M339G, September 8th, 2025.

Why $(\quad)^2$?

$$\mathbb{E}[(X-a)^2] \xrightarrow[a]{} \min$$

sol'n: $a = \mathbb{E}[X]$

The value is the $\underbrace{\mathbb{E}[(X-\mathbb{E}[X])^2]}_{= \ \mathrm{Var}[X]}$

Reducible vs. Irreducible.

Fact:

$$\mathbb{E}[(Y-\hat{f}(X))^2 \mid X=x] \stackrel{?}{=} \underbrace{(f(x)-\hat{f}(x))^2}_{\text{Reducible}} + \underbrace{\mathrm{Var}[\varepsilon]}_{\text{Irreducible}}$$

$\rightarrow$: By our model:

$Y = f(X) + \varepsilon$ w/ ε is independent from X

and $\boxed{\mathbb{E}[\varepsilon] = 0}$

$$\mathbb{E}[(f(X)+\varepsilon-\hat{f}(X))^2 \mid X=x] = (\text{linearity of } \mathbb{E})$$

$$= \mathbb{E}[(f(X)-\hat{f}(X))^2 \mid X=x]$$

$$+ \cancel{2\,\mathbb{E}[(f(X)-\hat{f}(X))\cdot\varepsilon \mid X=x]}$$

$$+ \mathbb{E}[\varepsilon^2 \mid X=x] \checkmark$$

ε is independent from X

$$= (f(x)-\hat{f}(x))^2 + \mathrm{Var}[\varepsilon] \quad \square$$

In general, for any r.v. ω,

$$\mathrm{Var}[\omega] = \mathbb{E}[\omega^2] - (\mathbb{E}[\omega])^2$$

$$\mathbb{E}[\omega^2] = \mathrm{Var}[\omega] + (\mathbb{E}[\omega])^2$$