M378K: February 19^{th}, 2025.

The F-Distribution.

Let Y_1 and Y_2 be two independent χ^2-distributed r.v.s w/ $\boxed{df = 1.}$

For both Y_1 and Y_2, the pdf is

$$\boxed{f_Y(y) = \frac{1}{\sqrt{2\pi y}} e^{-\frac{y}{2}} \mathbb{1}_{(0,\infty)}(y)} \longleftarrow$$

Define $\boxed{W = \frac{Y_2}{Y_1}}$, i.e., $W = g(Y_1, Y_2)$ w/ $g(y_1, y_2) = \frac{y_2}{y_1}$

Goal: Density of W : f_W !

Start by figuring out the cdf F_W.

$w > 0$: $F_W(w) = \mathbb{P}[W \le w] = \mathbb{P}\left[\frac{Y_2}{Y_1} \le w\right]$

$$= \mathbb{P}[Y_2 \le w \cdot Y_1]$$

$$= \int_0^\infty \int_0^{w \cdot y_1} f_{Y_1, Y_2}(y_1, y_2)\, dy_2\, dy_1$$

$$= \int_0^\infty \int_0^{w \cdot y_1} \frac{1}{\sqrt{2\pi y_1}} e^{-\frac{y_1}{2}} \cdot \frac{1}{\sqrt{2\pi y_2}} e^{-\frac{y_2}{2}}\, dy_2\, dy_1$$

$$= \int_0^\infty \frac{1}{\sqrt{2\pi y_1}} e^{-\frac{y_1}{2}} \underbrace{\int_0^{w \cdot y_1} \frac{1}{\sqrt{2\pi y_2}} e^{-\frac{y_2}{2}}\, dy_2}_{F_{Y_2}(w y_1)}\, dy_1$$

$$F_W(w) = \int_0^\infty \frac{1}{\sqrt{2\pi y_1}} e^{-\frac{y_1}{2}} \cdot F_{Y_2}(w y_1)\, dy_1$$

$$f_W(w) = \frac{d}{dw} \int_0^\infty \frac{1}{\sqrt{2\pi y_1}} e^{-\frac{y_1}{2}} \boxed{F_{Y_2}(w y_1)}\, dy_1$$

$$f_W(w) = \frac{d}{dw} F_W(w) = \lim_{\varepsilon \to 0} \frac{F_W(w+\varepsilon) - F(w)}{\varepsilon}$$

$$f_W(w) = \int_0^\infty \frac{1}{\sqrt{2\pi y_1}} e^{-\frac{y_1}{2}} \boxed{f_{Y_2}(w y_1)} \cdot y_1 \, dy_1$$

$$f_W(w) = \int_0^\infty \frac{1}{\sqrt{2\pi \cancel{y_1}}} e^{-\frac{y_1}{2}} \frac{1}{\sqrt{2\pi w \cancel{y_1}}} e^{-\frac{w y_1}{2}} \cdot \cancel{y_1} \, dy_1$$

$$f_W(w) = \frac{1}{2\pi \sqrt{w}} \boxed{\int_0^\infty e^{-\frac{(1+w)y_1}{2}} \, dy_1}$$

$$= \left(-\frac{2}{1+w} e^{-\frac{(1+w)y_1}{2}} \right)_{y_1=0}^{\infty}$$

$$f_W(w) = \frac{1}{\cancel{2}\pi \sqrt{w}} \cdot \frac{\cancel{2}}{1+w} = \frac{1}{\pi \sqrt{w}(1+w)} \mathbb{1}_{(0,\infty)}(w)$$

is the density $F(1,1)$, i.e.,

the F-distribution w/ 1 numerator df and 1 denominator df. $\square$

$$F(\nu_1, \nu_2) \stackrel{(d)}{=} \frac{\frac{\chi^2(\nu_1)}{\nu_1}}{\frac{\chi^2(\nu_2)}{\nu_2}}$$

M378K Introduction to Mathematical Statistics
Problem Set #9
Moment generating functions.

Definition 9.1. *The k^{th}* moment of a random variable Y taken about the origin *is defined as* $\mathbb{E}[Y^k]$ *provided that the expectation exists. We write*

$$\mu_k = \mathbb{E}[Y^k]$$

when there is no ambiguity about the random variable in question.

Remark 9.2. *μ_k is also referred to as the k^{th}* raw moment.

Remark 9.3. *In particular, $\mu_1 = \mu$ happens to be the* **mean** *of the random variable Y.*

Definition 9.4. *The k^{th}* central moment *of a random variable Y is defined as* $\mathbb{E}[(Y-\mu)^k]$ *provided that the expectation exists. We write*

$$\mu_k^c = \mathbb{E}[(Y-\mu)^k]$$

when there is no ambiguity about the random variable in question.

Remark 9.5. *μ_k is also referred to as the k^{th}* moment of a random variable Y taken about its mean.

Definition 9.6. *The* moment-generating function (mgf) m_Y *for a random variable Y is defined as*

$$m_Y(t) = \mathbb{E}[e^{tY}]$$

for all t for which the above expectation exists. In fact, we say that the moment-generating function **exists** *if there exists a positive number b such that $m_Y(t)$ is finite for all t such that $|t| \leq b$.*

Problem 9.1. *How much is $m_Y(0)$?*

$$m_Y(0) = \mathbb{E}[e^{0\cdot Y}] = 1$$

Remark 9.7. *On the choice of terminology ...*

Step 1.

$$\frac{d}{dt}m_Y(t) = ?$$

$$\frac{d}{dt}m_Y(t) = \frac{d}{dt}\mathbb{E}[e^{tY}] = \mathbb{E}\left[\frac{d}{dt}e^{tY}\right]$$

$$= \mathbb{E}[Ye^{tY}]$$

Step 2.

$$m'_Y(0) = ?$$

$$m'_Y(0) = \mathbb{E}[Y e^{0 \cdot Y}] = \mathbb{E}[Y] = \mu_Y$$

Step 3.

$$\frac{d^2}{dt^2} m_Y(t) = ?$$

$$\frac{d}{dt}\left(\frac{d}{dt} m_Y(t)\right) = \frac{d}{dt}\mathbb{E}[Y e^{tY}] = \mathbb{E}[Y^2 e^{tY}]$$

Step 4.

$$m''_Y(0) = ?$$

$m''_Y(0) = \mathbb{E}[Y^2]$, i.e., the 2^{nd} moment ☺

Step 5. *What do you suspect the* **generalization** *of the above would be?*