UNIVERSITY OF TEXAS AT AUSTIN

HW Assignment 2

Note: You **must** show all your work. Numerical answers without a proper explanation or a clearly written down path to the solution will be assigned zero points.

Problem 2.1. (20 points) Two coins are tossed and a (6-sided) die is rolled.

(i) (5 points) Describe a sample space (probability space), together with the probability, on which such a situation can be modelled.

(ii) (15 points)Find the probability mass function of the random variable whose value is the sum of the number on the die and the total number of heads.

Solution: Each elementary event ω should track the information about three things - the outcome of the first coin toss, the outcome of the second coin toss and the number on the die. This corresponds to triplets $\omega = (c_1, c_2, d)$, where $c_1, c_2 \in \{H, T\}$ and $d \in \{1, \dots, 10\}$. Therefore, $\Omega = \{H, T\} \times \{H, T\} \times \{1, \dots, 6\}$. Since all the instruments involved are fair, the independence requirements dictate that

$$\mathbb{P}[\omega = (c_1, c_2, d)] = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{6} = \frac{1}{24},$$

for any $(c_1, c_2, d) \in \Omega$. In words, all elementary events are equally likely. Let C_1 be the random variable which equals to 1 if the outcome of the first coin toss if H, so that

$$C_1(\omega) = \begin{cases} 1, & c_1 = H, \\ 0, & \text{otherwise,} \end{cases} \quad \text{where } \omega = (c_1, c_2, d).$$

In other words, $C_1 = \mathbf{1}_A$ is the indicator of the event

$$A = \{\omega = (c_1, c_2, d) \in \Omega \,:\, c_1 = H\}.$$

Let C_2 and D (the number on the die) be defined analogously. Then the total number of heads M is given by $M = C_1 + C_2$. Each C_1 and C_2 are independent Bernoulli random variables with $p = \frac{1}{2}$, so M is a binomial random variable with $n = 2$ and $p = \frac{1}{2}$. Therefore, the pmf of M is

	0	1	2
p	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Let X be the random variable from the text of the problem:

$$X = D + M.$$

The values random variable X can take are $\{1, 2, \dots, 8\}$, and they correspond to the following table (the table entry is the value of X, columns go with D and rows with M):

	1	2	3	4	5	6
0	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8

A bit of accounting gives the following pmf for X:

	1	2	3	4	5	6	7	8
p	$\frac{1}{24}$	$\frac{3}{24}$	$\frac{4}{24}$	$\frac{4}{24}$	$\frac{4}{24}$	$\frac{4}{24}$	$\frac{3}{24}$	$\frac{1}{24}$

Problem 2.2. (10 points) *Source: Sample P exam, Problem #126.* Under an insurance policy, a maximum of five claims may be filed per year by a policyholder. Let p_n be the probability that a policyholder files exactly n claims during a given year, where $n = 0, 1, 2, 3, 4, 5$. An actuary makes the following observations:

- $p_n \geq p_{n+1}$ for $n = 0, 1, 2, 3, 4$.
- The difference between p_n and p_{n+1} is the same for $n = 0, 1, 2, 3, 4$.
- Exactly 40% of policyholders file strictly fewer than two claims during a given year.

Calculate the probability that a random policyholder will file strictly more than three claims during a given year.

Solution: Let us denote the constant difference between p_n and p_{n+1} by a. Then, we have that for $n = 1, \dots, 5$, $p_n = p_0 + na$. Hence,

$$\begin{aligned} p_0 + p_1 + \cdots + p_5 = 1 \quad &\Rightarrow \quad p_0 + (p_0 + a) + \cdots + (p_0 + 5a) = 1 \\ &\Rightarrow \quad 6p_0 + a(1 + \cdots + 5) = 1 \quad \Rightarrow \quad 6p_0 + 15a = 1 \end{aligned}$$

On the other hand, we are given that

$$p_0 + p_1 = 0.4 \quad \Rightarrow \quad 2p_0 + a = 0.4.$$

From the two equations, we get $12a = -0.2$, i.e., $a = -\frac{1}{60}$. Thus, $p_0 = 5/24$. Our answer is

$$p_4 + p_5 = 2p_0 + 9a = 2\left(\frac{5}{24}\right) - \frac{9}{60} = \frac{5}{12} - \frac{3}{20} = \frac{25 - 9}{60} = \frac{16}{60} = \frac{4}{15}\,.$$

Problem 2.3. (10 points) A continuous random variable X has the probability density function f_X given by

$$f_X(x) = A - \tfrac{x}{50}, \quad 0 \leq x \leq 10.$$

(a) Find the value of the constant A.
(b) Find the value of the survival function of X at 7, i.e., calculate $S_X(7)$.

Solution:

(a) Necessarily, $\int_0^{10} f_X(x)\,dx = 1$. So,

$$10A = 1 + \frac{10^2}{2 \cdot 50} = 2 \quad \Rightarrow \quad A = 1/5.$$

(b) Note that f_X is piecewise linear. So, we can calculate $S_X(7) = \mathbb{P}[X > 7]$ as the area of a triangle (draw a picture if in doubt!). We get

$$S_X(7) = \frac{1}{2} \cdot f_X(7) \cdot (10 - 7) = \frac{3}{2} \cdot (\frac{1}{5} - \frac{7}{50}) = \frac{3(10 - 7)}{2 \cdot 50} = 9/100.$$

Problem 2.4. (10 points) The lifespan of a certain machine is exponentially distributed. The probability that the lifespan exceeds 4 years is p. Find the expression for the density of the lifespan in terms of p.

Solution: Let T be the random variable which stands for the lifespan in the problem. It is modelled as exponential. Let's denote its parameter, as usual, by θ. We can write

$$T \sim Exponential(mean = \theta).$$

The survival function of the random variable T is of the form, for all $t > 0$,

$$S_T(t) = \mathbb{P}[T > t] = e^{-t/\theta}.$$

In the present problem, we are given that

$$\mathbb{P}[T > 4] = p \quad \Rightarrow \quad e^{-4/\theta} = p \quad \Rightarrow \quad -4/\theta = \ln(p) \quad \Rightarrow \quad \theta = -\frac{4}{\ln(p)}\,.$$

The density of the exponential random variable T is of the form, for all $t > 0$,

$$f_T(t) = \frac{1}{\theta} e^{-t/\theta} = -\frac{\ln(p)}{4} e^{t\ln(p)/4} = -\frac{\ln(p)}{4} p^{t/4}.$$