M378K: October 28th, 2024.

More on Estimators.

Def'n. The bias of an estimator $\hat{\theta}$ for θ is

$$\text{bias}(\hat{\theta}) = \mathbb{E}[\hat{\theta} - \theta]$$

If $\text{bias}(\hat{\theta}) = 0$, we say that $\hat{\theta}$ is unbiased

Def'n. The mean squared error of $\hat{\theta}$ is

$$MSE(\hat{\theta}) = \mathbb{E}\left[(\hat{\theta} - \theta)^2\right]$$

Then,

$$\boxed{MSE(\hat{\theta}) = Var[\hat{\theta}] + (\text{bias}(\hat{\theta}))^2}$$

Def'n. An estimator $\hat{\theta}$ is UMVUE of θ if:

- $\hat{\theta}$ is unbiased
- $MSE[\hat{\theta}] \leq MSE[\hat{\theta}']$ for all other $\hat{\theta}'$ unbiased estimators of θ

Example. These are UMVUE :

- $\bar{Y}$ for μ where $(Y_1, \dots, Y_n)$ is a random sample from $N(\mu, 1)$
- $\bar{Y}$ for p where $(Y_1, \dots, Y_n)$ is a random sample from $B(p)$
- $\boxed{S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (Y_i - \bar{Y})^2}$ for σ^2 where $(Y_1, Y_2, \dots, Y_n)$ from $N(\mu, \sigma)$ (both unknown)

Def'n. An estimator $\hat{\theta}$ is said to be linear if it's of the form

$$\hat{\theta} = \alpha_1 Y_1 + \alpha_2 Y_2 + \dots + \alpha_n Y_n$$

where $\alpha_1, \dots, \alpha_n$ are all constants

Example. $\bar{Y}$ is a linear estimator

Def'n. A linear estimator $\hat{\Theta}$ is said to be the **best linear unbiased estimator (BLUE)** if

- $\hat{\Theta}$ is unbiased
- $MSE[\hat{\Theta}] \leq MSE[\hat{\Theta}']$ for all $\hat{\Theta}'$ unbiased linear estimators of Θ

Example 9.2.9. ← Look @ @ home

Problem 13.2. *Suppose that the two estimators $\hat{\theta}_1$ and $\hat{\theta}_2$ are unbiased. We know that* $\mathrm{Var}[\hat{\theta}_1] = \sigma_1^2$ *and* $\mathrm{Var}[\hat{\theta}_2] = \sigma_2^2$.

Consider ~~the estimator~~ all the estimators that can be obtained as convex combinations of $\hat{\theta}_1$ and $\hat{\theta}_2$, i.e., all the estimators of the form

$$\hat{\theta} = \alpha\hat{\theta}_1 + (1-\alpha)\hat{\theta}_2.$$

What can you say about the bias *of estimators $\hat{\theta}$ of the form above? Assuming that $\hat{\theta}_1$ and $\hat{\theta}_2$ are* **independent**, *for which weight α is the variance minimal?*

$$\longrightarrow: \quad \mathbb{E}[\hat{\theta}] = \mathbb{E}[\alpha\hat{\theta}_1 + (1-\alpha)\hat{\theta}_2] = \alpha\underbrace{\mathbb{E}[\hat{\theta}_1]}_{\theta} + (1-\alpha)\underbrace{\mathbb{E}[\hat{\theta}_2]}_{\theta} = \theta$$

$\Rightarrow$ $\hat{\theta}$ unbiased

$$\mathrm{Var}[\hat{\theta}] \longrightarrow \min$$

$$\mathrm{Var}[\alpha\hat{\theta}_1 + (1-\alpha)\hat{\theta}_2] \longrightarrow \min$$

independence

$$\mathrm{Var}[\alpha\hat{\theta}_1] + \mathrm{Var}[(1-\alpha)\hat{\theta}_2] \longrightarrow \min$$

$$\alpha^2\mathrm{Var}[\hat{\theta}_1] + (1-\alpha)^2\mathrm{Var}[\hat{\theta}_2] \longrightarrow \min$$

$$\alpha^2\sigma_1^2 + (1-\alpha)^2\sigma_2^2 \xrightarrow[\alpha]{} \min$$

PARABOLA $\cup$

$$\cancel{2}\alpha\sigma_1^2 + \cancel{2}(1-\alpha)(-1)\sigma_2^2 = 0$$

$$\alpha\sigma_1^2 + \alpha\sigma_2^2 - \sigma_2^2 = 0$$

$$\boxed{\alpha^* = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}}$$

$\square$

Problem 13.3. *Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from a continuous distribution with probability density function*

$$\boxed{f_Y(y) = \frac{\alpha y^{\alpha-1}}{\theta^\alpha}\mathbf{1}_{[0,\theta]}(y)}$$

with a known parameter $\alpha > 0$ and an unknown parameter $\theta > 0$. We propose the estimator $\hat{\theta} = \max(Y_1, \ldots, Y_n)$. Is this estimator unbiased*? If not, how would you modify it to create an* unbiased *estimator? What is the* **mean-squared error** *of the* unbiased *estimator you obtained?*

$$\hat{\Theta} = \max(Y_1, \ldots, Y_n) = Y_{(n)}$$

$$\mathbb{E}[\hat{\Theta}] = \mathbb{E}[Y_{(n)}] = ?$$

$$g_{(n)}(y) = n \cdot f_Y(y)\,\boxed{F_Y(y)}^{n-1}$$

$$\mathbb{E}[Y_{(n)}] = \int_0^\Theta y \cdot g_{(n)}(y)\,dy$$

$$F_Y(y) = \int_0^y f_Y(u)\,du$$

$$= \int_0^y \frac{\alpha u^{\alpha-1}}{\Theta^\alpha}\,du = \left(\frac{y}{\Theta}\right)^\alpha$$

$y \in [0,\Theta]$

$$\mathbb{E}[Y_{(n)}] = \int_0^\Theta y\, n \cdot \frac{\alpha y^{\alpha-1}}{\Theta^\alpha} \cdot \left(\left(\frac{y}{\Theta}\right)^\alpha\right)^{n-1} dy$$

$$= \frac{1}{\Theta^{\alpha n}} \cdot n \cdot \alpha \int_0^\Theta y^{\alpha n}\,dy = \frac{n\alpha}{\cancel{\Theta^{\alpha n}}} \cdot \frac{\Theta^{\cancel{\alpha n}+1}}{\alpha n+1}$$

$$= \boxed{\frac{\alpha n}{\alpha n+1} \cdot \Theta}$$

Not Unbiased!

Let's define: $\boxed{\hat{\Theta}' = \frac{\alpha n+1}{\alpha n}\hat{\Theta}}$

$$MSE[\hat{\Theta}'] = Var[\hat{\Theta}'] + (\underbrace{bias(\hat{\Theta}')}_{=0})^2 = Var[\hat{\Theta}'] =$$

$$= Var\left[\frac{\alpha n+1}{\alpha n}\hat{\Theta}\right] = \left(\frac{\alpha n+1}{\alpha n}\right)^2 \underbrace{Var[\hat{\Theta}]}_{?}$$

$$Var[\hat{\Theta}] = \underbrace{\mathbb{E}[\hat{\Theta}^2]}_{\int_0^{\Theta} y^2 g_{(n)}(y)\,dy} - (\mathbb{E}[\hat{\Theta}])^2$$

$$\int_0^{\Theta} y^2 g_{(n)}(y)\,dy = \dots\dots$$