M378K Introduction to Mathematical Statistics
Problem Set #17
Hypothesis testing.

Problem 17.1. *An instructor of a massive online course claims that students solve at most* 20 *problems per week (on average). To verify this conviction, the instructor intends to conduct a hypothesis test.*

What are the null and alternative hypotheses in this case?

Solution: *Let* μ *denote the population mean, i.e., the mean number of solved problems (in a week).*

$$H_0 : \mu = 20 \quad \textit{vs.} \quad H_a : \mu < 20.$$

With a sample size of 256*, what is the test statistics appropriate the test the above claim? What is its (approximate) distribution under the null?*

Solution: *With such a large sample size (*$n = 256$*), we can use the CLT and the appropriate test statistic is*

$$Z = \frac{\bar{Y} - \mu_0}{\frac{S}{\sqrt{n}}} \approx N(0,1)$$

Note that with $n = 256$*, I am comfortable using the standard normal above (instead of the - more conservative -* $t-$*distribution).*

Say that the sample average equals 19.7 *and that the sample variance equals* 9*. What is the* $p-$*value associated with these data?*

Solution:

$$\mathbb{P}\left[\frac{\bar{Y} - \mu_0}{\frac{S}{\sqrt{n}}} < \frac{19.7 - 20}{\frac{3}{\sqrt{256}}}\right] \approx \mathbb{P}[Z < -1.6] = 0.0548.$$

Assume that the given significance level is 5%*. What would the decision be?*

Solution: *Fail to reject the null hypothesis.*

Problem 17.2. *A candy-cane twisting machine is considered defective it at least 10% of the candy canes crack or break in the twisting process. A random sample of* 100 *candy canes was collected and it was found that it contained* 12 *cracked candy canes. You believe that the machine is defective. Formulate and conduct the relevant hypothesis test with a 2% significance level.*

Solution: Our hypotheses are

$$H_0 : p = 0.10 \quad \text{vs} \quad H_a : p > 0.10$$

The appropriate test statistic is

$$\frac{\hat{P} - p_0}{\frac{p_0(1-p_0)}{n}}$$

which is approximately standard normal (under the null hypothesis). So, to calculate the $p-$value, we do the following

$$\mathbb{P}\left[Z > \frac{0.12-0.1}{\sqrt{\frac{0.1(1-0.1)}{100}}} = 0.6667\right] = 0.2514.$$

Obviously, the obtained $p-$value is larger than 0.02, and so we fail to reject the null hypothesis.

Problem 17.3. *Consider a poll ahead of an election with two candidates: A and B. Let p denote the population proportion of voters who will vote for A. We want to conduct a hypothesis test on whether candidate A will win, i.e., our hypotheses are*

$$H_0 : p = 0.5 \quad vs. \quad H_a : p < 0.5$$

Let our significance level be 5%. What is the rejection region (RR) for a sample size of 10?
Solution: *Evidently, in this case, we cannot use the large-sample approach with the normal approximation to the binomial. Luckily, we have 'R' at our disposal. Under the null hypothesis, our sample count T is binomial with the number of trials equal to $n = 10$ and the success probability equal to 0.5.*

From the structure of the alternative hypothesis, we know that the rejection region (RR) will be of the form $[0, t^*]$ for some critical value t^* such that

$$\mathbb{P}[T \leq t^*] \leq \alpha < \mathbb{P}[T \leq t^* + 1]$$

Let's examine the probabilities that $T \leq t$ produced by 'R' (as we vary t from 0 to, say, $n = 4$). The "code" is `pbinom(0:4, size=10, p=0.5)`. We output is

$$0.0009765625, 0.0107421875, 0.0546875000, 0.1718750000, 0.3769531250$$

We conclude that our rejection region is $RR = [0, 1]$.