M339D: October 9th, 2023.

The Binomial Asset Pricing Model.

The No·Arbitrage Assumption.

Market Model.

- riskless asset: @ the ccrfir r
- risky asset: non·dividend·paying stock

Imagine investing in one share of this stock @ time·0:

		Wealth
$S(0)$	$S_u = u \cdot S(0)$	S_u
	$S_d = d \cdot S(0)$	S_d

$\underbrace{\qquad\qquad}_{h=T}$

At the risk·free rate $S(0)$ accumulates to $\underline{S(0)e^{rh}}$ @ time $T=h$

The No arbitrage Condition

$$S_d < S(0)e^{rh} < S_u$$

$$d \cdot \cancel{S(0)} < \cancel{S(0)}e^{rh} < u \cdot \cancel{S(0)}$$

$$\boxed{d < e^{rh} < u}$$

Half·a·Proof.

Say, to the contrary, $\underline{e^{rh} \leq d < u}$. ✗

Propose: Long one share of stock

Verify: Profit = Payoff − $FV_{0,T}$(Initial Cost) = $S(h) - S(0)e^{rh}$

In the down node: $S_d - S(0)e^{rh} = S(0) \cdot d - S(0)e^{rh} \geq 0$

In the up node: $S_u - S(0)e^{rh} = S(0) \cdot u - S(0)e^{rh} > 0$

Indeed, an arbitrage portfolio.

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Problem Set #8

Binomial option pricing.

Problem 8.1. In the setting of the one-period binomial model, denote by i the **effective** interest rate **per period**. Let u denote the "up factor" and let d denote the "down factor" in the stock-price model.
If

$$d < u \leq 1 + i$$

then there certainly is no possibility for arbitrage.

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Fixed statement: $\boxed{d < 1+i < u}$

Problem 8.2. In our usual notation, does this parameter choice create a binomial model with an arbitrage opportunity?

$u = 1.18$, $d = 0.87$, $r = 0.05$, $\cancel{\delta = 0}$, $h = 1/4$

→: $d = 0.87 \overset{?}{<} \underbrace{e^{rh} = e^{0.05(0.25)}}_{1.0125} \overset{?}{<} 1.18 = u$

Q: $\boxed{\tilde{d} = 1.01}$ still works.

Forward Binomial Trees.

$$\begin{tikzcd}[column sep=large] & S_u = u \cdot S(0) \\ S(0) \arrow[ur] \arrow[dr] & \\ & S_d = d \cdot S(0) \end{tikzcd}$$

$$\underbrace{\qquad\qquad}_{h = T}$$

$\boxed{u, d = ?}$

The no-arbitrage condition:

$$\boxed{d < e^{rh} < u}$$

"Def'n". The volatility σ is the standard deviation of realized returns on a continuously compounded scale and annualized.

Heuristics: $T = 1$

$$h_m = \frac{1}{m} \quad \text{(year)}$$

Q: What is the volatility for the time period of length h_m?
Call this volatility σ_{h_m}.

Realized Return: $R(t, t+s)$ satisfies

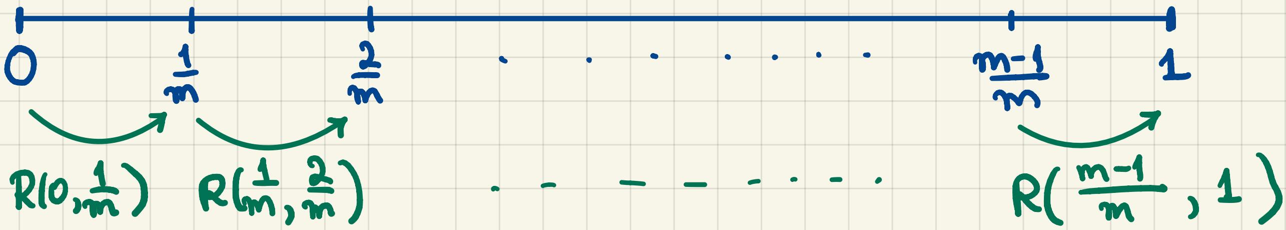
$$S(t+s) = S(t) e^{R(t,t+s)}$$

or, equivalently,

$$\boxed{R(t, t+s) = \ln\left(\frac{S(t+s)}{S(t)}\right)} \quad ✓$$

Compare to the simple return:

$$\frac{S(t+s) - S(t)}{S(t)} = \frac{S(t+s)}{S(t)} - 1$$

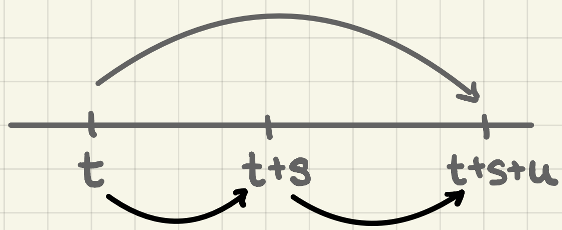


Note: $R\left(\frac{k-1}{m}, \frac{k}{m}\right)$ for $k = 1, 2, \dots, m$ are all random variables.

We make the following assumptions:

- all the returns above are identically distributed;
- the returns over disjoint intervals are independent.

We also know that the realized returns defined as above are additive, i.e.,



$$R(t, t+s+u) = \ln\left(\frac{S(t+s+u)}{S(t)}\right)$$

$$= \ln\left(\frac{S(t+s+u)}{S(t+s)} \cdot \frac{S(t+s)}{S(t)}\right)$$

$$= \boxed{\ln\left(\frac{S(t+s+u)}{S(t+s)}\right)} + \boxed{\ln\left(\frac{S(t+s)}{S(t)}\right)}$$

$$= R(t, t+s) + R(t+s, t+s+u)$$

$$\boxed{R\left(0,\frac{1}{m}\right) + R\left(\frac{1}{m},\frac{2}{m}\right) + \cdots + R\left(\frac{m-1}{m}, 1\right) = R(0,1)}$$

Q: $Var[R(0,1)] = ?$