M339J: February 25th, 2022.

Two-Point Mixtures.

Start w/ X_1 and X_2.

Take two positive constants a_1 and a_2 such that $a_1 + a_2 = 1$.

We want to create a r.v. X which is a two-point mixture of X_1 and X_2.

Imagination. You toss a coin w/ probab. of H equal to a_1 and prob. of T equal to a_2. If the coin comes up H, you "draw" a value from X_1.
If the coin comes up T, you "draw" a value from X_2.

The cdf of X is

$$\boxed{F_X(x) = a_1 \cdot F_{X_1}(x) + a_2 \cdot F_{X_2}(x)} \quad \text{for all } x \quad \checkmark$$

Problem. [Sample STAM Problem #169.]

The dist'n of a loss X is a two-pt mixture:

(i) w/ probab. 0.8, the loss has a two-parameter Pareto dist'n w/ $\alpha = 2$ and $\theta = 100$;
(ii) w/ probab. 0.2, the loss has a two-parameter Pareto dist'n w/ $\alpha = 4$ and $\theta = 3000$.

Find $\mathbb{P}[X \leq 200]$.

$\longrightarrow$: $F_X(200) = ?$

$$X \sim \begin{cases} X_1 \sim \text{Pareto}(\alpha_1 = 2, \theta_1 = 100) & \text{w/ prob. } 0.8 \\ X_2 \sim \text{Pareto}(\alpha_2 = 4, \theta_2 = 3000) & \text{w/ prob. } 0.2. \end{cases}$$

$$F_X(x) = a_1 \cdot F_{X_1}(x) + a_2 \cdot F_{X_2}(x)$$
$$= 0.8\left(1 - \left(\frac{100}{x+100}\right)^2\right) + 0.2\left(1 - \left(\frac{3000}{x+3000}\right)^4\right)$$

$$F_X(200) = 1 - 0.8\left(\frac{100}{300}\right)^2 - 0.2\left(\frac{3000}{3200}\right)^4 = 0.7566$$

k-point mixtures.

Let $X_1, X_2, \ldots, X_k$ be random variables w/ cdfs $F_{X_i}(\cdot)$

Let $a_1, a_2, \ldots, a_k$ be positive constants such that $a_1 + a_2 + \cdots + a_k = 1$.

Then, X is a k-point mixture if

$$\boxed{F_X(x) = a_1 \cdot F_{X_1}(x) + a_2 \cdot F_{X_2}(x) + \cdots + a_k \cdot F_{X_k}(x) \quad \text{for all } x}$$

Note:
- $\mathbb{E}[X] = a_1 \mathbb{E}[X_1] + a_2 \cdot \mathbb{E}[X_2] + \cdots + a_k \cdot \mathbb{E}[X_k]$
- $f_X(x) = a_1 \cdot f_{X_1}(x) + a_2 \cdot f_{X_2}(x) + \cdots + a_k f_{X_k}(x)$
- $\mathbb{E}[X^\ell] = a_1 \mathbb{E}[X_1^\ell] + a_2 \cdot \mathbb{E}[X_2^\ell] + \cdots + a_k \cdot \mathbb{E}[X_k^\ell]$