M378K Introduction to Mathematical Statistics
Homework assignment #3

Please, provide your **complete solutions** to the following problems. Final answers only, even if correct will earn zero points for those problems.

Problem 3.1. *(5 points) Let the density function of a random variable X be given as*

$$f_X(x) = cxI_{[0,1]}(x),$$

for some constant c. Find $\mathbb{E}[X^3]$.

Solution: Since the density function must integrate up to 1, we get $c = 2$. Whence,

$$\mathbb{E}[X^3] = 2\int_0^1 x^4\,dx = \frac{2}{5}\,.$$

Problem 3.2. *(5 points) Let X denote the outcome of a roll of a fair, regular dodecahedron (a polyhedron with 12 faces) with numbers $1, 2, \cdots, 12$ written on its sides. Find $\mathbb{E}[X]$.*

Solution:

$$\mathbb{E}[X] = \frac{1}{12}(1 + \cdots + 12) = \frac{13}{2}\,.$$

Problem 3.3. *(5 points) Let X be a random variable with mean $\mu = 2$ and standard deviation equal to $\sigma = 1$. Find $\mathbb{E}[X^2]$.*

Solution:

$$\mathbb{E}[X^2] = Var[X] + (\mathbb{E}[X])^2 = 1 + 2^2 = 5.$$

Problem 3.4. *(5 points) Let X denote the number of $1's$ in 100 throws of a fair die. Find $\mathbb{E}[X^2]$.*

Solution: Evidently, $X \sim b(100, 1/6)$. So,

$$\mathbb{E}[X^2] = Var[X] + (\mathbb{E}[X])^2 = 100 \cdot \frac{1}{6} \cdot \frac{5}{6} + (100 \cdot \frac{1}{6})^2 = \frac{500 + 10000}{36} = \frac{875}{3}\,.$$

Problem 3.5. *(15 points) Let X be a r.v. with an exponential distribution with mean $\tau = 1/\lambda > 0$, and let $Y = 1/X$. Compute the pdf of Y.* Optional: *compute $\mathbb{E}[Y]$. Is it true that $\mathbb{E}[Y] = 1/\mathbb{E}[X]$, i.e., that $\mathbb{E}[Y] = \lambda$?*

Solution: The pdf of X is $\frac{1}{\tau}e^{-\frac{x}{\tau}} = \lambda e^{-\lambda x}$, for $x \geq 0$, and the cdf if $F_X(x) = 1 - e^{-\frac{x}{\tau}} = 1 - e^{-\lambda x}$, for $x \geq 0$, and $F_X(x) = 0$, for $x < 0$. Therefore, for $y > 0$, we have

$$F_Y(y) = \mathbb{P}[Y \leq y] = \mathbb{P}[1/X \leq y] = \mathbb{P}[X \geq 1/y] = 1 - \mathbb{P}[X < 1/y] = 1 - F_X(1/y).$$

To go from the cdf to the pdf, we differentiate:

$$f_Y(y) = \tfrac{d}{dy}(1 - F_X(1/y)) = -F_X'(1/y)\tfrac{d}{dy}(1/y) = -f_X(1/y)\tfrac{-1}{y^2} = \lambda y^{-2} e^{-\lambda/y},$$

for $y > 0$, and $f_Y(y) = 0$ otherwise.

To compute the expectation of Y, now that we know the pdf of Y, we can use the formula $\mathbb{E}[Y] = \int y\, f_Y(y)\, dy$. We can also use the fact that $Y = g(X)$, and use the formula $\mathbb{E}[Y] = \mathbb{E}[g(X)] = \int g(x)\, f_X(x)\, dx$. The second approach is a bit faster, so

$$\mathbb{E}[Y] = \int_0^\infty \tfrac{1}{x}\lambda e^{-\lambda x}\, dx.$$

For $x \in [0, 1]$, we have $\lambda e^{-\lambda x} \geq \lambda e^{-\lambda}$, and so (the point being is that the integrand looks like a constant multiple of the function $1/x$, which is not integrable around 0)

$$\mathbb{E}[Y] \geq \int_0^1 \frac{1}{x}\lambda e^{-\lambda x}\, dx \geq \lambda e^{-\lambda} \int_0^1 \tfrac{1}{x}\, dx = \infty,$$

meaning that the expectation $\mathbb{E}[Y]$ *does not exist.* In particular, it is not equal to $1/\mathbb{E}[X]$.

Problem 3.6. *(20 points) Let X be a discrete random variable with the support $\mathcal{S}_X = \mathbb{N}$, such that $\mathbb{P}[X = n] = C\frac{1}{n^2}$, for $n \in \mathbb{N}$, where C is a constant chosen so that $\sum_n \mathbb{P}[X = n] = 1$. The distribution table of X is, therefore, given by*

1	2	3	...
$C\frac{1}{1^2}$	$C\frac{1}{2^2}$	$C\frac{1}{3^2}$	...

1. *(10 points) Show that $\mathbb{E}[X]$ does not exist.*
2. *(10 points) Construct a distribution of a similar random variable whose expectation does exist, but the variance does not. (Hint: Use the same support $\mathbb{N}$, but tweak the probabilities so that the sum for $\mathbb{E}[X]$ converges, while the sum for $\mathbb{E}[X^2]$ does not.)*

Solution:

1. The expression for $\mathbb{E}[X]$ is

$$\mathbb{E}[X] = \sum_{n=1}^\infty n\mathbb{P}[X = n] = C\sum_{n=1}^\infty \frac{1}{n} = +\infty,$$

because the *Harmonic series* $1 + 1/2 + 1/3 + \dots$ diverges.

2. The distribution of Y we need to construct should have the following properties

$$\sum_{n=1}^{\infty} n\mathbb{P}[Y = n] < \infty \text{ but } \sum_{n=1}^{\infty} n^2\mathbb{P}[Y = n] = \infty.$$

   We can try to achieve this by taking $\mathbb{P}[Y = n] = C'\frac{1}{n^3}$, where, as above, C' is simply a constant that ensures that $\sum_n \mathbb{P}[Y = n] = 1$. Indeed, in this case, we have

$$\mathbb{E}[X] = C'\sum_n \frac{1}{n^2} \text{ while } \mathbb{E}[X^2] = C'\sum_n \frac{1}{n}.$$

   The first sum converges, but the second one diverges.