Name:

M339D=M389D Introduction to Actuarial Financial Mathematics
University of Texas at Austin
Sample In-Term Exam III
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Notes: This is a closed book and closed notes exam. The maximal score on this exam is 100 points.

3.1. **FREE-RESPONSE PROBLEMS.**

Problem 3.1. Let the continuously compounded, risk-free interest rate be equal to 0.04. Consider a non-dividend-paying stock whose current price is 90. You model the evolution of this stock over the next six months using a two-period binomial tree assuming that the stock price can either increase or decrease by 10% in a single period. Consider a put-on-put option with the following characteristics:

- the exercise date of the compound option is in three months;
- the strike price of the compound option is \$2;
- the underlying put option is at-the-money at time-0 and its exercise date is in six months.

What is the price of the put-on-put option?

Solution: The risk-neutral probability is

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{e^{0.04(0.25)} - 0.9}{1.1 - 0.9} = 0.5502508.$$

The possible terminal asset prices are

$$S_{uu} = 90(1.1)^2 > 90, \quad S_{ud} = 90(1.1)(0.9) = 89.10, \quad S_{dd} = 90(0.9)^2 = 72.9.$$

So, the value of the underlying put at the up node equals

$$V_u^P = e^{-0.01}(1-p^*)(90 - 89.10) = 0.4007467.$$

So, the value of the underlying put at the down node equals

$$V_d^P = e^{-0.01}\left[p^*(90 - 89.10) + (1-p^*)(90 - 72.9)\right] = 8.104485.$$

Since the strike of the compound put option is \$2, the option is only exercised in the up node. Its payoff is $2 - 0.4007467 = 1.599253$. Finally, the time-0 price of the compound put is

$$V(0) = e^{-0.01}p^*(1.599253) = 0.8712342.$$

Problem 3.2. There are two European options on the same stock with the same time to expiration. The 90-strike call costs \$20 and the 100-strike call costs \$8.

Is there an arbitrage opportunity due to the above call prices?

Propose an arbitrage portfolio (if you concluded that it exists) and **verify** that your proposed portfolio is indeed an arbitrage portfolio.

Solution:
Diagnosis. The given call prices are obviously decreasing with respect to the strike price, so let us check if the amount of decrease is appropriately bounded from above. We have

$$V_C(90) - V_C(100) = 20 - 8 > 10 \geq PV_{0,T}(100-90). \tag{3.1}$$

We conclude that there exists an arbitrage oportunity to be exploited since the upper bound on the amount of decrease in call prices is violated.
Arbitrage-portfolio proposal. As we can see from inequality (3.1), the 100-strike call is relatively cheap when compared to the 90-strike call. So, we propose the following arbitrage portfolio:

- a **long** 100-strike call;
- a **short** 90-strike call.

The above position is commonly referred to as a **(call) bear spread**.

Verification. The initial cost of the above portfolio is

$$V_C(100) - V_C(90) = 8 - 20 = -12 < 0.$$

The payoff curve of the above call bear spread is bounded from below by -10.

As we can see, the payoff is bounded from below by -10. So, the profit of the above call bear spread is strictly greater than

$$-10 + FV_{0,T}(-12) \geq 2 > 0.$$

We have shown that the portfolio we proposed has a strictly positive profit in all states of the world. This is an even stronger requirement than the one from the definition of an arbitrage portfolio!

Problem 3.3. Consider a continuous-dividend-paying stock whose current price is \$40 and whose dividend yield is 0.02. The price of stock in three months is modeled using a one-period binomial tree.

The continuously compounded, risk-free interest rate is 0.06.

According to the above stock-price model, the replicating portfolio of an at-the-money, three-month European call option consists of:

- 0.6 shares of stock, and
- borrowing \$20 at the risk-free interest rate.

What is the risk-free portion of the replicating portfolio for the otherwise identical put option?

Solution: Let the stock price at the *up* node be denoted by S_u and the stock price at the *down* node by S_d. From the given value of the Δ in the replicating portfolio for the call, we conclude that the call is in-the-money at the *up* node and out-of-the-money at the *down* node. In fact,

$$0.6 = \Delta = e^{-0.02(0.25)}\frac{V_u}{S_u - S_d} = e^{-0.005}\frac{S_u - S(0)}{S(0)(u-d)} = e^{-0.005}\frac{u-1}{u-d}.$$

From the other component of the call's replicating portfolio, we get

$$20 = -B = e^{-0.06(0.25)}\frac{dV_u}{u-d} = e^{-0.015}\frac{dS(0)(u-1)}{u-d}.$$

Combining the above two equations, we get

$$20 = e^{-0.015}d(40)(0.6)e^{0.005} \quad \Rightarrow \quad d = \frac{20}{40(0.6)}e^{0.01} = 0.841708.$$

Reusing the equation for Δ, we get

$$0.6(u - 0.841708)e^{0.005} = u - 1 \quad \Rightarrow \quad u = \frac{1 - 0.6(0.841708)e^{0.005}}{1 - 0.6e^{0.005}} = 1.24044.$$

The put option is out-of-the-money at the *up* node and in-the-money at the *down* node where its payoff is $V_D^P = 40(1 - 0.841708) = 6.33168$. The risk-free portion of the put's replicating portfolio is

$$B_P =^{-0.015}\frac{1.24044(6.33168)}{1.24044 - 0.841708} = 19.4044.$$

3.2. **MULTIPLE CHOICE QUESTIONS.**

Problem 3.4. You construct an asymmetric butterfly spread using the following three types of European options on the same asset and with the same exercise date:

- a \$50-strike call,
- a \$60-strike call,
- a \$65-strike call.

You are told that there is exactly **one** short \$60-strike call in the asymmetric butterfly spread. What is the maximal payoff of the above butterfly spread?

(a) 0
(b) 10/3
(c) 5

(d) The payoff is not bounded from above.
(e) None of the above.

Solution: (b)
With the given strike prices, the asymmetric butterfly spread consists of the following components:

- 1/3 of a **long** \$50-strike call,
- one **short** \$60-strike call, and
- 2/3 of a **long** \$65-strike call.

The maximal payoff is attained for the final stock price equal to the "inner" strike of \$60. We get

$$\frac{1}{3}(60-50)_+ - (60-60)_+ + \frac{2}{3}(60-65)_+ = \frac{10}{3}.$$

Problem 3.5. The following two one-year European put options on the same asset are available in the market:

- a \$50-strike put with the premium of \$5,
- a \$55-strike put with the premium of \$10.

The continuously compounded, risk-free interest rate is 0.04.

Which of the following positions certainly exploits the arbitrage opportunity caused by the above put premia?

(a) Put bull spread.
(b) Put bear spread.
(c) Both of the above positions.
(d) There is no arbitrage opportunity.
(e) None of the above.

Solution: (a)

Problem 3.6. (5 points) Which one of the following positions always has an infinite upward potential in the sense that the payoff diverges to positive infinity as the argument s (standing for the final stock price) tends to positive infinity?

(a) A long call option.
(b) A bear spread.
(c) A bull spread.
(d) A long butterfly spread.
(e) None of the above.

Solution: (a)

Problem 3.7. A long strangle position...

(a) is equivalent to a short butterfly spread.
(b) can be replicated with a short call and a long put with the same strike, underlying asset and exercise date.
(c) is always strictly more expensive than the straddle on the same underlying asset and with the same exercise date.
(d) is a speculation on the stock's volatility.
(e) None of the above.

Solution: (d)

Problem 3.8. You are given that the price of:

- a \$50-strike, one-year European call equals \$8,
- a \$65-strike, one-year European call equals \$2.

Both options have the same underlying asset. What is the maximal price of a \$56-strike, one-year European call such that there is no arbitrage in our market model?

(a) \$4.40
(b) \$5
(c) \$5.60
(d) \$6.02
(e) None of the above.

Solution: (c)
Using the convexity of call price with respect to the strike, we get the following answer:

$$\frac{3}{5} \times 8 + \frac{2}{5} \times 2 = \frac{24 + 4}{5} = 5.60.$$

Problem 3.9. An investor bought a six-month, $(70, 80)$-put **bull** spread on an index. The \$70-strike, six-month put is currently valued at \$1, while the \$80-strike, six-month put is currently valued at \$8.

Assume that the continuously compounded, risk-free interest rate equals 0.02.

What is the **break-even** final index price for the above put bull spread?

(a) \$62.86
(b) \$71.84
(c) \$72.93
(d) \$73.23
(e) None of the above.

Solution: (c)
We need to solve for s such that $70 < s < 80$, in

$$80 - s = (8 - 1)e^{0.01} \quad \Rightarrow \quad s = 72.93$$

Problem 3.10. Consider a continuous-dividend-paying stock with the current price of \$50 and dividend yield 0.02.

The continuously compounded, risk-free interest rate is 0.05.

You are using a one-period binomial tree to model the stock price at the end of the next quarter. You assume that the stock price can either increase by 0.04 or decrease by 0.02. What is the risk-neutral probability associated with this tree?

(a) 0.3675
(b) 0.4588
(c) 0.5430
(d) 0.8409
(e) None of the above.

Solution: (b)
By defition, in our usual notation, we have

$$p^* = \frac{e^{(r-\delta)h} - d}{u - d} = \frac{e^{(0.05-0.02)(0.25)} - 0.98}{1.04 - 0.98} = 0.4588.$$

3.3. TRUE/FALSE QUESTIONS.

Problem 3.11. (2 points) The payoff curve of a **call bear** spread is never positive. *True or false?*

Solution: TRUE

Problem 3.12. (2 pts) Suppose that the European options with the same maturity and the same underlying assets have the following prices:

(1) a $50-$strike call costs \$9;

(2) a 55−strike call costs \$10;

Then, one should *acquire* a **call bear spread** to exploit the arbitrage since some of the monotonicity conditions for no-arbitrage are violated by the above premiums.

Solution: FALSE
We know that for strikes $K_1 < K_2$, the price of a call with strike K_1 should be greater than or equal to the price of a call with strike K_2. This condition is violated. One should acquire the **call <u>bull</u> spread** to construct an arbitrage portfolio.

Problem 3.13. (2 points) A **long** strangle has a non-negative payoff function. *True or false?*

Solution: TRUE

Problem 3.14. Suppose that prices of European calls on the same asset and with the same exercise date for varying strike prices are given by the following table

Strike	80	100	105
Call premium	22	9	5

Then, one can use a butterfly spread to exploit the violations of the no-arbitrage conditions exhibited by the prices in the above table. *True or false?*

Solution: TRUE

Problem 3.15. (2 points) In the setting of the binomial asset-pricing model, let d and u denote the up and down factors, respectively. Moreover, let r denote the continuously compounded, risk-free interest rate. Let h denote the length of a single period in our model.

Then, if,

$$e^{\delta h}d < e^{rh} < e^{\delta h}u$$

then there is no possibility for arbitrage. *True or false?*

Solution: TRUE

Problem 3.16. (2 pts) A European call option with strike K on a futures contract on a stock has the same value as the European call option with strike K on that same stock provided that the futures contract has the same expiration as the stock option. *True or false?*

Solution: TRUE

Problem 3.17. Assume a positive risk-free interest rate. It's never optimal to exercise an American option on a non-dividend-paying stock early. *True or false?*

Solution: TRUE
The justification is in your class notes.

Problem 3.18. (2 points) The price of a **up-and-in** option is increasing as a function of its barrier (with every other input held fixed).

Solution: FALSE

Problem 3.19. (2 points) The price of a geometric average price Asian call option is strictly greater the price of an otherwise identical arithmetic average price Asian call option. *True or false?*

Solution: FALSE

3.4. MORE MULTIPLE CHOICE QUESTIONS.

Problem 3.20. The current exchange rate is given to be \$1.25 per Euro and its volatility is given to be 0.15.

The continuously compounded, risk-free interest rate for the US dollar is 0.03, while the continuously compounded, risk-free interest rate for the Euro equals 0.06.

The evolution of the exchange rate over the following nine-month period is modeled using a three-period forward binomial tree.

What is the value of the so-called down factor in the above tree?

(a) $d \approx 0.8586$
(b) $d \approx 0.8982$
(c) $d \approx 0.9208$
(d) $d \approx 0.9347$
(e) None of the above.

Solution: (c)
In the forward binomial tree, the up and down factors are given as

$$u = e^{(0.03-0.06)\times 0.25+0.15\times\sqrt{0.25}} = 1.0698$$
$$d = e^{(0.03-0.06)\times 0.25-0.15\times\sqrt{0.25}} = 0.9208.$$

Problem 3.21. The current stock price is observed to be \$100 per share. The stock is projected to pay dividends continuously at the rate proportional to its price with the dividend yield of 0.03. The stock's volatility is given to be 0.23. You model the evolution of the stock price using a two-period forward binomial tree with each period of length one year.

The continuously compounded, risk-free interest rate is given to be 0.04.

What is the price of a two-year, \$101-strike **American** put option on the above stock consistent with the above stock-price tree?

(a) About \$6.62
(b) About \$8.34
(c) About \$8.83
(d) About \$11.11
(e) None of the above.

Solution: (d)
The up and down factors in the forward tree are

$$u = e^{0.01+0.23} = 1.2712, \quad \text{andd} = e^{-0.22} = 0.8025.$$

Of course, in the exam, you would not necessarily populate the entire stock-price tree, since you would want to work efficiently and only consider the nodes you need for pricing.

The risk-neutral probability equals

$$p^* = \frac{1}{1+e^{0.23}} = 0.4428.$$

Evidently, the option produces a positive payoff only in the down-down node; the value of this payoff is $V_{dd} = 101 - 64.40 = 36.60$.

The continuation value at the down node is, hence,

$$CV_d = e^{-0.04} \times (1 - 0.4428) \times 36.60 = 15.57.$$

On the other hand, the value of immediate exercise at the down node equals $IE_d = 101 - 80.25 = 20.75$. So, it is optimal to exercise the American put in the down node and the value of the American put equals $V_d^P = 20.75$.

In the up node, both the continuation value and the immediate exercise value are zero. So, the initial price of the American put is

$$V_P(0) = e^{-0.04} \times (1 - 0.4428) \times 20.75 = 11.11$$

Problem 3.22. The current price of a continuous-dividend-paying stock is \$100 per share. Its volatility is given to be 0.2 and its dividend yield is 0.03.

The continuously compounded risk-free interest rate equals 0.04.

Consider a \$95-strike European put option on the above stock with nine months to expiration. Using a three-period forward binomial tree, find the price of this put option.

(a) \$2.97
(b) \$3.06
(c) \$3.59
(d) \$4.32
(e) None of the above.

Solution: (d)
The up and down factors in the above model are

$$u = e^{0.01\times 0.25+0.2\sqrt{0.25}} = 1.10794,$$
$$d = e^{0.01\times 0.25-0.2\sqrt{0.25}} = 0.9071.$$

The relevant possible stock prices at the "leaves" of the binomial tree are

$$S_{ddd} = d^3S(0) = 100(0.9071)^3 = 74.6395,$$
$$S_{ddu} = d^2uS(0) = 91.1649.$$

The remaining two final states of the world result in the put option being out-of-the-money at expiration.

The risk-neutral probability of the stock price moving up in a single period is

$$p^* = \frac{1}{1+e^{0.2\sqrt{0.25}}} = 0.475.$$

So, the price of the European put option equals

$$V_P(0) = e^{-0.04(3/4)}\left[(95-74.6395)(1-0.475)^3 + (95-91.1649)(3)(1-0.475)^2(0.475)\right] = 4.32.$$