Value@Risk [cont'd].

M339W: February 14th, 2020.

α... probability

R... return random variable

Def'n. $VaR_\alpha(R)$ is the value π_α s.t.

$$\mathbb{P}[R \leq \pi_\alpha] = \alpha$$

In particular: Consider an R such that its density f_R is always positive (e.g., let R be normally dist'd).

Then, for any $a \in \mathbb{R}$:

$$F_R(a) = \mathbb{P}[R \leq a] = \int_{-\infty}^{a} f_R(x)\,dx$$

$\Rightarrow$ F_R is strictly increasing

$\Rightarrow$ F_R is one-to-one

$\Rightarrow$ F_R^{-1} exists

$$\Rightarrow \quad \pi_\alpha = F_R^{-1}(\alpha)$$

In particular, for normal returns, we use the std normal tables.

Note: If we are interested in the upper-tail probab. bounds, say if the random variable X signifies severity, we look @ $VaR_{1-\alpha}(X)$.

↙ profit random variable

34) Let X be the random gain from operations of a company. You are given:

(i) X is normally distributed with mean 42 and variance 6400.

$\sigma = 80$

(ii) p is the probability that X is negative.

(iii) K is the amount of capital such that the Value-at-Risk (VaR) at the 5th percentile for $X + K$ is zero.

Calculate p and K.

(A) $p = 0.7$; $K = 157$

(B) $p = 0.7$; $K = 131$

(C) $p = 0.5$; $K = 115$

(D) $p = 0.3$; $K = 115$

(E) $p = 0.3$; $K = 90$

(i): $X \sim N(\text{mean} = 42, \sigma^2 = (80)^2)$

(ii): $p = \mathbb{P}[X<0] = \mathbb{P}\Big[\overbrace{\frac{X-42}{80}}^{\sim N(0,1) \sim Z} < -\frac{42}{80}\Big]$

$p = \mathbb{P}[Z < -0.525] = N(-0.525)$

$p = 1 - N(0.525) = 1 - 0.7019$

↑ rounding up in the std normal tables

$\Rightarrow \quad p \approx 0.3$

$\Rightarrow$ (D) or (E)

(iii) $\mathbb{P}[X + K \leq 0] = 0.05$

$\mathbb{P}[X \leq -K] = 0.05$, i.e.,

$-K$ is the 5th percentile of X

$\Rightarrow$ w/ $z^*_{0.05}$ being the 5th percentile of $Z \sim N(0,1)$,

$42 + 80 \cdot z^*_{0.05} = -K$

$\Rightarrow \quad K = -(42 + 80(-1.645)) = 90 \quad \Rightarrow$ (E) ∎

(2.)

UNIVERSITY OF TEXAS AT AUSTIN

Log-normal stock prices: Tail probabilities.

Problem 4.1. (15 points)
You are considering an investment in a non-dividend-paying stock versus an investment in a savings account. According to your belief, the stock's mean rate of return is α and its volatility is σ.

The continuously compounded interest rate is equal to r.

What is the probability that the stock outperforms the savings account at time$-T$? You should leave your final answer in terms of the function N.

Investment amt : $S(0)$

- If it's a risk-free investment, then @ time$\cdot T$ the balance is $S(0)e^{rT}$
- If it's a stock investment, then @ time$\cdot T$ our wealth is $S(T)$

NOTE: This question is equivalent to asking whether the outright purchase has a positive profit.

In the log-normal stock-price model:

$$S(T) = S(0)e^{(\alpha-\delta-\frac{\sigma^2}{2})\cdot T+\sigma\sqrt{T}\cdot Z} \quad \text{w/ } Z\sim N(0,1)$$

$$\Rightarrow \mathbb{P}[S(0)e^{rT} < S(T)] =$$

$$= \mathbb{P}[S(0)e^{rT} < S(0)e^{(\alpha-\cancel{\delta}-\frac{\sigma^2}{2})\cdot T+\sigma\sqrt{T}\cdot Z}]$$

$$= \mathbb{P}[rT < (\alpha-\frac{\sigma^2}{2})\cdot T+\sigma\sqrt{T}\cdot Z]$$

$$= \mathbb{P}[\frac{1}{\sigma}(r-\alpha+\frac{\sigma^2}{2})\cdot\sqrt{T} < Z]$$

3.

$$= N\left(\frac{\sqrt{T}}{\sigma}(r-\alpha+\frac{\sigma^2}{2})\right)$$

Task: Think about the increase and the decrease of the above probab. for particular values of r, α, σ as $T \uparrow$.

Q: What about the risk-neutral probability?

→: Under $\mathbb{P}^*$, we have $\alpha \leftrightarrow r$, and so we get

$$\mathbb{P}^*\left[S(T) > S(0)e^{rT}\right] = N\left(\frac{\sqrt{T}}{\sigma} \cdot \frac{\sigma^2}{2}\right) = N\left(\frac{\sigma\sqrt{T}}{2}\right) \blacksquare$$

Q: Return to the physical probab. measure. What would happen if we reintroduced continuous dividends?

→: Due to continuous & immediate reinvestment of dividends in the same asset, we end up with the wealth:

$$e^{\delta \cdot T} \cdot S(T) =$$

$$= \cancel{e^{\delta T}} \cdot S(0)\, e^{(\alpha - \cancel{\delta} - \frac{\sigma^2}{2}) \cdot T + \sigma\sqrt{T} \cdot Z} \quad \text{w/ } Z \sim N(0,1)$$

Tail Probabilities of log-Normal Stock Prices.

$$S(T) = S(0)e^{(\alpha-\delta-\frac{\sigma^2}{2})\cdot T+\sigma\sqrt{T}\cdot Z} \quad \text{w/ } Z \sim N(0,1)$$

Q: Given a strike price K, what is the probab. that a European call option w/ exercise date T will be exercised?

→: $\mathbb{P}[S(T) > K]$ = (by our log-normal model)

$$= \mathbb{P}\left[S(0)e^{(\alpha-\delta-\frac{\sigma^2}{2})\cdot T+\sigma\sqrt{T}\cdot Z} > K\right]$$

$$= \mathbb{P}\left[e^{(\alpha-\delta-\frac{\sigma^2}{2})\cdot T+\sigma\sqrt{T}\cdot Z} > \left(\frac{K}{S(0)}\right)\right]$$

$$= \mathbb{P}\left[(\alpha-\delta-\frac{\sigma^2}{2})\cdot T+\sigma\sqrt{T}\cdot Z > \ln\left(\frac{K}{S(0)}\right)\right]$$

$$= \mathbb{P}\left[\sigma\sqrt{T}\cdot Z > \ln\left(\frac{K}{S(0)}\right) - (\alpha-\delta-\frac{\sigma^2}{2})\cdot T\right]$$

$$= \mathbb{P}\left[Z > \frac{1}{\sigma\sqrt{T}}\left(\ln\left(\frac{K}{S(0)}\right) - (\alpha-\delta-\frac{\sigma^2}{2})\cdot T\right)\right]$$

$$= \mathbb{P}\left[Z > -\frac{1}{\sigma\sqrt{T}}\left(\ln\left(\frac{S(0)}{K}\right) + (\alpha-\delta-\frac{\sigma^2}{2})\cdot T\right)\right]$$

$$= \mathbb{P}\left[Z < \boxed{\frac{1}{\sigma\sqrt{T}}\left(\ln\left(\frac{S(0)}{K}\right) + (\alpha-\delta-\frac{\sigma^2}{2})\cdot T\right)}\right]$$

$$=: \hat{d}_2$$

$$\Rightarrow \mathbb{P}[S(T) > K] = N(\hat{d}_2)$$