M378K: November 4th, 2024.

More on Confidence Intervals.

Example. $Y_1, \dots, Y_n \sim E(\tau)$

$$\bar{Y} = \frac{1}{n}(\underbrace{Y_1 + \dots + Y_n}_{\sim\Gamma(n,\tau)})$$ is not a pivotal quantity

We propose the pivotal quantity:

$$U = \frac{1}{\tau}\bar{Y} = \frac{1}{n\tau}\boxed{(\underbrace{Y_1 + \dots + Y_n}_{\Gamma(n,\tau)})}$$

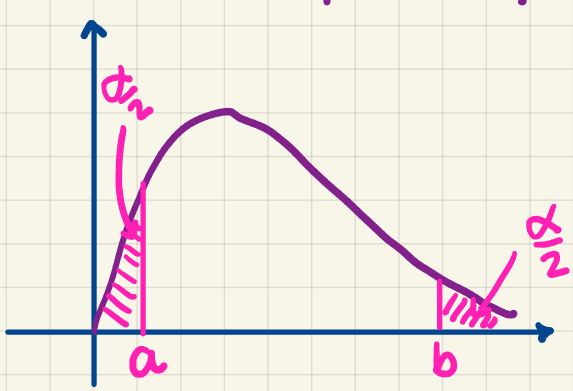
$m_{a\cdot X}(t) = \mathbb{E}[e^{aX\cdot t}]$
$= \mathbb{E}[e^{(at)\cdot X}]$
$= m_X(at)$

Because the second parameter of a Γ dist'n is a scale parameter, as we know from

$$m_X(t) = (1 - \tau t)^{-k} \quad \text{for } X \sim \Gamma(k, \tau)$$

$$\Rightarrow \quad U \sim \Gamma\left(n, \frac{\cancel{\tau}}{n\cancel{\tau}}\right) = \Gamma(n, 1/n)$$

The dist'n doesn't depend on τ, so this is a pivotal quantity



Pick the confidence level $C = 0.90$, i.e., $\alpha = 0.10$

In the lecture notes $\boxed{n=6}$

a = qgamma(0.05, shape=6, scale=1/6)

$\Rightarrow \quad a = 0.4355025 \approx 0.44$

b = qgamma(0.95, shape=6, scale=1/6)

$\Rightarrow \quad b = 1.752172 \approx 1.75$

We know that

$$\mathbb{P}[0.44 \leq U \leq 1.75] = 1-\alpha = 0.90$$

$$\mathbb{P}\left[0.44 \leq \frac{1}{\tau}\cdot\bar{Y} \leq 1.75\right] = 0.90$$

$$\mathbb{P}\left[\frac{0.44}{\bar{Y}} \leq \frac{1}{\tau} \leq \frac{1.75}{\bar{Y}}\right] = 0.90$$

$$\mathbb{P}\left[\underbrace{\frac{\bar{Y}}{1.75}}_{=\hat{\theta}_L} \leq \tau \leq \underbrace{\frac{\bar{Y}}{0.44}}_{=\hat{\theta}_R}\right] = 0.90$$

Choosing the Sample Size.

By def'n, the margin of error is $\frac{1}{2}(\hat{\theta}_R - \hat{\theta}_L)$

We can prescribe a margin of error m and a confidence level $1-\alpha$, and then seek the necessary sample size so that the margin of error is @ most m.

Example. $n = ?$

With a normal population $N(\mu, \sigma)$ w/ known σ,

i.e., $Y_1, Y_2, \dots, Y_n \sim N(\mu, \sigma)$

The form of the confidence interval for μ is

$$\bar{Y} \pm \underbrace{\boxed{z^* \cdot \frac{\sigma}{\sqrt{n}}}}_{\leq m}$$

$$z^* \cdot \frac{\sigma}{m} \leq \sqrt{n}$$

$$\boxed{\left(z^* \cdot \frac{\sigma}{m}\right)^2 \leq n}$$