# M339D: April 17th, 2024.

**Example.**

Black-Scholes Call price

$$V_c(0) = S(0)\boxed{N(d_1)}^{\to 1} - \boxed{Ke^{-rT} N(d_2)} \xrightarrow{T\to\infty} 0$$

w/ $d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)\cdot T\right]$

and $d_2 = d_1 - \sigma\sqrt{T}$

We want to look @ $V_c(0) \xrightarrow[T\to\infty]{} \cancel{?}\ S(0)$

$$d_1 = \underbrace{\frac{1}{\sigma\sqrt{T}}\ln\left(\frac{S(0)}{K}\right)}_{\xrightarrow{T\to\infty} 0} + \underbrace{\boxed{\frac{r+\frac{\sigma^2}{2}}{\sigma}\sqrt{T}}}_{\xrightarrow{T\to\infty} +\infty}$$

$$N(d_1) \xrightarrow[T\to\infty]{} 1 \ \checkmark$$

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**Example.** Risk-neutral pricing principle.

$$\boxed{V(0) = e^{-rT}\mathbb{E}^*[V(T)]}$$

$$\mathbb{E}^*[V(T)] = V(0)e^{rT}$$

Def'n. For any security $S$, and probability $\mathbb{P}$ the mean rate of return under $\mathbb{P}$ is an $\alpha$ which solves

$$\mathbb{E}[S(T)] = S(0)e^{\alpha T}$$

**Problem 14.3.** (5 pts) Assume the Black-Scholes framework. Let the current price of a non-dividend-paying stock be equal to $S(0) = 95$ and let its volatility be equal to 0.35. Consider a European call on that stock with strike 100 and exercise date in 9 months. Let the risk-free continuously compounded interest rate be 6% per annum.

Denote the price of the call by $V_C(0)$. Then,

(a) $V_C(0) < \$5.20$

(b) $\$5.20 \leq V_C(0) < \$7.69$

(c) $\$7.69 \leq V_C(0) < \$9.04$

(d) $9.04 \leq V_C(0) < \$11.25$

(e) None of the above.

→: 1st Calculate $d_1$ and $d_2$. ✓

2nd Use the standard normal tables. ✓

3rd Use the $\boxed{\text{BS pricing formula.}}$

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)\cdot T\right]$$

$$d_1 = \frac{1}{0.35\sqrt{3/4}}\left[\ln\left(\frac{95}{100}\right) + \left(0.06 + \frac{(0.35)^2}{2}\right)\cdot\left(\frac{3}{4}\right)\right] = \underline{0.1308} \approx 0.13$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.1308 - 0.35\sqrt{3/4} = \underline{-0.1723} \approx -0.17$$

$$N(d_1) = N(0.13) = \underline{0.5517}$$

$$N(d_2) = N(-0.17) = \underline{0.4325}$$

$$V_c(0) = S(0)N(d_1) - Ke^{-rT}N(d_2)$$

$$V_c(0) = 95 \cdot 0.5517 - 100e^{-0.06(0.75)} \cdot 0.4325 = \underline{11.06}$$

□

**Problem 14.4.** Assume the Black-Scholes setting. Let $S(0) = \$63.75$, $\sigma = 0.20$, $r = 0.055$. The stock pays no dividend and the option expires in 50 days (simplify the number of days in a year to 360).

What is the price of a \$60-strike European put?

(a) 0.66
(b) 0.55
(c) 0.44
(d) 0.37
(e) None of the above.

$\rightarrow$ : $V_P(0) = Ke^{-rT}N(-d_2) - S(0)N(-d_1)$

$$d_1 = \frac{1}{0.2\sqrt{\frac{50}{360}}}\left[\ln\left(\frac{63.75}{60}\right) + \left(0.055 + \frac{0.04}{2}\right)\left(\frac{50}{360}\right)\right]$$

$d_1 = 0.9531 \approx 0.95$

$d_2 = d_1 - \sigma\sqrt{T} = 0.8786 \approx 0.88$

$N(-d_1) = N(-0.95) = 0.1711$

$N(-d_2) = N(-0.88) = 0.1894$

$V_P(0) = 60e^{-0.055\left(\frac{50}{360}\right)} \cdot 0.1894 - 63.75 \cdot 0.1711$

$V_P(0) = 0.37$

$\square$

**8.** Let $S(t)$ denote the price at time $t$ of a stock that pays no dividends. The Black-Scholes framework holds. Consider a European call option with exercise date $T$, $T > 0$, and exercise price $\underbrace{\boxed{S(0)e^{rT}}}_{=K}$, where $r$ is the continuously compounded risk-free interest rate.

You are given:

(i) $S(0) = \$100$

(ii) $T = 10$

(iii) $\text{Var}[\ln S(t)] = 0.4t$, $t > 0$. $\longrightarrow$ $\boxed{\sigma = \sqrt{0.4}}$

$\ln(S(t)) = ?$

$S(t) = S(0)e^{(r - \frac{\sigma^2}{2})t + \sigma\sqrt{t}\cdot Z}$

$\ln(S(t)) = \boxed{\ln(S(0)) + (r - \frac{\sigma^2}{2})t} + \sigma\sqrt{t}\cdot Z$

$\text{Var}[\ln(S(t))] = \text{Var}[\sigma\sqrt{t}\cdot Z] = \sigma^2 \cdot t = 0.4\cdot t$

Determine the price of the call option.

(A) $7.96

(B) $24.82

(C) $68.26

(D) $95.44

(E) There is not enough information to solve the problem.

$\longrightarrow$: $V_c(0) = S(0)N(d_1) - Ke^{-rT}\cdot N(d_2)$

$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{\cancel{S(0)}}{\cancel{S(0)}e^{rT}}\right) + (r + \frac{\sigma^2}{2})\cdot T\right]$

$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\cancel{-rT} + \cancel{rT} + \frac{\sigma^2}{2}\cdot T\right] = \frac{\sigma\sqrt{T}}{2}$

$d_2 = d_1 - \sigma\sqrt{T} = -\frac{\sigma\sqrt{T}}{2}$

$\sigma = \sqrt{0.4}$, $T = 10$

$\Rightarrow d_1 = \frac{\sqrt{0.4}\cdot\sqrt{10}}{2} = 1$ and $d_2 = -1$

$V_c(0) = S(0)N(d_1) - S(0)\cancel{e^{rT}}\cdot\cancel{e^{-rT}}\cdot N(d_2)$

$$V_c(0) = S(0)\left(N(1) - \underbrace{N(-1)}_{1-N(1)}\right) =$$

$$V_c(0) = S(0)\left(2\underbrace{N(1)}_{0.8413} - 1\right) = 100\left(2 \cdot 0.8413 - 1\right) = 68.26.$$

$\square$