Homework assignment #2: Solutions

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2021-09-09

More probability review

Problem #1. (5 points)

Let E and F be any two events. If

$$\mathbb{P}[E] = \mathbb{P}[F] = \frac{2}{3}\,,$$

then E and F cannot be mutually exclusive. *True or false? Why?*

Solution: We will argue by contradiction. Assume that E and F have given probabilities and that they are mutually exclusive. Then,

$$1 \geq \mathbb{P}[E \cup F] = \mathbb{P}[E] + \mathbb{P}[F] = \frac{4}{3}\,.$$

A contradiction!

Problem #2. (5 points)

If events A and B are mutually exclusive, they are necessarily independent. *True or false? Why?*

Solution: Let A and B be two events with strictly positive probabilities such that $A \cap B = \emptyset$. Then,

$$\mathbb{P}[A \cap B] = \mathbb{P}[\emptyset] = 0$$

while

$$\mathbb{P}[A]\mathbb{P}[B] > 0.$$

Problem 3. (5 points)

A test is used to determine whether people exhibiting green spots have the *duckpox* or not. It is believed that at any given time 4% of people exhibiting green spots actually have the *duckpox*. The test is 99% accurate if a person actually has the *duckpox*. The test is 96% accurate if a person does **not** have the *duckpox*. What is the probability that a randomly selected person who tests positive for the *duckpox* actually has the *duckpox*?

Solution: The overall probability of obtaining a positive result on the {*duckpox} test is*

$$0.04(0.99) + 0.96(0.04) = 0.078.$$

The probability of both having the {duckpox} and testing positive is

$$0.04(0.99) = 0.0396$$

So, the answer is

$$\frac{0.0396}{0.078} = 0.5076923.$$

Textbook problems on the binomial

Problem 1. (a is 4 points; b is 2 points; c is 2 points; d is 3 points; e is 4 points=15 points total)

Solve **Problem 4.18** from the textbook.

Solution:

4.18

(a) In order to determine if we can we use the binomial distribution to calculate the probability of finding exactly 97 people out of a random sample of 100 American adults had chickenpox in childhood, we need to check if the binomial conditions are met:

1. Independent trials: In a random sample, whether or not one adult has had chickenpox does not depend on whether or not another one has.
2. Fixed number of trials: $n = 100$.
3. Only two outcomes at each trial: Have or have not had chickenpox.
4. Probability of a success is the same for each trial: $p = 0.90$.

(b) Let X be number of people who have had chickenpox in childhood, using a binomial distribution with $n = 100$ and $p = 0.90$:

$$P(X = 97) = \binom{100}{97} \times 0.90^{97} \times 0.10^{3} = 0.0059$$

(c) P(97 out of 100 did have chickenpox) = P(3 out of 100 did not have chickenpox in childhood) = 0.0059

(d) P(at least 1) = P(greater than or equal to 1):

$$\begin{aligned} P(X \geq 1) &= P(X = 1) + P(X = 2) + \cdots + P(X = 10) \\ &= 1 - P(X = 0) \\ &= 1 - 0.10^{10} \\ &\approx 1 \end{aligned}$$

(e) P(at most 3 did not have chickenpox) = P(less than or equal to 3 where p = 0.10)

$$\begin{aligned} P(X \leq 3) &= P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) \\ &= \binom{10}{0} \times 0.10^{0} \times 0.90^{10} + \binom{10}{1} \times 0.10^{1} \times 0.90^{9} + \binom{10}{2} \times 0.10^{2} \times 0.90^{8} + \binom{10}{3} \times 0.10^{3} \times 0.90^{7} \\ &= 0.3487 + 0.3874 + 0.1937 + 0.0574 \\ &= 0.9872 \end{aligned}$$

Problem 2. (a is 3 points; b is 2 points; c is 3 points = 8 points total)

Solve **Problem 4.22 (a, b, c)** from the textbook.

Solution:

4.22

(a) P(at least one afraid) = 1 - P(none afraid) $= 1 - (1 - 0.07)^{10}) = 1 - 0.484 = 0.516$

(b) P(exactly 2 afraid) $= \binom{10}{2} \times 0.07^2 \times 0.93^8 = 45 \times 0.07^2 \times 0.93^8 = 0.1234$

(c) P(at most 1 afraid) = P(none afraid) + P(1 afraid)
$= 0.4840 + \binom{10}{1} \times 0.07^1 \times 0.93^9$
$= 0.4840 + 0.3643$
$= 0.8483$

Problem 3. (2 points each)

Solve **Problem 4.24 (a, b, c)** from the textbook.

Solution:

4.24

(a) Using the binomial distribution with $n = 3$ and $p = 0.25$;

$$P(X = 2) = \binom{3}{2} \times 0.25^2 \times 0.75^1 = 3 \times 0.25^2 \times 0.75^1 = 0.1406$$

(b) Using the binomial distribution with $n = 3$ and $p = 0.25$;

$$P(X = 0) = \binom{3}{0} \times 0.25^0 \times 0.75^3 = 0.4219$$

(c) Using the binomial distribution with $n = 3$ and $p = 0.25$;

$$P(X \geq 1) = 1 - P(X = 0) = 1 - 0.4219 = 0.5781$$

Problem 4. (2 points each)

Solve **Problem 4.26** from the textbook.

Solution:

4.26

(a) Binomial distribution with $n = 3$ and $p = 0.51$;
$P(X = 2) = \binom{3}{2} \times 0.51^2 \times 0.49 = 3 \times 0.51^2 \times 0.49 = 0.3823$

(b) $P(B, B, G) = 0.51 \times 0.51 \times 0.49 = 0.12744$
$P(B, G, B) = 0.51 \times 0.49 \times 0.51 = 0.12744$
$P(G, B, B) = 0.49 \times 0.51 \times 0.51 = 0.12744$
$P(\text{2 out of 3 children are boys}) = 0.12744 + 0.12744 + 0.12744 = 3 \times 0.12744 = 0.3823$

(c) There are now $\binom{8}{3} = 56$ scenarios. It would be tedious to write them all out, and if we didn't know how many scenarios there are we might actually miss some of them. Using the binomial model to calculate this probability is a much more efficient approach.