Name:

M339D=M389D Introduction to Actuarial Financial Mathematics
University of Texas at Austin
Solution: Practice Problems for In-Term One
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1.1. DEFINITION.

Problem 1.1. Provide the definition of the **arbitrage portfolio**.

Solution: See your notes.

1.2. TRUE/FALSE QUESTIONS.

Problem 1.2. (2 points) Put-call parity applies only to European-style options. *True or false?*

Solution: **TRUE**

Problem 1.3. (2 points) The strike price at which the European call and the otherwise identical European put have the same premiums is the future value (on the exercise date) of the intial price of the underlying of the two options. *True or false?*

Solution: TRUE

Problem 1.4. (2 points) An agent is **only** allowed to long a forward contract if he/she is willing to take physical delivery of the underlying asset.

Solution: FALSE
It is possible to have *cash settlement* on the delivery date if the forward contract stipulates so.

Problem 1.5. (2 points) Denote the continuously compounded, risk-free interest rate by r and denote the equivalent annual effective interest rate by i. Then, $\ln(1+i) = r$. *True or false?*

Solution: TRUE

Problem 1.6. (2 pts) Two dice are rolled, the single most probable sum of the numbers of the upturned faces is 7. *True or false?*

Solution: TRUE

Problem 1.7. (2 pts) Consider a portfolio consisting of the following four European options with the same expiration date T on the underlying asset S:

- one long call with strike 40,
- two long calls with strike 50,
- one short call with strike 65.

Let $S(T) = 69$. Then, the payoff from the above position at time T is less than 60.

Solution: FALSE
The payoff is

$$(69-40)+2(69-50)-(69-65)=63.$$

Problem 1.8. Let the cumulative distribution function F_X of a random variable X satisfy the following conditions:

- $F_X(0) = 0$;
- $F_X(1) = 0.2$;
- $F_X(2) = 1$;
- F_X is linear on $(0, 1)$ and $(1, 2)$;
- F_X is continuous.

You use the inverse transform method to simulate values of X. The values given by the random number generator are

$$0.1 \quad 0.6 \quad 0.9.$$

Which simulated values from X were drawn based on the above three values?

Solution: From the given information about F_X, we can conclude that $F_X(0.5) = 0.1$. Therefore, the value of the random number generator equal to 0.1 maps into the simulated value 0.5 of the random variable X.

Similarly, $F_X(1.5) = 0.6$. So, the value of the random number generator equal to 0.6 maps into the simulated value 1.5 of the random variable X.

The last mapping is tricky. Between 1 and 2, the function F_X satisfies

$$F_X(x) = 0.2 + 0.8(x - 1).$$

Now, we need to solve for x in

$$F_X(x) = 0.9 \quad \Leftrightarrow \quad 0.2 + 0.8(x - 1) = 0.9 \quad \Leftrightarrow \quad 0.8(x - 1) = 0.7 \quad \Leftrightarrow \quad x - 1 = 0.875 \quad \Leftrightarrow \quad x = 1.875$$

1.3. MULTIPLE CHOICE QUESTIONS.

Problem 1.9. (5 pts) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be two functions given by

$$f(x) = 2x - 10$$

and

$$g(x) = \begin{cases} \min(x, 7) & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Then, $g(f(7))$ equals ...

(a) -4
(b) 0
(c) 4
(d) 7
(e) None of the above

Solution: (c)

Problem 1.10. *Source: Sample P exam, Problem #176.*
In a group of health insurance policyholders, 20% have high blood pressure and 30% have high cholesterol. Of the policyholders with high blood pressure, 25% have high cholesterol. A policyholder is randomly selected from the group. Calculate the probability that a policyholder has high blood pressure, **given** that the policyholder has high cholesterol.

(a) 1/6
(b) 1/5
(c) 1/4
(d) 2/3
(e) 5/6

Solution: (a)
Let E be the event containing all the policyholders with a high blood preasure and let F be the event which contains all the policyholders with high cholesterol. We are given the following

$$\mathbb{P}[E] = 0.2, \quad \mathbb{P}[F] = 0.3, \quad \mathbb{P}[F|E] = 0.25.$$

Then, the conditional probability we are looking for equals

$$\mathbb{P}[E|F] = \frac{\mathbb{P}[E \cap F]}{\mathbb{P}[F]} = \frac{\mathbb{P}[F|E]\mathbb{P}[E]}{\mathbb{P}[F]} = \frac{(0.25)(0.2)}{0.3} = \frac{1}{6}\,.$$

Problem 1.11. Harry plays a simple lottery in which the winnings are distributed as follows:

- \$5 with probability 0.2,
- \$10 with probability 0.4,
- \$20 with probability 0.4.

It turns out that Harry has to pay a fee to collect his winnings. If the actual amount he wins is smaller than \$9, then the fee is defined to equal the amount that Harry won – thus, he walks away with nothing. If the actual amount he wins is between \$9 and \$15, he does not have to pay anything in fees and gets a bonus of \$4. If the actual amount he wins is larger than \$15, then he pays the \$15-fee and pockets the remainder. What is the expected value of the net amount Harry collects?

(a) 3
(b) 6.4
(c) 7.6
(d) 15
(e) None of the above.

Solution: (c)
The actual amount that Harry gets is

- \$0 with probability 0.2,
- \$14 with probability 0.4,
- \$5 with probability 0.4.

So, his expected winnings are

$$14(0.4) + 5(0.4) = 7.6$$

Problem 1.12. Hermione sells short one share of non-dividend-paying stock. The stock is currently valued at \$80 per share.The continuously compounded risk-free interest rate is 0.04. Hermione intends to close the short sale in one year. What is the final stock price for which Hermione will break even?

Solution: In our usual notation, the break-even point is

$$S(0)e^{rT} = 80e^{0.04(1)} = 80e^{0.04} = 83.26486.$$

Problem 1.13. The current market price of widgets is \$4 per widget. The widget factory plans to sell their next batch of 100 widgets in half a year. The total aggregate costs of production of widgets will be equal to \$350.

The factory enters 100 short forward contracts on widgets for delivery in half a year. The forward price is \$4.20 per widget.

What is the factory's profit if the final price of widgets in half a year ends up being \$4.40?

(a) 30
(b) 50
(c) 70
(d) 90
(e) None of the above.

Solution: (c)
The factory will sell the widgets per the forward contract for \$420 total. The total aggregate costs are given to be \$350. Hence, the profit is \$70.

Problem 1.14. Maryam bakes batches of cupcakes for a cupcake convention. She buys forward 21 pounds of raspberries from a local farmer at the forward price of \$5.60 per pound.

She projects to bake 336 cupcakes and sell each for \$3. The total and aggregate non-raspberry costs of baking the cupcakes are \$200.

If the market price of raspberries on the day of the cupcake convention is \$5.40, what is Maryam's profit?

(a) \$690.40
(b) \$694.60
(c) \$890.40
(d) \$894.60
(e) None of the above.

Solution: (a)

$$336 \times 3 - 21 \times 5.60 - 200 = 690.40.$$

Problem 1.15. The **writer** of a call option has ...

(a) an obligation to sell the underlying asset at the strike price.
(b) a right, but **not** an obligation, to sell the underlying asset at the strike price.
(c) an obligation to buy the underlying asset at the strike price.
(d) a right, but **not** an obligation, to buy the underlying asset at the strike price.
(e) None of the above.

Solution: (a)

Problem 1.16. (5 points) Roger owns a cow named Elsie. Her estimated worth today is \$3,750. Roger enters into a forward agreement with Harry to sell him Elsie the cow in 6 months for \$4,000. On the delivery date, Roger changes his mind and wants cash settlement instead. Harry agrees. They look into the "Bovine Blue Book" and realize that Elsie's worth on that date is \$3,500.

What is the cash flow that has to take place as part of the cash settlement?

(a) \$500 from Roger to Harry
(b) \$500 from Harry to Roger
(c) \$250 from Roger to Harry
(d) \$250 from Harry to Roger
(e) None of the above.

Solution: (b)

Problem 1.17. (5 points) A farmer produces one thousand crates of apples. The total and aggregate costs of production are \$48,000. The farmer enters a forward contract for the entire harvest to hedge at a forward price of \$69 per crate at harvest time.

The market price of apples at harvest time is \$70 per crate.

What is the farmer's profit?

(a) 1000 loss
(b) 1000 gain
(c) 21000 gain
(d) 22000 gain
(e) None of the above.

Solution: (c)

$$69000 - 48000 = 21000$$

Problem 1.18. (5 points) **Pancakes, Inc.** produces strawberry pancakes for the pancake festival. It longed a forward contract on 100 pounds of strawberries at \$2.50 per pound to be delivered to the festival and added to the pancakes. According to the contract with the organizers, the total fixed revenue will be \$6,000 for the pancakes produced with the above strawberries. Costs other than strawberries total \$1200.

On the morning of the pancake festival, the market price of strawberries is \$2.25 per pound.

Find the company's profit.

(a) 2300
(b) 1550
(c) 2550
(d) 1500
(e) None of the above.

Solution: (e)

$$6000 - 1200 - 250 = 4550$$

Problem 1.19. (5 points) The current price of stock a certain type of stock is \$80. The premium for a 6−month, at-the-money call option is \$5.84. Let the continuously compounded, risk-free interest rate be 0.04. What is the break-even point of this call option?

(a) \$80
(b) \$85.72
(c) \$85.84
(d) \$85.96
(e) None of the above.

Solution: (d)

The break-even point is

$$80 + 5.84e^{0.04/2} = 85.958$$

Problem 1.20. The following nine-month European put options are available in the market:

- a \$120-strike put with the premium of \$12,
- a \$127-strike put with the premium of \$10,

The continuously compounded, risk-free interest rate is 0.04.

You construct a portfolio by buying the \$127-strike put and writing the \$120-strike put. Which of the following statements is correct?

(a) The minimum **profit** of this portfolio is -9.06.
(b) The minimum **profit** of this portfolio is -2.06.
(c) The minimum **profit** of this portfolio is -7.
(d) This is an arbitrage portfolio.
(e) None of the above.

Solution: (d)
The initial cost of this portfolio is $10 - 12 = -2$. The minimum payoff of this portfolio happens for the final asset price above 127. It is equal to 0. So, the minimal gain of this portfolio is

$$0 - (-2)e^{0.03} = 2.06.$$

Problem 1.21. Let the current price of a non-dividend-paying stock equal 100. The forward price for delivery of this stock in 3 months equals \$101.26

Consider a \$90-strike, six-month put option on this stock whose premium today equals \$2.22.

What will the profit of this long put option be if the stock price at expiration equals \$96?

(a) About \$2.28 loss.
(b) About \$2.22 loss.
(c) About \$2.28 gain.
(d) About \$2.22 gain.
(e) None of the above.

Solution: (a)
The option is out-of-the money at expiration, so its owner suffers a loss of the future value of its premium

$$2.22 \times \left(\frac{101.26}{100}\right)^2 = 2.2763.$$

Problem 1.22. (5 points) A derivative security has the payoff function given by

$$v(s) = (s^2 - 100)_+$$

Its exercise date is in one year. You model the time-1 price of the underlying asset as

$$S(1) \sim \begin{cases} 9.5 & \text{with probability } 1/4 \\ 10 & \text{with probability } 1/2 \\ 11 & \text{with probability } 1/4 \end{cases}$$

The continuously compounded, risk-free interest rate is 10%. What is the expected **payoff** of the above derivative security?

(a) 5.25

(b) 2.81

(c) 0.31

(d) 1.42

(e) None of the above.

Solution: (a)

$$(9.5^2 - 100)_+ \left(\frac{1}{4}\right) + (10^2 - 100)_+ \left(\frac{1}{2}\right) + (11^2 - 100)_+ \left(\frac{1}{4}\right) = 21 \left(\frac{1}{4}\right) = 5.25$$