
Please, provide your justification for your response to every question in this subsection. Just the final numerical answer will receive zero credit, even if it is correct. For the graphs, it is sufficient to carefully draw the graph correctly in a clearly labeled coordinate system.

Problem 3.1. (10 points) A population of insureds consists of three types of people: α, β and γ. There are half as many people of Type α as of Type β people in the population. The number of Type γ people is equal to the total number of the remaining two types of people. The probability that a Type α person makes at least one claim in a year is $1/5$. The probability that a Type β person makes at least one claim in a year is $2/5$. The probability that a Type γ person makes at least one claim in a year is $3/5$.

At the end of the year, a person is chosen at random from the population. It is observed that that person had at least one claim. What is the probability that the person was of Type β?

Solution: From the given breakdown of the population, we conclude that

$$\mathbb{P}[\alpha] = 1/6, \quad \mathbb{P}[\beta] = 1/3, \quad \mathbb{P}[\gamma] = 1/2. \tag{3.1}$$

Let E denote the event that there was at least one claim. By Bayes' Theorem, we have that

$$\begin{aligned} \mathbb{P}[\beta \mid E] &= \frac{\mathbb{P}[E \mid \beta] \times \mathbb{P}[\beta]}{\mathbb{P}[E \mid \alpha] \times \mathbb{P}[\alpha] + \mathbb{P}[E \mid \beta] \times \mathbb{P}[\beta] + \mathbb{P}[E \mid \gamma] \times \mathbb{P}[\gamma]} \\ &= \frac{(2/5)(1/3)}{(1/5)(1/6) + (2/5)(1/3) + (3/5)(1/2)} = \frac{2}{7}. \end{aligned} \tag{3.2}$$

Problem 3.2. (5 points) A continuous random variable X has the *probability density function* of the following form

$$f_X(x) = \kappa x^{-5} \quad \text{for } x > 1 \tag{3.3}$$

where κ is a positive constant. What is its 95^{th} percentile?

Solution: The cumulative distribution function is of the form

$$F_X(x) = \begin{cases} 0, & \text{for } x < 1, \\ \frac{\kappa}{4}(1 - x^{-4}), & \text{for } x \geq 1. \end{cases}$$

Since

$$\lim_{x \to \infty} F_X(x) = 1,$$

we conclude that $\kappa = 4$. Let x_* denote its 95^{th} percentile. Then,

$$F_X(x^*) = 0.95 \quad \Rightarrow \quad 1 - (x^*)^{-4} = 0.95 \quad \Rightarrow \quad (x^*)^4 = 20 \quad \Rightarrow \quad x^* = 2.114743.$$

$$x^* = 2.115 \tag{3.4}$$

Problem 3.3. (10 points) Consider a continuous random variable X whose probability density function is of the form

$$f_X(x) = \begin{cases} \kappa x^\alpha & \text{for } x \in (0,5) \\ 0 & \text{otherwise.} \end{cases}$$

You are given that the probability that X less than 3.75 is 0.4871. What is the value of the parameter α?

Solution: By anti-differentiation, the cumulative distribution function of X is

$$F_X(x) = \begin{cases} 0, & \text{for } x < 0 \\ \frac{\kappa}{\alpha+1} x^{\alpha+1}, & \text{for } 0 \leq x < 5 \\ 1, & \text{for } x \geq 5 \end{cases}$$

Since X is continuous, we know that

$$\frac{\kappa}{\alpha+1} 5^{\alpha+1} = 1.$$

On the other hand, due to the given probability

$$\frac{\kappa}{\alpha+1} (3.75)^{\alpha+1} = 0.4871.$$

Dividing the two equations, we get

$$\left(\frac{3.75}{5}\right)^{\alpha+1} = 0.4871 \quad \Rightarrow \quad (\alpha+1)\ln(0.75) = \ln(0.4871) \quad \Rightarrow \quad \alpha = \frac{\ln(0.4871)}{\ln(0.75)} - 1 = 1.50028.$$

Problem 3.4. (5 points) Consider the random variable X whose cumulative distribution function is given by

$$F_X(x) = \begin{cases} 0 & \text{for } x < 1 \\ \frac{x-1}{2} & \text{for } 1 \leq x < 2 \\ 1 & \text{for } 2 \leq x \end{cases}$$

What is the expectation of the random variable X?

Solution: This is a mixed distribution with a probability mass of $1/2$ at 2 and otherwise uniform on $(1,2)$. We have

$$\mathbb{E}[X] = \frac{1}{2}\left(\frac{3}{2}\right) + \frac{1}{2}(2) = \frac{7}{4}.$$

Problem 3.5. (10 points) Consider a continuous random variable X whose probability density function is of the form

$$f_X(x) = \begin{cases} \frac{p-1}{x^p} & \text{for } x > 1, \\ 0 & \text{otherwise} \end{cases}$$

for some parameter $p > 1$. Find the value of the parameter p such that the expected value of the random variable X equals 3.

Solution: By the definition of expectation, we have

$$\mathbb{E}[X] = \int_1^\infty x \left(\frac{p-1}{x^p}\right) dx = \int_1^\infty (p-1)x^{1-p}\, dx = \frac{(p-1)x^{2-p}}{2-p}|_{x=1}^\infty$$

The integral above exists only for $p > 2$. Then, it is equal to

$$\frac{p-1}{p-2} = 3 \quad \Rightarrow \quad p - 1 = 3(p-2) \quad \Rightarrow \quad p = \frac{5}{2}.$$

Problem 3.6. (2 points) Let X be an exponential random variable. Then, its mean and its variance are equal. *True or false? Why?*

Solution: FALSE
Let $X \sim Exponential(\theta)$. From our tables, we get

$$\mathbb{E}[X] = \theta,$$
$$Var[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \theta^2 \cdot 2! - \theta^2 = \theta^2.$$

Problem 3.7. (8 points) Let the random variable X denote the result of rolling a fair $8-$sided die (i.e., an octahedron) whose faces are numbered 1 through 8. Find the 60^{th} percentile of X.

Solution: In the notation introduced earlier in the course, $p = 0.6$ and we are supposed to find π_p such that

$$F_X(\pi_p-) \leq 0.6 \leq F_X(\pi_p).$$

Note that the p.m.f. of the random variable X is

$$p_X(k) = 1/8 \quad \text{for } k = 1, 2, \dots, 8.$$

The piecewise description of the c.d.f is, then,

$$F_X(x) = \begin{cases} 0 & \text{for } x < 1 \\ k/8 & \text{for } k \leq x < k+1, k = 1, \dots, 7 \\ 1 & \text{for } x \geq 8. \end{cases}$$

Solving for k in

$$\frac{k-1}{8} \leq 0.6 \leq \frac{k}{8} \quad \Leftrightarrow \quad 5(k-1) \leq 24 \leq 5k,$$

we get $k = 5$. Finally, $\pi_p = k = 5$.