**Name:**

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M362K Probability
University of Texas at Austin
**In-Term Exam II**
Instructor: Milica Čudina

**Notes**: This is a closed book and closed notes exam. The maximal score on this exam is 100 points.

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All written work handed in by the student is considered to be
**their own work, prepared without unauthorized assistance.**

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**The University Code of Conduct**

”The core values of The University of Texas at Austin are learning, discovery, freedom, leadership, individual opportunity, and responsibility. Each member of the university is expected to uphold these values through integrity, honesty, trust, fairness, and respect toward peers and community. As a student of The University of Texas at Austin, I shall abide by the core values of the University and uphold academic integrity.”

”I agree that I have complied with the UT Honor Code during my completion of this exam.”

**Signature:**

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## 2.1. DEFINITIONS.

**Problem 2.1.** (10 points) Complete the definition of a *random variable* on a finite outcome space below:

Let $\Omega$ be a finite outcome space. A *random variable* on $\Omega$ is ...

**Solution:** Let $\Omega$ be a finite outcome space. A *random variable* on $\Omega$ is any function $X : \Omega \to \mathbb{R}$.

**Problem 2.2.** (10 points) Write down the expression for the *standard normal density*.

**Solution:**

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \quad \text{for all } z \in \mathbb{R}$$

2.2. **FREE RESPONSE PROBLEMS.** Please, explain carefully all your statements and assumptions. Numerical results or single-word answers without an explanation (even if they're correct) are worth 0 points.

**Problem 2.3.** (10 points) Gussie is trying to capture newts in the English countryside. He is not a good hunter so the probability that he succeeds to net a singular newt is 0.1 in every try. He also does not learn from his mistakes, so his attempts are independent. What is the probability that Gussie catches his first newt in the $5^{th}$ try? Display your answer as a fully reduced fraction, please.

**Solution:** The total number of tries is geometric with success probability equal to 0.1. The probability we are looking for is

$$p_5 = \left(\frac{9}{10}\right)^4 \left(\frac{1}{10}\right) = \frac{9^4}{10^5}\,.$$

**Problem 2.4.** (10 points) An urn contains 10 balls. Two are blue, two are red, two are yellow, two are green and two are purple. Balls are drawn from the urn **without replacement** until the color appears that has appeared before. What is the probability that **exactly** three draws are needed?

*Hint:* Do **not** draw a probability tree; it is too time consuming!

**Solution:** The event that exactly three draws are needed is the same as the event that the first two balls are of different colors, and then that the next one matched the color of one of the balls that have already been drawn.

The choice of the first ball's color is irrelevant. But, the probability that the second ball's color does not match that of the first ball is $8/9$. Then, the probability that the third ball's color matches either one of the two already drawn is $2/8$. Altogether, our answer is

$$\frac{8}{9} \cdot \frac{2}{8} = \frac{2}{9}\,.$$

**Problem 2.5.** (20 points) Luka is practicing free throws. Given that he made 12 free throws in 20 attempts, what is the probability that at most two of the last 4 attempts were successful?

*Note:* Should your answer include binomial coefficients, there is no need to simplify them.

**Solution:** Let $p$ be the probability of success. We are looking for the following **conditional** probability

$$\mathbb{P}[\text{at most 2 of the last 4} \,|\, \text{12 in 20 trials}] = 1 - \mathbb{P}[\text{3 or 4 of the last 4} \,|\, \text{12 in 20 trials}].$$

We have that

$$\begin{aligned}\mathbb{P}[\text{3 of the last 4} \,|\, \text{12 in 20 trials}] &= \frac{\mathbb{P}[\text{3 of the last 4 AND 12 total in 20 trials}]}{\mathbb{P}[\text{12 in 20 trials}]}\\ &= \frac{\mathbb{P}[\text{3 of the last 4 AND 9 in the first 16 trials}]}{\mathbb{P}[\text{12 in 20 trials}]}\\ &= \frac{\mathbb{P}[\text{3 of the last 4}]\mathbb{P}[\text{9 in the first 16 trials}]}{\mathbb{P}[\text{12 in 20 trials}]}\\ &= \frac{4p^3(1-p)\binom{16}{9}p^9(1-p)^7}{\binom{20}{12}p^{12}(1-p)^8} = \frac{4\binom{16}{9}}{\binom{20}{12}}.\end{aligned}$$

Also, We have that

$$\begin{aligned}\mathbb{P}[\text{4 of the last 4} \,|\, \text{12 in 20 trials}] &= \frac{\mathbb{P}[\text{4 of the last 4 AND 12 total in 20 trials}]}{\mathbb{P}[\text{12 in 20 trials}]}\\ &= \frac{\mathbb{P}[\text{4 of the last 4 AND 8 in the first 16 trials}]}{\mathbb{P}[\text{12 in 20 trials}]}\\ &= \frac{\mathbb{P}[\text{4 of the last 4}]\mathbb{P}[\text{8 in the first 16 trials}]}{\mathbb{P}[\text{12 in 20 trials}]}\\ &= \frac{p^4\binom{16}{8}p^8(1-p)^8}{\binom{20}{12}p^{12}(1-p)^8} = \frac{\binom{16}{8}}{\binom{20}{12}}.\end{aligned}$$

Our final answer is

$$1 - \frac{4\binom{16}{9}}{\binom{20}{12}} - \frac{\binom{16}{8}}{\binom{20}{12}}.$$

**Problem 2.6.** (30 points) You roll a hypothetical fair 25−sided die 225 times. Its sides are numbered $1, 2, \ldots, 25$. If you roll at least 10, you get a garden gnome.

(i) (5 points) What is the distribution of the number of garden gnomes you collect? You need to write down its name and the numerical values of all of its parameters.

(ii) (10 points) What is the expression for the **exact** probability that you collect more than 145 garden gnomes? (Use sigma notation here.)

(iii) (15 points) Use the appropriate approximation to estimate the above probability. Remember to **round up** if the last digit is 5.

**Solution:**

(i) The distribution is binomial with parameters $n = 225$ and $p = \frac{16}{25}$. Symbolically written: $Binomial\left(n = 225, p = \frac{16}{25}\right)$.

(ii)

$$\mathbb{P}[\text{more than 145 garden gnomes}] = \sum_{i=146}^{225} \mathbb{P}[X = i] = \sum_{i=146}^{225} \binom{225}{i} \left(\tfrac{16}{25}\right)^i \left(\tfrac{9}{25}\right)^{1000-i}.$$

(iii) Let's check the usual conditions.

$$np = 225\left(\frac{16}{25}\right) = 144 > 10,$$
$$n(1-p) = 225\left(\frac{9}{25}\right) = 81 > 10.$$

Therefore, we can use the normal approximation to the binomial distribution. The parameters are

$$\mu = np = 144 \quad \text{and} \quad \sigma = \sqrt{225 \cdot \frac{16}{25} \cdot 925} = \sqrt{\frac{4^2 \cdot 9^2}{5^2}} = \frac{36}{5} = 7.2.$$

We must use the continuity correction. We get

$$\mathbb{P}[\text{more than 145 garden gnomes}] = \mathbb{P}[\text{at least 146 garden gnomes}]$$
$$\approx 1 - \Phi\left(\frac{146 - 0.5 - 144}{7.2}\right)$$
$$\approx 1 - \Phi(0.21) = 1 - 0.5832 = 0.4168.$$

**Problem 2.7.** (10 points) Bertie repeatedly plays the following game at the *Drones Club*. First, he rolls a fair icosahedron whose sides are numbered 1 through 20. If the outcome on the icosahedron is 13, then he independently rolls a fair dodecahedron with sides are numbered 1 through 12. If the result of this roll is a 7, Bertie wins a stuffed parrot. Using the Poisson approximation, what is the approximate probability that Bertie wins at least 2 stuffed parrots in 480 iterations of the game?

*Note:* Your answer should contain the exponential function. You should leave it in your expression, but otherwise simplify as much as possible.

**Solution:** Here, the probability of success is $p = \frac{1}{20} \cdot \frac{1}{12} = \frac{1}{240}$. Hence, with $n = 480$ iterations, we have $np = 2 < 10$. We should use the Poisson approximation.

$$\begin{aligned}\mathbb{P}[\text{at least 2 ''prizes'' were won}] &= 1 - \mathbb{P}[\text{at most 1 ''prize'' was won}] \\ &= 1 - e^{-2} - 2e^{-2} = 1 - 3e^{-2}.\end{aligned}$$