

**Problem 5.2.** (10 points)

At the *Hogwarts School of Witchcraft and Wizardry* the *Ordinary Wizarding Level (OWL)* exam is typically taken at the end of the fifth year. Based on historical data, we model the *OWL* scores as roughly normal with mean 100 and standard deviation 16.

(a) (5 points)

What is the range of scores for the bottom 15% of the *OWL* takers?

$$qnorm(0.15, 100, 16) = 83.41707$$

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w/ std normal tables:  $Z_{0.15}^* = -1.04$

$$x^* = 100 + 16 \cdot (-1.04) = 83.36$$

✓

(b) (5 points)

What is the probability that a randomly chosen OWL taker has a score higher than 125?

The raw score of 125 corresponds to the score  
in standard units :

$$z = \frac{125 - 100}{16} = \frac{25}{16} = 1.5625$$

$$1 - \text{pnorm}(1.5625) = 0.05908512$$

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$$1 - \text{pnorm}(125, \text{mean}=100, \text{sd}=16) = 0.05908512$$

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## Problem Set # 6

The Normal Approximation to the Binomial.For  $Y \sim \text{Binomial}(n, p)$  we know that its probability mass function is:

$$p_Y(k) = \binom{n}{k} p^k (1-p)^{n-k} \quad \text{for } k = 0, 1, \dots, n$$

Moreover, its expectation and its variance are

$$\mathbb{E}[Y] = np \quad \text{and} \quad \text{Var}[Y] = np(1-p).$$

Now, consider a sequence of binomial random variables  $Y_n \sim \text{Binomial}(n, p)$ . Note that, while the number of trials  $n$  varies, the probability of success in every trial  $p$  remains the same for all  $n$ . The *normal approximation to the binomial* is a theorem which states that

$$\frac{Y_n - np}{\sqrt{np(1-p)}} \xrightarrow{\mathcal{D}} N(0, 1)$$

Practically, this means that  $Y_n$  is "approximately" normal with mean  $np$  and variance  $np(1-p)$  for "large"  $n$ . The usual rule of thumb is that both  $np > 10$  and  $n(1-p) > 10$ .

Another practical adjustment needs to be made due to the fact that discrete distributions of  $Y_n$  are approximated by a continuous (normal) distribution. This adjustment is usually referred to as the **continuity correction**. More specifically, provided that the conditions above are satisfied, for every integer  $a < b$ , we have that

$$\begin{aligned} \mathbb{P}[a \leq Y_n \leq b] &= \mathbb{P}\left[a - \frac{1}{2} < Y_n < b + \frac{1}{2}\right] \\ &= \mathbb{P}\left[\frac{a - \frac{1}{2} - np}{\sqrt{np(1-p)}} < \frac{Y_n - np}{\sqrt{np(1-p)}} < \frac{b + \frac{1}{2} - np}{\sqrt{np(1-p)}}\right] \approx \Phi\left(\frac{b + \frac{1}{2} - np}{\sqrt{np(1-p)}}\right) - \Phi\left(\frac{a - \frac{1}{2} - np}{\sqrt{np(1-p)}}\right) \end{aligned}$$

where  $\Phi$ , as usual, stands for the cumulative distribution function of the standard normal distribution.

For more about the history of the theorem and ideas for its proof, go to: [Wikipedia: de Moivre-Laplace](#).

**Problem 6.1.** A student takes an exam with 200 TRUE/FALSE questions. Shirley knows the correct answer to exactly 100 questions. For the remaining questions, she guesses at random. The passing mark is 130 correct answers. What is the (approximate) probability she passes the exam?

$Y \dots \# \text{ of correct guesses}$

$Y \sim \text{Binomial}(n=100, p=0.5)$

$$\mathbb{P}[Y \geq 36] = ? \Rightarrow \mathbb{P}[Y \geq 36] = 1 - \overbrace{\mathbb{P}[Y \leq 35]}^{\text{pbinom}(35, 100, 0.5)}$$

- $n \cdot p = 50 > 10$  and  $n(1-p) = 50 > 10$

$$\mathbb{P}[Y \geq 36] = 1 - \Phi\left(\frac{35.5 - 50}{5}\right) = 1 - \Phi(-2.9)$$

$$\text{SD}[Y] = \sqrt{n \cdot p \cdot (1-p)} = \sqrt{100 \cdot 0.5 \cdot 0.5} = \sqrt{25} = 5$$

Std normal tables:  $1 - 0.0019 = 0.9981$

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$$1 - \text{pnorm}(-2.9) = 0.9981342 \text{ or}$$

$$1 - \text{pbinom}(35, 100, 0.5) = 0.9982412$$

**Problem 6.2.** A (hypothetical) 13-sided die is thrown 169 times (each of the numbers 1, 2, ..., 13 is equally likely on each throw). Every time 5 or a larger number is obtained, the player wins a candy bar. What is the probability that the player will receive at least 100 candy bars? (Use the normal approximation and note that  $169 = 13 \times 13$ ).

$$Y \sim \text{Binomial}(n=169, p = \frac{9}{13})$$

$$n \cdot p = 13 \cdot \frac{9}{13} = 117 > 10 \quad 13 \cdot 4 = 42 > 10 \quad \checkmark$$

$$n \cdot p \cdot q = 117 \cdot \frac{4}{13} = 36 \Rightarrow \text{SD}[Y] = \sqrt{36} = 6$$

$$1 - \Phi\left(\frac{99.5 - 117}{6}\right) = 0.9982$$