UNIVERSITY OF TEXAS AT AUSTIN

HW Assignment 1

Note: You **must** show all your work. Numerical answers without a proper explanation or a clearly written down path to the solution will be assigned zero points.

Problem 1.1. (5 points) Let $\Omega = \{a_1, a_2, a_3, a_4, a_5\}$ be an outcome space, and let $\mathbb{P}$ be a probability distribution on Ω. Assume that $\mathbb{P}[A] = 0.5$, $\mathbb{P}[B] = 0.4$, $\mathbb{P}[C] = 0.4$, and $\mathbb{P}[D] = 0.2$, where

$$A = \{a_1, a_2, a_3\},\ B = \{a_2, a_3, a_4\},$$
$$C = \{a_3, a_5\} \text{ and } D = \{a_4\}.$$

Are the events A and B independent?

Solution: We need to check whether $\mathbb{P}[A \cap B] = \mathbb{P}[A] \times \mathbb{P}[B]$. Since

$$\begin{aligned} \mathbb{P}[A \cap B] &= \mathbb{P}[\{a_2, a_3\}] \\ &= \mathbb{P}[\{a_2, a_3, a_4\}] - \mathbb{P}[\{a_4\}] \\ &= \mathbb{P}[B] - \mathbb{P}[D] = 0.2 \end{aligned}$$

Since $\mathbb{P}[A] \times \mathbb{P}[B] = 0.5 \times 0.4 = 0.2$, we conclude that A and B *are* independent.

Problem 1.2. (10 points) Consider a set-up in which a transmitter is transmitting either a 0 or a 1 and the receiver indicates that it received either a 0 or a 1. Denote the events that $i = 0, 1$ was transmitted by T_i, and the events that $i = 0, 1$ was indicated as received by R_i.
It is possible to have transmission errors. In fact, you are given the following data on accuracy and the frequency of transmitted signals:

$$\mathbb{P}[R_0 \,|\, T_0] = 0.99,\ \mathbb{P}[R_1 \,|\, T_1] = 0.98,$$

and

$$\mathbb{P}[T_0] = 0.75.$$

(a) Given that the receiver indicated 1, what is the probability that there was an error in the transmission?
(b) What is the overall probability that there was an error in transmission?

Solution:

(1) We need $\mathbb{P}[T_0|R_1]$. By the Bayes formula,

$$\begin{aligned} \mathbb{P}[T_0|R_1] &= \frac{\mathbb{P}[R_1|T_0]\mathbb{P}[T_0]}{\mathbb{P}[R_1|T_0]\mathbb{P}[T_0] + \mathbb{P}[R_1|T_1]\mathbb{P}[T_1]} \\ &= \frac{(1 - 0.99) \times 0.75}{(1 - 0.99) \times 0.75 + 0.98 \times 0.25} \\ &= \frac{3}{101} \cong 0.030. \end{aligned}$$

(2) An error will happen if $T_0 \cap R_1$ or $T_1 \cap R_0$ occur, i.e.,

$$\begin{aligned} \mathbb{P}[\text{error}] &= \mathbb{P}[T_0 \cap R_1] + \mathbb{P}[T_1 \cap R_0] \\ &= \mathbb{P}[R_1|T_0] \times \mathbb{P}[T_0] + \mathbb{P}[R_0|T_1] \times \mathbb{P}[T_1] \\ &= (1 - \mathbb{P}[R_0|T_0]) \times \mathbb{P}[T_0] \\ &\quad + (1 - \mathbb{P}[R_1|T_1]) \times (1 - \mathbb{P}[T_0]) \\ &= 0.01 \times 0.75 + 0.02 \times 0.25 \\ &= \tfrac{1}{80} \cong 0.013 \end{aligned}$$

Problem 1.3. (10 points) Two people are picked at random from a group of 50 and given \$10 each. After that, independently of what happened before, three people are picked from the same group - one or more people could have been picked both times - and given \$10 each. What is the probability that at least one person received \$20?

Solution: Define

$$A = \{\text{no person picked the first time was also picked the second time}\},$$

so that the probability that at least one person received \$20 is given by $\mathbb{P}[A^c] = 1 - \mathbb{P}[A]$. In order to compute $\mathbb{P}[A]$, we note that we can write

$$A = \bigcup_{1 \le i < j \le 50} A_{ij} \cap B_{ij},$$

where

$$A_{ij} = \{\text{ the first two people picked are } i \text{ and } j \text{ (not necessarily in that order)}\}, \text{ and}$$
$$B_{ij} = \{\ i \text{ and } j \text{ are not among the next three people picked}\}.$$

The sets $A_{ij} \cap B_{ij}$ and $A_{i'j'} \cap B_{i'j'}$ are mutually exclusive whenever $i \neq i'$ or $j \neq j'$, so we have

$$\mathbb{P}[A] = \sum_{1 \le i < j \le 50} \mathbb{P}[A_{ij} \cap B_{ij}].$$

Furthermore, A_{ij} and B_{ij} are independent by the assumption so $\mathbb{P}[A_{ij} \cap B_{ij}] = \mathbb{P}[A_{ij}]\mathbb{P}[B_{ij}]$.

Clearly, $\mathbb{P}[A_{ij}] = \frac{1}{\binom{50}{2}}$, since there are $\binom{50}{2}$ equally likely ways to choose 2 people out of 50, and only one of these corresponds to the choice (i, j). Similarly, $\mathbb{P}[B_{ij}] = \frac{\binom{48}{3}}{\binom{50}{3}}$, because there are $\binom{50}{3}$ ways to choose 3 people out of 50, and $\binom{48}{3}$ of those do not involve i or j. Therefore,

$$\mathbb{P}[A] = \sum_{1 \le i < j \le 50} \frac{1}{\binom{50}{2}} \frac{\binom{48}{3}}{\binom{50}{3}}.$$

The terms inside the sum are all equal and there are $\binom{50}{2}$ of them, so

$$\mathbb{P}[A] = \binom{50}{2} \frac{1}{\binom{50}{2}} \frac{\binom{48}{3}}{\binom{50}{3}} = \frac{\binom{48}{3}}{\binom{50}{3}},$$

and the required probability is

$$1 - \frac{\binom{48}{3}}{\binom{50}{3}}.$$

Problem 1.4. (5 points) Write down the definition of the *cumulative distribution function* of a random variable.

Solution: Denote the random variable by X. Then, its *cumulative distribution function* $F_X : \mathbb{R} \to [0, 1]$ is defined by

$$F_X(x) = \mathbb{P}[X \le x] \quad \text{for every } x \in \mathbb{R}. \tag{1.1}$$

Problem 1.5. (10 points) Two coins are tossed and a (6-sided) die is rolled. Describe a sample space (probability space), together with the probability, on which such a situation can be modelled. Find the probability mass function of the random variable whose value is the sum of the number on the die and the total number of heads.

Solution: Each elementary event ω should track the information about three things - the outcome of the first coin toss, the outcome of the second coin toss and the number on the die. This corresponds to triplets $\omega = (c_1, c_2, d)$, where $c_1, c_2 \in \{H, T\}$ and $d \in \{1, \dots, 10\}$. Therefore, $\Omega = \{H, T\} \times \{H, T\} \times \{1, \dots, 6\}$. Since all the instruments involved are fair, the independence requirements dictate that

$$\mathbb{P}[\omega = (c_1, c_2, d)] = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{6} = \frac{1}{24},$$

for any $(c_1, c_2, d) \in \Omega$. In words, all elementary events are equally likely. Let C_1 be the random variable which equals to 1 if the outcome of the first coin toss if H, so that

$$C_1(\omega) = \begin{cases} 1, & c_1 = H, \\ 0, & \text{otherwise}, \end{cases} \quad \text{where } \omega = (c_1, c_2, d).$$

In other words, $C_1 = \mathbf{1}_A$ is the indicator of the event

$$A = \{\omega = (c_1, c_2, d) \in \Omega \,:\, c_1 = H\}.$$

Let C_2 and D (the number on the die) be defined analogously. Then the total number of heads M is given by $M = C_1 + C_2$. Each C_1 and C_2 are independent Bernoulli random variables with $p = \frac{1}{2}$, so M is a binomial random variable with $n = 2$ and $p = \frac{1}{2}$. Therefore, the pmf of M is

	0	1	2
p	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Let X be the random variable from the text of the problem:

$$X = D + M.$$

The values random variable X can take are $\{1, 2, \dots, 8\}$, and they correspond to the following table (the table entry is the value of X, columns go with D and rows with M):

	1	2	3	4	5	6
0	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8

A bit of accounting gives the following pmf for X:

	1	2	3	4	5	6	7	8
p	$\frac{1}{24}$	$\frac{3}{24}$	$\frac{4}{24}$	$\frac{4}{24}$	$\frac{4}{24}$	$\frac{4}{24}$	$\frac{3}{24}$	$\frac{1}{24}$

Problem 1.6. (10 points) A continuous random variable X has the probability density function f_X given by

$$f_X(x) = A - \tfrac{x}{50}, \quad 0 \leq x \leq 10.$$

(a) Find the value of the constant A.
(b) Find the value of the survival function of X at 7, i.e., calculate $S_X(7)$.

Solution:

(a) Necessarily, $\int_0^{10} f_X(x)\,dx = 1$. So,

$$10A = 1 + \frac{10^2}{2 \cdot 50} = 2 \quad \Rightarrow \quad A = 1/5.$$

(b) Note that f_X is piecewise linear. So, we can calculate $S_X(7) = \mathbb{P}[X > 7]$ as the area of a triangle (draw a picture if in doubt!). We get

$$S_X(7) = \frac{1}{2} \cdot f_X(7) \cdot (10 - 7) = \frac{3}{2} \cdot (\frac{1}{5} - \frac{7}{50}) = \frac{3(10-7)}{2 \cdot 50} = 9/100.$$