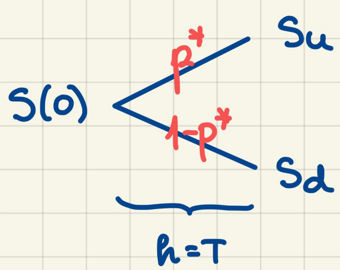
M339D: April 13th, 2022.

Risk-Neutral Pricing [cont'd].



$V_u = v(S_u)$

$V_d = v(S_d)$

Risk-neutral probability: $p^* = \dfrac{e^{(r-\delta)h} - d}{u-d}$

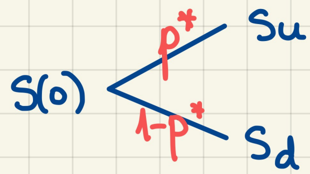
$$\boxed{V(0) = e^{-rT}\mathbb{E}^*[V(T)] = e^{-rT}\left[p^* \cdot V_u + (1-p^*) \cdot V_d\right]}$$

Example. Consider a stock which pays dividends continuously w/ the dividend yield δ.

Q: How much would I be charged today to receive one share @ time T?

$\longrightarrow$: $F^P_{0,T}(S) = e^{-\delta \cdot T} \cdot S(0)$

Say, we model the time-T stock price using a one-period binomial tree:



Payoff

$V_u = S_u$

$V_d = S_d$

By risk-neutral pricing:

$$V(0) = e^{-r\cdot T}\left[p^* \cdot V_u + (1-p^*)\cdot V_d\right]$$

$$= e^{-rT}\left[\frac{e^{(r-\delta)h} - d}{u-d} \cdot \underbrace{S_u}_{u\,S(0)} + \frac{u - e^{(r-\delta)h}}{u-d} \cdot \underbrace{S_d}_{d\,S(0)}\right]$$

$$= e^{-r\cdot T} \cdot \frac{S(0)}{u-d}\left[u \cdot \boxed{e^{(r-\delta)h}} \cancel{- u\cdot d} + \cancel{u\cdot d} - d\boxed{e^{(r-\delta)h}}\right]$$

$$= \cancel{e^{-r\cdot T}} \cdot \frac{S(0)}{\cancel{u-d}}\,\cancel{(u-d)}\,e^{(\cancel{r}-\delta)\cdot h} = S(0)\,e^{-\delta \cdot T}$$

Consistent w/ our model-free analysis!

Problem 9.5. Consider a non-dividend paying stock whose current price is \$95 per share. You model the evolution of this stock price over the following year using a one-period binomial tree under the assumption that the stock price can be either \$120, or \$75 in one year.

The continuously compounded risk-free interest rate is 0.06.

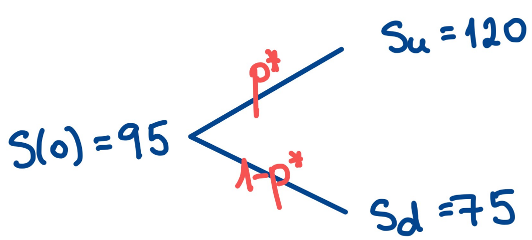
Consider a \$100-strike, one-year European **straddle** on the above stock. What is the straddle's price consistent with the above stock-price model?

(a) About \$10
(b) About \$10.83
(c) About \$15.45
(d) About \$20.84
(e) None of the above.

→ :

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{S(0)e^{(r-\delta)h} - S_d}{S_u - S_d} = \frac{95e^{(0.06)(1)} - 75}{120-75}$$

$p^* = 0.575$ $\boxed{K=100}$



$V_u = 20$

$V_d = 25$

$$V(0) = e^{-0.06}\left[0.575(20) + (1-0.575)(25)\right] = 20.84$$ □

Problem 9.6. (5 points) Consider the one-period binomial option pricing model. Let $V_C(0) > 0$ denote the price of a European call on a stock which pays continuous dividends. What is the impact on the value of European call option prices if the company decides to increase the dividend yield paid to the shareholders? δ

(a) The call option price will drop.

(b) The call option price will increase.

(c) The call option price will always remain constant.

(d) The impact on the price of the call cannot be determined using the binomial option pricing model.

(e) There is not enough information provided.

$\rightarrow$: $\boxed{\delta < \tilde{\delta}}$

$$V_d + p^* \overbrace{(V_u - V_d)}^{>0}$$

w/ δ :

$$V_c(0) = e^{-rT}\left[p^* \cdot (S_u - K)_+ + (1-p^*)(S_d - K)_+\right]$$

w/ $$p^* = \frac{e^{(r-\delta)h} - d}{u-d}$$

w/ $\tilde{\delta}$:

$$\tilde{V}_c(0) = e^{-rT}\left[\tilde{p}^*(S_u - K)_+ + (1-\tilde{p}^*)(S_d - K)_+\right]$$

$$\tilde{p}^* \underbrace{(V_u - V_d)}_{>0} + V_d$$

w/ $$\tilde{p}^* = \frac{e^{(r-\tilde{\delta})h} - d}{u-d}$$

$\delta < \tilde{\delta}$

$\Rightarrow \quad -\delta > -\tilde{\delta}$

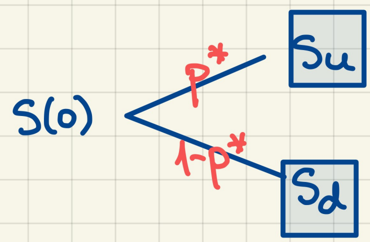
$$e^{(r-\delta)h} > e^{(r-\tilde{\delta})h} \quad \Rightarrow \quad p^* > \tilde{p}^* \Rightarrow \quad V_c(0) > \tilde{V}_c(0)$$

$\square$

Graphical Interpretation.

Consider a European call w/ the exercise date @ the end of the period and w/ the strike price K such that

$$S_d < K < S_u$$



Payoff

$V_u = S_u - K$

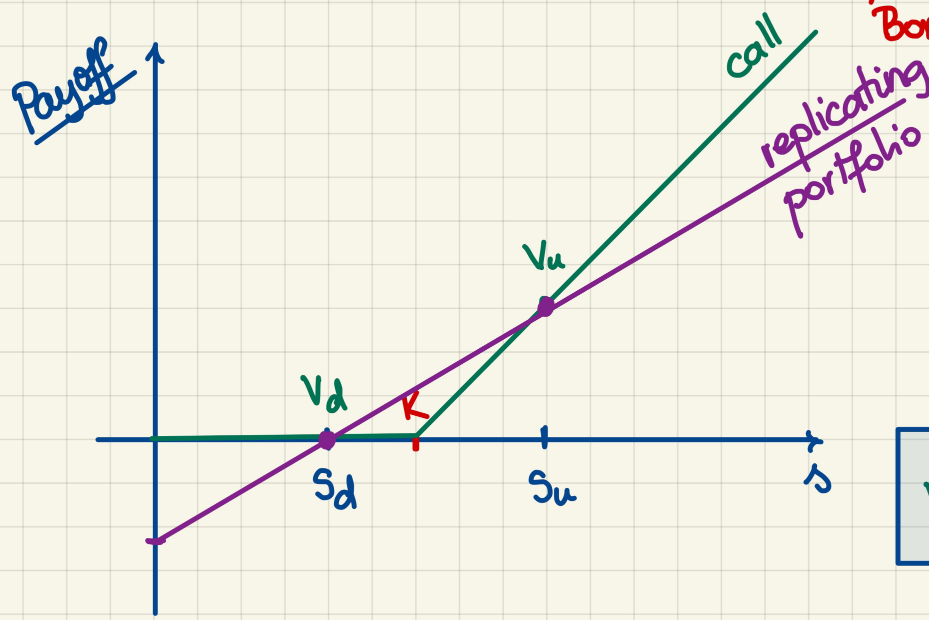
$\boxed{V_d = 0}$

In the replicating portfolio:

- $\Delta_c = e^{-\delta \cdot h} \cdot \dfrac{V_u}{S_u - S_d} = \boxed{e^{-\delta \cdot h} \cdot \dfrac{S_u - K}{S_u - S_d}} \in (0,1)$

- $B_c = e^{-r \cdot h} \cdot \dfrac{\cancel{u \cdot V_d} - d V_u}{u-d} = - e^{-r \cdot h} \cdot \dfrac{d \cdot V_u}{u-d}$

$\Downarrow$ Borrowing!



slope: $\dfrac{V_u - V_d}{S_u - S_d}$

intercept < 0

$\Rightarrow$ borrowing

$$\boxed{V_c(0) = e^{-r \cdot T} \cdot p^* \cdot V_u}$$