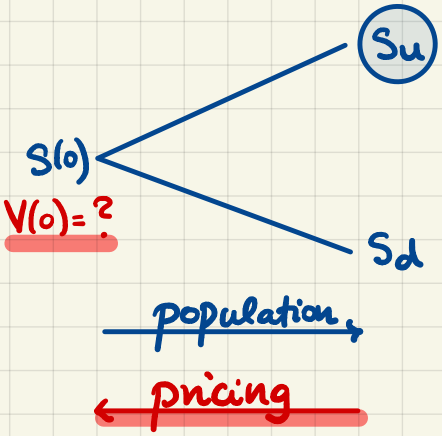
M339W: March 10^{th}, 2023.

Binomial Option Pricing [cont'd].



Payoff		Replicating Portfolio
$V_u := v(S_u)$	$=$	$\Delta \cdot S_u + Be^{rh}$
$V_d := v(S_d)$	$=$	$\Delta \cdot S_d + Be^{rh}$

In the binomial model, any derivative security can be **replicated** w/ a portfolio of this form:

- Δ shares of stock
- B @ the ccrfir r

$\uparrow$ time·0 holdings

$$\begin{cases} \Delta > 0 & \text{buying} \\ \Delta = 0 & \text{"nothing"} \\ \Delta < 0 & \text{shorting} \end{cases}$$

$$\begin{cases} B > 0 & \text{lending (buying a bond)} \\ B = 0 & \text{"nothing"} \\ B < 0 & \text{borrowing (issuing a bond)} \end{cases}$$

If we can find Δ and B, then $\boxed{V(0) = \Delta \cdot S(0) + B}$

We get a system of two equations w/ two unknowns:

$$\begin{aligned} \Delta \cdot S_u + \cancel{Be^{rh}} &= V_u \\ -\Delta \cdot S_d - \cancel{Be^{rh}} &= -V_d \end{aligned} \Big\} -$$

$$\Delta(S_u - S_d) = V_u - V_d$$

$$\boxed{\Delta = \frac{V_u - V_d}{S_u - S_d}}$$

unitless

$$\frac{V_u - V_d}{S_u - S_d} \cdot S_u + Be^{rh} = V_u$$

$$Be^{rh} = V_u - \frac{V_u - V_d}{\cancel{S(0)}(u-d)} \cdot \cancel{S(0)} \cdot u = \frac{\cancel{V_u \cdot u} - V_u \cdot d - \cancel{V_u \cdot u} + V_d \cdot u}{u-d}$$

$$\boxed{B = e^{-rh} \cdot \frac{u \cdot V_d - d \cdot V_u}{u-d}}$$ cash

By the Law of the Unique Price:

$$\boxed{V(0) = \Delta \cdot S(0) + B}$$ Pricing by Replication

Graphical Interpretation.

Consider a European call w/ exercise date @ the end of the period and w/ the strike price K such

$S_d < K < S_u$

$$S(0) \to S_u, \quad S(0) \to S_d$$

Payoff

$V_u = S_u - K$

$\boxed{V_d = 0}$

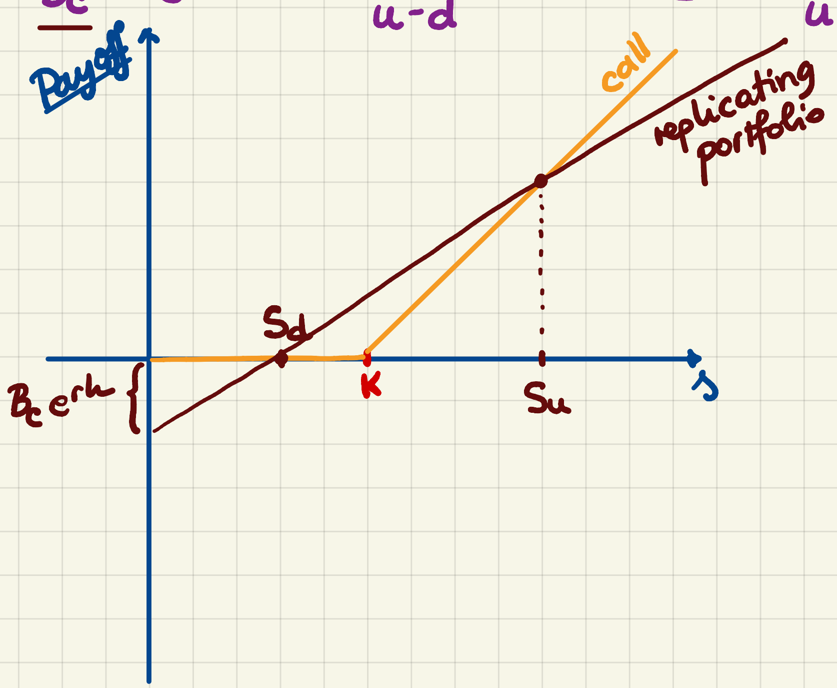
In the replicating portfolio:

- $\Delta_c = \frac{V_u - V_d}{S_u - S_d} = \boxed{\frac{S_u - K}{S_u - S_d}} \in (0,1)$

- $B_c = e^{-rh} \cdot \frac{u \cdot V_d - d \cdot V_u}{u-d} = -e^{-rh} \cdot \frac{d \cdot V_u}{u-d} < 0$

$\Downarrow$

Borrowing!



slope : Δ

intercept < 0

$\Rightarrow$ borrowing

Problem 8.3. Let the continuously compounded risk-free interest rate be equal to 0.04. Consider a one-period binomial tree with every period of length one year used to model the stock price of a stock whose current price is \$80 per share. In the model, it is assumed that the stock price can either go up by \$5 or down by \$4.

You use the binomial tree to construct a replicating portfolio for a 78−strike call option on the above stock. What is the stock investment in the replicating portfolio?

$\rightarrow$:

$$\begin{array}{ccc} & & S_u = 85 \\ S(0) = 80 & \begin{array}{c} \nearrow \\ \searrow \end{array} & \\ & & S_d = 76 \end{array}$$

$V_u = (85-78)_+ = 7$

$V_d = (76-78)_+ = 0$

$$\Delta = \frac{V_u - V_d}{S_u - S_d} = \frac{7-0}{85-76} = \frac{7}{9} \qquad \square$$

Problem 8.4. Let the continuously compounded risk-free interest rate be equal to 0.04. Consider a one-period binomial tree with every period of length one year used to model the stock price of a non-dividend-paying stock whose current price is $50 per share. In the model, it is assumed that the stock price can either go up by 5% or down by 10%.

You use the binomial tree to construct a replicating portfolio for a 45−strike call on the above stock. What is the risk-free investment in the replicating portfolio?

→ :

$S(0) = 50$

$S_u = 50(1.05) = 52.5$ $\quad V_u = (52.5-45)_+ = 7.5$

$S_d = 50(0.90) = 45$ $\quad V_d = (45-45)_+ = 0$

$$B = e^{-rh} \cdot \frac{u \cdot V_d - d \cdot V_u}{u-d} = e^{-0.04} \cdot \frac{1.05(0) - 0.90(7.5)}{1.05 - 0.90} = \underline{-43.2355}$$

Borrowing!