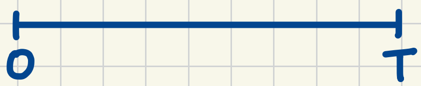
Replicating Portfolios.

Def'n. Consider a European-style derivative security. A static portfolio w/ the same payoff as that of the derivative security is called its replicating portfolio.

Note: Because we assume no arbitrage, the intial price of the derivative security is equal to the initial price of its replicating portfolio.

Example. Consider a forward contract on a non-dividend-paying stock/index.



Forward Contract: $S(T) - F$

Replicating Portfolio:
- long one share of stock
- issue a bond w/ redemption amount F and maturity date T

$$\text{Payoff (Portfolio)} = S(T) - F$$

no arbitrage

$\Rightarrow$ The forward contract and its replicating portfolio must have the same initial cost, i.e.,

$$0 = \underbrace{S(0)}_{\substack{\text{long}\\\text{stock}}} \underbrace{- PV_{0,T}(F)}_{\substack{\text{short}\\\text{bond}}}$$

$$\Rightarrow \quad PV_{0,T}(F) = S(0)$$

$$\Rightarrow \quad \boxed{F = FV_{0,T}(S(0)) = S(0)e^{rT}}$$

3. An insurance company sells single premium deferred annuity contracts with return linked to a stock index, the time-t value of one unit of which is denoted by $S(t)$. The contracts offer a minimum guarantee return rate of $g\%$. At time 0, a single premium of amount π is paid by the policyholder, and $\pi \times y\%$ is deducted by the insurance company. Thus, at the contract maturity date, T, the insurance company will pay the policyholder

$$\pi \times (1 - y\%) \times \text{Max}[S(T)/S(0), (1 + g\%)^T].$$

You are given the following information:

(i) The contract will mature in one year. $T=1$

(ii) The minimum guarantee rate of return, $g\%$, is 3%. $g=0.03$

(iii) Dividends are incorporated in the stock index. That is, the stock index is constructed with all stock dividends reinvested. NO DIVIDENDS!

(iv) $S(0) = 100$.

(v) The price of a one-year European $\boxed{\text{put option}}$ with strike price of \$103, on the stock index is \$15.21.

Determine $y\%$, so that the insurance company does not make or lose money on this contract.

(A) 12.8%.

(B) 13.0%

(C) 13.2%

(D) 13.4%

(E) 13.6%.

The Synopsis:

1. Focus on the insurance company's liability

2. Use our $\boxed{\text{data}}$

3. Algebraically simplify w/ an eye to "creating" the expresson including what's in the $\boxed{\text{data}}$

→: The Insurance Company's Liability:

$$\underbrace{\boxed{\pi(1-y)}}_{\text{const}} \cdot \text{Max}\left[\frac{S(T)}{\boxed{S(0)}}, (1+g)^T\right]$$

$\frac{1}{S(0)}$ const.

$$\text{Max}\Big[S(T), \underbrace{\underbrace{(1+g)^T}_{(1.03)^1} \cdot \underbrace{S(0)}_{100}}_{103}\Big]$$

$\text{Max}[S(T), 103] = ?$

$$V_P(T) = (103 - S(T))_+$$

a, b

$$\max(a,b) = a + \max(0, b-a) = a + (b-a)_+$$
$$= b + \max(0, a-b) = b + (a-b)_+$$

$$\text{Max}[S(T), 103] = \underbrace{\boxed{S(T)}}_{\text{Long stock}} + \underbrace{\boxed{(103 - S(T))_+}}_{\text{Payoff of the put w/ strike 103 and exercise date } T=1.}$$

The insurance company can **perfectly hedge** by:

- Longing/Buying $\frac{\pi(1-y)}{S(0)}$ units of the stock index;
- Buying $\frac{\pi(1-y)}{S(0)}$ European puts w/ $K = 103$ and $T = 1$.

If they receive the same amount of money @ time 0 as is the cost of this replicating portfolio, they break even.

$$\cancel{\pi} = \frac{\cancel{\pi}(1-y)}{S(0)}\left(S(0) + V_P(0)\right)$$

$$100 = (1-y)(100 + 15.21)$$

$$1 - y = \frac{100}{115.21} \Rightarrow y = \frac{15.21}{115.21} = 0.132$$

$\square$