M378K Introduction to Mathematical Statistics
Problem Set #12
Statistics.

Definition 12.1. *A* random sample *of size n from distribution D is a random vector*

$$(Y_1, Y_2, \ldots, Y_n)$$

such that

1. *$Y_1, Y_2, \ldots, Y_n$ are independent, and*
2. *each Y_i has the distribution D.*

Example 12.2. **Quality control**. *Times until a breaker trips under a particular load are modeled as exponential. The intended procedure is to choose n breakers at random from the assembly line, subject them to the load, and measure the time it takes for them to trip. The lifetime of a specific breaker indexed by i is a random variable Y_i with an exponential distribution with an unknown parameter $\theta = \tau$. Independence of $Y_i, i = 1, \ldots, n$ is assured by the random choice of breakers to test.*

Definition 12.3. *A* statistic *is a function of the (observable) random sample and known constants.*

Problem 12.1. *Give at least three examples of statistics of a certain random sample $Y_1, Y_2, \ldots, Y_n$.*

Solution:

Remark 12.4. *Statistics are random variables in their own right. We call their probability distributions* sampling distributions.

Example 12.5. **Quality control, cont'd.** *Let the random variable Y be the minimum of random variables $Y_1, \ldots, Y_n$, i.e., the shortest time until the breaker is tripped in the sample. We can write*

$$Y = \min(Y_1, \ldots, Y_n).$$

What is another name for this random variable?
Solution: *The first order statistic $Y_{(1)}$.*

Then, the sampling distribution *of Y can be figured out by looking at its cumulative distribution function. We have ...*
Solution: *For all $y \in \mathbb{R}$,*

$$F_{(1)}(y) = 1 - (1 - F_Y(y))^n = 1 - (1 - (1 - e^{-\frac{y}{\tau}}))^n = 1 - (e^{-\frac{y}{\tau}})^n = 1 - e^{-\frac{y}{\tau/n}}.$$

We can conclude that $Y \sim E(\tau/n)$.

Problem 12.2. *Let $Y_1, \ldots, Y_n$ be a random sample of size n from a normal distribution with mean μ and variance σ^2. What is the sampling distribution of*

$$\bar{Y}_n = \frac{1}{n}\sum_{k=1}^{n} Y_k \quad ?$$

Solution:

$$\bar{Y}_n \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right).$$