

M339D: March 26th, 2021.

Finite Probability Spaces

... serve as models for the possible paths that the price of our underlying asset can take.

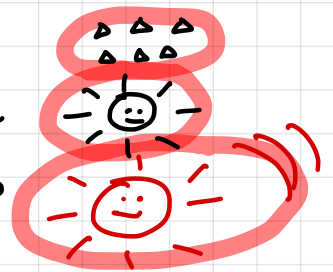
e.g.,

$$S(T) \sim \begin{cases} 100 \\ 110 \\ 115 \end{cases}$$

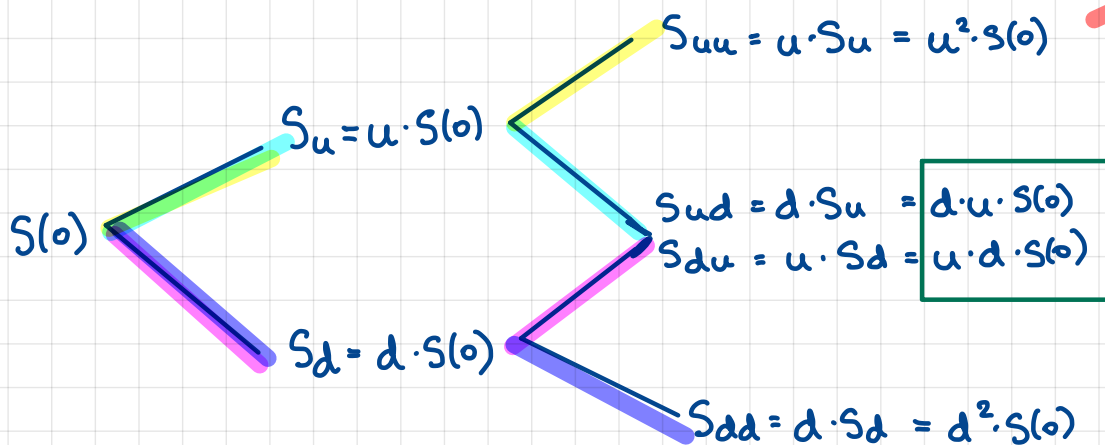
w/ probab. $1/6$

w/ probab. $1/2$

w/ probab. $1/3$



e.g.,



All of the finitely many possible scenarios are called states of the world.

We assume that:

- each one can happen, i.e., its probab > 0
- they all exhaust the possible scenarios, i.e.,
 $\sum \text{probab} = 1$

Arbitrage Portfolio.

Def'n. An **ARBITRAGE PORTFOLIO** is a portfolio whose **PROFIT** is:

- ✓ and • non-negative in all states of the world
- ✓ • strictly positive in at least one state of the world

Unless it is specified differently in a particular problem/example, we assume **NO ARBITRAGE**.

Law of the Unique Price.

For simplicity, focus on two static portfolios.

Assume that their payoffs are equal, i.e.,

$$V_A(T) = V_B(T)$$

In general, two random variables X and Y are said to be **equal** if $TP[X=Y]=1$.

On a finite probability space, they must take the same value for every **state of the world**, i.e., elementary outcome.

Our claim:

$$V_A(0) = V_B(0)$$



→: Assume, to the contrary, that

$$V_A(0) \neq V_B(0). \leftarrow$$

Without loss of generality, take

$$\boxed{\underbrace{V_A(0)}_{\text{relatively CHEAP}} < \underbrace{V_B(0)}_{\text{relatively EXPENSIVE}}}$$

Propose an arbitrage portfolio:

- LONG Portfolio A
 - SHORT Portfolio B
- } Total Portfolio

- Verify:
- Initial Cost: $V_A(0) - V_B(0) < 0$
 \Downarrow
Inflow of money at time 0
 - Payoff: $V_A(T) - V_B(T) = 0 \quad \checkmark$

$$\begin{aligned} \text{Profit} &= \text{Payoff} - FV_{0,T}(\text{Initial Cost}) \\ &= 0 - FV_{0,T}(V_A(0) - V_B(0)) > 0 \end{aligned}$$

We created
an arbitrage
portfolio!

$\Rightarrow \Leftarrow$

Def'n. Consider a European-style derivative security.
A static portfolio w/ the same payoff as that
of the derivative security is called its

REPLICATING PORTFOLIO.

Note: The initial price of the derivative security is
equal to the initial price of the replicating portfolio.

Ways to buy stock.



Overnight Purchase:

- pmt: $S(0)$
- delivery

Fully Leveraged:

- delivery

- pmt: $F_{0,T}(S(0))$

Forward Contract:

- pmt: $F_{0,T}$
- delivery

Prepaid Forward Contract:

- pmt: $F_{0,T}^P$

- delivery