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UNIVERSITY OF TEXAS AT AUSTIN

Problem Set # 7

The Central Limit Theorem.

Let $\{X_n, n = 1, 2, 3, \dots\}$ be a sequence of independent identically distributed random variables such that $\mu_X = \mathbb{E}[X_1] < \infty$ and $Var[X] = \sigma_X^2 < \infty$. For every $n = 1, 2, \dots$ define

$$\bar{X}_n = \frac{X_1 + X_2 + \cdots + X_n}{n}.$$

sample mean

**Problem 7.1.** Find the expected value of $\bar{X}_n$ for every $n$.

$$\mathbb{E}[\bar{X}_n] = \mathbb{E}\left[\frac{1}{n}(X_1 + X_2 + \cdots + X_n)\right]$$
$$= \frac{1}{n}\mathbb{E}[X_1 + X_2 + \cdots + X_n]$$
$$= \frac{1}{n}(\mathbb{E}[X_1] + \mathbb{E}[X_2] + \cdots + \mathbb{E}[X_n])$$
$$= \frac{1}{\cancel{n}}(\cancel{n} \cdot \mu_X) = \mu_X$$

accuracy

**Problem 7.2.** Find the variance and standard deviation of $\bar{X}_n$ for every $n$.

$$Var[\bar{X}_n] = Var\left[\frac{1}{n}(X_1 + X_2 + \cdots + X_n)\right]$$
$$= \frac{1}{n^2} Var[X_1 + X_2 + \cdots + X_n]$$

independent

$$= \frac{1}{n^2}(Var[X_1] + Var[X_2] + \cdots + Var[X_n])$$
$$= \frac{1}{n^{\cancel{2}}} \cdot (\cancel{n} \cdot \sigma_X^2) = \frac{\sigma_X^2}{n}$$

precision

$$SD[\bar{X}] = \sigma_X / \sqrt{n}$$

**Theorem 7.1. The Central Limit Theorem (CLT).** *If the above conditions are satisfied, we have that*

$$\boxed{\frac{\bar{X}_n - \mu_X}{\frac{\sigma_X}{\sqrt{n}}}} \xrightarrow{\mathcal{D}} N(0,1) \quad as\ n \to \infty.$$

Practically, for "large enough" $n$, $\bar{X}_n$ is approximately normal with mean $\mu_X$ and variance $\frac{\sigma_X^2}{n}$. The rule of thumb is that we use the theorem for $n \geq 30$. If that is the case, we have that for any real $a < b$,

$$\mathbb{P}[a < \bar{X}_n \leq b] = \mathbb{P}\left[\frac{a - \mu_X}{\frac{\sigma_X}{\sqrt{n}}} < \boxed{\frac{\bar{X}_n - \mu_X}{\frac{\sigma_X}{\sqrt{n}}}} \leq \frac{b - \mu_X}{\frac{\sigma_X}{\sqrt{n}}}\right] \approx \Phi\left(\frac{b - \mu_X}{\frac{\sigma_X}{\sqrt{n}}}\right) - \Phi\left(\frac{a - \mu_X}{\frac{\sigma_X}{\sqrt{n}}}\right).$$

FTC