M339D: April 10th, 2023.

LogNormal Tail Probabilities.

<u>Example</u>. Consider a non-dividend-paying stock. What is the probability that the stock <u>outperforms</u> a risk-free investment under the risk-neutral probability measure?

→: The initially invested amount: $S(0)$

- If it's the risk-free investment, the balance @ time T is $S(0)e^{rT}$
- If it's the stock investment, the wealth @ time T is $S(T)$

$$\boxed{\mathbb{P}^*[S(T) > S(0)e^{rT}] = ?}$$

This question is equivalent to the question of whether the profit is positive under $\mathbb{P}^*$.

$$\mathbb{P}^*[S(T) - S(0)e^{rT} > 0] = ?$$

In the Black-Scholes model:

$$\boxed{S(T) = S(0)e^{\left(r-\frac{\sigma^2}{2}\right)\cdot T + \sigma\sqrt{T}\, Z} \quad \text{with} \quad Z \sim N(0,1)}$$

$$\mathbb{P}^*\left[\cancel{S(0)}e^{\left(r-\frac{\sigma^2}{2}\right)\cdot T + \sigma\sqrt{T}\cdot Z} > \cancel{S(0)}e^{rT}\right] = \quad (\ln(\cdot) \text{ is increasing})$$

$$= \mathbb{P}^*\left[\cancel{rT} - \frac{\sigma^2}{2}\cdot T + \sigma\sqrt{T}\cdot Z > \cancel{rT}\right] =$$

$$= \mathbb{P}^*\left[\sigma\sqrt{T}\cdot Z > \frac{\sigma^2}{2}\cdot T\right] = \mathbb{P}^*\left[Z > \frac{\sigma\sqrt{T}}{2}\right] \quad (\text{symmetry of } N(0,1))$$

$$= N\left(-\frac{\sigma\sqrt{T}}{2}\right) \xrightarrow[T\to\infty]{} 0 \quad \square$$

Motivation.

Consider a European call option w/ strike K and exercise date T. Under the risk-neutral probability $\mathbb{P}^*$, what the probability that the option is in-the-money on the exercise date?

→: In the Black-Scholes model:

$$\boxed{S(T) = S(0)e^{(r-\frac{\sigma^2}{2})\cdot T + \sigma\sqrt{T}\cdot Z}} \quad \text{w/ } Z \sim N(0,1)$$

We are calculating:

$$\mathbb{P}^*[S(T) > K] =$$

$$= \mathbb{P}^*\left[S(0)e^{(r-\frac{\sigma^2}{2})\cdot T + \sigma\sqrt{T}\cdot Z} > K\right]$$

$$= \mathbb{P}^*\left[e^{(r-\frac{\sigma^2}{2})\cdot T + \sigma\sqrt{T}\cdot Z} > \frac{K}{S(0)}\right] \quad (\ln(\cdot) \text{ is increasing})$$

$$= \mathbb{P}^*\left[(r-\tfrac{\sigma^2}{2})\cdot T + \sigma\sqrt{T}\cdot Z > \ln\left(\frac{K}{S(0)}\right)\right]$$

$$= \mathbb{P}^*\left[\sigma\sqrt{T}\cdot Z > \ln\left(\frac{K}{S(0)}\right) - (r-\tfrac{\sigma^2}{2})\cdot T\right]$$

$$= \mathbb{P}^*\left[Z > \boxed{\frac{1}{\sigma\sqrt{T}}\left(\ln\left(\frac{K}{S(0)}\right) - (r-\tfrac{\sigma^2}{2})\cdot T\right)}\right] \quad (\text{symmetry of } N(0,1))$$

$$= \mathbb{P}^*\left[Z < \underbrace{\boxed{\frac{1}{\sigma\sqrt{T}}\left(\ln\left(\frac{S(0)}{K}\right) + (r-\tfrac{\sigma^2}{2})\cdot T\right)}}_{=: d_2}\right]$$

$$\boxed{\mathbb{P}^*[S(T) > K] = N(d_2)}$$

Consequently: The probability that the otherwise identical put is in-the-money is

$$\mathbb{P}^*[S(T) < K] = 1 - N(d_2) = N(-d_2)$$

Problem. Assume the Black-Scholes model.
Let the current stock price be \$100.
You are given:

(i) $\mathbb{P}^*[S(1/4) < 95] = 0.2358$

(ii) $\mathbb{P}^*[S(1/2) < \underline{110}] = 0.6026$.

What's the expected time-1 stock price under $\mathbb{P}^*$?

→ :

$$\mathbb{E}^*[S(T)] = S(0)e^{rT}$$

In this problem : $\mathbb{E}^*[S(1)] = 100e^{r}$

In the B-S model :

$$S(T) = S(0)e^{\overbrace{\left(r-\frac{\sigma^2}{2}\right)}^{=:\mu}\cdot T + \sigma\sqrt{T}\, Z}$$

$$\boxed{\mathbb{E}^*[S(1)] = 100\, e^{\mu + \frac{\sigma^2}{2}}}$$

✓

(i) : 95 is $\boxed{23.58^{th}}$ quantile of $S(1/4)$

The 23.58^{th} quantile of $N(0,1)$: standard normal tables: -0.72

or qnorm(0.2358) = −0.72

$$95 = 100e^{\mu\cdot(1/4) + \sigma\sqrt{1/4}\cdot(-0.72)} \quad /:100$$

$$0.95 = e^{\mu\cdot(1/4) + \sigma(1/2)(-0.72)}$$

$$\ln(0.95) = \frac{1}{4}\cdot\mu - 0.36\cdot\sigma$$

$$\underline{0.25\mu - 0.36\sigma = \ln(0.95)} \qquad \text{(i)}$$

(ii): 110 is the 60.26^{th} quantile of $S(1/2)$

The 60.26^{th} quantile of $N(0,1)$: std normal tables : 0.26

or qnorm(0.6026) = 0.26

$$110 = 100e^{\mu(1/2) + \sigma\sqrt{1/2}\cdot(0.26)}$$

$$1.1 = e^{\mu(1/2) + \sigma\sqrt{1/2}\,(0.26)}$$

$$\underline{0.5\mu + 0.26\sqrt{\tfrac{1}{2}}\cdot\sigma = \ln(1.1)} \qquad \text{(ii)}$$

We solve the system of two equations w/ two unknowns:

$$\mu = \underline{\quad ? \quad} \quad \text{and} \quad \sigma = \underline{\quad ? \quad}$$

$$2\cdot(\text{i}): \quad -\cancel{0.5\mu} \underset{+}{-} 0.72\sigma = -2\ln(0.95)$$

$$(\text{ii}): \quad \cancel{0.5\mu} + 0.26\sqrt{\tfrac{1}{2}}\cdot\sigma = \ln(1.1)$$

$$\left(0.26\sqrt{\tfrac{1}{2}} + 0.72\right)\cdot\sigma = \ln(1.1) - 2\ln(0.95)$$

$$\sigma = \underline{0.2189492}$$

$$4\cdot(\text{i}): \quad \mu = 1.44(0.2189492) + 4\ln(0.95) = \underline{0.11011}$$

Finally:

$$100e^{\mu + \frac{\sigma^2}{2}} = 100e^{0.11011 + \frac{(0.2189)^2}{2}} = \underline{114.3488}. \quad \square$$