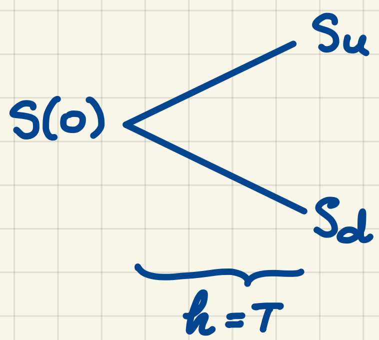
M339D: March 7th, 2025.

Binomial Option Pricing.



	e.g. $\boxed{\text{call}}$	Replicating P.
$V_u = v(S_u)$	$V_u = (S_u - K)_+$	$\Delta \cdot S_u + Be^{rh}$
$V_d = v(S_d)$	$V_d = (S_d - K)_+$	$\Delta \cdot S_d + Be^{rh}$

Replicating Portfolio: $\begin{cases} \Delta & \text{shares of stock} \\ B & \text{@ the ccrfir} \end{cases}$

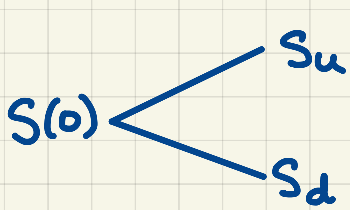
... solve for Δ and B ...

$$\boxed{\Delta = \frac{V_u - V_d}{S_u - S_d}} \quad \text{and} \quad \boxed{B = e^{-rh} \cdot \frac{u \cdot V_d - d \cdot V_u}{u - d}}$$

$\Rightarrow$ Time·0 price of the option is $\boxed{V(0) = \Delta \cdot S(0) + B}$

Graphical Interpretation.

Consider a European call w/ exercise date @ the end of the tree and the strike price K such that $\boxed{S_d < K < S_u}$



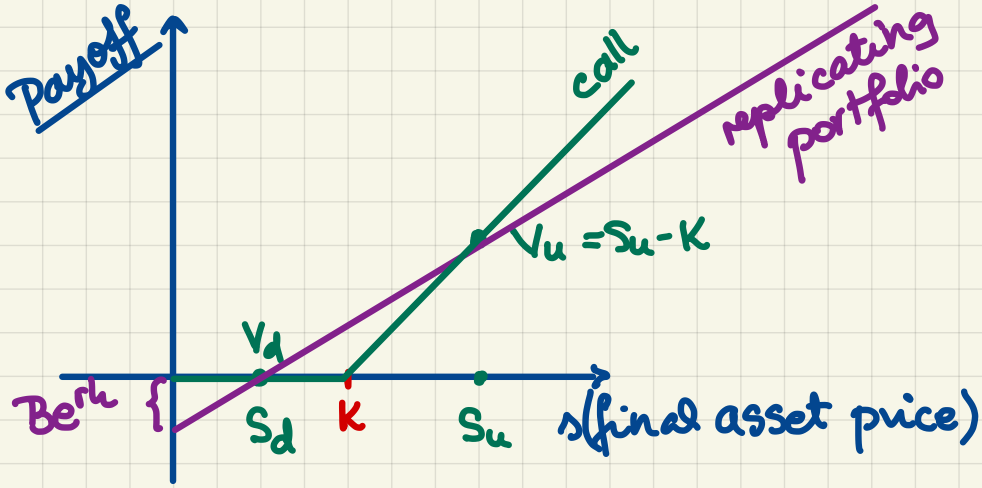
PAYOFF

$V_u = S_u - K$

$V_d = 0$

In the replicating portfolio:

- $\Delta_c = \frac{V_u - V_d}{S_u - S_d} = \frac{S_u - K}{S_u - S_d} \in (0,1)$ Buy a fraction of a stock!
- $B_c = e^{-rh} \cdot \frac{u \cdot V_d - d V_u}{u - d} = -e^{-rh} \cdot \frac{d(S_u - K)}{u - d} < 0$ Borrowing!



<u>slope</u>: $\Delta \in (0,1)$

$\Rightarrow$ buying a fraction of a <u>share</u>

<u>intercept</u> <0

$\Rightarrow$ borrowing <u>cash</u>

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Problem Set #9

Binomial option pricing.

Problem 9.1. In the setting of the one-period binomial model, denote by i the **effective** interest rate **per period**. Let u denote the "up factor" and let d denote the "down factor" in the stock-price model.
If

$$d < u \leq 1 + i$$

then there certainly is no possibility for arbitrage.

T/F ?

No-arbitrage condition

$$d < e^{rh} < u$$

$$d < 1+i < u$$

This is the fixed statement!

Problem 9.2. In our usual notation, does this parameter choice create a binomial model with an arbitrage opportunity?

$u = 1.18$, $d = 0.87$, $r = 0.05$, $\cancel{\delta = 0}$, $h = 1/4$

No, it doesn't :)

→: Check: $d = 0.87 < \underbrace{e^{rh} = e^{0.05/4}}_{e^{0.0125}} < u = 1.18$

≈ 1.0125

Taylor Expansion of e^x

$$\boxed{e^x = 1 + x} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$$

Q: What if $\tilde{d} = 1.01$?

This is still ok!

Problem 9.3. Let the continuously compounded risk-free interest rate be equal to 0.04. Consider a one-period binomial tree with every period of length one year used to model the stock price of a stock whose current price is \$80 per share. In the model, it is assumed that the stock price can either go up by \$5 or down by \$4.

You use the binomial tree to construct a replicating portfolio for a 78-strike call option on the above stock. What is the stock investment in the replicating portfolio?

$\rightarrow$:

$$\begin{array}{lll} & S_u = 85 & \\ S(0) & & \\ & S_d = 76 & \end{array}$$

Payoff

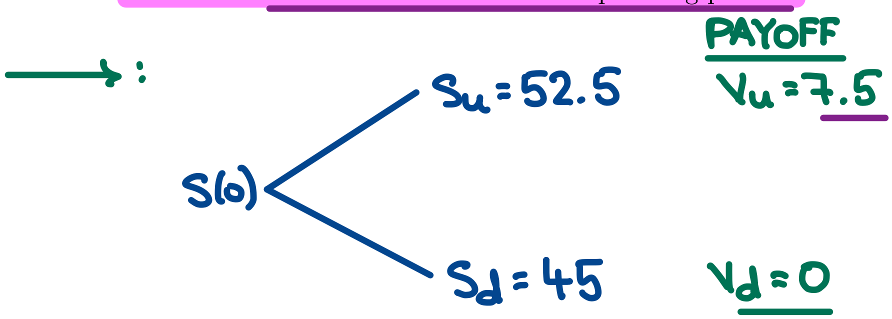
$$V_u = (85-78)_+ = 7$$

$$V_d = (76-78)_+ = 0$$

$$\Delta = \frac{V_u - V_d}{S_u - S_d} = \frac{7}{85-76} = \frac{7}{9} \quad \square$$

Problem 9.4. Let the continuously compounded risk-free interest rate be equal to 0.04. Consider a one-period binomial tree with every period of length one year used to model the stock price of a non-dividend-paying stock whose current price is \$50 per share. In the model, it is assumed that the stock price can either go up by 5% or down by 10%.

You use the binomial tree to construct a replicating portfolio for a 45−strike call on the above stock. What is the risk-free investment in the replicating portfolio?



$$B = e^{-rh} \cdot \frac{u \cdot V_d - d \cdot V_u}{u-d} = e^{-0.04} \cdot \frac{1.05 \cdot (0) - 0.90 \cdot 7.5}{1.05 - 0.90}$$

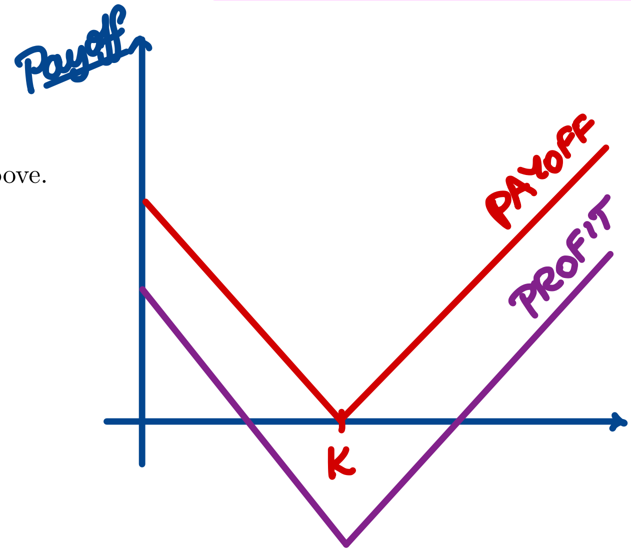
$$B = - e^{-0.04} \cdot \frac{6.75}{0.15} \quad \ldots.$$

Problem 9.5. Consider a non-dividend paying stock whose current price is \$95 per share. You model the evolution of this stock price over the following year using a one-period binomial tree under the assumption that the stock price can be either \$120, or \$75 in one year.

The continuously compounded risk-free interest rate is 0.06.

A **straddle** consists of a long call and a long otherwise identical put. Consider a \$100-strike, one-year European **straddle** on the above stock. What is the straddle's price consistent with the above stock-price model?

(a) About \$10
(b) About \$10.83
(c) About \$15.45
(d) About \$20.84
(e) None of the above.



Payoff f'tion of a straddle

$v(s) = |s - K|$

In this problem

$\boxed{K = 100}$