M339D: February 23rd, 2022.

Arbitrage Portfolios.

Def'n. An arbitrage portfolio is a portfolio whose $\boxed{\text{profit}}$ is:

? • non-negative in all states of the world
and
? • strictly positive in at least one state of the world.

> Unless it's specified otherwise in a particular problem/example we assume NO ARBITRAGE exists.

Law of the Unique Price.

For simplicity, focus on static portfolios.

Assume that the payoff of two static portfolios, A and B, are equal, i.e.,

$$\boxed{V_A(T) = V_B(T)}$$ ✓

In general, two random variables, X and Y, are said to be equal if $\boxed{\mathbb{P}[X = Y] = 1}$.

On a finite probability space, this means that they must take the same value on every elementary outcome.

Our claim: $\boxed{V_A(0) = V_B(0)}$

$\longrightarrow$: Assume, to the contrary, that

$$V_A(0) \neq V_B(0).$$

Without loss of generality, say,

$$\boxed{\underbrace{V_A(0)}_{\text{relatively cheap}} < \underbrace{V_B(0)}_{\text{relatively expensive}}}$$

Propose an arbitrage portfolio:

- long Portfolio A
- short Portfolio B

$\Big\}$ Total Portfolio

Verify:
- Initial Cost : $V_A(0) - V_B(0) < 0$

  Inflow of money @ time 0

- Payoff : $V_A(T) - V_B(T) = 0$

$$\text{Profit} = \text{Payoff} - FV_{0,T}(\text{Initial Cost})$$

$$= 0 - FV_{0,T}\big(V_A(0) - V_B(0)\big) > 0$$

We created an arbitrage portfolio!

$\Rightarrow\Leftarrow$

$\blacksquare$

Replicating Portfolios.

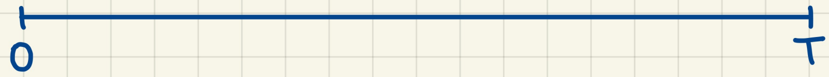
Def'n. Consider a European-style derivative security. A static portfolio w/ the same payoff as that of the derivative security is called its replicating portfolio.

Note: The initial price of the derivative security must be equal to the initial price of its replicating portfolio.

Ways to Buy Stock.

	0	T
Outright purchase:	• pmt: $S(0)$ • delivery	
Fully Leveraged Purchase:	• delivery	• pmt: $S(0)e^{rT}$
Forward Contract:		• pmt: $F_{0,T}$ • delivery
Prepaid Forward:	• pmt: $F^P_{0,T}$ ← prepaid forward price	• delivery

Relationship Between Prepaid Forwards and Forwards.



Forward Contract: Initial Cost = 0   Payoff = $S(T) - F_{0,T}$

Prepaid Forward: Initial Cost = $F^{P}_{0,T}$ Payoff = $S(T)$

Propose the following replicating portfolio for the prepaid forward:

PORT
- forward contract
- invest $PV_{0,T}(F_{0,T})$ @ the ccrfir r to be withdrawn @ time T

Note: This is, indeed, a static portfolio. So, we just need to check if the payoffs match.

$$\text{Payoff(PORT)} = \underbrace{S(T) - \cancel{F_{0,T}}}_{\text{forward contract}} + \underbrace{\cancel{FV_{0,T}(PV_{0,T}(F_{0,T}))}}_{\text{balance in the risk-free account}} = S(T)$$

$\Rightarrow$ We, indeed, obtained a replicating portfolio.

$\Rightarrow$ $\boxed{F^{P}_{0,T} = PV_{0,T}(F_{0,T})}$ $\Rightarrow$ $\boxed{F_{0,T} = FV_{0,T}(F^{P}_{0,T})}$

NO ARBITRAGE!