UNIVERSITY OF TEXAS AT AUSTIN

Homework Assignment #1

Prerequisite material.

Provide your **complete solution** to the following problems. Final answers only, without appropriate justification, will receive zero points even if correct.

Problem 1.1. (5 points) Roger initially deposits \$4,000 in an investment fund which pays him \$2,000 at time 1 and \$4,000 at time 2.

Sally gets \$2,000 at time 0 and \$4,000 at time 1, and deposits \$5,460 at time 2 in return.

Both investments are governed by compound interest with the same annual effective interest rate i and they have the same net present values.

Find i.

(a) About 9%

(b) About 10.0%

(c) About 11.5%

(d) About 12%

(e) None of the above

Solution: (b)

Problem 1.2. (5 pts) Find the total amount of interest that would be paid on a \$1,000 loan over a 10−year period, if the effective interest rate is 0.09 per annum under the following repayment method:

The entire loan plus entire accumulated interest is paid as one lump-sum at the end of the loan term.

Solution: Using compound interest, the accumulated value at the end of the 10 years is

$$1000 \cdot 1.09^{10} \approx 2367.36.$$

The total amount of interest is

$$2367.36 - 1000 = 1367.36.$$

Problem 1.3. (5 points) Let $\Omega = \{a_1, a_2, a_3, a_4\}$ be an outcome space, and let $\mathbb{P}$ be a probability distribution on Ω. Assume that $\mathbb{P}[\{a_1, a_2\}] = 1/3, \mathbb{P}[\{a_2, a_3\}] = 1/4$ and $\mathbb{P}[\{a_1, a_3\}] = 1/9$. How much is $\mathbb{P}[\{a_4\}]$?

Solution: For any outcome space Ω, from the axioms of probability, we must have that $\mathbb{P}[\Omega] = 1$. In this case, $\Omega = \{a_1, a_2, a_3, a_4\}$, and so

$$\mathbb{P}[\Omega] = \mathbb{P}[\{a_1, a_2, a_3, a_4\}] = \mathbb{P}[\{a_1, a_2, a_3\}] + \mathbb{P}[\{a_4\}] = \frac{1}{2}\left(\frac{1}{3} + \frac{1}{4} + \frac{1}{9}\right) + \mathbb{P}[\{a_4\}].$$

Hence,

$$\mathbb{P}[\{a_4\}] = 1 - \frac{25}{72} = \frac{47}{72}.$$

Problem 1.4. (10 points) Let $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$ be a probability space. We denote by p_k the probability of the elementary outcome ω_k, i.e., $p_k = \mathbb{P}[\{\omega_k\}]$ for $k = 1, \ldots, 5$. You are given that p_k/p_{k-1} is constant for $k = 2, 3, 4, 5$. You are also given that $p_1 = 16/31$. Find p_5.

(a) 1/31

(b) 2/31

(c) 4/31

(d) Not enough information is given.

(e) None of the above.

Solution: (a)
From the given recursive property, we know that, for some constant κ,

$$p_2 = \kappa p_1, \quad p_3 = \kappa^2 p_1, \quad p_4 = \kappa^3 p_1, \quad p_5 = \kappa^4 p_1.$$

We aslo know that

$$p_1 + p_2 + p_3 + p_4 + p_5 = 1.$$

So,

$$p_1(1 + \kappa + \kappa^2 + \kappa^3 + \kappa^4) = 1 \quad \Rightarrow \quad \frac{16}{31}\left(\frac{1-\kappa^5}{1-\kappa}\right) = 1 \quad \Rightarrow \quad \frac{1-\kappa^5}{1-\kappa} = \frac{31}{16} \quad \Rightarrow \quad \kappa = 1/2.$$

Finally, $p_5 = \frac{1}{2^4}\left(\frac{16}{31}\right) = \frac{1}{31}$.

Problem 1.5. (5 points) Emmanuel entered an extra special kind of game with his friend Fischer. First, they toss a fair coin. If the coin comes up heads, Emmanuel gives $\$5,000$ to Fischer. If the coin comes up tails, Fischer gives $\$2,000$ to Emmanuel. Then, regardless of the outcome of the first cointoss, they toss the same fair coin again. If it comes up heads, Emmanuel gives Fischer $4,000. If the coin comes up tails, Fischer gives $3,000 to Emmanuel. What is the expected cashflow, i.e., what is the expected amount of money that changes hands and who gives it to whom?

Solution: Let X_i be the cashflow from Emmanuel's perspective after the ith cointoss for $i = 1, 2$. Then, we are looking for

$$\begin{aligned}\mathbb{E}[X_1 + X_2] &= \mathbb{E}[X_1] + \mathbb{E}[X_2] \\ &= \frac{1}{2}(-5,000 + 2,000) + \frac{1}{2}(-4,000 + 3,000) = -2,000.\end{aligned}$$

So, the expected cashflow is $2,000 from Emmanuel to Fischer.

Problem 1.6. (5 points) The random variables (R_1, R_2) have the following moments:

$$\mathbb{E}[R_1] = 0.08, \quad SD[R_1] = 0.2,$$
$$\mathbb{E}[R_2] = 0.10, \quad SD[R_2] = 0.25.$$

The correlation coefficient between R_1 and R_2 is given to be 0.2. What is the standard deviation of the random variable $R = \frac{1}{2}(R_1 + R_2)$?

Solution: The variance of R is

$$\begin{aligned}Var[R] &= \frac{1}{4}Var[R_1 + R_2] = \frac{1}{4}\left(Var[R_1] + Var[R_2] + 2Cov[R_1, R_2]\right) \\ &= \frac{1}{4}\left(Var[R_1] + Var[R_2] + 2SD[R_1]SD[R_2]corr[R_1, R_2]\right) \\ &= \frac{1}{4}\left((0.2)^2 + (0.25)^2 + 2(0.2)(0.25)(0.2)\right) = 0.030625.\end{aligned}$$

So, the standard deviation of R is $\sigma = \sqrt{0.030625} = 0.175$.