**Example 10.1.1.** Consider the following two data sets, both consisting of measurements of the same quantity (say the distance to Proxima Centauri) and in the same units, but made with two different methods.

method 1:  4.51, 4.52, 4.48, 4.49, 4.47, 4.53
method 2:  14.12, 1.30, 0.40, 2.50, 1.00, 3.18

If we use the sample mean $\bar{Y}$ as the (point) estimator for the "true" mean $\mu$, both of these data sets yield the same result, namely $\bar{Y} = 4.5$. It is clear, however, that the first method is more accurate and that, in general, one should trust the results produced by method 1 more than those obtained by method 2.

## Point vs. Interval Estimators.

An interval estimator is a pair of point estimators

such that $\boxed{\hat{\Theta}_L \leq \hat{\Theta}_R}$

## Desirable traits:

- containing the true parameter $\Theta$
  "w/ a high probability"
- being "narrow"

First: Pick a confidence level $C = 0.95, 0.99,$ or another probability close to 1, or pick a significance level $\alpha = 0.05, 0.01,$ or another probab. close to 0.

$\boxed{\text{Convention:}}$ $\boxed{C = 1 - \alpha}$

The purpose:

$$\mathbb{P}[\hat{\Theta}_L \leq \Theta \leq \hat{\Theta}_R] = C = 1 - \alpha$$

Def'n. Consider a random sample $(Y_1, Y_2, \ldots, Y_n)$ from a dist'n $D$ which is parameterized by an unknown parameter $\Theta$.

A **PIVOTAL QUANTITY** is a function of $(Y_1, Y_2, \ldots, Y_n)$ and the parameter $\Theta$ whose distribution does not depend on $\Theta$.

**Example.** Say that $Y_i \sim N(\mu, 1)$, $i = 1, ..., n$, is our random sample.

$\bar{Y}$ is a great estimator for $\mu$

$\bar{Y} \sim \underline{\text{Normal}}(\mu, \frac{1}{\sqrt{n}}) \Rightarrow \bar{Y}$ is NOT a PIVOTAL QUANTITY

BUT

$$\bar{Y} - \mu \sim N(0, \frac{1}{\sqrt{n}})$$

is a PIVOTAL QUANTITY.

**Example.** Consider a random sample $(Y_1, ..., Y_n)$ from $N(\mu, \sigma)$ w/ **both parameters unknown**
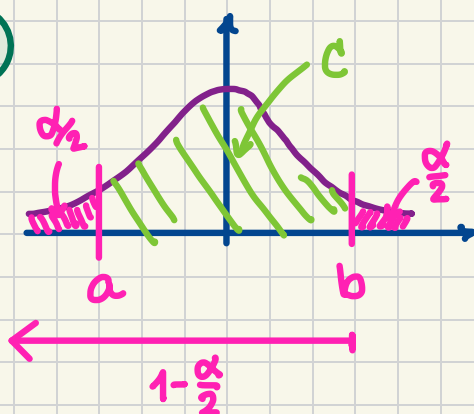
$$\frac{\bar{Y} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$$

is a PIVOTAL QUANTITY for $\theta = (\mu, \sigma)$

**Recipe.**

① Consider a pivotal quantity $U$

②



Define $a = q_U(\frac{\alpha}{2})$, $b = q_U(1 - \frac{\alpha}{2})$ w/ $q_U$ the quantile f'tion of $U$

Now we know:

$$\mathbb{P}\left[a \le \boxed{U} \le b\right] = C = 1-\alpha$$

Has $\Theta$ in its expression!

③. ⬭ALGEBRA⬭

"Undo" the "rule" for $U$
So that the final
expression looks like

$$\mathbb{P}\left[\hat{\Theta}_L \le \Theta \le \hat{\Theta}_R\right] = C = 1-\alpha$$

Works for 1D parameters!

Example. $Y_1, ..., Y_n$ a random sample from $N(\mu, \sigma)$
w/ $\mu$ unknown
and $\sigma$ known.

① Propose a pivotal quantity:

$$U = \frac{\bar{Y} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$$

② $a = q_U\left(\frac{\alpha}{2}\right)$

$b = q_U\left(1 - \frac{\alpha}{2}\right)$