Bivariate Normal Random Variables.

("Based on Pitman's "Probability")

Recall: In 1-D, the standard normal density is

$$\boxed{\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \quad \text{for all } z \in \mathbb{R}}$$

In 2-D, we start w/ X and Y that are independent and both are standard normal, i.e., $N(0,1)$

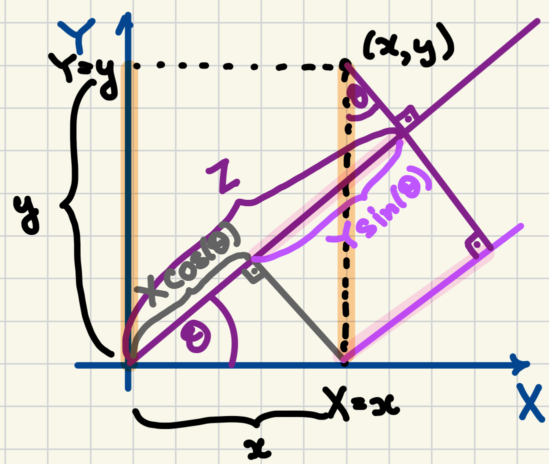
Then, the joint density of the pair (X, Y) is

$$f(x,y) = \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)} \quad \text{for all } (x,y) \in \mathbb{R}^2.$$

Standard.

Start w/ a pair of independent, standard normal variables.

Say, X and Y.



$\Rightarrow Z = X \cdot \cos(\theta) + Y \cdot \sin(\theta)$

We know: $Z \sim N(0,1)$

Q: What is the correlation coefficient between X and Z?

$\longrightarrow$:

$$\rho_{X,Z} = \frac{Cov[X,Z]}{\underbrace{SD[X]}_{=1} \cdot \underbrace{SD[Z]}_{=1}} = Cov[X,Z]$$

$$= \mathbb{E}[(X-\mu_X)(Z-\mu_Z)]$$

$$= \mathbb{E}[XZ] - \underbrace{\mu_X \mu_Z}_{=0}$$

$$= \mathbb{E}[XZ]$$

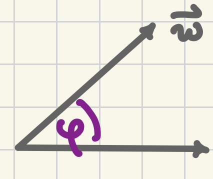
$$= \mathbb{E}[X(X\cos(\theta) + Y\sin(\theta))] =$$

$$= \cos\theta \underbrace{\mathbb{E}[X^2]}_{=1} + \sin\theta \underbrace{\mathbb{E}[XY]}_{=0}$$

$$\boxed{\rho_{X,Z} = \cos(\theta)}$$

$\square$

Linear Algebra



$$\vec{v} \cdot \vec{w} = \|\vec{v}\| \cdot \|\vec{w}\| \cdot \cos(\varphi)$$

$$\cos(\varphi) = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \cdot \|\vec{w}\|}$$

Special Cases:

$\theta = 0 \Rightarrow Z = X$

$\theta = \frac{\pi}{2} \Rightarrow Z = Y$ So, X and Z are independent.

$\theta = \pi \Rightarrow Z = -X$

In general: For each correlation coefficient

$$-1 \le \rho \le 1,$$

there exists an angle $\boxed{\Theta = \arccos(\rho)}$ such that X and Z defined as above have the correlation coefficient ρ.

Alternative: $\boxed{Z = \rho \cdot X + \sqrt{1-\rho^2} \cdot Y}$

w/ X and Y independent and $N(0,1)$.

Joint Density:

$$f_{X,Z}(x,z) = \frac{1}{2\pi(1-\rho^2)} \exp\left(-\frac{1}{2(1-\rho^2)}(x^2 - 2\rho xz + z^2)\right)$$

Marginal Dist'ns: $X \sim N(0,1)$, $Z \sim N(0,1)$

Conditional Dist'n: Given $\boxed{X = x}$, $Z \sim \text{Normal}(\rho \cdot x, \text{var} = 1-\rho^2)$

Given $Z = z$, $X \sim \text{Normal}(\rho \cdot z, \text{var} = 1-\rho^2)$

Independence:

$$\boxed{\begin{gathered} X \text{ and } Y \text{ independent} \\ \text{iff} \\ \rho_{X,Y} = 0 \end{gathered}}$$

Any Bivariate Normal.

A Random pair (U, V) is said to be bivariate normal w/ parameters $\mu_U, \mu_V, \sigma_U, \sigma_V$ and ρ iff

$$(X, Z) := \left(\frac{U - \mu_U}{\sigma_U}, \frac{V - \mu_V}{\sigma_V} \right)$$

has the standard normal dist'n w/ correlation ρ.

Note:

$$\rho_{U,V} = \frac{Cov[U,V]}{SD[U] \cdot SD[V]} = \left\{ \begin{array}{l} U = \mu_U + \sigma_U \cdot X \\ V = \mu_V + \sigma_V \cdot Z \end{array} \right\}$$

$$= \frac{Cov[\mu_U + \sigma_U \cdot X, \mu_V + \sigma_V \cdot Z]}{SD[\mu_U + \sigma_U \cdot X] \cdot SD[\mu_V + \sigma_V \cdot Z]}$$

$$= \frac{Cov[\sigma_U \cdot X, \sigma_V \cdot Z]}{SD[\sigma_U \cdot X] \cdot SD[\sigma_V \cdot Z]} =$$

$$= \frac{\cancel{\sigma_U} \cdot \cancel{\sigma_V} \cdot Cov[X, Z]}{\cancel{\sigma_U} \cdot SD[X] \cdot \cancel{\sigma_V} \cdot SD[Z]} = \rho_{X,Z} = \rho$$

$\Rightarrow$ $\boxed{U \text{ and } V \text{ are independent iff } \rho = 0}$