M339J: January 28th, 2022.

The Inverse Transformation (Simulation) Method.

Proposition.

(1) Let X be a continuous random variable. Denote its cumulative distribution f'tion by F_X and its probability density function by f_X.

Assume that $\boxed{f_X(x) > 0 \text{ for all } x}$

Set $\boxed{Y := F_X(X)}$ ✓

Then, $Y \sim \underline{U(0,1)}$

$\longrightarrow$: Support of Y will be contained in $[0,1]$.

Let $y \in [0,1]$.

$$F_Y(y) = \mathbb{P}[Y \leq y] = \mathbb{P}[\underline{F_X(X) \leq y}]$$

We assumed $f_X(x) > 0$ always

Also: $F_X(a) = \int_{-\infty}^{a} f_X(x)\,dx$

$\Rightarrow$ F_X is a strictly increasing function

$\Rightarrow$ F_X is one-to-one

$\Rightarrow$ F_X^{-1} exists and is also increasing

$$\underline{F_Y(y)} = \mathbb{P}[\cancel{F_X^{-1}}(\cancel{F_X}(X)) \leq F_X^{-1}(y)]$$

$$= \mathbb{P}[X \leq F_X^{-1}(y)] = \cancel{F_X}(\cancel{F_X^{-1}}(y)) = \underline{y}$$

$\boxed{Y \sim U(0,1)}$

(2) Let $U \sim U(0,1)$.

Let F be a cumulative dist'n function.

Set
$$X \sim F^{-1}(U)$$

Then, the random variable X has the cdf F.

Implementation.

1. Set F to be the cdf of the dist'n from which we want to simulate values.
"Figure out" F^{-1}.
2. Draw: $u_1, u_2, \ldots, u_n \sim U(0,1)$
3. Set $x_1 = F^{-1}(u_1)$, $x_2 = F^{-1}(u_2)$, ..., $x_n = F^{-1}(u_n)$

The $x_1, x_2, \ldots, x_n$ are the simulated values from your dist'n.

Example. In the exponential case: for $X \sim \text{Exponential}(\theta)$

$$F_X(x) = 1 - e^{-\frac{x}{\theta}} \qquad \text{for } x > 0$$

"For $x>0$", we "figure out" the F^{-1}

$$y = 1 - e^{-\frac{x}{\theta}}$$

$$e^{-\frac{x}{\theta}} = 1 - y$$

$$-\frac{x}{\theta} = \ln(1-y)$$

$$x = -\theta \ln(1-y)$$

$$\boxed{F^{-1}(y) = -\theta \ln(1-y)}$$

The simulated values from the exponential:

$$\boxed{-\theta \ln(1-u_i)}$$

Special Case: $\boxed{\text{Bernoulli}(q)}$