M378K Introduction to Mathematical Statistics
Problem Set #8
Transformations of Random Variables.

Problem 8.1. *Let X be a continuous random variable with the cumulative distribution function denoted by F_X and the probability density function denoted by f_X.*
Let the random variable $\boxed{Y = 2X}$ have the p.d.f. denoted by f_Y. Then,

(a) $f_Y(x) = 2f_X(2x)$

(b) $f_Y(x) = \frac{1}{2}f_X\left(\frac{x}{2}\right)$

(c) $f_Y(x) = f_X(2x)$

(d) $f_Y(x) = f_X\left(\frac{x}{2}\right)$

(e) *None of the above*

$\boxed{\text{The CDF Method}}$

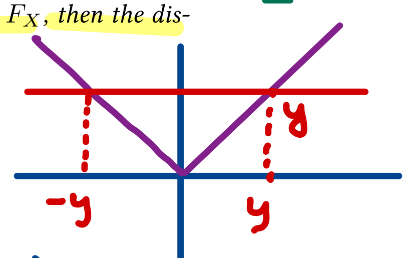
→: $y \in \mathbb{R}$: $F_Y(y) = \mathbb{P}[Y \le y] = \mathbb{P}[2X \le y] =$

$= \mathbb{P}[X \le \frac{y}{2}] = F_X\left(\frac{y}{2}\right)$

$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}F_X\left(\frac{y}{2}\right) = \frac{1}{2}f_X\left(\frac{y}{2}\right)$ □

Problem 8.2. *If the continuous random variable X has the distribution function F_X, then the distribution function of the random variable $Y = |X|$ equals*

$$F_Y(y) = ?$$



→: $F_Y(y) = \mathbb{P}[Y \le y] = \mathbb{P}[|X| \le y] =$

$= \mathbb{P}[-y \le X \le y]$

$= \mathbb{P}[X \le y] - \mathbb{P}[X \le -y] = F_X(y) - F_X(-y)$ □

$f_Y(y) = (f_X(y) + f_X(-y)) \cdot \mathbb{1}_{[0,\infty)}(y)$

Remark 8.1. *The goal is to figure out the distribution of the random variable*

$$X = g(Y_1, Y_2, \dots, Y_n)$$

where $Y_i, i = 1, \dots, n$ are a **random sample** *with a common density f_Y.*

Def'n. $Y_1, \dots, Y_n$ is a random sample from a dist'n D if:

(i) $Y_1, \dots, Y_n$ are independent

(ii) $Y_i \sim D$ for all $i = 1..n$

1. *Identify the objective: We want* f_X.
2. *Realize:* $f_X = F_X'$
3. *Recall the definition:* $F_X(x) = \mathbb{P}[X \le x] = \mathbb{P}[g(Y_1, \ldots, Y_n) \le x]$
4. *Identify the region* A_x *in* $\mathbb{R}^n$ *where*

$$g(y_1, \ldots, y_n) \le x$$

for every x, *i.e., express*

$$A_x = \{(y_1, \ldots, y_n) : g(y_1, \ldots, y_n) \le x\}$$

5. *Calculate*

$$F_X(x) = \int \cdots \int_{\mathbb{R}^n} \mathbf{1}_{A_x}(y_1, \ldots, y_n) f_Y(y_1) \ldots f_Y(y_n) dy_1 \ldots dy_n.$$

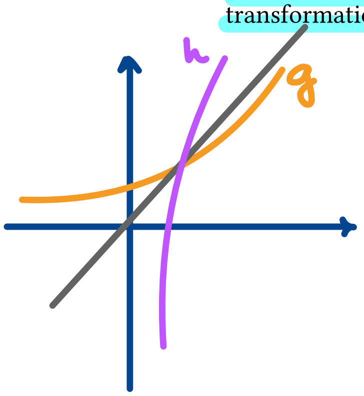
6. *Differentiate:* $f_X = F_X'$.
7. *Pat yourself on the back!*

Problem 8.3. **One-to-one transformations: Step-by-step** *Let* Y *be a random variable with density* f_Y. *Let* $g : \mathbb{R} \to \mathbb{R}$ *be a strictly increasing differentiable function. Define* $\boxed{\tilde{Y} = g(Y)}$. *What is the density function* $f_{\tilde{Y}}$ *of* $\tilde{Y}$ *expressed in terms of* f_Y *and* g?

1. Identify the objective: We want $f_{\tilde{Y}}$.
2. Realize: $f_{\tilde{Y}} = F_{\tilde{Y}}'$
3. Recall the definition:

$$F_{\tilde{Y}}(x) = \mathbb{P}[\tilde{Y} \le x] = \mathbb{P}[g(Y) \le x]$$

4. The function g is assumed to be **strictly increasing**. In which way can you modify the inequality in the probability you obtained above to *separate* the random variable Y from the transformation g?



There exists $h = g^{-1}$

this is g's INVERSE FUNCTION; h is also increasing

$$F_{\tilde{Y}}(x) = \mathbb{P}[g(Y) \le x] = \mathbb{P}[\cancel{h}(\cancel{g}(Y)) \le h(x)]$$

$$= \mathbb{P}[Y \le \boxed{h(x)}]$$

5. Express your result from above in terms of the c.d.f. F_Y of the r.v. Y.

$$F_{\tilde{Y}}(x) = \mathbb{P}[Y \leq \boxed{h(x)}] = F_Y(\boxed{h(x)})$$

6. Differentiate: $f_{\tilde{Y}} = F'_{\tilde{Y}}$.

$$f_{\tilde{Y}}(x) = \frac{d}{dx} F_{\tilde{Y}}(x) = \frac{d}{dx} F_Y(h(x)) \underset{\substack{\uparrow \\ \text{chain rule}}}{=} h'(x) \cdot f_Y(h(x))$$

Problem 8.4. *The time T that a manufacturing distribution system is out of operation is modeled by a distribution with the following c.d.f.*

$$F_T(t) = (1 - (2/t)^2)\mathbf{1}_{(2,\infty)}(t) = \begin{cases} 1 - 4 \cdot t^{-2}, & t > 2 \\ 0 & t \leq 2 \end{cases}$$

The resulting cost to the company is $Y = T^2$. Find the probability density function f_Y of the r.v. Y.

$\longrightarrow$: $g(t) = t^2,\ t > 2 \Rightarrow h(y) = \sqrt{y},\ y > 4 \Rightarrow h'(y) = \frac{1}{2\sqrt{y}},\ y > 4$

$$f_T(t) = \frac{d}{dt} F_T(t) = \frac{8}{t^3} \mathbb{1}_{(2,\infty)}(t)$$

$$f_Y(y) = \frac{1}{\cancel{2}\sqrt{y}} \cdot \frac{\cancel{8}\,4}{(\sqrt{y})^3} \mathbb{1}_{(4,\infty)}(y) = \frac{4}{y^2} \mathbb{1}_{(4,\infty)}(y) \quad \square$$

Problem 8.5. *What if h is $\boxed{\text{strictly decreasing}}$?*

$$F_{\tilde{Y}}(y) = \mathbb{P}[\tilde{Y} \leq y] = \mathbb{P}[g(Y) \leq y] \overset{!}{=} \mathbb{P}[Y \geq h(y)] =$$

$$= 1 - \mathbb{P}[Y \leq h(y)] = 1 - F_Y(h(y))$$

$$f_{\tilde{Y}}(y) = -\underbrace{h'(y)}_{<0} \cdot f_Y(h(y)) \quad :)$$

Problem 8.6. *The unifying formula?*

$$\boxed{f_{\tilde{Y}}(y) = |h'(y)| \cdot f_Y(h(y))}$$

<u>Def'n.</u> A function g is said to be (strictly) increasing if

$$x_1 < x_2 \Rightarrow g(x_1) \leq g(x_2)$$

<u>Example.</u> Let $Y \sim U(0,1)$

Let $\tilde{Y} = Y^{\varkappa}$ for $\varkappa > 1$

$$y \in (0,1): \; F_{\tilde{Y}}(y) = \mathbb{P}[Y^{\varkappa} \leq y]$$

$$= \mathbb{P}[Y \leq y^{1/\varkappa}] = F_Y(y^{1/\varkappa})$$

$\Rightarrow$ for $0 < y < 1$:

$$f_{\tilde{Y}}(y) = \frac{d}{dy} F_{\tilde{Y}}(y) = \frac{d}{dy} F_Y(y^{1/\varkappa}) =$$

$$= \frac{d}{dy}(y^{1/\varkappa}) = \frac{1}{\varkappa} y^{1/\varkappa - 1}$$

$\square$

Do not forget: it always makes sense to simply attack a problem without giving it a "label" .... Just look at the following problem:

Problem 8.7. *Let T_1 and T_2 be independent shifted geometric random variables with parameters* $\boxed{p_1 = 1/2}$ *and* $\boxed{p_2 = 1/3}$. *Compute* $\mathbb{E}[\min(T_1, T_2)]$.

$\longrightarrow$: $T = \min(T_1, T_2)$ ... counts the # of trials until the 1st success from either of the two coins

$\Rightarrow$ $T \sim$ shifted geometric

$p = ?$ ... the success probab. of this experiment

$p = \mathbb{P}[$@ least one of the two coins is a success$]$

$p = 1 - \mathbb{P}[$neither coin is a success$]$

$$p = 1 - \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{3} = 1 - \frac{1}{3} = \frac{2}{3}$$

$$\mathbb{E}[T] = ? = \frac{1}{p} = \frac{3}{2}$$

$\square$

Note:

$\mathbb{E}\Big/$ shifted geometric = 1 + geometric

$$\mathbb{E}[\text{shifted geometric}] = 1 + \mathbb{E}[\text{geometric}]$$
$$= 1 + \frac{q}{p} = \frac{p+q}{p} = \frac{1}{p}$$