M339G : October 23rd, 2024.

Bivariate Normal Random Variables.

(Based on Pitman's "Probability.")

Recall: In 1-D, the standard normal density is

$$\boxed{\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \quad \text{for } z \in \mathbb{R}}$$

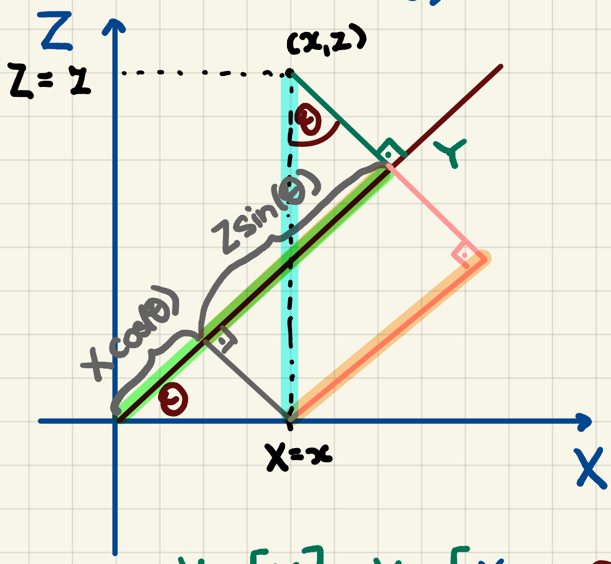
In 2-D, we start w/ X and Y that are independent and both standard normal, i.e., $N(0,1)$.

Then, their joint density, i.e., the density of the pair (X,Y) is

$$\boxed{f_{X,Y}(x,y) = \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)} \quad \text{for all } (x,y) \in \mathbb{R}^2}$$

Standard.

Start w/ a pair of independent, standard normal random variables. Say, X and Z.



$$\Rightarrow \boxed{Y = X \cdot \cos\theta + Z \cdot \sin\theta}$$

We know: Y is normally dist'd.

$$\mathbb{E}[Y] = \mathbb{E}[X \cdot \cos\theta + Z \cdot \sin\theta]$$

(linearity of $\mathbb{E}$) $= \underbrace{\mathbb{E}[X]}_{0} \cdot \cos\theta + \underbrace{\mathbb{E}[Z]}_{0} \cdot \sin\theta$

$= 0$

$\mathrm{Var}[Y] = \mathrm{Var}[X \cdot \cos\theta + Z \cdot \sin\theta] =$ (X and Z independent)

$= \cos^2(\theta)\underbrace{\mathrm{Var}[X]}_{1} + \sin^2(\theta) \cdot \underbrace{\mathrm{Var}[Z]}_{1} = 1$

$$\Rightarrow \boxed{Y \sim N(0,1)}$$

Q: What's the correlation coefficient between X and Y?

$\longrightarrow$: $\rho(X,Y) = \dfrac{Cov[X,Y]}{\underbrace{SD[X]}_{1}\underbrace{SD[Y]}_{1}} = Cov[X,Y]$

$= \mathbb{E}[(X-\mu_X)(Y-\mu_Y)] = \mathbb{E}[XY] - \underbrace{\mu_X \cdot \mu_Y}_{=0}$ (std normal X and Y)

$= \mathbb{E}[XY]$

$= \mathbb{E}[X(X\cdot\cos\theta + Z\cdot\sin\theta)] =$

$= \cos\theta \cdot \underbrace{\mathbb{E}[X^2]}_{\underbrace{Var[X] + (\mathbb{E}[X])^2}_{=1}} + \sin\theta \cdot \underbrace{\mathbb{E}[X\cdot Z]}_{\overset{\text{(independence)}}{=} \mathbb{E}[X]\cdot\mathbb{E}[Z] = 0}$

$$\boxed{\rho(X,Y) = \cos\theta}$$

$\square$

Special Cases: $\theta = 0 \Rightarrow Y = X$

$\theta = \frac{\pi}{2} \Rightarrow Y = Z$ (So, X and Y are independent)

$\theta = \pi \Rightarrow Y = -X$

In general: For each correlation coefficient $-1 \leq \rho \leq 1$, there exists an angle $\boxed{\theta = \arccos(\rho)}$

such that X and Y as above have the correlation coefficient ρ.

Alternatively, $\boxed{Y = \rho \cdot X + \sqrt{1-\rho^2} \cdot Z}$

w/ X and Z independent and $N(0,1)$

Joint Density: $$\boxed{f_{X,Y}(x,y) = \frac{1}{2\pi(1-\rho^2)} \exp\left(-\frac{1}{2(1-\rho^2)}(x^2-2\rho xy+y^2)\right)}$$

Marginal Dist'ns. $X \sim N(0,1)$, $Y \sim N(0,1)$

Conditional Dist'ns. Given $X=x$, $Y \sim \text{Normal}\left(\underline{\rho x}, \underbrace{1-\rho^2}_{}\right)$ variance

Given $Y=y$, $X \sim \text{Normal}\left(\underline{\rho y}, \underline{1-\rho^2}\right)$

Independence. $\boxed{\begin{array}{c}X \text{ and } Y \text{ are independent} \\ \text{iff} \\ \rho(X,Y) = 0\end{array}}$

Any Bivariate Normal.