UNIVERSITY OF TEXAS AT AUSTIN

Problem set 2

Loss elimination ratio. Policy modifications.

Problem 2.1. *Source: Sample STAM Exam Problem #87.*
Let X be the ground-up loss random variable whose density is given by

$$f_X(x) = \begin{cases} 0.01, & 0 \le x \le 80, \\ 0.03 - 0.00025x, & 80 < x \le 120. \end{cases}$$

Let there be an ordinary deductible of $d = 20$.
Calculate the **loss elimination ratio.**

Solution: The loss elimination ratio is defined as

$$\mathbb{E}[X \wedge d]/\mathbb{E}[X].$$

In this problem, with $d = 20$, we get

$$\begin{aligned} \mathbb{E}[X \wedge 20] &= \int_0^{120} (x \wedge 20) f_X(x)\, dx \\ &= \int_0^{20} x f_X(x)\, dx + \int_{20}^{120} 20 f_X(x)\, dx \\ &= \int_0^{20} x f_X(x)\, dx + 20 \int_0^{120} f_X(x)\, dx - 20 \int_0^{20} f_X(x)\, dx. \end{aligned}$$

We have

$$\int_0^{20} x\, f_X(x)\, dx = 0.01 \int_0^{20} x\, dx = 0.01 \cdot \frac{1}{2} x^2 |_{x=0}^{20} = 0.005 \cdot 20^2 = 0.005 \cdot 400 = 2.$$

Since f_X is a density function,

$$\int_0^{120} f_X(x)\, dx = 1.$$

As for the third integral,

$$\int_0^{20} f_X(x)\, dx = 0.01 \cdot 20 = 0.2.$$

Putting everything together,

$$\mathbb{E}[X \wedge 20] = 2 + 20 \cdot 1 - 20 \cdot 0.2 = 2 + 20 - 4 = 18.$$

Also,

$$\begin{aligned} \mathbb{E}[X] &= \int_0^{120} x f_X(x)\, dx \\ &= 0.01 \int_0^{80} x\, dx + 0.03 \int_{80}^{120} x\, dx - 0.00025 \int_{80}^{120} x^2\, dx \\ &= 0.005 x^2 |_{x=0}^{80} + 0.015 x^2 |_{x=80}^{120} - 0.00025 \cdot \frac{1}{3} x^3 |_{x=80}^{120} \\ &= 152/3. \end{aligned}$$

Finally,

$$\frac{\mathbb{E}[X \wedge 20]}{\mathbb{E}[X]} = \frac{18}{152/3} = \frac{54}{152} = \frac{27}{76} \approx 0.3553.$$

Problem 2.2. *Source: Sample STAM Exam Problem #127.*
Losses in 2003 follow a two-parameter Pareto distribution with $\alpha = 2$ and $\theta = 5$. Losses in 2004 are 20% uniformly higher than in 2003. An insurance covers each loss subject to a deductible of 10. Calculate the loss elimination ratio in 2004.

Solution: Let the losses in 2003 be represented by the random variable X and let the losses in 2004 be represented by the random variable $\tilde{X} = 1.2X$. We model X as follows:

$$X \sim Pareto(\alpha = 2, \theta = 5).$$

Since the two-parameter Pareto is a scale distribution with the scale parameter θ, we can conclude that

$$\tilde{X} \sim Pareto(\tilde{\alpha} = 2, \tilde{\theta} = (1.2)5 = 6).$$

By definition, with the ordinary deductible of $d = 10$, the loss elimination ratio in the year 2004 equals

$$\widetilde{LER} = \frac{\mathbb{E}[X \wedge d]}{\mathbb{E}[X]} = \frac{\frac{\tilde{\theta}}{\tilde{\alpha}-1}\left(1 - \left(\frac{\tilde{\theta}}{d+\tilde{\theta}}\right)^{\alpha-1}\right)}{\frac{\tilde{\theta}}{\tilde{\alpha}-1}} = 1 - \left(\frac{\tilde{\theta}}{d+\tilde{\theta}}\right)^{\alpha-1} = 1 - \left(\frac{6}{10+6}\right)^{2-1} = \frac{5}{8}\,.$$

Problem 2.3. *Source: Two old exams* 3*; I forgot to note the years.*
A jewelry store purchases two separate insurance policies that together provide full coverage. You are given:

- The expected ground-up loss is $11,100$.
- Policy A has an ordinary deductible of $5,000$ and **no** policy limit.
- Under policy A, the expected amount paid per loss is $6,500$.
- Under policy A, the expected amount paid per payment is $10,000$.
- Policy B has **no** deductible and has a policy limit of $5,000$.

i. **Given** that a loss has occurred, find the probability that the payment under policy B equals $5,000$.
ii. **Given** that a loss less than or equal to $5,000$ has occurred, what is the expected payment under policy B?

Solution:

i. Let X be the random variable denoting the ground-up loss. For simplicity, assume that X is continuous. We are looking for the following probability:

$$\mathbb{P}[X \geq 5000] = S_X(5000)$$

Using our usual notation (modified in an obvious way), we are given that with the deductible $d = 5000$ the following facts hold true for the policy A:

$$6,500 = \mathbb{E}[Y_A^L] = \mathbb{E}[(X-d)_+],$$
$$10,000 = \mathbb{E}[Y_A^P] = \mathbb{E}[X - d \mid X > d] = \frac{\mathbb{E}[(x-d)_+]}{S_X(d)}\,.$$

So, $S_X(d) = \frac{6500}{10000} = 0.65$.

ii. We are asked to calculate

$$\mathbb{E}[X \mid X \leq 5000].$$

By the definition of conditional expectation, the above equals

$$\frac{\mathbb{E}[X\mathbb{I}_{[X\leq 5000]}]}{\mathbb{P}[X \leq 5000]} = \frac{\mathbb{E}[X \wedge 5000] - 5000\mathbb{P}[X > 5000]}{\mathbb{P}[X \leq 5000]} = \frac{\mathbb{E}[X \wedge 5000] - 5000 S_X(5000)}{F_X(5000)}\,.$$

In the previous part of the problem we obtained $S_X(5000) = 0.65$. Hence, $F_X(5000) = 1 - 0.65 = 0.35$. The final necessary ingredient is

$$\mathbb{E}[X \wedge 5000] = \mathbb{E}[X] - \mathbb{E}[(X-5000)_+] = \mathbb{E}[X] - \mathbb{E}[Y_A^L] = 11100 - 6500 = 4600.$$

Our final answer is

$$\frac{4600 - 5000(0.65)}{0.35} = 3,857.14$$

Problem 2.4. Let the ground-up loss X be exponentially distributed with mean \$500. An insurance policy has an ordinary deductible of \$50 and a policy limit of \$2000. Find the expected value of the amount paid (by the insurance company) per positive payment.

Solution: We are given $X \sim \mathit{Exponential}(\theta = 500)$, the deductible $d = 50$ and the policy limit $u - d = 2000$. We need to calculate $\mathbb{E}[Y^P]$ where $Y^P = Y^L \,|\, , Y^L > 0$ and

$$\begin{aligned} Y^L &= \begin{cases} (X-d)_+, & X < u, \\ u-d, & X \geq u \end{cases} \\ &= (X \wedge u - d)_+. \end{aligned}$$

By the memoryless property of the exponential distribution, we have that

$$Y = X - d \,|\, X > d$$

is also exponential with mean 500. So, using our tables, we get

$$\mathbb{E}[Y^P] = \mathbb{E}[Y \wedge (u-d)] = \mathbb{E}[Y \wedge 2000] = 500(1 - e^{-2000/500}) = 490.84.$$

Alternatively, we could have used the theorem from class to get

$$\begin{aligned} \mathbb{E}[Y^L] &= \mathbb{E}[X \wedge 2050] - \mathbb{E}[X \wedge 50] \\ &= 500(1 - e^{-2050/500}) - 500(1 - e^{-50/500}) \\ &= 500\left[1 - e^{-2050/500} - 1 + e^{-50/500}\right] \\ &= 500e^{-50/500}\left[1 - e^{-2000/500}\right]. \end{aligned}$$

Then,

$$\mathbb{E}[Y^P] = \frac{1}{1 - F_X(50)}\mathbb{E}[Y^L] = e^{50/500} \cdot 500e^{-50/500}\left[1 - e^{-2000/500}\right] = 500\left[1 - e^{-2000/500}\right].$$

Problem 2.5. An insurance policy on a ground-up loss X has:

- no deductible;
- a coinsurance of 50%; and
- a maximum policy payment per loss of 5000

Let X be modeled using a two-parameter Pareto distribution with $\alpha = 2$ and $\theta = 10000$. What is the expected payment per loss for the insurer?

Solution: In our usual notation, we need to calculate $\mathbb{E}[Y^L]$. By our theorem, we have that

$$\mathbb{E}[Y^L] = \alpha(\mathbb{E}[X \wedge u] - \mathbb{E}[X \wedge d]).$$

In this problem, there is no deductible so $d = 0$. Hence,

$$\mathbb{E}[Y^L] = \alpha\mathbb{E}[X \wedge u].$$

With the coinsurance $\alpha = 0.5$, and the given maximum policy payment of 5000, we conclude that

$$\alpha u = 5000 \quad \Rightarrow \quad u = 10000.$$

So,

$$\mathbb{E}[Y^L] = 0.5\mathbb{E}[X \wedge 10000].$$

From the STAM tables, we have that

$$\begin{aligned} \mathbb{E}[X \wedge 10000] &= \left(\frac{\theta}{\alpha - 1}\right)\left[1 - \left(\frac{\theta}{10000 + \theta}\right)^{\alpha - 1}\right] \\ &= \left(\frac{10000}{2-1}\right)\left[1 - \left(\frac{10000}{10000 + 10000}\right)^{2-1}\right] = 10000(0.5) = 5000. \end{aligned}$$

Finally, $\mathbb{E}[Y^L] = 0.5(5000) = 2500$.

Problem 2.6. *Source: Sample STAM Problem #279.*
Loss amounts have the distribution function

$$F_X(x) = \begin{cases} \left(\frac{x}{100}\right)^2, & 0 \leq x \leq 100 \\ 1, & x > 100 \end{cases}$$

An insurance pays 80% of the amount of the loss in excess of an ordinary deductible of 20, subject to a maximum payment of 60 per loss. Calculate the conditional expected claim payment, given that a payment has been made.

Solution: As usual, let X represent the losses, let Y^P be the per payment random variable, and let Y^L be the per loss random variable. We are tasked with calculating, with $d = 20$,

$$\mathbb{E}[Y^P] = \mathbb{E}[Y^L \mid X > d] = \frac{\mathbb{E}[Y^L \mathbb{I}_{[X>d]}]}{S_X(d)} = \frac{\mathbb{E}[Y^L]}{S_X(d)}$$

Again, in our usual notation, the theorem from class implies that

$$\mathbb{E}[Y^L] = \alpha(\mathbb{E}[X \wedge u] - \mathbb{E}[X \wedge d]).$$

It is given in the problem that $\alpha = 0.8$. Moreover, we are given that $\alpha(u - d) = 60$. Easily, we get that $u = 95$. Hence,

$$\mathbb{E}[Y^L] = 0.8(\mathbb{E}[X \wedge 95] - \mathbb{E}[X \wedge 20]).$$

For any constant $c \in (0, 100)$, the tail formula for the expectation gives us

$$\mathbb{E}[X \wedge c] = \int_0^c S_X(x)\,dx = \int_0^c \left(1 - \left(\frac{x}{100}\right)^2\right) dx = \left[x - \frac{1}{100^2}\left(\frac{x^3}{3}\right)\right]_{x=0}^{c} = c - \frac{c^3}{3 \cdot 10^4}.$$

So, the expected value of the per loss random variable equals

$$\mathbb{E}[Y^L] = 0.8\left(\left(95 - \frac{95^3}{3 \cdot 10^4}\right) - \left(20 - \frac{20^3}{3 \cdot 10^4}\right)\right) = 37.35.$$

Our final answer is

$$\mathbb{E}[Y^P] = \frac{\mathbb{E}[Y^L]}{S_X(20)} = \frac{37.35}{1 - \left(\frac{20}{100}\right)^2} = 38.90625.$$