Name:

M339J: Probability models
University of Texas at Austin
Solution: Practice Problems for In-Term One
Instructor: Milica Čudina

Notes: This is a closed book and closed notes exam. The maximal score on this exam is 50 points.
Time: 50 minutes

1.1. **TRUE/FALSE QUESTIONS.** *Please, note your final answer on the front page of this exam.*

Problem 1.1. Let X denote the outcome of a roll of a fair, regular icosahedron (a polyhedron with 20 faces) with numbers $1, 2, \cdots, 20$ written on its sides. Then $\mathbb{E}[X] = 15/2$. *True or false? Why?*

Solution: FALSE
Straight from the definition of expectation, we have that

$$\mathbb{E}[X] = \frac{1}{20}(1 + 2 + \cdots + 20) = \frac{1}{20} \cdot \frac{20 \cdot 21}{2} = \frac{21}{2}\,.$$

Problem 1.2. (2 pts) Let X be an exponential random variable. Then, its mean and its standard deviation are equal. *True or false?*

Solution: TRUE
Let $X \sim Exponential(\theta)$. From our tables, we get

$$\begin{aligned} \mathbb{E}[X] &= \theta, \\ Var[X] &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \theta^2 \cdot 2! - \theta^2 = \theta^2. \end{aligned}$$

Since the standard deviation is the square root of the variance, we are done!

Problem 1.3. (2 points) For a random variable X and for a positive constant d, in our usual notation, we have

$$\mathbb{E}[X] = e_X(d)S_X(d) + \mathbb{E}[X \wedge d]\,. \tag{1.1}$$

True or false?

Solution: TRUE

1.2. **Free-response problems.** Please, explain carefully all your statements and assumptions. Numerical results or single-word answers without an explanation (even if they're correct) are worth 0 points.

Problem 1.4. (10 points) A population of insureds consists of three types of people: α, β and γ. There is an equal number of Type α and Type β people in the population. The number of Type γ people is equal to the total number of the remaining two types of people. The probability that a Type α person makes at least one claim in a year is $1/5$. The probability that a Type β person makes at least one claim in a year is $2/5$. The probability that a Type γ person makes at least one claim in a year is $3/5$.

At the end of the year, a person is chosen at random from the population. It is observed that that person had at least one claim. What is the probability that the person was of Type β?

Solution: From the given breakdown of the population, we conclude that

$$\mathbb{P}[\alpha] = \mathbb{P}[\beta] = 1/4, \quad \mathbb{P}[\gamma] = 1/2. \tag{1.2}$$

Let E denote the event that there was at least one claim. By Bayes' Theorem, we have that

$$\begin{aligned} \mathbb{P}[\beta \mid E] &= \frac{\mathbb{P}[E \mid \beta] \times \mathbb{P}[\beta]}{\mathbb{P}[E \mid \alpha] \times \mathbb{P}[\alpha] + \mathbb{P}[E \mid \beta] \times \mathbb{P}[\beta] + \mathbb{P}[E \mid \gamma] \times \mathbb{P}[\gamma]} \\ &= \frac{(2/5)(1/4)}{(1/5)(1/4) + (2/5)(1/4) + (3/5)(1/2)} = \frac{2}{9}. \end{aligned} \tag{1.3}$$

Problem 1.5. (15 points) Losses X follow a Pareto distribution with parameters $\alpha > 1$ and θ unspecified. For a positive constant c, determine the ratio of the mean excess loss function evaluated at $c\theta$ to the mean excess loss function evaluated at θ.

Solution: By definition, the *mean excess loss function* of the random variable X at a positive constant d such that $\mathbb{P}[X > d] > 0$ is given by

$$e_X(d) = \mathbb{E}[X - d \mid X > d]. \tag{1.4}$$

According to our class notes, we also have that

$$e_X(d) = \frac{\mathbb{E}[X] - \mathbb{E}[X \wedge d]}{S_X(d)}. \tag{1.5}$$

Using the STAM tables for the Pareto distribution, we get that in the present problem

$$e_X(d) = \frac{\frac{\theta}{\alpha-1} - \frac{\theta}{\alpha-1}\left[1 - \left(\frac{\theta}{d+\theta}\right)^{\alpha-1}\right]}{\left(\frac{\theta}{d+\theta}\right)^{\alpha}} = \frac{\frac{\theta}{\alpha-1}}{\frac{\theta}{d+\theta}} = \frac{d+\theta}{\alpha-1}. \tag{1.6}$$

Finally, our answer is

$$\frac{\frac{c\theta+\theta}{\alpha-1}}{\frac{\theta+\theta}{\alpha-1}} = \frac{c+1}{2}. \tag{1.7}$$

Problem 1.6. Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables. Assume that all the variables in the sequence have the two-parameter Pareto distribution with $\theta = 10$ and $\alpha = 3$. For each n, define the random variable

$$Y_n = \frac{X_1^2 + X_2^2 + \cdots + X_n^2}{n}.$$

Does the limit of the sequence $\{Y_n, n \geq 1\}$ as $n \to \infty$ exist? If so, how much is it? If not, why not?

Solution: Since the random variables $\{X_n, n \geq 1\}$ are independent and identically distributed, the random variables $\{X_n^2, n \geq 1\}$ are also independent and identically distributed. If the mean of the random variable X_1^2 exists and is finite, we can apply the Law of Large Numbers. Using the STAM tables, we get that

$$\mathbb{E}[X_1^2] = \frac{2\theta^2}{(\alpha-1)(\alpha-2)} = \frac{200}{2} = 100.$$

We see that the expected value is finite. So, not only does the Law of Large Numbers apply, but invoking it we can conclude that the limit of Y_n is 100.

1.3. **MULTIPLE CHOICE QUESTIONS.** *Please, note your final answers on the front page of this exam.*

Problem 1.7. (5 pts) Let X be exponential with variance 225. Let $a = \mathbb{E}[|20 - X|]$. Then,

(a) $0 \leq a < 50$
(b) $50 \leq a < 150$
(c) $150 \leq a < 325$
(d) $325 \leq a < 550$
(e) None of the above.

Solution: (a)
From the given data,

$$X \sim \mathit{Exponential}(\theta = 15).$$

So,

$$
\begin{aligned}
a = \mathbb{E}[|20 - X|] &= \int_0^\infty |20 - x| f_X(x)\,dx \\
&= \int_0^{20} (20 - x) f_X(x)\,dx + \int_{20}^\infty (-20 + x) f_X(x)\,dx \\
&= \int_0^\infty (20 - x) f_X(x)\,dx + 2\int_{20}^\infty (x - 20) f_X(x)\,dx \\
&= 20\int_0^\infty f_X(x)\,dx - \int_0^\infty x\, f_X(x)\,dx + 2\int_0^\infty y f_X(y + 20)\,dy \\
&= 20 - \theta + 2e^{-20/\theta}\int_0^\infty y f_X(y)\,dy \\
&= 20 - \theta + 2e^{-20/\theta}\theta \\
&= 20 - 15 + 30e^{-20/15} \approx 12.9079.
\end{aligned}
$$

Problem 1.8. Let E and F be two events on the same probability space. You know that $\mathbb{P}[E \cup F] = 0.75$ and $\mathbb{P}[E \cup F^c] = 0.85$. What is the probability of the event E?

(a) 0.5
(b) 0.6
(c) 0.65
(d) 0.7
(e) None of the above.

Solution: (b)
By the basic properties of probability, we know that

$$
\begin{aligned}
0.75 &= \mathbb{P}[E \cup F] = \mathbb{P}[E] + \mathbb{P}[F] - \mathbb{P}[E \cap F], \\
0.85 &= \mathbb{P}[E \cup F^c] = \mathbb{P}[E] + \mathbb{P}[F^c] - \mathbb{P}[E \cap F^c].
\end{aligned}
$$

We can sup up the above two equalities to get

$$1.6 = 2\mathbb{P}[E] + (\mathbb{P}[F] + \mathbb{P}[F^c]) - (\mathbb{P}[E \cap F] + \mathbb{P}[E \cap F^c]) = 2\mathbb{P}[E] + 1 - \mathbb{P}[E] = \mathbb{P}[E] + 1.$$

Finally, $\mathbb{P}[E] = 0.6$.

Problem 1.9. The time until the next bus arrives is a continuous random variable T with the density

$$f_T(t) = \begin{cases} \kappa(10 - t) & 0 < t < 10 \\ 0 & \text{otherwise} \end{cases}$$

for some constant κ. **Given** that you have already waited for 4 minutes, what is the probability that you will wait for at least another 4 minutes?

(a) 1/25
(b) 1/9
(c) 1/8
(d) 1/3
(e) None of the above.

Solution: (b)
For any t between 0 and 10, the survival function of the random variable T is

$$S_T(t) = \frac{\kappa}{2}(10-t)^2$$

So, the conditional probability in the problem is

$$\mathbb{P}[T > 8 \,|\, T > 4] = \frac{\mathbb{P}[T > 8, T > 4]}{\mathbb{P}[T > 4]} = \frac{\mathbb{P}[T > 8]}{\mathbb{P}[T > 4]} = \frac{\frac{\kappa}{2}(10-8)^2}{\frac{\kappa}{2}(10-4)^2} = \frac{4}{36} = \frac{1}{9}\,.$$

Problem 1.10. Let X_1, X_2, and X_3 be independent, identically distributed random variables with the probability mass function

$$p_X(x) = \begin{cases} \frac{1}{4} & x = 0 \\ \frac{1}{2} & x = 1 \\ \frac{1}{4} & x = 2 \end{cases}$$

Find $\mathbb{P}[X_1X_2X_3 = 0]$.

(a) 27/64
(b) 1/8
(c) 31/64
(d) 37/64
(e) None of the above.

Solution: (d)
The product $X_1X_2X_3$ equals zero if at least one of the random variables X_1, X_2 and X_3 equals zero. So,

$$\mathbb{P}[X_1X_2X_3 = 0] = 1 - \mathbb{P}[X_1 \neq 0, X_2 \neq 0, X_3 \neq 0].$$

Since the random variables are independent, the above probability equals

$$1 - \mathbb{P}[X_1 \neq 0]\mathbb{P}[X_2 \neq 0]\mathbb{P}[X_3 \neq 0] = 1 - \left(\frac{3}{4}\right)^3 = 1 - \frac{27}{64} = \frac{37}{64}\,.$$

Problem 1.11. A recent study indicates that the annual cost of fertilizing a Japanese plum tree in Austin has a mean 100 with a variance of 20. A tax of 10% is introduced on fertilizer, i.e., fertilizer is made 10% more expensive. What is the variance of the new annual cost of fertilizing a Japanese plum tree in Austin after the tax is introduced?

(a) 20
(b) 22
(c) 23.1
(d) 24.2
(e) None of the above.

Solution: (d)
Let X be the cost before the tax. Then, the cost after the tax equals $1.1X$. So, the new variance is

$$Var[1.1X] = 1.21Var[X] = 1.21(20) = 24.2.$$

Problem 1.12. Let the random variable X have the cumulative distribution function given by

$$F_X(x) = \begin{cases} 0 & \text{for } x < 1 \\ \frac{1}{2}(x^2 - 2x + 2) & \text{for } 1 \leq x < 2 \\ 1 & \text{for } 2 \leq x \end{cases}$$

What is the expectation of X?

(a) 2/3
(b) 5/6
(c) 7/6
(d) 4/3
(e) None of the above.

Solution: (d)
This is a mixed random variable with $\mathbb{P}[X = 1] = 1/2$. The pdf is $f_X(x) = x - 1$ for $1 < x < 2$. So,

$$\begin{aligned}\mathbb{E}[X] &= \frac{1}{2}(1) + \int_1^2 x(x-1)\,dx = \frac{1}{2} + \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_{x=1}^2 \\ &= \frac{1}{2} + \frac{8}{3} - \frac{4}{2} - \frac{1}{3} + \frac{1}{2} = 4/3.\end{aligned}$$

Problem 1.13. Let the independent random variables X and Y have the same mean. You are given that coefficient of variation of X equals 2 and the coefficient of variation of Y equals 4. What is the coefficient of variation of the average of X and Y?

(a) 3/2

(b) $\frac{\sqrt{13}}{2}$

(c) 5/2

(d) There is not enough information to answer this problem.

(e) None of the above.

Solution: (b)
Let $\mu = \mathbb{E}[X] = \mathbb{E}[Y]$. Then, $\sigma_X = SD[X] = 2\mu$ and $\sigma_Y = SD[Y] = 3\mu$. The variance of the average of the two random variables is

$$Var\left[\frac{1}{2}(X+Y)\right] = \frac{1}{4}(Var[X] + Var[Y]).$$

In terms of μ, the variance of the average can be rewritten as

$$Var\left[\frac{1}{2}(X+Y)\right] = \frac{1}{4}(4\mu^2 + 9\mu^2) = \frac{13\mu^2}{4}.$$

So, the standard deviation of the average can be expressed as $\frac{\mu\sqrt{13}}{2}$. Hence, the coefficient of variation of the average equals $\frac{\sqrt{13}}{2}$.