Exchange Options.

$$\begin{cases} T \dots \text{exercise date} \\ \text{two risky assets} \begin{cases} S \dots \text{underlying asset} \\ Q \dots \text{strike asset} \end{cases} \end{cases}$$

For an exchange call:

payoff: $\boxed{V_{EC}(T, S, Q) = (S(T) - Q(T))_+}$

For an exchange put:

payoff: $V_{EP}(T, S, Q) = (Q(T) - S(T))_+$

$\Rightarrow$ We have a special symmetry:

$$\boxed{V_{EC}(T, S, Q) = V_{EP}(T, Q, S)}$$

$\Rightarrow$ The time-0 prices must also be equal:

$$\boxed{V_{EC}(0, S, Q) = V_{EP}(0, Q, S)}$$

$\Rightarrow$ It suffices to develop the Black-Scholes pricing formula for exchange calls.

Black-Scholes.

- S... underlying asset : δ_S... dividend yield
  σ_S... volatility

Our goal is to price, so we have to consider S under the risk-neutral probability measure:

$$S(T) = S(0)\, e^{\left(r - \delta_S - \frac{\sigma_S^2}{2}\right)\cdot T + \sigma_S \cdot \sqrt{T} \cdot Z_S} \quad \text{w/ } \underline{Z_S \sim N(0,1)}$$

- Q.... strike asset : δ_Q ... dividend yield
  σ_Q ... volatility

Under the risk-neutral probability measure:

$$Q(T) = Q(0)\, e^{\left(r - \delta_Q - \frac{\sigma_Q^2}{2}\right)\cdot T + \sigma_Q \sqrt{T} \cdot Z_Q} \quad \text{w/ } \underline{Z_Q \sim N(0,1)}$$

w/ ρ... the correlation coefficient between Z_S and Z_Q.

Black-Scholes Price.

$$V_{EC}(0,S,Q) = F^P_{0,T}(S)\cdot N(d_1) - F^P_{0,T}(Q)\cdot N(d_2)$$

w/ $d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{F^P_{0,T}(S)}{F^P_{0,T}(Q)}\right) + \frac{1}{2}\cdot\sigma^2\cdot T\right]$

and $d_2 = d_1 - \sigma\sqrt{T}$

where $\boxed{\sigma^2 = \sigma_S^2 + \sigma_Q^2 - 2\rho\,\sigma_S\cdot\sigma_Q}$ ✓

Note: $\begin{cases} \cdot\ S(t),\ t \geq 0 \\ \cdot\ Q(t),\ t \geq 0 \end{cases}$

For every t:

$$\mathrm{Var}\left[\ln\left(\frac{S(t)}{Q(t)}\right)\right] = \boxed{\mathrm{Var}\left[\ln(S(t)) - \ln(Q(t))\right]}$$

$=$ (under $\mathbb{P}^*$)

$$= \mathrm{Var}\Big[\underbrace{\ln(S(0)) + \left(r - \delta_S - \frac{\sigma_S^2}{2}\right)\cdot t}_{\text{deterministic}} + \underline{\sigma_S\sqrt{t}\cdot Z_S} - \Big(\underbrace{\ln(Q(0)) + \left(r - \delta_Q - \frac{\sigma_Q^2}{2}\right)\cdot t}_{\text{deterministic}} + \underline{\sigma_Q\sqrt{t}\Big)\cdot Z_Q}\Big)\Big]$$

$$= \mathrm{Var}\left[\sigma_S\sqrt{t}\cdot Z_S - \sigma_Q\sqrt{t}\cdot Z_Q\right] =$$

$$= t\Big(\sigma_S^2 \cdot \overset{1}{\mathrm{Var}[Z_S]} + \sigma_Q^2 \cdot \overset{1}{\mathrm{Var}[Z_Q]} - 2\sigma_S\cdot\sigma_Q\cdot \underbrace{\mathrm{Cov}[Z_S, Z_Q]}_{=\rho}\Big)$$

$$= t\underbrace{\boxed{\left(\sigma_S^2 + \sigma_Q^2 - 2\sigma_S\cdot\sigma_Q\cdot\rho\right)}}_{=\sigma^2}$$

Favorite Special Case:

$$V_{EC}(0, S, Q) = S(0)e^{-\delta_S\cdot T}\cdot N(d_1) - Q(0)e^{-\delta_Q\cdot T}\cdot N(d_2)$$

w/ $d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)e^{-\delta_S\cdot T}}{Q(0)e^{-\delta_Q\cdot T}}\right) + \frac{1}{2}\sigma^2\cdot T\right]$

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{Q(0)}\right) + \left(\underset{\updownarrow\atop r}{\delta_Q} - \underset{\updownarrow\atop \delta}{\delta_S} + \frac{1}{2}\sigma^2\right)\cdot T\right]$$