Lesson 5

Variational Methods

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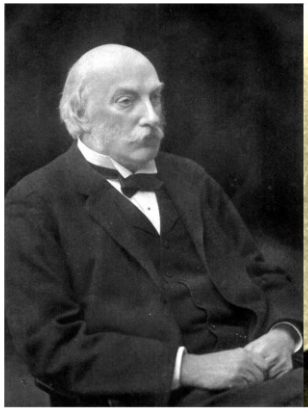
1. Rayleigh-Ritz Method – Background

- ❖ Minimum of a functional defined on a normed linear space is approximated by a linear combination of elements from that space.
- ❖ This method will yield solutions when an analytical form for the true solution may be intractable.
  - *Rayleigh, J. W. "In Finding the Correction for the Open End of an Organ-Pipe." Phil. Trans. 161, 77, 1870.*
  - *Ritz, W. "Uber eine neue Methode zur Losung gewisser Variations probleme der mathematischen Physik." J. reine angew. Math. 135, 1-61, 1908.*

$$u_i = U_i = \sum_{j=1}^{n} c_j \phi_j + \phi_0$$

- ❖ ϕ should satisfy the following conditions.
  - *Sufficiently differentiable.*
  - *Satisfy the homogeneous form of the specified essential boundary conditions.*
  - *Linearly independent and complete.*
- ❖ ϕ_0 should satisfy the specified essential boundary conditions.

1. Rayleigh-Ritz Method – Background




- Rayleigh: John William Strutt, 3rd Baron Rayleigh (1842-1919) was an English physicist who, with William Ramsay, discovered argon, an achievement for which he earned the Nobel Prize for Physics in 1904. He also discovered the phenomenon now called Rayleigh scattering, which can be used to explain why the sky is blue, and predicted the existence of the surface waves now known as Rayleigh waves
- Ritz: Walther Ritz (1878-1909) was a Swiss theoretical physicist.

R-R Formulation

- Consider a functional $$I(u)=\frac{1}{2}B(u,u)-l(u)=\frac{1}{2}B\left(\sum_{j=1}^{n}c_j\phi_j+\phi_0,\sum_{j=1}^{n}c_j\phi_j+\phi_0\right)-l\left(\sum_{j=1}^{n}c_j\phi_j+\phi_0\right)$$
- Minimize the functional $$0=\delta I\left(\sum_{j=1}^{n}c_j\phi_j+\phi_0\right)=\sum_{j=1}^{n}\frac{\partial I}{\partial c_j}\delta c_j \quad \rightarrow \quad \frac{\partial I}{\partial c_j}=0$$
- Alternative $$B\left(\phi_i,\sum_{j=1}^{n}c_j\phi_j+\phi_0\right)=l(\phi_i)$$

$$\sum_{j=1}^{n}B(\phi_i,\phi_j)c_j=l(\phi_i)-B(\phi_i,\phi_0)$$

or $$\sum_{j=1}^{n}B_{ij}c_j=F_i$$

1. Rayleigh-Ritz Method – 2nd-order Differential Equation (Dirichlet)

$$-\frac{d^2u}{dx^2}-u+x^2=0, \quad 0<x<1 \qquad \text{subjected to} \qquad u(0)=0, \;\; u(1)=0$$

❖ Weak Formulation $\quad 0=\int_0^1\left(-w\frac{d^2u}{dx^2}-wu+wx^2\right)dx=\int_0^1\left(\frac{dw}{dx}\frac{du}{dx}-wu+wx^2\right)dx=B(w,u)-l(w)$

❖ Choose approximate function (need to satisfy EBC only)

$$\phi_1=x(1-x); \quad \phi_2=x^2(1-x); \quad \phi_3=x^3(1-x); \quad ....; \quad \phi_n=x^n(1-x)$$

Therefore

$$U_N=\sum_{j=1}^{N}c_jx^j(1-x)$$

Approach 1

❖ R-R Functional (let w = u)

$$I(u)=\frac{1}{2}B(u,u)-l(u)=\frac{1}{2}\int_0^1\left[\left(\frac{du}{dx}\right)^2-u^2+2ux^2\right]dx$$

$$=\frac{1}{2}\int_0^1\left[\left(\sum_{j=1}^{N}c_j\frac{d\phi_j}{dx}\right)\left(\sum_{k=1}^{N}c_k\frac{d\phi_k}{dx}\right)-\left(\sum_{j=1}^{N}c_j\phi_j\right)\left(\sum_{k=1}^{N}c_k\phi_k\right)+2\left(\sum_{j=1}^{N}c_j\phi_j\right)x^2\right]dx$$

1. Rayleigh-Ritz Method – 2nd-order Differential Equation (Dirichlet)

Approach 1 (cont.)

- Weak Form $$0=\delta I(u)=\frac{\partial I}{\partial c_i}=\sum_{j=1}^{N}\int_0^1\left(\frac{d\phi_i}{dx}\frac{d\phi_j}{dx}-\phi_i\phi_j\right)c_j dx+\int_0^1 x^2\phi_i dx=\sum_{j=1}^{N}B(\phi_i,\phi_j)c_j-l(\phi_i)$$

with $$B_{ij}=B(\phi_i,\phi_j)=\int_0^1\left(\frac{d\phi_i}{dx}\frac{d\phi_j}{dx}-\phi_i\phi_j\right)dx$$

$$F_i=l(\phi_i)=-\int_0^1 x^2\phi_i dx$$

Approach 2

- Linear and Bilinear Form $$B(w,u)=\int_0^1\left(\frac{dw}{dx}\frac{du}{dx}-wu\right)dx,\qquad l(w)=-\int_0^1 wx^2\,dx$$
- Therefore

$$B\left(\phi_i,\sum_{j=1}^{N}c_j\phi_j+\phi_0\right)=\int_0^1\left[\frac{d\phi_i}{dx}\frac{d}{dx}\left(\sum_{j=1}^{N}c_j\phi_j\right)-\phi_i\left(\sum_{j=1}^{N}c_j\phi_j\right)\right]dx=\sum_{j=1}^{N}\int_0^1\left(\frac{d\phi_i}{dx}\frac{d\phi_j}{dx}-\phi_i\phi_j\right)dx\,c_j=\sum_{j=1}^{N}B(\phi_i,\phi_j)\,c_j$$

with $$B_{ij}=B(\phi_i,\phi_j)=\int_0^1\left(\frac{d\phi_i}{dx}\frac{d\phi_j}{dx}-\phi_i\phi_j\right)dx$$

$$F_i=l(\phi_i)=-\int_0^1 x^2\phi_i dx$$

1. Rayleigh-Ritz Method – 2^{nd}-order Differential Equation (Dirichlet)

Approach 2 (cont.)

- Evaluate the Linear and Bilinear Form

$$\phi_i = x^i(1-x) = x^i - x^{i+1} \quad \rightarrow \quad \frac{d\phi_i}{dx} = ix^{i-1} - (i+1)x^i$$

$$B_{ij} = \int_0^1 \left(\frac{d\phi_i}{dx}\frac{d\phi_j}{dx} - \phi_i\phi_j \right) dx = \int_0^1 \left[\left(ix^{i-1} - (i+1)x^i \right)\left(jx^{j-1} - (j+1)x^j \right) - \left(x^i - x^{i+1} \right)\left(x^j - x^{j+1} \right) \right] dx$$

$$F_i = -\int_0^1 x^2 \phi_i dx = -\int_0^1 x^2 \left(x^i - x^{i+1} \right) dx$$

- One-parameter solution (N=1)

$$\phi_1 = x(1-x) = x - x^2 \rightarrow \frac{d\phi_1}{dx} = 1-2x \rightarrow B_{11} = \int_0^1 \left[(1-2x)^2 - x^2(1-x)^2 \right] dx, \quad F_1 = -\int_0^1 x^2 x(1-x) dx$$

$$\rightarrow c_1 = F_1 / B_{11} = -0.1667 \quad \rightarrow \quad u = U_1 = -0.1667x(1-x)$$

- Two-parameter solution (N=2): $\phi_1 = x(1-x), \quad \phi_2 = x^2(1-x)$

$$B_{11} = \int_0^1 \left(\frac{d\phi_1}{dx}\frac{d\phi_1}{dx} - \phi_1\phi_1 \right) dx, \quad B_{12} = B_{21} = \int_0^1 \left(\frac{d\phi_1}{dx}\frac{d\phi_2}{dx} - \phi_1\phi_2 \right) dx, \quad B_{22} = \int_0^1 \left(\frac{d\phi_2}{dx}\frac{d\phi_2}{dx} - \phi_2\phi_2 \right) dx$$

$$F_1 = -\int_0^1 x^2 \phi_1 dx, \quad F_2 = -\int_0^1 x^2 \phi_2 dx$$

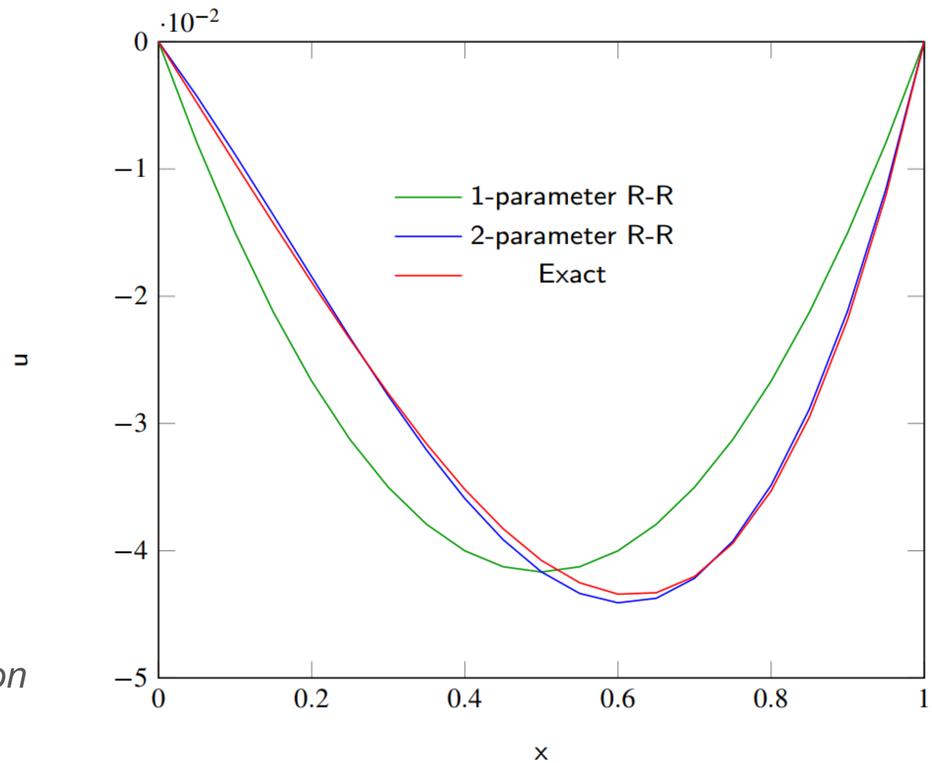
Approach 2 (cont.)

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$\therefore \quad U_2 = c_1\phi_1 + c_2\phi_2 = -0.0813x(1-x) - 0.1707x^2(1-x)$$

❖ Exact Solution

$$u_{exact} = \frac{\sin x + 2\sin(1-x)}{\sin 1} + x^2 - 2$$



Rayleigh-Ritz Solution

for 2nd-order Differential Equation

1. Rayleigh-Ritz Method – 2nd-order Diff. Equation (Mixed BVP)

$$-\frac{d^2u}{dx^2} - u + x^2 = 0, \quad 0 < x < 1 \quad \text{subjected to} \quad u(0) = 0, \quad \left.\frac{du}{dx}\right|_{x=1} = 1$$

❖ Linear and Bilinear Form
$$B(w,u) = \int_0^1 \left(\frac{dw}{dx}\frac{du}{dx} - wu \right) dx, \qquad l(w) = -\int_0^1 wx^2\, dx + w(1)$$

or
$$B_{ij} = B(\phi_i, \phi_j) = \int_0^1 \left(\frac{d\phi_i}{dx}\frac{d\phi_j}{dx} - \phi_i\phi_j \right) dx$$

$$F_i = l(\phi_i) = -\int_0^1 x^2 \phi_i dx + \phi_i(1)$$

❖ Choose Approximate Function

(Need to satisfy EBC only) $\phi_i = x^i$

❖ One-parameter Solution

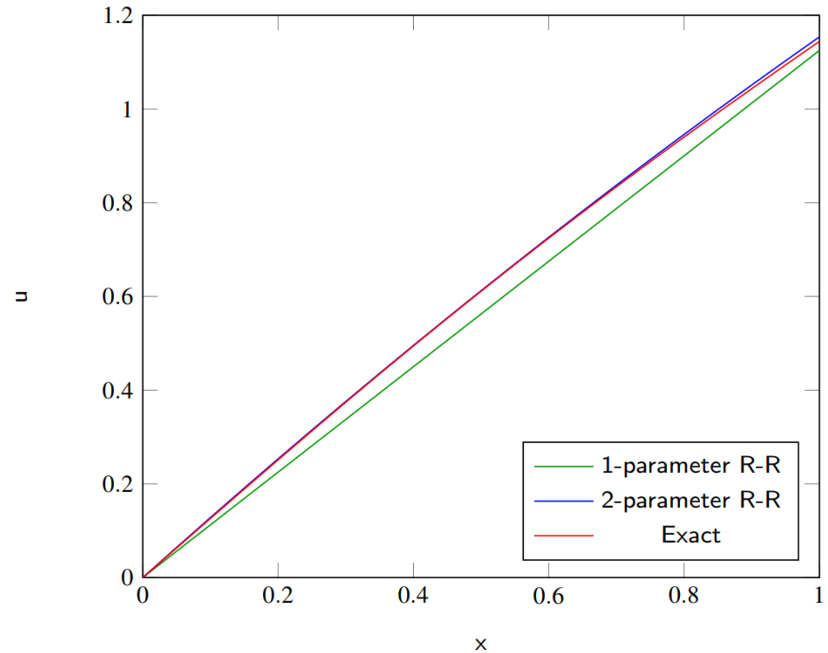
$$u = U_1 = 1.125x$$

❖ Two-parameter Solution

$$u = U_2 = 1.295x - 0.15108x^2$$

❖ Exact Solution

$$u_{exact} = \frac{2\cos(1-x) - \sin x}{\cos 1} + x^2 - 2$$



1. Rayleigh-Ritz Method – Bending of a Cantilever Beam

$$-\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right)+q=0 \quad \text{subjected to} \quad w(0)=\left.\frac{dw}{dx}\right|_{x=0}=0,\ \left.EI\frac{d^2w}{dx^2}\right|_{x=L}=M_0,\ \left.\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right)\right|_{x=L}=0$$

- Linear and Bilinear Form $B(w,\delta w)=\int_0^L EI\frac{d^2w}{dx^2}\frac{d^2\delta w}{dx^2}dx,\quad l(\delta w)=\int_0^L q\delta w dx+\left.\frac{d\delta w}{dx}\right|_{x=L}M_0$

  or $B_{ij}=B(\phi_i,\phi_j)=\int_0^L EI\frac{d^2\phi_i}{dx^2}\frac{d^2\phi_j}{dx^2}dx \quad F_i=l(\phi_i)=\int_0^L q\phi_i dx+\left.\frac{d\phi_i}{dx}\right|_{x=L}M_0$

- Choose Approximate Function (Need to satisfy EBC only) $\phi_i=x^{i+1}$

- Linear and Bilinear Form becomes

$$B_{ij}=\int_0^L EI(i+1)ix^{i-1}(j+1)jx^{j-1}dx=\frac{EIij(i+1)(j+1)L^{i+j-1}}{i+j-1},\qquad F_i=\frac{qL^{i+2}}{i+2}+M_0(i+1)L^i$$

- One-parameter Solution $4EILc_1=\frac{qL^3}{3}+2M_0L \rightarrow w_1=\frac{qL^2+6M_0}{12EI}x^2$

- Two-parameter Solution $EI\begin{bmatrix}4L & 6L^2\\ 6L^2 & 12L^3\end{bmatrix}\begin{Bmatrix}c_1\\ c_2\end{Bmatrix}=\frac{qL^3}{12}\begin{Bmatrix}4\\ 3L\end{Bmatrix}+M_0L\begin{Bmatrix}2\\ 3L\end{Bmatrix}$

$$w_3=\frac{qx^2}{24EI}\left(5L^2-2Lx\right)+\frac{M_0x^2}{2EI}$$

1. Rayleigh-Ritz Method – Bending of a Cantilever Beam

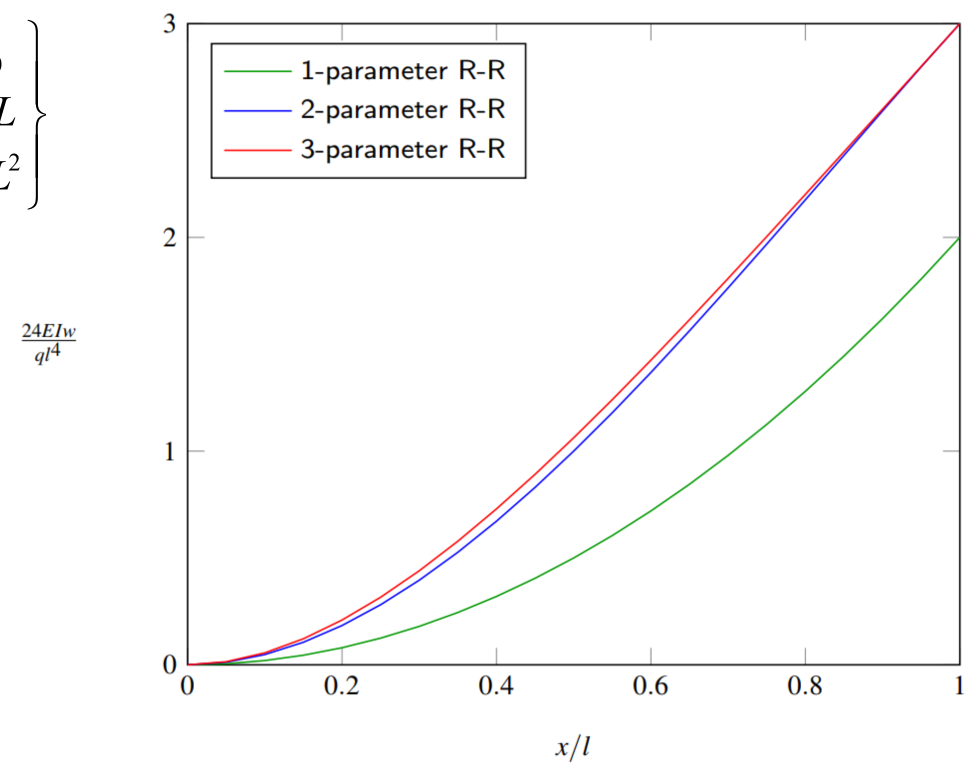
$$-\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right)+q=0 \quad \text{subjected to} \quad w(0)=\left.\frac{dw}{dx}\right|_{x=0}=0,\ \left.EI\frac{d^2w}{dx^2}\right|_{x=L}=M_0,\ \left.\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right)\right|_{x=L}=0$$

❖ Three-parameter Solution

$$EI\begin{bmatrix}4L & 6L & 8L^2\\ 6L & 12L^2 & 18L^3\\ 8L^2 & 18L^3 & 144/5L^4\end{bmatrix}\begin{Bmatrix}c_1\\ c_2\\ c_3\end{Bmatrix}=\begin{Bmatrix}1/3qL^2+2M_0\\ 1/4qL^3+3M_0L\\ 1/5qL^4+4M_0L^2\end{Bmatrix}$$

$$w_3=\frac{qx^2}{24EI}\left(6L^2-4Lx+x^2\right)+\frac{M_0x^2}{2EI}$$

(closed-form solution)



1. Rayleigh-Ritz Method – Axial Vibration of a Bar

- **Total Potential Energy** $$\Pi = \int_0^L \frac{EA}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx + \frac{k}{2}u(L)^2$$

- **Kinetic Energy** $$T = \frac{1}{2}\int_V \rho \dot{u}_i \dot{u}_i dV = \frac{1}{2}\int_0^L \rho A \dot{u}^2 dx$$

- **Hamilton's Principle** $$0 = -\int_{t_1}^{t_2}\left\{\int_0^L \left[\rho A \ddot{u}\delta u + EA\frac{\partial u}{\partial x}\frac{\partial \delta u}{\partial x}\right]dx + ku(L)\delta u(L)\right\}dt$$
$$= \int_{t_1}^{t_2}\left\{\int_0^L \left[\rho A \omega^2 u\delta u - EA\frac{\partial u}{\partial x}\frac{\partial \delta u}{\partial x}\right]dx - ku(L)\delta u(L)\right\}dt$$

- **Equations of Motion** $$\rho A \ddot{u} - \frac{\partial}{\partial x}\left(EA\frac{\partial u}{\partial x}\right) = 0$$

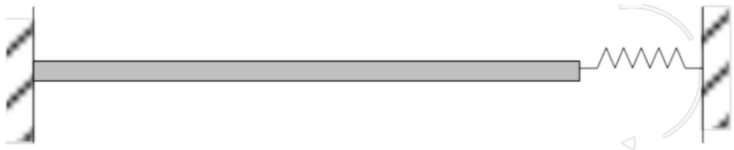
  For Harmonic Motion $$u(x,t) = u(x)e^{i\omega t} = \sum_{j=1}^{N} c_j \phi_j e^{i\omega t} \rightarrow \ddot{u} = -\omega^2 u \rightarrow \rho A \omega^2 u + \frac{\partial}{\partial x}\left(EA\frac{\partial u}{\partial x}\right) = 0$$

- **Boundary Conditions** $$u(0) = 0, \quad \left.\left(EA\frac{\partial u}{\partial x} + ku\right)\right|_{x=L} = 0$$

- **Weak Form** $$0 = \int_{t_1}^{t_2}\left\{\int_0^L \left[\rho A \omega^2 u\delta u - EA\frac{\partial u}{\partial x}\frac{\partial \delta u}{\partial x}\right]dx - ku(L)\delta u(L)\right\}dt = \omega^2 M(u,\delta u) - B(u,\delta u)$$

- **Bilinear Functionals** $$B_{ij} = B(\phi_i,\phi_j) = \int_0^L EA\frac{d\phi_i}{dx}\frac{d\phi_j}{dx}dx + k\phi_i(L)\phi_j(L), \quad M_{ij} = M(\phi_i,\phi_j) = \int_0^L \rho A \phi_i \phi_j dx$$

1. Rayleigh-Ritz Method – Axial Vibration of a Bar



❖ Choose Approximate Function

$$\phi_i = (x/L)^i \quad \rightarrow \quad B_{ij} = \frac{EA}{L}\frac{ij}{i+j-1} + k$$

$$M_{ij} = \rho AL \frac{1}{i+j+1}$$

with $k = EA/L$

❖ Two-parameter Solution

$$\left(\begin{bmatrix} 2 & 2 \\ 2 & 7/3 \end{bmatrix} - \lambda \begin{bmatrix} 1/3 & 1/4 \\ 1/4 & 1/5 \end{bmatrix} \right) \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Characteristic Equations

$$15\lambda^2 - 640\lambda + 2400 = 0 \quad \rightarrow \quad \lambda_1 = 4.1545,\ \lambda_2 = 38.512 \quad \rightarrow \quad \omega_1 = \frac{2.038}{L}\sqrt{\frac{E}{\rho}},\quad \omega_2 = \frac{6.206}{L}\sqrt{\frac{E}{\rho}}$$

❖ Another Approximate Function to satisfy both EBC and NBC

$$\hat{\phi}_1 = 3Lx - 2x^2 \quad \rightarrow \quad \lambda_1 = 4.1667 \quad \rightarrow \quad \omega_1 = \frac{2.041}{L}\sqrt{\frac{E}{\rho}}$$

❖ Closed-Form Solution

$$\lambda + \tan\lambda = 0 \quad \rightarrow \quad \omega_1 = \frac{2.02875}{L}\sqrt{\frac{E}{\rho}},\quad \omega_2 = \frac{4.91318}{L}\sqrt{\frac{E}{\rho}}$$

1. Rayleigh-Ritz Method – Free Vibration of a Cantilever Beam

- External Virtual Work $\delta W_E = 0$

- Internal Virtual Work $$\delta W_I = \int_V \sigma_{ij}\delta\varepsilon_{ij} dV = \int_0^L \int_A \sigma_x \delta\varepsilon_x dA dx = \int_0^L \int_A \sigma_x \left(\frac{d\delta u_0}{dx} - z\frac{d^2\delta w_0}{dx^2} \right) dA dx$$
$$= \int_0^L \left(N\frac{d\delta u_0}{dx} - M\frac{d^2\delta w_0}{dx^2} \right) dx = \int_0^L \left(EA\frac{\partial u}{\partial x}\frac{\partial \delta u}{\partial x} + EI\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 \delta w}{\partial x^2} \right) dx$$

- Kinetic Energy $$T = \frac{1}{2}\int_V \rho \dot{u}_i \dot{u}_i dV = \frac{1}{2}\int_0^L \rho A\left(\dot{u}^2 + \dot{w}^2\right) dx \rightarrow \delta T = -\int_0^L \rho A\left(\ddot{u}\delta u + \ddot{w}\delta w\right) dx$$

- Hamilton' Principle

$$0 = \int_{t_1}^{t_2} \left[\delta T - \left(\delta U + \delta V \right) \right] dt = -\int_{t_1}^{t_2}\int_0^L \left[\rho A\left(\ddot{u}\delta u + \ddot{w}\delta w\right) + EA\frac{\partial u}{\partial x}\frac{\partial \delta u}{\partial x} + EI\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 \delta w}{\partial x^2} \right] dxdt$$
$$= -\int_{t_1}^{t_2}\int_0^L \left[\rho A\ddot{w}\delta w + EI\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 \delta w}{\partial x^2} \right] dxdt = -\int_{t_1}^{t_2}\int_0^L \left[-\omega^2 \rho A w\delta w + EI\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 \delta w}{\partial x^2} \right] dxdt$$
$$= -\lambda M\left(w, \delta w\right) + B\left(w, \delta w\right)$$

with

$$B_{ij} = B\left(\phi_i, \phi_j\right) = \int_0^L \phi_i'' \phi_j'' \, dx, \quad M_{ij} = M\left(\phi_i, \phi_j\right) = \int_0^L \phi_i \phi_j \, dx, \quad \lambda = \frac{\rho A}{EI}\omega^2$$

1. Rayleigh-Ritz Method – Free Vibration of a Cantilever Beam

❖ Use Polynomial Function $\phi_j = x^{j+1}$ to satisfy the EBC.

❖ One-parameter Solution

$$\phi = x^2 \;\rightarrow\; B_{11} = 4L,\;\; M_{11} = L^5/5 \;\rightarrow\; \omega = \sqrt{20}\sqrt{\frac{EI}{\rho A L^4}} = 2.114^2\sqrt{\frac{EI}{\rho A L^4}}$$

❖ Two-parameter Solution

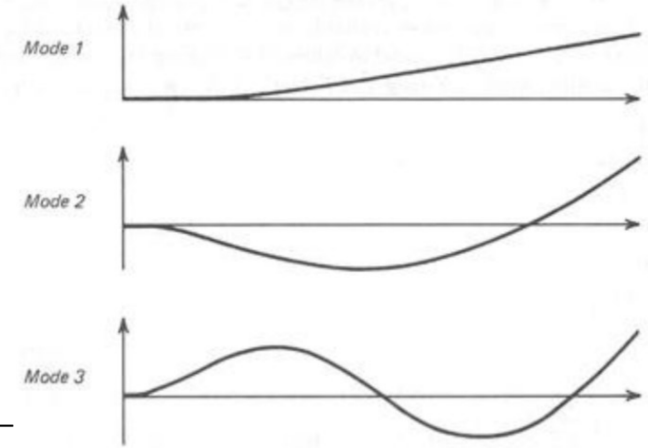
$$\phi_i = x^{i+1} \;\rightarrow\; \left(\begin{bmatrix} 4 & 6L \\ 6L & 12L^3 \end{bmatrix} - \lambda\begin{bmatrix} L^4/5 & L^5/6 \\ L^5/6 & L^6/7 \end{bmatrix}\right) \;\rightarrow\; \omega_1 = 1.880^2\sqrt{\frac{EI}{\rho A L^4}},\;\; \omega_2 = 5.899^2\sqrt{\frac{EI}{\rho A L^4}}$$

❖ Use Trigonometric Function $\phi_j = 1 - \cos\dfrac{j\pi x}{L}$ to satisfy the EBC.

Use Orthogonality Relations of the Trigonometric Functions

$$\int_0^\pi \sin mx \sin nx\, dx = \int_0^\pi \cos mx \cos nx\, dx = \frac{\pi}{2}\delta_{mn}$$

$$\int_0^L \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\, dx = \int_0^L \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\, dx = \frac{L}{2}\delta_{mn}$$



❖ Closed-Form Solution

$$\omega_1 = 1.875^2\sqrt{\frac{EI}{\rho A L^4}},\quad \omega_2 = 4.694^2\sqrt{\frac{EI}{\rho A L^4}},\quad \omega_3 = 7.855^2\sqrt{\frac{EI}{\rho A L^4}}$$

2. Weighted-Residual Methods – Description & Classification

	Rayleigh-Ritz	Weighted-Residual
Formulation	Weak Form	Weighted-Integral Form
Approximate Function	EBC	EBC+NBC

- ❖ Consider $A(u) = f \quad in \quad \Omega, \qquad B_1(u) = \hat{u} \quad in \quad S_1, \qquad B_2(u) = g \quad in \quad S_2$
- ❖ Seek $$u = U_N = \sum_{j=1}^{N} c_j \phi_j + \phi_0$$
- ❖ Residual $$R = A(U_N) - f = A\left(\sum_{j=1}^{N} c_j \phi_j + \phi_0\right) - f \neq 0$$
- ❖ Therefore $$\int_\Omega \psi_i R \, d\Omega = 0$$
- ❖ We have $$0 = \int_\Omega \psi_i \left(A\left(\sum_{j=1}^{N} c_j \phi_j + \phi_0\right) - f \right) d\Omega = \int_\Omega \psi_i \left(\sum_{j=1}^{N} A(\phi_j) c_j + A(\phi_0) - f \right) d\Omega$$
$$= \sum_{j=1}^{N} \int_\Omega \psi_i A(\phi_j) \, d\Omega \, c_j - \int_\Omega \psi_i \left(f - A(\phi_0) \right) d\Omega = A_{ij} c_j - q_i$$
with $A_{ij} = \int_\Omega \psi_i A(\phi_j) \, d\Omega, \quad q_i = \int_\Omega \psi_i \left(f - A(\phi_0) \right) d\Omega$

- ❖ Classification according to the choice of the weight function
  - (Petrov-)Galerkin Method
  - Least Square Method
  - Collocation Method
  - Subdomain Method
  - Kantorovich Method

2. Weighted-Residual Method – Methods

Galerkin Method

❖ The weight functions are approximate function

$$0 = \int_\Omega \phi_i \left[A\left(\sum_{j=1}^{N} c_j \phi_j + \phi_0 \right) - f \right] d\Omega = \int_\Omega \phi_i \left(\sum_{j=1}^{N} A(\phi_j) c_j + A(\phi_0) - f \right) d\Omega$$

$$= \sum_{j=1}^{N} \int_\Omega \phi_i A(\phi_j) d\Omega c_j - \int_\Omega \phi_i (f - A(\phi_0)) d\Omega = A_{ij} c_j - q_i \quad \textit{with} \quad A_{ij} = \int_\Omega \phi_i A(\phi_j) d\Omega, \; q_i = \int_\Omega \phi_i (f - A(\phi_0)) d\Omega$$

❖ For Dirichlet boundary value problem, Galerkin method becomes Rayleigh-Ritz method

Least-Square Method

❖ Determine the coefficients c_j by minimizing the integral of the square of the residual.

$$\min \left[I(c_i) = \int_\Omega R^2 d\Omega \right] \quad \rightarrow \quad 0 = \int_\Omega 2R \frac{\partial R}{\partial c_i} d\Omega$$

❖ This is a special case of the weighted residual method for the weight function

$$\psi_i = \frac{\partial R}{\partial c_i} = \frac{\partial}{\partial c_i} \left[A\left(\sum_{j=1}^{N} c_j \phi_j + \phi_0 \right) - f \right] = A(\phi_i)$$

$$0 = \int_\Omega A(\phi_i) \left[A\left(\sum_{j=1}^{N} c_j \phi_j + \phi_0 \right) - f \right] d\Omega = \sum_{j=1}^{N} \int_\Omega A(\phi_i) A(\phi_j) d\Omega c_j - \int_\Omega A(\phi_i)(f - A(\phi_0)) d\Omega = A_{ij} c_j - q_i$$

$$\textit{with} \quad A_{ij} = \int_\Omega A(\phi_i) A(\phi_j) d\Omega, \quad q_i = \int_\Omega A(\phi_i)(f - A(\phi_0)) d\Omega$$

2. Weighted-Residual Methods – Methods

Collocation Method

- The weight functions are taken from the family of Dirac Delta functions in the domain.

$$\psi_i(x) = \delta(x - x_i) = \begin{cases} 1 & x = x_i \\ 0 & \text{otherwise} \end{cases}$$

- The parameter c_j are determined by forcing the residual in the approximation of the governing equations to vanish *n* selected points ξ (xi)

$$0 = \int_{\Omega} \psi_i R d\Omega = \int_{\Omega} \delta(x - x_i) R(x, c_j) d\Omega = R(x_i, c_j)$$

- The selection of the collocation points x_j is crucial in obtaining well-conditioned system of equations and a convergent solution.
- The collocation points should be located as evenly as possible to avoid ill-conditioning of the resulting equations

Subdomain Method

- Modification of the collocation method. The idea is to force the weighted residual to zero not just at fixed points in the domain, but over various subsections of the domain.
- To accomplish this, the weight functions are set to unity, and the integral over the entire domain is broken into a number of subdomains sufficient to evaluate all unknown parameters.

2. Weighted-Residual Methods – 2nd order Differential Equation

$$-\frac{d^2u}{dx^2} - u + x^2 = 0, \quad 0 < x < 1 \quad \text{subjected to} \quad u(0) = 0, \quad \left.\frac{du}{dx}\right|_{x=1} = 1$$

(slide 8)

❖ Choose Approximate Functions

- ϕ_j are required to satisfy homogeneous form of all EBC and NBC: $\phi_j(0) = 0, \quad \left.\frac{d\phi_j}{dx}\right|_{x=1} = 0$
- ϕ_0 are required to satisfy all specified EBC and NBC: $\phi_0(0) = 0, \quad \left.\frac{d\phi_0}{dx}\right|_{x=1} = 1$

❖ Linear Approximation $\phi_0 = bx + a$: $\quad \phi_0(0) = a = 0, \quad \left.\frac{d\phi_0}{dx}\right|_{x=1} = b = 1 \quad \rightarrow \quad \phi_0 = x$

❖ Quadratic Approximation $\phi_1 = cx^2 + bx + a$

$$\phi_1(0) = a = 0, \quad \left.\frac{d\phi_1}{dx}\right|_{x=1} = b + 2c = 0 \quad \rightarrow \quad \phi_1 = x(2 - x)$$

❖ Cubic Approximation $\phi_2 = dx^3 + cx^2 + bx + a$

$$\phi_2(0) = a = 0, \quad \left.\frac{d\phi_2}{dx}\right|_{x=1} = b + 2c + 3d = 0 \quad \rightarrow \quad \begin{cases} b = 0, \ d = -2c/3 \rightarrow \phi_2 = x^2 - 2x^3/3 \\ c = 0, \ d = -b/3 \rightarrow \phi_2 = x - x^3/3 \end{cases}$$

❖ Therefore

$$U_N = \sum_{j=1}^{N} c_j \phi_j + \phi_0 = c_1\left(2x - x^2\right) + c_2\left(x^2 - 2x^3/3\right) + x$$

2. Weighted-Residual Methods – 2nd order Differential Equation

$$-\frac{d^2u}{dx^2} - u + x^2 = 0, \quad 0 < x < 1 \quad \text{subjected to} \quad u(0) = 0, \quad \left.\frac{du}{dx}\right|_{x=1} = 1$$

(slide 8)

❖ Residual Calculation

$$R = -\frac{d^2U_N}{dx^2} - U_N + x^2 = -\frac{d^2}{dx^2}\left(\sum_{j=1}^{N} c_j\phi_j + \phi_0\right) - \left(\sum_{j=1}^{N} c_j\phi_j + \phi_0\right) + x^2$$

$$= -\sum_{j=1}^{N} \frac{d^2\phi_j}{dx^2} c_j - \sum_{j=1}^{N} c_j\phi_j - \phi_0 + x^2 = c_1(2 - 2x + x^2) + c_2(-2 + 4x - x^2 + 2x^3/3) - x + x^2$$

Petrov-Galerkin Method

❖ Weight Function $\psi_1 = x, \quad \psi_2 = x^2$

❖ Weighted Integral

$$0 = \int_{\Omega} \psi_i R\, d\Omega \rightarrow \begin{cases} 0 = \int_0^1 x\left[c_1(2 - 2x + x^2) + c_2(-2 + 4x - x^2 + 2x^3/3) - x + x^2\right]dx \\ 0 = \int_0^1 x^2\left[c_1(2 - 2x + x^2) + c_2(-2 + 4x - x^2 + 2x^3/3) - x + x^2\right]dx \end{cases} \rightarrow \begin{cases} c_1 = 103/682 \\ c_2 = -15/682 \end{cases}$$

❖ Solution $U_{PG} = 1.302x - 0.173x^2 - 0.0146x^3$

2. Weighted-Residual Methods – 2nd order Differential Equation

Galerkin Method

❖ Weight Function $\psi_1 = \phi_1 = 2x - x^2, \quad \psi_2 = \phi_2 = x^2 - 2x^3/3$ (can use $\psi_1 = -\phi_1 = -2x + x^2$)

❖ Weighted Integral

$$\begin{cases} 0 = \int_0^1 \left(2x - x^2\right)\left[c_1(2 - 2x + x^2) + c_2(-2 + 4x - x^2 + 2x^3/3) - x + x^2\right]dx \\ 0 = \int_0^1 \left(x^2 - 2x^3/3\right)\left[c_1(2 - 2x + x^2) + c_2(-2 + 4x - x^2 + 2x^3/3) - x + x^2\right]dx \end{cases}$$

$$\rightarrow \begin{cases} \dfrac{4}{5}c_1 + \dfrac{17}{90}c_2 = \dfrac{7}{60} \\ \dfrac{17}{90}c_1 + \dfrac{29}{315}c_2 = \dfrac{1}{36} \end{cases} \quad \rightarrow \begin{cases} c_1 = 623/4306 \approx 0.14468 \\ c_2 = 21/4306 \approx 0.00488 \end{cases}$$

❖ Solution

$$\begin{aligned} U_G &= c_1\phi_1 + c_2\phi_2 + \phi_0 \\ &= \frac{623}{4306}\left(2x - x^2\right) + \frac{21}{4306}\left(x^2 - 2x^3/3\right) + x \\ &= \frac{2776}{2153}x - \frac{301}{2153}x^2 - \frac{7}{2153}x^3 \\ &\approx 1.28936x - 0.13981x^2 - 0.00325x^3 \end{aligned}$$

2. Weighted-Residual Methods – 2nd order Differential Equation

Least-Square Method

❖ Weight Function $\psi_1 = \dfrac{\partial R}{\partial c_1} = 2 - 2x + x^2, \qquad \psi_2 = \dfrac{\partial R}{\partial c_2} = -2 + 4x - x^2 + 2x^3/3$

❖ Weighted Integral

$$\begin{cases} 0 = \int_0^1 \left(2-2x+x^2\right)\left[c_1(2-2x+x^2)+c_2(-2+4x-x^2+2x^3/3)-x+x^2\right]dx \\ 0 = \int_0^1 \left(-2+4x-x^2+2x^3/3\right)\left[c_1(2-2x+x^2)+c_2(-2+4x-x^2+2x^3/3)-x+x^2\right]dx \end{cases}$$

$$\rightarrow \begin{cases} \dfrac{28}{15}c_1 - \dfrac{47}{90}c_2 = \dfrac{13}{60} \\ \dfrac{47}{90}c_1 - \dfrac{349}{315}c_2 = \dfrac{1}{36} \end{cases} \qquad \rightarrow \begin{cases} c_1 = 227/1807 \approx 0.12562 \\ c_2 = 134/3925 \approx 0.03414 \end{cases}$$

❖ Solution

$$\begin{aligned} U_G &= c_1\phi_1 + c_2\phi_2 + \phi_0 \\ &= \frac{227}{1807}\left(2x-x^2\right) + \frac{134}{3925}\left(x^2 - 2x^3/3\right) + x \\ &= \frac{2261}{1807}x - \frac{1175}{12844}x^2 - \frac{268}{11775}x^3 \\ &\approx 1.2512x - 0.09148x^2 - 0.02276x^3 \end{aligned}$$

2. Weighted-Residual Methods – 2nd order Differential Equation

Collocation Method

- For the collocation method, the residual is forced to zero at a number of discrete points. Since there are two unknown (c_1, c_2) , two collocation points are needed. We choose (arbitrarily, but from symmetry considerations) the collocation point $x = 1/3,\ 2/3$. Thus, the equations needed to evaluate the unknown (c_1, c_2)
- Weighted Integral

$$\begin{cases} 0 = \left[c_1(2-2x+x^2) + c_2(-2+4x-x^2+2x^3/3) - x + x^2 \right] \Big|_{x=1/3} \\ 0 = \left[c_1(2-2x+x^2) + c_2(-2+4x-x^2+2x^3/3) - x + x^2 \right] \Big|_{x=2/3} \end{cases} \rightarrow \begin{cases} c_1 = 1710/9468 \\ c_2 = 486/9468 \end{cases}$$

- Solution $U_C = 1.3612x - 0.12927x^2 - 0.03422x^3$

Sub-Domain Method

- Since we have two unknown constants, we choose two subdomain which covers the entire range of x. Therefore, the relation to evaluate the constant c1;c2
- Weighted Integral

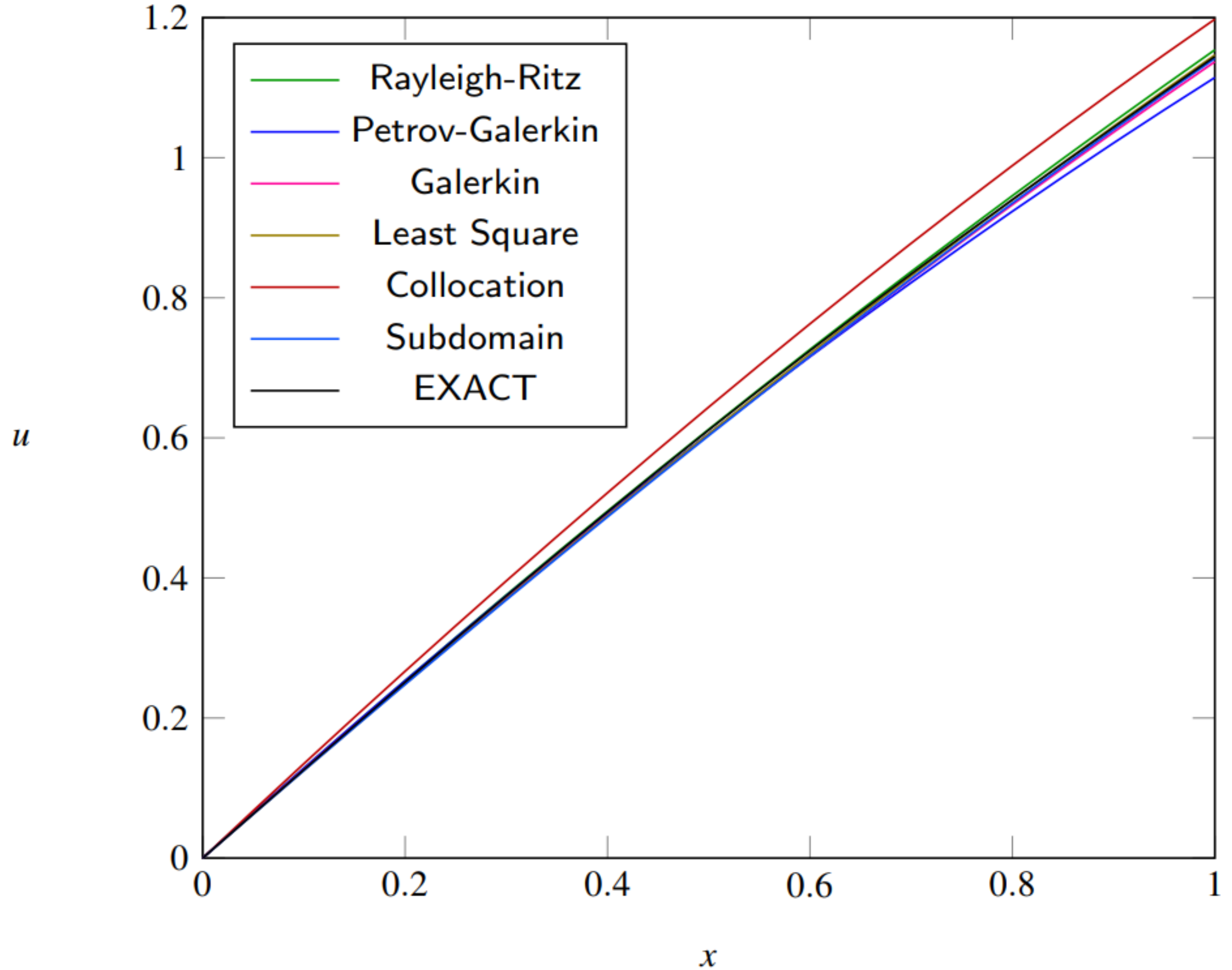
$$\begin{cases} 0 = \int_0^{1/2} \left[c_1(2-2x+x^2) + c_2(-2+4x-x^2+2x^3/3) - x + x^2 \right] dx \\ 0 = \int_{1/2}^{1} \left[c_1(2-2x+x^2) + c_2(-2+4x-x^2+2x^3/3) - x + x^2 \right] dx \end{cases} \rightarrow \begin{cases} c_1 = 0.129518 \\ c_2 = 0.036145 \end{cases}$$

- Solution $U_S = 1.259036x - 0.09337x^2 - 0.0241x^3$

2. Weighted-Residual Methods – 2nd order Diff. Eq. (Comparison)

	exact	r-r	pg	g	ls	c	s
0.00	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.05	0.063174	0.064372	0.064668	0.064120	0.062800	0.067673	0.062715
0.10	0.126198	0.127989	0.128460	0.127539	0.125175	0.134673	0.124946
0.15	0.188932	0.190851	0.191365	0.190254	0.187099	0.200976	0.186673
0.20	0.251251	0.252957	0.253372	0.252262	0.248547	0.266555	0.247880
0.25	0.313043	0.314308	0.314470	0.313562	0.309495	0.331386	0.308547
0.30	0.374210	0.374903	0.374648	0.374150	0.369917	0.395442	0.368657
0.35	0.434668	0.434743	0.433895	0.434025	0.429789	0.458697	0.428191
0.40	0.494347	0.493827	0.492199	0.493184	0.489085	0.521127	0.487133
0.45	0.553191	0.552156	0.549551	0.551624	0.547781	0.582705	0.545463
0.50	0.611159	0.609730	0.605938	0.609344	0.605851	0.643405	0.603163
0.55	0.668226	0.666548	0.661351	0.666340	0.663272	0.703202	0.660216
0.60	0.724379	0.722611	0.715777	0.722610	0.720017	0.762071	0.716603
0.65	0.779623	0.777919	0.769206	0.778152	0.776062	0.819986	0.772306
0.70	0.833975	0.832471	0.821627	0.832963	0.831382	0.876920	0.827308
0.75	0.887469	0.886268	0.873029	0.887041	0.885952	0.932849	0.881589
0.80	0.940151	0.939309	0.923402	0.940384	0.939747	0.987747	0.935133
0.85	0.992085	0.991595	0.972732	0.992989	0.992743	1.041587	0.987920
0.90	1.043345	1.043125	1.021011	1.044853	1.044913	1.094345	1.039934
0.95	1.094023	1.093900	1.068227	1.095974	1.096234	1.145994	1.091155
1.00	1.144224	1.143920	1.114369	1.146350	1.146680	1.196510	1.141566
RMS error		0.001202	0.012662	0.001287	0.003317	0.034773	0.005405

2. Weighted-Residual Methods – 2nd order Diff. Eq. (Comparison)



3. Eigenvalue Problem – Closed-Form, Rayleigh-Ritz

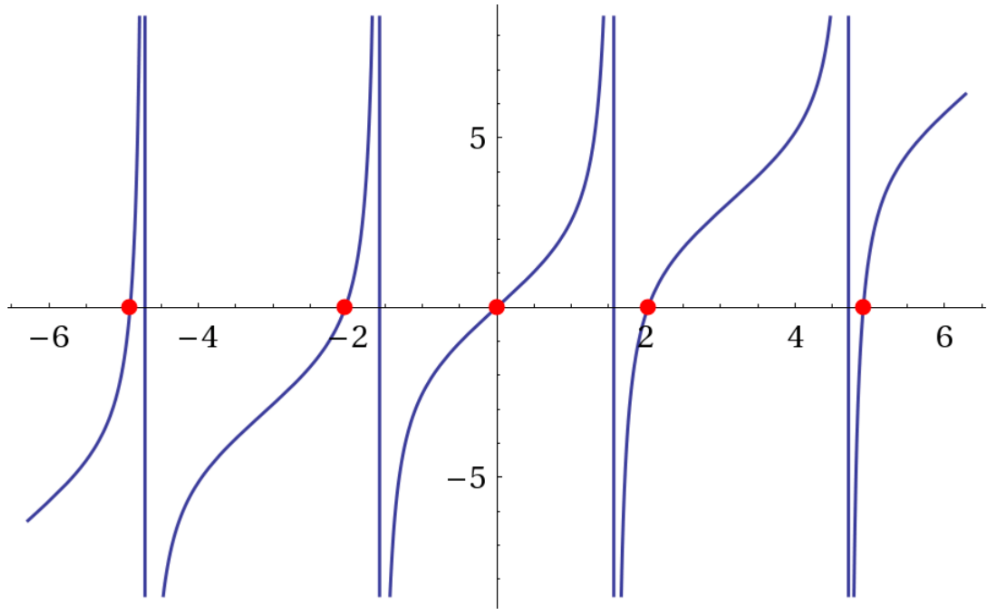
$$-\frac{d^2u}{dx^2} - \lambda u = 0, \quad 0 < x < 1 \quad \text{subjected to} \quad u(0) = 0, \quad \left.\frac{du}{dx} + u\right|_{x=1} = 0$$

❖ End up with transcendental equation

$$\tan\sqrt{\lambda} + \sqrt{\lambda} = 0$$

❖ Closed-Form Solution

$$\lambda_1 = 2.0287578^2 = 4.116$$
$$\lambda_2 = 4.9131804^2 = 24.1393$$
$$\lambda_3 = 7.9786657^2 = 63.6591$$



Rayleigh-Ritz Method

❖ Weak Form $0 = \int_0^1 \phi_i' \phi_j' dx + \phi_i \phi_j \Big|_{x=1} - \int_0^1 \lambda \phi_i \phi_j dx$

❖ Approximate Function $\phi_j = x^j$

❖ For 1-parameter R-R Solution $\lambda = 6$

3. Eigenvalue Problem – Weighted Residual

$$-\frac{d^2u}{dx^2} - \lambda u = 0, \quad 0 < x < 1 \quad \text{subjected to} \quad u(0) = 0, \quad \left.\frac{du}{dx} + u\right|_{x=1} = 0$$

❖ To determine approximate Function, try $\phi_1 = c_0 + c_1 x + c_2 x^2$ to satisfy both boundary conditions

$$\phi(0) = c_0 = 0, \qquad \phi' + \phi\big|_{x=1} = 2c_1 + 3c_2 = 0 \quad \rightarrow \quad \phi_1 = 3x - 2x^2$$

❖ Galerkin Method

$$0 = c_1 \int_0^1 \phi_1 \left(\frac{d^2\phi_1}{dx^2} + \lambda \phi_1 \right) dx \rightarrow 0 = (-10/3 + 4\lambda/5) c_1 \rightarrow \lambda = 4.167$$

❖ Collocation Method at $x = 0.5$

$$0 = c_1 \phi_1(0.5) \left(\left.\frac{d^2\phi_1}{dx^2}\right|_{x=0.5} + \lambda \phi_1(0.5) \right) \rightarrow 0 = (-4 + \lambda) c_1 \rightarrow \lambda = 4.0$$

❖ Least-Square Method

$$0 = c_1 \int_0^1 \frac{d^2\phi_1}{dx^2} \left(\frac{d^2\phi_1}{dx^2} + \lambda \phi_1 \right) dx \rightarrow 0 = (-4 + 5\lambda/6) c_1 \rightarrow \lambda = 4.8$$

❖ Subdomain Solution

$$0 = c_1 \int_0^1 \left(\frac{d^2\phi_1}{dx^2} + \lambda \phi_1 \right) dx \rightarrow 0 = (4 - 6\lambda/5) c_1 \rightarrow \lambda = 4.8$$

4. Bending of a Simple Beam – Weighted Residual Methods

❖ Simply-supported beam under uniformly distributed load

$$-\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right)+q=0 \quad \text{subjected to} \quad w(0)=w(L)=0, \quad \left.\frac{d^2w}{dx^2}\right|_{x=0}=\left.\frac{d^2w}{dx^2}\right|_{x=L}=0$$

❖ Approximate Function for two-parameter solution $w_2=c_1\sin\frac{\pi x}{L}+c_2\sin\frac{3\pi x}{L}$

❖ Residual Calculation

$$R=-\frac{d^2}{dx^2}\left(EI\frac{d^2w_N}{dx^2}\right)+q=-\frac{d^2}{dx^2}\left(EI\frac{d^2w_N}{dx^2}\left(\sum_{j=1}^{N}c_j\phi_j+\phi_0\right)\right)+q$$

$$=-EI\sum_{j=1}^{N}\frac{d^4\phi_j}{dx^4}c_j-EI\sum_{j=1}^{N}\frac{d^2\phi_0}{dx^2}+q=-EI\left[c_1\left(\frac{\pi}{L}\right)^4\sin\frac{\pi x}{L}+c_2\left(\frac{3\pi}{L}\right)^4\sin\frac{3\pi x}{L}\right]+q$$

Galerkin Method

❖ Weight Function $\psi_1=\phi_1=\sin\frac{\pi x}{L}, \quad \psi_2=\phi_2=\sin\frac{3\pi x}{L}$

❖ Weighted Integral

$$0=\int_0^L\sin\frac{\pi x}{L}\left\{-EI\left[c_1\left(\frac{\pi}{L}\right)^4\sin\frac{\pi x}{L}+c_2\left(\frac{3\pi}{L}\right)^4\sin\frac{3\pi x}{L}\right]+q\right\}dx$$

$$0=\int_0^L\sin\frac{3\pi x}{L}\left\{-EI\left[c_1\left(\frac{\pi}{L}\right)^4\sin\frac{\pi x}{L}+c_2\left(\frac{3\pi}{L}\right)^4\sin\frac{3\pi x}{L}\right]+q\right\}dx$$

4. Bending of a Simple Beam – Weighted Residual Methods

Least-Square Method

❖ Weight Function $\psi_1 = \dfrac{\partial R}{\partial c_1} = -EI\left(\dfrac{\pi}{L}\right)^4 \sin\dfrac{\pi x}{L}, \quad \psi_2 = \dfrac{\partial R}{\partial c_2} = -EI\left(\dfrac{3\pi}{L}\right)^4 \sin\dfrac{3\pi x}{L}$

❖ Weighted Integral

$$0 = \int_0^L -EI\left(\frac{\pi}{L}\right)^4 \sin\frac{\pi x}{L}\left\{-EI\left[c_1\left(\frac{\pi}{L}\right)^4 \sin\frac{\pi x}{L} + c_2\left(\frac{3\pi}{L}\right)^4 \sin\frac{3\pi x}{L}\right] + q\right\}dx$$

$$0 = \int_0^L -EI\left(\frac{3\pi}{L}\right)^4 \sin\frac{3\pi x}{L}\left\{-EI\left[c_1\left(\frac{\pi}{L}\right)^4 \sin\frac{\pi x}{L} + c_2\left(\frac{3\pi}{L}\right)^4 \sin\frac{3\pi x}{L}\right] + q\right\}dx$$

Collocation Method

❖ Collocation points at $x = L/4,\ L/2:$

$$0 = -EI\left[c_1\left(\frac{\pi}{L}\right)^4 \sin\frac{\pi}{4} + c_2\left(\frac{3\pi}{L}\right)^4 \sin\frac{3\pi}{4}\right] + q$$

$$0 = -EI\left[c_1\left(\frac{\pi}{L}\right)^4 \sin\frac{\pi}{2} + c_2\left(\frac{3\pi}{L}\right)^4 \sin\frac{3\pi}{2}\right] + q$$

❖ Solution

$$c_1 = \frac{\left(1+\sqrt{2}\right)qL^4}{2EI\pi^4}, \quad c_2 = \frac{\left(\sqrt{2}-1\right)qL^4}{162EI\pi^4}$$

$$w_2 = \frac{qL^4}{162EI\pi^4}\left(195.55\sin\frac{\pi x}{L} + 0.414\sin\frac{3\pi x}{L}\right)$$

4. Bending of a Simple Beam – Weighted Residual Methods

Sub-domain Method

❖ Consider two subdomain in the first half of the beam

$$0 = \int_0^{L/4} \left\{ -EI \left[c_1 \left(\frac{\pi}{L} \right)^4 \sin \frac{\pi x}{L} + c_2 \left(\frac{3\pi}{L} \right)^4 \sin \frac{3\pi x}{L} \right] + q \right\} dx$$

$$0 = \int_{L/4}^{L/2} \left\{ -EI \left[c_1 \left(\frac{\pi}{L} \right)^4 \sin \frac{\pi x}{L} + c_2 \left(\frac{3\pi}{L} \right)^4 \sin \frac{3\pi x}{L} \right] + q \right\} dx$$

❖ Solution

$$c_1 = \frac{\left(1+\sqrt{2}\right) qL^4}{4\sqrt{2} EI \pi^3}, \quad c_2 = \frac{\left(\sqrt{2}-1\right) qL^4}{108\sqrt{2} EI \pi^3}$$

$$w_2 = \frac{qL^4}{108\sqrt{2} EI \pi^3} \left(65.184 \sin \frac{\pi x}{L} + 0.414 \sin \frac{3\pi x}{L} \right)$$

thank you