Name:

UTeid:

M378K Introduction to Mathematical Statistics
Fall 2024
University of Texas at Austin
In-Term Exam I
Instructor: Milica Čudina

Notes: This is a closed book and closed notes exam. The maximal score on the exam is 100 points.
Time: 50 minutes

All written work handed in by the student is considered to be
their own work, prepared without unauthorized assistance.

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1.1. **Formulas.** If Y has the binomial distribution with parameters n and p, then $p_Y(k) = \mathbb{P}[Y = k] = \binom{n}{k} p^k (1-p)^{n-k}$, for $k = 0, \dots, n$, $\mathbb{E}[Y] = np$, $\text{Var}[Y] = np(1-p)$. The binomial coefficients are defined as follows for integers $0 \le k \le n$, $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

If Y has a geometric distribution with parameter p, then $p_Y(k) = p(1-p)^k$ for $k = 0, 1, \dots$, $\mathbb{E}[Y] = \frac{1-p}{p}$, $\text{Var}[Y] = \frac{1-p}{p^2}$.

If Y has a Poisson distribution with parameter λ, then $p_Y(k) = e^{-\lambda} \frac{\lambda^k}{k!}$ for $k = 0, 1, \dots$, $\mathbb{E}[Y] = \text{Var}[Y] = \lambda$.

If Y has a uniform distribution on $[l, r]$, its density is

$$f_Y(y) = \frac{1}{r-l} \mathbf{1}_{(l,r)}(y),$$

its mean is $\frac{l+r}{2}$, and its variance is $\frac{(r-l)^2}{12}$.

If Y has the standard normal distribution, then its mean is zero, its variance is one, and its density equals

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad x \in \mathbb{R}.$$

If Y has the exponential distribution with parameter τ, then its cumulative distribution function is $F_Y(y) = 1 - e^{-\frac{y}{\tau}}$ for $y \ge 0$, its probability density function is $f_Y(y) = \frac{1}{\tau} e^{-y/\tau}$ for $y \ge 0$. Also, $\mathbb{E}[Y] = SD[Y] = \tau$.

1.2. DEFINITIONS.

Problem 1.1. (10 points) Write down the definition of the **cumulative distribution function** of a random variable Y.

Solution:

$$F_Y(x) = \mathbb{P}[Y \leq x] \quad \text{for } x \in \mathbb{R}.$$

Problem 1.2. (10 points) Let Y be a continuous random variable with the probability density function denoted by f_Y. Let g be a function taking real values such that $g(Y)$ is well defined. How is $\mathbb{E}[g(Y)]$ evaluated using f_Y, if it exists?

Solution: We have that

$$\mathbb{E}[g(Y)] = \int_{-\infty}^{\infty} g(y) f_Y(y)\, dy$$

if the above integral is absolutely convergent.

1.3. **TRUE/FALSE QUESTIONS.**

Problem 1.3. (5 points)The pdf (probability density function) of the random variable Y is $f_Y(y) = c\exp(-2y)$ for $y > 0$ and $f(y) = 0$ for $y \leq 0$. The constant c is 2. *True or false? Why?*

Solution: TRUE
We can recognize Y as exponential with mean $\tau = \frac{1}{2}$. Also, we have $1 = \int_0^\infty ce^{-2y}\,dy = c \times \frac{1}{2}$.

Problem 1.4. (5 points) The random vector (X, Y) is jointly continuous with the joint probability density function given by

$$f_{(X,Y)}(x,y) = \begin{cases} (1/8)xe^{-(x+y)/2}, & x > 0, y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Then, random variables X and Y are independent. *True or false? Why?*

Solution: TRUE
The joint p.d.f. can be rewritten as

$$f_{(X,Y)}(x,y) = \frac{1}{4}xe^{-x/2} \times \frac{1}{2}e^{-y/2} = f_X(x)f_Y(y)\,.$$

So, the criterion for independence of jointly continuous random variables is satisfied. We conclude that X and Y are independent.

Problem 1.5. (5 points) Let Y be a random variable with mean $\mu = 1$ and standard deviation equal to $\sigma = 4$. Then, $\mathbb{E}[Y^2] = 5$. *True or false? Why?*

Solution: FALSE

$$\mathbb{E}[X^2] = Var[X] + (\mathbb{E}[X])^2 = 16 + 1^2 = 17.$$

Problem 1.6. (5 points) Let Y be a continuous random variable. Then, $\mathbb{P}[Y = y] = 0$ for every $y \in \mathbb{R}$. *True or false? Why?*

Solution: TRUE
For every y, we have that, in our usual notation,

$$\mathbb{P}[Y = y] = \mathbb{P}[y \leq Y \leq y] = \int_y^y f_Y(u)\,du = 0.$$

1.4. **Free-response problems.**

Please, explain carefully all your statements and assumptions. Numerical results or single-word answers without an explanation (even if they're correct) are worth 0 points.

Problem 1.7. (15 points) A random variable Y has the normal distribution with standard deviation 5 Its 0.8413−quantile is 8. What is its mean?

Solution: Since $Y \sim N(\mu, \sigma = 5)$, we know that Y can be expressed as

$$Y = \mu + \sigma Z$$

where Z is standard normal. We are also given that

$$\mathbb{P}[Y \leq 8] = 0.8413.$$

So,

$$\mathbb{P}[\mu + 5Z \leq 8] = 0.8413 \quad \Rightarrow \quad \mathbb{P}[Z \leq \tfrac{8-\mu}{5}] = 0.8413.$$

Using the standard normal tables, we see that

$$\frac{8-\mu}{5} = 1 \quad \Rightarrow \quad \mu = 8 - 5(1) = 3.$$

Problem 1.8. (10 points) Assume that the time T until the arrival of the bus at the bus stop is exponential with mean 5. You have been waiting at the bus stop for 3 minutes. What is the probability that your **total waiting time** will **exceed** 7 minutes? *Note: You can leave your response in the form that uses the exponential function, but you must* **simplify** *it as much as possible!*

Solution: By the memoryless property, the probability equals

$$\mathbb{P}[T > 4] = e^{-\frac{4}{5}}.$$

Problem 1.9. (15 points) Let $Y \sim b(n,p)$ such that its mean equals 8 and its variance equals 1.6. What is the probability of exactly 3 successes? *Note: Leave your answer in the form of a fraction containing only integers* **without** *any binomial coefficients.*

Solution: We are given that

$$\mathbb{E}[Y] = np = 8 \quad \text{and} \quad \mathrm{Var}[Y] = np(1-p) = 1.6 \quad \Rightarrow \quad 1-p = 0.2 \quad \Rightarrow \quad p = 0.8 \quad \Rightarrow \quad n = 10.$$

So,

$$\mathbb{P}[Y = 3] = \binom{10}{3}(0.8)^3(0.2)^7 = \frac{10 \cdot 9 \cdot 8}{3!} \cdot \frac{4^3}{5^{10}} = \frac{120 \cdot 4^3}{5^{10}} = \frac{24 \cdot 4^3}{5^9}.$$

Problem 1.10. (10 points) The number of jobs that arrive at a server is modeled as Poisson. You know that it's four times as likely that one job arrives as that two jobs arrive.

What is the probability that no jobs arrive? *Note: You can leave your response in the form that uses the exponential function, but you must* **simplify** *it as much as possible!*

Solution: Let the number of jobs be denoted by a random variable Y. Then, we know that

$$\mathbb{P}[Y=1] = 4\mathbb{P}[Y=2] \quad \Rightarrow \quad e^{-\lambda}\frac{\lambda^1}{1!} = 4e^{-\lambda}\frac{\lambda^2}{2!} \quad \Rightarrow \quad \lambda = 2\lambda^2 \quad \Rightarrow \quad \lambda = \tfrac{1}{2}\,.$$

The probability we are looking for is

$$\mathbb{P}[Y=0] = e^{-\lambda} = e^{-\frac{1}{2}}\,.$$

1.5. MULTIPLE CHOICE QUESTIONS.

Problem 1.11. (5 points) There are 10 green balls and 20 red balls in a box, well mixed together. We pick a ball at random, and then roll a die. If the ball was green, we write down the outcome of the die, but if the ball is red, we write down the number 3.

If the number 3 was written down, the probability that the ball was green is:

(a) 1/3
(b) 1/2
(c) 5/6
(d) 1
(e) none of the above

Solution: The correct answer is **(e)**.
Let R denote the event when the ball drawn was red, and $G = R^c$ the event corresponding to drawing a green ball, so that $\mathbb{P}[R] = 2/3$ and $\mathbb{P}[G] = 1/3$. If X denotes the number written down, we have

$$\mathbb{P}[X = 3|G] = 1/6 \text{ and } \mathbb{P}[X = 3|R] = 1.$$

Using Bayes formula,

$$\begin{aligned}\mathbb{P}[G|X = 3] &= \frac{\mathbb{P}[X = 3|G] \times \mathbb{P}[G]}{\mathbb{P}[X = 3|G] \times \mathbb{P}[G] + \mathbb{P}[X = 3|R] \times \mathbb{P}[R]} \\ &= \frac{1/6 \times 1/3}{1/6 \times 1/3 + 1 \times 2/3} = \frac{1}{13}.\end{aligned}$$

Problem 1.12. (5 points) The 6-th moment μ_6 of the uniform distribution $U(-2, 2)$ on $[-2, 2]$ is

(a) 0
(b) $\frac{256}{7}$
(c) $\frac{64}{7}$
(d) $\frac{1}{7}$
(e) none of the above

Solution: The correct answer is **(c)**.
The k-th moment μ_k is defined by $\mu_k = \mathbb{E}[Y^k] = \int_{-\infty}^{\infty} y^k f_Y(y)\, dy$. In our particular case we have

$$\mu_6 = \int_{-\infty}^{\infty} y^6 \tfrac{1}{4} \mathbf{1}_{\{-2 \le y \le 2\}}\, dy = \tfrac{1}{4} \int_{-2}^{2} y^6\, dy = \tfrac{1}{4} \left(\tfrac{1}{7}\right) y^7 \Big|_{-2}^{2} = \tfrac{64}{7}.$$