M339W: January 25^{th}, 2021

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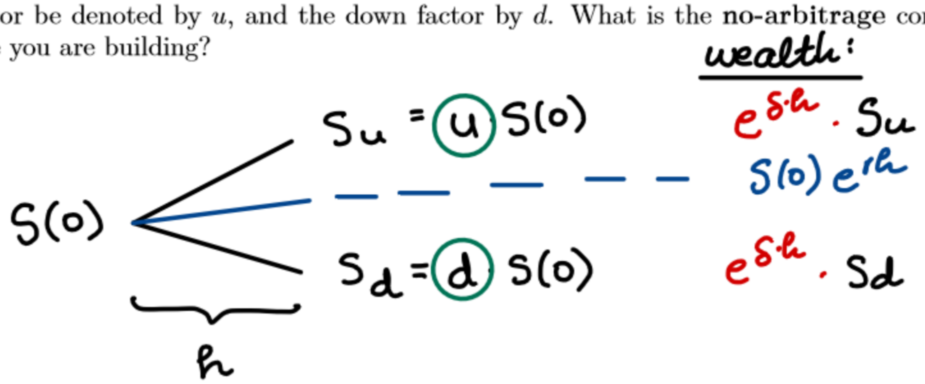
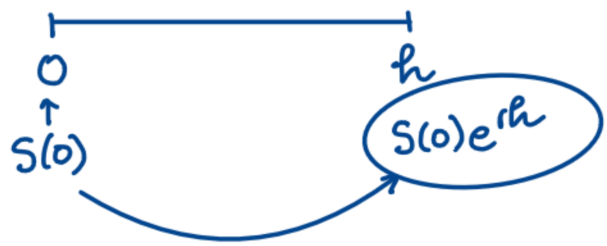
Binomial option pricing (review).

Problem 1.1. Let the continuously compounded risk-free interest rate be denoted by r. You are building a model for the price of a stock which pays dividends continuously with the dividend yield δ. Consider a binomial tree modeling the evolution of the stock price. Let the length of each period be h and let the up factor be denoted by u, and the down factor by d. What is the **no-arbitrage** condition for the binomial tree you are building?



- r... ccrfir
  Say, you invest $S(0)$ @ the risk-free interest rate.
  $\Rightarrow$ Balance @ time h.



- If we invest $S(0)$ @ time 0, that means we purchased 1 share of stock.
  Due to continuous and immediate reinvestment of dividend in the same stock we end up owning $\underline{e^{\delta h}}$ shares @ time h

To avoid creating arbitrage opportunities, we must have:

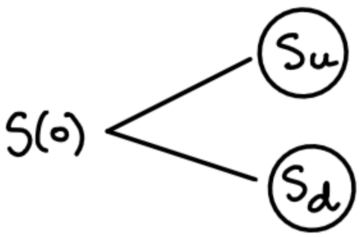
$$S_d \cdot e^{\delta \cdot h} < S(0) e^{rh} < S_u \cdot e^{\delta \cdot h}$$

$$d \cdot \cancel{S(0)}\, \underline{e^{\delta \cdot h}} < \cancel{S(0)}\, e^{rh} < u \cdot \cancel{S(0)} \cdot \underline{e^{\delta \cdot h}}$$

$$\boxed{d < e^{(r-\delta)h} < u}$$

Problem 1.2. Set up the framework for pricing by replication in a one-period binomial tree! What is the *risk-neutral pricing* formula?

Payoff function : $v(\cdot)$



$h = T$

exercise date

	Payoff		Worth of replicating port.
	$V_u = v(S_u)$	$=$	$e^{\delta \cdot h} \cdot S_u + Be^{rh}$
	$V_d = v(S_d)$	$=$	$e^{\delta \cdot h} \cdot S_d + Be^{rh}$

Replicating Portfolio:

- Δ ... shares of stock $\begin{cases} \Delta > 0 & \text{buy} \\ \Delta < 0 & \text{short shares} \end{cases}$
- B ... risk-free investment $\begin{cases} B > 0 & \text{Lending} \\ B < 0 & \text{Borrow} \end{cases}$

$$\boxed{\begin{aligned} \Delta &= e^{-\delta \cdot h} \cdot \frac{V_u - V_d}{S_u - S_d} \\ B &= e^{-r \cdot h} \cdot \frac{u \cdot V_d - d \cdot V_u}{u - d} \end{aligned}}$$

$$\Longrightarrow \quad V(0) = \Delta \cdot S(0) + B$$

$$= \dots$$

$$V(0) = e^{-r \cdot T}\left[V_u \cdot \underbrace{\frac{e^{(r-\delta)\cdot h} - d}{u-d}}_{= p^*} + V_d \cdot \overbrace{\frac{u - e^{(r-\delta)h}}{u-d}}^{= 1-p^*}\right]$$

risk-neutral probability

$$\Rightarrow \quad V(0) = e^{-rT}\left[V_u \cdot p^* + V_d \cdot (1-p^*) \right]$$

We generalize:

$$\boxed{V(0) = e^{-r \cdot T} \cdot \mathbb{E}^*[V(T)]}$$

We will use the same risk-neutral pricing principle in continuous time w/out proof.

Problem 1.3. The current price of a certain non-dividend-paying stock is \$100 per share. You are modeling the price of this stock at the end of a quarter year using a one-period binomial tree under the assumption that the stock price can either increase by 4%, or decrease by 2%.

The continuously compounded risk-free interest rate is 3%.

What is the price of a three-month, at-the-money European call option on the above stock consistent with the above binomial tree?

$$S(0) = 100 \quad \begin{array}{l} \xrightarrow{p^*} S_u = 104 \\ \xrightarrow{1-p^*} S_d = 98 \end{array}$$

Payoffs

$V_u = 4$

$V_d = 0$

$r = 0.03$

Call: $v(s) = (s-K)_+$

at-the-money: $K = S(0) = 100$

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{e^{0.03/4} - 0.98}{1.04 - 0.98} = 0.458803$$

$\delta = 0$

$h = 1/4$

$$V_c(0) = e^{-0.03/4} \cdot 4 \cdot p^* = 1.8215.$$

Problem 1.4. Let the continuously compounded risk-free interest rate be equal to 0.04.

The current price of a continuous-dividend-paying stock is \$80 and its dividend yield is 0.02. The stock's volatility is 0.25. You model the evolution of the stock price over the following half year using a two-period **forward** binomial tree.

What is the price of a six-month, \$82-strike European put option on the above stock consistent with the given binomial tree?

Length of each period : $h = 1/4$

In the forward tree:

$$p^* = \frac{1}{1+e^{\sigma\sqrt{h}}} = \frac{1}{1+e^{0.125}} = 0.4688$$

$$\begin{cases} u = e^{(r-\delta)h + \sigma\sqrt{h}} = e^{(0.04-0.02)/4 + 0.125} = e^{0.13} \\ d = e^{(r-\delta)h - \sigma\sqrt{h}} = e^{(0.04-0.02)4 - 0.125} = e^{-0.12} \end{cases}$$

Next: Draw & populate the tree.
Get the possible payoffs.
Use the risk-neutral pricing formula.

ans: 5.86.

Before next class: · Finish Problem Set #1 not to be handed in.

· Look into your ⊕ notes on currency options & futures options.