The Normal Distribution.

$Y \sim N(\mu, \sigma^2)$ where $\mu \in \mathbb{R}$ and $\sigma > 0$
is said to be normally distributed
w/ mean μ
and standard deviation σ

if

$$f_Y(y) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(y-\mu)^2}{2\sigma^2}} \quad \text{for all } y \in \mathbb{R}$$

If $\mu = 0$ and $\sigma = 1$, we say that Y is standard normal.

Its pdf is

$$\boxed{\varphi(y) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{y^2}{2}} \quad \text{for } y \in \mathbb{R}}$$

Q: Let $Y \sim N(\mu, \sigma^2)$.

$$\frac{Y-\mu}{\sigma} \sim N(0,1)$$

Q: Let $Z \sim N(0,1)$
Let α and β be two real constants.

$$\alpha \cdot Z + \beta \sim \underline{\text{Normal}}(\beta, \alpha^2)$$

Expectation [revisited].

In the discrete case:

$$\boxed{\mathbb{E}[Y] := \sum_{y \in S_Y} y \cdot p_Y(y)}$$ if it exists

Def'n. Let Y be a continuous r.v. w/ pdf f_Y.

We define the expectation of Y as

$$\boxed{\mathbb{E}[Y] := \int_{-\infty}^{\infty} y\, f_Y(y)\, dy}$$ if the integral exists

Task: Cauchy Dist'n.

Problem 5.3. *Consider a continuous random variable Y whose probability density function is given by*

$$f_Y(y) = 2y\mathbf{1}_{[0,1]}(y)$$

What is the expected value of this random variable?

$\rightarrow$: $E[Y] = \int_0^1 y \cdot f_Y(y)\, dy$

$= \int_0^1 y \cdot 2y \, dy$

$= \int_0^1 2y^2 \, dy$

$= \left. \frac{2y^3}{3} \right|_0^1$

$= \frac{2(1)^3}{3} - \cancel{\frac{2(0)^3}{3}}$

$\boxed{= \frac{2}{3}}$

Example. $Y \sim E(\tau)$, i.e.,

$$f_Y(y) = \frac{1}{\tau} e^{-\frac{y}{\tau}} \mathbb{1}_{[0,\infty)}(y)$$

$\longrightarrow$: $\mathbb{E}[Y] = \int_{-\infty}^{\infty} y f_Y(y)\, dy$

$$= \int_{-\infty}^{\infty} y \frac{1}{\tau} e^{-\frac{y}{\tau}} \mathbb{1}_{[0,\infty)}(y) dy$$

$$= \int_{0}^{\infty} \frac{1}{\tau} y e^{-\frac{y}{\tau}} dy =$$

$u = -\frac{y}{\tau}$ $\qquad$ $du = -\frac{1}{\tau} dy$

$\qquad\qquad\qquad$ $dy = -\tau du$

$$= +\int_{0}^{-\infty} u e^{u} (+\tau) du$$

$=$ $u = u$ $\qquad$ $du = du$

$e^u du = dv$ $\qquad$ $v = e^u$

$$= \tau \left(\underbrace{u e^{u} \Big|_{0}^{-\infty}}_{=0} - \int_{0}^{-\infty} e^u du \right)$$

$$= \tau \underbrace{\int_{-\infty}^{0} e^u du}_{1} = \tau \qquad \square$$

$Y \sim U(l, r)$ $\qquad$ $\mathbb{E}[Y] = \frac{l+r}{2}$

$\mathrm{Var}[Y] = ?$ $<$ $\mathbb{E}[Y^2] - (\mathbb{E}[Y])^2 \checkmark$