UNIVERSITY OF TEXAS AT AUSTIN

HW Assignment 4

Prerequisite material. Log-normal stock prices. Jensen's inequality.

Provide your **complete solution** to the following problems:

Problem 4.1. (10 points) A continuous-dividend-paying stock is valued at \$75.00 per share. Its dividend yield is 0.03. The time$-t$ realized (rate of) return is modeled as

$$R(0,t) \sim N(\text{mean} = 0.035t, \text{variance} = 0.09t)$$

An investor purchases a single share of stock at time-0 and continuously (and immediately) reinvests any dividends received in the same asset. What are the mean and median values of the investor's position at time-4?

Solution: The expected rate of return (per annum) is

$$0.035 + 0.03 + \frac{1}{2} \times 0.09 = 0.035 + 0.03 + 0.045 = 0.11.$$

The mean is

$$e^{4\delta}\mathbb{E}[S(T)] = 75e^{4\alpha} = 75e^{0.44} = 116.45.$$

Similarly, the median is

$$116.45 \times e^{-0.09\times 4/2} = 97.27.$$

Problem 4.2. (10 points) A continuous-dividend-paying stock is valued at \$75.00 per share. Its dividend yield is 0.03. The time$-t$ realized return is modeled as

$$R(0,t) \sim N(\text{mean} = 0.035t, \text{variance} = 0.09t)$$

Find the probability that the time-4 stock price exceeds today's stock price.

Solution: We need to find

$$\mathbb{P}[S(4) > S(0)]$$

with

$$S(t) = S(0)e^{R(0,t)}.$$

Since $R(0,t)$ follows the normal distribution with the above parameters, we have

$$\mathbb{P}[S(4) > S(0)] = \mathbb{P}[S(0)e^{R(0,4)} > S(0)] = \mathbb{P}[R(0,4) > 0] = 1 - N\left(-\frac{0.035 \times 4}{0.3 \times 2}\right) = N(0.23) = 0.591.$$

Problem 4.3. (10 points) A non-dividend-paying stock is valued at \$75.00 per share. The annual expected (rate of) return is 16.0% and the standard deviation of annualized returns is given to be 0.30. If the stock price is modeled using the lognormal distribution (as discussed in class), what is the constant $s^U_{1/2}$ such that

$$\mathbb{P}[S(1/2) > s^U_{1/2}] \leq 0.05.$$

Solution: Note that the 95^{th} percentile of the standard normal distribution equals 1.645. So,

$$s^U_{1/2} = 75e^{(0.16-\frac{1}{2}\times 0.3^2)\times\frac{1}{2}+0.3\times\frac{1}{\sqrt{2}}\times 1.645} = 112.61.$$

Provide your *final answer only* for the following problems:

Problem 4.4. (2 points) A time$-T$ exchange call with underlying $\mathbf{S}$ and strike asset $\mathbf{Q}$ is always worth strictly more than an exchange put option with underlying $\mathbf{Q}$ and strike asset $\mathbf{S}$. *True or false?*

Solution: FALSE

Problem 4.5. (2 points) A bear spread is a long position with respect to the underlying asset. *True or false?*

Solution: FALSE

Problem 4.6. (2 points) If the random variable X has the distribution function F_X, then the distribution function of the random variable $Y = |X|$ equals

$$F_Y(y) = 2F_X(y).$$

True or false?

Solution: FALSE

Problem 4.7. (2 points) Let $X_1, \dots, X_n$ be random variables with finite expectations and let $\alpha_1, \dots, \alpha_n$ be constants. Then, we always have that

$$\mathbb{E}[\alpha_1 X_1 + \cdots + \alpha_n X_n] = \sum_{i=1}^{n} \alpha_i \mathbb{E}[X_i]\,.$$

True or false?

Solution: TRUE

Problem 4.8. (2 points) Let the stock price be modeled by a lognormal distribution. Then, the expected payoff of a European put option with exercise date T and strike K greater than or equal to $\max(0, K - \mathbb{E}[S(T)])$. *True or false?*

Solution: TRUE

Problem 4.9. (5 points) The random vector (X_1, X_2) is jointly normal. Its marginal distributions are:

$$X_1 \sim N(\text{mean} = 0, \text{variance} = 4), \quad X_2 \sim N(\text{mean} = 1, \text{variance} = 1).$$

The correlation coefficient is given to be

$$corr[X_1, X_2] = -0.2.$$

What is the variance of the random variable $X = 3X_1 - 2X_2$?

(a) 32.8

(b) 47.2

(c) 54.4

(d) 58.2

(e) None of the above.

Solution: (e)

The variance of X is

$$\begin{aligned} Var[X] &= 9Var[X_1] + 4Var[X_2] - 2(3)(2)Cov[X_1, X_2] \\ &= 9(4) + 4(1) + 12(2)(1)(0.2) = 44.8. \end{aligned}$$

Problem 4.10. (5 points) Let the stochastic process $S = \{S(t); t \geq 0\}$ denote the stock price. The stock's rate of appreciation is 10% while its volatility is 0.30 Then,

(a) $Var[\ln(S(t))] = 0.3t$

(b) $Var[\ln(S(t))] = 0.09t^2$

(c) $Var[\ln(S(t))] = 0.09t$

(d) $Var[\ln(S(t))] = 0.09$

(e) None of the above.

Solution: (c)
The random variable $S(t)$ is lognormal so that the random variable $\ln(S(t))$ is normal with variance $0.3^2t = 0.09t$.