M339D: October 28th, 2024.

Risk-Neutral Probability [cont'd].

Method: Binomial Stock-Price Tree

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Pricing by Replication

$$\boxed{V(0) = \Delta \cdot S(0) + B}$$

⋮

Def'n. The risk-neutral probability:

$$\boxed{p^* = \frac{e^{rh} - d}{u - d}}$$

$\Rightarrow$ The risk-neutral pricing formula

$$\boxed{V(0) = e^{-rT}\left[V_u \cdot p^* + V_d(1 - p^*)\right]}$$

Special Case: Forward Binomial Tree

$$\boxed{\sigma \ldots \text{volatility}}$$

$$u := e^{rh + \sigma\sqrt{h}}$$

$$d := e^{rh - \sigma\sqrt{h}}$$

The risk-neutral probability:

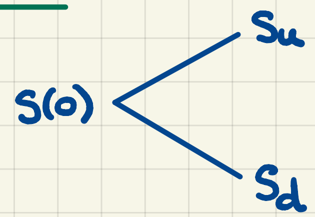
$$p^* = \frac{e^{rh} - d}{u - d} = \frac{\cancel{e^{rh}}^{\,1} - e^{\cancel{rh} - \sigma\sqrt{h}}}{e^{\cancel{rh} + \sigma\sqrt{h}} - e^{\cancel{rh} - \sigma\sqrt{h}}}$$

$$p^* = \frac{1 - e^{-\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}} \cdot \frac{e^{\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}}} = \frac{\cancel{e^{\sigma\sqrt{h}} - 1}^{\,1}}{\underbrace{e^{2\sigma\sqrt{h}} - 1}_{\cancel{(e^{\sigma\sqrt{h}} - 1)}^{\,1}(e^{\sigma\sqrt{h}} + 1)}}$$

$$\boxed{p^* = \frac{1}{1 + e^{\sigma\sqrt{h}}}} \xrightarrow{h \to 0} \frac{1}{2}$$

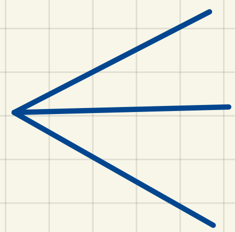
The shortcut **ONLY** for the forward binomial tree.

Inspiration.



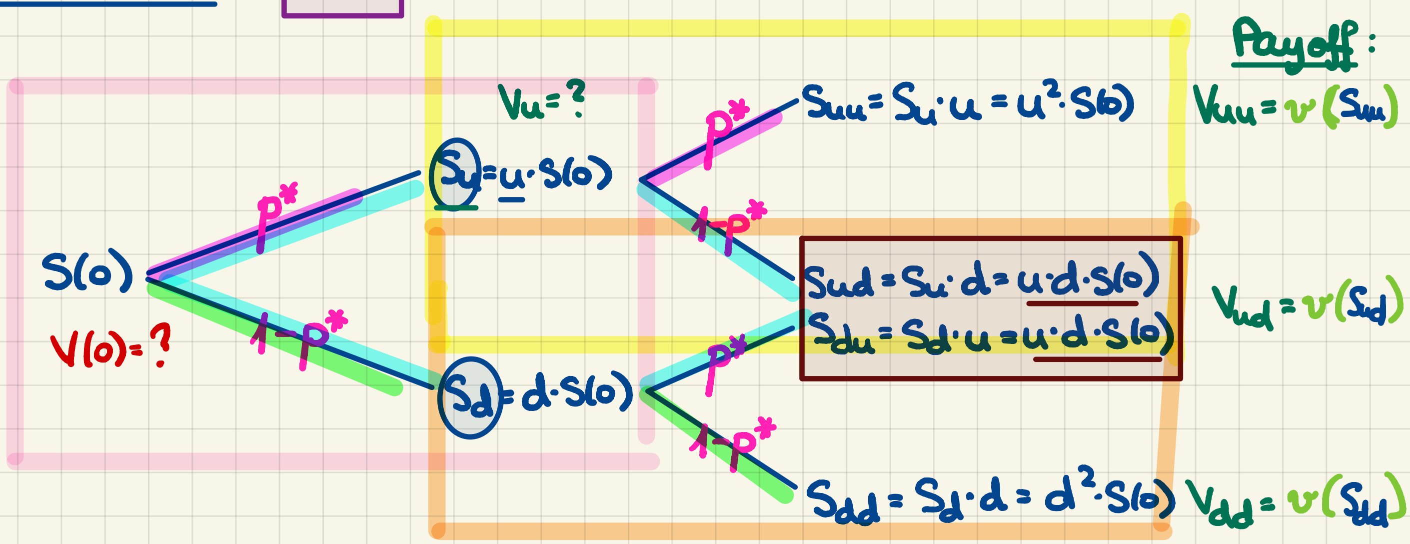
$\boxed{S(T)}$

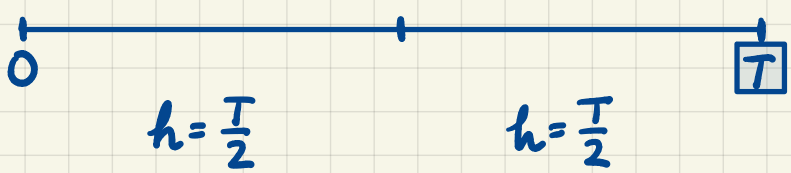
Q: How can we make the model for $S(T)$ more complex but still interpretable?



k-nary trees

Two Periods: $\boxed{n=2}$





populating the tree $\longrightarrow$

$\longleftarrow$ pricing the option

- **up node:** replicating portfolio for the option:

$$\Delta_u = \frac{V_{uu} - V_{ud}}{S_{uu} - S_{ud}}$$

$$B_u = e^{-rh} \cdot \frac{u V_{ud} - d \cdot V_{uu}}{u-d}$$

$\Rightarrow$ the option's value @ the up node:

$$V_u = \underline{\Delta_u \cdot S_u + B_u} = \underline{e^{-rh}\left[p^* \cdot V_{uu} + (1-p^*) \cdot V_{ud}\right]}$$

w/ $\boxed{p^* = \frac{e^{rh} - d}{u-d}}$

- **down node:** Δ_d, B_d

$\Rightarrow$ $V_d = \Delta_d \cdot S_d + B_d = e^{-rh}\left[p^* \cdot V_{ud} + (1-p^*) \cdot V_{dd}\right]$ w/

- **ROOT NODE:** $\Delta_0 = \frac{V_u - V_d}{S_u - S_d}$

$$B_0 = e^{-rh} \cdot \frac{u \cdot V_d - d \cdot V_u}{u-d}$$

$\Rightarrow$ $\boxed{V(0) = \Delta_0 \cdot S(0) + B_0}$

From the "risk·neutral perspective":

$$V(0) = e^{-rh} \cdot \left[p^* \cdot V_u + (1-p^*) \cdot V_d\right]$$

$$= e^{-rh}\left[p^* \cdot e^{-rh}\left(p^* \cdot V_{uu} + (1-p^*) \cdot V_{ud}\right) + (1-p^*) e^{-rh}\left(p^* \cdot V_{ud} + (1-p^*) \cdot V_{dd}\right)\right]$$

$$= \underbrace{e^{-r \overset{T}{(2h)}}}_{\text{Discounting}} \underbrace{\left[(p^*)^2 V_{uu} + 2 \cdot p^*(1-p^*) V_{ud} + (1-p^*)^2 \cdot V_{dd}\right]}_{\text{Risk·Neutral Expectation of the Payoff}}$$

Generally: $\boxed{V(0) = e^{-rT} E^*[V(T)]}$

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Problem Set #10

Binomial option pricing: Two or more periods.

Problem 10.1. For a two-period binomial model, you are given that:

(1) each period is one year;
(2) the current price of a non-dividend-paying stock S is $S(0) = \$20$;
(3) $u = 1.2$, with u as in the standard notation for the binomial model;
(4) $d = 0.8$, with d as in the standard notation for the binomial model;
(5) the continuously compounded risk-free interest rate is $r = 0.04$.

Consider a **special** call option which pays the excess above the strike price $K = 23$ (if any!) at the end of **every** binomial period.

Find the price of this option.

$\longrightarrow$: $1^{\underline{\underline{st}}}$ $p^* = ?$

$2^{\underline{\underline{nd}}}$ Tree + Payoffs

$3^{\underline{\underline{rd}}}$ Risk-Neutral Pricing.