M339J: January 29th, 2021.

Review.

Def'n. A random variable X is continuous if its cdf is
- continuous everywhere

and
- differentiable everywhere w/ the exception of at most countably many points.

$\Rightarrow$ We can differentiate F_X almost everywhere.

Def'n. The probability density function (pdf) of a continuous r.v. X is the function given by

$$f_X(x) = F_X'(x) \quad \text{wherever the derivative exists}$$

$\Rightarrow$ You can set f_X to be @ arbitrary values elsewhere.

Exponential Dist'n.

Any exponential r.v. X w/ parameter θ has the pdf

$$f_X(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}} \quad \text{for } x > 0$$

(and 0 otherwise)

In some sources: $f_X(x) = \lambda \cdot e^{-\lambda \cdot x}$, $x > 0$

Q: What's the support of X? All positive reals, i.e., $\mathbb{R}_+ = (0, +\infty)$

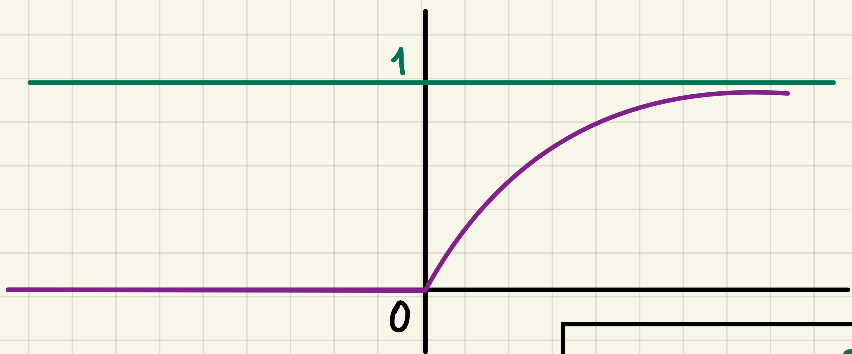
Its cumulative dist'n f'tion:

$$F_X(x) = \mathbb{P}[X \le x] = \int_0^x f_X(u)\,du$$

$$= \int_0^x \frac{1}{\theta} e^{-\frac{u}{\theta}}\,du =$$

$$= \frac{1}{\theta}\int_0^x e^{-\frac{u}{\theta}}\,du = \frac{1}{\cancel{\theta}}(-\cancel{\theta})\left[e^{-\frac{u}{\theta}}\right]_{u=0}^{x}$$

$$= -\left(e^{-\frac{x}{\theta}} - 1\right) = 1 - e^{-\frac{x}{\theta}}$$



Its survival f'tion is $\boxed{S_X(x) = e^{-x/\theta}}$

Example. Given that $X \sim$ Exponential (θ) is bigger than a, what is the probability that it's bigger than $a+b$ $(a>0, b>0)$?

→:
$$\mathbb{P}[X > a+b \mid X > a] =$$

$$= \frac{\mathbb{P}[X > a+b, X > a]}{\mathbb{P}[X > a]} =$$

$$= \frac{\mathbb{P}[X > a+b]}{\mathbb{P}[X > a]} = \frac{S_X(a+b)}{S_X(a)} =$$

$$= \frac{e^{-\frac{a+b}{\theta}}}{e^{-\frac{a}{\theta}}} = e^{-\frac{b}{\theta}} = S_X(b) = \mathbb{P}[X > b]$$

The Memoryless Property

Problem. The lifetime T of a printer modeled as exponential w/ parameter $\theta = 2$. The original price of the printer is 200. The manufacturer agrees to provide a full refund if the printer fails within a year of purchase. If it fails during the second year, the manufacturer refunds half the original price. If it fails afterwards, there's no refund. What's the expected refund per printer?

→ : Q: What are the possible values of the refund?

- $\text{Support}(X) = \{0, 100, 200\}$

  X ↑ refund amount

$$\mathbb{E}[X] = \cancel{0 \cdot \mathbb{P}[X=0]} + 100 \cdot \mathbb{P}[X=100] + 200 \cdot \mathbb{P}[X=200]$$

$$\mathbb{P}[X=200] = \mathbb{P}[T \leq 1] = 1 - e^{-\frac{1}{2}} = 0.3935$$

$$\mathbb{P}[X=100] = \mathbb{P}[1 < T \leq 2] = F_T(2) - F_T(1) = e^{-\frac{1}{2}} - e^{-\frac{2}{2}} = 0.23865.$$

$$\mathbb{E}[X] = 100 \cdot 0.23865 + 200 \cdot 0.3935 = 102.56 \quad \blacksquare$$

Review: Mean and Variance