M339J: February 24th, 2021.

The New In-Term Dates:

One: March 8th (Mon)

Mock Exam aka HW#3 From Thu March 4th to Sat March 6th

Two: April 9th

Three: May 3rd

Q: Another HW#2 due Wed 03/03?

!: Expect modified Syllabus on Friday!

UNIVERSITY OF TEXAS AT AUSTIN

Problem set 1

Problem 1.1. Let E and F be any two events. Then, $\mathbb{P}[E \cup F] \leq \mathbb{P}[E] + \mathbb{P}[F]$. *True or false? Why?*

Problem 1.2. Let E and F be any two events. If $\mathbb{P}[E] = \mathbb{P}[F] = \frac{2}{3}$, then E and F cannot be mutually exclusive. *True or false? Why?*

$\mathbb{P}[E \cup F] = \frac{4}{3} > 1$

Problem 1.3. Let E and F be any two events with positive probability. If $\mathbb{P}[E|F] < \mathbb{P}[E]$, then $\mathbb{P}[F|E] < \mathbb{P}[F]$. *True or false? Why?*

Problem 1.4. If events E and F are independent and events F and G are independent, then E and G are independent as well. *True or false? Why?*

Take $E = G$.

Problem 1.5. The four standard blood types are distributed in a populations as follows:

$A - 42\%$	$O - 33\%$
$B - 18\%$	$AB - 7\%$

Assuming that people choose their mates independently of their blood type, find the probability that the people in a randomly chosen couple from this population have different blood types.

Problem 1.6. Let X denote the outcome of a roll of a fair, regular icosahedron (a polyhedron with 20 faces) with numbers $1, 2, \cdots, 20$ written on its sides. Then $\mathbb{E}[X] = 15/2$. *True or false? Why?*

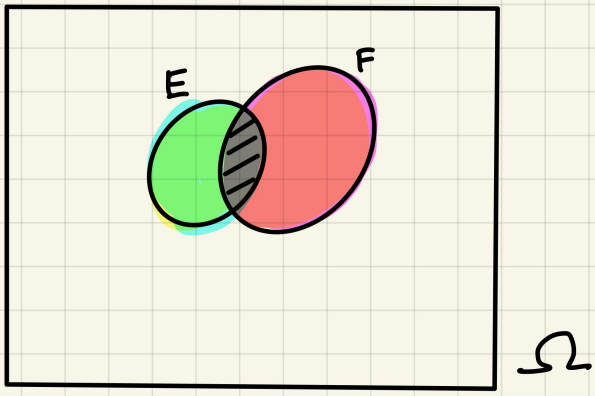
Problem 1.7. The minimum of two independent exponential random variables is also exponential. *True or false? Why?*

P#1.

$$\mathbb{P}[E \cup F] = \mathbb{P}[E] + \mathbb{P}[F] - \underbrace{\mathbb{P}[E \cap F]}_{\geq 0}$$

$$\leq \mathbb{P}[E] + \mathbb{P}[F]$$

Subadditivity



P#2.

$$\mathbb{P}[E \mid F] < \mathbb{P}[E]$$

$\Updownarrow$ by the def'n of conditional probabs.

$$\frac{\mathbb{P}[E \cap F]}{\mathbb{P}[F]} < \mathbb{P}[E] \quad / \cdot \mathbb{P}[F]$$

$\Updownarrow$

$$\mathbb{P}[E \cap F] < \mathbb{P}[E] \cdot \mathbb{P}[F] \quad / : \mathbb{P}[E]$$

$\Updownarrow$

$$\frac{\mathbb{P}[E \cap F]}{\mathbb{P}[E]} < \mathbb{P}[F]$$

$\Updownarrow$ by def'n of conditional probabs.

$$\mathbb{P}[F \mid E] < \mathbb{P}[F]$$

✓

TP#5.

$$\begin{aligned}
&\mathbb{P}[\text{a couple have different blood types}] = \\
&= 1 - \mathbb{P}[\text{have the same blood type}] \\
&= 1 - \mathbb{P}[\text{both A}] - \mathbb{P}[\text{both B}] - \mathbb{P}[\text{both O}] - \mathbb{P}[\text{both AB}] \\
&= 1 - (0.42)^2 - (0.18)^2 - (0.33)^2 - (0.07)^2 \\
&= 0.6774 \quad \checkmark
\end{aligned}$$

TP#6.

$$\mathbb{E}[X] = \frac{1}{20}\left(1 + 2 + 3 + \cdots + 20\right) =$$

$$= \frac{1}{\cancel{20}} \cdot \frac{\cancel{20}\cdot(20+1)}{2} = \frac{21}{2} = 10.5. \quad \blacksquare$$

TP#7.

$T_i \sim \text{Exponential}(\text{mean} = \theta_i)$, $i = 1, 2$.

Assume T_1 and T_2 are independent.

Define $\boxed{T := \min(T_1, T_2)}$ ✓

$\text{Support}(T) = (0; +\infty)$

Goal: To figure out the survival function of T!

Take $t > 0$.

$$\begin{aligned}
S_T(t) &= \mathbb{P}[T > t] = \mathbb{P}[\min(T_1, T_2) > t] \\
&= \mathbb{P}[T_1 > t, T_2 > t] \qquad \text{independence!} \\
&= \mathbb{P}[T_1 > t] \cdot \mathbb{P}[T_2 > t] \\
&= e^{-\frac{t}{\theta_1}} \cdot e^{-\frac{t}{\theta_2}} = e^{-t\left(\frac{1}{\theta_1} + \frac{1}{\theta_2}\right)} \qquad = \frac{1}{\theta}
\end{aligned}$$

↑ exponential r.v.s

$\Rightarrow$ T is exponential with parameter θ such that

$$\frac{1}{\theta} = \frac{1}{\theta_1} + \frac{1}{\theta_2} \quad \Rightarrow \quad \theta = \frac{1}{\frac{1}{\theta_1} + \frac{1}{\theta_2}}$$