Name:

M339D=M389D Introduction to Actuarial Financial Mathematics
University of Texas at Austin
In-Term One
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The maximum number of points on this exam is ??.

Problem 1.1. (10 points)The probability mass function p_X of a discrete random variable X is given by

$$p_X(x) = \begin{cases} 1/4, & \text{for } x = 10 \\ 2/3, & \text{for } x = 20 \\ 1/12, & \text{for } x = 40 \end{cases}$$

Find $\mathbb{E}[\max(X - 15, 0)]$.

Solution: The random variable $\max(X - 15, 0)$ has the following distribution

$$\max(X - 15, 0) \sim \begin{cases} 0 & \text{with probability } 1/4 \\ 5 & \text{with probability } 2/3 \\ 25 & \text{with probability } 1/12 \end{cases}$$

So, its expectation is

$$\mathbb{E}[\max(X - 15, 0)] = 5\left(\frac{2}{3}\right) + 25\left(\frac{1}{12}\right) = \frac{10}{3} + \frac{25}{12} = \frac{40 + 25}{12} = \frac{65}{12}.$$

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Problem 1.2. (5 points) Here is some information about two forward contracts with delivery dates in one year:

	Current price of underlying	Forward price
Forward I	100	105
Forward II	90	92

Alfur enters a long position in Forward I and a short position in Forward II. It turns out that the final price of the underlying asset for Forward I equals \$102, while the final price of the underlying asset for Forward II equals \$89.

Let the continuously compounded, risk-free interest rate be 0.03.

What is Alfur's profit?

Solution: The initial cost of any forward contract is zero, so the profit and the payoff are equal. For the long Forward I, Alfur's payoff is

$$102 - 105 = -3.$$

For the short Forward II, Alfur's payoff is

$$92 - 89 = 3.$$

Alfur's overall profit is zero.

Problem 1.3. (5 points) The current spot price of corn is \$3.60 per bushel. There is a forward contract on corn for delivery in six months with the forward price of \$3.65. In order to hedge, farmer Brown shorts a 1000-bushel forward contract.

At the delivery date, it turns out that the spot price of corn is \$3.80.

You know that farmer Brown's total aggregate costs of production for 1000 bushels of corn equal \$3,450.

What is farmer Brown's profit?

Solution:

$$1000(3.65) - 3450 = 200$$

Problem 1.4. (5 points) The current price of zinc is \$2.74 kilograms. David needs to buy zinc in three months for the purposes of galvanization of a component his company is making. Per kilogram of zinc, the total revenue from the sale of the finished component will be \$10.23, while the total aggregate costs of non-zinc inputs equal \$5.12.

To hedge, David enters a forward contract for delivery of zinc in three months with the forward price equal to \$2.80.

The market price of zinc turns out to be \$2.78 in three months.

What is David's total profit (per kilo of zinc used)?

Solution:

$$10.23 - 5.12 - 2.80 = 2.31$$

Problem 1.5. (5 points) The **owner** of a call option has . . .

(a) an obligation to sell the underlying asset at the strike price.

(b) a right, but **not** an obligation, to sell the underlying asset at the strike price.

(c) an obligation to buy the underlying asset at the strike price.

(d) a right, but **not** an obligation, to buy the underlying asset at the strike price.

(e) None of the above.

Solution: (d)

Problem 1.6. (10 points) Assume that the risk-free interest rate equals 0.04. The Sharpe ratio of asset S is given to be 1/4 while the Sharpe ratio of asset Q equals 1/3. You know that the volatility of S is three times the volatility of Q. If you build an equally weighted portfolio with assets S and Q as its two components, the expected return of this portfolio will be 0.10. What is the expected return of Q?

Solution: From the condition on the Sharpe ratio of S, we get

$$\frac{\mathbb{E}[R_S] - r_f}{\sigma_S} = \frac{1}{4} \quad \Rightarrow \quad \sigma_S = 4(\mathbb{E}[R_S] - 0.04).$$

From the condition on the Sharpe ratio of Q, we obtain

$$\frac{\mathbb{E}[R_Q] - r_f}{\sigma_Q} = \frac{1}{3} \quad \Rightarrow \quad \sigma_Q = 3(\mathbb{E}[R_Q] - 0.04).$$

Since $\sigma_S = 3\sigma_Q$, we have

$$\begin{aligned} 4(\mathbb{E}[R_S] - 0.04) = 3(3)(\mathbb{E}[R_Q] - 0.04) \quad &\Rightarrow \quad \mathbb{E}[R_S] - 0.04 = 2.25(\mathbb{E}[R_Q] - 0.04) \\ &\Rightarrow \quad \mathbb{E}[R_S] - 2.25\mathbb{E}[R_Q] = 0.04 - 0.09 = -0.05. \end{aligned}$$

On the other hand, from the given expected return on the equally-weighted portfolio, we know that

$$\frac{1}{2}(\mathbb{E}[R_S] + \mathbb{E}[R_Q]) = 0.10 \quad \Rightarrow \quad \mathbb{E}[R_S] + \mathbb{E}[R_Q] = 0.20.$$

Solving the above system of two equations with two unknowns, we get

$$\mathbb{E}[R_S] = 0.1231 \quad \text{and} \quad \mathbb{E}[R_Q] = 0.0769.$$

Problem 1.7. (5 points) The initial price of a non-dividend-paying market index equals \$1,000.

An investor simultaneosly purchases one unit of the index and a one-year, 975-strike European put option on the index for a premium of \$10.

In one year, the spot price of the index is observed to be \$950.

Given that the continuously compounded risk-free interest rate equals 0.03, what is the profit of the investor's portfolio?

Solution: In our usual notation,

$$S(T) + (K - S(T))_+ - (S(0) + V_P(0))e^{rT} = 950 + 25 - 1010e^{0.03} = -65.75908$$

Problem 1.8. (10 points) Assume the **Capital Asset Pricing Model** holds.

You are given the following information about stock X, stock Y, and the market:

- The expected return and volatility for the market portfolio are 0.12 and 0.2, respectively.
- The required return and volatility for the stock X are 0.0404 and 0.4, respectively.
- The correlation between the returns of stock X and the market is -0.25.
- The volatility of stock Y is 0.3.
- The correlation between the returns of stock Y and the market is 0.1.

Calculate the required return for stock Y.

Solution: The βs of stocks X and Y are

$$\beta_X = \frac{0.4(-0.25)}{0.2} = -0.5,$$
$$\beta_Y = \frac{0.3(0.1)}{0.2} = 0.15.$$

So, the required return of stock X must satisfy

$$\begin{aligned} 0.0404 = r_X = r_f + (-0.5)(0.12 - r_f) \quad &\Rightarrow \quad 0.0404 = r_f - 0.06 + 0.5r_f \\ &\Rightarrow \quad 1.5r_f = 0.1004 \quad \Rightarrow \quad r_f = 0.0669. \end{aligned}$$

Finally, the required return of stock Y equals

$$r_Y = 0.0669 + 0.15(0.12 - 0.0669) = 0.0749.$$

Problem 1.9. (5 points) Assume the **CAPM** holds.

Let the risk-free interest rate be 0.025 and let the expected return of the market portfolio be equal to 0.08.

Suppose that stock X has $\beta_X = 1.4$ and that stock Y has $\beta_Y = 0.8$. Using the risk-free asset, stock X, and stock Y, you create a portfolio such that you invest twice as much in asset X as in asset Y while the weight of the risk-free asset is 0.4. What is the expected return of this portfolio?

Solution: The β of the risk-free asset is zero. Hence, the β of the portfolio is

$$\beta_P = 0.4\beta_X + 0.2\beta_Y = 0.4(1.4) + 0.2(0.8) = 0.72.$$

So, realizing that the expected return of the portfolio equals its required return, we get

$$\mathbb{E}[R_P] = r_f + \beta_P(r_m - r_f) = 0.025 + 0.72(0.08 - 0.025) = 0.0646.$$

Problem 1.10. (5 points) You are given the following information about stock X and a portfolio P:

- The annual effective risk-free rate is 4%.
- The portfolio's expected return is 0.10 and its volatility is 0.2.
- The expected return of stock X is 0.03 and its volatility is 0.25.
- The correlation between the returns of stock X and the portfolio P is -0.2.

Then:

(a) The required return of stock X is 0.025 and the investor holding portfolio P should invest in stock X.

(b) The required return of stock X is 0.025 and the investor holding portfolio P should not invest in stock X.

(c) The required return of stock X is 0.055 and the investor holding portfolio P should invest in stock X.

(d) The required return of stock X is 0.055 and the investor holding portfolio P should not invest in stock X.

(e) None of the above.

You have to show your work. The final answer without a justification will earn zero points (even if it's correct).

Solution: (a)
The β for the stock X equals

$$\beta_X = \frac{0.25(-0.2)}{0.2} = -0.25.$$

So, the stock X has a required return equal to

$$r_X = r_f + \beta_X(\mathbb{E}[R_m] - r_f) = 0.04 + (-0.25)(0.10 - 0.04) = 0.04 - 0.015 = 0.025.$$

Since the expected return is bigger than the required return, one should invest in stock X.

Problem 1.11. (10 points) You are a pessimist and you model the state of the economy to be twice as likely to be *bad* as it is to be *good*. There are no other states of the economy in your model.

According to your model, if the economy is *good*, the return of stock S will be 0.09 and the return of stock Q will be 0.15. Also, if the economy is *bad*, the return of stock S will be -0.03 and the return of stock Q will be -0.045.

You build a portfolio out of the stocks S and Q. The expected return of this portfolio is 0.014. What is the volatility of your portfolio?

Solution: Let R_S and R_Q denote the returns of the two stocks S and Q, respectively. According to our model, we have

$$\mathbb{E}[R_S] = 0.09\left(\frac{1}{3}\right) + (-0.03)\left(\frac{2}{3}\right) = 0.01,$$
$$\mathbb{E}[R_Q] = 0.15\left(\frac{1}{3}\right) + (-0.045)\left(\frac{2}{3}\right) = 0.02.$$

Then, the return of the entire portfolio can be expressed as

$$R_P = w_s R_S + (1 - w_S)R_Q$$

where w_S stands for the weight of asset S. It's straightforward to see that the weight w_s needs to be equal to 0.6 so that the return of the entire portfolio equals the given 0.014. The distribution of the return of the entire portfolio satisfies

$$R_P \sim \begin{cases} 0.114, & \text{with probability } 1/3, \\ -0.036, & \text{with probability } 2/3. \end{cases}$$

So,

$$\mathbb{E}[R_P^2] = (0.114)^2\left(\frac{1}{3}\right) + (-0.036)^2\left(\frac{2}{3}\right) = 0.005196.$$

Thus,

$$Var[R_P] = 0.005196 - (0.014)^2 = 0.005.$$

Our answer is $\sqrt{0.005} = 0.0707$.

Problem 1.12. (5 points) Which one of the following statements is **not** correct?

(a) Any equally weighted portfolio contains only systematic risk.

(b) The volatility of an equally weighted portfolio is at most as large as the average of the volatilities of its components.

(c) Full diversification of an investment portfolio leaves only market risk.

(d) Adding another investment into your portfolio may increase the volatility of the portfolio.

(e) The Sharpe ratio reflects the reward-to-risk ratio of an investment.

Solution: (a)