Def'n. $Y \ldots$ r.v.

$m_Y(t) := \mathbb{E}[e^{t \cdot Y}]$ where the expectation exists.

Theorem 9.8. *If m_Y exists, then for $k \in \mathbb{N}$, we have*

$$m_Y^{(k)}(0) = \mu_k.$$

Example 9.9. *Let $Y \sim b(n=1,p)$, i.e., let Y model a Bernoulli trial with the probability of success denoted by p. Find m_Y.*

$\longrightarrow$: $m_Y(t) = \mathbb{E}[e^{tY}] = e^{t \cdot 0}(1-p) + e^{t \cdot 1} \cdot p$

$= \underline{1-p+pe^t}$ $\quad t \in \mathbb{R}$

Proposition 9.10. *Let Y_1 and Y_2 be $\boxed{\text{independent}}$ random variables with m.g.f.s denoted by m_{Y_1} and m_{Y_2}. Define $Y = Y_1 + Y_2$. Then, for every t for which both m_{Y_1} and m_{Y_2} are well defined, we have*

$$m_Y(t) =$$

Proof. By definition:

$$m_Y(t) = \mathbb{E}[e^{tY}]$$

Using $Y = Y_1 + Y_2$, we can subsitute $Y_1 + Y_2$ for Y in the expression above. So,

$$m_Y(t) = \mathbb{E}[e^{t(Y_1+Y_2)}]$$

$$= \mathbb{E}[e^{tY_1 + tY_2}]$$

One of the properties of the exponential function is that $e^{A+B} = e^A \times e^B$. Thus, the above becomes:

$$m_Y(t) = \mathbb{E}[e^{tY_1} \cdot e^{tY_2}]$$

Recall that Y_1 and Y_2 are assumed to be independent random variables. With this in mind, we get:

$$m_Y(t) = \mathbb{E}[e^{tY_1}] \cdot \mathbb{E}[e^{tY_2}]$$

Finally, using the definition of a m.g.f., we have

$$\boxed{m_Y(t) = m_{Y_1}(t) \cdot m_{Y_2}(t)}$$

$\square$

Example 9.11. *Let $Y \sim b(n, p)$. What is the moment generating function of Y?*

$\longrightarrow$: $m_Y(t) = ?$

$Y = I_1 + I_2 + \dots + I_n$ w/ I_j, $j = 1..n$, all $B(p)$ and independent

$$m_Y(t) = m_{I_1}(t) \cdot m_{I_2}(t) \cdots m_{I_n}(t) \underset{\text{identically dist'd}}{=} (m_{I_1}(t))^n = (1 - p + pe^t)^n$$

$(1 + p(e^t - 1))^n$

Example 9.12. *Let $N \sim Poisson(\lambda)$. What is the moment generating function m_N of N?*

$\longrightarrow$: $m_N(t) = \mathbb{E}[e^{t \cdot N}] = \sum_{n=0}^{\infty} e^{t \cdot n} \cdot p_N(n) = \sum_{n=0}^{\infty} \left(e^{t \cdot n} \cdot e^{-\lambda} \cdot \frac{\lambda^n}{n!}\right)$

$$= e^{-\lambda} \sum_{n=0}^{\infty} \frac{e^{tn} \lambda^n}{n!} = e^{-\lambda} \cdot \sum_{n=0}^{\infty} \frac{(e^t \cdot \lambda)^n}{n!} = e^{-\lambda} \cdot e^{e^t \cdot \lambda} = e^{\lambda(e^t - 1)}$$

Example 9.13. *Let $Z \sim N(0, 1)$. What is the moment generating function m_Z of Z?*

$\longrightarrow$: $m_Z(t) = \mathbb{E}[e^{t \cdot Z}] = \int_{-\infty}^{\infty} e^{t \cdot z} \cdot \varphi(z)\, dz = \int_{-\infty}^{\infty} e^{t \cdot z} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\, dz =$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2} + t \cdot z - \frac{t^2}{2}} e^{\frac{t^2}{2}}\, dz = e^{t^2/2} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-t)^2}{2}}\, dz}_{=1}$$

density of $N(t, \sigma = 1)$

$$\boxed{m_Z(t) = e^{t^2/2}}$$

Example 9.14. *Let the random variable Y have the mgf m_Y. Define $X = aY + b$ for some constants a and b. Express the mgf m_X of X in terms of m_Y, a and b.*

$\rightarrow$:
$$m_X(t) = \mathbb{E}[e^{tX}] = \mathbb{E}[e^{t(aY+b)}]$$
$$= \mathbb{E}[e^{taY} \cdot e^{tb}] = e^{tb}\,\mathbb{E}[e^{taY}] = e^{tb} \cdot m_Y(ta)$$
$\square$

Example 9.15. *Let $X \sim N(\mu, \sigma^2)$. What is the moment generating function m_X of X?*

$\rightarrow$:
$$X = \underbrace{\mu}_{b} + \underbrace{\sigma}_{a} \cdot Z \quad \text{w/ } Z \sim N(0,1)$$
$$m_X(t) = e^{t\mu} \cdot m_Z(t\sigma) = e^{t\mu} \cdot e^{\frac{t^2\sigma^2}{2}} = e^{t\mu + \frac{t^2\sigma^2}{2}}$$
$\square$

Problem 9.2. *A random variable Y is said to be* lognormal *if there exists a normally distributed random variable $X \sim N(\mu, \sigma^2)$ such that $Y \stackrel{(d)}{=} e^X$. Express the mean and the variance of the lognormal r.v. Y in terms of the parameters μ and σ.*

$\rightarrow$:
$$\mathbb{E}[Y] = \mathbb{E}[e^X] = \mathbb{E}[e^{X \cdot 1}] = m_X(1) = e^{\mu + \frac{\sigma^2}{2}}$$
$$\mathbb{E}[Y^2] = \mathbb{E}[(e^X)^2] = \mathbb{E}[e^{2X}] = m_X(2) = e^{2\mu + \frac{2^2\sigma^2}{2}} = e^{2(\mu+\sigma^2)}$$
$$\mathrm{Var}[Y] = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2 = e^{2(\mu+\sigma^2)} - e^{2\mu+\sigma^2} = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1)$$
$\square$

Proposition 9.16. 1. *If m_Y exists for a certain probability distribution, then it is unique.*

2. *If m_{Y_1} and m_{Y_2} are equal on an interval, then $Y_1 \stackrel{(d)}{=} Y_2$.*

Corollary 9.17. *Let Y_1 and Y_2 be independent and normally distributed. Define $Y = Y_1 + Y_2$. Then, the distribution of Y is ...*

$\rightarrow$:
$$m_Y(t) \boxed{=} m_{Y_1}(t) \cdot m_{Y_2}(t) =$$
$$= e^{\mu_1 t + \frac{\sigma_1^2 t^2}{2}} \cdot e^{\mu_2 t + \frac{\sigma_2^2 t^2}{2}} =$$
$$= e^{\underbrace{(\mu_1+\mu_2)}_{\mu} t + \frac{1}{2}\underbrace{(\sigma_1^2+\sigma_2^2)}_{\sigma^2} t^2}$$

Proof. $\square$

$Y \sim \text{Normal}(\text{mean} = \mu_1 + \mu_2, \text{var} = \sigma_1^2 + \sigma_2^2)$ $\square$

Dist'ns of Y_1, Y_2 $\longrightarrow$ mgfs of Y_1 and Y_2 $\longrightarrow$ mgf of Y $\longrightarrow$ Dist'n of Y

Corollary 9.18. *Let N_1 and N_2 be independent and Poisson distributed. Define $N = N_1 + N_2$. Then, the distribution of N is …*

$\longrightarrow$: $m_N(t) = m_{N_1}(t) \cdot m_{N_2}(t) = e^{\lambda_1(e^t-1)} \cdot e^{\lambda_2(e^t-1)}$

$= e^{\underbrace{(\lambda_1+\lambda_2)}_{\lambda}(e^t-1)}$

Proof. □

$N \sim \text{Poisson}(\lambda = \lambda_1 + \lambda_2)$ □