M378K: February 6th, 2026.

More on the expectation & variance.

The Uniform Distribution.

$$\boxed{Y \sim U(\ell, r)}$$

$$\mathbb{E}[Y] = \boxed{\frac{\ell + r}{2}}$$

$$\mathrm{Var}[Y] = ?$$

$$Y - \ell \sim U(0, r-\ell)$$

$$\boxed{U := \frac{Y-\ell}{r-\ell} \sim U(0,1)}$$

$$\mathrm{Var}[U] = \boxed{\mathbb{E}[U^2]} - (\boxed{\mathbb{E}[U]})^2$$

$$\mathbb{E}[U] = \frac{1}{2}$$

$$\mathbb{E}[U^2] \overset{?}{=}$$

$$\mathbb{E}[g(U)] = \int_{-\infty}^{\infty} g(u) f_U(u)\, du$$

$$\mathbb{E}[U^2] = \int_0^1 u^2\, du = \left(\frac{u^3}{3}\right)_{u=0}^{1} = \frac{1}{3}$$

$$\mathrm{Var}[U] = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12}$$

$$Y = (r-\ell) \cdot U + \ell$$

$$\mathrm{Var}[Y] = \mathrm{Var}[(r-\ell) \cdot U + \ell] = (r-\ell)^2 \cdot \mathrm{Var}[U] = \frac{(r-\ell)^2}{12} \quad \square$$

Moments.

Def'n. For a r.v. Y w/ pdf f_Y and for $k=1,2,\dots$,
we define • the k^{th} (raw) moment μ_k as

$$\boxed{\mu_k = \mathbb{E}[Y^k] = \int_{-\infty}^{\infty} y^k f_Y(y)\,dy}$$

$$\mu_Y = \mu = \mu_1 = \mathbb{E}[Y]$$

• the k^{th} central moment as

$$\boxed{\mu_k^c := \mathbb{E}\left[(Y-\mu_Y)^k\right] = \int_{-\infty}^{\infty} (y-\mu_Y)^k f_Y(y)\,dy}$$

Q: $\mu_2^c = \cancel{?}\ \mathrm{Var}[Y]$

The Cumulative Distribution Function.

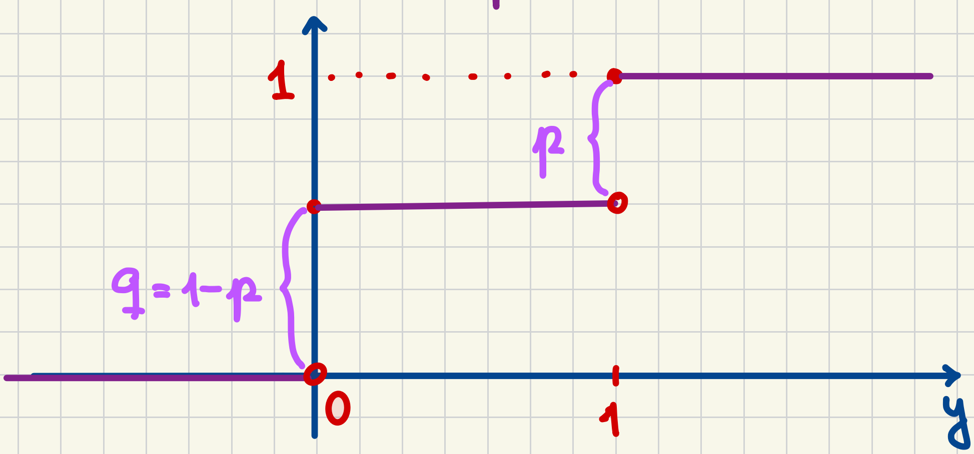
Def'n The cumulative dist'n f'tion (cdf) of a r.v. Y is a function $F_Y : \mathbb{R} \longrightarrow [0,1]$ defined as

$$\boxed{F_Y(y) = \mathbb{P}[Y \le y] \quad \text{for all } y \in \mathbb{R}}$$

Properties:
- $0 \le F_Y(y) \le 1$ for all y
- F_Y is nondecreasing
- $\lim_{y \to -\infty} F_Y(y) = 0$
- $\lim_{y \to +\infty} F_Y(y) = 1$

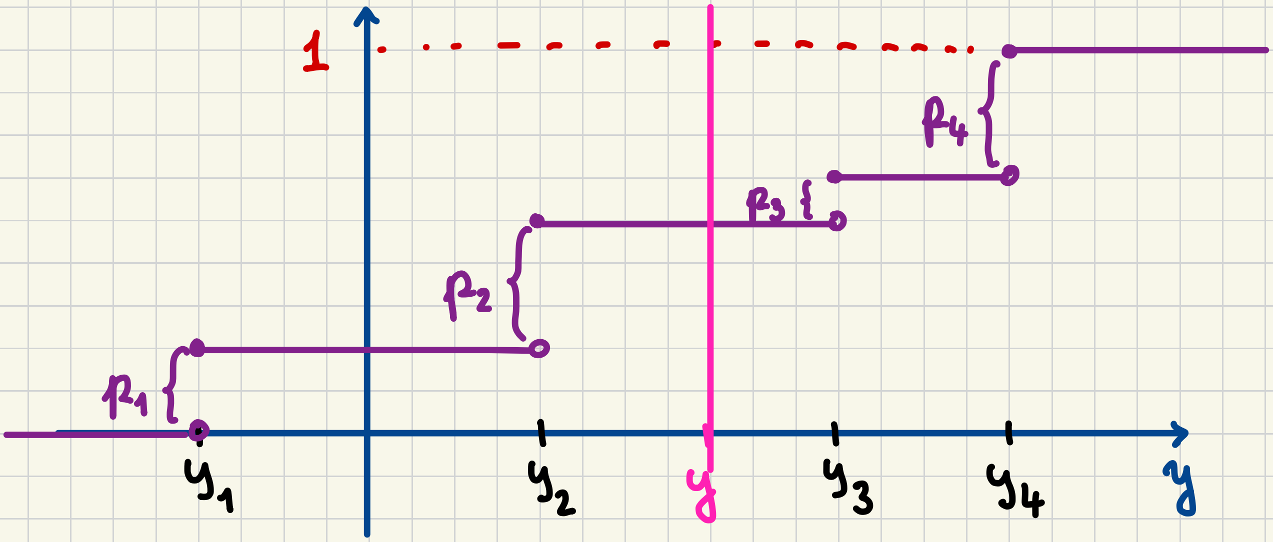
Example. Bernoulli dist'n.

$Y \sim B(p)$



Example. Discrete w/ finite support.

y	y_1	y_2	.....	y_m
$p_Y(y)$	p_1	p_2	....	p_m



The Discrete Case.

For Y discrete w/ pmf p_Y, we have

$$\boxed{F_Y(y) = \sum_{\substack{u \le y \\ u \in S_Y}} p_Y(u)}$$

M378K Introduction to Mathematical Statistics

Problem Set #6

Cumulative distribution functions.

Problem 6.1. Source: Sample P exam, Problem #342.
Consider a Poisson distributed random variable X. As usual, let's denote its cumulative distribution function by F_X. You are given that

$$\boxed{\frac{F_X(2)}{F_X(1)}} = 2.6$$

Calculate the expected value of the random variable X.

→ :

→ : $X \sim P(\lambda)$

$\mathbb{E}[X] = \lambda$

pmf of X: $k = 0, 1, 2, \dots$ $\boxed{p_X(k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!} =: p_k}$

$$\frac{\mathbb{P}[X \le 2]}{\mathbb{P}[X \le 1]} = 2.6$$

$$\frac{p_0 + p_1 + p_2}{p_0 + p_1} = 2.6$$

$$\frac{\cancel{e^{-\lambda}} + \cancel{e^{-\lambda}} \cdot \lambda + \cancel{e^{-\lambda}} \cdot \frac{\lambda^2}{2}}{\cancel{e^{-\lambda}} + \cancel{e^{-\lambda}} \cdot \lambda} = 2.6$$

$$1 + \lambda + \frac{\lambda^2}{2} = 2.6(1 + \lambda)$$

$$\frac{\lambda^2}{2} - 1.6\lambda - 1.6 = 0 \quad / \cdot 10$$

$$5\lambda^2 - 16\lambda - 16 = 0$$

$$(5\lambda + 4)(\lambda - 4) = 0$$

$\boxed{\lambda = 4}$ because positive

□

The Continuous Case.

Let Y be continuous w/ pdf f_Y.

Then,

$$F_Y(y) = \mathbb{P}[Y \le y]$$
$$= \mathbb{P}[-\infty \le Y \le y]$$
$$= \int_{-\infty}^{y} f_Y(u)\,du$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = F_Y'(y)$$ wherever the derivative exists.

$\boxed{\text{Fact:}}$ The cdf of a continuous r.v. is a continuous function w/ @ most countably many points @ which it's not differentiable.

Example. Uniform. $Y \sim U(l, r)$