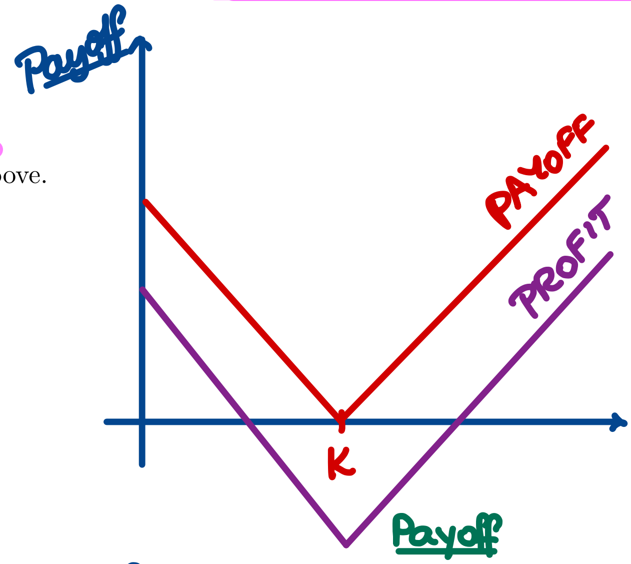
M339D: March 10^{th}, 2025.

Problem 9.5. Consider a non-dividend paying stock whose current price is \$95 per share. You model the evolution of this stock price over the following year using a one-period binomial tree under the assumption that the stock price can be either \$120, or \$75 in one year.

The continuously compounded risk-free interest rate is 0.06.

A **straddle** consists of a long call and a long otherwise identical put. Consider a \$100-strike, one-year European **straddle** on the above stock. What is the straddle's price consistent with the above stock-price model?

(a) About \$10
(b) About \$10.83
(c) About \$15.45
(d) About \$20.84
(e) None of the above.



Payoff f'tion of a straddle

$v(s) = |s-K|$

In this problem $\boxed{K=100}$

$S(0)=95$ → $S_u=120$, Payoff $V_u=20$; $S_d=75$, $V_d=25$

$$\Delta = \frac{V_u-V_d}{S_u-S_d} = \frac{20-25}{120-75} = -\frac{5}{45} = -\frac{1}{9}$$

$$B = e^{-rh}\frac{u\cdot V_d - d\cdot V_u}{u-d} =$$

$$B = e^{-0.06}\cdot\frac{\frac{120}{\cancel{95}}\cdot 25 - \frac{75}{\cancel{95}}\cdot 20}{\frac{120}{\cancel{95}} - \frac{75}{\cancel{95}}} = e^{-0.06}\,\frac{120(25)-75(20)}{45} = 31.392$$

$$V(0) = \Delta\cdot S(0) + B = -\frac{1}{9}(95) + 31.392 = 20.84$$

□

Risk-Neutral Probability.

Start w/ $V(0) = \Delta \cdot S(0) + B$

$$V(0) = \frac{V_u - V_d}{\underbrace{S_u - S_d}_{\cancel{S(0)}(u-d)}} \cdot \cancel{S(0)} + e^{-rh} \cdot \frac{u \cdot V_d - d V_u}{u-d}$$

$$V(0) = \frac{1}{u-d}\left[(V_u - V_d) + e^{-rh}(u \cdot V_d - d \cdot V_u)\right]$$

$$V(0) = e^{-rh} \cdot \frac{1}{u-d}\left[e^{rh} \cdot V_u - e^{rh} \cdot V_d + u \cdot V_d - d \cdot V_u\right]$$

$$V(0) = e^{-rh} \cdot \frac{1}{u-d}\left[(e^{rh} - d) \cdot V_u + (u - e^{rh}) \cdot V_d\right]$$

$$V(0) = e^{-rh}\left[\underbrace{\boxed{\frac{e^{rh}-d}{u-d}}}_{=p^*} \cdot V_u + \underbrace{\boxed{\frac{u-e^{rh}}{u-d}}}_{=1-p^*} \cdot V_d\right]$$

Both positive (duo to the no-arbitrage cond'n)

Add up to 1!

We choose to interpret the two fractions as probabilities!
We define the risk-neutral probability of the stock price going up in a single period as

$$\boxed{p^* := \frac{e^{rh}-d}{u-d}}$$

$\Rightarrow$ The risk-neutral pricing formula:

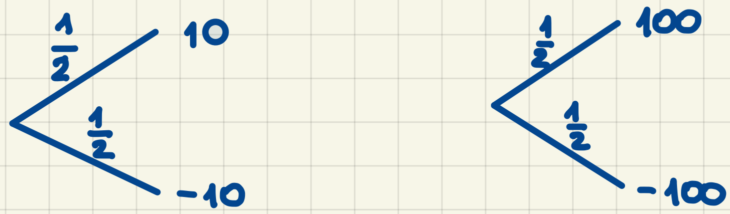
$$V(0) = \underbrace{e^{-rT}}_{\text{discounting}} \underbrace{\boxed{\left[V_u \cdot p^* + V_d(1-p^*)\right]}}_{\text{expected payoff under the risk-neutral probability}}$$

We can generalize this principle:

$$\boxed{V(0) = e^{-rT} E^*[V(T)]}$$

Q: Why "risk-neutral"?

Imagine bets:



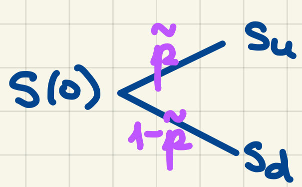
Consider a risk-neutral investor, i.e., one who is indifferent to risk and only cares about the expectation.

Q: What is the probability $\tilde{p}$ such that, for a specific stock-price tree, this investor is indifferent between investing in the stock and the risk-free investment?

→: Say, they start w/ $S(0)$.

If they invest @ the ccrfir r, then, their balance @ time h is $S(0)e^{rh}$

If they invest in the stock:



$$\mathbb{E}[\text{Wealth}] = \mathbb{E}[S(h)]$$

$$= \tilde{p} \cdot S_u + (1-\tilde{p}) \cdot S_d$$

$$= \boxed{\tilde{p} \cdot u \cdot S(0) + (1-\tilde{p}) \cdot d \cdot S(0)} = S(0)e^{rh}$$

$$\tilde{p}u + (1-\tilde{p})d = e^{rh}$$

$$\tilde{p}(u-d) = e^{rh} - d \implies \boxed{\tilde{p} = \frac{e^{rh}-d}{u-d}} = p^*$$

□

Problem 9.5. revisited.

$v(s) = |s - 100|$

$$S(0) = 95 \;\begin{cases} \xrightarrow{p^*} S_u = 120 \\ \xrightarrow{1-p^*} S_d = 75 \end{cases}$$

Payoff

$V_u = 20$

$V_d = 25$

w/ $$p^* = \frac{e^{rh} - d}{u - d} \cdot \frac{S(0)}{S(0)} = \frac{S(0)e^{rh} - S_d}{S_u - S_d} = \frac{95e^{0.06} - 75}{120 - 75} = \underline{0.5749}$$

$$V(0) = e^{-rT}\left[V_u \cdot p^* + V_d \cdot (1 - p^*)\right]$$
$$= e^{-0.06}\left[20p^* + 25 \cdot (1 - p^*)\right] = \underline{20.84} \quad \square$$

Special Case: Forward Binomial Tree.

$\boxed{\sigma \text{... volatility}}$

$u := e^{rh + \sigma\sqrt{h}}$

$d := e^{rh - \sigma\sqrt{h}}$

The risk-neutral probability:

$$p^* = \frac{e^{rh} - d}{u - d} = \frac{\overset{1}{\cancel{e^{rh}}} - e^{\cancel{rh} - \sigma\sqrt{h}}}{e^{\cancel{rh} + \sigma\sqrt{h}} - e^{\cancel{rh} - \sigma\sqrt{h}}}$$

$$= \frac{1 - e^{-\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}} \cdot \frac{e^{\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}}} = \frac{\overset{1}{\cancel{e^{\sigma\sqrt{h}} - 1}}}{\underbrace{e^{2\sigma\sqrt{h}} - 1}_{\cancel{(e^{\sigma\sqrt{h}} - 1)}(e^{\sigma\sqrt{h}} + 1)}}$$

$$\boxed{p^* = \frac{1}{1 + e^{\sigma\sqrt{h}}}} \xrightarrow[h \to 0]{} \frac{1}{2}$$

The shortcut $\boxed{\text{ONLY}}$ for the FORWARD binomial tree.