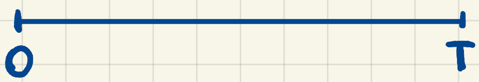
M339D: February 24th, 2023.

Replicating Portfolio.

Def'n. Consider a European-style derivative security.
A static portfolio w/ the same payoff as that of the derivative security is called its replicating portfolio.

Note: The initial price of the derivative security must be equal to the initial price of its replicating portfolio.

Example. Consider a $\boxed{\text{forward contract}}$ on a non-dividend-paying stock.



Forward contract: $S(T) - F$

Replicating Portfolio:
- long 1 share of stock
- issue a bond w/ redemption amt F and maturity date T

$$\text{Payoff(Portfolio)} = \underbrace{S(T)}_{\text{long stock}} - \underbrace{F}_{\text{short bond}}$$

=> The forward contract and its replicating portfolio have the same initial cost, i.e.,

$$\underbrace{S(0)}_{\text{long stock}} - \underbrace{PV_{0,T}(F)}_{\text{short bond}} = 0$$

$$\Rightarrow \quad PV_{0,T}(F) = S(0)$$

$$\Rightarrow \quad \boxed{F = S(0)e^{rT}}$$

3. An insurance company sells single premium deferred annuity contracts with return linked to a stock index, the time-t value of one unit of which is denoted by $S(t)$. The contracts offer a minimum guarantee return rate of g%. At time 0, a single premium of amount π is paid by the policyholder, and $\pi \times y$% is deducted by the insurance company. Thus, at the contract maturity date, T, the insurance company will pay the policyholder

$$\pi \times (1 - y\%) \times \text{Max}[S(T)/S(0), (1 + g\%)^T].$$

You are given the following information:

(i) The contract will mature in one year. $T=1$ ✓

(ii) The minimum guarantee rate of return, g%, is 3%. $g=0.03$ ✓

(iii) Dividends are incorporated in the stock index. That is, the stock index is constructed with all stock dividends reinvested. no dividends

(iv) $\boxed{S(0) = 100.}$ ✓

(v) The price of a one-year European put option, with strike price of \$103, on the stock index is \$15.21.

$V_p(0, T=1, K=103) = 15.21$

put payoff: $\text{Max}(K - S(T), 0)$

Determine y%, so that the insurance company does not make or lose money on this contract.

→ : The insurance company's liability:

$$\underbrace{\pi \cdot (1-y)}_{\text{constant}} \cdot \underbrace{\text{Max}\left(\frac{S(T)}{S(0)}, (1+g)^T\right)}$$

constant

$$= \frac{1}{S(0)} \underbrace{\text{Max}(S(T), \underbrace{S(0)(1+g)^T}_{\underbrace{100 \cdot (1.03)^1}_{103}})}_{\boxed{\text{Max}(S(T), 103)}}$$

(A) 12.8%.

(B) 13.0%

(C) 13.2%

(D) 13.4%

(E) 13.6%.

a, b

$\max(a,b) = a + \max(0, b-a) = a + (b-a)_+$

$= b + \max(a-b, 0) = b + (a-b)_+$

$$\mathrm{Max}(S(T), 103) = \boxed{S(T)} + \boxed{\mathrm{Max}(0, 103 - S(T))}$$

long stock index

The payoff of a put w/ strike 103 and exercise date @ time T.

The insurance company can perfectly hedge by:

- buying/longing $\frac{\pi(1-y)}{S(0)}$ shares of stock

and

- buying $\frac{\pi(1-y)}{S(0)}$ European puts w/ $K=103$ and $T=1$

The condition for the insurance company to break even:

> the amount they receive @ time 0 is EQUAL to the cost of the hedge.

$$\cancel{\pi} = \frac{\cancel{\pi}(1-y)}{S(0)}\left(S(0) + V_P(0, K=103, T=1)\right)$$

$$100 = (1-y)(100 + 15.21)$$

$$1-y = \frac{100}{115.21} \Rightarrow y = \frac{15.21}{115.21} = \underline{0.132}$$

$\square$