M378K: April 28th, 2025.

Example.

The Rayleigh density function is given by

$$f_Y(y) = c \cdot y e^{-\frac{y^2}{\tau}} \cdot \mathbb{1}_{[0,\infty)}(y)$$

Q: What is c?

$$\int_0^\infty y e^{-\frac{y^2}{\tau}} dy = \quad u = -\frac{y^2}{\tau} \Rightarrow du = -\frac{2}{\tau} y\,dy \Rightarrow y\,dy = -\frac{\tau}{2} du$$

$$= \int_0^{-\infty} e^u \left(-\frac{\tau}{2}\right) du = -\frac{\tau}{2} e^u \Big|_{u=0}^{-\infty} = \frac{\tau}{2}$$

$$\boxed{f_Y(y) = \frac{2}{\tau} y e^{-\frac{y^2}{\tau}} \mathbb{1}_{[0,\infty)}(y)}$$

Q: MLE?

Def'n. $Y_1, \dots, Y_n$ is a RANDOM SAMPLE from dist'n D if:

- $Y_1, \dots, Y_n$ are independent
- $Y_i \sim D$ for all $i = 1 \dots n$

Let $y_1, \dots, y_n \geq 0$ represent the observations of $Y_1, \dots, Y_n$.

$$L(\tau; y_1, \dots, y_n) = \prod_{i=1}^n f_Y^\tau(y_i) =$$

$$= \prod_{i=1}^n \left(\left(\frac{2}{\tau}\right) \cdot y_i \cdot e^{-\frac{y_i^2}{\tau}} \right)$$

$$= \left(\frac{2}{\tau}\right)^n \cdot \prod_{i=1}^n y_i \cdot e^{-\frac{1}{\tau} \sum_{i=1}^n y_i^2}$$

$$\ell(\tau; y_1, \dots, y_n) = n \cdot (\ln(2) - \ln(\tau)) + \sum_{i=1}^n \ln(y_i) - \frac{1}{\tau} \sum_{i=1}^n y_i^2$$

$$\ell'(\tau; y_1, \ldots, y_n) = -\frac{n}{\tau} + \left(+\frac{1}{\tau^2}\right)\sum_{i=1}^{n} y_i^2 = 0$$

$$\frac{1}{\tau^2}\sum_{i=1}^{n} y_i^2 = \frac{n}{\tau}$$

$$\hat{\tau}_{MLE} = \frac{\sum_{i=1}^{n} y_i^2}{n}$$

A sensible estimator for τ to propose:

$$\boxed{T = \frac{1}{n}\sum_{i=1}^{n} Y_i^2}$$

Def'n. The BIAS of the estimator $\hat{\theta}$ of the parameter θ is

$$\text{bias}(\hat{\theta}) = \mathbb{E}[\hat{\theta} - \theta]$$

In addition, we say that the estimator $\hat{\theta}$ is UNBIASED if

$$\text{bias}(\hat{\theta}) = 0, \text{ i.e.,}$$

$$\mathbb{E}[\hat{\theta}] = \theta$$

We want to check if T is unbiased for τ

$$\mathbb{E}[T] = \frac{1}{n}\sum_{i=1}^{n} \boxed{\mathbb{E}[Y_i^2]}$$

Q: If Y is Rayleigh, what is the dist'n of Y^2?

Def'n. The CUMULATIVE DIST'N F'TION of a random variable Y is defined as

$$F_Y : \mathbb{R} \longrightarrow [0,1]$$

$$F_Y(y) = \mathbb{P}[Y \leq y] \quad \text{for all } y \in \mathbb{R}$$

$\boxed{y>0}$:

$$F_{Y^2}(y) = \mathbb{P}[Y^2 \le y] = \mathbb{P}[Y \le \sqrt{y}] = F_Y(\sqrt{y})$$

$$= \int_0^{\sqrt{y}} \left(\frac{2}{\tau} \cdot u \, e^{-\frac{u^2}{\tau}}\right) du = \qquad \Big\lceil\ z = -\frac{u^2}{\tau} \Rightarrow dz = -\frac{2u\,du}{\tau} \Rightarrow$$

$$= -\int_0^{-\frac{y}{\tau}} e^z \, dz = 1 - e^{-\frac{y}{\tau}}$$

$$\Rightarrow \boxed{Y^2 \sim E(\tau)}$$

$$\mathbb{E}[T] = \frac{1}{n} \sum_{i=1}^n \mathbb{E}[Y_i^2] = \frac{1}{\cancel{n}} \cdot \cancel{n} \cdot \tau = \tau$$ unbiased ✓

Set $T_n = \frac{1}{n} \sum_{i=1}^n Y_i^2$. Is it consistent?

$\longrightarrow$: By our criterion: • unbiased ✓

• $Var[T_n] \xrightarrow[n\to\infty]{?} 0$

$$Var\left[\frac{1}{n} \sum_{i=1}^n Y_i^2\right] \overset{\text{(independence)}}{=} \frac{1}{n^2} \sum_{i=1}^n Var[Y_i^2] = \frac{\cancel{n} \cdot \tau^2}{n^{\cancel{2}}} \xrightarrow[n\to\infty]{} 0 \ ✓$$

Q: Is T sufficent?

$\longrightarrow$: By the F·N factorization criterion:

$$L(\tau; y_1, \ldots, y_n) = \underbrace{\left(\frac{2}{\tau}\right)^n e^{-\frac{1}{\tau}\overbrace{\sum_{i=1}^n y_i^2}^{n \cdot t}}}_{g(\tau;\, t)} \cdot \underbrace{\prod_{i=1}^n y_i}_{h(y_1, \ldots, y_n)}$$

Q: A pivotal quantity?

$$T = \frac{1}{n}\boxed{\sum_{i=1}^{n} Y_i^2}$$

$$Y_i^2 \sim E(\tau) \quad \text{for all } i = 1..n$$

$$\sum_{i=1}^{n} Y_i^2 \sim \Gamma(n, \tau)$$

$$T = \frac{1}{n}\sum_{i=1}^{n} Y_i^2 \sim \Gamma\left(n, \frac{\tau}{n}\right)$$

$$\frac{T}{\tau} \sim \Gamma\left(n, \frac{1}{n}\right)$$ is a PIVOTAL QUANTITY.