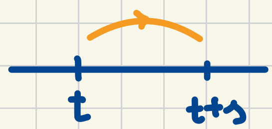
M339D: October 15th, 2025.

More on realized returns.

Review.



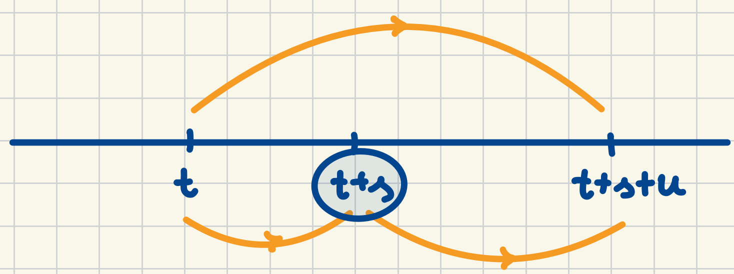
$$\boxed{R(t,t+s) := \ln\left(\frac{S(t+s)}{S(t)}\right)}$$

$R\left(\frac{k-1}{m},\frac{k}{m}\right)$ for $k=1..m$

are random variables.

We make the following assumptions:

- the returns over disjoint intervals are independent;
- all the returns over the same length intervals are identically distributed.



$$R(t, t+s+u) = \ln\left(\frac{S(t+s+u)}{S(t)}\right)$$

$$= \ln\left(\frac{S(t+s)}{S(t)} \cdot \frac{S(t+s+u)}{S(t+s)}\right)$$

$$= \boxed{\ln\left(\frac{S(t+s)}{S(t)}\right)} + \boxed{\ln\left(\frac{S(t+s+u)}{S(t+s)}\right)}$$

$$= R(t, t+s) + R(t+s, t+s+u)$$

Hence, realized returns are additive.

$$\boxed{R(0,\tfrac{1}{m}) + R(\tfrac{1}{m},\tfrac{2}{m}) + \dots + R(\tfrac{m-1}{m},1) = R(0,1)}$$

Q: $\mathrm{Var}[R(0,1)] = \sigma^2$

$$\Rightarrow \sigma^2 = \mathrm{Var}[R(0,\tfrac{1}{m}) + \dots + R(\tfrac{m-1}{m},1)] = \text{(independence)}$$

$$= \mathrm{Var}[R(0,\tfrac{1}{m})] + \dots + \mathrm{Var}[R(\tfrac{m-1}{m},1)] =$$

(identically dist'd)

$$= m \cdot \mathrm{Var}[R(0,\tfrac{1}{m})] = m \cdot \sigma_{h_m}^2$$

$$\sigma_{h_m}^2 = \frac{1}{m}\sigma^2 \quad \Rightarrow \quad \boxed{\sigma_{h_m} = \sigma\sqrt{\tfrac{1}{m}} = \sigma\sqrt{h_m}}$$

We generalize this identity to arbitrary lengths h:

$$\boxed{\sigma_h = \sigma\sqrt{h}}$$

Forward Binomial Tree [revisited].

$$S(0) \quad \begin{cases} S_u = u \cdot S(0) \\ F_{0,h}(S) \\ S_d = d \cdot S(0) \end{cases}$$

$\underbrace{\qquad\qquad}_{h}$

No arbitrage condition:

$$\boxed{d < e^{rh} < u}$$

Recall: $\boxed{F_{0,h}(S) = S(0)e^{rh}}$

$$S_u := F_{0,h}(S) \cdot e^{\sigma\sqrt{h}} = S(0)e^{rh} \cdot e^{\sigma\sqrt{h}} = S(0)\boxed{e^{rh+\sigma\sqrt{h}}} \quad (= u)$$

$$S_d := F_{0,h}(S) \cdot e^{-\sigma\sqrt{h}} = S(0)e^{rh} \cdot e^{-\sigma\sqrt{h}} = S(0)\boxed{e^{rh-\sigma\sqrt{h}}} \quad (= d)$$

Q: Do u and d satisfy the no-arbitrage cond'n?

$\longrightarrow$:
$$d < e^{rh} < u$$

$$\cancel{e^{rh}} \cdot e^{-\sigma\sqrt{h}} < \cancel{e^{rh}} < \cancel{e^{rh}} \cdot e^{\sigma\sqrt{h}}$$

$$\boxed{e^{-\sigma\sqrt{h}} < 1 < e^{\sigma\sqrt{h}}} \qquad \sigma, h > 0$$

$\square$

Q: What is $\frac{S_u}{S_d} = ?$

$\longrightarrow$:
$$\frac{S_u}{S_d} = \frac{\cancel{S(0)}\cancel{e^{rh}} \cdot e^{\sigma\sqrt{h}}}{\cancel{S(0)}\cancel{e^{rh}} \cdot e^{-\sigma\sqrt{h}}} = \boxed{e^{2\sigma\sqrt{h}}}$$

Example. Consider this one-period forward binomial tree w/ the period length of one quarter-year.

$$S(0) = 100 \quad \begin{cases} S_u = 105 \\ S_d = 98 \end{cases}$$

Q: What is the volatility of this stock?

$\longrightarrow$:

$$\frac{S_u}{S_d} = e^{2\sigma\sqrt{h}} \quad \Rightarrow \quad \sigma = \frac{1}{2\sqrt{h}} \ln\left(\frac{S_u}{S_d}\right)$$

$$\Rightarrow \sigma = \frac{1}{\cancel{2}\cdot\cancel{\sqrt{\frac{1}{4}}}} \ln\left(\frac{105}{98}\right) = \ln\left(\frac{105}{98}\right)$$

$\square$

Binomial Option Pricing

$$\begin{matrix} S(0) \\ V(0) = ? \end{matrix} \quad \begin{cases} S_u = u \cdot S(0) \\ S_d = d \cdot S(0) \end{cases}$$

$\underbrace{\qquad\qquad}_{h}$

populating the tree $\longrightarrow$

$\longleftarrow$

Goal: Pricing a European-style derivative security w/ exercise date @ the end of the tree, i.e., $\boxed{T = h}$

It is completely determined by its payoff function: $v(\cdot)$

e.g., for a call : $v_c(s)=(s-K)_+$

for a put : $v_p(s)=(K-s)_+$

for a power option : $v(s)=(s^2-K)_+$

The payoff of our derivative security is a random variable

$$\boxed{V(T):=v(S(T))}$$