Homework assignment #7: Solutions

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Problem #1 (5 points)

Let $Z \sim N(0,1)$. Given that Z is at least 2, what is the probability that Z is less than 3?

Solution: We need to calculate

$$\mathbb{P}[Z < 3 \,|\, Z > 2] = \frac{\mathbb{P}[2 < Z < 3]}{\mathbb{P}[Z > 2]} = \frac{\mathbb{P}[Z < 3] - \mathbb{P}[Z \leq 2]}{\mathbb{P}[Z > 2]}\,.$$

From the standard normal tables, we obtain

$$\begin{aligned}&\mathbb{P}[Z \leq 2] = \Phi(2) = 0.9772 \quad \Rightarrow \quad \mathbb{P}[Z > 2] = 1 - 0.9772 = 0.0228, \quad \text{and}\\ &\mathbb{P}[Z < 3] = \Phi(3) = 0.9987.\end{aligned}$$

So, our final answer is

$$\mathbb{P}[Z < 3 \,|\, Z > 2] = \frac{\mathbb{P}[Z < 3] - \mathbb{P}[Z \leq 2]}{\mathbb{P}[Z > 2]} = \frac{0.9987 - 0.9772}{0.0228} = 0.9429825.$$

Alternatively, in `R`, we have

```
(pnorm(3) - pnorm(2))/(1 - pnorm(2))
## [1] 0.9406642
```

Problem #2 (5 points)

Let $Z \sim N(0,1)$. Given that Z is at least 0.5, what is the probability that Z is at least 1?

Solution: By the definition of conditional probability, we are looking for

$$\mathbb{P}[Z \geq 1 \,|\, Z \geq 0.5] = \frac{\mathbb{P}[Z \geq 1, Z \geq 0.5]}{\mathbb{P}[Z \geq 0.5]} = \frac{\mathbb{P}[Z \geq 1]}{\mathbb{P}[Z \geq 0.5]}\,.$$

Using the standard normal tables, we get

$$\begin{aligned}\mathbb{P}[Z \geq 0.5] &= 1 - \mathbb{P}[Z < 0.5] = 1 - \Phi(0.5) = 1 - 0.6915 = 0.3085,\\ \mathbb{P}[Z \geq 1] &= 1 - \mathbb{P}[Z < 1] = 1 - \Phi(1) = 1 - 0.8413 = 0.1587.\end{aligned}$$

So, our answer is

$$\mathbb{P}[Z \geq 1 \,|\, Z \geq 0.5] = \frac{\mathbb{P}[Z \geq 1]}{\mathbb{P}[Z \geq 0.5]} = \frac{0.1587}{0.3085} = 0.5144.$$

Alternatively, in `R`, we have

```r
(1 - pnorm(1))/(1 - pnorm(0.5))
## [1] 0.514217
```

Problem #3 (5 points)

Let $Z \sim N(0,1)$. Given that Z is at most 2, what is the probability that Z is at least 0.5?

Solution: By the definition of conditional probability, we need to find

$$\mathbb{P}[Z \geq 0.5 \,|\, Z \leq 2] = \frac{\mathbb{P}[0.5 \leq Z \leq 2]}{\mathbb{P}[Z \leq 2]} = \frac{\mathbb{P}[Z \leq 2] - \mathbb{P}[Z < 0.5]}{\mathbb{P}[Z \leq 2]} = 1 - \frac{\mathbb{P}[Z < 0.5]}{\mathbb{P}[Z \leq 2]}.$$

From the standard normal tables, we get

$$\mathbb{P}[Z < 0.5] = \Phi(0.5) = 0.6915 \quad \text{and} \quad \mathbb{P}[Z \leq 2] = \Phi(2) = 0.9772.$$

So, our final answer is

$$\mathbb{P}[Z \geq 0.5 \,|\, Z \leq 2] = 1 - \frac{\mathbb{P}[Z < 0.5]}{\mathbb{P}[Z \leq 2]} = 1 - \frac{0.6915}{0.9772} = 0.2924.$$

Alternatively, in `R`, we have

```r
1 - pnorm(0.5)/pnorm(2)
## [1] 0.2924405
```

Problem #4 (5 points)

Let $Z \sim N(0,1)$. Given that Z is at most 3, what is the probability that Z is at most 2.5?

Solution: From the definition of conditional probability, we are looking for

$$\mathbb{P}[Z \leq 2.5 \,|\, Z \leq 3] = \frac{\mathbb{P}[Z \leq 3, Z \leq 2.5]}{\mathbb{P}[Z \leq 3]} = \frac{\mathbb{P}[Z \leq 2.5]}{\mathbb{P}[Z \leq 3]}.$$

From the standard normal tables, we obtain

$$\mathbb{P}[Z \leq 2.5] = \Phi(2.5) = 0.9938 \quad \text{and} \quad \mathbb{P}[Z \leq 3] = \Phi(3) = 0.9987.$$

So, our answer is

$$\mathbb{P}[Z \leq 2.5 \,|\, Z \leq 3] = \frac{\mathbb{P}[Z \leq 2.5]}{\mathbb{P}[Z \leq 3]} = \frac{0.9938}{0.9987} = 0.9951.$$

Alternatively, in `R`, we have

```r
pnorm(2.5)/pnorm(3)
## [1] 0.9951337
```

Problem #5 (10 points)

The distribution of lengths of adult bass in Cumberland Lake is modeled as normal with mean 32" and standard deviation 6".

At the annual Cumberland Lake bass fishing competition, you win a blue ribbon if you catch a bass that is over 38" in length. If you catch a bass over 42" in length you also win a gold medallion.

Assume that an angler (the person fishing) is only allowed to catch the first fish s/he reels in.

Which of the following is the closest to the probability that an angler has not won a gold medallion **given** that s/he has won a blue ribbon?

Solution: Let X denote the length of a randomly chosen adult bass in Cumberland Lake. We are given that

$$X \sim Normal(mean = 32, sd = 6).$$

We are looking for

$$\mathbb{P}[X < 42 \,|\, X > 38] = \frac{\mathbb{P}[38 < X < 42]}{\mathbb{P}[X > 38]} = \frac{\mathbb{P}[X < 42] - \mathbb{P}[X \leq 38]}{\mathbb{P}[X > 38]}.$$

Transitioning to standard units and using the standard normal table, we get

$$\mathbb{P}[X \leq 38] = \mathbb{P}\left[\frac{X-32}{6} \leq \frac{38-32}{6}\right] = \mathbb{P}[Z \leq 1] = \Phi(1) = 0.8413 \quad \Rightarrow \quad \mathbb{P}[X > 38] = 0.1587,$$
$$\mathbb{P}[X < 42] = \mathbb{P}\left[\frac{X-32}{6} \leq \frac{42-32}{6}\right] \approx \mathbb{P}[Z \leq 1.67] = \Phi(1.67) = 0.9525.$$

Finally, our answer is

$$\mathbb{P}[X < 42 \,|\, X > 38] = \frac{0.9525 - 0.8413}{0.1587} = 0.7007$$

Alternatively, in `R`, we have

```
(pnorm(42, 32, 6) - pnorm(38, 32, 6))/(1 - pnorm(38, 32, 6))
## [1] 0.6987786
```

Problem #6 (20 points)

Your diamond scale's measurements have a normally distributed error with mean 0 and standard deviation of 0.001 carats. Your procedure is to weigh a single diamond using your scale n times, average out the results, and report the average as the mass of the diamond. How many times n do you have to weigh your diamond so that your reported mass is at most 0.001 from the actual mass with probability 99%?

Solution: Let the error of measurement i be denoted by X_i. Then, the average error is going to be

$$\bar{X}_n = \frac{1}{n}(X_1 + X_2 + \cdots + X_n).$$

The random variable $\bar{X}_n$ is normally distributed with mean zero and standard deviation

$$SD[\bar{X}_n] = \frac{0.001}{\sqrt{n}}.$$

We are looking for n such that

$$\mathbb{P}[|\bar{X}_n| < 10^{-3}] \geq 99\%, \quad i.e., \quad \mathbb{P}[-10^{-3} < \bar{X}_n < 10^{-3}] \geq 99\%.$$

Rewriting $\bar{X}$ in standard units above, we obtain

$$\mathbb{P}\left[\frac{-10^{-3}-0}{\frac{0.001}{\sqrt{n}}} < \frac{\bar{X}_n - 0}{\frac{0.001}{\sqrt{n}}} < \frac{10^{-3}-0}{\frac{0.001}{\sqrt{n}}}\right] \geq 99\% \quad \Leftrightarrow \quad \mathbb{P}\left[\frac{-10^{-3}}{\frac{0.001}{\sqrt{n}}} < Z < \frac{10^{-3}}{\frac{0.001}{\sqrt{n}}}\right] \geq 99\%$$

where $Z \sim N(0,1)$. We can simplify the probability in the inequality above as follows

$$\mathbb{P}\left[\frac{-10^{-3}}{\frac{0.001}{\sqrt{n}}} < Z < \frac{10^{-3}}{\frac{0.001}{\sqrt{n}}}\right] = \mathbb{P}\left[-\sqrt{n} < Z < \sqrt{n}\right] = \mathbb{P}\left[Z < \sqrt{n}\right] - \mathbb{P}\left[Z \leq -\sqrt{n}\right] = \Phi\left(\sqrt{n}\right) - \Phi\left(-\sqrt{n}\right).$$

Due to the symmetry of the standard normal bell curve, we always have $\Phi(-x) = 1 - \Phi(x)$. Hence, the above probability simplifies to

$$\Phi\left(\sqrt{n}\right) - \Phi\left(-\sqrt{n}\right) = 2\Phi\left(\sqrt{n}\right) - 1.$$

So, our condition becomes

$$2\Phi\left(\sqrt{n}\right) - 1 \geq 0.99 \quad \Leftrightarrow \quad 2\Phi\left(\sqrt{n}\right) \geq 1.99 \quad \Leftrightarrow \quad \Phi\left(\sqrt{n}\right) \geq 0.995.$$

Remember that Φ is strictly increasing. Using the standard normal table, we invert Φ above and get

$$\sqrt{n} \geq 2.576 \quad \Leftrightarrow \quad n \geq (2.576)^2 = 6.6358 \quad \Leftrightarrow \quad n \geq 7.$$