M378K: March 7th, 2025.

Estimators.

Def'n. The **bias** of an estimator $\hat{\theta}$ of the parameter θ is defined as:

$$\text{bias}(\hat{\theta}) := \mathbb{E}(\hat{\theta} - \theta)$$

Notation from book: "$\mathbb{E}_\theta(\cdot)$, $\mathbb{E}^\theta(\cdot)$, $\mathbb{E}[\cdots / \theta]$"

We say that an estimator $\hat{\theta}$ is

unbiased for the parameter θ if

$$\boxed{\mathbb{E}[\hat{\theta}] = \theta \iff \text{bias}(\hat{\theta}) = 0}$$

for all possible values of θ.

Example. Consider a random sample $Y_1, Y_2, \dots, Y_n$ from $N(\mu, \sigma)$ w/ both $\mu \in \mathbb{R}$ and $\sigma > 0$ unknown

$$\hat{\mu} = \bar{Y} := \frac{Y_1 + Y_2 + \cdots + Y_n}{n}$$

sample mean

Then, $\boxed{\mathbb{E}[\hat{\mu}] = \mu}$, i.e., $\hat{\mu} = \bar{Y}$ is unbiased for μ.

Example. Let $Y_1, \dots, Y_n$ be a random sample from $N(\mu_0, \sigma)$ w/ μ_0 known and $\sigma > 0$ unknown

We propose this estimator for the variance σ^2:

$$S^2 := \frac{1}{n} \sum_{i=1}^{n} (Y_i - \mu_0)^2$$

Then, $$\mathbb{E}[S^2] = \frac{1}{n} \sum_{i=1}^{n} \underbrace{\boxed{\mathbb{E}[(Y_i - \mu_0)^2]}}_{\sigma^2} = \frac{1}{\cancel{n}} \cdot \cancel{n} \cdot \sigma^2 = \sigma^2$$

$\Rightarrow S^2$ is unbiased for σ^2.

Example. Let $Y_1, Y_2, \dots, Y_n$ be a random sample from $N(\mu, \sigma)$ with both μ and σ unknown.

Goal: Find a "good" estimator for σ^2!

Propose:

$$(S')^2 := \frac{1}{n}\boxed{\sum_{i=1}^{n}(Y_i - \bar{Y})^2}$$

Q: Is S' unbiased for σ^2?

$$\mathbb{E}[(S')^2] = \frac{1}{n}\sum_{i=1}^{n}\boxed{\mathbb{E}[(Y_i-\bar{Y})^2]}$$

$$\mathbb{E}[Y_i^2 - 2Y_i \cdot \bar{Y} + \bar{Y}^2]$$

$$= \frac{1}{n}\left(\sum_{i=1}^{n}\mathbb{E}[Y_i^2] - 2\boxed{\sum_{i=1}^{n}}\mathbb{E}[Y_i \cdot \bar{Y}] + \sum_{i=1}^{n}\mathbb{E}[(\bar{Y})^2]\right)$$

$$= \frac{1}{\cancel{n}} \cdot \cancel{n} \cdot \mathbb{E}[Y_1^2] - 2\,\mathbb{E}\left[\boxed{\frac{1}{n}\sum_{i=1}^{n}Y_i} \cdot \bar{Y}\right] + \frac{1}{\cancel{n}} \cdot \cancel{n} \cdot \mathbb{E}[(\bar{Y})^2]$$

$$= \mathbb{E}[Y_1^2] - 2 \cdot \mathbb{E}[(\bar{Y})^2] + \mathbb{E}[(\bar{Y})^2]$$

$$= \underbrace{\mathbb{E}[Y_1^2]}_{Var[Y_1] + (\mathbb{E}[Y_1])^2} - \boxed{\mathbb{E}[(\bar{Y})^2]}$$

$$\mathbb{E}[Y_1^2] = Var[Y_1] + (\mathbb{E}[Y_1])^2 = \sigma^2 + \mu^2$$

$$\mathbb{E}[(\bar{Y})^2] = Var[\bar{Y}] + (\mathbb{E}[\bar{Y}])^2 = \frac{\sigma^2}{n} + \mu^2$$

$$\mathbb{E}[(S')^2] = \sigma^2 + \cancel{\mu^2} - \left(\frac{\sigma^2}{n} + \cancel{\mu^2}\right) = \left(1 - \frac{1}{n}\right)\sigma^2 = \boxed{\left(\frac{n-1}{n}\right)}\sigma^2$$

$$\Rightarrow \boxed{bias((S')^2) = \mathbb{E}[(S')^2 - \sigma^2] = -\frac{\sigma^2}{n}}$$

$$\mathbb{E}\left[(S')^2 \cdot \frac{n}{n-1}\right] = \sigma^2$$

$$= \mathbb{E}\left[\frac{\cancel{n}}{n-1} \cdot \frac{1}{\cancel{n}}\sum(Y_i - \bar{Y})^2\right]$$

So, the UNBIASED estimator for σ^2 is:

$$\boxed{S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y})^2}$$

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Mean Squared Error.

Def'n. Let $\hat{\theta}$ be an estimator for the parameter θ.

1. the error of $\hat{\theta}$ is $\hat{\theta}-\theta$;
2. the absolute error of $\hat{\theta}$ is $|\hat{\theta}-\theta|$;
3. the squared error of $\hat{\theta}$ is $(\hat{\theta}-\theta)^2$;
4. the mean squared error of $\hat{\theta}$ is

$$\boxed{MSE(\hat{\theta}) = \mathbb{E}[(\hat{\theta}-\theta)^2]}$$

Proposition.

$$\boxed{MSE(\hat{\theta}) = (bias(\hat{\theta}))^2 + Var[\hat{\theta}]}$$

$\rightarrow$:
$$\begin{aligned}
MSE(\hat{\theta}) &= \mathbb{E}[(\hat{\theta}-\theta)^2] \\
&= \mathbb{E}[((\hat{\theta}-\mathbb{E}[\hat{\theta}])+(\mathbb{E}[\hat{\theta}]-\theta))^2] \\
&= \boxed{\mathbb{E}[(\hat{\theta}-\mathbb{E}[\hat{\theta}])^2]} + \\
&\qquad + 2\boxed{\mathbb{E}[(\hat{\theta}-\mathbb{E}[\hat{\theta}])(\mathbb{E}[\hat{\theta}]-\theta)]} \\
&\qquad + \boxed{\mathbb{E}[(\mathbb{E}[\hat{\theta}]-\theta)^2]} \\
&= Var[\hat{\theta}] + (bias(\hat{\theta}))^2 \\
&\qquad \cancel{+ 2\mathbb{E}[(\hat{\theta}-\mathbb{E}[\hat{\theta}])(\mathbb{E}[\hat{\theta}]-\theta)]}
\end{aligned}$$

Focus on:

$$\mathbb{E}[(\hat{\theta}-\mathbb{E}[\hat{\theta}])(\mathbb{E}[\hat{\theta}]-\theta)] = (\mathbb{E}[\hat{\theta}]-\theta)\cdot \underbrace{\underbrace{\mathbb{E}[\hat{\theta}-\mathbb{E}[\hat{\theta}]]}_{\mathbb{E}[\hat{\theta}]-\mathbb{E}[\hat{\theta}]}}_{=0}$$

$\square$

Def'n. The standard error of $\hat{\theta}$ is

$$\boxed{SE(\hat{\theta}) = \sqrt{Var[\hat{\theta}]}}$$

M378K Introduction to Mathematical Statistics
Problem Set #15
Bias. MSE.

Problem 15.1. Source: "Mathematical Statistics with Applications" by Wackerly, Mendenhall, Scheaffer.
Let $Y_1, , Y_2, Y_3$ be a random sample from $E(\tau)$. Consider the following five estimators of τ:

$$\hat{\theta}_1 = Y_1, \quad \hat{\theta}_2 = \frac{Y_1+Y_2}{2}, \quad \hat{\theta}_3 = \frac{Y_1+2Y_2}{3}, \quad \hat{\theta}_4 = 3Y_{(1)}, \quad \hat{\theta}_5 = \bar{Y}.$$

Which ones of these estimators are unbiased? Among the unbiased ones which one has the smallest variance?

→:

$$\mathbb{E}[\hat{\theta}_1] = \mathbb{E}[Y_1] = \tau \checkmark$$

$$\mathbb{E}[\hat{\theta}_2] = \mathbb{E}\left[\frac{Y_1+Y_2}{2}\right] = \tau \checkmark$$

$$\mathbb{E}[\hat{\theta}_3] = \mathbb{E}\left[\frac{Y_1+2Y_2}{3}\right] = \tau \checkmark$$

$$\mathbb{E}[\hat{\theta}_4] = \mathbb{E}[3 \cdot Y_{(1)}] = 3\boxed{\mathbb{E}[Y_{(1)}]} = 3 \cdot \frac{\tau}{3} = \tau \checkmark$$

$$\mathbb{E}[\hat{\theta}_5] = \mathbb{E}[\bar{Y}] = \tau \checkmark$$

$$Var[\hat{\theta}_1] = Var[Y_1] = \tau^2$$

independence

$$Var[\hat{\theta}_2] = Var\left[\frac{Y_1+Y_2}{2}\right] = \frac{1}{4} Var[Y_1+Y_2] = \frac{\tau^2}{2}$$

$$Var[\hat{\theta}_3] = Var\left[\frac{Y_1+2Y_2}{3}\right] = \frac{1}{9}\left(Var[Y_1] + 4Var[Y_2]\right) = \frac{5}{9}\tau^2$$

$$Var[\hat{\theta}_4] = Var[3Y_{(1)}] = 9 \cdot \left(\frac{\tau}{3}\right)^2 = \tau^2$$

$$Var[\hat{\theta}_5] = Var[\bar{Y}] = \boxed{\frac{\tau^2}{3}}$$

Remark: When we want to estimate the mean,

$\mathbb{E}[\bar{Y}] = \text{mean}$, i.e., unbiased,

$$MSE(\bar{Y}) = Var[\bar{Y}] = \frac{Var[Y_1]}{n}$$

$$SE[\bar{Y}] = \frac{SD[Y_1]}{\sqrt{n}}$$

Problem 15.2. *Suppose that the two estimators $\hat{\theta}_1$ and $\hat{\theta}_2$ are unbiased. We know that* $\mathrm{Var}[\hat{\theta}_1] = \sigma_1^2$ *and* $\mathrm{Var}[\hat{\theta}_2] = \sigma_2^2$.

Consider the estimator all the estimators that can be obtained as convex combinations of $\hat{\theta}_1$ and $\hat{\theta}_2$, i.e., all the estimators of the form

$$\hat{\theta} = \alpha\hat{\theta}_1 + (1-\alpha)\hat{\theta}_2.$$

What can you say about the bias *of estimators $\hat{\theta}$ of the form above? Assuming that $\hat{\theta}_1$ and $\hat{\theta}_2$ are* **independent**, *for which weight α is the variance minimal?*

$\rightarrow$: $\mathbb{E}[\hat{\theta}] = \mathbb{E}[\alpha\hat{\theta}_1 + (1-\alpha)\hat{\theta}_2] = \alpha\underbrace{\mathbb{E}[\hat{\theta}_1]}_{\theta} + (1-\alpha)\underbrace{\mathbb{E}[\hat{\theta}_2]}_{\theta} = \theta$

linearity

$\Rightarrow \hat{\theta}$ is unbiased

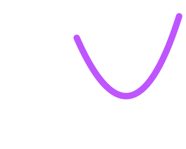
$\mathrm{Var}[\hat{\theta}] \longrightarrow \min$

$\mathrm{Var}[\alpha\hat{\theta}_1 + (1-\alpha)\hat{\theta}_2] \longrightarrow \min$

independence of $\hat{\theta}_1$ and $\hat{\theta}_2$

$\mathrm{Var}[\alpha\hat{\theta}_1] + \mathrm{Var}[(1-\alpha)\hat{\theta}_2] \longrightarrow \min$

$\alpha^2 \cdot \sigma_1^2 + (1-\alpha)^2\sigma_2^2 \longrightarrow \min$



$\cancel{2}\alpha\sigma_1^2 - \cancel{2}(1-\alpha)\sigma_2^2 = 0$

$\alpha\sigma_1^2 + \alpha\sigma_2^2 = \sigma_2^2$

$$\boxed{\alpha^* = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}}$$