UNIVERSITY OF TEXAS AT AUSTIN

HW Assignment 5

The Black-Scholes pricing formula.

Please, provide your **complete solution** to the following problem(s):

Problem 5.1. (2 points) Let the stock price $S(t)$ be modeled using te lognormal distribution. Define $Y(t) = S(t)^3$. Then, the random variable $Y(t)$ is lognormal itself. *True or false? Why?*

Solution: TRUE

Problem 5.2. (2 pts) Let the stochastic process $S = \{S(t), t \geq 0\}$ represent the stock price as in the Black-Scholes model. Let its volatility term be denoted by σ. Then, the volatility parameter of the process $Y(t) = 2S(t)$ is 4σ. *True or false? Why?*

Solution: FALSE
Note: The correct answer is relevant to the solution for the **Sample IFM Problem #54 (Derivatives: Advanced).** The volatility parameter of the process Y is σ.

Problem 5.3. (2 points) Under the risk-neutral probability measure, every option on a particular stock has the continuously compounded, risk-free interest rate as its mean rate of return. *True or false? Why?*

Solution: TRUE

Problem 5.4. (2 points) The Black-Scholes option pricing formula can **always** be used for pricing American-type call options on non-dividend-paying assets. *True or false? Why?*

Solution: TRUE

Problem 5.5. (2 points) The Black-Scholes option pricing formula can always be used for pricing American-type options. *True or false?*

Solution: FALSE

Problem 5.6. (20 points) Let $S(0) = \$100, K = \$120, \sigma = 0.3, r = 0.08$ and $\delta = 0$.

a. (8 pts) Let $V_C(0,T)$ denote the Black-Scholes European call price for the maturity T. Does the limit of $V_C(0,T)$ as $T \to \infty$ exist? If it does, what is it?
b. (8 pts) Now, set $\delta = 0.001$ and let $V_C(0,T,\delta)$ denote the Black-Scholes European call price for the maturity T. Again, how does $V_C(0,T,\delta)$ behave as $T \to \infty$?
c. (4 pts) Interpret in a sentence or two the differences, if there are any, between your answers to questions in a. and b.

Solution:

a. By the Black-Scholes pricing formula, the function $V_C(0,T)$ has the form

$$V_C(0,T) = S(0)N(d_1) - Ke^{-rT}N(d_2),$$

where N denotes the distribution function of the unit normal distribution and

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)T\right],$$
$$d_2 = d_1 - \sigma\sqrt{T}.$$

As $T \to \infty$, we have that

$$d_1 \to \infty \Rightarrow N(d_1) \to 1,$$
$$e^{-rT}N(d_2) \leq e^{-rT} \to 0.$$

Hence,

$$V_C(0,T) \to S(0), \text{ as } T \to \infty.$$

b. In this case, the price of the call option reads as

$$V_C(0,T,\delta) = S(0)e^{-\delta T}N(d_1) - Ke^{-rT}N(d_2),$$

with

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(r - \delta + \frac{1}{2}\sigma^2\right)T\right],$$
$$d_2 = d_1 - \sigma\sqrt{T}.$$

Since the function N is bounded between 0 and 1, we see that as $T \to \infty$, $V_C(0,T,\delta) \to 0$.

c. When the stock is paying the dividend, the benefit of owning the stock and opposed to owning the option on that stock lies precisely in the value of the issued dividend. As we can see from above, even a very small dividend yield is going to render the call options for very long maturities worthless.

Problem 5.7. (20 points) Let $S(0) = \$120, K = \$100, \sigma = 0.3, r = 0$ and $\delta = 0.08$.

a. (10 pts) Let $V_C(0,T)$ denote the Black-Scholes European call price for the maturity T. Does the limit of $V_C(0,T)$ as $T \to \infty$ exist? If it does, what is it?

b. (8 pts) Now, set $r = 0.001$ and let $V_C(0,T,r)$ denote the Black-Scholes European call price for the maturity T. Again, how does $V_C(0,T,r)$ behave as $T \to \infty$?

c. (2 pts) Interpret in a sentence or two the differences, if any, between your answers to questions in a. and b.

Solution:

a. By the Black-Scholes pricing formula, the function $V_C(0,T)$ has the form

$$V_C(0,T) = S(0)e^{-\delta T}N(d_1) - Ke^{-rT}N(d_2) = S(0)e^{-\delta T}N(d_1) - K\,N(d_2),$$

where N denotes the distribution function of the unit normal distribution and

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(-\delta + \frac{1}{2}\sigma^2\right)T\right],$$
$$d_2 = d_1 - \sigma\sqrt{T}.$$

As $T \to \infty$, we have that

$$e^{-\delta T}N(d_1) \le e^{-\delta T} \to 0,$$
$$d_2 \to -\infty \Rightarrow N(d_2) \to 0.$$

Hence,

$$V_C(0,T) \to 0, \text{ as } T \to \infty.$$

b. In this case, the price of the call option reads as

$$V_C(0,T,r) = Se^{-\delta T}N(d_1) - Ke^{-rT}N(d_2),$$

with

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(r - \delta + \frac{1}{2}\sigma^2\right)T\right],$$
$$d_2 = d_1 - \sigma\sqrt{T}.$$

As $T \to \infty$, we have that

$$e^{-\delta T}N(d_1) \le e^{-\delta T} \to 0,$$
$$d_2 \to -\infty \Rightarrow N(d_2) \to 0.$$

Since the function N is bounded between 0 and 1, we see that as $T \to \infty$, $V_C(0,T,r) \to 0$.

c. Until the call option is exercised, the owner of the option can earn interest on the strike price which he/she can invest at the risk-free rate. However, in forfeiting the physical ownership of the asset, he/she also forfeits the possible dividend payments. It would be interesting to see what happens for $r > 0$ and $\delta = 0$.