M378K: December 8^{th}, 2025.

Types of Errors.

H_0: ______ vs. H_a: ______

α ... significance level

Decision \ "Truth"	H_0 true	H_0 not true
Reject H_0	Type I Error	
Fail to Reject H_0		Type II Error

$\mathbb{P}[\text{Type I Error}] = \mathbb{P}_0[\text{Reject } H_0] = \alpha$ (significance level)

Example. The Rayleigh density function is given by

$$f_Y(y) = \frac{2}{\tau} y e^{-\frac{y^2}{\tau}} \cdot \mathbb{1}_{[0,\infty)}(y)$$

Q: Maximum likelihood estimation?

Def'n. $Y_1, \dots, Y_n$ is a RANDOM SAMPLE from dist'n D iff:

- $Y_1, \dots, Y_n$ are independent,
- $Y_i \sim D$ for all $i = 1, \dots, n$.

Let $y_1, \dots, y_n \geq 0$ represent the observations of $Y_1, \dots, Y_n$.

$$L(\tau; y_1, \dots, y_n) = \prod_{i=1}^{n} f_Y^{\tau}(y_i)$$

$$= \prod_{i=1}^{n} \left(\left(\frac{2}{\tau}\right) y_i e^{-\frac{y_i^2}{\tau}} \right)$$

$$= \left(\frac{2}{\tau}\right)^n \prod_{i=1}^{n} y_i \cdot e^{-\frac{1}{\tau}\sum y_i^2}$$

$$\ell(\tau; y_1, \dots, y_n) = n(\ln(2) - \ln(\tau)) + \sum_{i=1}^{n} \ln(y_i) - \frac{1}{\tau}\sum_{i=1}^{n} y_i^2$$

$$\ell'(\tau; y_1, \dots, y_n) = -\frac{n}{\tau} + \left(+\frac{1}{\tau^2}\right)\sum_{i=1}^{n} y_i^2 = 0$$

$$\frac{1}{\tau^2}\sum_{i=1}^{n} y_i^2 = \frac{n}{\tau}$$

$$\boxed{\hat{\tau}_{MLE} = \frac{1}{n}\sum_{i=1}^{n} Y_i^2}$$

Def'n. The BIAS of the estimator $\hat{\theta}$ of the parameter θ is

$$\text{bias}(\hat{\theta}) = \mathbb{E}[\hat{\theta}] - \theta$$

In addition, we say that the estimator $\hat{\theta}$ is UNBIASED if

$$\text{bias}(\hat{\theta}) = 0, \text{ i.e.,}$$

$$\boxed{\mathbb{E}[\hat{\theta}] = \theta}$$

We want to check if $\hat{\tau}_{MLE}$ is unbiased for τ!

$$\mathbb{E}[\hat{\tau}_{MLE}] \overset{?}{=} \tau$$

$$\frac{1}{n}\boxed{\sum_{i=1}^{n} \mathbb{E}[Y_i^2]} \overset{?}{=} \tau$$

Q: If Y is Rayleigh, what is the dist'n of Y^2?

Def'n. The CUMULATIVE DIST'N F'TION of a random variable Y is defined as

$$F_Y : \mathbb{R} \longrightarrow [0,1]$$

$$F_Y(y) = \mathbb{P}[Y \leq y] \text{ for all } y \in \mathbb{R}$$

$y > 0$

$$F_{Y^2}(y) = \mathbb{P}[Y^2 \leq y] = \mathbb{P}[Y \leq \sqrt{y}] = F_Y(\sqrt{y})$$

$$= \int_0^{\sqrt{y}} \frac{2}{\tau} u e^{-\frac{u^2}{\tau}} du \qquad z = \frac{u^2}{\tau} \qquad dz = \frac{2u\,du}{\tau}$$

$$= \int_0^{\frac{y}{\tau}} e^{-z} dz = 1 - e^{-\frac{y}{\tau}} \Rightarrow \boxed{Y^2 \sim E(\tau)}$$

$$\mathbb{E}[\hat{\tau}_{MLE}] = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}[Y_i^2] = \frac{1}{n} \cdot n \cdot \tau = \tau$$

unbiased!

Q: A pivotal quantity?

$$\boxed{\frac{1}{n}\sum_{i=1}^{n} Y_i^2 \qquad \sim E(\tau)}$$

$$\sim \Gamma(n, \tau)$$

$$U = \frac{1}{\tau}\cdot\frac{1}{n}\sum_{i=1}^{n} Y_i^2 \sim \boxed{\Gamma\left(n, \frac{1}{n}\right)} \quad ✓$$