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M339D=M389D Introduction to Actuarial Financial Mathematics
University of Texas at Austin
In-Term Exam II
Instructor: Milica Čudina

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The maximum number of points on this part of the exam is 65.

Provide your **complete solution** to the following problems. Final answers only, without appropriate justification, will receive zero points even if correct.

2.1. **FREE-RESPONSE PROBLEMS.**

Problem 2.1. (20 points) Consider a non-dividend-paying stock whose current price is $80 per share. The stock has the volatility equal to 0.25.

Let the continuously-compounded, risk-free interest rate be equal to 0.04.

You model the evolution of this stock over the next nine months with a three-period **forward** binomial tree.

What is the price of a $75-strike, nine-month call on the above stock consistent with this model?

Solution: In our usual notation, in the forward binomial tree, the risk-neutral probability is

$$p^* = \frac{1}{1 + e^{\sigma\sqrt{h}}} = \frac{1}{1 + e^{0.25\sqrt{1/4}}} = \frac{1}{1 + e^{0.125}} = 0.4687906.$$

The *up* and *down* factors are

$$u = e^{rh + \sigma\sqrt{h}} = e^{0.04(1/4) + 0.125} = e^{0.135} = 1.144537,$$

$$d = e^{rh - \sigma\sqrt{h}} = e^{0.04(1/4) - 0.125} = e^{-0.115} = 0.8913661.$$

Hence, the possible stock prices at the end of the three periods are

$$S_{uuu} = S(0)u^3 = 119.9442, \quad S_{uud} = S(0)u^2d = 93.41264,$$

$$S_{udd} = S(0)ud^2 = 72.74983, \quad S_{ddd} = S(0)d^3 = 56.65763.$$

So, the call option is in the money only in the two top nodes where the payoff equals

$$V_{uuu} = (S_{uuu} - K)_+ = 44.9442 \quad \text{and} \quad V_{uud} = (S_{uud} - K)_+ = 18.41264.$$

By the risk-neutral pricing formula, we have that

$$V_P(0) = e^{-0.04(3/4)} \left[44.9442(0.4687906)^3 + 18.41264(3)(0.4687906)^2(1 - 0.4687906) \right] = 10.75142.$$

Problem 2.2. (10 points) A squirrel is throwing hazelnuts at a picnicking family repeatedly. Assume that her attempts are independent and that the probability of hitting a member of the picnicking family in any single attempt equals 0.64. The total number of hazelnuts the squirrel is willing to waste in this exercise is 100. Using the *normal approximation to the binomial*, what is the approximate probability that the squirrel hits a family member at least 60 times?

Solution: *The following solution uses the continuity correction. The solutions without the continuity correction would also be graded as correct if otherwise accurate.*
The number of trials (throws) is 100. The probability of hitting a human in a single throw is 0.64. So, the total number of hits is, in our usual notation,

$$X \sim Binomial(n = 100, p = 0.64).$$

The probability we are looking for is $\mathbb{P}[X \geq 60]$. The mean of the random variable X is $np = 64$ and its standard deviation is $\sqrt{np(1-p)} = 4.8$. Using the normal approximation to the binomial, we get

$$\mathbb{P}[X \geq 60] = \mathbb{P}[X > 59.5] = \mathbb{P}\left[\frac{X - 64}{4.8} > \frac{59.5 - 64}{4.8}\right] = 1 - \Phi(-0.9375) \approx 0.8257493.$$

Using the standard normal tables, the final answer is

$$1 - \Phi(-0.94) = 0.8264.$$

Problem 2.3. (20 points) Consider a non-dividend-paying stock whose price $\mathbf{S} = \{S(t), t \geq 0\}$ is modeled using the Black-Scholes framework. Suppose that the current stock price equals $80 and that its volatility is given to be 0.20.

The continuously compounded, risk-free interest rate is assumed to equal 0.04.

Consider a derivative security which entitles its owner to obtain a European call option on the above stock six months from today, i.e., at time $t_* = 1/2$. The call option is to be $3-$month to expiration at time of delivery and be at-the-money at time$-t^*$. This contract is an example of a **forward start option**.

What is the price of this forward start option?

Solution: At time $t*$, the required Black-Scholes price of the call option equals
$$V_C(t^*) = S(t^*)N(d_1) - S(t^*)e^{-r(T-t^*)}N(d_2)$$
$$= S(t^*)(N(d_1) - e^{-0.01}N(d_2))$$

with

$$d_1 = \frac{1}{0.1}\left(0.04 + \frac{0.2^2}{2}\right) \times \frac{1}{4} = 0.15,$$

$$d_2 = d_1 - \sigma\sqrt{T - t^*} = 0.15 - 0.1 = 0.05.$$

So, $N(d_1) = 0.5596177$ and $N(d_2) = 0.5199388$. Hence,

$$V_C(t^*) = S(t^*)(0.5596177 - e^{-0.01} \times 0.5199388) = 0.04485238 S(t^*).$$

So, one would need to buy exactly 0.04485238 shares of stock to be able to buy the call option in question at time$-t^*$. This amount of shares costs $3.58819.

Problem 2.4. (15 points) Assume the Black-Scholes model. Consider a non-dividend-paying stock with the initial stock price of $120 and volatility equal to 0.25.

The continuously-compounded, risk-free interest rate is 0.10.

Answer the following questions under the risk-neutral probability measure $\mathbb{P}^*$.

(i) (3 points) What is the mean time-1 stock price?

(ii) (5 points) At what time t^* is the median stock price equal to 200?

(iii) (7 points) What is the probability that the time-1 stock price exceeds today's stock price?

Solution:

(i) $\mathbb{E}^*[S(1)] = 120e^{0.10} = 132.6205$

(ii) We need to solve for t^* in

$$120 \exp\left((0.10 - \tfrac{0.25^2}{2})t^*\right) = 200 \quad \Rightarrow \quad t^* = \frac{\ln\left(\frac{200}{120}\right)}{0.10 - \frac{0.25^2}{2}} = 7.430191.$$

(iii) This probability is precisely the probability that the return $R(0,1)$ is positive. We know that

$$R(0,1) \sim Normal\left(mean = 0.10 - \tfrac{0.25^2}{2} = 0.06875, sd = 0.25\right).$$

So,

$$\mathbb{P}^*[R(0,1) > 0] = \mathbb{P}^*\left[\frac{R(0,1) - 0.06875}{0.25} > \frac{0 - 0.06875}{0.25} = -0.275\right] = \Phi(0.275) = 0.6083419.$$

Using the standard normal tables, we get $\Phi(0.28) = 0.6103$.