M339W: February 3rd, 2021.

Subjective Probabilities.

Our agents form conclusions / models about what the relative likelihoods are of the price of a particular asset at a later date. Formally, they create a model for the distribution of the time-T asset price $S(T)$.

At least, the operate under certain assumptions on $\mathbb{E}[S(T)]$. For now, we focus on this simple case.

Assume: Agents invest in a portfolio (among those admissible in the market model) which has the highest expected profit according to their model.

Note: Your investors always have the option to invest the money @ the risk-free interest rate. $\Rightarrow$ The least the investors require of their investment is a strictly positive expected profit.

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Subjective expectations.

Problem 2.1. IFM Sample (Introductory) Problem #6.
The following relates to one share of XYZ stock:

- The current price is 100.
- The forward price for delivery in one year is 105.
- An investor who decides to long the forward contract denotes by P the expected stock price in one year.

Determine which of the following statements about P is **TRUE**.

(A) $P < 100$
(B) $P = 100$
(C) $100 < P < 105$
(D) $P = 105$
(E) $P > 105$

$$\mathbb{E}\left[\underbrace{Profit(Long\ Forward)}_{=}\right] > 0$$

$$\mathbb{E}\left[S(T) - F\right] > 0$$

$$\Rightarrow \mathbb{E}\left[S(T)\right] > F$$

$$\Rightarrow P > F = 105$$

Problem 2.2. IFM Sample (Introductory) Problem #38.

$\mathbb{E}[S(3)] = 90$

The current price of a medical company's stock is 75. The expected value of the stock price in three years is 90 per share. The stock pays no dividends. You are also given:

- The risk-free interest rate is positive. $\boxed{r>0}$
- There are no transaction costs.
- Investors require compensation for risk. $\Rightarrow \mathbb{E}[Profit] > 0$

The price of a three-year forward on a share of this stock is X and at this price an investor is willing to enter into the forward. Determine what can be concluded about X.

(A) $X < 75$
(B) $X = 75$
(C) $75 < X < 90$
(D) $X = 90$
(E) $X > 90$

- By the same reasoning as in Problem 2.1., we get $90 = \mathbb{E}[S(3)] > X$
- By the formula for the forward price:

$$X = F_{0,3}(S) = S(0)e^{(r-\delta)\cdot 3}$$

$$\Rightarrow X = S(0)e^{3\cdot r} > S(0) = 75$$

$r > 0$

Problem 2.3. **IFM Sample (Introductory) Problem #70.**
Investors in a certain stock demand to be compensated for risk. The current stock price is 100. The stock pays dividends at a rate proportional to its price. The dividend yield is 2%. The continuously compounded risk-free interest rate is 5%. Assume there are no transaction costs.

Let X represent the expected value of the stock price 2 years from today. Assume it is known that X is a whole number. Determine which of the following statements is true about X.

(A) The only possible value of X is 105.
(B) The largest possible value of X is 106.
(C) The smallest possible value of X is 107.
(D) The largest possible value of X is 110.
(E) The smallest possible value of X is 111.

$S(0) = 100, \quad \delta = 0.02$

$r = 0.05$

$X := \mathbb{E}[S(2)]$

Say, the investor invests in 1 share.
Their criterion is, at least, $\mathbb{E}[\text{Profit}] > 0$.

Initial Cost: $S(0) = 100$

Payoff: The investor's wealth @ time 2.
The investor owns $e^{\delta \cdot T} = e^{0.04}$ shares @ time 2.
$\Rightarrow$ Their wealth is $\boxed{e^{0.04} \cdot S(2).}$

$$\Rightarrow \text{Profit} = \text{Payoff} - FV(\text{Initial Cost})$$
$$= e^{0.04} \cdot S(2) - e^{0.05 \cdot 2} \cdot S(0)$$

$$\Rightarrow \mathbb{E}[e^{0.04} \cdot S(2) - e^{0.1} \cdot S(0)] > 0$$

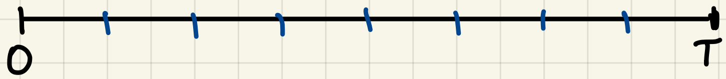
$$\Rightarrow e^{0.04} \cdot \mathbb{E}[S(2)] > S(0)\, e^{0.1}$$

linearity of expectation

$$\Rightarrow X = \mathbb{E}[S(2)] > S(0)\, e^{0.06} = 100\, e^{0.06} = 106.18$$

Realized Returns

In a binomial model:



Our time-line can now be any choice of an investment horizon (not just exercise or expiration date of an option!)

n periods $\Rightarrow h_n = \frac{T}{n}$ is the length of every period

Remember:
- returns are independent between periods
- returns are identically distributed for different periods (which are, by design, equal in length)

These same assumptions are made for a binomial tree w/ subjective probabilities.