The lognormal distribution.

Definition 1.1. Let $X \sim Normal(mean = m, variance = \nu^2)$. Define the random variable $Y = e^X$. We say that the random variable Y is *lognormally distributed.*

1.1. **First properties.**

- The expected value of the lognormally distributed random variable Y can be obtained as follows:
$$\mathbb{E}[Y] = \mathbb{E}[e^X] = M_X(1) = e^{m+\frac{\nu^2}{2}}.$$
- Let Y be a lognormal and let $a \neq 0$. Then, the random variable Y^a is also lognormal. *Note:* For $a = 0$, we get a degenerate random variable at 1 which can, technically, be interpreted as lognormal, but is not fun.
- Let Y_1 and Y_2 be independent and lognormally distributed. Then, Y_1Y_2 is also lognormal.

1.2. **Quantiles.**

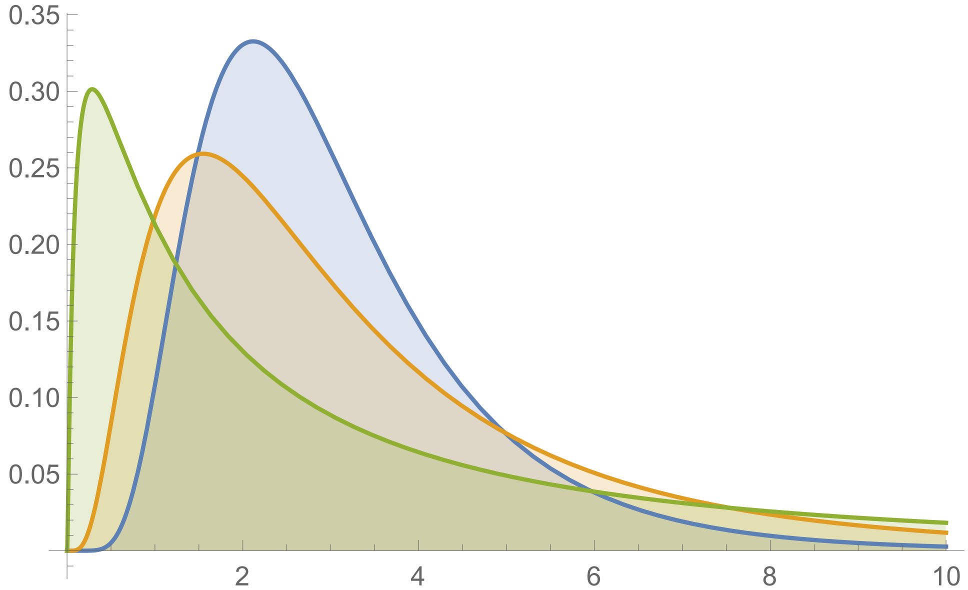
Definition 1.2. For p such that $0 < p < 1$, we define the $100p^{th}$ *quantile* of a random variable X as any value π_p such that
$$F_X(\pi_p-) \leq p \leq F_X(\pi_p).$$
In particular, the 50^{th} quantile is referred to as the *median.*

Note: When the random variable X is continuous, we can obtain the $100p^{th}$ quantile by simply solving for π_p in
$$F_X(\pi_p) = p.$$

Consider a probability p. Let z_p be the $100p^{th}$ quantile of the standard normal distribution. Let Y be lognormally distributed as above. My claim is that the value
$$y_p = e^{m+\nu z_p}$$
is the $100p^{th}$ quantile of Y. Let us simply verify this claim by calculating $F_Y(y_p)$. We have, with $Z \sim N(0,1)$,
$$F_Y(y_p) = \mathbb{P}[Y \leq y_p] = \mathbb{P}[e^X \leq y_p] = \mathbb{P}[e^{m+\nu Z} \leq e^{m+\nu z_p}].$$
Since the logarithmic function is increasing, we have that the above equals
$$F_Y(y_p) = \mathbb{P}[m + \nu Z \leq m + \nu z_p] = \mathbb{P}[Z \leq z_p] = p.$$
The above concludes our proof.

In particular, since the median of the standard normal distribution equals 0, the median of the lognormal distribution will be e^m.

Note: Since
$$e^m < e^{m+\frac{\nu^2}{2}}, \tag{1.1}$$
i.e., since the mean of a lognormal distribution always exceeds the median, we say that it's *right-skewed.* In fact, this is what its probability density function looks like.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