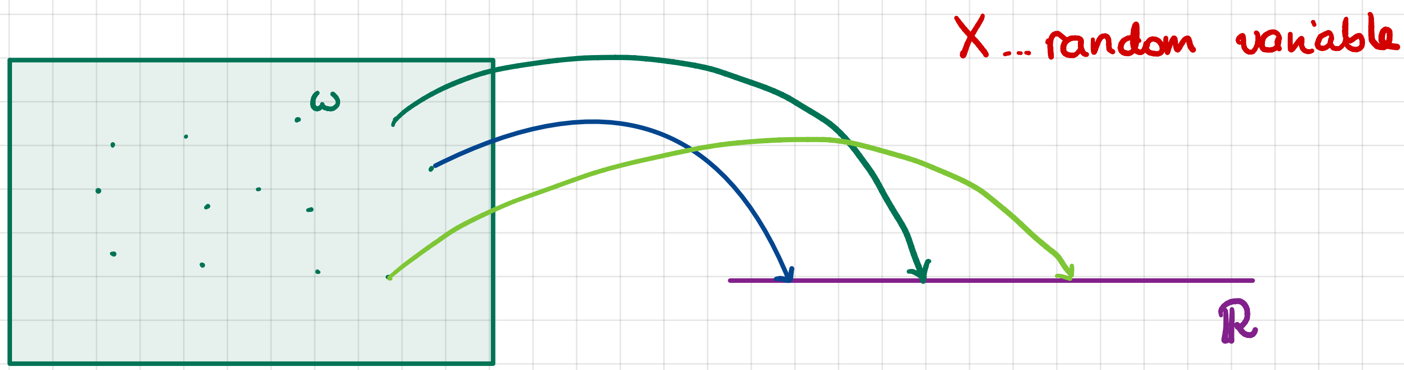
M339J: January 11th, 2023.

Probability Review.



Probability Space Ω
(aka outcome space, sample space S)

ω ... elementary outcomes

E ... event, i.e., a "nice" subset of the probability space

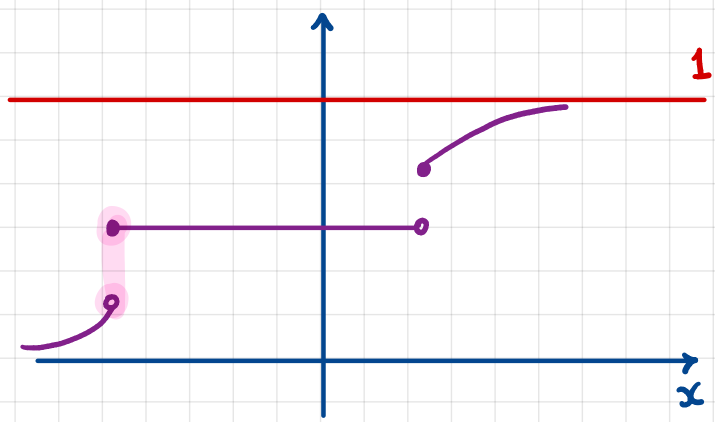
Def'n. The cumulative distribution function (cdf) of a random variable X is a function F_X such that

$$F_X : \mathbb{R} \longrightarrow [0,1]$$

given by

$$\boxed{F_X(x) = \mathbb{P}[X \leq x] \quad \text{for all } x \in \mathbb{R}}$$

Example.



Properties:

- Non-decreasing, i.e.,

$$\boxed{x_1 < x_2 \implies F_X(x_1) \leq F_X(x_2)}$$

- $\lim_{x \to -\infty} F_X(x) = \underline{0}$

- $\lim_{x \to +\infty} F_X(x) = 1$

- Right continuous w/ left limits

<u>Def'n</u>. The survival function of a random variable X is the function $S_X : \mathbb{R} \to [0,1]$ given by

$$\boxed{S_X(x) = 1 - F_X(x) = \mathbb{P}[X > x] \text{ for all } x \in \mathbb{R}}$$

<u>'Def'n"</u>. The support of a random variable X is the set of all the values it can take.

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Problem set 1

The cumulative distribution function.

Problem 1.1. The random variable X has the following cumulative distribution function:

$$F_X(x) = \begin{cases} \zeta & \text{for } x < 0 \\ \frac{1}{2} & \text{for } 0 \le x < 1 \\ \kappa x + \nu & \text{for } 1 \le x < 3 \\ \eta & \text{for } x \ge 3 \end{cases}$$

$\Rightarrow$ $\zeta = 0$, $\eta = 1$

The function F_X is continuous at 1 and 3. How much are η, κ and ν? What is the probability that X is less than or equal to 2? What is the probability that X is equal to 1? What is the probability that X is equal to 0?

$$\left.\begin{array}{l} \kappa(1) + \nu = \frac{1}{2} \\ -\ \kappa(3) \pm \nu = -1 \end{array}\right\} -$$

$$-2\kappa = -\frac{1}{2} \Rightarrow \kappa = \frac{1}{4} \Rightarrow \nu = \frac{1}{4}$$

$$\mathbb{P}[X \le 2] = F_X(2) = \kappa(2) + \nu = \frac{1}{4}(2) + \frac{1}{4} = \frac{3}{4}$$

$$\mathbb{P}[X = 1] = F_X(1) - F_X(1-) = 0$$

$$\mathbb{P}[X = 0] = F_X(0) - F_X(0-) = \frac{1}{2} - 0 = \frac{1}{2}$$

$\square$

Problem 1.2. The random variable X has the following cumulative distribution function:

$$F_X(x) = x^3 \quad \text{for } x \in (0,1)$$

and is defined in the obvious way outside of the interval $(0,1)$. What is the probability that X exceeds $1/2$, **given** that it exceeds $1/4$?

→:

E an event, $\mathbb{P}[E] > 0$

The conditional probability:

F an event

$$\mathbb{P}[F \mid E] := \frac{\mathbb{P}[E \cap F]}{\mathbb{P}[E]}$$

$$\mathbb{P}\left[X > \frac{1}{2} \,\middle|\, X > \frac{1}{4}\right] = \frac{\mathbb{P}\left[X > \frac{1}{2}, X > \frac{1}{4}\right]}{\mathbb{P}\left[X > \frac{1}{4}\right]}$$

$$= \frac{\mathbb{P}\left[X > \frac{1}{2}\right]}{\mathbb{P}\left[X > \frac{1}{4}\right]} = \frac{1 - \left(\frac{1}{2}\right)^3}{1 - \left(\frac{1}{4}\right)^3} = \frac{\frac{7}{8}}{\frac{63}{64}} = \frac{56}{63} \quad \square$$