M339D: September 25th, 2023.

Arbitrage Portfolios.

Def'n. An arbitrage portfolio is a portfolio whose $\boxed{\text{profit}}$ is:

- nonnegative in all states of the world

and

- strictly positive in $\boxed{\text{at least one}}$ state of the world.

> Unless it is specified otherwise in a particular problem/example, we assume NO ARBITRAGE.

Law of the Unique Price.

Assume that the payoffs of two static portfolios A and B are equal, i.e.,

$$\boxed{V_A(T) = V_B(T)} \quad ✓$$

> In general, two random variables, X and Y, are said to be equal if $\boxed{\mathbb{P}[X=Y]=1}$
>
> On a finite probability space, this means that they take the exact same value for every elementary outcome.

Our claim: $\boxed{V_A(0) = V_B(0)}$

Proof. Assume, to the contrary, that

$$V_A(0) \neq V_B(0) \quad ✗$$

Without loss of generality, say,

$$\boxed{\underbrace{V_A(0)}_{\text{relatively cheap}} < \underbrace{V_B(0)}_{\text{relatively expensive}}}$$

Diagnosis.

Propose an arbitrage portfolio:

- **Long** portfolio A
- **Short** portfolio B

} Total Portfolio

Verify:

- Payoff (Total Portfolio) $= V_A(T) - V_B(T) = 0$ ✓
- Initial Cost (Total Portfolio) $= V_A(0) - V_B(0) < 0$

Inflow of money @ time 0.

$$\text{Profit} = \text{Payoff} - FV_{0,T}(\text{Initial Cost})$$

$$\text{Profit} = 0 - FV_{0,T}\big(\underbrace{V_A(0) - V_B(0)}_{<0}\big) > 0$$

Indeed, this is an arbitrage portfolio!

$\Rightarrow\Leftarrow$ $\square$

Remark: If $V_A(T) \geq V_B(T)$,

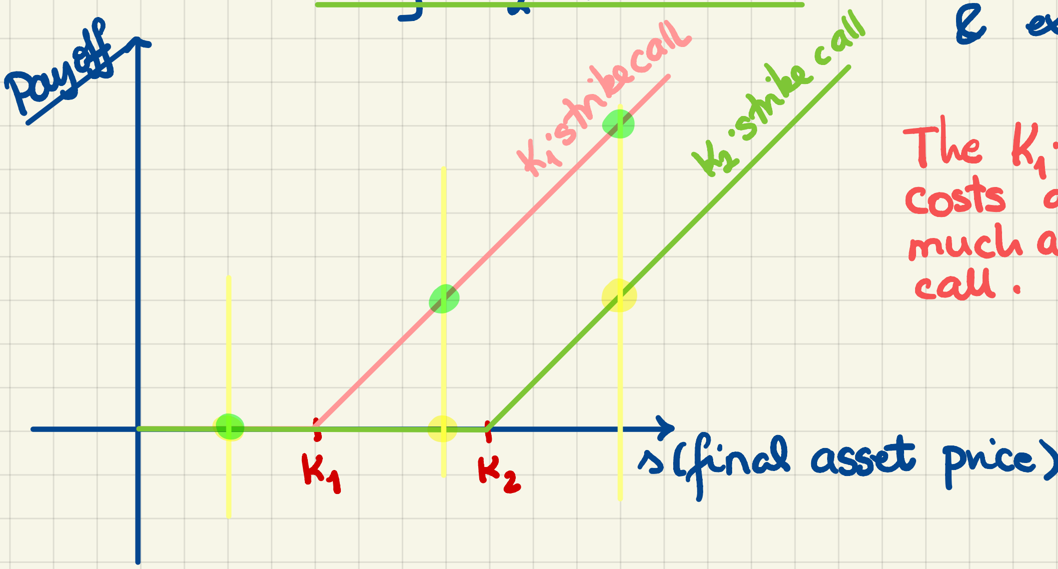
then $V_A(0) \geq V_B(0)$.

Example. $\boxed{K_1 < K_2}$

A: one long K_1-strike call

B: one long K_2-strike call

} w/ the same underlying asset & exercise date



The K_1-strike call costs at least as much as the K_2-strike call.

Replicating Portfolios.

Def'n. Consider a European-style derivative security. A static portfolio w/ the same payoff as that of the derivative security is called its **replicating portfolio**.

Note: The initial price of the derivative security is equal to the initial price of the replicating portfolio.

Example. Consider a forward contract on a non-dividend-paying stock.

$$0 \longrightarrow T$$

Forward contract: $\quad S(T) - F$

Replicating portfolio:
- long 1 share of stock
- issue a bond w/ redemption amount F and maturity date T

$$\text{Payoff(portfolio)} = S(T) - F \quad \checkmark$$

$\Rightarrow$ The forward contract and its replicating portfolio have the same initial cost, i.e.,

$$0 = \underbrace{S(0)}_{\text{long stock}} - \underbrace{PV_{0,T}(F)}_{\text{short bond}}$$

$$\Rightarrow \quad PV_{0,T}(F) = S(0)$$

$$\Rightarrow \quad \boxed{F = S(0)e^{rT}}$$

$\square$