Log Normal Stock Prices

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NOTATION & MODELLING ASSUMPTIONS

$R(0,T)$ ... realized return, i.e., $R(0,T) := \ln\left(\frac{S(T)}{S(0)}\right)$

$S(T) = S(0)e^{R(0,T)}$

We model $R(0,T)$ as normally dist'd, i.e.,

$$R(0,T) \sim Normal(mean = m, var = \tau^2)$$

$\Rightarrow$ $S(T)$ are lognormally dist'd

On the other hand, in our <u>market model</u>:

- r ... cont. compounded, risk-free i.r.
- <u>Stock</u> <u>parameters</u>:
  - * α ... (mean) rate of return
  - * δ ... dividend yield

  $\alpha - \delta$ (mean) rate of appreciation
  - * σ ... volatility

By its def'n: $SD[R(0,1)] = \sigma$

$$Var[R(0,1)] = \sigma^2$$

$$\Rightarrow Var[R(0,T)] = \sigma^2 \cdot T \iff \tau^2 = \sigma^2 \cdot T$$

$$\iff \tau = \sigma\sqrt{T}$$

Focus on the expected time-T stock price:

* due to the parametrization:

$$\mathbb{E}[S(T)] = S(0)e^{(\alpha-\delta)\cdot T} \quad (\star)$$

* due to the log-normality of $S(T)$:

$$\mathbb{E}[S(T)] = \mathbb{E}[S(0) \cdot e^{R(0,T)}]$$
$$= S(0) \cdot \mathbb{E}[e^{R(0,T)}]$$

<u>Recall:</u> $R(0,T) \sim \text{Normal}(\text{mean} = m, \text{var} = \tau^2 = \sigma^2 \cdot T)$

$$\mathbb{E}[S(T)] = S(0)\, e^{m + \frac{1}{2}\cdot\sigma^2 \cdot T} = S(0)\, e^{m + \frac{\sigma^2 T}{2}} \quad ★★$$

$\Rightarrow$ We equate ★ & ★★ and get

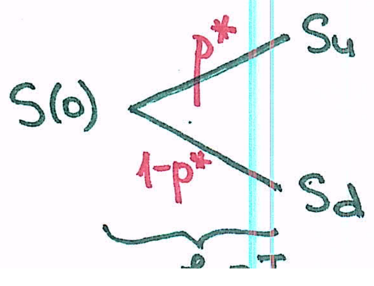
$$\mathbb{E}[S(T)] = S(0)\, e^{(\alpha - \delta)\cdot T} = S(0)\, e^{m + \frac{\sigma^2 T}{2}}$$

$$\Rightarrow \quad m = ?$$
$$(\alpha - \delta)\cdot T = m + \frac{\sigma^2 T}{2}$$
$$\Rightarrow \quad m = \left(\alpha - \delta - \frac{\sigma^2}{2}\right)\cdot T$$

$$\Rightarrow R(0,T) \sim N\left(\text{mean} = \left(\alpha - \delta - \frac{\sigma^2}{2}\right)\cdot T,\ \text{var} = \sigma^2 \cdot T\right)$$

<u>Note:</u> When we consider quality of investment, say, interms of the expected payoff of a stock investment, an option on a stock, or a portfolio, we look @ the PHYSICAL/SUBJECTIVE PROBABILITY, i.e., the one using the parameter α.

<u>Recall:</u> the one-period binomial tree



Under the <u>risk-neutral measure</u>, what is

$$\mathbb{E}^*[S(T)] = \mathbb{E}^*[S(h)]$$
$$= ?$$

$$\mathbb{E}^*[S(T)] = p^* \cdot S_u + (1-p^*) \cdot S_d$$

$$= p^* \cdot u \cdot S(0) + (1-p^*) \cdot d \cdot S(0)$$

$$= S(0)\left[p^* \cdot u + (1-p^*) \cdot d\right]$$

$$= S(0)\underbrace{\left[\frac{e^{(r-\delta)\cdot h} - d}{u-d} \cdot u + \frac{u - e^{(r-\delta)\cdot h}}{u-d} \cdot d\right]}_{\text{Simplify!}}$$

$$= S(0)\, e^{(r-\delta)\cdot h} = F_{0,h}(S) \ \ldots \text{ the forward price}$$

r plays the (mean) rate of return under the risk-neutral measure!

In the continuous model, we have that the risk-neutral probab. measure $\mathbb{P}^*$ is such that

$$R(0,T) \sim N\left(\text{mean} = \left(r - \delta - \frac{\sigma^2}{2}\right)\cdot T,\ \text{var} = \sigma^2 \cdot T\right)$$

We use this dist'n for pricing options!

Q: What is the expected time-T stock price under the risk-neutral probab. measure $\mathbb{P}^*$?

$$\mathbb{E}^*[S(T)] = S(0)\, e^{(r-\delta)\cdot T} = \underbrace{F_{0,T}(S)}_{\text{the forward price (again)}}$$

Q: Under the physical probab. measure, what is the median time-T stock price?

$\rightarrow$: $S(T) = S(0)e^{R(0,T)}$

w/ $R(0,T) \sim N(\text{mean} = (\alpha - \delta - \frac{\sigma^2}{2})\cdot T,\ \text{var} = \sigma^2 \cdot T)$

Introduce: $Z \sim N(0,1)$

$\Rightarrow$ $S(T) = S(0)\cdot e^{(\alpha-\delta-\frac{\sigma^2}{2})\cdot T + \sigma\sqrt{T}\cdot Z}$

$\Rightarrow$ the median of $S(T)$ is:

$$\boxed{S(0)e^{(\alpha-\delta-\frac{\sigma^2}{2})\cdot T}}$$

Q: Given the mean & the median of $S(T)$, find the volatility?

$$\nu := \frac{\text{mean}}{\text{median}} = \frac{S(0)e^{(\alpha-\delta)\cdot T}}{S(0)e^{(\alpha-\delta-\frac{\sigma^2}{2})\cdot T}} = e^{\frac{\sigma^2 \cdot T}{2}}$$

$\Rightarrow$ $\ln(\nu) = \frac{\sigma^2 \cdot T}{2}$

$\Rightarrow$ $\sigma = \sqrt{\frac{2\ln(\nu)}{T}}$