UNIVERSITY OF TEXAS AT AUSTIN

Quiz #11

Transformations of random variables.

Please, provide your **complete solutions** to the following questions:

Problem 11.1. (2 points) For a random variable X have the exponential distribution. Then, for a constant $\tau > 0$, the random variable $X^{1/\tau}$ has the Weibull distribution. *True or false? Why?*

Solution: TRUE
The justification is in the class notes.

Problem 11.2. (7 pts) Let X have the loglogistic distribution. Then, the random variable $X' = 1/X$ also has the loglogistic distribution. *True or false? Why?*

Solution: TRUE
From the tables, the cdf of X can be written as

$$F_X(x) = \frac{(x/\theta)^\gamma}{1 + (x/\theta)^\gamma}, \qquad x > 0,$$

for parameters γ and θ.

In class, we learned that for $y > 0$,

$$\begin{aligned} F_{X'}(y) &= 1 - F_X(1/y) \\ &= 1 - \frac{(1/y\theta)^\gamma}{1 + (1/y\theta)^\gamma} \\ &= \frac{1}{1 + (1/y\theta)^\gamma} \\ &= \frac{(y/\theta^*)^\gamma}{1 + (y/\theta^*)^\gamma} \end{aligned}$$

with $\theta^* = 1/\theta$.

Problem 11.3. (6 points) Once a tunnel drill breaks down, it takes at least a month to get a replacement. The waiting time T to get a new drill after that time has the following cumulative distribution function:

$$F_T(t) = \begin{cases} 1 - t^{-2} & \text{for } t > 1 \\ 0 & \text{otherwise} \end{cases}$$

The resulting cost to the construction company is $X = T^2$. Find the probability density function of the random variable X?

Solution: The support of the random variable X is $(1, \infty)$. Let us try to find the cumulative distribution function of X. For $x > 1$, we have that

$$F_X(x) = \mathbb{P}[X \le x] = \mathbb{P}[T^2 \le x] = \mathbb{P}[T \le \sqrt{x}] = F_T(\sqrt{x}) = 1 - (\sqrt{x})^{-2} = 1 - x^{-1}.$$

So, we can get the density of the random variable X as

$$f_X(x) = F_X'(x) = x^{-2} \quad \text{for } x > 1.$$