# M339J: April 17th, 2023.

## Mortality Laws [cont'd].

**Example.** $X \sim U(0,\theta)$

$h_X(x) = ?$

$x \in (0,\theta)$: $\quad h_X(x) = \boxed{\frac{f_X(x)}{S_X(x)}} = \frac{\frac{1}{\theta}}{1-\frac{x}{\theta}} = \boxed{\frac{1}{\theta - x}}$ ✓

**Example.** $X \sim \text{Exponential}(\theta)$

$\mu_x = ?$

$x > 0$: $\quad \mu_x = \boxed{\frac{f_X(x)}{S_X(x)}} = \frac{\frac{1}{\theta}\cancel{e^{-\frac{x}{\theta}}}}{\cancel{e^{-\frac{x}{\theta}}}} = \boxed{\frac{1}{\theta}} \qquad \boxed{\lambda = \frac{1}{\theta}}$

**Example.** Gompertz Distribution (1825)

$$\boxed{\underset{x>0}{\mu_x = B \cdot c^x} \qquad B>0, \; c>1 \quad \text{constant parameters}}$$

$$\Rightarrow \quad S_X(b) = \exp\left(-\int_0^b \mu_x \, dx\right)$$

$$= \exp\left(-\int_0^b B \cdot c^x \, dx\right) = \exp\left(-B\int_0^b c^x \, dx\right)$$

$$= \exp\left(-B \cdot \frac{1}{\ln(c)} \, c^x \Big|_{x=0}^{b}\right)$$

$$\boxed{S_X(b) = \exp\left(\frac{B}{\ln(c)}\left(1-c^b\right)\right)} \quad ✓$$

$$\boxed{f_X(x) = \mu_x \cdot S_X(x)} \quad ✓$$

## Notation.

$$\boxed{{}_kp_n} = \frac{S_x(n+k)}{S_x(n)} = \frac{\exp\left(\frac{B}{\ln(c)}(1-c^{n+k})\right)}{\exp\left(\frac{B}{\ln(c)}(1-c^{n})\right)}$$

$${}_kp_n = \exp\left(\frac{B}{\ln(c)}\left(\cancel{1}-c^{n+k}-(\cancel{1}-c^{n})\right)\right)$$

$$\boxed{{}_kp_n = \exp\left(\frac{B}{\ln(c)}c^n(1-c^k)\right)}$$

**18.6** You are doing a mortality study of insureds between ages 70 and 90. Two specific lives contributed this data to the study:

| Life | Age at Entry | Age at Exit | Cause of exit |
|---|---|---|---|
| 1 | 70.0 | $\boxed{90.0}$ | $\boxed{\text{End of study}}$ |
| 2 | 70.0 | Between 89.0 and 90.0 | Death |

You assume mortality follows $\boxed{\text{Gompertz law } \mu_x = B \times c^x}$ and plan to use maximum likelihood estimation.

$L$ is the likelihood function associated with these two lives.

$L^*$ denotes the value of $L$ if the Gompertz parameters are $B = 0.000003$ and $c = 1.1$.

Calculate $L^*$.

→ : Life 1.

Lived @ least 20 years after the age of 70:

$$_{20}p_{70} = \exp\left(\frac{B}{\ln(c)} \cdot c^{70}(1-c^{20})\right)$$

Life 2.

Lived between 19 and 20 years more after the age of 70:

$$_{19}p_{70} - {}_{20}p_{70} = \exp\left(\frac{B}{\ln(c)} \cdot c^{70}(1-c^{19})\right) - \exp\left(\frac{B}{\ln(c)} \cdot c^{70}(1-c^{20})\right)$$

(A) 0.0115

(B) 0.0131

(C) 0.0147

(D) 0.0163

(E) 0.0179

answer: Life 1 × Life 2 w/ B and c as given.

0.8672956 Life 2

answer = 0.0115 2314

**18.7** You are doing a mortality study of insureds between ages 60 and 90. Two specific lives contributed this data to the study:

| Life | Age at Entry | Age at Exit | Cause of exit |
|---|---|---|---|
| 1 | 60.0 | 74.5 | Policy lapsed |
| 2 | 60.0 | 74.5 | Death |

You assume mortality follows Gompertz law $\boxed{\mu_x = B \times c^x}$ and plan to use maximum likelihood estimation.

$\ell$ is the log-likelihood function (using natural logs) associated with these two lives.

$\ell^*$ denotes the value of $L$ if the Gompertz parameters are $B = 0.000004$ and $c = 1.12$.

Calculate $\ell^*$.

Life 1 Life 2.

$$L^*(B,c) = {}_{14.5}p_{60} \cdot {}_{14.5}p_{60} \cdot \mu_{74.5}$$

$$L^*(B,C) = ({}_{14.5}p_{60})^2 \cdot \mu_{74.5}$$

$$\ell^*(B,C) = 2 \cdot \ln({}_{14.5}p_{60}) + \ln(\mu_{74.5})$$

$$= 2 \cdot \frac{B}{\ln(c)} c^{60}(1-c^{14.5}) + \ln(B) + 74.5 \ln(c)$$

$$= \cdots = -4.250579$$

(A) -4,67

(B) -4.53

(C) -4.39

(D) -4.25

(E) -4.11

**18.8** You are given the following seriatim data on survival times for a group of 12 lives. The superscript $+$ indicates a right-censored value.

$25,\ 32^+,\ 35^+,\ 36,\ 40^+,\ 44,\ 48,\ 60,\ 62^+,\ 65,\ 67,\ 70^+$

Calculate the standard deviation of the estimate of $S(50)$ using the Nelson-Aalen estimator.

(A) 0.1455

(B) 0.1519

(C) 0.1547

(D) 0.1621

(E) 0.1650

[Question on October 2022 FAM-L Exam]