Name:

M339J: Probability models
University of Texas at Austin
In-Term Exam I
Instructor: Milica Čudina

Notes: This is a closed book and closed notes exam. The maximal score on this exam is 100 points.
Time: 50 minutes

Problem 1.1. (5 points) Let E and F be two events such that $\mathbb{P}[E] > 0$ and $\mathbb{P}[F] > 0$. You know that

$$\mathbb{P}[E \mid F] > \mathbb{P}[E].$$

Then,

$$\mathbb{P}[F \mid E] > \mathbb{P}[F].$$

True or false? Why?

Problem 1.2. (5 points) Provide the definition of the *cumulative distribution function*.

Problem 1.3. (10 points) Consider the following game: You toss a fair coin. If the coin comes up heads, you win \$10. If the coin comes up tails, you spin a perfectly balanced spinner which is equally likely to point to any points on the circumference of its base after being spun. You win the amound equal to the angle between the original position of the spinner and the final position of the spinner (modulo 2π, and in radians).

Let X be the random variable denoting the amount you win. Draw the graph of the cumulative distribution function of X.

Problem 1.4. (5 points) For any random variable X, we have that

$$\mathbb{E}[|X|] = |\mathbb{E}[X]|.$$

True or false? Why?

Problem 1.5. (2 points) Let the random variable X have the survival function $S_X(x) = e^{-x/80}$, for $x > 0$. Then the mean of that random variable equals $1/80$. *True or false? Why?*

Problem 1.6. (5 points) A simple experiment consists of drawing a single ball at random from each of two urns containing red and blue marbles. The first urn contains 1 red and 3 blue marbles. A second urn contains 12 red marbles and an unknown number of blue marbles. You are told that the probability that both marbles are the same color equals $9/20$. Calculate the number of blue marbles in the second urn.

Problem 1.7. (5 points) The local pool supply store is having an end-of-season sale. They have a seemingly infinite number of floaties lying around. They know that among those 1/4 are ancient floaties from seasons past (so, old) and that 3/4 are the last season's floaties (so, new). We know that 15% of old floaties leak, and that 5% of new floaties leak. When an order comes in, a floatie is chosen at random to fulfill the order. You are excited about the sale and you are the first one to show up at the door. You buy a floatie. You take it to the pool. It leaks. What's the probability that it was an old floatie?

Problem 1.8. (5 points) Let the random variables X_1, X_2 and X_3 be independent and identically distributed such that their probability mass function is

$$p_X(x) = \begin{cases} 1/4, & \text{for } x = 0, \\ 3/4, & \text{for } x = 1. \end{cases}$$

What is the expectation of $X_1X_2X_3$?

Problem 1.9. (8 points) A continuous random variable X has the probability density function f_X given by

$$f_X(x) = \frac{2}{5} - \kappa x, \quad 0 \leq x \leq 5.$$

Find the value of the survival function of X at 3.

Problem 1.10. (5 points) Consider the random variable X whose cumulative distribution function is given by

$$F_X(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{x}{2} & \text{for } 0 \leq x < 1 \\ 1 & \text{for } 1 \leq x \end{cases}$$

What is the expectation of the random variable X?

Problem 1.11. (10 points) The random variable T is exponentially distributed with mean 10. The probability that it takes a value between 2 and 5 equals the probability that it takes a value between 6 and t^*, with $t^* > 6$. What is t^*?

Problem 1.12. (10 points) Let a severity random variable X be uniform over $[0, 100]$. What is the value of its mean excess loss function at 40?

Problem 1.13. (10 points) Let X have the two-parameter Pareto distribution with $\alpha = 5$ and $\theta = 2$. Find the variance of X.

Problem 1.14. (10 points) Consider independent random variables X and Y. You are given that they have the same mean. Also, the coefficient of variation of X equals 36 and the coefficient of variation of Y equals 77. What is the coefficient of variation of the average of X and Y?

Problem 1.15. (5 points) Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables. Assume that all the variables in the sequence have the two-parameter Pareto distribution with $\theta = 10$ and $\alpha = 5$. For each n, define the random variable

$$Y_n = \frac{X_1^3 + X_2^3 + \cdots + X_n^3}{n}.$$

Does the limit of the sequence $\{Y_n, n \geq 1\}$ as $n \to \infty$ exist? If so, how much is it? If not, why not?