UNIVERSITY OF TEXAS AT AUSTIN

Problem Set #10

Binomial option pricing: Forward trees. Two periods.

Problem 10.1. (5 points) Assume that the a stock price is modeled using a one-period forward binomial tree with the length of a single period equal to three months. According to this model, the stock price can take either the value of \$50, or the value of \$40 in exactly three months. Calculate the volatility of the stock price.

(a) About 0.22
(b) About 0.28
(c) About 0.30
(d) About 0.32
(e) None of the above.

Solution: (a)

$$e^{2\sigma\sqrt{h}} = S_u/S_d \quad \Rightarrow \quad \sigma = \frac{1}{2\sqrt{h}}\ln(S_u/S_d) = \frac{1}{2\sqrt{1/4}}\ln(50/40) = \ln(50/40) = 0.2231$$

Problem 10.2. The current price of a continuous-dividend-paying stock is \$100 per share. Its volatility is given to be 0.30 and its dividend yield is 0.03.

The continuously compounded risk-free interest rate equals 0.06.

Consider a \$95-strike European put option on the above stock with three months to expiration. Using a one-period forward binomial tree, find the price of this put option.

(a) \$3.97
(b) \$4.52
(c) \$4.70
(d) \$4.97
(e) None of the above.

Solution: (b)
The risk-neutral probability of the stock price moving up in a single period is

$$p^* = \frac{1}{1+e^{0.3\sqrt{0.25}}} = 0.4626.$$

The up and down factors in the above model are

$$u = e^{0.03\times 0.25+0.3\sqrt{0.25}} = 1.1706,$$
$$d = e^{0.03\times 0.25-0.3\sqrt{0.25}} = 0.8672.$$

The possible stock prices at the "leaves" of the binomial tree are

$$S_u = S(0)u = 117.06$$
$$S_d = dS(0) = 86.72.$$

So, the price of the European put option equals

$$V_P(0) = e^{-0.06(1/4)}\left[(95-86.72)(1-0.4626)\right] = 4.5169.$$

Problem 10.3. The current price of a stock is \$100 per share. Its dividend yield is 0.01 and volatility is 0.3. In order to model the stock price at the end of a year, Bertie constructed a forward binomial tree. He then calculated the price of a one-year, 90-strike European put option on this stock and obtained \$8.50. What is the **positive** continuously compounded, risk-free interest rate Bertie used?

Solution: The risk-neutral probability is

$$p^* = \frac{1}{1+e^{\sigma\sqrt{h}}} = \frac{1}{1+e^{0.3}} = 0.4255575.$$

The up and down factors can be expressed as

$$u = e^{(r-\delta)h+\sigma\sqrt{h}} = e^{(r-0.01)+0.3} = e^{r+0.29},$$
$$d = e^{(r-\delta)h-\sigma\sqrt{h}} = e^{(r-0.01)-0.3} = e^{r-0.31}.$$

With r positive, we know that

$$S_u = uS(0) = e^{r+0.29}(100) > 100.$$

So, the put option is out-of-the-money at the *up* node. Hence, we can express the option price as

$$V_P(0) = e^{-r}(1-p^*)V_d = e^{-r}(1-0.4255575)(90-e^{r-0.31}) = 0.5744425(90e^{-r} - 100e^{-0.31}) = 8.50.$$

Hence,

$$90e^{-r} - 100e^{-0.31} = \frac{8.50}{0.5744425} = 14.79695 \quad \Rightarrow \quad 90e^{-r} = 14.79695 + 100e^{-0.31} = 88.14165$$

Finally,

$$r = \ln\left(\frac{90}{88.14165}\right) = 0.02086449.$$

Problem 10.4. For a two-period binomial model, you are given that:

(1) each period is one year;
(2) the current price of a non-dividend-paying stock S is $S(0) = \$20$;
(3) $u = 1.2$, with u as in the standard notation for the binomial model;
(4) $d = 0.8$, with d as in the standard notation for the binomial model;
(5) the continuously compounded risk-free interest rate is $r = 0.04$.

Consider a **special** call option which pays the excess above the strike price $K = 23$ (if any!) at the end of **every** binomial period.

Find the price of this option.

Solution: The risk-neutral probability is

$$p^* = \frac{e^{0.04}-0.8}{1.2-0.8} = 0.6020.$$

When one constructs the two-period binomial tree, one gets

$$S_u = 24,\ S_d = 16,$$
$$S_{uu} = 28.80,\ S_{ud} = S_{dd} = 19.2,\ S_{dd} = 12.8.$$

So, the payoffs at the end of the first period are

$$V_u = 1, V_d = 0.$$

The payoffs at the end of the second period are

$$V_{uu} = 5.80, \quad V_{ud} = 0, \quad V_{dd} = 0.$$

So, taking the expected value at time 0 of the payoff with respect to the risk-neutral probability, we get that the price of this call should be

$$\begin{aligned} &e^{-0.04} \times V_u \times p^* + e^{-0.04\times 2}[V_{uu} \times (p^*)^2 + V_{ud} \times 2p^*(1-p^*)] \\ &= e^{-0.04} \times 1 \times 0.6020 + e^{-0.08}[5.8 \times 0.6020^2] \\ &= 2.51893. \end{aligned}$$