M362K Probability
University of Texas at Austin
Practice Problems for In-Term Exam II
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Notes: This is a closed book and closed notes exam. The maximal score on this exam is 100 points. **There are many ways in which any single problem can be solved. The solutions herein are just one possible way to tackle the given problems.**

All written work handed in by the student is considered to be
their own work, prepared without unauthorized assistance.

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"I agree that I have complied with the UT Honor Code during my completion of this exam."

Signature:

2.1. **DEFINITIONS.**

Problem 2.1. (5 points) Complete the definition of a *random variable* on a finite outcome space below:

Let Ω be a finite outcome space. A *random variable* on Ω is ...

Solution: Let Ω be a finite outcome space. A *random variable* on Ω is any function $X : \Omega \to \mathbb{R}$.

2.2. **DEFINITIONS.**

Problem 2.2. (5 points) Write down the expression for the *standard normal density.*

Solution:

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \quad \text{for all } x \in \mathbb{R}$$

2.3. **TRUE/FALSE QUESTIONS.**

Problem 2.3. (2 pts) Let A, B and C be *pairwise independent events.* Then, they are necessarily *independent. True or false?*

Solution: FALSE
There were counterexamples in class and in the homework.

Problem 2.4. (2 pts) Two dice are rolled, the probability that the maximum (and **not** necessarily a strict maximum) of the upturned faces is achieved on the second die equals 1/2. *True or false?*

Solution: FALSE
Let X denote the outcome on the first die and let Y denote the outcome on the second die. Then, the probability we are considering is

$$\mathbb{P}[X \geq Y] = 1 - \mathbb{P}[X < Y] = 1 - \mathbb{P}[X \leq Y] + \mathbb{P}[X = Y].$$

Due to symmetry, it must be that $\mathbb{P}[X \leq Y] = \mathbb{P}[Y \leq X]$. So,

$$\mathbb{P}[X \geq Y] = \frac{1}{2}\left(1 + \frac{1}{6}\right).$$

2.4. **FREE RESPONSE PROBLEMS.**

Please, explain carefully all your statements and assumptions. Numerical results or single-word answers without an explanation (even if they're correct) are worth 0 points.

Problem 2.5. (10 points) Lady Eleanor Ward-Beaumont is learning to cook pasta. She tosses strands of (possibly) cooked spaghetti at the ceiling until one sticks. She is not a good cook (yet!), so we can assume that her tests are independent and that the probability of the pasta sticking to the ceiling is 0.2 in every trial. What is the probability that the strand of spaghetti sticks to the ceiling at exactly the 8^{th} try? Display your answer as a fully reduced fraction, please.

Solution: The total number of tries is geometric with success probability equal to 0.2. The probability we are looking for is

$$p_8 = \left(\frac{4}{5}\right)^7 \left(\frac{1}{5}\right) = \frac{4^7}{5^8}.$$

Problem 2.6. (25 points) A biased coin with the probability of "Heads" equal to 0.7 is tossed 1000 times. The number of "Heads" in the 1000 tosses is represented by the random variable X.

(i) (5 points) What is the distribution of the random variable X? You need to write down its name and the numerical values of all of its parameters.
(ii) (10 points) Write down the **exact** expression for the probability that more than 750 "Heads" have been observed.
(iii) (10 points) Use the normal approximation to estimate the above probability.

Solution:

(i) X is binomial with parameters $n = 1000$ and $p = 0.7$. Symbolically written: $X \sim b(1000, 0.7)$.
(ii)

$$\mathbb{P}[X > 750] = \sum_{i=751}^{1000} \mathbb{P}[X = i] = \sum_{i=751}^{1000} \binom{1000}{i} (0.7)^i \, (0.3)^{1000-i}.$$

(iii) **The solution below uses a calculator; in the exam, you will be given a problem designed so that a calculator is not needed.**
It is prudent to use the continuity correction. We get

$$\mathbb{P}[X > 750] \approx 1 - \Phi(3.48483) \approx 0.0002462249.$$

The number above is obtained using **R**. If you use the standard normal tables, your final answer will be 0.0003.

Problem 2.7. (25 points) Let the random variable X represent the number of odd numbers obtained in four rolls of a fair die.

(i) (5 points) What is the support of the random variable X?
(ii) (5 points) What is the distribution of the random variable X? State its **name** and **parameter values**.
(iii) (5 points) What is the probability mass function of the random variable X?
(iv) (5 points) Define the random variable $Y = |X - 1|$. What is the support of Y?
(v) (5 points) What is the probability mass function of the random variable Y?

Solution:

(i)

$$\{0, 1, 2, 3, 4\}.$$

(ii) It is simplest to write

$$X \sim b(n = 4, p = 1/2).$$

(iii)

$$p_X(0) = p_X(4) = 1/2^4 = 1/16, \quad p_X(1) = p_X(3) = 4/2^4 = 1/4, \quad p_X(2) = \binom{4}{2}/2^4 = 3/8.$$

(iv)

$$\{0, 1, 2, 3\}.$$

(v)

$$p_Y(0) = p_X(1) = 1/4, \quad p_Y(1) = p_X(0) + p_X(2) = 7/16, \quad p_Y(2) = p_X(3) = 1/4, \quad p_Y(3) = p_X(4) = 1/16.$$

Problem 2.8. (25 points) Seven Easter eggs are hidden in a backyard. Three of the seven eggs contain a toy train from the "Thomas the Tank Engine Series". A toddler is on an Easter egg hunt and only really cares about the "train eggs". He continues the egg hunt until he finds the first "train egg". Let the random variable X represent the number of regular eggs the toddler finds **before** discovering the first "train egg".

(i) (5 points) What is the support of the random variable X?
(ii) (10 points) What is the probability mass function of the random variable X?
(iii) (10 points) Now, imagine that the toddler wants to collect **all** of the "train eggs" and stops hunting eggs only after collecting all three. The eggs which remain in the backyard are the parents' responsibility to collect. Let Y denote the number of eggs collected by the parents. What is the support and the pmf of Y?

Solution:

(i)

$$\{0, 1, 2, 3, 4\}$$

(ii)

$$\begin{aligned}
p_X(0) &= 3/7,\\
p_X(1) &= (4/7)(3/6) = 2/7,\\
p_X(2) &= (4/7)(3/6)(3/5) = 6/35,\\
p_X(3) &= (4/7)(3/6)(2/5)(3/4) = 3/35,\\
p_X(4) &= (4/7)(3/6)(2/5)(1/4) = 1/35.
\end{aligned}$$

(iii) Evidently, the support of Y is $\{0, 1, 2, 3, 4\}$. One can argue using a symmetry argument that we can represent the random variable Y as $Y = 4 - X'$ where X' has the same p.m.f. as the r.v. X from above. So,

$$p_Y(0) = 1/35, \quad p_Y(1) = 3/35, \quad p_Y(2) = 6/35, \quad p_Y(3) = 2/7, \quad p_Y(4) = 3/7.$$

2.5. **MULTIPLE CHOICE QUESTIONS.**

Problem 2.9. (5 pts) You are given a TRUE/FALSE exam with 30 questions. Suppose that you need to answer 21 questions correctly in order to pass. You have no idea what the class is about and decide to toss a fair coin to answer all the questions; you circle TRUE if the outcome is tails and you circle FALSE if the outcome is heads. What is your estimate of the probability p that you manage to pass the exam using this strategy?

Hint: It is best to use the Normal Approximation to get the approximate probability.

(a) $p \leq 0.0005$

(b) $0.0005 < p \leq 0.006$
(c) $0.006 < p \leq 0.04$
(d) $0.04 < p$
(e) None of the above

Solution: (c)
The solution below uses a calculator; in the exam, you will be given a problem designed so that a calculator is not needed.
Let us denote the number of correct answers you get using the coin-toss strategy by X. Then, $X \sim b(30, 1/2)$. The mean of X is $30 \cdot \frac{1}{2} = 15$ and its variance is $30 \cdot \frac{1}{2} \cdot \frac{1}{2} = 7.5$. So, the standard deviation of X is $\sqrt{7.5} \approx 2.74$. We can express the probability p as

$$p = \mathbb{P}[X \geq 21] = \mathbb{P}[X \geq 20.5] = \mathbb{P}[\frac{X-15}{2.74} \geq \frac{20.5-15}{2.74}] \approx \mathbb{P}[\frac{X-15}{2.74} \geq 2].$$

By the DeMoivre-Laplace theorem, this probability is approximately equal to $\Phi(+\infty) - \Phi(2) = 1 - \Phi(2) = 1 - 0.97725 = 0.02275$.

Problem 2.10. (5 pts) An urn contains 20 balls, of which 19 are blue and one is red. If 7 of these balls are drawn, one at a time, with each selection being equally likely to be any of the balls that remain in the urn at the time, what is the probability that the red ball is chosen among those 7?
(a) 1/20
(b) 1/19
(c) 7/20
(d) 1/7
(e) None of the above

Solution: (c)
We are using the Multiplication Theorem.

$$\begin{aligned}\mathbb{P}[\{\text{red ball is chosen}\}] &= 1 - \mathbb{P}[\{\text{all the chosen balls are green}\}] \\ &= 1 - \frac{19}{20} \cdot \frac{18}{19} \cdot \frac{17}{18} \cdot \frac{16}{17} \cdot \frac{15}{16} \cdot \frac{14}{15} \cdot \frac{13}{14} \\ &= \frac{7}{20}.\end{aligned}$$

Problem 2.11. (5 pts) Find the probability of obtaining exactly two fives in six rolls of a fair die.
(a) $5^5/(2^3 \cdot 3^6)$
(b) $5^5/(2^6 \cdot 3^6)$
(c) $5^5/(2^6 \cdot 3^5)$
(d) $1/5$
(e) None of the above

Solution: (c)
The number of fives in six rolls of a die has the binomial distribution with parameters $n = 6$ and $p = 1/6$. The probability of getting exactly two fives is

$$\binom{6}{2} \cdot \frac{1}{6^2} \cdot \frac{5^4}{6^4} = 3 \cdot 5 \cdot \frac{5^4}{6^6} = \frac{5^5}{2^6 \cdot 3^5}.$$