M378K: February 21st, 2025.

Theorem 9.8. *If m_Y exists, then for $k \in \mathbb{N}$, we have*

$$m_Y^{(k)}(0) = \mu_k.$$

Example 9.9. *Let $Y \sim b(n = 1, p)$, i.e., let Y model a Bernoulli trial with the probability of success denoted by p. Find m_Y.*

$\rightarrow$: $m_Y(t) = \mathbb{E}[e^{tY}] = e^{t\cdot 0}(1-p) + e^{t\cdot 1}\cdot p$

$= \boxed{1-p+pe^t}, \quad t \in \mathbb{R}$

Proposition 9.10. *Let Y_1 and Y_2 be independent random variables with m.g.f.s denoted by m_{Y_1} and m_{Y_2}. Define $Y = Y_1 + Y_2$. Then, for every t for which both m_{Y_1} and m_{Y_2} are well defined, we have*

$$m_Y(t) = \ ?$$

Proof. By definition:

$$m_Y(t) = \mathbb{E}[e^{t\cdot Y}]$$

Using $Y = Y_1 + Y_2$, we can subsitute $Y_1 + Y_2$ for Y in the expression above. So,

$$m_Y(t) = \mathbb{E}[e^{t(Y_1+Y_2)}]$$

One of the properties of the exponential function is that $e^{A+B} = e^A \times e^B$. Thus, the above becomes:

$$m_Y(t) = \mathbb{E}[e^{t\cdot Y_1}\cdot e^{t\cdot Y_2}]$$

Recall that Y_1 and Y_2 are assumed to be independent random variables. With this in mind, we get:

$$m_Y(t) = \underbrace{\mathbb{E}[e^{tY_1}]}\cdot\underbrace{\mathbb{E}[e^{tY_2}]}$$

Finally, using the definition of a m.g.f., we have

$$\boxed{m_Y(t) = m_{Y_1}(t)\cdot m_{Y_2}(t)}$$

□

Example 9.11. *Let $Y \sim b(n,p)$. What is the moment generating function of Y?*

→: $m_Y(t) = ?$

$Y = I_1 + I_2 + \dots + I_n$ w/ I_j, $j=1..n$ all $B(p)$ and independent

$$m_Y(t) = m_{I_1}(t)\cdot m_{I_2}(t)\cdots m_{I_n}(t) \underset{\text{identically dist'd}}{=} (m_{I_1}(t))^n = (\underbrace{(1-p)+pe^t}_{})^n$$

$$(1+p(e^t-1))^n \quad \square$$

Example 9.12. *Let $N \sim Poisson(\lambda)$. What is the moment generating function m_N of N?*

$$\to: m_N(t) = \mathbb{E}[e^{t\cdot N}] = \sum_{n=0}^{\infty} e^{t\cdot n} p_N(n) = \sum_{n=0}^{\infty}\left(e^{t\cdot n} e^{-\lambda}\frac{\lambda^n}{n!}\right)$$

$$= e^{-\lambda}\cdot\sum_{n=0}^{\infty}\frac{e^{tn}\lambda^n}{n!} = e^{-\lambda}\cdot\sum_{n=0}^{\infty}\frac{(e^t\cdot\lambda)^n}{n!} = e^{-\lambda}e^{e^t\cdot\lambda} = e^{\lambda(e^t-1)}$$

□

Example 9.13. *Let $Z \sim N(0,1)$. What is the moment generating function m_Z of Z?*

$$\to: m_Z(t) = \mathbb{E}[e^{t\cdot Z}] = \int_{-\infty}^{\infty} e^{t\cdot z}\varphi(z)\,dz = \int_{-\infty}^{\infty} e^{t\cdot z}\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}\,dz$$

$$= \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\,e^{-\frac{z^2}{2}+t\cdot z-\frac{t^2}{2}}\,e^{\frac{t^2}{2}}\,dz$$

density of $N(t, \sigma=1)$

$$= e^{\frac{t^2}{2}}\int_{-\infty}^{\infty}\underbrace{\frac{1}{\sqrt{2\pi}}\,e^{-\frac{1}{2}(z-t)^2}\,dz}_{=1}$$

$$\boxed{m_Z(t) = e^{\frac{t^2}{2}}} \quad \square$$

Example 9.14. *Let the random variable Y have the mgf m_Y. Define $X = aY + b$ for some constants a and b. Express the mgf m_X of X in terms of m_Y, a and b.*

→: $m_X(t) = \mathbb{E}[e^{tX}] = \mathbb{E}[e^{t(aY+b)}] =$

$= \mathbb{E}[e^{taY} e^{tb}] = e^{tb}\,\mathbb{E}[e^{taY}] = e^{tb}\, m_Y(ta)$ □

Example 9.15. *Let $X \sim N(\mu, \sigma^2)$. What is the moment generating function m_X of X?*

→: $X = \underset{b}{\mu} + \underset{a}{\sigma} \cdot Z$ w/ $Z \sim N(0,1)$

$$m_X(t) = e^{\mu t} \cdot m_Z(\sigma \cdot t) = e^{\mu t} \cdot e^{\frac{\sigma^2 \cdot t^2}{2}} = \underline{e^{\mu t + \frac{\sigma^2 t^2}{2}}}$$ □

Problem 9.2. *A random variable Y is said to be* lognormal *if there exists a normally distributed random variable $X \sim N(\mu, \sigma^2)$ such that $Y \overset{(d)}{=} e^X$. Express the mean and the variance of the lognormal r.v. Y in terms of the parameters μ and σ.*

→: $\mathbb{E}[Y] = \mathbb{E}[e^X] = \mathbb{E}[e^{1 \cdot X}] = m_X(1) = e^{\mu + \frac{\sigma^2}{2}}$

$\mathbb{E}[Y^2] = \mathbb{E}[(e^X)^2] = \mathbb{E}[e^{2X}] = m_X(2) = e^{2\mu + \frac{\sigma^2 \cdot 4}{2}} = e^{2(\mu + \sigma^2)}$

$\mathrm{Var}[Y] = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2 = e^{2(\mu+\sigma^2)} - e^{2\mu+\sigma^2} = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1)$ □

Proposition 9.16. 1. *If m_Y exists for a certain probability distribution, then it is unique.*

2. *If m_{Y_1} and m_{Y_2} are equal on an interval, then $Y_1 \overset{(d)}{=} Y_2$.*

Corollary 9.17. *Let Y_1 and Y_2 be independent and normally distributed. Define $Y = Y_1 + Y_2$. Then, the distribution of Y is ...*

→: $m_Y(t) = m_{Y_1}(t) \cdot m_{Y_2}(t)$

$= e^{\mu_1 \cdot t + \frac{\sigma_1^2 t^2}{2}} \cdot e^{\mu_2 \cdot t + \frac{\sigma_2^2 \cdot t^2}{2}}$

$= e^{\overset{\mu}{(\mu_1+\mu_2)} \cdot t + \frac{\overset{\sigma^2}{(\sigma_1^2+\sigma_2^2)} \cdot t^2}{2}}$

Proof. □

$Y \sim \text{Normal}(\text{mean} = \mu = \mu_1 + \mu_2,\ \text{var} = \sigma^2 = \sigma_1^2 + \sigma_2^2)$

$\text{sd} = \sqrt{\sigma_1^2 + \sigma_2^2}$ □

Dist'n Y_1, Y_2 → mgfs → mgf of Y → Dist'n of Y