H378K: April 4th, 2025.

M378K Introduction to Mathematical Statistics Problem Set #17 Relative efficiency.

Definition 17.1. Given two unbiased estimators $\hat{\theta}_1$ and $\hat{\theta}_2$, the efficiency of $\hat{\theta}_1$ relative to $\hat{\theta}_2$ is defined as

$$extit{eff}(\hat{ heta}_1,\hat{ heta}_2) = rac{ ext{Var}[\hat{ heta}_2]}{ ext{Var}[\hat{ heta}_1]}\,.$$

Problem 17.1. Let Y_1, Y_2 be a random sample from the exponential distribution with the unknown parameter θ .

- (i) The estimator $\hat{\theta}_1 = (Y_1 + Y_2)/2$ for θ is proposed. What is its variance?
- (ii) The estimator $\hat{\theta}_2 = cY_{(1)}$ for θ is proposed. Find the constant c such that $\hat{\theta}_2$ is an unbiased estimator of θ . What is its variance?

iii) Calculate the efficiency of
$$\hat{\theta}_1$$
 relative to $\hat{\theta}_2$.

Var $\begin{bmatrix} \hat{\theta}_A \end{bmatrix} = Var \begin{bmatrix} \frac{1}{2}(Y_A + Y_2) \end{bmatrix} = \frac{Var \begin{bmatrix} Y_A \end{bmatrix}}{2} = \frac{\Theta^2}{2}$

E $\begin{bmatrix} \hat{\theta}_2 \end{bmatrix} = \hat{\Theta} \implies \mathbb{C} = \mathbb{Z}$

Var $\begin{bmatrix} \hat{\theta}_2 \end{bmatrix} = Var \begin{bmatrix} 2 \cdot Y_{(1)} \end{bmatrix} = 4 \cdot Var \begin{bmatrix} Y_{(1)} \end{bmatrix} = 4 \cdot \frac{\Theta^2}{4} = \frac{\Theta^2}{4}$

iii.

eff $(\hat{\theta}_A, \hat{\theta}_2) = \frac{\Theta^2}{2} = 2$

M378K Introduction to Mathematical Statistics Problem Set #18 Consistency.

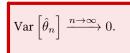
Definition 18.1. $\hat{\theta}_n$ is said to be a consistent estimator of θ if

 $\hat{ heta}_n o heta$ in probability as $n o \infty$,

i.e., if for any $\varepsilon > 0$,



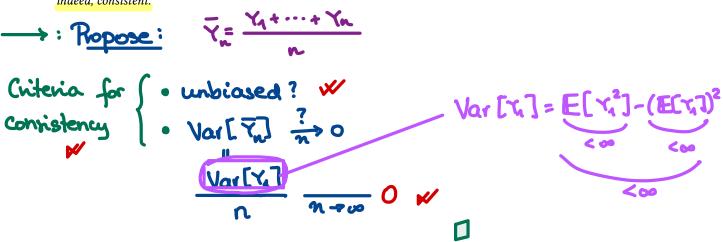
Theorem 18.2. Let $\hat{\theta}_n$ be unbiased and such that



Markov Inequality Chebyshev Inequality

Then, $\hat{\theta}_n$ is a consistent estimator.

Problem 18.1. Let Y_1, Y_2, \ldots, Y_n be a random sample from any distribution with finit first and second moments. Propose a consistent estimator for the population mean μ and **prove** that it is, indeed, consistent.



Google Cauchy Dist'n

Problem 18.2. Consider a random sample Y_1, Y_2, \ldots, Y_n from a power distribution with the density of the form

$$f_Y(y) = \theta y^{\theta - 1} \mathbf{1}_{(0,1)}(y).$$

What is a consistent estimator for $\frac{\theta}{\theta+1}$ **Prove** that your choice is indeed consistent.

$$E[Y_{1}] = \int_{0}^{1} y^{\theta-1}y \, dy = 0 \int_{0}^{1} y^{\theta} dy = 0 \cdot \frac{y^{\theta+1}}{\theta+1} \Big|_{y=0}^{1}$$

$$E[Y_{1}] = \underbrace{\theta}_{\theta+1}$$
We propose Y_{n} : • unbiased \checkmark

$$Y_{n} = \frac{1}{n} (Y_{1} + \dots + Y_{n})$$

$$Var[Y_{n}] = \underbrace{E[Y_{1}]}_{0}^{2} \cdot \underbrace{E[Y_{1}]^{2}}_{0}^{2} \cdot \underbrace{E[Y_{1}]^{2}}_{0}^{2}$$

$$Var[Y_{1}] = \underbrace{E[Y_{1}]}_{0}^{2} \cdot \underbrace{E[Y_{1}]^{2}}_{0}^{2} \cdot \underbrace{e^{2}}_{0}^{2}$$

$$0 \cdot \underbrace{e^{2}}_{0}^{2} \cdot \underbrace{e^{2}}_$$

Example. $Y_1, ..., Y_n$ random sample from $E(T=\theta)$. $\hat{\theta}_n = \frac{n}{Y_{(1)}} \quad \text{unbiased for all } n$ Consider $Var[\hat{\theta}_n] = Var[n \cdot Y_{(1)}] = n^2 \cdot \left(\frac{\theta}{n}\right)^2 = \theta^2 \stackrel{>}{\to} 0$ $v = n^2 \cdot Var[Y_{(1)}] = n^2 \cdot \left(\frac{\theta}{n}\right)^2 = \theta^2 \stackrel{>}{\to} 0$

The criterion is not satisfied.

We would still have to explore whether

On is consistent.

Task: find an unbiased estimator for θ in $Y_1, ..., Y_n \in U(0, \theta)$ Is it consistent?