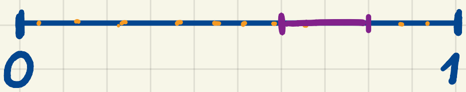
M378K: September 9th, 2024.

More Review: Continuous Distributions.

Example.



Imagine a r.v. Y on $[0,1]$

The probability of it landing between a and where $0 \leq a < b \leq 1$ is

$$\boxed{\mathbb{P}[Y \in [a,b]] = b-a}$$

Note: $\mathbb{P}[Y = y] = 0$ for all y

Def'n. A r.v. Y is said to be continuous is there exists a function

$$f_Y : \mathbb{R} \longrightarrow [0,\infty)$$

such that

$$\mathbb{P}[Y \in [a,b]] = \int_a^b f_Y(y)\,dy \quad \text{for all } a<b.$$

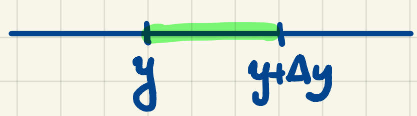
The function f_Y is called the probability density function (pdf) of Y.

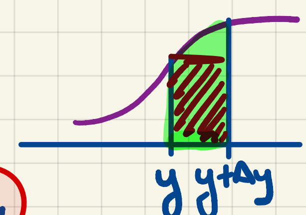
Properties:
- $f_Y(y) \geq 0$ for all y
- $\int_{-\infty}^{\infty} f_Y(y)\,dy = \underline{1}$

Q: Is it possible that $p_Y(k) > 1$ where p_Y is a pmf?

Q: Is it possible that $f_Y(y) > 1$ at some y?

Note:





$$\mathbb{P}[Y \in [y, y+\Delta y]] \approx f_Y(y)\,\Delta y$$

Caveat: There are r.v.s that are neither discrete, nor continuous.

Example. $Y \sim U(0, 1/4)$, i.e., Y is uniformly distributed between 0 and 1/4

$$\boxed{f_Y(y) = c \cdot \mathbb{1}_{[0,1/4]}(y)}$$

where $\mathbb{1}_A : \mathbb{R} \to \mathbb{R}$

$$\mathbb{1}_A(y) = \begin{cases} 1 & y \in A \\ 0 & \text{otherwise} \end{cases}$$

is the indicator function

$$\int_{-\infty}^{\infty} c \cdot \mathbb{1}_{[0,1/4]}(y)\, dy = 1$$

$$c \int_0^{1/4} dy = 1$$

$$c \cdot \left(\frac{1}{4}\right) = 1 \quad \Rightarrow \quad \boxed{c = 4}$$

$\square$

M378K Introduction to Mathematical Statistics
Problem Set #3
Continuous distributions.

Problem 3.1. Source: Sample P exam, Problem #33.
The lifetime of a machine part has a continuous distribution on the interval $(0, 40)$ *with probability density function* f_X*, where*

$$f_X(x) \propto \frac{1}{(10+x)^2}$$

on the interval.

Calculate the probability that the lifetime of the machine part is less than 6.

→ :

$$f_X(x) = \kappa (10+x)^{-2} \mathbb{1}_{[0,40]}(x)$$

$$\kappa \boxed{\int_0^{40} (10+x)^{-2}\,dx} = 1$$

$$\int_0^{40} (10+x)^{-2}\,dx = \frac{(10+x)^{-1}}{-1}\Bigg|_{x=0}^{40} = -\frac{1}{10+x}\Bigg|_{x=0}^{40}$$

$$= \frac{1}{10} - \frac{1}{50} = \frac{4}{50} = \frac{2}{25} \Rightarrow \boxed{\kappa = \frac{25}{2}}$$

$$\mathbb{P}[Y \leq 6] = \mathbb{P}[Y \in [0,6]]$$

$$= \frac{25}{2}\int_0^6 (10+x)^{-2}\,dx$$

$$= \frac{25}{2}\left(\frac{1}{10} - \frac{1}{16}\right) = \frac{25}{2} \cdot \frac{8-5}{80} = \frac{15}{32}$$

□

Example. $Y \sim U(\ell, r)$

$$\mathbb{P}[Y \in [a,b]] = \frac{b-a}{r-\ell} \quad \text{for } \ell \leq a < b \leq r$$

$$f_Y(y) = \boxed{\frac{1}{r-\ell}} \mathbb{1}_{[\ell,r]}(y)$$

Problem 3.2. Source: Sample P exam, Problem #419.
A customer purchases a lawnmower with a two-year warranty. The number of years before the lawnmower needs a repair is uniformly distributed on $[0, 5]$*. Calculate the probability that the lawnmower needs no repairs within* 4.5 *years after the purchase, given that the lawnmower needs no repairs within the warranty period*

$\longrightarrow$: T... lifetime of lawnmower

$T \sim U(0,5)$

$$\mathbb{P}[T > 4.5 \mid T > 2] = \frac{\mathbb{P}[T > 4.5, T > 2]}{\mathbb{P}[T > 2]}$$

$$= \frac{\mathbb{P}[T > 4.5]}{\mathbb{P}[T > 2]} = \frac{\frac{5-4.5}{\cancel{5-0}}}{\frac{5-2}{\cancel{5-0}}} = \frac{1}{6} \quad \square$$

Example. $Y \sim N(\mu, \sigma)$ where $\mu \in \mathbb{R}$ and $\sigma > 0$
is normally distributed w/ mean μ
and standard
deviation σ

If $$f_Y(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-\mu)^2}{2\sigma^2}} \quad \text{for all } y \in \mathbb{R}$$

If $\mu = 0$ and $\sigma = 1$, we say that Y is
standard normal and write $Y \sim N(0,1)$
w/ pdf $$\boxed{f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}} \quad \text{for all } y \in \mathbb{R}$$

Q: Let $Y \sim N(0,1)$
Let α and β be two real constants

$$\alpha Y + \beta \sim \text{Normal}(\mu = \beta, \sigma = |\alpha|)$$

Q: Let $Y \sim N(\mu, \sigma)$

$$\frac{Y - \mu}{\sigma} \sim N(0,1)$$

Problem 3.3. *Consider a continuous random variable Y whose probability density function is given by*

$$f_Y(y) = 2y\mathbf{1}_{[0,1]}(y)$$

What is the expected value of this random variable?