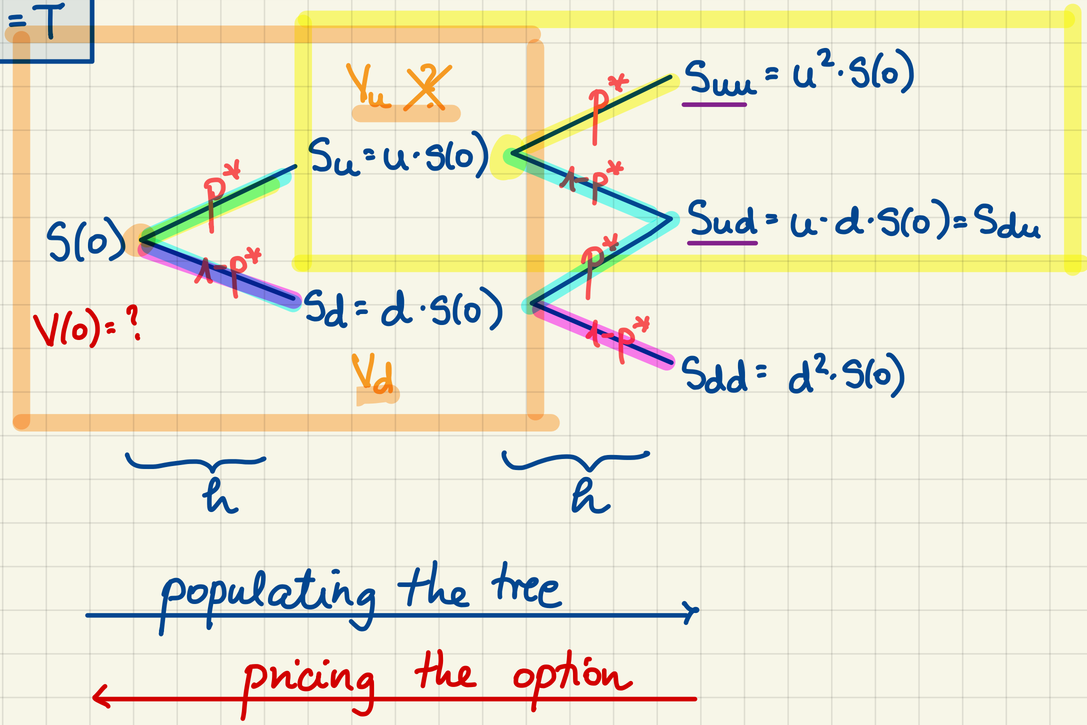
M339D: October 31st, 2022.

Two Binomial Periods.

$\boxed{2\cdot h = T}$



Payoff

$V_{uu} = v(S_{uu})$

$V_{ud} = v(S_{ud})$

$V_{dd} = v(S_{dd})$

- up node:

replicating portfolio for the option @ the up node:

$$\begin{cases} \Delta_u = \dfrac{V_{uu} - V_{ud}}{S_{uu} - S_{ud}} \\ B_u = e^{-rh}\cdot \dfrac{u\cdot V_{ud} - d\cdot V_{uu}}{u-d} \end{cases}$$

$\Rightarrow$ the option's value @ the up node:

$$V_u = \Delta_u\cdot S_u + B_u = e^{-rh}\left[p^*\cdot V_{uu} + (1-p^*)\cdot V_{ud}\right]$$

w/ $\boxed{p^* = \dfrac{e^{rh} - d}{u-d}}$

- down node: Δ_d, B_d

$$\Rightarrow \quad V_d = \Delta_d\cdot S_d + B_d = e^{-rh}\left[p^*\cdot V_{ud} + (1-p^*)\cdot V_{dd}\right]$$

- ROOT node: the replicating portfolio:

$$\Delta_0 = \frac{V_u - V_d}{S_u - S_d}$$

$$B_0 = e^{-rh} \frac{u \cdot V_d - d \cdot V_u}{u - d}$$

$$\Rightarrow \boxed{V(0) = \Delta_0 \cdot S(0) + B_0}$$

From the risk·neutral perspective:

$$V(0) = e^{-rh}\left[p^* \cdot V_u + (1-p^*) \cdot V_d\right]$$

$$V(0) = e^{-rh}\Big[p^* \cdot e^{-rh}\left(p^* \cdot V_{uu} + (1-p^*) \cdot V_{ud}\right) + (1-p^*) \cdot e^{-rh}\left(p^* \cdot V_{ud} + (1-p^*) \cdot V_{dd}\right)\Big]$$

$$V(0) = e^{-r \overset{T}{\overset{\shortparallel}{2h}}} \underbrace{\left[(p^*)^2 \cdot V_{uu} + 2 \cdot p^*(1-p^*) \cdot V_{ud} + (1-p^*)^2 \cdot V_{dd}\right]}_{\text{Risk·Neutral Expectation of the Payoff}}$$

Generally: $\boxed{V(0) = e^{-rT}\, \mathbb{E}^*[V(T)]}$

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Problem Set #7

Binomial option pricing: Two or more periods.

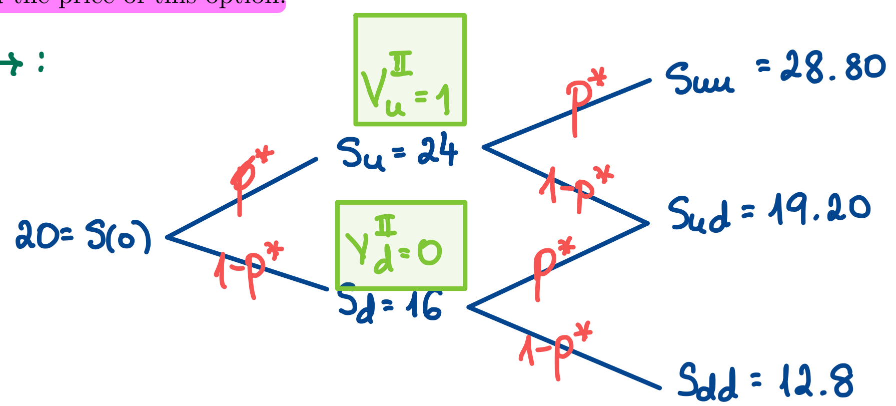
Problem 7.1. For a two-period binomial model, you are given that:

(1) each period is one year; $h=1$
(2) the current price of a non-dividend-paying stock S is $S(0) = \$20$;
(3) $u = 1.2$, with u as in the standard notation for the binomial model;
(4) $d = 0.8$, with d as in the standard notation for the binomial model;
(5) the continuously compounded risk-free interest rate is $r = 0.04$.

Consider a **special** call option which pays the excess above the strike price $K = 23$ (if any!) at the end of **every** binomial period.

Find the price of this option.

$\longrightarrow$:



$V_{uu}^{I} = (28.8-23)_+ = 5.8$

$V_{ud}^{I} = (19.20-23)_+ = 0$

$V_{dd}^{I} = (12.8-23)_+ = 0$

The Risk-Neutral Probability.

$$p^* = \frac{e^{rh} - d}{u-d} = \frac{e^{0.04} - 0.8}{1.2 - 0.8} = 0.602027$$

$$V^{I}(0) = \boxed{e^{-2r}} \cdot \boxed{(p^*)^2 \cdot V_{uu}^{I}} = \boxed{e^{-0.08}} \boxed{(p^*)^2 (5.8)} = 1.94$$

discounting

Expected risk-neutral payoff

$$V^{II}(0) = e^{-0.04} \cdot (p^*) \cdot 1 = 0.576$$

$$V(0) = 1.94 + 0.576 \cong 2.52 \quad \square$$