M339W: October 8th, 2021.

More on the Delta.

Example. [European Put]

K... strike

T... exercise date

Recall: the call Δ

$$\boxed{\Delta_c(\cdots) = e^{-\delta(T-t)}N(d_1(\cdots))} \checkmark$$

Put-call Parity.

$$\frac{\partial}{\partial s}\Big| \quad v_c(\cdots) - v_p(\cdots) = \underline{se^{-\delta(T-t)}} - \underline{Ke^{-r(T-t)}}$$

$$\Delta_c(\cdots) - \Delta_p(\cdots) = e^{-\delta(T-t)}$$

$$\begin{aligned}\Delta_p(\cdots) &= \Delta_c(\cdots) - e^{-\delta(T-t)}\\ &= e^{-\delta(T-t)}\cdot N(d_1(\cdots)) - e^{-\delta(T-t)}\\ &= e^{-\delta(T-t)}\left(N(d_1(\cdots)) - 1\right)\\ &= -e^{-\delta(T-t)}N(-d_1(\cdots))\end{aligned}$$

$$\boxed{\Delta_p(\cdots) = -e^{-\delta(T-t)}N(-d_1(\cdots))} < 0$$

Makes sense that it's negative since puts are short w.r.t. the underlying.

Unless otherwise stated in the examination question, assume:

- The market is frictionless. There are no taxes, transaction costs, bid/ask spreads or restrictions on short sales. All securities are perfectly divisible. Trading does not affect prices. Information is available to all investors simultaneously. Every investor acts rationally and there are no arbitrage opportunities.
- The risk-free interest rate is constant.
- The notation is the same as used in *Derivatives Markets*, by Robert L. McDonald.

When using the standard normal distribution table, do not interpolate.

- Use the nearest z-value in the table to find the probability. Example: Suppose that you are to find $\Pr(Z < 0.759)$, where Z denotes a standard normal random variable. Because the z-value in the table nearest to 0.759 is 0.76, your answer is $\Pr(Z < 0.76) = 0.7764$.
- Use the nearest probability value in the table to find the z-value. Example: Suppose that you are to find z such that $\Pr(Z < z) = 0.7$. Because the probability value in the table nearest to 0.7 is 0.6985, your answer is 0.52.

In *Derivatives Markets*, $\Pr(Z < x)$ is written as $N(x)$.

The standard normal density function is

$$\boxed{f_Z(x) = N'(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}} = \frac{e^{-x^2/2}}{\sqrt{2 \times 3.14159}} = \frac{e^{-x^2/2}}{2.50663}, \qquad -\infty < x < \infty.}$$

Let Y be a lognormal random variable. Assume that $\ln(Y)$ has mean m and standard deviation v. Then, the density function of Y is

$$f_Y(x) = \frac{1}{xv\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\ln(x) - m}{v}\right)^2\right], \qquad x > 0.$$

The distribution function of Y is

$$F_Y(x) = N\left(\frac{\ln(x) - m}{v}\right), \qquad x > 0.$$

Also,

$$\mathrm{E}[Y^k] = \exp\left(km + \frac{1}{2}k^2v^2\right),$$

which is the same as the moment-generating function of the random variable $\ln(Y)$ evaluated at the value k.

FORMULAS FOR OPTION GREEKS:

Delta (Δ)

Call: $e^{-\delta(T-t)}N(d_1)$,

Put: $-e^{-\delta(T-t)}N(-d_1)$

Gamma (Γ)

Call and Put: $\dfrac{e^{-\delta(T-t)}N'(d_1)}{S\sigma\sqrt{T-t}}$

Theta (θ)

Call: $\delta Se^{-\delta(T-t)}N(d_1)-rKe^{-r(T-t)}N(d_2)-\dfrac{Ke^{-r(T-t)}N'(d_2)\sigma}{2\sqrt{T-t}}$,

Put: Call Theta + $rKe^{-r(T-t)}-\delta Se^{-\delta(T-t)}$

Vega

Call and Put: $Se^{-\delta(T-t)}N'(d_1)\sqrt{T-t}$

Rho (ρ)

Call: $(T-t)Ke^{-r(T-t)}N(d_2)$,

Put: $-(T-t)Ke^{-r(T-t)}N(-d_2)$

Psi (ψ)

Call: $-(T-t)Se^{-\delta(T-t)}N(d_1)$,

Put: $(T-t)Se^{-\delta(T-t)}N(-d_1)$

(A) 7.32 million

(B) 7.42 million

(C) 7.52 million

(D) 7.62 million

(E) 7.72 million

8. You are considering the purchase of a 3-month 41.5-strike American call option on a nondividend-paying stock.

$\boxed{T = 1/4}$ $\boxed{K = 41.5}$

Never optimal to exercise it early. ⇒ Equivalent to a European call.

You are given:

(i) The Black-Scholes framework holds. ✓

(ii) The stock is currently selling for 40. $\boxed{S(0) = 40}$

(iii) The stock's volatility is 30%. $\sigma = 0.30$

(iv) The current call option delta is 0.5. $\Delta_C(S(0), 0) = 0.5$

No dividends!

$\boxed{N(d_1(S(0),0)) = 0.5}$

Determine the current price of the option.

$v_c(S(0),0) = ?$

(A) $20 - 20.453\int_{-\infty}^{0.15} e^{-x^2/2}dx$

(B) $20 - 16.138\int_{-\infty}^{0.15} e^{-x^2/2}dx$

(C) $20 - 40.453\int_{-\infty}^{0.15} e^{-x^2/2}dx$

(D) $16.138\int_{-\infty}^{0.15} e^{-x^2/2}dx - 20.453$

(E) $40.453\int_{-\infty}^{0.15} e^{-x^2/2}dx - 20.453$

?

$v_c(S(0),0) = S(0) \cdot e^{-\delta \cdot T} N(d_1(S(0),0)) - Ke^{-r\cdot T} \boxed{N(d_2(S(0),0))}$

$r = ?$

$N(d_1(S(0),0)) = 0.5$

$\Rightarrow \boxed{d_1(S(0),0) = 0}$

$\Rightarrow \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)\cdot T\right] = 0$

$$\ln\left(\frac{S(0)}{K}\right) + \left(r+\frac{\sigma^2}{2}\right)\cdot T = 0$$

$$\left(r+\frac{\sigma^2}{2}\right)\cdot T = -\ln\left(\frac{S(0)}{K}\right) = \ln\left(\frac{K}{S(0)}\right)$$

$$\left(r+\frac{0.09}{2}\right)\frac{1}{4} = \ln\left(\frac{41.5}{40}\right)$$

$$\Rightarrow \quad r = 4\cdot\ln\left(\frac{41.5}{40}\right) - 0.045 = 0.10226$$

$$d_2(S(0),0) = d_1(S(0),0) - \sigma\sqrt{T} = 0 - 0.3\sqrt{\tfrac{1}{4}} = -0.15 \checkmark$$

$$v_c(S(0),0) = 40(0.5) - \underline{41.5\,e^{-0.10226(1/4)}}\cdot N(-0.15)$$

$$= 20 - 40.453\cdot N(-0.15)$$

$$= 20 - 40.453\,(1-N(0.15))$$

$$= 40.453\,\underbrace{N(0.15)}_{} - \underline{20.453}$$

$$\int_{-\infty}^{0.15} f_Z(z)\,dz$$

$$\parallel$$

$$\int_{-\infty}^{0.15} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\,dz$$

$$\underbrace{\frac{40.453}{\sqrt{2\pi}} \int_{-\infty}^{0.15} e^{-\frac{z^2}{2}}\,dz}$$

$$\parallel$$

$$16.138 \quad \Rightarrow$$ We choose (D) as the correct answer!