UNIVERSITY OF TEXAS AT AUSTIN

Extra-credit homework assignment 6

Binomial pricing of American options.

Please, provide your **complete solutions** to the following problems:

Problem 6.1. (10 points) Assume a nonnegative continuously compounded, risk-free interest rate. It is never optimal to exercise an American call option on a non-dividend paying stock early. *True or false? Why?*

Solution: TRUE
Say, to the contrary, that it is optimal to exercise an American call option on a non-dividend-paying stock at a (random) time t^* strictly prior to the exercise date T. Then, we know that at that time, in our usual notation,

$$v_C^A(t^*) = S(t^*) - K > CV(t^*)$$

where $CV(t^*)$ stands for the continuation value at time t^*. However, the continuation value at any time is greater than or equal to the value of the otherwise identical European option at that time, i.e.,

$$S(t^*) - K > v_C^E(t^*).$$

The lower bound based on put-call parity, which we know to hold for European style call options is

$$v_C^E(t^*) \geq F_{t^*,T}^P(S) - Ke^{r(T-t^*)}.$$

Since the stock pays no dividends, the prepaid forward price equals the current price of the stock at any time. So,

$$S(t^*) - K > v_C^E(t^*) \geq S(t^*) - Ke^{r(T-t^*)}.$$

We have reached a contradiction since $K > Ke^{r(T-t^*)}$.

Problem 6.2. (10 points) The current stock price is observed to be \$100 per share. The stock is projected to pay dividends continuously at the rate proportional to its price with the dividend yield of 0.03. The stock's volatility is given to be 0.23. You model the evolution of the stock price using a two-period forward binomial tree with each period of length one year.

The continuously compounded, risk-free interest rate is given to be 0.04.

What is the price of a two-year, \$101-strike **American** put option on the above stock consistent with the above stock-price tree?

Solution: The up and down factors in the forward tree are

$$u = e^{0.01+0.23} = 1.2712, \quad \text{andd} = e^{-0.22} = 0.8025.$$

The risk-neutral probability equals

$$p^* = \frac{1}{1+e^{0.23}} = 0.4428.$$

Of course, in the exam, you would not necessarily populate the entire stock-price tree, since you would want to work efficiently and only consider the nodes you need for pricing.

Evidently, the option produces a positive payoff only in the down-down node; the value of this payoff is $V_{dd} = 101 - 64.40 = 36.60$.

The continuation value at the down node is, hence,

$$CV_d = e^{-0.04} \times (1 - 0.4428) \times 36.60 = 15.57.$$

On the other hand, the value of immediate exercise at the down node equals $IE_d = 101 - 80.25 = 20.75$. So, it is optimal to exercise the American put in the down node and the value of the American put equals $V_d^P = 20.75$.

In the up node, both the continuation value and the immediate exercise value are zero. So, the initial price of the American put is

$$V_P(0) = e^{-0.04} \times (1 - 0.4428) \times 20.75 = 11.11$$

Problem 6.3. (10 points) The current price of a non-dividend-paying stock is \$100 per share. A two-period binomial stock-price tree is used to model the movements of the stock price during the following year. The up and down factors are given to be $u = 1.2$ and $d = 0.9$.

The continuously compounded, risk-free interest rate equals 0.06.

Consider a \$110-strike, one-year American put on the above stock. Use the two-period binomial stock-price tree to calculate the current price of the American put.

Solution: With the given up and down factors, the stock-prices tree looks like this:

$$\begin{array}{ccc} & & S_{uu} > 110 \\ & S_u = 120 > 100 & \\ S_0 = 100 & & S_{ud} = 108 \\ & S_d = 90 & \\ & & S_{dd} = 81 \end{array}$$

The risk-neutral probability of the stock price going up in a single period equals

$$p^* = \frac{e^{(r-\delta)h-d}}{u-d} = \frac{e^{0.03} - 0.9}{1.2 - 0.9} = 0.4348.$$

Should the American option not be exercised early the possible payoffs would be

$$V_{uu} = 0, \quad V_{ud} = 110 - 108 = 2, \quad V_{dd} = 110 - 81 = 29.$$

It is not sensible to exercise the American put at the up node, so the value of the American put equals the continuation value at the up node. We get

$$V_u^A = CV_u = e^{-0.03}(1 - 0.4348) \times 2 = 1.09699.$$

At the down node, the value of immediate exercise is

$$IE_d = 110 - 90 = 20.$$

On the other hand, the continuation value at the down node equals

$$CV_d = e^{-0.03}[0.4348 \times 2 + (1 - 0.4348) \times 29] = 16.7503.$$

We conclude that the American put's value at the down node equals the value of immediate exercise, i.e., $V_d^A = 20$.

Should the option be exercised at time-0, the payoff would be 10. The continuation value at the root node is

$$CV_0 = e^{-0.03}[0.4348 \times 1.09699 + (1 - 0.4348) \times 20] = 11.4328.$$

So, the price we were looking for is \$13.38.

Problem 6.4. (10 points) The current price of a non-dividend-paying stock is \$100 per share and its volatility is given to be 0.25.

The continuously compounded, risk-free interest rate equals 0.04.

Consider a \$110-strike, one-year American put on the above stock. Use a two-period forward binomial stock-price tree to calculate the current price of the American put.

Solution: By the definition of the forward binomial tree, we obtain

$$u = e^{(r-\delta)h + \sigma\sqrt{h}} = e^{0.02 + 0.25\sqrt{\frac{1}{2}}} \approx 1.2175,$$

$$d = e^{(r-\delta)h - \sigma\sqrt{h}} = e^{0.02 - 0.25\sqrt{\frac{1}{2}}} \approx 0.8549.$$

in our usual notation.

The risk-neutral probability of the stock price going up in a single period equals

$$p^* = \frac{1}{1+e^{\sigma\sqrt{h}}} = 0.4559.$$

Should the American put option not be exercised early the possible payoffs would be

$$V_{uu} = 0, V_{ud} = 110 - 104.081 = 5.9189, V_{dd} = 110 - 73.0845 = 36.9155.$$

Since it is out of the money, it is not sensible to exercise the American put at the up node, so the value of the American put equals the continuation value at the up node. We get

$$V_u^A = CV_u = e^{-0.02}(1-0.4559) \times 5.1989 = 3.1566.$$

At the down node, the value of immediate exercise is

$$IE_d = 24.5105.$$

On the other hand, the continuation value at the down node equals

$$CV_d = e^{-0.02}[0.4559 \times 5.9189 + (1-0.4559) \times 36.9155] = 22.3324.$$

We conclude that the American put's value at the down node equals the value of immediate exercise, i.e., $V_d^A = 24.5105$.

Should the option be exercised at time-0, the payoff would be 10. The continuation value at the root node is

$$CV_0 = e^{-0.02}[0.4559 \times 3.1566 + (1-0.4559) \times 24.5105] = 14.4823.$$

So, the price we were looking for is \$14.48.

Problem 6.5. (10 points) A certain non-dividend-paying stock is currently priced at \$100 per share. You assume that each year the price can either increase or decrease by 25%. Using this assumption, you construct a $2-$period binomial tree modeling the evolution of the stock price over the next two years.

Assume that the continuously compounded risk-free interest rate equals 5%.

Consider a two-year, \$90$-$strike American call option on the above stock. What is its time-0 value $V_C^A(0)$ as calculated using the above model for the stock price?

Solution: The values of the stock price in the binomial tree are

$$\begin{aligned} S(0) = 100, \quad S_u &= 125, \quad S_{uu} = 156.25,\\ S_d &= 75, \quad S_{ud} = 93.75,\\ & \quad\quad\;\; S_{dd} = 56.25. \end{aligned}$$

The risk-neutral probability associated with the above asset is

$$p^* = \frac{e^{0.05} - 0.75}{1.25 - 0.75} = 0.6025.$$

Since we are pricing an Americal **call** on a **non-dividend-paying** asset, we can price it as European (since early exercise is never optimal). We get

$$V_C^A(0) = e^{-0.05\times 2}[(0.6025)^2(156.25-90) + 2(0.6025)(1-0.6025)(93.75-90)] = 23.39$$