M378K Introduction to Mathematical Statistics
Problem Set #7
Cumulative distribution functions: Named continuous distributions.

Problem 7.1. Source: Sample P exam, Problem #126.
A company's annual profit is normally distributed with mean 100 and variance 400. Then, we can express the probability that the company's profit in a year is at most 60, given that the profit in the year is positive in terms of the standard normal cumulative distribution function denoted by Φ as

$$1 - \frac{\Phi(2)}{\Phi(5)}$$

Solution: Let's denote the company's profit by a random variable Y. We are given that $Y \sim N(\mu = 100, \sigma = \sqrt{400} = 20)$. We know that when we express Y in standard units, we obtain a standard normal distribution, i.e.,

$$Z = \frac{Y - 100}{20} \sim N(0, 1)$$

So, we can calculate our conditional probability as

$$\begin{aligned}
\mathbb{P}[Y \leq 60 \,|\, Y > 0] &= \frac{\mathbb{P}[0 < Y \leq 60]}{\mathbb{P}[Y > 0]} = \frac{\mathbb{P}\left[\frac{0-100}{20} < \frac{Y-100}{20} \leq \frac{60-100}{20}\right]}{\mathbb{P}\left[\frac{Y-100}{20} > \frac{0-100}{20}\right]} \\
&= \frac{\mathbb{P}[-5 < Z \leq -2]}{\mathbb{P}[-5 < Z]} = \frac{\mathbb{P}[Z \leq -2] - \mathbb{P}[Z \leq -5]}{1 - \mathbb{P}[Z \leq -5]} \\
&= \frac{\Phi(-2) - \Phi(-5)}{1 - \Phi(-5)} = \frac{(1 - \Phi(2)) - (1 - \Phi(5))}{\Phi(5)} = \frac{\Phi(5) - \Phi(2)}{\Phi(5)} = 1 - \frac{\Phi(2)}{\Phi(5)}.
\end{aligned}$$

Problem 7.2. Source: Sample P exam, Problem #267.
The lifetime of a certain electronic device has an exponential distribution with mean 0.50. *Calculate the probability that the lifetime of the device is greater than* 0.70, *given that it is greater than* 0.40.

Solution: This problem is an excellent opportunity to study a unique property of the exponential distribution. Consider $T \sim E(\tau)$ and two positive constants, s and t. Then, we could be interested in finding the probability that T exceeds $t + s$ given that it exceeds s.

$$\mathbb{P}[T > t + s \,|\, T > s] = \frac{\mathbb{P}[T > t + s, T > s]}{\mathbb{P}[T > s]} = \frac{\mathbb{P}[T > t + s]}{\mathbb{P}[T > s]} = \frac{e^{-\frac{t+s}{\tau}}}{e^{-\frac{s}{\tau}}} = e^{-\frac{t}{\tau}} = \mathbb{P}[T > t].$$

In a sense, the random variable T "forgets" that it already waited for s time units. This property is, hence, called the **memoryless property**.

In the present problem, we have $T \sim E(\tau = 0.5)$. So,

$$\mathbb{P}[T > 0.70 \,|\, T > 0.40] = \mathbb{P}[T > 0.3] = e^{-\frac{0.3}{0.5}} = e^{-0.6}.$$