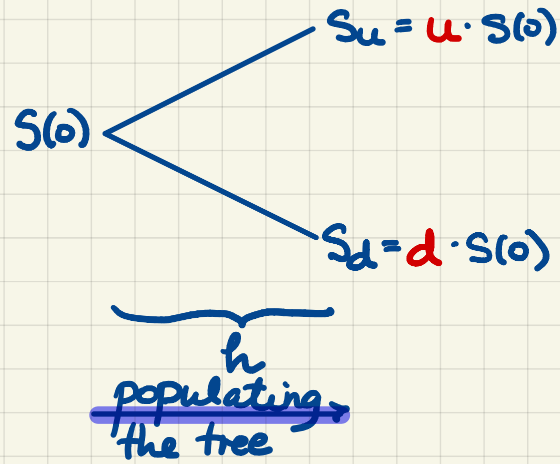
Binomial Option Pricing.

10/23/2024.

Stock Price Tree.



Goal: Pricing a European-style derivative security w/ exercise date @ the end of the tree, i.e., $T=h$.

It is completely determined by its payoff f'tion: $v(\cdot)$

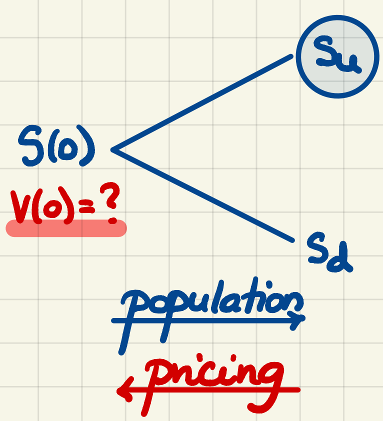
e.g., for a call: $v_c(s) = (s-K)_+$,

for a put: $v_p(s) = (K-s)_+$,

for a power option: $\boxed{v(s) = (s^2-K)_+}$

The payoff of a derivative security is a random variable

$$\boxed{V(T) := v(S(T))}$$



PAYOFF		REPLICATING PORTFOLIO
$V_u := v(S_u)$	$=$	$\Delta \cdot S_u + Be^{rh}$
$V_d := v(S_d)$	$=$	$\Delta \cdot S_d + Be^{rh}$

In the binomial model, any derivative security can be REPLICATED w/ a portfolio of this form:

@ time 0

- Δ shares of stock $\begin{cases} \Delta > 0 & \text{buying} \\ \Delta = 0 & \text{"nothing"} \\ \Delta < 0 & \text{short selling} \end{cases}$

and

- B @ the ccrfir r $\begin{cases} B > 0 & \text{lending (buying a bond)} \\ B = 0 & \text{"nothing"} \\ B < 0 & \text{borrowing (issuing a bond)} \end{cases}$

If we can calculate Δ and B, then $\boxed{V(0) = \Delta \cdot S(0) + B}$

We get a system of two eq'ns w/ two unknowns:

$$\left.\begin{aligned} \Delta \cdot S_u + \cancel{Be^{rh}} &= V_u \\ -\ \Delta \cdot S_d + \cancel{Be^{rh}} &= V_d \end{aligned}\right\} -$$

$$\Delta (S_u - S_d) = V_u - V_d$$

$$\boxed{\Delta = \frac{V_u - V_d}{S_u - S_d}} \qquad \text{unitless / shares of stock}$$

$$\frac{V_u - V_d}{S_u - S_d} \cdot S_u + Be^{rh} = V_u$$

$$Be^{rh} = V_u - \frac{V_u - V_d}{\cancel{S(0)}(u-d)} \cdot \cancel{S(0)} \cdot u = \frac{\cancel{u \cdot V_u} - d \cdot V_u - \cancel{u \cdot V_u} + u \cdot V_d}{u-d}$$

$$\boxed{B = e^{-rh} \cdot \frac{u \cdot V_d - d \cdot V_u}{u-d}} \qquad \text{cash (\$)}$$

Graphical Interpretation.

Consider a European call w/ exercise date @ the end of the period and strike price K such that $\boxed{S_d < K < S_u}$

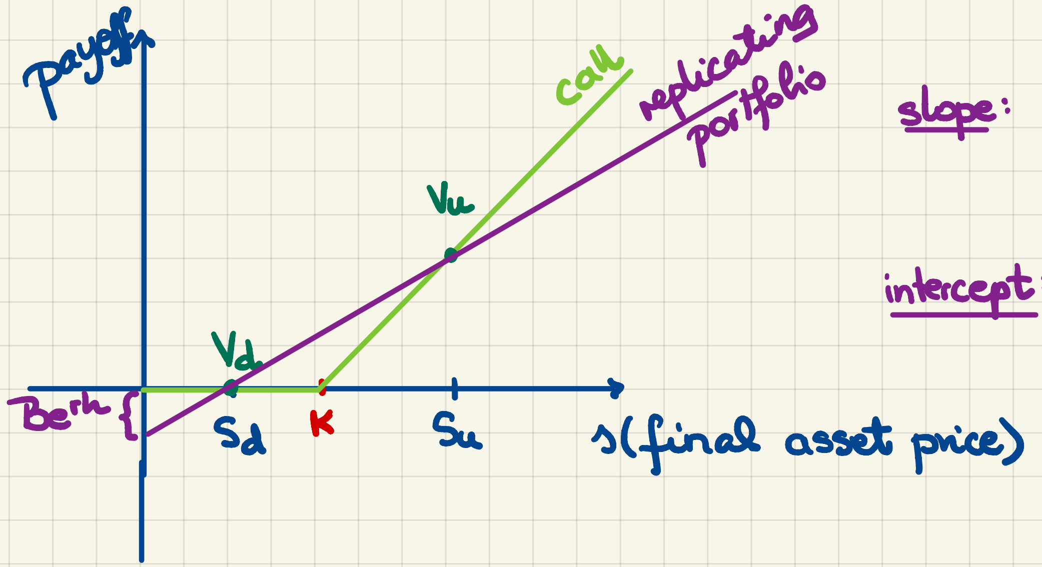
PAYOFF:

$S(0) \to S_u$: $\quad V_u = S_u - K$

$S(0) \to S_d$: $\quad V_d = 0$

In the replicating portfolio:

- $\Delta_C = \dfrac{V_u - V_d}{S_u - S_d} = \dfrac{S_u - K}{S_u - S_d} \in (0, 1)$ Buy a fraction of a stock!
- $B_C = e^{-rh} \cdot \dfrac{u \cdot V_d - d \cdot V_u}{u-d} = -e^{-rh} \cdot \dfrac{d \cdot V_u}{u-d} < 0$ Borrowing!



Recall: $v_c(s) = (s-K)_+$

slope: $\Delta \in (0,1)$
$\Rightarrow$ buying a fraction of a share

intercept: <0 $\Rightarrow$ borrowing

Problem 9.3. Let the continuously compounded risk-free interest rate be equal to 0.04. Consider a one-period binomial tree with every period of length one year used to model the stock price of a stock whose current price is \$80 per share. In the model, it is assumed that the stock price can either go up by \$5 or down by \$4.

You use the binomial tree to construct a replicating portfolio for a $\boxed{78\text{−strike call}}$ option on the above stock. What is the stock investment in the replicating portfolio?

→ :

Payoff

$S(0)$

$S_u = 85$ $\quad V_u = (85-78)_+ = 7$

$S_d = 76$ $\quad V_d = (76-78)_+ = 0$

$$\Delta = \frac{V_u - V_d}{S_u - S_d} = \frac{7}{85-76} = \frac{7}{9}$$