University of Texas at Austin

Lecture 3

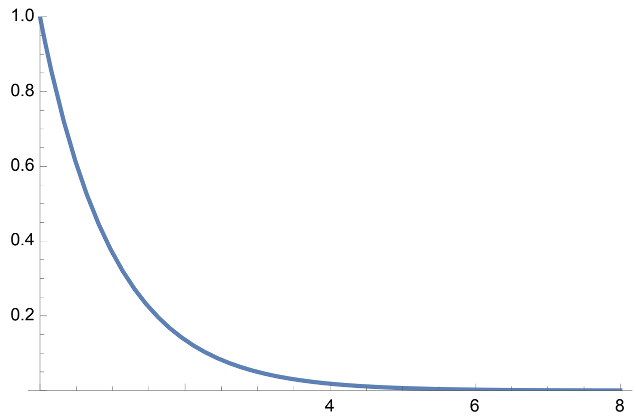
The Exponential Distribution

An **exponential** random variable X with parameter θ has the *probability density function* given by

$$f_X(x) = \frac{1}{\theta}e^{-\frac{x}{\theta}} \quad \text{for } x > 0.$$

We write $X \sim Exponential(\theta)$.

The graph of the probability density function of an exponential random variable with parameter $\theta = 1$ is shown below.



Remark 3.1. We choose the parameterization above because we are focused on modeling the *time* until some event of interest happens or we are interested in the extent of a loss.

In other sources, one might be emphasizing the *rate* at which some events of interest occur. There, you would encounter the parameterization with $\lambda = \frac{1}{\theta}$. So, the probability density function would be expressed as

$$f_X(x) = \lambda e^{-\lambda x} \quad \text{for } x > 0.$$

The support of the exponential distribution is $[0, \infty)$.

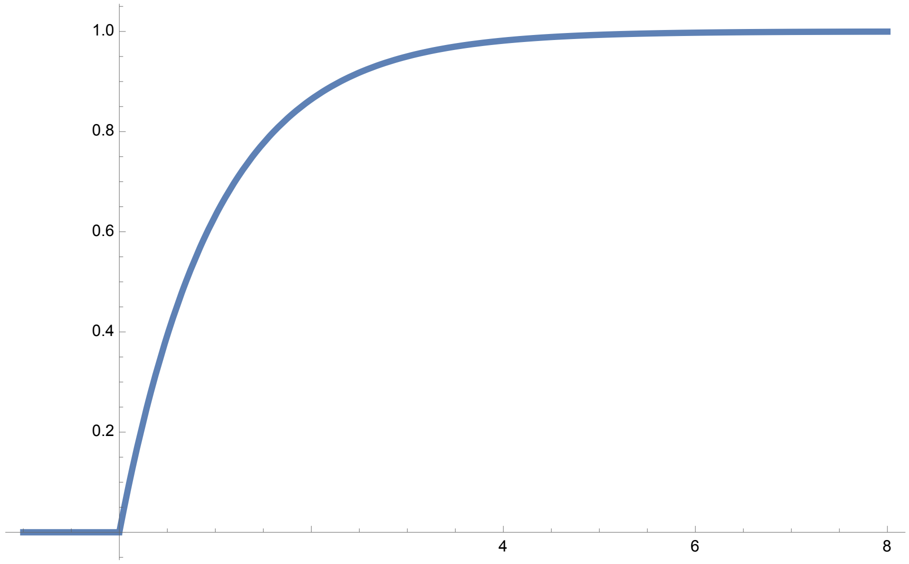
The *cumulative distribution function* is

$$F_X(x) = 1 - e^{-\frac{x}{\theta}} \quad \text{for } x > 0.$$

The *survival function* is

$$S_X(x) = e^{-\frac{x}{\theta}} \quad \text{for } x > 0.$$

The graph of the cumulative distribution function of an exponential random variable with parameter $\theta = 1$ is shown below.



Proposition 3.2. **Memoryless property.**
Let $X \sim Exponential(\theta)$. *For* $a, b > 0$, *we have*

$$\mathbb{P}[X > a + b \,|\, X > a] = \mathbb{P}[X > b].$$

Proof.

$$\begin{aligned}
\mathbb{P}[X > a + b \,|\, X > a] &= \frac{\mathbb{P}[X > a + b, X > a]}{\mathbb{P}[X > a]} \quad \text{(by definition of \textbf{conditional probability})}\\
&= \frac{\mathbb{P}[X > a + b]}{\mathbb{P}[X > a]} \quad (\text{since } \{X > a + b\} \subseteq \{X > a\})\\
&= \frac{S_X(a+b)}{S_X(a)} \quad \text{(by the definition of \textbf{survival function})}\\
&= \frac{e^{-\frac{a+b}{\theta}}}{e^{-\frac{a}{\theta}}} \quad (\text{since } X \sim Exponential(\theta))\\
&= e^{-\frac{b}{\theta}} = \mathbb{P}[X > b]
\end{aligned}$$

□