M339D: October 11th, 2023.

More on Forward Trees.

$Var[R(0,1)] = ?$

$$\underline{\sigma}^2 = Var[R(0,1)] = Var\left[R(0,\tfrac{1}{m}) + R(\tfrac{1}{m},\tfrac{2}{m}) + \dots + R(\tfrac{m-1}{m},1)\right]$$

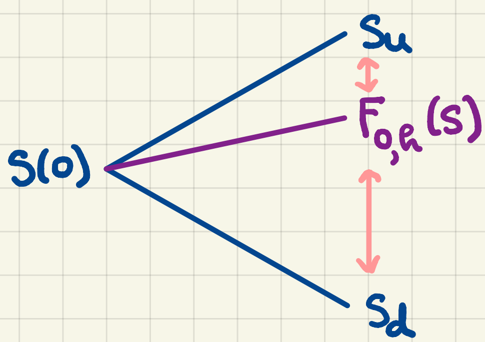
$$\overset{\text{independence}}{=} Var[R(0,\tfrac{1}{m})] + Var[R(\tfrac{1}{m},\tfrac{2}{m})] + \dots + Var[R(\tfrac{m-1}{m},1)]$$

$$\overset{\text{identically distributed}}{=} m \cdot Var[R(0,\tfrac{1}{m})] = \underline{m \cdot \sigma_{h_m}^2}$$

$$\Rightarrow \sigma^2 = m \cdot \sigma_{h_m}^2$$

$$\Rightarrow \boxed{\sigma_{h_m} = \sigma\sqrt{\tfrac{1}{m}} = \sigma\sqrt{h_m}}$$

<u>We generalize this identity to arbitrary lengths h</u>: $\boxed{\sigma_h = \sigma\sqrt{h}}$



<u>Recall:</u> $\boxed{F_{0,h}(S) = S(0)e^{rh}}$

$$S_u := F_{0,h}(S)\cdot e^{\sigma\sqrt{h}} = S(0)\underline{e^{rh}\cdot e^{\sigma\sqrt{h}}} = S(0)\boxed{e^{rh+\sigma\sqrt{h}}} = u$$

$$S_d := F_{0,h}(S)\cdot e^{-\sigma\sqrt{h}} = S(0)\underline{e^{rh}\cdot e^{-\sigma\sqrt{h}}} = S(0)\boxed{e^{rh-\sigma\sqrt{h}}} = d$$

Q: Do u and d satisfy the no-arbitrage condition?

$\rightarrow$: Yes. Always.

Q: What is S_u/S_d ?

$$\rightarrow: \boxed{\frac{S_u}{S_d}} = \frac{u\cdot \cancel{S(0)}}{d\cdot \cancel{S(0)}} = \frac{\cancel{e^{rh}}\cdot e^{\sigma\sqrt{h}}}{\cancel{e^{rh}}\cdot e^{-\sigma\sqrt{h}}} = \boxed{e^{2\sigma\sqrt{h}}}$$

Example. Consider this one-period tree w/ the time-horizon of one quarter.

$$\begin{tikzcd}[row sep=small] & S_u = 105 \\ 100 = S(0) \arrow[ur] \arrow[dr] & \\ & S_d = 98 \end{tikzcd}$$

Q: If this is a forward tree, what is the volatility?

→:

$$\frac{S_u}{S_d} = e^{2\sigma\sqrt{h}}$$

$$\frac{105}{98} = e^{2\sigma\sqrt{1/4}} = e^{\sigma}$$

$$\sigma = \ln\left(\frac{105}{98}\right) = 0.0689 \quad \square$$

Binomial Option Pricing.

Stock-Price Tree.

$$\begin{tikzcd}[row sep=small] & S_u = u \cdot S(0) \\ S(0) \arrow[ur, "h"] \arrow[dr] & \\ & S_d = d \cdot S(0) \end{tikzcd}$$

populating the tree

We want to price a European-style derivative security w/ exercise date @ the end of the tree, i.e., $T = h$

It is completely determined by its payoff f'tion: $v(\cdot)$

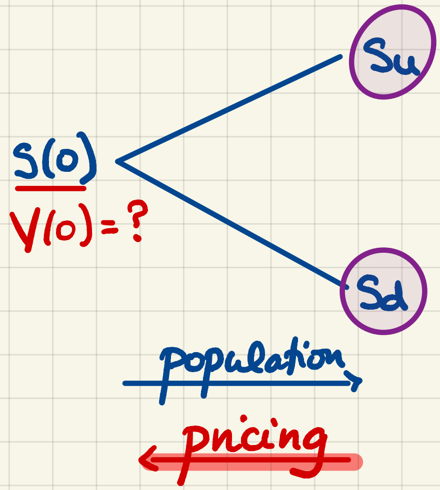
e.g., for a call: $v_c(s) = (s-K)_+$,

or for a put: $v_p(s) = (K-s)_+$,

or something "quite different": $\boxed{v(s) = (s^2-K)_+}$

The payoff of the derivative security is a random variable:

$$\boxed{V(T) = v(S(T))}$$



PAYOFF | REPLICATING PORTFOLIO

$$V_u := v(S_u) = \Delta\cdot S_u + Be^{rh}$$

$$V_d := v(S_d) = \Delta\cdot S_d + Be^{rh}$$

In the binomial model, any derivative security can be **REPLICATED** w/ a portfolio of this form:

- Δ shares of stock
  - $\Delta > 0$ buying
  - $\Delta = 0$ "nothing"
  - $\Delta < 0$ short-selling

and

- B @ the ccrfir r
  - $B > 0$ lending (buying a bond)
  - $B = 0$ "nothing"
  - $B < 0$ borrowing (issuing a bond)

time-0 holdings

If we can find Δ and B, then $\boxed{V(0) = \Delta\cdot S(0) + B}$

we get a system of two equations w/ two unknowns:

$$\begin{cases} \Delta\cdot S_u + \cancel{Be^{rh}} = V_u \\ -\Delta\cdot S_d - \cancel{Be^{rh}} = -V_d \end{cases} -$$

$$\Delta(S_u - S_d) = V_u - V_d$$

$$\boxed{\Delta = \frac{V_u - V_d}{S_u - S_d}}$$

unitless

$$\frac{V_u - V_d}{S_u - S_d} \cdot S_u + Be^{rh} = V_u$$

$$Be^{rh} = V_u - \frac{V_u - V_d}{(u-d)\cdot \cancel{S(0)}} \cdot u \cdot \cancel{S(0)} = \frac{\cancel{u \cdot V_u} - d \cdot V_u - \cancel{u \cdot V_u} + u \cdot V_d}{u-d}$$

$$\boxed{B = e^{-rh} \cdot \frac{u \cdot V_d - d \cdot V_u}{u-d}}$$

cash