M339J: April 18th, 2022.

125. Two types of insurance claims are made to an insurance company. For each type, the number of claims follows a Poisson distribution and the amount of each claim is uniformly distributed as follows:

Type of Claim	Poisson Parameter λ for Number of Claims in one year	Range of Each Claim Amount
I	12 $=\lambda_I$	(0, 1) $X^I \sim U(0,1)$
II	4 $=\lambda_{II}$	(0, 5) $X^{II} \sim U(0,5)$

The numbers of claims of the two types are independent and the claim amounts and claim numbers are independent.

Calculate the normal approximation to the probability that the total of claim amounts (S) in one year exceeds 18.

$\rightarrow$:

$$S_I = X_1^I + X_2^I + \cdots + X_{N_I}^I$$

$$S_{II} = X_1^{II} + X_2^{II} + \cdots + X_{N_{II}}^{II}$$

$$\boxed{S = S_I + S_{II}}$$

$$\mathbb{P}[S > 18] = ?$$

(A) 0.37

(B) 0.39

(C) 0.41

(D) 0.43

(E) 0.45

The normal approximation requires that we express S in standard units. So, we need $\boxed{\mathbb{E}[S]}$ and $\boxed{Var[S]}$.

$$\mathbb{E}[S] = \mathbb{E}[S_I] + \mathbb{E}[S_{II}] = 12 \cdot \left(\frac{1}{2}\right) + 4\left(\frac{5}{2}\right) = 16$$

$$Var[S] = Var[S_I + S_{II}]$$ Independence!

$$= Var[S_I] + Var[S_{II}]$$

$$Var[S_I] = \mathbb{E}[N_I]\, Var[X^I] + Var[N_I]\,(\mathbb{E}[X^I])^2$$

$$= 12 \cdot \frac{1}{12} + 12 \cdot \left(\frac{1}{2}\right)^2$$

$$= 1 + 3 = 4$$

$$\checkmark\ Var[S_{II}] = \mathbb{E}[N_{II}] \cdot Var[X^{II}] + Var[N_{II}] \cdot (\mathbb{E}[X^{II}])^2$$
$$= 4 \cdot \frac{25}{12} + 4 \cdot \left(\frac{5}{2}\right)^2$$
$$= \frac{25}{3} + 25 = \frac{100}{3}$$

$$Var[S] = 4 + \frac{100}{3} = 37.333\dot{3} \Rightarrow SD[S] = \sigma_S = \underline{6.1101}$$

Method #1.

$$S \approx \text{Normal}(\text{mean} = 16,\ \text{sd} = 6.1101)$$

$$\mathbb{P}[S > 18] = 1 - \text{pnorm}(18, \text{mean} = 16, \text{sd} = \text{sqrt}(100/3)) =$$
$$= 0.3645172$$

Method #2.

$$\mathbb{P}[S > 18] = \mathbb{P}\left[\frac{S-16}{6.1101} > \frac{18-16}{6.1101}\right]$$

$\sim N(0,1) \sim Z$

$$= \mathbb{P}[Z > 0.33] = 1 - \Phi(0.33)$$
$$= 1 - 0.6293 = 0.3707 \quad \square$$

Problem. Medical and dental claims are assumed to be independent w/ compound Poisson dist'ns.

Claim Type	Poisson rate	Claim Amt Dist.
Medical	2	$U(0,1000) \sim X_M$
Dental	3	$U(0,200) \sim X_D$

Let X be a random variable which denotes a randomly chosen claim under a policy which covers both medical and dental claims.

Find $\boxed{\mathbb{E}[(X-100)_+]}$.

$\longrightarrow$: The combined claim count is $N \sim$ Poisson$(\lambda = 2+3 = 5)$.

For an individual claim amount X, its cdf is:

$$F_X(x) = \frac{2}{5} \cdot F_{X_M}(x) + \frac{3}{5} \cdot F_{X_D}(x)$$

$\Rightarrow$ its pdf:

$$f_X(x) = \frac{2}{5} \cdot f_{X_M}(x) + \frac{3}{5} f_{X_D}(x)$$

$$= \begin{cases} \frac{2}{5} \cdot \frac{1}{1000} + \frac{3}{5} \cdot \frac{1}{200} & x \in (0,200) \\ \frac{2}{5} \cdot \frac{1}{1000} & x \in (200,1000) \end{cases}$$

$$= \begin{cases} \frac{17}{5000} & x \in (0,200) \\ \frac{1}{2500} & x \in (200,1000) \end{cases}$$

$$\mathbb{E}[(X-100)_+] = \int_{100}^{1000} (x-100) f_X(x)\,dx$$

$$= \int_{100}^{1000} x f_X(x)\,dx - 100 \int_{100}^{1000} f_X(x)\,dx$$

$$\int_{100}^{1000} f_X(x)\,dx = 1 - \int_{0}^{100} f_X(x)\,dx = 1 - \frac{17}{5000} \cdot 100 = 1 - \frac{17}{50} = \frac{33}{50}$$

$$= \int_{100}^{200} x\left(\frac{17}{5000}\right)dx + \int_{200}^{1000} x\left(\frac{1}{2500}\right)dx - 100 \cdot \frac{33}{50}$$

$$= \frac{17}{5000} \cdot \frac{x^2}{2}\Big|_{x=100}^{200} + \frac{1}{2500} \cdot \frac{x^2}{2}\Big|_{x=200}^{1000} - 66$$

$$= \underline{177} \quad \square$$