5. Express your result from above in terms of the c.d.f. F_Y of the r.v. Y.

$$F_{\tilde{Y}}(x) = \mathbb{P}[Y \le h(x)] = F_Y(h(x))$$

6. Differentiate: $f_{\tilde{Y}} = F'_{\tilde{Y}}$.

$$f_{\tilde{Y}}(x) = \frac{d}{dx} F_{\tilde{Y}}(x) = \frac{d}{dx} F_Y(h(x)) = \text{(chain rule)}$$

$$= \boxed{f_Y(h(x)) \cdot h'(x)} \quad \square$$

02/20/2026

Problem 8.4. *The time T that a manufacturing distribution system is out of operation is modeled by a distribution with the following c.d.f.*

$$F_T(t) = (1 - (2/t)^2)\mathbf{1}_{(2,\infty)}(t)$$

The resulting cost to the company is $\boxed{Y = T^2}$. *Find the probability density function f_Y of the r.v. Y.*

$\rightarrow$: $g(t) = t^2$, $t > 2$ $\Rightarrow$ $h(y) = \sqrt{y}$, $y > 4$ $\Rightarrow$ $h'(y) = \frac{1}{2\sqrt{y}}$, $y > 4$

$$f_T(t) = \frac{d}{dt} F_T(t) = \frac{8}{t^3} \mathbb{1}_{(2,\infty)}(t)$$

$$f_Y(y) = \frac{\cancel{8}\,4}{(\sqrt{y})^3} \cdot \frac{1}{\cancel{2}\sqrt{y}} \mathbb{1}_{(4,\infty)}(y) = \frac{4}{y^2} \mathbb{1}_{(4,\infty)}(y) \quad \square$$

Problem 8.5. *What if h is strictly decreasing?*

$$F_{\tilde{Y}}(y) = \mathbb{P}[\tilde{Y} \le y] = \mathbb{P}[g(Y) \le y] \overset{!}{=} \mathbb{P}[\cancel{h}(\cancel{g}(Y)) \ge h(y)]$$

$$= \mathbb{P}[Y \ge h(y)] = 1 - F_Y(h(y))$$

$$f_{\tilde{Y}}(y) = - f_Y(h(y)) \underbrace{h'(y)}_{<0}$$

Problem 8.6. *The unifying formula?*

$$\boxed{f_{\tilde{Y}}(y) = f_Y(h(y)) |h'(y)|}$$

Do not forget: it always makes sense to simply attack a problem without giving it a "label" .... Just look at the following problem:

Problem 8.7. *Let T_1 and T_2 be independent $\boxed{\text{shifted}}$ geometric random variables with parameters $p_1 = 1/2$ and $p_2 = 1/3$. Compute $\mathbb{E}[\min(T_1, T_2)]$.*

$\longrightarrow$: $T = \min(T_1, T_2)$... counts the # of trials until the 1st success happens from either of the two coins.

$\Rightarrow$ $T \sim$ shifted geometric

p... success probability of this experiment

$p = \mathbb{P}$[@ least one of the two coins is a success]

$p = 1 - \mathbb{P}$[neither coin is a success]

$p = 1 - \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{3} = \frac{2}{3}$

$\mathbb{E}[T] = ?$

$\mathbb{E}[\text{geometric}] = \frac{q}{p}$

$$\mathbb{E}[\text{shifted geometric}] = 1 + \mathbb{E}[\text{geometric}]$$
$$= 1 + \frac{q}{p} = \frac{p+q}{p} = \boxed{\frac{1}{p}}$$

$\mathbb{E}[T] = \frac{3}{2}$

$\square$

The CDF Method in 2D.

Goal: We want to find the density f_W of a r.v.

$$\boxed{W = g(Y_1, Y_2)}$$

where (Y_1, Y_2) are jointly continuous w/ pdf f_{Y_1,Y_2}

$\longrightarrow$: $F_W(w) = \mathbb{P}[W \le w] = \mathbb{P}[\underline{g(Y_1,Y_2) \le w}] = \dots$

$$A = \{(y_1, y_2) \in \mathbb{R}^2 : g(y_1,y_2) \le w\}$$

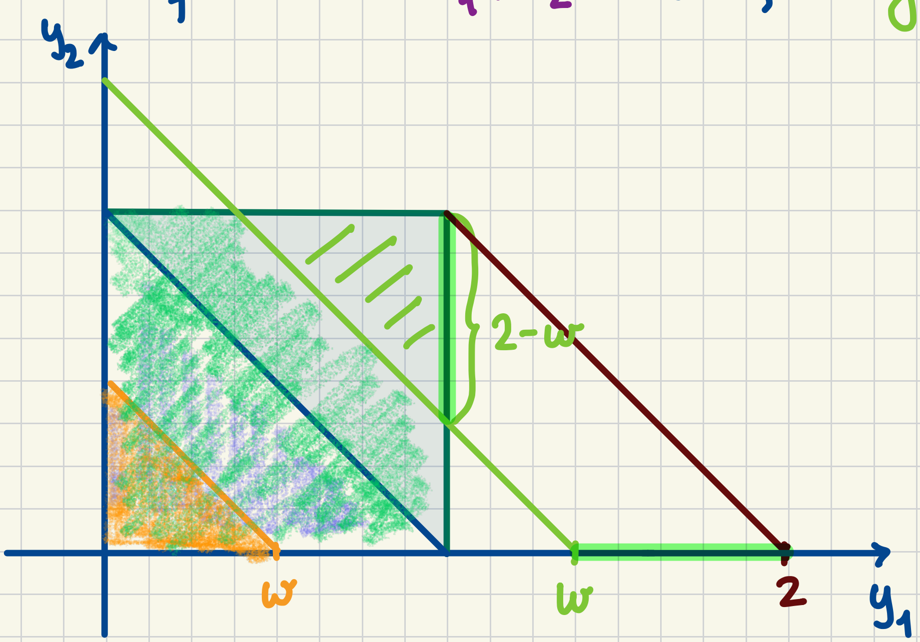
$$\dots = \mathbb{P}[(Y_1, Y_2) \in A]$$

$$\boxed{F_W(w) = \iint_A f_{Y_1,Y_2}(y_1,y_2)\, dy_2\, dy_1}$$

Example. Say (Y_1, Y_2) represent points chosen @ random in a unit square $[0,1] \times [0,1] = [0,1]^2$

$$\boxed{f_{Y_1,Y_2}(y_1,y_2) = \mathbb{1}_{[0,1]^2}(y_1, y_2)}$$

Define $W = Y_1 + Y_2$ i.e., $g(y_1,y_2) = y_1 + y_2$



for $w < 0$: $F_W(w) = 0$

for $w \in [0,1]$: $F_W(w) = \mathbb{P}[W \leq w] = \frac{1}{2}w^2$

for $w = 1$: $F_W(w) = \frac{1}{2}$

for $w \in (1,2)$: $F_W(w) = 1 - \frac{(2-w)^2}{2}$

for $w > 2$: $F_W(w) = 1$

$$f_W(w) = \begin{cases} 0 & w < 0 \\ \frac{1}{\cancel{2}} \cdot \cancel{2} \cdot w = w & w \in [0,1] \\ -\frac{1}{\cancel{2}}(-1)(2-w) \cdot \cancel{2} = 2-w & w \in [1,2] \\ 0 & w > 2 \end{cases}$$