University of Texas at Austin

The binomial asset-pricing model. Extra-credit homework assignment 5

Binomial pricing of European options.

Please, provide your **complete solutions** to the following problems:

Problem 5.1. (2 points) In the setting of the one-period binomial model, denote by i the **effective** interest rate **per period**. Let u denote the "up factor" and let d denote the "down factor" in the stock-price model. If

$$d < u \leq 1 + i$$

then there certainly is no possibility for arbitrage. *True or false? Why?*

Solution: FALSE
The no-arbitrage condition is $d < 1 + i < u$.

Problem 5.2. (2 points) In the binomial asset pricing model, the number of shares Δ of the underlying asset in the replicating portfolio for a **put** option is always positive. *True or false? Why?*

Solution: FALSE
In our usual notation, the formula for the put delta is

$$\Delta_P = e^{-\delta h} \frac{V_u - V_d}{S_u - S_d}\,.$$

Since $V_u = (K - S_u)_+$ and $V_d = (K - S_d)_+$ with $S_u > S_d$, we conclude that the right-hand side above must be between -1 and 0.

Problem 5.3. (2 points) You are using a binomial asset-pricing model to model the evolution of the price of a particular stock. Then, the Δ in the replicating portfolio of a single call option on that stock never exceeds 1. *True or false? Why?*

Solution: TRUE
The call's Δ will always be between 0 and 1.

Problem 5.4. (2 points) In the binomial asset pricing model, the replicating portfolio for a put option has a bond investment which is equivalent to borrowing at the risk-free interest rate. *True or false? Why?*

Solution: FALSE
It's actually lending.

Problem 5.5. (2 points) In our usual notation, let $S(0) = 40, r = 0.08, \sigma = 0.3, \delta = 0$. You need to construct a $2-$period forward binomial tree for the above stock with every period in the tree of length $h = 0.5$. Then, $u > 1.45$. *True or false? Why?*

Solution: FALSE

$$u = \exp\{(0.08 - 0) \cdot 0.5 + 0.3\sqrt{0.5}\} \approx 1.29.$$

Problem 5.6. (10 points) In our usual notation, which of the parameter choices below creates a binomial model with an arbitrage opportunity?

(a) $u = 1.18, \quad d = 0.87, \quad r = 0.05, \quad \delta = 0, \quad h = 1/4$
(b) $u = 1.23, \quad d = 0.80, \quad r = 0.05, \quad \delta = 0.06, \quad h = 1/2$
(c) $u = 1.08, \quad d = 1, \quad r = 0.05, \quad \delta = 0.04, \quad h = 1$
(d) $u = 1.28, \quad d = 0.78, \quad r = \delta, \quad h = 2$
(e) None of the above.

Solution: (e)

Problem 5.7. (4 pts) Consider a non-dividend-paying stock currently priced at \$100 per share.

The price of this stock in one year is modeled using a one-period binomial tree under the assumption that the stock price can either go up to 110 or down to 90.

Let the continuously compounded, risk-free interest rate equal 0.04. What is the risk-neutral probability of the stock price going up?

Solution:

$$p^* = \frac{100e^{0.04} - 90}{110 - 90} = 0.7041.$$

Problem 5.8. (5 points) The current stock price is 40 per share. The price at the end of a three-month period is modeled with a one-period binomial tree so that the stock price can either increase by \$10, or decrease by \$4. The stock pays dividends continuously with the dividend yield 0.04.

The continuously compounded, risk-free interest rate is 0.05.

What is the stock investment in a replicating portfolio for three-month, \$40-strike European **straddle** on the above stock?

Solution: In our usual notation,

$$\Delta = e^{-\delta h}\frac{V_u - V_d}{S_u - S_d} = e^{-0.04/4}\left(\frac{10-4}{14}\right) \approx 0.4243$$

Problem 5.9. (6 points) Consider a non-dividend paying stock whose current price is \$95 per share. You model the evolution of this stock price over the following year using a one-period binomial tree under the assumption that the stock price can be either \$120, or \$75 in one year.

The continuously compounded, risk-free interest rate is 0.06.

Consider a \$100-strike, one-year European **straddle** on the above stock. What is the straddle's price consistent with the above stock-price model?

Solution: The risk-neutral probability of an up movement is

$$p^* = \frac{95e^{0.06} - 75}{120 - 75} = 0.575.$$

So, the price of our straddle is

$$V(0) = e^{-0.06}[0.575 \times (120 - 100) + (1 - 0.575) * (100 - 75)] = 20.8366.$$

Problem 5.10. (5 points) Assume that the a stock price is modeled using a one-period forward binomial tree with the length of a single period equal to three months. According to this model, the stock price can take either the value of \$55, or the value of \$40 in exactly three months. Calculate the volatility of the stock price.

Solution:

$$e^{2\sigma\sqrt{h}} = S_u/S_d \quad \Rightarrow \quad \sigma = \frac{1}{2\sqrt{h}}\ln(S_u/S_d) = \frac{1}{2\sqrt{1/4}}\ln(55/40) = \ln(55/40) = 0.3185.$$

Problem 5.11. (10 points) The current price of a continuous-dividend-paying stock is \$100 per share. Its volatility is given to be 0.2 and its dividend yield is 0.03.

The continuously compounded risk-free interest rate equals 0.06.

Consider a \$95-strike European put option on the above stock with nine months to expiration. Using a three-period forward binomial tree, find the price of this put option.

Solution: The up and down factors in the above model are

$$u = e^{0.03\times 0.25+0.2\sqrt{0.25}} = 1.1135,$$
$$d = e^{0.03\times 0.25-0.2\sqrt{0.25}} = 0.9116.$$

The relevant possible stock prices at the "leaves" of the binomial tree are

$$S_{ddd} = d^3S(0) = 100(0.9116)^3 = 75.7553,$$
$$S_{ddu} = d^2uS(0) = 92.5335.$$

The remaining two final states of the world result in the put option being out-of-the-money at expiration.

The risk-neutral probability of the stock price moving up in a single period is

$$p^* = \frac{1}{1+e^{0.2\sqrt{0.25}}} = 0.475.$$

So, the price of the European put option equals

$$V_P(0) = e^{-0.06(3/4)}\left[(95-75.7553)(1-0.475)^3 + (95-92.5335)(3)(1-0.475)^2(0.475)\right] = 3.5884.$$