M339D: November 19^{th}, 2025.

8. Let $S(t)$ denote the price at time t of a stock that pays no dividends. The Black-Scholes framework holds. Consider a European call option with exercise date T, $T>0$, and exercise price $\underset{K}{S(0)e^{rT}}$, where r is the continuously compounded risk-free interest rate.

You are given:

(i) $S(0)=\$100$

(ii) $T=10$

(iii) $\text{Var}\left[\ln S(t)\right]=0.4t$, $t>0$. $\Rightarrow$ $\sigma^2 t = 0.4t$ $\boxed{\sigma=\sqrt{0.4}}$

Determine the price of the call option.

(A) $7.96

(B) $24.82

(C) $68.26

(D) $95.44

(E) There is not enough information to solve the problem. :(

$$V_c(0)=S(0)\cdot N(d_1)-Ke^{-rT}\cdot N(d_2)$$

$$d_1=\frac{1}{\sigma\sqrt{T}}\left[\underbrace{\ln\left(\frac{\cancel{S(0)}}{\cancel{S(0)}e^{rT}}\right)}_{\cancel{-rT}}+\left(\cancel{r}+\frac{\sigma^2}{2}\right)\cdot T\right]=\frac{\sigma\sqrt{T}}{2}=\frac{\sqrt{0.4}\cdot\sqrt{10}}{2}=1$$

$$d_2=d_1-\sigma\sqrt{T}=-\frac{\sigma\sqrt{T}}{2}$$

$$V_c(0)=S(0)N(d_1)-S(0)\cancel{e^{rT}}\cdot\cancel{e^{-rT}}\cdot N(-d_1)$$

$$V_c(0)=S(0)\left(N(d_1)-\underbrace{N(-d_1)}_{1-N(d_1)}\right)=S(0)\left(2N(d_1)-1\right)$$

$$V_c(0)=100\left(2\cdot N(1)-1\right)=100\left(2\cdot 0.8413-1\right)=68.26.$$

Problem. Assume the Black-Scholes framework.
For a European call, the strike is $S(0)e^{rT}$ where T is the exercise date.
(The price of such a call w/ one year to exercise is $0.6 \cdot S(0)$).

Find the price of such a call option w/ three months to exercise in terms of $S(0)$.

→: From the previous problem:

$$V_c(0,T) = S(0)\left(2N\left(\frac{\sigma\sqrt{T}}{2}\right) - 1\right)$$

For $T=1$:

$$V_c(0,T=1) = S(0)\left(2N\left(\frac{\sigma}{2}\right) - 1\right) = 0.6 \cdot S(0)$$

$$\Rightarrow 2N\left(\frac{\sigma}{2}\right) - 1 = 0.6$$

$$\Rightarrow N\left(\frac{\sigma}{2}\right) = 0.8$$

$$\Rightarrow \frac{\sigma}{2} = 0.84 \quad \Rightarrow \boxed{\sigma = 1.68}$$

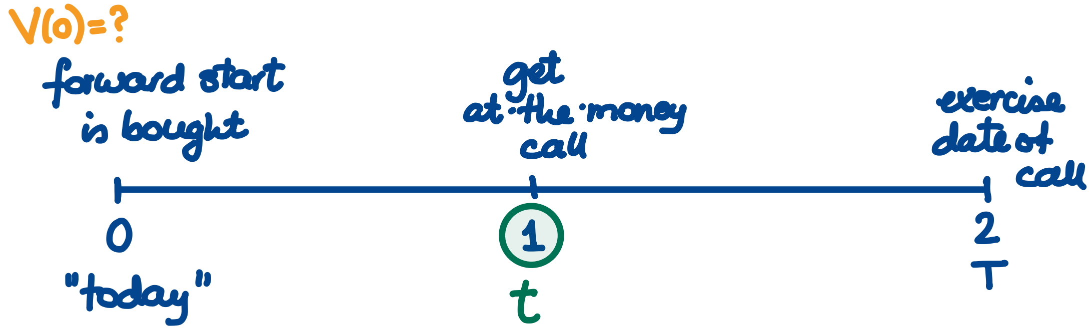
For $T=\frac{1}{4}$:

$$V_c\left(0,T=\tfrac{1}{4}\right) = S(0)\left(2N\left(\frac{\sigma\sqrt{\frac{1}{4}}}{2}\right) - 1\right)$$

0.42

$$= S(0)\left(2 \cdot 0.6628 - 1\right) = 0.3256 \cdot S(0)$$

□



(A) 586

(B) 594

(C) 684

(D) 692

(E) 797

19. Consider a *forward start option* which, 1 year from today, will give its owner a 1-year European call option with a strike price equal to the stock price at that time.

You are given:

(i) The European call option is on a stock that pays no dividends.

(ii) The stock's volatility is 30%.

(iii) The forward price for delivery of 1 share of the stock 1 year from today is 100.

$F_{0,1} = S(0)e^{r\cdot 1} = 100 \Rightarrow S(0) = 100e^{-0.08}$

(iv) The continuously compounded risk-free interest rate is 8%.

Under the Black-Scholes framework, determine the price today of the forward start option.

(A) 11.90

(B) 13.10

(C) 14.50

(D) 15.70

(E) 16.80

At $t<T$:

$$V_c(t) = S(t)\cdot N(d_1(t)) - Ke^{-r(T-t)}\cdot N(d_2(t))$$

w/

$$d_1(t) = \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S(t)}{K}\right) + \left(r+\frac{\sigma^2}{2}\right)(T-t)\right]$$

and

$$d_2(t) = d_1(t) - \sigma\sqrt{T-t}$$

In this problem: $t=1$ and $K=S(1)$

$$V_c(1) = S(1)N(d_1(1)) - S(1)e^{-r(2-1)}\cdot N(d_2(1))$$

$$V_c(1) = S(1)\boxed{\left(N(d_1(1)) - e^{-r}\cdot N(d_2(1))\right)}$$

w/ $d_1(1) = \frac{1}{0.3\sqrt{2-1}}\left[\ln\left(\cancel{\frac{S(1)}{S(1)}}\right) + \left(0.08 + \frac{0.09}{2}\right)(2-1)\right]$

@ the money

$$d_1(1) = \frac{0.08+0.045}{0.3} = \frac{0.125}{0.3} = 0.42$$

$$d_2(1) = d_1 - 0.3\sqrt{2-1} = 0.12$$

$N(d_1(1)) = 0.6628$

$N(d_2(1)) = 0.5478$

$$V_c(1) = S(1)\left(0.6628 - e^{-0.08} \cdot 0.5478\right) = S(1) \cdot 0.1571$$

At time 0, our forward start option is worth

$$S(0) \cdot 0.1571$$

$\Rightarrow$

answer: $100\, e^{-0.08} \cdot 0.1571 = 14.50$ $\square$