Motivation.

Consider a European call option w/ strike K and exercise date T.

By our risk-neutral pricing

$$V_C(0) = e^{-rT}\mathbb{E}^*[V_C(T)]$$
$$= e^{-rT}\mathbb{E}^*[(S(T)-K)_+]$$
$$= e^{-rT}\mathbb{E}^*[(S(T)-K)\cdot \mathbb{I}_{[S(T)\geq K]}]$$
$$= e^{-rT}\mathbb{E}^*[S(T)\cdot \mathbb{I}_{[S(T)\geq K]}] - e^{-rT}\boxed{\mathbb{E}^*[K\cdot \mathbb{I}_{[S(T)\geq K]}]}$$

$$\boxed{\mathbb{E}^*[K\cdot \mathbb{I}_{[S(T)\geq K]}]} = K\cdot\boxed{\mathbb{E}^*[\mathbb{I}_{[S(T)\geq K]}]} = \boxed{\mathbb{P}^*[S(T)\geq K]}$$

LogNormal Tail Probabilities.

Example. Consider a non-dividend-paying stock. What is the probability that the stock outperforms a risk-free investment under the risk-neutral probability measure?

$\rightarrow$: The initially invested amount is $S(0)$.

- If it's a risk-free investment, the balance @ time-T is $S(0)e^{rT}$
- If it's a stock investment, the wealth @ time-T is $S(T)$

$$\mathbb{P}^*[S(T) > S(0)e^{rT}] = ?$$

This question is equivalent to the one of whether the profit for the stock investment is positive under $\mathbb{P}^*$.

In the Black-Scholes model:

$$\boxed{S(T) = S(0)\, e^{\left(r-\frac{\sigma^2}{2}\right)\cdot T + \sigma\sqrt{T}\cdot Z} \qquad \text{w/ } Z \sim N(0,1)}$$

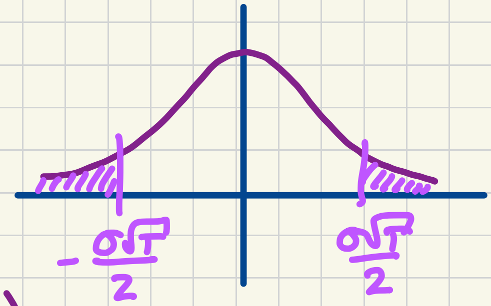
$$\mathbb{P}^*\left[\cancel{S(0)}\, e^{\left(\cancel{r}-\frac{\sigma^2}{2}\right)\cdot T + \sigma\sqrt{T}\cdot Z} > \cancel{S(0)}\,\cancel{e^{rT}}\right] =$$

$\ln(\quad) \qquad \ln(1)$

$$= \mathbb{P}^*\left[-\frac{\sigma^2}{2}\cdot T + \sigma\sqrt{T}\cdot Z > 0\right]$$

$$= \mathbb{P}^*\left[\sigma\sqrt{T}\cdot Z > \frac{\sigma^2}{2}\cdot T\right]$$

$$= \mathbb{P}^*\left[Z > \frac{\sigma\sqrt{T}}{2}\right] =$$



$$= \mathbb{P}^*\left[Z < -\frac{\sigma\sqrt{T}}{2}\right] = N\left(-\frac{\sigma\sqrt{T}}{2}\right) \xrightarrow[T\to\infty]{} 0$$

$\square$

Example. Consider a European call w/ strike K and exercise date T. Under the risk-neutral probability measure, what is the probability that the call is in-the-money @ expiration?

$\longrightarrow$: $\mathbb{P}^*[S(T) > K] =$

$$= \mathbb{P}^*\left[S(0)\, e^{\left(r-\frac{\sigma^2}{2}\right)\cdot T + \sigma\sqrt{T}\cdot Z} > K\right]$$

$$= \mathbb{P}^*\left[e^{\left(r-\frac{\sigma^2}{2}\right)\cdot T + \sigma\sqrt{T}\cdot Z} > \frac{K}{S(0)}\right] \qquad (\ln(\cdot) \text{ is increasing})$$

$$= \mathbb{P}^*\left[\left(r-\frac{\sigma^2}{2}\right)\cdot T + \sigma\sqrt{T}\cdot Z > \ln\left(\frac{K}{S(0)}\right)\right]$$

$$= \mathbb{P}^*\left[\sigma\sqrt{T}\cdot Z > \ln\left(\frac{K}{S(0)}\right) - \left(r-\frac{\sigma^2}{2}\right)\cdot T\right]$$

$$= \mathbb{P}^*\left[Z > \frac{1}{\sigma\sqrt{T}}\left(\ln\left(\frac{K}{S(0)}\right) - \left(r-\frac{\sigma^2}{2}\right)\cdot T\right)\right]$$

symmetry of $N(0,1)$

$$= \mathbb{P}^*\left[Z < \boxed{\frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)\cdot T\right]}\right]$$

$$=: d_2$$

$$\boxed{\mathbb{P}^*[S(T) > K] = N(d_2)}$$

$\square$

Consequently. The probability under $\mathbb{P}^*$ that the otherwise identical put is in-the-money @ expiration is

$$\mathbb{P}^*[S(T) < K] = 1 - N(d_2) = N(-d_2)$$

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Problem Set 14

Black-Scholes pricing.

Problem 14.1. Let the stock prices be modeled using the lognormal distribution. Under the risk-neutral probability measure, the mean stock price at time−1 equals 120 and the median stock price 115. What is the risk-neutral probability that the time−1 stock price exceeds 100?

→ :

$$\mathbb{P}^*[S(1) > 100] = ?$$

1st Figure out σ.

$$\frac{\text{mean}}{\text{median}} = \frac{S(0)e^{rT}}{S(0)e^{(r-\frac{\sigma^2}{2})\cdot T}} = e^{\frac{\sigma^2}{2}\cdot T}$$

$$\frac{120}{115} = e^{\frac{\sigma^2}{2}\cdot T}$$

$$\ln\left(\frac{120}{115}\right) = \frac{\sigma^2}{2}\cdot T = \frac{\sigma^2}{2}$$

$$\sigma = \sqrt{2\cdot\ln\left(\frac{120}{115}\right)} = \underline{0.2918}$$

2nd $\mathbb{P}^*[\underbrace{\boxed{S(0)e^{(r-\frac{\sigma^2}{2})\cdot T}}}_{\text{median of } S(1)\ 115} + \sigma\sqrt{T}\cdot Z > 100]$

$$\mathbb{P}^*[115 e^{\sigma\cdot Z} > 100] = \mathbb{P}^*\left[e^{\sigma\cdot Z} > \frac{100}{115}\right]$$

$$= \mathbb{P}^*\left[Z > \frac{1}{\sigma}\ln\left(\frac{100}{115}\right)\right] = \dots = \underline{0.6844}$$