University of Texas at Austin

Problem Set #11

Binomial option pricing: Currency options. Futures options.

Problem 11.1. Your goal is to price a call option on a futures contract. The movements of the futures price are modeled by a binomial tree. You are given:

(i) Each period is three months.
(ii) $u_F/d_F = 5/4$, where u_F is one plus the rate of gain on the futures price if it goes up, and d_F is one plus the rate of loss if it goes down.
(iii) The risk-neutral probability of an up move is $1/2$.
(iv) The initial futures price is 80.
(v) The continuously compounded risk-free interest rate is 5%.

Find the price of a half-year, 85-strike European put option on the futures contract.

(a) \$8.23
(b) \$13.06
(c) \$13.27
(d) \$13.36
(e) None of the above.

Solution: (a)
We are given that

$$\frac{1}{2} = \frac{1-d_F}{u_F - d_F} = \frac{d_F^{-1}-1}{\frac{u_F}{d_F}-1} = \frac{d_F^{-1}-1}{\frac{5}{4}-1} \quad \Rightarrow \quad d_F^{-1} = \frac{9}{8} \quad \Rightarrow \quad d_F = \frac{8}{9} \quad \Rightarrow \quad u_F = \frac{10}{9}.$$

The prices in the futures-price tree are, thus,

$$\begin{array}{ccc} & & F_{uu} = 98.77 \\ & F_u = 88.89 & \\ F_0 = 80 & & F_{ud} = 79.01 \\ & F_d = 71.11 & \\ & & F_{dd} = 63.21 \end{array}$$

The option's price is

$$V_P(0) = e^{-0.025}\left[\frac{1}{2}(85-79.01) + \frac{1}{4}(85-63.21)\right] = 8.234.$$

Problem 11.2. The current futures price is given to be \$80. The evolution of this futures price over the following year is modeled using a two-period binomial tree such that the ratio of the up factor to the down factor equals $4/3$. Moreover, you are given that the risk-neutral probability of an up movement in the tree in any single step equals $1/3$.

The continuously compounded risk-free interest rate is 0.05.

What is the price of a one-year, \$85-strike European put option on the above futures contract consistent with our model?

(a) About \$2.24.
(b) About \$8.12.
(c) About \$8.54.
(d) About \$8.98.
(e) None of the above.

Solution: (c)
We are given that, in our usual notation,

$$u_F/d_F = 4/3 \quad \text{and} \quad p^* = \frac{1-d_F}{u_F-d_F} = 1/3.$$

So, $u_F = 1.2$ and $d_F = 0.9$. Hence, the possible futures prices at the end of the two periods are

$$F_{uu} = 80 \times (1.2)^2 = 115.20, \quad F_{ud} = 86.4, \quad F_{dd} = 64.8$$

Finally, the put-price is

$$V_P(0) = e^{-0.05} \times (2/3)^2 \times (85 - 64.8) = 8.54$$

Problem 11.3. The current exchange rate is given to be \$1.11 per Euro and its volatility is given to be 0.16. The continuously compounded risk-free interest rate for the US dollar is 0.02, while the continuously compounded risk-free interest rate for the Euro equals 0.04.

The evolution of the exchange rate over the following nine-months is modeled using a three-period forward binomial tree. What is the value of the so-called up factor in the above tree?

(a) $u \approx 1.0779$
(b) $u \approx 1.0887$
(c) $u \approx 1.1503$
(d) $u \approx 1.1972$
(e) None of the above.

Solution: (a) In the forward binomial tree, the up and down factors are given as

$$u = e^{(0.02-0.04)\times 0.25 + 0.16\times\sqrt{0.25}} = 1.0779$$
$$d = e^{(0.02-0.04)\times 0.25 - 0.16\times\sqrt{0.25}} = 0.9185.$$

Problem 11.4. The current exchange rate is given to be \$1.25 per Euro and its volatility is given to be 0.15.

The continuously compounded risk-free interest rate for the US dollar is 0.03, while the continuously compounded risk-free interest rate for the Euro equals 0.06.

The evolution of the exchange rate over the following nine-month time-horizon is modeled using a three-period forward binomial tree.

What is the price of an at-the-money, nine-month European call option on the Euro?

(a) 0.0376
(b) 0.0531
(c) 0.0543
(d) 0.0602
(e) None of the above.

Solution: (c)
In the forward binomial tree, the up and down factors are given as

$$u = e^{(0.03-0.06)\times 0.25 + 0.15\times\sqrt{0.25}} = 1.0698$$
$$d = e^{(0.03-0.06)\times 0.25 - 0.15\times\sqrt{0.25}} = 0.9208.$$

The risk-neutral probability is

$$p^* = \frac{1}{1+e^{\sigma\sqrt{h}}} = \frac{1}{1+e^{0.15\sqrt{0.25}}} = \frac{1}{1+e^{0.15\sqrt{0.25}}} = 0.4813.$$

The possible final values of the exchange rate in our tree are

$$x_{uuu} = x(0)u^3 = 1.5304, \quad x_{uud} = x(0)u^2d = 1.3173, \quad x_{udd} = x(0)ud^2 = 1.1338, \quad x_{ddd} = x(0)d^3 = 0.9759.$$

Thus, the call option is in-the-money in the two topmost final nodes. The payoff values at those two nodes are.

$$V_{uuu} = 0.2804, \quad \text{and} \quad V_{uud} = 0.0673.$$

Finally, the call option's price is

$$\begin{aligned} V(0) &= e^{-0.03(0.75)}\left[(p^*)^3 V_{uuu} + 3(p^*)^2(1-p^*)V_{uud}\right] \\ &= e^{-0.03(0.75)}\left[(0.4813)^3(0.2804) + 3(0.4813)^2(1-0.4813)(0.0673)\right] = 0.05427236. \end{aligned}$$