Lesson 8

Trusses in 2D & 3D Space, Python codes

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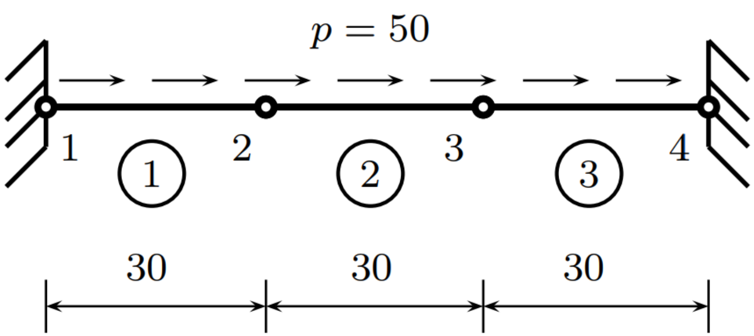
1. Python Codes – Isoparametric Bar under Uniform Load

❖ Python code: https://github.com/mctrinh/fem-class



- An isotropic bar is clamped at two ends.
- Modulus of elasticity $E = 30 \cdot 10^6$
- Cross-section area $A = 1$
- Uniform axial load $p = 50$
- Exact solution $u = \dfrac{pLx}{2EA}\left(1 - \dfrac{x}{L}\right), \quad \sigma_x = \dfrac{pL}{A}\left(\dfrac{1}{2} - \dfrac{x}{L}\right)$

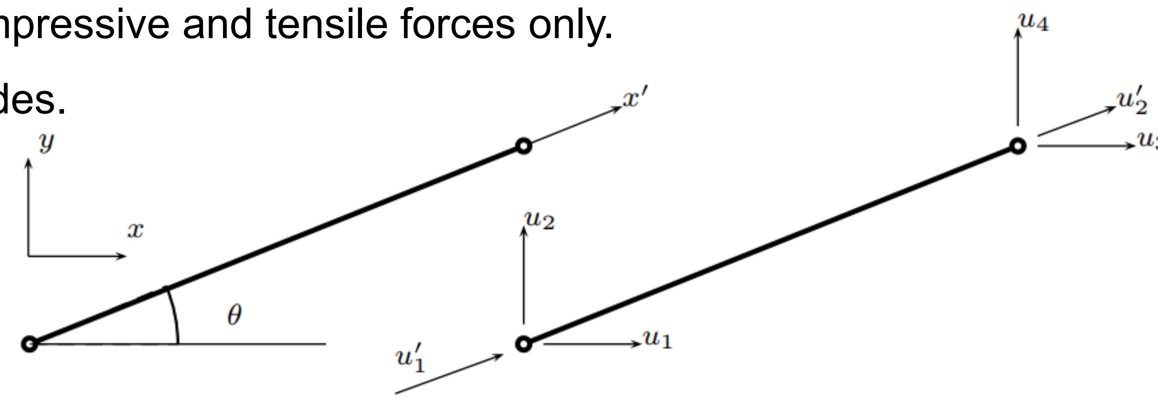
2. Trusses in 2D Space – Formulation - DOFs

- A transformation of coordinate basis is necessary to translate the local element matrices into the structural coordinate system.
- Trusses and Bars supports compressive and tensile forces only.
- All forces are applied at the nodes.
- 2 DOFs in local coordinate

$$\mathbf{u}'^T = \begin{bmatrix} u'_1 & u'_2 \end{bmatrix}$$

- 4 DOFs in global coordinate

$$\mathbf{u}^T = \begin{bmatrix} u_1 & u_2 & u_3 & u_4 \end{bmatrix}$$



2D truss element: local and global DOFs

- Relation between local and global DOFs

$$\begin{cases} u'_1 = u_1 \cos\theta + u_2 \sin\theta \\ u'_2 = u_3 \cos\theta + u_4 \sin\theta \end{cases} \quad \leftrightarrow \quad \mathbf{u}' = \mathbf{L}\mathbf{u}$$

$$\mathbf{L} = \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}$$ *in which* $l = \cos\theta = \dfrac{x_2 - x_1}{L_e}$; $m = \sin\theta = \dfrac{y_2 - y_1}{L_e}$; $L_e = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

with θ *is the angle between the local axis* x' *and global axis* x.

2. Trusses in 2D Space – Formulation (Stiffness, Mass)

❖ The local stiffness matrix of 2D truss elements is like bar elements:

$$\mathbf{K}' = \frac{EA}{L_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

❖ The strain energy of this element is transformed from local to global coordinate system

$$U^e = \frac{1}{2}\mathbf{u}'^T\mathbf{K}'\mathbf{u}' = \frac{1}{2}\mathbf{u}^T\left[\mathbf{L}^T\mathbf{K}'\mathbf{L}\right]\mathbf{u} = \frac{1}{2}\mathbf{u}^T\mathbf{K}\mathbf{u}$$

with the global stiffness matrix $\mathbf{K} = \frac{EA}{L_e}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$

❖ The local mass matrix of 2D truss elements

The consistent mass matrix $\mathbf{M}'_C = \frac{\rho A L_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

The lumped mass matrix $\mathbf{M}'_L = \frac{\rho A L_e}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

2. Trusses in 2D Space – Formulation (Stiffness, Mass, Stress)

❖ The kinetic energy of this element is transformed from local to global coordinate system

$$K^e = \frac{1}{2}\dot{\mathbf{u}}'^T \mathbf{M}' \dot{\mathbf{u}}' = \frac{1}{2}\dot{\mathbf{u}}^T \left[\mathbf{L}^T \mathbf{M}' \mathbf{L}\right] \dot{\mathbf{u}} = \frac{1}{2}\dot{\mathbf{u}}^T \mathbf{M} \dot{\mathbf{u}}$$

$$\mathbf{M} = \frac{\rho A L_e}{6}\begin{bmatrix} 2l^2 & 2lm & l^2 & lm \\ 2lm & 2m^2 & lm & m^2 \\ l^2 & lm & 2l^2 & 2lm \\ lm & m^2 & 2lm & 2m^2 \end{bmatrix}$$

(the global consistent mass matrix)

$$\mathbf{M} = \frac{\rho A L_e}{2}\begin{bmatrix} l^2 & lm & 0 & 0 \\ lm & m^2 & 0 & 0 \\ 0 & 0 & l^2 & lm \\ 0 & 0 & lm & m^2 \end{bmatrix}$$

(the global lumped mass matrix)

❖ The translational masses never vanish and must be retained in the local mass matrix. Setting $lm = 0$ and other terms are set to 1.

$$\mathbf{M} = \frac{\rho A L_e}{6}\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

(the global consistent mass matrix)

$$\mathbf{M} = \frac{\rho A L_e}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(the global lumped mass matrix)

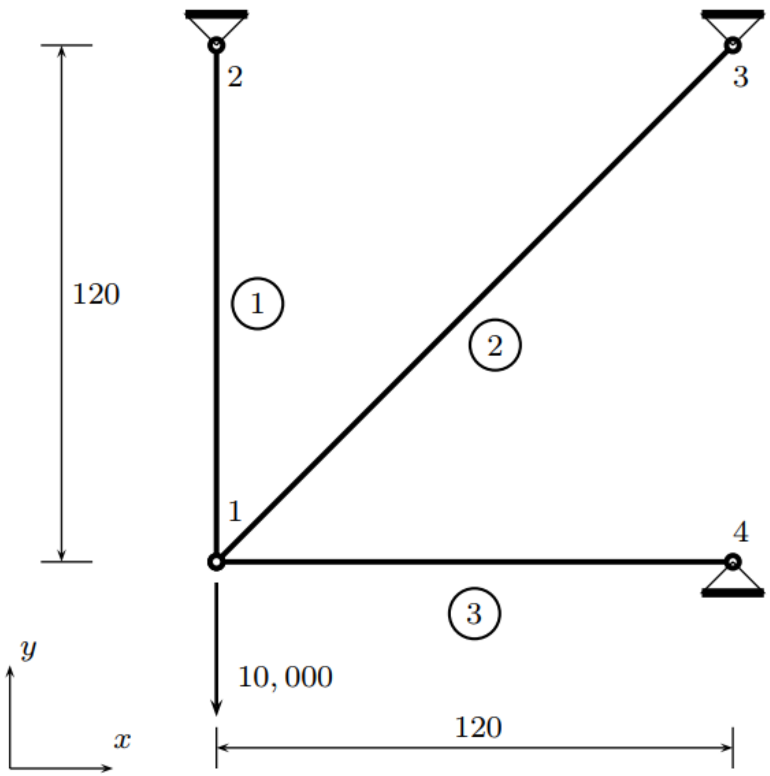
❖ The stress is transformed from local to global coordinate system

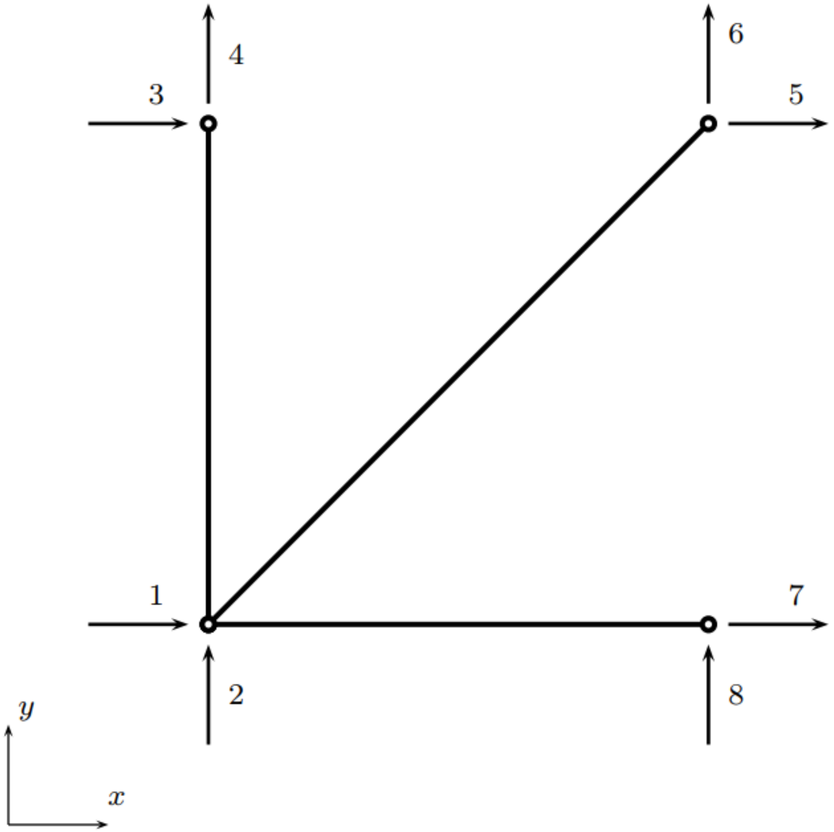
$$\sigma_x = E\varepsilon_x = E\frac{u_2' - u_1'}{L_e} = \frac{E}{L_e}\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{E}{L_e}\begin{bmatrix} -1 & 1 \end{bmatrix}\mathbf{u}' = \frac{E}{L_e}\begin{bmatrix} -1 & 1 \end{bmatrix}\mathbf{L}\mathbf{u} = \frac{E}{L_e}\begin{bmatrix} -l & -m & l & m \end{bmatrix}\mathbf{u}$$

2. Trusses in 2D Space – 2D Truss Problem - Python

❖ Python code: https://github.com/mctrinh/fem-class

- Modulus of elasticity $E = 30 \cdot 10^6$
- Cross-section area $A = 2$
- Applied load $P = 10000$





3. Trusses in 3D Space – Formulation

- Generalize the 2D truss model in 3D Cartesian space.
- The element has 2 DOFs in local coordinate

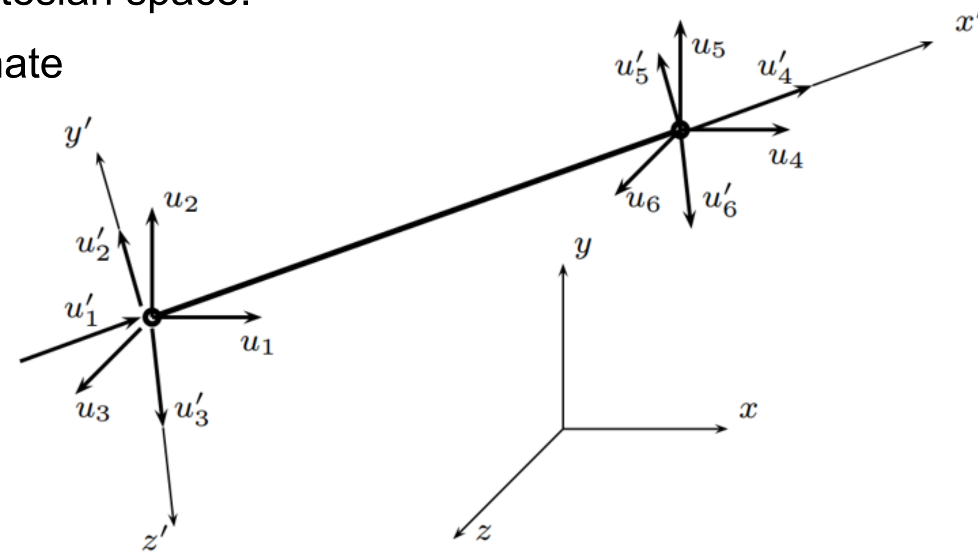
$$\mathbf{u}'^T = \begin{bmatrix} u_1' & u_2' & u_3' & u_4' & u_5' & u_6' \end{bmatrix}$$

- Each node has 3 global DOFs.
- Totally, 6 DOFs in global coordinate

$$\mathbf{u}^T = \begin{bmatrix} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{bmatrix}$$

- Relation between local and global DOFs

$$\mathbf{u}' = \mathbf{L}\mathbf{u}$$



3D truss element: local and global DOFs

The direction cosines matrix

$$\mathbf{L} = \begin{bmatrix} l_x & l_y & l_z & 0 & 0 & 0 \\ 0 & 0 & 0 & l_x & l_y & l_z \end{bmatrix} \quad \textit{with} \quad l_x = \frac{x_2 - x_1}{L_e}, \quad l_y = \frac{y_2 - y_1}{L_e}, \quad l_z = \frac{z_2 - z_1}{L_e}$$

- The local stiffness matrix of 3D truss elements is like before

$$\mathbf{K}^e = \frac{EA}{L_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

3. Trusses in 3D Space – Formulation

❖ The global stiffness matrix is

$$\mathbf{K} = \mathbf{L}^T \mathbf{K}^e \mathbf{L} = \frac{EA}{L_e} \begin{bmatrix} l_x^2 & l_x l_y & l_x l_z & -l_x^2 & -l_x l_y & -l_x l_z \\ & l_y^2 & l_y l_z & -l_x l_y & -l_y^2 & -l_y l_z \\ & & l_z^2 & -l_x l_z & -l_y l_z & -l_z^2 \\ & & & l_x^2 & l_x l_y & l_x l_z \\ & & & & l_y^2 & l_y l_z \\ sym & & & & & l_z^2 \end{bmatrix}$$

❖ The global mass matrix is

$$\mathbf{M} = \mathbf{L}^T \mathbf{M}^e \mathbf{L} = \frac{\rho A L_e}{6} \begin{bmatrix} 2 & 0 & 0 & 1 & 0 & 0 \\ & 2 & 0 & 0 & 1 & 0 \\ & & 2 & 0 & 0 & 1 \\ & & & 2 & 0 & 0 \\ & & & & 2 & 0 \\ sym & & & & & 2 \end{bmatrix}$$

The global consistent mass matrix

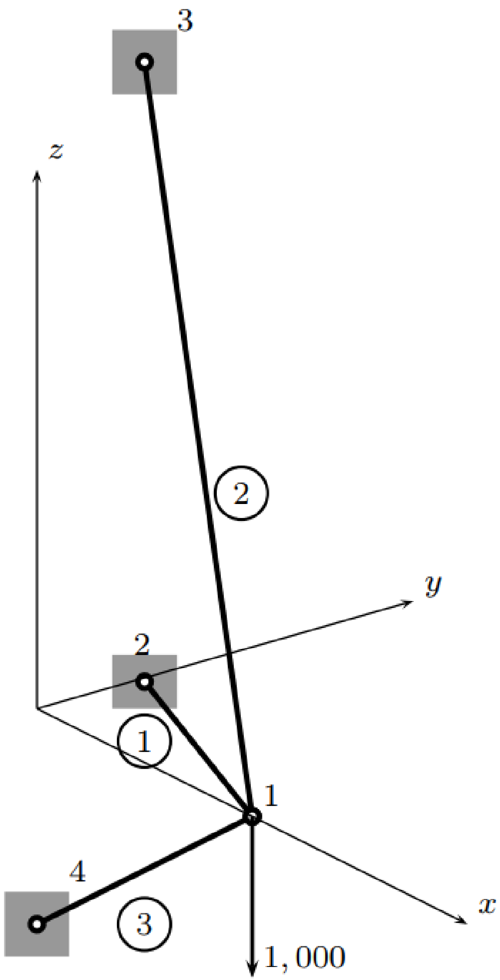
$$\mathbf{M} = \mathbf{L}^T \mathbf{M}^e \mathbf{L} = \frac{\rho A L_e}{2} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 1 & 0 \\ sym & & & & & 1 \end{bmatrix}$$

The global lumped mass matrix

3. Trusses in 3D Space – 3D Truss Problem - Python

- Python code: https://github.com/mctrinh/fem-class
  - Given

$$
\begin{aligned}
E &= 1.2 \cdot 10^6 \\
P_1 &= (72, 0, 0) \\
P_2 &= (0, 36, 0) \\
P_3 &= (0, 36, 72) \\
P_4 &= (0, 0, -48) \\
U_2 &= U_3 = U_4 = (0, 0, 0) \\
v_1 &= 0 \\
A_1 &= 0.302 \\
A_2 &= 0.729 \\
A_3 &= 0.187
\end{aligned}
$$



  - Evaluate displacements, reactions, stresses of elements.

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thank you