Name:

M339W/389W Financial Mathematics for Actuarial Applications
University of Texas at Austin
The Prerequisite In-Term Exam
Instructor: Milica Čudina

Notes: This is a closed book and closed notes exam. The maximum number of points on this exam is 100.
Time: 50 minutes

1.1. **Free-response problems.** Please, explain carefully all your statements and assumptions. Numerical results or single-word answers without an explanation (even if they're correct) are worth 0 points.

Problem 1.1. (15 points)
Two scales are used to measure the mass m of a precious stone. The first scale makes an error in measurement which we model by a normally distributed random variable with mean $\mu_1 = 0$ and standard deviation $\sigma_1 = 0.04m$. The second scale is more accurate. We model its error by a normal random variable with mean $\mu_2 = 0$ and standard deviation $\sigma_2 = 0.03m$.

We assume that the measurements made using the two different scales are independent.

To get our final estimate of the mass of the stone, we take the average of the two results from the two different scales.

What is the probability that the value we get is within $0.005m$ of the actual mass of the stone?

Solution: Let us denote the random variable modeling the error from the first scale by $X_1 \sim N(0, \sigma_1^2)$ and the random variable modeling the error from the second scale by $X_2 \sim N(0, \sigma_2^2)$.

Then, if Y denotes the average of the two measurements, we have that

$$Y = \frac{1}{2}(X_1 + X_2) \sim N(0, \frac{1}{4}(\sigma_1^2 + \sigma_2^2)),$$

i.e.,

$$Y \sim N(0, \sigma^2)$$

with

$$\sigma^2 = \frac{1}{4}(\sigma_1^2 + \sigma_2^2) = \frac{1}{4}(0.04^2 m^2 + 0.03^2 m^2) = \frac{1}{4} \cdot 0.01^2 m^2 (4^2 + 3^2) = \frac{1}{4} 0.05^2 m^2 = \left(\frac{0.05m}{2}\right)^2.$$

The probability we are looking for can be expressed as

$$\begin{aligned}\mathbb{P}[Y \in (-0.005m, 0.005m)] &= \mathbb{P}[-0.005m < Y < 0.005m] \\ &= \mathbb{P}[-\frac{2 \cdot 0.005m}{0.05m} < \frac{Y}{\sigma} < \frac{2 \cdot 0.005m}{0.05m}] \\ &= \mathbb{P}[-0.2 < \frac{Y}{\sigma} < 0.2].\end{aligned}$$

Since $\frac{Y}{\sigma} \sim N(0,1)$, the above probability equals

$$2\Phi(0.2) - 1 \approx 2 \cdot 0.57926 - 1 \approx 0.16.$$

Problem 1.2. (10 points) Assume that $Y_1 = e^X$ where X is a standard normal random variable.

(i) (2 points) What is the probability that Y_1 exceeds 5?
(ii) $(3+5$ points) Find the mean and the variance of Y_1.

Hint: It helps if you use the expression for the moment generating function of a standard normal random variable.

Solution:

(i)

$$\mathbb{P}[Y_1 > 5] = \mathbb{P}[e^X > 5] = \mathbb{P}[X > \ln(5)] = 1 - N(\ln(5)) \approx 1 - N(1.61) = 1 - 0.9463 = 0.0537.$$

(ii)

$$\mathbb{E}[Y_1] = \mathbb{E}[e^X] = \mathbb{E}[e^{1 \cdot X}] = M_X(1)$$

where M_X denotes the moment generating function of X. In class, we recalled the following expression for M_X:

$$M_X(t) = e^{t^2/2}.$$

So, $\mathbb{E}[Y_1] = e^{1/2} = \sqrt{e}$.

The second moment of Y_1 is obtained similarly as

$$\mathbb{E}[Y_1^2] = \mathbb{E}[e^{2 \cdot X}] = M_X(2) = e^2.$$

So,

$$Var[Y_1] = \mathbb{E}[Y_1^2] - (\mathbb{E}[Y_1])^2 = e^2 - e = e(e-1).$$

Problem 1.3. (15 points) The random vector (X_1, X_2, X_3) is jointly normal. Its marginal distributions are:

$$X_1 \sim N(\text{mean} = 0, \text{variance} = 4), X_2 \sim N(\text{mean} = 1, \text{variance} = 1), X_3 \sim N(\text{mean} = -1, \text{variance} = 9).$$

The correlation coefficients are given to be

$$corr[X_1, X_2] = 0.3, corr[X_2, X_3] = 0.4, corr[X_1, X_3] = -0.3.$$

What is the distribution of the random variable $X = X_1 - X_2 + 2X_3$? Please, provide the **name** of the distribution, as well as the **values** of its parameters.

Solution:

(2 points) The linear combination of jointy normal random variables in normally distributed itself. Now, we need to identify the mean and the variance of this normal distribution.

(3 points) The mean of X is

$$\mathbb{E}[X] = \mathbb{E}[X_1] - \mathbb{E}[X_2] + 2\mathbb{E}[X_3] = 0 - 1 + 2(-1) = -3.$$

(10 points) The variance of X is

$$\begin{aligned} Var[X] &= Var[X_1] + Var[X_2] + 4Var[X_3] - 2Cov[X_1, X_2] + 4Cov[X_1, X_3] - 4Cov[X_2, X_3] \\ &= 4 + 1 + 36 - 2(2)(1)(0.3) + 4(2)(3)(-0.3) - 4(1)(3)(0.4) = 27.8. \end{aligned}$$

Problem 1.4. (10 pts) For a two-period binomial model, you are given that:

(1) each period is one year;
(2) the current price of a non-dividend-paying stock S is $S(0) = \$20$;
(3) $u = 1.3$, with u as in the standard notation for the binomial model;
(4) $d = 0.9$, with d as in the standard notation for the binomial model;
(5) the continuously compounded risk-free interest rate is $r = 0.05$.

Consider a special call option which pays the excess above the strike price $K = 23$ (if any!) at the end of **every** binomial period.

Find the price of this option.

Solution: The risk-neutral probability is

$$p^* = \frac{e^{0.05} - 0.9}{1.3 - 0.9} = 0.3782.$$

When one constructs the two-period binomial tree, one gets

$$S_u = 26,\ S_d = 17,$$
$$S_{uu} = 33.80,\ S_{ud} = S_{dd} = 23.4,\ S_{dd} = 16.2.$$

So, the payoffs at the end of the first period are

$$V_u = 3, V_d = 0.$$

The payoffs at the end of the second period are

$$V_{uu} = 10.80, \quad V_{ud} = 0.4, \quad V_{dd} = 0.$$

So, taking the expected value at time 0 of the payoff with respect to the risk-neutral probability, we get that the price of this call should be

$$\begin{aligned} & e^{-0.05} \times V_u \times p^* + e^{-0.05\times 2}[V_{uu} \times (p^*)^2 + V_{ud} \times 2p^*(1 - p^*)] \\ &= e^{-0.05} \times 3 \times 0.3782 + e^{-0.1}[10.8 \times 0.3782^2 + 0.4 \times 2 \times 0.3782 \times (1 - 0.3782)] \\ &= 1.079 + 1.568 = 2.647. \end{aligned}$$

Problem 1.5. (10 points) Let the stock prices be modeled using the lognormal distribution. The mean stock price at time-1 equals 120 and the median stock price 115. What is the probability that the time-1 stock price exceeds 100?

Solution: The stock price at time-1 is lognormally distributed. In fact, using our usual parameters, we can rewrite it as

$$S(1) = S(0)e^{(\alpha-\delta-\frac{1}{2}\sigma^2)+\sigma Z(1)}.$$

Recall that the median of $S(1)$ equals $S(0)e^{(\alpha-\delta-\frac{1}{2}\sigma^2)}$. So, the required probability can be expressed as

$$\begin{aligned}\mathbb{P}[S(1) > 100] = \mathbb{P}[115e^{\sigma Z(1)} > 100] &= \mathbb{P}\left[Z(1) > \frac{1}{\sigma}\ln\left(\frac{100}{115}\right)\right] \\ &= \mathbb{P}\left[Z(1) < \frac{1}{\sigma}\ln\left(\frac{115}{100}\right)\right] = N\left(\frac{1}{\sigma}\ln\left(\frac{115}{100}\right)\right).\end{aligned}$$

Since the mean of $S(1)$ equals $S(0)e^{(\alpha-\delta)}$, we have

$$e^{\frac{1}{2}\sigma^2} = \frac{120}{115} \quad \Rightarrow \quad \sigma = \sqrt{2\ln(1.04348)} = 0.2918.$$

So, our final answer is

$$\mathbb{P}[S(1) > 100] = N(0.48) = 0.6844.$$

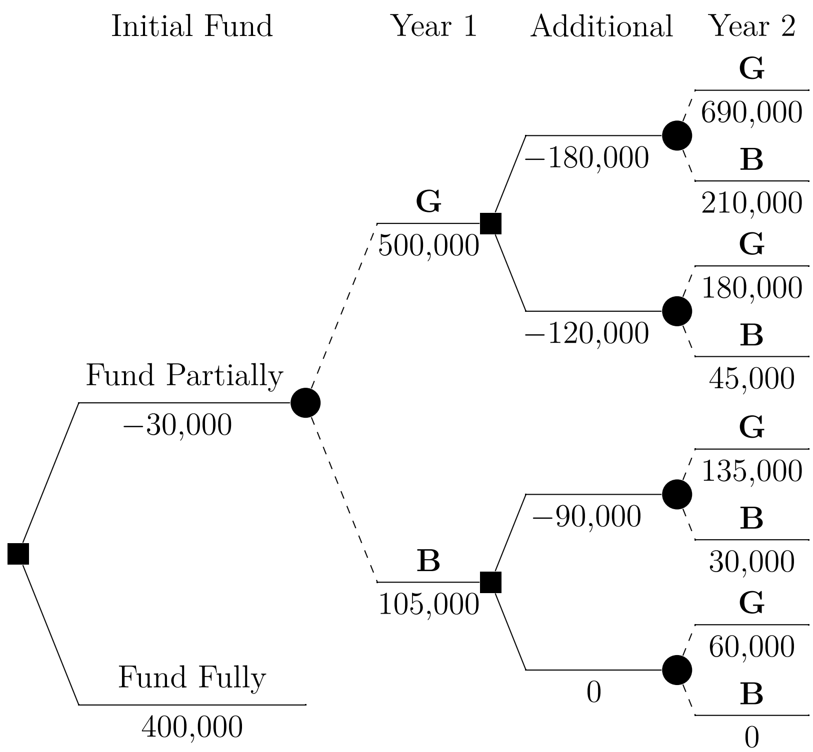
Problem 1.6. (5 points) Netflix is considering a cartoon series. When the production of two seasons is fully funded at time-0 the project has a net present value of 400,000.

The decision tree below shows the cash flows of the series when the promotion at the beginning of the Year 1 (i.e., at $t = 0$) is only partial with an option to provide different amounts of funding at the beginning of Year 2 (i.e., at $t = 1$) depending on how well the first season did.

This tree reflects two possible receptions of the two seasons at each information node (**G** = good, **B** = bad). The probability of the series being a success is given to be 2/3 and the probability of it being merely watchable is 1/3.

Assume the interest rate is 0%.

Find the **initial** (i.e., at $t = 0$) value of the option to fund partially.



Solution: As usual, when pricing options, we are moving backwards through the tree.

- In the *uppermost final* information node, the possible cashflows are 690,000 with probability 2/3 and 210,000 with probability 1/3. So, the value of the project at that node equals

$$690000\left(\frac{1}{3}\right) + 210000\left(\frac{1}{3}\right) = 530000.$$

- In the *second-by-height final* information node, the possible cashflows are 180,000 with probability 1/3 and 45,000 with probability 1/3. So, the value of the project at that node equals

$$180000\left(\frac{2}{3}\right) + 45000\left(\frac{1}{3}\right) = 135000.$$

- In the *third-by-height final* information node, the possible cashflows are 135,000 with probability 2/3 and 30,000 with probability 1/3. So, the value of the project at that node equals

$$135000\left(\frac{2}{3}\right)+30000\left(\frac{1}{3}\right)=100000.$$

- In the *lowest final* information node, the possible cashflows are 60,000 with probability 2/3 and 0 with probability 1/3. So, the value of the project at that node equals

$$60000\left(\frac{2}{3}\right)=40000.$$

We continue working backwards, at the **upper decision** node at the end of Year 1, we can go "up" or "down" in the tree.

- We go "up" by investing 180,000; combining this cashflow with the average revenue at the *uppermost final* node, we get the total effect of going "up" to be

$$530000-180000=350000.$$

- We go "down" by investing 120,000; combining this cashflow with the average revenue at the *second-by-height final* node, we get the total effect of going "down" to be

$$135000-120000=15000.$$

Comparing the two values we obtained, we conclude that the **optimal** decision at this node is to go "up" and we keep the value of this project at this node to be

$$350000+500000=850000.$$

Here, we took into account that the first season was a success resulting in 500,000 in revenue in Year 1.

Similarly, at the **lower decision** node at the end of Year 1, we can go "up" or "down" in the tree.

- We go "up" by investing 90,000; combining this cashflow with the average revenue at the *third-by-height final* node, we get the total effect of going "up" to be

$$100000-90000=10000.$$

- We go "down" by investing nothing; so, the total effect of going "down" is 40000. Comparing the two values we obtained, we conclude that the **optimal** decision at this node is to go "down" and we keep the value of this project at this node to be

$$40000+105000=145000.$$

  Here, we took into account that the first season was "meh" resulting in 105,000 in revenue in Year 1.

  Altogether, at the information node corresponding to Year 1, we have that the expected value of the project is

$$850000\left(\frac{2}{3}\right)+105000\left(\frac{1}{3}\right)=601666.67.$$

Now, we take into account that we funded the series partially with 30,000. So, the total expected present value of the cashflows we get should we decide to fund partially is

$$601666.67 - 30000 = 571666.7$$

The total value of the option is

$$571666.7 - 400000 = 171666.7$$

Note: I know that the numbers are ridiculous. My aim was to make them so that it's easy to calculate with and write, not so that they are realistic.

Problem 1.7. (5 points) Assume the Black-Scholes model. The initial price of a continuous-dividend-paying stock is \$100. Its dividend yield is 0.03 and its volatility is 0.15. According to your model, the mean rate of return is 0.08.

The continuously compounded risk-free interest rate is 0.04.

Calculate the probability that the realized return for the time period $[0,2]$ exceeds 0.06.

Solution: In our usual notation, the realized returns are normally distributed as

$$R(0,t) \sim Normal(mean = (\alpha - \delta - \tfrac{\sigma^2}{2})t, variance = \sigma^2 t).$$

In the present problem, we are focused on

$$R(0,2) \sim Normal(mean = (0.08 - 0.03 - \tfrac{(0.15)^2}{2})(2) = 0.0775, variance = (0.15)^2(2) = 0.045).$$

Finally, we calculate

$$\begin{aligned}\mathbb{P}[R(0,2) > 0.06] &= \mathbb{P}\left[\frac{R(0,2) - 0.0775}{\sqrt{0.045}} > \frac{0.06 - 0.0775}{\sqrt{0.045}}\right] \\ &= \mathbb{P}[Z > -0.08] = N(0.08) = 0.5319.\end{aligned}$$

Problem 1.8. (5 points) Assume the Black-Scholes model. Consider a non-dividend-paying stock with the intial stock price of \$100 and volatility equal to 0.30. According to your model, the stock's mean rate of return is 0.10. Find

$$\mathbb{E}[S(1)\mathbb{I}_{[S(1)\geq 105]}].$$

Solution: According to the work done in class,

$$\mathbb{E}[S(1)\mathbb{I}_{[S(1)\geq 105]}] = \mathbb{E}[S(1)]N(\hat{d}_1)$$

where

$$\hat{d}_1 = \frac{1}{\sigma\sqrt{1}}\left[\ln\left(\frac{100}{105}\right) + \left(0.10 + \frac{(0.3)^2}{2}\right)(1)\right] \approx 0.32.$$

So,

$$\mathbb{E}[S(1)\mathbb{I}_{[S(1)\geq 105]}] = 100e^{0.10}N(0.32) = 69.12844.$$

1.2. MULTIPLE CHOICE QUESTIONS. *Please note your answers on the front page.*

Problem 1.9. (5 pts) Consider a non-dividend-paying stock currently priced at \$100 per share.

The price of this stock in one year is modeled using a one-period binomial tree under the assumption that the stock price can either go up to 110 or down to 90.

Let the continuously compounded risk-free interest rate equal 0.04. What is the risk-neutral probability of the stock price going up?

(a) About 0.2969
(b) About 0.3039
(c) About 0.5000
(d) About 0.7041
(e) None of the above.

Solution: (d)

$$p^* = \frac{100e^{0.04} - 90}{110 - 90} = 0.7041.$$

Problem 1.10. (5 points) The current stock price is 40 per share. The price at the end of a three-month period is modeled with a one-period binomial tree so that the stock price can either increase by \$10, or decrease by \$4. The stock pays dividends continuously with the dividend yield 0.04.

The continuously compounded risk-free interest rate is 0.05.

What is the stock investment in a replicating portfolio for three-month, \$40-strike European **straddle** on the above stock?

(a) Long 0.42 shares
(b) Long 0.71 shares
(c) Short 0.71 shares
(d) Short 0.42 shares
(e) None of the above.

Solution: (a)
In our usual notation,

$$\Delta = e^{-\delta h} \frac{V_u - V_d}{S_u - S_d} = e^{-0.04/4} \left(\frac{10 - 4}{14} \right) \approx 0.4243$$

Problem 1.11. The following relates to one share of XYZ stock:

- The current price is 80.
- The forward price for delivery in two years is 88.
- An investor who decides to long the forward contract denotes by P the expected stock price in two years.

Determine which of the following statements about P is **TRUE**.

(a) $P < 80$
(b) $P = 80$
(c) $80 < P < 88$
(d) $P = 88$
(e) $P > 88$

Solution: (e)
Since the investor decided to long the forward contract, the payoff/profit will be

$$S(T) - 88$$

where $S(T)$ denotes the stock price on the delivery date T. The reason the investor chose to long the forward was the belief that the expected profit would be positive, i.e.,

$$\mathbb{E}[S(T)] = P > 88.$$

Problem 1.12. The current futures price is given to be \$80. The evolution of this futures price over the following year is modeled using a two-period binomial tree such that the ratio of the up factor to the down factor equals 4/3. Moreover, you are given that the risk-neutral probability of an up movement in the tree in any single step equals 1/3.

The continuously compounded risk-free interest rate is 0.05.

What is the price of a one-year, \$85-strike European put option on the above futures contract consistent with our model?

(a) About \$2.24.
(b) About \$8.12.
(c) About \$8.54.
(d) About \$8.98.
(e) None of the above.

Solution: (c)
We are given that, in our usual notation,

$$u_F/d_F = 4/3 \quad \text{and} \quad p^* = \frac{1-d_F}{u_F - d_F} = 1/3.$$

So, $u_F = 1.2$ and $d_F = 0.9$. Hence, the possible futures prices at the end of the two periods are

$$F_{uu} = 80 \times (1.2)^2 = 115.20, \quad F_{ud} = 86.4, \quad F_{dd} = 64.8$$

Finally, the put-price is

$$V_P(0) = e^{-0.05} \times (2/3)^2 \times (85 - 64.8) = 8.54$$

Problem 1.13. The current exchange rate is given to be \$1.11 per Euro and its volatility is given to be 0.16. The continuously compounded risk-free interest rate for the US dollar is 0.02, while the continuously compounded risk-free interest rate for the Euro equals 0.04.

The evolution of the exchange rate over the following nine-months is modeled using a three-period forward binomial tree. What is the value of the so-called up factor in the above tree?

(a) $u \approx 1.0779$
(b) $u \approx 1.0887$
(c) $u \approx 1.1503$
(d) $u \approx 1.1972$
(e) None of the above.

Solution: (a) In the forward binomial tree, the up and down factors are given as

$$u = e^{(0.02-0.04)\times 0.25+0.16\times\sqrt{0.25}} = 1.0779$$
$$d = e^{(0.02-0.04)\times 0.25-0.16\times\sqrt{0.25}} = 0.9185.$$

Problem 1.14. (5 points) Let the current price of a continuous-dividend-paying stock be denoted by $S(0)$. We model the time$-T$ stock price as lognormal. The mean rate of return on the stock is 0.10, its dividend yield is 0.01, and its volatility is 0.20. The continuously compounded risk-free interest rate is 0.03. You invest in one share of stock at time-0. and simultaneously deposit an amount $\varphi S(0)$ in a savings account. Assume continuous and immediate reinvestment of all dividends in the same stock. What should the proportion φ be so that the VaR at the level 0.05 of your total wealth at time-1 equals today's stock price $S(0)$?

(a) $\varphi = 0.0573$
(b) $\varphi = 0.1966$
(c) $\varphi = 0.2139$
(d) $\varphi = 0.5$
(e) None of the above.

Solution: (c)
The total wealth at time-1 is equal to $e^{\delta}S(1) + \varphi S(0)e^{r}$. So, our condition on the VaR is

$$\mathbb{P}[e^{\delta}S(1) + \varphi S(0)e^{r} < S(0)] = 0.05.$$

The above is equivalent to

$$\mathbb{P}[S(0)e^{\alpha-\frac{\sigma^2}{2}+\sigma Z} + \varphi S(0)e^{r} < S(0)] = 0.05$$

with $Z \sim N(0,1)$. We have that the above is, in turn, equivalent to

$$\mathbb{P}[e^{\alpha-\frac{\sigma^2}{2}+\sigma Z} + \varphi e^{r} < 1] = 0.05.$$

The constant z^* such that $\mathbb{P}[Z < z^*] = 0.05$ equals -1.645. Therefore,

$$e^{\alpha-\frac{\sigma^2}{2}+\sigma(-1.645)} + \varphi e^{r} = 1.$$

So,

$$\varphi = e^{-r}\left(1 - e^{\alpha-\frac{\sigma^2}{2}+\sigma(-1.645)}\right) = e^{-0.03}\left(1 - e^{0.10-\frac{(0.2)^2}{2}+0.2(-1.645)}\right) = 0.2139.$$