M378K: September 29th, 2025.

More on Random Vectors.

Jointly Continuous.

Say that the random vector $(Y_1, Y_2, \dots, Y_n)$ is

jointly continuous w/ density $f_{Y_1, Y_2, \dots, Y_n}$.

Then,

$$\boxed{\begin{aligned}&\mathbb{P}[Y_1 \in [a_1, b_1], Y_2 \in [a_2, b_2], \dots, Y_n \in [a, b_n]] = \\ &= \int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_n}^{b_n} f_{Y_1,\dots,Y_n}(y_1, y_2, \dots, y_n)\, dy_n \cdots dy_2\, dy_1\end{aligned}}$$

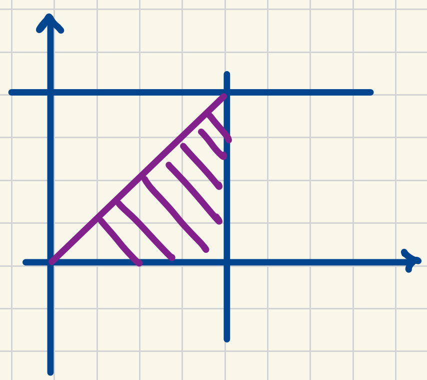
For any $\boxed{\text{"nice"}}$ region $\boxed{A} \subseteq \mathbb{R}^n$,

$$\boxed{\mathbb{P}[(Y_1, Y_2, \dots, Y_n) \in A] = \underbrace{\int \cdots \int}_{\boxed{A}} f_{Y_1,\dots,Y_n}(y_1, \dots, y_n)\, dy_n \dots dy_1}$$

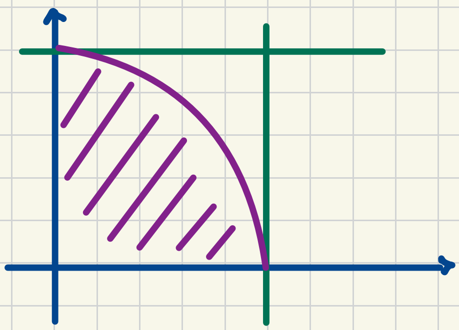
Example. (Y_1, Y_2) ... represents a point chosen $\boxed{@\ \text{random}}$ in a unit square $[0,1]^2$

$$\boxed{f_{Y_1,Y_2}(y_1, y_2) = 1 \cdot \mathbb{1}_{[0,1]\times[0,1]}(y_1, y_2)}$$

$\mathbb{P}[Y_1 > Y_2] = \cancel{?}\ \frac{1}{2}$



$\mathbb{P}[Y_1^2+Y_2^2 \le 1] = ?$



$$\boxed{A = \{(y_1,y_2) \in [0,1]^2 : y_1^2+y_2^2 \le 1\}}$$

$\mathbb{P}[(Y_1,Y_2) \in A] =$

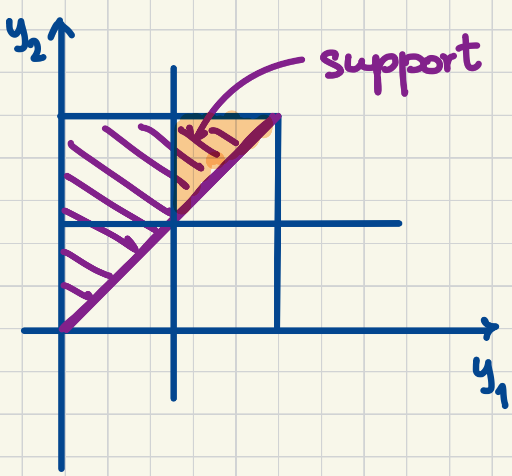
$$= \iint_A f_{Y_1,Y_2}(y_1,y_2)\,dy_2\,dy_1 = \cdots = \frac{\pi}{4}$$

Example. Let (Y_1,Y_2) be jointly continuous w/ pdf

$$\boxed{f_{Y_1,Y_2}(y_1,y_2) = 6y_1 \mathbb{1}_{[0 \le y_1 \le y_2 \le 1]}}$$

OR

$$f_{Y_1,Y_2}(y_1,y_2) = \begin{cases} 6y_1 & \text{for } 0 \le y_1 \le y_2 \le 1 \\ 0 & \text{otherwise} \end{cases}$$



$\mathbb{P}[Y_1 > \frac{1}{2}, Y_2 > \frac{1}{2}] =$

$$= \int_{\frac{1}{2}}^{1} \int_{\frac{1}{2}}^{1} 6y_1 \mathbb{1}_{[0 \le y_1 \le y_2 \le 1]}\,dy_2\,dy_1 =$$

$$= \int_{\frac{1}{2}}^{1} 6y_1 \int_{\cancel{\frac{1}{2}}\, y_1}^{1} \mathbb{1}_{[0 \leq y_1 \leq y_2 \leq 1]} \, dy_2 \, dy_1 =$$

$$= \int_{\frac{1}{2}}^{1} 6y_1 \underbrace{\int_{y_1}^{1} dy_2}_{= 1-y_1} \, dy_1 =$$

$$= \int_{\frac{1}{2}}^{1} 6y_1 (1-y_1) \, dy_1 =$$

$$= 6 \int_{\frac{1}{2}}^{1} (y_1 - y_1^2) \, dy_1 =$$

$$= 6 \left(\frac{y_1^2}{2} \Big|_{y_1=\frac{1}{2}}^{1} - \frac{y_1^3}{3} \Big|_{y_1=\frac{1}{2}}^{1} \right)$$

$$= 6 \left(\frac{1}{2} - \frac{1}{8} - \left(\frac{1}{3} - \frac{1}{24} \right) \right) = \cancel{6} \cdot \frac{12-3-8+1}{\cancel{24}\, 4} = \frac{1}{2} \quad \square$$

Functions of Random Vectors.

Theorem. Let $(Y_1, \dots, Y_n)$ be a continuous random vector w/ the joint pdf $f_{Y_1,\dots,Y_n}(\cdot, \dots, \cdot)$

Let g be a function of n variables such that we can define

$$W = g(Y_1, \dots, Y_n)$$

Then,

$$\mathbb{E}[W] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(y_1, \dots, y_n) \cdot f_{Y_1,\dots,Y_n}(y_1, \dots, y_n)\, dy_n \cdots dy_1$$

if the integral is well defined.

Example. (previous cont'd)

(Y_1, Y_2)

$$f_{Y_1,Y_2}(y_1, y_2) = \boxed{6y_1 \mathbb{1}_{[0 \le y_1 \le y_2 \le 1]}}$$

$$\mathbb{E}[Y_1^2 + Y_2^2] = ?$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (y_1^2 + y_2^2) \cdot 6y_1 \mathbb{1}_{[0 \le y_1 \le y_2 \le 1]}\, dy_2\, dy_1$$

$$= 6 \int_0^1 \int_{y_1}^1 (y_1^2 + y_2^2) \cdot y_1\, dy_2\, dy_1$$

$$= 6 \int_0^1 \int_{y_1}^1 (y_1^3 + y_1 y_2^2)\, dy_2\, dy_1$$

$$= 6 \int_0^1 \left(y_1^3 y_2 + y_1 \frac{y_2^3}{3}\right)\Big|_{y_2 = y_1}^{1}\, dy_1$$

$$= 6 \int_0^1 \left(y_1^3(1 - y_1) + y_1 \cdot \frac{1}{3} \cdot (1 - y_1^3)\right) dy_1$$

$$= 6\int_0^1 \left(y_1^3 - y_1^4 + \frac{y_1}{3} - \frac{y_1^4}{3}\right) dy_1$$

$$= \int_0^1 \left(\underbrace{6y_1^3} - \underbrace{8y_1^4} + \underbrace{2y_1}\right) dy_1$$

$$= 6\cdot\frac{1}{4} - 8\cdot\frac{1}{5} + 2\cdot\frac{1}{2} = \frac{3}{2} - \frac{8}{5} + 1 = \frac{9}{10} \quad \square$$