M378K Introduction to Mathematical Statistics
Problem Set #5
Continuous distributions.

Problem 5.1. Source: Sample P exam, Problem #33.
The lifetime of a machine part has a continuous distribution on the interval $(0, 40)$ *with probability density function* f_X*, where*

$$f_X(x) \propto \frac{1}{(10+x)^2}$$

on the interval.

Calculate the probability that the lifetime of the machine part is less than 6.

Solution: Using the indicator function, we can write our probability density function as

$$f_X(x) = \kappa(10+x)^{-2}\mathbf{1}_{(0,40)}(x)$$

where κ is the proportionality constant. Evidently, we need to figure out κ first. To accomplish this, we use the fact that the area under the density must equal 1, i.e.,

$$\kappa \int_0^{40} (10+x)^{-2}\,dx = 1.$$

We have

$$\int_0^{40} (10+x)^{-2}\,dx = \frac{(10+x)^{-1}}{-1}\Big|_{x=0}^{40} = \frac{1}{10} - \frac{1}{50} = \frac{5-1}{50} = \frac{2}{25}\,.$$

So, $\kappa = \frac{25}{2}$. The probability we seek is

$$\tfrac{25}{2}\int_0^{6} (10+x)^{-2}\,dx = \frac{25}{2}\left(\frac{1}{10} - \frac{1}{16}\right) = \frac{25}{2}\cdot\frac{8-5}{80} = \frac{15}{32}\,.$$

Problem 5.2. Source: Sample P exam, Problem #419.
A customer purchases a lawnmower with a two-year warranty. The number of years before the lawnmower needs a repair is uniformly distributed on $[0, 5]$*. Calculate the probability that the lawnmower needs no repairs within* 4.5 *years after the purchase, given that the lawnmower needs no repairs within the warranty period*

Solution: Let the lifetime of the lawnmower be $T \sim U(0,5)$. We have

$$\mathbb{P}[T > 4.5 \,|\, T > 2] = \frac{\mathbb{P}[T > 4.5, T > 2]}{\mathbb{P}[T > 2]} = \frac{\mathbb{P}[T > 4.5]}{\mathbb{P}[T > 2]} = \frac{5-4.5}{5-2} = \frac{1}{6}\,.$$

Problem 5.3. *Consider a continuous random variable Y whose probability density function is given by*

$$f_Y(y) = 2y\mathbf{1}_{[0,1]}(y)$$

What is the expected value of this random variable?

Solution: Straight from the definition, we get

$$\mathbb{E}[Y] = \int_0^1 y(2y)\,dy = 2\int_0^1 y^2\,dy = 2\left(\frac{1^3}{3} - \frac{0^3}{3}\right) = \frac{2}{3}\,.$$