

M378K Introduction to Mathematical Statistics

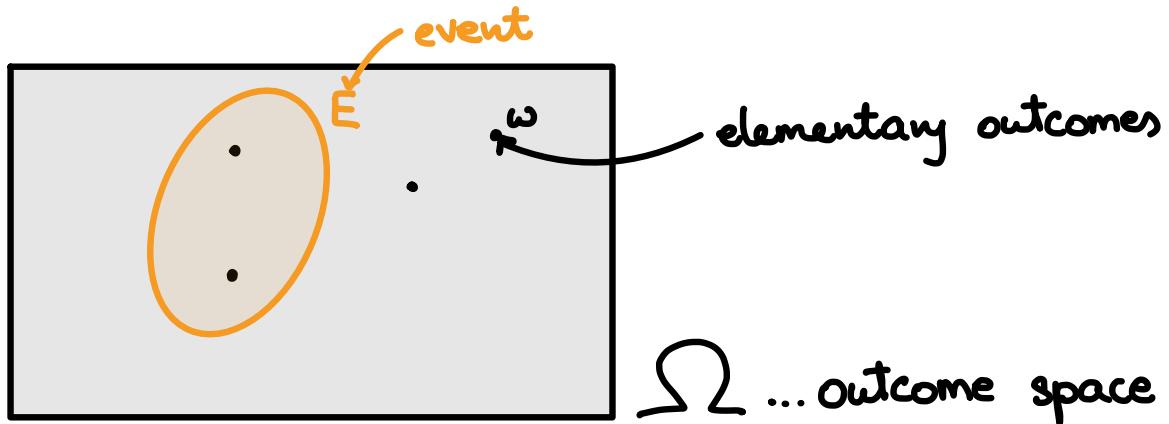
Problem Set #1

Probability spaces. Random variables.

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**1.1. Probability distributions.** Consider an **outcome space** (also known as a **sample space**)  $\Omega$ . In statistics, it's convenient to imagine it as containing your entire target population. The individual elements  $\omega \in \Omega$  are known in probability as **elementary outcomes**; in statistics, they can frequently (but not always!) be understood as individuals in your target population.

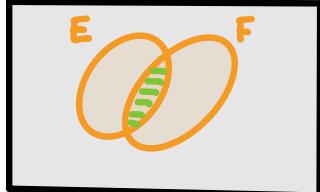
We are usually not interested that much in individual  $\omega$ , but want to consider **events**  $E$  in  $\Omega$ . In full mathematical generality, the set  $\Omega$  can be complicated so that the family of all of its subsets is also a complicated object. Due to mathematical difficulties, it is sometimes impossible to measure in any meaningful way absolutely all subsets of  $\Omega$ <sup>1</sup>. For the purposes of this course, we will not have to dwell on these matters. So, we will refer to any (nice) subset of  $\Omega$  as an **event**.



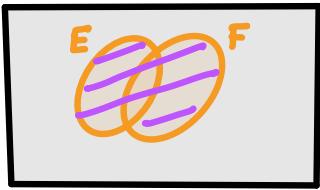
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<sup>1</sup>See [https://en.wikipedia.org/wiki/Banach-Tarski\\_paradox](https://en.wikipedia.org/wiki/Banach-Tarski_paradox)

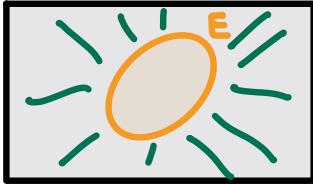
We will inherit all the notation and rules from set theory while adding some more vocabulary and context. Most notably, we will consider intersections, unions, and complements of events. These are best understood via Venn diagrams.



$E \cap F$ ... both happened



$E \cup F$ ... @ least one happened



$E^c$ ... did not happen

Moreover, in a probabilistic setting, we have the following definition:

**Definition 1.1.** Let  $E$  and  $F$  be two events on the same  $\Omega$  such that

$$E \cap F = \emptyset.$$

Then, we say that  $E$  and  $F$  are mutually exclusive (or disjoint).

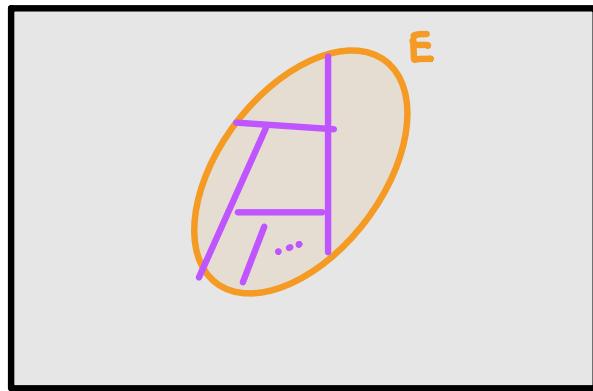


Now, we are ready for the following (crucial!) definition:

**Definition 1.2.** Consider a mapping  $\mathbb{P}$  from the set of all events on  $\Omega$  to  $\mathbb{R}$ . We say that  $\mathbb{P}$  is a probability (distribution, measure) if it satisfies the following three conditions:

- $\mathbb{P}[E] \geq 0$  for all events  $E$ ;
- $\mathbb{P}[\Omega] = 1$ ;
- for all pairwise disjoint sequences of events  $\{E_j : j = 1, 2, \dots\}$ , we have that

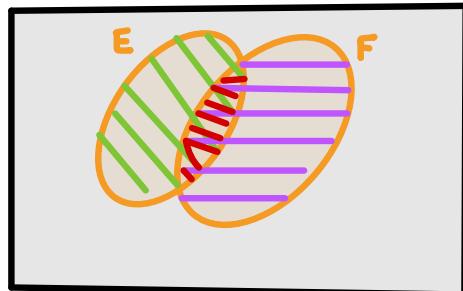
$$\mathbb{P}\left[\bigcup_{j=1}^{\infty} E_j\right] = \sum_{j=1}^{\infty} \mathbb{P}[E_j].$$



One immediately useful consequence is the **inclusion-exclusion formula**. This is its simplest version.

**Proposition 1.3.** Let  $E$  and  $F$  be two events on  $\Omega$ . Then,

$$\mathbb{P}[E \cup F] = \underline{\mathbb{P}[E]} + \underline{\mathbb{P}[F]} - \underline{\mathbb{P}[E \cap F]}.$$



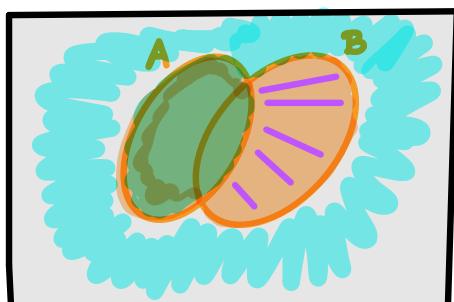
Of course, the above formula can be generalized to arbitrary unions of finitely many events. Try to figure it out!

**Problem 1.1.** Source: An old P exam problem.

For two events  $A$  and  $B$ , you are given that

$$\mathbb{P}[A \cup B] = 0.7 \quad \text{and} \quad \mathbb{P}[A \cup B^c] = 0.9$$

Calculate  $\mathbb{P}[A]$ .



$$\begin{aligned} \mathbb{P}[B \cap A^c] &= 1 - 0.9 = 0.1 \\ \mathbb{P}[A^c \cap B^c] &= \\ &= \mathbb{P}[(A \cup B)^c] = 1 - 0.7 = 0.3 \end{aligned} \quad ) +$$

$$\mathbb{P}[A^c] = 0.1 + 0.3 = 0.4$$

$$\mathbb{P}[A] = 1 - 0.4 = 0.6 \quad \square$$

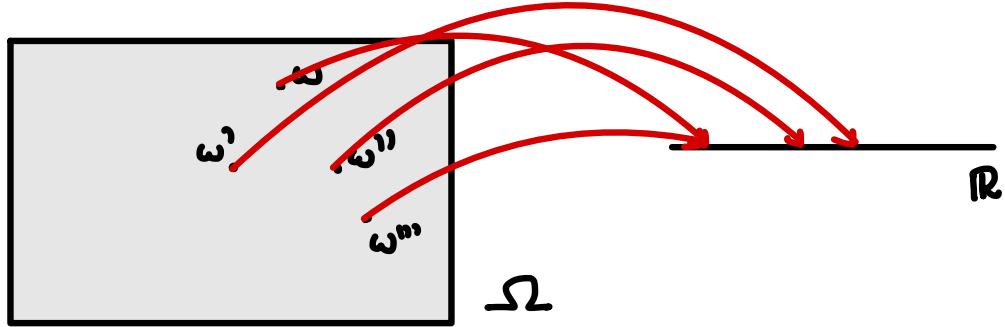
Alternatively:

$$\begin{aligned} \mathbb{P}[A \cup B] &= \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B] = 0.7 \\ \mathbb{P}[A \cup B^c] &= \mathbb{P}[A] + \mathbb{P}[B^c] - \mathbb{P}[A \cap B^c] = 0.9 \end{aligned} \quad ) +$$

$$2\mathbb{P}[A] + 1 - \mathbb{P}[A] = 1.6$$

$$\mathbb{P}[A] = 0.6 \quad \square$$

**1.2. Random variables.** Informally speaking, any "nice" mapping/function from  $\Omega$  to a target set  $S$  is a *random element*<sup>2</sup>. When  $S$  is  $\mathbb{R}$ , we like to use the term *random variable*. When  $S$  is  $\mathbb{R}^n$  for some  $n$ , we like to use the term *random vector*.



Let's consider a classroom of students as our  $\Omega$  and give examples of a

- random element

Weaning a jacket or not  $\begin{cases} \text{TRUE} \\ \text{FALSE} \end{cases}$   
 Emotions  
 Hairstyle  
 Major

- random variable

Weaning a jacket or not  $\begin{cases} 1 \\ 0 \end{cases}$

Height

- random vector

(Height, Weight)

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<sup>2</sup>In practice, people like to use the term *random variable* even in more general context when there is no source of confusion. We will habitually do this.

To keep track of what values a random variable is "allowed" to take, we use the following terminology<sup>3</sup>:

**Definition 1.4.** *Given a set  $B$ , we say that a random variable  $Y$  is  $B$ -valued if*

$$\mathbb{P}[Y \in B] = 1.$$

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<sup>3</sup>Read your lecture notes: [https://web.ma.utexas.edu/users/gordanz/notes/discrete\\_probability\\_color.pdf](https://web.ma.utexas.edu/users/gordanz/notes/discrete_probability_color.pdf)