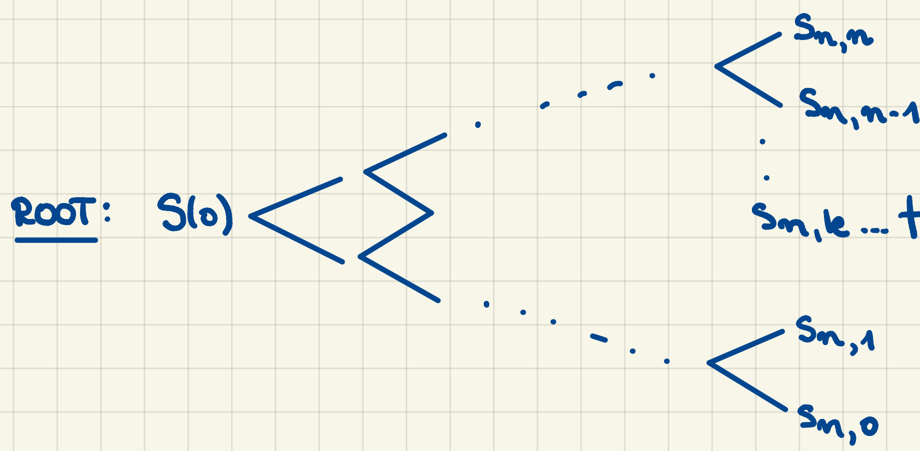
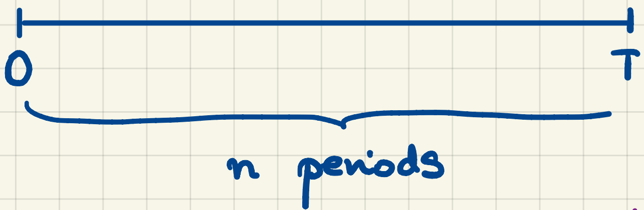
M339D: October 30th, 2023.

The Pre-limit n-period Binomial Tree.



$S_{n,k}$ ... the stock price resulting from k upsteps.



$$\boxed{h = \frac{T}{n}}$$

u_n ... up factor

d_n ... down factor

e.g., in the forward tree

$$u_n = \exp\left(r \cdot \frac{T}{n} + \sigma\sqrt{\frac{T}{n}}\right)$$

$$d_n = \exp\left(r \cdot \frac{T}{n} - \sigma\sqrt{\frac{T}{n}}\right)$$

$$S_{n,k} = S(0) \cdot u_n^k \cdot d_n^{n-k} = S(0)\left(\frac{u_n}{d_n}\right)^k \cdot d_n^n$$

k corresponds to the realization of the binomial dist'n w/ n trial and $p_n^* = \dfrac{e^{r\left(\frac{T}{n}\right)} - d_n}{u_n - d_n}$

de Moivre · Laplace

e.g., in the forward tree:

$$\boxed{p_n^* = \frac{1}{1 + e^{\sigma\sqrt{\frac{T}{n}}}}}$$

Say, X_n ... # of upsteps in n periods

$$\boxed{X_n \sim \text{Binomial}(\#\text{ of trials } n, \text{ probab. of success} = p_n^*)}$$

On Pricing.

When we're pricing, we work under the risk-neutral measure.

$$\begin{tikzcd}[row sep=small] & S_u = u\cdot S(0) \\ S(0) \arrow[ur, "p^*"] \arrow[dr, "1-p^*"'] & \\ & S_d = d\cdot S(0) \end{tikzcd}$$

$$\underbrace{\qquad}_{h=T}$$

$$\boxed{p^* = \frac{e^{rh}-d}{u-d}}$$

Q: If we invest in one share of non-dividend-paying stock @ time 0, what is our expected wealth @ time T under the risk-neutral probability measure?

$\longrightarrow$:
$$\mathbb{E}^*[S(T)] = S_u \cdot p^* + S_d \cdot (1-p^*)$$
$$= u\cdot S(0)\,\frac{e^{rh}-d}{u-d} + d\cdot S(0)\cdot\frac{u-e^{rh}}{u-d}$$
$$= S(0)\left(\frac{1}{u-d}\right)\left(ue^{rh} - \cancel{ud} + \cancel{ud} - de^{rh}\right)$$
$$= S(0)\left(\cancel{\frac{1}{u-d}}\right) e^{rh}\cancel{(u-d)} \underset{h=T}{=} S(0)\,e^{rT}$$

In Contrast:

There can be a subjective probability $\mathbb{P}$. We can think about the quality of our investment under that probability, e.g.,

$$\mathbb{E}[S(T)] = S(0)e^{\alpha T}$$

We usually refer to α as the mean rate of return.

In a binomial tree, we can talk about the "true" probab. of a step up

$$\boxed{p = \frac{e^{\alpha h}-d}{u-d}}$$

$\square$