M362K Probability
Spring 2024
University of Texas at Austin
Practice for In-Term Exam I
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Notes: This is a closed book and closed notes exam. The maximal score on the real exam will be 100 points.
There are many ways in which any single problem can be solved. The solutions herein are just one possible way to tackle the given problems.
Time: 50 minutes

All written work handed in by the student is considered to be
their own work, prepared without unauthorized assistance.

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”I agree that I have complied with the UT Honor Code during my completion of this exam.”

Signature:

1.1. **DEFINITIONS.**

Problem 1.1. (10 points) Provide the definition of *mutually exclusive* events.

Solution: Consult your notes.

Problem 1.2. (10 points) Provide the definition of *independent* events.

Solution: Consult your notes.

1.2. **TRUE/FALSE QUESTIONS.**

Problem 1.3. (2 pts) A party of n people can arrange themselves around a circular table in $n!$ ways assuming rotations are interchangeable. *True or false?*

Solution: FALSE
The correct answer is $(n-1)!$ ways and it is sometimes referred to as the number of *circular permutations.* One can verify this result by noting that the number of "linear" permutations is $n!$. Then, every one of these permutations can be chosen to start from any of the n seats at the round table and all of these arrangements will be equivalent (they are just the rotations of the original linear arrangement). So, the total number of distinct arrangements is

$$\frac{n!}{n} = (n-1)!\,.$$

Problem 1.4. (2 pts) For every positive integer n, the number

$$\frac{(n+2)!}{n!}$$

is even. *True or false?*

Solution: TRUE
Note that

$$\frac{(n+2)!}{n!} = (n+2)(n+1).$$

Then, one of the numbers $n+1$ and $n+2$ is even, so their product is even as well.

Problem 1.5. (2 pts) If events E and F are disjoint, they are necessarily independent. *True or false?*

Solution: FALSE
Let A and B be two events with strictly positive probabilities such that $A \cap B = \emptyset$. Then,

$$\mathbb{P}[A \cap B] = \mathbb{P}[\emptyset] = 0$$

while

$$\mathbb{P}[A]\mathbb{P}[B] > 0.$$

Problem 1.6. (2 pts) If events E and F are independent, then E^c and F are independent as well. *True or false?*

Solution: TRUE See your lecture notes.

Problem 1.7. (2 pts) Let E and F be any two events. Then

$$\mathbb{P}[E \cup F] \geq \mathbb{P}[E] + \mathbb{P}[F].$$

True or false?

Solution: FALSE
From the axioms of probability, we have that

$$\mathbb{P}[E \cap F] \geq 0.$$

Due to the inclusion-exclusion identity, we get

$$\mathbb{P}[E \cup F] = \mathbb{P}[E] + \mathbb{P}[F] - \mathbb{P}[E \cap F] \leq \mathbb{P}[E] + \mathbb{P}[F].$$

For any E and F that are not mutually disjoint, a strict inequality holds above.

1.3. **Free-response problems.** Please, explain carefully all your statements and assumptions. Numerical results or single-word answers without an explanation (even if they're correct) are worth 0 points.

Problem 1.8. (4 pts) One tosses a coin three times and observes the sequence of heads (H) and tails (T) that appears. Write down the appropriate sample space S for this "experiment".

(2 points) Let A be the event that two or more heads appear consecutively. Write down the event A as a set of elementary outcomes from S .

(2 points) Let B be the event that all tosses are the same. Write down the event B as a set of elementary outcomes from S.

(2 points) Let C denote the event that only heads appear. Express C in terms of A and B.

Solution:

$$\begin{aligned} S &= \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\} \\ A &= \{HHH, HHT, THH\} \\ B &= \{HHH, TTT\} \\ C &= A \cap B \end{aligned}$$

Problem 1.9. (8 points) A certain soccer team wins (W) with probability 0.6, loses (L) with probability 0.3 and ties (T) with probability 0.1. The team plays three games over one weekend. The outcomes of the games are assumed to be independent.

(i) (3 pts) Determine the elements (elementary outcomes) of the event A that the team wins at least twice and does not lose.
(ii) (5 pts) Find $\mathbb{P}[A]$.

Solution:

$$A = \{WWT, WTW, TWW, WWW\}.$$

$$\begin{aligned} \mathbb{P}[A] &= \mathbb{P}[\{WWT\}] + \mathbb{P}[\{WTW\}] + \mathbb{P}[\{TWW\}] + \mathbb{P}[\{WWW\}] \\ &= 0.6 \cdot 0.6 \cdot 0.1 + 0.6 \cdot 0.1 \cdot 0.6 + 0.1 \cdot 0.6 \cdot 0.6 + 0.6 \cdot 0.6 \cdot 0.6 \\ &= 0.6^2(3 \cdot 0.1 + 0.6) \\ &= 0.36 \cdot 0.9 = 0.324. \end{aligned}$$

Problem 1.10. (10 points) A box contains three coins; one coin is fair, one coin is two-headed and one coin is weighted so that the probability of heads is 1/3. A coin is selected at random and tossed. Find the probability that the outcome of this coin-toss is heads.

Solution: Let us label the three coins by $i = 1, 2, 3$ in the order in which they are listed in the problem and denote by C_i the event that the coin i was drawn from the box at random. Since we are not given any extra information, we conclude that the events of choosing any single one of the three coins are equally likely, i.e.,

$$\mathbb{P}[C_1] = \mathbb{P}[C_2] = \mathbb{P}[C_3] = \frac{1}{3} \, .$$

Now, we can formalize the properties of the coins in the following way:

$$\mathbb{P}[H|C_1] = \frac{1}{2}, \; \mathbb{P}[H|C_2] = 1, \; \mathbb{P}[H|C_3] = \frac{1}{3}.$$

where H denotes the event that the outcome of the single coin-toss is heads. By the law of total probability, we get that

$$\begin{aligned}
\mathbb{P}[H] &= \mathbb{P}[H|C_1]\mathbb{P}[C_1] + \mathbb{P}[H|C_2]\mathbb{P}[C_2] + \mathbb{P}[H|C_3]\mathbb{P}[C_3] \\
&= \frac{1}{2} \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{3} \\
&= \frac{1}{3}(\frac{1}{2} + 1 + \frac{1}{3}) \\
&= \frac{1}{3} \cdot \frac{11}{6} = \frac{11}{18}.
\end{aligned}$$

Problem 1.11. (30 points) A piggy bank contains coins of three different types: T_1, T_2 and T_3.

There are twice as many type T_1 coins as type T_2 coins, and twice as many type T_2 coins as type T_3 coins. The coins are indistiguishable to touch.

(i) (10 points) A coin is extracted from the piggy bank at random. Let the probability that the coin is of type T_i be denoted by p_i for $i = 1, 2, 3$. Find p_1, p_2 and p_3.

(ii) (10 points) Coins of type T_1 are fair, coins of type T_2 come up heads (H) when tossed with probability 3/10, and coins of type T_3 come up heads (H) when tossed with probability 1/10. A coin is drawn from the piggy bank at random and tossed. What is the probability that the result of the coin toss was heads?

(iii) (10 points) A coin is drawn from the piggy bank at random and tossed. It is observed that the result of the coin toss was tails (T). What is the probability that the coin was of type T_3?

Solution: From the problem statement, we have that

$$p_1 = 2p_2 = 4p_3.$$

Since $p_1 + p_2 + p_3 = 1$, we have that $p_3 = 1/7$, $p_2 = 2/7$ and $p_1 = 4/7$.

By the rule of average conditional probabilities, we get

$$\begin{aligned}
\mathbb{P}[H] &= \mathbb{P}[T_1]\mathbb{P}[H \mid T_1] + \mathbb{P}[T_2]\mathbb{P}[H \mid T_2] + \mathbb{P}[T_3]\mathbb{P}[H \mid T_3] \\
&= \frac{4}{7} \cdot \frac{1}{2} + \frac{2}{7} \cdot \frac{3}{10} + \frac{1}{7} \cdot \frac{1}{10} = \frac{27}{70} \, .
\end{aligned}$$

From our solution to part (ii), we see that $\mathbb{P}[T] = 1 - \mathbb{P}[H] = 43/70$. Using Bayes' Theorem,

$$\mathbb{P}[T_3 \,|\, T] = \frac{\mathbb{P}[T_3]\mathbb{P}[T \,|\, T_3]}{\mathbb{P}[T]} = \frac{\frac{1}{7} \cdot \frac{9}{10}}{\frac{43}{70}} = \frac{9}{43}\,.$$

1.4. MULTIPLE CHOICE QUESTIONS.

Problem 1.12. (5 pts) A class has 12 boys and 4 girls. If three students are selected at random from this class, what is the probability that they are all boys?

(a) 1/4
(b) 5/9
(c) 11/28
(d) 17/36
(e) None of the above

Solution: (c)
Let A_i stand for the event of choosing a boy in the i^{th} selection with $i = 1, 2, 3$. The probability we are seeking is

$$\mathbb{P}[A_1 \cap A_2 \cap A_3].$$

By the multiplication rule,

$$\begin{aligned}\mathbb{P}[A_1 \cap A_2 \cap A_3] &= \mathbb{P}[A_1]\mathbb{P}[A_2|A_1]\mathbb{P}[A_3|A_2 \cap A_1] \\ &= \frac{12}{16} \cdot \frac{11}{15} \cdot \frac{10}{14} = \frac{11}{2 \cdot 14} = \frac{11}{28}\,.\end{aligned}$$

Problem 1.13. (5 pts) A pair of dice is thrown. Find the probability that the sum of the outcomes is 10 or greater if a 5 appears on the first die.

(a) 1/6
(b) 1/4
(c) 1/3
(d) 1/2
(e) None of the above

Solution: (c)
Let A_i denote the event that i was the outcome on the first die for $i = 1, 2, \ldots 6$. Let E denote the event that the sum of the outcomes on both of the dies was greater than or equal to 10. Formally,

$$\begin{aligned}E &= \{(i,j) : 1 \leq i, j \leq 6 \text{ and } i + j \geq 10\} \\ &= \{(i,j) : 1 \leq i, j \leq 6 \text{ and } i + j = 10\} \cup \{(i,j) : 1 \leq i, j \leq 6 \text{ and } i + j = 11\} \\ &\quad \cup \{(i,j) : 1 \leq i, j \leq 6 \text{ and } i + j = 12\}.\end{aligned} \tag{1.1}$$

We want to find the probability $\mathbb{P}[E|A_5]$. Directly from the definition of conditional probability, we get

$$\mathbb{P}[E|A_5] = \frac{\mathbb{P}[E \cap A_5]}{\mathbb{P}[A_5]}.$$

From the representation in (1.1), we get that

$$\begin{aligned}
\mathbb{P}[E \cap A_5] &= \mathbb{P}[\{(i,j) : 1 \leq i,j \leq 6 \text{ and } i+j=10\} \cap \{(i,j) : i=5 \text{ and } 1 \leq j \leq 6\}] \\
&\quad + \mathbb{P}[\{(i,j) : 1 \leq i,j \leq 6 \text{ and } i+j=11\} \cap \{(i,j) : i=5 \text{ and } 1 \leq j \leq 6\}] \\
&\quad + \mathbb{P}[\{(i,j) : 1 \leq i,j \leq 6 \text{ and } i+j=12\} \cap \{(i,j) : i=5 \text{ and } 1 \leq j \leq 6\}] \\
&= \mathbb{P}[\{(i,j) : i=5 \text{ and } i+j=10\}] \\
&\quad + \mathbb{P}[\{(i,j) : i=5 \text{ and } i+j=11\}] \\
&\quad + \mathbb{P}[\{(i,j) : i=5 \text{ and } i+j=12\}] \\
&= \mathbb{P}[\{(5,5)\}] + \mathbb{P}[\{(5,6)\}] + \mathbb{P}[\emptyset] \\
&= \frac{1}{36} + \frac{1}{36} + 0 = \frac{1}{18}\,.
\end{aligned}$$

On the other hand, $\mathbb{P}[A_5] = \frac{1}{6}$, and so $\mathbb{P}[E|A_5] = \frac{1}{3}$.

Note: The above solution is very formal. You could have solved this problem correctly by straightforward counting of “good” outcomes quite fast and in many ways.