Name:

M339J: Probability models
University of Texas at Austin
Solution: Sample In-Term Exam I
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Notes: This is a closed book and closed notes exam. The maximal score on this exam is 50 points.
Time: 50 minutes

Problem 1.1. (5 pts)Let X be the ground-up loss random variable. Assume that X has the exponential distribution with mean $5,000$. Let B denote the expected payment per loss on behalf of an insurer which wrote a policy with a deductible of $1,500$ and with no maximum policy payment. What is the value of B?

(a) $About 1,700$
(b) $About 2,700$
(c) $About 3,700$
(d) $About 4,700$
(e) None of the above

Solution: (c)
Using our tables,

$$B = \mathbb{E}[(X-1500)_+] = \mathbb{E}[X] - \mathbb{E}[X \wedge 1500] = \theta - \theta(1-e^{-1500/\theta}) = \theta\, e^{-1500/\theta} = 5000e^{-3/10} \approx 3704.$$

Problem 1.2. (5 points) A simple experiment consists of drawing a single ball at random from each of two urns containing red and blue marbles. The first urn contains 4 red and 6 blue marbles. A second urn contains 16 red marbles and an unknown number of blue marbles. You are told that the probability that both marbles are the same color equals 0.44. Calculate the number of blue marbles in the second urn.

(a) 4
(b) 5
(c) 10
(d) 16
(e) None of the above.

Solution: (a)

Problem 1.3. (5 points) A particular disease is known to afflict 1% of the population at any given time. There is a blood test for the disease. We know that:

- the test shows positive for people who actually have the disease 95% of the time, and
- the test shows positive for people who actually don't have the disease 0.5% of the time.

Find the probability that a randomly chosen person whose test shows positive actually does have the disease.

(a) 0.2213

(b) 0.3451

(c) 0.5671

(d) 0.6574

(e) None of the above.

Solution: (d)
0.6574394

Problem 1.4. (5 points) *Source: Sample P exam, Problem #126.* Under an insurance policy, a maximum of five claims may be filed per year by a policyholder. Let p_n be the probability that a policyholder files exactly n claims during a given year, where $n = 0, 1, 2, 3, 4, 5$. An actuary makes the following observations:

- $p_n \geq p_{n+1}$ for $n = 0, 1, 2, 3, 4$.
- The difference between p_n and p_{n+1} is the same for $n = 0, 1, 2, 3, 4$.
- Exactly 40% of policyholders file strictly fewer than two claims during a given year.

Calculate the probability that a random policyholder will file strictly more than three claims during a given year.

(a) About 0.14

(b) About 0.16

(c) About 0.27

(d) About 0.29

(e) About 0.33

Solution: (c)

Problem 1.5. (5 points) Consider a continuous random variable X whose probability density function is of the form

$$f_X(x) = \begin{cases} \kappa x^{\alpha} & \text{for } x \in (0,5) \\ 0 & \text{otherwise.} \end{cases}$$

You are given that the probability that X less than 3.75 is 0.4871. What is the value of the parameter α?

(a) 1

(b) 1.5

(c) 2

(d) 2.5

(e) None of the above.

Solution: (b)

Problem 1.6. (5 points) Consider the random variable X whose cumulative distribution function is given by

$$F_X(x) = \begin{cases} 0 & \text{for } x < 1 \\ \frac{x-1}{2} & \text{for } 1 \leq x < 2 \\ 1 & \text{for } 2 \leq x \end{cases}$$

What is the expectation of the random variable X?

(a) 3/4

(b) 5/4

(c) 3/2

(d) 7/4

(e) None of the above.

Solution: (d)
This is a mixed distribution with a probability mass of 1/2 at 2 and otherwise uniform on $(1, 2)$. We have

$$\mathbb{E}[X] = \frac{1}{2}\left(\frac{3}{2}\right) + \frac{1}{2}(2) = \frac{7}{4}.$$

Problem 1.7. (5 points) *Source: Sample P Exam, Problem #147.* The severity random variable covered by a car insurance company follows an exponential distribution. By imposing a deductible of d, the insurance company reduces the expected claim payment by 10%. In other words, the expected value of the per payment random variable is by 10% lower than the expected value of the severity. Calculate the percentage reduction on the variance of the claim payment.

(a) No reduction.

(b) 1%

(c) 5%

(d) 10%

(e) None of the above.

Solution: (b)

Problem 1.8. (5 points) Consider a continuous random variable X whose probability density function is of the form

$$f_X(x) = \begin{cases} \frac{p-1}{x^p} & \text{for } x > 1, \\ 0 & \text{otherwise} \end{cases}$$

for some parameter $p > 1$. Find the value of the parameter p such that the expected value of the random variable X equals 3.

(a) 5/2
(b) 3
(c) 5
(d) Such a p does not exist.
(e) None of the above.

Solution: (a)

Problem 1.9. (5 points) The manufacturer claims that the lifetime of an espresso machine is uniform between 0 and 4. The coffee shop replaces the machine either at the time of failure or at time 3, whichever occurs first. What is the variance of the replacement time?

(a) About 0.98
(b) About 1.12
(c) About 1.27
(d) About 1.33
(e) None of the above.

Solution: (a)
Let the lifetime of the machine be $T \sim U(0,4)$. We need to calculate $\mathrm{Var}[T \wedge 3]$. We have

$$\mathbb{E}[T \wedge 3] = \int_0^3 t\left(\frac{1}{4}\right)\,dt + \int_3^4 3\left(\frac{1}{4}\right)\,dt = \frac{1}{4}\left[\frac{t^2}{2}\right]_{t=0}^3 + \frac{3}{4} = \frac{1}{4}\left(\frac{9}{2}\right) + \frac{3}{4} = \frac{15}{8},$$

$$\mathbb{E}[(T \wedge 3)^2] = \int_0^3 t^2\left(\frac{1}{4}\right)\,dt + \int_3^4 3^2\left(\frac{1}{4}\right)\,dt = \frac{1}{4}\left[\frac{t^3}{3}\right]_{t=0}^3 + \frac{9}{4} = \frac{9}{4} + \frac{9}{4} = \frac{9}{2}.$$

So, the variance is

$$\mathrm{Var}[T \wedge 3] = \frac{9}{2} - \left(\frac{15}{8}\right)^2 = 0.984375.$$

Problem 1.10. (5 points) Consider two independent random variables X and Y which have the same mean. You are given that coefficient of variation of X equals 5 and the coefficient of variation of Y equals 12. What is the coefficient of variation of the sum of X and Y?

(a) 13
(b) 15
(c) 17
(d) There is not enough information to solve this problem.
(e) None of the above.

Solution: (e)
Let $\mu = \mathbb{E}[X] = \mathbb{E}[Y]$. Then, $\sigma_X = SD[X] = 5\mu$ and $\sigma_Y = SD[Y] = 12\mu$. TDue to their independence, the variance of the sum of the two random variables is

$$Var\,[X + Y] = Var[X] + Var[Y].$$

In terms of μ, the variance of the sum can be rewritten as

$$Var\,[X + Y] = 25\mu^2 + 144\mu^2 = 169\mu^2.$$

So, the standard deviation of the sum can be expressed as 13μ. Hence, the coefficient of variation of the sum equals 13/2..

Problem 1.11. (5 points) Let the severity random variable X be modelled using the Pareto distribution with parameters $\theta = 0.5$ and $\alpha = 6$. For a particular value of the ordinary deductible d, the expected value of the per-payment random variable Y^P is 10. What is the value of the deductible?

(a) 12.5
(b) 25.6
(c) 37.5
(d) 50.5
(e) None of the above.

Solution: (e)
We can calculate that $\mathbb{E}[Y^P] = \frac{d+\theta}{\alpha-1}$. So,

$$\mathbb{E}[Y^P] = \frac{d+\theta}{\alpha - 1} = \frac{d + 0.5}{6 - 1} = 10 \quad \Rightarrow \quad d = 49.5.$$

Problem 1.12. (5 points) *Source: Sample P exam, Problem #46.* A device that continuously measures and records seismic activity is placed in a remote region. The time T to failure of this device is exponentially distributed with mean 3 years. Since the device will not be monitored during its first two years of service, the time to discovery of its failure is $X = \max(T, 2)$. Calculate the expected time until discovery of failure.

(a) $2 + \frac{1}{3}e^{-6}$

(b) $2 - 2e^{-\frac{2}{3}} + 5e^{-\frac{4}{3}}$

(c) 3

(d) $2 + 3e^{-\frac{2}{3}}$

(e) 5

Solution: (d)