M339J : February 5th, 2021.

Problem. Let X be a two-parameter Pareto random variable w/ $\alpha = 3$ and $\theta = 10$.
Find $Var[X]$.

→: $X \sim Pareto(\alpha = 3, \theta = 10)$

$$Var[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$ the computational formula for the variance

From my STAM Tables: for k a positive integer

$$\mathbb{E}[X^k] = \frac{\theta^k \cdot k!}{(\alpha-1)\cdots(\alpha-k)}$$

For $k=1$, we get the mean:

$$\mathbb{E}[X] = \frac{\theta^1 \cdot 1!}{\alpha - 1} = \frac{\theta}{\alpha-1}$$

For $k=2$, we get the 2nd raw moment:

$$\mathbb{E}[X^2] = \frac{\theta^2 \cdot 2!}{(\alpha-1)(\alpha-2)}$$

$$\Rightarrow Var[X] = \frac{2\theta^2}{(\alpha-1)(\alpha-2)} - \left(\frac{\theta}{\alpha-1}\right)^2 \longleftarrow$$

$$= \frac{2\theta^2}{(\alpha-1)(\alpha-2)} - \frac{\theta^2}{(\alpha-1)^2}$$

$$= \frac{\theta^2}{\alpha-1}\left(\frac{2}{\alpha-2} - \frac{1}{\alpha-1}\right)$$

For $\alpha = 3$:

$$Var[X] = \frac{10^2}{3-1}\left(\frac{2}{3-2} - \frac{1}{3-1}\right) =$$

$$= \frac{10^2}{2} \cdot \left(2 - \frac{1}{2}\right) = \frac{10 \cdot 10}{2} \cdot \frac{3}{2} = 75 \blacksquare$$

Def'n. The coefficient of variation is:

$$\frac{\sigma}{\mu}.$$

Excess Loss (Random) Variable.

Def'n. Let X be a random variable and let d be a (positive) constant such that

$$\boxed{\mathbb{P}[X>d]>0.}$$

The excess loss (random) variable is usually denoted by Y^P and it's defined as

$$Y^P = X-d \quad \text{given that} \quad X>d$$

Note:
- all the values of X less than d are discarded $\Rightarrow$ LEFT TRUNCATED
- d is subtracted $\Rightarrow$ SHIFTED
- We usually write $\boxed{Y^P = X-d \mid X>d}$

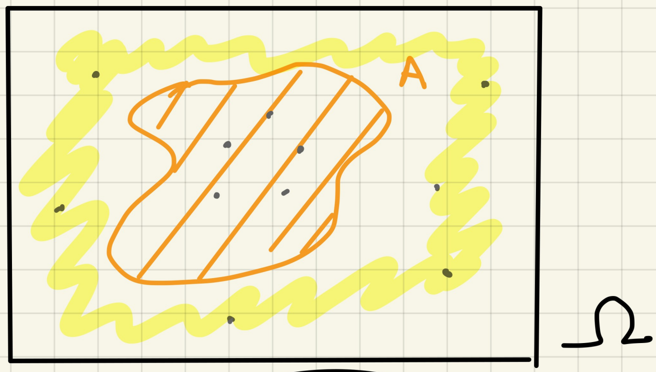
Def'n. The mean excess loss function, denoted by $e_X(d)$, is defined as the expectation of Y^P, i.e.,

$$e_X(d) = \mathbb{E}[Y^P] = \mathbb{E}[X-d \mid X>d]$$

This is what we do w/ conditional expectation :)

Consider a r.v. G and an event A such that $\mathbb{P}[A]>0$.

$$\mathbb{E}[G \mid A] = ?$$



$$\mathbb{E}[G|A] := \frac{\mathbb{E}[G\cdot \mathbb{I}_A]}{\mathbb{P}[A]}$$

In our situation:

$$e_X(d) = \mathbb{E}[\,X-d \mid \underbrace{X>d}_{}\,] = \frac{\mathbb{E}[(X-d)\cdot \mathbb{I}_{[X>d]}]}{\mathbb{P}[X>d]}$$

Note: We can use the tail formula for the expectation in the numerator:

$$e_X(d) = \frac{\int_d^\infty S_X(x)\,dx}{S_X(d)}$$

Think about:
- Dig out an insurance policy!
- What is the dist'n of Y^P for $X \sim$ Exponential(θ)?