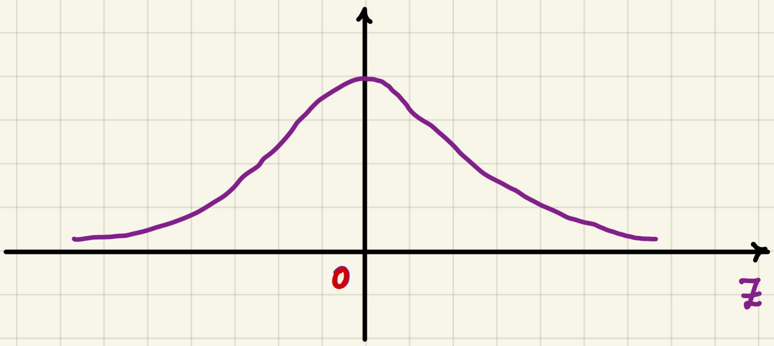
M358K: September 29th, 2021.

Standard Normal Distribution.

We say that a random variable Z has a standard normal distribution if its pdf has the following form:

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \quad \text{for all } z \in \mathbb{R}$$



- even (symmetric about the vertical axis), i.e., $\varphi(z) = \varphi(-z)$
- median of $Z = 0$

The cdf of the standard normal is:

$$\Phi(x) = \mathbb{P}[Z \leq x] = \int_{-\infty}^{x} \varphi(z)\,dz = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\,dz$$

$\boxed{\text{No analytic form!}}$

There are standard normal tables.

We can also use the built-in commands in R.

- $\mathbb{E}[Z] = \underbrace{\int_{-\infty}^{\infty} z \underbrace{\varphi(z)}_{\text{even}}}_{\text{odd}} dz = 0$ for odd functions $g(z) = -g(-z)$
- $\mathrm{Var}[Z] = 1 \Rightarrow SD[Z] = 1$

We write: $\boxed{Z \sim \mathrm{Normal}(\text{mean} = 0, \text{var} = 1) \sim N(0,1)}$

Normal Distribution.

We completely specify any normal distribution by its mean and its variance (or standard deviation).

We write : $\boxed{X \sim \text{Normal}(\text{mean} = \mu_X, \text{variance} = \sigma_X^2)}$

which means that X can be written as a linear transform of a standard normal Z :

$$\boxed{X = \mu_X + \sigma_X \cdot Z}$$

We can check:

- $\mathbb{E}[X] = \mathbb{E}[\mu_X + \sigma_X \cdot Z] =$ (linearity of expectation)

  $= \mu_X + \sigma_X \cdot \underbrace{\mathbb{E}[Z]}_{=0} = \mu_X$

- $\text{Var}[X] = \text{Var}[\mu_X + \sigma_X \cdot Z] =$

  deterministic (added does affect the variance)

  $= \text{Var}[\sigma_X \cdot Z] = \sigma_X^2 \cdot \underbrace{\text{Var}[Z]}_{=1}$

  $= \sigma_X^2$