M339J: February 21^{st}, 2022.

Generating Functions.

Def'n. Let X be a random variable.

- The moment generating f'tion (mgf) of X is denoted by M_X and defined by:

$$\boxed{M_X(t) := \mathbb{E}\left[e^{t\cdot X}\right]}$$

for all $t \in \mathbb{R}$ such that the expectation exists.

Note: At least $t=0$ is in the domain!

- The probability generating f'tion (pgf) of X is denoted by P_X and defined by:

$$\boxed{P_X(s) := \mathbb{E}\left[s^X\right]}$$

for all $s > 0$ and such that the expectation exists.

Note: At least $s=1$ is in the domain!

Note:

$$s \longleftrightarrow e^t$$

$$P_X(s) = M_X(\ln(s))$$

and

$$M_X(t) = P_X(e^t)$$

Sums of Independent Random Variables.

Theorem. Let $\{X_1, X_2, \dots, X_n\}$ be independent random variables.

Define $S = X_1 + X_2 + \dots + X_n$.

Then,

$$\boxed{M_S(t) = \prod_{i=1}^{n} M_{X_i}(t)} \quad \text{and} \quad \boxed{P_S(s) = \prod_{i=1}^{n} P_{X_i}(s)}$$

Consequences.

1. Let $X_i \sim \text{Normal}(\text{mean} = \mu_i, \text{variance} = \sigma_i^2)$, $i = 1..n$.
   Assume they are independent.

$$M_{X_i}(t) = e^{\mu_i t + \frac{\sigma_i^2 t^2}{2}} \quad \text{for all } i$$

By our Theorem:

$$M_S(t) = \prod_{i=1}^{n} M_{X_i}(t)$$
$$= \prod_{i=1}^{n} e^{\mu_i \cdot t + \frac{\sigma_i^2 \cdot t^2}{2}}$$
$$= \exp\left(\sum_{i=1}^{n}\left(\mu_i \cdot t + \frac{\sigma_i^2 \cdot t^2}{2}\right)\right)$$
$$= \exp\left(\underbrace{\left(\sum_{i=1}^{n}\mu_i\right)}_{=:\mu} \cdot t + \underbrace{\left(\sum_{i=1}^{n}\sigma_i^2\right)}_{=:\sigma^2}\left(\frac{t^2}{2}\right)\right)$$

$$\boxed{M_S(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}}$$

$\Rightarrow$ $\boxed{S \sim \text{Normal}(\text{mean} = \mu = \mu_1 + \mu_2 + \cdots + \mu_n,\ \text{var} = \sigma^2 = \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2)}$

2. Let $X_i \sim \text{Gamma}(\alpha_i, \theta)$, $i = 1..n$ be independent r.v.s.

$$M_{X_i}(t) = (1 - \theta t)^{-\alpha_i} \quad \text{for } t < \frac{1}{\theta}$$

By our theorem:

$$M_S(t) = \prod_{i=1}^{n} (1 - \theta t)^{-\alpha_i} = (1 - \theta t)^{-\sum_{i=1}^{n}\alpha_i}$$

$\Rightarrow$ $\boxed{S \sim \text{Gamma}(\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_n, \theta)}$

Parametric Distributions.

Terminology. A parametric distribution is a collection of distribution functions which share the same finite and fixed family of parameters.

Challenge. Try to think of a non-parametric model!

Terminology. A parametric dist'n is said to be a scale distribution if:

> when any of random variables in that family is multiplied by a positive constant the resulting r.v. remains in the same family.

Example. $X \sim \text{Exponential}(\theta)$

Let $\kappa > 0$ be a constant.

Define $\boxed{\tilde{X} := \kappa \cdot X}$

$\rightarrow$: The support of $\tilde{X}$ is $[0, +\infty)$.

Let $x > 0$:

$$F_{\tilde{X}}(x) = \mathbb{P}[\tilde{X} \leq x] = \mathbb{P}[\kappa \cdot X \leq x]$$

$$= \mathbb{P}\left[X \leq \frac{x}{\kappa}\right] = 1 - \exp\left(-\frac{\frac{x}{\kappa}}{\theta}\right)$$

$$= 1 - \exp\left(-\frac{x}{\kappa \cdot \theta}\right)$$

$$\Rightarrow \boxed{\tilde{X} \sim \text{Exponential}(\tilde{\theta} = \kappa \cdot \theta = \text{mean})}$$