MAY 5TH

Bond Options, Caps and the Black Model

The "master" Black-Scholes pricing formula:

$$V_c(0) = F^P_{0,T}(S)N(d_1) - \underbrace{F^P_{0,T}(K)}N(d_2)$$

$$V_p(0) = F^P_{0,T}(K)N(-d_2) - F^P_{0,T}(S)N(-d_1)$$

w/ $$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{F^P_{0,T}(S)}{F^P_{0,T}(K)}\right) + \frac{\sigma^2 T}{2}\right],$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

For futures options: $\boxed{\text{BLACK FORMULA}}$

$$V_c(0) = e^{-rT}\left(F_{0,T_F}N(d_1) - K\cdot N(d_2)\right)$$

$$V_p(0) = e^{-rT}\left(K\cdot N(-d_2) - F_{0,T_F}\cdot N(-d_1)\right)$$

w/ $$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{F_{0,T_F}}{K}\right) + \frac{\sigma^2 T}{2}\right],$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Sections 14.6 and 24.3

Example: What if the futures contract is on a $\boxed{\text{BOND}}$?

For any underlying:

$$F_{t,T_F}[S] = \frac{S(t)}{P(t,T_F)}$$

Argument:

- **At time-t:**
  - → SHORT the forward contract
  - → SHORT $\frac{S(t)}{P(t,T_F)}$ of zero-coupon bonds redeemable for \$1.
    - ⇒ obtain $S(t)$
  - → LONG 1 unit of underlying S (no dividends) ⇒ pay $S(t)$

the net effect is 0. :)

- **At time-T:**
  - → DELIVER the 1 unit of the underlying that you and RECEIVE F (the forward price)
  - → have to pay out the redemption value for all the bonds you shorted (issued):
    - ⇒ give up $\frac{S(t)}{P(t,T_F)}$

Since we started w/ zero investment

$$\Rightarrow F_{t,T_F}[S] = \frac{S(t)}{P(t,T_F)}$$

- As a special case:
  - S a non-dividend-paying stock
  - r forever constant continuously compounded rate

$$P(t,T_F) = e^{-r(T_F-t)}$$

$$\Rightarrow \quad F_{t,T_F}[S] = S(t)\cdot e^{r(T_F-t)}$$

:) In agreement w/ D-course!

- In a special case relevant today:

S is another bond with maturity @ T_F+s w/ floating (non-deterministic) short rates:

$$F_{t,T_F}\Big[\underbrace{P(T_F,T_F+s)}_{\substack{\text{the bond}\\ \text{redeemable @ } T_F+s\\ \text{for \$1}}}\Big] = \frac{P(t,T_F+s)}{P(t,T_F)}$$

Q: What is the time-t prepaid-forward price for the bond?

Equal to the bond price itself.

$$P(t,T_F+s)$$

⇒ In the Black formula:

- the prepaid forward price of the underlying (i.e., the bond) will be

$$P(t, T_F+s)$$

- the prepaid forward price of the strike K will be

$$K \cdot P(t, T_F)$$

Assume that the bond-forward prices are GBMs (i.e., the Black-Scholes model applies)

$$Var\left[F_{t,T_F}[P(T_F, T_F+s)] \right] = Var\left[\frac{P(t, T_F+s)}{P(t, T_F)} \right] = \sigma^2$$

$\downarrow$

To go into the pricing formula.

⇒ Black formula:

$$F_{0,T}[P(T, T+s)] = F_{0,T}$$

$$V_C(0) = P(0, T) \left[F_{0,T} N(d_1) - K N(d_2) \right]$$

$\uparrow$ The exercise date of the call.

w/ $$d_1 = \frac{1}{\sigma\sqrt{T}} \left[\ln\left(\frac{F_{0,T}}{K}\right) + \frac{\sigma^2 T}{2} \right]$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

7. You are given the following information:

Bond maturity (years)	1	2
Zero-coupon bond price	0.9434	0.8817

A European call option, that expires in 1 year, gives you the right to purchase a 1-year bond for 0.9259.

$K = 0.9259$ $\quad T = 1$ $\quad s = 1$

The bond forward price is lognormally distributed with volatility $\sigma = 0.05$.

Using the Black formula, calculate the price of the call option.

(A) 0.011

(B) 0.014

(C) 0.017

(D) 0.020

(E) 0.022

$$V_C(0) = P(0,1)\left[F_{0,1} N(d_1) - K N(d_2)\right]$$

Recall: $F_{0,1} \cdot P(0,1) = P(0,2)$

$$\Rightarrow V_C(0) = P(0,2) N(d_1) - P(0,1) \cdot K N(d_2)$$

w/ $$d_1 = \frac{1}{0.05\sqrt{1}}\left[\ln\left(\frac{F_{0,1}}{K}\right) + \frac{(0.05)^2}{2}\right]$$

$$F_{0,1} = \frac{0.8817}{0.9434} = 0.9346$$

$$\Rightarrow d_1 = 20\left[\ln\left(\frac{0.9346}{0.9259}\right) + \frac{0.0025}{2}\right] = \cdots = 0.212$$

$$d_2 = 0.212 - 0.05 = 0.162.$$

Fill in the blanks ü

$\Rightarrow$ answer $= 0.022 \Rightarrow$ E.

Exercise.

Spot prices for zero-coupon bonds are:

Years	2	5
Price	0.90	0.72

Volatility of the price of a 2-year forward on a bond w/ 3 years to maturity is 0.1

Find the Black price of a 2-year, 0.75-strike European call option on a bond w/ 3 year to maturity.

$$F_{0,2} = F_{0,2}[P(2, 2+3)]$$

Shorthand :)

Caplets.

realized floating rate

$$\frac{(R_T - K_R)_+}{\underbrace{1+R_T}_{\text{1-year interest rate}}} = \frac{((1+R_T)-(1+K_R))_+}{1+R_T} = \left(1 - \frac{1+K_R}{1+R_T}\right)_+$$

$$= (1+K_R)\underbrace{\left(\frac{1}{1+K_R} - \frac{1}{1+R_T}\right)_+}_{\text{Put payoff}}$$

~~aff~~ can play the role of the strike

underlying: bond w/ one year to maturity @ time-T

Put payoff

for a ~~E.~~ put w/ exercise date T and strike $\frac{1}{1+K_R}$ for a bond w/ one year to maturity (w/ $s=1$)