UNIVERSITY OF TEXAS AT AUSTIN

Problem Set # 14

Power of Test.

Provide your **complete solution** for the following problems.

Problem 14.1. As the sample size increases, the power of a test will increase. *True or false? Why?*

Solution: TRUE
For a proof of concept, let's look at the right-sided alternative, i.e., the test

$$H_0 : \mu = \mu_0 \quad vs. \quad H_a : \mu > \mu_0.$$

In this situation, the rejection region will be of the form $[x^*, \infty)$ where

$$x^* = \mu_0 + z_{1-\alpha}\left(\frac{\sigma}{\sqrt{n}}\right)$$

with $z_{1-\alpha} = \Phi^{-1}(1-\alpha)$. Let $\mu_a > \mu_0$ be a value from the alternative. Under that alternative, the distribution of the sample mean is

$$\bar{X}_n \sim Normal(mean = \mu_a, variance = \frac{\sigma^2}{n}).$$

The power of the test, $1-\beta$, for this particular μ_a is

$$1-\beta = \mathbb{P}_{\mu_a}\left[\bar{X}_n \geq x^*\right] = \mathbb{P}_{\mu_a}\left[\frac{\bar{X}_n - \mu_a}{\frac{\sigma}{\sqrt{n}}} \geq \frac{x^* - \mu_a}{\frac{\sigma}{\sqrt{n}}}\right] = \mathbb{P}_{\mu_a}\left[Z \geq \frac{\mu_0 + z_{1-\alpha}\left(\frac{\sigma}{\sqrt{n}}\right) - \mu_a}{\frac{\sigma}{\sqrt{n}}}\right]$$

where $Z \sim N(0,1)$. Simplifying the above expression, we get

$$1-\beta = \mathbb{P}\left[Z \geq z_{1-\alpha} - \frac{(\mu_a - \mu_0)\sqrt{n}}{\sigma}\right].$$

As n grows, the value $\frac{(\mu_a-\mu_0)\sqrt{n}}{\sigma}$ increases. Hence, as n grows, the value $-\frac{(\mu_a-\mu_0)\sqrt{n}}{\sigma}$ decreases. So, as n grows, the probability above increases.

Problem 14.2. (2 points) Consider a two-sided hypothesis test for the population mean of a normal population. Then, the power of the test is symmetric with respect to the null mean. *True or false? Why?*

Solution: TRUE
Look back to the picture of the power of the test we analyzed in class (posted on the course website). The symmetry is evident.

Problem 14.3. (2 points) Let μ denote the population mean μ of a normally distributed population model with a known σ. At a given significance level α, we are testing

$$H_0 : \mu = \mu_0 \quad vs. \quad H_a : \mu < \mu_0.$$

Let μ_a and μ_a' be two values in the alternative such that $\mu_a < \mu_a'$. Then, the power of the test at the alternative μ_a exceeds the power of the test at the alternative μ_a'. *True or false? Why?*

Solution: TRUE
The analysis is analogous to the one in the first problem in this problem set.

Problem 14.4. The time needed for college students to complete a certain mirror-symmetry puzzle is modeled using a normal distribution with a mean of 30 seconds and a standard deviation of 3 seconds. You wish to see if the population mean time μ is changed by vigorous exercise, so you have a group of nine college students exercise vigorously for 30 minutes and then complete the puzzle.

i. What are your null and alternative hypotheses?

ii. What is the rejection region at the significance level 0.01?
iii. What is the power of your test at $\mu = 28$ seconds?

Solution:

i. $H_0 : \mu = 30$ vs. $H_a : \mu \neq 30$

ii. The $z-$values corresponding to the two-sided hypothesis test at the 0.01 significance level are $z^* = \pm 2.576$. So, the rejection region is the complement of the interval

$$\left(30 - 2.576 \times \frac{3}{\sqrt{9}}, 30 + 2.576 \times \frac{3}{\sqrt{9}}\right) = (27.424, 32.576).$$

iii. We are looking for the probability of the event

$$27.424 < \bar{X}_n < 32.576$$

when

$$\bar{X}_n \sim N(mean = 28, variance = 1).$$

In standard units, we are looking for the probability that

$$27.424 - 28 < Z < 32.576 - 28.$$

We get

$$\Phi(4.576) - \Phi(-0.576) = 1 - (1 - \Phi(0.58)) = \Phi(0.58) = 0.7190$$

Hence, the power of the test is $1 - 0.7190 = 0.2810$

Problem 14.5. (10 points) You believe that the mean pancake consumption at the pancake jamboree is more than 16 per person. So, you decide to test your hypothesis. You model the pancake consumption as normally distributed with an unknown mean and with variance equal to 4. The plan is to collect the information on the number of pancakes consumed from a sample of 64 people. Since you want to have everything ready for the big day, you work out the rejection region right away and you get $(16.4375, \infty)$.

(i) (5 points) What is the significance level used to obtain the above rejection region?
Solution: From the lower bound of the rejection region, we get

$$16.4375 = 16 + z^* \left(\frac{2}{\sqrt{64}}\right) \quad \Rightarrow \quad z^* = 1.75.$$

Hence, the significance level is the upper tail probability associated with the critical value $z^* = 1.75$, i.e., 0.0401.

(ii) (5 points) What is the power of the above test at the alternative mean of 17?
Solution:

$$\mathbb{P}_{\mu=17}\left[\bar{X} > 16.435\right] = \mathbb{P}_{\mu=17}\left[\frac{\bar{X} - 17}{1/4} > \frac{16.435 - 17}{1/4}\right] = 1 - \Phi(-2.25) = \Phi(2.25) = 0.9878.$$