M339D: October 16th, 2023.

Risk-Neutral Pricing.

Start w/

$$V(0) = \Delta \cdot S(0) + B$$

$$V(0) = \frac{V_u - V_d}{\underbrace{S_u - S_d}_{\cancel{S(0)}(u-d)}} \cdot \cancel{S(0)} + \boxed{e^{-rh} \cdot \frac{u \cdot V_d - d \cdot V_u}{u-d}} \longleftarrow B$$

$$V(0) = \frac{1}{u-d}\left[V_u - V_d + e^{-rh}(u \cdot V_d - d \cdot V_u)\right]$$

$$V(0) = e^{-rh} \cdot \frac{1}{u-d}\left[e^{rh} \cdot V_u - e^{rh} \cdot V_d + u \cdot V_d - d \cdot V_u\right]$$

$$V(0) = e^{-rh} \cdot \frac{1}{u-d}\left[(e^{rh} - d) \cdot V_u + (u - e^{rh}) \cdot V_d\right]$$

$$V(0) = e^{-rh}\left[\boxed{\frac{e^{rh} - d}{u-d}} \cdot V_u + \boxed{\frac{u - e^{rh}}{u-d}} \cdot V_d\right]$$

Both positive (Due to the no-arbitrage condition!)

Add up to 1!

We choose to interpret the two quantities as probabilities!

We define the risk-neutral probability of the stock price going up in a single period as:

$$\boxed{p^* := \frac{e^{rh} - d}{u-d}}$$

$\Rightarrow$ The risk-neutral pricing formula

$$\boxed{V(0) = e^{-rT}\left[V_u \cdot p^* + V_d(1-p^*)\right]}$$

We can generalize this principle:

$$\boxed{V(0) = e^{-rT}\, \mathbb{E}^*[V(T)]}$$

$\boxed{8.5.}$ w/ the risk-neutral pricing approach:

$$p^* = \frac{e^{rh}-d}{u-d} = \frac{S(0)e^{rh}-S_d}{S_u-S_d} = \frac{95e^{0.06}-75}{120-75} = \underline{0.5749}$$

$$V(0) = e^{-rT}\left[V_u \cdot p^* + V_d \cdot (1-p^*)\right]$$

$$V(0) = e^{-0.06}\left[20 \cdot p^* + 25\,(1-p^*)\right] = \underline{20.837}$$

□

More about the Forward Binomial Tree.

σ... volatility

✓ $u := e^{rh+\sigma\sqrt{h}}$

✓ $d := e^{rh-\sigma\sqrt{h}}$

The risk-neutral probability in this special case.

$$p^* = \frac{e^{rh}-d}{u-d} = \frac{e^{rh}-e^{rh-\sigma\sqrt{h}}}{e^{rh+\sigma\sqrt{h}}-e^{rh-\sigma\sqrt{h}}}$$

$$p^* = \frac{\cancel{e^{rh}}(1-e^{-\sigma\sqrt{h}})}{\cancel{e^{rh}}(e^{\sigma\sqrt{h}}-e^{-\sigma\sqrt{h}})} = \frac{1-e^{-\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}}-e^{-\sigma\sqrt{h}}}$$

$$p^* = \frac{\cancel{e^{\sigma\sqrt{h}}-1}}{\underbrace{e^{2\sigma\sqrt{h}}-1}_{(\cancel{e^{\sigma\sqrt{h}}-1})(e^{\sigma\sqrt{h}}+1)}}$$

$$\boxed{p^* = \frac{1}{1+e^{\sigma\sqrt{h}}}} \xrightarrow{h\to 0} \frac{1}{2}$$

The shortcut only for the FORWARD BINOMIAL TREE.

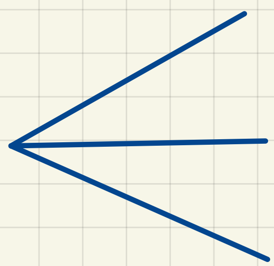
Inspiration.

One Period.

$$S(0) \nearrow S_u, \quad S(0) \searrow S_d$$

$\boxed{S(T)}$

Q: How can you make the model for $S(T)$ more complex?



k-nary trees ☹

Two Periods: $n=2$

$S_{uu} = u \cdot S_u = u^2 \cdot S(0)$

$S_u = u \cdot S(0)$

$S(0)$

$V(0) = ?$

$$\boxed{\begin{aligned} S_{ud} &= d \cdot S_u = d \cdot u \cdot S(0) \\ S_{du} &= u \cdot S_d = u \cdot d \cdot S(0) \end{aligned}}$$

$S_d = d \cdot S(0)$

$S_{dd} = d \cdot S_d = d^2 \cdot S(0)$

$\underbrace{\qquad}_{h}$ $\underbrace{\qquad}_{h}$

Payoff:

$V_{uu} := v(S_{uu})$

$V_{ud} := v(S_{ud})$

$V_{dd} := v(S_{dd})$

Possible number of values of $S(T)$ is $n+1=3$.

$0 \quad h = \frac{T}{2} \quad h = \frac{T}{2} \quad T$

populating the tree →

← pricing the option