UNIVERSITY OF TEXAS AT AUSTIN

HW Assignment 2

Prerequisite material.

Please, provide your justification for your response to every question in this subsection. Just the final numerical answer will receive zero credit, even if it is correct. For the graphs, it is sufficient to carefully draw the graph correctly in a clearly labeled coordinate system.

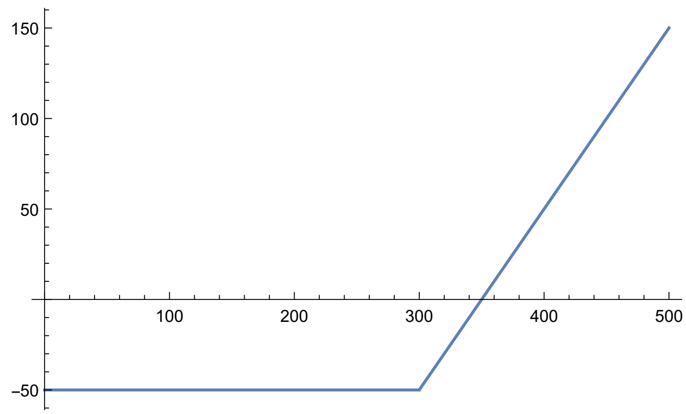
Problem 2.1. (5 points) Let the function f be given by

$$f(x) = \begin{cases} x - 300 & \text{for } x \geq 300 \\ 0 & \text{otherwise} \end{cases}$$

Draw the graph of the function g defined as

$$g(x) = f(x) - 50.$$

Solution:



Problem 2.2. (5 points) Let the function f be defined as

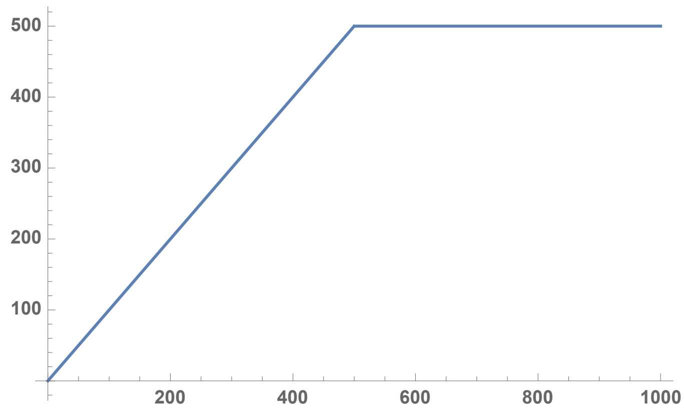
$$f(x) = x$$

Let the function g be defined as

$$g(x) = \begin{cases} 0 & \text{for } x < 500 \\ x - 500 & \text{for } x \geq 500 \end{cases}$$

Draw the graph of the function $f - g$.

Solution:



Problem 2.3. (5 points) Let $x > 0$. Then, we always have $e^x > 1 + x$. *True or false? Why?*

Solution: TRUE
By the Taylor expansion of the exponential function, we have

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

Since $x > 0$, the higher-order terms are all positive and the proposed inequality is correct.

Problem 2.4. (5 points)
We define the minimum of two values in the usual way, i.e.,

$$\min(x, y) = \begin{cases} x & \text{if } x \leq y \\ y & \text{if } x \geq y \end{cases}$$

We define the maximum of two values in the usual way, i.e.,

$$\max(x, y) = \begin{cases} x & \text{if } x \geq y \\ y & \text{if } x \leq y \end{cases}$$

Then, for every x and y we have that

$$x - \min(x - y, 0) = \max(x, y)$$

True or false? Why?

Solution: TRUE

$$\begin{aligned} x - \min(x - y, 0) &= \begin{cases} x - 0 = x, & \text{if } x \geq y \\ x - (x - y) = y, & \text{if } x < y \end{cases} \\ &= \max(x, y) \end{aligned}$$

Problem 2.5. (5 points)
We define the maximum of two values in the usual way, i.e.,

$$\max(x, y) = \begin{cases} x & \text{if } x \geq y \\ y & \text{if } x \leq y \end{cases}$$

Then, for every x and y we have that

$$\max(x, y) = \max(x - y, 0) + y$$

True or false? Why?

Solution: TRUE
If $x \geq y$, then the left-hand side of the proposed equality equals x. On the other hand, we also have that $x - y \geq 0$. So, the right-hand side equals

$$\max(x-y,0)+y = x-y+y = x.$$

If $x < y$, then the left-hand side of the equality equals y. On the other hand, we also have that $x - y < 0$. So, the right-hand side equals

$$\max(x-y,0)+y = 0+y = y.$$

Therefore, the proposed equality is always true.

Problem 2.6. (5 points) Let Y be a random variable such that $\mathbb{P}[Y = 2] = 1/2$, $\mathbb{P}[Y = 3] = 1/3$ and $\mathbb{P}[Y = 6] = 1/6$. What is $\mathbb{E}[\min(Y,5)]$?

Solution:

$$\mathbb{E}[\min(Y,5)] = \frac{1}{2}(2) + \frac{1}{3}(3) + \frac{1}{6}(5) = \frac{17}{6}\,.$$

Problem 2.7. (5 points) A coin for which *Heads* is twice as likely as *Tails* is tossed twice, independently. If the outcomes are two consecutive *Heads*, Bertie gets a \$15 reward; if the outcomes are different, Bertie gets a \$10 reward; if the outcomes are two consecutive *Tails*, Bertie has to pay \$5. What is the expected value of Bertie's winnings?

Solution: Since *Heads* is twice as likely as *Tails*, *Heads* appears with probability $2/3$, while *Tails* appears with probability $1/3$.

Let X denote the amount Bertie wins. Then, X has the following distribution:

$$X \sim \begin{cases} 15, & \text{with probability } 4/9, \\ 10, & \text{with probability } 4/9, \\ -5, & \text{with probability } 1/9. \end{cases}$$

$$\mathbb{E}[X] = \frac{4}{9}(15) + \frac{4}{9}(10) + \frac{1}{9}(-5) = \frac{95}{9}\,.$$

Problem 2.8. (5 points) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be two functions given by

$$f(x) = |x - 10|$$

and

$$g(x) = \begin{cases} \min(x,4) & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Then, $g(f(7))$ equals ...

(a) 0
(b) 3
(c) 4
(d) 7
(e) None of the above

Solution: (b)

$$f(7) = |7 - 10| = |-3| = 3$$
$$g(3) = \min(3,4) = 3$$

Problem 2.9. (5 points) Let the accumulation function be given by

$$a(t) = (1 + 0.05)^{2t}(1 + 0.02)^{t/3}$$

Then, we can say the following about the continuously compounded, risk-free interest rate r associated with the above accumulation function:

(a) $r = 0.11$

(b) $r = (\ln(1.05))^2 + (\ln(1.02))^{1/3}$

(c) $r = 2\ln(1.05) + \frac{1}{3}\ln(1.02)$

(d) The continuously compounded, risk-free interest rate is not constant.

(e) None of the above

Solution: (c)

$$r = \frac{d}{dt}\ln(a(t)) = \frac{d}{dt}\ln[(1 + 0.05)^{2t}(1 + 0.2)^{t/3}] = \frac{d}{dt}[2t\ln(1.05) + \frac{t}{3}\ln(1.02)] = 2\ln(1.05) + \frac{1}{3}\ln(1.02).$$

Problem 2.10. (5 points) *Source: Sample P Exam, Problem #198.*
In a certain group of cancer patients, each patient's cancer is classified in exactly one of the following five stages: stage 0, stage 1, stage 2, stage 3, or stage 4. You know the following:

(i) 75% of the patients in the group have stage 2 or lower;
(ii) 80% of the patients in the group have stage 1 or higher;
(iii) 80% of the patients in the group have stage 0, 1, 3, or 4.

One patient from the group is randomly selected. Calculate the probability that the selected patient's cancer is stage 1.

(a) 1/4

(b) 1/5

(c) 7/20

(d) Not enough information is given.

(e) None of the above.

Solution: (c)
Define $p_k = \mathbb{P}[\text{a randomly chosen patien has cancer stage } i]$, for $i = 0, 1, 2, 3, 4$. By the definition of probability, we have that

$$p_0 + p_1 + p_2 + p_3 + p_4 = 1.$$

From condition (i), we conclude that

$$p_0 + p_1 + p_2 = 0.75.$$

From condition (ii), we conclude that

$$p_1 + p_2 + p_3 + p_4 = 1 - p_0 = 0.80 \quad \Rightarrow \quad p_0 = 0.2.$$

From condition (iii), we conclude that

$$p_0 + p_1 + p_3 + p_4 = 1 - p_2 = 0.80 \quad \Rightarrow \quad p_2 = 0.2.$$

So, $p_1 = 0.75 - 0.2 - 0.2 = 0.35 = 7/20$.