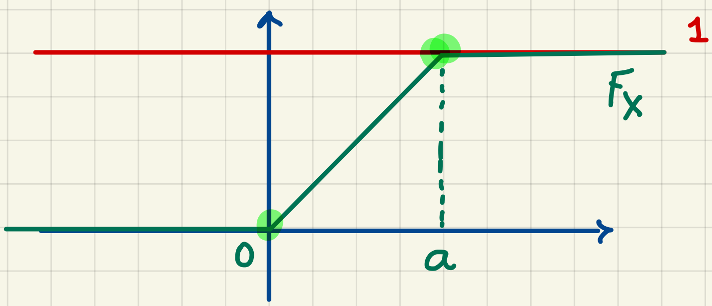
M358K: Applied Statistics.

Continuous Random Variables [Review].

Def'n. A random variable X is said to be continuous if its cumulative distribution function F_X is:

(i) continuous everywhere;

(ii) differentiable everywhere except @ at most countably many points.

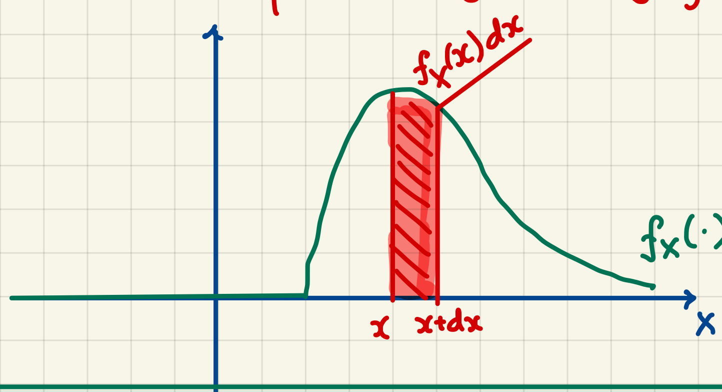


$X \sim Uniform(0, a)$

Def'n. Any function $f_X : \mathbb{R} \to [0, +\infty)$ such that

$f_X(x) = F_X'(x)$ for all x where the derivative exists

is called the probability density function (pdf) of X.



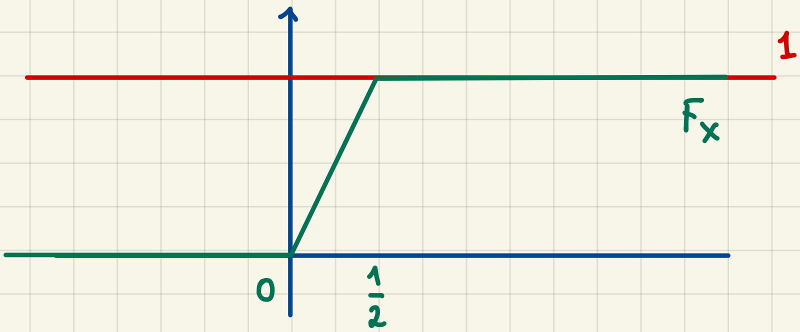
Q: $$\boxed{\mathbb{P}[a < X \leq b] = \int_a^b f_X(x)\,dx = F_X(b) - F_X(a)}$$

Q: X is continuous $\Rightarrow$ $\boxed{\mathbb{P}[X = x] = 0}$

Q: $$\boxed{\int_{-\infty}^{+\infty} f_X(x)\,dx = 1}$$

Q: Is it possible that $f_X(x) > 1$ for some x? Yes.

Example. $\boxed{X \sim U(0, \tfrac{1}{2})}$



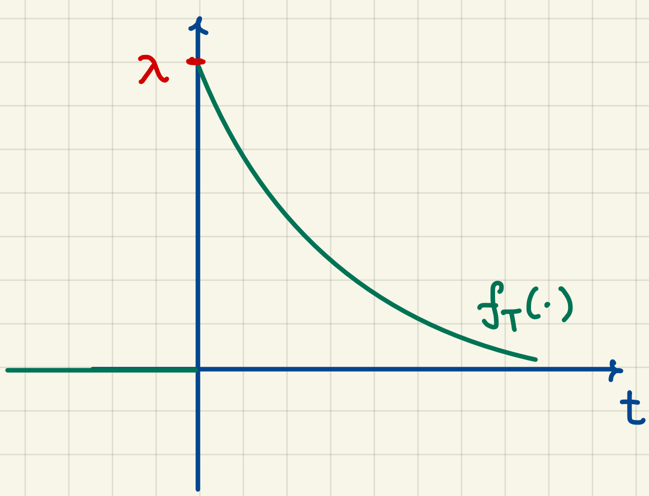
$$F_X(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ 2x & \text{for} \quad 0 \leq x \leq \frac{1}{2} \\ 1 & \text{for} \quad x > \frac{1}{2} \end{cases}$$

$$f_X(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ 2 & \text{for} \quad 0 < x < \frac{1}{2} \\ 0 & \text{for} \quad x > \frac{1}{2} \end{cases}$$

Example. Exponential Distribution. $T \sim Exp(\lambda)$

w/ a positive parameter λ

Its pdf is: $$f_T(t) = \begin{cases} \lambda e^{-\lambda t} & t > 0 \\ 0 & \text{otherwise} \end{cases}$$



Note: If $\lambda = 177$, then $f_T(t) \approx 177$ for $t \approx 0$

Expected Value.

Def'n. For a discrete r.v. X, its expected value (expectation/mean) is given by:

$$\boxed{\mathbb{E}[X] := \sum_{x} p_X(x) \cdot x}$$

when the sum exists

For a continuous r.v. X, its expected value (expectation/mean) is given by:

$$\boxed{\mathbb{E}[X] := \int_{-\infty}^{+\infty} x \cdot f_X(x)\,dx}$$

when the integral exists

Example. $T \sim Exp(\lambda)$

$$\Rightarrow \boxed{\mathbb{E}[T] = \frac{1}{\lambda}}$$

Def'n. For any r.v. X, its variance is defined as:

$$\boxed{Var[X] := \mathbb{E}\left[\left(X - \mathbb{E}[X]\right)^2\right]}$$

if it exists

Note: Set $\mu_X := \mathbb{E}[X]$

$$\Rightarrow Var[X] = \mathbb{E}\left[(X - \mu_X)^2\right] =$$

$$= \mathbb{E}\left[X^2 - 2 \cdot \mu_X \cdot X + \mu_X^2\right]$$

$$= \mathbb{E}[X^2] - 2\mu_X \cdot \underbrace{\mathbb{E}[X]}_{= \mu_X} + \mu_X^2$$

$$\boxed{Var[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2}$$

Def'n. The standard deviation of the r.v. X is:

$$\boxed{SD[X] = \sqrt{Var[X]}}$$