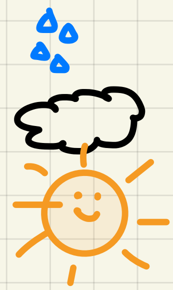
M339D : February 14th, 2025.

Finite Probability Spaces.

... serve as environments for the possible paths that asset prices can take.

e.g.,

$$S(T) \sim \begin{cases} 120 & \text{w/ probab. } 1/6 \\ 80 & \text{w/ probab. } 1/2 \\ 50 & \text{w/ probab. } 1/3 \end{cases}$$



Q: What is the expected put payoff w/ strike equal to 105?

$\longrightarrow$:

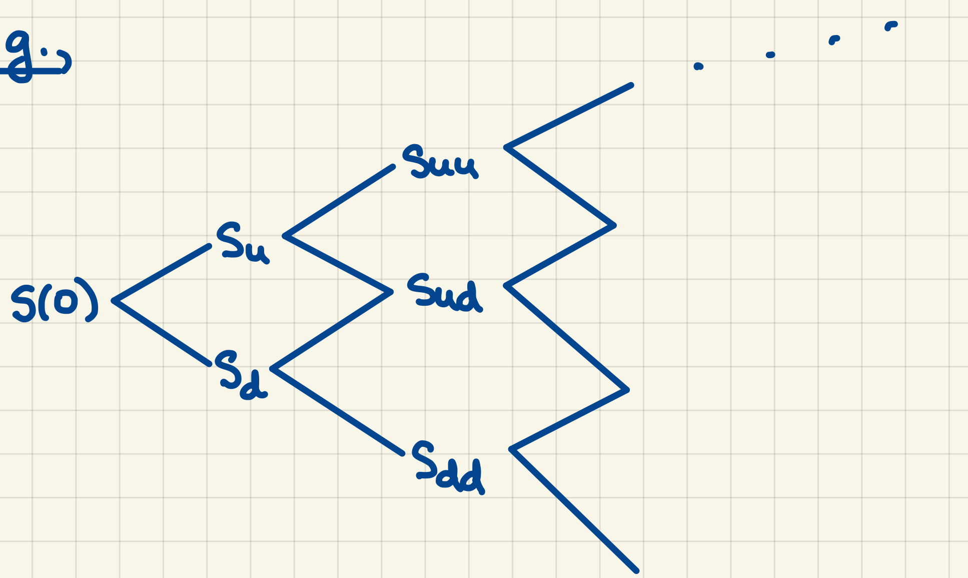
$$\boxed{V_P(T) = (K - S(T))_+}$$

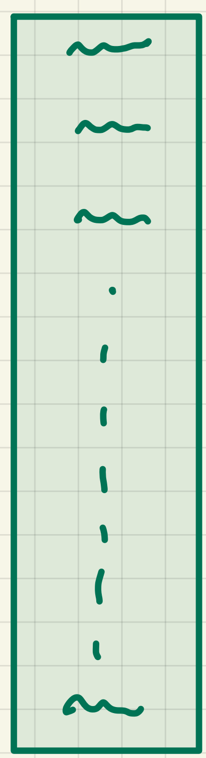
$$V_P(T) \sim \begin{cases} 0 & \text{w/ probab. } 1/6 \\ 25 & \text{w/ probab. } 1/2 \\ 55 & \text{w/ probab. } 1/3 \end{cases}$$

$$\mathbb{E}[V_P(T)] = 25 \cdot \frac{1}{2} + 55 \cdot \frac{1}{3} = \dots \quad \square$$

Caveat: $\mathbb{E}[g(X)] \neq g(\mathbb{E}[X])$

e.g.,





All the finitely many scenarios are called **states of the world**.

We assume that:

- each can happen, i.e., probab > 0

and

- they exhaust all possibilities, i.e., $\sum$ probab = 1

Arbitrage Portfolio.

Def'n. An **arbitrage portfolio** is a portfolio whose profit is:

- nonnegative in ALL states of the world, i.e., w/ probability 1,

and

- strictly positive in AT LEAST ONE state of the world, i.e., w/ probability > 0.

> Unless it's specified otherwise in a specific problem/example, we assume NO ARBITRAGE.

Law of the Unique Price.

Assume that the payoffs of two static portfolios A and B are **equal**, i.e.,

$$\boxed{V_A(T) = V_B(T)}$$

random variables

T... time horizon (temporarily fixed)

Claim. $\boxed{V_A(0) = V_B(0)}$

Proof. Assume, to the contrary, that

$$V_A(0) \neq V_B(0)$$

Without loss of generality,

$$\boxed{\underbrace{V_A(0)}_{\text{relatively cheap}} < \underbrace{V_B(0)}_{\text{relatively expensive}}}$$

Diagnosis.

Propose an arbitrage portfolio:

- Long Portfolio A
- Short Portfolio B

} Total Portfolio

Verify that this is, indeed, an arbitrage portfolio.

- Initial Cost (Total Portfolio) = $V_A(0) - V_B(0) < 0$

Inflow of money @ time 0.

- Payoff (Total Portfolio) = $V_A(T) - V_B(T) = 0$

$$\text{Profit} = \text{Payoff} - FV_{0,T}(\text{Initial Cost})$$
$$= 0 - FV_{0,T}(\underbrace{V_A(0) - V_B(0)}_{<0}) > 0$$

Indeed, this is an ARBITRAGE PORTFOLIO!

$\lightning$ $\Rightarrow\Leftarrow$

$\square$

Corollary. If $V_A(T) \geq V_B(T)$, then $\boxed{V_A(0) \geq V_B(0)}$