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Log-normal stock prices: Tail probabilities.

Problem 3.1. (15 points)

You are considering an investment in a non-dividend-paying stock versus an investment in a savings account. According to your belief, the stock's mean rate of return is α and its volatility is σ.

The continuously compounded interest rate is equal to r.

What is the probability that the stock outperforms the savings account at time$-T$? You should leave your final answer in terms of the function N.

If risk-free investment, then your time-T balance is

$$S(0)e^{rT}$$

If invest in the stock, then you own 1 share @ time-T and your wealth $S(T)$.

$$\mathbb{P}[S(T) > S(0)e^{rT}] = \mathbb{P}[S(T) - \underbrace{S(0)}_{\text{Init. Cost}}e^{rT} > 0]$$

FV(Init. Cost)

* Equivalent to asking whether the outright purchase has a positive profit.! *

LogNormal Stock price: (no dividends $\Rightarrow \delta = 0$)

$$S(T) = S(0)\cdot e^{(\alpha - \frac{\sigma^2}{2})\cdot T + \sigma\sqrt{T}\cdot Z}$$

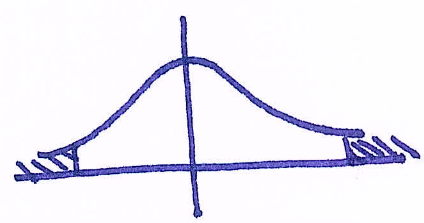
$Z \sim N(0,1)$

$$\mathbb{P}[\cancel{S(0)}e^{(\alpha - \frac{\sigma^2}{2})\cdot T + \sigma\sqrt{T}\cdot Z} > \cancel{S(0)}e^{rT}] =$$

$$= \mathbb{P}[(\alpha - \frac{\sigma^2}{2})\cdot T + \sigma\sqrt{T}\cdot Z > rT]$$

$$= \mathbb{P}[\sigma\sqrt{T}\cdot Z > (r - \alpha + \frac{\sigma^2}{2})\cdot T]$$

$$= \mathbb{P}[Z > \frac{1}{\sigma\sqrt{T}}(r - \alpha + \frac{\sigma^2}{2})\cdot T]$$



$$= \mathbb{P}\left[Z < -\frac{1}{\sigma\cancel{\sqrt{T}}}\left(r-\alpha+\frac{\sigma^2}{2}\right)\cancel{T}^{\sqrt{T}} \right]$$

$$= N\left(\frac{\sqrt{T}}{\sigma}\left(\alpha - r - \frac{\sigma^2}{2}\right)\right) \blacksquare$$

Qs: · Think about ↑ and ↓ w.r.t. T of this probab. as you vary the possible values of α, r, σ.

· If we are looking for the risk-neutral probab., then we get

$$\mathbb{P}^*[S(T) > S(0)e^{rT}] = N\left(\frac{\sqrt{T}}{\sigma}\left(-\frac{\sigma^2}{2}\right)\right) = N\left(-\frac{\sigma\sqrt{T}}{2}\right)$$

· What changes if we reintroduce continuous dividends?

Tail Probabilities of LogNormal Stock Prices

$$S(T) = S(0)\cdot e^{(\alpha-\delta-\frac{\sigma^2}{2})\cdot T + \sigma\sqrt{T}\cdot Z}$$

w/ $Z \sim N(0,1)$

Q: Given a strike price K, what is the probability that a call option will be exercised?

$$\mathbb{P}[S(T) > K] = ?$$

$\rightarrow$: $\mathbb{P}[S(T) > K] =$ (by the log-normal model for $S(T)$)

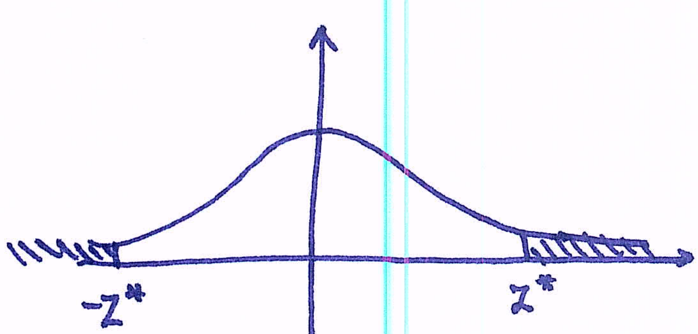
$$= \mathbb{P}\left[S(0)e^{(\alpha-\delta-\frac{\sigma^2}{2})\cdot T+\sigma\sqrt{T}\cdot Z} > K\right]$$

$$= \mathbb{P}\left[e^{(\alpha-\delta-\frac{\sigma^2}{2})\cdot T+\sigma\sqrt{T}\cdot Z} > \frac{K}{S(0)}\right]$$ (ln of both sides)

$$= \mathbb{P}\left[\overbrace{(\alpha-\delta-\frac{\sigma^2}{2})\cdot T}+\sigma\sqrt{T}\cdot Z > \ln\left(\frac{K}{S(0)}\right)\right]$$

$$= \mathbb{P}\left[\underbrace{\sigma\sqrt{T}}\cdot Z > \ln\left(\frac{K}{S(0)}\right) - (\alpha-\delta-\frac{\sigma^2}{2})\cdot T\right]$$

$$= \mathbb{P}\left[Z > \underbrace{\frac{1}{\sigma\sqrt{T}}\left(\ln\left(\frac{K}{S(0)}\right)-(\alpha-\delta-\frac{\sigma^2}{2})\cdot T\right)}_{\overset{..}{z^*}}\right]$$



$\mathbb{P}[S(T) > K] =$

$$= \mathbb{P}\left[Z < -\frac{1}{\sigma\sqrt{T}}\left(\ln\left(\frac{K}{S(0)}\right) - \left(\alpha - \delta - \frac{\sigma^2}{2}\right)\cdot T\right)\right]$$

$$= \mathbb{P}\left[Z < \boxed{\frac{1}{\sigma\sqrt{T}}\left(\ln\left(\frac{S(0)}{K}\right) + \left(\alpha - \delta - \frac{\sigma^2}{2}\right)\cdot T\right)}\right]$$

$$=: \hat{d}_2$$

$$(= \hat{d}_-)$$

$$\mathbb{P}[S(T) > K] = N(\hat{d}_2)$$

Q: What is the probab. that the put is in the money?

$$\mathbb{P}[S(T) < K] = 1 - \mathbb{P}[S(T) > K]$$
$$= 1 - N(\hat{d}_2)$$
$$= N(-\hat{d}_2)$$

$$\mathbb{P}[S(T) < K] = N(-\hat{d}_2)$$

Problem 3.2. Assume the Black-Scholes framework. You are given the following information for a stock that pays dividends continuously at a rate proportional to its price:

(i) The current stock price is \$250. $S(0) = 250$

(ii) The stocks volatility is 0.3. $\sigma = 0.30$

(iii) The continuously compounded expected rate of stock-price appreciation is 15%. $\alpha - \delta = 0.15$

Find the value s^* such that

$$\mathbb{P}[S(4) > s^*] = 0.05.$$

(a) \$861.65
(b) \$874.18
(c) \$889.94
(d) \$905.48
(e) None of the above.

$$S(T) = S(0)e^{(\alpha - \delta - \frac{\sigma^2}{2})\cdot T + \sigma\sqrt{T}\,Z}$$

Value z^* such that $\mathbb{P}[Z > z^*] = 0.05 \Rightarrow \underline{z^* = 1.645}$

then:

$$s^* = S(0)e^{(\alpha - \delta - \frac{\sigma^2}{2})\cdot T + \sigma\sqrt{T}\cdot z^*}$$

$$\Rightarrow s^* = 250 \cdot e^{(0.15 - \frac{0.09}{2})\cdot 4 + 0.3\sqrt{4}\cdot(1.645)}$$

$$\boxed{s^* = 1020.92}$$

16. You are given the following information about a nondividend-paying stock:

(i) The current stock price is 100.

(ii) The stock-price process is a geometric Brownian motion.
↳ Black·Scholes model or lognormal

(iii) The continuously compounded expected return on the stock is 10%.
$\alpha = 0.10$

(iv) The stock's volatility is 30%.

$\sigma = 0.30$

Consider a $\underbrace{\text{nine-month}}_{T = 3/4}$ $\underbrace{\text{125-strike}}_{K = 125}$ European call option on the stock.

Calculate the probability that the call will be exercised.

(A) 24.2%

(B) 25.1%

(C) 28.4%

(D) 30.6%

(E) 33.0%

$$\mathbb{P}[S(T) > K] = ?$$

$$\hat{d}_2 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(\alpha - \delta - \frac{\sigma^2}{2}\right)\cdot T\right]$$

$$\hat{d}_2 = \frac{1}{0.3\sqrt{3/4}}\left[\ln\left(\frac{100}{125}\right) + \left(0.10 - 0 - \frac{0.09}{2}\right)\cdot\frac{3}{4}\right]$$

$$\hat{d}_2 = -0.70$$

$$N(\hat{d}_2) = N(-0.70) = 1 - N(0.70)$$

$$= 1 - 0.7580 = 0.2420. \Rightarrow (A)$$