M378K Introduction to Mathematical Statistics
Problem Set #16
Consistency.

Definition 16.1. *$\hat{\theta}_n$ is said to be a* consistent estimator *of θ if*

$$\hat{\theta}_n \to \theta \quad \textit{in probability as } n \to \infty,$$

i.e., if for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left[|\hat{\theta}_n - \theta| > \varepsilon\right] = 0.$$

Theorem 16.2. *Let $\hat{\theta}_n$ be* unbiased *and such that*

$$\mathrm{Var}\left[\hat{\theta}_n\right] \xrightarrow{n\to\infty} 0.$$

Then, $\hat{\theta}_n$ is a **consistent estimator**.

Problem 16.1. *Let $Y_1, Y_2, \dots, Y_n$ be a random sample from any distribution with finite first and second moments. Propose a consistent estimator for the population mean μ and* **prove** *that it is, indeed, consistent.*

Solution: An obvious choice is the sample mean

$$\bar{Y}_n = \frac{1}{n}(Y_1 + Y_2 + \dots + Y_n).$$

We have already verified earlier in class that $\bar{Y}$ is *unbiased* for μ. According to the theorem above, it suffices to show that

$$\mathrm{Var}[\bar{Y}_n] \xrightarrow{n\to\infty} 0.$$

However, we know that

$$\mathrm{Var}[\bar{Y}_n] = \frac{\sigma^2}{n} \xrightarrow{n\to\infty} 0$$

which completes our proof.

Problem 16.2. *Consider a random sample $Y_1, Y_2, \dots, Y_n$ from a power distribution with the density of the form*

$$f_Y(y) = \theta y^{\theta-1}\mathbf{1}_{(0,1)}(y).$$

What is a consistent estimator for $\frac{\theta}{\theta+1}$? **Prove** *that your choice is indeed consistent.*

Solution: The expression $\frac{\theta}{\theta+1}$ stands for the population mean. Indeed,

$$\mathbb{E}[Y_1] = \int_0^1 y f_Y(y)\,dy = \int_0^1 \theta y^\theta\,dy = \frac{\theta}{\theta+1}.$$

So, a sensible choice for a consistent estimator is $\bar{Y}_n$. It's evidently unbiased. Let's try to use the previous criterion in terms of the vanishing variance. We know that

$$\mathrm{Var}[\bar{Y}_n] = \frac{\mathrm{Var}[Y_1]}{n}.$$

So, it suffices to verify that the variance of Y_1 is finite. We have

$$\mathrm{Var}[Y_1] = \mathbb{E}[Y_1^2] - \left(\frac{\theta}{\theta+1}\right)^2.$$

On the other hand,

$$\mathbb{E}[Y_1^2] = \int_0^1 y^2\left(\theta y^{\theta-1}\right)\,dy = \theta\int_0^1 y^{\theta+1}\,dy = \frac{\theta}{\theta+2}.$$

Clearly, $\mathrm{Var}[Y_1]$ is finite and the theorem applies. Hence, $\bar{Y}_n$ is, indeed, consistent.