April 4th, 2025.

Moment Generating Function.

For a random variable Y,
and for an independent argument denoted by t, we define the moment generating function (mgf) of Y as this function of t:

$$M_Y(t) := \mathbb{E}\left[e^{Y \cdot t}\right]$$

for all t such that the expectation exists

Note:

- $M_Y(0) = \underline{1}$ $\Rightarrow$ @ least $t=0$ is in the domain
- We say that the mgf exists is if it's finite for t such that $|t|<b$ for some $b>0$.

Goal: To understand e^X w/ $X \sim Normal(mean = m, var = \nu^2)$

Recall: In terms of $Z \sim N(0,1)$,

$$\boxed{X = m + \nu \cdot Z}$$

Fact: $\boxed{M_Z(t) = e^{\frac{t^2}{2}}}$ for all $t \in \mathbb{R}$

$\Rightarrow$ For any normal X:

$$\begin{aligned}
M_X(t) = \mathbb{E}\left[e^{Xt}\right] &= \mathbb{E}\left[e^{(m+\nu \cdot Z)t}\right] \\
&= \mathbb{E}\left[e^{mt} \cdot e^{\nu t \cdot Z}\right] \\
&= e^{mt}\, \mathbb{E}\left[e^{\nu t \cdot Z}\right] \\
&= e^{mt} \cdot M_Z(\nu t) \\
&= e^{mt} \cdot e^{\frac{\nu^2 t^2}{2}} = e^{mt + \frac{\nu^2 t^2}{2}}
\end{aligned}$$

The lognormal distribution.

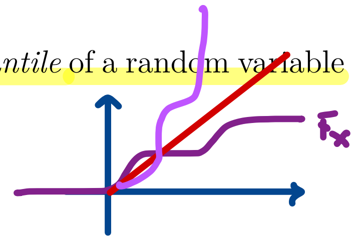
Definition 1.1. Let $X \sim Normal(mean = m, variance = \nu^2)$. Define the random variable $Y = e^X$. We say that the random variable Y is *lognormally distributed.*

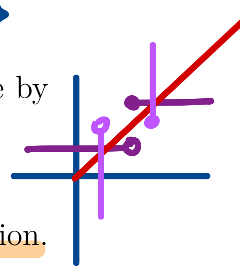
1.1. First properties.

- The expected value of the lognormally distributed random variable Y can be obtained as follows:
$$\boxed{\mathbb{E}[Y] = \mathbb{E}[e^X] = M_X(1) = e^{m+\frac{\nu^2}{2}}.}$$
- Let Y be a lognormal and let $a \neq 0$. Then, the random variable Y^a is also lognormal. *Note:* For $a = 0$, we get a degenerate random variable at 1 which can, technically, be interpreted as lognormal, but is not fun.
- Let Y_1 and Y_2 be independent and lognormally distributed. Then, Y_1Y_2 is also lognormal.

1.2. Quantiles.

Definition 1.2. For p such that $0 < p < 1$, we define the $100p^{th}$ *quantile* of a random variable X as any value π_p such that
$$\boxed{F_X(\pi_p-) \leq p \leq F_X(\pi_p).}$$
In particular, the 50^{th} quantile is referred to as the *median.*



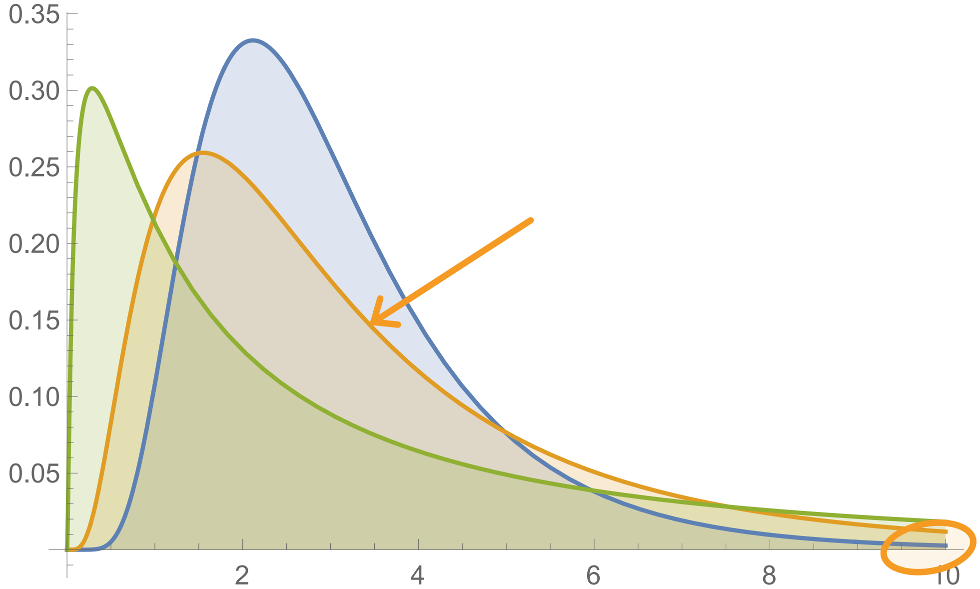


Note: When the random variable X is continuous, we can obtain the $100p^{th}$ quantile by simply solving for π_p in
$$F_X(\pi_p) = p.$$

Consider a probability p. Let z_p be the $100p^{th}$ quantile of the standard normal distribution. Let Y be lognormally distributed as above. My claim is that the value
$$y_p = e^{m+\nu z_p}$$
is the $100p^{th}$ quantile of Y. Let us simply verify this claim by calculating $F_Y(y_p)$. We have, with $Z \sim N(0,1)$,
$$F_Y(y_p) = \mathbb{P}[Y \leq y_p] = \mathbb{P}[e^X \leq y_p] = \mathbb{P}[e^{m+\nu Z} \leq e^{m+\nu z_p}].$$
Since the logarithmic function is increasing, we have that the above equals
$$F_Y(y_p) = \mathbb{P}[m + \nu Z \leq m + \nu z_p] = \mathbb{P}[Z \leq z_p] = p.$$
The above concludes our proof.

In particular, since the median of the standard normal distribution equals 0, the median of the lognormal distribution will be e^m.

Note: Since
$$e^m < e^{m+\frac{\nu^2}{2}}, \tag{1.1}$$
i.e., since the mean of a lognormal distribution always exceeds the median, we say that it's *right-skewed*. In fact, this is what its probability density function looks like.


0.35
0.30
0.25
0.20
0.15
0.10
0.05
2
4
6
8
10

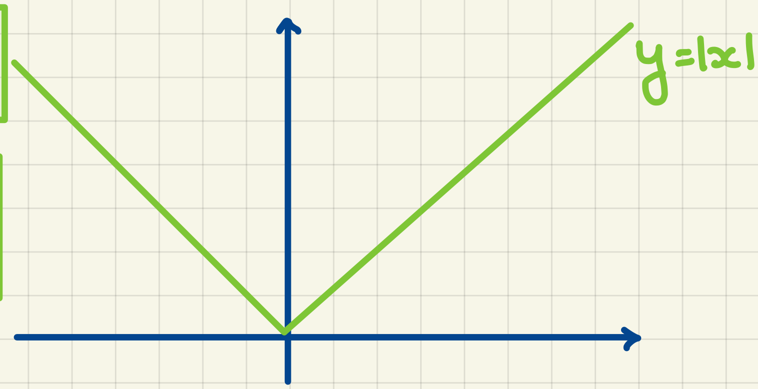
Jensen's Inequality.

Caveat: $\boxed{\mathbb{E}[e^X] \geq e^{\mathbb{E}[X]}}$

Theorem. Let X be a random variable, and let g be a convex function such that $g(X)$ is well-defined and $\mathbb{E}[g(X)]$ exists.

Then, $\boxed{\mathbb{E}[g(X)] \geq g(\mathbb{E}[X])}$

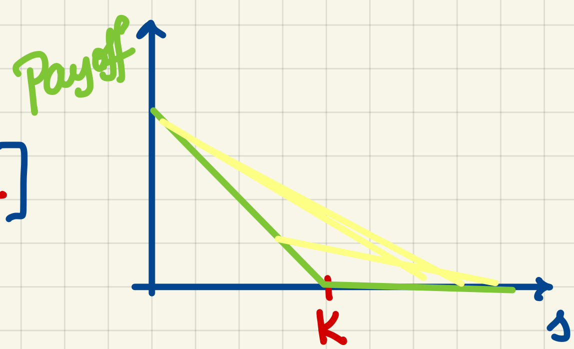
Examples. i. $\boxed{g(x) = |x|}$



$\boxed{\mathbb{E}[|X|] \geq |\mathbb{E}[X]|}$

ii. Consider a European put w/ strike K.

Its payoff function: $v_p(s) = (K-s)_+$



The expected payoff

$$\mathbb{E}[v_p(S(T))] = \mathbb{E}[(K-S(T))_+]$$

By Jensen's inequality

it's $\geq (K - \mathbb{E}[S(T)])_+$

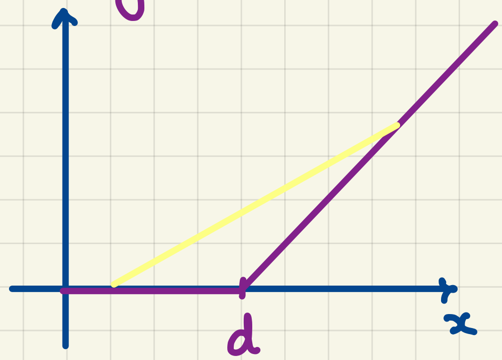
iii. In classical insurance:

$$\begin{cases} X \dots \text{(ground-up) loss, i.e., severity r.v.} \\ d \dots \text{deductible} \end{cases}$$

The insurer pays $(X-d)_+$, i.e., $g(x) = (x-d)_+$

By Jensen's inequality:

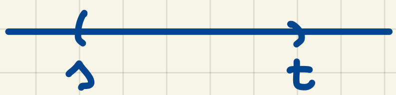
$$\mathbb{E}[(X-d)_+] \geq (\mathbb{E}[X]-d)_+$$



Log-Normal Stock Prices.

Temporarily fix a time horizon T.

$S(t), t \in [0,T]$ ... time-t stock price



Recall:

$$R(s,t) = \ln\left(\frac{S(t)}{S(s)}\right) \iff S(t) = S(s) \cdot e^{R(s,t)}$$

In particular: $R(0,T)$ ... realized return for $[0,T]$

$$\boxed{\text{We model realized returns as normal.}}$$

$$R(0,T) \sim \text{Normal}(\text{mean} = m, \text{var} = \nu^2)$$

$\Rightarrow$ $S(T)$ is lognormal

and $\boxed{\mathbb{E}^*[S(T)] = S(0)e^{m+\frac{\nu^2}{2}}}$

Market model:

- Riskless asset w/ ccrfir r
- Risky asset: a non-dividend paying stock w/ volatility σ