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Problem Set # 13

Mean and median of the log-normal stock prices.

$\mathbb{E}^*[S(T)] = S(0)e^{rT}$

Problem 13.1. The current price of a non-dividend-paying stock is \$80 per share. Under the risk-neutral probability measure, its mean rate of return is 12% and its volatility is 30%.

Let $R(0,t)$ denote the realized return of this stock over the time period $[0,t]$ for any $t > 0$. Calculate $\mathbb{E}^*[R(0,2)]$.

$\rightarrow$: $\boxed{R(0,2) \sim Normal\left(mean = \left(r - \frac{\sigma^2}{2}\right)\cdot 2, \ var = \sigma^2 \cdot 2\right)}$

$\boxed{r = 0.12}$

$$\mathbb{E}^*[R(0,2)] = \left(0.12 - \frac{0.3^2}{2}\right)\cdot 2 = (0.12 - 0.045)\cdot 2 = 0.15$$

$\square$

$\mathbb{E}[S(T)] = S(0)e^{\alpha T}$; α... mean rate of return

Problem 13.2. A stock is valued at \$75.00. The continuously compounded, risk-free interest rate is 10.0% and the standard deviation of annualized returns is 25.0%. If the stock is lognormally distributed, what is the expected stock price after 2 years under the risk-neutral probability measure?

$\rightarrow$: $\mathbb{E}^*[S(2)] = S(0)e^{2r} = 75\cdot e^{2\cdot 0.1} = 75e^{0.2} = \underline{91.61}$

$\square$

Problem 13.3. A non-dividend-paying stock is valued at \$55.00 per share. Its standard deviation of annualized returns is given to be 22.0%. The continuously compounded risk-free interest rate is 12%. If the stock price is modeled using the lognormal distribution (as discussed in class), what is the median of the stock price in 3 years under the risk-neutral probability measure?

$\rightarrow$: $R(0,T) \sim Normal\left(mean = \left(r - \frac{\sigma^2}{2}\right)\cdot T, \ var = \sigma^2 \cdot T\right)$

$S(T) = S(0)e^{R(0,T)}$

median of $S(T)$: $\underline{S(0)e^{\left(r - \frac{\sigma^2}{2}\right)\cdot T}}$

$55e^{\left(0.12 - \frac{0.22^2}{2}\right)\cdot 3} = \underline{73.31}$

$\square$

Problem 13.4. Assume that the stock price is modeled using the lognormal distribution. The stock pays no dividends. Under $\mathbb{P}^*$, the annual mean rate of return on the stock is given to be 12%. Also under $\mathbb{P}^*$, the median time$-t$ stock price is evaluated to be $S(0)e^{0.1t}$. What is the volatility parameter of this stock price?

$\rightarrow$:

$$\cancel{S(0)}\cancel{e}^{(r-\frac{\sigma^2}{2})\cancel{t}} = \cancel{S(0)}\cancel{e}^{0.1\cancel{t}}$$

$$r-\frac{\sigma^2}{2} = 0.1 \qquad \frac{\sigma^2}{2} = 0.02$$

$$0.12-\frac{\sigma^2}{2} = 0.1 \qquad \sigma^2 = 0.04 \Rightarrow \boxed{\sigma = 0.2}$$

$\square$

Problem 13.5. The current stock price is \$100 per share. The stock price at any time $t > 0$ is modeled using the lognormal distribution. Assume that the continuously compounded risk-free interest rate equals 8%. Let its volatility equal 20%.

Find the value t^* at which the median stock price equals \$120, under the risk-neutral probability measure.

$\rightarrow$:

$$S(0)e^{(r-\frac{\sigma^2}{2})\cdot t^*} = 120$$

$$10\cancel{0}\, e^{(0.08-\frac{0.04}{2})\cdot t^*} = 12\cancel{0}$$

$$e^{0.06t^*} = 1.2$$

$$0.06t^* = \ln(1.2)$$

$$t^* = \frac{\ln(1.2)}{0.06} = \underline{3.039}$$

$\square$

Problem 13.6. The volatility of the price of a non-dividend-paying stock is 0.2. The stock price is modeled using a log-normal distribution. Under $\mathbb{P}^*$, the expected time-2 stock price is \$120. What is the median of the time-2 stock price under $\mathbb{P}^*$?

$\rightarrow$:

$$\text{median of } S(T) = S(0)e^{(r-\frac{\sigma^2}{2})\cdot T} = \underbrace{\boxed{S(0)e^{rT}}}_{\text{mean}} \cdot e^{-\frac{\sigma^2}{2}\cdot T}$$

$$120\cdot e^{-\frac{0.04}{2}\cdot 2} = 120e^{-0.04} = \underline{115.29}$$

$\square$

LogNormal Tail Probabilities.

Example. Consider a non-dividend-paying stock.
What is the probability that the stock outperforms a risk-free investment under the risk-neutral probability measure?

→: The initially invested amount is: $S(0)$

- If it's the risk-free investment, the balance @ time T is

$$S(0)e^{rT}$$

- If it's the stock investment, the wealth @ time T is

$$S(T)$$

$$\mathbb{P}^*[S(T) > S(0)e^{rT}] = ?$$

This question is equivalent to the one of whether the profit of a stock investment is positive under $\mathbb{P}^*$.

In the **Black-Scholes model**

$$S(T) = S(0)\, e^{(r-\frac{\sigma^2}{2})T + \sigma\sqrt{T}\cdot Z} \qquad Z \sim N(0,1)$$

$$\mathbb{P}^*\left[\cancel{S(0)}e^{(r-\frac{\sigma^2}{2})\cdot T + \sigma\sqrt{T}\cdot Z} > \cancel{S(0)}e^{rT}\right] =$$

ln is increasing

$$= \mathbb{P}^*\left[(\cancel{r}-\frac{\sigma^2}{2})\cdot T + \sigma\sqrt{T}\cdot Z > \cancel{rT}\right]$$

$$= \mathbb{P}^*\left[\sigma\sqrt{T}\cdot Z > \frac{\sigma^2 \cdot T}{2}\right] = \mathbb{P}^*\left[Z > \frac{\sigma\sqrt{T}}{2}\right]$$

$$= \mathbb{P}^*\left[Z < -\frac{\sigma\sqrt{T}}{2}\right]$$

(symmetry of $N(0,1)$)

$$= N\left(-\frac{\sigma\sqrt{T}}{2}\right) \xrightarrow[T\to\infty]{} 0$$

$\square$

Motivation.

Consider a European call option w/ strike K and exercise date T. Under the risk-neutral probability measure $\mathbb{P}^*$, what is the probability that the call is in-the-money @ expiration?