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M339D=M389D Introduction to Actuarial Financial Mathematics
University of Texas at Austin
In-Term Exam II
Instructor: Milica Čudina

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The maximum number of points on this part of the exam is 65.

Provide your **complete solution** to the following problems. Final answers only, without appropriate justification, will receive zero points even if correct.

2.1. **FREE-RESPONSE PROBLEMS.**

Problem 2.1. (20 points) Consider a non-dividend-paying stock whose current price is \$120 per share. The stock has the volatility equal to 0.20.

Let the continuously-compounded, risk-free interest rate be equal to 0.05.

You model the evolution of this stock over the next quarter with a **forward** binomial tree.

What is the price of a \$122-strike, three-month put on the above stock consistent with this model?

Solution: In our usual notation, in the forward binomial tree, the risk-neutral probability is

$$p^* = \frac{1}{1+e^{\sigma\sqrt{h}}} = \frac{1}{1+e^{0.2\sqrt{1/4}}} = \frac{1}{1+e^{0.1}} = 0.4750208$$

The *up* and *down* factors are

$$u = e^{rh+\sigma\sqrt{h}} = e^{0.05(1/4)+0.1} = e^{0.1125},$$
$$d = e^{rh-\sigma\sqrt{h}} = e^{0.05(1/4)-0.1} = e^{-0.0875}.$$

Hence, the two possible stock prices at the end of the period are $S_u = 120e^{0.1125} = 134.2887$ and $S_d = 120e^{-0.0875} = 109.9463$. So, the option is in the money only in the *down* node where the payoff equals

$$V_d = (K - S_d)_+ = 12.05374.$$

By the risk neutral pricing formula, we have that

$$V_P(0) = e^{-0.05(1/4)}(1-0.4750208)(12.05374) = 6.249356.$$

Alternatively, the replicating portfolio has the following components

$$\Delta = \frac{V_u - V_d}{S_u - S_d} = -\frac{12.05374}{134.2887 - 109.9463} = -0.4951747,$$
$$B = e^{-rh}\frac{uV_d - dV_u}{u-d} = -e^{0.0125}\frac{e^{-0.0875}(12.05374)}{e^{0.1125} - e^{-0.0875}} = 65.6703.$$

So,

$$V_P(0) = \Delta S(0) + B = 6.249336.$$

Problem 2.2. (10 points) An archer shoots at a target repeatedly. Assume that their attempts are independent and that the probability of hitting bull's eye in any single attempt equals 1/3. The total number of times the archer shoots at the target is 81. Using the *normal approximation to the binomial*, what is the approximate probability that the archer hits bull's eye at least 26 times?

Solution: *The following solution uses the continuity correction. The solutions without the continuity correction would also be graded as correct if otherwise accurate.*
The number of trials is 81. The probability of hitting bull's eye in a single trial is 1/3. So, the total number of hits is, in our usual notation,

$$X \sim Binomial(n = 81, p = 1/3).$$

The probability we are looking for is $\mathbb{P}[X \geq 26]$. The mean of the random variable X is $np = 27$ and its standard deviation is $\sqrt{np(1-p)} = 4.242641$. Using the normal approximation to the binomial, we get

$$\mathbb{P}[X \geq 26] = \mathbb{P}[X > 25.5] = \mathbb{P}\left[\frac{X - 27}{4.242641} > \frac{25.5 - 27}{4.242641}\right] = 1 - \Phi\left(-0.3535534\right) \approx 0.638163.$$

Problem 2.3. (20 points) Consider a non-dividend-paying stock whose price $\mathbf{S} = \{S(t), t \geq 0\}$ is modeled using the Black-Scholes framework. Suppose that the current stock price equals \$80 and that its volatility is given to be 0.20.

The continuously compounded, risk-free interest rate is assumed to equal 0.04.

Consider a derivative security which entitles its owner to obtain a European call option on the above stock six months from today, i.e., at time $t_* = 1/2$. The call option is to be 3−month to expiration at time of delivery and be at-the-money at time−t^*. This contract is an example of a **forward start option**.

What is the price of this forward start option?

Solution: At time $t*$, the required Black-Scholes price of the call option equals

$$\begin{aligned} V_C(t^*) &= S(t^*)N(d_1) - S(t^*)e^{-r(T-t^*)}N(d_2) \\ &= S(t^*)(N(d_1) - e^{-0.01}N(d_2)) \end{aligned}$$

with

$$\begin{aligned} d_1 &= \frac{1}{0.1}\left(0.04 + \frac{0.2^2}{2}\right) \times \frac{1}{4} = 0.15, \\ d_2 &= d_1 - \sigma\sqrt{T - t^*} = 0.15 - 0.1 = 0.05. \end{aligned}$$

So, $N(d_1) = 0.5596177$ and $N(d_2) = 0.5199388$. Hence,

$$V_C(t^*) = S(t^*)(0.5596177 - e^{-0.01} \times 0.5199388) = 0.04485238S(t^*).$$

So, one would need to buy exactly 0.04485238 shares of stock to be able to buy the call option in question at time−t^*. This amount of shares costs \$3.58819.

Problem 2.4. (15 points) Assume the Black-Scholes model. Consider a non-dividend-paying stock with the initial stock price of \$120 and volatility equal to 0.25.

The continuously-compounded, risk-free interest rate is 0.10.

Find

$$\mathbb{E}^*[S(1)\mathbb{I}_{[S(1)\geq 105]}].$$

Solution: According to the work done in class,

$$\mathbb{E}^*[S(1)\mathbb{I}_{[S(1)\geq 105]}] = \mathbb{E}^*[S(1)]N(d_1)$$

where

$$d_1 = \frac{1}{0.25\sqrt{1}}\left[\ln\left(\frac{120}{105}\right) + \left(0.10 + \frac{(0.25)^2}{2}\right)(1)\right] \approx 1.059126.$$

So,

$$\mathbb{E}[S(1)\mathbb{I}_{[S(1)\geq 105]}] = 120e^{0.10}N(1.059126) = 113.4209.$$