M339D: September 29th, 2023.

3. An insurance company sells single premium deferred annuity contracts with return linked to a stock index, the time-t value of one unit of which is denoted by $S(t)$. The contracts offer a minimum guarantee return rate of $g\%$. At time 0, a single premium of amount π is paid by the policyholder, and $\boxed{\pi \times y\%}$ is deducted by the insurance company. Thus, at the contract maturity date, T, the insurance company will pay the policyholder

$$\pi \times (1 - y\%) \times \text{Max}[\boxed{S(T)/S(0)}, \boxed{(1 + g\%)^T}].$$

You are given the following information:

(i) The contract will mature in one year. $\boxed{T=1}$

(ii) The minimum guarantee rate of return, $g\%$, is 3%. $\boxed{g=0.03}$

(iii) Dividends are incorporated in the stock index. That is, the stock index is constructed with all stock dividends reinvested. no dividends

(iv) $S(0) = 100$

(v) The price of a one-year European put option, with strike price of \$103, on the stock index is $\boxed{\$15.21}$

Put Payoff: $(K-S(T))_+ = \boxed{Max(K-S(T),0)}$

Determine $\cancel{y\%}$, so that the insurance company does not make or lose money on this contract.

$\longrightarrow$:

(A) 12.8%.

(B) 13.0%

(C) 13.2%

(D) 13.4%

(E) 13.6%.

The insurance company's liability:

$$\underbrace{\boxed{\pi(1-y)}}_{\text{constant}} \cdot Max\left(\frac{S(T)}{S(0)}, (1+g)^T\right) = \underbrace{\frac{1}{S(0)}}_{\text{Constant}} \boxed{Max[S(T), S(0)(1+g)^T]}$$

$$= Max[S(T), 100(1.03)^1]$$

$$= \boxed{Max[S(T), 103]}$$

a, b

$\max(a,b) = a + \max(0, b-a) = a + (b-a)_+$

$= b + \max(a-b, 0) = b + (a-b)_+$

$$\text{Max}(S(T), 103) = \boxed{S(T)} + \boxed{(103 - S(T))_+}$$

Long stock index

The payoff of a put w/ strike 103 and exercise date @ time T.

The insurance company can perfectly hedge by:

- longing/buying $\frac{\pi(1-y)}{S(0)}$ units of stock index

and

- buying $\frac{\pi(1-y)}{S(0)}$ European puts w/ $K=103$ and $T=1$

The condition for the insurance company to break even:

The amount they receive @ time 0 is EQUAL to the cost of the hedge.

$$\cancel{\pi} = \frac{\cancel{\pi}(1-y)}{S(0)}\left(S(0) + V_p(0)\right)$$

$$100 = (1-y)(100 + 15.21)$$

$$1 - y = \frac{100}{115.21} \Rightarrow \boxed{y = \frac{15.21}{115.21} = \underline{0.132}}$$

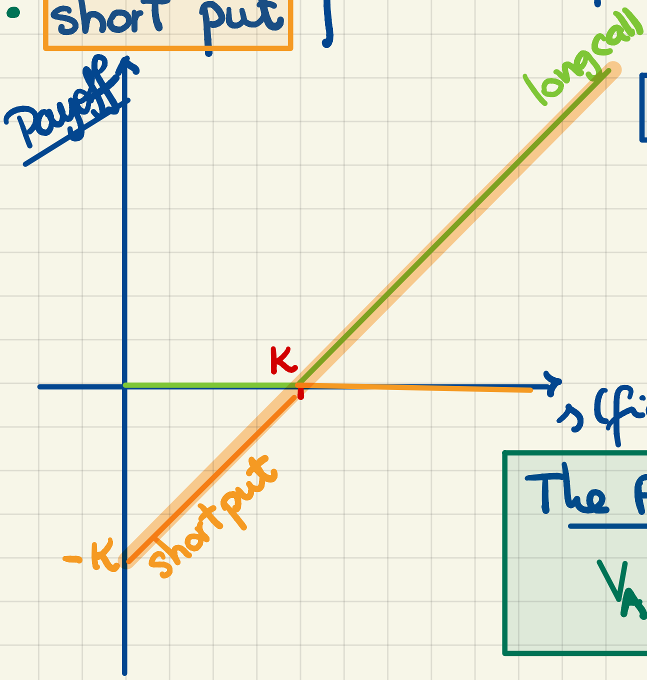
$\square$

Put·Call Parity.

Portfolio A:

- long call
- short put

} both European & otherwise identical



$\boxed{s-K}$ ... the payoff f'tion of portfolio A

The Payoff:

$$V_A(T) = V_c(T) - V_p(T) = \underline{S(T)} - \underline{K}$$

Portfolio B:

- long non·dividend·paying stock
- borrow $PV_{0,T}(K)$ @ the risk·free interest rate r to be repaid @ time T

$$\Rightarrow \boxed{V_B(T) = S(T) - K}$$

Note: $\boxed{V_A(T) = S(T) - K = V_B(T)}$

$\Rightarrow$ NO ARBITRAGE! $V_A(0) = V_B(0)$

$$\Rightarrow \boxed{V_c(0) - V_p(0) = S(0) - PV_{0,T}(K)}$$

Put·Call Parity.

More generally: for any $t \in [0,T]$:

$$\boxed{V_c(t) - V_p(t) = S(t) - PV_{t,T}(K)}$$

Remarks:
- The no-arbitrage assumption is sufficient.
- Only works for European options.
- With portfolio A, we constructed a replicating portfolio for an "off-market forward" aka a "synthetic forward".

Special case:

strike = forward price on the stock

$$\Longleftrightarrow$$

$$K = F_{0,T}(S)$$

$$\Longleftrightarrow$$

$$0 = S(0) - PV_{0,T}(K) = S(0) - PV_{0,T}(F_{0,T}(S))$$

$$\Longleftrightarrow$$

$$\boxed{V_c(0) = V_p(0)}$$

By put-call parity :)