University of Texas at Austin

Problem Set # 3

Problem 3.1. An investor wants to hold 200 euros two years from today. The spot exchange rate is \$1.31 per euro. If the euro denominated annual interest rate is 3.0% what is the price of a currency prepaid forward?

Solution:

$$F^P_{0,T}(x) = 200e^{-0.03 \cdot 2} \cdot 1.31 = 246.67.$$

Problem 3.2. You produce tiramisu cakes. You plan to sell 1,000 cakes in a month. Your (unhedged) payoff will be $\$10,000 - S(1)$, where $S(1)$ denotes the price of the amount of belgian chocolate required to dust the 1,000 cakes.

Assume that the continuously compounded annual risk-free interest rate equals 6%.

Your hedge consists of the following two components:

(1) one **long** one-month, \$9,000-strike call option on the amount of chocolate you need; it's premium is $V_C(0) = \$60.00$,
(2) one **written** one-month, \$8,500-strike put option on the amount of chocolate you need; it's premium is $V_P(0) = \$200.00$.

Calculate the profit of the hedged portoflio if the final price of the amount of chocolate you need turns out to be $\$8,800$.

Solution: The hedged portfolio consists of the following components:

(1) the **payoff** from the tiramisu cake sales,
(2) one **long** one-year, \$9,000-strike call option on the chocolate whose premium was \$60.00,
(3) one **written** one-year, \$8,500-strike put option on the chocolate whose premium was \$200.00.

The initial cost for this portfolio is the cost of hedging (all other accumulated production costs are incorporated in the revenue expression $10,000 - S(1)$). Their future value is

$$(60 - 200)e^{0.005} \approx -140.70.$$

As usual, the negative initial cost signifies an initial influx of money for the principal character (in this case, you: the producer of tiramisu cakes).

The profit for $S(1) = 8,800$ is

$$10,000 - 8,800 + 140.70 = 1,340.70.$$

Problem 3.3. (5 points) A stock is currently priced at \$118 per share. It is scheduled to pay a continuous dividend in the amout proportional to its price with the dividend yield of 2.0% per annum.
A nine-month 120-strike European call and put options on this stock have equal prices. Let the continuously-compounded annual risk-free rate of interest be denoted by r. How much is r?

Solution: By the put-call parity, we have

$$0 = V_C(0) - V_P(0) = S(0)e^{-\delta T} - Ke^{-rT} \quad \Rightarrow \quad -rT = \ln\left(\frac{S(0)}{K}\right) - \delta T$$
$$\Rightarrow \quad r = \frac{1}{T}\ln\left(\frac{K}{S(0)}\right) + \delta = 0.042.$$

Problem 3.4. (2 points)

In the setting of the binomial asset-pricing model, let d and u denote the up and down factors, respectively. Moreover, let r denote the continuously compounded, risk-free interest rate. Let h denote the length of a single period in our model.

Then, if,

$$e^{\delta h}d < e^{rh} < e^{\delta h}u$$

then there is no possibility for arbitrage. *True or false?*

Solution: TRUE

Problem 3.5. The current exchange rate of one Swiss franc to euros is 0.90. The volatility of the exchange rate is given to be 0.10.

The continuously compounded risk-free interest rate for the Swiss franc is 0.06 while the continuously compounded risk-free interest rate for the euro equals 0.02.

You want to price a euro-denominated, at-the-money one-year European call option on the Swiss franc using a twelve-period forward binomial tree. What is the up factor u in this tree?

(a) 1.02586
(b) 1.03101
(c) 1.03272
(d) 1.03445
(e) None of the above.

Solution: (a)

The length of a singe period in our tree is $h = 1/12$. So,

$$u = e^{(0.02-0.06)/12+0.1\sqrt{1/12}} = 1.02586.$$

Problem 3.6. Assume that the a stock price is modeled using a one-period forward binomial tree with the length of a single period equal to three months. According to this model, the stock price can take either the value of \$55, or the value of \$40 in exactly three months. Calculate the volatility of the stock price.

Solution:

$$e^{2\sigma\sqrt{h}} = S_u/S_d \quad \Rightarrow \quad \sigma = \frac{1}{2\sqrt{h}}\ln(S_u/S_d) = \frac{1}{2\sqrt{1/4}}\ln(55/40) = \ln(55/40) = 0.3185.$$

Problem 3.7. In the setting of the binomial asset-pricing model, let d and u denote the up and down factors, respectively. Moreover, let r denote the continuously compounded, risk-free interest rate. Let h denote the length of a single period in our model.

Then, if,

$$d < e^{rh} < u$$

then there is no possibility for arbitrage. *True or false?*

Solution: FALSE

Dividends!

Problem 3.8. Let the current exchange rate of euros (€) to USD (\$) be denoted by $x(0)$, i.e., currently, 1 € $= \$X(0)$.

Let $r_\$$ denote the continuously compounded, risk-free interest rate for the \$, and let r_e denote the continuously compounded, risk-free interest rate for the €.

Denote the price of a \$-denominated European call option with strike K and exercise date T by $V_C(0)$ and the price of an otherwise identical put option by $V_P(0)$. Then,

$$V_C(0) - V_P(0) = x(0)e^{-r_\$ T} - Ke^{-r_e T}.$$

True or false?

Solution: FALSE
The two interest rates have switched places.

Problem 3.9. (5 points) Let the current price of a non-dividend-paying stock be \$100 per share. The price of this stock in one year is modeled by a one-period binomial model. The two possible prices that the stock can attain in this model are \$130 and \$75. Assume that the continuously compounded risk-free interest rate equals 0.05.

An investor wants to construct a replicating portfolio for a \$100-strike, one-year European put on the above stock. What is the risk-free component of the replicating portfolio?

(a) Borrow 10.75
(b) Borrow 56.20
(c) Lend 10.75
(d) Lend 56.20
(e) None of the above.

Solution: (d)
The risk-neutral probability of the stock-price going up is

$$p^* = \frac{e^{0.05} - 1.3}{1.3 - 0.75} = 0.5478.$$

The risk-neutral price is

$$V_P(0) = e^{-0.05}(1 - 0.5478) \times 25 = 10.75.$$

The Δ is

$$\Delta = \frac{-25}{130 - 75} = -\frac{5}{11}.$$

So,

$$B = 10.75 - \left(-\frac{5}{11}\right) \times 100 = 56.20.$$

Problem 3.10. The current price of a continuous-dividend-paying stock is \$100 per share. Its dividend yield is 0.02 and its volatility is given to be 0.2.

The continuously compounded risk-free interest rate equals 0.06.

Consider a \$110-strike, half-year American put on the above stock. Use a two-period forward binomial stock-price tree to calculate the current price of the American put.

Solution: This is a forward binomial tree, so

$$u = e^{(r-\delta)h+\sigma\sqrt{h}} = e^{(0.06-0.02)(0.25)+0.2\sqrt{h}(0.5)} = e^{0.11} \approx 1.1163,$$
$$d = e^{(r-\delta)h-\sigma\sqrt{h}} = e^{(0.06-0.02)(0.25)-0.2\sqrt{h}(0.5)} = e^{-0.09} \approx 0.9139.$$

Note: You can notice that $e^{0.11} > 1.10$ so that the option is out of the money at the up and the $up - up$ nodes. You do not need to get the explicit value for u. Hence,

$$
\begin{array}{lll}
 & & S_{uu} > K, \\
 & S_u > K, & \\
 & & S_{ud} = udS(0) = 100e^{0.02} = 102.02, \\
S_d = 91.39, & & \\
 & & S_{dd} = 83.53.
\end{array}
$$

Note: Again, you could have noticed that $S_{ud} > 100e^{0.2} > 120 > K$. There was no need to get the stock price at the $up - down$ node explicitly!

The risk-neutral probability of an up movement in a single step is

$$p^* = \frac{1}{1 + e^{0.1}} = 0.475.$$

Now we work backwards through the tree.

the up node:

$$\begin{aligned} CV_u &= e^{-0.06/4}(1-0.475)(110-102.02) = 4.1271, \\ IE_u &= 0, \\ V_u^A &= 4.1271. \end{aligned}$$

the down node:

$$\begin{aligned} CV_d &= e^{-0.06/4}\left[0.475 * (110-102.02) + (1-0.475)(110-83.53)\right] 17.4239, \\ IE_d &= 110 - 91.39 = 18.61. \end{aligned}$$

We conclude that early exercise is optimal and that the American put value in the down node equals $V_d^A = 18.61$.

the **root** *node:*

$$\begin{aligned} CV_0 &= e^{-0.06/4}\left[0.475 * 4.1271 + (1-0.475)(18.61)\right] = 11.556, \\ IE_0 &= 110 - 100 = 10. \end{aligned}$$

So, the American option is worth 11.56 at time-0.

Problem 3.11. A non-dividend-paying market index currently sells for 1,000. An investor wants to lock in the ability to buy this index in one year for a price of 1,028. He can do this by buying or selling European put and call options with a strike price of 1,028. The continuously compounded risk-free interest rate is 4%.

Which of the following gives the strategy that will achieve this investor's objective and also give the cost today of establishing this position.

(a) Buy the put and sell the call, receive 12.31.
(b) Buy the put and sell the call, spend 12.31.
(c) Buy the put and sell the call, no cost.
(d) Buy the call and sell the put, receive 12.31.
(e) Buy the call and sell the put, spend 12.31.

Solution: (e)
The investor needs to long the call option and write a put option to achieve the above guaranteed price. The actual forward price for delivery in one year of the underlying asset consistent with today's market price of the index and the continuously compounded risk-free interest rate is

$$1000e^{0.04} = 1040.81.$$

Since the synthetic forward ensures that the investor will be able to purchase the index at a smaller price of \$1028, he needs to spend the present value of the difference today, i.e.,

$$12.81e^{-0.04} = 12.3077.$$

Problem 3.12. The current stock price is 40 per share. The price at the end of a three-month period is modeled with a one-period binomial tree so that the stock price can either increase by \$10, or decrease by \$4. The stock pays dividends continuously with the dividend yield 0.04.

The continuously compounded risk-free interest rate is 0.05.

What is the stock investment in a replicating portfolio for three-month, \$40-strike European **straddle** on the above stock?

(a) Long 0.42 shares
(b) Long 0.71 shares
(c) Short 0.71 shares
(d) Short 0.42 shares
(e) None of the above.

Solution: (a)
In our usual notation,

$$\Delta = e^{-\delta h} \frac{V_u - V_d}{S_u - S_d} = e^{-0.04/4} \left(\frac{10 - 4}{14} \right) \approx 0.4243$$