Problem 14.9. *(20 points)*
A random sample of size 10 *is drawn from a normal distribution with* **both** *mean and standard deviation* **unknown.** *The generated sample has the sample mean* $\bar{y}_{10} = 14$ *and the (unbiased) estimate of the variance* $\boxed{s^2 = 25.}$

$= 1-\alpha \Rightarrow \alpha = 0.10$

(i) (10 points) *Construct a (symmetric) 90%-confidence interval for* μ.

→ : Critical values of the t-dist'n w/ df = 10-1 = 9

$t_L^* = qt(0.05, df=9)$

$t_R^* = qt(0.95, df=9) = -t_L^*$

$\mu = \bar{y} \pm t_R^* \cdot \left(\frac{s}{\sqrt{n}}\right)$

$\mu = 14 \pm 1.833 \frac{5}{\sqrt{10}}$

$\boxed{qt(0.05, df=9)}$

(ii) (10 points) *Construct a (symmetric) 90%-confidence interval for* σ^2.
Hint: *Remember that you know the distribution of* $(n-1)S^2/\sigma^2$.

→ :

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(df = \underline{10-1})$$

$s^2 = 25$

$$a = \boxed{\text{qchisq}(0.05, df=9)} = 3.325$$

$$b = \boxed{\text{qchisq}(0.95, df=9)} = 16.92$$

The CI is :

$$\left(\frac{9 \cdot 25}{\underbrace{16.92}_{b}} , \frac{9 \cdot 25}{\underbrace{3.325}_{a}} \right)$$

□

<u>M378K Introduction to Mathematical Statistics</u>
<u>Problem Set #15</u>
<u>Relative efficiency.</u>

Definition 15.1. *Given two unbiased estimators $\hat{\theta}_1$ and $\hat{\theta}_2$, the efficiency of $\hat{\theta}_1$ relative to $\hat{\theta}_2$ is defined as*

$$\boxed{\mathit{eff}(\hat{\theta}_1, \hat{\theta}_2) = \frac{\mathrm{Var}[\hat{\theta}_2]}{\mathrm{Var}[\hat{\theta}_1]}.}$$

Problem 15.1. *Let Y_1, Y_2 be a random sample from the exponential distribution with the unknown parameter θ.*

$$Y_1, Y_2 \sim E(\tau = \theta)$$

(i) *The estimator $\hat{\theta}_1 = (Y_1 + Y_2)/2$ for θ is proposed. What is its variance?*

(ii) *The estimator $\hat{\theta}_2 = cY_{(1)}$ for θ is proposed. Find the constant c such that $\hat{\theta}_2$ is an unbiased estimator of θ. What is its variance?*

(iii) *Calculate the efficiency of $\hat{\theta}_1$ relative to $\hat{\theta}_2$.*

$$\Rightarrow \mathit{eff}(\hat{\theta}_1, \hat{\theta}_2) = \frac{\theta^2}{\theta^2/2} = 2$$

$\longrightarrow$: (i)

$$\mathrm{Var}[\hat{\theta}_1] = \mathrm{Var}\left[\tfrac{1}{2}(Y_1+Y_2)\right] =$$

$$= \tfrac{1}{4}\mathrm{Var}[Y_1+Y_2] = \quad \text{independence}$$

$$= \tfrac{1}{4}\left(\mathrm{Var}[Y_1]+\mathrm{Var}[Y_2]\right) \quad \text{identically dist'd}$$

$$= \frac{1}{\cancel{4}_{2}}(\cancel{2}\cdot\mathrm{Var}[Y_1]) = \frac{\mathrm{Var}[Y_1]}{2}$$

$$= \boxed{\frac{\theta^2}{2}}$$

$$\boxed{\mathrm{Var}[\bar{Y}] = \frac{\mathrm{Var}[Y_1]}{n}}$$

(ii) $Y_{(1)} = \min(Y_1, Y_2) \sim E(\theta/2)$

$$\mathbb{E}[\hat{\theta}_2] = \theta \iff \mathbb{E}[c \cdot Y_{(1)}] = \theta$$

$$\iff c \cdot \mathbb{E}[Y_{(1)}] = c \cdot \frac{\theta}{2} = \theta \Rightarrow \boxed{c = 2}$$

$$\Rightarrow \mathrm{Var}[\hat{\theta}_2] = \mathrm{Var}[c \cdot Y_{(1)}] = 4 \cdot \mathrm{Var}[Y_{(1)}]$$

$$= \cancel{4} \cdot \frac{\theta^2}{\cancel{4}} = \boxed{\theta^2}$$

□

Example. $Y_1, Y_2, \dots, Y_n \sim E(\tau)$

$$\boxed{Y_{(1)} \sim E(\tau/n)}$$