

M378K Introduction to Mathematical Statistics

Problem Set #15

Relative efficiency.

Definition 15.1. Given two unbiased estimators $\hat{\theta}_1$ and $\hat{\theta}_2$, the efficiency of $\hat{\theta}_1$ relative to $\hat{\theta}_2$ is defined as

$$\text{eff}(\hat{\theta}_1, \hat{\theta}_2) = \frac{\text{Var}[\hat{\theta}_2]}{\text{Var}[\hat{\theta}_1]}.$$

Problem 15.1. Let Y_1, Y_2 be a random sample from the exponential distribution with the unknown parameter θ .

- (i) The estimator $\hat{\theta}_1 = (Y_1 + Y_2)/2$ for θ is proposed. What is its variance?
- (ii) The estimator $\hat{\theta}_2 = cY_{(1)}$ for θ is proposed. Find the constant c such that $\hat{\theta}_2$ is an unbiased estimator of θ . What is its variance?
- (iii) Calculate the efficiency of $\hat{\theta}_1$ relative to $\hat{\theta}_2$.

Solution:

(i)

$$\text{Var}[\hat{\theta}_1] = \text{Var}\left[\frac{Y_1 + Y_2}{2}\right] = \frac{1}{4} (\text{Var}[Y_1] + \text{Var}[Y_2]) = \frac{2\tau^2}{4} = \frac{\tau^2}{2}.$$

(ii) By definition, $Y_{(1)} = \min(Y_1, Y_2)$. As we have proved earlier in the class

$$Y_{(1)} \sim E\left(\frac{\tau}{2}\right)$$

So, $\mathbb{E}[Y_{(1)}] = \frac{\tau}{2}$. In order to have $\hat{\theta}_2$ be unbiased, we must set $c = 2$. Since, $\text{Var}[Y_{(1)}] = \frac{\tau^2}{4}$, we have that

$$\text{Var}[\hat{\theta}_2] = \text{Var}[2Y_{(1)}] = 4\left(\frac{\tau^2}{4}\right) = \tau^2.$$

(iii)

$$\text{eff}(\hat{\theta}_1, \hat{\theta}_2) = \frac{\text{Var}[\hat{\theta}_2]}{\text{Var}[\hat{\theta}_1]} = \frac{\tau^2}{\frac{\tau^2}{2}} = 2.$$