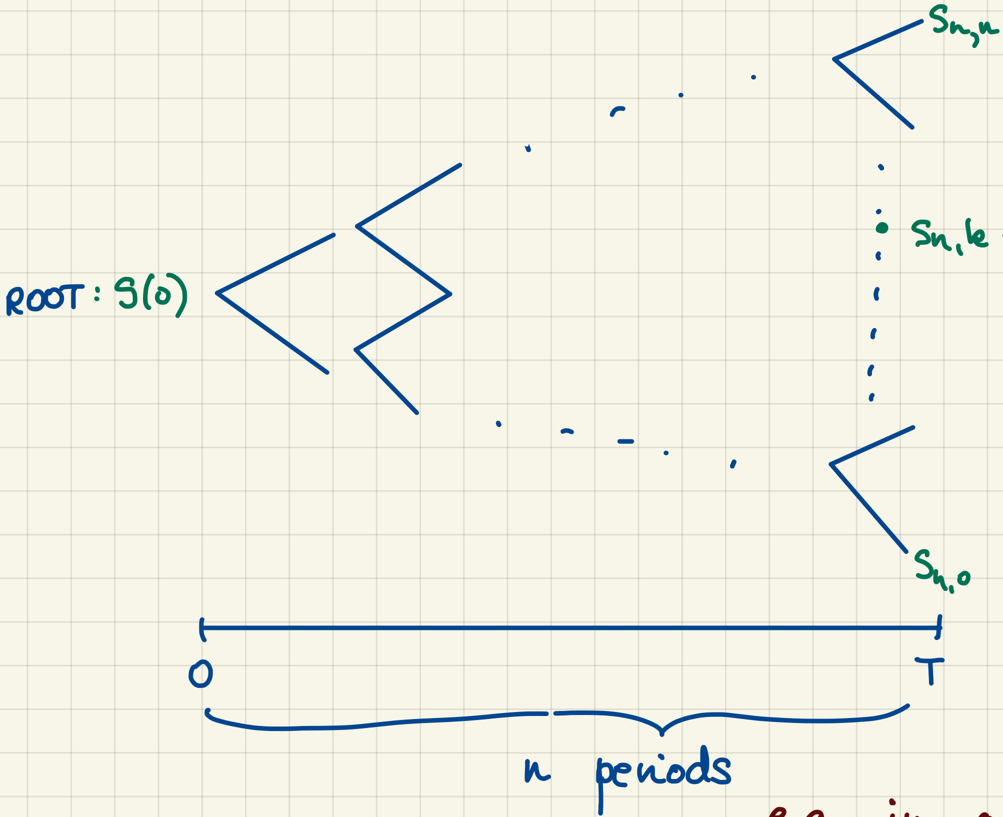
The Pre-Limit: n-period Binomial Tree.



$S_{n,k}$ ... the stock price resulting from k upsteps.

u_n ... up factor

d_n ... down factor

e.g., in a forward tree

$$u_n = e^{r(T/n) + \sigma\sqrt{T/n}}$$

$$d_n = e^{r(T/n) - \sigma\sqrt{T/n}}$$

$$S_{n,k} = S(0)\, u_n^k \cdot d_n^{n-k} = S(0) \left(\frac{u_n}{d_n}\right)^k \cdot d_n^n$$

k corresponds to a realization of the binomial distribution w/ n trials

and

p_n^* as the probability of success in every trial

$$p_n^* = \frac{e^{r(T/n)} - d_n}{u_n - d_n}$$

Say, X_n ... # of upsteps in n periods

Then, $X_n \sim$ Binomial(# of trials $= n$, probab of success $= p_n^*$)

$$S(T) = S(0) \left(\frac{u_n}{d_n}\right)^{X_n} \cdot d_n^n$$

de Moivre-Laplace has fixed

Black-Scholes Model

... lognormal stock prices

Temporarily fix a time horizon T.

$S(T)$ ... time-T stock price, a r.v.

$R(0,T)$ ... realized return over $(0,T)$

$$R(0,T) = \ln\left(\frac{S(T)}{S(0)}\right) \iff S(T) = S(0)e^{R(0,T)}$$

Market Model:

- RISKLESS ASSET w/ the ccrfir r
- RISKY ASSET : a nondividend paying stock w/ volatility σ

Under the risk-neutral probability $\mathbb{P}^*$:

$$R(0,T) \sim Normal\left(mean = \left(r - \frac{1}{2}\sigma^2\right)T,\ var = \sigma^2 \cdot T\right)$$

Say that $Z \sim N(0,1)$. Then, we can express $R(0,T)$ as

$$\boxed{R(0,T) = \left(r - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T} \cdot Z}$$ ✓

Hence, $$\boxed{S(T) = S(0)e^{\left(r - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T} \cdot Z}}$$ ✓✓

Note:

- $\mathbb{E}^*[S(T)] = S(0)e^{rT}$ ✓
- median: $S(0)\, e^{\left(r - \frac{1}{2}\sigma^2\right) \cdot T}$
- $$\frac{mean}{median} = e^{\frac{\sigma^2 \cdot T}{2}}$$