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M339D=M389D Introduction to Actuarial Financial Mathematics
University of Texas at Austin
In-Term Exam II
Instructor: Milica Čudina

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The maximum number of points on this exam is 100.

Provide your **complete solution** to the following problems. Final answers only, without appropriate justification, will receive zero points even if correct.

2.1. **FREE-RESPONSE PROBLEMS.**

Problem 2.1. (20 points) Assume the Black-Scholes setting. Let the initial price of a non-dividend-paying stock be 60 and let its volatility be 0.32.

The continuously compounded, risk-free interest rate equals 0.04.

Consider a \$45-strike European put option which expires in four months. What is the price of the put?

Solution: In our usual notation, the price is

$$V_P(0) = Ke^{-r \cdot T} N(-d_2) - S(0)N(-d_1)$$

with

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T\right] = 1.72167, \\ d_2 &= d_1 - \sigma\sqrt{T} = 1.536918. \end{aligned}$$

So, $V_P(0) = 0.2061258$.

Problem 2.2. (10 points) The current stock price is given to be $S(0) = 30$ and its volatility is 0.3 The continuously-compounded, risk-free interest rate is 0.12.

(i) (2 points) What is the expected stock price in three months under the risk-neutral probability measure?

(ii) (3 points) What is the median stock price in three months under the risk-neutral probability measure?

(iii) (5 points) Find the risk-neutral probability that the stock price in three months is less than \$32.

Solution:

(i)

$$\mathbb{E}^*[S(1/4)] = S(0)e^{r/4} = 30.91364$$

(ii)

$$S(0)e^{(r-\frac{\sigma^2}{2})/4} = 30.56781$$

(iii) First, we calculate d_2. We get

$$d_2 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T\right] = \frac{1}{0.3\sqrt{\frac{1}{4}}}\left[\ln\left(\frac{30}{32}\right) + \left(0.12 - \frac{0.09}{2}\right) \times \frac{1}{4}\right] = -0.30525.$$

Then, we use the standard normal tables to obtain

$$\mathbb{P}[S(1/4) < 32] = N(-d_2) \approx N(0.31) = 0.6217 \tag{2.1}$$

Problem 2.3. (20 points) Consider a non-dividend-paying stock whose current price is \$120 per share. The stock has the volatility equal to 0.20.

Let the continuously-compounded, risk-free interest rate be equal to 0.05.

You model the evolution of this stock over the next quarter with a **forward** binomial tree.

What is the price of a \$122-strike, three-month put on the above stock consistent with this model?

Solution: In our usual notation, in the forward binomial tree, the risk-neutral probability is

$$p^* = \frac{1}{1+e^{\sigma\sqrt{h}}} = \frac{1}{1+e^{0.2\sqrt{1/4}}} = \frac{1}{1+e^{0.1}} = 0.4750208$$

The *up* and *down* factors are

$$u = e^{rh+\sigma\sqrt{h}} = e^{0.05(1/4)+0.1} = e^{0.1125},$$
$$d = e^{rh-\sigma\sqrt{h}} = e^{0.05(1/4)-0.1} = e^{-0.0875}.$$

Hence, the two possible stock prices at the end of the period are $S_u = 120e^{0.1125} = 134.2887$ and $S_d = 120e^{-0.0875} = 109.9463$. So, the option is in the money only in the *down* node where the payoff equals

$$V_d = (K - S_d)_+ = 12.05374.$$

By the risk neutral pricing formula, we have that

$$V_P(0) = e^{-0.05(1/4)}(1 - 0.4750208)(12.05374) = 6.249356.$$

Alternatively, the replicating portfolio has the following components

$$\Delta = \frac{V_u - V_d}{S_u - S_d} = -\frac{12.05374}{134.2887 - 109.9463} = -0.4951747,$$
$$B = e^{-rh}\frac{uV_d - dV_u}{u-d} = -e^{0.0125}\frac{e^{-0.0875}(12.05374)}{e^{0.1125} - e^{-0.0875}} = 65.6703.$$

So,

$$V_P(0) = \Delta S(0) + B = 6.249336.$$

Problem 2.4. (15 points) An archer shoots at a target repeatedly. Assume that their attempts are independent and that the probability of hitting bull's eye in any single attempt equals $1/3$. The total number of times the archer shoots at the target is 81. Using the *normal approximation to the binomial*, what is the approximate probability that the archer hits bull's eye at least 26 times?

Solution: *The following solution uses the continuity correction. The solutions without the continuity correction would also be graded as correct if otherwise accurate.*
The number of trials is 81. The probability of hitting bull's eye in a single trial is $1/3$. So, the total number of hits is, in our usual notation,

$$X \sim Binomial(n = 81, p = 1/3).$$

The probability we are looking for is $\mathbb{P}[X \geq 26]$. The mean of the random variable X is $np = 27$ and its standard deviation is $\sqrt{np(1-p)} = 4.242641$. Using the normal approximation to the binomial, we get

$$\mathbb{P}[X \geq 26] = \mathbb{P}[X > 25.5] = \mathbb{P}\left[\frac{X - 27}{4.242641} > \frac{25.5 - 27}{4.242641}\right] = 1 - \Phi\left(-0.3535534\right) \approx 0.638163.$$