UNIVERSITY OF TEXAS AT AUSTIN

Lecture 10

An introduction to Pricing Forward Contracts.

10.1. **Different ways to buy an asset.**

(1) **Outright purchase**: investor buys the asset with own funds
(2) **Fully leveraged purchase**: investor borrows the full amount needed to buy the asset
(3) **Forward contract**: just the agreement today, both pay the **forward price** and receive the asset on the *delivery date*
(4) **Prepaid forward contract**: pay the **prepaid forward price** today, receive the asset on the *delivery date*

Example 10.1. **Sample IFM (Derivatives: Intro) Problem #7**
A non-dividend paying stock currently sells for 100. One year from now the stock sells for 110. The risk-free rate, compounded continuously, is 6%. The stock is purchased in the following manner:

(1) You pay 100 today
(2) You take possession of the security in one year.

Which of the following describes this arrangement?

A. Outright purchase
B. Fully leveraged purchase
C. Prepaid forward contract
D. Forward contract
E. This arrangement is not possible due to arbitrage opportunities

Solution: C. Simply the definition of a prepaid forward contract!

10.2. **Outright purchase.** We have talked about the payoff structure of a simple long position in an underlying asset. The profit is straightforward for non-dividend-paying assets. Let us look into profits in the case of dividend-paying assets.

Discrete dividends. The natural examples of these kinds of assets are dividend-paying stocks. Let the company whose shares the prepaid forward contract is on is projected to pay discrete dividends in the amounts $D_1, \dots, D_k, \dots, D_n$ at times $0 < t_1 < \dots < t_k < \dots < t_n \leq T$. Then, the owner of the asset is entitled to the dividend payments, and they have to be incorporated into the profit calculation. Taking into account the time-value-of-money, the investor's profit is

$$S(T) + \sum_{k=1}^{n} FV_{t_k,T}(D_k) - FV_{0,T}(S(0)). \tag{10.1}$$

If the continuously compounded interest rate equals r, the above equation becomes

$$S(T) + \sum_{k=1}^{n} D_k e^{r(T-t_k)} - S(0)e^{rT}.$$

Continuous dividends. The examples of assets in this category would be market indices paying continuous dividends, stocks, and (foreign) currencies. In the case of indices and stocks, the dividend yield is denoted by δ. If the underlying is a foreign currency, then the role of δ is played by that currencies continuously compounded interest rate r_f. Assume that the investor's goal is to own exactly **one** unit of the asset on the delivery date T. Then, taking into account the continuous immediate reinvestment of dividends paid, the number of units he/she must acquire at time-0 equals $e^{-\delta T}$. So, the inital cost of this trade is $e^{-\delta T}S(0)$. The profit is

$$S(T) - FV_{0,T}(e^{-\delta T}S(0)). \tag{10.2}$$

If the prevailing continuously compounded interest rate is r, then the above profit can be expressed as

$$S(T) - e^{(r-\delta)T}S(0).$$

10.3. **Fully-leveraged purchase.** We have studied the cashflows associated with an outright purchase of an asset already. Let us focus on the fully leveraged purchase next. With a fully-leveraged purchase, the investor does not wish to invest his/her own funds in a risky asset. So, he/she borrows the required amount at the risk-free interest rate. Here is the breakdown of the cashflows:

time-0:
- **borrow** $S(0)$ at the risk-free rate,
- **purchase** one unit of the asset for $S(0)$;

time$-T$:
- **repay** $FV_{0,T}(S(0))$,
- one unit of the asset is now **worth** $S(T)$.

So, the investor's portfolio consists of two components:

i. the loan for the amount needed to invest in the asset, and
ii. the purchased asset itself.

The initial cost of a fully-leveraged position is **zero**. In fact, this is the **definition** of "fully leveraged". The payoff of the portfolio is

$$S(T) - FV_{0,T}(S(0)).$$

Because the above portfolio is fully leveraged, the profit equals the payoff. Note that the profits of an outright purchase and a fully leveraged purchase are equal. This is necessarily true so that arbitrage is avoided.

10.4. **Forward Contracts.** Recalling the forward contracts, we realize that they are another example of fully leveraged financial positions. The initial cost is zero, so that the payoff/profit equals

$$S(T) - F$$

where F denotes the forward price. Let us compare the above with the prepaid forward contract.

10.5. **Prepaid Forward Contracts.** With a prepaid forward contract, there is an initial contract from the buyer of the contract to the writer. We call the amount of this cashflow the **prepaid forward price** and we denote it by F^P. The payoff of a prepaid forward contract is simply $S(T)$. So, the profit equals

$$S(T) - FV_{0,T}(F^P). \tag{10.3}$$

The prepaid forward price and the forward price are completely dependent on each other in a no-arbitrage market-model. Comparing the profits of the forward and the prepaid forward contracts, we see that in order to avoid arbitrage, it must be that

$$F = FV_{0,T}(F^P). \tag{10.4}$$

The above equality is model-free. It also will be true regardless of the underlying asset-type.

10.5.1. *The prepaid forward price.* As we will learn very soon, F^P is a unique amount which can be found using the no-arbitrage principle. It depends on:

(1) the current stock price $S(0)$,
(2) the prevailing risk-free interest rate, and
(3) the asset's projected dividends/interest in the period until the delivery date T.

To emphasize the above dependence, as well as the uniqueness of the "fair", no-arbitrage prepaid forward price, we will henceforth denote it by $F^P_{0,T}(S)$.

The plan is to first find the prepaid forward price in the case that the underlying asset is a stock. To figure out the cases of contracts on foreign currencies and on commodities, we will argue analogously.

No dividends. Comparing the profit of the prepaid forward contract to the profit of the outright purchase of the underyling, we see that

$$F^P_{0,T}(S) = S(0).$$

Discrete dividends. Let the company whose shares the prepaid forward contract is on be projected to pay discrete dividends in the amounts $D_1, \dots, D_k, \dots, D_n$ at times $0 < t_1 < \cdots < t_k < \cdots < t_n \leq T$. Then, the comparison of profit from the case of an outright purchase (see equation (10.1)) to the profit in this case (see equation (10.3)) yields

$$F^P_{0,T}(S) = S(0) - \sum_{k=1}^{n} PV_{0,t_k}(D_k).$$

In words, the investor needs to be compensated for the "loss of dividend payments" that he/she would have received in case of an outright purchase. In terms of the continuously compounded, risk-free interest rate r, we have

$$F^P_{0,T}(S) = S(0) - \sum_{k=1}^{n} D_k e^{-rt_k}.$$

Continuous dividends. Let the dividend yield be δ. Again, comparing the profit equation from (10.2) to the profit for the prepaid forward contract (10.3), we get

$$F^P_{0,T}(S) = e^{-\delta T} S(0).$$

10.6. **Pricing forwards on stocks.** We will denote the no-arbitrage forward price for the underlying S and the delivery date T by $F_{0,T}(S)$. From equation (10.4) and the above three cases for prepaid forward prices, we get these expressions for forward prices if the continuously compounded interest rate equals r.

- *No dividends:* $F_{0,T}(S) = e^{rT}S(0)$
- *Discrete dividends:* $F_{0,T}(S) = e^{rT}S(0) - \sum_{k=1}^{n} D_k e^{r(T-t_k)}$
- *Continuous dividends:* $F_{0,T}(S) = e^{(r-\delta)T}S(0)$

10.7. **The annualized forward premium.** The *forward premium* is meant to reflect the ratio of the current forward price on a stock to the stock price. The *annualized forward premium (rate)* also normalizes the forward premium using the length of time to the delivery date of the forward. Both measures are useful to try to infer the stock price in markets that do not have frequent trades in the underlying asset (so that the traders are not confident in the stock prices that were last observed a relatively long time ago).

10.7.1. *Definition.* As usual, let $F_{0,T}(S)$ denote the forward price for the delivery of asset S at time T. Then, the *forward premium* is defined as

$$\frac{F_{0,T}(S)}{S(0)}.$$

The *annualized forward premium* is defined as

$$\frac{1}{T}\ln\left(\frac{F_{0,T}(S)}{S(0)}\right).$$

10.7.2. *Interpretation.* Let us temporarily write $\alpha(S)$ for the annualized forward premium of the asset S. Then, for every T, we have

$$\alpha(S) = \frac{1}{T}\ln\left(\frac{F_{0,T}(S)}{S(0)}\right) \quad \Rightarrow \quad F_{0,T}(S) = S(0)e^{\alpha(S)T}.$$

Let us look at the simple case of an asset which pays continuous dividends at the rate δ. We still denote the continuously compounded interest rate by r. Then, the above equality gives us

$$S(0)e^{(r-\delta)T} = S(0)e^{\alpha(S)T} \quad \Rightarrow \quad r-\delta = \alpha(S).$$

So, in this case the annualized forward premium rate reflects "mean appreciation" of the stock itself.

Problem 10.1. The current price of a stock is $S(0) = \$125$ per share. Let the stock pay continuous dividends at the continuous dividend rate δ. Assume that the continuously compounded interest rate equals $r = 0.3$. The prepaid forward pricefor delivery of the above stock in two years is \$83.79. Calculate the annualized forward premium (rate).

Solution: Based on the above discussion, we conclude that the answer equals

$$r-\delta = 0.3-\delta.$$

We use the prepaid forward price to calculate the δ.

$$F^P_{0,T}(S) = S(0)e^{-\delta T} \quad \Rightarrow \quad \delta = -\frac{1}{T}\ln\left(\frac{F^P_{0,T}(S)}{S(0)}\right) = -\frac{1}{2}\ln\left(\frac{83.79}{125}\right) \approx 0.2.$$

So, the final answer is about 0.1.

10.8. **Forwards and arbitrage: An example.** Suppose that the current price of a dividend-paying stock equals $\$1,000$. Let $r = 0.25$ and $\delta = 0.15$. You notice that a forward price for delivery of this stock in two-years equals $F = \$1,200$. You suspect that this forward price creates an arbitrage opportunity. The reason for this suspicion is that the forward price based on the initial stock price, r and δ equals

$$F_{0,T}(S) = S(0)e^{(r-\delta)T} = 1000e^{(0.25-0.15)\cdot 2} \approx 1,221.4 > F = 1,200.$$

The conclusion is that the observed forward price is "too low". One way to exploit this arbitrage opportunity would be to do the following:

(1) engage in the **long** forward contract,
(2) **short-sell** $e^{-\delta T}$ shares of stock,
(3) invest the proceeds from the short sale at the risk-free rate.

So, the initial cost of this portfolio is zero.

During the time period $(0, T]$, all of the continuously paid dividends are automatically reinvested in the asset S. So, at the end, one share of stock needs to be returned. Thus, at time$-T$, the payoff is

$$(S(T) - F) - S(T) + e^{(r-\delta)T}S(0) = 21.4 > 0.$$

The portfolio we constructed is, indeed, an arbitrage portfolio.