right)^2 dx$$

❖ Complementary strain energy density. $$U_0^* = \frac{1}{2}\sigma_x \varepsilon_x = \frac{1}{2}\frac{\sigma^2}{E} = \frac{1}{2E}\left(\frac{P}{A}\right)^2$$

❖ Complementary strain energy. $$U^* = \int_V \frac{P^2}{2EA^2} dV = \int_V \frac{P^2}{2EA} dx$$

1. Virtual Work – Definition



- The set of configuration that satisfies the geometric constraints is called the set of admissible configurations.
- When mechanical system experiences such variations in its displacement configuration, it is said to undergo virtual displacements from its equilibrium configuration.
- The work done by the actual forces through a virtual displacement of the actual configuration is called virtual work.

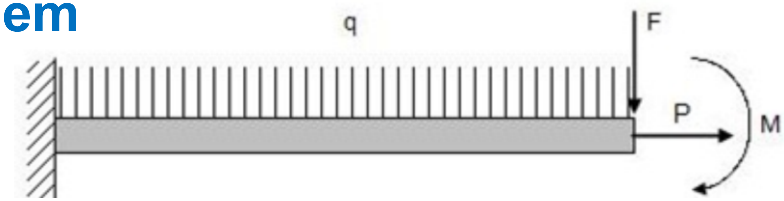
$$\delta W = \int_V F \cdot \delta u \, dV$$

- External work done subjected to body forces and surface forces.

$$\delta W_E = -\int_V f_i \cdot \delta u_i \, dV - \int_{S_2} \hat{t}_i \cdot \delta u_i \, dS$$

- Internal virtual work $\delta W_I = \int_V \delta U_0 \, dV = \int_V \delta \left(\int_0^{\varepsilon_{ij}} \sigma_{ij} d\varepsilon_{ij} \right) dV = \int_V \delta \left(\frac{1}{2} \sigma_{ij} \varepsilon_{ij} \right) dV = \int_V \sigma_{ij} \delta\varepsilon_{ij} \, dV$

1. Virtual Work – Cantilever Beam Problem



- Displacements $u = u_0 - z\dfrac{dw_0}{dx}\;;\qquad w = w_0$

- Strains $\varepsilon_x = \dfrac{\partial u}{\partial x} = \dfrac{du_0}{dx} - z\dfrac{d^2 w_0}{\partial x^2} \quad\rightarrow\quad \delta\varepsilon_x = \dfrac{d\delta u_0}{dx} - z\dfrac{d^2\delta w_0}{\partial x^2}$

- Stress resultants $N = \int_A \sigma_x dA \qquad M = \int_A \sigma_x z dA$

- External virtual work

$$\delta W_E = -\int_V f_i\cdot\delta u_i\,dV - \int_{S_2}\hat{t}_i\cdot\delta u_i\,dS = -\int_0^L\int_A\left(f\delta u + q\delta w\right)dA\,dx - P\delta u(L) - F\delta w(L)$$

$$= -\int_0^L\left(f\delta u_0 + q\delta w_0\right)dx - P\delta u_0(L) - F\delta w_0(L)$$

- Internal virtual work

$$\delta W_I = \int_V \sigma_{ij}\delta\varepsilon_{ij}\,dV = \int_0^L\int_A \sigma_x\delta\varepsilon_x dA\,dx = \int_0^L\int_A\sigma_x\left(\frac{d\delta u_0}{dx} - z\frac{d^2\delta w_0}{\partial x^2}\right)dA\,dx = \int_0^L\left[N\frac{d\delta u_0}{dx} - M\frac{d^2\delta w_0}{\partial x^2}\right]dx$$

- Total virtual work

$$\delta W = \delta W_I + \delta W_E = \int_0^L\left[N\frac{d\delta u_0}{dx} + M\left(\frac{-d^2\delta w_0}{\partial x^2}\right) - f\delta u_0 - q\delta w_0\right]dx - P\delta u_0(L) - F\delta w_0(L)$$

1. Virtual Work – Complementary Virtual Work

❖ The work done by the virtual forces through an actual displacement is called complementary virtual work.

$$\delta W^{*} = \int_{V} \delta F \cdot u \, dV$$

❖ External work done subjected to body forces and surface forces.

$$\delta W_{E}^{*} = -\int_{V} \delta f_{i} \cdot u_{i} \, dV - \int_{S_{1}} \delta t_{i} \cdot \hat{u}_{i} \, dS$$

❖ Internal Complementary Virtual Work

$$\delta W_{I}^{*} = \int_{V} \delta U_{0}^{*} \, dV = \int_{V} \delta \left(\int_{0}^{\sigma_{ij}} \varepsilon_{ij} d\sigma_{ij} \right) dV = \int_{V} \delta \left(\frac{1}{2} \sigma_{ij} \varepsilon_{ij} \right) dV = \int_{V} \varepsilon_{ij} \delta \sigma_{ij} \, dV$$

2. Calculus of Variation – Definition

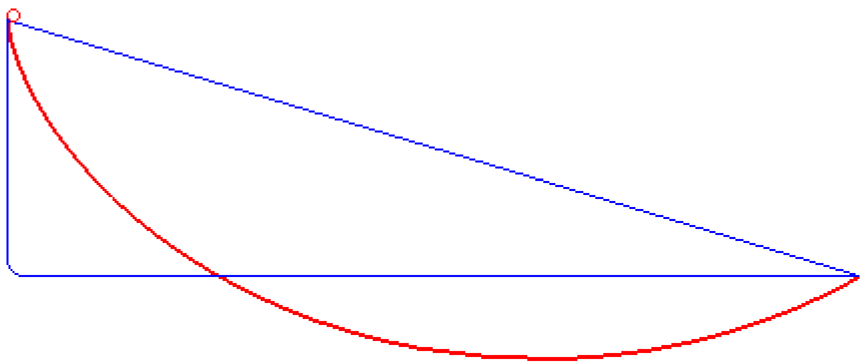
- ❖ Calculus of variations is a field of mathematical analysis that deals with maximizing or minimizing functionals, which are mappings from a set of functions to the real numbers.
- ❖ The interest is in extremal functions that make the functional attain a maximum or minimum value or stationary functions those where the rate of change of the functional is zero.
- ❖ A simple example of such a problem is to find the curve of shortest length connecting two points. If there are no constraints, the solution is obviously a straight line between the points. However, if the curve is constrained to lie on a surface in space, then the solution is less obvious, and possibly many solutions may exist.

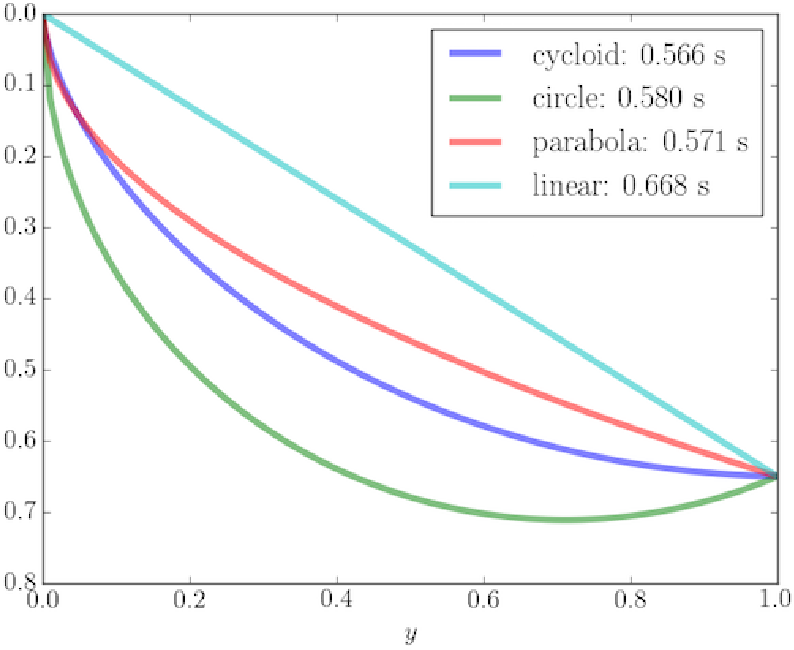
Historical Review

- Johann Bernoulli (1696): Raised Brachistochrone curve
- Leonhard Euler (1733) : Elaborate the subject for Brachistochrone curve
- Joseph Louis Lagrange (1811): Euler-Lagrange Equation
- Simeon Poisson (1831): Contributes
- William Rowan Hamilton (1834) : Hamilton's principle

2. Calculus of Variation – Brachistochrone curve

❖ In mathematics, a brachistochrone curve, meaning shortest time, or curve of fastest descent, is the curve that would carry an idealized point-like body, starting at rest and moving along the curve, without friction, under constant gravity, to a given end point in the shortest time. For a given starting point, the brachistochrone curve is the same as the tautochrone curve.





The curve of fastest descent is not a straight or polygonal line (blue) but a cycloid (red).

2. Calculus of Variation – Variational Operator

- The first variation of *u*: δu
- Consider a function of dependent variable *u* and its derivative *u'(x)*: $F = F\left(x,\ u(x),\ u'(x)\right)$
- Consider the change in *F* for fixed *x*

$$\Delta F = F(x,\ u+\delta u,\ u'+\delta u') - F(x, u, u')$$
$$= F + \frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u' + \frac{1}{2!}\frac{\partial^2 F}{\partial u^2}(\delta u)^2 + \frac{1}{2!}\frac{\partial^2 F}{\partial u \partial u'}(\delta u)(\delta u') + \frac{1}{2!}\frac{\partial^2 F}{\partial u'^2}(\delta u')^2 + \cdots - F$$
$$= \frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u' \ \left(+\ \cdots\ H.O.T\right)$$

- Comparison between δF and dF

$$\delta F = \frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u'$$
$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial u}du + \frac{\partial F}{\partial u'}du'$$

- δ acts link d with respect to dependent variable (i.e., u(x) and u'(x)) when independent variable (i.e., x) is fixed.

2. Calculus of Variation – What is a Functional?

- **Functional**: Functions of dependent variables, also function of other parameters

$$I(u)=\int_a^b F\left(x, u, u'\right)dx$$

- Mathematically, functional *I(u)* is a mapping from a vector space *u* into the real number field; operator *I* mapping *u* into scalar.
- First variation of a functional $$\delta I(u)=\delta\int_a^b F\left(x, u, u'\right)dx=\int_a^b \delta F\, dx=\int_a^b\left[\frac{\partial F}{\partial u}\delta u+\frac{\partial F}{\partial u'}\delta u'\right]dx$$
- A functional $I(u)$ is said to have a minimum at u_0 if and only if $I(u)\geq I(u_0)$
- From elementary calculus, we know that a differentiable function *f(x)* has an extremum at a point xo only if (necessary condition): $$\left.\frac{df}{dx}\right|_{x=x_0}=0$$
- The sufficient condition for maximum: $$\frac{d^2 f}{dx^2}<0$$
- The sufficient condition for minimum: $$\frac{d^2 f}{dx^2}>0$$
- The functional $I(u)$ is said to have a minimum at $u=u_0$ if it first variation $\delta I(u)=0$

  and its second variation $\delta^2 I(u)$ is strongly positive at $u=u_0$.

2. Calculus of Variation – Euler-Lagrange Equations

- Determine the minimum of the functional $I(u) = \int_a^b F\left(x, u, u'\right) dx$
- Subjected to the end conditions $u\left(a\right) = u_a, \quad u\left(b\right) = u_b$
- Let $u = u_0 + \delta u$: $\quad \delta u\left(a\right) = \delta u\left(b\right) = 0$
- Seek particular function *u* that makes the integral a minimum

$$0 = \delta I(u) = \int_a^b \left[\frac{\partial F}{\partial u} \delta u + \frac{\partial F}{\partial u'} \delta u' \right] dx = \int_a^b \left[\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'} \right) \right] \delta u dx + \underbrace{\left[\frac{\partial F}{\partial u'} \delta u \right]_a^b}_{=0}$$

- Euler-Lagrange Equation $\quad \dfrac{\partial F}{\partial u} - \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial u'} \right) = 0 \quad$ in $\quad a < x < b$

- Essential Boundary Conditions (Displacement) - Specifying dependent variables:

$$\delta u = 0 \ \left(u = \hat{u}\right) \text{ at } \ x = a, b \ \text{ thus } \ \left.\frac{\partial F}{\partial u'}\right|_{x=b} \delta u\left(b\right) - \left.\frac{\partial F}{\partial u'}\right|_{x=a} \delta u\left(a\right) = 0$$

- Natural Boundary Conditions (Force) - Specifying the coefficients of dependent variables

$$\frac{\partial F}{\partial u'} = 0 \quad \text{at} \quad x = a, b$$

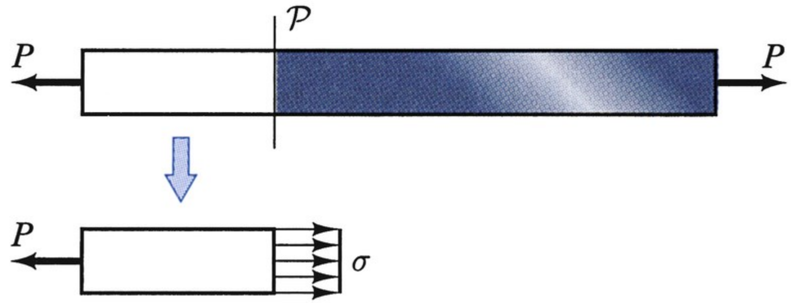
2. Calculus of Variation – Euler-Lagrange Equations

- Either $u(a)$ or $\left.\dfrac{\partial F}{\partial u'}\right|_{x=a}$ is specified.
- Either $u(b)$ or $\left.\dfrac{\partial F}{\partial u'}\right|_{x=b}$ is specified.

Types of Boundary Value Problems

- All Essential Boundary Conditions: Dirichlet Boundary Value Problem
- All Natural Boundary Conditions: Neumann Boundary Value Problem
- Essential and Natural Boundary Conditions: Mixed Boundary Value Problem

2. Calculus of Variation – Axially-loaded Bar Problem



❖ Internal Energy (Strain Energy):

$$W_I = U = \int_0^L \frac{EA}{2}\left(\frac{du}{dx}\right)^2 dx$$

❖ External Energy: $W_E = -Pu(L)$

❖ Total Work (Functional): $W = W_I + W_E = \int_0^L \frac{EA}{2}\left(\frac{du}{dx}\right)^2 dx - Pu(L) = I(u)$

❖ Minimization (first variation of a functional - mathematically or virtual work - physically)

$$\delta I(u) = \delta W = \int_0^L EA\frac{du}{dx}\frac{d\delta u}{dx} dx - P\delta u(L)$$

$$0 = -\int_0^L \frac{d}{dx}\left(EA\frac{du}{dx}\right)\delta u\, dx + \left[\left(EA\frac{du}{dx} - P\right)\delta u\right]_{x=L} + \left[\left(EA\frac{du}{dx}\right)\delta u\right]_{x=0}$$

❖ Euler Equations $-\frac{d}{dx}\left(EA\frac{du}{dx}\right) = 0$

❖ Boundary Conditions

At $x = 0$: $u = 0 \ \ (\text{EBC}) \rightarrow \delta u = 0 \rightarrow \left(EA\frac{du}{dx}\right)\delta u = 0$

At $x = L$: $u = unknown \rightarrow \delta u = arbitrary \rightarrow \left(EA\frac{du}{dx} - P\right)\delta u = 0 \rightarrow P = EA\frac{du}{dx} \ \ (\text{NBC})$

2. Calculus of Variation – Axial Bar with Spring Problem

- An elastic bar fixed at the left end, spring supported at the right end, and subjected to distributed load. Total work is

$$\Pi(u) = \int_0^L \left[\frac{EA}{2}\left(\frac{du}{dx}\right)^2 - fu \right] dx + \frac{k}{2} u(L)^2$$

- The first variation is given by

$$\delta\Pi(u) = \int_0^L \left(EA \frac{du}{dx}\frac{d\delta u}{dx} - f\,\delta u \right) dx + ku(L)\,\delta u(L)$$

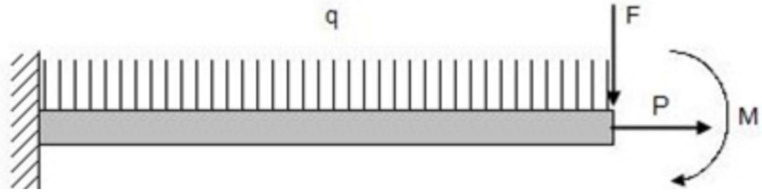
- Euler Equation (after using the integration by parts)

$$-\frac{d}{dx}\left(EA\frac{du}{dx}\right) - f = 0, \quad 0 < x < L$$

- Natural Boundary Condition (NBC)

$$EA\frac{du}{dx} + ku(L) = 0 \quad \text{at} \quad x = L$$

2. Calculus of Variation – Cantilever Beam Problem



- First variation of the functional (refer to page 5)

$$\delta W = \int_0^L \left[N\frac{d\delta u}{dx} + M\left(\frac{-d^2\delta w}{\partial x^2}\right) - f\delta u - q\delta w \right] dx - P\delta u(L) - F\delta w(L) = 0$$

u and w now represent mid-plane displacements.

- Stress resultants $\quad N = \int_A \sigma_x dA = EA\frac{du}{dx}; \qquad M = \int_A \sigma_x z dA = -EI\frac{d^2 w}{dx^2}$

- Integration by parts

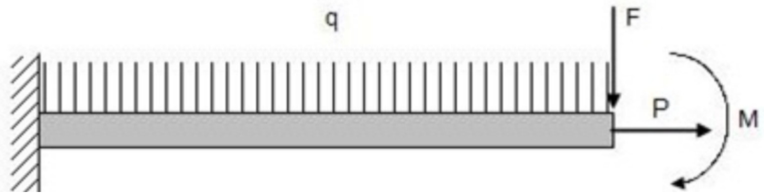
$$\delta W = -\int_0^L \left[\frac{d}{dx}\left(EA\frac{du}{dx}\right)\delta u + \frac{d}{dx}\left(EI\frac{d^2 w}{dx^2}\right)\frac{d\delta w}{dx} + f\delta u + q\delta w \right] dx$$

$$+\left[\left(EA\frac{du}{dx}\right)\delta u\right]_0^L + \left[\left(EI\frac{d^2 w}{dx^2}\right)\frac{d\delta w}{dx}\right]_0^L - P\delta u(L) - F\delta w(L) = 0$$

$$\delta W = -\int_0^L \left[\frac{d}{dx}\left(EA\frac{du}{dx}\right)\delta u - \frac{d^2}{dx^2}\left(EI\frac{d^2 w}{dx^2}\right)\delta w + f\delta u + q\delta w \right] dx$$

$$+\left[\left(EA\frac{du}{dx}\right)\delta u + \left(EI\frac{d^2 w}{dx^2}\right)\frac{d\delta w}{dx} - \frac{d}{dx}\left(EI\frac{d^2 w}{dx^2}\right)\delta w\right]_0^L - P\delta u(L) - F\delta w(L) = 0$$

2. Calculus of Variation – Cantilever Beam Problem



- Euler Equations

$$\frac{d}{dx}\left(EA\frac{du}{dx}\right)+f=0$$

$$-\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right)+q=0$$

- Boundary Conditions

At x = 0

Either $\left(EA\frac{du}{dx}\right)$ or δu

Either $\left(EI\frac{d^2w}{dx^2}\right)$ or $\frac{d\delta w}{dx}$

Either $\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right)$ or δw

At x = L

Either $\left(EA\frac{du}{dx}\right)-P$ or δu

Either $\left(EI\frac{d^2w}{dx^2}\right)$ or $\frac{d\delta w}{dx}$

Either $\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right)+F$ or δw

3. Hamilton's Principle – Principle of Virtual Displacements

- A continuous body is in equilibrium if and only if the virtual work of all forces acting on the body is zero in virtual displacement.

$$\delta W = \delta W_I + \delta W_E = 0$$

- Applicable to any continuous bodies (elastic and inelastic)

$$\int_V \sigma_{ij} \cdot \delta\varepsilon_{ij}\, dV - \int_V f_i \cdot \delta u_i\, dV - \int_{S_2} \hat{t}_i \cdot \delta u_i\, dS = 0$$

- Divergence theorem $\int_\Omega \nabla \cdot \mathbf{A}\, d\Omega = \oint_\Gamma \hat{\mathbf{n}} \cdot \mathbf{A}\, ds$

$$\int_V \sigma_{ij}\delta\varepsilon_{ij}\, dV = \int_V \sigma_{ij}\delta u_{i,j}\, dV = -\int_V \sigma_{ij,j}\delta u_i\, dV + \int_V \left(\sigma_{ij}\delta u_i\right)_{,j} dV = -\int_V \sigma_{ij,j}\delta u_i\, dV + \oint_S n_j\sigma_{ij}\delta u_i dS$$

- The boundary integral can be expressed as the sum of two integral, one on S1 (EBC) and the other on S2 (NBC). The integral on S1 is zero.

$$-\int_V \left(\sigma_{ij,j} + f_i\right)\delta u_i\, dV + \oint_{S_2} \left(n_j\sigma_{ij} - \hat{t}_i\right)\delta u_i dS = 0$$

- Equilibrium Equations and Stress Boundary Conditions

$$\sigma_{ij,j} + f_i = 0 \quad \text{in} \quad V$$

$$n_j\sigma_{ij} - \hat{t}_i = 0 \quad \text{in} \quad S_2$$

3. Hamilton's Principle – Principle of Virtual Forces

- The strains and displacements in a deformable body are compatible and consistent with the constraints if and only if the total complementary virtual work is zero

$$\delta W^* = \delta W_I^* + \delta W_E^* = 0$$

- Applicable to any continuous bodies (elastic and inelastic)

$$\int_V \varepsilon_{ij} \cdot \delta\sigma_{ij}\, dV - \int_V \delta f_i \cdot u_i\, dV - \int_{S_1} \delta t_i \cdot \hat{u}_i\, dS = 0$$

- Equilibrium Equations and Stress BCs (page 17) $\delta f_i = -\delta\sigma_{ij,j}$; $\delta t_i = n_j \delta\sigma_{ji}$

$$\int_V \varepsilon_{ij}\delta\sigma_{ij}\, dV + \int_V \delta\sigma_{ij,j}u_i\, dV - \int_{S_1} \delta\sigma_{ji}n_j\hat{u}_i\, dS = 0$$

Integration by parts $\leftrightarrow$ $$\int_V \varepsilon_{ij}\delta\sigma_{ij}\, dV + \int_V \left(\delta\sigma_{ij}u_i\right)_{,j} dV - \int_V \delta\sigma_{ij}u_{i,j}\, dV - \int_{S_1} \delta\sigma_{ji}n_j\hat{u}_i\, dS = 0$$

Divergence theorem $\leftrightarrow$ $$\int_V \varepsilon_{ij}\delta\sigma_{ij}\, dV + \int_{S_1} \delta\sigma_{ij}u_i n_j\, dS - \int_V \delta\sigma_{ij}u_{i,j}\, dV - \int_{S_1} \delta\sigma_{ji}n_j\hat{u}_i\, dS = 0$$

Symmetry feature $\leftrightarrow$ $$\int_V \left[\varepsilon_{ij} - \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)\right]\delta\sigma_{ij}\, dV + \int_{S_1} \left(u_i - \hat{u}_i\right)n_j\delta\sigma_{ij}dS = 0$$

- Strain-displacement Equations and Displacement Boundary Conditions

$$\varepsilon_{ij} - \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) = 0 \quad \text{in} \quad V$$

$$u_i - \hat{u}_i = 0 \quad \text{on} \quad S_1$$

3. Hamilton's Principle – Principle of Potential Energy

Principle of Total Potential Energy

❖ A special case of the principle of virtual work that deals with elastic bodies is known as the principle of total potential energy.

$$\delta\Pi = \delta U + \delta V = 0$$

with

$$\delta U = \delta W_I, \quad \delta V = \delta W_E$$

Principle of Complementary Potential Energy

❖ A special case of the principle of complementary virtual work that deals with elastic bodies is known as the principle of complementary potential energy.

$$\delta\Pi^* = \delta U^* + \delta V^* = 0$$

with

$$\delta U^* = \delta W_I^*, \quad \delta V^* = \delta W_E^*$$

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thank you