UNIVERSITY OF TEXAS AT AUSTIN

Problem set 2

Problem 2.1. *Source: Sample P exam, Problem #192.*
Losses covered by a flood insurance policy are uniformly distributed on the interval $[0, 2]$. The insurer pays the amount of the loss in excess of a deductible d. The probability that the insurer pays at least 1.20 on a random loss is 0.30. Calculate the probability that the insurer pays at least 1.44 on a random loss.

Solution: Let X denote the loss amount. We are modelling it so that $X \sim U(0, 2)$. We are given the following probability:

$$\mathbb{P}[\text{the insurer pays at least } 1.20] = 0.3.$$

The deductible is, as usual, denoted by d. Since the insurer pays the excess above the deductible, we can rewrite the above probability as

$$\mathbb{P}[X - d > 1.20] = 0.30 \quad \Rightarrow \quad \mathbb{P}[X > 1.20 + d] = 0.30.$$

Recall that $X \sim U(0, 2)$. We have that

$$\frac{2 - (1.2 + d)}{2} = 0.3 \quad \Rightarrow \quad 0.8 - d = 0.6 \quad \Rightarrow \quad d = 0.2.$$

Calculating the probability that the insurer pays at least 1.44 is similar to the above work. We have

$$\mathbb{P}[\text{the insurer pays at least } 1.44] = \mathbb{P}[X \geq d + 1.44] = \mathbb{P}[X \geq 1.64] = \frac{2 - 1.64}{2} = \frac{0.36}{2} = 0.18.$$

Problem 2.2. The lifespan of a certain machine is exponentially distributed. The probability that the lifespan exceeds 4 years is p. Find the expression for the density of the lifespan in terms of p.

Solution: Let T be the random variable which stands for the lifespan in the problem. It is modelled as exponential. Let's denote its parameter, as usual, by θ. We can write

$$T \sim Exponential(mean = \theta).$$

The survival function of the random variable T is of the form, for all $t > 0$,

$$S_T(t) = \mathbb{P}[T > t] = e^{-t/\theta}.$$

In the present problem, we are given that

$$\mathbb{P}[T > 4] = p \quad \Rightarrow \quad e^{-4/\theta} = p \quad \Rightarrow \quad -4/\theta = \ln(p) \quad \Rightarrow \quad \theta = -\frac{4}{\ln(p)}.$$

The density of the exponential random variable T is of the form, for all $t > 0$,

$$f_T(t) = \frac{1}{\theta} e^{-t/\theta} = -\frac{\ln(p)}{4} e^{t \ln(p)/4} = -\frac{\ln(p)}{4} p^{t/4}.$$

Problem 2.3. *Source: Sample P exam, Problem #206.*
An insurance company issues policies covering damage to automobiles. The amount of damage is modeled by a uniform distribution on $[0, b]$. The policy payout is subject to a deductible of $b/10$. A policyholder experiences automobile damage. Calculate the ratio of the standard deviation of the policy payout to the standard deviation of the amount of the damage.

Solution: The variance of a unit uniform random variable $U \sim U(0, 1)$ is $\frac{1}{12}$. So, the variance of the uniform random variable $X \sim U(0, b)$ is $\frac{b^2}{12}$. It standard deviation is, in turn, $\frac{b}{\sqrt{12}}$. This is the standard deviation of the amount of damage. Let $Y^L = (X - \frac{b}{10})_+$ be the per-loss random variable, i.e., the policy payout. We need to calculate $Var[Y^L] = \mathbb{E}[(Y^L)^2] - (\mathbb{E}[Y^L])^2$ in order to be able to find its standard deviation. We have

$$\mathbb{E}[Y^L] = \int_{b/10}^{b} (y - \tfrac{b}{10})\frac{1}{b}\,dy = \frac{y^2}{2b} - \frac{y}{10}\Big]_{y=b/10}^{b} = \frac{b}{2} - \frac{b}{10} - \left(\frac{b}{200} - \frac{b}{100}\right) = 0.405b,$$

$$\mathbb{E}[(Y^L)^2] = \int_{b/10}^{b} (y - \tfrac{b}{10})^2\frac{1}{b}\,dy = \frac{y^3}{3b} - \frac{y^2}{10} + \frac{yb}{100}\Big]_{y=b/10}^{b} = \frac{b^2}{3} - \frac{b^2}{10} + \frac{b^2}{100} - \left(\frac{b^2}{3000} - \frac{b^2}{1000} + \frac{b^2}{1000}\right)$$
$$= 0.243b^2.$$

So, $\mathrm{Var}[Y^L] = 0.243b^2 - (0.405b)^2 = 0.078975b^2$. By taking the square root, we obtain the standard deviation of $SD[Y^L] = 0.2810249b$. The ratio we were supposed to calculate is, therefore, $0.2810249\sqrt{12} = 0.9734988$.

Problem 2.4. *Source: Sample P exam, Problem #209.*
A policy covers a gas furnace for one year. During that year, only one of three problems can occur:

(i) The igniter switch may need to be replaced at a cost of 60. There is a 0.10 probability of this.
(ii) The pilot light may need to be replaced at a cost of 200. There is a 0.05 probability of this.
(iii) The furnace may need to be replaced at a cost of 3000. There is a 0.01 probability of this.

Calculate the deductible that would produce an expected claim payment of 30.

Solution: The condition on the deductible d is that

$$(60-d)_+(0.1) + (200-d)_+(0.05) + (3000-d)_+(0.01) = 30.$$

Let's take a look at the possible cases for the values of d.

If $d < 60$, the above equation becomes

$$\begin{aligned}(60-d)(0.1) + (200-d)(0.05) + (3000-d)(0.01) = 30 \quad &\Rightarrow \quad 6 - 0.1d + 10 - 0.05d + 30 - 0.01d = 30\\ &\Rightarrow \quad 0.16d = 16 \quad \Rightarrow \quad d = 100.\end{aligned}$$

This is not an acceptable solution since we assumed that $d < 60$.

Onto the next case! If $60 \le d < 200$, the equation above reads as

$$\begin{aligned}(60-d)_+(0.1) + (200-d)_+(0.05) + (3000-d)_+(0.01) = 30 \quad &\Rightarrow \quad (200-d)(0.05) + (3000-d)(0.01) = 30\\ &\Rightarrow \quad 10 - 0.05d + 30 - 0.01d = 30\\ &\Rightarrow \quad 0.06d = 10 \quad \Rightarrow \quad d = \frac{500}{3}.\end{aligned}$$

This is an acceptable solution. Lastly, if $200 \le d < 3000$, we get

$$\begin{aligned}(60-d)_+(0.1) + (200-d)_+(0.05) + (3000-d)_+(0.01) = 30 \quad &\Rightarrow \quad (3000-d)(0.01) = 30\\ &\Rightarrow \quad 30 - 0.01d = 30\\ &\Rightarrow \quad d = 0.\end{aligned}$$

Again, this is not an acceptable solution. The case $d \ge 3000$ obviously yields no solutions.

Problem 2.5. *Source: Sample P exam, Problem #214.*
Losses due to accidents at an amusement park are exponentially distributed. An insurance company offers the park owner two different policies, with different premiums, to insure against losses due to accidents at the park. Policy A has a deductible of 1.44. For a random loss, the probability is 0.640 that under this policy, the insurer will pay some money to the park owner. Policy B has a deductible of d. For a random loss, the probability is 0.512 that under this policy, the insurer will pay some money to the park owner. Calculate d.

Solution: Let the losses be denoted by the random variable $X \sim Exponential(mean = \theta)$. Under Policy A, we are given that

$$S_X(1.44) = 0.64 \quad \Rightarrow \quad e^{-1.44/\theta} = 0.64 \quad \Rightarrow \quad e^{-1/\theta} = (0.64)^{1/1.44} \quad \Rightarrow \quad -\frac{1}{\theta} = \frac{1}{1.44}\ln(0.64).$$

Under Policy B, we have that

$$S_X(d) = 0.512 \quad \Rightarrow \quad e^{-d/\theta} = 0.512 \quad \Rightarrow \quad -\frac{d}{\theta} = \ln(0.512) \quad \Rightarrow \quad d\left(\frac{1}{1.44}\ln(0.64)\right) = \ln(0.512)$$

$$\Rightarrow \quad d = \frac{1.44\ln(0.512)}{\ln(0.64)} = 2.16.$$