UNIVERSITY OF TEXAS AT AUSTIN

Lecture 4

The Inverse Transformation (Simulation) Method

Proposition 4.1. *Let X be a continuous random variable with the cumulative distribution function F_X and probability density function f_X.*
Assume that $f(x) > 0$ for all positive x and zero elsewhere.
Define $Y = F_X(X)$.
Then, $Y \sim U(0,1)$.

Proof: The support of Y will be contained in $[0,1]$. Let $y \in (0,1)$. Then,

$$F_Y(y) = \mathbb{P}[Y \leq y] = \mathbb{P}[F_X(X) \; ley].$$

By the Fundamental Theorem of Calculus

$$F_X(a) = \int_0^a f_X(x)\,dx$$

We assumed that $f(x) > 0$ for all positive x. So, F_X is a **strictly increasing function** on $\mathbb{R}_+$. Hence, F_X is one-to-one on $\mathbb{R}_+$. Thus, its inverse function F_X^{-1} exists and it is also increasing. Therefore,

$$F_Y(y) = \mathbb{P}[F_X(X) \leq y] = \mathbb{P}[X \leq F_X^{-1}(y)] = F_X(F_X^{-1}(y)) = y.$$

We conclude that $Y \sim U(0,1)$.

Proposition 4.2. *Let $U \sim U(0,1)$ and let F be a cumulative distribution function.*
Define $X = F^{-1}(U)$.
Then, the random variable X has the cumulative distribution function F.

Proof: See Section 6.1.2 in the textbook.

An Informal Implementation.

1. Set F to be the cdf of the distribution from which we want to simulate values. "Figure out" F^{-1}; this can be analytic or numerical.
2. Draw the simulated values from the unit uniform $U(0,1)$:
$$u_1, u_2, \ldots, u_n$$
3. Apply F^{-1} to the simulated values to obtain
$$x_1 = F^{-1}(u_1), x_2 = F^{-1}(u_2), \ldots, x_n = F^{-1}(u_n)$$
The $x_1, x_2, \ldots, x_n$ are the simulated values from your target distribution.

Example 4.3. In the exponential case $X \sim \mathit{Exponential}(\theta)$, we have already obtained the analytic expression for the quantile function F_X^{-1}. It is

$$F_X^{-1}(y) = -\theta \ln(1-y)$$

So, with $\{u_i, i = 1, \ldots, n\}$ generated from the unit uniform, the x_i defined as

$$-\theta \ln(1-u_1) \quad \text{for } i = 1, \ldots, n$$

will be simulated values from the exponential distribution with parameter θ.