M339D: October 9th, 2024.

The Inverse Transform Method.

Proposition.

(1) Let X be a continuous random variable, i.e., let X have a probability density function f_X.

Assume that $\boxed{f_X(x)>0 \text{ for all } x}$

Denote its cumulative distribution function by F_X.

$F_X : \mathbb{R} \to [0,1]$

Set: $\boxed{\tilde{X} := F_X(X)}$ ✓

Then, $\tilde{X} \sim U(0,1)$

→: Support of $\tilde{X}$ will be contained $[0,1]$

$F_{\tilde{X}}(u) = \underline{0}$ for $u<0$

$F_{\tilde{X}}(u) = \underline{1}$ for $u>1$

Focus on $u \in [0,1]$:

$$F_{\tilde{X}}(u) = \mathbb{P}[\tilde{X} \leq u] = \mathbb{P}[F_X(X) \leq u] = \cdots$$

$f_X(x)>0$ for all x

Recall: $F_X(a) = \int_{-\infty}^{a} f_X(x)\,dx$

$\Rightarrow$ the cdf F_X is strictly increasing

$\Rightarrow$ F_X is one-to-one

$\Rightarrow$ F_X^{-1} exists and is increasing

$$F_{\tilde{X}}(u) = \mathbb{P}[\cancel{F_X^{-1}}(\cancel{F_X}(X)) \leq F_X^{-1}(u)] = \mathbb{P}[X \leq F_X^{-1}(u)] = \cancel{F_X}(\cancel{F_X^{-1}}(u))$$

$$\boxed{F_{\tilde{X}}(u) = u}$$

$\square$

(2) Let $U \sim U(0,1)$.

Let F be a cumulative distribution function.

Set : $\boxed{Y := F^{-1}(U)}$

Then, the cumulative dist'n f'tion of Y is F.

Implementation:

1. F... the cdf of the dist'n you want to draw simulated values from
2. Find an "expression" for F^{-1}
3. Draw: $u_1, u_2, \dots, u_n \sim U(0,1)$ from your random number generator (rng)
4. Set: $\boxed{x_i = F^{-1}(u_i)\ ,\quad i = 1, .., n}$

These will be our simulated values from the target distribution.

Problem 7.2. Let the random variable X have the following density function:

$$f_X(x) = 3x^{-4}, \quad x > 1$$

You use the *inverse transform method* to simulate values from X. Let the simulated value of the unit uniform be equal to 0.25. What is the corresponding value of X?

→: The cdf of X:

$$F_X(x) = \begin{cases} 0 & \text{for } x < 1 \\ \boxed{\int_1^x f_X(u)\,du} & \text{for } x \geq 1 \end{cases}$$

$$\int_1^x 3u^{-4}\,du = \cancel{3}\frac{u^{-3}}{(-\cancel{3})}\Big|_{u=1}^{x} = 1 - x^{-3}$$

The inverse of the cdf (the quantile f'tion ☺)

$$y = 1 - x^{-3} \iff 1 - y = x^{-3}$$

$$\iff x^3 = \frac{1}{1-y}$$

$$\iff \boxed{x = \frac{1}{\sqrt[3]{1-y}}}$$

So, our answer is:

$$x = \frac{1}{\sqrt[3]{1-0.25}} = \underline{\quad ? \quad}$$

□