M378K: November 3rd, 2025.

More on Estimators.

Def'n. The bias of an estimator $\hat{\theta}$ for θ is

$$\text{bias}(\hat{\theta}) = \mathbb{E}[\hat{\theta}] - \theta$$

If $\boxed{\text{bias}(\hat{\theta}) = 0}$, we say that $\hat{\theta}$ is unbiased for θ.

Def'n. The mean squared error of $\hat{\theta}$ is

$$MSE(\hat{\theta}) = Var(\hat{\theta}) + (\text{bias}(\hat{\theta}))^2$$
$$= \mathbb{E}[(\hat{\theta} - \theta)^2]$$

Def'n. An estimator $\hat{\theta}$ is uniformly minimum variance unbiased estimator (UMVUE) of θ if:
- $\hat{\theta}$ is unbiased
- $MSE(\hat{\theta}) \leq MSE(\hat{\theta}')$ for all other unbiased estimators $\hat{\theta}'$ of θ.

Example. These are UMVUE:
- $\bar{Y}$ for μ where $(Y_1, \dots, Y_n)$ is a random sample from $N(\mu, \sigma_0^2)$
- $\bar{Y}$ for p where $(Y_1, \dots, Y_n)$ is a random sample from $B(p)$
- $$\boxed{S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (Y_i - \bar{Y})^2}$$
  for σ^2 where $(Y_1, \dots, Y_n)$ is a random sample from $N(\mu, \sigma)$ w/ both parameters unknown

Def'n. An estimator $\hat{\theta}$ is said to be linear if it's of the form

$$\hat{\theta} = \alpha_1 Y_1 + \alpha_2 Y_2 + \dots + \alpha_n Y_n$$

where α_i are all constants

e.g., $\bar{Y}$

Problem 15.3. *Let $Y_1, Y_2, \dots, Y_n$ be a random sample from a continuous distribution with probability density function*

$$\boxed{f_Y(y) = \frac{\alpha y^{\alpha-1}}{\theta^\alpha}\mathbf{1}_{[0,\theta]}(y)}$$

with a known parameter $\alpha > 0$ and an unknown parameter $\theta > 0$. We propose the estimator $\hat{\theta} = \max(Y_1, \dots, Y_n)$. Is this estimator unbiased*? If not, how would you modify it to create an* unbiased *estimator? What is the* **mean-squared error** *of the* unbiased *estimator you obtained?*

$\longrightarrow$: $\hat{\theta} = Y_{(n)} = \max(Y_1, \dots, Y_n)$

$$\mathbb{E}[\hat{\theta}] = \boxed{\mathbb{E}[Y_{(n)}]} = ?$$

$$g_{(n)}(y) = n \cdot \boxed{(F_Y(y))}^{n-1} \cdot f_Y(y)$$

$$\mathbb{E}[Y_{(n)}] = \int_0^\theta y\, g_{(n)}(y)\, dy = ?$$

$y \in (0,\theta)$:

$$F_Y(y) = \int_0^y f_Y(u)\,du = \int_0^y \frac{\alpha u^{\alpha-1}}{\theta^\alpha}\,du = \frac{\alpha}{\theta^\alpha}\int_0^y u^{\alpha-1}\,du = \frac{\cancel{\alpha}}{\theta^\alpha}\cdot\frac{y^\alpha}{\cancel{\alpha}} = \left(\frac{y}{\theta}\right)^\alpha$$

$$\mathbb{E}[Y_{(n)}] = \int_0^\theta y \cdot n \cdot \left(\left(\frac{y}{\theta}\right)^\alpha\right)^{n-1} \cdot \frac{\alpha y^{\alpha-1}}{\theta^\alpha}\,dy$$

$$= n \cdot \alpha \cdot \frac{1}{\theta^{n\alpha}} \int_0^\theta y \cdot y^{\alpha(n-1)} \cdot y^{\alpha-1}\,dy$$

$$= \frac{n\alpha}{\theta^{n\alpha}} \cdot \int_0^\theta y^{n\alpha}\,dy = \frac{n\alpha}{\cancel{\theta^{n\alpha}}} \cdot \frac{\theta^{\cancel{n\alpha}+1}}{n\alpha+1} = \boxed{\frac{n\alpha}{n\alpha+1}} \cdot \theta$$

Not unbiased!

Let's define: $\boxed{\hat{\Theta}' = \frac{n\alpha+1}{n\alpha} \cdot \hat{\Theta} = \frac{n\alpha+1}{n\alpha} \cdot Y_{(n)}}$

unbiased :)

$$MSE[\hat{\Theta}'] = Var[\hat{\Theta}'] + \underbrace{(bias(\hat{\Theta}'))^2}_{=0} = Var[\hat{\Theta}']$$

$$= Var\left[\frac{n\alpha+1}{n\alpha} \cdot Y_{(n)}\right] = \left(\frac{n\alpha+1}{n\alpha}\right)^2 \cdot \boxed{Var[Y_{(n)}]}$$

$$Var[Y_{(n)}] = \boxed{\mathbb{E}[Y_{(n)}^2]} - (\mathbb{E}[Y_{(n)}])^2$$

$$\int_0^\theta y^2 \cdot g_{(n)}(y)\,dy = \int_0^\theta y^2 \cdot n \cdot \frac{\alpha y^{\alpha-1}}{\theta^\alpha} \cdot \left(\left(\frac{y}{\theta}\right)^\alpha\right)^{n-1} dy$$

$$= \frac{n\alpha}{\theta^{n\alpha}} \int_0^\theta y^{\alpha n+1}\,dy = \frac{n\alpha}{\cancel{\theta^{n\alpha}}} \cdot \frac{\theta^{\cancel{n\alpha}+2}}{n\alpha+2} = \frac{n\alpha}{n\alpha+2} \cdot \theta^2$$

$$Var[Y_{(n)}] = \frac{n\alpha}{n\alpha+2}\theta^2 - \left(\frac{n\alpha}{n\alpha+1}\right)^2 \cdot \theta^2$$

$$MSE[\hat{\Theta}'] = \frac{(n\alpha+1)^2}{(n\alpha)^2}\left(\frac{n\alpha}{n\alpha+2} - \frac{(n\alpha)^2}{(n\alpha+1)^2}\right)\theta^2$$

$$= \left(\frac{(n\alpha+1)^2}{n\alpha(n\alpha+2)} - 1\right)\theta^2$$

$$= \frac{(n\alpha+1)^2 - n\alpha(n\alpha+2)}{n\alpha(n\alpha+2)} \cdot \theta^2$$

$$= \frac{\cancel{(n\alpha)^2} + \cancel{2n\alpha} + 1 - \cancel{(n\alpha)^2} - \cancel{2n\alpha}}{n\alpha\,(n\alpha+2)}\,\theta^2$$

$$= \frac{1}{n\alpha\,(n\alpha+2)} \cdot \theta^2 \qquad \square$$