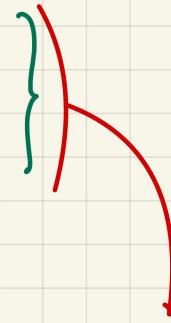


M339g: March 25<sup>th</sup>, 2022.

## Poisson-Gamma Mixing.

Let  $N$  have the following mixture distribution w/ mixing parameter  $\Delta$ :

$$\begin{aligned} N | \Delta &\sim \text{Poisson}(\text{mean} = \Delta) \\ \Delta &\sim \text{Gamma}(\alpha, \theta) \end{aligned}$$



Q: What is the support of  $N$ ?

$$\rightarrow: N_0 = \{0, 1, 2, \dots\}$$

$$N \sim \text{NegBinomial}(r = \underline{\alpha}, \beta = \underline{\theta})$$

✓

### Problem.

$$\begin{aligned} N | \Delta &\sim \text{Poisson}(\text{mean} = \Delta) \\ \Delta &\sim \text{Gamma}(\alpha = \frac{5}{2}, \theta = 4) \end{aligned}$$

What is the (unconditional) probability  $\text{P}[N=3]$ ?

$\rightarrow:$

$$N \sim \text{NegBin}(r = \frac{5}{2}, \beta = 4)$$

$$\text{P}[N=3] = \binom{\frac{5}{2} + 3 - 1}{3} \left( \frac{1}{1+\beta} \right)^{\frac{5}{2}} \left( \frac{\beta}{1+\beta} \right)^3$$

w/  $\beta = 4$  ✓

by our generalization  
from last class

$$\frac{\Gamma(3 + \frac{5}{2})}{3! \cdot \Gamma(\frac{5}{2})}$$

Hint: •  $\Gamma(x) = (x-1) \cdot \Gamma(x-1)$

\*  
for  $k \in \mathbb{N}$  \*\*

•  $\Gamma(k) = (k-1)!$

•  $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

\*\*\*

$$\Gamma\left(3 + \frac{5}{2}\right) = \left(2 + \frac{5}{2}\right) \cdot \Gamma\left(2 + \frac{5}{2}\right) = \left(2 + \frac{5}{2}\right) \cdot \left(1 + \frac{5}{2}\right) \Gamma\left(1 + \frac{5}{2}\right)$$

$$= \left(2 + \frac{5}{2}\right) \left(1 + \frac{5}{2}\right) \left(\frac{5}{2}\right) \Gamma\left(\frac{5}{2}\right)$$

$$\Gamma\left(\frac{5}{2}\right) = \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}$$

$$\frac{\Gamma\left(3 + \frac{5}{2}\right)}{3! \cdot \Gamma\left(\frac{5}{2}\right)} = \frac{(2 + \frac{5}{2})(1 + \frac{5}{2})(\frac{5}{2}) \Gamma(\frac{5}{2})}{3! \cdot \Gamma(\frac{5}{2})} = \frac{\frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2}}{6} = \frac{105}{16}$$

$$\text{PP}[N=3] = \frac{105}{16} \cdot \frac{4^3}{5^{3+\frac{5}{2}}} = 0.0601 \quad \square$$

## Binomial Distribution.

Consider  $m$  independent, identically distributed risks w/ probability of making a claim denoted by  $q$ .

Formally, for  $j = 1, \dots, m$ , we set

$$I_j = \begin{cases} 1 & \text{if risk } j \text{ makes a claim} \\ 0 & \text{if risk } j \text{ does not make claim} \end{cases}$$

Then, for all  $j$ ,

$$I_j \sim \text{Bernoulli}(q)$$

and  $\{I_j : j = 1 \dots m\}$  are independent.

N... the total number of claims made

$$N = I_1 + I_2 + \dots + I_m$$

Q: What is the pmf of  $N$ ?

→:

First, we figure out the pgf of  $N$ .

$$P_N(z) = ?$$

Note:  
There are many approaches to this.  
I'm choosing this one so that  
we can practise a particular  
technique!

$$P_N(z) = P_{I_1}(z) \cdot P_{I_2}(z) \cdots P_{I_m}(z) = \left( P_{I_1}(z) \right)^m$$

↑ independence      ↑ identically dist'd

For a single Bernoulli trial w/ probability of success  $q$ .

$$P_{I_1}(z) = \mathbb{E}[z^{I_1}] = p_{I_1}(0) \cdot z^0 + p_{I_1}(1) \cdot z^1 = 1-q + q \cdot z$$

↑  
by def'n  
of pgf

$$= 1+q(z-1)$$

$$P_N(z) = (1+q(z-1))^n$$

Using the binomial formula on

$P_N(z) = ((1-q) + q \cdot z)^m$ , we get, for  $k=0, 1, \dots, m$ ,

$$P_N(k) = \binom{m}{k} (1-q)^{m-k} \cdot q^k$$