The University of Texas at Austin
Homework assignment 4

Introduction to Mathematical Statistics

February 27, 2026

Instructions: Provide your complete solution to the following problems. Final answers only, without appropriate justification, will receive zero points even if correct.

Problem 4.1. (15 points) Let (Y_1, Y_2) be a random vector with the joint pdf

$$f_{Y_1,Y_2}(y_1, y_2) = \frac{1}{4} 1_{\{-1 \le y_1 \le 1\}} 1_{()\{-1 \le y_2 \le 1\}}.$$

Find $\mathbb{P}[|Y_1| + |Y_2| \le 1/2]$.

Solution. The pair (Y_1, Y_2) is uniformly distributed over the square $[-1, 1] \times [-1, 1]$, while the region $\{(y_1, y_2) \in [-1, 1]^2 : y_1^2 \le y_2^2\}$ corresponds to the square with vertices $(\frac{1}{2}, 0)$, $(0, \frac{1}{2})$, $(-\frac{1}{2}, 0)$ and $(0, -\frac{1}{2})$. The side length of this square is $1/\sqrt{2}$, so its total area is $\frac{1}{2}$. The total area of the square $[-1, 1]$ is 4, and, since we are dealing with a geometric-probability problem, the answer is $\frac{1}{2}/4 = \frac{1}{8}$.

Problem 4.2. (15 points) Two random numbers, Y_1 and Y_2 are chosen independently of each other, according to the uniform distribution $U(-1, 2)$ on $[-1, 2]$. What is the probability that their product is positive?

Solution. The product $Y_1 Y_2$ is positive if and only if both Y_1 and Y_2 are positive or if both are negative. We could integrate the uniform density

$$f_{Y_1,Y_2}(y_1, y_2) = \frac{1}{9} \mathbb{1}_{\{-1 \le y_1, y_2 \le 2\}}$$

over the set

$$\{(y_1, y_2) : y_1 \le 0, y_2 \le 0\} \cup \{(y_1, y_2) : y_1 \ge 0, y_2 \ge 0\},$$

or use independence as in

$$\begin{aligned}
\mathbb{P}[Y_1 \le 0, Y_2 \le 0 \ \text{ or } \ Y_1 \ge 0, Y_2 \ge 0] &= \\
&= \mathbb{P}[Y_1 \le 0, Y_2 \le 0] + \mathbb{P}[Y_1 \ge 0, Y_2 \ge 0] \\
&= \mathbb{P}[Y_1 \le 0] \times \mathbb{P}[Y_2 \le 0] + \mathbb{P}[Y_1 \ge 0] \times \mathbb{P}[Y_2 \ge 0] \\
&= \frac{1}{3} \times \frac{1}{3} + \frac{2}{3} \times \frac{2}{3} = \frac{5}{9}.
\end{aligned}$$

Problem 4.3. (20 points) Three (fair and independent) coins are thrown; let Y_1, Y_2 and Y_3 be the outcomes (encoded as H or T). Player 1 gets \$1 if H shows on coin 1 ($Y_1 = H$) and/or \$2 if H shows on coin 2 ($Y_2 = H$). Player 2, on the other hand, gets \$1 when $Y_2 = H$ and/or \$2 when $Y_3 = H$. With W_1 and W_2 denoting the total amount of money given to Player 1 and Player 2, respectively,

1. (5 points) Write down the marginal distributions (pmfs) of W_1 and W_2,
2. (10 points) Write down the joint distribution table of (W_1, W_2).

3. (5 points) Are W_1 and W_2 independent?

Solution.

1. The support (the set of possible values) of W_1 is $\{0, 1, 2, 3\}$ and these values correspond to the events

$$\{Y_1 = T, Y_2 = T\}, \{Y_1 = H, Y_2 = T\}, \{Y_1 = T, Y_2 = H\}, \{Y_1 = H, Y_2 = H\},$$

in this order. Each of these events has probability $1/4$ and, so, the distribution of W_1 is uniform on $\{0, 1, 2, 3\}$, i.e., its table looks like this

0	1	2	3
$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

The distribution of W_2 is the same.

2. The joint distribution table of (W_1, W_2) will be a 4×4 table and each entry will correspond either to a specific coin pattern (like HTH) is possible, or its value will be 0. For example

$$\mathbb{P}[W_1 = 1, W_2 = 2] = \mathbb{P}[Y_1 = H, Y_2 = T \text{ and } Y_2 = T, Y_3 = H] = \frac{1}{8},$$

while

$$\mathbb{P}[W_1 = 1, W_2 = 1] = \mathbb{P}[Y_1 = H, Y_2 = T \text{ and } Y_2 = H, Y_3 = Y] = 0.$$

Going through all 16 pairs, we obtain the following table, with the values in the top row correspond to W_1 and the values in the left-most column to W_2:

	0	1	2	3
0	$\frac{1}{8}$	0	$\frac{1}{8}$	0
1	$\frac{1}{8}$	0	$\frac{1}{8}$	0
2	0	$\frac{1}{8}$	0	$\frac{1}{8}$
3	0	$\frac{1}{8}$	0	$\frac{1}{8}$

3. W_1 and W_2 are not independent. One way to see that is to use the factorization criterion: if they were the entries in the joint distribution table would be products of corresponding marginal pmfs. As calculated in the first part, these are all $1/4$ and their products would produce a table with all entries equal to $1/16$.

   Another way of arguing this is that the information that $W_1 = 0$ (for example) would make it impossible for W_2 to take values 1 or 3.