

## M378K Introduction to Mathematical Statistics

### Homework assignment #8

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Please, provide your **final answer only** to the following problems.

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**Problem 8.1.** (5 points) Let  $Y_1, \dots, Y_n$  be a random sample from  $U(0, \theta)$ , with an unknown  $\theta > 0$ . For what value of the constant  $c$  is the estimator  $\hat{\theta} = c \sum_{i=1}^n Y_i$  unbiased for  $\theta$ ?

- (a) 1
- (b)  $1/n$
- (c)  $2/n$
- (d)  $n$
- (e) **None of the above.**

**Solution:** The correct answer is (c).

We have

$$\mathbb{E}[c \sum_{i=1}^n Y_i] = c \sum_{i=1}^n \frac{\theta}{2} = \frac{nc}{2} \theta,$$

so  $c$  must be equal to  $2/n$ .

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Please, provide your **complete solutions** to the following problems. Final answers only, even if correct will earn zero points for those problems.

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**Problem 8.2.** (20 points) Source: "Probability" by Pitman

Four people agree to meet at a cafe at noon. Suppose that each person arrives at a time normally distributed with mean 12noon and standard deviation of 5 minutes, independently of all the others.

1. (5 points) What is the chance that the first person to get to the cafe arrives before 11:50am?
2. (5 points) What is the chance that some of the four have still not arrived at 12:15pm?
3. (10 points) Approximately, what is the chance that the second person to arrive gets there within 10 seconds on 12noon?

**Solution:** Let the random variables  $X_1, X_2, X_3$  and  $X_4$  denote the arrival times of the four people centered around 12noon and measured in minutes. Then,  $X_i, i = 1, \dots, 4$  are independent and all have distribution  $N(0, \sigma = 5)$ . When we create the order statistics of this random sample, we will have ordered the people in order of arrival from the earliest to the latest.

1.

$$\mathbb{P}[X_{(1)} < -10] = 1 - (\mathbb{P}[X_1 \geq -10])^4 = 1 - (\mathbb{P}[X_1/5 \geq -2])^4 = 1 - (\Phi(2))^4 = 1 - (0.9772)^4 = 0.0881.$$

2.

$$\mathbb{P}[X_{(4)} > 15] = 1 - \mathbb{P}[X_{(4)} \leq 15] = 1 - (\mathbb{P}[X_1 \leq 15])^4 = 1 - (\mathbb{P}[X_1/5 \leq 3])^4 = 1 - (0.9986)^4 = 0.0056.$$

3.

$$\mathbb{P}[-1/6 < X_{(2)} < 1/6] = \int_{-1/6}^{1/6} g_{(2)}(x) dx$$

with  $g_{(2)}$  denoting the density function of the second order statistic, i.e.,

$$g_{(2)}(x) = 4 \binom{4-1}{2-1} f_X(x) F_X(x) (1 - F_X(x))^{4-2}.$$

We can approximate the required probability as:

$$\begin{aligned} \mathbb{P}[-1/6 < X_{(2)} < 1/6] &\approx \frac{1}{3} g_{(2)}(0) = \frac{1}{3} \times 12 f_X(0) F_X(0) (1 - F_X(0))^2 \\ &= 4 \times \frac{1}{5\sqrt{2\pi}} (1/2) (1/2)^2 \\ &= \frac{1}{10\sqrt{2\pi}} \approx 0.0399. \end{aligned}$$

**Problem 8.3.** (5 points) Consider an estimator  $\hat{\theta}$  for a parameter  $\theta$ . Let's say that

$$\mathbb{E}[\hat{\theta}] = \kappa_1 \theta + \kappa_2$$

for some constants  $\kappa_i \neq 0, i = 1, 2$ . Is the estimator  $\hat{\theta}$  unbiased? If so, justify your answer; if not, how would you transform the estimator  $\hat{\theta}$  to obtain an unbiased estimator?

**Solution:** The condition for unbiasedness is

$$\mathbb{E}[\hat{\theta}] = \theta.$$

So, the only scenario in which  $\hat{\theta}$  is unbiased is if  $\kappa_2 = 0$  and  $\kappa_1 = 1$ . In general, we could define a new estimator  $\hat{\theta}'$  as follows:

$$\hat{\theta}' = \frac{\hat{\theta} - \kappa_2}{\kappa_1}.$$

Indeed, we would have

$$\mathbb{E}[\hat{\theta}'] = \mathbb{E}\left[\frac{\hat{\theta} - \kappa_2}{\kappa_1}\right] = \frac{1}{\kappa_1} (\mathbb{E}[\hat{\theta}] - \kappa_2) = \frac{1}{\kappa_1} (\kappa_1 \theta + \kappa_2 - \kappa_2) = \theta.$$

**Problem 8.4.** (20 points) Let  $Y_1, \dots, Y_n$  be a random sample from the uniform distribution on  $[0, \theta]$ , where  $\theta > 0$  is an unknown parameter. We consider the estimator

$$\hat{\theta} = c \frac{1}{n} \sum_{i=1}^n Y_i^2.$$

where  $c$  is a constant (not dependent on  $\theta$  or on  $Y_1, \dots, Y_n$ ).

1. (10 points) For what value of the constant  $c$  will  $\hat{\theta}$  be an unbiased estimator for  $\theta^2$ ? Is there such a value if  $\hat{\theta}$  is used as an estimator for  $\theta$  instead of  $\theta^2$ ?
2. (10 points) Using the value of  $c$  obtained above, compute the mean squared error of  $\hat{\theta}$  (when interpreted as an estimator of  $\theta^2$ ).

**Solution:**

1. For any  $i$ , we have  $\mathbb{E}[Y_i^2] = \frac{1}{\theta} \int_0^\theta y^2 dy = \frac{1}{3} \theta^2$ . Therefore

$$\mathbb{E}[\hat{\theta}] = \frac{c}{n} \mathbb{E}[\sum_{i=1}^n Y_i^2] = \frac{c}{n} \sum_{i=1}^n \mathbb{E}[Y_i^2] = \frac{c}{3} \theta^2.$$

It follows that  $\hat{\theta}$  is unbiased for  $\theta^2$  when  $c = 3$ , but that there is no  $c$  that will make it unbiased for  $\theta$ .

2. We take  $c = 3$  as above. Since  $\hat{\theta}$  is unbiased, its mean-squared error is the same as its variance, so that

$$MSE(\hat{\theta}) = \text{Var}[\hat{\theta}] = c^2 \frac{1}{n^2} \sum_{i=1}^n \text{Var}[Y_i^2].$$

Since  $\mathbb{E}[Y_i^2] = \frac{1}{3} \theta^2$ , we have  $\text{Var}[Y_i^2] = \mathbb{E}[Y_i^4] - \mathbb{E}[Y_i^2]^2 = \frac{1}{\theta} \int_0^\theta y^4 dy - \frac{1}{9} \theta^4 = \frac{4}{45} \theta^4$ . Hence

$$MSE(\hat{\theta}) = c^2 \frac{1}{n^2} \times n \frac{4}{45} \theta^4 = \frac{4}{5n} \theta^4.$$