The Binomial Asset Pricing Model.

$S(0)$ ... the observable initial asset price

0 ———— T

T time horizon (e.g., the exercise date of an option)

$\boxed{S(0)}$ ROOT NODE

$\boxed{S_u = u \cdot S(0)}$ up node

$\boxed{S_d = d \cdot S(0)}$ down node

h .. length of a single period

By convention:

$\boxed{u > d}$

u ... up factor
d ... down factor

one period: $\boxed{S(T) = S(h)}$ ... a random variable denoting the time T stock price w/ two possible values: S_u and S_d

As a random variable : SIMPLE RETURN

$$\boxed{\frac{S(T) - S(0)}{S(0)}}$$

up:
$$\frac{S_u - S(0)}{S(0)} = \frac{S_u}{S(0)} - 1 = u - 1$$

down:
$$\frac{S_d - S(0)}{S(0)} = \frac{S_d}{S(0)} - 1 = d - 1$$

No-Arbitrage Condition.

Market Model.

- riskless asset : @ the ccrfir r
- risky asset : non-dividend-paying stock

Imagine investing in one share of this stock @ time-0.

$$
\begin{tikzcd}
 & S_u = u \cdot S(0) \\
S(0) \arrow[ur] \arrow[dr] & \\
 & S_d = d \cdot S(0)
\end{tikzcd}
\qquad
\begin{array}{c}
\underline{\text{Wealth}} \\
S_u \\
\\
S_d
\end{array}
$$

$h = T$

At the risk-free rate, the amount $S(0)$ accumulates to $\underline{S(0)e^{rh}}$ in the same time period.

The No-Arbitrage Condition:

$$S_d < S(0)e^{rh} < S_u$$

$$d \cdot \cancel{S(0)} < \cancel{S(0)}e^{rh} < u \cdot \cancel{S(0)}$$

$$\boxed{d < e^{rh} < u}$$

Half-a-Proof.

Say, to the contrary, $\underline{e^{rh} \leq d < u}$

Propose. Long one share of stock.

Verify. Profit = Payoff $-$ $FV_{0,T}$(Initial Cost)

$$= S(h) - S(0)e^{rh}$$

down node: $S_d - S(0)e^{rh} = d \cdot S(0) - S(0)e^{rh} = S(0)\overbrace{(d - e^{rh})}^{\geq 0}$

up node: $S_u - S(0)e^{rh} = u \cdot S(0) - S(0)e^{rh} = S(0)\underbrace{(u - e^{rh})}_{>0}$

$\boxed{\text{Indeed, this is an arbitrage portfolio.}}$

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Problem Set #9

Binomial option pricing.

Problem 9.1. In the setting of the one-period binomial model, denote by i the **effective** interest rate **per period**. Let u denote the "up factor" and let d denote the "down factor" in the stock-price model.
If

$$d < u \leq 1 + i$$

then there certainly is no possibility for arbitrage.

F

Fixed statement:

$$\boxed{d < 1+i < u}$$

Problem 9.2. In our usual notation, does this parameter choice create a binomial model with an arbitrage opportunity?

$u = 1.18$, $d = 0.87$, $r = 0.05$, $\cancel{\delta = 0}$, $h = 1/4$

No, it doesn't!

$$d = 0.87 \overset{?}{\cancel{<}} \underbrace{e^{rh} = e^{0.05/4}}_{= e^{0.0125}} \overset{?}{\cancel{<}} 1.18 = u$$

$$e^{0.0125} \overset{??}{=} \boxed{1.0125}$$

Taylor Expansion of e^x

$$\boxed{e^x = 1 + x} + \frac{x^2}{2} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$$

Q: What if $\tilde{d} = 1.01$?

It's still OK!

Forward Binomial Tree.

"Def'n". The **volatility σ** is the standard deviation of realized returns on a continuously compound scale and annualized.

Heuristics. $T = 1$

$$h_m = \frac{1}{m} \quad \text{(of a year)}$$

Q: What is the volatility for the time period of length h_m?
Call this volatility σ_{h_m}