Name:

All written work handed in by the student is considered to be
their own work, prepared without unauthorized assistance.

The maximum number of points on this part of the exam is 65.

Provide your **complete solution** to the following problems. Final answers only, without appropriate justification, will receive zero points even if correct.

2.1. **FREE-RESPONSE PROBLEMS.**

Problem 2.1. (20 points) Consider a non-dividend-paying stock whose current price is $80 per share. The stock has the volatility equal to 0.25.

Let the continuously-compounded, risk-free interest rate be equal to 0.04.

You model the evolution of this stock over the next nine months with a three-period **forward** binomial tree.

What is the price of a $75-strike, nine-month call on the above stock consistent with this model?

Solution: In our usual notation, in the forward binomial tree, the risk-neutral probability is

$$p^* = \frac{1}{1 + e^{\sigma\sqrt{h}}} = \frac{1}{1 + e^{0.25\sqrt{1/4}}} = \frac{1}{1 + e^{0.125}} = 0.4687906.$$

The *up* and *down* factors are

$$u = e^{rh+\sigma\sqrt{h}} = e^{0.04(1/4)+0.125} = e^{0.135} = 1.144537,$$

$$d = e^{rh-\sigma\sqrt{h}} = e^{0.04(1/4)-0.125} = e^{-0.115} = 0.8913661.$$

Hence, the possible stock prices at the end of the three periods are

$$S_{uuu} = S(0)u^3 = 119.9442, \quad S_{uud} = S(0)u^2d = 93.41264,$$

$$S_{udd} = S(0)ud^2 = 72.74983, \quad S_{ddd} = S(0)d^3 = 56.65763.$$

So, the call option is in the money only in the two top nodes where the payoff equals

$$V_{uuu} = (S_{uuu} - K)_+ = 44.9442 \quad \text{and} \quad V_{uud} = (S_{uud} - K)_+ = 18.41264.$$

By the risk-neutral pricing formula, we have that

$$V_P(0) = e^{-0.04(3/4)} \left[44.9442(0.4687906)^3 + 18.41264(3)(0.4687906)^2(1 - 0.4687906)\right] = 10.75142.$$

Problem 2.2. (10 points) A squirrel is throwing hazelnuts at a picnicking family repeatedly. Assume that her attempts are independent and that the probability of hitting a member of the picnicking family in any single attempt equals 0.64. The total number of hazelnuts the squirrel is willing to waste in this exercise is 100. Using the *normal approximation to the binomial*, what is the approximate probability that the squirrel hits a family member at least 60 times?

Solution: *The following solution uses the continuity correction. The solutions without the continuity correction would also be graded as correct if otherwise accurate.*
The number of trials (throws) is 100. The probability of hitting a human in a single throw is 0.64. So, the total number of hits is, in our usual notation,

$$X \sim Binomial(n = 100, p = 0.64).$$

The probability we are looking for is $\mathbb{P}[X \geq 60]$. The mean of the random variable X is $np = 64$ and its standard deviation is $\sqrt{np(1-p)} = 4.8$. Using the normal approximation to the binomial, we get

$$\mathbb{P}[X \geq 60] = \mathbb{P}[X > 59.5] = \mathbb{P}\left[\frac{X - 64}{4.8} > \frac{59.5 - 64}{4.8}\right] = 1 - \Phi(-0.9375) \approx 0.8257493.$$

Using the standard normal tables, the final answer is

$$1 - \Phi(-0.94) = 0.8264.$$

Problem 2.3. (20 points) Consider a non-dividend-paying stock whose price $\mathbf{S} = \{S(t), t \geq 0\}$ is modeled using the Black-Scholes framework. Suppose that the current stock price equals \$100 and that its volatility is given to be 0.25.

The continuously compounded, risk-free interest rate is assumed to equal 0.05.

Consider a derivative security which entitles its owner to obtain a European call option on the above stock one year from today, i.e., at time $t_* = 1$. The call option is to be $3-$month to expiration at time of delivery and be at-the-money at time$-t^*$. This contract is an example of a **forward start option**.

What is the price of this forward start option?

Solution: At time $t*$, the required Black-Scholes price of the call option equals

$$V_C(t^*) = S(t^*)N(d_1(t^*)) - S(t^*)e^{-r(T-t^*)}N(d_2(t^*))$$
$$= S(t^*)(N(d_1(t^*)) - e^{-0.01}N(d_2(t^*)))$$

with

$$d_1(t^*) = \frac{1}{0.25\sqrt{0.25}}\left(0.05 + \frac{0.25^2}{2}\right) \times \frac{1}{4} = 0.1625,$$
$$d_2(t^*) = d_1 - \sigma\sqrt{T - t^*} = 0.0375.$$

So, $N(d_1(t^*)) = 0.5645439$ and $N(d_2(t^*)) = 0.5149568$. Hence,

$$V_C(t^*) = S(t^*)(0.5645439 - e^{-0.05(0.25)} \times 0.5149568) = 0.055984S(t^*).$$

So, one would need to buy exactly 0.055984 shares of stock to be able to buy the call option in question at time$-t^*$. This amount of shares costs \$5.5984.

Problem 2.4. (15 points) Assume the Black-Scholes model. Consider a non-dividend-paying stock with the initial stock price of \$120 and volatility equal to 0.25.

The continuously-compounded, risk-free interest rate is 0.10.

Answer the following questions under the risk-neutral probability measure $\mathbb{P}^*$.

(i) (3 points) What is the mean time-1 stock price?

(ii) (5 points) At what time t^* is the median stock price equal to 200?

(iii) (7 points) What is the probability that the time-1 stock price exceeds today's stock price?

Solution:

(i) $\mathbb{E}^*[S(1)] = 120e^{0.10} = 132.6205$

(ii) We need to solve for t^* in

$$120 \exp\left((0.10 - \tfrac{0.25^2}{2})t^*\right) = 200 \quad \Rightarrow \quad t^* = \frac{\ln\left(\frac{200}{120}\right)}{0.10 - \frac{0.25^2}{2}} = 7.430191.$$

(iii) This probability is precisely the probability that the return $R(0,1)$ is positive. We know that

$$R(0,1) \sim Normal\left(mean = 0.10 - \tfrac{0.25^2}{2} = 0.06875, sd = 0.25\right).$$

So,

$$\mathbb{P}^*[R(0,1) > 0] = \mathbb{P}^*\left[\frac{R(0,1) - 0.06875}{0.25} > \frac{0 - 0.06875}{0.25} = -0.275\right] = \Phi(0.275) = 0.6083419.$$

Using the standard normal tables, we get $\Phi(0.28) = 0.6103$.