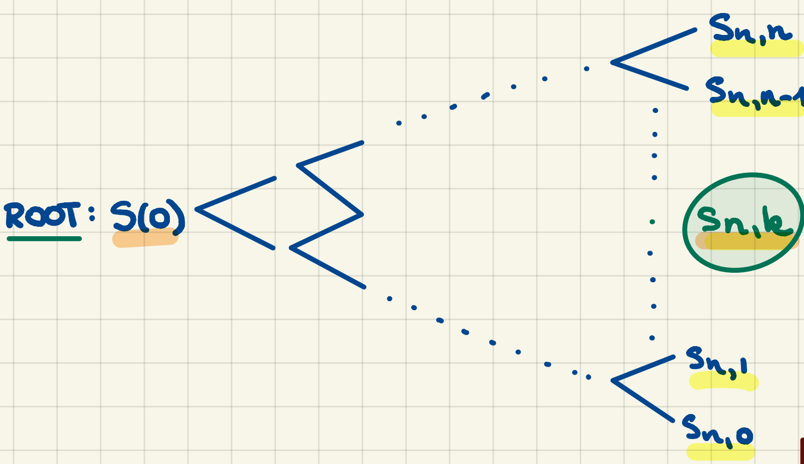
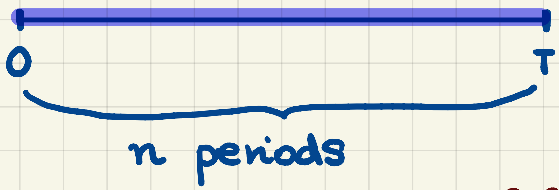
M339D: April 3rd, 2023.

The Pre-Limit n-period Binomial Tree.



$S_{n,k}$ ... the stock price resulting from k upsteps



$$\boxed{h = \frac{T}{n}}$$

u_n ... up factor

d_n ... down factor

e.g., in the forward tree

$$u_n = \exp\left(r \cdot \frac{T}{n} + \sigma\sqrt{\frac{T}{n}}\right)$$

$$d_n = \exp\left(r \cdot \frac{T}{n} - \sigma\sqrt{\frac{T}{n}}\right)$$

$$S_{n,k} = S(0) \cdot u_n^k \cdot d_n^{n-k} = S(0)\left(\frac{u_n}{d_n}\right)^k \cdot d_n^n$$

k corresponds to the realization of the binomial distribution w/ n trials

and p_n^* as the probability of success in each

w/ $$p_n^* = \frac{e^{r\left(\frac{T}{n}\right)} - d_n}{u_n - d_n}$$

e.g., in the forward tree: $\boxed{p_n^* = \frac{1}{1+e^{\sigma\sqrt{\frac{T}{n}}}}}$

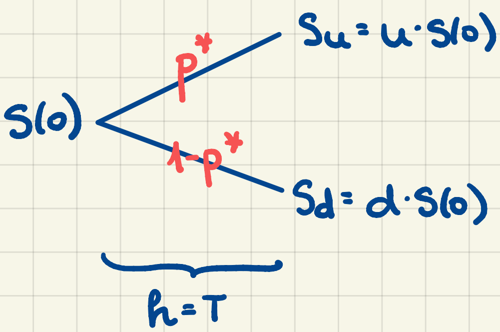
Say, X_n ... # of upsteps in n periods

$$\boxed{X_n \sim \text{Binomial}(\#\text{ of trials } n, \text{ probab. of success} = p_n^*)}$$

de Moivre-Laplace does not apply

On Pricing.

When we're pricing, we work under the risk-neutral measure.



$$\boxed{p^* = \frac{e^{rh} - d}{u-d}}$$

Q: If we invest in one share of non-dividend paying stock @ time 0, what is our expected wealth @ time T under the risk-neutral probability measure?

$\rightarrow$:

$$\mathbb{E}^*[S(T)] = S_u \cdot p^* + S_d \cdot (1-p^*)$$
$$= u \cdot S(0) \cdot \frac{e^{rh}-d}{u-d} + d \cdot S(0) \cdot \frac{u-e^{rh}}{u-d}$$
$$= S(0)\left(\frac{1}{u-d}\right)\left(u e^{rh} - \cancel{u d} + \cancel{d u} - d e^{rh}\right)$$
$$= S(0)\left(\cancel{\frac{1}{u-d}}\right) e^{rh} \cancel{(u-d)} = S(0)e^{rT}$$

In Contrast:

There can be a subjective probability $\mathbb{P}$. We can think about the quality of our investment under that probability.
e.g.,

$$\mathbb{E}[S(T)] = S(0)e^{\alpha \cdot T}$$

We usually refer to α as the mean rate of return.

In a binomial tree, we can talk about the "true" probability of a step up

$$\boxed{p = \frac{e^{\alpha h} - d}{u-d}}$$

Moment Generating Functions.

For any random variable Y,
and for independent arguments denoted by t,
we define the moment generating ftion (mgf) of Y
as this function of t:

$$\boxed{M_Y(t) := \mathbb{E}\left[e^{Y\cdot t}\right]}$$

for all t such that the expectation exists, i.e., if it's finite

Note: • $M_Y(0) = \underline{1}$ => @ least $t=0$ is in the domain of M_Y.

Goal: To understand e^X w/ $X \sim Normal(mean = m, var = \nu^2)$

Recall. In terms of $Z \sim N(0,1)$,

$$\boxed{X = m + \nu Z}$$

Recall. $\boxed{M_Z(t) = e^{\frac{t^2}{2}}}$ ✓

=> For any normal X:

$$M_X(t) = \mathbb{E}\left[e^{Xt}\right] = \mathbb{E}\left[e^{(m+\nu Z)t}\right] = \mathbb{E}\left[e^{m\cdot t} e^{\nu t\cdot Z}\right]$$

$$= e^{mt}\,\mathbb{E}\left[e^{\nu t\cdot Z}\right] = e^{mt}\cdot M_Z(\nu t)$$

$$= e^{m\cdot t} e^{\frac{\nu^2 t^2}{2}} = e^{mt + \frac{\nu^2 t^2}{2}}$$