UNIVERSITY OF TEXAS AT AUSTIN

HW Assignment 3

Prerequisite material. Subjective expectations.

Provide your **complete solution** to the following problems. Final answers only, without appropriate justification, will receive zero points even if correct.

Problem 3.1. (5 points) In the setting of the one-period binomial model, denote by i the **effective** interest rate **per period**. Let u denote the "up factor" and let d denote the "down factor" in the stock-price model for a non-dividend-paying stock. Which of the following statements is the correct no-arbitrage condition for the binomial asset-pricing model?

(a) $d < 1+i < u$
(b) $d < 1 < u$
(c) $d < e^i < u$
(d) $d = \frac{i}{1+i}$
(e) None of the above.

Solution: (a)

Problem 3.2. (5 points) The current price of a continuous-dividend paying stock is \$100. Its dividend yield is 0.02. Its evolution over the following year is modeled using a four-period binomial tree under the assumption that the price can increase by 1% or decrease by 0.5% over each period.

The continuously compounded, risk-free interest rate is 0.02.

What is the risk-neutral probability of the stock price going up in a single period?

Solution: The length of every period is $h = 1/4$. In our usual notation, we have that the definition of the risk-neutral probability reads as

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{1-0.995}{1.01-0.995} = 1/3.$$

Problem 3.3. (5 points) The evolution of a market index over the following year is modeled using a four-period binomial tree. We are given that the current value of the market index equals \$144, that its volatility equals 0.25, and that it pays dividends continuously.

You are tasked with constructing a four-period forward tree for the evolution over the following year of the forward price of the above market index with delivery at time-2.

What is the down factor d_F in the forward price tree for the futures prices on the stock?

Solution: In our usual notation,

$$d_F = de^{-(r-\delta)h} = e^{-\sigma\sqrt{h}} = e^{-0.25\sqrt{1/4}} = 0.8825.$$

Problem 3.4. (5 points) The current exchange rate is given to be \$1.25 per Euro and its volatility is given to be 0.15.

The continuously-compounded, risk-free interest rate for the US dollar is 0.03, while the continuously-compounded, risk-free interest rate for the Euro equals 0.06.

The evolution of the exchange rate over the following nine-month period is modeled using a three-period forward binomial tree.

What is the value of the so-called down factor in the above tree?

Solution: In the forward binomial tree, the up and down factors are given as

$$u = e^{(0.03-0.06)\times 0.25+0.15\times\sqrt{0.25}} = 1.0698$$
$$d = e^{(0.03-0.06)\times 0.25-0.15\times\sqrt{0.25}} = 0.9208.$$

Problem 3.5. (5 points) The current price of a continuous-dividend-paying stock is \$80 per share. The stock's dividend yield is 0.02. According to your model, the expected value of the stock price in two years is \$90 per share. You are also given:

The risk-free interest rate exceeds the dividend yield.

The two-year forward price on a share of this stock is denoted by F. At this price you are willing to enter into the forward. What is the smallest range of values F can take according to the above information?

Solution: Using the fact that the investor is willing to enter a forward contract, we conclude that the forward contract's profit is positive. So,

$$\mathbb{E}[S(T)] > F \quad \Rightarrow \quad 90 > F$$

On the other hand, we know that

$$F = S(0)e^{(r-\delta)T} = 80e^{2(r-0.02)} > 80.$$

So, the most we can say about F is that $80 < F < 90$.

Problem 3.6. (5 points) The current price of a non-dividend-paying stock is \$100 per share. According to your model, the expected value of the stock price in two years is \$90 per share. You are also given:

The risk-free interest rate is strictly positive.

The two-year forward price on a share of this stock is denoted by F. At this price you are willing to short the forward contract. What is the smallest range of values F can take according to the above information?

Solution: Using the fact that the investor is willing to enter a **short** forward contract, we conclude that the **short** forward contract's profit is positive. So,

$$\mathbb{E}[S(T)] < F \quad \Rightarrow \quad 90 < F$$

On the other hand, we know that

$$F = S(0)e^{(r-\delta)T} = 100e^{2r} > 100.$$

So, the most we can say about F is that $F > 100$.

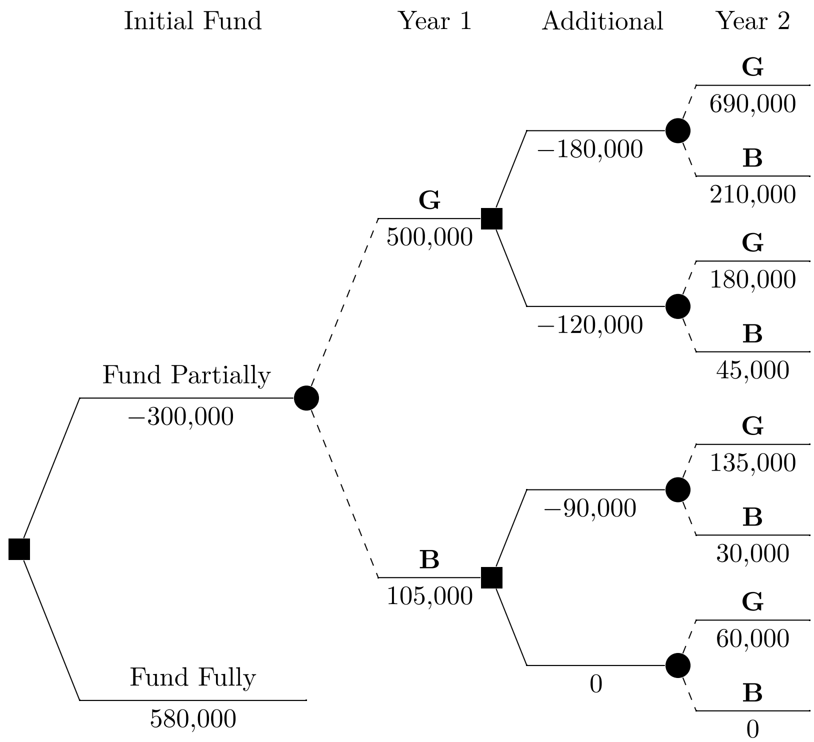
Problem 3.7. (15 points) Netflix is considering a cartoon series. When the production of two seasons is fully funded at time-0 the project has a net present value of 580,000.

The decision tree below shows the cash flows of the series when the promotion at the beginning of the Year 1 (i.e., at $t = 0$) is only partial with an option to provide different amounts of funding at the beginning of Year 2 (i.e., at $t = 1$) depending on how well the first season did.

This tree reflects two possible receptions of the two seasons at each information node (**G** = good, **B** = bad). The probability of the series being a success is given to be 1/2 and the probability of it being merely watchable is 1/2.

Assume the interest rate is 0%.

Find the **initial** (i.e., at $t = 0$) value of the option to fund partially.



Solution: As usual, when pricing options, we are moving backwards through the tree.

- In the *uppermost final* information node, the possible cashflows are 690,000 with probability 1/2 and 210,000 with probability 1/2. So, the value of the project at that node equals

$$690000\left(\frac{1}{2}\right) + 210000\left(\frac{1}{2}\right) = 450000.$$

- In the *second-by-height final* information node, the possible cashflows are 180,000 with probability 1/2 and 45,000 with probability 1/2. So, the value of the project at that node equals

$$180000\left(\frac{1}{2}\right) + 45000\left(\frac{1}{2}\right) = 112500.$$

- In the *third-by-height final* information node, the possible cashflows are 135,000 with probability 1/2 and 30,000 with probability 1/2. So, the value of the project at that node equals

$$135000\left(\frac{1}{2}\right) + 30000\left(\frac{1}{2}\right) = 82500.$$

- In the *lowest final* information node, the possible cashflows are 60,000 with probability 1/2 and 0 with probability 1/2. So, the value of the project at that node equals

$$60000\left(\frac{1}{2}\right) = 30000.$$

We continue working backwards, at the **upper decision** node at the end of Year 1, we can go "up" or "down" in the tree.

- We go "up" by investing 180,000; combining this cashflow with the average revenue at the *uppermost final* node, we get the total effect of going "up" to be

$$450000 - 180000 = 270000.$$

- We go "down" by investing 120,000; combining this cashflow with the average revenue at the *second-by-height final* node, we get the total effect of going "down" to be

$$112500 - 120000 = -7500.$$

Comparing the two values we obtained, we conclude that the **optimal** decision at this node is to go "up" and we keep the value of this project at this node to be

$$270000 + 500000 = 770000.$$

Here, we took into account that the first season was a success resulting in 500,000 in revenue in Year 1.

Similarly, at the **lower decision** node at the end of Year 1, we can go "up" or "down" in the tree.

- We go "up" by investing 90,000; combining this cashflow with the average revenue at the *third-by-height final* node, we get the total effect of going "up" to be

$$82500 - 90000 = -7500.$$

- We go "down" by investing nothing; so, the total effect of going "down" is 30000. Comparing the two values we obtained, we conclude that the **optimal** decision at this node is to go "down" and we keep the value of this project at this node to be

$$30000 + 105000 = 135000.$$

  Here, we took into account that the first season was "meh" resulting in 105,000 in revenue in Year 1.

  Altogether, at the information node corresponding to Year 1, we have that the expected value of the project is

$$770000\left(\frac{1}{2}\right) + 135000\left(\frac{1}{2}\right) = 452500.$$

Now, we take into account that we funded the series partially with 300,000. So, the total expected present value of the cashflows we get should we decide to fund partially is

$$452500 - 300000 = 152500$$

The total value of the option is

$$152500 - 580000 = -427500.$$

Problem 3.8. (5 points) A certain type of lightbulb has a useful life that is normally distributed with mean 4 years and standard deviation of a year. The useful lives of different lightbulbs are independent. Calculate the probability that the total useful life of two randomly selected lightbulbs exceeds 1.2 times the useful life of a third randomly selected light bulb.

Solution: Let $X_i, i = 1, 2, 3$ be the useful lifetimes of the three lightbulbs. We are looking for

$$\mathbb{P}[X_1 + X_2 > 1.2X_3] = \mathbb{P}[X_1 + X_2 - 1.2X_3 > 0].$$

We are given that, for $i = 1, 2, 3$,

$$X_i \sim N(mean = 4, sd = 1).$$

So, since $X_i, i = 1, 2, 3$ are independent,

$$Y := X_1 + X_2 - 1.2X_3 \sim N(mean = 3.2, var = 3.44).$$

Finally,

$$\mathbb{P}[Y > 0] = \mathbb{P}\left[\frac{Y - 3.2}{\sqrt{3.44}} > \frac{0 - 3.2}{\sqrt{3.44}} = -1.73\right] = N(1.73) = 0.9582.$$