M378K Introduction to Mathematical Statistics
Homework assignment #6

Please, provide your **final answer only** to the following problems.

Problem 6.1. *($3 \times 7 = 21$ points) Identify the distributions with the following mgfs:*

- $\frac{2}{2-t}$.
- e^{2e^t-2},
- $e^{t(t-2)}$,
- $(3-2e^t)^{-1}$
- $\frac{1}{9} + \frac{4}{9}e^t + \frac{4}{9}e^{2t}$.
- $\frac{1}{t}(1-e^{-t})$.
- $\frac{1}{4}(e^{4t}+3e^{-t})$

If the distribution has a name, give the name and the parameters. If it does not, give the pdf or the pmf (table).

Solution: All but the last one are named distribution and can be found in the table/notes:

1. $E(1/2)$.
2. $P(2)$,
3. Since $t(t-2) = (-2)t + \frac{1}{2}(\sqrt{2})^2t^2$, this is $N(-2, \sqrt{2})$.
4. Since $\frac{1}{3-2e^t} = \frac{\frac{1}{3}}{1-\frac{2}{3}e^t}$, this is $g(\frac{1}{3})$.
5. $B(2, \frac{2}{3})$.
6. $U(-1, 0)$
7. The last one corresponds to the discrete distribution with support $\{-1, 4\}$ with the pmf $p(-1) = \frac{3}{4}, p(4) = \frac{1}{4}$.

Please, provide your **complete solutions** to the following problems. Final answers only, even if correct will earn zero points for those problems.

Problem 6.2. *(30 points) Solve* **Problem 7.6.11** *from the* Lecture notes.

Solution:

1. $\Gamma(1) = \int_0^\infty e^{-y}\, dy = 1$. Using the fact that $\lim_{y\to 0} ye^{-y} = 0$ and $\lim_{y\to\infty} ye^{-y} = 0$, for $n = 2$, we have

$$\Gamma(2) = \int_0^\infty ye^{-y}\, dy = (-ye^{-y})|_0^\infty + \int_0^\infty e^{-y}\, dy = (0-0) + \int_0^\infty e^{-y}\, dy = \Gamma(1) = 1.$$

Similarly,

$$\Gamma(3) = \int_0^\infty y^2 e^{-y}\, dy = (-y^2 e^{-y})|_0^\infty + \int_0^\infty 2ye^{-y}\, dy = (0-0) + 2\int_0^\infty ye^{-y}\, dy = 2\Gamma(2) = 2,$$
$$\Gamma(4) = \int_0^\infty y^3 e^{-y}\, dy = (-y^3 e^{-y})|_0^\infty + \int_0^\infty 3y^2 e^{-y}\, dy = (0-0) + 3\int_0^\infty y^2 e^{-y}\, dy = 3\Gamma(3) = 6.$$

In general,

$$\begin{aligned}\Gamma(n+1) &= \int_0^\infty y^n e^{-y}\, dy = (-y^n e^{-y})|_0^\infty + \int_0^\infty ny^{n-1} e^{-y}\, dy = (0-0) + n\int_0^\infty y^{n-1} e^{-y}\, dy \\ &= n\Gamma(n).\end{aligned}$$

From here it follows immediately that $\Gamma(n) = (n-1)!$.

2. $c = 1/d$, where $d = \int_0^\infty y^{k-1} e^{-y/\tau}\, dy$. We change the variables in the integral by setting $z = y/\tau$, so that $dy = \tau\, dz$ and

$$d = \int_0^\infty \tau^{k-1} z^{k-1} e^{-z} \tau\, dz = \tau^k \int_0^\infty z^{k-1} e^{-z}\, dz = \tau^k \Gamma(k).$$

3. For $t < 1/\tau$, we have

$$\begin{aligned} m(t) &= \int_0^\infty e^{ty} \frac{1}{\tau^k \Gamma(k)} y^{k-1} e^{-y/\tau}\, dy = \ [z = (1/\tau - t)y] \\ &= \frac{1}{\tau^k \Gamma(k)} \int_0^\infty e^{-z} z^{k-1} (1/\tau - t)^{-k}\, dz \\ &= (1-\tau t)^{-k} \frac{1}{\Gamma(k)} \int_0^\infty e^{-z} z^{k-1}\, dz = (1-\tau t)^{-k}, \end{aligned}$$

so, yes, our guess was correct:

$$f_Y(y) = \frac{1}{\tau^k \Gamma(k)} y^{k-1} e^{-y/\tau} 1_{\{y>0\}}$$

is indeed the pdf of the gamma distribution with parameters k and τ.

Problem 6.3. *(18 points)* Source: "Mathematical Statistics with Applications" by Wackerly, Mendenhal, Scheaffer.
Suppose that a random variable Y has a probability density function given by

$$f_Y(y) = \kappa y^3 e^{-y/2} \mathbf{1}_{(0,\infty)}(y)$$

(i) (5 points) Find the value of κ that makes $f_Y(y)$ a density function.

(ii) (3 points) Does Y have a χ^2-distribution? If so, how many degrees of freedom?

(iii) (5 points) What are the mean and standard deviation of Y?

(iv) (5 points) (Extra credit) Using **R***, find the probability that Y lies within 2 standard deviations of its mean?*

Solution:

(i) Using the findings of the previous problem, we first realise that Y is Gamma distributed with $k = 4$ and $\tau = 2$ in our usual parameterization. Then, we conclude that

$$\kappa^{-1} = \tau^k \Gamma(k) = 2^4 \Gamma(4) = 2^4 \cdot 3! = 96.$$

So, $\kappa = 1/96$.

(ii) Since $\tau = 2$, this is a χ^2-distribution with $2\kappa = 8$ degrees of freedom.

(iii) The mean is $\mathbb{E}[Y] = k\tau = 8$ and the standard deviation is $SD[Y] = \tau\sqrt{k} = 4$.

(iv) We are looking for the probability

$$\mathbb{P}[0 \leq Y \leq 16].$$

In **R**, we can use ‘pchisq(16, df=8)‘ to get 0.9576199.