M339D : April 3rd, 2024.

UNIVERSITY OF TEXAS AT AUSTIN

Problem Set 12

The normal approximation to the binomial.

Problem 12.1. According to the Pew research center, 64% of Americans say that social media have a mostly negative effect on things (see https://pewrsr.ch/3dsV7uR). You take a sample of 1000 randomly chosen Americans. What is the $\boxed{\text{approximate}}$ probability that at least 650 of them say that social media have a mostly negative effect on things?

→ : Y... a r.v. denoting the # of surveyed people who claim that social media is negative

$$Y \sim \text{Binomial}(n=1000, p=0.64)$$

$$n \cdot p = 1000(0.64) = 640 \geq 10 \quad \text{and} \quad n(1-p) = 360 \geq 10 \quad ✓$$

$$\mu_Y = \mathbb{E}[Y] = n \cdot p = 640$$

$$\sigma_Y = \sqrt{np(1-p)} = \sqrt{640(0.36)} = 15.18$$

$$\mathbb{P}[Y \geq 650] = \mathbb{P}\left[\frac{Y-640}{15.18} \geq \frac{650-640}{15.18}\right]$$

$\overset{\sim}{\sim} N(0,1) \sim Z$

$$\approx \mathbb{P}[Z \geq \underbrace{0.6587615}_{0.66}] = 1-\text{pnorm}(z) = 0.2550245$$

$$\text{OR} \quad 1 - N(0.66) = 1 - 0.7454 = 0.2546$$

□

Just for laughs:

$$\mathbb{P}[Y \geq 650] = \mathbb{P}[Y > 649] = 1 - \mathbb{P}[Y \leq 649]$$

$$= 1 - \text{pbinom}(649, \text{size}=1000, \text{prob}=0.64)$$

$$= 0.2663257$$

Problem 12.2. According to a Gallup survey, only 22% of American young adults rate their mental health as *excellent*:

https://www.gallup.com/education/328961/why-higher-education-lead-wellbeing-revolution.aspx.

You sample 6000 randomly chosen American young adults. What is the $\boxed{\text{approximate}}$ probability that at most 1400 of them rate their mental health as *excellent*?

→: Y... # of sampled people who said excellent

$Y \sim$ Binomial$(n = 6000, p = 0.22)$

Check: $n \cdot p = 6000(0.22) = \boxed{1320} \geq 10$

$n(1-p) = 6000(0.78) = 4680 \geq 10$

$\mu_Y = n \cdot p = 1320$

$\sigma_Y = \sqrt{np(1-p)} = \sqrt{1320(0.78)} = 32.08738$

$$\mathbb{P}[Y \leq 1400] = \mathbb{P}\left[\boxed{\frac{Y - 1320}{32.09}}^{\sim Z} \leq \frac{1400-1320}{32.09}\right]$$

$$\approx \mathbb{P}[Z \leq \underbrace{2.493192}_{2.49}] = \text{pnorm}(2.493192) = 0.99367$$

$N(2.49) = 0.9936$

$\square$

Just for fun:

pbinom(1400, size = 6000, prob = 0.22) = 0.9936818

Problem 12.3. You toss a fair coin 10,000 times. What is the approximate probability that the number of *Heads* exceeds the number of *Tails* by between 200 and 500 (inclusive)?

→ : $Y \ldots$ # of Hs

$Y \sim Binomial(n = 10{,}000, p = 0.5)$ $np = n(1-p) = 5000 \geq 10$ ✓

$$\mathbb{P}[200 \leq \underbrace{Y}_{\#\text{ of Hs}} - \underbrace{(10000 - Y)}_{\#\text{ of tails}} \leq 500] =$$

$$= \mathbb{P}[10200 \leq 2Y \leq 10500]$$

$$= \mathbb{P}[5100 \leq Y \leq 5250]$$

$\mu_Y = 5000$ ✓

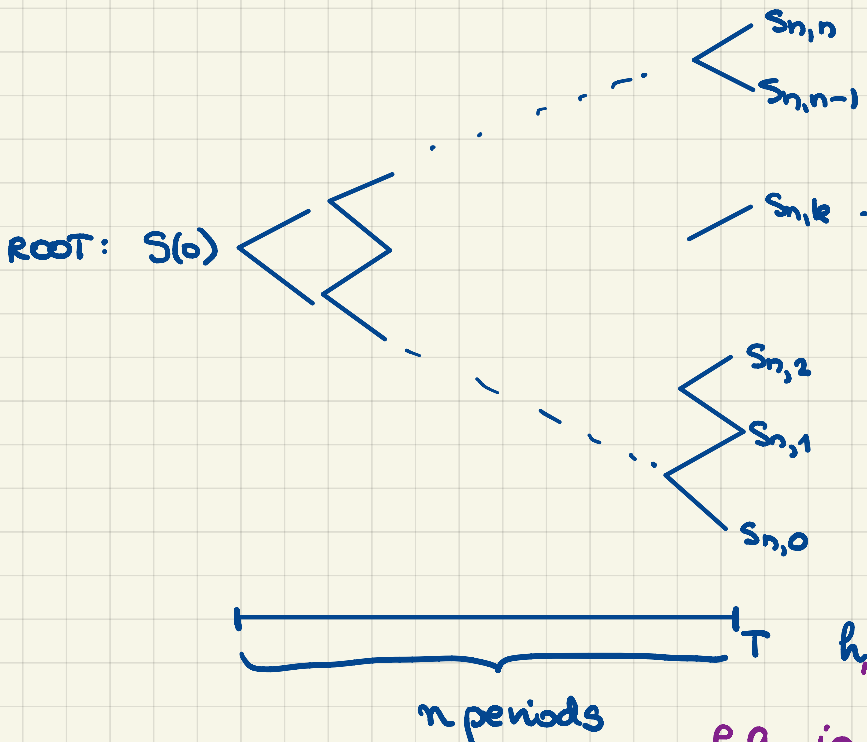
$\sigma_Y = \sqrt{2500} = 50$ ✓

$$\mathbb{P}\left[\frac{5100 - 5000}{50} \leq \frac{Y - 5000}{50} \leq \frac{5250 - 5000}{50}\right]$$

$$\approx \mathbb{P}[2 \leq Z \leq 5] = N(5) - N(2) = 1 - 0.9772 = 0.0228$$

□

The Pre-limit n-Period Binomial Tree.



$S_{n,k}$ ... the stock price resulting from k upsteps

$h_n = \frac{T}{n}$

u_n ... up factor

d_n ... down factor

e.g., in the forward tree

$u_n = \exp\left(r\cdot\frac{T}{n} + \sigma\sqrt{\frac{T}{n}}\right)$

$d_n = \exp\left(r\cdot\frac{T}{n} - \sigma\sqrt{\frac{T}{n}}\right)$

$$S_{n,k} = S(0)\cdot u_n^k \cdot d_n^{n-k} = S(0)\left(\frac{u_n}{d_n}\right)^k \cdot d_n^n$$

k corresponds to the realization of the binomial random variable w/ n trials

and success probab. $p_n^* = \frac{e^{r\left(\frac{T}{n}\right)} - d_n}{u_n - d_n}$

e.g., in the forward tree

$$\boxed{p_n^* = \frac{1}{1+e^{\sigma\sqrt{T/n}}}} \xrightarrow[n\to\infty]{} \frac{1}{2}$$

Say, X_n ... # of upsteps in n periods

$$\boxed{X_n \sim \text{Binomial}(\text{\# of trials } n,\ \text{probab. of success} = p_n^*)}$$

Q: Can we simply use the normal approximation to the binomial?

Subjective Probability.

When pricing, we use the risk·neutral measure.

$$\begin{tikzcd} & S_u = u\cdot S(0) \\ S(0) \arrow[ur, "p^*"] \arrow[dr, "1-p^*"'] & \\ & S_d = d\cdot S(0) \end{tikzcd}$$

$$\underbrace{\qquad\qquad}_{h=T}$$

$$\boxed{p^* = \frac{e^{rh}-d}{u-d}}$$

☆

Q: If we invest in one share of non·dividend·paying stock @ time·0, what is our expected wealth @ time·T, under the risk·neutral probability measure?

$\longrightarrow$: $\mathbb{E}^*[S(T)] = S_u \cdot p^* + S_d \cdot (1-p^*)$

$$= S_u \cdot \frac{e^{rh}-d}{u-d} + S_d \frac{u-e^{rh}}{u-d} = \cdots$$