M378K Introduction to Mathematical Statistics Problem Set #9

Moment generating functions.

Definition 9.1. The k^{th} moment of a random variable Y taken about the origin is defined as $\mathbb{E}[Y^k]$ provided that the expectation exists. We write

$$\mu_k = \mathbb{E}[Y^k]$$

when there is no ambiguity about the random variable in question.

Remark 9.2. μ_k is also referred to as the k^{th} raw moment.

Remark 9.3. In particular, $\mu_1 = \mu$ happens to be the **mean** of the random variable Y.

Definition 9.4. The k^{th} central moment of a random variable Y is defined as $\mathbb{E}[(Y-\mu)^k]$ provided that the expectation exists. We write

$$\mu_k^c = \mathbb{E}[(Y - \mu)^k]$$

when there is no ambiguity about the random variable in question.

Remark 9.5. μ_k is also referred to as the k^{th} moment of a random variable Y taken about its mean.

Definition 9.6. The moment-generating function (mgf) m_Y for a random variable Y is defined as

$$m_Y(t) = \mathbb{E}[e^{tY}]$$

for all t for which the above expectation exists. In fact, we say that the moment-generating function **exists** if there exists a positive number b such that $m_Y(t)$ is finite for all t such that $|t| \le b$.

Problem 9.1. How much is $m_Y(0)$?

Remark 9.7. On the choice of terminology ...

Step 1.

$$\frac{d}{dt}m_Y(t) = ?$$

$$m_Y'(0) = ?$$

$$\frac{d^2}{dt^2}m_Y(t) = ?$$

$$m_Y''(0) = ?$$

 $\underline{Step~5.}~\textit{What do you suspect the}~\textbf{generalization}~\textit{of the above would be?}$

Theorem 9.8. If m_Y exists, then for $k \in \mathbb{N}$, we have

$$m_Y^{(k)}(0) = \mu_k.$$

Example 9.9. Let $Y \sim b(n = 1, p)$, i.e., let Y model a Bernoulli trial with the probability of success denoted by p. Find m_Y .

Proposition 9.10. Let Y_1 and Y_2 be independent random variables with m.g.f.s denoted by m_{Y_1} and m_{Y_2} . Define $Y = Y_1 + Y_2$. Then, for every t for which both m_{Y_1} and m_{Y_2} are well defined, we have

$$m_Y(t) =$$

Proof. By definition:

$$m_Y(t) =$$

Using $Y = Y_1 + Y_2$, we can substitute $Y_1 + Y_2$ for Y in the expression above. So,

$$m_Y(t) =$$

One of the properties of the exponential function is that $e^{A+B}=e^A\times e^B$. Thus, the above becomes:

$$m_Y(t) =$$

Recall that Y_1 and Y_2 are assumed to be independent random variables. With this in mind, we get:

$$m_Y(t) =$$

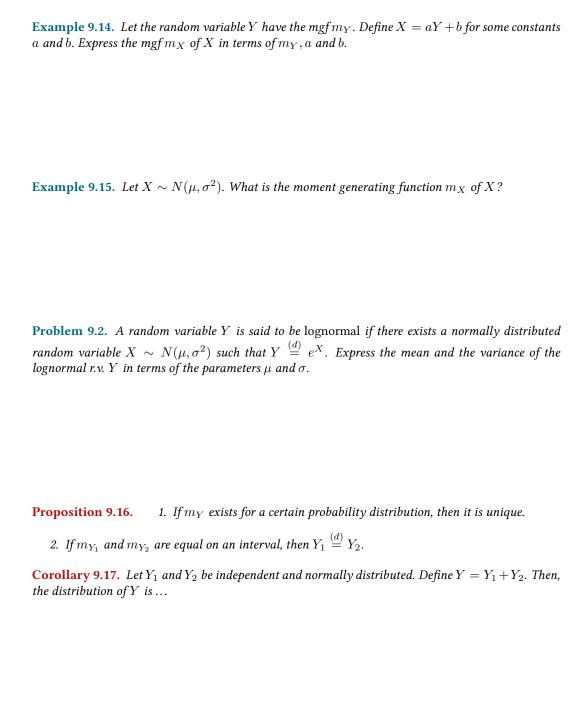
Finally, using the definition of a m.g.f., we have

$$m_Y(t) =$$

Example 9.11. Let $Y \sim b(n, p)$. What is the moment generating function of Y?

Example 9.12. Let $N \sim Poisson(\lambda)$. What is the moment generating function m_N of N?

Example 9.13. Let $Z \sim N(0,1)$. What is the moment generating function m_Z of Z?



Proof.

