UNIVERSITY OF TEXAS AT AUSTIN

Quiz # 11

The goodness-of-fit test. The χ^2-test of independence.

Provide your **complete solution** to the following problems.

Problem 11.1. (7 points)
A die is rolled 60 times and the face values are recorded. The results are as follows:

Up face	1	2	3	4	5	6
Number of occurrences	8	11	0	12	20	9

Using your χ^2-tables, at the 5% significance level, test the hypothesis that the die is fair.

Solution: Let $p_k, k = 1, \ldots, 6$ denote the probability that the die is going to come up k when rolled. We are testing

$$H_0: \quad p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = 1/6$$
$$vs.$$
$$H_a: \quad \text{At least one of the probabilities is different from } 1/6.$$

The expected count is $E_k = 60(1/6) = 10$ for all $k = 1, \ldots, 6$. The observed counts are as noted in the table in the problem. The observed value of our test statistic is

$$\begin{aligned} q^2 &= \frac{(8-10)^2}{10} + \frac{(11-10)^2}{10} + \frac{(0-10)^2}{10} + \frac{(12-10)^2}{10} + \frac{(20-10)^2}{10} + \frac{(9-10)^2}{10} \\ &= \frac{1}{10}(2^2 + 1^2 + 10^2 + 2^2 + 10^2 + 1^2) = \frac{210}{10} = 21. \end{aligned}$$

The number of degrees of freedom for our χ^2-distribution is $df = 6 - 1 = 5$. According to the χ^2-tables, the $p-$value is less than 0.01. We reject the null hypothesis.

Problem 11.2. (8 points) *Source: "Probability and Statistics for Engineers and Scientists" by Walople, Myers, Myers, and Ye.*
Suppose that we wish to determine whether the opinions of the voting residents of the state of Illinois concerning the new tax reform are independent of their levels of income. Members of a random sample of 1000 registered voters from the state of Illinois were classified as to whether they are in a *low, medium*, or *high* income bracket and whether or not they favor the tax reform. Here are the results:

		Income level		
Tax reform	Low	Medium	High	Total
For	182	213	203	598
Against	154	138	110	402
Total	336	351	313	1000

Using your $\chi^2 - tables$, at the significance level of 0.01, test the null hypothesis of independence.

Solution: The null hypothesis is

$$H_0: \text{the tax opinion is independent from the income level} \tag{11.1}$$

while the alternative is

$$H_a: \text{the tax opinion is } \textbf{not} \text{ independent from the income level} \tag{11.2}$$

The expected counts in the 6 possible combinations, under the null hypothesis, are

$$\begin{aligned} E_{FL} &= \frac{598(336)}{1000} = 200.928, \quad E_{FM} = \frac{598(351)}{1000} = 209.898, \quad E_{FH} = \frac{598(313)}{1000} = 187.174, \\ E_{AL} &= \frac{402(336)}{1000} = 135.072, \quad E_{AM} = \frac{402(351)}{1000} = 141.102, \quad E_{AH} = \frac{402(313)}{1000} = 125.826. \end{aligned} \tag{11.3}$$

So, the observed value of our test statistic equals

$$Q^2 = \frac{(182-200.928)^2}{200.928} + \frac{(213-209.898)^2}{213.898} + \frac{(203-187.174)^2}{187.174} + \frac{(154-135.072)^2}{135.072} + \frac{(138-141.102)^2}{141.102} + \frac{(110-125.826)^2}{125.826} = 7.87821. \tag{11.4}$$

At the 0.01 significance level, the critical value of the χ^2-distribution with $(3-1)(2-1) = 2$ degrees of freedom is 9.21. Since the observed value of the test statistic is below the critical value, we **fail to reject** the null hypothesis.