M358K: October 12th, 2022.

Confidence Intervals.

We are still in the normal model.
The same logic will apply to other models as well.

Let $X_1, X_2, \dots, X_n$ be a normal random sample, i.e.,
$\{X_i, i=1..n\}$ are all independent, and
$X_i \sim Normal(mean = \mu, sd = \sigma)$

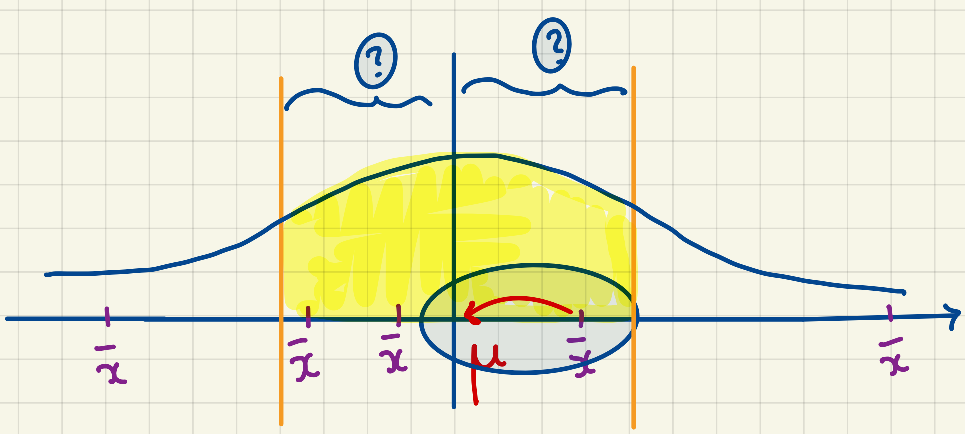
We know exactly the distribution of the sample mean:

$$\boxed{\bar{X}_n \sim Normal(mean = \mu, sd = \frac{\sigma}{\sqrt{n}})}$$

For now: assume that σ is known.

We know that $\bar{X}_n$ is a "good" estimator for the population mean μ.

Q: How $\boxed{\text{CONFIDENT}}$ are we about the value that we get?
What does "confidence" even mean?



Let C be a "large" probability, i.e., a confidence level.

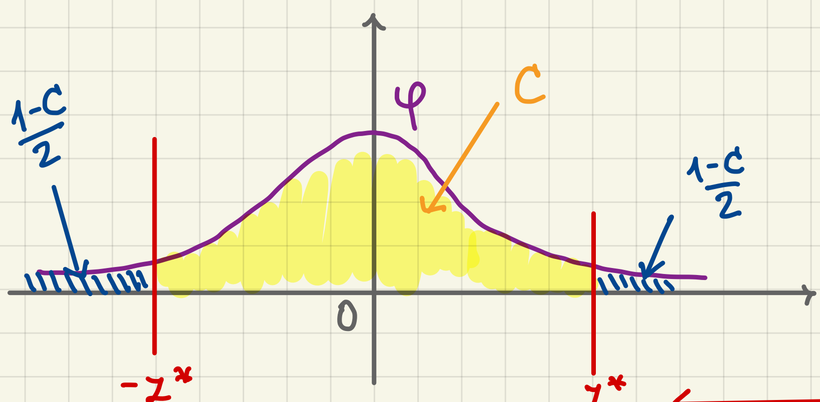
Say $C = 0.95, 0.90, 0.99, 0.80$

Look @

$$\mathbb{P}\left[\,|\bar{X}_n - \mu| < ?\,\right] = C$$

$$\mathbb{P}\left[-? < \bar{X}_n - \mu < ?\right] = C$$

$$\mathbb{P}\left[\underbrace{\boxed{-\frac{?}{\frac{\sigma}{\sqrt{n}}}}}_{= -z^*} < \underbrace{\frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}}}_{= Z \,\sim\, N(0,1)} < \underbrace{\boxed{\frac{?}{\frac{\sigma}{\sqrt{n}}}}}_{= z^*}\right] = C \longleftarrow \times$$



$z^* \longleftarrow \checkmark$

qnorm$\left(\frac{1+C}{2}\right)$

$$\frac{1-C}{2} + C = \frac{1+C}{2}$$

$$\boxed{z^* = \Phi^{-1}\left(\frac{1+C}{2}\right)}$$

z^* is the CRITICAL VALUE of $N(0,1)$ such that

$$\boxed{\mathbb{P}[-z^* < Z < z^*] = C}$$

(ii) $\Rightarrow$ $z^* = \dfrac{?}{\frac{\sigma}{\sqrt{n}}}$ $\Rightarrow$ $\boxed{? = z^* \cdot \frac{\sigma}{\sqrt{n}}}$

(iii) $\Rightarrow$ $\mathbb{P}\left[-z^* \cdot \frac{\sigma}{\sqrt{n}} < \bar{X}_n - \mu < z^* \cdot \frac{\sigma}{\sqrt{n}} \right] = C$

$$\mathbb{P}\left[\bar{X}_n - z^* \cdot \frac{\sigma}{\sqrt{n}} < \mu < \bar{X}_n + z^* \cdot \frac{\sigma}{\sqrt{n}} \right] = C$$

my RANDOM INTERVAL
which we call our CONFIDENCE INTERVAL

- Every time that you collect a sample and construct a confidence interval, you obtain a DIFFERENT INTERVAL.
- With a probability C, the confidence interval w/ contain the mean parameter μ w/ probability C
  and w/ probability 1-C it will NOT.