

M378K Introduction to Mathematical Statistics

Problem Set #8

The Normal Distribution.

Definition 8.1. The moment-generating function (mgf) m_Y for a random variable Y is defined as

$$m_Y(t) = \mathbb{E}[e^{tY}]$$

for all t for which the above expectation exists. In fact, we say that the moment-generating function **exists** there exists a positive number b such that $m_Y(t)$ is finite for all t such that $|t| \leq b$.

Proposition 8.2. 1. If m_Y exists for a certain probability distribution, then it is unique.

2. If m_{Y_1} and m_{Y_2} are equal on an interval, then $Y_1 \stackrel{(d)}{=} Y_2$.

Corollary 8.3. Let Y_1 and Y_2 be independent and normally distributed. Define $Y = Y_1 + Y_2$. Then, the distribution of Y is ...

Proof. Note that $Y_i \sim N(\mu = \mu_i, \sigma_i)$ for $i = 1, 2$. Now, let's look at the mgf of Y . Then, since Y_1 and Y_2 are independent, we have

$$m_Y(t) = m_{Y_1}(t)m_{Y_2}(t).$$

We can now use the fact that for any $X \sim N(\mu, \sigma)$,

$$m_X(t) = e^{\mu t} m_Z(\sigma t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

Hence,

$$m_Y(t) = e^{\mu_1 t + \frac{\sigma_1^2 t^2}{2}} \times e^{\mu_2 t + \frac{\sigma_2^2 t^2}{2}} = e^{(\mu_1 + \mu_2)t + \frac{(\sigma_1^2 + \sigma_2^2)t^2}{2}}$$

We can conclude that $Y \sim N(\mu_1 + \mu_2, \sqrt{\sigma_1^2 + \sigma_2^2})$. □

Problem 8.1. Two scales are used to measure the mass m of a precious stone. The first scale makes an error in measurement which we model by a normally distributed random variable X_1 with mean $\mu_1 = 0$ and standard deviation $\sigma_1 = 0.04m$. The second scale is more accurate. We model its error by a normal random variable X_2 with mean $\mu_2 = 0$ and standard deviation $\sigma_2 = 0.03m$.

We assume that the measurements made using the two different scales are independent, i.e., that the random variables X_1 and X_2 are independent.

To get our final estimate of the mass of the stone, we take the average of the two results from the two different scales, i.e., we define $Y = \frac{X_1 + X_2}{2}$.

- (i) What is the distribution of the random variable Y ? State the **name** of its distribution and the **values** of the parameters.
- (ii) What is the probability that the error Y we get is within $0.005m$ of the actual mass of the stone? Namely, calculate

$$\mathbb{P}[|Y| < 0.005m].$$

Solution: Let us denote the random variable modeling the error from the first scale by $X_1 \sim N(0, \sigma_1^2)$ and the random variable modeling the error from the second scale by $X_2 \sim N(0, \sigma_2^2)$.

Then, if Y denotes the average of the two measurements, we have that

$$Y = \frac{1}{2}(X_1 + X_2) \sim N(0, \frac{1}{4}(\sigma_1^2 + \sigma_2^2)),$$

i.e.,

$$Y \sim N(0, \sigma^2)$$

with

$$\sigma^2 = \frac{1}{4}(\sigma_1^2 + \sigma_2^2) = \frac{1}{4}(0.04^2 m^2 + 0.03^2 m^2) = \frac{1}{4} \cdot 0.01^2 m^2 (4^2 + 3^2) = \frac{1}{4} 0.05^2 m^2 = \left(\frac{0.05m}{2}\right)^2.$$

The probability we are looking for can be expressed as

$$\begin{aligned} \mathbb{P}[Y \in (-0.005m, 0.005m)] &= \mathbb{P}[-0.005m < Y < 0.005m] \\ &= \mathbb{P}\left[-\frac{2 \cdot 0.005m}{0.05m} < \frac{Y}{\sigma} < \frac{2 \cdot 0.005m}{0.05m}\right] \\ &= \mathbb{P}\left[-0.2 < \frac{Y}{\sigma} < 0.2\right]. \end{aligned}$$

Since $\frac{Y}{\sigma} \sim N(0, 1)$, the above probability equals

$$2\Phi(0.2) - 1 \approx 2 \cdot 0.57926 - 1 \approx 0.16.$$

Corollary 8.4. Let Y_1, \dots, Y_n be independent and identically distributed. Assume that $Y_1 \sim N(\mu, \sigma)$. Define

$$S = Y_1 + Y_2 + \dots + Y_n$$

Then, the distribution of S is ...

Proof. **Solution:** Using the same reasoning as above, we have that

$$m_S(t) = \prod_{i=1}^n m_{Y_i}(t) = \prod_{i=1}^n e^{\mu t + \frac{\sigma^2 t^2}{2}} = e^{n\mu t + \frac{n\sigma^2 t^2}{2}}$$

We can conclude that $S \sim N(n\mu, \sqrt{n}\sigma)$. □