M378K: January 30th, 2026.

Variance.

Def'n. The variance of a random variable Y is defined as

$$\boxed{Var[Y] := \mathbb{E}\left[(Y - \mathbb{E}[Y])^2\right]} \quad \text{If "finite"}$$

The standard deviation of Y is defined as

$$\boxed{SD[Y] := \sqrt{Var[Y]}}$$

Formula. $Var[Y] = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2$

$\longrightarrow$: $\mu_Y := \mathbb{E}[Y]$

$$Var[Y] = \mathbb{E}[(Y-\mu_Y)^2] =$$

$$= \mathbb{E}[Y^2 - 2\mu_Y \cdot Y + \mu_Y^2] =$$

linearity

$$= \mathbb{E}[Y^2] - 2\mu_Y \underbrace{\mathbb{E}[Y]}_{=\mu_Y} + \mu_Y^2$$

$$= \mathbb{E}[Y^2] - 2\mu_Y^2 + \mu_Y^2 = \mathbb{E}[Y^2] - \mu_Y^2 \quad \square$$

Theorem. Let Y be a r.v. w/ a finite variance, and let α be a real constant.

$$\boxed{Var[\alpha \cdot Y] = \alpha^2 \cdot Var[Y]}$$

Q: Say that Y_1 and Y_2 are r.v.s w/ finite variances.

$$Var[Y_1 + Y_2] = ?$$

Def'n. If two r.v.s Y_1 and Y_2 satisfy that

$$\mathbb{P}[Y_1 \in B_1, Y_2 \in B_2] = \mathbb{P}[Y_1 \in B_1] \cdot \mathbb{P}[Y_2 \in B_2] \text{ for "all" } B_1, B_2 \subseteq \mathbb{R},$$

then, we say that Y_1 and Y_2 are independent.

Theorem. If Y_1 and Y_2 are independent,
then, $\boxed{Var[Y_1+Y_2] = Var[Y_1] + Var[Y_2]}$.

Example.

- **Bernoulli.**

  $Y \sim B(p)$

  $Var[Y] = \mathbb{E}[Y^2] - (\underbrace{\mathbb{E}[Y]}_{=p})^2 = \boxed{\mathbb{E}[Y^2]} - p^2$

  $\mathbb{E}[Y^2] = \cancel{?}\ p$

  $Y \sim \begin{cases} 1 & \text{w/ prob } p \\ 0 & \text{w/ prob } 1-p \end{cases}$

  $Y^2 \sim \begin{cases} 1 & \text{w/ prob } p \\ 0 & \text{w/ prob. } 1-p \end{cases}$

  $Var[Y] = p - p^2 = p(1-p) = pq$

- **Binomial.**

  $Y \sim b(n,p)$

  $Var[Y] = ?$

  $I_j,\ j=1..n$ are independent and $I_j \sim B(p)$.

  $Y = I_1 + \dots + I_n$

  $Var[Y] = Var[I_1 + \dots + I_n]$

  independence

  $= Var[I_1] + \dots + Var[I_n] = n \cdot p \cdot (1-p)$

- **Geometric.** $Y \sim g(p)$

  $\mathbb{E}[Y] = \frac{q}{p}$

  $Var[Y] = \frac{q}{p^2} \implies SD[Y] = \frac{\sqrt{q}}{p}$

- **Poisson.** $Y \sim P(\lambda)$ $\boxed{\mathbb{E}[Y] = Var[Y] = \lambda}$

Problem 4.2. Source: Sample P exam, Problem #458.

An investor wants to purchase a total of ten units of two assets, A and B, with annual payoffs **per unit** *purchased of X and Y, respectively. Each asset has the same purchase price per unit. The payoffs are independent random variables with equal expected values and with $\mathrm{Var}(X) = 30$ and $\mathrm{Var}(Y) = 20$.*

Calculate the number of units of asset A the investor should purchase to minimize the variance of the total payoff.

→: n ... # of units of asset A that is bought

10-n ... # of units of B bought

$$\mathrm{Var}[n \cdot X + (10-n) \cdot Y] \xrightarrow[n]{} \min$$

independence

$$n^2 \cdot \mathrm{Var}[X] + (10-n)^2 \cdot \mathrm{Var}[Y] \xrightarrow[n]{} \min$$

$$30n^2 + 20(10-n)^2 \xrightarrow[n]{} \min$$

$$30 \cdot 2n + 20 \cdot 2 \cdot (-1)(10-n) = 0 \quad / :20$$

$$3n - 2(10-n) = 0$$

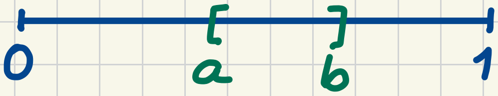
$$3n - 20 + 2n = 0$$

$$5n = 20$$

$$\boxed{n = 4}$$

Continuous Distributions.

The Uniform Distribution.



Imagine a r.v. Y on $[0,1]$ such that the probability of Y landing between a and b where $0 \leq a \leq b \leq 1$ is

$$\mathbb{P}[a \leq Y \leq b] = \mathbb{P}[Y \in [a,b]] = b-a$$

Note: $\mathbb{P}[Y=y] = \mathbb{P}[y \leq Y \leq y] = y-y=0$ for <u>all $y \in [0,1]$</u>.

Def'n. A r.v. Y is said to be **continuous** if there exists a function

$$f_Y : \mathbb{R} \longrightarrow [0,\infty)$$

such that

$$\mathbb{P}[Y \in [a,b]] = \int_a^b f_Y(y)\,dy \quad \text{for all } a \leq b.$$

The function f_Y is called the

probability density function (pdf) of Y.

Properties.
- $f_Y(y) \geq 0$ for all y
- $\int_{-\infty}^{\infty} f_Y(y)\,dy = \underline{1}$

Note:
- For a pmf p_Y, we have $p_Y(y) \leq 1$ for all $y \in S_Y$.
- For a pdf f_Y, it's possible to have $f_Y(y) > 1$ for some y