Problem 13.4. Assume that the stock price is modeled using the lognormal distribution. The stock pays no dividends. Under $\mathbb{P}^*$, the annual mean rate of return on the stock is given to be 12%. Also under $\mathbb{P}^*$, the median time$-t$ stock price is evaluated to be $S(0)e^{0.1t}$. What is the volatility parameter of this stock price?

→:

$$\cancel{S(0)}e^{(r-\frac{\sigma^2}{2})t} = \cancel{S(0)}e^{0.1t}$$

$$r-\frac{\sigma^2}{2} = 0.1$$

$$0.12-\frac{\sigma^2}{2} = 0.1$$

$$\frac{\sigma^2}{2} = 0.12-0.1 = 0.02 \Rightarrow \sigma^2 = 0.04 \Rightarrow \boxed{\sigma = 0.2}$$

Problem 13.5. The current stock price is \$100 per share. The stock price at any time $t > 0$ is modeled using the lognormal distribution. Assume that the continuously compounded risk-free interest rate equals 8%. Let its volatility equal 20%.

Find the value t^* at which the median stock price equals \$120, under the risk-neutral probability measure.

→: median time-t stock price $= S(0)e^{(r-\frac{\sigma^2}{2})\cdot t}$

$$1.2\cancel{0} = 1\cancel{00}\, e^{(0.08-\frac{0.04}{2})\cdot t^*}$$

$$t^* = \frac{\ln(1.2)}{0.06} = 3.039$$

Problem 13.6. The volatility of the price of a non-dividend-paying stock is 0.2. The stock price is modeled using a log-normal distribution. Under $\mathbb{P}^*$, the expected time-2 stock price is \$120. What is the median of the time-2 stock price under $\mathbb{P}^*$?

M339D: April 12th, 2024.

→: mean $= \mathbb{E}^*[S(T)] = \boxed{S(0)e^{rT}}$

$$S(T) = S(0)e^{(r-\frac{\sigma^2}{2})\cdot T + \sigma\sqrt{T}\cdot Z}$$

$Z \sim N(0,1)$

median $= \boxed{S(0)e^{(r-\frac{\sigma^2}{2})\cdot T}}$

$\sigma = 0.2$

$$\mathbb{E}^*[S(2)] = 120 = \boxed{S(0)e^{r\cdot 2}}$$

$$\boxed{S(0)e^{r\cdot 2}}\cdot e^{-\frac{\sigma^2}{\cancel{2}}\cdot\cancel{2}} = 120\cdot e^{-\sigma^2} = 120e^{-0.04} = 115.29$$

Problem 11.3.

$X \sim \text{Normal}(\text{mean} = -0.35, \text{sd} = 0.2)$

$Y = e^X$

$\mathbb{P}[Y > t^*] = 0.95 \qquad t^* = ?$

$\mathbb{P}[e^X > t^*] = 0.95$

$\mathbb{P}[X > \ln(t^*)] = 0.95$

$\mathbb{P}[-0.35 + 0.2 \cdot Z > \ln(t^*)] = 0.95$

$Z \sim N(0,1)$

$X \sim N(\mu, \sigma^2)$

$Z := \frac{X - \mu}{\sigma} \sim N(0,1)$

$\Leftrightarrow$

$X = \mu + \sigma \cdot Z$

LogNormal Tail Probabilities.

Example. Consider a non-dividend-paying stock. What is the probability that the stock outperforms a risk-free investment under the risk-neutral probability measure?

$\rightarrow$: The initially invested amount is: $S(0)$

- If it's the risk-free investment, the balance @ time T is:

$$S(0)e^{rT}$$

- If it's the stock investment, the wealth @ time T is:

$$S(T)$$

$$\boxed{\mathbb{P}^*[S(T) > S(0)e^{rT}] = ?}$$

This question is equivalent to the one of whether the profit for the stock investment is positive under $\mathbb{P}^*$.

In the Black-Scholes model:

$$\boxed{S(T) = S(0)e^{(r - \frac{\sigma^2}{2}) \cdot T + \sigma\sqrt{T} \cdot Z}} \qquad \text{w/ } Z \sim N(0,1)$$

$$\mathbb{P}^*\left[\cancel{S(0)}e^{(r-\frac{\sigma^2}{2})\cdot T+\sigma\sqrt{T}\cdot Z} > \cancel{S(0)}e^{rT}\right] =$$

$$= \mathbb{P}^*\left[(\cancel{r}-\tfrac{\sigma^2}{2})\cdot T+\sigma\sqrt{T}\cdot Z > \cancel{rT}\right]$$ (ln(·) is increasing)

$$= \mathbb{P}^*\left[\sigma\sqrt{T}\cdot Z > \frac{\sigma^2}{2}\cdot T\right] = \mathbb{P}^*\left[Z > \frac{\sigma\sqrt{T}}{2}\right]$$ (symmetry of $N(0,1)$)

$$= \mathbb{P}^*\left[Z < -\frac{\sigma\sqrt{T}}{2}\right] = N\left(-\frac{\sigma\sqrt{T}}{2}\right) \xrightarrow[T\to\infty]{} 0$$

□

Motivation.

Consider a European call option w/ strike K and exercise date T. Under the risk-neutral probability measure $\mathbb{P}^*$, what is the probability that the option is in-the-money on the exercise date?

→: In the Black-Scholes model:

$$\boxed{S(T) = S(0)e^{(r-\frac{\sigma^2}{2})\cdot T+\sigma\sqrt{T}\cdot Z} \quad \text{w/ } Z\sim N(0,1)}$$

We are calculating:

$$\mathbb{P}^*[\boxed{S(T)} > K] =$$

$$= \mathbb{P}^*\left[S(0)e^{(r-\frac{\sigma^2}{2})\cdot T+\sigma\sqrt{T}\cdot Z} > K\right]$$

$$= \mathbb{P}^*\left[e^{(r-\frac{\sigma^2}{2})\cdot T+\sigma\sqrt{T}\cdot Z} > \frac{K}{S(0)}\right]$$ (ln(·) is increasing)

$$= \mathbb{P}^*\left[(r-\tfrac{\sigma^2}{2})\cdot T+\sigma\sqrt{T}\cdot Z > \ln\left(\frac{K}{S(0)}\right)\right]$$

$$= \mathbb{P}^*\left[\sigma\sqrt{T}\cdot Z > \ln\left(\frac{K}{S(0)}\right) - (r-\tfrac{\sigma^2}{2})\cdot T\right]$$

$$= \mathbb{P}^*\left[Z > \boxed{\frac{1}{\sigma\sqrt{T}}\left(\ln\left(\frac{K}{S(0)}\right) - (r-\tfrac{\sigma^2}{2})\cdot T\right)}\right]$$ (symmetry of $N(0,1)$)

$$= \mathbb{P}^*\left[Z < \underbrace{\boxed{\frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + (r-\tfrac{\sigma^2}{2})\cdot T\right]}}_{=:d_2}\right]$$

$$\boxed{\mathbb{P}^*[S(T)>K] = N(d_2)}$$

Consequently: The probability that the otherwise identical put is in the money is

$$\mathbb{P}^*[S(T)<K] = 1-N(d_2) = N(-d_2)$$

Why?

Goal: Price calls and puts in the Black-Scholes model,

By risk-neutral pricing:

$$V_c(0) = e^{-rT}\boxed{\mathbb{E}^*[V_c(T)]}$$

$$\begin{aligned}\mathbb{E}^*[V_c(T)] &= \mathbb{E}^*[(S(T)-K)_+] \\ &= \mathbb{E}^*\left[(S(T)-K)\cdot \mathbb{I}_{[S(T)>K]}\right] \\ &= \mathbb{E}^*\left[S(T)\cdot \mathbb{I}_{[S(T)>K]}\right] - K\cdot \boxed{\mathbb{E}^*\left[\mathbb{I}_{[S(T)>K]}\right]}\end{aligned}$$

$$\mathbb{E}^*\left[\mathbb{I}_{[S(T)>K]}\right] = \mathbb{P}^*[S(T)>K] = N(d_2)$$