UNIVERSITY OF TEXAS AT AUSTIN

HW Assignment 9

Hedging. Exchange options.

Please, provide your **complete solutions** to the following problems:

Problem 9.1. (15 points) There are two stocks present in our market: **S** and **Q**. Their current prices are $S(0) = 60$ and $Q(0) = 65$. Both stocks pay dividends continuously. The dividend yield for **S** is 0.02 while the dividend yield for **Q** equals 0.03.

You are given that for $t \geq 0$

$$\mathrm{Var}[\ln(S(t)/Q(t))] = 0.04t.$$

What is the Black-Scholes price of a one-year **exchange call** with underlying **S** and the strike asset **Q**?

Solution: In our usual notation, the volatility of the difference of the stocks' realized returns is $\sigma = 0.2$. So,

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right)T\right] = \frac{1}{0.2}\left[\ln\left(\frac{60}{65}\right) + \left(0.03 - 0.02 + \frac{0.04}{2}\right)\right] \\ &= 5\left[\ln\left(\frac{60}{65}\right) + 0.03\right] = -0.25, \\ d_2 &= -0.25 - 0.2 = -0.45. \end{aligned}$$

So,

$$N(d_1) = 1 - N(0.25) = 0.4013, \quad N(d_2) = 1 - N(0.45) = 0.3264.$$

Finally,

$$V_{EC}(0, \mathbf{S}, \mathbf{Q}) = 60e^{-0.02}(0.4013) - 65e^{-0.03}(0.3264) = 3.01225.$$

Problem 9.2. (15 points) Assume the Black-Scholes framework for the pair of stocks **S** and **Q**.

For the stock **S**, you are given that

- the current stock price is \$80 per share;
- the stock pays dividends in the amount $0.05S(t)\,dt$ during the time period $(t, t+dt)$;
- the stock's volatility is 0.2.

For the stock **Q**, you are given that

- the current stock price is \$50 per share;
- the stock pays no dividends;
- the stock's volatility is 0.4.

The correlation between the returns of **S** and **Q** is -0.4.

The continuously compounded, risk-free interest rate is 0.055.

What is the price of the maximum option on **S** and **Q** with exercise date at time-4?

Solution: The payoff of the maximum option can be expressed as follows

$$V_{max}(T) = \max(S(T), Q(T)) = Q(T) + \max(0, S(T) - Q(T)).$$

So, we can replicate our maximum option using the prepaid forward contract on **Q** and an exchange call option with underlying **S** and strike asset **Q**. Hence, taking into account that **Q** pays no dividends, the time-0 price of our maximum option is

$$Q(0) + V_{EC}(0, \mathbf{S}, \mathbf{Q})$$

In order to price the exchange call, we first need to find the "relative" volatility between **S** and **Q**. We get

$$\begin{aligned} \sigma^2 &= \sigma_S^2 + \sigma_Q^2 - 2\sigma_S\sigma_Q\rho \\ &= 0.04 + 0.16 - 2(0.2)(0.4)(-0.4) = 0.264 \quad \Rightarrow \quad \sigma = 0.5138. \end{aligned}$$

Next, we calculate the terms in the Black-Scholes price of the exchange call. We obtain

$$d_1 = \frac{1}{0.5138\sqrt{4}}\left[\ln\left(\frac{80}{50}\right) + \left(0 - 0.05 + \frac{0.5138^2}{2}\right)(4)\right] = 0.78,$$
$$d_2 = 0.78 - 0.5138\sqrt{4} = -0.25.$$

From the standard normal tables, we get

$$N(d_1) = 0.7823, \quad N(d_2) = 1 - N(0.25) = 1 - 0.5987 = 0.4013.$$

So,

$$V_{EC}(0, \mathbf{S}, \mathbf{Q}) = 80e^{-0.05(4)}(0.7823) - 50(0.4013) = 31.1744.$$

Our answer is 81.1744.

Problem 9.3. (20 points) Assume the Black-Scholes framework. A market maker writes an option (call it option I) on a non-dividend-paying stock whose price is equal to $S(0)$ and receives $V_I(0)$ for its sale at time-0. Moreover, the market-maker delta-gamma hedges the commitment using another option (call it option II) on the same stock and the stock itself. Denote the time-0 price of option II by $V_{II}(0)$.

(i) (2 points) Let the current gamma of the written option be equal to Γ_I and let the gamma of the option used for hedging be equal to Γ_{II}. What is the number of units of option II which the market-maker has in the total hedged portfolio?
Solution: Let the number of units of option II be denoted by n_{II}. To ensure gamma-neutrality, we have

$$-\Gamma_I + n_{II}\Gamma_{II} = 0 \quad \Rightarrow \quad n_{II} = \frac{\Gamma_I}{\Gamma_{II}}.$$

(ii) (3 points) In addition to the above notation, let the delta of option I be denoted by Δ_I and let the delta of option II be denoted by Δ_{II}. What is the number of shares of stock needed in the total hedged portfolio? Express this number in terms of deltas and gammas of the two stocks and nothing else.
Solution: Let the number of shares of stock be denoted by n_S. Then, delta-neutrality of the total hedged porfolio implies

$$-\Delta_I + n_{II}\Delta_{II} + n_S = 0 \quad \Rightarrow \quad n_s = \Delta_I - n_{II}\Delta_{II} = \Delta_I - \frac{\Gamma_I}{\Gamma_{II}}\Delta_{II}.$$

(iii) (3 points) Using the above notation, what is the time-0 value of the total hedged portfolio?
Solution:

$$-V_I(0) + \frac{\Gamma_I}{\Gamma_{II}}V_{II}(0) + (\Delta_I - \frac{\Gamma_I}{\Gamma_{II}}\Delta_{II})S(0).$$

(iv) (4 points) Denote the theta of option I by Θ_I and the theta of option II by Θ_{II}. Using the delta-gamma-theta approximation, approximate the value after one day of option I and option II if the stock price changes by ds. Feel free to denote one day by dt.
Solution: Let v_I be the value of option I and let v_{II} be the value of option II. The

$$v_I(S(0) + ds, dt) = V_I(0) + \Delta_I\, ds + \frac{1}{2}\Gamma_I(ds)^2 + \Theta_I\, dt$$
$$v_{II}(S(0) + ds, dt) = V_{II}(0) + \Delta_{II}\, ds + \frac{1}{2}\Gamma_{II}(ds)^2 + \Theta_{II}\, dt.$$

(v) (8 points) What is the approximate value after one day, i.e., at time dt, of the entire delta-gamma-neutral portfolio acording the the delta-gamma-theta approximation?
Solution: You notice that the total portfolio is delta and gamma neutral and use the delta-gamma-theta approximation directly on it. You get

$$-V_I(0) + \frac{\Gamma_I}{\Gamma_{II}}V_{II}(0) + (\Delta_I - \frac{\Gamma_I}{\Gamma_{II}}\Delta_{II})S(0) + (-\Theta_I + \frac{\Gamma_I}{\Gamma_{II}}\Theta_{II})\, dt.$$

Alternatively, you calculate te following

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\begin{aligned}
-v_I(S(0)+ds,dt) &+ \frac{\Gamma_I}{\Gamma_{II}}v_{II}(S(0)+ds,dt) + (\Delta_I - \frac{\Gamma_I}{\Gamma_{II}}\Delta_{II})(S(0)+ds) \\
&= -V_I(0) - \Delta_I\,ds - \frac{1}{2}\Gamma_I(ds)^2 - \Theta_I\,dt + \frac{\Gamma_I}{\Gamma_{II}}(V_{II}(0) + \Delta_{II}\,ds + \frac{1}{2}\Gamma_{II}(ds)^2 + \Theta_{II}\,dt) \\
&\quad + (\Delta_I - \frac{\Gamma_I}{\Gamma_{II}}\Delta_{II})(S(0)+ds) \\
&= -V_I(0) + \frac{\Gamma_I}{\Gamma_{II}}V_{II}(0) + (\Delta_I - \frac{\Gamma_I}{\Gamma_{II}}\Delta_{II})S(0) \\
&\quad + (-\Theta_I + \frac{\Gamma_I}{\Gamma_{II}}\Theta_{II})\,dt
\end{aligned}
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