M339D: November 16th, 2022.

Black-Scholes Pricing.

- Under the risk-neutral probability measure $\mathbb{P}^*$:

$$S(T) = S(0)e^{(r-\frac{\sigma^2}{2})\cdot T + \sigma\sqrt{T}\cdot Z} \quad w/ \quad Z \sim N(0,1)$$

- By the risk-neutral pricing principle:

$$\boxed{V(0) = e^{-rT}\,\mathbb{E}^*[\underbrace{V(T)}_{\text{the payoff of a European option}}]}$$

Call Options.

$$V_c(0) = S(0)\cdot N(d_1) - PV_{0,T}(K)\cdot N(d_2)$$

w/ $$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(r+\frac{\sigma^2}{2}\right)\cdot T\right]$$

and $$d_2 = d_1 - \sigma\sqrt{T}$$

N... cdf of N(0,1)

Put Options.

By put-call parity:

$$V_c(0) - \boxed{V_p(0)} = S(0) - PV_{0,T}(K)$$

$$\begin{aligned}
V_p(0) &= V_c(0) - S(0) + PV_{0,T}(K) \\
&= S(0)N(d_1) - PV_{0,T}(K)\cdot N(d_2) \\
&\quad - S(0) \quad + PV_{0,T}(K) \\
&= S(0)\underbrace{(N(d_1)-1)}_{-N(-d_1)} + PV_{0,T}(K)\underbrace{(1-N(d_2))}_{N(-d_2)}
\end{aligned}$$

symmetry of N(0,1)

$$\boxed{V_p(0) = PV_{0,T}(K)N(-d_2) - S(0)N(-d_1)}$$

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Problem Set 10

Black-Scholes pricing.

Problem 10.1. Let the stock prices be modeled using the lognormal distribution. Under the risk-neutral probability measure, the mean stock price at time−1 equals 120 and the median stock price 115. What is the risk-neutral probability that the time−1 stock price exceeds 100?

→: In general: $\mathbb{E}^*[S(T)] = S(0)e^{rT}$ ... the mean

$S(T) = S(0)e^{(r-\frac{\sigma^2}{2})\cdot T + \sigma\sqrt{T}Z}$ $Z \sim N(0,1)$

the median ... $S(0)e^{(r-\frac{\sigma^2}{2})\cdot T}$

$$\frac{\text{mean}}{\text{median}} = e^{\frac{\sigma^2}{2}\cdot T} \underset{T=1}{=} \frac{120}{115} \Rightarrow \frac{\sigma^2}{2}\cdot T = \ln\left(\frac{120}{115}\right)$$

$$\sigma^2 = 2\ln\left(\frac{120}{115}\right)$$

$$\sigma = \sqrt{2\ln\left(\frac{120}{115}\right)}$$

$$\sigma = 0.2918$$

$$\mathbb{P}^*[Z \leq 0] = \frac{1}{2}$$

$$\mathbb{P}^*[\sigma\sqrt{T}\cdot Z \leq 0] = \frac{1}{2}$$

$$\mathbb{P}^*\left[(r-\frac{\sigma^2}{2})\cdot T + \sigma\sqrt{T}\cdot Z \leq (r-\frac{\sigma^2}{2})\cdot T\right] = \frac{1}{2}$$

$$\mathbb{P}^*\left[e^{(r-\frac{\sigma^2}{2})\cdot T + \sigma\sqrt{T}\cdot Z} \leq e^{(r-\frac{\sigma^2}{2})\cdot T}\right] = \frac{1}{2}$$

$$\mathbb{P}^*\left[\underbrace{\boxed{S(0)e^{(r-\frac{\sigma^2}{2})\cdot T + \sigma\sqrt{T}\cdot Z}}}_{= S(T)} \leq S(0)e^{(r-\frac{\sigma^2}{2})\cdot T}\right] = \frac{1}{2}$$

$$\mathbb{P}^*\left[S(T) \leq \underbrace{S(0)e^{(r-\frac{\sigma^2}{2})\cdot T}}_{\text{median}}\right] = \frac{1}{2}$$

$$\mathbb{E}[g(X)] \neq g(\mathbb{E}[X])$$

$$\mathbb{P}^*[S(1) > 100] = \mathbb{P}^*\Big[\underbrace{S(0)\, e^{(r-\frac{\sigma^2}{2})}}_{\text{median}} e^{\sigma \cdot Z} > 100\Big]$$

$$= \mathbb{P}^*\left[115\, e^{\sigma \cdot Z} > 100\right]$$

$$= \mathbb{P}^*\left[e^{\sigma \cdot Z} > \frac{100}{115}\right]$$

$$= \mathbb{P}^*\left[\sigma Z < \ln\left(\frac{115}{100}\right)\right] = \mathbb{P}^*\Big[Z < \overbrace{\frac{1}{0.2918}\ln\left(\frac{115}{100}\right)}^{0.48}\Big]$$

$$= N(0.48) = 0.6844$$

$\square$

Problem 10.2. (5 pts) Let the stochastic process $S = \{S(t); t \geq 0\}$ denote the stock price. The stock's
r ~~rate of appreciation is~~ 10% while its volatility is 0.30 Then,

(a) $Var[\ln(S(t))] = 0.3t$

(b) $Var[\ln(S(t))] = 0.09t^2$

(c) $Var[\ln(S(t))] = 0.09t$

(d) $Var[\ln(S(t))] = 0.09$

(e) None of the above.

→ :

$$S(t) = S(0)\,e^{(r-\frac{\sigma^2}{2})\cdot t + \sigma\sqrt{t}\cdot Z} \qquad Z \sim N(0,1)$$

deterministic

$$\ln(S(T)) = \boxed{\ln(S(0)) + (r-\frac{\sigma^2}{2})\cdot t} + \sigma\sqrt{t}\cdot Z$$

$$Var[\ln(S(T))] = Var[\sigma\sqrt{t}\cdot Z] = \sigma^2\cdot t\cdot \underbrace{Var[Z]}_{=1} = 0.09t$$

□

Problem 10.3. (5 pts) Assume the Black-Scholes framework. Let the current price of a non-dividend-paying stock be equal to $S(0) = 95$ and let its volatility be equal to 0.35. Consider a European call on that stock with strike 100 and exercise date in 9 months. Let the risk-free continuously compounded interest rate be 6% per annum.

Denote the price of the call by $V_C(0)$. Then,

(a) $V_C(0) < \$5.20$

(b) $\$5.20 \leq V_C(0) < \$7.69$

(c) $\$7.69 \leq V_C(0) < \$9.04$

(d) $9.04 \leq V_C(0) < \$11.25$

(e) None of the above.

$\longrightarrow$:

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)\cdot T\right]$$

$$d_1 = \frac{1}{0.35\sqrt{3/4}}\left[\ln\left(\frac{95}{100}\right) + \left(0.06 + \frac{(0.35)^2}{2}\right)\cdot \frac{3}{4}\right]$$

$$d_1 = \underline{0.13079 \approx 0.13} \implies N(0.13) = 0.5517$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.13079 - 0.35\sqrt{3/4} = \underline{-0.1723} \approx -0.17$$

$$\implies N(-0.17) = 0.4325$$

$$V_c(0) = S(0)\cdot N(d_1) - Ke^{-rT}\cdot N(d_2)$$

$$V_c(0) = 95 \cdot 0.5517 - 100\, e^{-0.06(3/4)} \cdot 0.4325$$

$$V_c(0) = \underline{11.06} \quad \square$$