M339J: May 2nd, 2022.

**276.** For a group of policies, you are given:

(i) Losses follow the distribution function

$$\boxed{F(x) = 1 - \theta / x,} \quad x > 0.$$

(ii) A sample of 20 losses resulted in the following:

| Interval | Number of Losses |
|---|---|
| (0,10] | 9 |
| (10, 25] | 6 |
| (25,∞) | 5 |

Calculate the maximum likelihood estimate of $\theta$.

(A) 5.00

(B) 5.50

(C) 5.75

(D) 6.00

(E) 6.25

Likelihood ftion:

$$L(\theta) = (F(10))^9 (F(25) - F(10))^6 (1 - F(25))^5$$

$$L(\theta) = \left(1 - \frac{\theta}{10}\right)^9 \left(\cancel{1} - \frac{\theta}{25} - \left(\cancel{1} - \frac{\theta}{10}\right)\right)^6 \left(\cancel{1} - \left(\cancel{1} - \frac{\theta}{25}\right)\right)^5$$

$$L(\theta) = \underbrace{\left(1 - \frac{\theta}{10}\right)^9}_{\frac{(10-\theta)^9}{10^9}} \theta^6 \left(\frac{1}{10} - \frac{1}{25}\right)^6 \frac{\theta^5}{25^5}$$

$$L(\theta) \propto (10-\theta)^9 \cdot \theta^{11}$$

proportional to

$\Rightarrow$ Up to a constant, our log-likelihood is

$$\ell(\theta) = 9 \ln(10-\theta) + 11 \ln(\theta)$$

$$\Rightarrow \ell'(\theta) = 9 \cdot (-1) \cdot \frac{1}{10-\theta} + 11 \cdot \frac{1}{\theta} = 0$$

$$\frac{9}{10-\theta} = \frac{11}{\theta}$$

$$9\theta = 110 - 11\theta \Rightarrow 20\theta = 110$$

$$\Rightarrow \hat{\theta}_{MLE} = 5.5.$$

# MLE: Censored and truncated date.

## Censoring.

If the $j^{th}$ data point is censored @ $u_j$, then we set

$$A_j = [u_j, +\infty) \left(= (u_j, +\infty)\right)$$

## Truncation.

If the $j^{th}$ data point is truncated @ $d_j$, then we can do:

- shifting: let $x_j$ be the value of the actual loss; construct $L(\theta)$ based on the actual losses w/ the model for actual losses;
- conditioning: omit $x_j < d_j$;

for the remainder

$$L(\theta) = \prod_{j=1}^{n} \frac{f_X(x_j;\theta)}{S_X(d_j;\theta)}$$

for individual observations.

**218.** The random variable $X$ has survival function:

$$\boxed{S_X(x) = \frac{\theta^4}{(\theta^2 + x^2)^2}}$$

Two values of $X$ are observed to be 2 and 4. One other value exceeds 4.

Calculate the maximum likelihood estimate of $\theta$.

$\longrightarrow$ : $L(\theta) = f(2) \cdot f(4) \cdot S(4)$

$f(x) = -S'(x) = +\theta^4(+2)\cdot(\theta^2+x^2)^{-3}\cdot(2x)$

$f(x) = \dfrac{4\theta^4 \cdot x}{(x^2+\theta^2)^3}$

(A) Less than 4.0

(B) At least 4.0, but less than 4.5

(C) At least 4.5, but less than 5.0

(D) At least 5.0, but less than 5.5

(E) At least 5.5

**219.** For a portfolio of policies, you are given:

(i) The annual claim amount on a policy has probability density function:

$$f(x \mid \theta) = \frac{2x}{\theta^2}, \quad 0 < x < \theta$$

(ii) The prior distribution of $\theta$ has density function:

$$\pi(\theta) = 4\theta^3, \quad 0 < \theta < 1$$

(iii) A randomly selected policy had claim amount 0.1 in Year 1.

Calculate the Bühlmann credibility estimate of the claim amount for the selected policy in Year 2.

(A) 0.43

(B) 0.45

(C) 0.50

(D) 0.53

(E) 0.56

$$L(\theta) = \frac{4\theta^4 \cdot 2}{(\theta^2+4)^3} \cdot \frac{4\theta^4 \cdot 4}{(\theta^2+16)^3} \cdot \frac{\theta^4}{(\theta^2+16)^2} \propto \frac{\theta^{12}}{(\theta^2+4)^3(\theta^2+16)^5}$$

$$l(\theta) \text{"="} \ 12 \ln(\theta) - 3\ln(\theta^2+4) - 5\ln(\theta^2+16)$$

$$l'(\theta) = 12 \cdot \frac{1}{\theta} - 3 \cdot \frac{1}{\theta^2+4}(2\theta) - 5 \cdot \frac{1}{\theta^2+16}(2\theta) = 0$$

$$\frac{12}{\theta} = \frac{6\theta}{\theta^2+4} + \frac{10\theta}{\theta^2+16}$$

$$12(\theta^2+4)(\theta^2+16) = 6\theta^2(\theta^2+16) + 10\theta^2(\theta^2+4)$$

$$12\theta^4 + 12 \cdot 16 \cdot \theta^2 + 12 \cdot 4 \cdot \theta^2 + 12 \cdot 4 \cdot 16 = 6\theta^4 + 96\theta^2 + 10\theta^4 + 40\theta^2$$

$$4\theta^4 + \underbrace{(136 - 12 \cdot 20)}_{-104}\theta^2 - 768 = 0 \qquad /:4$$

$$\theta^4 - 26\theta^2 - 192 = 0$$

$$\theta^2_{1,2} = \frac{26 \pm \sqrt{26^2 + 4 \cdot 192}}{2} = 32 \Rightarrow \hat{\theta}_{MLE} = \sqrt{32}$$

$$= 5.6569 \quad \square$$

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**Example.** Truncated data w/ a common $d$.

The likelihood function is:

$$L(\theta) = \prod_{j=1}^{n} \frac{f_X(x_j;\theta)}{S_X(d;\theta)} = \frac{1}{(S_X(d;\theta))^n} \cdot \prod_{j=1}^{n} f_X(x_j;\theta)$$

$\Rightarrow$ the log-likelihood ftion:

$$l(\theta) = \sum_{j=1}^{n} \ln(f_X(x_j;\theta)) - n \cdot \ln(S_X(d;\theta))$$