UNIVERSITY OF TEXAS AT AUSTIN, DEPARTMENT OF MATHEMATICS
M358K - Applied Statistics

THE MOCK IN-TERM THREE

Problem 1.1. (5 points) Alice performs a $z-$test. The $z-$score she obtains is equal to -1.76. Which decision does she make?

(a) Reject the null hypothesis.

(b) Fail to reject the null hypothesis.

(c) Reject the alternative hypothesis.

(d) Not enough information is given to answer this question.

(e) None of the above.

Solution: (d)
The significance level is not given.

Problem 1.2. (5 points) A manufacturer of scented candles claims that their luxury candles last at least 12 hours. You suspect that this might not be entirely true and you decide to test their claim. You model the candle burn times as normal with a known standard deviation of 2 hours (based on the last holiday season's study). You purchase and burn 16 candles recording the sample average of 11 hours and 45 minutes. What is your decision?

(a) Reject at the 1% significance level.

(b) Fail to reject at the 1% significance level; reject at the 5% significance level.

(c) Fail to reject at the 5% significance level; reject at the 10% significance level.

(d) Fail to reject at the 10% significance level.

(e) None of the above.

Solution: (d)
Let μ denote the unknown population mean. According to our model, the distribution of the sample mean is

$$\bar{X} \sim Normal\left(mean = \mu, sd = \frac{2}{\sqrt{16}} = \frac{1}{2}\right).$$

We are testing

$$H_0 : \mu = 12 \quad vs. \quad H_a : \mu < 12.$$

Under the null, the $z-$score corresponding to the observed sample average is

$$z = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{11.75 - 12}{\frac{1}{2}} = -2(0.25) = -0.5.$$

With $Z \sim N(0,1)$, the $p-$value is

$$\mathbb{P}[Z < -0.5] = 0.3085.$$

We fail to reject at the 10% significance level.

Problem 1.3. (5 points) *Organically Produced* claims that their supplements contain 65 mg of iron per capsule. To be able to continue to maintain their claim, they periodically test the contents of a batch of 100 randomly chosen capsules from their production line. They model the iron content as normally distributed with a known standard deviation of 5 mg. In the last test, the sample average was 64 mg. What is the $p-$value?

(a) 0.0228

(b) 0.0384

(c) 0.0418

(d) 0.0456

(e) None of the above.

Solution: (d)
Let μ denote the mean iron content per capsule. They are testing

$$H_0 : \mu = 65 \quad vs. \quad H_a : \mu \neq 65.$$

Under the null hypothesis, the $z-$score corresponding to the observed sample average of 64 mg is, in our usual notation,

$$z = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{64 - 65}{\frac{5}{\sqrt{100}}} = -\frac{1}{\frac{1}{2}} = -2.$$

With $Z \sim N(0,1)$, the corresponding $z-$score is

$$\mathbb{P}[Z < -2] + \mathbb{P}[Z > 2] = 2\mathbb{P}[Z < -2] = 0.0456.$$

Problem 1.4. (5 points) Suppose that we have a random sample of size 25 from a normal population with an unknown mean μ and a standard deviation of 4. Bertram Wooster was in charge of testing the hypotheses

$$H_0 : \mu = 10 \quad vs. \quad H_a : \mu > 10.$$

Before he went on vacation, he obtained the rejection region $[11.2, \infty)$. However, he forgot to tell anyone which significance level α he used. Calculate α.

(a) 0.0401

(b) 0.0495

(c) 0.05

(d) 0.0668

(e) None of the above.

Solution: (d)
In our usual notation, the rejection region of this righ-sided hypothesis test is of the form

$$\left[\mu_0 + z_{1-\alpha}\left(\frac{\sigma}{\sqrt{n}}\right), \infty\right)$$

with $z_{1-\alpha} = \Phi^{-1}(1-\alpha)$ unknown in this problem. So, we have

$$10 + z_{1-\alpha}\left(\frac{4}{\sqrt{25}}\right) = 11.2 \quad \Leftrightarrow \quad z_{1-\alpha}\left(\frac{4}{5}\right) = 1.2 \quad \Leftrightarrow \quad z_{1-\alpha} = 1.2\left(\frac{5}{4}\right) = 1.5.$$

Therefore, $1 - \alpha = \Phi(1.5) = 0.9332$. Finally, $\alpha = 0.0668$.

Problem 1.5. (5 points) *Source: "Mathematical Statistics with Applications in R" by Ramachandran and Tsokos.* A cross is hypothesized to result in a $3:1$ phenotypic ratio of red-flowered to white-flowered plants. You set up a hypothesis test to test this claim. Suppose your cross actually produces 170 red- and 30 white-flowered plants. What is the $p-$value you obtain?

a. Less than 0.005.
b. Between 0.005 and 0.01.
c. Between 0.01 and 0.025.
d. Between 0.025 and 0.05.
e. None of the above.

Solution: a.
Let p denote the population proportion of red-flowered plants. We are testing

$$H_0 : p = 3/4 \quad \text{vs.} \quad H_a : p \neq 3/4.$$

Under the null, the observed value of the $z-$statistic is

$$z = \frac{\frac{170}{200} - \frac{3}{4}}{\sqrt{\frac{(3/4)(1/4)}{200}}} = 3.266$$

The $p-$value is approximately than $2(1 - \Phi(3.27)) = 2\Phi(-3.27) = 2(0.0005) = 0.001$.

Problem 1.6. (5 points) A die is rolled 60 times and the face values are recorded. The results are as follows:

Up face	1	2	3	4	5	6
Number of occurrences	8	11	5	12	15	9

You test the hypothesis that the die is fair. What can you say about the $p-$value?

(a) Less that 0.05.

(b) Between 0.05 and 0.10.

(c) Between 0.10 and 0.20.

(d) Between 0.20 and 0.30.

(e) It's greater than 0.30.

Solution: (e)
Let $p_i, i = 1, \ldots, 6$ be the probability that the die falls on i. We are testing

$$\begin{aligned} H_0: &\quad p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = \frac{1}{6} \\ &\quad vs. \\ H_a: &\quad \text{At least one of the probabilities } p_i \text{ is different from } \frac{1}{6}. \end{aligned}$$

The observed value of the χ^2-statistic is

$$q^2 = \frac{1}{10}((8-10)^2 + (11-10)^2 + (5-10)^2 + (12-10)^2 + (15-10)^2 + (9-10)^2) = 6.$$

With $6 - 1 = 5$ degrees of freedom, using the χ^2-table, we see that the $p-$value is more than 0.30.

Problem 1.7. (5 points) An improvement is a process for manufacturing is being considered. Samples are taken both from parts manufactured using the existing method and the new method. It is found that 50 out of a sample of 1000 items produced using the existing method are defective. It is also found that 40 out of a sample of 1600 items produced using the new method are defective. The two samples are independent.

Find the 80%-confidence interval for the true difference in the proportions of defectives produced using the existing and the new method. *Note: Round your point estimate and the margin of error to four places after the decimal point.*

(a) $(0.0149, 0.0351)$

(b) $(0.0171, 0.0329)$

(c) $(0.0120, 0.0380)$

(d) $(0.0095, 0.0405)$

(e) None of the above.

Solution: (a)
Let p_1 denote the proportion of defectives resulting from the existing method and let p_2 denote the proportion of defectives resulting from the new method. We are supposed to find the 80%-confidence interval for $p_1 - p_2$.

The sample proportion of defectives for the existing method is $\hat{p}_1 = 50/1000 = 0.05$ and the sample proportion of defectives for the new method is $\hat{p}_2 = 40/1600 = 0.025$. So, the standard error equals

$$\sqrt{\frac{0.05(0.95)}{1000} + \frac{0.025(0.975)}{1600}} = 0.0079.$$

So, with the critical value corresponding to the 80%-confidence being $z^* = 1.28$, we get that the margin of error is

$$1.28(0.0079) = 0.0101.$$

Hence, the confidence interval is

$$0.025 \pm 0.0101 = (0.0149, 0.0351).$$

Problem 1.8. (5 points) Let the random sample $X_1, \dots, X_{10}$ be drawn from a normal distribution with mean 2 and variance 1. Define

$$Y = \sum_{i=1}^{10} (X_i - 2)^2.$$

Find the constant q such that

$$\mathbb{P}[Y \leq q] = 0.975.$$

(a) 18.31

(b) 19.02

(c) 20.48

(d) 21.92

(e) None of the above.

Solution: (c)
The random variable Y has the χ^2-distribution with 10 degrees of freedom. In the χ^2-tables, we find that $q = 20.48$.

Problem 1.9. (5 points) In a simple random sample of 1000 Austinites owning televisions, it is found that 480 do not have cable (but do have Netflix or some such or just game on the big screen). Find an 92% confidence interval for the true proportion of Austinites with television who do not have cable. *Note: Round the margin of error to four places after the decimal point.*

(a) 0.48 ± 0.0277
(b) 0.48 ± 0.0260
(c) 0.48 ± 0.0310
(d) 0.48 ± 0.0158
(e) None of the above.

Solution: (a)
The observed proportion is $\hat{p} = 480/1000 = 0.48$. So, the standard error equals

$$\sqrt{\frac{0.48(0.52)}{1000}} = 0.0158.$$

With the critical value associated with the 92%-confidence level equal to 1.75, we get the margin of error equal to

$$1.75(0.0158) = 0.02765 = 0.0277.$$

Hence, our confidence interval is 0.48 ± 0.0277.

Problem 1.10. Let p_m and p_f be the population proportions of male and female warblers who return to their hatching site. You want to test whether the two proportions are different. The observed number of males who returned is 135 out of 900, while the observed number of females who returned is 84 out of 700. What is your decision for this hypothesis test? *Note: Keep four significant places after the decimal point for all your point estimates.*

(a) Reject at the 0.01 significance level.

(b) Fail to reject at the 0.01 significance level; reject at the 0.025 significance level.

(c) Fail to reject at the 0.025 significance level; reject at the 0.05 significance level.

(d) Fail to reject at the 0.05 significance level; reject at the 0.10 significance level.

(e) Fail to reject at the 0.10 significance level.

Solution: (d)
We are testing

$$H_0 : p_m = p_f \quad vs. \quad H_a : p_m \neq p_f.$$

The observed proportions are

$$\hat{p}_m = \frac{135}{900} = 0.15 \quad \text{and} \quad p_f = \frac{84}{700} = 0.12.$$

The pooled proportion estimate is

$$\hat{p} = \frac{135+84}{900+700} = 0.136875 = 0.1369.$$

The observed value of the $z-$statistic is

$$z = \frac{\hat{p}_m - \hat{p}_f}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}} = \frac{0.15-0.12}{\sqrt{0.1369(1-0.1369)\left(\frac{1}{900}+\frac{1}{700}\right)}} \approx 1.7318.$$

Since this is a two-tailed test, we have that the $p-$value equals about $2\Phi(-1.7318)$. This value is between $2\Phi(-1.74)$ and $2\Phi(-1.73)$. Using the standard normal tables, we conclude that the $p-$value is between $2(0.0409)$ and $2(0.0418)$, i.e., between 0.0818 and 0.0836.

Problem 1.11. (5 points) *Source: "Probability and Statistical Inference" by Hogg, Tanis, and Zimmerman.*
A study was conducted to determine whether there is an association between the choice of the most credible media source for reporting news and the education level. The results are displayed in the following table:

	Newspaper	Television	Radio	Total
Grade School	45	22	6	73
High School	94	115	30	239
College	49	52	13	114
Total	188	189	49	426

Your goal is to test whether the choice of the most credible medium is independent from the education level. The observed value of the relevant test statistic is 11.399. What is your decision?

(a) Reject at the 0.01 significance level.

(b) Fail to reject at the 0.01 significance level; reject at the 0.025 significance level.

(c) Fail to reject at the 0.025 significance level; reject at the 0.05 significance level.

(d) Fail to reject at the 0.05 significance level; reject at the 0.10 significance level.

(e) Fail to reject at the 0.10 significance level.

Solution: (b)
The distribution of the test statistic is approximately χ^2 with $(3-1)(3-1) = 4$ degrees of freedom. Consulting the χ^2-table, we see that the given observed value of the test statistic is between the critical values $\chi^2_{0.01}(df = 4)$ and $\chi^2_{0.025}(df = 4)$. So, the $p-$value is between 0.01 and 0.025.