M339J : January 20 ; 2023.

Definition 1.1. Random variables X and Y with cumulative distribution functions F_X and F_Y (resp.) are said to be *independent* if

$$\mathbb{P}[X \leq x, Y \leq y] = F_X(x)F_Y(y) \quad \text{for all } x \text{ and } y.$$

Problem 1.4. Let T_1 and T_2 be two independent random variables with cumulative distributions functions denoted by F_1 and F_2, respectively. Define the random variables $T_\wedge$ and $T_\vee$ in the following fashion:

$$T_\wedge = \min(T_1, T_2), \quad T_\vee = \max(T_1, T_2).$$

Express the cumulative distribution functions of $T_\wedge$ and $T_\vee$ in terms of F_1 and F_2.

$\longrightarrow$: $t \in \mathbb{R}$

$$\begin{aligned}
F_\wedge(t) &= \mathbb{P}[T_\wedge \leq t] = \mathbb{P}[\min(T_1,T_2) \leq t] \\
&= 1 - \mathbb{P}[\min(T_1,T_2) > t] \\
&= 1 - \mathbb{P}[T_1 > t, T_2 > t] \quad \text{(independence)} \\
&= 1 - \mathbb{P}[T_1 > t] \cdot \mathbb{P}[T_2 > t] \\
&= 1 - (1 - \mathbb{P}[T_1 \leq t])(1 - \mathbb{P}[T_2 \leq t]) \\
&= 1 - (1 - F_1(t))(1 - F_2(t))
\end{aligned}$$

$$\begin{aligned}
F_\vee(t) &= \mathbb{P}[T_\vee \leq t] = \mathbb{P}[\max(T_1,T_2) \leq t] \\
&= \mathbb{P}[T_1 \leq t, T_2 \leq t] \quad \text{(independence)} \\
&= \mathbb{P}[T_1 \leq t] \cdot \mathbb{P}[T_2 \leq t] \\
&= F_1(t) \cdot F_2(t)
\end{aligned}$$

$\square$

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Lecture 3

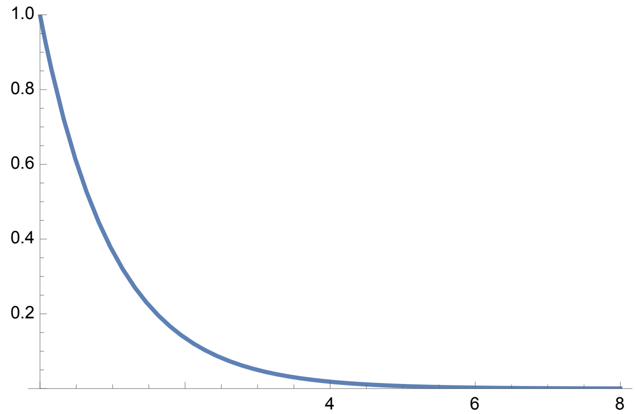
The Exponential Distribution

An **exponential** random variable X with parameter θ has the *probability density function* given by

$$f_X(x) = \frac{1}{\theta}e^{-\frac{x}{\theta}} \quad \text{for } x > 0.$$

We write $X \sim Exponential(\theta)$.

The graph of the probability density function of an exponential random variable with parameter $\theta = 1$ is shown below.



Remark 3.1. We choose the parameterization above because we are focused on modeling the *time* until some event of interest happens or we are interested in the extent of a loss.

In other sources, one might be emphasizing the *rate* at which some events of interest occur. There, you would encounter the parameterization with $\lambda = \frac{1}{\theta}$. So, the probability density function would be expressed as

$$f_X(x) = \lambda e^{-\lambda x} \quad \text{for } x > 0.$$

The support of the exponential distribution is $[0, \infty)$.

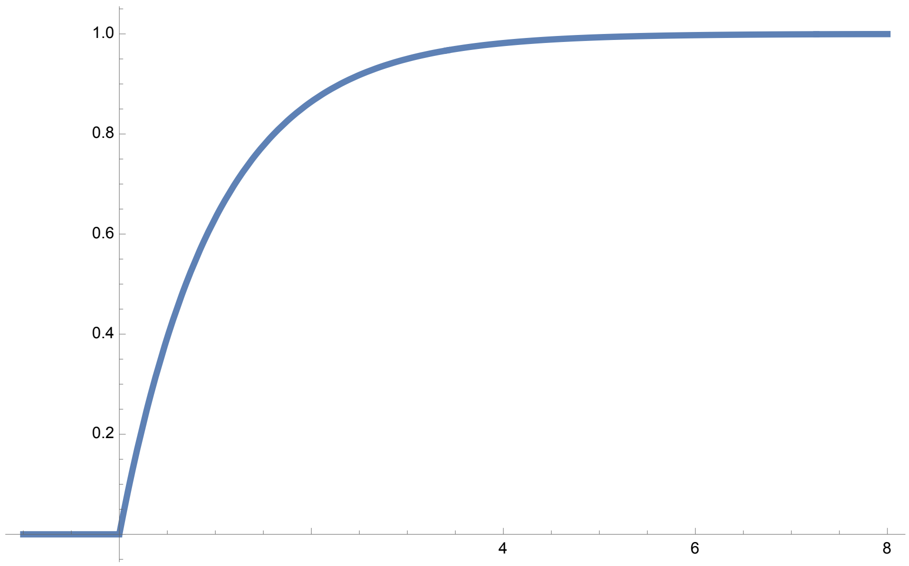
The *cumulative distribution function* is

$$F_X(x) = 1 - e^{-\frac{x}{\theta}} \quad \text{for } x > 0.$$

The *survival function* is

$$S_X(x) = e^{-\frac{x}{\theta}} \quad \text{for } x > 0.$$

The graph of the cumulative distribution function of an exponential random variable with parameter $\theta = 1$ is shown below.



Proposition 3.2. **Memoryless property.**
Let $X \sim Exponential(\theta)$. For $a, b > 0$, we have

$$\mathbb{P}[X > a + b \mid X > a] = \mathbb{P}[X > b].$$

$\rightarrow$:

$$\mathbb{P}[X > a+b \mid X > a] = \frac{\mathbb{P}[X > a+b, X > a]}{\mathbb{P}[X > a]}$$

$$= \frac{\mathbb{P}[X > a+b]}{\mathbb{P}[X > a]}$$

$$= \frac{e^{-\frac{a+b}{\theta}}}{e^{-\frac{a}{\theta}}} = e^{-\frac{b}{\theta}} = \mathbb{P}[X > b]$$

$\square$

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Problem set 2

The Exponential Distribution.

Problem 2.1. The lifetime T of a printer is modeled by an exponential distribution with parameter $\theta = 2$.
There is a warranty on the printer with the following stipulations:

- If the printer fails within the first year, a full refund of 200 is issued.
- If the printer fails within the second year, a half refund is issued.
- If the printer fails after two years or longer, no refund is issued.

What is the *probability mass function* of the refund?

$\rightarrow$: Y... the refund

$\text{Support}(Y) = \{0, 100, 200\}$

$p_Y(200) = \mathbb{P}[T \leq 1] = 1 - e^{-\frac{1}{2}} \approx 0.3935,$

$p_Y(100) = \mathbb{P}[1 < T \leq 2] = e^{-\frac{1}{2}} - e^{-\frac{2}{2}} \approx 0.23865,$

$p_Y(0) = \mathbb{P}[T > 2] = e^{-\frac{2}{2}} = e^{-1} = 0.36788.$

$\square$

Problem 2.2. The waiting time until a driver is involved in an accident is modeled as exponential with an unknown parameter. We know that 30% of the drivers will be involved in an accident in the first two months. What is the probability that the driver is involved in an accident in the first three months?

→ : $T \sim \text{Exponential}(\theta)$

$\mathbb{P}[T \leq \frac{2}{12}] = 0.30$

$\mathbb{P}[T \leq \frac{1}{4}] = \cancel{?} = 1 - e^{-\frac{1}{4\theta}} = 1 - \left(e^{-\frac{1}{\theta}}\right)^{1/4} = 1 - (0.7)^{6/4} =$

$= 0.41434$

$1 - e^{-\frac{\frac{1}{6}}{\theta}} = 0.3$

$1 - e^{-\frac{1}{6\theta}} = 0.3$

$\boxed{e^{-\frac{1}{6\theta}} = 0.7} \longrightarrow e^{-\frac{1}{\theta}} = (0.7)^6$

ln

$-\frac{1}{6\theta} = \ln(0.7)$

$\boxed{\theta = -\frac{1}{6\ln(0.7)}}$

□