

M339W: March 29<sup>th</sup>, 2021.

## Focus on the Delta.

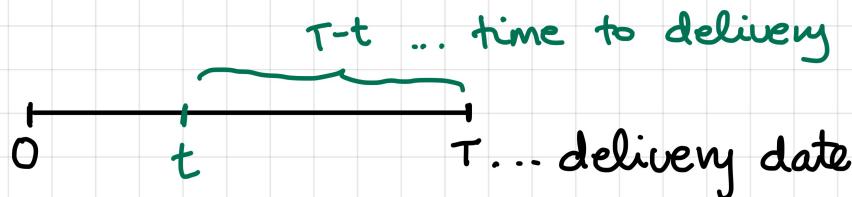
We represent the worth of our portfolio via its value function:

$$v(s, t, r, \delta, \sigma)$$

Def'n. The delta ( $\Delta$ ) of a portfolio is:

$$\Delta(s, t, r, \delta, \sigma) = \frac{\partial}{\partial s} v(s, t, r, \delta, \sigma)$$

Example. [PREPAID FORWARD CONTRACT ON A CONTINUOUS-DIVIDEND-PAYING STOCK]



$$v(s, t, r, \delta, \sigma) = s \cdot e^{-\delta(T-t)}$$

$$\Rightarrow \Delta(\dots) = e^{-\delta(T-t)} > 0 \quad \Rightarrow \Gamma(\dots) = \frac{\partial^2}{\partial s^2} v(\dots) = 0$$

$$\Theta(\dots) = \frac{\partial}{\partial t} v(\dots) = \delta s \cdot e^{-\delta(T-t)} > 0$$

$$\Psi(\dots) = \frac{\partial}{\partial \delta} v(\dots) = -(T-t) s e^{-\delta(T-t)} < 0$$

Example. [EUROPEAN CALL w/ STRIKE K & EXERCISE DATE T]

The Black-Scholes price:

$$v_c(\dots) = s e^{-\delta(T-t)} \cdot N(d_1(\dots)) - K e^{-r(T-t)} \cdot N(d_2(\dots))$$

w/

$$d_1 = \frac{1}{\sigma \sqrt{T-t}} \left[ \ln\left(\frac{s}{K}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right)(T-t) \right]$$

$$\text{and } d_2 = d_1 - \sigma \sqrt{T-t}$$

By def'n,

$$\Delta_C(\dots) = \frac{\partial}{\partial S} v_C(\dots)$$

After the chain rule and the product rule:

$$\boxed{\Delta_C(\dots) = e^{-\delta(T-t)} N(d_1(\dots)) > 0} \quad \checkmark$$

The positivity of  $\Delta_C$  makes sense since the call is long w.r.t. the underlying.

Example. [EUROPEAN PUT w/ STRIKE K & EXERCISE DATE T]

→: Put-call Parity:

$$v_C(\dots) - v_P(\dots) = \gamma e^{-\delta(T-t)} - K e^{-r(T-t)}$$

$\frac{\partial}{\partial S}$  |

$$\Delta_C(\dots) - \Delta_P(\dots) = e^{-\delta(T-t)}$$

$$\Rightarrow \Delta_P(\dots) = \Delta_C(\dots) - e^{-\delta(T-t)} = e^{-\delta(T-t)} \cdot N(d_1(\dots)) - e^{-\delta(T-t)}$$
$$= e^{-\delta(T-t)} (N(d_1(\dots)) - 1)$$
$$= -e^{-\delta(T-t)} N(-d_1(\dots)) < 0$$

$$\boxed{\Delta_P(\dots) = -e^{-\delta(T-t)} N(-d_1(\dots))}$$

Sensible, since puts are short with respect to the underlying 😊



$$d_2(S(0), 0) = d_1(S(0), 0) - \sigma\sqrt{T} = 0 - 0.3\sqrt{\frac{1}{4}} = -0.15$$

Recall: the pdf of the std normal:

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \quad z \in \mathbb{R}$$

the cdf of the std normal:

$$N(x) = \int_{-\infty}^x \varphi(z) dz = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

$$v_c(S(0), 0) = 40 \cdot 0.5 - 41.5 e^{-0.1023(\frac{1}{4})} \cdot \underbrace{N(-0.15)}_{1 - N(0.15)}$$

$$= 20 - 40.453(1 - N(0.15))$$

$$= 40.453 \cdot N(0.15) - 20.453$$

$$= \frac{40.453}{\sqrt{2\pi}} \int_{-\infty}^{0.15} e^{-\frac{z^2}{2}} dz - 20.453$$

$$" 16.138 \Rightarrow \text{the correct answer is (D).}$$

At home, calculate the price. :)

Unless otherwise stated in the examination question, assume:

- The market is frictionless. There are no taxes, transaction costs, bid/ask spreads or restrictions on short sales. All securities are perfectly divisible. Trading does not affect prices. Information is available to all investors simultaneously. Every investor acts rationally and there are no arbitrage opportunities.
- The risk-free interest rate is constant.
- The notation is the same as used in *Derivatives Markets*, by Robert L. McDonald.

When using the standard normal distribution table, do not interpolate.

- Use the nearest  $z$ -value in the table to find the probability. Example: Suppose that you are to find  $\Pr(Z < 0.759)$ , where  $Z$  denotes a standard normal random variable. Because the  $z$ -value in the table nearest to 0.759 is 0.76, your answer is  $\Pr(Z < 0.76) = 0.7764$ .
- Use the nearest probability value in the table to find the  $z$ -value. Example: Suppose that you are to find  $z$  such that  $\Pr(Z < z) = 0.7$ . Because the probability value in the table nearest to 0.7 is 0.6985, your answer is 0.52.

In *Derivatives Markets*,  $\Pr(Z < x)$  is written as  $N(x)$ .

The standard normal density function is

$$f_Z(x) = N'(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}} = \frac{e^{-x^2/2}}{\sqrt{2 \times 3.14159}} = \frac{e^{-x^2/2}}{2.50663}, \quad -\infty < x < \infty.$$

Let  $Y$  be a lognormal random variable. Assume that  $\ln(Y)$  has mean  $m$  and standard deviation  $v$ . Then, the density function of  $Y$  is

$$f_Y(x) = \frac{1}{xv\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\ln(x)-m}{v}\right)^2\right], \quad x > 0.$$

The distribution function of  $Y$  is

$$F_Y(x) = N\left(\frac{\ln(x)-m}{v}\right), \quad x > 0.$$

Also,

$$\mathbb{E}[Y^k] = \exp\left(km + \frac{1}{2}k^2v^2\right),$$

which is the same as the moment-generating function of the random variable  $\ln(Y)$  evaluated at the value  $k$ .

