M378K: March 14th, 2025.

More on Estimators.

Def'n. The **bias** of an estimator $\hat{\theta}$ for θ is

$$\text{bias}(\hat{\theta}) = \mathbb{E}[\hat{\theta} - \theta]$$

If $\text{bias}(\hat{\theta}) = 0$, we say that $\hat{\theta}$ is **unbiased** for θ.

Def'n. The **mean squared error** of $\hat{\theta}$ is

$$MSE(\hat{\theta}) = \mathbb{E}[(\hat{\theta} - \theta)^2]$$

Then,

$$\boxed{MSE(\hat{\theta}) = Var[\hat{\theta}] + (\text{bias}(\hat{\theta}))^2}$$

Def'n. An estimator $\hat{\theta}$ is **UMVUE** of θ if

- $\hat{\theta}$ is unbiased
- $MSE[\hat{\theta}] \leq MSE[\hat{\theta}']$ for all other unbiased estimators $\hat{\theta}'$ for θ

Example. These are **UMVUE**:

- $\bar{Y}$ for μ where $(Y_1, \dots, Y_n)$ is a random sample from $N(\mu, 1)$
- $\bar{Y}$ for p where $(Y_1, \dots, Y_n)$ is a random sample from $B(p)$
- $$\boxed{S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (Y_i - \bar{Y})^2}$$
  for σ^2 where $(Y_1, \dots, Y_n)$ is a random sample from $N(\mu, \sigma)$ w/ both parameters unknown.

Def'n. An estimator $\hat{\theta}$ is said to be **linear** if it's of the form

$$\hat{\theta} = \alpha_1 Y_1 + \alpha_2 Y_2 + \dots + \alpha_n Y_n$$

where $\alpha_1, \dots, \alpha_n$ are all constants

Example. $\bar{Y}$ is a linear estimator.

Problem 15.3. *Let $Y_1, Y_2, \dots, Y_n$ be a random sample from a continuous distribution with probability density function*

$$f_Y(y) = \frac{\alpha y^{\alpha-1}}{\theta^\alpha}\mathbf{1}_{[0,\theta]}(y)$$

with a known parameter $\alpha > 0$ and an unknown parameter $\theta > 0$. We propose the estimator $\hat{\theta} = \max(Y_1, \dots, Y_n)$. Is this estimator unbiased*? If not, how would you modify it to create an* unbiased *estimator? What is the* **mean-squared error** *of the* unbiased *estimator you obtained?*

$\rightarrow$: $\hat{\theta} = Y_{(n)} = \max(Y_1, \dots, Y_n)$

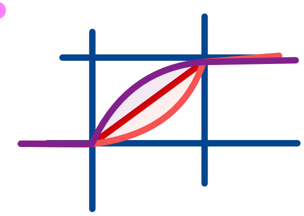
$$\mathbb{E}[\hat{\theta}] = \mathbb{E}[Y_{(n)}] = ?$$

$$g_{(n)}(y) = n \cdot f_Y(y) \boxed{(F_Y(y))}^{n-1}$$

$$\mathbb{E}[Y_{(n)}] = \int_0^\theta y\, g_{(n)}(y)\, dy = ?$$

$$y \in [0,\theta]: \quad F_Y(y) = \int_0^y f_Y(u)\,du = \int_0^y \frac{\alpha u^{\alpha-1}}{\theta^\alpha}\,du = \frac{\alpha}{\theta^\alpha}\int_0^y u^{\alpha-1}\,du = \frac{\cancel{\alpha}}{\theta^\alpha}\cdot\frac{y^\alpha}{\cancel{\alpha}} = \left(\frac{y}{\theta}\right)^\alpha$$

[0, θ]



$$\mathbb{E}[Y_{(n)}] = \int_0^\theta y\, n \frac{\alpha y^{\alpha-1}}{\theta^\alpha}\left(\frac{y^\alpha}{\theta^\alpha}\right)^{n-1} dy$$

$$= n \cdot \alpha \cdot \frac{1}{\theta^{n\alpha}} \int_0^\theta y^{\alpha n}\, dy = \frac{n\alpha}{\cancel{\theta^{n\alpha}}} \cdot \frac{\theta^{\cancel{\alpha n}+1}}{\alpha n + 1}$$

$$= \boxed{\frac{n\alpha}{n\alpha+1} \cdot \theta}$$

Not unbiased!

Let's define: $\boxed{\hat{\theta}' = \frac{n\alpha+1}{n\alpha} \cdot \hat{\theta}}$ unbiased ☺

$$MSE[\hat{\theta}'] = Var[\hat{\theta}'] + \underbrace{(bias(\hat{\theta}'))^2}_{=0} = Var[\hat{\theta}']$$

$$= Var\left[\frac{n\alpha+1}{n\alpha}\cdot\hat{\theta}\right] = \left(\frac{n\alpha+1}{n\alpha}\right)^2 \cdot \boxed{Var[\hat{\theta}]}$$

$$Var[\hat{\theta}] = \boxed{E[\hat{\theta}^2]} - (E[\hat{\theta}])^2$$

$$\int_0^\theta y^2 g_{(n)}(y)\,dy = \int_0^\theta y^2 \cdot n \cdot \frac{\alpha y^{\alpha-1}}{\theta^\alpha}\cdot\left(\left(\frac{y}{\theta}\right)^\alpha\right)^{n-1} dy$$

$$= \frac{n\alpha}{\theta^{n\alpha}}\int_0^\theta y^{\alpha n+1}\,dy = \frac{n\alpha}{\cancel{\theta^{n\alpha}}}\cdot\frac{\theta^{\cancel{\alpha n}+2}}{\alpha n+2}$$

$$= \frac{n\alpha}{n\alpha+2}\theta^2$$

$$Var[\hat{\theta}] = \frac{n\alpha}{n\alpha+2}\theta^2 - \left(\frac{n\alpha}{n\alpha+1}\right)^2\theta^2$$

$$MSE[\hat{\theta}'] = \left(\frac{n\alpha+1}{n\alpha}\right)^2\left(\frac{n\alpha}{n\alpha+2} - \left(\frac{n\alpha}{n\alpha+1}\right)^2\right)\theta^2$$

$$= \boxed{\left(\frac{(n\alpha+1)^2}{n\alpha(n\alpha+2)} - 1\right)}\theta^2 = \frac{1}{n\alpha(n\alpha+2)}\theta^2 \quad \square$$

$$\frac{\cancel{(n\alpha)^2} + \cancel{2n\alpha} + 1 - \cancel{(n\alpha)^2} - \cancel{2n\alpha}}{n\alpha(n\alpha+2)}$$