UNIVERSITY OF TEXAS AT AUSTIN

Homework Assignment #1

Prerequisite material. Transaction costs. Continuously compounded interest.

1.1. **Prerequisite material.** Please, provide your **final answer only** to the following problems.

Problem 1.1. (5 pts) You invest an amount A into an account at time-0. The account is governed by a continuously compounded risk-free interest rate equal to 0.04.

At time-4, you deposit an additional amount $3A$ into the account and the continuously compounded risk-free interest ratechanges to 0.06.

Which of the following best describes your balance at time 8?

(a) $A(e^{0.16} + 3e^{0.24})$
(b) $A(e^{0.32} + 3e^{0.24})$
(c) $A(e^{0.40} + 3e^{0.24})$
(d) $A(e^{0.40} + 3e^{0.48})$
(e) None of the above

Solution: (c)
The balance is

$$Ae^{0.04\cdot 4+0.06\cdot 4} + 3Ae^{0.06\cdot 4}.$$

Problem 1.2. (5 pts) Roger initially deposits \$4,000 in an investment fund which pays him \$2,000 at time 1 and \$4,000 at time 2.

Sally gets \$2,000 at time 0 and \$4,000 at time 1, and deposits \$5,460 at time 2 in return.

Both investments are governed by compound interest with the same annual effective interest rate i and they have the same net present values.

Find i.

(a) About 9%
(b) About 10.0%
(c) About 11.5%
(d) About 12%
(e) None of the above

Solution: (b)

Problem 1.3. (5 pts) Roger makes an inital deposit of K into an account governed by the time-varying continuously compounded risk-free interest rate $r(t) = \frac{9}{10}\sqrt{t}$ (per annum).

At the same time, Harry makes an initial deposit at the same amount into an account governed by the constant annual discount rate d.

There are no subsequent deposits to or withdrawals from either of the two accounts.

After 4 years, Roger and Harry realize that the balances in their accounts are equal. Which of the following is the closest to d?

(a) $e^{-6/5}$
(b) $e^{-1/5}$
(c) $1 - e^{-1/5}$
(d) $1 - e^{-6/5}$
(e) 1

Solution: (d)
Without loss of generality, we can set $K = 1$. The balance in Roger's account at time 4 can be expressed as

$$e^{\int_0^4 r(t)\,dt} = e^{\frac{9}{10}\cdot\frac{2}{3}4^{3/2}} = e^{24/5}.$$

So, the balance in Harry's account is

$$e^{24/5} = (1-d)^{-4} \quad \Rightarrow \quad d = 1 - e^{-6/5}.$$

Please provide your **complete solution** to the following problems.

Problem 1.4. (10 points) By scenario A there is an offer to pay at the rate of $10,000 per annum, continuously, for the next 10 years. By scenario B it is offered to pay the amount X at the end of each of the next 10 years. The force of interest applying to both scenarios is 12%. Find the value of X such that you are indifferent between these two scenarios in the sense that they have the same present values.

Solution: The equality of present values of the two annuities translates into

$$10000\bar{a}_{\overline{10|}\,\delta=0.12} = Xa_{\overline{10|}\,\delta=0.12} \quad \Rightarrow \quad 10000\frac{1-e^{-1.2}}{0.12} = X\frac{1-e^{-1.2}}{e^{0.12}-1}.$$

So,

$$X = \frac{10000(e^{0.12}-1)}{0.12} \approx 10624.74.$$

Problem 1.5. (5 pts) Find the total amount of interest that would be paid on a $1,000 loan over a 10−year period, if the effective interest rate is 0.09 per annum under the following repayment method:

The entire loan plus entire accumulated interest is paid as one lump-sum at the end of the loan term.

Solution: Using compound interest, the accumulated value at the end of the 10 years is

$$1000 \cdot 1.09^{10} \approx 2367.36.$$

The total amount of interest is

$$2367.36 - 1000 = 1367.36.$$

Problem 1.6. (2 points) Assume that the force of interest is constant and denoted by r. Express the accumulation function $a(t)$ in terms of r for $t \geq 0$.

Solution:

$$a(t) = e^{rt}$$

Example 1.1. **A warm-up example**
Source: "Calculus" by James Stewart.
"One model of *population growth* is based on the assumption that the population grows at a rate proportional to the size of the population." Let us denote the proportionality constant by k and let the function $P(\cdot)$ stand for the size of the population. Then, P must satisfy the following (ordinary differential) equation:

$$\frac{dP(t)}{dt} = kP(t)$$

Let the initial population size be p_0. Then, the population size $P(t)$ at time $t \geq 0$ is explicitly given by:

$$P(t) = p_0 e^{kt}$$

Please, provide your **complete solution** to the following problem:

Problem 1.7. (8 points) **Continuously compounded interest**
Assume that the balance in a savings account is growing so that its rate of growth is proportional to the current balance at any time. Let us denote the proportionality constant by r and let the function $B(\cdot)$ stand for the balance as a function of time. Then, B must satisfy which (ordinary differential) equation?

Solution:

$$\frac{dB(t)}{dt} = rB(t)$$

If the initial balance in the account is b_0, then what is the expression for the balance as a function of time $t \geq 0$?

Solution:

$$B(t) = b_0 e^{rt}$$

1.2. **Transaction costs.** Please, read the following lecture note prior to attempting the remaining problems:

`https://www.ma.utexas.edu/users/mcudina/m339d-lecture-two-transaction-costs.pdf`

Provide your **final answer** only for the following problems.

Problem 1.8. (5 points) What is the cost of purchasing 100 shares of Jiffy, Inc. stock given that the bid-ask prices are \$31.25 – \$32.00 and that there is a \$15.00 commission per transaction?

(a) \$1,293
(b) \$3,215
(c) \$3,504
(d) \$3,264
(e) None of the above.

Solution: (b)

$$100 \times 32 + 15 = 3215$$

Problem 1.9. (5 points) *Source: Prof. Jim Daniel (personal communication).*
The bid-ask spread on a share of stock is \$98-\$102. A 5% commission is paid for either buying or selling. Calculate the round-trip transaction cost.

(a) \$14
(b) \$10
(c) \$6
(d) \$4
(e) None of the above.

Solution: (a)
You spend $102 \times (1 + 0.05) = 107.10$ to buy the asset, and receive $98 \times (1 - 0.05) = 93.10$ when you sell the asset. The round-trip transaction cost is $107.10 - 93.10 = 14$.