

**The University of Texas at Austin**  
**IN-CLASS WORK 9**

*M378K Introduction to Mathematical Statistics*

February 28, 2026

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**MOMENT-GENERATING FUNCTIONS.**

**DEFINITION 9.1:** The  $k^{\text{th}}$  moment of a random variable  $Y$  taken about the origin is defined as  $\mathbb{E}[Y^k]$  provided that the expectation exists. We write

$$\mu_k = \mathbb{E}[Y^k]$$

when there is no ambiguity about the random variable in question.

**Remark.**  $\mu_k$  is also referred to as the  $k^{\text{th}}$  raw moment.

**Remark.** In particular,  $\mu_1 = \mu$  happens to be the mean of the random variable  $Y$ .

**DEFINITION 9.2:** The  $k^{\text{th}}$  central moment of a random variable  $Y$  is defined as  $\mathbb{E}[(Y - \mu)^k]$  provided that the expectation exists. We write

$$\mu_k^c = \mathbb{E}[(Y - \mu)^k]$$

when there is no ambiguity about the random variable in question.

**Remark.**  $\mu_k^c$  is also referred to as the  $k^{\text{th}}$  moment of a random variable  $Y$  taken about its mean.

**DEFINITION 9.3:** The moment-generating function (mgf)  $m_Y$  for a random variable  $Y$  is defined as

$$m_Y(t) = \mathbb{E}[e^{tY}]$$

for all  $t$  for which the above expectation exists. In fact, we say that the moment-generating function **exists** if there exists a positive number  $b$  such that  $m_Y(t)$  is finite for all  $t$  such that  $|t| \leq b$ .

**Problem 9.1.** How much is  $m_Y(0)$ ?

**Solution.**

$$m_Y(0) = 1.$$

**Remark.** On the choice of terminology ...

Step 1.

$$\frac{d}{dt}m_Y(t) = ?$$

**Solution.**

$$\mathbb{E}[Y e^{tY}]$$

Step 2.

$$m'_Y(0) = ?$$

**Solution.**

$$\mu_Y = \mathbb{E}[Y]$$

Step 3.

$$\frac{d^2}{dt^2}m_Y(t) = ?$$

**Solution.**

$$\mathbb{E}[Y^2 e^{tY}]$$

Step 4.

$$m''_Y(0) = ?$$

**Solution.**

$$\mu_2 = \mathbb{E}[Y^2]$$

Step 5. What do you suspect the **generalization** of the above would be?

**THEOREM 9.4:** If  $m_Y$  exists, then for  $k \in \mathbb{N}$ , we have

$$m_Y^{(k)}(0) = \mu_k.$$

**EXAMPLE 9.5:** Let  $Y \sim b(n = 1, p)$ , i.e., let  $Y$  model a Bernoulli trial with the probability of success denoted by  $p$ . Find  $m_Y$ .

**Solution.**

$$m_Y(t) = \mathbb{E}[e^{tY}] = (1-p)e^0 + pe^t = (1-p) + pe^t, \quad t \in \mathbb{R}.$$

**PROPOSITION 9.6:** Let  $Y_1$  and  $Y_2$  be independent random variables with m.g.f.s denoted by  $m_{Y_1}$  and  $m_{Y_2}$ . Define  $Y = Y_1 + Y_2$ . Then, for every  $t$  for which both  $m_{Y_1}$  and  $m_{Y_2}$  are well defined, we have

$$m_Y(t) =$$

*Proof:* By definition:

$$m_Y(t) =$$

**Solution.**

$$\mathbb{E}[e^{tY}]$$

Using  $Y = Y_1 + Y_2$ , we can substitute  $Y_1 + Y_2$  for  $Y$  in the expression above. So,

$$m_Y(t) =$$

**Solution.**

$$\mathbb{E}[e^{t(Y_1+Y_2)}]$$

One of the properties of the exponential function is that  $e^{A+B} = e^A \times e^B$ . Thus, the above becomes:

$$m_Y(t) =$$

**Solution.**

$$\mathbb{E}[e^{tY_1}] \times \mathbb{E}[e^{tY_2}]$$

Recall that  $Y_1$  and  $Y_2$  are assumed to be independent random variables. With this in mind, we get:

$$m_Y(t) =$$

**Solution.**

$$\mathbb{E}[e^{tY_1}] \times \mathbb{E}[e^{tY_2}]$$

Finally, using the definition of a moment generating function, we have

$$m_Y(t) =$$

**Solution.**

$$m_{Y_1}(t)m_{Y_2}(t)$$

■

**EXAMPLE 9.7:** Let  $Y \sim b(n, p)$ . What is the moment generating function of  $Y$ ?

**Solution.**

$$Y \stackrel{(d)}{=} X_1 + \dots + X_n$$

with  $X_i \sim b(1, p)$ ,  $i = 1, \dots, n$  independent random variables. Then,

$$m_Y(t) = m_{X_1}(t) \times \dots \times m_{X_n}(t) = (m_{X_1}(t))^n = (1 - p + pe^t)^n.$$

**EXAMPLE 9.8:** Let  $N \sim \text{Poisson}(\lambda)$ . What is the moment generating function  $m_N$  of  $N$ ?

**Solution.**

$$\begin{aligned} m_N(t) &= \mathbb{E}[e^{tN}] \\ &= \sum_{n=0}^{\infty} e^{tn} p_N(n) \\ &= \sum_{n=0}^{\infty} e^{tn} e^{-\lambda} \frac{\lambda^n}{n!} \\ &= e^{-\lambda} \sum_{n=0}^{\infty} \frac{(e^t \lambda)^n}{n!} = e^{-\lambda} \times e^{\lambda e^t} = e^{\lambda(e^t - 1)}. \end{aligned}$$

**EXAMPLE 9.9:** Let  $Z \sim N(0, 1)$ . What is the moment generating function  $m_Z$  of  $Z$ ?

**Solution.**

$$m_Z(t) = e^{t^2/2} \quad t \in \mathbb{R}.$$

**EXAMPLE 9.10:** Let the random variable  $Y$  have the mgf  $m_Y$ . Define  $X = aY + b$  for some constants  $a$  and  $b$ . Express the mgf  $m_X$  of  $X$  in terms of  $m_Y$ ,  $a$  and  $b$ .

**Solution.**

$$m_X(t) = e^{bt} m_Y(at)$$

**EXAMPLE 9.11:** Let  $X \sim N(\mu, \sigma^2)$ . What is the moment generating function  $m_X$  of  $X$ ?

**Solution.** Since  $X$  can be expressed as a linear transform of  $Z \sim N(0, 1)$  in the following way

$$X = \mu + \sigma Z,$$

we get that

$$m_X(t) = e^{\mu t} m_Z(\sigma t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

**Problem 9.2.** A random variable  $Y$  is said to be *lognormal* if there exists a normally distributed random variable  $X \sim N(\mu, \sigma^2)$  such that  $Y \stackrel{(d)}{=} e^X$ . Express the mean and the variance of the lognormal r.v.  $Y$  in terms of the parameters  $\mu$  and  $\sigma$ .

**Solution.**

$$\mathbb{E}[Y] = \mathbb{E}[e^X] = \mathbb{E}[e^{1 \times X}] = m_X(1) = \exp\left\{\frac{1}{2}\sigma^2 + \mu\right\}.$$

$$\mathbb{E}[Y^2] = \mathbb{E}\left[(e^X)^2\right] = \mathbb{E}[e^{2 \times X}] = m_X(2) = \exp\left\{\frac{1}{2}\sigma^2 \times 4 + \mu \times 2\right\} = \exp\{2(\sigma^2 + \mu)\}.$$

$$\text{Var}[Y] = \mathbb{E}\left[(e^X)^2\right] - (\mathbb{E}[e^X])^2 = \exp\{2(\sigma^2 + \mu)\} - \exp\{\sigma^2 + 2\mu\}.$$

**PROPOSITION 9.12:**

- If  $m_Y$  exists for a certain probability distribution, then it is unique.
- If  $m_{Y_1}$  and  $m_{Y_2}$  are equal on an interval, then  $Y_1 \stackrel{(d)}{=} Y_2$ .

**COROLLARY 9.13:** Let  $Y_1$  and  $Y_2$  be independent and normally distributed. Define  $Y = Y_1 + Y_2$ . Then, the distribution of  $Y$  is ...

*Proof:*

**Solution.** Note that  $Y_i \sim N(\mu = \mu_i, \sigma_i)$  for  $i = 1, 2$ . Now, let's look at the mgf of  $Y$ . Then, since  $Y_1$  and  $Y_2$  are independent, we have

$$m_Y(t) = m_{Y_1}(t)m_{Y_2}(t).$$

We can now use the fact that for any  $X \sim N(\mu, \sigma)$ ,

$$m_X(t) = e^{\mu t} m_Z(\sigma t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

Hence,

$$m_Y(t) = e^{\mu_1 t + \frac{\sigma_1^2 t^2}{2}} \times e^{\mu_2 t + \frac{\sigma_2^2 t^2}{2}} = e^{(\mu_1 + \mu_2)t + \frac{(\sigma_1^2 + \sigma_2^2)t^2}{2}}$$

We can conclude that  $Y \sim N(\mu_1 + \mu_2, \sqrt{\sigma_1^2 + \sigma_2^2})$ . ■

**COROLLARY 9.14:** Let  $N_1$  and  $N_2$  be independent and Poisson distributed. Define  $N = N_1 + N_2$ . Then, the distribution of  $N$  is ...

*Proof:*

**Solution.** We are given that  $N_i \sim \text{Poisson}(\lambda_i)$  for  $i = 1, 2$ . We saw in a previous example that

$$m_{N_i}(t) = e^{\lambda_i(e^t - 1)}.$$

Hence,

$$m_N(t) = m_{N_1}(t)m_{N_2}(t) = e^{\lambda_1(e^t - 1)} \times e^{\lambda_2(e^t - 1)} = e^{(\lambda_1 + \lambda_2)(e^t - 1)}$$

We can conclude that  $N \sim \text{Poisson}(\lambda_1 + \lambda_2)$ . ■