M339D: October 2nd, 2024.

3. An insurance company sells single premium deferred annuity contracts with return linked to a stock index, the time-t value of one unit of which is denoted by $S(t)$. The contracts offer a minimum guarantee return rate of $g\%$. At time 0, a single premium of amount π is paid by the policyholder, and $\pi \times y\%$ is deducted by the insurance company. Thus, at the contract maturity date, T, the insurance company will pay the policyholder

$$\pi \times (1 - y\%) \times \text{Max}[S(T)/S(0), (1 + g\%)^T].$$

You are given the following information:

(i) The contract will mature in one year.

(ii) The minimum guarantee rate of return, $g\%$, is 3%. $g=0.03$

(iii) Dividends are incorporated in the stock index. That is, the stock index is constructed with all stock dividends reinvested. no dividends !

(iv) $\boxed{S(0) = 100.}$ ✓

(v) The price of a one-year European put option, with strike price of \$103, on the stock index is \$15.21. ✓ $\boxed{V_P(T) = (K - S(T))_+}$

Determine $y\%$, so that the insurance company does not make or lose money on this contract.

(A) 12.8%.

(B) 13.0%

(C) 13.2%

(D) 13.4%

(E) 13.6%.

→: The Synopsis:

1. Focus on the insurance company's liability
2. Use our data
3. Simplify

creating the put payoff along the way

The insurance company's liability:

$$\underbrace{\boxed{\pi \times (1-y)}}_{\text{const.}} \times \text{Max}\left(\frac{S(T)}{S(0)}, (1+g)^T\right)$$

$$\underbrace{\frac{1}{S(0)}}_{\text{Const.}} \boxed{\text{Max}(S(T), S(0)(1+g)^T)}$$

$$= \text{Max}(S(T), 100(1+0.03)^1)$$

$$= \boxed{\text{Max}(S(T), 103)}$$ ✓

a, b

$$\max(a,b) = a + \max(0, b-a) = a + (b-a)_+$$
$$= b + \max(a-b, 0) = b + (a-b)_+$$

$$\rightarrow \max(S(T), 103) = \underbrace{\boxed{S(T)}}_{\text{Long Stock Index}} + \underbrace{\boxed{(103 - S(T))_+}}_{\text{Payoff of put w/ strike 103 and exercise date @ T=1}}$$

The insurance company can perfectly hedge by:

- longing/buying $\frac{\pi(1-y)}{S(0)}$ units of the stock index

and

- buying $\frac{\pi(1-y)}{S(0)}$ European puts w/ $K=103$ and $T=1$

If the amount they receive @ time 0 exactly mathes the cost of this portfolio, they neither lose nor gain money.

$$\cancel{\pi} = \frac{\cancel{\pi}(1-y)}{S(0)}\left(S(0) + V_p(0)\right)$$

$$100 = (1-y)(100 + 15.21)$$

$$1 - y = \frac{100}{115.21} \Rightarrow \boxed{y = \frac{15.21}{115.21} = 0.132}$$

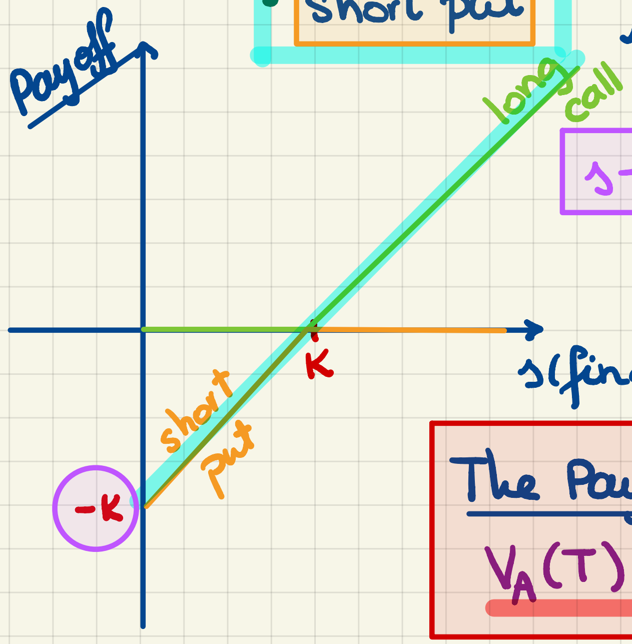
$\square$

Put-Call Parity.

Portfolio A:

- long call
- short put

} both European & otherwise identical



s (final asset price)

$\boxed{s-K}$ ... payoff function of the portfolio

The Payoff:

$$V_A(T) = V_c(T) - V_p(T) = S(T) - K$$

Portfolio B:

- long non-dividend-paying stock
- borrow $PV_{0,T}(K)$ @ the risk-free interest rate r to be repaid @ time T

$$\Rightarrow \boxed{V_B(T) = S(T) - K}$$

$\Rightarrow$ NO ARBITRAGE!

$$V_A(0) = V_B(0)$$

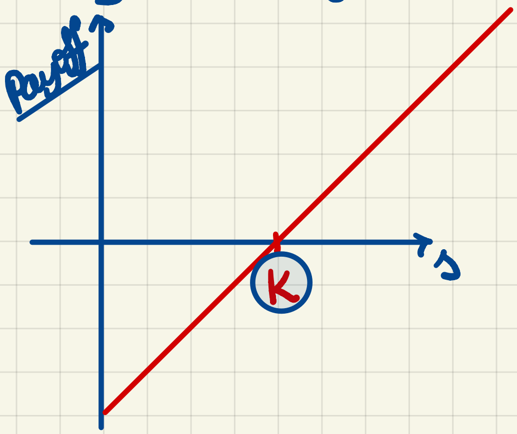
$$\Rightarrow \boxed{V_c(0) - V_p(0) = S(0) - PV_{0,T}(K)}$$

Put-Call Parity.

More generally: for any $t \in [0, T]$:

$$\boxed{V_c(t) - V_p(t) = S(t) - PV_{t,T}(K)}$$

Remarks:

- The no·arbitrage assumption is sufficient.
- Only works for European options.
- 

With Portfolio A, we have a replicating portfolio for an "off·market forward" aka "synthetic forward".

Special Case:

strike = forward price on stock

$\Leftrightarrow$

$$K = F_{0,T}(S) = S(0)e^{rT}$$

$\Leftrightarrow$

$$PV_{0,T}(K) = S(0)$$

$\Leftrightarrow$

$$V_C(0) - V_P(0) = 0 = S(0) - PV_{0,T}(K)$$

$\Leftrightarrow$

$$\boxed{V_C(0) = V_P(0)}$$

By put·call parity.

:)