M378K Introduction to Mathematical Statistics
Homework assignment #5

Please, provide your **final answer only** to the following problems.

Problem 5.1. *(5 points) Let $Z_1 \sim N(1,1)$, $Z_2 \sim N(2,2)$ and $Z_3 \sim N(3,3)$ be independent random variables. The distribution of the random variable $W = Z_1 + \frac{1}{2}Z_2 + \frac{1}{3}Z_3$ is …*

(a) $N(5/3, 7/6)$

(b) $N(3,3)$

(c) $N(3,\sqrt{3})$

(d) $N(3,\sqrt{5/3})$

(e) *None of the above*

(Note: *In our notation $N(\mu,\sigma)$ means normal with mean μ and* standard deviation σ.)

Solution: The correct answer is **(c)**. As a linear combination of independent normals, the random variable W is normally distributed itself. To compute its parameters, we compute its mean and its variance:

$$\mathbb{E}[W] = \mathbb{E}[Z_1] + \tfrac{1}{2}\mathbb{E}[Z_2] + \frac{1}{3}\mathbb{E}[Z_3] = 1 + \frac{1}{2}\times 2 + \frac{1}{3}\times 3 = 3,$$

$$\mathrm{Var}[W] = \mathrm{Var}[Z_1] + \frac{1}{4}Var[Z_2] + \frac{1}{9}\mathrm{Var}[Z_3] = 1 + \frac{1}{4}\times 4 + \frac{1}{9}\times 9 = 3.$$

Therefore $W \sim N(3,\sqrt{3})$.

Problem 5.2. *(5 points) Let $Y_1,\dots,Y_{100}$ be independent random variables with the Bernoulli $B(p)$ distribution, with $p = 0.2$ The best approximation to $\bar{Y} = \frac{1}{n}(Y_1 + \dots + Y_n)$ (among the offered answers) is*

(a) $N(0,1)$

(b) $N(100,20)$

(c) $N(0.2,0.04)$

(d) $N(20,4)$

(e) $N(20,20)$

(Note: *In our notation $N(\mu,\sigma)$ means normal with mean μ and* standard deviation σ.)

Solution: The correct answer is **(c)**.
The sum $W = Y_1 + \dots + Y_n$ is binomially distributed with mean $np = 20$ and variance $np(1-p) = 16$, i.e., standard deviation 4. It is well approximated by a normal $N(20,4)$. Since $\bar{Y} = \frac{1}{n}W$, its best approximation will a normal with mean $\frac{1}{100}20 = 0.2$ and standard deviation $\sigma = \frac{1}{100}4 = 0.04$.

Problem 5.3. *(5 points) Use the uniqueness of moment-generating functions to give the distribution of a random variable Y with moment-generating function $m_Y(t) = (0.7e^t + 0.3)^3$.*

(a) $Y \sim b(3, 0.7)$

(b) $Y \sim b(3, 0.3)$

(c) $Y \sim B(0.7)$

(d) $Y \sim P(0.7)$

(e) None of the above.

Solution: The correct answer is **(a)**.
See *Example 6.2.2* from the lecture notes.

Please, provide your **complete solutions** to the following problems. Final answers only, even if correct will earn zero points for those problems.

Problem 5.4. *(10 points) The moment generating function of a certain random variable Y is given to be equal to*

$$m_Y(t) = (1 - 2500t)^{-4}.$$

Calculate the standard deviation of the random variable Y.

Solution: The mean of the given distribution is

$$\mathbb{E}[Y] = m'_Y(0)$$

with

$$m'_Y(t) = -4(1 - 2500t)^{-5} \times (-2500) = 10000(1 - 2500t)^{-5}.$$

So, $\mathbb{E}[Y] = 10,000$.
The second raw moment of the given distribution is

$$\mathbb{E}[Y^2] = m''_Y(0)$$

with

$$m''_Y(t) = 10000 \times (-5)(1 - 2500t)^{-6}(-2500) = 125 \times 10^6(1 - 2500t)^{-6}.$$

So, $\mathbb{E}[Y^2] = 125 \times 10^6$.
Hence,

$$Var[Y] = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2 = 125 \times 10^6 - 100 \times 10^6 = 25 \times 10^6 \quad \Rightarrow \quad \sigma_Y = 5,000.$$

Problem 5.5. *(10 points) Let Y be a geometric random variable with parameter p. What is its moment generating function m_Y? Do not forget to explicitly state the domain of m_Y!*

Solution: See *Example 6.1.23* from the lecture notes.

Problem 5.6. *(15 points) Let $Y \sim E(\tau)$. Find the moment generating function on Y not forgetting to explicitly state the domain. Using the moment generating function, recalculate the mean and the variance of the random variable Y.*

Solution: We are given that $Y \sim E(\tau)$. Then,

$$\begin{aligned} m_Y(t) = \mathbb{E}[e^{tY}] &= \int_0^\infty e^{ty} \frac{1}{\tau} e^{-y/\tau}\, dy \\ &= \frac{1}{\tau} \int_0^\infty e^{(t-\frac{1}{\tau})y}\, dy. \end{aligned}$$

The indefinite integral converges only if $t < \frac{1}{\tau}$. In that case, we get

$$m_Y(t) == \frac{1}{\tau} \times \frac{1}{t - \frac{1}{\tau}} e^{(t-\frac{1}{\tau})y} \Big]_{y=0}^{\infty} = \frac{1}{1 - t\tau}.$$

As for the moments, we have

$$m_Y'(t) = -\frac{-\tau}{(1-t\tau)^2} \quad \Rightarrow \quad \mathbb{E}[Y] = m_Y'(0) = \tau$$

$$m_Y''(t) = -\frac{-2\tau^2}{(1-t\tau)^2} \quad \Rightarrow \quad \mathbb{E}[Y^2] = m_Y''(0) = 2\tau^2 \quad \Rightarrow \quad Var[Y] = \tau^2$$