UNIVERSITY OF TEXAS AT AUSTIN

Problem set 6

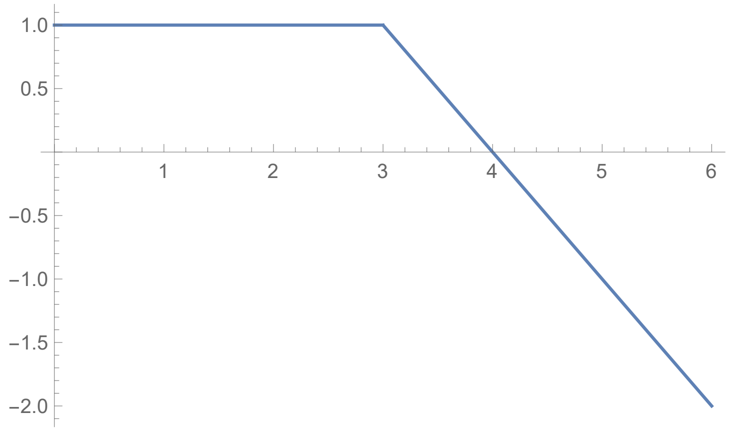
Problem 6.1. Let the function f be given by

$$f(x) = \begin{cases} 3-x & \text{for } x \geq 3 \\ 0 & \text{otherwise} \end{cases}$$

Draw the graph of the function g defined as

$$g(x) = f(x) + 1.$$

Solution:



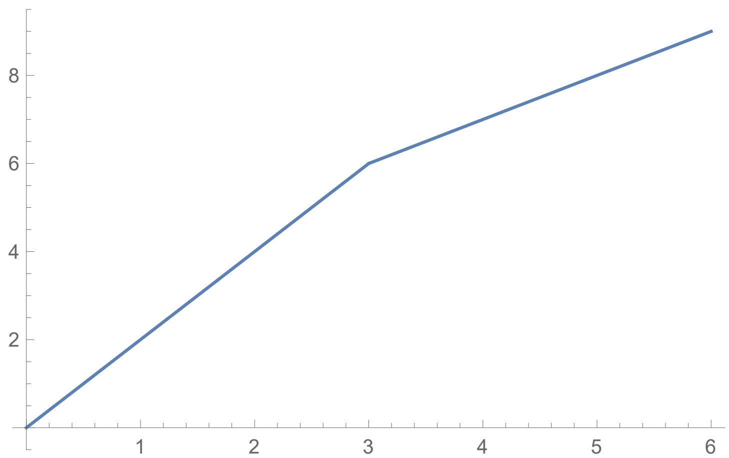
Problem 6.2. Let the function f be given by

$$f(x) = \begin{cases} 3-x & \text{for } x \geq 3 \\ 0 & \text{otherwise} \end{cases}$$

Draw the graph of the function g defined as

$$g(x) = f(x) + 2x.$$

Solution:



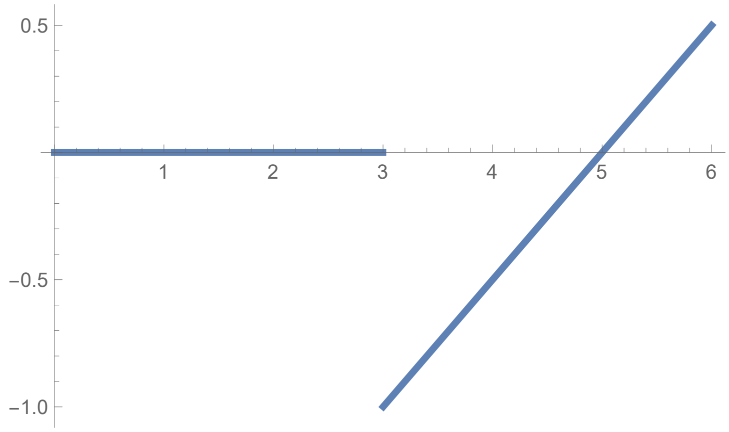
Problem 6.3. Let the function f be given by

$$f(x) = \begin{cases} x - 5 & \text{for } x \geq 3 \\ 0 & \text{otherwise} \end{cases}$$

Draw the graph of the function g defined as

$$g(x) = f(x)/2.$$

Solution:



Problem 6.4. (2 pts)

We define the minimum of two values in the usual way, i.e.,

$$\min(x, y) = \begin{cases} x & \text{if } x \leq y \\ y & \text{if } x \geq y \end{cases}$$

We define the maximum of two values in the usual way, i.e.,

$$\max(x, y) = \begin{cases} x & \text{if } x \geq y \\ y & \text{if } x \leq y \end{cases}$$

Then, for every x and y we have that

$$\min(x, y) + \min(x - y, 0) = y$$

True or false? Why?

Solution: FALSE

When $x \geq y$, we have that $x - y \geq 0$

$$\min(x, y) + \min(x - y, 0) = y + 0 = y.$$

When $x < y$, we have that $x - y < 0$

$$\min(x, y) + \min(x - y, 0) = x + x - y = 2x - y.$$

Problem 6.5. (2 pts)

We define the minimum of two values in the usual way, i.e.,

$$\min(x, y) = \begin{cases} x & \text{if } x \leq y \\ y & \text{if } x \geq y \end{cases}$$

We define the maximum of two values in the usual way, i.e.,

$$\max(x, y) = \begin{cases} x & \text{if } x \geq y \\ y & \text{if } x \leq y \end{cases}$$

Then, for every x and y we have that

$$\max(x, y) + \min(x, y) = x + y.$$

True or false?

Solution: TRUE
When $x \geq y$, we have

$$\max(x, y) + \min(x, y) = x + y.$$

When $x < y$, we have

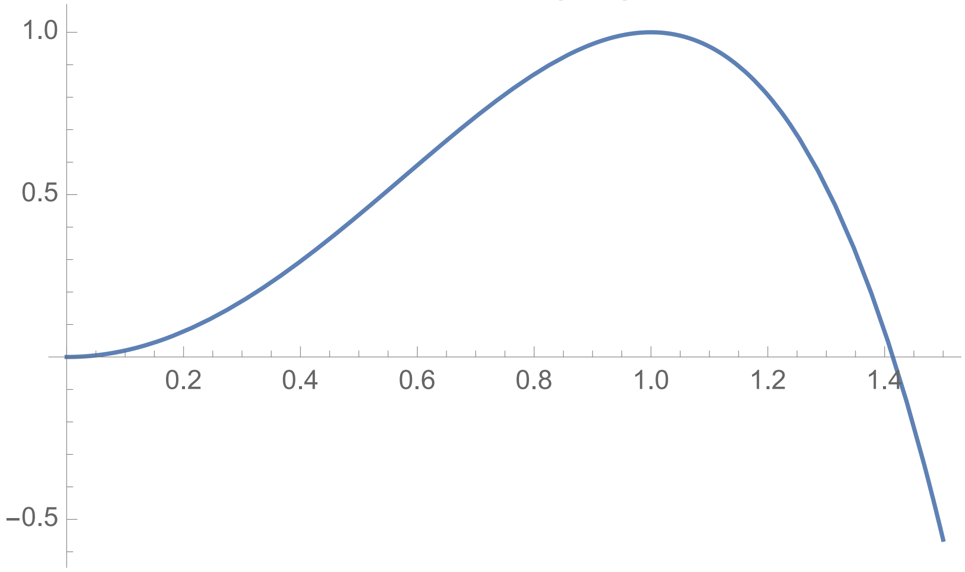
$$\max(x, y) + \min(x, y) = y + x = x + y.$$

Problem 6.6. (5 pts) Which of the following formulas hold for the exponential function:

(a) $e^x + e^y = e^{x+y}$
(b) $e^x e^y = e^x + e^y$
(c) $e^{x+y} = e^x e^y$
(d) $e^{x-y} = e^x - e^y$
(e) none of the above

Solution: The correct answer is **(c)**.
This is just the product rule for the exponential function.

Problem 6.7. (5 points) Consider a function $f : [0, 1.5] \to \mathbb{R}$ with the following graph



Then, this function is ...

(a) ...increasing.
(b) ...decreasing.
(c) ...both increasing and decreasing
(d) ...neither increasing, nor decreasing.
(e) None of the above.

Solution: (d)
The monotonicity statement is by default for the entire domain.

Problem 6.8. If events E and F are independent, then E^c and F^c are independent as well. *True or false?*

Solution: TRUE
We are given that E and F are independent, i.e.,

$$\mathbb{P}[E \cap F] = \mathbb{P}[E]\mathbb{P}[F].$$

Let us start with $\mathbb{P}[E^c \cap F^c]$. By de Morgan's laws, we have

$$\mathbb{P}[E^c \cap F^c] = \mathbb{P}[(E \cap F)^c].$$

Since the probability of the entire probability space Ω is 1, by the addition rule for the probability, we have

$$\mathbb{P}[(E \cap F)^c] = 1 - \mathbb{P}[E \cap F].$$

Using the inclusion-exclusion formula, the above equals

$$1 - \mathbb{P}[E \cap F] = 1 - (\mathbb{P}[E] + \mathbb{P}[F] - \mathbb{P}[E \cap F]) = 1 - \mathbb{P}[E] - \mathbb{P}[F] + \mathbb{P}[E \cap F].$$

Since E and F are independent, we have that the above equals

$$1 - \mathbb{P}[E] - \mathbb{P}[F] + \mathbb{P}[E]\mathbb{P}[F] = (1 - \mathbb{P}[E]) - \mathbb{P}[F](1 - \mathbb{P}[E]) = (1 - \mathbb{P}[E])(1 - \mathbb{P}[F]) = \mathbb{P}[E^c]\mathbb{P}[F^c].$$

Problem 6.9. (5 pts) Let Y be a random variable such that $\mathbb{P}[Y = 2] = 1/2$, $\mathbb{P}[Y = 3] = 1/3$ and $\mathbb{P}[Y = 6] = 1/6$. Then $\mathbb{E}[Y^2] = \dots$

(a) 1
(b) 2
(c) 3
(d) 11
(e) None of the above.

Solution: The correct answer is **(d)**.
Using the expression for the expectation of a function of a discrete random variable, we have

$$\mathbb{E}[Y^2] = 2^2\left(\frac{1}{2}\right) + 3^2\left(\frac{1}{3}\right) + 6^2\left(\frac{1}{6}\right) = 2 + 3 + 6 = 11.$$

Problem 6.10. Four fair coins are tossed. The probability that at least one for them fell on *heads* is

(a) $\frac{15}{16}$
(b) $\frac{1}{16}$
(c) $\frac{1}{4}$
(d) $\frac{3}{4}$
(e) None of the above.

Solution: The correct answer is **(a)**.
The probability we are looking for is

$$\mathbb{P}[\text{at least one of four coins was } \textit{heads}] = 1 - \mathbb{P}[\text{all four coins were } \textit{tails}] = 1 - \left(\frac{1}{2}\right)^4 = 1 - \frac{1}{16} = \frac{15}{16}.$$

Problem 6.11. Find the total amount of interest that would be paid on a \$1,000 loan over a 10−year period, if the effective interest rate is 0.09 per annum under the following repayment method:
The entire loan plus entire accumulated interest is paid as one lump-sum at the end of the loan term.

(a) \$900
(b) \$990
(c) \$1,367

(d) \$1,557

(e) None of the above

Solution: (c)
Using compound interest, the accumulated value at the end of the 10 years is

$$1000 \cdot 1.09^{10} \approx 2367.36.$$

The total amount of interest is

$$2367.36 - 1000 = 1367.36.$$

Problem 6.12. (5 pts) *Source: Sample FM Problem #26.*
A 5-year loan for 10,000 is charged a nominal interest rate of 12% compounded semiannually.
The loan is to be repaid so that interest is repaid at the end of every 6 month period as it accrues and the principal is repaid in total at the end of the 5 years.

Denote the total amount of interest paid on this loan by I. Then,

(a) $I \approx 2,750$

(b) $I \approx 3,000$

(c) $I \approx 3,250$

(d) $I \approx 3,500$

(e) None of the above

Solution: (e)

$$10 \cdot \frac{0.12}{2} \cdot 10,000 = 6,000.$$

Problem 6.13. Roger deposits opens a savings account at time-0. He does not make any subsequent withdrawals or deposits. The account earns at a continuously compounded, risk-free interest rate r.

After 15 years and 3 months, the balance in his account has doubled. Then,

(a) $0 \leq r < 0.0150$

(b) $0.0150 \leq r < 0.0250$

(c) $0.0250 \leq r < 0.0550$

(d) $0.0550 \leq r < 0.0650$

(e) None of the above

Solution: (c)
The unknown continuously compounded, risk-free interest rate r must satisfy

$$2 = e^{15.25r}.$$

So,

$$r = \ln(2)/15.25 \approx 0.04545227.$$