UNIVERSITY OF TEXAS AT AUSTIN, DEPARTMENT OF MATHEMATICS
M358K - Applied Statistics

IN-TERM EXAM III

Free-response problems.

Problem 3.1. (10 points) Every jar of *Nocciolata* is labeled to contain 270 mg of delicious hazelnut spread. You suspect that the mean contents of every jar is actually **less**. You do some research and you are comfortable assuming that the amount of spread per jar is normally distributed with a known standard deviation of 10 mg. Your plan is to randomly collect 64 jars and weigh their contents. What is the form of your rejection region with the significance level of 1%?

Solution: We are testing

$$H_0 : \mu = \mu_0 = 270 \quad vs. \quad H_a : \mu < \mu_0 = 270.$$

So, remembering that the amount of spread in the jar cannot be negative, the form of our rejection region is, in our usual notation,

$$RR = \left[0, \mu_0 + z_\alpha \left(\frac{\sigma}{\sqrt{n}}\right)\right]$$

where $z_\alpha = \Phi^{-1}(0.01) = -2.33$. We conclude that

$$RR = \left[0, 270 - 2.33 \left(\frac{10}{\sqrt{64}}\right)\right] = [0, 270 - 2.9125] = [0, 267.0875] \, .$$

Problem 3.2. (10 points) A medicine dispensing machine is supposed to be calibrated to dispense 20 ml of medication. Of course, the amount dispensed is not exact. You model the amount actually dispensed using the normal distribution with standard deviation 1.5 ml. Periodically, the machine is tested to see if it's correctly calibrated. Each time, a sample of 9 doses is taken and measured carefully. With a particular significance level, the rejection region is the complement of the interval $(19.1, 20.9)$. What is the power of the test at the alternative mean $\mu_a = 21.5$?

Solution: Under the alternative $\mu_a = 21.5$, the distribution of the sample mean is

$$\bar{X} \sim Normal \left(mean = 21.5, sd = \frac{1.5}{\sqrt{9}} = \frac{1.5}{3} = 0.5\right).$$

The probability of making a Type II Error is

$$\begin{aligned}\beta = \mathbb{P}[19.1 < \bar{X} < 20.9] &= \mathbb{P}\left[\frac{19.1 - 21.5}{0.5} < \frac{\bar{X} - 21.5}{0.5} < \frac{20.9 - 21.5}{0.5}\right] \\ &= \mathbb{P}[-4.8 < Z < -1.2] = \mathbb{P}[Z < -1.2] - \mathbb{P}[Z \leq -4.8] = \Phi(-1.2) - 0 = 0.1151.\end{aligned}$$

So, the power of the test is $1 - \beta = 1 - 0.1151 = 0.8849$.

Problem 3.3. (5 points) In a random sample of 1000 homes in a particular large city, it is found that 228 have fireplaces. Find the 99%-confidence interval for the proportion of homes in this city that have fireplaces.

Solution: The point estimate is $\hat{p} = 228/1000 = 0.228$. The critical value z^* associated with the 99% confidence level is $z^* = 2.576$. So, the confidence interval is

$$0.228 \pm 2.576\sqrt{\frac{0.228(1 - 0.228)}{1000}} = 0.228 \pm 0.0342$$

Problem 3.4. (10 points) We want to compare the proportions of people from town A (pop. 25000) and people from town B (pop. 50000) whose favorite music genre is country music. Two polls are conducted. In town A, 120 out of the total of 200 people like country music best. In town B, 240 out of the total of 500 people like country music best. You conduct a hypothesis test for whether there is a difference in population proportions. Which $p-$value would you report?

Solution: Let p_A be the probability that a randomly chosen person from town A prefers country music and let p_B be the probability that a randomly chosen person from town B prefers country music. We are testing

$$H_0 : p_A = p_B \quad vs. \quad H_a : p_A \neq p_B.$$

The point estimates for the two proportions are $\hat{p}_A = \frac{120}{200} = 0.60$ and $\hat{p}_B = \frac{240}{500} = 0.48$. The pooled estimate of the proportion is

$$\hat{p} = \frac{120 + 240}{200 + 500} = 0.5143$$

The observed value of the $z-$statistic, under the null, is

$$z = \frac{0.60 - 0.48}{\sqrt{(0.5143)(1 - 0.5143)\left(\frac{1}{200} + \frac{1}{500}\right)}} = 2.87$$

So, the $p-$value is $2\Phi(-2.87) = 2(0.0021) = 0.0042$.

Problem 3.5. (5 points) You want to test whether a six-sided die is fair. Here are the observed counts in 120 rolls of the die:

Side	1	2	3	4	5	6
Count	20	22	17	19	24	18

What is your $p-$value? *Give bounds from the χ^2-table if not using R.*

Solution: Let $p_i, i = 1, \dots, 6$ be the probability that the die falls on i. We are testing

$$\begin{aligned} H_0 : &\quad p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = \frac{1}{6} \\ &\quad vs. \\ H_a : &\quad \text{At least one of the probabilities } p_i \text{ is different from } \frac{1}{6}. \end{aligned}$$

The observed value of the χ^2-statistic is

$$q^2 = \frac{1}{20}((20-20)^2 + (22-20)^2 + (17-20)^2 + (19-20)^2 + (24-20)^2 + (18-20)^2) = 1.7.$$

With $6 - 1 = 5$ degrees of freedom, using the χ^2-table, we see that the $p-$value is more than 0.30.

Problem 3.6. (10 points) A survey of 1000 students in Austin found that 274 chose Pikachu as their favorite Pokemon. In another survey of 760 students in Pittsburgh, 240 chose Pikachu. Find the 95%-confidence interval for the difference in the two population proportions.

Solution: Let the proportion of Pikachu fans in Austin be p_A and let the proportion of Pikachu fans in Pittsburgh be p_P. Then, the point estimates of the two proportions equal

$$\hat{p}_A = \frac{274}{1000} = 0.274 \quad \text{and} \quad \hat{p}_P = \frac{240}{760} = 0.3158.$$

The critical value associated with the 95%-confidence level is 1.96. The standard error equals

$$\sqrt{\frac{0.274(1-0.274)}{1000} + \frac{0.3158(1-0.3158)}{760}} = 0.022.$$

So, the confidence interval is $(0.274 - 0.3158) \pm 1.96(0.022)$, i.e., $(-0.0849, 0.0013)$

Multiple-choice problems.

Problem 3.7. (5 points) Public policy researchers are studying whether a new school lunch program reduces obesity amongst elementary school children. The authors compute the $p-$value for their sample to be 0.10. Which of the following interpretations of the $p-$value is correct?

(a) The probability that the policy is effective.
(b) The probability that the policy is *not* effective.
(c) The probability of determining the policy is not effective when it actually is.
(d) The probability of getting results as extreme or more extreme than the ones in the study if the policy is actually effective.
(e) The probability of getting results as extreme or more extreme than the ones in the study if the policy is actually *not* effective.

Solution: (e)

Problem 3.8. (5 points) *Source: "Probability and Statistics for Engineers and Scientists" by Walpole, Myers, Myers, and Ye.*
A study was conducted to determine whether there is an association between being for or against tax reform and income level. The results are displayed in the following table:

	Low	Medium	High	Total
For	182	213	203	598
Against	154	138	110	402
Total	336	351	313	1000

Your goal is to test whether being for or against the tax reform is independent from income level. The observed value of the relevant test statistic is 7.85. What is your decision for this hypothesis test?

(a) Reject at the 0.01 significance level.
(b) Fail to reject at the 0.01 significance level; reject at the 0.025 significance level.
(c) Fail to reject at the 0.025 significance level; reject at the 0.05 significance level.
(d) Fail to reject at the 0.05 significance level; reject at the 0.10 significance level.
(e) Fail to reject at the 0.10 significance level.

Solution: (b)
The distribution of the test statistic is approximately χ^2 with $(3-1)(2-1) = 2$ degrees of freedom. Consulting the χ^2-table, we see that the given observed value of the test statistic is between the critical values $\chi^2_{0.01}(df = 2)$ and $\chi^2_{0.025}(df = 2)$. So, the $p-$value is between 0.01 and 0.025.