M378K Introduction to Mathematical Statistics

Homework assignment #1

Please, provide your **final answer only** to the following problems.

Problem 1.1. *(4 points) Evaluate the limit* $\lim_{n\to\infty}\left(1-\frac{2}{n}\right)^n$.

Solution: e^{-2}.

Problem 1.2. *(2 points) Evaluate the limit* $\lim_{t\to\infty} e^{-t}$.

Solution: 0.

Problem 1.3. *(4 points) Find the sum* $\sum_{i=0}^{\infty}\frac{4^i}{i!}$.

Solution: This is the Maclaurin series for the function $f(x) = e^x$ at $x = 4$, so its sum evaluates to e^4.

Problem 1.4. *(5 points) A class has 7 female and 13 male students. It is also known that there are 15 blue-eyed and 5 brown-eyed students in that class. The probability that a student picked at random is a brown-eyed female is*

(a) $\frac{7}{80}$

(b) $\frac{13}{80}$

(c) $\frac{21}{80}$

(d) $\frac{39}{80}$

(e) Not enough information is given.

Solution: The correct answer is **(e)**.
We do not know how the eye color is distributed among male/female students, so we cannot compute the probability

$$\mathbb{P}[\{\text{ brown-eyed }\}\cap\{\text{ female }\}]$$

(*Note:* One thing we *do know* is that these two traits cannot be independent. If they were, $5/20 = 1/4$ of the female students would be brown-eyed, but that cannot be the case as there are 7 female students, and 7 is not divisible by 4.)

Problem 1.5. *(5 points) Let A and B be two events, and the only thing we know about them is that $\mathbb{P}[A] = \mathbb{P}[B] = \frac{2}{3}$. Then, it is* **necessarily** *true that*

(a) $A = B$

(b) $A \subseteq B$ *or* $B \subseteq A$

(c) A *and* B *are independent*

(d) A *and* B^c *are mutually exclusive*

(e) *All of the above are possible, but not necessarily true.*

Solution: The correct answer is **(e)**.

Problem 1.6. *(5 points) Which of the following formulas hold for the exponential function:*

(a) $e^x + e^y = e^{x+y}$

(b) $e^x e^y = e^x + e^y$

(c) $e^{x+y} = e^x e^y$

(d) $e^{x-y} = e^x - e^y$

(e) *None of the above.*

Solution: The correct answer is **(c)**.

Problem 1.7. *(5 points) A coin is tossed, and, independently, a 6-sided die is rolled. Let*

$$A = \{4 \textit{ is obtained on the die}\} \textit{ and}$$
$$B = \{\text{Heads} \textit{ is obtained on the coin and an even number is obtained on the die}\}.$$

Then

(a) A *and* B *are mutually exclusive*

(b) A *and* B *are independent*

(c) $A \subseteq B$

(d) $A \cap B = B$

(e) *None of the above.*

Solution: The correct answer is **(e)**.

Problem 1.8. *(5 points) If $n!$ is the factorial function $n! = n \times (n-1) \times \cdots \times 2 \times 1$, then* $\log(\sqrt[n]{n!})$ *equals ...*

(a) $\sum_{i=1}^n \log(n/i)$

(b) $\frac{1}{n}\sum_{i=1}^{n}\log(i)$

(c) $\sqrt[n]{\prod_{i=1}^{n}\log(n)}$

(d) $\frac{1}{n}\prod_{i=1}^{n}\log(i)$

(e) *None of the above.*

Solution: The correct answer is **(b)**.

$$\log(\sqrt[n]{n!}) = \frac{1}{n}\log(n!) = \frac{1}{n}\sum_{i=1}^{n}\log(i)$$

Please, provide your **complete solutions** to the following problems. Final answers only, even if correct will earn zero points for those problems.

Problem 1.9. (5 *points) Every possible combination of a letter in the English alphabet (i.e., chosen from the 26-element set* $\{A, B, C, \ldots, X, Y, Z\}$*) and a number from the set* $\{1, 2, \ldots, 19, 20\}$ *is written on a card. The cards are otherwise identical, and well shuffled in a deck. If a single card is drawn from that deck, what is the probability that the number on it is odd or that the letter is a vowel (i.e., in the set* $\{A, E, I, O, U\}$*)?*

Solution: By the inclusion-exclusion formula, we have

$$\frac{5}{26} + \frac{1}{2} - \frac{5}{26 \cdot 2} = \frac{10 + 26 - 5}{52} = \frac{31}{52}\,.$$

Problem 1.10. (5 *points) Four fair coins are tossed independently. What is the probability that at least one for them came up* heads*?*

Solution: The answer is

$$1 - \mathbb{P}[\text{all the coins were } \textit{tails}] = 1 - \left(\frac{1}{2}\right)^4 = \frac{15}{16}\,.$$

Problem 1.11. (5 *points) How much is*

$$\sum_{i=1}^{99}\log_{10}\left(\tfrac{i}{i+1}\right)$$

when simplified completely?

Solution: The sum of logs is the log of the product, so the expression above equals $\log_{10}(\prod_{i=1}^{99} \frac{i}{i+1})$. The product inside is

$$\frac{1}{2} \times \frac{2}{3} \times \cdots \times \frac{98}{99} \times \frac{99}{100} = \frac{1}{100},$$

and $\log_{10}(1/100) = -2$.