M339D: February 22nd, 2023.

Arbitrage Portfolios.

Def'n. An **arbitrage portfolio** is a portfolio whose **profit** is:

- **nonnegative** in **all** states of the world

and

- **strictly positive** in **at least one** state of the world.

Unless it's specified otherwise in a particular problem/example, we assume **NO ARBITRAGE**.

Law of the Unique Price.

Assume that the payoffs of the two static portfolios A and B are equal, i.e.,

$$V_A(T) = V_B(T)$$

In general, two random variables, X and Y, are said to be **equal** if

$$P[X = Y] = 1$$

On a finite probability space, this means that they must take the exact same value on every elementary outcome.

Our claim:

$$V_A(0) = V_B(0)$$

Proof: Assume, to the contrary, that

$$V_A(0) \neq V_B(0) \qquad \text{X}$$

Without loss of generality, say,

$$V_A(0) < V_B(0)$$

$V_A(0)$: relatively cheap; $V_B(0)$: relatively expensive

Propose an arbitrage portfolio.

- Long portfolio A
- short portfolio B

} Total Portfolio

Verify:

- Payoff (Total Portfolio) = $V_A(T) - V_B(T) = 0$
- Initial Cost (Total Portfolio) = $V_A(0) - V_B(0) < 0$

↑ Inflow of money @ time 0

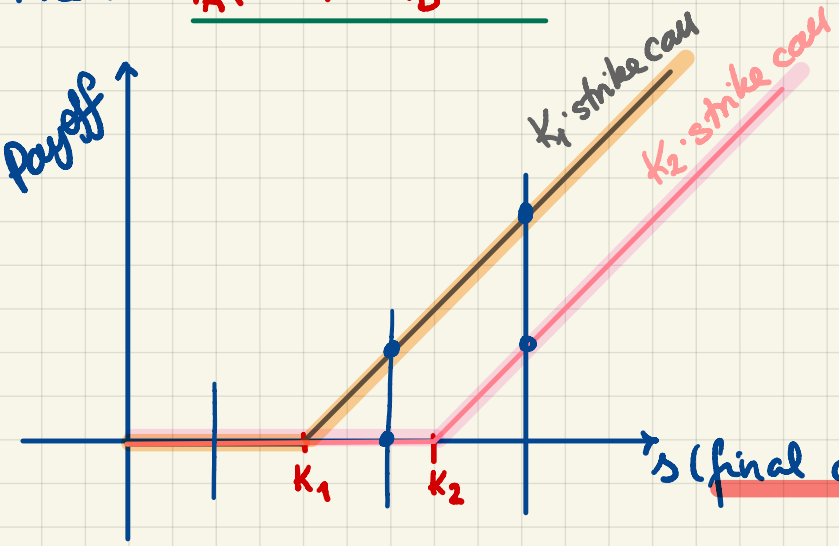
$$\text{Profit} = \text{Payoff} - FV_{0,T}(\text{Initial Cost})$$
$$= 0 - FV_{0,T}\left(V_A(0) - V_B(0)\right) > 0$$

↑ Indeed, this is an arbitrage portfolio!

$\Rightarrow\Leftarrow$ □

Remark: If $V_A(T) \geq V_B(T)$, then $V_A(0) \geq V_B(0)$.

Example.



- A = one long K_1-strike call
- B = one long K_2-strike call

w/ same underlying and exercise date

The K_1-strike call costs @ least as much as the K_2-strike call.

Replicating Portfolio.

Def'n. Consider a European-style derivative security. A static portfolio w/ the same payoff as that of the derivative security is called its **replicating portfolio**.

Note: The initial price of the derivative security must be equal to the initial price of its replicating portfolio.