M378K: February 5th, 2025.

More on the cdf.

Example. The Normal Distribution.

$$Y \sim N(\mu, \sigma)$$

$$F_Y(y) = \mathbb{P}[Y \leq y] = \cdots$$

$$\cdots = \int_{-\infty}^{y} f_Y(u)\,du$$

$$\text{w/}\quad f_Y(u) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(u-\mu)^2}{2\sigma^2}}$$

No analytic cdf!

Fact.

$$\frac{Y - \mu_Y}{\sigma_Y} \sim N(0,1) \sim Z$$

$$Y = \mu_Y + \sigma_Y \cdot Z$$

$$\mathbb{P}[Y \leq y] = \mathbb{P}\left[Z \leq \frac{y-\mu_Y}{\sigma_Y}\right] =: \Phi\left(\frac{y-\mu_Y}{\sigma_Y}\right)$$

$$\boxed{\Phi \ldots \text{ cdf of } N(0,1)}$$

Def'n. Let Y be a random variable w/ the cdf F_Y.
For $\alpha \in (0,1)$ the α-quantile of the dist'n of the random variable Y is defined as the number

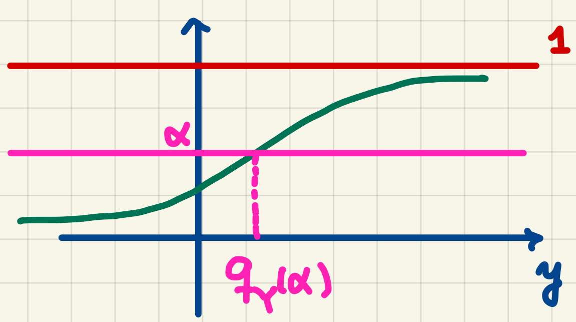
$$q_Y(\alpha) \in \mathbb{R}$$

which satisfies

$$\mathbb{P}[Y \leq \boxed{q_Y(\alpha)}] = \alpha$$

$$\iff$$

$$F_Y(\boxed{q_Y(\alpha)}) = \alpha$$



Note: If F_Y^{-1} exists, then $q_Y(\alpha) = F_Y^{-1}(\alpha)$.

This is, for example, the case w/ the normal.

M378K Introduction to Mathematical Statistics

Problem Set #7

Cumulative distribution functions: Named continuous distributions.

Problem 7.1. Source: Sample P exam, Problem #126.

A company's annual profit is normally distributed with mean 100 and variance 400. Then, we can express the probability that the company's profit in a year is at most 60, given that the profit in the year is positive, in terms of the standard normal cumulative distribution function denoted by Φ as

$$\boxed{1 - \frac{\Phi(2)}{\Phi(5)}}$$

True or false?

$\rightarrow$:

$$Y \sim N(\mu = 100, sd = \sigma = 20)$$

$$Y = 100 + 20 \cdot Z \quad w/\ Z \sim N(0,1)$$

$$\mathbb{P}[Y \leq 60 \mid Y > 0] = \frac{\mathbb{P}[0 < Y \leq 60]}{\mathbb{P}[Y > 0]}$$

$$= \frac{\mathbb{P}[0 < 100 + 20Z \leq 60]}{\mathbb{P}[100 + 20Z > 0]} =$$

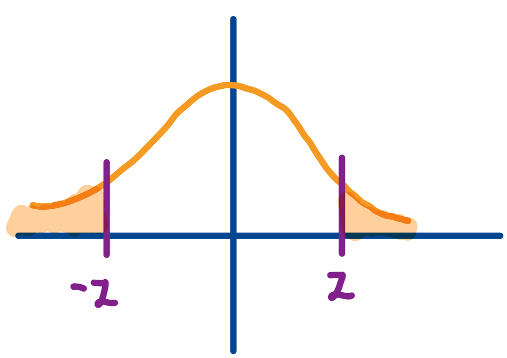
$$= \frac{\mathbb{P}[-5 < Z \leq -2]}{\mathbb{P}[-5 < Z]}$$

$$= \frac{\Phi(-2) - \Phi(-5)}{1 - \Phi(-5)}$$

$$= \frac{\cancel{1} - \Phi(2) - (\cancel{1} - \Phi(5))}{\Phi(5)}$$

$$= \frac{\Phi(5) - \Phi(2)}{\Phi(5)}$$

$$= 1 - \frac{\Phi(2)}{\Phi(5)} \quad \boxed{T}$$



$$\Phi(-z) = \mathbb{P}[Z > z]$$

$$= 1 - \mathbb{P}[Z \leq z] = 1 - \Phi(z)$$

Example. The Exponential Dist'n.

$$Y \sim E(\tau)$$

$$f_Y(y) = \frac{1}{\tau} e^{-\frac{y}{\tau}} \mathbb{1}_{[0,\infty)}(y)$$

$F_Y(y) = ?$

Evidently, $F_Y(y) = \underline{0}$ for $y \leq 0$

for $y > 0$:

$$F_Y(y) = \mathbb{P}[Y \leq y] = \int_0^y \frac{1}{\tau} e^{-\frac{u}{\tau}} du$$

$$= \frac{1}{\cancel{\tau}} (-\cancel{\tau}) e^{-\frac{u}{\tau}} \Big|_{u=0}^{y} = -\left(e^{-\frac{y}{\tau}} - 1\right)$$

$$\boxed{F_Y(y) = 1 - e^{-\frac{y}{\tau}}}$$