

## The Monte Carlo Method.

1.1. **The Strong Law of Large Numbers.** Let  $\{X_k : k = 1, 2, \dots\}$  be a sequence of **independent, identically distributed** random variables with a finite first moment

$$\mu_X := \mathbb{E}[X_1]$$

Then,

$$\frac{X_1 + X_2 + \dots + X_n}{n} \xrightarrow{n \rightarrow \infty} \mu_X$$

Moreover, let  $g$  be a function such that  $g(X_1)$  is well defined. *What does this mean?* Assume that  $\mathbb{E}[g(X_1)]$  is finite.

Then,

$$\frac{X_1 + X_2 + \dots + X_n}{n} \xrightarrow{n \rightarrow \infty} \mathbb{E}[g(X_1)]$$

1.2. **Monte Carlo.** The idea of Monte Carlo is to "marry" our ability to simulate pseudo-random numbers using software with the *Strong Law of Large Numbers*.

1.2.1. *Recipe.* Here is the algorithm for estimating "complicated" expectations.

- ☐ Draw simulated values from a specific distribution.
- ☐ Apply the function to the simulated values.
- ☐ Calculate the arithmetic average of the obtained quantities.

The result is a value **close to** the theoretical expectation.

1.2.2. **Precision.** Notice that our estimates are merely simulated realizations of a random variable (the **sample mean** in the language of statistics). The measure of how **close** we expect our estimate to be to the theoretical mean is its **variance** or the **standard deviation**.

We have

$$\begin{aligned} \text{Var} \left[ \frac{X_1 + X_2 + \dots + X_n}{n} \right] &= \frac{1}{n^2} \text{Var} [X_1 + X_2 + \dots + X_n] \\ &= \textbf{(independence)} \frac{1}{n^2} (\text{Var}[X_1] + \text{Var}[X_2] + \dots + \text{Var}[X_n]) \\ &= \textbf{(identical distribution)} \frac{1}{n^2} (n \text{Var}[X_1]) \\ &= \frac{\text{Var}[X_1]}{n}. \end{aligned}$$

Ergo,

$$SD \left[ \frac{X_1 + X_2 + \dots + X_n}{n} \right] = \frac{SD[X_1]}{\sqrt{n}}.$$

In words: **to increase our precision by a factor of  $\eta$ , we must increase our number of variates (draws) by a factor of  $\eta^2$ .**