Homework assignment #1

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Definitions

Problem 1. (5 points)

Write down the definition of *independence* of two *events*.

Solution: Two events A and B are said to be *independent* if

$$\mathbb{P}[A \cap B] = \mathbb{P}[A] \cdot \mathbb{P}[B]$$

Problem 2. (5 points)

Write down the definition of the *cumulative distribution function* of a random variable.

Solution: Let X be a random variable. Its *cumulative distribution function* is a function $F_X : \mathbb{R} \to [0, 1]$ defined by

$$F_X(x) = \mathbb{P}[X \leq x], \quad \text{for every } x \in \mathbb{R}.$$

Problems

Problem 3. (9 points)

Let $\Omega = \{a_1, a_2, a_3, a_4, a_5\}$ be an outcome space, and let $\mathbb{P}$ be a probability on Ω. Assume that $\mathbb{P}[A] = 0.5$, $\mathbb{P}[B] = 0.4$, $\mathbb{P}[C] = 0.4$, and $\mathbb{P}[D] = 0.2$, where

$$\begin{aligned} &A = \{a_1, a_2, a_3\},\ B = \{a_2, a_3, a_4\}, \\ &C = \{a_3, a_5\} \text{ and } D = \{a_4\}. \end{aligned}$$

Are the events A and B independent? Why?

Solution: We need to check whether $\mathbb{P}[A \cap B] = \mathbb{P}[A] \times \mathbb{P}[B]$. Since

$$\begin{aligned} \mathbb{P}[A \cap B] &= \mathbb{P}[\{a_2, a_3\}] \\ &= \mathbb{P}[\{a_2, a_3, a_4\}] - \mathbb{P}[\{a_4\}] \\ &= \mathbb{P}[B] - \mathbb{P}[D] = 0.2 \end{aligned}$$

and $\mathbb{P}[A] \times \mathbb{P}[B] = 0.5 \times 0.4 = 0.2$, we conclude that A and B *are* independent.

Textbook problems

Problem 4. (4 points)

Solve **Problem 3.4** from the textbook.

Solution: For any $k = 1, \dots, 6$, we have the following probability of any roll

$$\mathbb{P}[k \text{ on both dice}] = \mathbb{P}[k \text{ on first die and } k \text{ on the second die}].$$

Due to **independence**, the above probability is equal to

$$\mathbb{P}[k \text{ on first die}] \times \mathbb{P}[k \text{ on second die}] = \left(\frac{1}{6}\right)\left(\frac{1}{6}\right) = \frac{1}{36}.$$

So, the probability of *your* two independent rolls is $\left(\frac{1}{36}\right)^2$.

Similarly, the probability of your *friend's* rolls is $\left(\frac{1}{36}\right)^2$.

Problem 5. (4 points)

Solve **Problem 3.6** from the textbook.

Part a.

It's impossible to obtain a sum of 1 since the minimum on each die is a 1. So, the required probability is **zero**.

Part b.

We can get a sum of 5 as a $1+4$ or $2+3$ or $3+2$ or $4+1$ – **order matters!**. So, the total probability of obtaining a sum of 5 is

$$4\left(\frac{1}{6}\right)\left(\frac{1}{6}\right) = \left(\frac{4}{36}\right) = \left(\frac{1}{9}\right).$$

Part c.

The event that the sum on the two dice is 12 is exactly the event that both rolls result in a 6. So, the required probability is $\left(\frac{1}{6}\right)\left(\frac{1}{6}\right) = \frac{1}{36}$.

Problem 6. (a-e are 2 points each; f is 5 points=15 points total)

Solve **Problem 3.8** from the textbook.

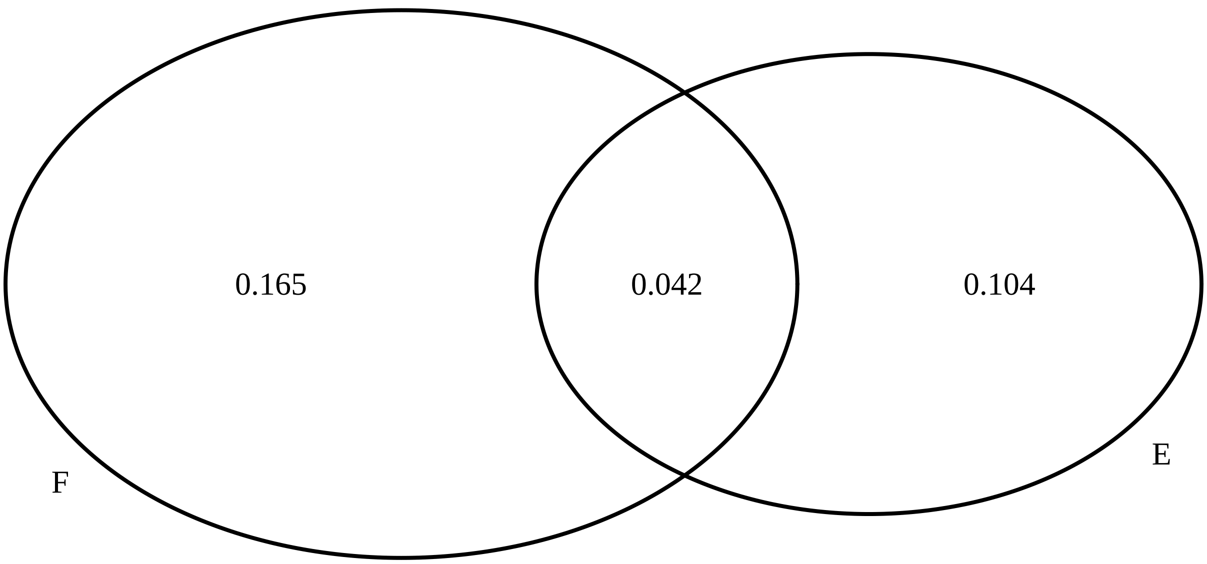
Solution: Let E denote the event that a person lives below poverty line and let F denote the event that a person speaks a language other than English at home. We are given that $\mathbb{P}[E] = 0.146$, $\mathbb{P}[F] = 0.207$, and $\mathbb{P}[E \cap F] = 0.042$

Part a.

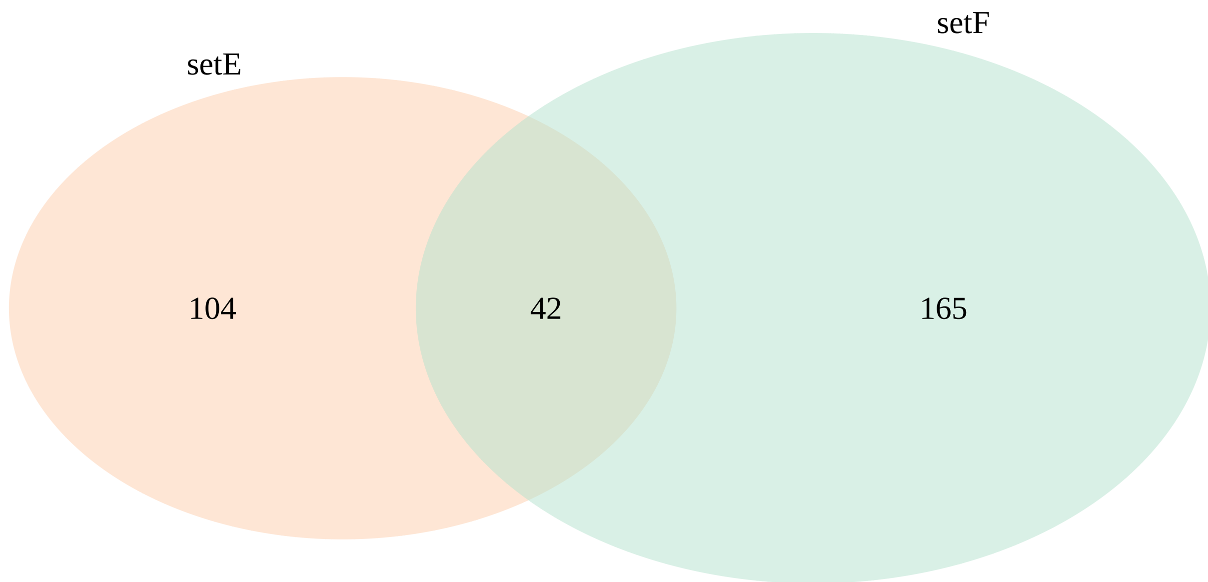
Since we are given that the probability $\mathbb{P}[E \cap F] = 0.042 > 0$, these two events **cannot be disjoint** (in other words, they **cannot be mutually exclusive**).

Part b.

```r
library(VennDiagram)
## Loading required package: grid
## Loading required package: futile.logger
venn1<-draw.pairwise.venn(0.146, 0.207, 0.042, category=c("E","F"))
```



I drew another Venn diagram in a different style, just for laughs.



Part c.

$$\mathbb{P}[E \setminus F] = 0.146 - 0.042 = 0.104$$

Part d.

Using the *Inclusion-Exclusion Formula*:

$$\mathbb{P}[E \cup F] = \mathbb{P}[E] + \mathbb{P}[F] - \mathbb{P}[E \cap F] = 0.146 + 0.207 - 0.042 = 0.311.$$

Part e.

This is the probability of the complement of the event from **Part d.**. We have

$$\mathbb{P}[E^c \cap F^c] = 1 - 0.311 = 0.689.$$

Part f.

By the definition of independence, we would need to have

$$\mathbb{P}[E \cap F] = \mathbb{P}[E] \times \mathbb{P}[F].$$

However, it is given that $\mathbb{P}[E \cap F] = 0.042$. On the other hand,

$$\mathbb{P}[E] \times \mathbb{P}[F] = 0.146(0.207) = 0.030222.$$

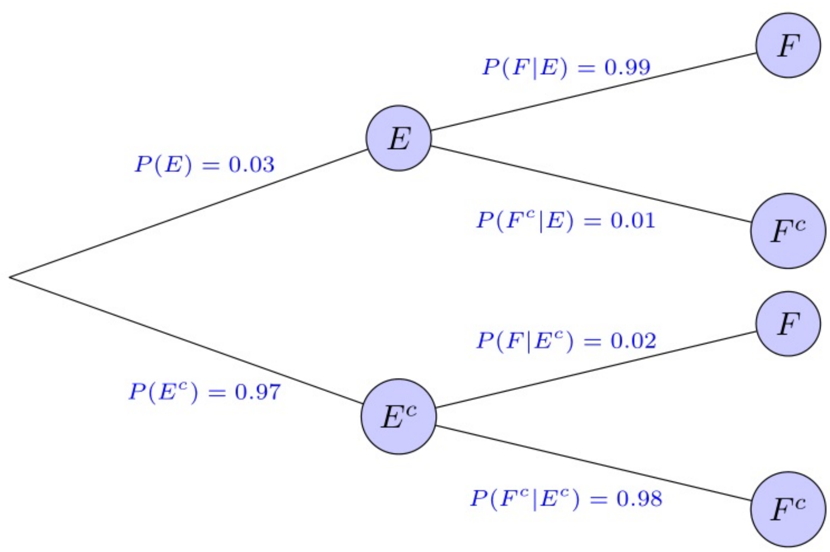
Since the two values are different, the events E and F are **not independent**.

Problem 7. (4 points)

Solve **Problem 3.20** from the textbook.

Solution: Let E denote the event that a randomly chosen individual has the *predisposition*. Let F denote the event that a randomly chosen individual *tests positive*. We are given that $\mathbb{P}[E] = 0.03$, $\mathbb{P}[F \mid E] = 0.99$, and $\mathbb{P}[F^c \mid E^c] = 0.98$.
Here is the probability tree generated by the above probabilities.

```
knitr::include_graphics("tree.jpg")
```



We are asked to calculate $\mathbb{P}[E \mid F]$. By Bayes' Theorem and using the probabilities in the above tree, we have

$$\mathbb{P}[E \mid F] = \frac{\mathbb{P}[E \cap F]}{\mathbb{P}[F]} = \frac{0.03(0.99)}{0.03(0.99) + 0.97(0.02)} = 0.604888.$$

Problem 8. (4 points)

Solve **Problem 3.22** from the textbook.

Solution: Using similar reasoning to the one in the previous problem, we have

$$\mathbb{P}[Walker \mid college] = \frac{0.53(0.37)}{0.53(0.37) + 0.47(0.44)} = 0.4867213.$$