M378K Introduction to Mathematical Statistics
Problem Set #9
Moment generating functions.

Definition 9.1. *The* k^{th} moment of a random variable Y taken about the origin *is defined as* $\mathbb{E}[Y^k]$ *provided that the expectation exists. We write*

$$\mu_k = \mathbb{E}[Y^k]$$

when there is no ambiguity about the random variable in question.

Remark 9.2. μ_k *is also referred to as the* k^{th} raw moment.

Remark 9.3. *In particular,* $\mu_1 = \mu$ *happens to be the* **mean** *of the random variable* Y.

Definition 9.4. *The* k^{th} central moment *of a random variable* Y *is defined as* $\mathbb{E}[(Y-\mu)^k]$ *provided that the expectation exists. We write*

$$\mu_k^c = \mathbb{E}[(Y-\mu)^k]$$

when there is no ambiguity about the random variable in question.

Remark 9.5. μ_k *is also referred to as the* k^{th} moment of a random variable Y taken about its mean.

Definition 9.6. *The* moment-generating function (mgf) m_Y *for a random variable* Y *is defined as*

$$m_Y(t) = \mathbb{E}[e^{tY}]$$

for all t *for which the above expectation exists. In fact, we say that the moment-generating function* **exists** *if there exists a positive number* b *such that* $m_Y(t)$ *is finite for all* t *such that* $|t| \leq b$.

Problem 9.1. *How much is* $m_Y(0)$*?*

Solution:

$$m_Y(0) = 1.$$

Remark 9.7. *On the choice of terminology ...*

Step 1.

$$\frac{d}{dt} m_Y(t) = ?$$

Solution:

$$\mathbb{E}[Ye^{tY}]$$

Step 2.

$$m_Y'(0) = ?$$

Solution:

$$\mu_Y = \mathbb{E}[Y]$$

Step 3.

$$\frac{d^2}{dt^2} m_Y(t) = ?$$

Solution:

$$\mathbb{E}[Y^2 e^{tY}]$$

Step 4.

$$m_Y''(0) = ?$$

Solution:

$$\mu_2 = \mathbb{E}[Y^2]$$

Step 5. *What do you suspect the* **generalization** *of the above would be?*

Theorem 9.8. *If m_Y exists, then for $k \in \mathbb{N}$, we have*

$$m_Y^{(k)}(0) = \mu_k.$$

Example 9.9. *Let $Y \sim b(n = 1, p)$, i.e., let Y model a Bernoulli trial with the probability of success denoted by p. Find m_Y.*

Solution:

$$m_Y(t) = \mathbb{E}[e^{tY}] = (1-p)e^0 + pe^t = (1-p) + pe^t, \quad t \in \mathbb{R}.$$

Proposition 9.10. *Let Y_1 and Y_2 be independent random variables with m.g.f.s denoted by m_{Y_1} and m_{Y_2}. Define $Y = Y_1 + Y_2$. Then, for every t for which both m_{Y_1} and m_{Y_2} are well defined, we have*

$$m_Y(t) =$$

Proof. By definition:

$$m_Y(t) =$$

Solution:

$$\mathbb{E}[e^{tY}]$$

Using $Y = Y_1 + Y_2$, we can subsitute $Y_1 + Y_2$ for Y in the expression above. So,

$$m_Y(t) =$$

Solution:

$$\mathbb{E}[e^{t(Y_1+Y_2)}]$$

One of the properties of the exponential function is that $e^{A+B} = e^A \times e^B$. Thus, the above becomes:

$$m_Y(t) =$$

Solution:

$$\mathbb{E}[e^{tY_1} \times e^{tY_2}]$$

Recall that Y_1 and Y_2 are assumed to be independent random variables. With this in mind, we get:

$$m_Y(t) =$$

Solution:

$$\mathbb{E}[e^{tY_1}] \times \mathbb{E}[e^{tY_2}]$$

Finally, using the definition of a m.g.f., we have

$$m_Y(t) =$$

Solution:

$$m_{Y_1}(t)m_{Y_2}(t)$$

□

Example 9.11. *Let $Y \sim b(n, p)$. What is the moment generating function of Y?*

Solution:

$$Y \overset{(d)}{=} X_1 + \cdots + X_n$$

with $X_i \sim b(1, p), i = 1, \ldots, n$ independent random variables. Then,

$$m_Y(t) = m_{X_1}(t) \times \cdots \times m_{X_n}(t) = (m_{X_1}(t))^n = (1 - p + pe^t)^n.$$

Example 9.12. *Let $N \sim Poisson(\lambda)$. What is the moment generating function m_N of N?*

Solution:

$$\begin{aligned}
m_N(t) &= \mathbb{E}[e^{tN}] \\
&= \sum_{n=0}^{\infty} e^{tn} p_N(n) \\
&= \sum_{n=0}^{\infty} e^{tn} e^{-\lambda} \frac{\lambda^n}{n!} \\
&= e^{-\lambda} \sum_{n=0}^{\infty} \frac{(e^t \lambda)^n}{n!} = e^{-\lambda} \times e^{\lambda e^t} = e^{\lambda(e^t - 1)}.
\end{aligned}$$

Example 9.13. *Let $Z \sim N(0, 1)$. What is the moment generating function m_Z of Z?*

Solution:

$$m_Z(t) = e^{t^2/2} \quad t \in \mathbb{R}.$$

Example 9.14. *Let the random variable Y have the mgf m_Y. Define $X = aY + b$ for some constants a and b. Express the mgf m_X of X in terms of m_Y, a and b.*

Solution:

$$m_X(t) = e^{bt} m_Y(at)$$

Example 9.15. *Let $X \sim N(\mu, \sigma^2)$. What is the moment generating function m_X of X?*

Solution: Since X can be expressed as a linear transform of $Z \sim N(0,1)$ in the following way

$$X = \mu + \sigma Z,$$

we get that

$$m_X(t) = e^{\mu t} m_Z(\sigma t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

Problem 9.2. *A random variable Y is said to be* lognormal *if there exists a normally distributed random variable $X \sim N(\mu, \sigma^2)$ such that $Y \overset{(d)}{=} e^X$. Express the mean and the variance of the lognormal r.v. Y in terms of the parameters μ and σ.*

Solution:

$$\mathbb{E}[Y] = \mathbb{E}[e^X] = \mathbb{E}[e^{1\times X}] = m_X(1) = \exp(\{)\frac{1}{2}\sigma^2 + \mu\}.$$

$$\mathbb{E}[Y^2] = \mathbb{E}[(e^X)^2] = \mathbb{E}[e^{2\times X}] = m_X(2) = \exp(\{)\frac{1}{2}\sigma^2 \times 4 + \mu \times 2\} = \exp(\{)2(\sigma^2 + \mu)\}.$$

$$Var[Y] = \mathbb{E}[(e^X)^2] - (\mathbb{E}[e^X])^2 = \exp(\{)2(\sigma^2 + \mu)\} - \exp(\{)\sigma^2 + 2\mu\}.$$

Proposition 9.16.

1. *If m_Y exists for a certain probability distribution, then it is unique.*
2. *If m_{Y_1} and m_{Y_2} are equal on an interval, then $Y_1 \overset{(d)}{=} Y_2$.*

Corollary 9.17. *Let Y_1 and Y_2 be independent and normally distributed. Define $Y = Y_1 + Y_2$. Then, the distribution of Y is ...*

Proof. **Solution:** Note that $Y_i \sim N(\mu = mu_i, \sigma_i)$ for $i = 1, 2$. Now, let's look at the mgf of Y. Then, since Y_1 and Y_2 are independent, we have

$$m_Y(t) = m_{Y_1}(t) m_{Y_2}(t).$$

We can now use the fact that for any $X \sim N(\mu, \sigma)$,

$$m_X(t) = e^{\mu t} m_Z(\sigma t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

Hence,

$$m_Y(t) = e^{\mu_1 t + \frac{\sigma_1^2 t^2}{2}} \times e^{\mu_2 t + \frac{\sigma_2^2 t^2}{2}} = e^{(\mu_1 + \mu_2)t + \frac{(\sigma_1^2 + \sigma_2^2)t^2}{2}}$$

We can conclude that $Y \sim N(\mu_1 + \mu_2, \sqrt{\sigma_1^2 + \sigma_2^2})$. □

Corollary 9.18. *Let N_1 and N_2 be independent and Poisson distributed. Define $N = N_1 + N_2$. Then, the distribution of N is ...*

Proof. **Solution:** We are given that $N_i \sim Poisson(\lambda_i)$ for $i = 1, 2$. We saw in a previous example that

$$m_{N_i}(t) = e^{\lambda_i(e^t-1)}.$$

Hence,

$$m_N(t) = m_{N_1}(t)m_{N_2}(t) = e^{\lambda_1(e^t-1)} \times e^{\lambda_2(e^t-1)} = e^{(\lambda_1+\lambda_2)(e^t-1)} \tag{9.1}$$

We can conclude that $N \sim Poisson(\lambda_1 + \lambda_2)$. $\square$