UNIVERSITY OF TEXAS AT AUSTIN

HW Assignment 1

The prerequisite material.

Please, provide just the final answer to the True/False and Multiple-choice questions. For the free-response questions, provide your complete solutions if you expect partial credit. For the free-response questions, even if your final answer is correct and your work is not shown, you should not expect **any** partial credit.

Problem 1.1. (10 points)
Provide the definition of an **arbitrage portfolio**.

Solution: See your lecture notes from *M339D*.

Problem 1.2. (2 points) It is never optimal to exercise an American call option on a non-dividend paying stock early. *True or false?*

Solution: TRUE

Problem 1.3. (2 points) Gap put options always have a nonnegative payoff. *True or false?*

Solution: FALSE

Problem 1.4. (6 pts) Let Z be a standard normal random variable. Using the standard normal tables, calculate the following probabilities:

(i) (2 points) $\mathbb{P}[-1.23 < Z < 2.37]$
(ii) (2 points) $\mathbb{P}[|Z| < 0.5]$
(iii) (2 points) $\mathbb{P}[Z^2 > 2.56]$

Solution:

(i)

$$\begin{aligned}\mathbb{P}[-1.23 < Z < 2.37] &= \mathbb{P}[Z < 2.37] - \mathbb{P}[Z < -1.23] = \Phi(2.37) - (1 - \Phi(1.23)) \\ &= 0.9911 + 0.8907 - 1 = 0.8818.\end{aligned}$$

(ii)

$$\mathbb{P}[|Z| < 0.5] = \mathbb{P}[Z < 0.5] - \mathbb{P}[Z < -0.5] = 2\Phi(0.5) - 1 = 2(0.6915) - 1 = 0.383.$$

(iii)

$$\mathbb{P}[Z^2 > 2.56] = \mathbb{P}[|Z| > 1.6] = 2(\mathbb{P}[Z > 1.6]) = 2(1 - \Phi(1.6)) = 0.1096.$$

Problem 1.5. (5 points) An investor short sells one share of a non-dividend-paying stock and buys an at-the-money, T-year, European call option on this stock. The call premium is denoted by $V_C(0)$. Assume that there are no transaction costs. The continuously compounded, risk-free interest rate is denoted by r. Let the argument s represent the stock price at time T.

(i) (3 points) Determine an algebraic expression for the investor's profit at expiration T in terms of $V_C(0), r, T$ and the strike K.
(ii) (2 points) In particular, how does the expression you obtained in (i) simplify if the call is in-the-money on the exercise date?

Solution:

$$-s + (s-K)_+ + (S(0) - V_C(0))e^{rT} = -s + (s-K)_+ + (K - V_C(0))e^{rT}.$$

For the option to be in-the-money at expiration, we must have $s < K$. So, the profit simplifies to

$$-s + (s-K) + (K - V_C(0))e^{rT} = -K + (K - V_C(0))e^{rT}.$$

Problem 1.6. (10 pts) For a two-period binomial model, you are given that:

(1) each period is one year;
(2) the current price of a non-dividend-paying stock S is $S(0) = \$20$;
(3) $u = 1.2$, with u as in the standard notation for the binomial model;
(4) $d = 0.8$, with d as in the standard notation for the binomial model;
(5) the continuously compounded risk-free interest rate is $r = 0.04$.

Consider a **special** call option which pays the excess above the strike price $K = 23$ (if any!) at the end of **every** binomial period.

Find the price of this option.

Solution: The risk-neutral probability is

$$p^* = \frac{e^{0.04} - 0.8}{1.2 - 0.8} = 0.6020.$$

When one constructs the two-period binomial tree, one gets

$$S_u = 24,\ S_d = 16,$$
$$S_{uu} = 28.80,\ S_{ud} = S_{dd} = 19.2,\ S_{dd} = 12.8.$$

So, the payoffs at the end of the first period are

$$V_u = 1, V_d = 0.$$

The payoffs at the end of the second period are

$$V_{uu} = 5.80, \quad V_{ud} = 0, \quad V_{dd} = 0.$$

So, taking the expected value at time 0 of the payoff with respect to the risk-neutral probability, we get that the price of this call should be

$$\begin{aligned} e^{-0.04} \times V_u \times p^* + e^{-0.04\times 2}[V_{uu} \times (p^*)^2 + V_{ud} \times 2p^*(1-p^*)] \\ = e^{-0.04} \times 1 \times 0.6020 + e^{-0.08}[5.8 \times 0.6020^2] \\ = 2.51893. \end{aligned}$$

Problem 1.7. (5 points) A discrete-dividend-paying stock sells today for \$100 per share. The continuously compounded, risk-free interest rate is 0.04. The first dividend will be paid at in three months in the amount of \$2. The remaining dividends will be equal to \$1 and continue to be paid out quarterly. What is the **prepaid forward price** of this stock for delivery in seven months?

(a) \$73.02
(b) \$97.04
(c) \$98.02
(d) \$100
(e) None of the above.

Solution: The correct answer is **(b)**.

$$F^P_{0,7/12}(S) = 100 - 2e^{-0.01} - 1e^{-0.02} = 97.0397.$$

Problem 1.8. (5 points) The random vector (X_1, X_2) is jointly normal. Its marginal distributions are:

$$X_1 \sim N(\text{mean} = 0, \text{variance} = 4), \quad X_2 \sim N(\text{mean} = 1, \text{variance} = 1).$$

The correlation coefficient is given to be

$$corr[X_1, X_2] = 0.3.$$

What is the variance of the random variable $X = 3X_1 - 2X_2$?

(a) 32.8
(b) 47.2
(c) 54.4
(d) 58.2
(e) None of the above.

Solution: (a)

The variance of X is

$$\begin{aligned} Var[X] &= 9Var[X_1] + 4Var[X_2] - 2(3)(2)Cov[X_1, X_2] \\ &= 9(4) + 4(1) - 12(2)(1)(0.3) = 32.8. \end{aligned}$$

Problem 1.9. (5 points) A coin is tossed and, independently, a 6-sided die is rolled. Let

$A = \{4 \text{ is obtained on the die}\}$ and
$B = \{\textit{Heads} \text{ is obtained on the coin and}$
$\text{an even number is obtained on the die}\}$.

Then

(a) A and B are mutually exclusive
(b) A and B are independent
(c) $A \subseteq B$
(d) $A \cap B = B$
(e) None of the above.

Solution: The correct answer is (e).