Lesson 10

Timoshenko Beam

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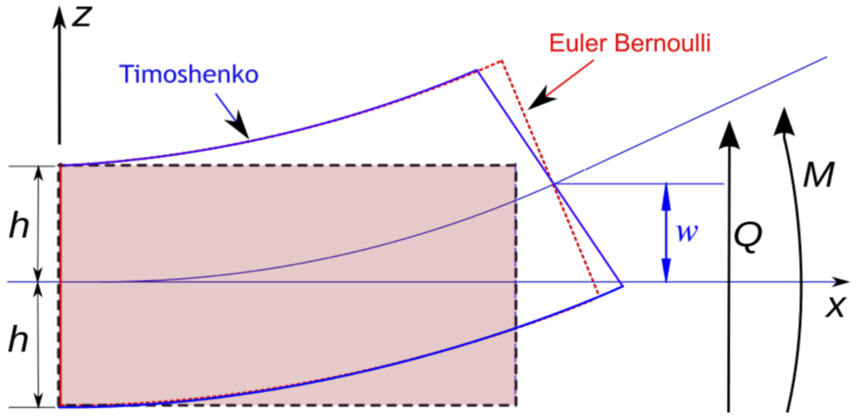
Content

1. Timoshenko Beam – Static Analysis

❖ The Timoshenko beam theory accounts for transverse shear deformation and rotational inertia effects. Thus, it is capable of modeling thin and thick beams.



❖ The deformed cross-section planes remain plane but not normal to the middle axis.

❖ The beam lays in the $x-z$ plane, the displacement field is

$$u_1(x,z,t) = z\theta_y(x,t), \quad u_3(x,z,t) = w(x,t)$$

$u_1,\ u_3$ *are the axial and transverse displacements of the 3D fibers of the beam.*

$w,\ \theta_y$ *denote the kinematic parameters of the theory as constant transverse displacement and rotation of cross-section plane about a normal to the middle axis x.*

❖ The normal and transverse shear strain are

$$\varepsilon_x = \frac{\partial u_1}{\partial x} = z\frac{\partial \theta_y}{\partial x}; \qquad \gamma_{xz} = \frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x} = \theta_y + \frac{\partial w}{\partial x}$$

1. Timoshenko Beam – Static Analysis

- The strain energy considers both bending and shear contributions,

$$U=\frac{1}{2}\int_V \sigma_x\varepsilon_x dV+\frac{1}{2}\int_V \tau_{xz}\gamma_{xz}dV=\frac{1}{2}\int_V E\varepsilon_x^2 dV+\frac{1}{2}\int_V kG\gamma_{xz}^2 dV=\frac{1}{2}\int_{-a}^{a} EI_y\left(\frac{\partial\theta_y}{\partial x}\right)^2 dx+\frac{1}{2}\int_{-a}^{a} kGA\left(\frac{\partial w}{\partial x}+\theta_y\right)^2 dx$$

$\tau_{xz}=kG\gamma_{xz}$ *is the transverse shear stress,* $k=5/6$ *is the shear correction factor.*

$G=\dfrac{E}{2(1+\nu)}$ *is the shear modulus.*

- Each node of this 2-node element has 2 DOFs (i.e., w and θ_y)

$$\mathbf{u}^{eT}=\begin{bmatrix} w_1 & w_2 & \theta_{y1} & \theta_{y2}\end{bmatrix}=\begin{bmatrix}\mathbf{w}^{eT} & \boldsymbol{\theta}_y^{eT}\end{bmatrix} \leftrightarrow \mathbf{u}^e=\begin{bmatrix}\mathbf{w}^e \\ \boldsymbol{\theta}_y^e\end{bmatrix}$$

- In opposition to Bernoulli beam, here the interpolation of displacements is independent for both w and θ_y .

$$w=\mathbf{N}\mathbf{w}^e;\quad \theta_y=\mathbf{N}\boldsymbol{\theta}_y^e$$

with the shape functions $\mathbf{N}=\begin{bmatrix}\frac{1}{2}(1-\xi) & \frac{1}{2}(1+\xi)\end{bmatrix}$ *in natural coordinates* $\xi\in[-1,1]$

- The strain energy then becomes

$$U=\frac{1}{2}\boldsymbol{\theta}_y^{eT}\int_{-a}^{a} EI_y\mathbf{N}'^T\mathbf{N}'dx\,\boldsymbol{\theta}_y^e+\frac{1}{2}\int_{-a}^{a} kGA\left(\mathbf{N}'\mathbf{w}^e+\mathbf{N}\boldsymbol{\theta}_y^e\right)^T\left(\mathbf{N}'\mathbf{w}^e+\mathbf{N}\boldsymbol{\theta}_y^e\right)dx \quad \textit{with} \quad \mathbf{N}'=\frac{d\mathbf{N}}{dx}$$

1. Timoshenko Beam – Static Analysis

- The coordinate transformation is applied to have the integrals in natural coordinates as

$$U = U_b + U_s = \frac{1}{2}\boldsymbol{\theta}_y^{eT}\int_{-1}^{1}\frac{EI_y}{a^2}\mathbf{N}'^T\mathbf{N}'\,a\,d\xi\,\boldsymbol{\theta}_y^e + \frac{1}{2}\int_{-1}^{1}kGA\left(\frac{1}{a}\mathbf{N}'\mathbf{w}^e + \mathbf{N}\boldsymbol{\theta}_y^e\right)^T\left(\frac{1}{a}\mathbf{N}'\mathbf{w}^e + \mathbf{N}\boldsymbol{\theta}_y^e\right)a\,d\xi$$

U_b is the bending part, U_s is the shear part.

- The bending part can be directly computed, the shear part should be reordered.

$$\begin{aligned}
U_s &= \frac{1}{2}\int_{-1}^{1}kGA\left(\begin{bmatrix}\frac{1}{a}\mathbf{N}' & \mathbf{N}\end{bmatrix}\begin{bmatrix}\mathbf{w}^e \\ \boldsymbol{\theta}_y^e\end{bmatrix}\right)^T\left(\begin{bmatrix}\frac{1}{a}\mathbf{N}' & \mathbf{N}\end{bmatrix}\begin{bmatrix}\mathbf{w}^e \\ \boldsymbol{\theta}_y^e\end{bmatrix}\right)a\,d\xi \\
&= \frac{1}{2}\int_{-1}^{1}kGA\begin{bmatrix}\mathbf{w}^{eT} & \boldsymbol{\theta}_y^{eT}\end{bmatrix}\begin{bmatrix}\frac{1}{a}\mathbf{N}'^T \\ \mathbf{N}^T\end{bmatrix}\begin{bmatrix}\frac{1}{a}\mathbf{N}' & \mathbf{N}\end{bmatrix}\begin{bmatrix}\mathbf{w}^e \\ \boldsymbol{\theta}_y^e\end{bmatrix}a\,d\xi \\
&= \frac{1}{2}\mathbf{u}^{eT}\int_{-1}^{1}kGA\begin{bmatrix}\frac{1}{a}\mathbf{N}'^T \\ \mathbf{N}^T\end{bmatrix}\begin{bmatrix}\frac{1}{a}\mathbf{N}' & \mathbf{N}\end{bmatrix}a\,d\xi\,\mathbf{u}^e \\
&= \frac{1}{2}\mathbf{u}^{eT}\int_{-1}^{1}kGA\begin{bmatrix}\frac{1}{a^2}\mathbf{N}'^T\mathbf{N}' & \frac{1}{a}\mathbf{N}'^T\mathbf{N} \\ \frac{1}{a}\mathbf{N}^T\mathbf{N}' & \mathbf{N}^T\mathbf{N}\end{bmatrix}a\,d\xi\,\mathbf{u}^e
\end{aligned}$$

1. Timoshenko Beam – Static Analysis

- Finally, the strain energy becomes

$$U = \frac{1}{2}\mathbf{u}^{eT}\int_{-1}^{1}\left(\frac{EI_y}{a}\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{N}'^T\mathbf{N}'\end{bmatrix} + \frac{kGA}{a}\begin{bmatrix}\mathbf{N}'^T\mathbf{N}' & a\mathbf{N}'^T\mathbf{N}\\ a\mathbf{N}^T\mathbf{N}' & a^2\mathbf{N}^T\mathbf{N}\end{bmatrix}\right)d\xi\,\mathbf{u}^e$$

- Thus, the stiffness matrix for a generic element (size 4 x 4) is

$$\mathbf{K}^e = \int_{-1}^{1}\left(\frac{EI_y}{a}\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{N}'^T\mathbf{N}'\end{bmatrix} + \frac{kGA}{a}\begin{bmatrix}\mathbf{N}'^T\mathbf{N}' & a\mathbf{N}'^T\mathbf{N}\\ a\mathbf{N}^T\mathbf{N}' & a^2\mathbf{N}^T\mathbf{N}\end{bmatrix}\right)d\xi$$

- Due to shear locking in thin beams, it is strongly recommended to use exact integration with 2 Gauss points for the bending stiffness and use reduce integration with 1 Gauss points for the shear stiffness.

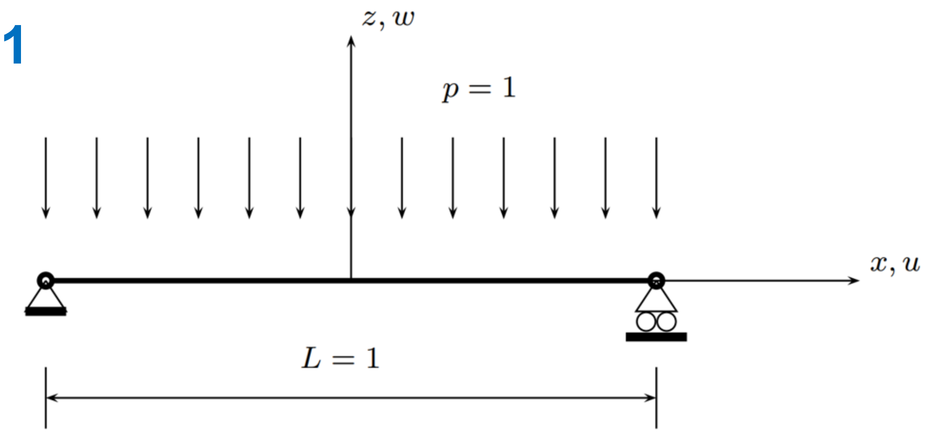
- The external work is

$$W_E = \int_{-a}^{a} p\,w\,dx = \int_{-a}^{a} p\,\mathbf{N}\mathbf{w}^e\,dx = \mathbf{w}^{eT}\int_{-a}^{a} p\,\mathbf{N}^T dx = \mathbf{u}^{eT}\int_{-a}^{a}\begin{bmatrix}p\,\mathbf{N}^T\\ \mathbf{0}\end{bmatrix}dx = \mathbf{u}^{eT}\int_{-1}^{1}\begin{bmatrix}p\,\mathbf{N}^T\\ \mathbf{0}\end{bmatrix}a\,d\xi$$

- The force vector is given by

$$\mathbf{f}^e = \int_{-1}^{1}\begin{bmatrix}p\,\mathbf{N}^T\\ \mathbf{0}\end{bmatrix}a\,d\xi$$

1. Timoshenko Beam – Example 1



- A simply-supported (SS) Timoshenko beam, under uniform load $p=1$, beam width $b=1$, thus $I = h^3/12$.
  $E = 18^8, \quad \upsilon = 0.3$

- Exact solutions based on the first order shear deformation theory of SS Timoshenko beam:

$$w(x) = \frac{PL^4}{24D}\left(\frac{x}{L} - \frac{2x^3}{L^3} + \frac{x^4}{L^4}\right) + \frac{PL^2}{2S}\left(\frac{x}{L} - \frac{x^3}{L^3}\right)$$

with the shear stiffness $S = kGA$ *and the flexural stiffness* $D = \dfrac{Eh^3}{12\left(1-v^2\right)}$

$$w_{\max} = \frac{5}{384}\frac{pL^4}{EI}$$

- Exact solutions for cantilever (CF) Timoshenko beam:

$$w(x) = \frac{PL^4}{24D}\left(6\frac{x}{L} - \frac{4x^3}{L^3} + \frac{x^4}{L^4}\right) + \frac{PL^2}{2S}\left(2\frac{x}{L} - \frac{x^2}{L^2}\right)$$

$$w_{\max} = \frac{1}{8}\frac{pL^4}{EI}$$

1. Timoshenko Beam – Example 1 - Python

- Python code: https://github.com/mctrinh/fem-class

1. Timoshenko Beam – Free Vibrations

- The kinematic energy has two parts (i.e., translation and rotations related)

$$K = \frac{1}{2}\int_{-a}^{a} \rho A \dot{w}^2 dx + \frac{1}{2}\int_{-a}^{a} \rho I_y \dot{\theta}_y^2 dx$$

- After coordinate transformation and linear interpolation

$$K = \frac{1}{2}\dot{\mathbf{w}}^{eT}\int_{-1}^{1} \rho A \mathbf{N}^T\mathbf{N}\, a d\xi \dot{\mathbf{w}}^e + \frac{1}{2}\dot{\boldsymbol{\theta}}_y^{eT}\int_{-1}^{1} \rho I_y \mathbf{N}^T\mathbf{N} a d\xi \dot{\boldsymbol{\theta}}_y^e$$

- Collecting the terms of the displacement vector leads to

$$K = \frac{1}{2}\mathbf{u}^{eT}\int_{-1}^{1}\left(\rho A a\begin{bmatrix}\mathbf{N}^T\mathbf{N} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix} + \rho I_y a\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{N}^T\mathbf{N}\end{bmatrix}\right)d\xi\, \mathbf{u}^e$$

- The element mass matrix is

$$\mathbf{M}^e = \int_{-1}^{1}\begin{bmatrix}\rho A a \mathbf{N}^T\mathbf{N} & \mathbf{0}\\ \mathbf{0} & \rho I_y a \mathbf{N}^T\mathbf{N}\end{bmatrix} d\xi$$

- The element stiffness matrix is like that in the static analysis.

1. Timoshenko Beam – Example 2

- Consider a thin Timoshenko cantilever beam with $(L = 1, h = 0.001)$.
- Results are presented in the nondimensional natural frequencies as

$$\bar{\omega} = \omega L^2 \sqrt{\frac{\rho A}{EI_y}}$$

- Python codes: https://github.com/mctrinh/fem-class

1. Timoshenko Beam – Buckling Analysis

- The work (energy) due to the applied compression load is

$$W_g = \frac{1}{2}\int_{-a}^{a} P\left(\frac{\partial w}{\partial x}\right)^2 dx = \frac{1}{2}\mathbf{w}^{eT}\int_{-1}^{1}\frac{P}{a^2}\mathbf{N}'^T\mathbf{N}' a\, d\xi\, \mathbf{w}^e$$

- The relation is written in terms of the displacement vector

$$W_g = \frac{1}{2}\mathbf{u}^{eT}\int_{-1}^{1}\frac{P}{a}\begin{bmatrix}\mathbf{N}'^T\mathbf{N}' & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix} d\xi\, \mathbf{u}^e$$

- The geometric stiffness matrix is

$$\mathbf{K}_g = \int_{-1}^{1}\frac{P}{a}\begin{bmatrix}\mathbf{N}'^T\mathbf{N}' & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix} d\xi$$

- The buckling analysis of Timoshenko beams considers the solution of the eigen-problem

$$\left[\mathbf{K} - \lambda\mathbf{K}_g\right]\mathbf{X} = \mathbf{0}$$

λ *is the critical buckling load,* $\mathbf{X}$ *is the buckling modes.*

- The exact solution for simply-supported (SS) and clamped (CC) Timoshenko is

$$P_{cr} = \frac{\pi^2 EI}{L_{eff}^2}\left[1 + \frac{\pi^2 EI}{L_{eff}^2 kGA}\right]^{-1}$$

with $L_{eff} = L$ *for SS (pinned-pinned) beams,*
$L_{eff} = L/2$ *for CC (fixed-fixed) beams.*

1. Timoshenko Beam – Problem 3

- Python code: https://github.com/mctrinh/fem-class

Content

1. Timoshenko Beam
    - Static Analysis
    - Example 1 – Python code
    - Free Vibrations
    - Example 2 – Python code
    - Buckling Analysis
    - Example 3 – Python code

thank you