On Market-Making and Delta-Hedging

1. Market Makers

2. Market-Making and Bond-Pricing

Sections 13.1,2

What to market makers do?

- Provide **immediacy** by standing ready to sell to buyers (at ask price) and to buy from sellers (at bid price)
- Generate **inventory** as needed by short-selling
- **Profit** by charging the bid-ask spread
- Their position is determined by the order flow from customers
- In contrast, proprietary trading relies on an investment strategy to make a profit

Risk Exposure
=
$\Uparrow$ Job Description.

Recall the meaning of Delta

- An option written on an undelying asset S is most sensitive to the changes in the value of S
- The largest part of the risk comes from the price movements of asset S - which is reflected in the delta of the option, i.e., if C is the price of a call the most pronounced effect comes from

$$\Delta_C := \frac{\partial C}{\partial S}$$

- The replicating protfolio will always contain Δ_C shares of the underlying stock
- The portfolio which contains the option, along with Δ_C shares of stock will have the value of its Delta equal to zero - we say it is **delta neutral**

SAMPLE MFE

23. Consider a European call option on a nondividend-paying stock with exercise date T, $T > 0$. Let $S(t)$ be the price of one share of the stock at time t, $t \geq 0$. For $0 \leq t \leq T$, let $\boxed{C(s, t)}$ be the price of one unit of the call option at time t, if the stock price is s at that time. You are given:

value function of the call option

(i) $\boxed{\dfrac{dS(t)}{S(t)} = 0.1dt + \sigma dZ(t)}$, where σ is a positive constant and $\{Z(t)\}$ is a Brownian motion.

Black-Scholes model

(ii) $\dfrac{dC(S(t),t)}{C(S(t),t)} = \underline{\gamma(S(t),t)}dt + \underline{\sigma_C(S(t),t)}dZ(t), \qquad 0 \leq t \leq T$

STOCHASTIC PROCESSES

(iii) $C(S(0), 0) = 6$.

INITIAL PRICE OF THE CALL

(iv) At time $t = 0$, the cost of shares required to delta-hedge one unit of the call option is 9.

(v) The continuously compounded risk-free interest rate is 4%.

Determine $\gamma(S(0), 0)$.

(A) 0.10

(B) 0.12

(C) 0.13

(D) 0.15

(E) 0.16

Δ-hedge a call:

- call
- investment in the underlying asset; # of shares $N(t)$

THE ENTIRE PORTFOLIO IS $\boxed{\Delta\text{-NEUTRAL}}$

The worth of this portfolio, as a function of s is:

$$C(s,t) + N(t)\cdot s$$

LONG

Δ-neutral means: $\dfrac{\partial}{\partial s}C(s,t) + N(t) = 0$

$$\Rightarrow \boxed{N(t) = -\Delta_C(t)}$$

Short shares!

Part (iv) $\Rightarrow$ cost of Δ-hedging one written call is $\boxed{9 = \Delta_c(0) \cdot S(0)}$

Option elasticity: $\Omega = \dfrac{\Delta \cdot S}{\underset{\uparrow\ \text{price}}{P}}$

The call & the underlying asset are driven by the same std BM $\Rightarrow$ their Sharpe ratios are equal:

$$\underline{\underline{\text{Constant}}} = \frac{\alpha_S - r}{\sigma_S} = \frac{\gamma(s,t) - r}{\sigma_c(s,t)}$$

$$\Rightarrow \gamma(S(0),0) - \underbrace{r}_{0.04} = \underbrace{\Omega_c}_{= \frac{9}{6}} (\alpha_S - r)$$

$$\Rightarrow \gamma(S(0),0) = 0.04 + \frac{3}{2}(0.1 - 0.04) = 0.13 \Rightarrow \text{C.}$$

SAMPLE MFE

65. Assume the Black-Scholes framework.

You are given:

(i) $S(t)$ is the time-t price of a stock, $t \geq 0$.

(ii) The stock pays dividends continuously at a rate proportional to its price.

(iii) Under the true probability measure, $\ln[S(2)/S(1)]$ is a normal random variable with mean 0.10.

(iv) Under the risk-neutral probability measure, $\ln[S(5)/S(3)]$ is a normal random variable with mean 0.06.

(v) The continuously compounded risk-free interest rate is 4%.

(vi) The time-0 price of a European put option on the stock is 10.

(vii) For delta-hedging at time 0 one unit of the put option with shares of the stock, the cost of stock shares is 20.

Calculate the absolute value of the time-0 continuously compounded expected rate of return on the put option.

(A) 4%

(B) 5%

(C) 10%

(D) 11%

(E) 18%

under $\mathbb{P}$:

$$S(t) = S(0)\, e^{(\alpha - \delta - \frac{\sigma^2}{2})t + \sigma Z(t)}$$

$$\ln\left[\frac{S(t+h)}{S(t)}\right] = (\alpha - \delta - \frac{\sigma^2}{2})h + \sigma[Z(t+h) - Z(t)]$$

$$\Rightarrow \mathbb{E}\left[\ln\left(\frac{S(t+h)}{S(t)}\right)\right] = (\alpha - \delta - \frac{\sigma^2}{2})h$$

under $\mathbb{P}^*$: $\mathbb{E}^*\left[\ln\left(\frac{S(t+h)}{S(t)}\right)\right] = (r - \delta - \frac{\sigma^2}{2})h$

(iii) $\Rightarrow \left(\underbrace{\alpha - \delta - \frac{\sigma^2}{2}}\right)\cdot 1 = 0.1 \Longleftrightarrow \alpha - \left(\delta + \frac{\sigma^2}{2}\right) = 0.1$

(iv) $\Rightarrow \left(r - \delta - \frac{\sigma^2}{2}\right)\cdot 2 = 0.06$

(v) $\Rightarrow 0.04 - \left(\delta + \frac{\sigma^2}{2}\right) = 0.03$

$\Rightarrow \delta + \frac{\sigma^2}{2} = 0.01$

$\alpha = 0.1 + 0.01$

$\alpha = 0.11$

$\Downarrow$

$\boxed{\alpha - r = 0.07}$

$$\gamma_P(0) - r = \Omega_P \underbrace{(\alpha - r)}$$

$$\Omega_P = \frac{\Delta_P(0)\cdot S(0)}{P(0)} \overset{(vii)}{=} \frac{-20}{10} = -2$$

$$\gamma_P(0) = 0.04 + (-2)\cdot(0.07) = -0.1 \Rightarrow \text{(C.)}$$

Just for laughs: $\gamma = 0.04 + 2\cdot 0.07 = 0.18$

Exactly offered answer (D.) :(

$$\left.\begin{aligned}
\frac{dS(t)}{S(t)} &= \alpha_s\,dt + \sigma_s\,dZ(t) \\[2ex]
\frac{dV_p(t)}{V_p(t)} &= \gamma_p(t)\,dt + \underbrace{\sigma_p(t)}_{\Omega\cdot\sigma_s}\,dZ(t)
\end{aligned}\right\}\ \underline{\underline{\text{System}}}$$

sign of Ω <u>matters</u>

positive Ω

positive correlation between the underlying and the option price

negative Ω

negative correlation.

Introduce $Z_p = -Z_{stock}$.

Taylor-like expansions

$W(t)$ ... total worth of a portfolio @ time-t in the market model w/ :

- **risky asset** : w/ $S(t), t \geq 0$ its price @ time-t
- **risk-free** : @ a constant, continuously compounded ⓡ

$\Rightarrow$ • Derivative securities on S also available.

$$W(t+dt, S(t)+dS(t)) \approx$$

$$\approx W(t,S(t)) + \underbrace{\boxed{\frac{\partial}{\partial t} W(t,S(t))}}_{\text{THETA ... time decay}} dt + \underbrace{\boxed{\frac{\partial}{\partial s} W(t,S(t))}}_{\text{DELTA}} dS(t) + \frac{1}{2} \underbrace{\boxed{\frac{\partial^2}{\partial s^2} W(t,S(t))}}_{\text{GAMMA}} (dS(t))^2$$

+ smaller/negligible terms :)

$$\boxed{W(t+dt, S(t)+dS(t)) \approx \approx W(t,S(t)) + \Theta_W(t)\,dt + \Delta_W(t)\,dS(t) + \frac{1}{2}\Gamma_W(t)(dS(t))^2}$$

Delta-Gamma-Theta Approximation.

If we exclude Θ, we get:

$$\boxed{W(t+dt, S(t)+dS(t)) = W(t, S(t)) + \Delta_W(t)\,dS(t) + \frac{1}{2}\Gamma_W(t)(dS(t))^2}$$

Delta-Gamma Approximation

19. Assume that the Black-Scholes framework holds. The price of a nondividend-paying stock is \$30.00. The price of a put option on this stock is \$4.00.

You are given:

(i) $\Delta = -0.28$

(ii) $\Gamma = 0.10$

Using the delta-gamma approximation, determine the price of the put option if the stock price changes to \$31.50.

(A) \$3.40

(B) \$3.50

(C) \$3.60

(D) \$3.70

(E) \$3.80

$$V_p(S(t) + dS(t)) \approx V_p(S(t)) + \Delta_p \, dS(t) + \frac{1}{2}\Gamma_p (dS(t))^2$$

$$= 4 + (-0.28)\cdot(1.50) + \frac{1}{2}\cdot 0.10 \cdot (1.50)^2 = 3.69$$

****END OF EXAMINATION****

Market maker: wrote a call;

Δ-hedges using a stock investment.

Q: What is the portfolio's value in terms of $S(t)$ and t?

Q: What does our market maker do after a little bit of time (say, a day) has passed?