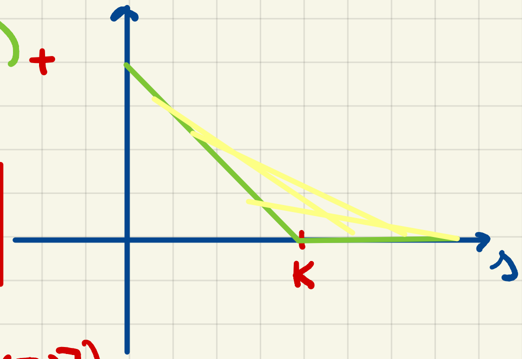
ii. Look @ a European put w/ strike K.

Its payoff f'tion: $v_p(s) = (K-s)_+$

The expected payoff

$$\mathbb{E}[v_p(S(T))] = \boxed{\mathbb{E}\left[(K-S(T))_+\right]}$$



$$\geq$$

By Jensen, its lower bound is $(K-\mathbb{E}[S(T)])_+$

$\boxed{\text{November } 13^{th}, 2024.}$

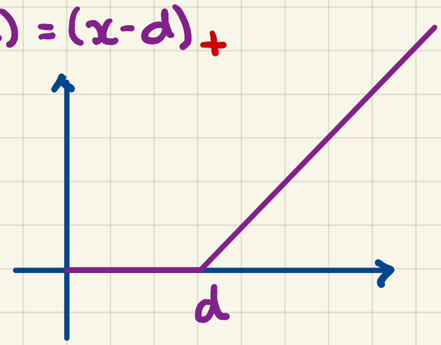
iii. In classical insurance:

$$\begin{cases} X \ldots \text{(ground-up) loss, i.e., severity r.v.} \\ d \ldots \text{deductible} \end{cases}$$

The insurer pays $(X-d)_+$, i.e., $g(x) = (x-d)_+$

By Jensen's inequality

$$\mathbb{E}[(X-d)_+] \geq (\mathbb{E}[X]-d)_+$$



The median = ?

Find $\pi_{0.5}$ such that $\mathbb{P}[Y \leq \pi_{0.5}] = 0.5$

$Y = e^X$ w/ $X \sim Normal(mean = m, var = \nu^2)$

$$\mathbb{P}[e^X \leq \pi_{0.5}] = 0.5$$

$$\mathbb{P}[X \leq \underbrace{\ln(\pi_{0.5})}_{= m}] = 0.5$$

$\leftarrow$ m is the median of X

$$\Updownarrow$$

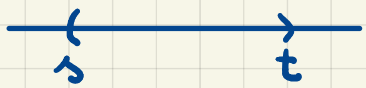
$$\boxed{\pi_{0.5} = e^m}$$

Last class, we did this for ALL the quantiles.

Log-Normal Stock Prices.

Temporarily fix a time-horizon T.

$S(t)$, $t \in [0,T]$ .... time-t stock price



Define: $\boxed{R(s,t) := \ln\left(\frac{S(t)}{S(s)}\right)}$

Analogously: $\boxed{S(t) = S(s)e^{R(s,t)}}$

In particular: $R(0,T)$ ... realized return over $(0,T)$

We model realized returns as normal

$R(0,T) \sim \text{Normal}(\text{mean} = m, \boxed{\text{var} = \nu^2})$

$\Rightarrow$ $S(T)$ is lognormal

and $\boxed{\mathbb{E}^*[S(T)] = S(0)e^{m+\frac{\nu^2}{2}}}$ ☆

Market Model.

- **Riskless Asset** w/ ccrfir r
- **Risky Asset**: a non-dividend paying stock w/

σ... volatility

Under the risk-neutral probability measure $\mathbb{P}^*$, we have

$$\boxed{\mathbb{E}^*[S(T)] = S(0)e^{rT}}$$ ☆☆

Equating ☆ & ☆☆, we get $\boxed{m + \frac{\nu^2}{2} = rT}$

Recall: $Var[R(0,1)] = \sigma^2$, i.e., $SD[R(0,1)] = \sigma$

$$\Rightarrow Var[R(0,T)] = \boxed{\sigma^2 \cdot T = \nu^2}$$

$$m = rT - \frac{\nu^2}{2} = rT - \frac{\sigma^2 \cdot T}{2} = \boxed{\left(r - \frac{\sigma^2}{2}\right) \cdot T}$$

Finally,

$$\boxed{R(0,T) \sim Normal\left(mean = \left(r - \frac{\sigma^2}{2}\right) \cdot T,\ var = \sigma^2 \cdot T\right)}$$

Say, $Z \sim N(0,1)$

Then, we can express $R(0,T)$ as

$$\boxed{R(0,T) = \left(r - \frac{\sigma^2}{2}\right) \cdot T + \sigma\sqrt{T} \cdot Z}$$

Thus,

$$\boxed{S(T) = S(0) e^{\left(r - \frac{\sigma^2}{2}\right) \cdot T + \sigma\sqrt{T} \cdot Z}}$$

Q: What is the median of $S(T)$ under the risk-neutral probability measure $\mathbb{P}^*$?

$$\longrightarrow : \boxed{S(0) e^{\left(r - \frac{\sigma^2}{2}\right) \cdot T}}$$

Note:

$$\frac{mean}{median} = \frac{\cancel{S(0)} e^{\cancel{r}T}}{\cancel{S(0)} e^{(\cancel{r} - \frac{\sigma^2}{2}) \cdot T}} = e^{\frac{\sigma^2 \cdot T}{2}} \quad \square$$