M339J: February 11th, 2022.

Raw Moments.

Def'n. The k^{th} raw moment of a random variable X is given by

$$\boxed{\mu_k' := \mathbb{E}[X^k]}$$

Note: The 1^{st} raw moment is the mean we frequently denote by $\mu_X \equiv \mu$.

Central Moments.

Def'n. The k^{th} central moment of a r.v. X is

$$\mu_k := \mathbb{E}\left[(X-\mu)^k\right]$$

Q: What is the 2^{nd} central moment?

$\longrightarrow$: $\mu_2 = Var[X] = \mathbb{E}\left[(X-\mu)^2\right] = \mathbb{E}[X^2] - \mu^2$

$= \mu_2' - \mu^2$

Problem. Let X be a two-parameter Pareto w/ $\alpha = 3$ and $\theta = 10$.

Find $Var[X]$.

$\longrightarrow$: $X \sim Pareto(\alpha = 3, \theta = 10)$

Using my STAM tables:

$$\boxed{\mathbb{E}[X^k] = \frac{\theta^k \cdot k!}{(\alpha-1)\cdots\cdots(\alpha-k)}}$$

k integer
$k < \alpha$

$$\mathbb{E}[X] = \frac{\theta^1 \cdot 1!}{\alpha - 1} = \frac{\theta}{\alpha-1} = \frac{10}{3-1} = 5$$

$$\mathbb{E}[X^2] = \frac{\theta^2 \cdot 2!}{(\alpha-1)(\alpha-2)} = \frac{2\theta^2}{(\alpha-1)(\alpha-2)} = \frac{\cancel{2} \cdot 10^2}{\cancel{(3-1)}(3-2)} = 100$$

$$\Rightarrow \quad Var[X] = 100 - 5^2 = 75$$

$\blacksquare$

Task: The expression for the variance of the exponential & the gamma in terms if it's "simple."

Def'n. The coefficient of variation of a random variable X is:

$$\boxed{\frac{SD[X]}{\mathbb{E}[X]}}$$

w/ $SD[X] = \sqrt{Var[X]}$.

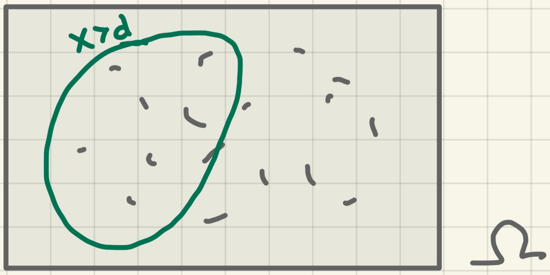
Excess Loss (Random) Variable.

Def'n. Let X be a random variable.
Let d be a (positive) constant such that

$$\boxed{\mathbb{P}[X > d] > 0} \checkmark$$

The excess loss random variable is usually denoted by Y^P and it's defined as

$$\boxed{Y^P = \underline{X - d} \text{ given that } \underline{X > d}} \checkmark$$

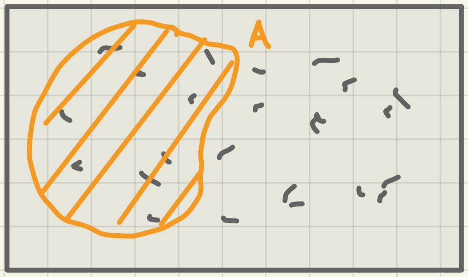


Note:
- All the values of X less than d are "discarded"

  $\Rightarrow$ left truncated
- d is subtracted $\Rightarrow$ shifted
- We usually write: $\boxed{Y^P = X - d \mid X > d}$

Def'n. The mean excess loss function, denoted by $e_X(d)$, is defined as the expectation of Y^P, i.e.,

$$\boxed{e_X(d) = \mathbb{E}[X - d \mid X > d]}$$



Ω

Consider a random variable G and an event A such that $\mathbb{P}[A] > 0$.

$$\boxed{\mathbb{E}[G \mid A] := \frac{\mathbb{E}[G \cdot \mathbb{I}_A]}{\mathbb{P}[A]}}$$

In particular,

$$e_X(d) = \mathbb{E}[\overbrace{X-d}^{G} \mid \overbrace{X>d}^{A}] = \frac{\mathbb{E}[(X-d)\cdot \mathbb{I}_{[X>d]}]}{\mathbb{P}[X>d]}$$

It is frequently convenient to write:

$$\boxed{e_X(d) = \frac{\int_d^{+\infty} S_X(x)\,dx}{S_X(d)}} \quad \checkmark$$

Example. Let $X \sim \text{Exponential}(\theta)$.
Let $d > 0$.

Note: $\mathbb{P}[X>d] = S_X(d) = e^{-\frac{d}{\theta}} > 0$

Define $\boxed{Y^P = X - d \mid X > d}$

$$e_X(d) = \boxed{\frac{\int_d^{+\infty} e^{-\frac{x}{\theta}}\,dx}{e^{-\frac{d}{\theta}}}} = e^{\frac{d}{\theta}} \cdot (-\theta)\, e^{-\frac{x}{\theta}} \Big|_{x=d}^{+\infty}$$

$$= \cancel{e^{\frac{d}{\theta}}} \cdot (+\theta) \cdot \left(0 + \cancel{e^{-\frac{d}{\theta}}}\right) = \underline{\theta}$$

Task: Figure out the distribution of Y^P in this example. :)