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Example. $Y_1, \dots, Y_n$ a random sample from

$$f^{\theta}(y) = (\theta+1)\cdot y^{\theta} \cdot \mathbb{1}_{[0,1]}(y)$$

where $\boxed{\theta > -1}$ is the unknown parameter

i. $\boxed{\text{MLE}}$

$$L(\theta; y_1, \dots, y_n) = f^{\theta}(y_1)\cdot f^{\theta}(y_2)\cdot \dots \cdot f^{\theta}(y_n)$$
$$= \prod_{i=1}^{n}\left((\theta+1)\cdot y_i^{\theta}\right)$$
$$= (\theta+1)^n \cdot \left(\prod_{i=1}^{n} y_i\right)^{\theta}$$

$$\ell(\theta; y_1, \dots, y_n) = \ln\left(L(\theta; y_1, \dots, y_n)\right)$$
$$= n\cdot \ln(\theta+1) + \theta \cdot \sum_{i=1}^{n} \ln(y_i)$$

$$\ell'(\theta; y_1, \dots, y_n) = n\cdot \frac{1}{\theta+1} + \sum_{i=1}^{n} \ln(y_i) = 0$$

$$\frac{n}{\theta+1} = -\sum_{i=1}^{n} \ln(y_i)$$

$$\frac{\theta+1}{n} = -\frac{1}{\sum_{i=1}^{n} \ln(y_i)}$$

$$\theta + 1 = -\frac{n}{\sum_{i=1}^{n} \ln(y_i)}$$

$$\boxed{\hat{\theta}_{MLE} = -\frac{n}{\sum \ln(Y_i)} - 1}$$

ii. Moment Matching.

$$\underbrace{\bar{Y}}_{\text{empirical}} = \underbrace{E[Y]}_{\text{theoretical}}$$

$$E[Y] = \int_{-\infty}^{\infty} y f^{\theta}(y)\, dy = \int_0^1 y \cdot (\theta+1) \cdot y^{\theta} dy$$

$$= (\theta+1) \int_0^1 y^{\theta+1} dy =$$

$$= (\theta+1) \cdot \frac{y^{\theta+2}}{\theta+2} \Big|_{y=0}^{1} = \frac{\theta+1}{\theta+2}$$

$$\bar{Y} = \frac{\theta+1}{\theta+2}$$

$$\bar{Y}(\theta+2) = \theta+1$$

$$\bar{Y} \cdot \theta + 2\bar{Y} = \theta+1$$

$$\theta(1-\bar{Y}) = 2\bar{Y} - 1$$

$$\boxed{\hat{\theta}_{MM} = \frac{2\bar{Y}-1}{1-\bar{Y}}}$$

$\square$

Example. $Y_1, \dots, Y_n$ from

$$f^{\theta,\alpha}(y) = \begin{cases} \frac{1}{\Gamma(\alpha)\theta^\alpha} \cdot y^{\alpha-1} \cdot e^{-\frac{y}{\theta}} & , y>0 \\ 0 & , y \le 0 \end{cases}$$

w/ α known and θ is of interest.

i. $\boxed{\text{MM}}$

$$\bar{Y} = \mathbb{E}[Y] = \alpha \cdot \theta$$

$$\hat{\theta}_{MM} = \frac{\bar{Y}}{\alpha}$$

ii. $\boxed{\text{MLE}}$

$$L(\theta; y_1, \dots, y_n) = \prod_{i=1}^{n} \left(\frac{1}{\Gamma(\alpha) \cdot \theta^\alpha} \cdot y_i^{\alpha-1} \cdot e^{-\frac{y_i}{\theta}} \right)$$

$$= \left(\frac{1}{\Gamma(\alpha)}\right)^n \cdot \theta^{-n\alpha} \cdot \left(\prod_{i=1}^{n} y_i\right)^{\alpha-1} \cdot e^{-\frac{1}{\theta}\sum_{i=1}^{n} y_i}$$

$$\propto \theta^{-n\alpha} \cdot e^{-\frac{1}{\theta}\sum y_i}$$

$$\ell(\theta; y_1, \dots, y_n) = \text{const} + (-n\alpha) \cdot \ln(\theta) - \frac{1}{\theta}\sum_{i=1}^{n} y_i$$

$$\ell'(\theta; y_1, \dots, y_n) = -n \cdot \alpha \cdot \frac{1}{\theta} + (+1) \cdot \frac{1}{\theta^2}\sum_{i=1}^{n} y_i = 0 \quad / \cdot \theta^2$$

$$-n \cdot \alpha \cdot \theta + \sum_{i=1}^{n} y_i = 0$$

$$\boxed{\hat{\theta}_{MLE} = \frac{\bar{Y}}{\alpha}}$$

$\square$

Example. The Rayleigh density function is given by

$$f^{\tau}(y) = c \cdot y \cdot e^{-\frac{y^2}{\tau}} \mathbb{1}_{[0,\infty)}(y)$$

Q: What is c?

$$\int_{-\infty}^{\infty} f^{\tau}(y)\,dy = 1$$

$$c \cdot \boxed{\int_0^{\infty} y e^{-\frac{y^2}{\tau}} dy} = 1$$

$$\left[\; u = -\frac{y^2}{\tau} \implies du = -\frac{2}{\tau} y\,dy \right.$$

$$\left. \implies y\,dy = -\frac{\tau}{2} du \;\right]$$

$$c \cdot \int_0^{-\infty} e^{u} \left(-\frac{\tau}{2}\right) du = 1$$

$$c \cdot \left(\frac{\tau}{2}\right) \int_{-\infty}^{0} e^{u} du = 1$$

$$c \cdot \left(\frac{\tau}{2}\right) e^{u} \Big|_{u=-\infty}^{0} = 1 \implies \boxed{c = \frac{2}{\tau}}$$

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$$\boxed{f^{\tau}(y) = \frac{2}{\tau} y e^{-\frac{y^2}{\tau}} \mathbb{1}_{[0,\infty)}(y)}$$

$\boxed{\text{MM}}$

$$\mathbb{E}[Y] = \int_0^{\infty} y \cdot \frac{2}{\tau} y e^{-\frac{y^2}{\tau}} dy =$$

$$= \frac{2}{\tau} \int_0^{\infty} y^2 e^{-\frac{y^2}{\tau}} dy$$

$$u = y \qquad du = dy$$

$$dv = \frac{2}{\tau} y e^{-\frac{y^2}{\tau}} dy \qquad v = -e^{-\frac{y^2}{\tau}}$$

$$\frac{d}{dy} e^{-\frac{y^2}{\tau}} = -\frac{2y}{\tau} e^{-\frac{y^2}{2}}$$

$$= y \cdot \left(-e^{-\frac{y^2}{\tau}}\right)\Big|_{y=0}^{\infty} + \int_0^\infty + e^{-\frac{y^2}{\tau}} dy$$

$$\int_0^\infty \frac{2}{\tau} y^2 e^{-\frac{y^2}{\tau}} dy = \left[u = y^2 \Rightarrow du = 2y\,dy \Rightarrow dy = \frac{du}{2y} \right]$$

$$\int_0^\infty \frac{1}{\tau} \sqrt{u}\, e^{-\frac{u}{\tau}} du \qquad ???$$

BUT!!! USING PROBABILITY !!!

$$\int_0^\infty y^2 e^{-\frac{y^2}{\tau}} dy = \left[u = \frac{\sqrt{2} \cdot y}{\sqrt{\tau}} \Rightarrow du = \sqrt{\frac{2}{\tau}} dy \right]$$

$$= \int_0^\infty \frac{\tau}{2} u^2 e^{-\frac{u^2}{2}} \sqrt{\frac{\tau}{2}} du$$

$$= \frac{\tau}{2} \cdot \sqrt{\frac{\tau}{2}} \int_0^\infty u^2 e^{-\frac{u^2}{2}} du$$

$$= \frac{\tau}{2} \sqrt{\frac{\tau}{2}} \cdot \sqrt{2\pi} \int_0^\infty u^2 \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} \right) du$$

$$= \frac{\tau}{2}\sqrt{\tau\pi} \cdot \frac{1}{2} \underbrace{\int_{-\infty}^{\infty} u^2 \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} \right) du}_{= \operatorname{Var}[Z] = 1}$$

w/ $Z \sim N(0,1)$

$$\Rightarrow \mathbb{E}[Y] = \frac{\cancel{2}}{\cancel{\tau}} \cdot \frac{\cancel{\tau}}{\cancel{2}} \cdot \sqrt{\tau\pi} \cdot \frac{1}{2}$$

$$\Rightarrow \boxed{\mathbb{E}[Y] = \frac{\sqrt{\tau\pi}}{2}}$$