Binomial Option Pricing.

M339D: October 26th, 2022.

Stock Price Tree.

$$\begin{tikzcd}[column sep=large] & S_u = uS(0) \\ S(0) \arrow[ur, "h"] \arrow[dr] & \\ & S_d = dS(0) \end{tikzcd}$$

$\xrightarrow{\text{populating the tree}}$

No-arbitrage condition

$$\boxed{d < e^{rh} < u}$$

We want to price a European-style derivative security w/ the exercise date @ the end of the period.

It is completely determined by its payoff function: $v(\cdot)$

e.g., for a call: $v_c(s) = (s-K)_+$

for a put: $v_p(s) = (K-s)_+$

The payoff of the derivative security is a random variable:

$$\boxed{V(T) := v(S(T))}$$

$$\begin{tikzcd}[column sep=large] & S_u \\ S(0) \arrow[ur, "h"] \arrow[dr] & \\ & S_d \end{tikzcd}$$

$V(0) = ?$

$\xrightarrow{\text{population}}$

$\xleftarrow{\text{pricing}}$

Payoff		Replicating Portfolio
$V_u := v(S_u)$	$=$	$\Delta \cdot S_u + Be^{rh}$
$V_d := v(S_d)$	$=$	$\Delta S_d + Be^{rh}$

In the binomial model, any derivative security can be replicated w/ a portfolio of this form:

- $\underline{\Delta}$ shares of stock $\begin{cases} \Delta > 0 & \text{buying} \\ \Delta = 0 & \text{"nothing"} \\ \Delta < 0 & \text{shorting} \end{cases}$
- $\underline{B}$ @ the ccrfir r $\begin{cases} B > 0 & \text{lending (buying a bond)} \\ B = 0 & \text{"nothing"} \\ B < 0 & \text{borrowing (issuing a bond)} \end{cases}$

$\boxed{\text{time-0 holdings}}$

If we can find Δ and B, then:

$$\boxed{V(0) = \Delta \cdot S(0) + B}$$

We get a system of two eq'ns w/ two unknowns:

$$\left.\begin{aligned} \Delta \cdot S_u + \cancel{Be^{rh}} &= V_u \\ -\Delta \cdot S_d - \cancel{Be^{rh}} &= -V_d \end{aligned}\right\} -$$

$$\Delta (S_u - S_d) = V_u - V_d$$

$$\boxed{\Delta = \frac{V_u - V_d}{S_u - S_d}}$$

unitless

$$\frac{V_u - V_d}{S_u - S_d} \cdot S_u + Be^{rh} = V_u$$

$$Be^{rh} = V_u - \frac{V_u - V_d}{\cancel{S(0)}(u-d)} \cdot \cancel{S(0)} \cdot u = \frac{\cancel{u \cdot V_u} - d \cdot V_u - \cancel{u \cdot V_u} + u \cdot V_d}{u-d}$$

$$\boxed{B = e^{-rh} \frac{u \cdot V_d - d \cdot V_u}{u-d}}$$

By the Law of the Unique Price:

$$\boxed{V(0) = \Delta \cdot S(0) + B}$$

Pricing by Replication

Problem 6.3. Let the continuously compounded risk-free interest rate be equal to 0.04. Consider a one-period binomial tree with every period of length one year used to model the stock price of a stock whose current price is \$80 per share. In the model, it is assumed that the stock price can either go up by \$5 or down by \$4.

You use the binomial tree to construct a replicating portfolio for a 78−strike call option on the above stock. What is the stock investment in the replicating portfolio?

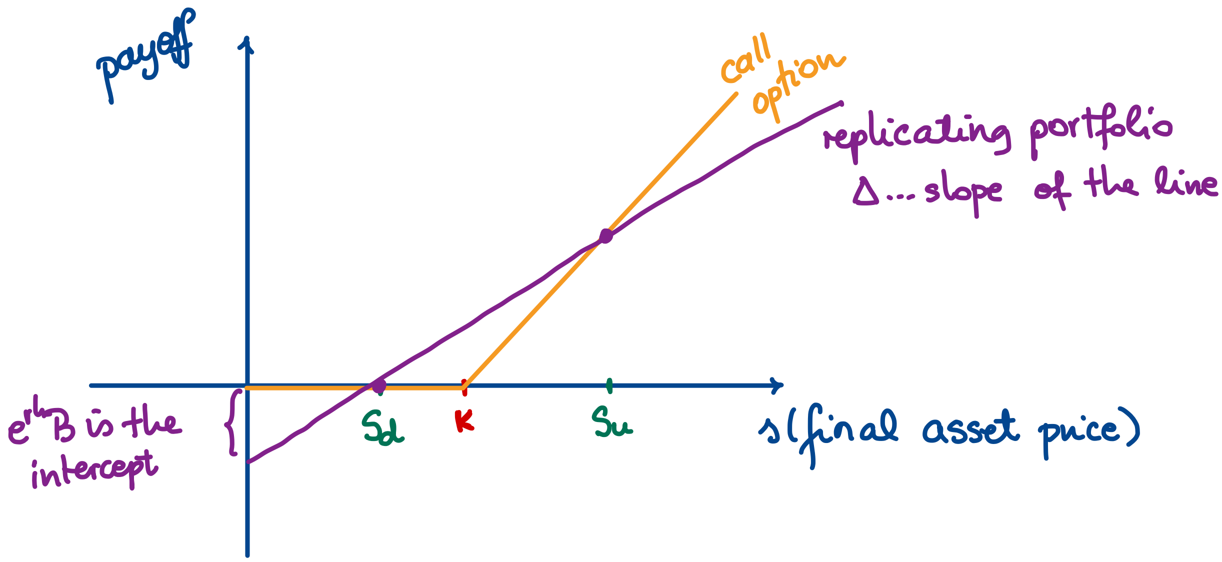
→ :

$S(0) = 80$ — $S_u = 85$, $S_d = 76$

$V_u = (85-78)_+ = 7$

$V_d = (76-78)_+ = 0$

$$\Delta = \frac{V_u - V_d}{S_u - S_d} = \frac{7-0}{85-76} = \frac{7}{9} \quad \square$$



$r = 0.04$

Problem 6.4. Let the continuously compounded risk-free interest rate be equal to $\boxed{0.04}$. Consider a one-period binomial tree with every period of length one year used to model the stock price of a non-dividend-paying stock whose current price is \$50 per share. In the model, it is assumed that the stock price can either go up by 5% or down by 10%.

You use the binomial tree to construct a replicating portfolio for a 45−strike call on the above stock. What is the risk-free investment in the replicating portfolio?

borrowing

$$B = e^{-rh}\frac{u \cdot V_d - d \cdot V_u}{u-d} = e^{-0.04}\frac{1.05(0) - 0.90(7.5)}{1.05 - 0.90} = \underline{-43.2355}$$

$S_u = 50(1.05) = 52.5$ $V_u = (52.5-45)_+ = 7.5$

$S(0) = 50$

$S_d = 50(0.90) = 45$ $V_d = (45-45)_+ = 0$