M378K: November 19th, 2025.

More About Maximum Likelihood Estimation.

Def'n. An estimator $\hat{\theta} = \hat{\theta}(y_1, y_2, \dots, y_n)$ is called the **maximum likelihood estimator (MLE)** if it satisfies that for any $\hat{\theta}' = \hat{\theta}'(y_1, \dots, y_n)$ we have

$$\boxed{L(\hat{\theta}; y_1, \dots, y_n) \geq L(\hat{\theta}'; y_1, \dots, y_n) \quad \text{for all } y_1, \dots, y_n}$$

Example. $Y_1, Y_2, \dots, Y_n \sim U(0, \theta)$ $\quad \theta > 0$ unknown

The pdf of $U(0,\theta)$:

$$f^{\theta}(y) = \frac{1}{\theta} \mathbb{1}_{[0,\theta]}(y) = \boxed{\frac{1}{\theta} \mathbb{1}_{\{0 \leq y \leq \theta\}}}$$

The likelihood:

$$L(\theta; y_1, \dots, y_n) = f^{\theta}(y_1) \cdot f^{\theta}(y_2) \cdots f^{\theta}(y_n)$$

$$= \left(\frac{1}{\theta} \cdot \mathbb{1}_{\{0 \leq y_1 \leq \theta\}}\right) \cdots \left(\frac{1}{\theta} \cdot \mathbb{1}_{\{0 \leq y_n \leq \theta\}}\right)$$

$$= \left(\frac{1}{\theta}\right)^n \mathbb{1}_{\{0 \leq y_1, \dots, y_n \leq \theta\}}$$

$$= \left(\frac{1}{\theta}\right)^n \mathbb{1}_{\{0 \leq \min(y_1, \dots, y_n)\}} \cdot \mathbb{1}_{\{\max(y_1, \dots, y_n) \leq \theta\}}$$

"assume $y_1, \dots, y_n \geq 0$"

$$= \theta^{-n} \cdot \mathbb{1}_{\{\max(y_1, \dots, y_n) \leq \theta\}}$$

$$L(\theta; y_1, \dots, y_n) = \theta^{-n} \cdot \mathbb{1}_{\{y_{(n)} \leq \theta\}}$$

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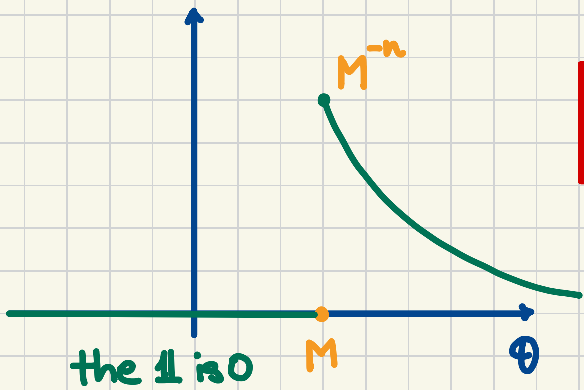

$$\boxed{\hat{\theta}_{MLE} = \max(Y_1, \dots, Y_n) = Y_{(n)}}$$

Q: Is this the same as the moment matching estimator?

→: Moment matching:

theoretical mean = sample mean

$$\frac{\theta}{2} = \bar{Y}$$

$$\Rightarrow \boxed{\hat{\theta}_{MM} = 2\bar{Y}}$$

Example. Let $Y_1, \dots, Y_n$ be a random sample from a dist'n w/ density

$$f^{\theta}(y) = \boxed{\theta \cdot y^{\theta-1} \cdot \mathbb{1}_{[0,1]}(y)}$$

for some unknown positive parameter θ. Find the MLE for θ.

→:

$$L(\theta; y_1, \dots, y_n) = \prod_{i=1}^{n} f^{\theta}(y_i)$$

$$= \prod_{i=1}^{n} \left(\theta y_i^{\theta-1}\right) = \theta^n \left(\prod_{i=1}^{n} y_i\right)^{\theta-1}$$

$$\ell(\theta; y_1, \dots, y_n) = n \cdot \ln(\theta) + (\theta - 1) \cdot \ln\left(\prod_{i=1}^{n} y_i\right)$$

$$\ell'(\theta; y_1, \dots, y_n) = n \cdot \frac{1}{\theta} + \sum_{i=1}^{n} \ln(y_i) = 0$$

$$\frac{n}{\theta} = -\sum_{i=1}^{n} \ln(y_i)$$

$$\boxed{\hat{\theta}_{MLE} = -\frac{n}{\sum_{i=1}^{n} \ln(y_i)}}$$

$\square$

Example. CAS Exam 3, Spring 2007.

Consider a random sample

$Y_1, \dots, Y_n$ from a dist'n w/ the pdf

$$f^{\theta}(y) = e^{-y+\theta} \cdot \mathbb{1}_{(\theta, \infty)}(y).$$

Find the MLE for θ.

$\longrightarrow$:

$$L(\theta; y_1, \dots, y_n) = \prod_{i=1}^{n} \left(e^{-y_i+\theta} \cdot \mathbb{1}_{\{\theta < y_i\}} \right)$$

$$= \boxed{\prod_{i=1}^{n} e^{-y_i+\theta}} \cdot \mathbb{1}_{\{\theta < \underbrace{\min(y_1, \dots, y_n)}_{y_{(1)}}\}}$$

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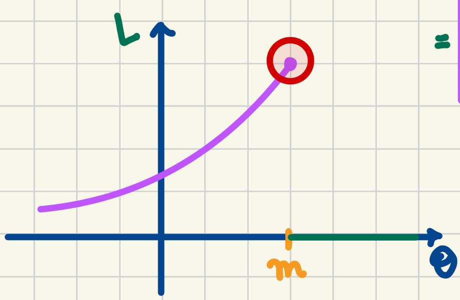

$$\boxed{\hat{\theta}_{MLE} = Y_{(1)}}$$