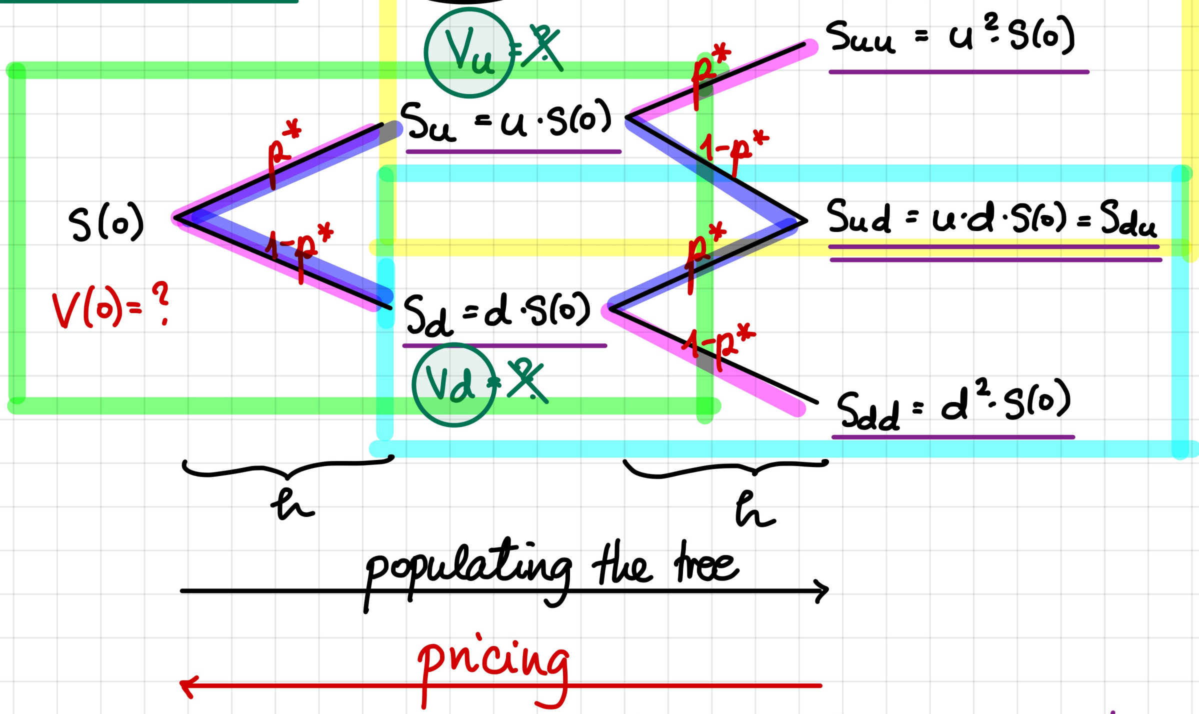
Two Periods.

$2\cdot h = T$ ✓



Payoff.

$V_{uu} = v(S_{uu})$

$V_{ud} = v(S_{ud})$

$V_{dd} = v(S_{dd})$

- **up node:** replicating portfolio for the option @ the up node:

$$\begin{cases} \Delta_u = e^{-\delta\cdot h}\cdot \dfrac{V_{uu}-V_{ud}}{S_{uu}-S_{ud}} \\ B_u = e^{-r\cdot h}\cdot \dfrac{u\cdot V_{ud} - d\cdot V_{uu}}{u-d} \end{cases}$$

$\Rightarrow$ the option's value @ the up node:

$$V_u = \Delta_u \cdot S_u + B_u = e^{-rh}\left[p^*\cdot V_{uu} + (1-p^*)\cdot V_{ud}\right]$$

w/ $\boxed{p^* = \dfrac{e^{(r-\delta)h} - d}{u-d}}$

M339D: May 5th, 2021.

- **down node:**

replicating portfolio for the option @ the down node:

$$\begin{cases} \Delta_d = e^{-\delta\cdot h}\dfrac{V_{ud}-V_{dd}}{S_{ud}-S_{dd}} \\ B_d = e^{-rh}\cdot \dfrac{u\cdot V_{dd} - d\cdot V_{ud}}{u-d} \end{cases}$$

$\Rightarrow$ the option's value @ the down node:

$$V_d = \Delta_d\cdot S_d + B_d = e^{-r\cdot h}\cdot\left[p^*\cdot V_{ud} + (1-p^*)\cdot V_{dd}\right]$$

w/ $\boxed{p^* = \dfrac{e^{(r-\delta)h} - d}{u-d}}$

• ROOT node : the replicating portfolio :

$$\begin{cases} \Delta_0 = e^{-\delta \cdot h} \cdot \dfrac{V_u - V_d}{S_u - S_d} \\ B_0 = e^{-r \cdot h} \cdot \dfrac{u \cdot V_d - d V_u}{u-d} \end{cases}$$

$\Rightarrow$ The time $\cdot$ 0 price is

$$V(0) = \Delta_0 \cdot S(0) + B_0 = e^{-r\cdot h}\left[p^* \cdot V_u + (1-p^*) \cdot V_d\right]$$ w/ p^* as above

$$V(0) = e^{-rh}\Big[p^* e^{-rh}\left(p^* \cdot V_{uu} + (1-p^*) \cdot V_{ud}\right) + (1-p^*) e^{-r\cdot h}\left(p^* \cdot V_{ud} + (1-p^*) \cdot V_{dd}\right)\Big]$$

$$V(0) = \underbrace{e^{-r(\overbrace{2h}^{T})}}_{\text{discounting}} \underbrace{\left[(p^*)^2 \cdot V_{uu} + 2 \cdot p^*(1-p^*) \cdot V_{ud} + (1-p^*)^2 \cdot V_{dd}\right]}_{\text{Risk neutral expectation of the payoff}}$$

Generally :

$$\boxed{V(0) = e^{-r\cdot T}\, \mathbb{E}^*[V(T)]}$$

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Problem Set #10

Binomial option pricing: Forward trees. Two periods.

Problem 10.1. (5 points) Assume that the a stock price is modeled using a one-period forward binomial tree with the length of a single period equal to three months. According to this model, the stock price can take either the value of \$50, or the value of \$40 in exactly three months. Calculate the volatility of the stock price.

(a) About 0.22
(b) About 0.28
(c) About 0.30
(d) About 0.32
(e) None of the above.

$h = \frac{1}{4}$

$S_u = 50$

$S_d = 40$

In the forward tree:

$$u = e^{(r-\delta)h + \sigma\sqrt{h}}$$

$$d = e^{(r-\delta)h - \sigma\sqrt{h}}$$

$$S_u / S_d = \frac{50}{40} = e^{2\sigma\sqrt{h}}$$

$$1.25 = e^{\cancel{2}\sigma\sqrt{\cancel{\tfrac{1}{4}}}} = e^{\sigma}$$

$$\sigma = \ln(1.25) = 0.223$$

Problem 10.2. The current price of a continuous-dividend-paying stock is \$100 per share. Its volatility is given to be 0.30 and its dividend yield is 0.03.

The continuously compounded risk-free interest rate equals 0.06.

Consider a \$95-strike European put option on the above stock with three months to expiration. Using a one-period forward binomial tree, find the price of this put option.

(a) \$3.97
(b) \$4.52
(c) \$4.70
(d) \$4.97
(e) None of the above.

$h = 1/4$

$$p^* = \frac{1}{1+e^{\sigma\sqrt{h}}} = \frac{1}{1+e^{0.3\sqrt{1/4}}} = 0.4626$$

$S_u = 117.06$ $V_u = 0$

$S(0) = 100$ $1-p^*$

$S_d = 86.72$ $V_d = 8.28$

$$u = e^{(r-\delta)h+\sigma\sqrt{h}} = e^{(0.06-0.03)(0.25)+0.15} = 1.1706$$

$$d = e^{(r-\delta)h-\sigma\sqrt{h}} = e^{0.03(0.25)-0.15} = 0.8672$$

$$V(0) = e^{-0.06(0.25)} \cdot 8.28\,(1-p^*) = 4.38$$