M358K: October 20th, 2023.



University of Texas at Austin

Problem Set # 12

Hypothesis testing: The normal case.

**Problem 12.1.** *Source: Ramachandran, Tsokos.*
The management of the local health club claims that its members lose on average 15 pounds or more within the first three months of their membership. A consumer agency took a simple random sample of 45 members and found the sample average of 13.8 in pounds lost. Assume that we model the weight loss as normal with an **unknown** mean $\mu$ and the **known** standard deviation of 4.2 pounds. What is the $p-$value corresponding to the gathered data? What would your decision be at the 0.05 significance level?

→: The population model.

X... weight loss of a randomly chosen gym member

X ~ Normal (mean = $\mu$, sd = $\sigma = 4.2$)

Our hypotheses:

$H_0: \mu = \mu_0 = 15$ vs. $H_a: \mu < \mu_0 = 15$

The observed value of the z-statistic, i.e., our z-score:

$$z = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{13.8 - 15}{\frac{4.2}{\sqrt{45}}} = -1.9166$$

The p-value is:

$$P[Z < z] = \Phi(-1.9166) = \text{pnorm}(-1.9166) = 0.0276$$

Less than $\alpha = 0.05$, so we REJECT the null hypothesis. □

**Problem 12.2.** *Source: Ramachandran, Tsokos.*
It is claimed that sports-car owners drive on the average 20,000 miles per year. A consumer firm believes that the mean annual mileage is actually lower. To check, the consumer firm decided to test this hypothesis.

The modeling assumptions are that the annual mileage is normally distributed with an unknown mean $\mu$ and with the standard deviation of 1200.

The consumer firm obtained information from 36 randomly selected sports-car owners that resulted in a sample average of 19,530 miles. What is the decision of this hypothesis test at the signficance level of 0.01?

→:

$H_0: \mu = 20000$ vs. $H_a: \mu < \mu_0 = 20000$

The observed value of the z-statistic:

$$z = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{19530 - 20000}{\frac{1200}{\sqrt{36}}} = -2.35$$

The p-value:

$$P[Z < z] = \Phi(-2.35) = 0.0094$$

(std normal tables)

$\Rightarrow$ Reject the null hypothesis since $0.0094 < 0.01$

□

# Types of Errors.

| Decision \ "Truth" | $H_0: \mu = \mu_0$ | $H_a: \begin{cases} \mu < \mu_0 & \text{left-sided} \\ \mu \neq \mu_0 & \text{two-sided} \\ \mu > \mu_0 & \text{right-sided} \end{cases}$ |
|---|---|---|
| Reject $H_0$ | Type I Error |  |
| Fail to Reject $H_0$ |  | Type II Error |

$$\mathbb{P}[\text{Type I Error}] = \mathbb{P}_0[\text{Reject } H_0] = \alpha \text{ (significance level)}$$

$\mathbb{P}_0$: under the null, i.e., if $\mu = \mu_0$

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# Problem Set # 13

## Types of errors.

Provide your **final answer only** for the following problems.

**Problem 13.1.** You perform 2000 significance tests using a significance level 0.10. Under the assumption that all of the null hypotheses for the 2000 significance tests are true, how many of the 2000 significance tests would you expect to **not** result in a Type I error?

**a.** 200
**b.** 1800
**c.** 2000
**d.** 0
**e.** None of the above.

**Problem 13.2.** A medical researcher is working on a new treatment for a certain type of cancer. The average survival time after diagnosis on the standard treatment is 2 years. In an early trial, she tries the new treatment on three subjects who have an average survival time after diagnosis of 4 years. Although the survival time has doubled, the results are not statistically significant, even at the 0.10 significance level.

Suppose, in fact, that the new treatment **does** increase the mean survival time in the population of all patients with this particular type of cancer. Which of the following statements is TRUE?

**a.** A Type I error occurred.
**b.** A Type II error occurred.
**c.** No error occurred.

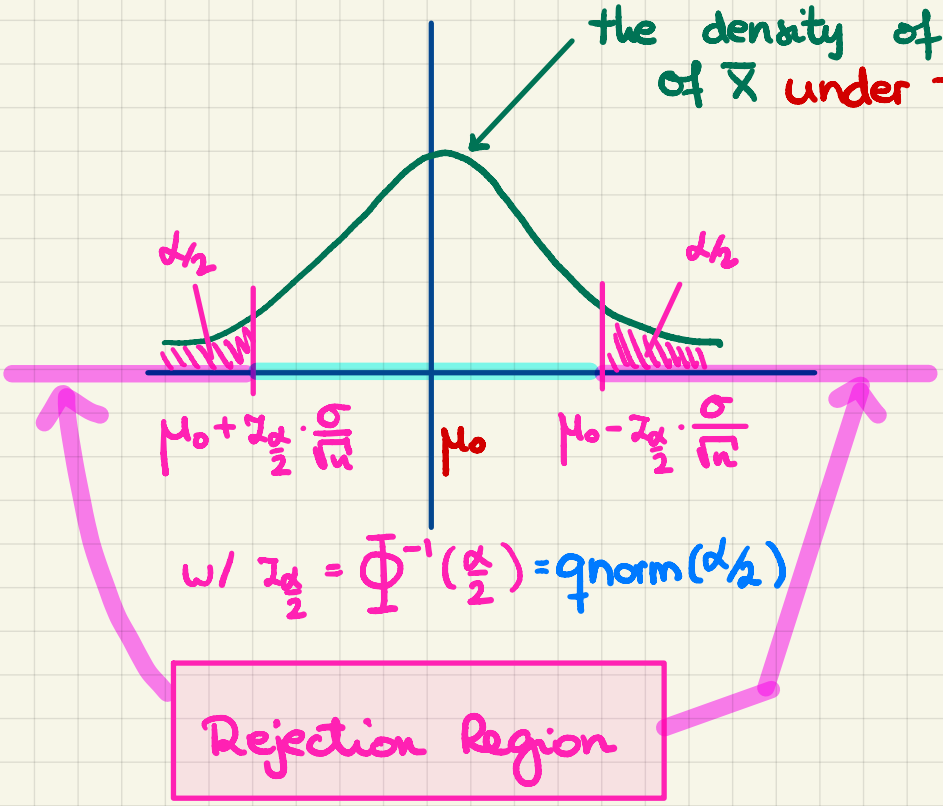
**Problem 13.3.** An engineer has designed an improved light bulb. The previous design had a mean lifetime of 1200 hours. Using a sample of 2000 of the new bulbs, the sample average lifetime of this improved light bulb is found to be 1201 hours. Although the difference is quite small, the effect was statistically significant at the 0.05 level. Suppose that, in fact, there is no difference between the mean lifetimes of the previous design and the new design. Which of the following statements is TRUE?

**a.** A Type I error occurred.
**b.** A Type II error occurred.
**c.** No error occurred.

# Power of Test.

Temporarily, we focus on the two-sided alternative;
We will consider the other two structures of the alternative hypothesis by analogy.

$\alpha$ ... significance level




Focus on the "fail-to-reject" region, i.e., for a two-sided test

$$\left(\mu_0 + z_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}},\ \mu_0 - z_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}\right)$$