M378K Introduction to Mathematical Statistics
Problem Set #1
Named discrete random variables.

Problem 1.1. Source: Sample P exam, Problem #125.
An electronic system contains three cooling components that operate independently. The probability of each component's failure is 0.05. *The system will overheat if and only if at least two components fail. Calculate the probability that the system will overheat.*

Solution: Let us denote the number of components that fail by N. Since the components operate independently, and their failure probabilities are equal, we can model N using a binomial distribution with the number of trials equal to the number of components (i.e., $n = 3$) and the common "success" probability equal to the probability of failure (i.e., $p = 0.05$). In short,

$$N \sim b(n = 3, p = 0.05)$$

The probability that the system overheats is

$$\begin{aligned}\mathbb{P}[N \geq 2] &= \mathbb{P}[N = 2] + \mathbb{P}[N = 3] \\ &= \binom{3}{2}(0.05)^2(0.95) + \binom{3}{3}(0.05)^3 = 0.00725.\end{aligned}$$

Problem 1.2. Source: Sample P exam, Problem #462.
Each person in a large population independently has probability p *of testing positive for diabetes where* $0 < p < 1$. *People are tested for diabetes, one person at a time, until a test is positive. Individual tests are independent. Determine the probability that* m *or fewer people are tested, given that* n *or fewer people are tested, where* $1 \leq m \leq n$.

Solution: Let N' be the total number of people tested. This is a **shifted** geometric random variable with parameter p, i.e., we have that

$$N = N' - 1 = g(p)$$

We need to figure out the following probability:

$$\begin{aligned}\mathbb{P}[N' \leq m \,|\, N' \leq n] &= \mathbb{P}[N + 1 \leq m \,|\, N + 1 \leq n] \\ &= \mathbb{P}[N \leq m - 1 \,|\, N \leq n - 1] \\ &= \frac{\mathbb{P}[N \leq m - 1, N \leq n - 1]}{\mathbb{P}[N \leq n - 1]} \\ &= \frac{\mathbb{P}[N \leq m - 1]}{\mathbb{P}[N \leq n - 1]} \\ &= \frac{1 - \mathbb{P}[N > m - 1]}{1 - \mathbb{P}[N > n - 1]} \\ &= \frac{1 - q^m}{1 - q^n}.\end{aligned}$$

Problem 1.3. Source: Sample P exam, Problem #355.
The number of calls received by a certain emergency unit in a day is modeled by a Poisson distribution with parameter 2*. Calculate the probability that on a particular day the unit receives at least two calls.*

Solution: Let N denote the number of calls. We are told that $N \sim P(\lambda = 2)$. So, we have

$$\mathbb{P}[N \geq 2] = 1 - \mathbb{P}[N \leq 1] = 1 - p_0 - p_1 = 1 - e^{-\lambda} + e^{-\lambda}\lambda = 1 - 3e^{-2}.$$