Name:

M339J: Probability models
University of Texas at Austin
Solution: In-Term Exam I
Instructor: Milica Čudina

Notes: This is a closed book and closed notes exam. The maximal score on this exam is 100 points.
Time: 50 minutes

Problem 1.1. (5 points) Let E and F be two events such that $\mathbb{P}[E] > 0$ and $\mathbb{P}[F] > 0$. You know that

$$\mathbb{P}[E \,|\, F] > \mathbb{P}[E].$$

Then,

$$\mathbb{P}[F \,|\, E] > \mathbb{P}[F].$$

True or false? Why?

Solution: By definition,

$$\mathbb{P}[E|F] > \mathbb{P}[E] \quad \Leftrightarrow \quad \frac{\mathbb{P}[E \cap F]}{\mathbb{P}[F]} > \mathbb{P}[E] \quad \Leftrightarrow \quad \frac{\mathbb{P}[E \cap F]}{\mathbb{P}[E]} > \mathbb{P}[F] \quad \Leftrightarrow \quad \mathbb{P}[F \,|\, E] > \mathbb{P}[F].$$

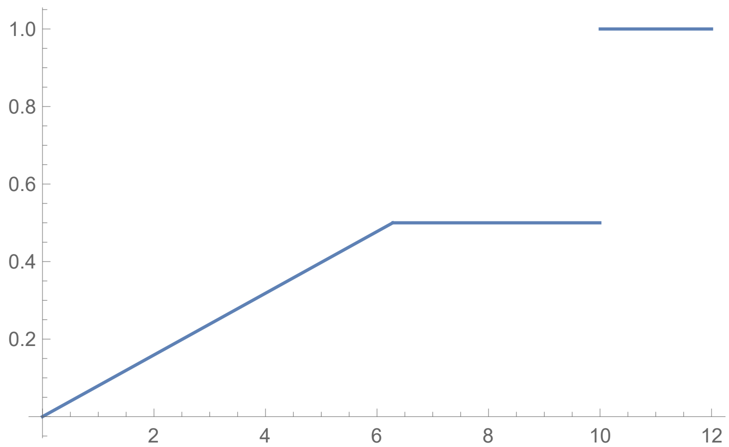
Problem 1.2. (5 points) Provide the definition of the *cumulative distribution function*.

Solution: Check your notes.

Problem 1.3. (10 points) Consider the following game: You toss a fair coin. If the coin comes up heads, you win \$10. If the coin comes up tails, you spin a perfectly balanced spinner which is equally likely to point to any points on the circumference of its base after being spun. You win the amound equal to the angle between the original position of the spinner and the final position of the spinner (modulo 2π, and in radians).

Let X be the random variable denoting the amount you win. Draw the graph of the cumulative distribution function of X.

Solution:



Problem 1.4. (5 points) For any random variable X, we have that

$$\mathbb{E}[|X|] = |\mathbb{E}[X]|.$$

True or false? Why?

Solution: FALSE
Try the random variable with values -1 and 1 being equally likely.

Problem 1.5. (2 points) Let the random variable X have the survival function $S_X(x) = e^{-x/80}$, for $x > 0$. Then the mean of that random variable equals $1/80$. *True or false? Why?*

Solution: FALSE
The mean is actually 80. You can recognize the distribution as exponential with mean 80. Or, you can calculate the expected value outright.

Problem 1.6. (5 points) A simple experiment consists of drawing a single ball at random from each of two urns containing red and blue marbles. The first urn contains 1 red and 3 blue marbles. A second urn contains 12 red marbles and an unknown number of blue marbles. You are told that the probability that both marbles are the same color equals $9/20$. Calculate the number of blue marbles in the second urn.

Solution: Let b denote the number of blue marbles in the second urn. Then,

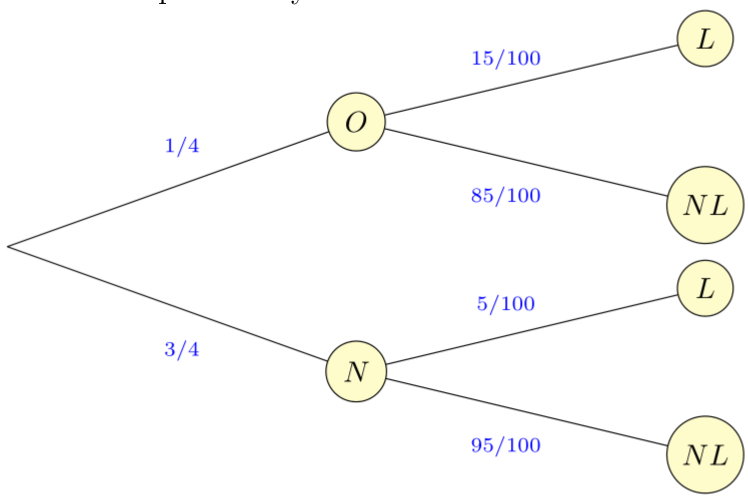
$$\frac{1}{4} \cdot \frac{12}{12+b} + \frac{3}{4} \cdot \frac{b}{12+b} = \frac{9}{20}.$$

So,

$$12 + 3b = \frac{9}{20}(4)(12+b) \quad \Leftrightarrow \quad 5(12+3b) = 9(12+b) \quad \Leftrightarrow \quad 60 + 15b = 108 + 9b$$
$$\Leftrightarrow \quad 6b = 48 \quad \Leftrightarrow \quad b = 8.$$

Problem 1.7. (5 points) The local pool supply store is having an end-of-season sale. They have a seemingly infinite number of floaties lying around. They know that among those $1/4$ are ancient floaties from seasons past (so, old) and that $3/4$ are the last season's floaties (so, new). We know that 15% of old floaties leak, and that 5% of new floaties leak. When an order comes in, a floatie is chosen at random to fulfill the order. You are excited about the sale and you are the first one to show up at the door. You buy a floatie. You take it to the pool. It leaks. What's the probability that it was an old floatie?

Solution: This probability tree describes the situation in the problem:



We use the Bayes' theorem here.

$$\mathbb{P}[O \,|\, \text{floatie leaks}] = \frac{\mathbb{P}[\text{floatie leaks} \,|\, O]\mathbb{P}[O]}{\mathbb{P}[\text{floatie leaks}]}.$$

Using our tree, we get

$$\mathbb{P}[\text{floatie leaks}] = 0.25(0.15) + 0.75(0.05) = 0.075.$$

So,

$$\mathbb{P}[O \,|\, \text{floatie leaks}] = \frac{0.25(0.15)}{0.075} = 0.5.$$

Problem 1.8. (5 points) Let the random variables X_1, X_2 and X_3 be independent and identically distributed such that their probability mass function is

$$p_X(x) = \begin{cases} 1/4, & \text{for } x = 0, \\ 3/4, & \text{for } x = 1. \end{cases}$$

What is the expectation of $X_1 X_2 X_3$?

Solution: Let $Y = X_1 X_2 X_3$. Its probability mass function is

$$p_Y(x) = \begin{cases} 1 - (3/4)^3, & \text{for } x = 0, \\ (3/4)^3, & \text{for } x = 1. \end{cases}$$

So, $\mathbb{E}[Y] = (\frac{3}{4})^3 = \frac{27}{64}$.

Problem 1.9. (8 points) A continuous random variable X has the probability density function f_X given by

$$f_X(x) = \frac{2}{5} - \kappa x, \quad 0 \leq x \leq 5.$$

Find the value of the survival function of X at 3.

Solution: Necessarily, $\int_0^5 f_X(x)\, dx = 1$. So,

$$\int_0^5 \left(\frac{2}{5} - \kappa x\right) dx = 1 \quad \Rightarrow \quad 5\left(\frac{2}{5}\right) - \kappa \left[\frac{x^2}{2}\right]_{x=0}^{5} = 1 \quad \Rightarrow \quad \kappa = \frac{2}{25}.$$

Note that f_X is piecewise linear. So, we can calculate $S_X(3) = \mathbb{P}[X > 3]$ as the area of a triangle (draw a picture if in doubt!). We get

$$S_X(7) = \frac{1}{2} \cdot f_X(3) \cdot (5-3) = f_X(3) = \frac{2}{5} - \frac{2}{25}(3) = \frac{2}{5} - \frac{6}{25} = \frac{10-6}{25} = \frac{4}{25}.$$

Problem 1.10. (5 points) Consider the random variable X whose cumulative distribution function is given by

$$F_X(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{x}{2} & \text{for } 0 \leq x < 1 \\ 1 & \text{for } 1 \leq x \end{cases}$$

What is the expectation of the random variable X?

Solution: This is a mixed distribution with a probability mass of 1/2 at 1 and otherwise uniform on $(0, 1)$. We have

$$\mathbb{E}[X] = \frac{1}{2}\left(\frac{1}{2}\right) + \frac{1}{2}(1) = \frac{3}{4}.$$

Problem 1.11. (10 points) The random variable T is exponentially distributed with mean 10. The probability that it takes a value between 2 and 5 equals the probability that it takes a value between 6 and t^*, with $t^* > 6$. What is t^*?

Solution: Let $T \sim \mathit{Exponential}(\theta = 10)$. We know that

$$\mathbb{P}[2 < T < 5] = \mathbb{P}[T > 2] - \mathbb{P}[T > 5] = \frac{1}{10}e^{-\frac{2}{10}} - \frac{1}{10}e^{-\frac{5}{10}} = \frac{1}{10}(e^{-\frac{1}{5}} - e^{-\frac{1}{2}}).$$

On the other hand, using the same reasoning,

$$\mathbb{P}[6 < T < t^*] = \frac{1}{10}(e^{-\frac{6}{10}} - e^{-\frac{t^*}{10}}).$$

Equating the two, we get

$$\begin{aligned}\frac{1}{10}(e^{-\frac{1}{5}} - e^{-\frac{1}{2}}) = \frac{1}{10}(e^{-\frac{6}{10}} - e^{-\frac{t^*}{10}}) \quad &\Rightarrow \quad e^{-\frac{1}{5}} - e^{-\frac{1}{2}} = e^{-\frac{6}{10}} - e^{-\frac{t^*}{10}} \\ &\Rightarrow \quad e^{-\frac{t^*}{10}} = e^{-\frac{3}{5}} - e^{-\frac{1}{5}} + e^{-\frac{1}{2}} = 0.3366115 \\ &\Rightarrow \quad -\tfrac{t^*}{10} = \ln(0.3366115) = -1.088826 \\ &\Rightarrow \quad t^* = 10.88826\end{aligned}$$

Problem 1.12. (10 points) Let a severity random variable X be uniform over $[0, 100]$. What is the value of its mean excess loss function at 40?

Solution: It's easy to see that $X - d|X > d \sim U(0, 100 - d)$. So,

$$e_X(40) = \frac{100 - 40}{2} = 30.$$

Problem 1.13. (10 points) Let X have the two-parameter Pareto distribution with $\alpha = 5$ and $\theta = 2$. Find the variance of X.

Solution: From the STAM tables

$$\mathbb{E}[X] = \frac{\theta}{\alpha - 1} = \frac{2}{4} = \frac{1}{2} \quad \text{and} \quad \mathbb{E}[X^2] = \frac{2\theta^2}{(\alpha-1)(\alpha-2)} = \frac{2(4)}{4(3)} = \frac{2}{3}.$$

So,

$$Var[X] = \frac{2}{3} - \left(\frac{1}{2}\right)^2 = \frac{2}{3} - \frac{1}{4} = \frac{8-3}{12} = \frac{5}{12}.$$

Problem 1.14. (10 points) Consider independent random variables X and Y. You are given that they have the same mean. Also, the coefficient of variation of X equals 36 and the coefficient of variation of Y equals 77. What is the coefficient of variation of the average of X and Y?

Solution: Let $\mu = \mathbb{E}[X] = \mathbb{E}[Y]$. Then, $\sigma_X = SD[X] = 36\mu$ and $\sigma_Y = SD[Y] = 77\mu$. The variance of the average of the two random variables is

$$Var\left[\frac{1}{2}(X+Y)\right] = \frac{1}{4}(Var[X] + Var[Y]).$$

In terms of μ, the variance of the average can be rewritten as

$$Var\left[\frac{1}{2}(X+Y)\right] = \frac{1}{4}(36^2\mu^2 + 77^2\mu^2) = \frac{7225\mu^2}{4} = \frac{85^2\mu^2}{4}.$$

So, the standard deviation of the average can be expressed as $\frac{85\mu}{2}$. Hence, the coefficient of variation of the average equals $\frac{85}{2} = 42.5$.

Problem 1.15. (5 points) Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables. Assume that all the variables in the sequence have the two-parameter Pareto distribution with $\theta = 10$ and $\alpha = 5$. For each n, define the random variable

$$Y_n = \frac{X_1^3 + X_2^3 + \cdots + X_n^3}{n}.$$

Does the limit of the sequence $\{Y_n, n \geq 1\}$ as $n \to \infty$ exist? If so, how much is it? If not, why not?

Solution: Since the random variables $\{X_n, n \geq 1\}$ are independent and identically distributed, the random variables $\{X_n^3, n \geq 1\}$ are also independent and identically distributed. If the mean of the random variable X_1^2 exists and is finite, we can apply the Law of Large Numbers. Using the STAM tables, we get that

$$\mathbb{E}[X_1^3] = \frac{6\theta^3}{(\alpha-1)(\alpha-2)(\alpha-3)} = \frac{6000}{4 \cdot 3 \cdot 2} = 250.$$

We see that the expected value is finite. So, not only does the Law of Large Numbers apply, but invoking it we can conclude that the limit of $\{Y_n\}_{n\in\mathbb{N}}$ is 250.