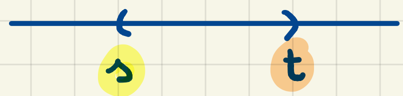
M339D : April 7th, 2023.

# Log-Normal Stock Prices.

Temporarily fix a time-horizon $T$.

$S(t), t \in [0,T]$ ... time-$t$ stock price



Define
$$R(s,t) := \ln\left(\frac{S(t)}{S(s)}\right)$$

In other words: $S(t) = S(s)e^{R(s,t)}$

In particular: $R(0,T)$ ... realized return over $(0,T)$

We model realized returns as normal

$R(0,T) \sim Normal(mean = m, \boxed{variance = \nu^2})$

$\Rightarrow$ $S(T)$ is lognormal

and $\mathbb{E}^*[S(T)] = S(0)e^{m+\frac{\nu^2}{2}}$ ☆

## Market model.

- Riskless Asset w/ ccrfir $r$
- Risky Asset : a non-dividend-paying stock
  - $\boxed{\sigma \text{.. volatility}}$

Under the risk-neutral measure $\boxed{\mathbb{E}^*[S(T)] = S(0)e^{rT}}$ ☆☆

Equating: ☆ & ☆☆

$$\boxed{m + \frac{\nu^2}{2} = rT} \Rightarrow \boxed{m = rT - \frac{\nu^2}{2}}$$

Consider: $Var[R(0,T)] = \nu^2 = ?$

Recall: $Var[R(0,1)] = \sigma^2$, i.e., $SD[R(0,1)] = \sigma$

$Var[R(0,T)] = \sigma^2 \cdot T$, i.e., $SD[R(0,T)] = \sigma\sqrt{T}$

$\Rightarrow \boxed{\nu^2 = \sigma^2 \cdot T}$

Finally:

$$\boxed{R(0,T) \sim \text{Normal}\left(\text{mean} = \left(r - \frac{\sigma^2}{2}\right)\cdot T,\ \text{var} = \sigma^2 \cdot T\right)}$$

Say that $Z \sim N(0,1)$.

Then, I can express $R(0,T)$ as

$$\boxed{R(0,T) = \left(r - \frac{\sigma^2}{2}\right)\cdot T + \sigma\sqrt{T}\cdot Z}$$

Hence,

$$\boxed{S(T) = S(0)\cdot e^{\left(r - \frac{\sigma^2}{2}\right)\cdot T + \sigma\sqrt{T}\cdot Z}}$$

Q: What is the median of $S(T)$ under $\mathbb{P}^*$?

$\longrightarrow$ : $\boxed{S(0)e^{\left(r - \frac{\sigma^2}{2}\right)\cdot T}}$

Note:

$$\frac{\text{mean}}{\text{median}} = \frac{\cancel{S(0)}e^{\cancel{rT}}}{\cancel{S(0)}\,e^{(\cancel{r} - \frac{\sigma^2}{2})\cdot T}} = e^{\frac{\sigma^2 \cdot T}{2}}$$

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# Problem Set # 100

## Mean and median of the log-normal stock prices.

**Problem 100.1.** The current price of a non-dividend-paying stock is \$80 per share. Under the risk-neutral probability measure, its mean rate of return is 12% and its volatility is 30%.

Let $R(0,t)$ denote the realized return of this stock over the time period $[0,t]$ for any $t > 0$. Calculate $\mathbb{E}^*[R(0,2)]$.

$\rightarrow$ : $r = 0.12$

$$R(0,2) \sim Normal\left(\boxed{mean = \left(r - \frac{\sigma^2}{2}\right) \cdot 2}, var = \sigma^2 \cdot 2\right)$$

$$\mathbb{E}^*[R(0,2)] = \left(0.12 - \frac{0.09}{2}\right) \cdot 2 = 0.075 \cdot 2 = 0.15$$

$\square$

**Problem 100.2.** A stock is valued at \$75.00. The continuously compounded, risk-free interest rate is 10.0% and the standard deviation of annualized returns is 25.0%. If the stock is lognormally distributed, what is the expected stock price after 2 years under the risk-neutral probability measure?

$\rightarrow$ : $$\mathbb{E}^*[S(2)] = S(0)e^{2r} = 75e^{2 \cdot 0.10} = 75e^{0.20} \approx 91.605$$

$\square$

**Problem 100.3.** A non-dividend-paying stock is valued at \$55.00 per share. Its standard deviation of annualized returns is given to be 22.0%. The continuously compounded risk-free interest rate is 12%. If the stock price is modeled using the lognormal distribution (as discussed in class), what is the median of the stock price in 3 years under the risk-neutral probability measure?

$\rightarrow$ : $$\underline{S(0)e^{\left(r - \frac{\sigma^2}{2}\right) \cdot T}} = 55e^{\left(0.12 - \frac{0.22^2}{2}\right) \cdot 3}$$

$$= \underline{73.313}$$

$\square$

**Problem 100.4.** Assume that the stock price is modeled using the lognormal distribution. The stock pays no dividends. Under $\mathbb{P}^*$, the annual mean rate of return on the stock is given to be 12%. Also under $\mathbb{P}^*$, the median time$-t$ stock price is evaluated to be $S(0)e^{0.1t}$. What is the volatility parameter of this stock price?

→ :

$$\left(r - \frac{\sigma^2}{2}\right) \cdot \cancel{t} = 0.1 \cdot \cancel{t}$$

$$0.12 - \frac{\sigma^2}{2} = 0.1$$

$$\sigma^2 = 2 \cdot 0.02 = 0.04 \Rightarrow \boxed{\sigma = 0.2}$$

□

**Problem 100.5.** The current stock price is $100 per share. The stock price at any time $t > 0$ is modeled using the lognormal distribution. Assume that the continuously compounded risk-free interest rate equals 8%. Let its volatility equal 20%.

Find the value $t^*$ at which the median stock price equals $120, under the risk-neutral probability measure.

→ :

median time-t stock price $= S(0)e^{\left(r - \frac{\sigma^2}{2}\right)t}$

$$\overset{1.2}{\cancel{120}} = 1\cancel{00} \cdot e^{\left(0.08 - \frac{0.04}{2}\right) \cdot t^*}$$

$$t^* = \frac{\ln(1.2)}{0.06} = \underline{3.039}$$

□

**Problem 100.6.** The volatility of the price of a non-dividend-paying stock is 0.2. The stock price is modeled using a log-normal distribution. Under $\mathbb{P}^*$, the expected time$-2$ stock price is $120. What is the median of the time$-2$ stock price under $\mathbb{P}^*$?

→ :

the median time·2 stock price:

$$\mathbb{E}^*[S(2)] \cdot e^{-\frac{2\sigma^2}{2}} = 120\, e^{-0.04} = \underline{115.295}$$

□

# Log Normal Tail Probabilities.

**Example.** Consider a non-dividend-paying stock. What is the probability that the stock outperforms a risk-free investment under the risk-neutral probability measure?

→: The initially invested amount: $S(0)$

- If it's the risk-free investment, the balance @ time $T$ is $S(0)e^{rT}$
- If it's the stock investment, the wealth @ time $T$ is $S(T)$

$$\boxed{\mathbb{P}^*[S(T) > S(0)e^{rT}] = ?}$$

This question is equivalent to the question of whether the profit is positive under $\mathbb{P}^*$.

$$\mathbb{P}^*[S(T) - S(0)e^{rT} > 0] = ?$$

In the Black-Scholes model:

$$\boxed{S(T) = S(0)e^{(r-\frac{\sigma^2}{2})\cdot T + \sigma\sqrt{T}Z} \quad \text{with } Z \sim N(0,1)}$$