M339W : December 7^{th}, 2020.

Strong Law of Large Numbers (SLLN).

A sequence of random variables

$$\{X_k, k = 1, 2, \dots\}$$ independent, identially distributed

Assume $\mu_X := \mathbb{E}[X_1] < 0$.

Then,

$$\boxed{\frac{X_1 + X_2 + \dots + X_n}{n} \xrightarrow[n \to \infty]{} \mu_X}$$

Also, if a function g is such that

$g(X_1)$ is well defined

and

$\mathbb{E}[g(X_1)] < \infty$,

then,

$$\frac{g(X_1) + g(X_2) + \dots + g(X_n)}{n} \xrightarrow[n \to \infty]{} \mathbb{E}[g(X_1)]$$

Monte Carlo

Recipe:
- Create simulated values of your random variable from a particular dist'n.
- Apply a function to the simulate value.
- Average the quantities that you get.

We get a value which is "close to" the theoretical value. To increase precision by a fator of η, we must increase the number of variates by η^2.

$$\mathrm{Var}\left[\frac{X_1+X_2+\cdots+X_n}{n}\right] \overset{\text{independent}}{=} \frac{1}{n^2}\left(\mathrm{Var}[X_1]+\cdots+\mathrm{Var}[X_n]\right)$$
$$\overset{\text{identically dist'd}}{=} \frac{1}{n^{\cancel{2}}}\left(\cancel{n}\cdot \mathrm{Var}[X_1]\right) = \frac{\mathrm{Var}[X_1]}{n}$$

The Inverse Transformation (Simulation) Method.

Proposition.

(1) Let X be a continuous random variable, i.e., let X have a density function f_X.

Assume that $\boxed{f_X(x) > 0 \text{ always (i.e., for all } x).}$

Set $\boxed{\tilde{X} := F_X(X)}$ w/ F_X the cumulative distribution function.

Then, $\boxed{\tilde{X} \sim U(0,1)}$

→: Support of $\tilde{X}$ will be contained in $[0,1]$.

Let $u \in [0,1]$.

$$F_{\tilde{X}}(u) = \mathbb{P}[\tilde{X} \le u]$$
$$= \mathbb{P}[\underline{F_X(X) \le u}]$$

$f_X(x) > 0$ always

Then, $F_X(a) = \int_{-\infty}^{a} f_X(x)\,dx$

$\Rightarrow F_X$ is a strictly increasing function

$\Rightarrow F_X$ is one-to-one

$\Rightarrow F_X^{-1}$ exists and is increasing

$$F_{\tilde{X}}(u) = \mathbb{P}\left[\cancel{F_X^{-1}}(\cancel{F_X}(X)) \leq F_X^{-1}(u)\right]$$

$$= \mathbb{P}\left[X \leq F_X^{-1}(u)\right] =$$

$$= \cancel{F_X}(\cancel{F_X^{-1}}(u)) = u \implies \tilde{X} \sim U(0,1).$$

(2) Let $U \sim U(0,1)$ and let

F be a (strictly increasing) cumulative dist'n f'tion.

Set $\boxed{Y := F^{-1}(U)}$

Then, the cdf of the r.v. Y is F.

Implementation:

1. F ... the cdf of the dist'n we want to draw from, e.g., $F = N =$ cdf of $N(0,1)$

2. Draw: $u_1, u_2, \dots, u_n \sim U(0,1)$

3. Set $x_1 = F^{-1}(u_1)$; $x_2 = F^{-1}(u_2)$; ...; $x_n = F^{-1}(u_n)$

They will be the simulated values from your desired dist'n.