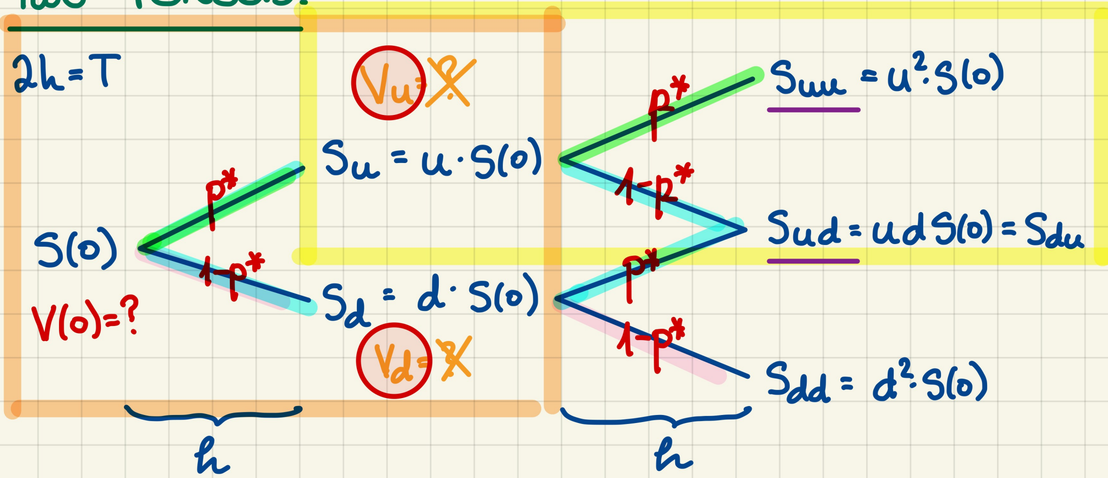
M339D : April 18th, 2022.

Two Periods.

$2h = T$



Payoff

$V_{uu} = v(S_{uu})$

$V_{ud} = v(S_{ud})$

$V_{dd} = v(S_{dd})$

populating the tree →

← pricing the option

- up node:

replicating portfolio for the option @ the up node:

$$\begin{cases} \Delta_u = e^{-\delta \cdot h} \dfrac{V_{uu} - V_{ud}}{S_{uu} - S_{ud}} \\ B_u = e^{-rh} \dfrac{u \cdot V_{ud} - d \cdot V_{uu}}{u-d} \end{cases}$$

$\Rightarrow$ the option's value @ the up node:

$$V_u = \Delta_u \cdot S_u + B_u = \boxed{e^{-rh}\left[p^* \cdot V_{uu} + (1-p^*) \cdot V_{ud}\right]}$$

$$\boxed{p^* = \frac{e^{(r-\delta)h} - d}{u-d}}$$

- down node: Δ_d, B_d

$$\Rightarrow \quad V_d = \Delta_d \cdot S_d + B_d = \boxed{e^{-rh}\left[p^* \cdot V_{ud} + (1-p^*) \cdot V_{dd}\right]}$$

- **ROOT node:** the replicating portfolio:

$$\Delta_0 = e^{-\delta h} \cdot \frac{V_u - V_d}{S_u - S_d}$$

$$B_0 = e^{-rh} \, \frac{u \cdot V_d - d \cdot V_u}{u-d}$$

$$\Rightarrow \boxed{V(0) = \Delta_0 \cdot S(0) + B_0}$$

From the risk-neutral perspective: w/ same p^*

$$V(0) = e^{-rh} \cdot \left[p^* \cdot V_u + (1-p^*) \cdot V_d \right]$$

$$V(0) = e^{-rh} \cdot \Big[p^* \, e^{-rh} \left(p^* \cdot V_{uu} + (1-p^*) \cdot V_{ud} \right) + (1-p^*) \, e^{-rh} \left(p^* \cdot V_{ud} + (1-p^*) \cdot V_{dd} \right) \Big]$$

$$V(0) = \underbrace{e^{-r \overset{T}{\overset{\parallel}{(2h)}}}}_{\text{discounting}} \underbrace{\left[(p^*)^2 \cdot V_{uu} + 2 \cdot p^*(1-p^*) \cdot V_{ud} + (1-p^*)^2 \cdot V_{dd} \right]}_{\text{Risk-neutral Expectation of the Payoff}}$$

Generally:

$$\boxed{V(0) = e^{-rT} \, \mathbb{E}^*[V(T)]}$$

Problem 10.4. For a two-period binomial model, you are given that:

(1) each period is one year; $h=1$ $\delta=0$

(2) the current price of a non-dividend-paying stock S is $S(0) = \$20$;

(3) $\boxed{u = 1.2,}$ with u as in the standard notation for the binomial model;

(4) $\boxed{d = 0.8,}$ with d as in the standard notation for the binomial model;

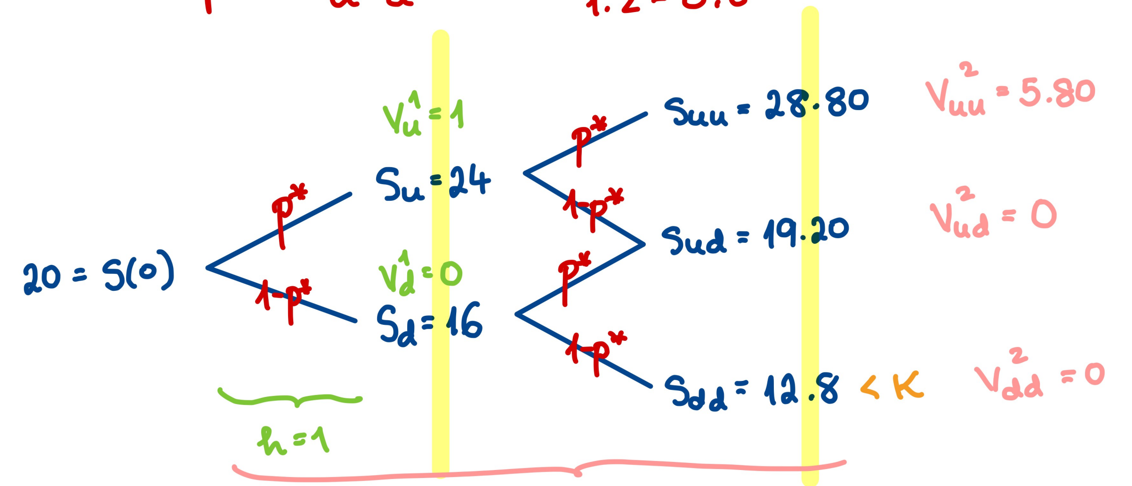
(5) the continuously compounded risk-free interest rate is $r = 0.04$.

Consider a **special** call option which pays the excess above the strike price $\boxed{K = 23}$ (if any!) at the end of **every** binomial period.

Find the price of this option.

→: The risk-neutral probability:

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{e^{0.04} - 0.8}{1.2 - 0.8} = 0.602027$$



We can think about our option as:

- call w/ exercise date @ $T_1 = 1$: $V_1(0) = e^{-r} \cdot p^* \cdot V_u^1$

and

- call w/ exercise date @ $T_2 = 2$: $V_2(0) = e^{-2r} \cdot (p^*)^2 \cdot V_{uu}^2$

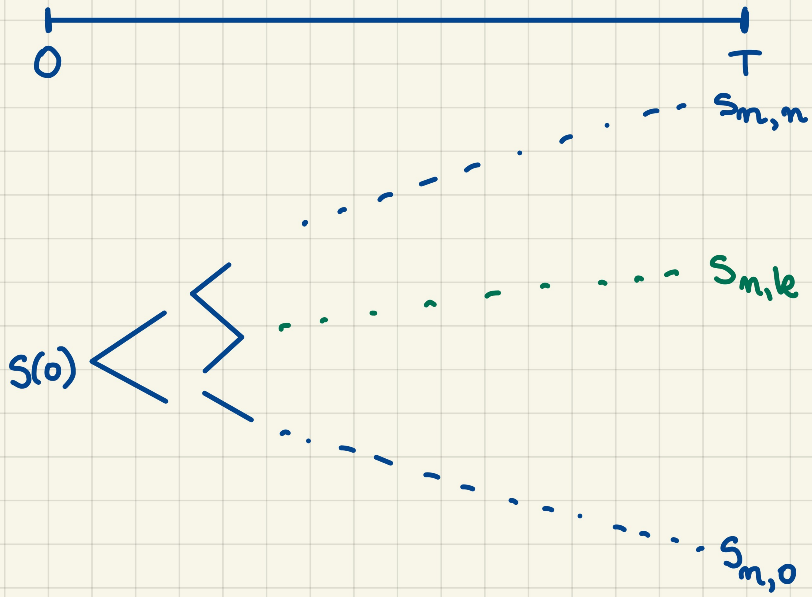
\+

$$V_{SC}(0) = V_1(0) + V_2(0) = 2.5189$$

Multiple Binomial Periods.

T... exercise date of a European option
n... # of periods

} the length of each period $\boxed{h = \frac{T}{n}}$



... k = # of steps up

These $(n+1)$ values are the support of the r.v. $S(T)$.