M378K Introduction to Mathematical Statistics
Problem Set #10
The Normal Distribution.

Definition 10.1. *The* moment-generating function (mgf) m_Y *for a random variable* Y *is defined as*

$$m_Y(t) = \mathbb{E}[e^{tY}]$$

for all t *for which the above expectation exists. In fact, we say that the moment-generating function* **exists** *there exists a positive number* b *such that* $m_Y(t)$ *is finite for all* t *such that* $|t| \leq b$.

Proposition 10.2.

1. *If* m_Y *exists for a certain probability distribution, then it is unique.*
2. *If* m_{Y_1} *and* m_{Y_2} *are equal on an interval, then* $Y_1 \stackrel{(d)}{=} Y_2$.

Corollary 10.3. *Let* Y_1 *and* Y_2 *be independent and normally distributed. Define* $Y = Y_1 + Y_2$. *Then, the distribution of* X *is . . .*

Proof. Note that $Y_i \sim N(\mu = mu_i, \sigma_i)$ for $i = 1, 2$. Now, let's look at the mgf of Y. Then, since Y_1 and Y_2 are independent, we have

$$m_Y(t) = m_{Y_1}(t)m_{Y_2}(t).$$

We can now use the fact that for any $X \sim N(\mu, \sigma)$,

$$m_X(t) = e^{\mu t}m_Z(\sigma t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

Hence,

$$m_Y(t) = e^{\mu_1 t + \frac{\sigma_1^2 t^2}{2}} \times e^{\mu_2 t + \frac{\sigma_2^2 t^2}{2}} = e^{(\mu_1 + \mu_2)t + \frac{(\sigma_1^2 + \sigma_2^2)t^2}{2}}$$

We can conclude that $Y \sim N(\mu_1 + \mu_2, \sqrt{\sigma_1^2 + \sigma_2^2})$. $\square$

Problem 10.1. *Two scales are used to measure the mass m of a precious stone. The first scale makes an error in measurement which we model by a normally distributed random variable X_1 with mean $\mu_1 = 0$ and standard deviation $\sigma_1 = 0.04m$. The second scale is more accurate. We model its error by a normal random variable X_2 with mean $\mu_2 = 0$ and standard deviation $\sigma_2 = 0.03m$.*

We assume that the measurements made using the two different scales are independent, i.e., that the random variables X_1 and X_2 are independent.

To get our final estimate of the mass of the stone, we take the average of the two results from the two different scales, i.e., we define $Y = \frac{X_1+X_2}{2}$.

(i) What is the distribution of the random variable Y? State the **name** *of its distribution and the* **values** *of the parameters.*

(ii) What is the probability that the error Y we get is within $0.005m$ of the actual mass of the stone? Namely, calculate

$$\mathbb{P}[|Y| < 0.005m].$$

Solution: Let us denote the random variable modeling the error from the first scale by $X_1 \sim N(0, \sigma_1^2)$ and the random variable modeling the error from the second scale by $X_2 \sim N(0, \sigma_2^2)$.

Then, if Y denotes the average of the two measurements, we have that

$$Y = \frac{1}{2}(X_1 + X_2) \sim N(0, \frac{1}{4}(\sigma_1^2 + \sigma_2^2)),$$

i.e.,

$$Y \sim N(0, \sigma^2)$$

with

$$\sigma^2 = \frac{1}{4}(\sigma_1^2+\sigma_2^2) = \frac{1}{4}(0.04^2m^2+0.03^2m^2) = \frac{1}{4}\cdot 0.01^2m^2(4^2+3^2) = \frac{1}{4}\,0.05^2m^2 = \left(\frac{0.05m}{2}\right)^2.$$

The probability we are looking for can be expressed as

$$\begin{aligned}\mathbb{P}[Y \in (-0.005m, 0.005m)] &= \mathbb{P}[-0.005m < Y < 0.005m]\\ &= \mathbb{P}[-\frac{2\cdot 0.005m}{0.05m} < \frac{Y}{\sigma} < \frac{2\cdot 0.005m}{0.05m}]\\ &= \mathbb{P}[-0.2 < \frac{Y}{\sigma} < 0.2].\end{aligned}$$

Since $\frac{Y}{\sigma} \sim N(0,1)$, the above probability equals

$$2\Phi(0.2) - 1 \approx 2\cdot 0.57926 - 1 \approx 0.16.$$

Corollary 10.4. *Let $Y_1, \ldots, Y_n$ be independent and identically distributed. Assume that $Y_1 \sim N(\mu, \sigma)$. Define*

$$S = Y_1 + Y_2 + \cdots + Y_n$$

Then, the distribution of S is …

Proof. **Solution:** Using the same reasoning as above, we have that

$$m_S(t) = \prod_{i=1}^{n} m_{Y_i}(t) = \prod_{i=1}^{n} e^{\mu t + \frac{\sigma^2 t^2}{2}} = e^{n\mu t + \frac{n\sigma^2 t^2}{2}}$$

We can conclude that $S \sim N(n\mu, \sqrt{n}\sigma)$. □