

M378K: November 7th, 2025.

More on Pivotal Quantities.

Sketch of def'n.

Pivotal quantity is a function of the random sample (Y_1, \dots, Y_n) and the parameter θ of the distribution such that its distribution doesn't depend on θ .

Example. $Y_1, \dots, Y_n \sim N(\mu, \sigma)$ w/ μ unknown and σ known

1.

$$U = \frac{\bar{Y} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0, 1)$$

2.

C ... confidence level or $\alpha = 1 - C$, i.e., the significance level

$$\left. \begin{aligned} -z^* &= a = q_u\left(\frac{\alpha}{2}\right) \\ z^* &= b = q_u\left(1 - \frac{\alpha}{2}\right) \end{aligned} \right\} \text{ so that } \boxed{a = -b}$$

3.

$$\mathbb{P}[-z^* \leq U \leq z^*] = C$$

$$\mathbb{P}\left[-z^* \leq \frac{\bar{Y} - \mu}{\frac{\sigma}{\sqrt{n}}} \leq z^*\right] = C$$

$$\mathbb{P}\left[-z^* \cdot \frac{\sigma}{\sqrt{n}} \leq \bar{Y} - \mu \leq z^* \cdot \frac{\sigma}{\sqrt{n}}\right] = C$$

$$\mathbb{P}\left[-\bar{Y} - z^* \cdot \frac{\sigma}{\sqrt{n}} \leq -\mu \leq -\bar{Y} + z^* \cdot \frac{\sigma}{\sqrt{n}}\right] = C$$

$$P\left[\bar{Y} + z^* \cdot \frac{\sigma}{\sqrt{n}} \geq \mu \geq \bar{Y} - z^* \cdot \frac{\sigma}{\sqrt{n}}\right] = C$$

$$P\left[\underbrace{\bar{Y} - z^* \cdot \frac{\sigma}{\sqrt{n}}}_{\hat{\theta}_L} \leq \mu \leq \underbrace{\bar{Y} + z^* \cdot \frac{\sigma}{\sqrt{n}}}_{\hat{\theta}_R}\right] = C$$

The anatomy of a normal confidence interval for mean

point estimate \pm margin of error

z^* std error

$\frac{\sigma}{\sqrt{n}}$... population std

\sqrt{n} ... sample size

M378K Introduction to Mathematical Statistics

Problem Set #16

Confidence intervals.

Problem 16.1. Suppose that the thumb sizes of the US males are following a normal distribution with an unknown mean μ and standard deviation $\sigma = 20$ on the LDI - scale (Lauretski's Digital Index - LDI - from 50 to 280). The US Department of Thumbs and Toes (DTT) reports that the mean thumb size in the country is $\mu = 150$. Being the chairman of the Faculty of Thumbs of the local university you see an excellent opportunity here and decide to conduct your own study of the size of the average American thumb.

- (i) After carefully collecting a random sample of 100 American thumbs you obtain the following sample mean: $\bar{x} = 153$. This result doesn't seem to be compatible with the DTT report so you decide to construct a 95%-confidence interval for the unknown parameter μ based on your study. What is your confidence interval?
- (ii) Now, you dream about achieving fame and fortune by being the first person ever to estimate the mean thumb size up to ± 0.1 . How large a sample size do you need for that?

i. pt estimate \pm margin of error

A diagram illustrating the components of a confidence interval. It shows the formula $\bar{x} \pm z \cdot \frac{\sigma}{\sqrt{n}}$. Arrows point from the text labels to the corresponding parts of the formula: 'pt estimate' points to \bar{x} , 'margin of error' points to the entire \pm term, '20' points to σ , '100' points to n , and '1.96' points to z .

$$153 \pm 1.96 \cdot \frac{20}{\sqrt{100}} = 153 \pm 3.92$$
$$= (149.08, 156.92)$$



Choosing the Sample Size.

By def'n, the margin of error is $\frac{1}{2}(\hat{\theta}_R - \hat{\theta}_L)$.

We can prescribe a margin of error m and a confidence level C .

We seek a sample size n such that the margin of error is @ most m .

Example. $n = ?$

$$Y_1, \dots, Y_n \sim N(\mu, \sigma)$$

unknown known

$$\bar{Y} \pm z^* \cdot \frac{\sigma}{\sqrt{n}} \leq m$$

$$z^* \cdot \frac{\sigma}{\sqrt{n}} \leq \sqrt{n} \quad /^2$$

$$n \geq \left(z^* \frac{\sigma}{m} \right)^2$$

Always round up!

Back to the problem:

ii.

$$n \geq \left(1.96 \frac{20}{0.1} \right)^2 = (1.96 \cdot 200)^2 = (392)^2 = \underline{\hspace{2cm}}$$

□

Example. $Y_1, \dots, Y_n \sim E(\tau)$ w/ τ unknown