# M339G: March 14th, 2025.

## Any Bivariate Normal.

Random variables $U$ and $V$ are said to be bivariate normal w/ parameters $\mu_U, \mu_V, \sigma_U, \sigma_V$ and $\rho$ if

$$\boxed{(X,Y) := \left(\frac{U-\mu_U}{\sigma_U}, \frac{V-\mu_V}{\sigma_V}\right)}$$

has the standard normal dist'n w/ correlation $\rho$.

**Note:**

- $\rho(U,V) = ?$

By def'n:

$$\boxed{\begin{aligned} U &= \mu_U + \sigma_U \cdot X \\ V &= \mu_V + \sigma_V \cdot Y \end{aligned}}$$

$$\begin{aligned}
\rho(U,V) &= \frac{Cov[U,V]}{SD[U] \cdot SD[V]} \\
&= \frac{Cov[\cancel{\mu_U +} \sigma_U \cdot X, \cancel{\mu_V +} \sigma_V \cdot Y]}{SD[\cancel{\mu_U +} \sigma_U \cdot X] \cdot SD[\cancel{\mu_V +} \sigma_V \cdot Y]} \\
&= \frac{Cov[\sigma_U \cdot X, \sigma_V \cdot Y]}{SD[\sigma_U \cdot X] \cdot SD[\sigma_V \cdot Y]} \\
&= \frac{\cancel{\sigma_U} \cdot \cancel{\sigma_V} Cov[X,Y]}{\cancel{\sigma_U} \cdot SD[X] \cdot \cancel{\sigma_V} SD[Y]} = \rho(X,Y) = \rho
\end{aligned}$$

$\mu_U$ and $\mu_V$ are deterministic

- $\boxed{U \text{ and } V \text{ are independent iff } \rho = 0}$

## Example. Midterm and Final.

Midterm & final scores in a large class have an (approximately) bivariate normal dist'n w/ parameters

| | mean | sd |
|---|---|---|
| midterm scores: | 65 | 18 |
| final scores: | 60 | 20 |

correlation : 0.76

Q: What is the "estimated" mean final score of the students who were above the mean on the midterm?

→: Let $U$ be the midterm score, and let $V$ be the final score.

Let $X$ and $Y$ be $U$ and $V$ in std units, resp.

$Y = \rho X + \sqrt{1-\rho^2} \cdot Z$

$Y \mid X = x \sim \text{Normal}(\text{mean} = \rho x, \text{var} = 1-\rho^2)$

Our task is to find:

$$\mathbb{E}[V \mid U > \mu_U] = ?$$

Our ancillary task is to find:

$$\mathbb{E}[Y \mid X>0] = \int_{\cancel{-\infty}\ 0}^{\infty} \underbrace{\mathbb{E}[Y \mid X=x]}_{=\rho x} \cdot f_X(x \mid X>0)\, dx$$

The Law of Total Probability

for $x > 0$:

$$f_X(x \mid X>0)\,dx = \mathbb{P}[X \in (x, x+dx) \mid X>0]$$

$$= \frac{\mathbb{P}[X \in (x, x+dx) \text{ and } X>0]}{\mathbb{P}[X>0]}$$

$$= \frac{\mathbb{P}[X \in (x, x+dx)]}{\frac{1}{2}} = 2 \cdot f_X(x)\,dx$$

$$= 2 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx$$

$$\mathbb{E}[Y \mid X>0] = \int_0^\infty \rho x \cdot 2 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= \frac{2\rho}{\sqrt{2\pi}} \int_0^\infty x\, e^{-\frac{x^2}{2}} dx = \left[ \begin{array}{l} u = \frac{x^2}{2} \\ du = \frac{2x}{2} dx = x\,dx \end{array} \right]$$

$$= \frac{2\rho}{\sqrt{2\pi}} \int_0^\infty e^{-u} du = \frac{2\rho}{\sqrt{2\pi}} \underbrace{\left( -e^{-u} \Big|_{u=0}^{\infty} \right.}_{=1} = \frac{2\rho}{\sqrt{2\pi}}$$

In this problem: $\frac{1.52}{\sqrt{2\pi}} = \underline{0.6063923}$

$\Rightarrow$ Our final answer: $60 + 20(0.6063923) = \underline{72.12785}$

$\square$