M339D: March 8th, 2023.

Forward Binomial Trees.

$$S(0) \begin{cases} S_u = u \cdot S(0) \\ S_d = d \cdot S(0) \end{cases}$$

$$\underbrace{\qquad}_{h=T}$$

$\boxed{u, d = ?}$

The no-arbitrage condition:

$$d < e^{rh} < u$$

"Def'n." The **volatility σ** is the standard deviation of the realized returns on a continuously compounded scale and annualized.

Heuristics: $T=1$

$h = \frac{1}{m}$ (year)

Q: What is the volatility for a time period of length h?

Call this volatility σ_h

$$\underset{t}{|}\qquad\underset{t+s}{|}$$

$R(t, t+s)$ satisfies $S(t+s) = e^{R(t,t+s)} S(t)$

Compare to simple:

$$\frac{S(t+s) - S(t)}{S(t)} = \frac{S(t+s)}{S(t)} - 1$$

$$\boxed{R(t,t+s) := \ln\left(\frac{S(t+s)}{S(t)}\right)} \checkmark$$

$$0 \quad \frac{1}{m} \quad \frac{2}{m} \quad \cdots \quad \frac{m-1}{m} \quad 1$$

$$R(0, \tfrac{1}{m}) \quad R(\tfrac{1}{m}, \tfrac{2}{m}) \quad \cdots \quad R(\tfrac{m-1}{m}, 1)$$

Note: $R\left(\frac{k-1}{m}, \frac{k}{m}\right)$ for $k = 1, 2, \dots, m$

are all random variables.

We make the following assumptions:

- all the returns above are identically distributed;
- the returns over disjoint intervals are independent.

We also know that the returns defined as above are additive, i.e.,

$$\boxed{R\left(0,\tfrac{1}{m}\right) + R\left(\tfrac{1}{m},\tfrac{2}{m}\right) + \cdots + R\left(\tfrac{m-1}{m},1\right) = R(0,1)}$$

$$\Rightarrow \quad \sigma^2 = \mathrm{Var}\left[R(0,1)\right] = \mathrm{Var}\left[R\left(0,\tfrac{1}{m}\right) + \cdots + R\left(\tfrac{m-1}{m},1\right)\right]$$

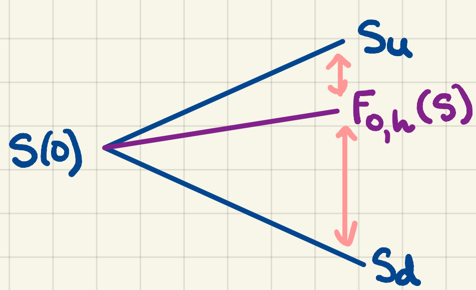
$$\overset{\text{independence}}{=} \mathrm{Var}\left[R\left(0,\tfrac{1}{m}\right)\right] + \cdots + \mathrm{Var}\left[R\left(\tfrac{m-1}{m},1\right)\right]$$

$$\overset{\text{identically dist'd}}{=} m \cdot \mathrm{Var}\left[R\left(0,\tfrac{1}{m}\right)\right] = m \cdot \sigma_h^2 \qquad \boxed{h = \tfrac{1}{m}}$$

$$\Rightarrow \quad \sigma^2 = m \cdot \sigma_h^2$$

$$\Rightarrow \quad \boxed{\sigma_h = \sigma\sqrt{h}}$$

We generalize this identity to arbitrary lengths h.



Recall: $\boxed{F_{0,h}(S) = S(0)e^{rh}}$

$$S_u := F_{0,h}(S)\cdot e^{\sigma\sqrt{h}} = S(0)e^{rh}\cdot e^{\sigma\sqrt{h}} = S(0)\overset{u}{\overset{\shortparallel}{\boxed{e^{rh+\sigma\sqrt{h}}}}}$$

$$S_d := F_{0,h}(S)\cdot e^{-\sigma\sqrt{h}} = S(0)e^{rh}\cdot e^{-\sigma\sqrt{h}} = S(0)\underset{d}{\underset{\shortparallel}{\boxed{e^{rh-\sigma\sqrt{h}}}}}$$

Note: u and d immediately satisfy the no-arbitrage condition.

Q: What is S_u/S_d?

$$\longrightarrow: \boxed{\frac{S_u}{S_d}} = \frac{u\cdot \cancel{S(0)}}{d\cdot \cancel{S(0)}} = \frac{\cancel{e^{rh}}\cdot e^{\sigma\sqrt{h}}}{\cancel{e^{rh}}\cdot e^{-\sigma\sqrt{\ }}} = \boxed{e^{2\sigma\sqrt{h}}} \qquad \square$$

Example. One-period tree
w/ time-horizon of one quarter

$$
\begin{tikzcd}[row sep=small]
 & S_u = 105 \\
100 = S(0) \arrow[ur] \arrow[dr] & \\
 & S_d = 98
\end{tikzcd}
$$

Q: If this is a forward tree, what is the volatility?

$\longrightarrow$:

$$\frac{S_u}{S_d} = \frac{105}{98} = e^{2\sigma\sqrt{\frac{1}{4}}} = e^{\sigma}$$

$$\sigma = \ln\left(\frac{105}{98}\right) = \underline{0.0689}$$

Binomial Option Pricing.

Stock Price Tree.

$$
\begin{tikzcd}[row sep=small]
 & S_u = u \cdot S(0) \\
S(0) \arrow[ur, "h"] \arrow[dr] & \\
 & S_d = d \cdot S(0)
\end{tikzcd}
$$

$\xrightarrow{\text{populating}}$ the tree

We want to price a European-style derivative security w/ the exercise date @ the end of the tree.

It is completely determined by its payoff f'tion: $v(\cdot)$,

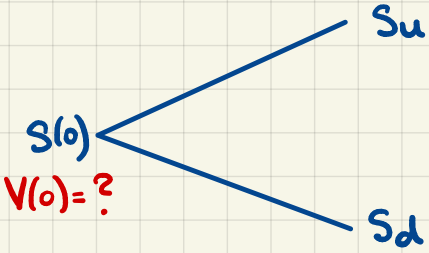
e.g., for a call: $v_c(s) = (s-K)_+$,

or for a put: $v_p(s) = (K-s)_+$,

or something "completely" different: $v(s) = (s^2-K)_+$

The payoff of the derivative security is a random variable:

$$\boxed{V(T) = v(S(T))}$$



Payoff

$V_u := v(S_u)$

$V_d := v(S_d)$

population $\longrightarrow$

$\longleftarrow$ pricing