M339J *Probability models*
University of Texas at Austin
HW IV
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Note: You **must** show all your work. Numerical answers without a proper explanation or a clearly written down path to the solution will be assigned zero points.

Problem 4.1. (5 pts) Let the severity random variable X be continuous such that $f_X(x) > 0$ for all $x > 0$. Let Y^P denote the per payment random variable associated with X for some ordinary deductible $d > 0$. Then the random variable Y^P is

(a) continuous.
(b) discrete, but not degenerate (constant).
(c) degenerate (constant).
(d) mixed.
(e) None of the above

Solution: (a)

Problem 4.2. (5 pts) Let the severity random variable X be continuous such that $f_X(x) > 0$ for all $x > 0$. Let Y^L denote the per loss random variable associated with X for some ordinary deductible $d > 0$. Then the random variable Y^L is

(a) continuous.
(b) discrete, but not degenerate (constant).
(c) degenerate (constant).
(d) mixed.
(e) None of the above

Solution: (d)

Problem 4.3. (5 pts) The ground-up loss X is modeled by an exponential distribution with mean \$500. There is an ordinary deductible of $d = 200$. What is the expected value of the **per-loss** random variable?

Solution: Let

$$Y^L = (X - d)_+$$

with $X \sim Exp(\theta = 500)$ and $d = 200$. Then,

$$\begin{aligned}
\mathbb{E}[Y^L] &= \mathbb{E}[(X-d)\mathbb{I}_{[X>d]}] \\
&= \int_d^\infty (x-d)\frac{1}{\theta}e^{-\frac{x}{\theta}}\,dx \\
&= \int_0^\infty y\frac{1}{\theta}e^{-\frac{y+d}{\theta}}\,dx \\
&= e^{-\frac{d}{\theta}}\int_0^\infty y\frac{1}{\theta}e^{-\frac{y}{\theta}}\,dx \\
&= \theta e^{-\frac{d}{\theta}} = 500e^{-2/5} \approx 335.16.
\end{aligned}$$

Problem 4.4. (5 points) Let the severity random variable X be modelled using the Pareto distribution with parameters $\theta = 0.5$ and $\alpha = 6$. For a particular value of the ordinary deductible d, the expected value of the per-payment random variable Y^P is 10. What is the value of the deductible?

Solution: We can calculate that $\mathbb{E}[Y^P] = \frac{d+\theta}{\alpha-1}$. So,

$$\mathbb{E}[Y^P] = \frac{d+\theta}{\alpha-1} = \frac{d+0.5}{6-1} = 10 \quad \Rightarrow \quad d = 49.5.$$

Problem 4.5. (5 points) For a random variable X and for a positive constant d, in our usual notation, we have

$$\mathbb{E}[X] = e_X(d)S_X(d) + \mathbb{E}[X \wedge d]\,. \tag{4.1}$$

True or false? Why?

Solution: TRUE
See your class notes.

Problem 4.6. (5 points) Let $X \sim Pareto(\alpha = 3, \theta = 3000)$. Assume that there is a deductible of $d = 5000$. Find $\mathbb{E}[X \wedge d]$.

Solution: Using the tables, we get

$$\mathbb{E}[X \wedge d] = \frac{\theta}{\alpha-1}\left[1 - (\tfrac{\theta}{d+\theta})^{\alpha-1}\right] = \frac{3000}{3-1}\left[1 - (\tfrac{3000}{5000+3000})^{3-1}\right] = 1500[1 - (\tfrac{3}{8})^2].$$

Problem 4.7. (10 points) Let X be the ground-up loss random variable whose density is given by

$$f_X(x) = \begin{cases} 0.01, & 0 \le x \le 80, \\ 0.03 - 0.00025x, & 80 < x \le 120. \end{cases}$$

Let there be an ordinary deductible of $d = 20$.
Calculate $\mathbb{E}[X \wedge d]$.

Solution: In this problem, with $d = 20$, we get

$$\begin{aligned}
\mathbb{E}[X \wedge 20] &= \int_0^{120} (x \wedge 20) f_X(x)\, dx \\
&= \int_0^{20} x f_X(x)\, dx + \int_{20}^{120} 20 f_X(x)\, dx \\
&= \int_0^{20} x f_X(x)\, dx + 20 \int_0^{120} f_X(x)\, dx - 20 \int_0^{20} f_X(x)\, dx.
\end{aligned}$$

We have

$$\int_0^{20} x\, f_X(x)\, dx = 0.01 \int_0^{20} x\, dx = 0.01 \cdot \frac{1}{2} x^2 |_{x=0}^{20} = 0.005 \cdot 20^2 = 0.005 \cdot 400 = 2.$$

Since f_X is a density function,

$$\int_0^{120} f_X(x)\, dx = 1.$$

As for the third integral,

$$\int_0^{20} f_X(x)\, dx = 0.01 \cdot 20 = 0.2.$$

Putting everything together,

$$\mathbb{E}[X \wedge 20] = 2 + 20 \cdot 1 - 20 \cdot 0.2 = 2 + 20 - 4 = 18.$$

Problem 4.8. (10 points) *Source: Problem 4.3 from "Loss Models".*
Assume that the claims r.v. X has a Pareto distribution with $\alpha = 2$ and θ unknown. Claims for the following year are denoted by Y and will experience uniform inflation of 6%.

(i) (2 points) Find the expression for the probability $\mathbb{P}[X > d]$ in terms of d and θ.
Solution: Using our tables, we get

$$\mathbb{P}[X > d] = 1 - F_X(d) = \left(\frac{\theta}{\theta + d}\right)^2.$$

(ii) (3 points) Find the expression for the probability $\mathbb{P}[Y > d]$ in terms of d and θ.
Solution: We established in class that the Pareto is scale with the scale parameter θ. So, taking into account the inflation rate of 0.06, we get

$$Y \sim Pareto(\alpha = 2, \theta^* = 1.06\theta).$$

Again, we use our tables and obtain

$$\mathbb{P}[Y > d] = 1 - F_Y(d) = \left(\frac{\theta^*}{\theta^* + d}\right)^2 = \left(\frac{1.06\theta}{1.06\theta + d}\right)^2.$$

(iii) (5 points) Find the expression for the ratio $\rho(d) = \frac{\mathbb{P}[X>d]}{\mathbb{P}[Y>d]}$ in terms of d and θ. Find the limit of $\rho(d)$ as $d \to \infty$.

Solution:

$$\rho(d) = \frac{\left(\frac{\theta}{\theta+d}\right)^2}{\left(\frac{1.06\theta}{1.06\theta+d}\right)^2} = 1.06^{-2}\left(\frac{1.06\theta + d}{\theta + d}\right)^2.$$

As $d \to \infty$,

$$\rho(d) \to 1.06^{-2} \cdot 1^2 = 1.06^{-2} = 1/1.1236 \approx 0.89.$$