M362K Homework Assignment #2

Please, provide your **complete solutions** to the following problems. Final answers only, even if correct will earn zero points for those problems.

Problem 2.1. (5 points) Complete the definition of *mutual exclusivity* of events below:
Events $A, B \subset \Omega$ are said to be *mutually exclusive* if ...

Solution:

$$A \cap B = \emptyset.$$

Problem 2.2. (5 points) An urn contains 1 red ball and 10 blue balls. Other than their color, the balls are indistinguishable, so if one is to draw a ball from the urn without peeking - all the balls will be equally likely to be selected. If we draw 5 balls from the urn at once and without peeking. What is the probability that this collection of 5 balls contains the red ball?

Solution:

$$\mathbb{P}[\text{"the red ball is selected"}] = \frac{\binom{10}{4}}{\binom{11}{5}} = \frac{5}{11}\,.$$

Problem 2.3. (15 points) Consider a an ordinary deck of 52 cards. It consists of 4 suits, each containing 13 cards, and it has 13 kinds (or ranks) of cards - one of every kind in each suit. A poker hand is a set of 5 cards from the above deck. It does not matter in which order the cards were dealt. Assume that all poker hands are equally likely.

What is the probability that in a poker hand, one gets exactly 3 of a kind, i.e., 3 of the same rank, with the remaining two cards being of different ranks?

Solution: There are altogether $\binom{52}{5}$ different possibilities for a poker hand. Out of those, we have to count the number of "good" poker hands, i.e., the ones containing exactly 3 cards of a kind and having the remaining two cards of different kinds.

We can choose the rank which will be represented by 3 cards in 13 ways, and we can choose the remaining 2 ranks that will be represented with a single card each in $\binom{12}{2}$ ways (this is the number of subsets of 2 elements out of a set of 12).

Then, the number of choices of the triplet of cards of the same rank is $\binom{4}{3}$. The number of choices for the first of the remaining cards is 4, and finally the number of possibilities for the last card in a "good" poker hand is 4, again.

So, there are

$$13 \cdot \binom{12}{2} \cdot \binom{4}{3} \cdot 4 \cdot 4$$

"good" poker hands. Since all the poker hands are assumed to be equally likely, the probability of a randomly dealt poker hand being a "good" one is

$$\frac{13 \cdot \binom{12}{2} \cdot \binom{4}{3} \cdot 4 \cdot 4}{\binom{52}{5}}\,.$$

Problem 2.4. (5 points) Solve Problem **1.4.6** from the textbook.

Solution: Let the events A and B be given by

$$A = \{\text{the first card is black}\}$$

and

$$B = \{\text{the second card is a spade}\}.$$

We are looking for the conditional probability $\mathbb{P}[B|A] = \mathbb{P}[B \cap A]\mathbb{P}[A]$. Clearly, $\mathbb{P}[A] = 1/2$. To compute $\mathbb{P}[B \cap A]$, we partition A into $A = A_1 \cup A_2$, where

$$A_1 = \{\text{the first card is a spade}\}$$

and

$$A_2 = \{\text{the first card is a club}\}.$$

Then

$$\begin{aligned}\mathbb{P}[B \cap A] &= \mathbb{P}[B \cap A_1] + \mathbb{P}[B \cap A_2] \\ &= \mathbb{P}[B|A_1]\mathbb{P}[A_1] + \mathbb{P}[B|A_2]\mathbb{P}[A_2].\end{aligned}$$

It remains to note that $\mathbb{P}[A_1] = \mathbb{P}[A_2] = 1/4$, $\mathbb{P}[B|A_1] = 12/51$ and $\mathbb{P}[B|A_2] = 13/51$ to obtain

$$\mathbb{P}[B|A] = \tfrac{25}{102}.$$

Problem 2.5. (10 points) Solve Problem **1.4.8** from the textbook.

Solution: Let

$$A_1 = \{\text{W/W card is drawn}\},$$
$$A_2 = \{\text{W/B card is drawn}\},$$

and

$$A_3 = \{\text{B/B card is drawn}\}.$$

So that $\mathbb{P}[A_1] = 0.3$, $\mathbb{P}[A_2] = 0.5$ and $\mathbb{P}[A_3] = 0.2$. Imagine that each card has the front side and the back side (e.g., one side is shiny and the other is flat) and set

$$B = \{\text{the top side is shiny}\}.$$

For definiteness, let us assume that the shiny side of the W/B card is white. The observed event is $C = \{\text{top side is black}\}$, and note that

$$C = (A_2 \cap B) \cup A_3.$$

Given C, the other side will be white if (and only if) the card on the table is the second card. Therefore, we are looking for the conditional probability $\mathbb{P}[A_2|C]$. The definition yields:

$$\mathbb{P}[A_2|C] = \tfrac{\mathbb{P}[A_2 \cap C]}{\mathbb{P}[C]} = \tfrac{\mathbb{P}[A_2 \cap B]}{\mathbb{P}[A_3] + \mathbb{P}[A_2 \cap B]}.$$

The probability that the top side is shiny is $\frac{1}{2}$, no matter what card is chosen, so $\mathbb{P}[A_2 \cap B] = \mathbb{P}[B|A_2]\mathbb{P}[A_2] = \frac{1}{2} \times 0.5 = 0.25$, so

$$\mathbb{P}[A_2|C] = \tfrac{0.25}{0.2+0.25} = \tfrac{5}{9}.$$

Problem 2.6. (5 points) Solve Problem **1.4.12** from the textbook.

Solution:

$$\mathbb{P}[F|G^c] = \tfrac{\mathbb{P}[F \cap G^c]}{\mathbb{P}[G^c]} = \tfrac{\mathbb{P}[F] - \mathbb{P}[F \cap G]}{1 - \mathbb{P}[G]}.$$

Problem 2.7. (5 points) A pair of dice is thrown. Find the probability that the sum of the outcomes is 10 or greater **if** a 5 appears on at least one of the dice.

Solution: Here is an incredibly formal solution, the approach with the *restricted probability space* is much shorter.

Let

$$\begin{aligned} E &:= \{\text{the sum of the outcomes is greater than or equal to } 10\} \\ &= \{(i,j) : 1 \le i, j \le 6 \,\text{and}\, i+j \ge 10\}, \\ A &:= \{\text{the outcome on the first die is equal to } 5\} \\ &= \{(i,j) : 1 \le j \le 6 \,\text{and}\, i = 5\}, \\ B &:= \{\text{the outcome on the second die is equal to } 5\} \\ &= \{(i,j) : 1 \le i \le 6 \,\text{and}\, j = 5\}. \end{aligned}$$

Then, we are looking for the following conditional probability:

$$\mathbb{P}[E|A \cup B] = \frac{\mathbb{P}[E \cap (A \cup B)]}{\mathbb{P}[A \cup B]}\,.$$

We have that

$$E \cap (A \cup B) = \{(5,5), (5,6), (6,5)\},$$

and so $\mathbb{P}[E \cap (A \cup B)] = 3/36 = 1/12$. On the other hand,

$$\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B] = \frac{1}{6} + \frac{1}{6} - \frac{1}{36} = \frac{11}{36}\,.$$

Finally, Then, we are looking for the following conditional probability:

$$\mathbb{P}[E|A \cup B] = \frac{\mathbb{P}[E \cap (A \cup B)]}{\mathbb{P}[A \cup B]} = \frac{\frac{3}{36}}{\frac{11}{36}} = \frac{3}{11}\,.$$