UNIVERSITY OF TEXAS AT AUSTIN

Quiz #16

Discrete distributions.

Provide your **complete solution** to the following problems. Final answers only, without appropriate justification, will receive zero points even if correct.

Problem 16.1. (5 points) Let the number of floods in a calendar year be denoted by N and modeled using the Poisson distribution with mean 5. We say that a flood is "minor" if the damages associated with it do not exceed \$1,000,000. Otherwise, a flood is designated as "major". The number of floods and the damages caused by individual floods are assumed to be independent.

Assume that the probability that an observed flood is "major" equals 1/5.

Find the probability that the number of "major" floods is 2, given that the **total** number of floods in that year equals 5.

Solution: Let N_1 denote the r.v. which stands for the number of "major"floods, and let N_2 be the number of "minor" floods. According to the *"Thinning" theorem*, N_1 and N_2 are independent and

$$N_1 \sim Poisson(\frac{1}{5} \cdot 5 = 1),$$
$$N_2 \sim Poisson(\frac{4}{5} \cdot 5 = 4).$$

We are ready to calculate the conditional probability

$$\begin{aligned}
\mathbb{P}[N_1 = 2 \,|\, N = 5] &= \frac{\mathbb{P}[N_1 = 2, N = 5]}{\mathbb{P}[N = 5]} \\
&= \frac{\mathbb{P}[N_1 = 2, N_1 + N_2 = 5]}{\mathbb{P}[N = 5]} \\
&= \frac{\mathbb{P}[N_1 = 2, N_2 = 3]}{\mathbb{P}[N = 5]}.
\end{aligned}$$

Since N_1 and N_2 are independent, this probability equals

$$\begin{aligned}
\frac{\mathbb{P}[N_1 = 2]\,\mathbb{P}[N_2 = 3]}{\mathbb{P}[N = 5]} &= \frac{e^{-1}\,\frac{1^2}{2!} \cdot e^{-4}\,\frac{4^3}{3!}}{e^{-5}\,\frac{5^5}{5!}} \\
&= \frac{\frac{1^2}{2!} \cdot \frac{4^3}{3!}}{\frac{5^5}{5!}} \\
&= \frac{4^3 \cdot 5!}{5^5 \cdot 2! \cdot 3!} = \frac{2^7}{5^4} = 0.2048.
\end{aligned}$$

Of course, we obtained the binomial conditional distribution above. This is a fact we have shown in class and you could have just used it directly.

Problem 16.2. (5 points) Suppose that the number N of customers visiting a fast food restaurant in a given morning is Poisson with mean 20. Assume that each customer purchases a drink with probability 3/4, independently from other customers, and independently from the value of N. Let N_1 be the number of customers who purchase drinks in that time interval and let N_2 be the number of customers that do not purchase drinks.

What is the probability that exactly 3 customers purchase a drink in a given morning, **given** that there is a total of 10 customers on that particular morning?

Solution: We have established in class that

$$N_1 \,|\, N = 10 \sim Binomial(m = 10, q = 3/4).$$

Hence,

$$\mathbb{P}[N_1 = 3 \,|\, N = 10] = \binom{10}{3}\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^7 = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2} \cdot \frac{3^3}{4^{10}} = 0.0030899.$$

Problem 16.3. (5 points) Let us denote the claim count r.v. by N. We are given that N is a mixture random variable such that

$$N \,|\, \Lambda = \lambda \sim Poisson(\lambda)$$

while Λ is Gamma distributed with mean 2 and variance equal to 4. What is the probability that N is at most 1?

Solution: We have shown in class that the distribution of N is negative binomial. Let us find its parameters r and β. Using the fact that $N \,|\, \Lambda$ is Poisson, we get

$$r\beta = \mathbb{E}[N] = \mathbb{E}[\mathbb{E}[N \,|\, \Lambda]] = \mathbb{E}[\Lambda] = 2,$$
$$r\beta(1+\beta) = Var[N] = \mathbb{E}[Var[N \,|\, \Lambda]] + Var[\mathbb{E}[N \,|\, \Lambda]] = \mathbb{E}[\Lambda] + Var[\Lambda] = 6.$$

So, $\beta = 2$ and $r = 1$. Finally, using our tables, we get

$$F_N(1) = p_N(0) + p_N(1) = \frac{1}{3} + \frac{2}{9} = \frac{5}{9}.$$