M339J: January 26th, 2022.

Exponential Distribution.

Any exponential r.v. X w/ parameter θ has the probability density function given by

$$\boxed{f_X(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}} \quad \text{for } x>0}$$

In other sources: $f_X(x) = \lambda e^{-\lambda x}$ for $x>0$

Q: What is the support of the exponential distribution?

$\longrightarrow$: $[0, +\infty)$

Its cumulative dist'n function:

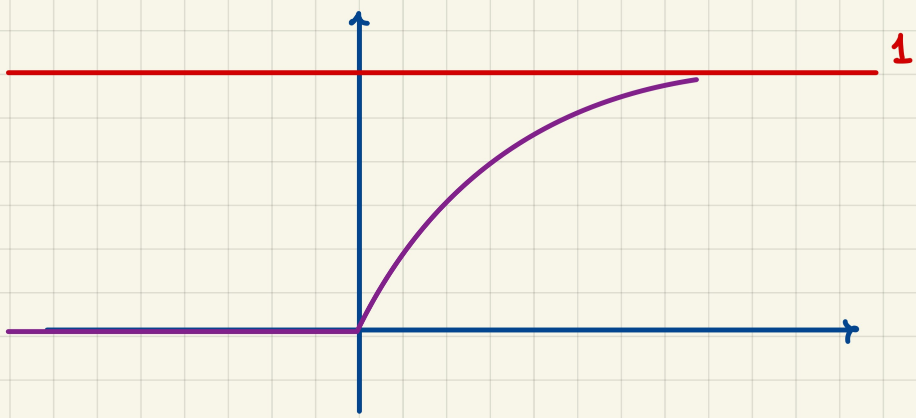
For $x>0$:

$$F_X(x) = \mathbb{P}[X \le x] = \int_0^x f_X(u)\,du = \int_0^x \frac{1}{\theta} e^{-\frac{u}{\theta}}\,du$$

$$= \frac{1}{\theta}\int_0^x e^{-\frac{u}{\theta}}\,du = \frac{1}{\cancel{\theta}}\cdot(-\cancel{\theta})\, e^{-\frac{u}{\theta}}\Big|_{u=0}^{x}$$

$$= -\left(e^{-\frac{x}{\theta}} - e^{-\frac{0}{\theta}}\right) = 1 - e^{-\frac{x}{\theta}}$$

$$\boxed{F_X(x) = 1 - e^{-\frac{x}{\theta}} \quad \text{for } x>0}$$ ✓



Its survival function is : $\boxed{S_X(x) = e^{-\frac{x}{\theta}}}$ ✓

Property: Given that $X \sim$ Exponential (θ) is bigger than $a > 0$, what is the probability that it's bigger than $a+b$ $(a>0, b>0)$?

$\longrightarrow$: $\mathbb{P}[X > a+b \mid X > a] =$

$$= \frac{\mathbb{P}[X > a+b, X > a]}{\mathbb{P}[X>a]}$$

$\{X > a+b\} \cap \{X > a\}$

$$= \frac{\mathbb{P}[X > a+b]}{\mathbb{P}[X>a]}$$

$$= \frac{S_X(a+b)}{S_X(a)} = \frac{e^{-\frac{\cancel{a}+b}{\theta}}}{\cancel{e^{-\frac{a}{\theta}}}}$$

$$= e^{-\frac{b}{\theta}} = \mathbb{P}[X > b]$$

This is the memoryless property.

Problem. The lifetime of a printer T is modeled by an exponential distribution w/ parameter $\theta = 2$.
There is a warranty on the printer.

- If the printer fails w/in the first year, a full refund of 200 is issued.
- If the printer fails w/in the second year, a half refund is issued.
- If the printer fails after two years or longer, there's no refund.

What is the pmf of the refund?

→: Denote the refund by Y.
What's the support of Y?

$$\{0, 100, 200\}$$

pmf of Y

$$p_Y(200) = \mathbb{P}[T \le 1] = F_T(1) = 1 - e^{-\frac{1}{2}} \approx 0.3935$$

$$p_Y(100) = \mathbb{P}[1 < T \le 2] = F_T(2) - F_T(1) =$$

$$= e^{-\frac{1}{2}} - e^{-\frac{2}{2}} = e^{-\frac{1}{2}} - e^{-1} = 0.23865$$

$$p_Y(0) = \mathbb{P}[T > 2] = S_T(2) = e^{-1} = 0.36788$$

<u>Problem.</u> The waiting time until a driver is involved in an accident is modeled as exponential w/ an unknown parameter.

We know that 30% of drivers will be involved in an accident in the first two months.

What's the probability that the driver is involved in an accident in the first three months?

→: T... waiting time until the first accident

$T \sim Exponential(\theta)$

$$\mathbb{P}\left[T \le \frac{2}{12}\right] = 0.3$$

$$1 - e^{-\frac{\frac{1}{6}}{\theta}} = 0.3$$

$$1 - e^{-\frac{1}{6\theta}} = 0.3$$

$$e^{-\frac{1}{6\theta}} = 0.7 \quad \Rightarrow \quad -\frac{1}{6\theta} = \ln(0.7)$$

$$\Rightarrow \quad \theta = -\frac{1}{6\ln(0.7)}$$

$$\boxed{e^{-\frac{1}{\theta}} = (0.7)^6}$$

$$\mathbb{P}\left[T \le \frac{1}{4}\right] = 1 - e^{-\frac{\frac{1}{4}}{\theta}}$$

$$= 1 - e^{-\frac{1}{4\theta}} = \dots$$

$$= 1 - \left(e^{-\frac{1}{\theta}}\right)^{1/4}$$

$$= 1 - \left((0.7)^6\right)^{1/4} = 1 - (0.7)^{3/2} = 0.41434$$