UNIVERSITY OF TEXAS AT AUSTIN

Lecture 18

The forward binomial tree.

18.1. **Introduction.** Now that we have learned how to use the one-period binomial asset-pricing model to price European-style derivative securities, the question arises on how to set up the asset-pricing model itself. More precisely, how do we define u and d?

We still model a market in which there is the possibility of a riskless investment. The interest rate governing the riskless investment is the continuously-compounded, risk-free interest rate r.

For now, the risky asset is a continuous-dividend-paying stock $\mathbf{S}$ with the dividend yield equal to δ. In the future, we will be able to handle similar risky assets – such as foreign currencies, market indices, or futures contracts – in an analogous way. The new parameter we introduce is a measure of the variability of stoch prices. More precisely, it is understood as the annualized standard deviation of the realized returns on the stock. We call it the **volatility**, and denote it by σ. It is customary to assume that $\sigma > 0$.

The length of a single period in our binomial tree is still equal to h. However, the volatility parameter above corresponds to the length of time equal to one year. We assume that realized returns are

- identically distributed for time periods of the same length (this is *time homogeneity*), and
- *independent* over disjoint time intervals (or time intervals only touching at an end-point).

It is straightforward, then, to show that the appropriate rescaling of the volatility parameter to a time period of length h is

$$\sigma_h = \sigma\sqrt{h}.$$

18.2. **The forward tree definition.** Consider the forward contract of stock $\mathbf{S}$ for delivery at time h.

Question 18.1. What is the **forward price** for delivery of one share of $\mathbf{S}$ at time h?

Solution:

$$F_{0,h}(S) = S(0)e^{(r-\delta)h}$$

The S_u and S_d in the forward tree are modeled so that the return of the forward contract is, in a sense, centered between the returns for the "up" and "down" states-of-the-world. We set

$$S_u = F_{0,h}(S)e^{\sigma\sqrt{h}} = S(0)e^{(r-\delta)h+\sigma\sqrt{h}},$$
$$S_d = F_{0,h}(S)e^{-\sigma\sqrt{h}} = S(0)e^{(r-\delta)h-\sigma\sqrt{h}}.$$

In other words, u and d are explicitly given by

$$u = e^{(r-\delta)h+\sigma\sqrt{h}} \quad \text{and} \quad d = e^{(r-\delta)h-\sigma\sqrt{h}}.$$

Question 18.2. What is the ratio S_u/S_d?

Solution:

$$\frac{S_u}{S_d} = e^{2\sigma\sqrt{h}}$$

Question 18.3. What additional conditions need to be made on u and d so that the *no-arbitrage condition* for the binomial asset pricing model is

Solution: None, since

$$d < e^{(r-delta)h} < u$$

is equivalent to

$$e^{-\sigma\sqrt{h}} < 1 < e^{\sigma\sqrt{h}}$$

and true for every $\sigma > 0$.

18.3. **The risk-neutral probability.**

Question 18.4. What is the expression for the risk-neutral probability in the forward tree?

Solution:

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d} = \cdots = \frac{1}{1+e^{\sigma\sqrt{h}}}\,.$$

Question 18.5. What is the limit of p^* as $h \to 0$?

Solution: 1/2

18.4. **Exercises.**

Problem 18.1. Consider a non-dividend-paying stock with a current price of \$70 per share. Its volatility is given to be 0.25.

The continuously-compounded, risk-free interest rate equals 4%.

We use a one-period **forward** binomial tree to model the stock price at the end of the one year.

What is the price of a one-year, at-the-money European call option on this stock consistent with the above stock-price model?

Solution: The up and down factors are

$$u = e^{0.04+0.25} = 1.3364, d = e^{-0.21} < 1.$$

The risk-neutral probability equals

$$p^* = \frac{1}{1+e^{0.25}} = 0.4378.$$

So,

$$V_C(0) = e^{-0.04} \times 0.4378 \times 70 \times (1.3364 - 1) = 9.9056.$$