$$F_Y(y) = 1 - e^{-y/\tau}$$

Then, the sampling distribution *of* Y *can be figured out by looking at its cumulative distribution function. We have ...*

$$g_{(1)}(y) = n \cdot f_Y(y)\left(1 - F_Y(y)\right)^{n-1} = n \cdot \frac{1}{\tau} e^{-y/\tau}\left(e^{-y/\tau}\right)^{n-1}$$

$$= \left(\frac{n}{\tau}\right) e^{-\frac{ny}{\tau}} \qquad Y_{(1)} \sim E\left(\tau/n\right)$$

10/21/24.

Problem 12.2. *Let* $Y_1, \ldots, Y_n$ *be a random sample of size* n *from a normal distribution with mean* μ *and variance* σ^2*. What is the sampling distribution of*

$$\bar{Y}_n = \frac{1}{n}\sum_{k=1}^{n} Y_k \quad ?$$

SAMPLE MEAN

$\longrightarrow$: $\bar{Y}_n \sim$ Normal (mean $= \mu$, sd $= \frac{\sigma}{\sqrt{n}}$)

Estimators.

Def'n. The bias of an estimator $\hat{\theta}$ of the parameter θ is defined as:

$$\text{bias}(\hat{\theta}) := \mathbb{E}\left(\hat{\theta} - \theta\right)$$

Notation from books: "$\mathbb{E}_\theta$; $\mathbb{E}^\theta$; $\mathbb{E}[\cdots \mid \theta]$"

We say that an estimator $\hat{\theta}$ is unbiased for the parameter θ if

$$\boxed{\mathbb{E}[\hat{\theta}] = \theta} \iff \boxed{\text{bias}(\hat{\theta}) = 0}$$

Means: FOR ALL POSSIBLE VALUES θ

Example. Consider a random sample $Y_1, Y_2, \ldots, Y_n$ from $N(\mu, \sigma)$ w/ both $\mu \in \mathbb{R}$ and $\sigma > 0$ unknown

$$\hat{\mu} = \bar{Y} = \frac{Y_1 + \cdots + Y_n}{n}$$

sample mean

Then, $\boxed{\mathbb{E}[\hat{\mu}] = \mu}$, i.e., $\hat{\mu} = \bar{Y}$ is unbiased for μ

Example. Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from $N(\mu_0, \sigma)$ w/ μ_0 known and $\sigma > 0$ unknown

We propose this estimator for the variance σ^2:

$$S^2 = \frac{1}{n} \sum_{i=1}^{n} (Y_i - \mu_0)^2$$

Then, $$\mathbb{E}[S^2] = \frac{1}{n} \sum_{i=1}^{n} \underbrace{\boxed{\mathbb{E}[(Y_i - \mu_0)^2]}}_{\sigma^2} = \frac{1}{\cancel{n}} \cdot \cancel{n} \cdot \sigma^2 = \sigma^2$$

$\Rightarrow$ S^2 is unbiased in this case.

Example. Let $Y_1, \dots, Y_n$ be a random sample from $N(\mu, \sigma)$ where both μ and σ unknown.

Goal: Find a "good" estimator for σ^2!

Propose:

$$S'^2 = \frac{1}{n} \sum_{i=1}^{n} (Y_i - \bar{Y})^2$$

Q: Is S'^2 unbiased for σ^2?

$$\mathbb{E}[S'^2] = \frac{1}{n} \sum_{i=1}^{n} \boxed{\mathbb{E}[(Y_i - \bar{Y})^2]}$$

$$\mathbb{E}[(Y_i - \bar{Y})^2] = \mathbb{E}[Y_i^2 - 2 \cdot Y_i \cdot \bar{Y} + \bar{Y}^2]$$

$$\mathbb{E}[S'^2] = \frac{1}{n}\left(\sum_{i=1}^{n} \mathbb{E}[Y_i^2] - 2 \cdot \sum_{i=1}^{n} \mathbb{E}[Y_i \cdot \bar{Y}] + n \cdot \mathbb{E}[\bar{Y}^2]\right)$$

$$= \frac{1}{n} \cdot n \cdot \mathbb{E}[Y_1^2] - 2\, \mathbb{E}\Big[\underbrace{\frac{1}{n}\sum_{i=1}^{n} Y_i}_{\bar{Y}} \cdot \bar{Y}\Big] + \mathbb{E}[\bar{Y}^2]$$

$$= \underbrace{\mathbb{E}[Y_1^2]}_{\overset{?}{=}\ Var[Y_1] + (\mathbb{E}[Y_1])^2 \ = \ \sigma^2 + \mu^2} - \underbrace{\mathbb{E}[\bar{Y}^2]}_{\overset{?}{=}\ Var[\bar{Y}] + (\mathbb{E}[\bar{Y}])^2 \ = \ \frac{\sigma^2}{n} + \mu^2}$$

$$\mathbb{E}[S'^2] = \sigma^2 + \cancel{\mu^2} - \left(\frac{\sigma^2}{n} + \cancel{\mu^2}\right) = \left(1 - \frac{1}{n}\right)\sigma^2$$

$$\Rightarrow \boxed{bias(S'^2) = -\frac{\sigma^2}{n}}$$

So, the unbiased estimator for σ^2 is:

$$\boxed{S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (Y_i - \bar{Y})^2}$$