M339J: March 5th, 2021.

Transformation II. Raising to a Power.

Let X be a positive continuous random variable w/ probability density function $f_X(\cdot)$.

Let $\tau \neq 0$ be a constant.

Define $\boxed{Y := X^{1/\tau}}$.

Then,

• for $\tau > 0$:

$y>0$: $F_Y(y) = \mathbb{P}[Y \leq y] = \mathbb{P}[X^{1/\tau} \leq y]$

$= \mathbb{P}[X \leq y^\tau] = F_X(y^\tau)$

$\Rightarrow f_Y(y) = F_X'(y^\tau) = \tau \cdot y^{\tau-1} \cdot f_X(y^\tau)$

chain rule

• for $\tau < 0$:

$y>0$: $F_Y(y) = \mathbb{P}[Y \leq y] = \mathbb{P}[X^{1/\tau} \leq y]$

$= \mathbb{P}[X \geq y^\tau] = 1 - F_X(y^\tau)$

$\Rightarrow f_Y(y) = -\tau \cdot y^{\tau-1} \cdot f_X(y^\tau)$

Example. Start w/ $X \sim$ Exponential (mean $= \theta$).

Define $Y := X^{-1}$ (i.e., $\tau = -1$).

For $y>0$: $F_Y(y) = 1 - F_X(y^{-1}) = S_X(\frac{1}{y})$

$= e^{-\frac{1}{y \cdot \theta}}$

In the STAM TABLES for the inverse exponential dist'n w/ parameter θ : $F(x) = e^{-\frac{\theta}{x}}$

$\Rightarrow Y \sim InvExp(\text{parameter} = \frac{1}{\theta})$

mean $= \frac{1}{\theta}$

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Example. Let $X \sim Exp(\text{mean} = \theta)$.
Let $\tau > 0$. Define $Y := X^{\frac{1}{\tau}}$

For $y > 0$: $F_Y(y) = F_X(y^\tau) = 1 - e^{-\frac{y^\tau}{\theta}} = \ldots$

$X \sim$ exponential

$$\ldots = 1 - e^{-\frac{y^\tau}{(\theta^{1/\tau})^\tau}} = 1 - e^{-\left(\frac{y}{\theta^{1/\tau}}\right)^\tau}$$

STAM Tables:

Weibull (θ, τ) has the cdf of the form:

$$F(x) = 1 - e^{-\left(\frac{x}{\theta}\right)^\tau}$$

$$\Rightarrow \boxed{Y \sim \text{Weibull}(\theta' = \theta^{1/\tau}, \tau)}$$

Transformation III. Exponentiation.

Let X be a continuous r.v. w/ $f_X(x) > 0$, e.g., normal.
Define $Y := e^X$
Then, $F_Y(y) = F_X(\ln(y))$
and $f_Y(y) = \frac{1}{y} f_X(\ln(y))$.

Example. If X is normal, we say that Y is
lognormally distributed.

Two-point mixture.

Start w/ two random variables X_1 and X_2.
Take two positive constants a_1 and a_2 such that $a_1 + a_2 = 1$.

We want to create a r.v. Y which will be a
two-point mixture of X_1 and X_2; in a sense,

you toss a coin such that the probability of Heads is a_1. If the coin comes up Heads, then you draw a value from X_1. If the coin comes up Tails, then you draw a value from X_2.

In fact, the cdf of Y is constructed as:

$$F_Y(y) = a_1 \cdot \underline{F_{X_1}(y)} + a_2 \cdot \underline{F_{X_2}(y)} \quad \text{for all } y.$$

At home: • Think about what a k-point mixture would be.

• Think about examples on your own.