M339J: January 27th, 2023.

Expected Value

Def'n. The **expected value** or **expectation** or **mean** of a random variable X is given as follows:

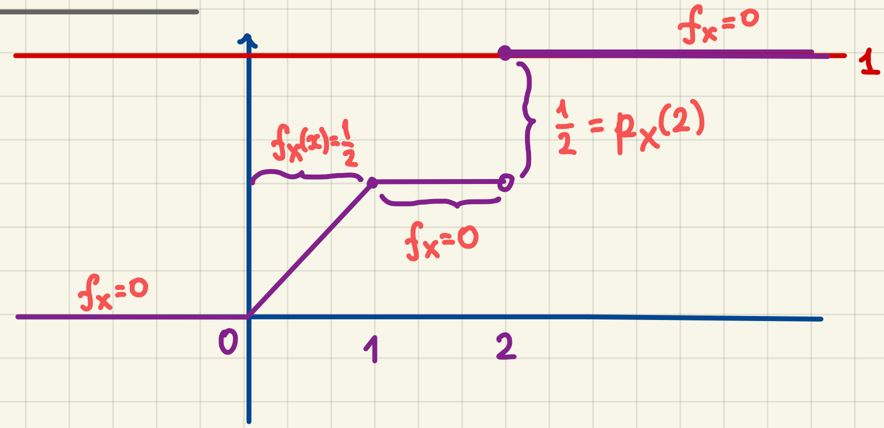
- if X is a discrete random variable:

$$\boxed{\mathbb{E}[X] = \sum_x x \cdot p_X(x)}$$ if the sum exists

- if X is a continuous random variable:

$$\boxed{\mathbb{E}[X] = \int_{-\infty}^{+\infty} x \cdot f_X(x)\,dx}$$ if the integral exists

- if X is a mixed random variable:



$$\boxed{\mathbb{E}[X] = \sum_x x \cdot p_X(x) + \int_{-\infty}^{+\infty} x\, f_X(x)\,dx}$$ if everything is convergent

Problem. An insurance policy pays 100 per day of hospitalization for up to three days. After that, they pay 50 per day for up to five days total.
The number of days of hospitalization is modeled by a random variable N whose pmf is:

$$p_N(k) = \frac{6-k}{15}, \quad k = 1,2,3,4,5$$

and 0 otherwise.

Find the expected pmt per hospitalization under the policy.

→:

hosp. length	probab.	pmt amount
1	1/3	100
2	4/15	200
3	1/5	300
4	2/15	350
5	1/15	400

The expected pmt. per hospitalization:

$$\frac{1}{3}\cdot 100 + \frac{4}{15}\cdot 200 + \frac{1}{5}\cdot 300 + \frac{2}{15}\cdot 350 + \frac{1}{15}\cdot 400 = 220 \quad \blacksquare$$

Problem. Let X be a continuous random variable w/ the pdf

$$f_X(x) = \begin{cases} \frac{p-1}{x^p} & x > 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the value of p such that $\mathbb{E}[X] = 2$.

→: By def'n:

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} x \cdot f_X(x)\,dx$$

$$= \int_1^{+\infty} x \cdot (p-1) \cdot x^{-p}\,dx$$

$$= (p-1)\int_1^{+\infty} x^{1-p}\,dx \qquad \text{take } \underline{p>2}$$

$$= (p-1)\cdot\left(\frac{1}{2-p}\right) x^{2-p}\Big|_{x=1}^{+\infty}$$

$$= \frac{p-1}{2-p}\,(0-1)$$

$$= \frac{p-1}{p-2} = 2 \quad \Rightarrow \quad p = 3$$

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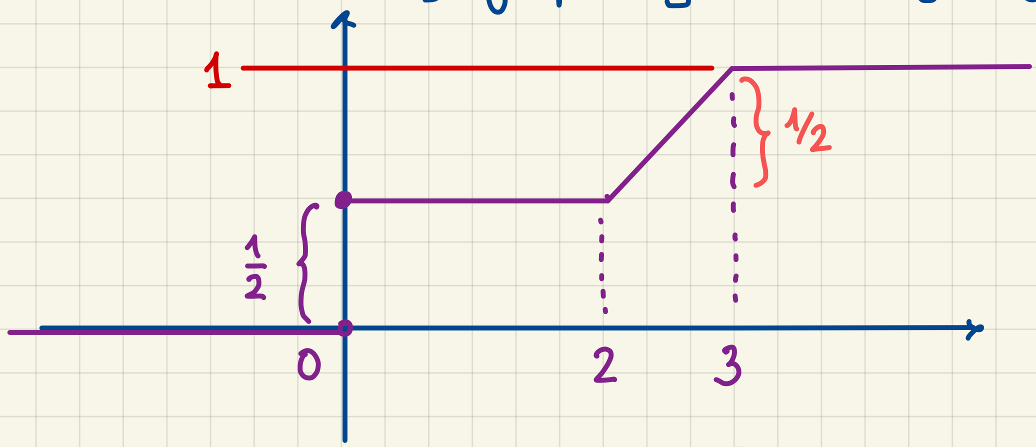
Problem. Consider the following graph of the cdf of X:



Find $\mathbb{E}[X]$.

$\longrightarrow$:

- There is a jump @ zero. It does not affect the expectation.
- $(-\infty,0)$, $(0,2)$, $(3,+\infty)$ are all impossible.
- Between 2 and 3 the dist'n is uniform:

$$f_X(x) = \frac{1}{2} \quad \text{for } x \in (2,3)$$

$$\mathbb{E}[X] = 0\cdot \cancel{p_X(0)} + \int_2^3 x f_X(x)\,dx =$$

$$= \frac{1}{2}\int_2^3 x\,dx = \frac{1}{2}\cdot\frac{x^2}{2}\Big|_2^3 = \frac{1}{4}(9-4) = \frac{5}{4}$$

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Tail Formula for the Expectation.

Let Y be a nonnegative continuous random variable.
Then, we have that

$$\boxed{\mathbb{E}[Y] = \int_0^{+\infty} S_Y(y)\,dy} \quad ☆$$

→: We know that, by definition,

$$\mathbb{E}[Y] = \int_0^{+\infty} y \cdot f_Y(y)\,dy.$$

If we can show that the right-hand side in ☆ equals the integral above, we're done.

$$\int_0^{+\infty} S_Y(y)\,dy \underset{\text{by defn.}}{=} \int_0^{+\infty} \mathbb{P}[Y > y]\,dy = \int_0^{+\infty}\int_y^{+\infty} f_Y(u)\,du\,dy$$

Now, we switch the integrals!

$$\int_0^{+\infty}\int_0^{u} f_Y(u)\,dy\,du = \int_0^{+\infty} f_Y(u) \underbrace{\left(\int_0^{u} dy\right)}_{=u} du = \int_0^{+\infty} f_Y(u)\cdot u\,du = \mathbb{E}[Y] \quad \blacksquare$$

For discrete random variables, we focus on $\mathbb{N}_0$-valued r.v.s.

$$\boxed{\mathbb{E}[Y] = \sum_{k=0}^{+\infty} S_X(k)} \quad ✓$$