M339D: March 5^{th}, 2025.

Review:

Def'n. An arbitrage portfolio is a portfolio whose profit is

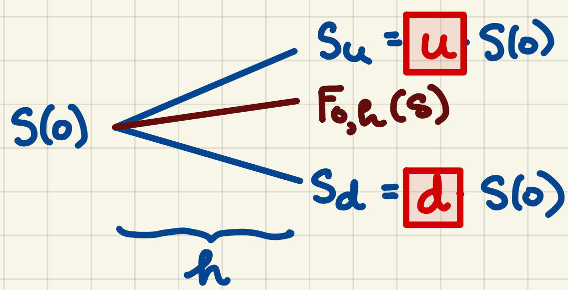
- nonnegative in all states of the world,

and

- strictly positive in @ least one state of the world.

Def'n. Consider a European-style derivative security. If a static portfolio has the same payoff as the derivative security, we say that it is its replicating portfolio.

Forward Binomial Tree.



No-arbitrage Condition.

$$\boxed{d < e^{r \cdot h} < u}$$

Recall: $\boxed{F_{0,h}(S) = S(0)e^{rh}}$

$$S_u := F_{0,h}(S) \cdot e^{\sigma\sqrt{h}} = S(0)\underline{e^{rh} \cdot e^{\sigma\sqrt{h}}} = S(0)\boxed{e^{rh+\sigma\sqrt{h}}} =: u$$

$$S_d := F_{0,h}(S) \cdot e^{-\sigma\sqrt{h}} = S(0)\underline{e^{rh} \cdot e^{-\sigma\sqrt{h}}} = S(0)\boxed{e^{rh-\sigma\sqrt{h}}} =: d$$

Q: Do u and d satisfy the no-arbitrage condition?

$\longrightarrow$:

$$d < e^{rh} < u$$

$$\cancel{e^{rh}} \cdot e^{-\sigma\sqrt{h}} < \cancel{e^{rh}} < \cancel{e^{rh}} \cdot e^{\sigma\sqrt{h}}$$

$$\boxed{e^{-\sigma\sqrt{h}} < 1 < e^{\sigma\sqrt{h}}}$$

✓ $\sigma > 0$, $h > 0$

□

Q: What is S_u/S_d?

$$\rightarrow: \quad \frac{S_u}{S_d} = \frac{\cancel{S(0)}\, e^{\cancel{rh}+\sigma\sqrt{h}}}{\cancel{S(0)}\, e^{\cancel{rh}-\sigma\sqrt{h}}} = \underline{e^{2\sigma\sqrt{h}}}$$

Example. Consider this one-period binomial tree w/ the time horizon of one quarter year.

$$\begin{tikzcd}[column sep=small]
 & S_u = 105 \\
100 = S(0) \arrow[ur, no head] \arrow[dr, no head] & \\
 & S_d = 98
\end{tikzcd}$$

Q: What is the volatility in this forward tree?

$$\rightarrow: \quad \frac{S_u}{S_d} = e^{2\sigma\sqrt{h}} \quad \Rightarrow \ln\left(\frac{105}{98}\right) = 2\sigma\sqrt{\tfrac{1}{4}} = \sigma \qquad \square$$

Binomial Option Pricing.

$$\underbrace{\begin{tikzcd}[column sep=small]
 & S_u = u \cdot S(0) \\
S(0) \arrow[ur, no head] \arrow[dr, no head] & \\
 & S_d = d \cdot S(0)
\end{tikzcd}}_{h}$$

populating the tree

Goal: Pricing a European-style derivative security w/ exercise date @ the end of the tree, i.e., $T=h$.

It is completely determined by its payoff function: $v(\cdot)$

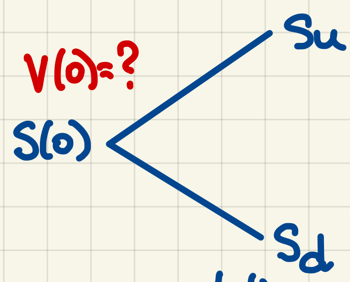
e.g., for a call: $v_c(s) = (s-K)_+$

for a put: $v_p(s) = (K-s)_+$

for a power option: $v(s) = (s^2-K)_+$

The payoff of a derivative security is a random variable

$$\boxed{V(T) := v(S(T))}$$



population →

← pricing

PAYOFF

$V_u := v(S_u)$ = $\Delta \cdot S_u + B e^{rh}$

$V_d := v(S_d)$ = $\Delta \cdot S_d + B e^{rh}$

REPLICATING PORTFOLIO

In the binomial model, any derivative security can be **REPLICATED** w/ a portfolio of this form:

@ time 0
- Δ shares of stock

and
- B @ the ccrfir

$$\begin{cases} \Delta > 0 & \text{buying} \\ \Delta = 0 & \text{"nothing"} \\ \Delta < 0 & \text{short-selling} \end{cases}$$

$$\begin{cases} B > 0 & \text{lending (buying a bond)} \\ B = 0 & \text{"nothing"} \\ B < 0 & \text{borrowing (issuing a bond)} \end{cases}$$

If we can calculate Δ and B, then $\boxed{V(0) = \Delta \cdot S(0) + B}$ ✓

We get a system of two eq'ns w/ two unknowns:

$$\left.\begin{aligned} \Delta \cdot S_u + \cancel{B e^{rh}} &= V_u \\ -\ \Delta \cdot S_d + \cancel{B e^{rh}} &= V_d \end{aligned}\right\} -$$

$$\Delta (S_u - S_d) = V_u - V_d$$

$$\boxed{\Delta = \frac{V_u - V_d}{S_u - S_d}}$$ unitless / shares of stock

$$\frac{V_u - V_d}{S_u - S_d} \cdot S_u + B e^{rh} = V_u$$

$$B e^{rh} = V_u - \frac{V_u - V_d}{\cancel{S(0)}(u-d)} \cdot u \cdot \cancel{S(0)} = \frac{\cancel{u \cdot V_u} - d \cdot V_u - \cancel{u \cdot V_u} + u \cdot V_d}{u-d}$$

$$\boxed{B = e^{-rh} \cdot \frac{u \cdot V_d - d \cdot V_u}{u-d}}$$ cash($)