MAY 7TH

Monte Carlo Simulation of Option Prices

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Monte Carlo

- **Theorem** [Law of Large Numbers (LLN)]
  Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of i.i.d. random variables and let $g:\mathbb{R}\longrightarrow\mathbb{R}$ be a function such that $\mu := \mathbb{E}[g(X_1)]$ exists. Then,

$$\frac{1}{n}(g(X_1)+g(X_2)+\cdots+g(X_n)) \longrightarrow \mu \text{ as } n\longrightarrow\infty.$$

- The **idea** behind Monte Carlo simulation (integration) is the following:

  Suppose that you are interested in a quantity y which can be expressed as $y=\mathbb{E}[g(X)]$ for some random variable X and a real function g. Then, draw the simulated values $x_1, x_2, \ldots, x_n$ from the distribution X. To approximate y, calculate the average

$$\frac{1}{n}(g(x_1)+g(x_2)+\cdots+g(x_n))$$

- The **accuracy** of this approximation can be shown to be of the order $1/\sqrt{n}$. This means that you have to *quadruple* the number of simulations n if you wish to *double* the precision of the approximation.

Monte Carlo Simulation of Option Prices

Introduction

- Let $V(S_T, T)$ represent the payoff of the option on S with maturity T.
- Then, the time-0 Monte Carlo price $V(S_0, 0)$ is given by

$$V(S_0, 0) = \frac{1}{N} e^{-rT} \sum_{i=1}^{N} V(S_T^i, T)$$

where r denotes the interest rate and $S_T^i, i = 1, 2, \ldots, N$ are simulated time$-T$ stock prices

MONTE CARLO SIMULATION OF OPTION PRICES

Example. A European call

$$v(s,T) = (s-K)_+$$ ... terminal condition, i.e., the payoff

LogNormal stock-price simulations

$$s_i = S(0)e^{(r-\delta-\frac{\sigma^2}{2})T+\sigma\sqrt{T}\cdot z_i}$$

RISK-NEUTRAL PRICING!

w/ $z_i = N^{-1}(u_i)$ and u_i a draw from the unit uniform dist'n.

The Monte Carlo method produces this "approximation" for the call price:

$$\bar{C} = e^{-rT}\cdot\frac{1}{N}\sum_{i=1}^{N}(s_i-K)_+ = e^{-rT}\cdot\frac{1}{N}\sum_{i=1}^{N}(s_i-K)_+$$

N... constant # of simulation

In order to double the accuracy (i.e., to halve the std dev of the simulated price), we need to quadruple the number of draws.

Accuracy of Monte Carlo

- To gauge the accuracy of the Monte Carlo method, we calculate the standard deviation of the estimate.
- Let $\bar{C}_n$ denote the Monte Carlo estimate based on n trials. Then,

$$\bar{C}_n = \frac{1}{n}\sum_{i=1}^{n} C(\tilde{S}_i)$$

where $C(\tilde{S}_i), i = 1, \ldots, n$ denote the call prices obtained using the simulated values $\tilde{S}_i$.

- Then,

$$\sigma_n^2 = \frac{1}{n}\sigma_C^2$$

where σ_n denotes the standard deviation of the call price based on n draws and σ_C denotes the standard deviation of the call price based on a single draw

Accuracy of Monte Carlo (cont'd)

- So,

$$\sigma_n = \frac{1}{\sqrt{n}}\sigma_C$$

which implies that, in order to double the accuracy, one needs to quadruple the number of draws.

Arithmetic Asian option

- Here, the arithmetic average of the stock price replaces the actual stock price at expiration.
- The recipe is to simulate paths of the stock - only at discrete times used in the averaging - and calculate the payoff for each of them.
- Do the above N times (N will depend on the desired accuracy).
- Calculate the average of the discounted values you obtained in the previous step.

Arithmetic Asian option (cont'd)

- Let the Asian option have the payoff

$0 \quad \frac{T}{M} \quad \frac{2T}{M} \quad \dots \quad \frac{kT}{M} \quad \dots \quad T$

$$V(S,T) = \left(\frac{1}{M}\sum_{k=1}^{M} S_{\frac{kT}{M}} - K\right)^{+}$$

- For each simulated path $S^i, i = 1, \dots, N$, you find:

$$S^i_{\frac{T}{M}} = S_0 \exp\{(r - \delta - \frac{1}{2}\sigma^2)\frac{T}{M} + \sigma\sqrt{\frac{T}{M}} z^i_1\}$$

$$\dots$$

$$S^i_{\frac{kT}{M}} = S_{\frac{(k-1)T}{M}} \exp\{(r - \delta - \frac{1}{2}\sigma^2)\frac{T}{M} + \sigma\sqrt{\frac{T}{M}} z^i_k\}$$

$$\dots$$

where $z^i_k, k = 1, \dots, M$ are draws from the standard normal distribution

- Calculate

$$C_i = e^{-rT} V(S^i, T)$$

Arithmetic Asian option (cont'd)

- Do the above N times for $i = 1, \ldots, N$ and get C_i for $i = 1, \ldots, N$
- Your estimate will be

$$\bar{C}_A = \frac{1}{N}\sum_{i=1}^{N} C_i$$

- See Figure 19.2 and Table 19.3

Monte Carlo Simulation of Option Prices

Efficiency Improvement

CONTROL VARIATE METHOD

The Idea

Use the price of a correlated product which HAS an analytic pricing formula to estimate the error in pricing by simulation of the actual option of interest.

The Implementation

Simulate X; get $\widehat{\mathbb{E}[v(X)]} = \bar{\mu}_v$

get $\widehat{\mathbb{E}[g(X)]} = \bar{\mu}_g$

simulated

any random variable

Assume that we do know how to explicitly get $\mu_g = \mathbb{E}[g(X)]$ (the theoretical mean!)

But we do not know of an expression for $\mu_v = \mathbb{E}[v(X)]$

Instead of looking at:

$\bar{\mu}_v$ as the estimator, we look at

$$\bar{\mu}_v + \hat{a}\left(\bar{\mu}_g - \mu_g\right)$$

The optimal $\hat{a}$ is:

$$\hat{a} = -\frac{\mathrm{Cov}[v(X), g(X)]}{\mathrm{Var}[g(X)]}$$

Illustration

- The illustrative example is pricing arithmetic Asian options using geometric Asian options
- Read the relevant appendix to Chapter 14
- Let $\bar{A}$ be the simulated price of an arithmetic Asian option and let $\bar{G}$ be the simulated price of a geometric Asian option. It is **essential** that the same simulated paths be used to calculate both $\bar{A}$ and $\bar{G}$.
- Let G represent the true geometric price (see Appendices 14.B and 19. A)
- Choosing $\hat{a} = -1$, we get a price estimate

$$A^* = \bar{A} + (G - \bar{G})$$

with a (possibly) smaller variance since

$$Var[A^*] = Var[\bar{A}] + Var[\bar{G}] - 2Cov[\bar{A}, \bar{G}]$$

Illustration (cont'd)

- As we have seen already, it is optimal to choose

$$\hat{a} = -\frac{Cov[A(S), G(S)]}{Var[G(S)]}$$

with $A(S)$ and $G(S)$ the true payoffs as functions of the paths of the stock price S

- However, it is not always clear how to calculate the above
- One remedy is to **estimate** it as

$$\hat{a} = -\frac{Cov[\bar{A}, \bar{G}]}{Var[\bar{G}]}$$

which you should recognize as the (negative) of the slope when one runs the linear regression of $\bar{G}$ versus $\bar{A}$

For Questions **58** and **59**, you are to assume the Black-Scholes framework.

Let $C(K)$ denote the Black-Scholes price for a 3-month K-strike European call option on a nondividend-paying stock. "THEORETICAL" /ANALYTIC PRICES

Let $\hat{C}(K)$ denote the Monte Carlo price for a 3-month K-strike European call option on the stock, calculated by using 5 random 3-month stock prices simulated under the risk-neutral probability measure.

SIMULATED PRICES

You are to estimate the price of a 3-month 42-strike European call option on the stock using the formula

$\beta = -\hat{a}$ from McDonald

$$C^*(42) = \hat{C}(42) + \beta[C(40) - \hat{C}(40)],$$

$$\beta = \frac{Cov[\hat{C}(42), \hat{C}(40)]}{Var[\hat{C}(40)]}$$

where the coefficient β is such that the variance of $C^*(42)$ is minimized.

You are given:

(i) The continuously compounded risk-free interest rate is 8%. $r = 0.08$

(ii) $C(40) = 2.7847.$ THE ANALYTIC PRICE

(iii) Both Monte Carlo prices, $\hat{C}(40)$ and $\hat{C}(42)$, are calculated using the following 5 random 3-month stock prices:

33.29, 37.30, 40.35, 43.65, 48.90

We do not have the actual covariances & variances

=> we estimate!

58. Based on the 5 simulated stock prices, estimate β.

(A) Less than 0.75

(B) At least 0.75, but less than 0.8

(C) At least 0.8, but less than 0.85

(D) At least 0.85, but less than 0.9

(E) At least 0.9

59. Based on the 5 simulated stock prices, compute $C^*(42)$.

(A) Less than 1.7

(B) At least 1.7, but less than 1.9

(C) At least 1.9, but less than 2.2

(D) At least 2.2, but less than 2.6

(E) At least 2.6

$C^*(42) = ?$

$\hat{C}(42) = e^{-0.08 \cdot \frac{1}{4}} \cdot 1.71 = 1.676$

$\hat{C}(40) = e^{-0.02} \cdot 2.58 = 2.529$

$C^*(42) = 1.676 + 0.7642(2.7847 - 2.529)$

$= 1.87$