M339 J: April 2nd, 2021.

Poisson Thinning.

Thm. Let $N \sim Poisson(\lambda)$ be a counting random variable for some events of interest.
Suppose that independently of N, each event falls into a particular category indexed by $i = 1 \ldots m$ w/ probability p_i $(i = 1..m)$

Let N_i be the number of events of type i, for all $i = 1 \ldots m$

Then:
- $N_i \sim Poisson(\lambda_i = p_i \cdot \lambda)$ for all $i = 1..m$
- $N_1, N_2, \ldots, N_m$ are independent random variables

Sample STAM

$N \sim \text{Poisson}(\lambda = 12)$

111. The number of accidents follows a Poisson distribution with mean 12. Each accident generates 1, 2, or 3 claimants with probabilities 1/2, 1/3, and 1/6, respectively.

ASSUME INDEPENDENCE BETWEEN N and # of claimants.

Calculate the variance of the total number of claimants.

S... the total # of claimants

$Var[S] = ?$

(A) 20

(B) 25

(C) 30

(D) 35

(E) 40

for $i = 1,2,3$

N_i... # of accidents w/ i claimants

$N_i \sim \text{Poisson}(\lambda_i = \lambda \cdot p_i)$

$\Rightarrow$ $N_1 \sim \text{Poisson}(\lambda_1 = 6)$

$N_2 \sim \text{Poisson}(\lambda_2 = 4)$

$N_3 \sim \text{Poisson}(\lambda_3 = 2)$

112. In a clinic, physicians volunteer their time on a daily basis to provide care to those who are not eligible to obtain care otherwise. The number of physicians who volunteer in any day is uniformly distributed on the integers 1 through 5. The number of patients that can be served by a given physician has a Poisson distribution with mean 30.

Determine the probability that 120 or more patients can be served in a day at the clinic, using the normal approximation with continuity correction.

(A) $1-\Phi(0.68)$

(B) $1-\Phi(0.72)$

(C) $1-\Phi(0.93)$

(D) $1-\Phi(3.13)$

(E) $1-\Phi(3.16)$

$$\Rightarrow \quad S = 1\cdot N_1 + 2\cdot N_2 + 3\cdot N_3$$

$$Var[S] = Var[N_1 + 2N_2 + 3N_3] =$$

$\boxed{N_1, N_2, N_3 \text{ independent by our thinning thm}}$

$$= Var[N_1] + Var[2N_2] + Var[3N_3]$$

$$= Var[N_1] + 4\,Var[N_2] + 9\,Var[N_3]$$

$$= \lambda_1 + 4\lambda_2 + 9\lambda_3$$

$$= 6 + 4\cdot 4 + 9\cdot 2 = 40 \quad \blacksquare$$

Negative Binomial Dist'n.

Binomial Coefficients.

For $n, k \in \mathbb{N}_0$ w/ $n \geq k$:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

We can generalize to $x \in \mathbb{R}_+$ and $k \in \mathbb{N}_0$.

If $x > k-1$, then

$$\boxed{\binom{x}{k} = \frac{\Gamma(x+1)}{\Gamma(k+1)\cdot\Gamma(x-k+1)}}$$

A Modelling Problem.

Imagine a sequence of Bernoulli trials such that:

- each trial results in success or failure;
- the probability of success in every trial is p;
- the trials are independent

The experiment continues until a fixed number r of successes is achieved.

N… random variable denoting the total # of failures before the r^{th} success.

Q: What is the pmf of N?

$\longrightarrow$: for $k = 0, 1, 2, \ldots$

$$p_N(k) := \mathbb{P}[\underbrace{N=k}] = \binom{r+k-1}{k} p^r (1-p)^k$$

What's the probab. of seing k failures before reaching the r^{th} success?

The very last trial must be a success.

In this class, and in the STAM tables, we use the parametrisation:

$$\boxed{p = \frac{1}{1+\beta}} \quad \text{for some } \beta > 0$$

With this parametrisation:

✓ $$p_N(k) = \binom{r+k-1}{k}\left(\frac{1}{1+\beta}\right)^r\left(\frac{\beta}{1+\beta}\right)^k \quad k = 0, 1, 2, \ldots$$

Returning to the generalization of the binomial coefficients, we generalize the pmf to be well-defined for any pair of parameters $r > 0$ and $\beta > 0$

We write $N \sim NegBinomial(r, \beta)$.

✓ • $\mathbb{E}[N] = r \cdot \beta$

✓ • $Var[N] = r \cdot \beta(1+\beta) > \mathbb{E}[N]$

✓ • $P_N(z) = \left(1 - \beta(z-1)\right)^{-r}$

Note: The geometric dist'n is a special case: $r = 1$.

We write $N \sim Geometric(p)$.

Do you remember (or can you find) one nifty property of the geometric dist'n?