M378K: September 30th, 2024.

F Distribution.

Let Y_1 and Y_2 be two **independent** χ^2-distributed r.v.s. w/ $df = 1$

For both Y_1 and Y_2, the pdf is

$$\boxed{f_Y(y) = \frac{1}{\sqrt{2\pi y}} e^{-\frac{y}{2}} \mathbb{1}_{(0,+\infty)}(y)} \leftarrow$$

Define $W = \frac{Y_2}{Y_1}$, i.e., $W = g(Y_1, Y_2)$ w/ $\boxed{g(y_1, y_2) = \frac{y_2}{y_1}}$

Goal: Density of W : f_W !

Start by figuring out the cdf F_W.

$w > 0$: $F_W(w) = \mathbb{P}[W \leq w] = \mathbb{P}\left[\frac{Y_2}{Y_1} \leq w\right] = \mathbb{P}[Y_2 \leq w \cdot Y_1]$

$$= \int_0^{\infty} \int_0^{w \cdot y_1} \boxed{f_{Y_1,Y_2}(y_1, y_2)}\, dy_2\, dy_1$$

$$= \int_0^{\infty} \int_0^{w \cdot y_1} \frac{1}{\sqrt{2\pi y_1}} e^{-\frac{y_1}{2}} \cdot \frac{1}{\sqrt{2\pi y_2}} e^{-\frac{y_2}{2}}\, dy_2\, dy_1$$

$$= \int_0^{\infty} \frac{1}{\sqrt{2\pi y_1}} e^{-\frac{y_1}{2}} \boxed{\int_0^{w \cdot y_1} \frac{1}{\sqrt{2\pi y_2}} e^{-\frac{y_2}{2}}\, dy_2}\, dy_1$$

$F_{Y_2}(w \cdot y_1)$

$$F_W(w) = \int_0^{\infty} \frac{1}{\sqrt{2\pi y_1}} e^{-\frac{y_1}{2}} \boxed{F_{Y_2}(w \cdot y_1)}\, dy_1$$

$f_W(w)$
$\shortparallel$
$F_W'(w) = \frac{d}{dw} F_W(w)$

$$\longrightarrow f_W(w) = F_W'(w) = \int_0^{\infty} \frac{1}{\sqrt{2\pi y_1}} e^{-\frac{y_1}{2}} \boxed{f_Y(w \cdot y_1)} \cdot y_1\, dy_1$$

$$f_W(w) = \int_0^\infty \frac{1}{\sqrt{2\pi \cancel{y_1}}} e^{-\frac{y_1}{2}} \cdot \frac{1}{\sqrt{2\pi w \cdot \cancel{y_1}}} e^{-\frac{w \cdot y_1}{2}} \cdot \cancel{y_1}\, dy_1$$

$$= \frac{1}{2\pi\sqrt{w}} \boxed{\int_0^\infty e^{-\frac{1}{2} y_1 (1+w)}\, dy_1} =$$

$$= \frac{1}{2\pi\sqrt{w}} \left(-\frac{2}{1+w} e^{-\frac{1}{2} y_1 (1+w)} \right) \Bigg|_{y_1=0}^{\infty}$$

$$= \frac{1}{\cancel{2}\pi\sqrt{w}} \cdot \frac{\cancel{2}}{1+w} = \frac{1}{\pi\sqrt{w}(1+w)}$$

is the density $F(1,1)$, i.e.,

the F distribution w/ 1 numerator degree of freedom and 1 denominator degree of freedom

$\square$

M378K Introduction to Mathematical Statistics
Problem Set #7
Moment generating functions.

Definition 7.1. *The* k^{th} moment of a random variable Y taken about the origin *is defined as* $\mathbb{E}[Y^k]$ *provided that the expectation exists. We write*

$$\mu_k = \mathbb{E}[Y^k]$$

when there is no ambiguity about the random variable in question.

Remark 7.2. μ_k *is also referred to as the* k^{th} raw moment.

Remark 7.3. *In particular,* $\mu_1 = \mu$ *happens to be the* **mean** *of the random variable* Y.

Definition 7.4. *The* k^{th} central moment *of a random variable* Y *is defined as* $\mathbb{E}[(Y-\mu)^k]$ *provided that the expectation exists. We write*

$$\mu_k^c = \mathbb{E}[(Y-\mu)^k]$$

when there is no ambiguity about the random variable in question.

Remark 7.5. μ_k^c *is also referred to as the* k^{th} moment of a random variable Y taken about its mean.

Definition 7.6. *The* moment-generating function (mgf) m_Y *for a random variable* Y *is defined as*

$$m_Y(t) = \mathbb{E}[e^{tY}]$$

$\mathbb{E}[e^{t \cdot Y}]$

for all t *for which the above expectation exists. In fact, we say that the moment-generating function* **exists** *if there exists a positive number* b *such that* $m_Y(t)$ *is finite for all* t *such that* $|t| \leq b$.

Problem 7.1. *How much is* $m_Y(0)$?

$$m_Y(0) = \mathbb{E}[e^{0 \cdot Y}] = 1$$

Remark 7.7. *On the choice of terminology ...*

Step 1.

$$\boxed{\frac{d}{dt} m_Y(t)} = ?$$

$$\frac{d}{dt} m_Y(t) = \frac{d}{dt}\mathbb{E}[e^{t \cdot Y}] = \mathbb{E}\left[\frac{d}{dt} e^{tY}\right] =$$

$$= \underline{\mathbb{E}[Y e^{tY}]}$$

Step 2.

$$m'_Y(0) = ?$$

$$m'_Y(0) = \mathbb{E}[Ye^{0\cdot Y}] = \mathbb{E}[Y] = \mu_Y$$

Step 3.

$$\frac{d^2}{dt^2} m_Y(t) = ?$$

$$\frac{d}{dt}\left(\frac{d}{dt} m_Y(t)\right) = \frac{d}{dt}\mathbb{E}[Ye^{tY}] = \mathbb{E}[Y^2 e^{tY}]$$

Step 4.

$$m''_Y(0) = ?$$

$$m''_Y(0) = \mathbb{E}[Y^2]$$

Step 5. *What do you suspect the* **generalization** *of the above would be?*

Theorem 7.8. *If m_Y exists, then for $k \in \mathbb{N}$, we have*

$$\boxed{m_Y^{(k)}(0) = \mu_k.}$$

Example 7.9. *Let $Y \sim b(n = 1, p)$, i.e., let Y model a Bernoulli trial with the probability of success denoted by p. Find m_Y.*

$$: m_Y(t) = \mathbb{E}[e^{tY}] = e^{t \cdot 0} \cdot (1-p) + e^{t \cdot 1} \cdot p$$
$$= 1-p + pe^t \quad t \in \mathbb{R}$$

Proposition 7.10. *Let Y_1 and Y_2 be independent random variables with m.g.f.s denoted by m_{Y_1} and m_{Y_2}. Define $Y = Y_1 + Y_2$. Then, for every t for which both m_{Y_1} and m_{Y_2} are well defined, we have*

$$m_Y(t) = ?$$

Proof. By definition:

$$m_Y(t) = \mathbb{E}[e^{tY}]$$

Using $Y = Y_1 + Y_2$, we can subsitute $Y_1 + Y_2$ for Y in the expression above. So,

$$m_Y(t) = \mathbb{E}[e^{t(Y_1+Y_2)}]$$

One of the properties of the exponential function is that $e^{A+B} = e^A \times e^B$. Thus, the above becomes:

$$m_Y(t) = \mathbb{E}[e^{tY_1} \cdot e^{tY_2}]$$

Recall that Y_1 and Y_2 are assumed to be independent random variables. With this in mind, we get:

$$m_Y(t) = \boxed{\mathbb{E}[e^{tY_1}]} \cdot \boxed{\mathbb{E}[e^{tY_2}]}$$

Finally, using the definition of a m.g.f., we have

$$\boxed{m_Y(t) = m_{Y_1}(t) \cdot m_{Y_2}(t)}$$

□

Example 7.11. *Let $Y \sim b(n, p)$. What is the moment generating function of Y?*

Example 7.12. *Let $N \sim Poisson(\lambda)$. What is the moment generating function m_N of N?*

Example 7.13. *Let $Z \sim N(0, 1)$. What is the moment generating function m_Z of Z?*