M362K Probability
University of Texas at Austin
Practice Problems for the Final Exam
Instructor: Milica Čudina

Notes: This is a closed book and closed notes exam. The maximal score on this exam is 100 points. **There are many ways in which any single problem can be solved. The solutions herein are just one possible way to tackle the given problems.**

All written work handed in by the student is considered to be
their own work, prepared without unauthorized assistance.

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"I agree that I have complied with the UT Honor Code during my completion of this exam."

Signature:

4.1. DEFINITIONS.

Problem 4.1. (5 points) Complete the following definition:
Let X and Y be any two random variables on the same outcome space Ω, we say that X and Y are *independent* if ...

Problem 4.2. (5 points) Complete the following definition:
Let X be a continuous random variable with the density function denoted by f_X. The *expected value* of X is defined as ...

4.2. TRUE/FALSE QUESTIONS.

Problem 4.3. (3 points) We say that a function $g : \mathbb{R} \to \mathbb{R}$ is *even* if its graph is symmetric about the vertical axis, i.e., if $g(x) = g(-x)$ for all $x \in \mathbb{R}$.

It is possible that a cumulative distribution function be even. *True or false? Why?*

Problem 4.4. (3 points) If the continuous random variable X has the distribution function F_X, then the distribution function of the random variable $Y = |X|$ equals

$$F_Y(y) = 2F_X(y).$$

True or false? Why?

Problem 4.5. (2 points) The minimum of two exponential random variables is also exponential. *True or false?*

Problem 4.6. (2 points) Assume that **only** the marginal probability density functions f_X and f_Y are given for a random pair X, Y, then we can **always** calculate the joint probability density function $f_{X,Y}$ for the pair X, Y. *True or false?*

4.3. FREE RESPONSE PROBLEMS.

Please, explain carefully all your statements and assumptions. Numerical results or single-word answers without an explanation (even if they're correct) are worth 0 points.

Problem 4.7. (5 points) Let X and Y be random variables such that the random pair (X, Y) denotes the coordinates of a point uniformly chosen in a circle of radius 1 centered at the origin.

Write the expression for the joint density function of the pair (X, Y).

Problem 4.8. (6 points) Let $Z \sim N(0,1)$. Find the following probabilities:

$$\mathbb{P}[Z \leq 1.11] =$$

$$\mathbb{P}[1 \leq Z \leq 1.11] =$$

$$\mathbb{P}[Z \leq -1.11] =$$

Problem 4.9. Let X be an exponential random variable with parameter $\lambda > 0$. Compute the probability density function f_Y of the random variable $Y = \ln(X)$.

Problem 4.10. In a certain state, tax returns are audited for everyone whose income is in the top 15% of all incomes. Assume that the income in the state is modeled by a normally distributed random variable with mean $\mu = 54,000$ and standard deviation $\sigma = 15,000$.

Find the minimum income for which the tax-payer certainly gets audited in this state.

Problem 4.11. An insurance company is handling claims from two categories of drivers: the good drivers and the bad drivers. The waiting time for the first claim from a **good** driver is modeled by an exponential random variable T_g with mean 6 (in years). The waiting time for the first claim from a **bad** driver is modeled by an exponential random variable T_b with mean 3 (in years). We assume that the random variables T_g and T_b are independent.

(i) (10 points) What is the distribution of the waiting time T until the first claim occurs (regardless of the type of driver this claim was filed by)?

(ii) (10 points) What is the probability that no claims from a good driver will be filed in the next 3 years **and** that the first claim from a bad driver will be filed within 2 years?

4.4. **MULTIPLE CHOICE QUESTIONS.**

Problem 4.12. (5 points) Let $X \sim U(0,1)$. Calculate $\mathbb{E}[X^3]$

(a) 1/6
(b) 1/4
(c) 1/3
(d) 1/2
(e) None of the above

Problem 4.13. (5 points) Let the density function of a random variable X be given as

$$f_X(x) = cxI_{[0,1]}(x),$$

for some constant c. Find $\mathbb{E}[X^3]$.

(a) 2/3
(b) 2/5
(c) 2/7
(d) 2/9
(e) None of the above

Problem 4.14. (5 points) Let X and Y be independent Poisson random variables with parameters $\lambda_1 = 1$ and $\lambda_2 = 3$, respectively. Define $Z = X + Y$. Find $\mathbb{E}[Z^2]$.

(a) 10
(b) 20
(c) 25
(d) 30
(e) None of the above

Problem 4.15. (5 points) Let X_1 and X_2 be independent normal random variables with a common mean μ and a common standard deviation σ. Then, the random variable $X = X_1 + X_2$ has the following distribution:

(a) $Normal(mean = \mu, sd = \sigma)$
(b) $Normal(mean = 2\mu, sd = 2\sigma)$
(c) $Normal(mean = 2\mu, sd = \sigma\sqrt{2})$
(d) $Normal(mean = \mu, sd = \sigma\sqrt{2})$
(e) None of the above.

Note: In addition to these problems, you should also work on the following: suggested problems from the textbook, past in-term exams, practice problem sets for the past in-term exams, homework problems, problems done in class, any other textbook problems.