M378K Introduction to Mathematical Statistics

Problem Set #2

Discrete random variables.

2.1. **Probability mass function.** Recall the following definition from the last class:

Definition 2.1. *Given a set B, we say that a random variable Y is $B-$*valued *if*

$$\mathbb{P}[Y \in B] = 1.$$

We reserve special terminology for random variables Y depending on the cardinality of the set B from the above definition. In particular, we have the following definition:

Definition 2.2. *A random variable Y is said to be* discrete *if there exists a set S such that :*

- *Y is $S-$valued, and*
- *S is either* **finite** *or* **countable**.

Problem 2.1. *Provide an example of a* **discrete** *random variable.*

Our next task is to try to keep track of the probabilities that Y takes specific values from S. In order to be more "economical", we introduce the following concept:

Definition 2.3. *The* support *S_Y of a random variable Y is the* **smallest** *set S such that Y is $S-$valued.*

Problem 2.2. *What is the* **support** *of the random variable you provided as an example in the above problem?*

Problem 2.3. *Let $y \in S_Y$ where Y is a discrete random variable. Is it possible to have $\mathbb{P}[Y = y] = 0$?*

Usually, we are interested in calculating and modeling probabilities that look like this

$$\mathbb{P}[Y \in A] \quad \text{for some } A \subseteq S_Y.$$

Note that, if we know the probabilities of the form

$$\mathbb{P}[Y = y] \quad \text{for all } y \in S_Y,$$

then we can calculate any probability of the above form. *How?*

So, if we "tabulate" the probabilities of the form $\mathbb{P}[Y = y]$ for all $y \in S_Y$, we have sufficient information to calculate any probability of interest to do with the random variable Y. This observation motivates the following definition:

Definition 2.4. *The* probability mass function (pmf) *of a* **discrete** *random variable* Y *is the function* $p_Y : S_Y \to \mathbb{R}$ *defined as*

$$p_Y(y) = \mathbb{P}[Y = y] \quad \textit{for all } y \in S_Y.$$

Can you think of different ways in which to display the pmf?

What is the pmf of the random variable which you provided as an example above?

What are the immediate properties of every pmf?

Does the "reverse" hold, i.e., if a function p_Y satisfies you stated, is it always a pmf of **some** random variable?

Problem 2.4. *The number of pieces of gossip that break out in a particular high school in a week is modeled by a random variable Y with the following probability mass function:*

$$p_Y(n) = \frac{1}{(n+1)(n+2)} \quad \textit{for all } n \in \mathbb{N}_0.$$

Is the above a well-defined probability mass function?

2.2. **Conditional probability.** In order to "build" more complicated (and useful!) random variables, it helps to review a bit more probability.

Definition 2.5. *Let E and F be two events on the same Ω such that $\mathbb{P}[E] > 0$. The* conditional probability of F given E *is defined as*

$$\mathbb{P}[F \mid E] = \frac{\mathbb{P}[E \cap F]}{\mathbb{P}[E]} .$$

Let's spend a moment with the geometric/informational perspective on this definition.

By far, the most popular problems relying on the notion of **conditional probability** are those to do with **specificity** and **sensitivity**[1] of medical tests.

Problem 2.5. *At any given time, 2% of the population actually has a particular disease.*

A test indicates the presence of a particular disease 96% of the time in people who actually have the disease. The same test is positive 1% of the time when actually healthy people are tested.

Calculate the probability that a particular person actually has the disease **given** *that they tested positive.*

[1] https://en.wikipedia.org/wiki/Sensitivity_and_specificity

Moreover, now that we remember the definition of **conditional probability**, we can solve interesting problems such as this one:

Problem 2.6. *The number of pieces of gossip that break out in a particular high school in a week is modeled by a random variable* Y *with the following probability mass function:*

$$p_Y(n) = \frac{1}{(n+1)(n+2)} \quad \textit{for all } n \in \mathbb{N}_0.$$

Calculate the probability that at least one piece of gossip occurred in a week **given** *that at most four pieces of gossip occurred.*

3. Independent events

What if knowing that an event happened in fact does **not** give any information about the probability of another event?

Definition 3.1. *We say that events E and F on Ω are* independent *if*

$$\mathbb{P}[E \cap F] = \mathbb{P}[E]\mathbb{P}[F].$$

In the case when E or F have a positive probability, it's possible to rewrite the above condition in a different (illustrative!) way. *How?*

Now that we know the notion of **independence**, we can construct random variables in many creative ways.

Example 3.2. *A fair coin is tossed repeatedly and* **independently** *until the first* Heads*. Let the random variable Y represent the total number of* Tails *observed by the end of the procedure.*

What is the support of the random variable Y?

What is the **probability mass function** *of the random variable Y?*