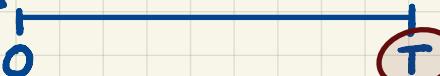
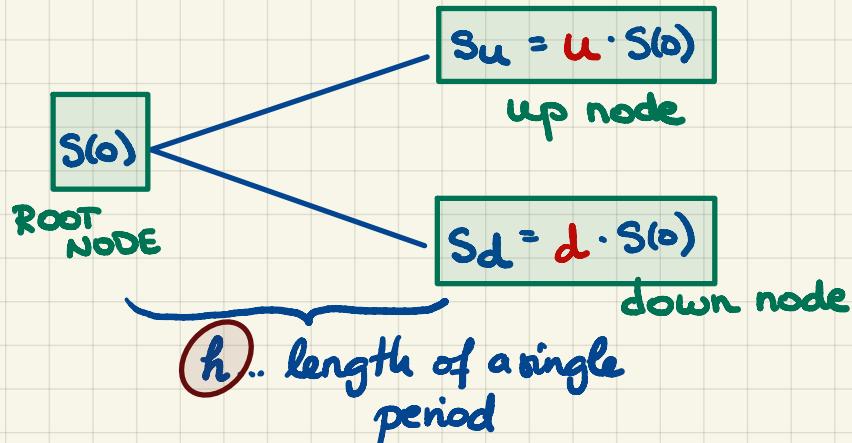


## The Binomial Asset Pricing Model.

$S(0)$ ... the observable initial asset price

0  T  
time horizon (e.g., the exercise date of an option)



By convention:

$$u > d$$

u... up factor  
d... down factor

one period:

$$S(T) = S(h)$$

... a random variable denoting the time  $T$  stock price w/ two possible values:  
 $S_u$  and  $S_d$

As a random variable : SIMPLE RETURN

$$\frac{S(T) - S(0)}{S(0)}$$

up •  $\frac{S_u - S(0)}{S(0)} = \frac{S_u}{S(0)} - 1 = u - 1$

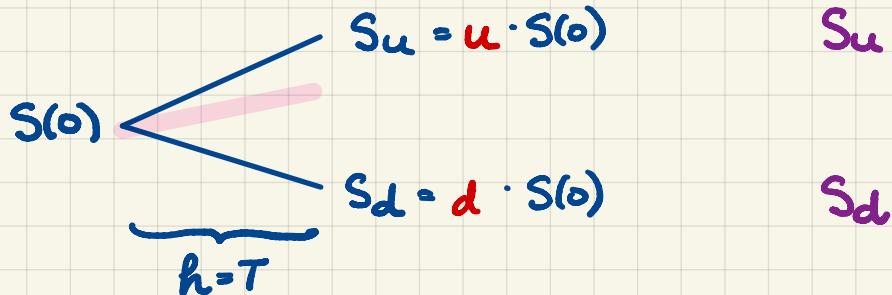
down •  $\frac{S_d - S(0)}{S(0)} = \frac{S_d}{S(0)} - 1 = d - 1$

## No Arbitrage Condition.

### Market Model.

- riskless asset: @ the ccfir  $r$
- risky asset: non-dividend-paying stock

Imagine investing in one share of this stock @ time 0.  
Wealth



At the risk-free rate, the amount  $S(0)$  accumulates to  $\underline{S(0)e^{rh}}$  in the same time period.

### The No Arbitrage Condition:

$$\begin{aligned} S_d &< S(0)e^{rh} < S_u \\ d \cdot S(0) &< S(0)e^{rh} < u \cdot S(0) \\ d &< e^{rh} < u \end{aligned}$$

### Half-a-Proof.

Say, to the contrary,  $e^{rh} \leq d < u$

Propose. Long one share of stock.

Verify. Profit = Payoff - FV<sub>0,T</sub>(Initial Cost)

$$= S(h) - S(0)e^{rh}$$

$$\text{down node: } S_d - S(0)e^{rh} = d \cdot S(0) - S(0)e^{rh} = S(0)(d - e^{rh}) \geq 0$$

$$\text{up node: } S_u - S(0)e^{rh} = u \cdot S(0) - S(0)e^{rh} = S(0)(u - e^{rh}) > 0$$

Indeed, this is an arbitrage portfolio.

UNIVERSITY OF TEXAS AT AUSTIN

Problem Set #9

Binomial option pricing.

**Problem 9.1.** In the setting of the one-period binomial model, denote by  $i$  the **effective interest rate per period**. Let  $u$  denote the “up factor” and let  $d$  denote the “down factor” in the stock-price model. If

$$d < u \leq 1 + i$$

then there certainly is no possibility for arbitrage.



fixed statement :

$$d < 1+i < u$$

**Problem 9.2.** In our usual notation, does this parameter choice create a binomial model with an arbitrage opportunity?

$$u = 1.18, \quad d = 0.87, \quad r = 0.05, \quad \delta = 0, \quad h = 1/4$$

No, it doesn't!

$$\rightarrow: \quad d = 0.87 \quad < \quad e^{rh} = e^{0.05/4} \quad < \quad 1.18 = u$$

$\underbrace{e^{0.0125}}$

$$1.0125$$

Taylor Expansion of  $e^x$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$$

Q: What if  $\tilde{d} = 1.01$ ?

It's still OK!

## Forward Binomial Tree.

"Def'n". The volatility  $\sigma$  is the standard deviation of realized returns on a continuously compound scale and annualized.

Heuristics.  $T = 1$

$$h_m = \frac{1}{m} \text{ (of a year)}$$

Q: What is the volatility for the time period of length  $h_m$ ?  
Call this volatility

$$\sigma_{h_m}$$