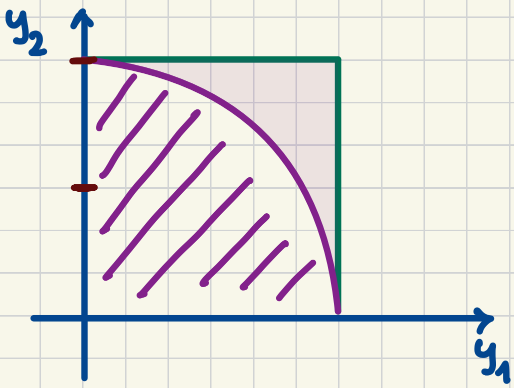
M378K: February 16^{th}, 2026.

Random Vectors [cont'd].

Example. (Y_1, Y_2) jointly uniform $[0,1]^2$

$$\boxed{f_{Y_1,Y_2}(y_1,y_2) = 1 \cdot \mathbb{1}_{[0,1]^2}(y_1,y_2)}$$



$$\mathbb{P}[Y_1^2 + Y_2^2 \leq 1] = \cancel{X}\ \frac{\pi}{4}$$

$$\boxed{A = \{(y_1,y_2) \in [0,1]^2 : y_1^2 + y_2^2 \leq 1\}}$$

$$\mathbb{P}[(Y_1,Y_2) \in A] =$$

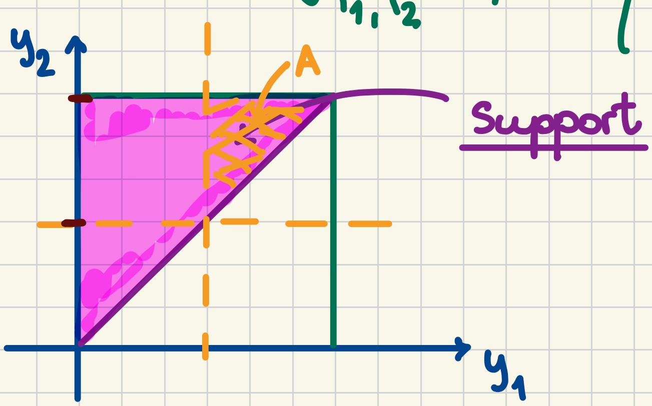
$$= \iint_A f_{Y_1,Y_2}(y_1,y_2)\,dy_2\,dy_1 = \dots = \frac{\pi}{4}$$

Example. Let (Y_1,Y_2) be jointly continuous w/ joint pdf

$$\boxed{f_{Y_1,Y_2}(y_1,y_2) = 6y_1 \cdot \mathbb{1}_{[0 \leq y_1 \leq y_2 \leq 1]}}$$

OR

$$f_{Y_1,Y_2}(y_1,y_2) = \begin{cases} 6y_1 & \text{for } 0 \leq y_1 \leq y_2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$



$$\mathbb{P}\left[Y_1 > \tfrac{1}{2}, Y_2 > \tfrac{1}{2}\right] = \cancel{?}$$

$$= \iint_A f_{Y_1,Y_2}(y_1,y_2)\, dy_2\, dy_1$$

$$= \int_{\frac{1}{2}}^{1} \int_{\frac{1}{2}}^{1} 6y_1\, \mathbb{1}_{[0 \le y_1 \le y_2 \le 1]}\, dy_2\, dy_1$$

$$= \int_{\frac{1}{2}}^{1} 6y_1 \int_{\cancel{\frac{1}{2}}\; y_1}^{1} \mathbb{1}_{[0 \le y_1 \le y_2 \le 1]}\, dy_2\, dy_1$$

$$= \int_{\frac{1}{2}}^{1} 6y_1 \overbrace{\int_{y_1}^{1} dy_2}^{=1-y_1}\, dy_1$$

$$= \int_{\frac{1}{2}}^{1} 6y_1 (1-y_1)\, dy_1 =$$

$$= 6 \int_{\frac{1}{2}}^{1} (y_1 - y_1^2)\, dy_1 =$$

$$= 6 \left(\frac{y_1^2}{2} \Big|_{y_1=\frac{1}{2}}^{1} - \frac{y_1^3}{3} \Big|_{y_1=\frac{1}{2}}^{1} \right)$$

$$= 6 \left(\frac{1}{2} - \frac{1}{8} - \frac{1}{3} + \frac{1}{24} \right) = 6 \cdot \frac{12-3-8+1}{24} = \frac{1}{2} \quad \square$$

Functions of Random Vectors.

Theorem. Let $(Y_1, Y_2, \dots, Y_n)$ be a jointly continuous random vector w/ the joint pdf

$$f_{Y_1,\dots,Y_n}(\cdot, \dots, \cdot).$$

Let g be a real function of n variables such that we can define

$$W = g(Y_1, \dots, Y_n)$$

Then,

$$\mathbb{E}[W] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(y_1, \dots, y_n) f_{Y_1,\dots,Y_n}(y_1, \dots, y_n)\, dy_n \cdots dy_1$$

$$\mathbb{E}[g(Y)] = \int_{-\infty}^{\infty} g(y) f_Y(y)\, dy \quad \text{in 1D}$$

Example. (previous cont'd)

$$\mathbb{E}[Y_1^2 + Y_2^2] = \cancel{?}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (y_1^2 + y_2^2)\, 6y_1 \cdot \mathbb{1}_{[0 \le y_1 \le y_2 \le 1]}\, dy_2\, dy_1$$

$$= 6 \int_0^1 \int_{y_1}^1 (y_1^2 + y_2^2)\, y_1\, dy_2\, dy_1$$

$$= 6 \int_0^1 \int_{y_1}^1 (y_1^3 + y_1 y_2^2)\, dy_2\, dy_1$$

$$= 6 \int_0^1 \left(y_1^3 \cdot y_2 + y_1 \frac{y_2^3}{3} \right)\Big|_{y_2 = y_1}^{1}\, dy_1$$

$$= 6\int_0^1 \left(y_1^3 + \frac{y_1}{3} - y_1^4 - \frac{y_1^4}{3} \right) dy_1$$

$$= \int_0^1 (6y_1^3 + 2y_1 - 8y_1^4)\, dy_1$$

$$= \cancel{6}^{3} \cdot \frac{1}{\cancel{4}_{2}} + 2 \cdot \frac{1}{2} - 8 \cdot \frac{1}{5} = \frac{9}{10} \qquad \square$$

Marginal Distributions & Independence.

Theorem. Say the $(Y_1, \dots, Y_n)$ has the joint pdf $f_{Y_1,\dots,Y_n}(\cdot, \dots, \cdot)$

Then, for every $i = 1..n$, the random variable Y_i is also continuous w/ its marginal density

$$f_{Y_i}(y) = \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{n-1} f_{Y_1,\dots,Y_n}(y_1, \dots, y_{i-1}, y, y_{i+1}, \dots, y_n)\, dy_n \dots dy_{i+1}\, dy_{i-1} \cdots dy_1$$

Example. [cont'd]

Marginal of $Y_1 = ?$

$$f_{Y_1}(y) = \int_{-\infty}^{\infty} f_{Y_1,Y_2}(y, y_2)\, dy_2$$

$$= \int_{-\infty}^{\infty} 6y\, \mathbb{1}_{[0 \le y \le y_2 \le 1]}\, dy_2$$

$$= 6y \int_y^1 dy_2 = 6y(1-y) \cdot \mathbb{1}_{[0,1]}(y)$$

Marginal of Y_2 = ?

$$f_{Y_2}(y) = \int_{-\infty}^{\infty} f_{Y_1,Y_2}(y_1, y)\, dy_1$$

$$= \int_{-\infty}^{\infty} 6y_1 \mathbb{1}_{[0 \le y_1 \le y \le 1]}\, dy_1$$

$$= 6\int_0^y y_1\, dy_1 \cdot \mathbb{1}_{[0,1]}(y)$$

$$= 6 \cdot \frac{y^2}{2} \cdot \mathbb{1}_{[0,1]}(y) = 3y^2 \mathbb{1}_{[0,1]}(y)$$