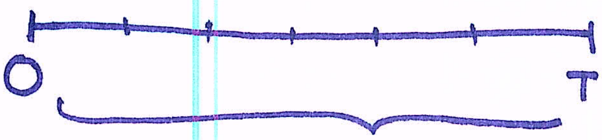
Realized Returns.

W: 02/08/2019

Recall:



n ... # of periods

$\Rightarrow$ length of every period is $h_n = \frac{T}{n}$

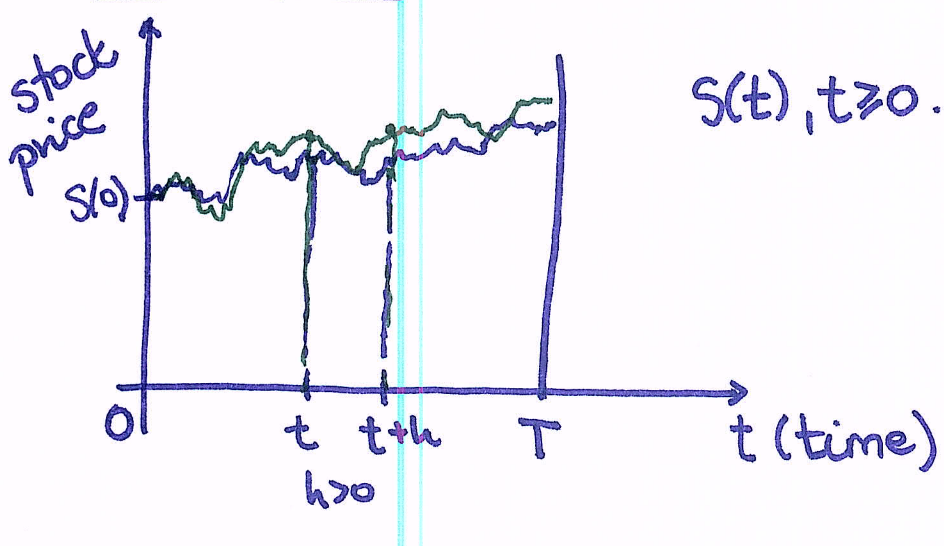
$\Rightarrow$ u_n, d_n, p^*_n dependent on n

Facts about our model:

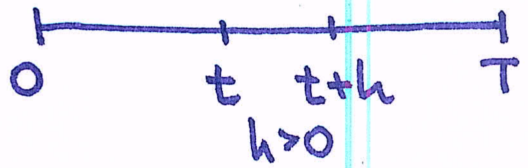
- $\rightarrow$ realized returns are <u>independent</u> between periods;
- $\rightarrow$ realized returns are <u>identically</u> <u>distributed</u> over different periods of the same length.

We want to <u>inherit</u> these properties in a continuous-time model.

In <u>continuous</u> time:



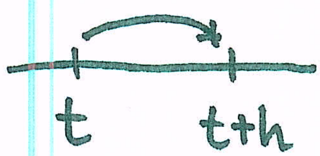
$S(t), t \geq 0$ ... time-t stock price

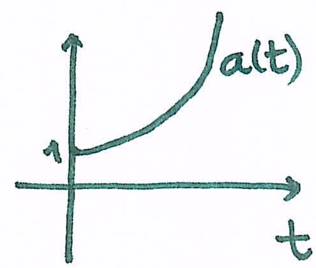


For every t, h, we define the realized return as

$$\underbrace{R(t, t+h)}_{\text{random variable}} := \ln\left(\frac{S(t+h)}{S(t)}\right)$$

Think back to the accumulation f'tion and the c.c.r.f.i.r. r:





$a(t) \quad a(t+h)$

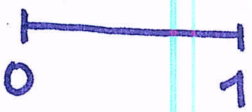
$$a(t+h) = a(t) \cdot e^{r \cdot h}$$

$$r \cdot h = \ln\left(\frac{a(t+h)}{a(t)}\right)$$

$$\boxed{S(t+h) = S(t)e^{R(t,t+h)}}$$

Recall: σ... volatility parameter;
the standard deviation of the realized return over a time period of length one year

We should have

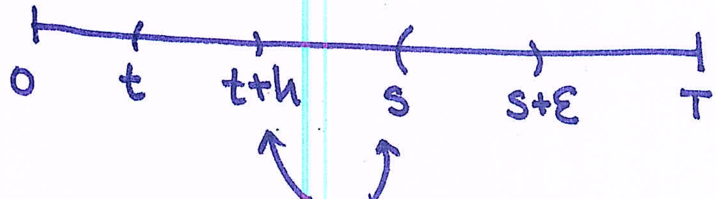


$$Var[R(0,1)] = \sigma^2, \text{ i.e., } \quad SD[R(0,1)] = \sigma$$

1^{st}

We require that $R(t,t+h)$ and $R(s,s+h)$ be identically distributed

2^{nd}

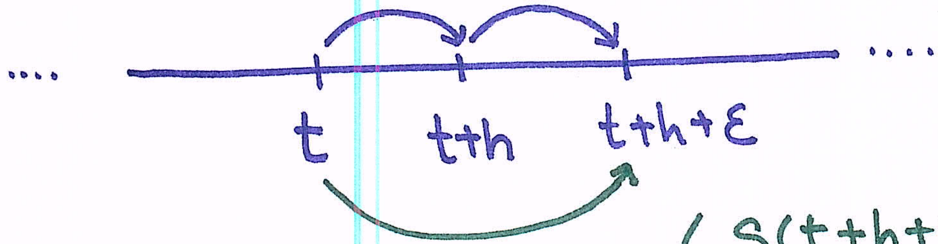


it's allowed to have $t+h=s$
(the intervals can touch @ the boundary)

$R(t,t+h)$ and $R(s,s+\varepsilon)$ are required to be independent.

These are the requirements we place on our model which are "inherited" from the discrete model.

3^{rd} Just from the def'n of realized returns, we have:



$$R(t, t+h+\varepsilon) = \ln\left(\frac{S(t+h+\varepsilon)}{S(t)}\right)$$

by def'n

$$= \ln\left(\frac{S(t+h+\varepsilon)}{S(t+h)} \cdot \frac{S(t+h)}{S(t)}\right)$$

$$= \ln\left(\frac{S(t+h+\varepsilon)}{S(t+h)}\right) + \ln\left(\frac{S(t+h)}{S(t)}\right)$$

$$= R(t, t+h) + R(t+h, t+h+\varepsilon)$$

Realized returns are additive. ∎

We decide to model

$R(t, t+h)$ as normally distributed for every choice of t, h.

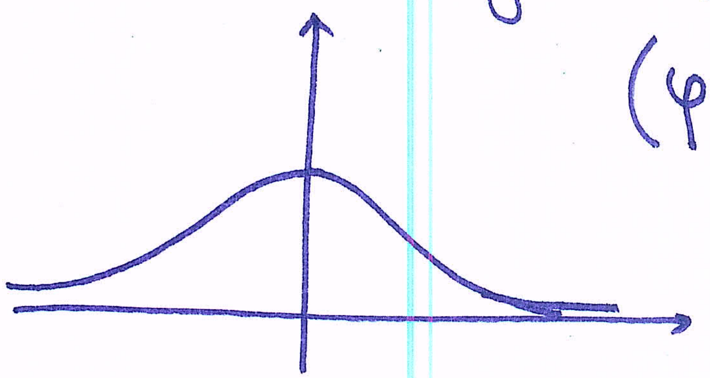
$$S(t+h) = S(t) \cdot \underbrace{e^{R(t,t+h)}}$$

Normal Distribution.

*Standard Normal random variable:

$$Z \sim N(\text{mean}=0, \text{sd}=1)$$

Its dist'n is given by its density (pdf):



$$\left(\varphi(z)=\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}, \quad z \in \mathbb{R}$$

$$=: f_Z(z)$$

$\Rightarrow$ Its cumulative dist'n f'tion (cdf):

$$N(a) = \mathbb{P}[Z \leq a]$$

$$= \int_{-\infty}^{a} f_Z(z)\,dz = \cdots \text{ we find in the std normal tables or using an online calculator}$$

$$f_Z(z) = N'(z)$$

* Normal random variable:

$$\boxed{X \sim N(\text{mean}=m, \text{sd}=\tau)}$$

is given through its relationship w/ the $\boxed{Z \sim N(0,1)}$

Since any normal random variable is a linear transform of the std normal, we get

$$X \overset{(d)}{=} m + \tau \cdot Z \quad \Leftrightarrow \quad \frac{X-m}{\tau} \overset{(d)}{=} Z$$

Moment Generating Function

For any random variable Y, its moment generating f'tion is:

$$M_Y(t) := \mathbb{E}\left[e^{Y \cdot t}\right]$$ wherever it's well-defined