Types of Errors.

"Truth" → Decision ↓	$H_0: \mu = \mu_0$	$H_a: \begin{cases} \mu < \mu_0 & \text{left-sided} \\ \mu \neq \mu_0 & \text{two-sided} \\ \mu > \mu_0 & \text{right-sided} \end{cases}$
Reject H_0	Type I Error	☺
Fail to Reject H_0	☺	Type II Error

$$\mathbb{P}\left[\text{Type I Error}\right] = \mathbb{P}_0\left[\text{Reject } H_0\right] = \alpha$$

$\mathbb{P}_0$ → under the null, i.e., if $\mu = \mu_0$

α → significance level

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Problem Set # 13
Types of errors.

Provide your **final answer only** for the following problems.

Problem 13.1. You perform 2000 significance tests using a significance level 0.10. Under the assumption that all of the null hypotheses for the 2000 significance tests are true, how many of the 2000 significance tests would you expect to **not** result in a Type I error?
a. 200
b. 1800
c. 2000
d. 0
e. None of the above.

Problem 13.2. A medical researcher is working on a new treatment for a certain type of cancer. The average survival time after diagnosis on the standard treatment is 2 years. In an early trial, she tries the new treatment on three subjects who have an average survival time after diagnosis of 4 years. Although the survival time has doubled, the results are not statistically significant, even at the 0.10 significance level.
Suppose, in fact, that the new treatment **does** increase the mean survival time in the population of all patients with this particular type of cancer. Which of the following statements is TRUE?
a. A Type I error occurred.
b. A Type II error occurred.
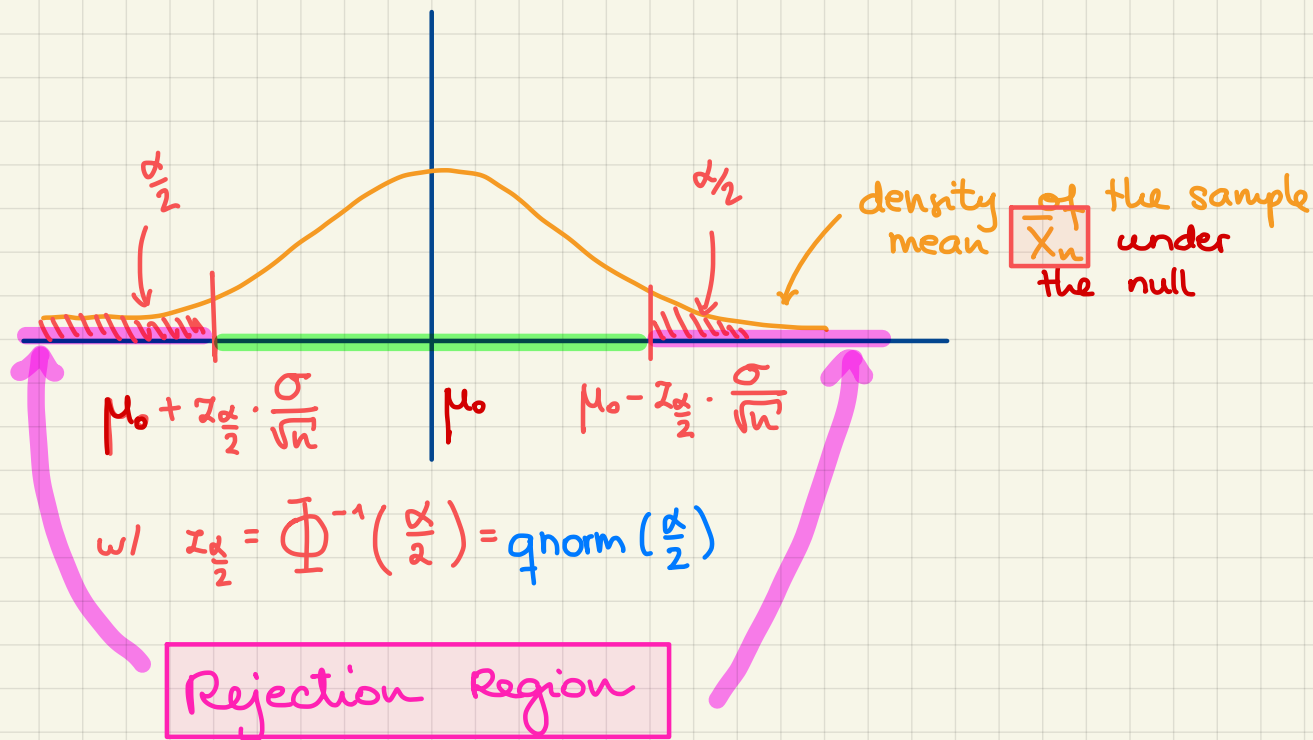**c.** No error occurred.

Problem 13.3. An engineer has designed an improved light bulb. The previous design had a mean lifetime of 1200 hours. Using a sample of 2000 of the new bulbs, the sample average lifetime of this improved light bulb is found to be 1201 hours. Although the difference is quite small, the effect was statistically significant at the 0.05 level. Suppose that, in fact, there is no difference between the mean lifetimes of the previous design and the new design. Which of the following statements is TRUE?
a. A Type I error occurred.
b. A Type II error occurred.
c. No error occurred.

Power of Test.

Temporarily, we focus on the two-sided alternative;
we will look @ the other two structures for the alternative hypothesis
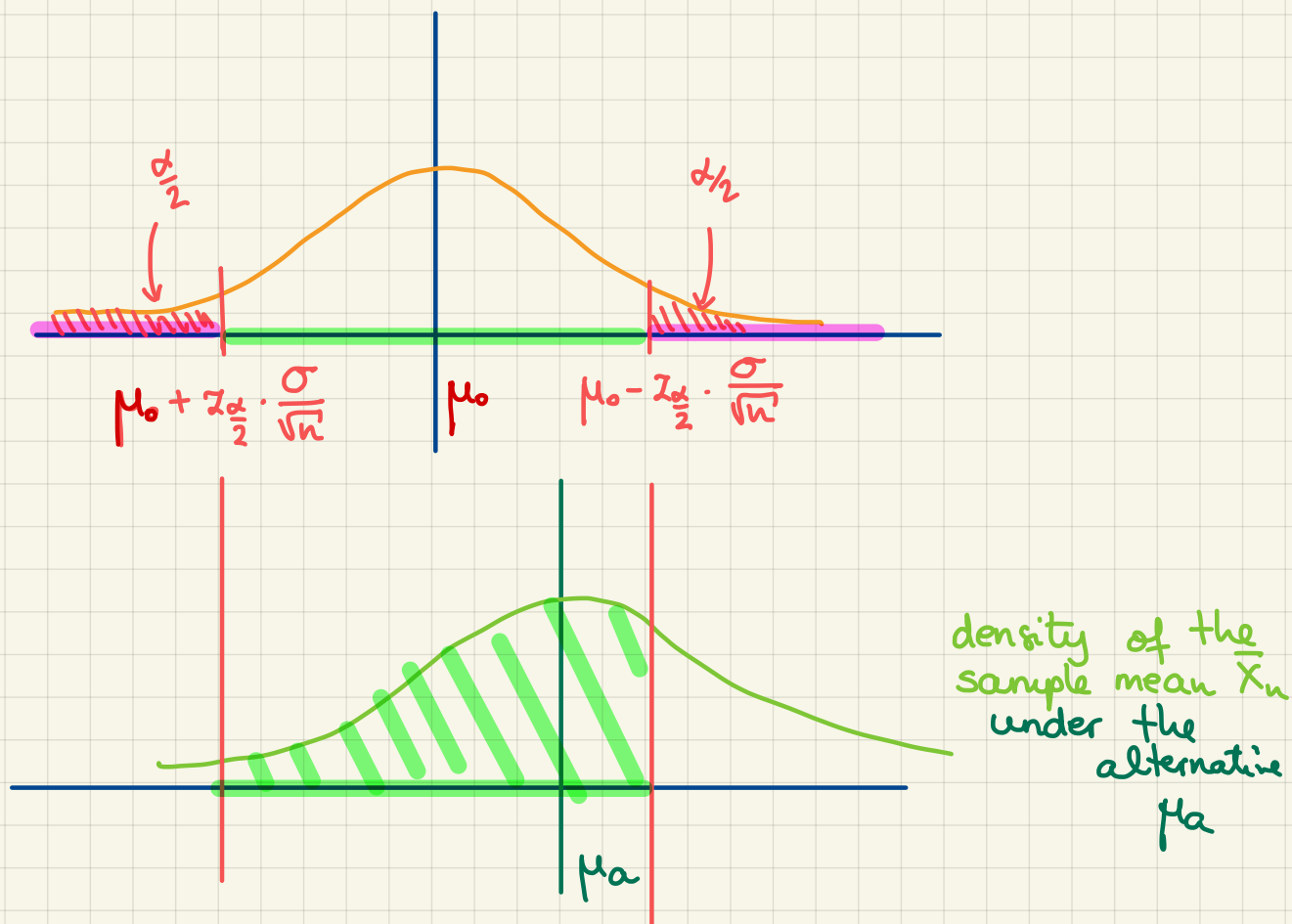using analogy.

α ... significance level

$\frac{\alpha}{2}$ $\frac{\alpha}{2}$

density of the sample
mean $\boxed{\bar{X}_n}$ under
the null

$\mu_0 + Z_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}$ μ_0 $\mu_0 - Z_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}$

w/ $Z_{\frac{\alpha}{2}} = \Phi^{-1}\left(\frac{\alpha}{2}\right) = qnorm\left(\frac{\alpha}{2}\right)$

Rejection Region

Focus on the "fail·to·reject" region, i.e., for a two-sided test

$$\left(\mu_0 + Z_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}} \,,\, \mu_0 + \left|Z_{\frac{\alpha}{2}}\right| \cdot \frac{\sigma}{\sqrt{n}}\right)$$

Focus on a particular value from the alternative, say $\boxed{\mu_a}$.

Q: What is the probability of failing to reject if this
particular alternative μ_a is the "true" value of the parameter μ?

The top graph shows a normal distribution (orange) centered at μ_0 with shaded rejection regions of area $\frac{\alpha}{2}$ in each tail. The critical values are labeled:

$$\mu_0 + z_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}$$

$$\mu_0$$

$$\mu_0 - z_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}$$

The bottom graph shows the density of the sample mean $\overline{X}_n$ under the alternative μ_a, centered at μ_a.

$$\mathbb{P}_{\mu_a}\left[\text{Fail to Reject } H_0\right] = \mathbb{P}\left[\text{Type II Error}\right] =: \beta$$

Note: The <u>smaller</u> the value of β, the better the test!

Def'n. The power of the test @ $\mu = \mu_a$ is defined as $1 - \beta$.