M339D: February 21st, 2025.

3. An insurance company sells single premium deferred annuity contracts with return linked to a stock index, the time-t value of one unit of which is denoted by $S(t)$. The contracts offer a minimum guarantee return rate of $g\%$. At time 0, a single premium of amount π is paid by the policyholder, and $\pi \times y\%$ is deducted by the insurance company. Thus, at the contract maturity date, T, the insurance company will pay the policyholder

$$\boxed{\pi \times (1 - y\%)} \times \text{Max}[S(T)/S(0), (1 + g\%)^T].$$

You are given the following information:

(i) The contract will mature in $\boxed{\text{one year}}$.

(ii) The minimum guarantee rate of return, $g\%$, is 3%.

(iii) Dividends are incorporated in the stock index. That is, the stock index is constructed with all stock dividends reinvested. NO DIVIDENDS!

(iv) $\boxed{S(0) = 100}$. ✓

(v) The price of a one-year European $\boxed{\text{put option}}$, with strike price of $\boxed{\$103}$, on the stock index is $\boxed{\$15.21}$.

$$V_P(T) = (K - S(T))_+$$

Determine $y\%$, so that the insurance company does not make or lose money on this contract.

(A) 12.8%.

(B) 13.0%

(C) 13.2%

(D) 13.4%

(E) 13.6%.

The Synopsis:

1. Focus on the insurance company's liability

2. Use our $\boxed{\text{data}}$

3. Algebraically simplify w/ an eye on the $\boxed{\text{data}}$

→:

The Insurance Company's Liability:

$$\boxed{\pi(1-y)} \times \text{Max}\left[\frac{S(T)}{\boxed{S(0)}}, (1+g)^T\right]$$

const

$$\frac{1}{S(0)} \boxed{\text{Max}[S(T), \underbrace{S(0)(1+g)^T}_{100\cdot(1.03)^1}]}$$

$\text{Max}[S(T), 103] = ?$

$$\boxed{V_p(T) = (103 - S(T))_+}$$

a, b

$$\max(a,b) = a + \max(0, b-a) = a + (b-a)_+$$
$$= b + \max(a-b, 0) = b + (a-b)_+$$

$$\text{Max}[S(T), 103] = \boxed{S(T)} + \boxed{(103 - S(T))_+}$$

Long stock Index

Payoff of the Put w/ strike of 103 and exercise date T=1

The insurance company can perfectly hedge by:

- Longing/Buying $\frac{\pi(1-y)}{S(0)}$ units of the stock index

and

- buying $\frac{\pi(1-y)}{S(0)}$ European puts w/ K=103 and T=1

If they receive the same amount of money @ time 0 as is the cost of this replicating portfolio, they break even.

$$\cancel{\pi} = \frac{\cancel{\pi}(1-y)}{S(0)} \left(S(0) + V_p(0) \right)$$

$$100 = (1-y)\left(100 + 15.21\right)$$

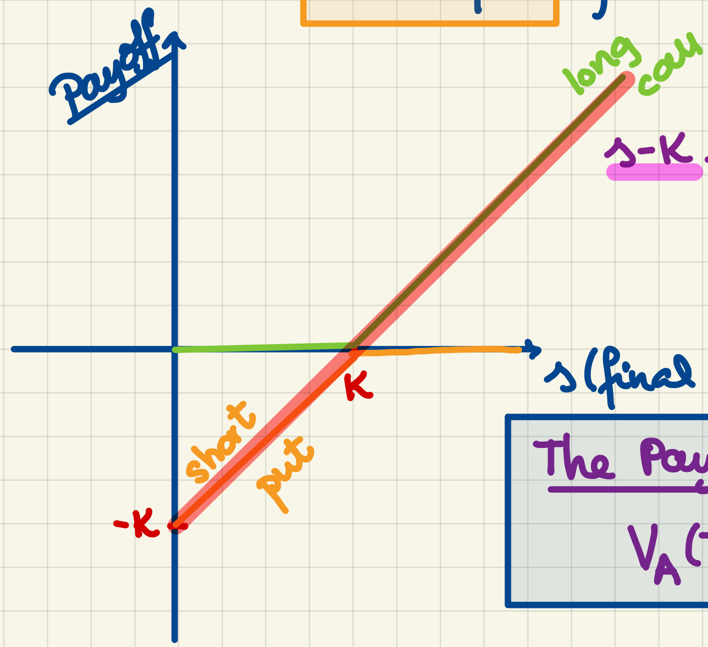
$$1-y = \frac{100}{115.21} \Rightarrow y = \frac{15.21}{115.21} = 0.132$$

$\square$

Put·Call Parity.

Portfolio A:
- long call
- written put

} both European & otherwise identical



$s-K$ ... payoff function of the portfolio

The Payoff:

$$V_A(T) = V_c(T) - V_p(T) = S(T) - K$$

Portfolio B:
- long non·dividend·paying stock
- borrow $PV_{0,T}(K)$ @ the risk·free interest rate to be repaid @ time·T

$$\Rightarrow \boxed{V_B(T) = S(T) - K}$$

$\Rightarrow$ NO ARBITRAGE!

$$V_A(0) = V_B(0)$$

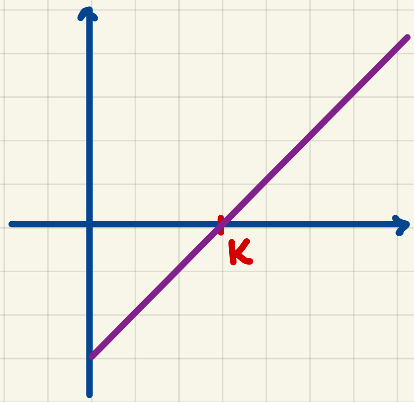
$$\boxed{V_c(0) - V_p(0) = \boxed{S(0) - PV_{0,T}(K)}}$$ Put·Call Parity.

More generally: for all $t \in [0,T]$:

$$\boxed{V_c(t) - V_p(t) = S(t) - PV_{t,T}(K)}$$

Remarks: • The no-arbitrage assumption is sufficient.

• Only works for European options.

• 

With Portfolio A, we construct a "synthetic" forward or "off-market" forward

Special Case:

strike price K = forward price of the stock F

$$\Longleftrightarrow$$

$$K = F_{0,T}(S) = S(0)e^{rT} = FV_{0,T}(S(0))$$

$$\Longleftrightarrow$$

$$PV_{0,T}(K) = S(0)$$

$$\Longleftrightarrow$$

$$V_c(0) - V_p(0) = 0 = S(0) - PV_{0,T}(K)$$

$$\Longleftrightarrow$$

$$\boxed{V_c(0) = V_p(0)}$$

By Put-Call Parity.