Confidence Intervals for the μ w/ variance unknown.

Focus on the normal model $N(\mu, \sigma)$ w/ both parameters unknown, but w/ target parameter μ

Theorem. In the above setting, let

$$\bar{Y} = \frac{1}{n}(Y_1 + \cdots + Y_n)$$

and

$$Q^2 = \frac{1}{\sigma^2} \sum_{i=1}^{n} (Y_i - \bar{Y})^2$$

Then,

- $\bar{Y} \sim \text{Normal}(\text{mean} = \mu, \text{sd} = \frac{\sigma}{\sqrt{n}})$
- $Q^2 \sim \chi^2(df = n-1)$
- $\boxed{\bar{Y} \text{ and } Q^2 \text{ are independent}}$

Goal: Confidence interval for μ.

Idea: Use

$$\boxed{\frac{\bar{Y} - \mu}{\frac{S}{\sqrt{n}}}} \quad \text{w/} \quad S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (Y_i - \bar{Y})^2$$

as a pivotal quantity.

$$\frac{\overbrace{\frac{\bar{Y} - \mu}{\sigma/\sqrt{n}}}^{\sim N(0,1) \sim Z}}{S/\sigma} = \frac{Z}{\sqrt{S^2/\sigma^2}} = \frac{Z}{\boxed{\sqrt{\frac{Q^2}{n-1}}}}$$

t-distribution.

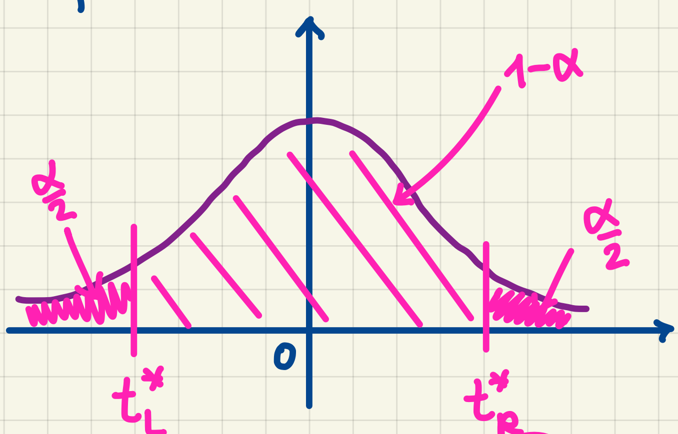
Def'n. A Student t-distribution w/ k degrees of freedom is the dist'n of the random variable

$$T = \frac{Z}{\sqrt{Q^2/k}}$$

w/
- $Z \sim N(0,1)$
- $Q^2 \sim \chi^2(df=k)$
- Z and Q^2 are independent

We write $\boxed{T \sim t(df=k)}$

To construct a confidence interval w/ the confidence level $1-\alpha$.



$$-t_L^* = +t_R^* = t^*$$

$$t^* = qt((1+C)/2,\ df=n-1)$$

$$t^* = qt(1-\alpha/2,\ df=n-1) \qquad \boxed{C = 1-\alpha}$$

$$\mathbb{P}\left[-t^* \le \frac{\bar{Y}-\mu}{S/\sqrt{n}} \le t^*\right] = 1-\alpha$$

$$\mathbb{P}\left[-t^* \cdot \frac{S}{\sqrt{n}} \le \bar{Y}-\mu \le t^* \cdot \frac{S}{\sqrt{n}}\right] = 1-\alpha$$

$$\mathbb{P}\Big[\underbrace{\bar{Y}-t^{*}\cdot\frac{S}{\sqrt{n}}}_{\hat{\theta}_L}\leq \mu \leq \underbrace{\bar{Y}+t^{*}\cdot\frac{S}{\sqrt{n}}}_{\hat{\theta}_R}\Big]=1-\alpha$$

$$\boxed{\mu = \bar{Y} \pm t^{*}\cdot\frac{S}{\sqrt{n}}}$$

std error

pt. estimate

critical value

Problem 16.9. *(20 points)*
A random sample of size 10 is drawn from a normal distribution with **both** *mean and standard deviation* **unknown**. *The generated sample has the sample mean* $\bar{y}_{10} = 14$ *and the (unbiased) estimate of the variance* $s^2 = 25$.

(i) *(10 points) Construct a (symmetric) 90%-confidence interval for* μ.

(ii) *(10 points) Construct a (symmetric) 90%-confidence interval for* σ^2.
Hint: *Remember that you know the distribution of* $(n-1)S^2/\sigma^2$.

Critical value t^* of the t-dist'n w/ df = 10 − 1 = 9.

$$t^* = qt(0.95, df=9) = 1.833$$

$$\mu = 14 \pm 1.833 \cdot \frac{5}{\sqrt{10}} \quad \square$$

$$Q^2 = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2(df=9)$$

$$\chi^2_L = qchisq(0.05, df=9) = 3.325$$

$$\chi^2_R = qchisq(0.95, df=9) = 16.92$$

The confidence interval is

$$\left(\frac{9 \cdot 25}{\chi^2_R}, \frac{9 \cdot 25}{\chi^2_L} \right)$$

$$= \left(\frac{225}{16.92}, \frac{225}{3.325} \right) \quad \square$$