Name:

M339D/M389D Introduction to Financial Mathematics for Actuaries
University of Texas at Austin
Practice for In-Term Three
Instructor: Milica Čudina

Notes: This is a closed book and closed notes exam. The maximum number of points on this exam is 100.
Time: 50 minutes

3.1. **Free-response problems.** Please, explain carefully all your statements and assumptions. Numerical results or single-word answers without an explanation (even if they're correct) are worth 0 points.

Please, solve the final two problems from the practice problem set for in-term two.

Problem 3.1. (15 points) Two scales are used to measure the mass m of a precious stone. The first scale makes an error in measurement which we model by a normally distributed random variable with mean $\mu_1 = 0$ and standard deviation $\sigma_1 = 0.04m$. The second scale is more accurate. We model its error by a normal random variable with mean $\mu_2 = 0$ and standard deviation $\sigma_2 = 0.03m$.

We assume that the measurements made using the two different scales are independent.

To get our final estimate of the mass of the stone, we take the average of the two results from the two different scales.

What is the probability that the value we get is within $0.005m$ of the actual mass of the stone?

Solution: Let us denote the random variable modeling the error from the first scale by $X_1 \sim N(0, \sigma_1^2)$ and the random variable modeling the error from the second scale by $X_2 \sim N(0, \sigma_2^2)$.

Then, if Y denotes the average of the two measurements, we have that

$$Y = \frac{1}{2}(X_1 + X_2) \sim N(0, \frac{1}{4}(\sigma_1^2 + \sigma_2^2)),$$

i.e.,

$$Y \sim N(0, \sigma^2)$$

with

$$\sigma^2 = \frac{1}{4}(\sigma_1^2 + \sigma_2^2) = \frac{1}{4}(0.04^2 m^2 + 0.03^2 m^2) = \frac{1}{4} \cdot 0.01^2 m^2 (4^2 + 3^2) = \frac{1}{4}\, 0.05^2 m^2 = \left(\frac{0.05m}{2}\right)^2.$$

The probability we are looking for can be expressed as

$$\begin{aligned}\mathbb{P}[Y \in (-0.005m, 0.005m)] &= \mathbb{P}[-0.005m < Y < 0.005m] \\ &= \mathbb{P}[-\frac{2 \cdot 0.005m}{0.05m} < \frac{Y}{\sigma} < \frac{2 \cdot 0.005m}{0.05m}] \\ &= \mathbb{P}[-0.2 < \frac{Y}{\sigma} < 0.2].\end{aligned}$$

Since $\frac{Y}{\sigma} \sim N(0,1)$, the above probability equals

$$2\Phi(0.2) - 1 \approx 2 \cdot 0.57926 - 1 \approx 0.16.$$

Problem 3.2. (10 points) Assume that $Y_1 = e^X$ where X is a standard normal random variable.

(i) (2 points) What is the probability that Y_1 exceeds 5?

(ii) $(3+5$ points) Find the mean and the variance of Y_1.

Hint: It helps if you use the expression for the moment generating function of a standard normal random variable.

Solution:

(i)

$$\mathbb{P}[Y_1 > 5] = \mathbb{P}[e^X > 5] = \mathbb{P}[X > \ln(5)] = 1 - N(\ln(5)) \approx 1 - N(1.61) = 1 - 0.9463 = 0.0537.$$

(ii)

$$\mathbb{E}[Y_1] = \mathbb{E}[e^X] = \mathbb{E}[e^{1 \cdot X}] = M_X(1)$$

where M_X denotes the moment generating function of X. In class, we recalled the following expression for M_X:

$$M_X(t) = e^{t^2/2}.$$

So, $\mathbb{E}[Y_1] = e^{1/2} = \sqrt{e}$.

The second moment of Y_1 is obtained similarly as

$$\mathbb{E}[Y_1^2] = \mathbb{E}[e^{2 \cdot X}] = M_X(2) = e^2.$$

So,

$$Var[Y_1] = \mathbb{E}[Y_1^2] - (\mathbb{E}[Y_1])^2 = e^2 - e = e(e-1).$$

Problem 3.3. The final exam in a particular course has 100 multiple-choice questions: for each question there are five offered answers exactly one of which is correct. Out of the 100 questions, 36 questions come from a public problem bank. A student diligently memorizes the correct answers to all of those questions. However, since the student learned by rote, they are not able to do any work on the remaining questions. So, in the exam, they are able to answer exactly 36 questions correctly. For the remaining questions, the student guesses completely at random and independently between problems. Approximately, what is the probability that the student achieves a passing score of 65?

Solution: *The following solution uses the continuity correction. The solutions without the continuity correction would also be graded as correct if otherwise accurate.*
The number of problems that the student guesses on at random is 64. The probability of guessing correctly for a single problem is 1/5. So, the total number of problems that the student guesses correctly is, in our usual notation,

$$X \sim Binomial(n = 64, p = 0.2).$$

Out of the problems that the student guesses on at random, they need to guess correctly on at least $65 - 36 = 29$. The probability of passing is $\mathbb{P}[X \geq 29]$. The mean of the random variable X is $np = 12.8$ and its standard deviation is $\sqrt{np(1-p)} = 3.2$ Using the normal approximation to the binomial, we get

$$\mathbb{P}[X \geq 29] = \mathbb{P}[X > 28.5] = \mathbb{P}\left[\frac{X - 12.8}{3.2} > \frac{28.5 - 12.8}{3.2}\right] = 1 - \Phi(4.90625) \approx 0.$$

Problem 3.4. (10 points) Let the stock prices be modeled using the lognormal distribution. Under the risk-neutral probability, the mean stock price at time-1 equals 120 and the median stock price 115. What is the risk-neutral probability that the time-1 stock price exceeds 100?

Solution: The stock price at time-1 is lognormally distributed. In fact, using our usual parameters, we can rewrite it as

$$S(1) = S(0)e^{(r-\frac{1}{2}\sigma^2)+\sigma Z}.$$

Recall that the median of $S(1)$ equals $S(0)e^{(r-\frac{1}{2}\sigma^2)}$. So, the required probability can be expressed as

$$\begin{aligned}\mathbb{P}[S(1) > 100] = \mathbb{P}[115e^{\sigma Z} > 100] &= \mathbb{P}\left[Z > \frac{1}{\sigma}\ln\left(\frac{100}{115}\right)\right] \\ &= \mathbb{P}\left[Z < \frac{1}{\sigma}\ln\left(\frac{115}{100}\right)\right] = N\left(\frac{1}{\sigma}\ln\left(\frac{115}{100}\right)\right).\end{aligned}$$

Since the mean of $S(1)$ equals $S(0)e^{r}$, we have

$$e^{\frac{1}{2}\sigma^2} = \frac{120}{115} \quad \Rightarrow \quad \sigma = \sqrt{2\ln(1.04348)} = 0.2918.$$

So, our final answer is

$$\mathbb{P}[S(1) > 100] = N(0.48) = 0.6844.$$

Problem 3.5. (5 points) Assume the Black-Scholes model. The initial price of a non-dividend-paying stock is \$100. Its volatility is 0.15. The continuously compounded risk-free interest rate is 0.05.

Calculate the risk-neutral probability that the realized return for the time period $[0, 2]$ exceeds 0.06.

Solution: In our usual notation, under the risk-neutral probability measure, the realized returns are normally distributed as

$$R(0,t) \sim Normal(mean = (r - \tfrac{\sigma^2}{2})t, variance = \sigma^2 t).$$

In the present problem, we are focused on

$$R(0,2) \sim Normal(mean = (0.05 - \tfrac{(0.15)^2}{2})(2) = 0.0775, variance = (0.15)^2(2) = 0.045).$$

Finally, we calculate

$$\begin{aligned}\mathbb{P}[R(0,2) > 0.06] &= \mathbb{P}\left[\frac{R(0,2)-0.0775}{\sqrt{0.045}} > \frac{0.06-0.0775}{\sqrt{0.045}}\right]\\ &= \mathbb{P}[Z > -0.08] = N(0.08) = 0.5319.\end{aligned}$$

Problem 3.6. (5 points) Assume the Black-Scholes model. Consider a non-dividend-paying stock with the initial stock price of \$100 and volatility equal to 0.30.

The continuously compounded risk-free interest rate is 0.10. Find

$$\mathbb{E}^*[S(1)\mathbb{I}_{[S(1)\geq 105]}].$$

Solution: According to the work done in class,

$$\mathbb{E}[S(1)\mathbb{I}_{[S(1)\geq 105]}] = \mathbb{E}[S(1)]N(\hat{d}_1)$$

where

$$\hat{d}_1 = \frac{1}{\sigma\sqrt{1}}\left[\ln\left(\frac{100}{105}\right) + \left(0.10 + \frac{(0.3)^2}{2}\right)(1)\right] \approx 0.32.$$

So,

$$\mathbb{E}[S(1)\mathbb{I}_{[S(1)\geq 105]}] = 100e^{0.10}N(0.32) = 69.12844.$$

Problem 3.7. (10 points) Consider a non-dividend-paying stock whose price $\mathbf{S} = \{S(t), t \geq 0\}$ is modeled using the Black-Scholes framework. Suppose that the current stock price equals \$100 and that its volatility is given to be 0.25.

The continuously compounded, risk-free interest rate is assumed to equal 0.04.

Consider a derivative security which entitles its owner to obtain a European call option on the above stock six months from today, i.e., at time $t_* = 1/2$. The call option is to be 3−month to expiration at time of delivery and have the strike equal to 105% of the time−t^* price of the underlying asset. This contract is called a **forward start option**.

What is the price of the forward start option?

Solution: At time $t*$, the required Black-Scholes price of the call option equals

$$\begin{aligned}V_C(t^*) &= S(t^*)N(d_1) - 1.05S(t^*)e^{-r(T-t^*)}N(d_2)\\ &= S(t^*)(N(d_1) - 1.05e^{-0.01}N(d_2))\end{aligned}$$

with

$$d_1 = \frac{1}{0.125}\left[-\ln(1.05) + \left(0.04 - \frac{0.25^2}{2}\right) \times \frac{1}{4}\right] = -0.2478,$$
$$d_2 = d_1 - \sigma\sqrt{T - t^*} = -0.2478 - 0.125 = -0.3728.$$

So, $N(d_1) = 1 - N(0.25) = 1 - 0.5987 = 0.4013$ and $N(d_2) = 1 - N(0.37) = 1 - 0.6443 = 0.3557$ and Hence,

$$V_C(t^*) = S(t^*)(0.4013 - 1.05e^{-0.01} \times 0.3557) = S(t^*)0.31531.$$

So, one would need to buy exactly 0.031531 shares of stock to be able to buy the call option in question at time$-t^*$. This amount of shares costs \$3.1531.

3.2. MULTIPLE CHOICE QUESTIONS.

Problem 3.8. Assume the Black-Scholes model. Under the risk-neutral probability, you expect the stock price in half a year to be \$84.10. The median stock price in half a year is \$83.26 according to that same model. What is the stock's volatility?

(a) 0.15
(b) 0.20
(c) 0.25
(d) Not enough information is given.
(e) None of the above.

Solution: (b)
In our usual notation, we have that

$$\frac{\text{mean stock price}}{\text{median stock price}} = \frac{S(0)e^{rT}}{S(0)e^{(r-\frac{\sigma^2}{2})T}} = e^{\frac{\sigma^2 T}{2}}$$

So, in this problem,

$$\frac{84.10}{83.26} = e^{\frac{\sigma^2}{4}} \quad \Rightarrow \quad \frac{\sigma^2}{4} = \ln\left(\frac{84.10}{83.26}\right) \quad \Rightarrow \quad \sigma = \sqrt{4\ln\left(\frac{84.10}{83.26}\right)} = 0.2004.$$

Problem 3.9. Assume the Black-Scholes setting. Assume $S(0) = \$28.50, \sigma = 0.32, r = 0.04$. The stock pays no dividends. Consider a \$30-strike put option which expires in 110 days (simplify the number of days in a year to 360). What is the price of the put?

(a) 2.75
(b) 2.10
(c) 1.80
(d) 1.20
(e) None of the above.

Solution: (a)
In our usual notation, the price is

$$V_P(0) = Ke^{-r\cdot T}\,N(-d_2) - S(0)N(-d_1)$$

with

$$d_1 = -0.1324386, \quad d_2 = -0.3093253.$$

So, $V_P(0) = 0.2011571$.

Problem 3.10. The current price of a non-dividend-paying stock is given to be \$92. The stock's volatility is 0.35.

The continuously compounded risk-free interest rate is 0.05.

Consider a \$90-strike European call option on the above stock with exercise date in a quarter-year.

What is the Black-Scholes price of this call option?

(a) 5.05
(b) 7.66
(c) 7.71
(d) 7.99
(e) None of the above.

Solution: (d)

$$d_1 = 0.2845223, d_2 = 0.1095223.$$

So,

$$V_C(0) = 7.986754.$$

Problem 3.11. (5 pts) Let the stochastic process $S = \{S(t); t \geq 0\}$ denote the stock price. The stock's rate of appreciation is 10% while its volatility is 0.30 Then,

(a) $Var[\ln(S(t))] = 0.3t$
(b) $Var[\ln(S(t))] = 0.09t^2$
(c) $Var[\ln(S(t))] = 0.09t$
(d) $Var[\ln(S(t))] = 0.09$
(e) None of the above.

Solution: (c)
The random variable $S(t)$ is lognormal so that the random variable $\ln(S(t))$ is normal with variance $0.3^2 t = 0.09t$.