M339G: February 23rd, 2024.

Singular Value Decomposition.

For a matrix A, its singular value decomposition is the factorization

$$\boxed{A = U \Sigma V^T}$$

where:

- U and V have orthonormal columns
- Σ is diagonal w/ positive entries

Outline:

Let A be an $n\times m$ matrix. Then, U is $n\times n$,
Σ is $n\times m$,
V is $m\times m$.

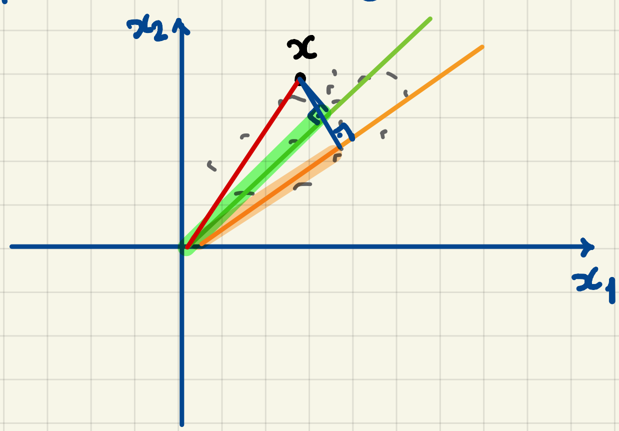
$$n\left\{\underset{m}{\begin{bmatrix} \\ A \\ \\ \end{bmatrix}}\right. = \underbrace{\begin{bmatrix} \\ U \\ \\ \end{bmatrix}}_{n}\Bigg\}n \; n\left\{\underset{m}{\begin{bmatrix} \sigma_1 & & 0 \\ & \sigma_2 & \\ 0 & & \ddots \ \sigma_m \\ & 0 & \end{bmatrix}}\right. \underset{m}{\begin{bmatrix} V^T \end{bmatrix}}\Big\}m$$

$$\Sigma$$

$$\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \cdots \geq \sigma_m \geq 0$$

Geometry.

The worth of the singular value decomposition is in figuring out which directions, i.e., linear combinations, take up most of the variability in the matrix A.



Implementation.

The algorithm is similar to working through the above geometry one line @ a time until we exhaust the dimension.

However, computationally, we start from

$$A = U \Sigma V^T \quad \Rightarrow \quad \boxed{A^T = V \Sigma^T U^T}$$

$$A^T A = (V \Sigma^T \underbrace{\boxed{U^T)(U}}_{I} \Sigma V^T)$$

(because orthonormal columns)

$$A^T A = V \underbrace{\boxed{\Sigma^T \Sigma}}_{\Sigma^2} V^T$$

(because Σ diagonal)

$$\boxed{\underbrace{A^T A}_{} = V \begin{bmatrix} \sigma_1^2 & & & \\ & \sigma_2^2 & & 0 \\ 0 & & \ddots & \\ & & & \sigma_m^2 \end{bmatrix} V^T}$$

Start w/ this side;

then, diagonalize AA^T to get $V \Sigma^2 V^T$;

then, get U by setting

$$AV = U\Sigma,$$

i.e., for v_i ... i^{th} column in V,

set $u_i = \frac{1}{\sigma_i} A v_i$ as the i^{th} column in U

Summary.

$$A = \sigma_1 \cdot u_1 \cdot v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_m \cdot u_m \cdot v_m^T$$

$$\underbrace{\sigma_1 \quad \sigma_2 \quad \sigma_m}_{\text{decreasing}}$$