M339J: February 23rd, 2022.

Transformations of Random Variables.

Transformation I. Multiplying by a Constant.

Say that X is a continuous random variable w/ a pdf f_X.
Let $\varkappa$ be a constant.
Define $\underline{\tilde{X} = \varkappa \cdot X}$.

Q: If it exists, what is the pdf of $\tilde{X}$?

$\longrightarrow$: If $\varkappa = 0$, then $\tilde{X}$ is degenerate, i.e., it's a constant.

If $\varkappa > 0$,

then for all $x \in \mathbb{R}$, we have

$$F_{\tilde{X}}(x) = \mathbb{P}[\tilde{X} \le x] = \mathbb{P}[\varkappa \cdot X \le x]$$
$$= \mathbb{P}\left[X \le \frac{x}{\varkappa}\right] = F_X\left(\frac{x}{\varkappa}\right)$$

$$\Rightarrow f_{\tilde{X}}(x) = \frac{\partial}{\partial x} F_X\left(\frac{x}{\varkappa}\right) = \frac{1}{\varkappa} f_X\left(\frac{x}{\varkappa}\right)$$

If $\varkappa < 0$,

then for all $x \in \mathbb{R}$,

$$F_{\tilde{X}}(x) = \mathbb{P}[\varkappa \cdot X \le x] = \mathbb{P}\left[X \ge \frac{x}{\varkappa}\right] = 1 - F_X\left(\frac{x}{\varkappa}\right)$$

$$\Rightarrow f_{\tilde{X}}(x) = \frac{\partial}{\partial x}\left(1 - F_X\left(\frac{x}{\varkappa}\right)\right) = -\frac{1}{\varkappa} f_X\left(\frac{x}{\varkappa}\right) \quad \square$$

Example.

$X \sim Gamma(\alpha, \theta)$

Then, the pdf of X is
$$\boxed{f_X(x) = \frac{\left(\frac{x}{\theta}\right)^{\alpha} e^{-\frac{x}{\theta}}}{x\,\Gamma(\alpha)}}$$

Let $\varkappa > 0$.
Then, the pdf of $\tilde{X} = \varkappa \cdot X$ is:

$$f_{\tilde{X}}(x) = \frac{1}{\varkappa} \cdot f_X\left(\frac{x}{\varkappa}\right) = \frac{1}{\cancel{\varkappa}} \cdot \frac{\left(\frac{x}{\varkappa\theta}\right)^{\alpha} e^{-\frac{x}{\varkappa\theta}}}{\frac{x}{\cancel{\varkappa}}\,\Gamma(\alpha)}$$

$$\Rightarrow \boxed{\tilde{X} \sim Gamma(\alpha, \tilde{\theta} = \varkappa \cdot \theta)}$$

Terminology. Let X be a random variable w/ nonnegative support w/ a scale distribution.

If a parameter of this scale dist'n satisfies:

1. When X is multiplied by a positive constant, the parameter is multiplied by the same constant;
2. All other parameters stays the same,

then the parameter is called a **scale parameter**.

Transformation II. Raising to a Power.

Let X be positive continuous random variable w/ pdf f_X.
Let $\tau \neq 0$ be a constant.

Define $\boxed{\tilde{X} := X^{1/\tau}}$

For $\tau > 0$: for any $x > 0$:

$$F_{\tilde{X}}(x) = \mathbb{P}[\tilde{X} \leq x] = \mathbb{P}[X^{1/\tau} \leq x]$$
$$= \mathbb{P}[X \leq x^{\tau}] = F_X(x^{\tau}) \checkmark$$

$$\Rightarrow \quad f_{\tilde{X}}(x) = \tau \cdot x^{\tau-1} \cdot f_X(x^{\tau})$$

For $\tau < 0$: for any $x > 0$:

$$F_{\tilde{X}}(x) = \mathbb{P}[X \geq x^{\tau}] = 1 - F_X(x^{\tau})$$

$$\Rightarrow \quad f_{\tilde{X}}(x) = -\tau \cdot x^{\tau-1} \cdot f_X(x^{\tau}) \checkmark$$

Example.

$X \sim \text{Exponential}(\text{mean} = \theta)$

Define $\tilde{X} := X^{-1}$ (i.e., $\tau = -1$)

$$F_{\tilde{X}}(x) = 1 - F_X(x^{-1}) = \cancel{1-}\left(\cancel{1} - e^{-\frac{1/x}{\theta}}\right) = e^{-\frac{1}{x\theta}}$$

$$\Rightarrow \quad \tilde{X} \sim \text{InvExponential}\left(\tilde{\theta} = \frac{1}{\theta}\right)$$

STAM Tables

Example.

$X \sim \text{Exponential}(\text{mean} = \theta)$

Let $\tau > 0$. Define $\tilde{X} := X^{1/\tau}$

For $x > 0$: $F_{\tilde{X}}(x) = F_X(x^\tau) = 1 - e^{-\frac{x^\tau}{\theta}}$

$$= 1 - \exp\left(-\left(\frac{x}{\theta^{1/\tau}}\right)^\tau\right)$$

STAM TABLES : $\boxed{\tilde{X} \sim \text{Weibull}(\tilde{\theta} = \theta^{1/\tau}, \tau)}$

Transformation III. **Exponentiation.**

Let X be a continuous r.v. w/ $f_X(x) > 0$, e.g., normal.

Define $\boxed{\tilde{X} := e^X}$

Then,

$$F_{\tilde{X}}(x) = F_X(\ln(x))$$

$$f_{\tilde{X}}(x) = \frac{1}{x} \cdot f_X(x)$$

Example. $X \sim \text{Normal}(\text{mean} = \mu, \text{variance} = \sigma^2)$

Define $\boxed{\tilde{X} = e^X}$

We say that $\tilde{X}$ is lognormally distributed.

Q: What's the expected value of $\tilde{X}$?

→: $\mathbb{E}[e^X] = M_X(1) = e^{\mu + \frac{\sigma^2}{2}}$ ■

Let $\varkappa > 0$.

Define $X' := \varkappa \cdot \tilde{X}$.

Q: Is the lognormal a scale distribution?

→: We need to check if $X' = \varkappa \cdot \tilde{X}$ is still lognormal.

$$X' = \varkappa \cdot \tilde{X} = \varkappa \cdot e^X = e^{\ln(\varkappa)} \cdot e^X$$

$$X' = e^{\boxed{\ln(\varkappa) + X}}$$

Note: $\ln(\varkappa) + X \sim \text{Normal}(\text{mean} = \mu + \ln(\varkappa), \text{var} = \sigma^2)$

The lognormal is a scale distribution, but it does not have, in this parametrization, a scale parameter.

Q: Can the parametrization be changed so that there is a scale parameter? How?

Task: Recall exam P problems w/ various types of drivers.