Name:

UTeid:

M362K Probability
University of Texas at Austin
In-Term Exam III
Instructor: Milica Čudina

Notes: This is a closed book and closed notes exam. The maximal score on this exam is 100 points.

All written work handed in by the student is considered to be
their own work, prepared without unauthorized assistance.

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Signature:

3.1. **DEFINITIONS.**

Problem 3.1. (10 points) Complete the definition of a *cumulative distribution function* below:

Let X be a random variable on an outcome space Ω. The *cumulative distribution function* of X is ...

Solution: ...the function $F_X : \mathbb{R} \to [0,1]$ given by

$$F_X(x) = \mathbb{P}[X \le x] \quad \text{for all } x \in \mathbb{R}.$$

Problem 3.2. (10 points) Complete the definition of the *probability mass function* of a random variable X on a finite outcome space Ω.

Let X be a random variable on a finite outcome space Ω. The *probability mass function* of X is ...

Solution: ...the function $p_X : \mathit{Support}(X) \to [0,1]$ given by

$$p_X(x) = \mathbb{P}[X = x] \quad \text{for all } x \in \mathit{Support}(X).$$

3.2. **TRUE/FALSE QUESTIONS.**

Problem 3.3. (3 points) Let X denote the outcome of a roll of a fair, regular icosahedron (a polyhedron with 20 faces) with numbers $1, 2, \cdots, 20$ written on its sides. Then $\mathbb{E}[X] = 21/2$. *True or false? Why?*

Solution: TRUE
Since each outcome is equally likely, by the definition of the expected value

$$\mathbb{E}[X] = \frac{1}{20} \cdot 1 + \frac{1}{20} \cdot 2 + \cdots + \frac{1}{20} \cdot 20 = \frac{1}{20}(1 + 2 + \cdots + 20) = \frac{1}{20} \cdot \frac{20 \cdot 21}{2} = 10.5\,.$$

Problem 3.4. (3 points) Let X and Y be two independent random variables with $\mathbb{E}[X] = 1, \mathrm{Var}[X] = 4, \mathbb{E}[Y] = 2$ and $\mathrm{Var}[Y] = 9$. Define $S = X + Y$. Then, $\mathrm{Var}[S] = 13$. *True or false? Why*

Solution: TRUE
Since X and Y are independent, they must be uncorrelated. Hence, the *addition formula for variances* applies and we have

$$\mathrm{Var}[S] = \mathrm{Var}[X] + \mathrm{Var}[Y] = 4 + 9 = 13.$$

3.3. **FREE RESPONSE PROBLEMS.** Please, explain carefully all your statements and assumptions. Numerical results or single-word answers without an explanation (even if they're correct) are worth 0 points.

Problem 3.5. (14 points) Let X and Y be two random variables on the same outcome space such that

$$\begin{aligned} \mu_X = \mathbb{E}[X] = 0, &\quad \sigma_X^2 = \mathrm{Var}[X] = 1, \\ \mu_Y = \mathbb{E}[Y] = -2, &\quad \sigma_Y^2 = \mathrm{Var}[Y] = 4, \quad \rho_{X,Y} = corr[X,Y] = 0.5. \end{aligned}$$

Calculate $\mathrm{Var}[2X + Y - 3]$.

Solution: Since -3 is just a deterministic shift, the variance we need to calculate is equal to $\mathrm{Var}[2X + Y]$. By the covariance formula and the definition of the correlation coefficient,

$$\begin{aligned} \mathrm{Var}[2X + Y] &= 4\,\mathrm{Var}[X] + 2 \cdot 2\,\mathrm{Cov}[X,Y] + \mathrm{Var}[Y] \\ &= 4\sigma_X^2 + 4\sigma_X\sigma_Y\rho_{X,Y} + \sigma_Y^2 \\ &= 4 \cdot 1 + 4 \cdot 1 \cdot 2 \cdot 0.5 + 4 = 12. \end{aligned}$$

Problem 3.6. (20 points) Let X and Y be independent geometric random variables with the same parameter p. Define the random variable $Z = X+Y$. Find the probability mass function of the random variable Z. *Hint:* If you want to solve the problem quickly - do the probabilistic method. If you want to brute force the solution - use the convolution.

Solution: For every $n \geq 1$, we have

$$\begin{aligned}
p_Z(n) &= \mathbb{P}[Z = n] \\
&= \mathbb{P}[X + Y = n] \\
&= \mathbb{P}[\text{“there was one “heads” in the first } n-1 \text{ flips and the } n^{th} \text{ flip came out “heads” ” }] \\
&= \sum_{i=1}^{n-1} \mathbb{P}[\text{“the } i^{th} \text{ and the } n^{th} \text{ flips were “heads” and the rest were “tails” ”}] \\
&= (n-1)p^2(1-p)^{n-2}.
\end{aligned}$$

Problem 3.7. (25 points) A new addition of Kafka's "Metamorphosis" has 72 pages. The printing press often malfunctions and introduces typos. The number of typos on each page has a Poisson distribution with mean $\ln(3)$ and is independent of the number of typos on other pages (or other books). A book is thrown away if it contains typos on more than 32 pages. Use the normal approximation (i.e., the Central Limit Theorem) to estimate the proportion of books that get thrown away. *Hint: Start by figuring out the Poisson probability that a particular page has at least one typo. Then, use the normal approximation to the binomial to figure out the approximate probability.*

Solution: *Note: Compare to problem 3.5.3 from the textbook that was suggested.*
By the formula for the Poisson distribution, the probability that a page contains no typos is $p = e^{-\ln(3)} = \frac{1}{3}$. So, the probability that a page contains at least one typo is $\frac{2}{3}$. The number of pages with at least one typo (per book) is therefore binomially distributed with the mean $\mu = 72 \times \frac{2}{3} = 48$ and standard deviation $\sigma = \sqrt{72 \times \frac{2}{3} \times \frac{1}{3}} = 4$.

In the normal approximation, we are looking for the probability

$$\begin{aligned}\mathbb{P}\left[Z \geq \frac{32 + \frac{1}{2} - 48}{4}\right] &= \mathbb{P}[Z \geq -3.875] \\ &= 1 - \Phi(-3.875) \cong 1.\end{aligned}$$

We would, therefore, expect almost all of the books to be thrown away.

3.4. **MULTIPLE CHOICE QUESTIONS.**

Problem 3.8. (5 points) Let X and Y be independent and both uniformly distributed on $\{1, 2, \ldots, n\}$. Define $\tilde{p} = \mathbb{P}[X = Y]$. Then,

(a) $\tilde{p} = 1/n$
(b) $\tilde{p} = 1/n^2$
(c) $\tilde{p} = (n^2 - 1)/n^2$
(d) $\tilde{p} = 1/(n-1)$
(e) None of the above

Solution: (a)

$$\mathbb{P}[X = Y] = \sum_{k=1}^{n} \mathbb{P}[X = k, Y = k] = n \cdot \frac{1}{n^2} = \frac{1}{n}, .$$

Problem 3.9. (5 points) *Source: Problem 3.1.9 from Pitman.*
A box contains 8 balls. Two are red, two are yellow, two are green and two are purple.
Balls are drawn from the box **without replacement** until the color appears that has appeared before. Let X be the random variable denoting the number of draws that are made. Let $p_X(2) = \mathbb{P}[X = 2]$. Then,

(a) $p_X(2) = 1/5$
(b) $p_X(2) = 1/7$
(c) $p_X(2) = 4/35$
(d) $p_X(2) = 1/35$
(e) None of the above

Solution: (b)

$$\begin{aligned} p_X(2) &= \mathbb{P}[\{\text{first two balls are red}\}] + \mathbb{P}[\{\text{first two balls are yellow}\}] \\ &\quad + \mathbb{P}[\{\text{first two balls are green}\}] + \mathbb{P}[\{\text{first two balls are purple}\}] \\ &= 4 \cdot \frac{1}{4} \cdot \frac{1}{7} = \frac{1}{7}. \end{aligned}$$

Problem 3.10. (5 points) Let $X \sim N(\mu_X = 5, \sigma_X^2 = 2)$. Find $\mathbb{E}[X^2]$.

(a) 27
(b) 30
(c) 33
(d) 46
(e) None of the above

Solution: (a)
From the given parameters, we can conclude that $\mathbb{E}[X] = \mu = 5$ and $Var[X] = \sigma^2 = 2$. So,

$$\mathbb{E}[X^2] = Var[X] + \mathbb{E}[X]^2 = 2 + 25 = 27.$$