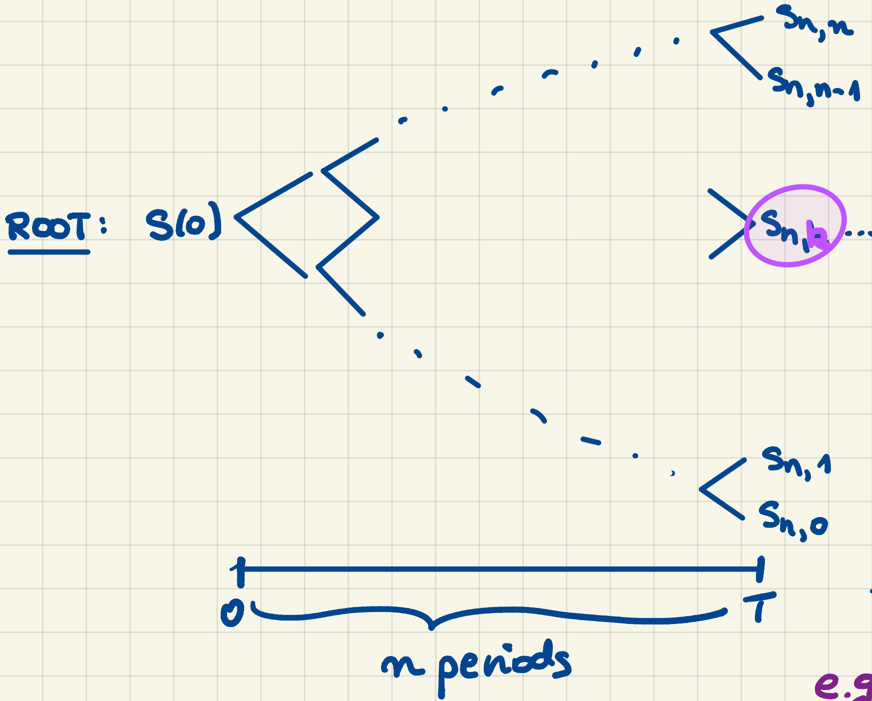
M339D: November 8th, 2024.

The Pre-Limit n-Period Binomial Tree.



$S_{n,k}$ ... the stock price resulting from k upsteps.

$h_n = \frac{T}{n}$

u_n ... up factor

d_n ... down factor

e.g., in the forward tree

$$u_n = \exp\left(r \cdot \frac{T}{n} + \sigma\sqrt{\frac{T}{n}}\right)$$

$$d_n = \exp\left(r \cdot \frac{T}{n} - \sigma\sqrt{\frac{T}{n}}\right)$$

$$S_{n,k} = S(0) \cdot u_n^k \cdot d_n^{n-k} = S(0)\left(\frac{u_n}{d_n}\right)^k d_n^n$$

k ... corresponds to a realization of the binomial random variable w/ n trials and success probability

$$p_n^* = \frac{e^{r(T/n)} - d_n}{u_n - d_n}$$

e.g., in the forward tree

$$\boxed{p_n^* = \frac{1}{1 + e^{\sigma\sqrt{T/n}}}} \xrightarrow[n \to \infty]{} \frac{1}{2}$$

Say, X_n ... # of upsteps in n periods

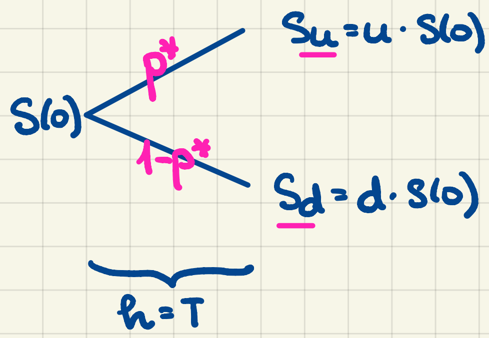
$$\boxed{X_n \sim Binomial(\#\text{ of trials} = n,\ \text{success prob} = p_n^*)}$$

Q : Can we simply use the normal approximation to the binomial?

$\boxed{\text{Nope! } p_n^* \text{ varies w/}}$

Subjective Probability.

When pricing, we use the $\mathbb{P}^*$ risk-neutral probability measure.



$$\boxed{p^* = \frac{e^{rh} - d}{u - d}}$$

Q : If we invest in one share of non-dividend-paying stock @ time 0, what is our expected wealth @ time T, under the risk-neutral probability measure?

$\longrightarrow$:

$$\mathbb{E}^*[S(T)] = S_u \cdot p^* + S_d \cdot (1-p^*)$$

$$= S_u \cdot \frac{e^{rh} - d}{u-d} + S_d \cdot \frac{u - e^{rh}}{u-d} =$$

$$= \frac{1}{u-d}\left(S_u \cdot e^{rh} - \cancel{d \cdot S_u} + \cancel{S_d \cdot u} - S_d e^{rh}\right)$$

$$= \frac{1}{\cancel{u-d}} \cdot e^{rh} \cdot S(0)\left(\cancel{u-d}\right) = S(0)e^{rh} = S(0)e^{rT}$$

In Contrast:

There can be a subjective probability measure $\mathbb{P}$

We can think about the quality of our investment under that probability measure, i.e.,

$$\mathbb{E}[S(T)] = S(0)e^{\alpha \cdot T}$$

We refer to α as the mean rate of return.

In a binomial tree, we can talk about the "true" probability of a step up

$$\boxed{p = \frac{e^{\alpha h} - d}{u - d}}$$