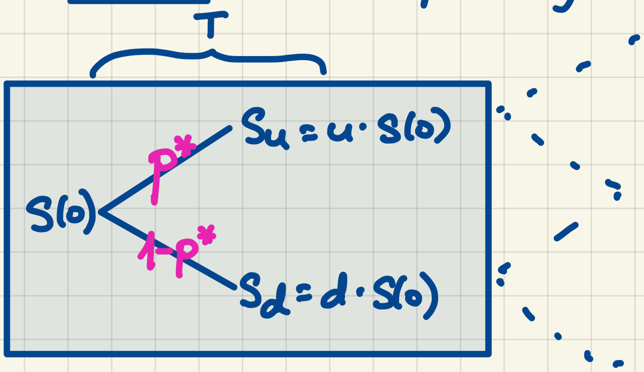
Subjective Probability.

Recall: When pricing we use $\mathbb{P}^*$ risk-neutral probability measure.

$$\boxed{p^* = \frac{e^{rh} - d}{u - d}}$$



Q: If we invest in one share of non-dividend-paying stock @ time 0, what is the expected wealth @ time T under $\mathbb{P}^*$?

$$\longrightarrow: \quad \mathbb{E}^*[S(T)] = S(0)e^{rT} \qquad \square$$

The mean rate of return under the risk neutral measure is r.

In Contrast:

There can be a subjective probability measure $\mathbb{P}$.
We can think about the quality of our investment under that probability measure, i.e.,

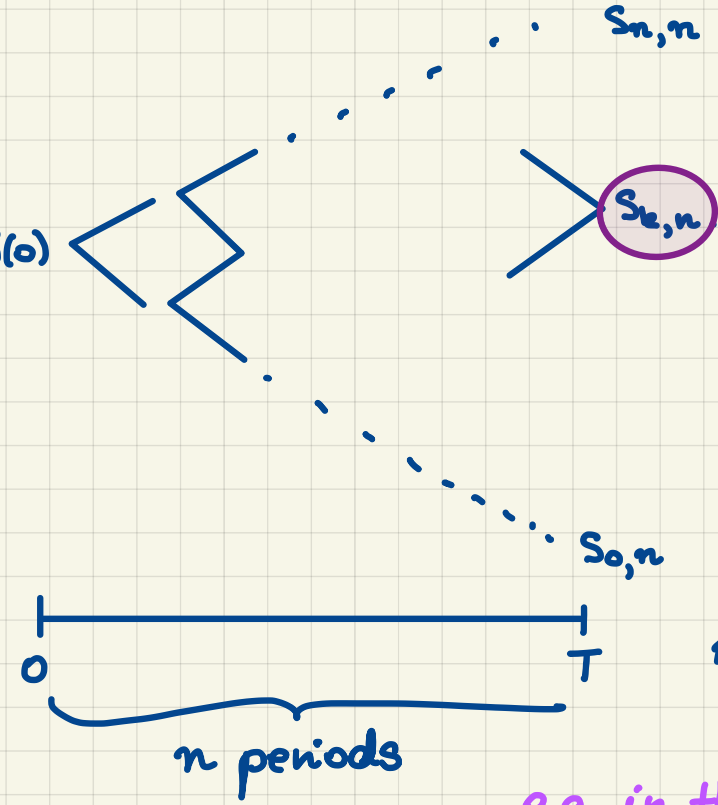
$$\mathbb{E}[S(T)] = S(0)e^{\alpha \cdot T}$$

We call this α the mean rate of return.
In a binomial tree, we can express the "true" probability of a step up as

$$\boxed{p = \frac{e^{\alpha h} - d}{u - d}}$$

The Pre-Limit n-Period Binomial Tree.

ROOT: $S(0)$



$S_{k,n}$ ... the stock price resulting from k steps up

$h_n = \frac{T}{n}$

u_n ... upfactor

d_n ... down factor

e.g. in the forward tree

$u_n = \exp\left(r \cdot \frac{T}{n} + \sigma\sqrt{\frac{T}{n}}\right)$

$d_n = \exp\left(r \cdot \frac{T}{n} - \sigma\sqrt{\frac{T}{n}}\right)$

$$S_{k,n} = S(0) \cdot u_n^k \cdot d_n^{n-k} = S(0)\left(\frac{u_n}{d_n}\right)^{k} \cdot d_n^n$$

k ... corresponds to a realization of the binomial random variable w/ n trials and success probability

$$p_n^* = \frac{e^{r(T/n)} - d}{u - d}$$

e.g., in the forward tree

$$\boxed{p_n^* = \frac{1}{1 + e^{\sigma\sqrt{T/n}}}} \xrightarrow[n \to \infty]{} \frac{1}{2}$$

Say, X_n ... # of steps up in n periods

$X_n \sim \text{Binomial}(\#\text{ of trials} = n, \text{ success probability} = p_n^*)$

Q: Can we simply use the normal approximation to the binomial?

Nope! p_n^* depends on n.

Moment Generating Function.

For a random variable Y,
and for an independent argument denoted by t, we define the moment generating function (mgf) of Y as this function of t:

$$M_Y(t) := \mathbb{E}\left[e^{Y \cdot t}\right]$$ for all t such that the expectation exists

Note:

- $M_Y(0) = \underline{1}$ => @ least $t=0$ is in the domain
- We say that the mgf exists is if it's finite for t such that $|t|<b$ for some $b>0$.