The University of Texas at Austin

HOMEWORK ASSIGNMENT 3

Prerequisite material. Long/short positions. Short sales.

February 07, 2026

Instructions: Provide your complete solution to the following problems. Final answers only, without appropriate justification, will receive zero points even if correct.

Probability

Problem 3.1. (10 points) *Source: Sample P exam, Problem #201.* A theme park conducts a study of families that visit the park during a year. The fraction of such families of size m is $\frac{8-m}{28}$, for $m = 1, 2, 3, 4, 5, 6$, and 7.

For a family of size m that visits the park, the number of members of the family that ride the roller coaster follows a discrete uniform distribution on the set $\{1, ..., m\}$.

Calculate the probability that a family visiting the park has exactly six members, f given that exactly five members of the family ride the roller coaster.

Solution. Let M be the size of a randomly chosen family and let N denote the number of family members wh o ride the roller coaster. We seek $\mathbb{P}[M = 6 \mid N = 5]$. By the definition of conditional probability,

$$\mathbb{P}[M = 6 \mid N = 5] = \frac{\mathbb{P}[M = 6, N = 5]}{\mathbb{P}[N = 5]}.$$

Again, by the definition of conditional probability

$$\mathbb{P}[M = 6, N = 5] = \mathbb{P}[M = 6]\mathbb{P}[N = 5 \mid M = 6] = \frac{8-6}{28}\left(\frac{1}{6}\right) = \frac{2}{28}\left(\frac{1}{6}\right).$$

By the *Law of Total Probability*, we have

$$\begin{aligned}
\mathbb{P}[N = 5] = {} & \mathbb{P}[M = 1] \times \mathbb{P}[N = 5 \mid M = 1] + \mathbb{P}[M = 2] \times \mathbb{P}[N = 5 \mid M = 2] \\
& + \mathbb{P}[M = 3] \times \mathbb{P}[N = 5 \mid M = 3] + \mathbb{P}[M = 4] \times \mathbb{P}[N = 5 \mid M = 4] \\
& + \mathbb{P}[M = 5] \times \mathbb{P}[N = 5 \mid M = 5] + \mathbb{P}[M = 6] \times \mathbb{P}[N = 5 \mid M = 6] \\
& \quad + \mathbb{P}[M = 7] \times \mathbb{P}[N = 5 \mid M = 7] \\
= {} & 0 + 0 + 0 + 0 + \frac{8-5}{28}\left(\frac{1}{5}\right) + \frac{8-6}{28}\left(\frac{1}{6}\right) + \frac{8-7}{28}\left(\frac{1}{7}\right) \\
= {} & \frac{3}{28}\left(\frac{1}{5}\right) + \frac{2}{28}\left(\frac{1}{6}\right) + \frac{1}{28}\left(\frac{1}{7}\right).
\end{aligned}$$

Finally,

$$\mathbb{P}[M = 6 \mid N = 5] = \frac{\mathbb{P}[M = 6, N = 5]}{\mathbb{P}[N = 5]} = \frac{\frac{2}{28}\left(\frac{1}{6}\right)}{\frac{3}{28}\left(\frac{1}{5}\right) + \frac{2}{28}\left(\frac{1}{6}\right)\frac{1}{28}\left(\frac{1}{7}\right)} = \frac{\frac{2}{6}}{\frac{3}{5} + \frac{2}{6} + \frac{1}{7}} \approx 0.3097345.$$

Monotonicity

Problem 3.2.

1. (3 points) Write down the definition of an *increasing* real-valued function whose domain are all nonnegative real numbers.
2. (3 points) Write down the definition of a *decreasing* real-valued function whose domain are all nonnegative real numbers.

Solution.

1. A function $f : [0, \infty) \to \mathbb{R}$ is said to be an *increasing* function if
$$x_1 < x_2 \Rightarrow f(x_1) \le f(x_2).$$
2. A function $f : [0, \infty) \to \mathbb{R}$ is said to be a *decreasing* function if
$$x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2).$$

Problem 3.3. (1 point) Draw the graph of an *increasing* function.

Solution. Correct answers vary. $f(x) = x$ works.

Problem 3.4. (1 point) Draw the graph of a *decreasing*.

Solution. Correct answers vary. $f(x) = -x$ works.

Problem 3.5. (2 points) Draw the graph of a function which is neither decreasing nor increasing.

Solution. Correct answers vary. $f(x) = |x - 3|$ works.

Problem 3.6. (4 points) Consider the functions $f : [0, \infty) \to \mathbb{R}$ and $g : [0, \infty) \to \mathbb{R}$. Let f be given by

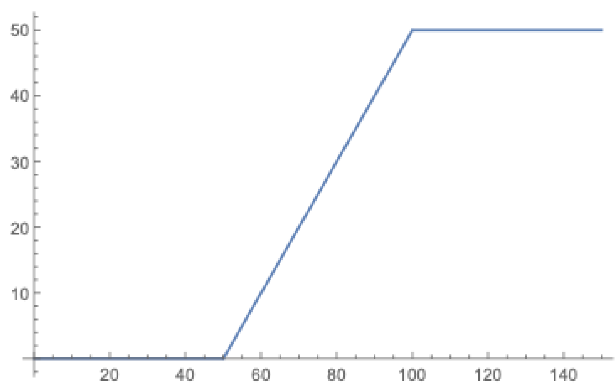
$$f(x) = \max(x - 50, 0).$$

Let g be given by

$$g(x) = \max(x - 100, 0).$$

What can you say about the monotonicity of the function $f - g$? Remember to justify your answer!

Solution. Here is the graph of the function $f - g$:



For any $x_1 < x_2$, we have $f(x_1) - g(x_1) \le f(x_2) - g(x_2)$.

Problem 3.7. (5 points) Complete the following definition:
A financial portfolio is said to be *long with respect to an underlying asset* if

Solution. its payoff/profit function is increasing as a function of the final asset price.

Problem 3.8. (5 points) Complete the following definition:
A financial portfolio is said to be *short with respect to an underlying asset* if

Solution. its payoff/profit function is decreasing as a function of the final asset price.

Payoff/Profit of an Investment

Problem 3.9. (3 points) Consider an outright purchase of a share of non-dividend-paying stock whose current price is \$80 per share. Let the be equal to 0.04. What is the time−2 break-even stock price for this investment?

Solution. In our usual notation, the break-even price is

$$S(0)e^{rT} = 80e^{0.04(2)} = 80e^{0.08} = 86.66297$$

Problem 3.10. (3 points) Bertram sells short 10 shares of a non-dividend-paying stock. The time−0 price of this stock is \$100. Assume that the continuously compounded, risk-free interest rate equals 0.06. If Bertram closes the short sale in six months, what is his break-even final stock price?

Solution. In our usual notation, the break-even price for Bertram's short sale equals

$$S(0)e^{rT} = 100e^{0.06(0.5)} = 100e^{0.03} = 103.0455$$

Problem 3.11. (5 points) Let the current price of a non-dividend-paying stock be \$40.

You model the distribution of the time−1 price of the above stock as follows:

$$S(1) \sim \begin{cases} 42, \text{with probability} & 1/4 \\ 38, \text{with probability} & 1/2 \\ 36, \text{with probability} & 1/4. \end{cases}$$

The continuously compounded, risk-free interest rate is 0.04.

What is your expected profit under the above model, if you short sell one share of stock at time−0 and intend to close the short sale at time−1?

Solution. The initial cost is $-S(0)$ and the payoff is $-S(1)$. So, with $T = 1$, the profit equals

$$-S(T) + S(0)e^{rT}.$$

Thus, the expected profit equals

$$-\mathbb{E}[S(T)] + S(0)e^{rT}.$$

According to the given model for the stock price, we have

$$\mathbb{E}[S(T)] = 42\left(\frac{1}{4}\right) + 38\left(\frac{1}{2}\right) + 36\left(\frac{1}{4}\right) = 38.5.$$

Finally, the expected profit is

$$-38.5 + 40e^{0.04} = 3.132431.$$

Problem 3.12. (5 points) The current price of a non-dividend paying stock is \$100 per share. You purchase one share of this stock. You do not intend to make any further trades over the next year. You intend to liquidate your investment at the end of the year.

You model the stock price at the end of the year to be distributed as follows:

$$S(T) \sim \begin{cases} 90, & \text{with probability } 1/10 \\ 100, & \text{with probability } 1/2 \\ 110, & \text{with probability } 2/5 \end{cases}$$

The continuously compounded, risk-free interest rate is 0.01.

What is the expected profit of your investment?

Solution. The expected stock price is

$$\mathbb{E}[S(T)] = 90\left(\frac{1}{10}\right) + 100\left(\frac{1}{2}\right) + 110\left(\frac{2}{5}\right) = 9 + 50 + 44 = 103.$$

So, the expected wealth/payoff is 103. Finally, the expected profit is

$$103 - 100e^{0.01} = 1.994983.$$