University of Texas at Austin

Problem set 2

The Exponential Distribution.

Problem 2.1. The lifetime T of a printer is modeled by an exponential distribution with parameter $\theta = 2$.
There is a warranty on the printer with the following stipulations:

- If the printer fails within the first year, a full refund of 200 is issued.
- If the printer fails within the second year, a half refund is issued.
- If the printer fails after two years or longer, no refund is issued.

What is the *probability mass function* of the refund?

Solution: Let's denote the refund by Y. The support of Y is $\{0, 100, 200\}$. The *pmf* of Y is

$$\begin{aligned} p_Y(200) &= \mathbb{P}[T \leq 1] = F_T(1) = 1 - e^{-\frac{1}{2}} \approx 0.3935, \\ p_Y(100) &= \mathbb{P}[1 < T \leq 2] = F_T(2) - F_T(1) = e^{-\frac{1}{2}} - e^{-1} \approx 0.23865, \\ p_Y(0) &= \mathbb{P}[T > 2] = S_T(2) = e^{-1} \approx 0.36788. \end{aligned}$$

Problem 2.2. The waiting time until a driver is involved in an accident is modeled as exponential with an unknown parameter. We know that 30% of the drivers will be involved in an accident in the first two months. What is the probability that the driver is involved in an accident in the first three months?

Solution: Let T denote the waiting time until the first accident. We are given that

$$T \sim Exponential(\theta)$$

We are also given that $\mathbb{P}[T \leq \frac{2}{12}] = 0.3$. So,

$$1 - e^{-\frac{\frac{1}{6}}{\theta}} = 0.3 \quad \Rightarrow \quad e^{-\frac{1}{6\theta}} = 0.7 \quad \Rightarrow \quad e^{-\frac{1}{\theta}} = (0.7)^6.$$

We are looking for

$$\mathbb{P}\left[T \leq \frac{1}{4}\right] = 1 - e^{-\frac{\frac{1}{4}}{\theta}} = 1 - e^{-\frac{1}{4\theta}} = 1 - \left(e^{-\frac{1}{\theta}}\right)^{1/4} = 1 - ((0.7)^6)^{1/4} = 1 - (0.7)^{3/2} = 0.41434.$$

Problem 2.3. Find the ratio of the 90^{th} percentile to the median of the exponential distribution with parameter θ.

Solution: Let $X \sim Exponential(\theta)$.

Consider a probability $p \in (0,1)$. We will find an expression for π_p of an exponential distribution. Since X is a continuous distribution, we have

$$F_X(\pi_p) = p \quad \Rightarrow \quad 1 - e^{-\frac{\pi_p}{\theta}} = p \quad \Rightarrow \quad e^{-\frac{\pi_p}{\theta}} = 1 - p \quad \Rightarrow \quad -\frac{\pi_p}{\theta} = \ln(1-p) \quad \Rightarrow \quad \pi_p = -\theta \ln(1-p)$$

So, in the present problem, we get

$$\frac{\pi_{0.9}}{\pi_{0.5}} = \frac{-\theta \ln(1-0.9)}{-\theta \ln(1-0.5)} = \frac{\ln(0.1)}{\ln(0.5)} = 3.3219.$$

Remark 2.1. Based on the above calculation, we can also conclude that $VaR_p(X) = -\theta \ln(1-p)$.