UNIVERSITY OF TEXAS AT AUSTIN

Problem Set # 4

Normal distribution.

Problem 4.1. Let Z be a standard normal random variable. Find the following probabilities:

i. $\mathbb{P}[-1.33 < Z \leq 0.24]$
ii. $\mathbb{P}[0.49 < |Z|]$
iii. $\mathbb{P}[Z^4 < 0.0256]$
iv. $\mathbb{P}[e^{2Z} < 2.25]$
v. $\mathbb{P}\left[\frac{1}{Z} < 2\right]$

Solution:

i.
$$\begin{aligned}\mathbb{P}[-1.33 < Z \leq 0.24] &= \mathbb{P}[Z \leq 0.24] - \mathbb{P}[Z \leq -1.33] = \mathbb{P}[Z \leq 0.24] - (1 - \mathbb{P}[Z \leq 1.33]) \\ &= 0.5948 - 1 + 0.9082 = 0.503\end{aligned}$$

ii.
$$\begin{aligned}\mathbb{P}[0.49 < |Z|] &= \mathbb{P}[Z < -0.49] + \mathbb{P}[0.49 < Z] = 2\mathbb{P}[Z > 0.49] \\ &= 2(1 - \mathbb{P}[Z \leq 0.49]) = 2(1 - 0.6879) = 0.6242\end{aligned}$$

iii.
$$\begin{aligned}\mathbb{P}[Z^4 < 0.0256] &= \mathbb{P}[|Z| < \sqrt[4]{0.0256}] = \mathbb{P}[|Z| < 0.4] = \mathbb{P}[Z < 0.4] - \mathbb{P}[Z < -0.4] \\ &= 2\mathbb{P}[Z < 0.4] - 1 = 2(0.6554) - 1 = 0.3108\end{aligned}$$

iv.
$$\mathbb{P}[e^{2Z} < 2.25] = \mathbb{P}[2Z < \ln(2.25)] = \mathbb{P}[Z < 0.5\ln(2.25)] \approx \mathbb{P}[Z \leq 0.41] = 0.6591$$

v.
$$\begin{aligned}\mathbb{P}\left[\frac{1}{Z} < 2\right] &= \mathbb{P}[\tfrac{1}{Z} < 0] + \mathbb{P}[0 < \tfrac{1}{Z} < 2] \\ &= \mathbb{P}[Z < 0] + \mathbb{P}[Z > 0.5] = 0.5 + (1 - \mathbb{P}[Z \leq 0.5]) = 0.5 + (1 - 0.6915) = 0.8085.\end{aligned}$$

Problem 4.2. (10 points)

At the *Hogwarts School of Witchcraft and Wizardry* the *Ordinary Wizarding Level (OWL)* exam is typically taken at the end of the fifth year. Based on hystorical data, we model the *OWL* scores as roughly normal with mean 100 and standard deviation of 16.

(a) (5 points)

What is the range of scores for the bottom 15% of the *OWL* takers?

Solution:

The z-score corresponding to 15% is -1.04. So, we solve for x in
$$-1.04 = \frac{x - 100}{16},$$
and get $x = 100 - 16(1.04) = 87.52$. Therefore, the range of scores for the bottom 15% of OWL takers is $[0, 83.36]$.

(b) (5 points)
What is the probability that a randomly chosen *OWL* taker has a score higher than 125?

Solution: For the raw score of 125, the corresponding score in standard units equals

$$z = \frac{125 - 100}{16} \approx 1.56$$

So, the probability is approximately

$$1 - \Phi(1.56) = 1 - 0.9406 = 0.0594.$$