Expectation.

In the discrete case:

$$\mathbb{E}[Y] := \sum_{y \in S_Y} y \cdot p_Y(y) \qquad \text{if it exists}$$

Def'n. Let Y be a continuous random variable w/ pdf f_Y.

We define the expected value of Y as

$$\boxed{\mathbb{E}[Y] := \int_{-\infty}^{\infty} y f_Y(y)\, dy} \qquad \text{if the integral exists}$$

Task: Cauchy Dist'n.

Problem 5.3. *Consider a continuous random variable Y whose probability density function is given by*

$$f_Y(y) = 2y\mathbf{1}_{[0,1]}(y)$$

What is the expected value of this random variable?

by def'n.

$\rightarrow$:

$$\mathbb{E}[Y] = \int_{-\infty}^{\infty} y f_Y(y)\,dy$$

$$= \int_{-\infty}^{\infty} y \cdot \left(2y \boxed{\mathbb{1}_{[0,1]}(y)}\right) dy \qquad \begin{cases} 1 & \text{for } y \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

$$= 2\int_0^1 y^2\,dy = 2 \cdot \left(\frac{y^3}{3}\right)_{y=0}^{1} = \frac{2}{3} \qquad \square$$

Def'n. $\mathrm{Var}[Y] = \mathbb{E}[(Y-\mu_Y)^2]$ w/ $\mu_Y = \mathbb{E}[Y]$

$\mathrm{SD}[Y] = \sqrt{\mathrm{Var}[Y]}$

Example. $\boxed{Y \sim U(\ell, r)}$

$\mathbb{E}[Y] = \frac{\ell + r}{2}$

$\mathrm{Var}[Y] = ?$

$Y - \ell \sim U(0, r-\ell)$

$\boxed{U := \frac{Y-\ell}{r-\ell} \sim U(0,1)}$

$\mathrm{Var}[U] = \cancel{?}\ \boxed{\mathbb{E}[U^2]} - (\boxed{\mathbb{E}[U]})^2$; $\mathbb{E}[U] = \frac{1}{2}$

$\mathbb{E}[U^2] = ?$

$\mathbb{E}[g(U)] = \int_{-\infty}^{\infty} g(u) f_U(u)\, du$

$\mathbb{E}[U^2] = \int_0^1 u^2\, du = \left(\frac{u^3}{3}\right)_{u=0}^{1} = \frac{1}{3}$

$\mathrm{Var}[U] = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$

$Y = (r-\ell) \cdot U + \ell$

$\mathrm{Var}[Y] = \mathrm{Var}[(r-\ell) \cdot U + \ell]$

$= (r-\ell)^2 \cdot \mathrm{Var}[U] = \frac{(r-\ell)^2}{12}$ $\square$

Example. $\boxed{Y \sim N(\mu, \sigma^2)}$

μ: $\mathbb{E}[Y]$

σ^2: $\mathrm{Var}[Y]$

Example. $Y \sim E(\tau)$, i.e., Y is exponential w/ parameter τ

$$f_Y(y) = \frac{1}{\tau} e^{-\frac{y}{\tau}} \cdot \mathbb{1}_{[0,\infty)}(y)$$

$$\mathbb{E}[Y] = \int_{-\infty}^{\infty} y \, f_Y(y) \, dy$$

$$= \int_{-\infty}^{\infty} y \cdot \frac{1}{\tau} e^{-\frac{y}{\tau}} \cdot \mathbb{1}_{[0,\infty)}(y) \, dy$$

$$= \int_{0}^{\infty} \left(\frac{y}{\tau}\right) e^{-\frac{y}{\tau}} \, dy$$

$$u = \frac{y}{\tau} \qquad du = \frac{dy}{\tau} \qquad dy = \tau \, du$$

$$= \tau \int_0^{\infty} u e^{-u} \, du$$

$$= \begin{bmatrix} u = u & du = du \\ dv = e^{-u} du & v = -e^{-u} \end{bmatrix}$$

$$= \tau \left(\underbrace{u(-e^{-u}) \Big|_{u=0}^{\infty}}_{0} + \underbrace{\int_0^{\infty} (+e^{-u}) \, du}_{1} \right) = \tau$$

□