Notes on regrasping as an underactuated pivot

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July 24, 2014

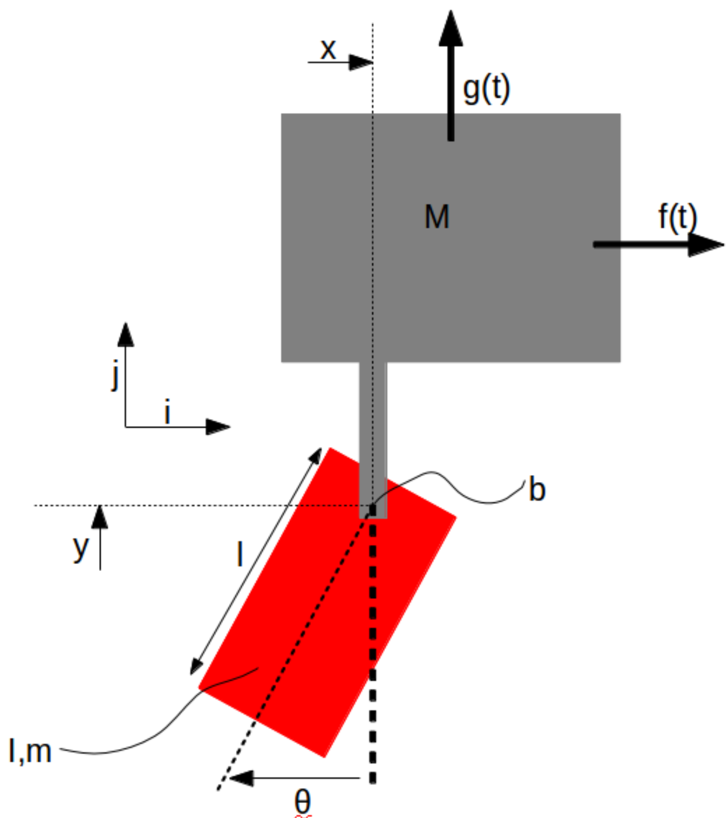

Figure 1: A rough sketch of the system. The hand is in gray with gripper coming down and the block in red.

As a first pass at the problem of pivoting a block into any angle using the dynamics of the system, I am approaching it as an underactuated system where the grasp acts as a pivot joint with a certain rotation damping due to the friction of the grasp, b. The block has a mass m, moment of inertia about the block's center of mass I, and a length l. The hand has a mass, M and its can move in the $\hat{i}$ and $\hat{j}$ directions as functions $f(t)$ and $g(t)$.

1 Generalized Coordinate System and Forces

There are three degrees of freedom here (assuming no motion in the $\hat{k}$ is relevant), which are x,y, and θ. Positive x points to the right, y up, and θ clockwise measured from the downward position.

$$\zeta_j : x, y, \theta$$

with the variation $\zeta_j : \delta x, \delta y, \delta\theta$

The generalized forces, Ξ_j can be obtained fromt the work where

$$\delta W = \sum_{i=1}^{n} \mathbf{F_i} \cdot \delta \mathbf{r_i} = \sum_{j=1}^{n} \Xi_j \delta\zeta_j$$

Nonconservative forces from the inputs $g(t)$ and $f(t)$ and the damping b around the grasp result in,

$$\delta W = f(t)\delta x + g(t)\delta y - b\dot{\theta}\delta\theta$$

So we get:

$$\Xi_x = f(t)\Xi_y = g(t)\Xi_\theta = -b\dot{\theta}$$

2 Kinetic Energy

The kinetic energy of the hand (the cart in simple cases):

$$K_{hand} = \frac{1}{2}M\mathbf{v}^2$$

Where M is the mass of the hand and v is a planar velocity vector of the hand (lets just assume x,y for now since movement in the same axis as the pivot axis doesnt do anything).

The kinetic energy for the block:

$$K_{block} = \frac{1}{2}m\mathbf{v_c}^2 + \frac{1}{2}I\omega^2$$

where v_c is the velocity of the center of mass of the block, m is the mass of the block, I is the moment of inertia around the block's center of mass, and ω is the angular velocity.
The position vector can then be written,

$$\mathbf{r_c} = (x - l\sin(\theta))\hat{\imath} + (y - l\cos(\theta))\hat{\jmath}$$

$$\mathbf{v_c} = \frac{d\mathbf{r_c}}{dt} = (\dot{x} - l\cos(\theta)\dot{\theta})\hat{\imath} + (\dot{y} + l\sin(\theta)\dot{\theta}\hat{\jmath}$$

Then, $\omega = \dot{\theta}$
So plugging in we get,

$$K_{block} = \frac{1}{2}m(\dot{x}^2 - 2\dot{x}lcos(\theta)\dot{\theta} + l^2cos^2(\theta)\dot{\theta}^2 + \dot{y}^2 + 2\dot{y}lsin(\theta)\dot{\theta} + l^2sin^2(\theta)\dot{\theta}^2) + \frac{1}{2}I\dot{\theta}^2$$

$$= \frac{1}{2}m(\dot{x}^2 - 2\dot{x}l\cos(\theta)\dot{\theta} + \dot{y}^2 + 2\dot{y}l\sin(\theta)\dot{\theta} + l^2\dot{\theta}^2) + \frac{1}{2}I\dot{\theta}^2$$

Then our total kinetic energy is,

$$K = K_{hand} + K_{block} = \frac{1}{2}M(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}m(\dot{x}^2 - 2\dot{x}l\cos(\theta)\dot{\theta} + \dot{y}^2 + 2\dot{y}l\sin(\theta)\dot{\theta} + l^2\dot{\theta}^2) + \frac{1}{2}I\dot{\theta}^2$$

3 Potential Energy

$$P = -mg(y + l)cos(\theta)$$

4 Lagrangian

Fromt he kinetic and potential energies, the lagrangians is given by

$$L = K - P$$

which becomes

$$L = \frac{1}{2}M(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}m(\dot{x}^2 - 2\dot{x}l\cos(\theta)\dot{\theta} + \dot{y}^2 + 2\dot{y}l\sin(\theta)\dot{\theta} + l^2\dot{\theta}^2) + \frac{1}{2}I\dot{\theta}^2 + mg(y + l)\cos(\theta)$$

4.1 More Lagrange Shit

The equations for x,

$$\Xi_x = \frac{d}{dt}(\frac{\partial L}{\partial \dot{x}}) - \frac{\partial L}{\partial x}$$

$$\frac{d}{dt}(M\dot{x} + m\dot{x} - ml\cos(\theta)\dot{\theta}) = f(t)$$

$$(M + m)\ddot{x} - ml\cos(\theta)\ddot{\theta} + ml\sin(\theta)\dot{\theta}^2 = f(t)$$

The same for y,

$$\Xi_y = \frac{d}{dt}(\frac{\partial L}{\partial \dot{y}}) - \frac{\partial L}{\partial y}$$

$$\frac{d}{dt}(M\dot{y} + m\dot{y} + ml\sin(\theta)\dot{\theta}) - mg\cos(\theta) = g(t)$$

$$(M + m)\ddot{y} + ml\sin(\theta)\ddot{\theta} - ml\cos(\theta)\dot{\theta}^2 - mg\cos(\theta) = g(t)$$

And for θ,

$$\Xi_\theta = \frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta}}) - \frac{\partial L}{\partial \theta}$$

$$\frac{d}{dt}(-m\dot{x}l\cos(\theta) + m\dot{y}l\sin(\theta) + ml^2\dot{\theta} + I\dot{\theta}) - (ml\dot{x}\sin(\theta)\dot{\theta} + ml\dot{y}\cos(\theta)\dot{\theta} - mg\sin(\theta)) = -b\dot{\theta}$$

$$(ml^2+I)\ddot{\theta} - ml\ddot{x}\cos(\theta) + ml\dot{x}\sin(\theta)\dot{\theta} + ml\ddot{y}\sin(\theta) + ml\dot{y}\cos(\theta)\dot{\theta} - (ml\dot{x}\sin(\theta)\dot{\theta} + ml\dot{y}\cos(\theta)\dot{\theta} - mg\sin(\theta)) = -b\dot{\theta}$$

where b is a rotational damping coefficient around the grasp.

Now we have to solve for $f(t)$ (motion in the $\hat{\imath}$ direction) and $g(t)$ (motion in the $\hat{\jmath}$ direction)

$$(M + m)\ddot{x} - ml\cos(\theta)\ddot{\theta} + ml\sin(\theta)\dot{\theta}^2 = f(t) \tag{1}$$

$$(M + m)\ddot{y} + ml\sin(\theta)\ddot{\theta} - ml\cos(\theta)\dot{\theta}^2 - mg\cos(\theta) = h(t) \tag{2}$$

$$(ml^2 + I)\ddot{\theta} + b\dot{\theta} - ml\ddot{x}\cos(\theta) + ml\ddot{y}\sin(\theta) + mg\sin(\theta) = 0 \tag{3}$$

5 Energy Regulation using PFLs

5.1 Collocated Partial Feedback Linearization (PFL)

Our goal is for $\ddot{x} = \ddot{x_d}$ and $\ddot{y} = \ddot{y_d}$. We can solve for $\ddot{\theta}$ using 3

$$\ddot{\theta} = \frac{1}{ml^2}(-mgs - mls\ddot{y} + mlc\ddot{x} - b\dot{\theta})$$

Where $s = \sin(\theta)$ and $c = \cos(\theta)$.
If we plug this into 2

$$(M+m)\ddot{y} + mls(\frac{1}{ml^2}(-mgs - mls\ddot{y} + mlc\ddot{x} - b\dot{\theta})) - mlc\dot{\theta}^2 - mgc = h(t)$$

$$(M+m)\ddot{y} + \frac{s}{l}(-mgs - mls\ddot{y} + mlc\ddot{x} - b\dot{\theta}) - mlc\dot{\theta}^2 - mgc = h(t)$$

$$\ddot{y} = \frac{1}{M+mc^2}(h(t) + \frac{mgs^2}{l} - msc\ddot{x} + \frac{bs\dot{\theta}}{l} + mlc\dot{\theta}^2 + mgc)$$

Then into 1

$$(M+m)\ddot{x} - ml\cos(\theta)\ddot{\theta} + ml\sin(\theta)\dot{\theta}^2 = f(t)$$

etc. etc. etc.

5.2 Energy Regulation

If $\ddot{x} = \bar{u}$ and $\ddot{y} = \bar{v}$, then

$$\ddot{\theta} = \frac{1}{ml^2}(-mgs - mls\bar{v} + mlc\bar{u} - b\dot{\theta})$$

From before our energy is given by,

$$E = \frac{1}{2}ml^2\dot{\theta}^2 - mg(y+l)c$$

$$\dot{E} = ml^2\dot{\theta}\ddot{\theta} + mg(y+l)s\dot{\theta}$$

$$\dot{E} = ml^2\dot{\theta}(\frac{1}{ml^2}(-mgs - mls\bar{v} + mlc\bar{u} - b\dot{\theta})) + mg(y+l)s\dot{\theta}$$

$$\dot{E} = \dot{\theta}(-mgs - mls\bar{v} + mlc\bar{u} - b\dot{\theta}) + mg(y+l)s\dot{\theta}$$

So to increase the energy, we can chose and $\bar{u}$ and $\bar{v}$ to make this expression positive. For example,

$$\bar{u} = kmlc\dot{\theta}\bar{E}$$

$$\bar{v} = -kmls\dot{\theta}\bar{E}$$

Where $\bar{E}$ is the difference between the desired energy and the actual energy.

5.3 now wat

We have two problems: we have no feedback for $\dot{\theta}$, and since our desired energy is not at an equilibrium point, it won't have zero kinetic energy, so we don't really know what $\bar{E}$ is.

The first is easier to deal with. If we just put in just enough energy to get to θ before it begins decreasing, we will be at zero velocity at the apex of a path arc at θ. So our $E_d = mg(y+l)\cos(\theta)$.

To handle no feedback (which is inherent in this problem since the block is always connected at the fingertips where there is no sensing normally), we can predict what θ is ideally doing based on the equations of motion.

Let's plug our expression for $\ddot{\theta}$ into 1 (or we could use 2)

$$\ddot{\theta} = \frac{1}{ml^2}(-mgs - mls\bar{v} + mlc\bar{u} - b\dot{\theta})$$

We know that $f(t) = \bar{u}$ and that $\ddot{x} = \frac{f(t)}{m} = klc\dot{\theta}$. Similarly for $\ddot{y}$

$$\ddot{\theta} = \frac{1}{ml^2}(-mgs - mls(-kls\dot{\theta}) + mlc(klc\dot{\theta}) - b\dot{\theta})$$

$$\ddot{\theta} = \frac{1}{ml^2}(-mgs + kml^2\dot{\theta} - b\dot{\theta})$$

From 1, $(M+m)\ddot{x} - ml\cos(\theta)\ddot{\theta} + ml\sin(\theta)\dot{\theta}^2 = f(t)$

$$(M+m)(klc\dot{\theta}) - ml\cos(\theta)(\frac{1}{ml^2}(-mgs + kml^2\dot{\theta} - b\dot{\theta})) + ml\sin(\theta)\dot{\theta}^2 = kmlc\dot{\theta}$$

$$(M-m)(klc\dot{\theta}) + \frac{cb\dot{\theta}}{l} + \frac{mgsc}{l} + mls\dot{\theta}^2 = 0 \tag{4}$$

From 2, $(M+m)\ddot{y} + ml\sin(\theta)\ddot{\theta} - ml\cos(\theta)\dot{\theta}^2 - mg\cos(\theta) = h(t)$

$$(M+m)(-kls\dot{\theta}) + ml\sin(\theta)(\frac{1}{ml^2}(-mgs + kml^2\dot{\theta} - b\dot{\theta})) - ml\cos(\theta)\dot{\theta}^2 - mg\cos(\theta) = -kmls\dot{\theta}$$

$$(m-M)(kls\dot{\theta}) - \frac{bs\dot{\theta}}{l} - \frac{mgs^2}{l} - mlc\dot{\theta}^2 - mgc = 0 \tag{5}$$

To solve for θ and $\dot{\theta}$, we can multiply 4 by $\cos(\theta)$, 5 by $\sin(\theta)$, and add.

$$(M-m)(klc^2\dot{\theta}) + \frac{c^2b\dot{\theta}}{l} + \frac{mgsc^2}{l} + mlsc\dot{\theta}^2 = 0$$

$$(m-M)(kls^2\dot{\theta}) - \frac{bs^2\dot{\theta}}{l} - \frac{mgs^3}{l} - mlsc\dot{\theta}^2 - mgsc = 0$$

Giving

$$(M-m)(kl\dot{\theta}) + \frac{b\dot{\theta}}{l} + \frac{mgs}{l} + mgsc = 0$$

Then we can solve for $\dot{\theta}$ and integrate to get θ

$$\dot{\theta} = \frac{1}{(M-m)kl + \frac{b}{l}}(-mgsc - \frac{mgs}{l})$$

$$\theta = \frac{1}{(M-m)kl + \frac{b}{l}}(-\frac{mgc^2}{2} + \frac{mgc}{l})$$

If we substitute these expressions back into our control laws $\bar{u}$ and $\bar{v}$, we get

$$\bar{u} = \frac{kml}{(M-m)kl + \frac{b}{l}} * c(-mgsc - \frac{mgs}{l}))$$

$$\bar{v} = \frac{-kml}{(M-m)kl + \frac{b}{l}} * s(-mgsc - \frac{mgs}{l}))$$

Ignoring costants those expressions roughly look like

$$\bar{u} = \cos(\theta) * (-\sin(\theta)\cos(\theta) - \sin(\theta))$$

$$\bar{v} = \sin(\theta) * (-\sin(\theta)\cos(\theta) - \sin(\theta))$$

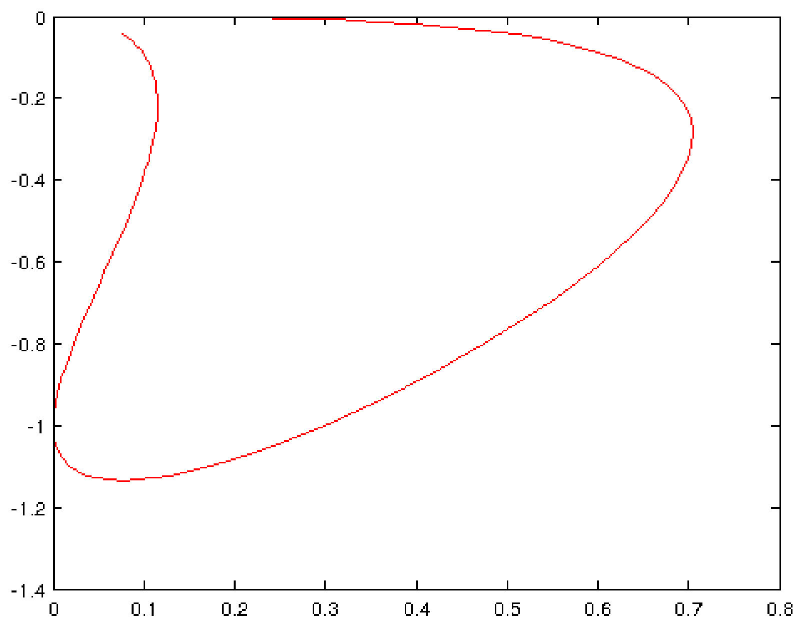

Figure 2