Q: What is the estimated mean final score of the students who were above the mean on the midterm?

$\longrightarrow$: Let U be the midterm score and V be the final score.

Let X and Y be U and V in standard units, resp.

Our first task is to find:

$$\mathbb{E}[Y \mid X>0] = \int_{\cancel{0}}^{+\infty} \boxed{\mathbb{E}[Y \mid X=x]} \boxed{f_X(x \mid X>0)\,dx} \quad \checkmark$$

$0 - \cancel{\infty}$

$\mathbb{E}[Y \mid X=x] = \rho x$

The Law of Total probability

and for $x>0$:

$$f_X(x \mid X>0)dx = \mathbb{P}[X \in dx \mid X>0] = \frac{\mathbb{P}[X \in dx \text{ and } X>0]}{\mathbb{P}[X>0]}$$

$$= \frac{f_X(x)dx}{\frac{1}{2}} = 2 f_X(x)dx$$

$$= 2 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

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$$\mathbb{E}[Y \mid X>0] = \int_0^{+\infty} \rho x \cdot 2 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx =$$

$$= \frac{2\rho}{\sqrt{2\pi}} \int_0^{+\infty} x e^{-\frac{x^2}{2}} dx = \quad u = \frac{x^2}{2} \quad du = x\,dx$$

$$= \frac{2\rho}{\sqrt{2\pi}} \int_0^{+\infty} e^{-u} du = \frac{2\rho}{\sqrt{2\pi}} \left[-e^{-u}\right]_{u=0}^{+\infty} = \frac{2\rho}{\sqrt{2\pi}}$$

In this problem: $\frac{1.52}{\sqrt{2\pi}} = 0.6063923$

$\Rightarrow$ Our answer is: $60 + 20(0.6063923) = 72.12785$ $\square$

Matrix Notation.

In two dimensions, we can place the means in a vector $\begin{pmatrix} \mu_U \\ \mu_V \end{pmatrix}$ and the variances/covariances in a matrix:

$$\Sigma = \begin{bmatrix} \sigma_U^2 & \sigma_U\sigma_V\rho \\ \sigma_U\sigma_V\rho & \sigma_V^2 \end{bmatrix}$$ (positive definite)

Then, the joint density of (U,V) can be written as:

$$f_{U,V}(u,v) = \frac{1}{2\pi} \cdot \frac{1}{(\det(\Sigma))^{1/2}} \exp\left(-\frac{1}{2}\begin{pmatrix} u-\mu_U \\ v-\mu_V \end{pmatrix}^T \Sigma^{-1} \begin{pmatrix} u-\mu_U \\ v-\mu_V \end{pmatrix}\right)$$

Multivariate Normal Density.

Let $\mathbb{X} = (X_1, X_2, \dots, X_p)^T$ be $N(\text{mean} = \mu = (\mu_1, \mu_2, \dots, \mu_p)^T$,

$$\Sigma = \begin{bmatrix} \sigma_1^2 & & \text{Cov} \\ & \ddots & \\ \text{Cov} & & \sigma_p^2 \end{bmatrix})$$

w/ Σ positive definite

Then,

$$f_{\mathbb{X}}(x_1, x_2, \dots, x_p) = \frac{1}{(2\pi)^{p/2}} \cdot \frac{1}{(\det(\Sigma))^{1/2}} \exp\left(-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)\right)$$