M378K Introduction to Mathematical Statistics
Fall 2024
University of Texas at Austin
Practice for In-Term Exam III
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Notes: This is a closed book and closed notes exam. The maximal score on the real exam will be 100 points.
There are many ways in which any single problem can be solved. The solutions herein are just one possible way to tackle the given problems.
Time: 50 minutes

All written work handed in by the student is considered to be
their own work, prepared without unauthorized assistance.

The University Code of Conduct

”The core values of The University of Texas at Austin are learning, discovery, freedom, leadership, individual opportunity, and responsibility. Each member of the university is expected to uphold these values through integrity, honesty, trust, fairness, and respect toward peers and community. As a student of The University of Texas at Austin, I shall abide by the core values of the University and uphold academic integrity.”

”I agree that I have complied with the UT Honor Code during my completion of this exam.”

Signature:

3.1. **Formulas.** If Y has the binomial distribution with parameters n and p, then $p_Y(k) = \mathbb{P}[Y = k] = \binom{n}{k} p^k (1-p)^{n-k}$, for $k = 0, \ldots, n$, $\mathbb{E}[Y] = np$, $\mathrm{Var}[Y] = np(1-p)$. The binomial coefficients are defined as follows for integers $0 \le k \le n$, $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. The moment generating function of Y is given by $m_Y(t) = (pe^t + q)^n$.

If Y has a geometric distribution with parameter p, then $p_Y(k) = p(1-p)^k$ for $k = 0, 1, \ldots,$ $\mathbb{E}[Y] = \frac{1-p}{p}$, $\mathrm{Var}[Y] = \frac{1-p}{p^2}$. Its mgf is $m_Y(t) = \frac{p}{1-qe^t}$ for t such that $qe^t < 1$.

If Y has a Poisson distribution with parameter λ, then $p_Y(k) = e^{-\lambda} \frac{\lambda^k}{k!}$ for $k = 0, 1, \ldots,$ $\mathbb{E}[Y] = \mathrm{Var}[Y] = \lambda$. Its mgf is $m_Y(t) = e^{\lambda(e^t - 1)}$.

If Y has a uniform distribution on $[l, r]$, its density is

$$f_Y(y) = \frac{1}{r-l} \mathbf{1}_{(l,r)}(y),$$

its mean is $\frac{l+r}{2}$, and its variance is $\frac{(r-l)^2}{12}$. Let $U \sim U(0,1)$. The mgf of U is $m_U(t) = \frac{1}{t}(e^t - 1)$.

If Y has the standard normal distribution, then its mean is zero, its variance is one, and its density equals

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad x \in \mathbb{R}.$$

Its mgf is $m_Y(t) = e^{\frac{t^2}{2}}$.

If Y has the exponential distribution with parameter τ, then its cumulative distribution function is $F_Y(y) = 1 - e^{-\frac{y}{\tau}}$ for $y \ge 0$, its probability density function is $f_Y(y) = \frac{1}{\tau} e^{-y/\tau}$ for $y \ge 0$. Also, $\mathbb{E}[Y] = SD[Y] = \tau$. Its mgf is $m_Y(t) = \frac{1}{1-\tau t}$.

The mgf of $Y \sim \Gamma(k, \tau)$ is

$$m_Y(t) = \frac{1}{(1-\tau t)^k} \text{ for } t < 1/\tau.$$

Its expectation is $k\tau$ and its variance is $k\tau^2$. The χ^2-distribution with n degrees of freedom is the special case $\Gamma\left(\frac{n}{2}, 2\right)$

3.2. DEFINITIONS.

Problem 3.1. (10 points) Write down the definition of the **bias** of an estimator $\hat{\theta}$ of a parameter θ.

Problem 3.2. (10 points) Write down the definition of the **mean squared error** of an estimator $\hat{\theta}$ of a parameter θ.

3.3. TRUE/FALSE QUESTIONS.

Problem 3.3. (5 points) Let $n \geq 2$ and let $Y_1, Y_2, \dots, Y_n$ be a random sample from $E(\tau)$. Then, in our usual notation,

$$Y_{(n)} \sim E(\tau/n).$$

True or false? Why?

Problem 3.4. (5 points) Let Z be a standard normal random variable and let Q^2 have the χ^2-distribution with $\nu \geq 2$ degrees of freedom. Assume that Z and Q^2 are independent. Set

$$T = \frac{Z}{Q^2}\,.$$

Then, T has a $t-$distribution with ν degrees of freedom. *True or false? Why?*

3.4. Free-response problems.

Please, explain carefully all your statements and assumptions. Numerical results or single-word answers without an explanation (even if they're correct) are worth 0 points.

Problem 3.5. (20 points) Consider a random sample $Y_1, Y_2, \dots, Y_n$ from the Weibull distribution with parameters m and α, i.e., the distribution whose density is given by

$$f_Y(y) = \frac{m}{\alpha} y^{m-1} e^{-\frac{y^m}{\alpha}} \mathbf{1}_{(0,\infty)}(y)$$

where m and α are positive constants. What is the distribution of $Y_{(1)}$? If you recognize it as a named distribution, provide its name and its parameters in terms of n, m, and α. If not, provide its density.

Problem 3.6. (10 points) Let $Y_1, Y_2, \dots, Y_n$ be a random sample from a population with mean μ and standard deviation σ. Let $\tilde{Y}_1, \tilde{Y}_2, \dots, \tilde{Y}_n$ be a random sample from a population with mean $\tilde{\mu}$ and $\tilde{\sigma}$. Assume that the two random samples are mutually independent random. Show that

$$\hat{\theta}_n = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \tilde{Y}_i)$$

is a consistent estimator for $\mu - \tilde{\mu}$.

Problem 3.7. (10 points) Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from $N(\mu, \sigma)$ for $n \geq 3$. Consider the following two estimators for μ:

$$\hat{\theta}_1 = \bar{Y} \quad \text{and} \quad \hat{\theta}_2 = \frac{Y_1 + Y_n}{2}$$

Are these estimators unbiased? If so, find the efficiency of $\hat{\theta}_1$ relative to $\hat{\theta}_2$.

Problem 3.8. (10 points) Let $Y_1, \ldots, Y_n$ be a random sample from a Poisson distribution with an unknown parameter λ. What is the maximum likelihood estimator for λ? Make sure that you prove your claim!

Problem 3.9. (10 points) Let $Y_1, \ldots, Y_n$ be a random sample from $E(\tau)$. Find a sufficient statistic for τ and justify your answer.

3.5. MULTIPLE CHOICE QUESTIONS.

Problem 3.10. (5 points) In a sample $Y_1, \ldots, Y_n$ from the exponential distribution $E(\tau)$ with parameter $\tau > 0$, $U = c\bar{Y}$ is a pivotal quantity if the value of the constant c is

(a) 1 (b) 2 (c) $\frac{2n}{\tau}$ (d) τ (e) none of the above

Problem 3.11. (5 points) In a random sample of 100 voters 80 prefer candidate A and the rest prefer candidate B. The (approximate) $(1-\alpha)$-confidence interval for the parameter p (the population proportion of A voters) is of the form

$$[0.8 - z_{\alpha/2} \times c, 0.8 + z_{\alpha/2} \times c],$$

where $z_{\alpha/2} = \texttt{qnorm}(1 - \alpha/2, 0, 1)$. The value of c is:

(a) 0.01 (b) 0.02 (c) 0.03 (d) 0.04 (e) ≥ 0.05

Problem 3.12. (5 points) A sample of size $n = 2$ from normal distribution with unknown μ and σ is collected and the data are

$$y_1 = 1 \text{ and } y_2 = 5.$$

The left end-point of a symmetric 95% confidence interval for σ^2 is

(a) 8/`qchisq(0.975,2)` (b) 16/`qchisq(0.975,1)` (c) 8/`qchisq(0.975,1)` (d) 16/`qchisq(0.975,2)` (e) **None of the above**

Problem 3.13. (5 points) Let $Y_1, \ldots, Y_n$ be a random sample from the uniform distribution $U(0, \theta)$, with parameter $\theta > 0$. The MSE (mean-squared error) of the estimator $\hat{\theta} = c\bar{Y}$ for θ is the smallest when the constant c equals

(a) $\frac{1}{2}$ (b) 2 (c) $\frac{6n}{3n+1}$ (d) $\frac{3n}{6n+1}$ (e) none of the above

Problem 3.14. (5 points) Let $Y_1, \dots, Y_5$ be a random sample from the normal distribution $N(\mu, \sigma)$, with an unknown mean μ and an unknown standard deviation σ. The collected data turn out to be

$$y_1 = 2, y_2 = 5, y_3 = 1, y_4 = 4, y_5 = 3.$$

The right end-point of the one-sided 90%-confidence interval $(-\infty, \hat{\mu}_R]$ for μ is

(a) $3 - \frac{1}{2}\texttt{qnorm}(0.1, 0, 1)$.

(b) $3 - \frac{\sqrt{5}}{\sqrt{8}}\texttt{qt}(0.1, 4)$.

(c) $3 - \frac{1}{\sqrt{2}}\texttt{qt}(0.1, 5)$.

(d) $3 - \frac{1}{\sqrt{2}}\texttt{qt}(0.1, 4)$.

(e) none of the above

Problem 3.15. (5 points) Let $Y_1, \dots, Y_5$ be a random sample from the normal distribution $N(\mu, 2)$, with an unknown mean μ and the known standard deviation $\sigma = 2$. The collected data turn out to be

$$y_1 = 2,\ y_2 = 5,\ y_3 = 1,\ y_4 = 4,\ y_5 = 3.$$

The left end-point $\hat{\mu}_L$ of the symmetric 90%-confidence interval $[\hat{\mu}_L, \hat{\mu}_R]$ for μ is

(a) $3 + \frac{2}{\sqrt{5}}\texttt{qnorm}(0.9, 0, 1)$.

(b) $3 - \frac{2}{\sqrt{5}}\texttt{qnorm}(0.95, 0, 1)$.

(c) $3 - \frac{1}{\sqrt{5}}\texttt{qt}(0.95, 4)$.

(d) $3 + \frac{1}{5}\texttt{qnorm}(0.9, 5)$.

(e) none of the above