M378K: February 3rd, 2025.

Moments.

Def'n. For a r.v. Y w/ pdf f_Y and for $k = 1, 2, \ldots$, we define the k^{th} (raw) moment μ_k as

$$\mu_k = \mathbb{E}[Y^k] = \int_{-\infty}^{\infty} y^k f_Y(y)\,dy$$

$$\mu = \boxed{\mu_1 = \mathbb{E}[Y]}$$

The k^{th} central moment is

$$\mu_k^c = \mathbb{E}[(Y-\mu)^k] = \int_{-\infty}^{\infty} (y-\mu)^k f_Y(y)\,dy$$

Q: $\mu_2^c = \cancel{?}\ \mathrm{Var}[Y]$

The Cumulative Distribution Function.

Def'n. The cumulative distribution function (cdf) of a r.v. Y is a function

$$F_Y : \mathbb{R} \longrightarrow [0,1]$$

defined as $F_Y(y) = \mathbb{P}[Y \leq y]$ for all $y \in \mathbb{R}$.

Properties:

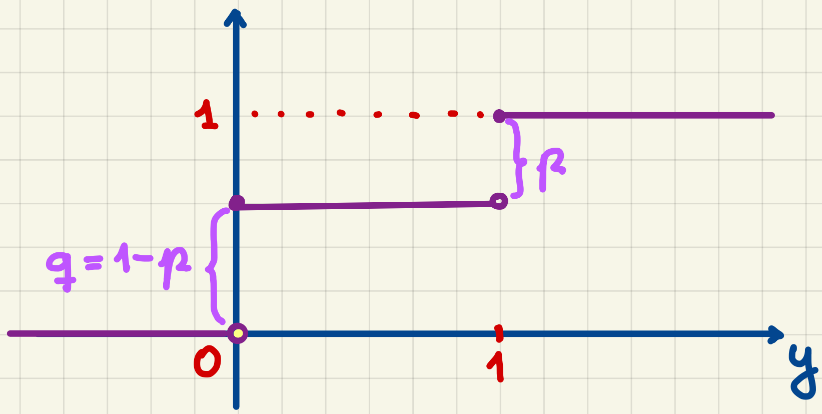
- $0 \leq F_Y(y) \leq 1$
- F_Y is non decreasing
- $\lim_{y \to -\infty} F_Y(y) = 0$

  $\lim_{y \to +\infty} F_Y(y) = 1$

Example. • Bernoulli Dist'n.

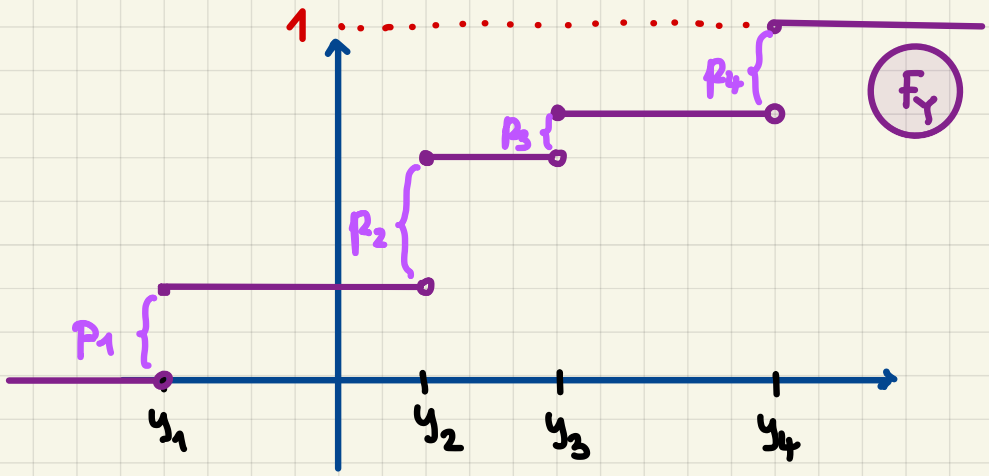
$\boxed{Y \sim B(p)}$

$\boxed{F_Y \ldots \text{cdf}}$



• Discrete r.v. w/ a finite support.

	y_1	y_2	....	y_m
	p_1	p_2	....	p_m



The Discrete Case.

Say that Y is discrete w/ pmf p_Y.

Then,

$$\boxed{F_Y(y) = \sum_{\substack{u \leq y \\ u \in S_Y}} \boxed{p_Y(u)}}$$

M378K Introduction to Mathematical Statistics
Problem Set #6
Cumulative distribution functions.

Problem 6.1. Source: Sample P exam, Problem #342.
Consider a Poisson distributed random variable X. As usual, let's denote its cumulative distribution function by F_X. You are given that

$$\boxed{\frac{F_X(2)}{F_X(1)} = 2.6}$$

Calculate the expected value of the random variable X.

→: $\mathbb{E}[X] = \lambda$

pmf of X: $k = 0, 1, 2, \dots$ $\quad \boxed{p_X(k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}}$

$$\frac{\mathbb{P}[X \leq 2]}{\mathbb{P}[X \leq 1]} = 2.6$$

$$\frac{p_X(0) + p_X(1) + p_X(2)}{p_X(0) + p_X(1)} = 2.6$$

$$\frac{\cancel{e^{-\lambda}} + \cancel{e^{-\lambda}} \cdot \lambda + \cancel{e^{-\lambda}} \cdot \frac{\lambda^2}{2}}{\cancel{e^{-\lambda}} + \cancel{e^{-\lambda}} \cdot \lambda} = 2.6$$

$$1 + \lambda + \frac{\lambda^2}{2} = 2.6(1 + \lambda) \quad / \cdot 10$$

$$5\lambda^2 - 16\lambda - 16 = 0$$

$$\lambda_{1,2} = \frac{16 \pm \sqrt{256 + 320}}{10} = \frac{16 \pm \sqrt{576}}{10} = \frac{16 \pm 24}{10}$$

Only keep the positive solution: $\boxed{\lambda = 4}$

□

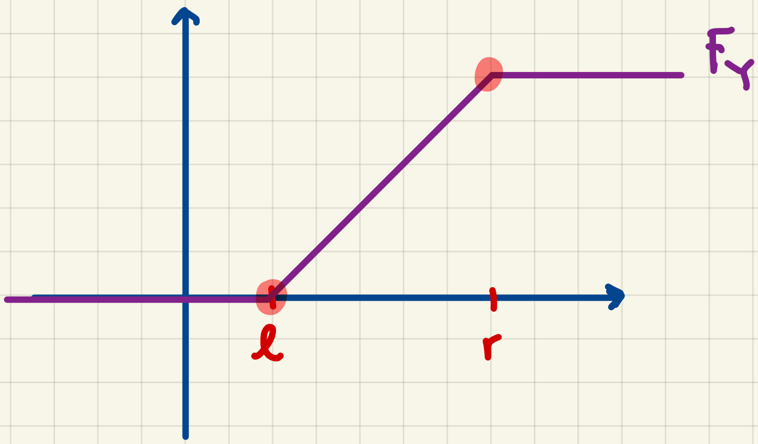
The Continuous Case.

Let Y be continuous w/ pdf f_Y.

Then,
$$F_Y(y) = \mathbb{P}[Y \leq y] = \int_{-\infty}^{y} f_Y(u)\,du$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = F_Y'(y)$$ wherever the derivative exists.

Example. Uniform $Y \sim U(l, r)$



Fact. The cdf of a continuous r.v. is a continuous function w/ @ most countably many points where it's not differentiable.

Problem 6.2. *Consider a random variable Y whose cumulative distribution function is given by*

$$F_Y(y) = \begin{cases} 0, & \text{for } y < 0 \\ y^4, & \text{for } 0 \leq y < 1 \\ 1, & \text{for } 1 \leq y \end{cases}$$

Calculate the expectation of the random variable Y.

$\longrightarrow$:

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \begin{cases} 0 & \text{for } y<0 \\ 4y^3 & \text{for } 0 \leq y<1 \\ 0 & \text{for } y>1 \end{cases}$$

$$\boxed{f_Y(y) = 4y^3 \mathbb{1}_{(0,1)}(y)}$$

$$\mathbb{E}[Y] = \int_{-\infty}^{\infty} y f_Y(y)\,dy = \int_0^1 y \cdot 4y^3\,dy$$

$$= 4\int_0^1 y^4\,dy = 4 \cdot \left(\frac{y^5}{5}\right)_{y=0}^{1} = \frac{4}{5} \quad \square$$