M358K : October 5^{th}, 2020.

University of Texas at Austin

Problem Set # 4

Normal distribution.

Problem 4.1. Let Z be a standard normal random variable. Find the following probabilities:

$Z \sim N(0,1)$

- ✓ i. $\mathbb{P}[-1.33 < Z \leq 0.24]$
- ✓ ii. $\mathbb{P}[0.49 < |Z|]$
- ✓ iii. $\mathbb{P}[Z^4 < 0.0256]$
- ✓ iv. $\mathbb{P}[e^{2Z} < 2.25]$
- ✓ v. $\mathbb{P}\left[\frac{1}{Z} < 2\right]$

Problem 4.2. (10 points)

At the *Hogwarts School of Witchcraft and Wizardry* the *Ordinary Wizarding Level (OWL)* exam is typically taken at the end of the fifth year. Based on hystorical data, we model the *OWL* scores as roughly normal with mean 100 and standard deviation of 16.

$X \sim N(\text{mean}=100, \text{sd}=16)$

(a) (5 points)

What is the range of scores for the bottom 15% of the *OWL* takers?

$X = \mu_X + \sigma_X \cdot Z$

$\underbrace{\frac{X-\mu_X}{\sigma_X}}_{\text{standard units}} = Z$ ☺

$$\mathbb{P}[X \leq x_*] = 0.15$$

$$\mathbb{P}\left[\underbrace{\frac{X-100}{16}}_{= Z \sim N(0,1)} \leq \frac{x_*-100}{16}\right] = 0.15$$

$$\mathbb{P}\left[Z \leq \frac{x_*-100}{16}\right] = 0.15$$

← the 15^{th} percentile of the std normal.

$$\frac{x_*-100}{16} = -1.04$$

$$\Rightarrow x_* = 100 + 16(-1.04) = 83.36$$

(b) (5 points)

What is the probability that a randomly chosen *OWL* taker has a score higher than 125?

$$\mathbb{P}[X > 125] = ?$$

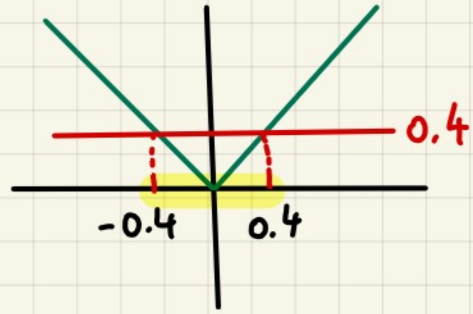
z-score : $\frac{125-100}{16} = 1.5625$

$1 - \Phi(1.56) = 1 - 0.9406 = 0.0594$

Using R:

0.0591

(iii) $\mathbb{P}[Z^4 < 0.0256] =$



$$= \mathbb{P}[|Z| < 0.4]$$

$$= \mathbb{P}[-0.4 < Z < 0.4]$$

$$= \mathbb{P}[Z \leq 0.4] - \underbrace{\mathbb{P}[Z \leq -0.4]}_{(1 - \mathbb{P}[Z \leq 0.4])}$$

$$= 2 \cdot \mathbb{P}[Z \leq 0.4] - 1$$

$$= 2 \cdot \Phi(0.4) - 1 = 2 \cdot (0.6554) - 1$$

$$= 0.3108 \checkmark$$

(iv) $\mathbb{P}[e^{2Z} < 2.25] =$

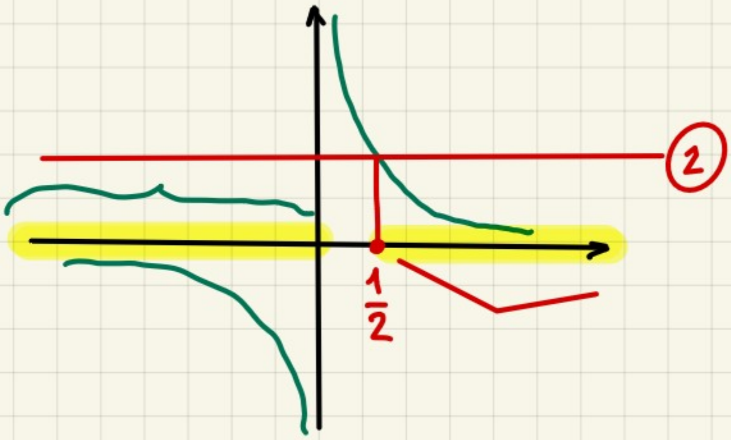
$$= \mathbb{P}[2 \cdot Z < \ln(2.25)]$$

$$= \mathbb{P}[Z < \tfrac{1}{2}\ln(2.25)] =$$

$$= 0.6574$$

If using tables 0.6591 ✓

(v) $\mathbb{P}\left[\frac{1}{Z} < 2\right] = \mathbb{P}[Z<0] + \mathbb{P}\left[Z > \frac{1}{2}\right]$



$= 0.5 + (1 - \mathbb{P}[Z \leq 0.5])$

$= 0.5 + (1 - 0.6915)$

$= 0.8085$ ✓

Normal Dist'n.

We completely specify any normal dist'n by its mean and its variance (or standard deviation).

$$X \sim \text{Normal}(\text{mean} = \mu_X, \text{variance} = \sigma_X^2).$$

is equivalent to saying

$$\boxed{X = \mu_X + \sigma_X \cdot Z} \quad \text{w/ } Z \sim N(0,1)$$

To check:

- $\mathbb{E}[X] = \mathbb{E}[\mu_X + \sigma_X \cdot Z]$

$$= \mu_X + \sigma_X \underbrace{\mathbb{E}[Z]}_{=0} = \mu_X \quad ✓$$

linearity of expectation

- $\text{Var}[X] = \text{Var}[\mu_X + \sigma_X \cdot Z]$

deterministic

$$= \text{Var}[\sigma_X \cdot Z] = \sigma_X^2 \cdot \overbrace{\text{Var}[Z]}^{=1}$$

$$= \sigma_X^2 \quad ✓$$