University of Texas at Austin, Department of Mathematics
M358K - Applied Statistics

The Mock In-term One

Problem 1.1. (5 points) The standard deviation of the zinc concentration in a certain river is given to be 0.45 grams per milliliter. You want to create a 90%-confidence interval for the mean zinc concentration. Your goal is to have the margin of error of at most 0.04. How large should your sample be?

a. 342
b. 343
c. 376
d. 388
e. None of the above.

Solution: b.
We need to have

$$n \geq \left(\frac{1.645(0.45)}{0.04}\right)^2 = 342.481. \tag{1.1}$$

Problem 1.2. (5 points) A medical researcher thinks that adding calcium to the diet will help reduce blood pressure. She believes that the effect is different for men and women. 20 men and 20 women are willing to participate in the study. The researcher chooses 10 of the men and 10 of the women at random. These chosen 20 men and women take a calcium pill every day. The other 20 men and women take a placebo. This is a ...

a.: stratified random sample design.
b.: simple random sample design.
c.: randomized block experimental design.
d.: completely randomized experimental design.
e.: None of the above is correct.

Solution: c.

Problem 1.3. (5 points) To estimate a population mean, our resident statistician Martyn Rivera plans to pick two simple random samples, each of size 100, from the population. He also plans to calculate the confidence interval with level C for each sample. What is the probability that at least one of his confidence intervals will cover the population mean?

a.: C^2
b.: $1 - C^2$
c.: $2C$
d.: $1 - (1 - C)^2$
e.: None of the above

Solution: d.

Problem 1.4. (5 points) Let $Z \sim N(0, 1)$. Given that $Z > 0$, find the probability that $Z < 2$.

(a) 0.4772
(b) 0.6800
(c) 0.9544

(d) 0.9772

(e) None of the above.

Solution: (c)
We are looking for the probability

$$\mathbb{P}[Z < 2 \,|\, Z > 0].$$

By the definition of conditional probability,

$$\mathbb{P}[Z < 2 \,|\, Z > 0] = \frac{\mathbb{P}[0 < Z < 2]}{\mathbb{P}[Z > 0]}.$$

By the symmetry of the bell curve, $\mathbb{P}[Z > 0] = 0.5$. As for the numerator in the above expression, we have

$$\mathbb{P}[0 < Z < 2] = \mathbb{P}[Z < 2] - \mathbb{P}[Z \leq 0].$$

Again, by the symmetry of the bell curve, $\mathbb{P}[Z \leq 0] = 0.5$. On the other hand, using the standard normal table, we get $\mathbb{P}[Z < 2] = \Phi(2) = 0.9772$. Finally, our answer is

$$\mathbb{P}[Z < 2 \,|\, Z > 0] = \frac{0.9772 - 0.5}{0.5} = 2(0.4772) = 0.9544.$$

Problem 1.5. (5 points) Let the monthly profit of a local cupcakery be normally distributed with mean \$20,000 and standard deviation of \$4,000. What is the probability that the combined profit in the months of October and November exceeds \$36,000?

(a) 0.4052

(b) 0.7611

(c) 0.7642

(d) 0.8023

(e) None of the above.

Solution: (b)
Independence is implicit. Let X_1 be the profit for October and let X_2 be the profit for November. Then

$$X_1 + X_2 \sim Normal(mean = 40000, variance = 32,000,000).$$

The probability we are looking for is

$$\mathbb{P}[X_1 + X_2 > 36000] = \mathbb{P}\left[\frac{X_1 + X_2 - 40000}{\sqrt{32000000}} > \frac{36000 - 40000}{\sqrt{32000000}}\right] \approx \mathbb{P}[Z > -0.71]$$

where $Z \sim N(0,1)$. Using our standard normal tables we obtain the answer of $\Phi(0.71) = 0.7611$.

Problem 1.6. (5 points) Your diamond scale's measurement have a normally distributed error with mean 0 and standard deviation of 0.001 carats. Your procedure is to weigh a single diamond using your scale n times, average out the results, and report the average as the mass of the diamond. How many times n do you have to weigh your diamond so that your reported mass is at most 0.001 from the actual mass with probability 99%?

(a) 6

(b) 7

(c) 8

(d) 9

(e) None of the above.

Solution: (b)
Let the error of measurement i be denoted by X_i. Then, the average error is going to be

$$\bar{X}_n = \frac{1}{n}(X_1 + X_2 + \cdots + X_n).$$

The random variable $\bar{X}_n$ is normally distributed with mean zero and standard deviation

$$SD[\bar{X}_n] = \frac{0.001}{\sqrt{n}}.$$

We are looking for n such that

$$\mathbb{P}[|\bar{X}_n| < 10^{-3}] \geq 99\%, \quad i.e., \quad \mathbb{P}[-10^{-3} < \bar{X}_n < 10^{-3}] \geq 99\%.$$

Rewriting $\bar{X}$ in standard units above, we obtain

$$\mathbb{P}\left[\frac{-10^{-3}-0}{\frac{0.001}{\sqrt{n}}} < \frac{\bar{X}_n - 0}{\frac{0.001}{\sqrt{n}}} < \frac{10^{-3}-0}{\frac{0.001}{\sqrt{n}}}\right] \geq 99\% \quad \Leftrightarrow \quad \mathbb{P}\left[\frac{-10^{-3}}{\frac{0.001}{\sqrt{n}}} < Z < \frac{10^{-3}}{\frac{0.001}{\sqrt{n}}}\right] \geq 99\%$$

where $Z \sim N(0,1)$. We can simplify the probability in the inequality above as follows

$$\mathbb{P}\left[\frac{-10^{-3}}{\frac{0.001}{\sqrt{n}}} < Z < \frac{10^{-3}}{\frac{0.001}{\sqrt{n}}}\right] = \mathbb{P}\left[-\sqrt{n} < Z < \sqrt{n}\right] = \mathbb{P}\left[Z < \sqrt{n}\right] - \mathbb{P}\left[Z \leq -\sqrt{n}\right] = \Phi\left(\sqrt{n}\right) - \Phi\left(-\sqrt{n}\right).$$

Due to the symmetry of the standard normal bell curve, we always have $\Phi(-x) = 1 - \Phi(x)$. Hence, the above probability simplifies to

$$\Phi\left(\sqrt{n}\right) - \Phi\left(-\sqrt{n}\right) = 2\Phi\left(\sqrt{n}\right) - 1.$$

So, our condition becomes

$$2\Phi\left(\sqrt{n}\right) - 1 \geq 0.99 \quad \Leftrightarrow \quad 2\Phi\left(\sqrt{n}\right) \geq 1.99 \quad \Leftrightarrow \quad \Phi\left(\sqrt{n}\right) \geq 0.995.$$

Remember that Φ is strictly increasing. Using the standard normal table, we invert Φ above and get

$$\sqrt{n} \geq 2.576 \quad \Leftrightarrow \quad n \geq (2.576)^2 = 6.6358 \quad \Leftrightarrow \quad n \geq 7.$$

Problem 1.7. (5 points) Your friend Cyril works as a work-study for the statistics department. The chair of the department decides Cyril's weekly pay by spinning a spinner which is equally likely to land on red, yellow, or blue. If the spinner lands on yellow, Cyril gets \$0. If the spinner lands on red, Cyril gets \$400. If the spinner lands on blue, Cyril gets \$500. What is the probability that Cyril's average pay in the following three weeks is \$300?

(a) 0

(b) 1/27

(c) 1/9

(d) 2/9

(e) None of the above.

Solution: (d)
If the average pay is \$300, that means that the total pay is \$900. The only way that can happen is if Cyril gets the payments of $(0, 400, 500)$ or some permutation thereof. Every one of these combinations of payments has the probability $1/27$ and there are $3! = 6$ of them. So, the total probability is $6/27 = 2/9$.

Problem 1.8. (5 points) Alice performs a $z-$test. The $z-$score she obtains is equal to -1.76. Which decision does she make?

(a) Reject the null hypothesis.

(b) Fail to reject the null hypothesis.

(c) Reject the alternative hypothesis.

(d) Not enough information is given to answer this question.

(e) None of the above.

Solution: (d)
The significance level is not given.

Problem 1.9. (5 points) A manufacturer of scented candles claims that their luxury candles last at least 12 hours. You suspect that this might not be entirely true and you decide to test their claim. You model the candle burn times as normal with a known standard deviation of 2 hours (based on the last holiday season's study). You purchase and burn 16 candles recording the sample average of 11 hours and 45 minutes. What is your decision?

(a) Reject at the 1% significance level.

(b) Fail to reject at the 1% significance level; reject at the 5% significance level.

(c) Fail to reject at the 5% significance level; reject at the 10% significance level.

(d) Fail to reject at the 10% significance level.

(e) None of the above.

Solution: (d)
Let μ denote the unknown population mean. According to our model, the distribution of the sample mean is

$$\bar{X} \sim Normal\left(mean = \mu, sd = \frac{2}{\sqrt{16}} = \frac{1}{2}\right).$$

We are testing

$$H_0 : \mu = 12 \quad vs. \quad H_a : \mu < 12.$$

Under the null, the $z-$score corresponding to the observed sample average is

$$z = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{11.75 - 12}{\frac{1}{2}} = -2(0.25) = -0.5.$$

With $Z \sim N(0,1)$, the $p-$value is

$$\mathbb{P}[Z < -0.5] = 0.3085.$$

We fail to reject at the 10% significance level.

Problem 1.10. (5 points) *Organically Produced* claims that their supplements contain 65 mg of iron per capsule. To be able to continue to maintain their claim, they periodically test the contents of a batch of 100 randomly chosen capsules from their production line. They model the iron content as normally distributed with a known standard deviation of 5 mg. In the last test, the sample average was 64 mg. What is the $p-$value?

(a) 0.0228

(b) 0.0384

(c) 0.0418

(d) 0.0456

(e) None of the above.

Solution: (d)
Let μ denote the mean iron content per capsule. They are testing

$$H_0 : \mu = 65 \quad vs. \quad H_a : \mu \neq 65.$$

Under the null hypothesis, the $z-$score corresponding to the observed sample average of 64 mg is, in our usual notation,

$$z = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{64 - 65}{\frac{5}{\sqrt{100}}} = -\frac{1}{\frac{1}{2}} = -2.$$

With $Z \sim N(0,1)$, the corresponding $z-$score is

$$\mathbb{P}[Z < -2] + \mathbb{P}[Z > 2] = 2\mathbb{P}[Z < -2] = 0.0456.$$

Problem 1.11. (5 points) Suppose that we have a random sample of size 25 from a normal population with an unknown mean μ and a standard deviation of 4. Bertram Wooster was in charge of testing the hypotheses

$$H_0 : \mu = 10 \quad vs. \quad H_a : \mu > 10.$$

Before he went on vacation, he obtained the rejection region $[11.2, \infty)$. However, he forgot to tell anyone which significance level α he used. Calculate α.

(a) 0.0401
(b) 0.0495
(c) 0.05
(d) 0.0668
(e) None of the above.

Solution: (d)
In our usual notation, the rejection region of this righ-sided hypothesis test is of the form

$$\left[\mu_0 + z_{1-\alpha}\left(\frac{\sigma}{\sqrt{n}}\right), \infty\right)$$

with $z_{1-\alpha} = \Phi^{-1}(1-\alpha)$ unknown in this problem. So, we have

$$10 + z_{1-\alpha}\left(\frac{4}{\sqrt{25}}\right) = 11.2 \quad \Leftrightarrow \quad z_{1-\alpha}\left(\frac{4}{5}\right) = 1.2 \quad \Leftrightarrow \quad z_{1-\alpha} = 1.2\left(\frac{5}{4}\right) = 1.5.$$

Therefore, $1 - \alpha = \Phi(1.5) = 0.9332$. Finally, $\alpha = 0.0668$.

Problem 1.12. (5 points) Suppose that we have a random sample of size 25 from a normal population with an unknown mean μ and a standard deviation of 4. Bertram Wooster was in charge of testing the hypotheses

$$H_0 : \mu = 10 \quad vs. \quad H_a : \mu > 10.$$

Before he went on vacation, he obtained the rejection region $[11.2, \infty)$. What is the power of the above test at the alternative mean $\mu_a = 11$?

(a) 0.4013
(b) 0.4503
(c) 0.5120
(d) 0.6368
(e) None of the above.

Solution: (a)
Under the given alternative mean, the distribution of the sample mean is

$$\bar{X} \sim Normal\left(mean = 11, sd = \frac{4}{\sqrt{25}} = \frac{4}{5}\right).$$

The power of the test at the alternative mean $\mu_a = 11$ is the probability that $\bar{X}$ falls into the given rejection region $[11.2, \infty)$. We have

$$\mathbb{P}[\bar{X} \geq 11.2] = \mathbb{P}\left[\frac{\bar{X} - 11}{0.8} \geq \frac{11.2 - 11}{0.8}\right] = \mathbb{P}\left[Z \geq 0.25\right]$$

where $Z \sim N(0,1)$. Our answer is

$$\mathbb{P}[\bar{X} \geq 11.2] = \mathbb{P}\left[Z \geq 0.25\right] = 1 - \Phi(0.25) = 1 - 0.5987 = 0.4013.$$