UNIVERSITY OF TEXAS AT AUSTIN

# Quiz #5

Log-normal stock prices: Tail probabilities.

**Problem 5.1.** (5 points) The current stock price is given to be $S(0) = 30$. The stock has the rate of appreciation 0.12 and volatility 0.3

Find the probability that the stock price in three months is less than \$32.

**Solution:**

$$N(-\hat{d}_2) = N(0.3052568) = N(0.31) = 0.6217.$$

**Problem 5.2.** (10 points) Let $\mathbf{S} = \{S(t), t \geq 0\}$ denote the stock-price process. For any time$-t$, the stock price is modeled as lognormal. The mean stock price at time$-2$ equals 140 and the median stock price at time$-2$ equals 130. What is the probability that the time$-2$ stock price exceeds 140?

**Solution:** Since $\mathbf{S} = \{S(t), t \geq 0\}$ is the stock-price process modeled by a geometric Brownian motion, the stock price at time$-1$ is lognormally distributed. In fact, using our usual parameters, we can rewrite it as

$$S(2) = S(0)e^{(\alpha-\delta-\frac{1}{2}\sigma^2)2+\sigma\sqrt{2}Z}$$

where $Z \sim N(0,1)$. Since $S(2)$ is log-normally distributed, the median of $S(2)$ equals $S(0)e^{(\alpha-\delta-\frac{1}{2}\sigma^2)2}$. So, the required probability can be expressed as

$$\begin{aligned}
\mathbb{P}[S(2) > 140] = \mathbb{P}[130e^{\sigma\sqrt{2}Z} > 140] &= \mathbb{P}\left[Z > \frac{1}{\sigma\sqrt{2}}\ln\left(\frac{140}{130}\right)\right] \\
&= \mathbb{P}\left[Z < \frac{1}{\sigma\sqrt{2}}\ln\left(\frac{130}{140}\right)\right] = N\left(\frac{1}{\sqrt{2}\sigma}\ln\left(\frac{130}{140}\right)\right).
\end{aligned}$$

Since the mean of $S(2)$ equals $S(0)e^{(\alpha-\delta)2}$, we have

$$e^{\sigma^2} = \frac{140}{130} \quad \Rightarrow \quad \sigma = \sqrt{\ln(140/130)}.$$

So, our final answer is

$$\begin{aligned}
\mathbb{P}[S(2) > 140] &= N\left(\frac{1}{\sqrt{2} \times \sqrt{\ln(140/130)}}\ln\left(\frac{130}{140}\right)\right) \\
&= N\left(-\sqrt{\frac{\ln(14/13)}{2}}\right) = N(-0.1925) = 1 - N(0.19) = 1 - 0.5753 = 0.4247.
\end{aligned}$$