

## M378K Introduction to Mathematical Statistics

### Problem Set #2

#### Discrete random variables.

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**2.1. Probability mass function.** Recall the following definition from the last class:

**Definition 2.1.** Given a set  $B$ , we say that a random variable  $Y$  is  $B$ -valued if

$$\mathbb{P}[Y \in B] = 1.$$

We reserve special terminology for random variables  $Y$  depending on the cardinality of the set  $B$  from the above definition. In particular, we have the following definition:

**Definition 2.2.** A random variable  $Y$  is said to be discrete if there exists a set  $S$  such that :

- $Y$  is  $S$ -valued, and
- $S$  is either **finite** or **countable**.

**Problem 2.1.** Provide an example of a **discrete** random variable.

Our next task is to try to keep track of the probabilities that  $Y$  takes specific values from  $S$ . In order to be more "economical", we introduce the following concept:

**Definition 2.3.** The support  $S_Y$  of a random variable  $Y$  is the **smallest** set  $S$  such that  $Y$  is  $S$ -valued.

**Problem 2.2.** What is the **support** of the random variable you provided as an example in the above problem?

**Problem 2.3.** Let  $y \in S_Y$  where  $Y$  is a discrete random variable. Is it possible to have  $\mathbb{P}[Y = y] = 0$ ?

Usually, we are interested in calculating and modeling probabilities that look like this

$$\mathbb{P}[Y \in A] \quad \text{for some } A \subset S_Y.$$

Note that, if we know the probabilities of the form

$$\mathbb{P}[Y = y] \quad \text{for all } y \in S_Y,$$

then we can calculate any probability of the above form. *How?*

So, if we "tabulate" the probabilities of the form  $\mathbb{P}[Y = y]$  for all  $y \in S_Y$ , we have sufficient information to calculate any probability of interest to do with the random variable  $Y$ . This observation motivates the following definition:

**Definition 2.4.** *The probability mass function (pmf) of a **discrete** random variable  $Y$  is the function  $p_Y : S_Y \rightarrow \mathbb{R}$  defined as*

$$p_Y(y) = \mathbb{P}[Y = y] \quad \text{for all } y \in S_Y.$$

Can you think of different ways in which to display the pmf?

What are the immediate properties of every pmf? Does the "reverse" hold, i.e., if a function  $p_Y$  satisfies you stated, is it always a pmf of **some** random variable?

What is the pmf of the random variable which you provided as an example above?

**2.2. Conditional probability.** In order to "build" more complicated (and useful!) random variables, it helps to review a bit more probability.

**Definition 2.5.** Let  $E$  and  $F$  be two events on the same  $\Omega$  such that  $\mathbb{P}[E] > 0$ . The conditional probability of  $F$  given  $E$  is defined as

$$\mathbb{P}[F | E] = \frac{\mathbb{P}[E \cap F]}{\mathbb{P}[E]}.$$

Let's spend a moment with the geometric/informational perspective on this definition.

By far, the most popular problems relying on the notion of **conditional probability** are those to do with **specificity** and **sensitivity**<sup>1</sup> of medical tests.

**Problem 2.4.** At any given time, 2% of the population actually has a particular disease.

A test indicates the presence of a particular disease 96% of the time in people who actually have the disease. The same test is positive 1% of the time when actually healthy people are tested.

Calculate the probability that a particular person actually has the disease **given** that they tested positive.

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<sup>1</sup>[https://en.wikipedia.org/wiki/Sensitivity\\_and\\_specificity](https://en.wikipedia.org/wiki/Sensitivity_and_specificity)

### 3. INDEPENDENT EVENTS

What if knowing that an event happened in fact does **not** give any information about the probability of another event?

**Definition 3.1.** We say that events  $E$  and  $F$  on  $\Omega$  are independent if

$$\mathbb{P}[E \cap F] = \mathbb{P}[E]\mathbb{P}[F].$$

In the case when  $E$  or  $F$  have a positive probability, it's possible to rewrite the above condition in a different (illustrative!) way. *How?*

Now that we know the notion of **independence**, we can construct random variables in many creative ways.

**Example 3.2.** A fair coin is tossed repeatedly and **independently** until the first Heads. Let the random variable  $Y$  represent the total number of Tails observed by the end of the procedure.

*What is the support of the random variable  $Y$ ?*

*What is the **probability mass function** of the random variable  $Y$ ?*

Moreover, now that we remember the definition of **conditional probability**, we can solve interesting problems such as this one:

**Problem 3.1.** *The number of pieces of gossip that break out in a particular high school in a week is modeled by a random variable  $Y$  with the following probability mass function:*

$$p_Y(n) = \frac{1}{(n+1)(n+2)} \quad \text{for all } n \in \mathbb{N}_0.$$

- (i) *Is the above a well-defined probability mass function?*
- (ii) *Calculate the probability that at least one piece of gossip occurred in a week **given** that at most four pieces of gossip occurred.*