Name:

M339D/M389D Introduction to Financial Mathematics for Actuaries
University of Texas at Austin
Practice for In-Term Three
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Notes: This is a closed book and closed notes exam. The maximum number of points on this exam is 80.
Time: 50 minutes

1.1. **Free-response problems.** Please, explain carefully all your statements and assumptions. Numerical results or single-word answers without an explanation (even if they're correct) are worth 0 points.

Problem 1.1. (15 points) Two scales are used to measure the mass m of a precious stone. The first scale makes an error in measurement which we model by a normally distributed random variable with mean $\mu_1 = 0$ and standard deviation $\sigma_1 = 0.04m$. The second scale is more accurate. We model its error by a normal random variable with mean $\mu_2 = 0$ and standard deviation $\sigma_2 = 0.03m$.

We assume that the measurements made using the two different scales are independent.

To get our final estimate of the mass of the stone, we take the average of the two results from the two different scales.

What is the probability that the value we get is within $0.005m$ of the actual mass of the stone?

Solution: Let us denote the random variable modeling the error from the first scale by $X_1 \sim N(0, \sigma_1^2)$ and the random variable modeling the error from the second scale by $X_2 \sim N(0, \sigma_2^2)$.

Then, if Y denotes the average of the two measurements, we have that

$$Y = \frac{1}{2}(X_1 + X_2) \sim N(0, \frac{1}{4}(\sigma_1^2 + \sigma_2^2)),$$

i.e.,

$$Y \sim N(0, \sigma^2)$$

with

$$\sigma^2 = \frac{1}{4}(\sigma_1^2 + \sigma_2^2) = \frac{1}{4}(0.04^2 m^2 + 0.03^2 m^2) = \frac{1}{4} \cdot 0.01^2 m^2 (4^2 + 3^2) = \frac{1}{4} 0.05^2 m^2 = \left(\frac{0.05m}{2}\right)^2.$$

The probability we are looking for can be expressed as

$$\begin{aligned}\mathbb{P}[Y \in (-0.005m, 0.005m)] &= \mathbb{P}[-0.005m < Y < 0.005m] \\ &= \mathbb{P}[-\frac{2 \cdot 0.005m}{0.05m} < \frac{Y}{\sigma} < \frac{2 \cdot 0.005m}{0.05m}] \\ &= \mathbb{P}[-0.2 < \frac{Y}{\sigma} < 0.2].\end{aligned}$$

Since $\frac{Y}{\sigma} \sim N(0,1)$, the above probability equals

$$2\Phi(0.2) - 1 \approx 2 \cdot 0.57926 - 1 \approx 0.16.$$

Problem 1.2. (10 points) Assume that $Y_1 = e^X$ where X is a standard normal random variable.

(i) (2 points) What is the probability that Y_1 exceeds 5?

(ii) $(3+5$ points) Find the mean and the variance of Y_1.

Hint: It helps if you use the expression for the moment generating function of a standard normal random variable.

Solution:

(i)

$$\mathbb{P}[Y_1 > 5] = \mathbb{P}[e^X > 5] = \mathbb{P}[X > \ln(5)] = 1 - N(\ln(5)) \approx 1 - N(1.61) = 1 - 0.9463 = 0.0537.$$

(ii)

$$\mathbb{E}[Y_1] = \mathbb{E}[e^X] = \mathbb{E}[e^{1 \cdot X}] = M_X(1)$$

where M_X denotes the moment generating function of X. In class, we recalled the following expression for M_X:

$$M_X(t) = e^{t^2/2}.$$

So, $\mathbb{E}[Y_1] = e^{1/2} = \sqrt{e}$.

The second moment of Y_1 is obtained similarly as

$$\mathbb{E}[Y_1^2] = \mathbb{E}[e^{2 \cdot X}] = M_X(2) = e^2.$$

So,

$$Var[Y_1] = \mathbb{E}[Y_1^2] - (\mathbb{E}[Y_1])^2 = e^2 - e = e(e-1).$$

Problem 1.3. The final exam in a particular course has 100 multiple-choice questions: for each question there are five offered answers exactly one of which is correct. Out of the 100 questions, 36 questions come from a public problem bank. A student diligently memorizes the correct answers to all of those questions. However, since the student learned by rote, they are not able to do any work on the remaining questions. So, in the exam, they are able to answer exactly 36 questions correctly. For the remaining questions, the student guesses completely at random and independently between problems. Approximately, what is the probability that the student achieves a passing score of 65?

Solution: *The following solution uses the continuity correction. The solutions without the continuity correction would also be graded as correct if otherwise accurate.*

The number of problems that the student guesses on at random is 64. The probability of guessing correctly for a single problem is 1/5. So, the total number of problems that the student guesses correctly is, in our usual notation,

$$X \sim Binomial(n = 64, p = 0.2).$$

Out of the problems that the student guesses on at random, they need to guess correctly on at least $65 - 36 = 29$. The probability of passing is $\mathbb{P}[X \geq 29]$. The mean of the random variable X is $np = 12.8$ and its standard deviation is $\sqrt{np(1-p)} = 3.2$ Using the normal approximation to the binomial, we get

$$\mathbb{P}[X \geq 29] = \mathbb{P}[X > 28.5] = \mathbb{P}\left[\frac{X - 12.8}{3.2} > \frac{28.5 - 12.8}{3.2}\right] = 1 - \Phi\left(4.90625\right) \approx 0.$$

Problem 1.4. (10 points) The current price of a non-dividend-paying stock is \$100 per share. Its volatility is given to be 0.2.

The continuously compounded risk-free interest rate equals 0.03.

Consider a \$95-strike European put option on the above stock with nine months to expiration. Using a three-period forward binomial tree, find the price of this put option.

Solution: The up and down factors in the above model are

$$u = e^{0.03\times 0.25 + 0.2\sqrt{0.25}} = 1.1135,$$
$$d = e^{0.03\times 0.25 - 0.2\sqrt{0.25}} = 0.9116.$$

The relevant possible stock prices at the "leaves" of the binomial tree are

$$S_{ddd} = d^3 S(0) = 100(0.9116)^3 = 75.7553,$$
$$S_{ddu} = d^2 u S(0) = 92.5335.$$

The remaining two final states of the world result in the put option being out-of-the-money at expiration.

Convince yourselves that the following shortcut formula for the forward tree is correct!
The risk-neutral probability of the stock price moving up in a single period is

$$p^* = \frac{1}{1 + e^{0.2\sqrt{0.25}}} = 0.475.$$

So, the price of the European put option equals

$$V_P(0) = e^{-0.03(3/4)}\left[(95 - 75.7553)(1 - 0.475)^3 + (95 - 92.5335)(3)(1 - 0.475)^2(0.475)\right] = 3.670013.$$

1.2. MULTIPLE CHOICE QUESTIONS.

Problem 1.5. Assume the Black-Scholes model. Under the risk-neutral probability, you expect the stock price in half a year to be \$84.10. The median stock price in half a year is \$83.26 according to that same model. What is the stock's volatility?

(a) 0.15
(b) 0.20
(c) 0.25
(d) Not enough information is given.

(e) None of the above.

Solution: (b)
In our usual notation, we have that

$$\frac{\text{mean stock price}}{\text{median stock price}} = \frac{S(0)e^{rT}}{S(0)e^{(r-\frac{\sigma^2}{2})T}} = e^{\frac{\sigma^2 T}{2}}$$

So, in this problem,

$$\frac{84.10}{83.26} = e^{\frac{\sigma^2}{4}} \quad \Rightarrow \quad \frac{\sigma^2}{4} = \ln\left(\frac{84.10}{83.26}\right) \quad \Rightarrow \quad \sigma = \sqrt{4\ln\left(\frac{84.10}{83.26}\right)} = 0.2004.$$

Problem 1.6. Assume the Black-Scholes framework. For an at-the-money, $T-$year European call option on a non-dividend-paying stock you are given that its delta equals 0.5832. What is the delta of an otherwise identical option with exercise date at time $2T$?

(a) 0.62
(b) 0.66
(c) 0.70
(d) 0.74
(e) None of the above.

Solution: (a)
For the first option,

$$N(d_1) = 0.5832 \quad \Rightarrow \quad d_1 = 0.21.$$

On the other hand, by definition, and since the option is in-the-money

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T\right] = \frac{r + \frac{\sigma^2}{2}}{\sigma}\sqrt{T}$$

Using the same reasoning, for the $2T-$option, we have

$$\tilde{d}_1 = \frac{1}{\sigma\sqrt{2T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(2T)\right] = \frac{r + \frac{\sigma^2}{2}}{\sigma}\sqrt{2T}$$

Hence,

$$\tilde{d}_1 = d_1\sqrt{2} = 0.21\sqrt{2} = 0.2969848.$$

So, the delta of the second option equals

$$N(0.2969848) = 0.616761.$$

Problem 1.7. Assume the Black-Scholes setting. Assume $S(0) = \$28.50, \sigma = 0.32, r = 0.04$. The stock pays no dividends. Consider a \$30-strike put option which expires in 110 days (simplify the number of days in a year to 360). What is the price of the put?

(a) 2.75
(b) 2.10

(c) 1.80
(d) 1.20
(e) None of the above.

Solution: (a)
In our usual notation, the price is

$$V_P(0) = Ke^{-r \cdot T} N(-d_2) - S(0)N(-d_1)$$

with

$$d_1 = -0.1324386, \quad d_2 = -0.3093253.$$

So, $V_P(0) = 0.2011571$.

Problem 1.8. The current price of a non-dividend-paying stock is given to be \$92. The stock's volatility is 0.35.

The continuously compounded risk-free interest rate is 0.05.

Consider a \$90-strike European call option on the above stock with exercise date in a quarter-year.

What is the Black-Scholes price of this call option?

(a) 5.05
(b) 7.66
(c) 7.71
(d) 7.99
(e) None of the above.

Solution: (d)

$$d_1 = 0.2845223, d_2 = 0.1095223.$$

So,

$$V_C(0) = 7.986754.$$