

UNIVERSITY OF TEXAS AT AUSTIN

HW Assignment 2

Note: You **must** show all your work. Numerical answers without a proper explanation or a clearly written down path to the solution will be assigned zero points.

Problem 2.1. (20 points) Two coins are tossed and a (6-sided) die is rolled.

- (i) (5 points) Describe a sample space (probability space), together with the probability, on which such a situation can be modelled.
- (ii) (15 points) Find the probability mass function of the random variable whose value is the sum of the number on the die and the total number of heads.

Solution: Each elementary event ω should track the information about three things - the outcome of the first coin toss, the outcome of the second coin toss and the number on the die. This corresponds to triplets $\omega = (c_1, c_2, d)$, where $c_1, c_2 \in \{H, T\}$ and $d \in \{1, \dots, 10\}$. Therefore, $\Omega = \{H, T\} \times \{H, T\} \times \{1, \dots, 6\}$. Since all the instruments involved are fair, the independence requirements dictate that

$$\mathbb{P}[\omega = (c_1, c_2, d)] = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{6} = \frac{1}{24},$$

for any $(c_1, c_2, d) \in \Omega$. In words, all elementary events are equally likely. Let C_1 be the random variable which equals to 1 if the outcome of the first coin toss is H , so that

$$C_1(\omega) = \begin{cases} 1, & c_1 = H, \\ 0, & \text{otherwise,} \end{cases} \text{ where } \omega = (c_1, c_2, d).$$

In other words, $C_1 = \mathbf{1}_A$ is the indicator of the event

$$A = \{\omega = (c_1, c_2, d) \in \Omega : c_1 = H\}.$$

Let C_2 and D (the number on the die) be defined analogously. Then the total number of heads M is given by $M = C_1 + C_2$. Each C_1 and C_2 are independent Bernoulli random variables with $p = \frac{1}{2}$, so M is a binomial random variable with $n = 2$ and $p = \frac{1}{2}$. Therefore, the pmf of M is

	0	1	2
p	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Let X be the random variable from the text of the problem:

$$X = D + M.$$

The values random variable X can take are $\{1, 2, \dots, 8\}$, and they correspond to the following table (the table entry is the value of X , columns go with D and rows with M):

	1	2	3	4	5	6
0	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8

A bit of accounting gives the following pmf for X :

	1	2	3	4	5	6	7	8
p	$\frac{1}{24}$	$\frac{3}{24}$	$\frac{4}{24}$	$\frac{4}{24}$	$\frac{4}{24}$	$\frac{3}{24}$	$\frac{1}{24}$	

Problem 2.2. (10 points) *Source: Sample P exam, Problem #126.* Under an insurance policy, a maximum of five claims may be filed per year by a policyholder. Let p_n be the probability that a policyholder files exactly n claims during a given year, where $n = 0, 1, 2, 3, 4, 5$. An actuary makes the following observations:

- $p_n \geq p_{n+1}$ for $n = 0, 1, 2, 3, 4$.
- The difference between p_n and p_{n+1} is the same for $n = 0, 1, 2, 3, 4$.
- Exactly 40% of policyholders file strictly fewer than two claims during a given year.

Calculate the probability that a random policyholder will file strictly more than three claims during a given year.

Solution: Let us denote the constant difference between p_n and p_{n+1} by a . Then, we have that for $n = 1, \dots, 5$, $p_n = p_0 + na$. Hence,

$$\begin{aligned} p_0 + p_1 + \dots + p_5 = 1 &\Rightarrow p_0 + (p_0 + a) + \dots + (p_0 + 5a) = 1 \\ &\Rightarrow 6p_0 + a(1 + \dots + 5) = 1 \Rightarrow 6p_0 + 15a = 1 \end{aligned}$$

On the other hand, we are given that

$$p_0 + p_1 = 0.4 \Rightarrow 2p_0 + a = 0.4.$$

From the two equations, we get $12a = -0.2$, i.e., $a = -\frac{1}{60}$. Thus, $p_0 = 5/24$. Our answer is

$$p_4 + p_5 = 2p_0 + 9a = 2 \left(\frac{5}{24} \right) - \frac{9}{60} = \frac{5}{12} - \frac{3}{20} = \frac{25-9}{60} = \frac{16}{60} = \frac{4}{15}.$$

Problem 2.3. (10 points) A continuous random variable X has the probability density function f_X given by

$$f_X(x) = A - \frac{x}{50}, \quad 0 \leq x \leq 10.$$

- Find the value of the constant A .
- Find the value of the survival function of X at 7, i.e., calculate $S_X(7)$.

Solution:

- Necessarily, $\int_0^{10} f_X(x) dx = 1$. So,

$$10A = 1 + \frac{10^2}{2 \cdot 50} = 2 \Rightarrow A = 1/5.$$

- Note that f_X is piecewise linear. So, we can calculate $S_X(7) = \mathbb{P}[X > 7]$ as the area of a triangle (draw a picture if in doubt!). We get

$$S_X(7) = \frac{1}{2} \cdot f_X(7) \cdot (10 - 7) = \frac{3}{2} \cdot \left(\frac{1}{5} - \frac{7}{50} \right) = \frac{3(10-7)}{2 \cdot 50} = 9/100.$$

Problem 2.4. (10 points) The lifespan of a certain machine is exponentially distributed. The probability that the lifespan exceeds 4 years is p . Find the expression for the density of the lifespan in terms of p .

Solution: Let T be the random variable which stands for the lifespan in the problem. It is modelled as exponential. Let's denote its parameter, as usual, by θ . We can write

$$T \sim \text{Exponential}(\text{mean} = \theta).$$

The survival function of the random variable T is of the form, for all $t > 0$,

$$S_T(t) = \mathbb{P}[T > t] = e^{-t/\theta}.$$

In the present problem, we are given that

$$\mathbb{P}[T > 4] = p \Rightarrow e^{-4/\theta} = p \Rightarrow -4/\theta = \ln(p) \Rightarrow \theta = -\frac{4}{\ln(p)}.$$

The density of the exponential random variable T is of the form, for all $t > 0$,

$$f_T(t) = \frac{1}{\theta} e^{-t/\theta} = -\frac{\ln(p)}{4} e^{t \ln(p)/4} = -\frac{\ln(p)}{4} p^{t/4}.$$