University of Texas at Austin

HW Assignment 10

Binomial option pricing.

10.1. **The forward binomial tree.** Please, provide your *final answer only* to the following problem.

Problem 10.1. (5 points) Assume that the a stock price is modeled using a one-period forward binomial tree with the length of a single period equal to three months. According to this model, the stock price can take either the value of \$50, or the value of \$40 in exactly three months. Calculate the volatility of the stock price.

(a) About 0.22
(b) About 0.28
(c) About 0.30
(d) About 0.32
(e) None of the above.

Solution: (a)

$$e^{2\sigma\sqrt{h}} = S_u/S_d \quad \Rightarrow \quad \sigma = \frac{1}{2\sqrt{h}}\ln(S_u/S_d) = \frac{1}{2\sqrt{1/4}}\ln(50/40) = \ln(50/40) = 0.2231$$

10.2. **Alternative binomial trees.** Please, provide your **complete** solutions to the following problem(s):

Problem 10.2. Cox-Ross-Rubinstein (CRR)
The Cox-Ross-Rubinstein model is a binomial tree in which the up and down factors are given as

$$u = e^{\sigma\sqrt{h}}, \quad d = e^{-\sigma\sqrt{h}},$$

where σ denotes the volatility parameter and h stands for the length of a single period in a tree.

a. (2 points) What is the ratio S_u/S_d?
Solution: $S_u/S_d = e^{2\sigma\sqrt{h}}$.

b. (2 points) What is the (as simplified as possible) expression for the risk-neutral probability of the stock price going up in a single step?
Solution:

$$p^* = \frac{e^{(r-\delta)h} - e^{-\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}} = \frac{e^{(r-\delta)h+\sigma\sqrt{h}} - 1}{e^{2\sigma\sqrt{h}} - 1}$$

Substantial further simplification is impossible.

c. (2 points) Express S_{ud} in terms of $S(0), \sigma$ and h in a CRR tree.
Solution: $S_{ud} = S(0)$

d. (5 points) As was the case with the forward tree, the *no-arbitrage* condition for the binomial asset-pricing model is satisfied for the CRR tree regardless of the specific values of σ, δ, r and h. *True or false?*
Solution: FALSE
Counterexamples will vary.

Problem 10.3. The Jarrow-Rudd model.
The **Jarrow-Rudd** model (aka, the lognormal binomial tree) is a binomial tree in which the up and down factors are defined as follows

$$u = e^{\left(r-\delta-\frac{\sigma^2}{2}\right)h+\sigma\sqrt{h}}, \quad d = e^{\left(r-\delta-\frac{\sigma^2}{2}\right)h-\sigma\sqrt{h}},$$

where

- r stands for the continuously-compounded, risk-free interest rate,

- δ is the stock's dividend yield,
- σ denotes the volatility parameter, and
- h stands for the length of a single period in a tree.

Answer the following questions:

a. (2 points) What is the ratio S_u/S_d?
Solution: $S_u/S_d = e^{2\sigma\sqrt{h}}$.

b. (2 points) What is the (as simplified as possible) expression for the risk-neutral probability of the stock price going up in a single step?
Solution:

$$p^* = \frac{e^{(r-\delta)h} - e^{\left(r-\delta-\frac{\sigma^2}{2}\right)h-\sigma\sqrt{h}}}{e^{\left(r-\delta-\frac{\sigma^2}{2}\right)h+\sigma\sqrt{h}} - e^{\left(r-\delta-\frac{\sigma^2}{2}\right)h-\sigma\sqrt{h}}} = \frac{1 - e^{-\frac{\sigma^2 h}{2}-\sigma\sqrt{h}}}{e^{-\frac{\sigma^2 h}{2}+\sigma\sqrt{h}} - e^{-\frac{\sigma^2 h}{2}-\sigma\sqrt{h}}}.$$

Substantial further simplification is impossible.

c. (5 points) As was the case with the forward tree, the *no-arbitrage* condition for the binomial asset-pricing model is satisfied for the Jarrow-Rudd tree regardless of the specific values of σ, δ, r and h. *True or false?*
Solution: FALSE
Counterexamples will vary.

10.3. **Multi-period binomial option pricing: European options.** Please, provide your <u>**complete**</u> solutions to the following problem:

Problem 10.4. (10 points) The current price of a continuous-dividend-paying stock is \$100 per share. Its volatility is given to be 0.2 and its dividend yield is 0.03.

The continuously compounded risk-free interest rate equals 0.04.

Consider a \$95-strike European put option on the above stock with nine months to expiration. Using a three-period forward binomial tree, find the price of this put option.

Solution: The up and down factors in the above model are

$$u = e^{0.01\times 0.25+0.2\sqrt{0.25}} = 1.10794,$$
$$d = e^{0.01\times 0.25-0.2\sqrt{0.25}} = 0.9071.$$

The relevant possible stock prices at the "leaves" of the binomial tree are

$$S_{ddd} = d^3 S(0) = 100(0.9071)^3 = 74.6395,$$
$$S_{ddu} = d^2 u S(0) = 91.1649.$$

The remaining two final states of the world result in the put option being out-of-the-money at expiration.

The risk-neutral probability of the stock price moving up in a single period is

$$p^* = \frac{1}{1+e^{0.2\sqrt{0.25}}} = 0.475.$$

So, the price of the European put option equals

$$V_P(0) = e^{-0.04(3/4)}\left[(95-74.6395)(1-0.475)^3 + (95-91.1649)(3)(1-0.475)^2(0.475)\right] = 4.32.$$

10.4. **Two binomial periods: American options.** Please, provide your **complete** solutions to the following problem:

Problem 10.5. (15 points) Find the current price of a one-year, \$110-strike American put option on a non-dividend-paying stock whose current price is $S(0) = \$100$. Assume that the continuously compounded interest rate equals $r = 0.06$.
Use a two-period binomial tree with $u = 1.23$, and $d = 0.86$ to calculate the price $V_P(0)$ of the put option.

Solution:
We are pricing a one-year option, so $T = 1$. We are using a two-period tree, so $n = 2$. Hence, $h = T/n = 1/2$. The risk-neutral probability of the stock-price going up is

$$p^* = \frac{e^{0.03} - 0.86}{1.23 - 0.86} \approx 0.46.$$

The put has the following payoffs at the final leaves of the tree provided that it is kept alive throughout the two periods:

$$V_{uu} = 0, V_{ud} = 4.22, V_{dd} = 36.04.$$

So, the value at the "down" node is

$$\begin{aligned} V_d &= \max[K - S_d, e^{-rh}[p^* V_{ud} + (1-p^*)V_{dd}]] = \max[110 - 86, e^{-0.03}[0.46 \cdot 4.22 + 0.54 \cdot 36.04]] \\ &= \max[24, 20.77] = 24. \end{aligned}$$

Note that there is *early exercise* at this node. At the "up" node, there can be no early exercise since the put is out-of-the-money. So,

$$V_u = e^{-rh}[p^* V_{uu} + (1-p^*)V_{ud}] = e^{-0.03} \cdot 0.54 \cdot 4.22 = 2.21. \cdot 4.22$$

Finally,

$$V_P(0) = \max[110 - 100, e^{-0.03}[0.46 \cdot 2.21 + 0.54 \cdot 24] = 13.56.$$