M362K Probability
Spring 2024
University of Texas at Austin
In-Term Exam I
Instructor: Milica Čudina

Notes: This is a closed book and closed notes exam. The maximal score on the real exam will be 100 points.
Time: 50 minutes

All written work handed in by the student is considered to be
their own work, prepared without unauthorized assistance.

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"I agree that I have complied with the UT Honor Code during my completion of this exam."

Signature:

1.1. DEFINITIONS.

Problem 1.1. (10 points) Provide the definition of *mutually exclusive* events.

Solution: Consult your notes.

Problem 1.2. (10 points) Provide the definition of *independent* events.

Solution: Consult your notes.

1.2. **TRUE/FALSE QUESTIONS.**

Problem 1.3. (2 pts) Let E and F be **any** two events. Then

$$\mathbb{P}[E \cup F] = \mathbb{P}[E] + \mathbb{P}[F].$$

True or false?

Solution: FALSE
Let E and F be two events such that

$$\mathbb{P}[E \cap F] > 0.$$

If you are worried about the existence of E and F above, simply take $E = F = \Omega$.
Due to the inclusion-exclusion formula, we get

$$\mathbb{P}[E \cup F] = \mathbb{P}[E] + \mathbb{P}[F] - \mathbb{P}[E \cap F] < \mathbb{P}[E] + \mathbb{P}[F].$$

Problem 1.4. (2 points) Consider two events E and F such that $\mathbb{P}[E] = 2/3$ and $\mathbb{P}[F] = 3/4$. Then, it's possible that E and F be mutually exclusive. *True or false?*

Solution: FALSE
If E and F were mutually exclusive, we would have

$$\mathbb{P}[E \cup F] = \mathbb{P}[E] + \mathbb{P}[F] = \frac{2}{3} + \frac{3}{4} > 1.$$

This is impossible.

1.3. **Free-response problems.** Please, explain carefully all your statements and assumptions. Numerical results or single-word answers without an explanation (even if they're correct) are worth 0 points.

Problem 1.5. (5 points) Two dice are rolled, what is the probability that the sum of the upturned faces equals 7?

Solution: The outcome space is $\Omega = \{(i, j) : 1 \leq i, j \leq 6\}$ and the event E that the sum of upturned faces equals 7 can be written as

$$E = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}.$$

The probability of E is

$$\mathbb{P}[E] = \frac{\#(E)}{\#(\Omega)} = \frac{1}{6}\,.$$

Problem 1.6. (15 points) Harry's cafe offers a lunch special consisting of a protein, a starchy side and a vegetable side. Today, the following are on the menu:

P tofu(T) OR chicken(C) OR salmon(S);
S mashed potatoes (P) OR jasmine rice (R);
V roasted beets (B) OR asparagus (A).

A patron is to choose a single item from each category.

(i) (5 points) Write down the sample space of all the different meals that can be ordered from the above menu.

(ii) (3 points) How many items are there in the sample space?

(iii) (3 points) Let A be the event that tofu is chosen as the protein, how many outcomes are there in A?

(iv) (4 points) Let B denote the event that the beets are chosen. List all the outcomes in the event $A \cap B$.

Solution:

$$\Omega = \{TPB, TPA, TRB, TRA, CPB, CPA, CRB, CRA, SPB, SPA, SRB, SRA\}$$

$$3 \cdot 2 \cdot 2 = 12$$

$$2 \cdot 2 = 4$$

$$B = \{TPB, TRB\}.$$

Problem 1.7. (10 pts) Three balls are randomly drawn from a bowl containing 6 white and 4 black balls. What is the probability that one of the drawn balls is white and the remaining two are black?

Solution: There are altogether $\binom{10}{3}$ possibilities to extract 3 balls from the urn containing altogether 10 balls. On the other hand, there are

$$\binom{6}{1} \cdot \binom{4}{2}$$

possibilities to extract one white ball and two black balls.

All the possible triples are equally likely, so the probability we are looking for is

$$\frac{\binom{6}{1} \cdot \binom{4}{2}}{\binom{10}{3}} = \frac{6 \cdot \frac{4 \cdot 3}{2}}{\frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1}} = \frac{3 \cdot 4 \cdot 3}{10 \cdot 3 \cdot 4} = \frac{3}{10}\,.$$

Problem 1.8. (30 points) In a certain college, 25% of the students failed mathematics, 15% of the students failed chemistry, and 10% of the students failed both mathematics and chemistry. A student is selected at random.

(i) (10 pts) If they failed chemistry, what is the probability that they failed mathematics?

(ii) (10 pts) If they failed mathematics, what is the probability that they failed chemistry?

(iii) (10 pts) What is the probability that they failed either chemistry or mathematics (or, maybe, both)?

Solution: Let us introduce the following notation for the remainder of this problem:

$$M := \{\text{all students who failed mathematics}\},$$
$$C := \{\text{all students who failed chemistry}\}.$$

The problem gives us that $\mathbb{P}[M] = 1/4, \mathbb{P}[C] = 3/20$ and $\mathbb{P}[M \cap C] = 1/10$.

In the first part of the problem, we seek the following probability:

$$\mathbb{P}[M|C] = \frac{\mathbb{P}[M \cap C]}{\mathbb{P}[C]} = \frac{\frac{1}{10}}{\frac{3}{20}} = \frac{2}{3}\,.$$

In the second part of the problem, we need to calculate

$$\mathbb{P}[C|M] = \frac{\mathbb{P}[M \cap C]}{\mathbb{P}[M]} = \frac{\frac{1}{10}}{\frac{1}{4}} = \frac{2}{5}\,.$$

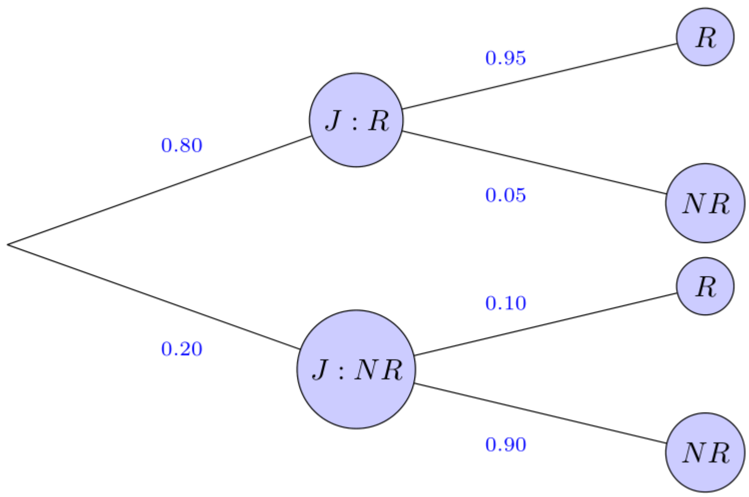
For the final part of the problem, by the inclusion-exclusion formula, we get:

$$\mathbb{P}[M \cup C] = \mathbb{P}[M] + \mathbb{P}[C] - \mathbb{P}[M \cap C] = \frac{1}{4} + \frac{3}{20} - \frac{1}{10} = \frac{5+3-2}{20} = \frac{3}{10}\,.$$

This is the probability that the students failed **either** one of the courses (maybe both).

Problem 1.9. (16 points) Most mornings, Bertie Wooster asks Jeeves whether it is going to rain that day. It being England, Jeeves forecasts rain 80% of the time and dry weather the remaining 20% of the time. If Jeeves forecasts rain, the chance if it actually raining is 95%. If Jeeves forecasts no rain, the chance of it not raining is 90%. Suppose that one day Bertie forgot to ask Jeeves if it would rain. It rained. What is the probability that Jeeves would have predicted rain?

Solution: This probability tree describes the situation in the problem:



We use the Bayes' theorem here.

$$\mathbb{P}[\text{Jeeves would have said rain} \mid R] = \frac{\mathbb{P}[R \mid \text{Jeeves would have said rain}]\mathbb{P}[\text{Jeeves would have said rain}]}{\mathbb{P}[R]}.$$

Using our tree, we get

$$\mathbb{P}[R] = 0.8(0.95) + 0.2(0.10) = 0.78.$$

So,

$$\mathbb{P}[\text{Jeeves would have said rain} \mid R] = \frac{0.8(0.95)}{0.78} = \frac{38}{39}.$$