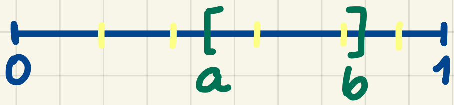
M378K: January 29th, 2025.

Continuous Distributions.

Example.



Imagine a r.v. Y on $[0,1]$.

The probability of Y landing between a and b, where $0 \leq a \leq b \leq 1$ is

$$\boxed{\mathbb{P}[a \leq Y \leq b] = \mathbb{P}[Y \in [a,b]] = b-a}$$

$$\boxed{Y = y \iff y \geq Y \text{ and } y \leq Y}$$

Note: $\boxed{\mathbb{P}[Y=y] = 0 \text{ for all } y \in [0,1]}$ Choose $a=b=y$

Def'n. A random variable Y is said to be continuous if there exists a function

$$f_Y : \boxed{\mathbb{R}} \to [0,\infty)$$

such that

$$\mathbb{P}[Y \in [a,b]] = \int_a^b f_Y(y)\,dy \quad \text{for all } a \leq b$$

The function f_Y is called the

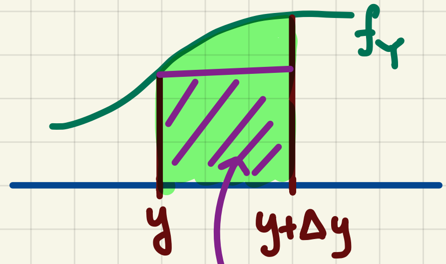
probability density function (pdf) of Y.

Properties:

- $f_Y(y) \geq 0$ for all $y \in \mathbb{R}$
- $\int_{-\infty}^{\infty} f_Y(y)\,dy = \underline{1}$

Q: Is it possible for a pmf p_Y to have a y such that $p_Y(y) > 1$? No!

Q: Is it possible for a pdf f_Y to have a y such that $f_Y(y) > 1$? **Yes!**



$$\mathbb{P}[Y \in [y, y+\Delta y]] \approx f_Y(y)\Delta y \approx f_Y(y)\,dy$$

__Caveat:__ There are r.v.s that are neither discrete nor continuous!

__Example.__ Y is uniformly distributed between 0 and 1/4

$$Y \sim U(0, 1/4)$$

$$f_Y(y) = \begin{cases} c & \text{for } y \in [0, 1/4] \\ 0 & \text{otherwise} \end{cases}$$

We introduce, for any subset $A \subseteq \mathbb{R}$

$$\mathbb{1}_A : \mathbb{R} \longrightarrow \mathbb{R}$$

as

$$\mathbb{1}_A(y) = \begin{cases} 1 & \text{if } y \in A \\ 0 & \text{if } y \notin A \end{cases}$$

This f'tion is called the indicator function.

$$\boxed{f_Y(y) = c \cdot \mathbb{1}_{[0,1/4]}(y)}$$

BTW $c = 4$

$\square$

M378K Introduction to Mathematical Statistics
Problem Set #5
Continuous distributions.

Problem 5.1. Source: Sample P exam, Problem #33.
The lifetime of a machine part has a continuous distribution on the interval $(0, 40)$ *with probability density function* f_X*, where*

is proportional to

$$f_X(x) \propto \frac{1}{(10+x)^2}$$

on the interval.

Calculate the probability that the lifetime of the machine part is less than 6.

$\rightarrow$:

$$f_X(x) = \kappa \cdot \frac{1}{(10+x)^2} \mathbb{1}_{[0,40]}(x)$$

We know: $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

So, $\kappa \cdot \boxed{\int_0^{40} (10+x)^{-2}\,dx} = 1$

$$\int_0^{40} (10+x)^{-2}\,dx = \frac{(10+x)^{-1}}{-1}\Bigg|_{x=0}^{40} = \boxed{-\frac{1}{10+x}}\Bigg|_{x=0}^{40}$$

$$= \frac{1}{10} - \frac{1}{50} = \frac{5-1}{50} = \frac{2}{25} \Rightarrow \boxed{\kappa = \frac{25}{2}}$$

$$\mathbb{P}[X \le 6] = \mathbb{P}[X \in [\underset{a}{0}, \underset{b}{6}]]$$

$$= \int_0^6 f_X(x)\,dx = \frac{25}{2}\int_0^6 (10+x)^{-2}\,dx$$

$$= \frac{25}{2}\left(\frac{1}{10} - \frac{1}{16}\right) = \frac{25}{2} \cdot \frac{8-5}{80} = \frac{15}{32} \quad \square$$

Example. $Y \sim U(\ell, r)$

$$\mathbb{P}[Y \in [a,b]] = \frac{b-a}{r-\ell} \quad \text{for } \ell \leq a \leq b \leq r$$

$$\boxed{f_Y(y) = \frac{1}{r-\ell} \mathbb{1}_{[\ell,r]}(y)}$$

$\mathbb{1}$,

$\mathbb{I}$, δ, χ