The Black-Scholes model: On pricing.

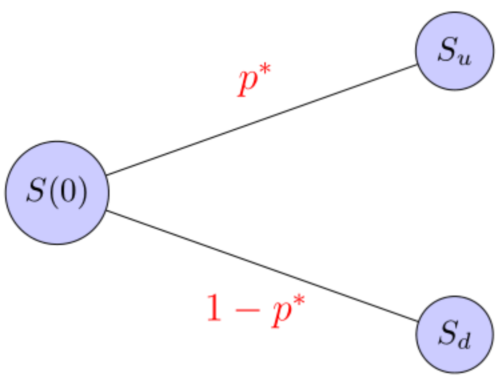
In many situations, our focus is on the **quality of an investment**. Say that we are considering investing in a call option on a particular stock. We would be interested in having a gauge on the probability that the option is in-the-money on the exercise date, or on the expected payoff, or a similar quantity. In this case, we would be using the **physical/subjective probability measure**. Plainly said, in the Black-Scholes model, this is the probability measure under which the mean rate of return of your stock is equal to α. Formally, you can write your model for the stock price at time$-T$ as

$$S(T) = S(0)e^{\left(\alpha-\delta-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}Z}$$

where Z is standard normal.

Whenever we are interested in **pricing**, though, we have to consider our stock price under the **risk-neutral probability measure**. The question is: how do we adapt the above model so that we are looking at the stock price under the **risk-neutral probability measure**. Let's look to the binomial tree for inspiration.

For simplicity, consider a one-period binomial tree. Any such tree will be of the following form:



Let the length of the period h be equal to our time-horizon T. Now, let's consider an investment in one share of a continuous-dividend-paying stock at time-0 and see what the expected wealth at time$-T$ turns out to me under the **risk-neutral probability measure**. With one share purchased at time-0, through continuous and immediate reinvestment of the dividends in the same asset, we end up owning $e^{\delta T}$ shares of stock at time$-T$. Therefore, our wealth can be expressed as $e^{\delta T}S(T)$. Under the **risk-neutral probability measure**, our expected wealth is

$$\mathbb{E}^*[e^{\delta T}S(T)] = e^{\delta T}\mathbb{E}[S(T)] = e^{\delta T}[p^*S_u + (1-p^*)S_d].$$

Recall that the risk-neutral probability in the binomial tree equals, by definition,

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d}.$$

So, the expected wealth equals, recalling that in one period $h = T$,

$$
\begin{aligned}
e^{\delta h} & \left[\left(\frac{e^{(r-\delta)h} - d}{u-d}\right) S_u + \left(\frac{u - e^{(r-\delta)h}}{u-d}\right) S_d\right] \\
&= \frac{1}{u-d}\left[e^{rh} uS(0) - e^{\delta h} duS(0) + e^{\delta h} udS(0) - e^{rh} dS(0)\right] \\
&= \frac{1}{u-d} e^{rh}(u-d)S(0) = e^{rh}S(0).
\end{aligned}
$$

In summation, we get that

$$\mathbb{E}^*[e^{\delta T}S(T)] = S(0)e^{rT}.$$

In words, the continuously compounded, risk-free interest rate r is the mean rate of return under the **risk-neutral probability measure**. Moreover, we have

$$\mathbb{E}^*[S(T)] = S(0)e^{(r-\delta)T} = F_{0,T}(S).$$

Remember that the Black-Scholes stock-price model can be understood as a continuous limit of binomial-tree models. So, in the Black-Scholes framework, under the **risk-neutral probability measure**, we model realized returns as follows:

$$R(t, t+h) \sim Normal\left(mean = \left(r - \delta - \frac{\sigma^2}{2}\right)h, var = \sigma^2 h\right).$$

In particular,

$$R(0, T) \sim Normal\left(mean = \left(r - \delta - \frac{\sigma^2}{2}\right)T, var = \sigma^2 T\right).$$

Therefore,

$$S(T) = S(0)e^{\left(r-\delta-\frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}Z}$$

with $Z \sim N(0,1)$. We use this model to **price options** in the Black-Scholes framework.

For instance, the prepaid forward price would be

$$e^{-rT}\mathbb{E}^*[S(T)] = e^{-rT}S(0)e^{(r-\delta)T} = S(0)e^{-\delta T}.$$

Note that the expression we got agrees with what we obtained when we were studying (prepaid) forward prices in a model-free way just assuming no-arbitrage.