UNIVERSITY OF TEXAS AT AUSTIN

Problem Set # 4

Problem 4.1. *Source: An old ASM Manual for Exam MFE.*
A one-year American, GBP-denominated put option on euros allows the sale of 100 euros for 90 GBP. The evolution of the exchange rate over the next year is modelled with a two-period forward binomial tree. You have the following:

- The current spot exchange rate is 0.80 GBP per euro.
- The continuously compounded risk-free interest rate for the GBP is 0.06.
- The continuously compounded risk-free interest rate for the euros is 0.04.
- The volatility of the exchange rate of GBP to euros is 0.10.

What is the price of the above put option?

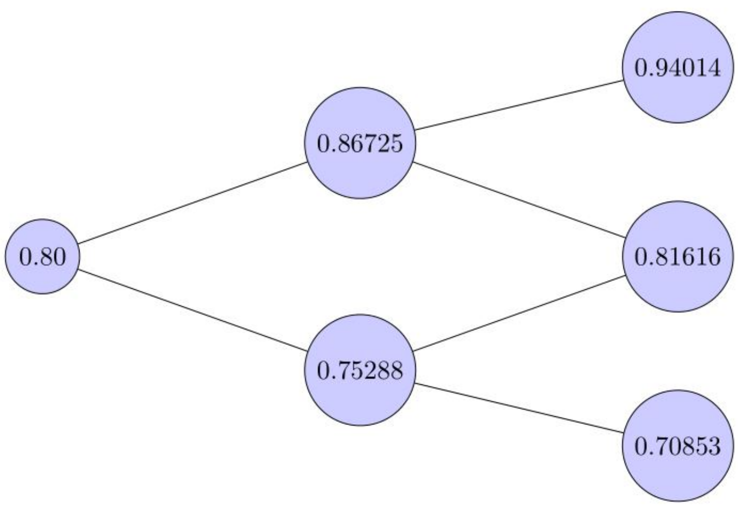
Solution: Remember that this is a **forward** binomial tree. So, we can use the "shortcut" formula for the risk-neutral probability. The length of a single period is $h = 1/2$. In our usual notation, the risk-neutral probability is

$$p^* = \frac{1}{1 + e^{\sigma\sqrt{h}}} = \frac{1}{1 + e^{0.1\sqrt{1/2}}} = 0.4823297.$$

For the GBP-denominated put option, the pounds are the domestic currency. This leaves the euros as the foreign currency. The up and down factors are, respectively,

$$u = e^{(r_p - r_e)h + \sigma\sqrt{h}} = e^{(0.06-0.04)/2+0.1\sqrt{1/2}} = 1.084057,$$
$$d = e^{(r_p - r_e)h - \sigma\sqrt{h}} = e^{(0.06-0.04)/2-0.1\sqrt{1/2}} = 0.9410955.$$

Here is the resulting binomial tree:



The put's strike is 0.9, so the possible payoffs should the put be unexercised until the expiration date are

$$V_{uu} = 0, \quad V_{ud} = 0.08384, \quad V_{dd} = 0.19147.$$

The continuation value at the *up* node is

$$CV_u = e^{-0.06/2}(1 - p^*)V_{ud} = 0.04212.$$

The value of immediate exercise at the same node is

$$IE_u = (0.9 - 0.86725)_+ = 0.03275.$$

Hence, it's optimal to hold onto the put option at this node and the value of the American put at the *up* node is $V_u^A = CV_u = 0.04212$.

Focusing on the *down* node, we see that the continuation value equals

$$CV_d = e^{-0.06/2}(p^* V_{ud} + (1-p^*)V_d d) = 0.1354323.$$

The value of immediate exercise is

$$IE_d = (0.9 - 0.75288)_+ = 0.14712.$$

This value exceeds the continuation value, so it's optimal to exercise early at this node. The value of the American put equals $V_d^A = 0.14712$.

At the *root* node, the continuation value equals

$$CV_0 = e^{-0.06/2}(p^* V_u^A + (1-p^*)V_d^A) = 0.0936241.$$

The value of immediate exercise at the *root* node is

$$IE_0 = (0.9 - 0.8)_+ = 0.1.$$

So, it's optimal to exercise the option right away. Its price is 0.10 pence per euro.

Problem 4.2. *Source: An old ASM Manual for Exam MFE.*
An American call option expiring in one year on a futures contract on a market index is modelled with a two-period binomial tree based on forward prices. You are given that:

- The futures price is currently 500.
- The option's strike price is 500.
- The volatility of the market index is 0.30.
- The continuously compounded risk-free interest rate is 0.06.
- The dividend yield of the market index is 0.02.

What is the amount of money lent in the replicating portfolio for the call at the beginning of the year?

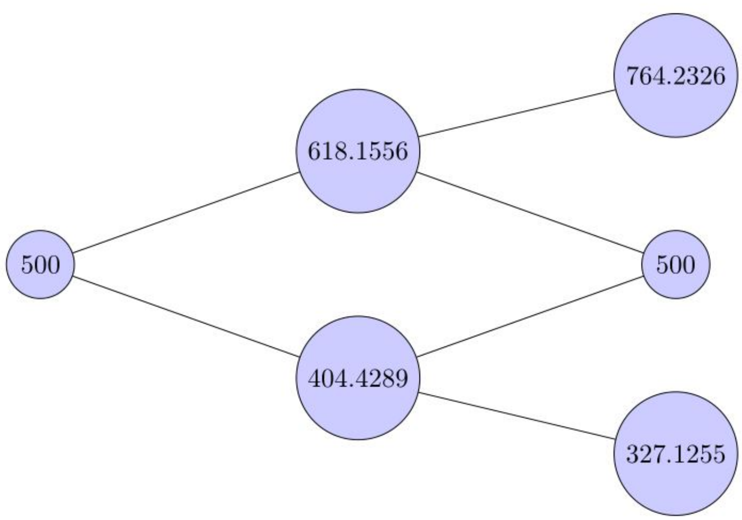
Solution: Since this is a forward binomial tree, we can use the "shortcut" for the risk-neutral probability. The length of a single period is $h = 1/2$. The risk-neutral probability is

$$p^* = \frac{1}{1+e^{\sigma\sqrt{h}}} = \frac{1}{1+e^{0.3\sqrt{1/2}}} = 0.447165.$$

The up and down factors for the **futures** tree are

$$u_F = e^{\sigma\sqrt{h}} = 1.236311 \quad \text{and} \quad d_F = 1/u_F = e^{-\sigma\sqrt{h}} = 0.8088579.$$

Here is the **futures** tree:



The possible payoffs at the leaves of the tree, should the option be held until the expiration date are

$$V_{uu} = 264.2326, \quad V_{ud} = V_{dd} = 0.$$

As usual, working backwards through the tree, we get the continuation value at the *up* node first:

$$CV_u = e^{-0.06/2} p^* V_{uu} = 114.6635.$$

The value of immediate exercise at that node is

$$IE_u = 618.1556 - 500 = 118.1556.$$

We conclude that it's best to exercise early at the *up* node and that the value of the American call at that node equals the immediate-exercise value, i.e., $V_u^A = 118.1556$.

The *down* node is trivial since the option's continuation value and immediate exercise are zero.

Since the futures in the replicating portfolio do not cost anything at time-0, the value of the call is exactly the amount lent in the replicating portfolio, i.e.,

$$B = V_C^A(0) = e^{-0.06/2} p^* V_u^A = 51.27353.$$

Problem 4.3. You are fiven a TRUE/FALSE exam with 30 questions. Suppose that you need to answer 21 questions correctly in order to pass. You have no idea what the class is about and decide to toss a fair coin to answer all the questions; you circle TRUE if the outcome is tails and you circle FALSE if the outcome is heads. What is your estimate of the probability p that you manage to pass the exam using this strategy?

Hint: It is best to use the **normal approximation** to get the approximate probability. There is no need to use the continuity correction.

Solution: Let us denote the number of correct answers you get using the coin-toss strategy by X. Then, $X \sim b(30, 1/2)$. The mean of X is $30 \cdot \frac{1}{2} = 15$ and its variance is $30 \cdot \frac{1}{2} \cdot \frac{1}{2} = 7.5$. So, the standard deviation of X is $\sqrt{7.5} \approx 2.74$. We can express the probability p as

$$p = \mathbb{P}[X \geq 21] = \mathbb{P}[\frac{X-15}{2.74} \geq \frac{21-15}{2.74}] \approx \mathbb{P}[\frac{X-15}{2.74} \geq 2.189781].$$

This probability is approximately $\Phi(+\infty) - \Phi(2.19) = 1 - \Phi(2.19) = 1 - 0.9857 = 0.0143$.

Problem 4.4. The current price of a continuous-dividend-paying stock is \$100 per share. The stock's dividend yield is 0.02. According to your model, the expected value of the stock price in one year is \$110 per share. You are also given:

The risk-free interest rate exceeds the dividend yield.

The one-year forward price on a share of this stock is denoted by F. At this price you are willing to enter into the forward. What is the smallest range of values F can take according to the above information?

Solution: Using the fact that the investor is willing to enter a forward contract, we conclude that the forward contract's profit is positive. So,

$$\mathbb{E}[S(2)] > F \quad \Rightarrow \quad 110 > F$$

On the other hand, we know that

$$F = S(0)e^{(r-\delta)1} = 100e^{1(r-0.02)} > 100.$$

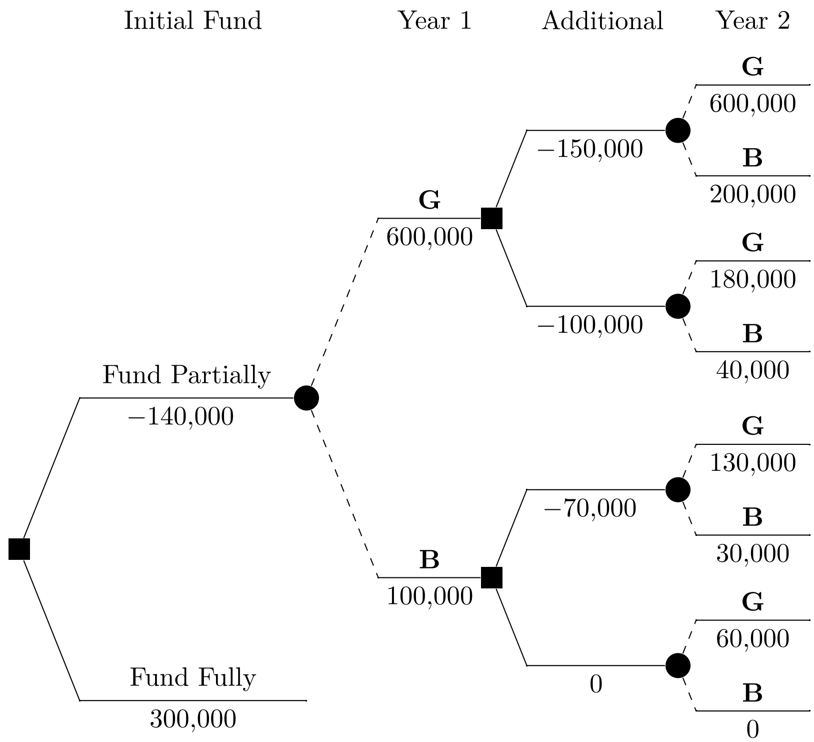
So, the most we can say about F is that $110 > F > 100$.

Problem 4.5. A marketing department is considering a marketing campaign. If the marketing campaign is fully funded at time-0 the project has a net present value of 300,000.

The decision tree below shows the cash flows associated with the extra revenue due to the marketing campaign being conducted in two stages. If the funding in the beginning of the Year 1 (i.e., at $t = 0$) is only partial with an option to provide different amounts of funding at the beginning of Year 2 (i.e., at $t = 1$) depending on how well the marketed product is doing.

This tree reflects two possible receptions of the marketed product at each information node ($\mathbf{G}$ = good, $\mathbf{B}$ = bad). The probability of the product being a success is given to be 1/2 and the probability of it being mediocre is 1/2. Failure is not an option.

Assume the interest rate is 0%.
Find the **initial** (i.e., at $t = 0$) value of the option to fund partially.



Solution: As usual, when pricing (real) options, we are moving backwards through the tree.

- In the *uppermost final* information node, the possible cashflows are 600,000 with probability 1/2 and 200,000 with probability 1/2. So, the value of the project at that node equals
$$600000\left(\frac{1}{2}\right) + 200000\left(\frac{1}{2}\right) = 400000.$$
- In the *second-by-height final* information node, the possible cashflows are 180,000 with probability 1/2 and 40,000 with probability 1/2. So, the value of the project at that node equals
$$180000\left(\frac{1}{2}\right) + 40000\left(\frac{1}{2}\right) = 110000.$$
- In the *third-by-height final* information node, the possible cashflows are 130,000 with probability 1/2 and 30,000 with probability 1/2. So, the value of the project at that node equals
$$130000\left(\frac{1}{2}\right) + 30000\left(\frac{1}{3}\right) = 80000.$$
- In the *lowest final* information node, the possible cashflows are 60,000 with probability 1/2 and 0 with probability 1/2. So, the value of the project at that node equals
$$60000\left(\frac{1}{2}\right) = 30000.$$

We continue working backwards, at the **upper decision** node at the end of Year 1, we can go "up" or "down" in the tree.

- We go "up" by investing 150,000; combining this cashflow with the average revenue at the *uppermost final* node, we get the total effect of going "up" to be
$$400000 - 150000 = 250000.$$

- We go ”down” by investing 100,000; combining this cashflow with the average revenue at the *second-by-height final* node, we get the total effect of going ”down” to be

$$110000 - 100000 = 10000.$$

Comparing the two values we obtained, we conclude that the **optimal** decision at this node is to go ”up” and we keep the value of this project at this node to be

$$250000 + 600000 = 850000.$$

Here, we took into account that the first year the campaign worked resulting in 600,000 in extra revenue in Year 1.

Similarly, at the **lower decision** node at the end of Year 1, we can go ”up” or ”down” in the tree.

- We go ”up” by investing 70,000; combining this cashflow with the average revenue at the *third-by-height final* node, we get the total effect of going ”up” to be

$$80000 - 70000 = 10000.$$

- We go ”down” by investing nothing; so, the total effect of going ”down” is 30000. Comparing the two values we obtained, we conclude that the **optimal** decision at this node is to go ”down” and we keep the value of this project at this node to be

$$30000 + 100000 = 130000.$$

Here, we took into account that the first year was mediocre resulting in 100,000 in extra revenue.

Altogether, at the information node corresponding to Year 1, we have that the expected value of the project is

$$850000\left(\frac{1}{2}\right) + 130000\left(\frac{1}{2}\right) = 490,000.$$

Now, we take into account that we funded the marketing campaign partially with 140,000. So, the total expected present value of the cashflows we get should we decide to fund partially is

$$490000 - 140000 = 350000$$

The total value of the option is

$$350000 - 300000 = 50000$$

Problem 4.6. For Company A there is a p_A chance that no claim is made during the coming year. If one or more claims are made, the total claim amount is normally distributed with mean μ_A and standard deviation σ_A.

For Company B there is a p_B chance that no claim is made during the coming year. If one or more claims are made, the total claim amount is normally distributed with mean μ_B and standard deviation σ_B.

The total claim amounts of the two companies are independent. Calculate the probability that, in the coming year, Company B's total claim amount will strictly exceed Company A's total claim amount. In the expression you get, you are allowed to use the parameters given and the standard normal cumulative distribution function N.

Solution: There are two possible ways in which the total claims to Company B exceed the total claims to Company A:

- Company B has at least one claim and Company A has no claims, or
- both companies have claims, but the ones to Company B exceed the ones to Company A.

The probability of the first scenario is $p_A(1-p_B)$. As for the second scenario, more work is needed. The probability that both companies have claims is $(1-p_A)(1-p_B)$. Let the random variable X_A denote the total claims to Company A and let the random variable X_B denote the total claims to Company B in this case. Then,

$$\mathbb{P}[X_B > X_A] = \mathbb{P}[X_B - X_A > 0].$$

We know that the random variable $X_B - X_A$ is also normally distributed. Its mean is $\mu = \mu_B - \mu_A$ while its variance equals $\sigma^2 = \sigma_A^2 + \sigma_B^2$. So, the probability in the above display equals

$$\mathbb{P}[X_B - X_A > 0] = \mathbb{P}\left[\frac{X_B - X_A - (\mu_B - \mu_A)}{\sqrt{\sigma_A^2 + \sigma_B^2}} > \frac{0 - (\mu_B - \mu_A)}{\sqrt{\sigma_A^2 + \sigma_B^2}}\right]$$
$$= 1 - N\left(-\frac{\mu_B - \mu_A}{\sqrt{\sigma_A^2 + \sigma_B^2}}\right) = N\left(\frac{\mu_B - \mu_A}{\sqrt{\sigma_A^2 + \sigma_B^2}}\right).$$

Problem 4.7. *Source: Sample P exam Problem #81.*
Claims filed under auto insurance policies follow a normal distribution with mean $19,400$ and standard deviation $5,000$. Calculate the probability that the average of 25 randomly selected claims exceeds $20,000$.

Solution: Let $X_i, i = 1, \dots, 25$ denote the individual claim amount for our 25 randomly selected claims. We are given that, for every i,

$$X_i \sim Normal(mean = 19400, sd = 5000).$$

Let $\bar{X}$ denote the average of our 25 random variables. Then,

$$\bar{X} \sim Normal\left(mean = 19400, sd = \frac{5000}{\sqrt{25}} = 1000\right).$$

Then,

$$\mathbb{P}[\bar{X} > 20000] = \mathbb{P}\left[\frac{\bar{X} - 19400}{1000} > \frac{20000 - 19400}{1000} = 0.6\right] = 1 - N(0.6) = 1 - 0.7257 = 0.2743.$$

Problem 4.8. A brand of light bulb has a lifetime (in months) that is modelled as normally distributed with mean 4 and variance 1. Roger wants to purchase a number of these bulbs with the intention of replacing them successively as they burn out. The light bulbs have mutually independent lifetimes.

What the smallest number of bulbs Roger needs to buy so that the succession of light bulbs produces light for at least 96 months with probability at least 0.9772?

Solution: Let $X_i, i = 1, 2, \dots$ be a sequence of independent normal random variables each with mean 4 and variance 1. For every $n = 1, 2, \dots,$ let

$$S_n = X_1 + X_2 + \cdots + X_n.$$

Each S_n is normally distributed with mean $4n$ and variance n. We are looking for the smallest n such that

$$\mathbb{P}[S_n > 96] \geq 0.9772.$$

The probability on the left-hand side above can, through transition to standard units, be rewritten as

$$\mathbb{P}[S_n > 96] = \mathbb{P}\left[\frac{S_n - 4n}{\sqrt{n}} > \frac{96 - 4n}{\sqrt{n}}\right] = 1 - N\left(\frac{96 - 4n}{\sqrt{n}}\right) = N\left(\frac{4n - 96}{\sqrt{n}}\right).$$

Since the standard normal tables tell us that $N(2) = 0.9772$. So, we would like to solve for n in

$$\frac{4n - 96}{\sqrt{n}} = 2 \quad \Rightarrow \quad 4n - 96 = 2\sqrt{n} \quad \Rightarrow \quad 2n - 48 = \sqrt{n} \quad \Rightarrow \quad (2n - 48)^2 = n.$$

Now, we solve for n in the following quadratic

$$4(n - 24)^2 = n \quad \Rightarrow \quad 4(n^2 - 48n + 576) = n \quad \Rightarrow \quad 4n^2 - 193n + 576 = 0.$$

We get

$$n_{1,2} = \frac{193 \pm \sqrt{193^2 - 4(4)(576)}}{8} = \frac{193 \pm \sqrt{28033}}{8} = \frac{193 \pm 167.4306}{8}.$$

We get $n_1 = 3.196175$ and $n_2 = 45.05383$. We discard the smaller option since it corresponds to the quantile of $1 - 0.9772$. We keep the latter solution. This gives us $n \geq 46$.

Problem 4.9. The profit of an individual franchise which sells Pokemon plushies is normally distributed with mean 100000 and standard deviation 30000. The profits of individual franchises are assumed to be independent.

Calculate the probability that the total profit of two randomly selected franchises will exceed 1.6 times the profit a third randomly selected franchise.

Solution: Let $X_i, i = 1, 2, 3$ be the profits of the three randomly chosen franchises. I will express everything in tens-of-thousands and write, for all i,

$$X_i \sim Normal(mean = 10, sd = 3).$$

We are tasked with finding the probability

$$\mathbb{P}[X_1 + X_2 > 1.6X_3] = \mathbb{P}[X_1 + X_2 - 1.6X_3 > 0].$$

The random variable $X_1 + X_2 - 1.6X_3$ is normally distributed with the following mean and variance:

$$\mathbb{E}[X_1 + X_2 - 1.6X_3] = 10 + 10 - 1.6(10) = 4,$$

$$\mathrm{Var}[X_1 + X_2 - 1.6X_3] = 3^2 + 3^2 + (1.6)^2(3)^2 = 41.04.$$

So,

$$X_1 + X_2 - 1.6X_3 \sim Normal(mean = 4, sd = 6.406247).$$

The probability we are looking for is now

$$\mathbb{P}[X_1 + X_2 - 1.6X_3 > 0] = \mathbb{P}\left[\frac{X_1 + X_2 - 1.6X_3 - 4}{6.406247} > \frac{0 - 4}{6.406247} = -0.6243905\right]$$
$$\approx 1 - N(-0.62) = N(0.62) = 0.7324.$$