NAME:

M339W/389W Financial Mathematics for Actuarial Applications
University of Texas at Austin
In-Term Exam III
Instructor: Milica Čudina

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The maximum number of points on this exam is 100.

Problem 3.1. (5 points) Assume that the risk-free interest rate equals 0.04. The Sharpe ratio of asset S is given to be $1/4$ while the Sharpe ratio of asset Q equals $1/3$. You know that the volatility of S is three times the volatility of Q. If you build an equally weighted portfolio with assets S and Q as its two components, the expected return of this portfolio will be 0.10. What is the expected return of Q?

Solution: From the condition on the Sharpe ratio of S, we get

$$\frac{\mathbb{E}[R_S] - r_f}{\sigma_S} = \frac{1}{4} \quad \Rightarrow \quad \sigma_S = 4(\mathbb{E}[R_S] - 0.04).$$

From the condition on the Sharpe ratio of Q, we obtain

$$\frac{\mathbb{E}[R_Q] - r_f}{\sigma_Q} = \frac{1}{3} \quad \Rightarrow \quad \sigma_Q = 3(\mathbb{E}[R_Q] - 0.04).$$

Since $\sigma_S = 3\sigma_Q$, we have

$$\begin{aligned} 4(\mathbb{E}[R_S] - 0.04) = 3(3)(\mathbb{E}[R_Q] - 0.04) \quad &\Rightarrow \quad \mathbb{E}[R_S] - 0.04 = 2.25(\mathbb{E}[R_Q] - 0.04) \\ &\Rightarrow \quad \mathbb{E}[R_S] - 2.25\mathbb{E}[R_Q] = 0.04 - 0.09 = -0.05. \end{aligned}$$

On the other hand, from the given expected return on the equally-weighted portfolio, we know that

$$\frac{1}{2}(\mathbb{E}[R_S] + \mathbb{E}[R_Q]) = 0.10 \quad \Rightarrow \quad \mathbb{E}[R_S] + \mathbb{E}[R_Q] = 0.20.$$

Solving the above system of two equations with two unknowns, we get

$$\mathbb{E}[R_S] = 0.1231 \quad \text{and} \quad \mathbb{E}[R_Q] = 0.0769.$$

Problem 3.2. (5 points) Assume the **Capital Asset Pricing Model** holds.

You are given the following information about stock X, stock Y, and the market:

- The expected return and volatility for the market portfolio are 0.12 and 0.2, respectively.
- The required return and volatility for the stock X are 0.0404 and 0.4, respectively.
- The correlation between the returns of stock X and the market is -0.25.
- The volatility of stock Y is 0.3.
- The correlation between the returns of stock Y and the market is 0.1.

Calculate the required return for stock Y.

Solution: The βs of stocks X and Y are

$$\beta_X = \frac{0.4(-0.25)}{0.2} = -0.5,$$
$$\beta_Y = \frac{0.3(0.1)}{0.2} = 0.15.$$

So, the required return of stock X must satisfy

$$\begin{aligned} 0.0404 = r_X = r_f + (-0.5)(0.12 - r_f) \quad &\Rightarrow \quad 0.0404 = r_f - 0.06 + 0.5r_f \\ &\Rightarrow \quad 1.5r_f = 0.1004 \quad \Rightarrow \quad r_f = 0.0669. \end{aligned}$$

Finally, the required return of stock Y equals

$$r_Y = 0.0669 + 0.15(0.12 - 0.0669) = 0.0749.$$

Problem 3.3. (5 points) Assume the **CAPM** holds.

Let the risk-free interest rate be 0.025 and let the expected return of the market portfolio be equal to 0.08.

Suppose that stock X has $\beta_X = 1.4$ and that stock Y has $\beta_Y = 0.8$. Using the risk-free asset, stock X, and stock Y, you create a portfolio such that you invest twice as much in asset X as in asset Y while the weight of the risk-free asset is 0.4. What is the expected return of this portfolio?

Solution: The β of the risk-free asset is zero. Hence, the β of the portfolio is

$$\beta_P = 0.4\beta_X + 0.2\beta_Y = 0.4(1.4) + 0.2(0.8) = 0.72.$$

So, realizing that the expected return of the portfolio equals its required return, we get

$$\mathbb{E}[R_P] = r_f + \beta_P(r_m - r_f) = 0.025 + 0.72(0.08 - 0.025) = 0.0646.$$

Problem 3.4. (5 points) You are given the following information about stock X and a portfolio P:

- The annual effective risk-free rate is 4%.
- The portfolio's expected return is 0.10 and its volatility is 0.2.
- The expected return of stock X is 0.03 and its volatility is 0.25.
- The correlation between the returns of stock X and the portfolio P is -0.2.

Then:

(a) The required return of stock X is 0.025 and the investor holding portfolio P should invest in stock X.

(b) The required return of stock X is 0.025 and the investor holding portfolio P should not invest in stock X.

(c) The required return of stock X is 0.055 and the investor holding portfolio P should invest in stock X.

(d) The required return of stock X is 0.055 and the investor holding portfolio P should not invest in stock X.

(e) None of the above.

You have to show your work. The final answer without a justification will earn zero points (even if it's correct).

Solution: (a)
The β for the stock X equals

$$\beta_X = \frac{0.25(-0.2)}{0.2} = -0.25.$$

So, the stock X has a required return equal to

$$r_X = r_f + \beta_X(\mathbb{E}[R_m] - r_f) = 0.04 + (-0.25)(0.10 - 0.04) = 0.04 - 0.015 = 0.025.$$

Since the expected return is bigger than the required return, one should invest in stock X.

Problem 3.5. (5 points) You are a pessimist and you model the state of the economy to be twice as likely to be *bad* as it is to be *good*. There are no other states of the economy in your model.

According to your model, if the economy is *good*, the return of stock S will be 0.09 and the return of stock Q will be 0.15. Also, if the economy is *bad*, the return of stock S will be -0.03 and the return of stock Q will be -0.045.

You build a portfolio out of the stocks S and Q. The expected return of this portfolio is 0.014. What is the volatility of your portfolio?

Solution: Let R_S and R_Q denote the returns of the two stocks S and Q, respectively. According to our model, we have

$$\mathbb{E}[R_S] = 0.09\left(\frac{1}{3}\right) + (-0.03)\left(\frac{2}{3}\right) = 0.01,$$
$$\mathbb{E}[R_Q] = 0.15\left(\frac{1}{3}\right) + (-0.045)\left(\frac{2}{3}\right) = 0.02.$$

Then, the return of the entire portfolio can be expressed as

$$R_P = w_s R_S + (1 - w_S)R_Q$$

where w_S stands for the weight of asset S. It's straightforward to see that the weight w_s needs to be equal to 0.6 so that the return of the entire portfolio equals the given 0.014. The distribution of the return of the entire portfolio satisfies

$$R_P \sim \begin{cases} 0.114, & \text{with probability } 1/3, \\ -0.036, & \text{with probability } 2/3. \end{cases}$$

So,

$$\mathbb{E}[R_P^2] = (0.114)^2\left(\frac{1}{3}\right) + (-0.036)^2\left(\frac{2}{3}\right) = 0.005196.$$

Thus,

$$Var[R_P] = 0.005196 - (0.014)^2 = 0.005.$$

Our answer is $\sqrt{0.005} = 0.0707$.

Problem 3.6. (5 points) Consider two assets X and Y such that:

- their expected returns are $\mathbb{E}[R_X] = 0.10$ and $\mathbb{E}[R_Y] = 0.08$;
- their volatilities are $\sigma_X = 0.2$ and $\sigma_Y = 0.25$;
- the correlation coefficient of their returns is $\rho_{X,Y} = -1$.

You construct a portfolio consisting of shares of X and Y with a risk-free return. What is the value of the risk-free interest rate?

Solution:

$$w_Y = \frac{\sigma_X}{\sigma_X + \sigma_Y} = \frac{0.2}{0.2 + 0.25} = \frac{4}{9}\,.$$

The risk-free interest rate has the value

$$w_x \mathbb{E}[R_X] + w_Y \mathbb{E}[R_Y] = \frac{5}{9}(0.10) + \frac{4}{9}(0.08) = 0.0911.$$

Problem 3.7. (5 points) Which one of the following statements is **not** correct?

(a) Any equally weighted portfolio contains only systematic risk.

(b) The volatility of an equally weighted portfolio is at most as large as the average of the volatilities of its components.

(c) Full diversification of an investment portfolio leaves only market risk.

(d) Adding another investment into your portfolio may increase the volatility of the portfolio.

(e) The Sharpe ratio reflects the reward-to-risk ratio of an investment.

Solution: (a)

Problem 3.8. (5 points) You are given the following information about the return of a security, using a three-factor model:

Factor	Beta	Expected Return
T	0.32	12%
U	0.40	16%
V	0.50	10%

The expected return of this security using the given three-factor model is equal to 14.7%. What is the annual effective risk-free rate of return?

Solution: By our three-factor model, we have that the expected return of our security S satisfies

$$\begin{aligned} \mathbb{E}[R_S] &= r_f + \beta^T(\mathbb{E}[R_T] - r_f) + \beta^U(\mathbb{E}[R_U] - r_f) + \beta^V(\mathbb{E}[R_V] - r_f) \\ &= \beta_T \mathbb{E}[R_T] + \beta_U \mathbb{E}[R_U] + \beta_V \mathbb{E}[R_V] + r_f(1 - \beta_T - \beta_U - \beta_V). \end{aligned} \tag{3.1}$$

So,

$$\begin{aligned} r_f &= \frac{\mathbb{E}[R_S] - \beta_T \mathbb{E}[R_T] - \beta_U \mathbb{E}[R_U] - \beta_V \mathbb{E}[R_V]}{1 - \beta_T - \beta_U - \beta_V} \\ &= \frac{0.147 - 0.32(0.12) - 0.4(0.16) - 0.5(0.1)}{1 - 0.32 - 0.4 - 0.5} = 0.0245. \end{aligned}$$

Problem 3.9. (5 points) A market-maker sells option I for \$8. This option's delta is 0.6557 and its gamma is 0.01. The market maker proceeds to delta-gamma hedge this commitment by trading in the underlying and also in option II on the same stock. The latter option's price is \$2.37, its delta is 0.5794 and its gamma is 0.04.

What is the market-maker's resulting position in option II?

Solution: With n_{II} denoting the position in option II, to achieve gamma-neutrality we need

$$-0.01 + n_{II}(0.04) = 0 \quad \Rightarrow \quad n_{II} = 0.25.$$

Hence, the investor should buy 0.25 options II.

Problem 3.10. (15 points) Assume the Black-Scholes framework for the pair of stocks **S** and **Q**.

For the stock **S**, you are given that

- the current stock price is \$50 per share;
- the stock pays dividends in the amount $0.02S(t)\,dt$ during the time period $(t, t+dt)$;
- the stock's volatility is 0.20.

For the stock **Q**, you are given that

- the current stock price is \$40 per share;
- the stock pays no dividends;
- the stock's volatility is 0.30.

The correlation between the returns of **S** and **Q** is -0.25.

The continuously compounded, risk-free interest rate is 0.055.

What is the price of the exchange call option on **S** with the strike asset **Q** with exercise date in a quarter year?

Solution: In order to price the exchange call, we first need to find the "relative" volatility between **S** and **Q**. We get

$$\begin{aligned}\sigma^2 &= \sigma_S^2 + \sigma_Q^2 - 2\sigma_S\sigma_Q\rho \\ &= 0.04 + 0.09 - 2(0.2)(0.3)(-0.25) = 0.16 \quad \Rightarrow \quad \sigma = 0.40.\end{aligned}$$

Next, we calculate the terms in the Black-Scholes price of the exchange call. We obtain

$$\begin{aligned}d_1 &= \frac{1}{0.4\sqrt{1/4}}\left[\ln\left(\frac{50}{40}\right) + \left(0 - 0.02 + \frac{0.16}{2}\right)\left(\frac{1}{4}\right)\right] = 1.190718, \\ d_2 &= 1.190718 - 0.4\sqrt{\frac{1}{4}} = 0.990718\end{aligned}$$

Using **R**, we get

$$N(d_1) = 0.8831178 \quad \text{and} \quad N(d_2) = 0.8390883.$$

So,

$$V_{EC}(0, \mathbf{S}, \mathbf{Q}) = 50e^{-0.02(1/4)}(0.8831178) - 40(0.8390883) = 10.37213.$$

Problem 3.11. (5 points)Consider a two-year project. There are only three cash flows for this project:

- The first occurs at $t = 0$, and is -50.
- The second occurs at $t = 1$, and is 40.
- The third occurs at $t = 2$, and is 11.50.

Determine r, the cost of capital, that leads to the project breaking even.

Solution: The break-even value of the cost of capital must satisfy

$$-50(1+r)^2 + 40(1+r) + 11.50 = 0 \quad \Leftrightarrow \quad (1+r)^2 - 0.8(1+r) - 0.23 = 0.$$

Solving the quadratic equation, we obtain

$$(1+r)_{1,2} = \frac{0.8 \pm \sqrt{0.8^2 + 4(0.23)}}{2} = \frac{0.8 \pm \sqrt{1.56}}{2} = \frac{0.8 \pm 1.249}{2}.$$

Our acceptable solution is $1 + r = 1.0245$, i.e., $r = 0.0245$.

Problem 3.12. (10 points)Assume that the continuously-compounded, risk-free interest rate equals 0.04

Consider a fund which is modeled using the Black-Scholes framework. The fund pays no dividends and its volatility is 0.20.

A variable annuity is linked to this fund. It has a *guaranteed minimum accumulation benefit (GMAB)*. More precisely, it is guaranteed that the amount paid to the policyholder (if they are still living in five years) will be at least 5% greater than the originally invested amount.

According to your tables, the policyholder will still be alive in five years with probability 0.95.

Assume that there are no lapses.

If the initial investment the policyholder makes is equal to \$10,000, what is the value of the guarantee?

Solution: The total value of the guarantee is

$$(0.95)v_P(S(0), 0, T = 5)$$

where $v_P(S(0), 0, T = 5)$ denotes the Black-Scholes price of the European put with strike $10,500$ and with the exercise date in five years. Of course, $S(0) = 10,000$ is the amount of initial investment. In our usual notation,

$$d_1 = \frac{1}{0.2\sqrt{5}}\left[\ln\left(\frac{10000}{10500}\right) + \left(0.04 + \frac{0.04}{2}\right)(5)\right] = 0.5617223,$$
$$d_2 = d_1 - \sigma\sqrt{T} = 0.5617223 - 0.2\sqrt{5} = 0.1145087.$$

Using **R**, we obtain

$$N(-d_1) = 0.2871526 \quad \text{and} \quad N(-d_2) = 0.4544173.$$

So,

$$v_P(S(0), 0, T = 5) = 10500e^{-0.04(5)}(0.4544173) - 10000(0.2871526) = 1034.95.$$

Hence, the value of the guarantee is $0.95(1034.95) = 983.2029$.

Problem 3.13. (5 points) Which of the following exotic options would be useful for hedging an inflation-indexed pension?

(a) Standard lookback call.

(b) Extrema lookback put.

(c) Rainbow option.

(d) Shout option.

(e) Chooser option.

Solution: (a)

Problem 3.14. (5 points) Consider a company whose current stock price is \$50 per share. The company has $100,000$ shares outstanding. The company also has \$5,000,000 in debt. Let the equity beta be equal to 1 and let the debt beta be equal to 0.4.

The company sells some shares and uses the proceeds to repay some of the debt. This lowers the equity beta to 0.8.

Assuming that the debt beta remains the same, how much of the debt was repaid?

Solution: We will use the second Miller-Modigliani proposition. The unlevered beta is

$$\beta_U = \frac{1}{2}(1) + \frac{1}{2}(0.4) = 0.7.$$

The change in the capital structure does not change the unlevered beta. Let w be the weight of equity. Then,

$$\beta_u = 0.7 = w(0.8) + (1-w)(0.4) \quad \Rightarrow \quad 0.4w = 0.3 \quad \Rightarrow \quad w = \frac{3}{4}.$$

So, the new amount of equity is $0.75(10000000) = 7500000$. The company repaid 2500000.

Problem 3.15. (5 points) A company has X in revenue this year. The revenue growth is projected to be 2% annually. The company maintains a unit debt-to-equity ratio at all times. The cost of equity is 12% and the cost of debt is 4%.

The corporate tax rate is 35%.

What is the ratio of the pre-tax value of the company to the post-tax value of the company?

Solution: The unlevered cost of capital is

$$r_U = \frac{1}{2}(r_E + r_D) = \frac{1}{2}(0.12 + 0.04) = 0.08.$$

So, the total pre-tax value is

$$\frac{X}{r_U - 0.02} = \frac{X}{0.06}.$$

The (post-tax) weighted average cost of capital is

$$r_{wacc} = \frac{1}{2}(r_E + r_D(1-\tau_C)) = \frac{1}{2}(0.12 + 0.04(1-0.35)) = 0.073.$$

So, the total post-tax value is

$$\frac{X}{0.073 - 0.02} = \frac{X}{0.053}.$$

The ratio is $\frac{0.06}{0.053} = 1.132075$.

Problem 3.16. (5 points) For stock S_1, you are given that its expected return equals 0.16 and its β is 1.2. For stock S_2, you are given that its expected return equals 0.08 and its β is 0.32. Both of these stocks lie on the *Security Market Line.* For stock S_3, you are given that its expected return equals 0.12 and its β is 0.8. What is the α of stock S_3?

Solution: Since both S_1 and S_2 are on the **SML**, we know that

$$0.16 = r_f + 1.2(r_m - r_f),$$
$$0.08 = r_f + 0.32(r_m - r_f),$$

where r_f stands for the risk-free interest rate and r_m stands for the expected return of the market. Subtracting the second equation from the first one, we get

$$0.08 = 0.88(r_m - r_f) \quad \Rightarrow \quad r_m - r_f = \frac{0.08}{0.88} = 0.0909.$$

The risk-free interest rate r_f can, then, be calculated as

$$r_f = 0.16 - 1.2(0.0909) = 0.05092.$$

Hence, the α of stock S_3 is

$$0.12 - 0.05092 - 0.8(0.0909) = -0.00364.$$

Problem 3.17. (5 points) Which of the following statements is not correct?

(a) Familiarity bias generally does not result in a systematic trading bias.

(b) The disposition effect results in longing the stocks with positive returns and shorting the stocks with negative returns.

(c) According to the weak formulation of the efficient market hypothesis, one cannot consistently make gains by trading based on the information contained in past prices.

(d) In the semi-strong form of the efficient market theory, one cannot consistently make gains by trading based on public information.

(e) Herd behavior results in a systematic trading bias.

Solution: (b)