M339W/389W Financial Mathematics for Actuarial Applications
University of Texas at Austin
Practice Problems for In-Term Exam #3
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Note: This is a closed book and closed notes exam.
Time: 50 minutes

3.1. **TRUE/FALSE QUESTIONS.**

Problem 3.1. (2 points) According to the weak formulation of the efficient market hypothesis, one cannot consistently make gains by trading based on the information contained in past prices. *True or false?*

Solution: TRUE

Problem 3.2. (2 points) Under the **CAPM**, the expected return and the required return of the market portfolio are equal. *True or false?*

Solution: TRUE

Problem 3.3. (2 points) Assume the assumptions of CAPM. Then, the **capital market line (CML)** is the tangent line of the feasible set going through the market portfolio. *True or false?*

Solution: TRUE

Problem 3.4. (2 points) The variability of an investment portfolio that is balanced evenly between the stocks it contains is lower than the average variability of the individual stocks it contains. *True or false?*

Solution: TRUE

Problem 3.5. (2 points) Consider the feasible set for two stocks. The higher the correlation of the two stocks' returns, the flatter the curve of the feasible set.*True or false?*

Solution: TRUE

3.2. **FREE-RESPONSE PROBLEMS.** Please, explain carefully all your statements and assumptions. Numerical results or single-word answers without an explanation (even if they're correct) are worth 0 points.

Problem 3.6. (5 points) Consider a company which has one million shares, each worth \$100. Its equity cost of capital is 12%. The same company has twenty million dollars in debt at the interest rate of 4%.

The company borrows more at the same interest rate with the aim of purchasing back some equity to establish a 1 : 1 debt-to-equity ratio. What is the new return on equity after this change in capital structure?

Solution: Originally, the amount of debt is $D = 20 \cdot 10^6$ while the amount of equity equals $E = 100 \cdot 10^6$. So, the original weights are

$$\frac{D}{D+E} = \frac{20}{120} = \frac{1}{6} \quad \text{and} \quad \frac{E}{D+E} = \frac{100}{120} = \frac{5}{6}.$$

The cost of capital is

$$r_U = \frac{1}{6}(0.04) + \frac{5}{6}(0.12) = 0.106667.$$

After the change, with $\tilde{r}_E$ denoting the new return on equity,

$$r_U = \frac{1}{2}(\tilde{r}_E + 0.04) \quad \Rightarrow \quad \tilde{R}_E = 0.2133333 - 0.04 = 0.1733333.$$

Problem 3.7. (5 points) An unlevered company is currently priced at \$40 per share. There are 50 million shares outstanding. Its after-tax return on equity equals 12%. Its corporate tax rate is 0.35.

The company wants to increase its return on equity. So, it plans to borrow an amout D at the interest rate of 5%. The target return on equity is 16%. How much should the company borrow?

Solution: To begin with, the total amount of equity is $40(50,000,000) = 2,000,000,000$. With the return on equity equal to 12%, we can calculate the total earnings as $0.12(2,000,000,000) = 240,000,000$. So, the new debt D must satisfy:

$$\begin{aligned}
&0.16(2 \cdot 10^9 - D) = 24 \cdot 10^7 - D(0.05)(1-0.35) \\
&\qquad \Rightarrow \quad 32 \cdot 10^7 - 0.16D = 24 \cdot 10^7 - 0.0325D \quad \Rightarrow \quad 0.1275D = 8 \cdot 10^7 \\
&\qquad \Rightarrow \quad D = 62.7451 \cdot 10^7 = 627,451,000.
\end{aligned}$$

Problem 3.8. (5 points) A variable annuity has a GMDB and a GMAB. The GMDB expires in 20 years. The GMAB guarantees that in 20 years, the account will be worth at least 115% of the purchase price. The inital investment is $100,000$. Express the total value of these guarantees in terms of:

- $v_P(0,T,K)$, i.e., the price of the put with exercise date T and strike K;
- $f(\cdot)$, i.e., the probability density function of the random variable denoting the time until death of the annuitant; and

- $F(\cdot)$, i.e., the cumulative function of the random variable denoting the time until death of the annuitant;

Solution:

$$\int_0^{20} v_P(0,t,100000)f(t)\,dt + (1-F(20))v_P(0,20,100000).$$

Problem 3.9. (5 points) Let the current stock price of a non-dividend-paying stock be 100. The continuously compounded, risk-free interest rate is 0.

You are given that the price of a one-year, \$90-strike call option on the above stock equals \$15.

A chooser option on the above stock with the choice date on one year and the exercise date in 3 years is currently priced at \$24.

What is the price of a three-year, \$90-strike put option on the same stock?

Solution: By the pricing formula for chooser options, we know that

$$24 = 15 + v_P(0, T = 3, K = 90)$$

where $v_P(0,T,K)$ denotes the time-0 price of a European put option on the same stock with the exercise date T and strike price K. Hence, our answer is \$9.

3.3. MULTIPLE CHOICE QUESTIONS.

Problem 3.10. (5 points) There are two stocks present in our market: **S** and **Q**. Their current prices are $S(0) = 50$ and $Q(0) = 55$. Both stocks pay dividends continuously. The dividend yield for **S** is 0.02 while the dividend yield for **Q** equals 0.03.

You are given that for $t \geq 0$

$$\mathrm{Var}[\ln(S(t)/Q(t))] = 0.09t.$$

What is the Black-Scholes price of a one-year **exchange call** with underlying **S** and the strike asset **Q**?

(a) \$2.89
(b) \$3.01
(c) \$3.57
(d) \$4.36
(e) None of the above.

Solution: (d)
In our usual notation, the volatility of the difference of the stocks' realized returns is $\sigma = 0.3$. So,

$$\begin{aligned} d_1 &= \frac{1}{0.3}\left[\ln\left(\frac{50}{55}\right) + \left(0.03 - 0.02 + \frac{0.09}{2}\right)\right] \approx -0.13, \\ d_2 &= -0.13 - 0.3 = -0.43. \end{aligned}$$

So,

$$N(d_1) = 1 - N(0.13) = 0.4483, \quad N(d_2) = 1 - N(0.44) = 0.33.$$

Finally,

$$V_{EC}(0, \mathbf{S}, \mathbf{Q}) = 50e^{-0.02}(0.4483) - 55e^{-0.03}(0.33) = 4.357567.$$

Problem 3.11. A market-maker sells option I for \$10. This option's delta is 0.6557 and its gamma is 0.02. The market maker proceeds to delta-gamma hedge this commitment by trading in the underlying and also in option II on the same stock. The latter option's price is \$4.70, its delta is 0.5794 and its gamma is 0.04.

What is the market-maker's resulting position in option II?

(a) Buy 0.5 of option II.
(b) Write 0.5 of option II.
(c) Buy 2 of option II.
(d) Write 2 of option II.
(e) None of the above.

Solution: (a)
With n_{II} denoting the position in option II, to achieve gamma-neutrality we need

$$-0.02 + n_{II}(0.04) = 0 \quad \Rightarrow \quad n_{II} = 0.5.$$

Problem 3.12. (5 points) Consider a two-year project. There are only three cash flows for this project:

- The first occurs at $t = 0$, and is -80.
- The second occurs at $t = 1$, and is 40.
- The third occurs at $t = 2$, and is 44.30.

Determine r, the cost of capital, that leads to the project breaking even.

(a) 0.035
(b) 0.04
(c) 0.045
(d) 0.05
(e) None of the above.

Solution: (a)
The break-even value of the cost of capital must satisfy

$$-80(1+r)^2 + 40(1+r) + 44.30 = 0 \quad \Leftrightarrow \quad (1+r)^2 - 0.5(1+r) - 0.55375 = 0.$$

Solving the quadratic equation, we obtain

$$(1+r)_{1,2} = \frac{0.5 \pm \sqrt{0.5^2 + 4(0.55375)}}{2} = \frac{0.5 \pm \sqrt{2.465}}{2} = \frac{0.5 \pm 1.57}{2}.$$

Our acceptable solution is $1 + r = 1.035$, i.e., $r = 0.035$.

Problem 3.13. You are given the following information about the return of a security, using a three-factor model:

Factor	Beta	Expected Return
T	0.10	12%
U	0.15	15%
V	0.20	10%

The expected return of this security using the given three-factor model is equal to 0.09. What is the annual effective risk-free rate of return?

(a) About 0.0375
(b) About 0.0415
(c) About 0.0485
(d) About 0.06455
(e) None of the above.

Solution: (d)
By our three-factor model, we have that the expected return of our security S satisfies

$$\begin{aligned}\mathbb{E}[R_S] &= r_f + \beta^T(\mathbb{E}[R_T] - r_f) + \beta^U(\mathbb{E}[R_U] - r_f) + \beta^V(\mathbb{E}[R_V] - r_f)\\ &= \beta_T\mathbb{E}[R_T] + \beta_U\mathbb{E}[R_U] + \beta_V\mathbb{E}[R_V] + r_f(1 - \beta_T - \beta_U - \beta_V).\end{aligned} \tag{3.1}$$

So,

$$\begin{aligned}r_f &= \frac{\mathbb{E}[R_S] - \beta_T\mathbb{E}[R_T] - \beta_U\mathbb{E}[R_U] - \beta_V\mathbb{E}[R_V]}{1 - \beta_T - \beta_U - \beta_V}\\ &= \frac{0.09 - 0.10(0.12) - 0.15(0.15) - 0.2(0.1)}{1 - 0.10 - 0.15 - 0.2} = 0.06454545.\end{aligned}$$

Problem 3.14. (5 points) Assume the **Capital Asset Pricing Model** holds.
You are given the following information about stock X, stock Y, and the market:

- The required return and volatility for the market portfolio are 0.10 and 0.25, respectively.
- The required return and volatility for the stock X are 0.08 and 0.4, respectively.
- The correlation between the returns of stock X and the market is -0.2.
- The volatility of stock Y is 0.25.
- The correlation between the returns of stock Y and the market is 0.4.

Calculate the required return for stock Y.

(a) About 0.075.
(b) About 0.08.
(c) About 0.085.
(d) About 0.09.
(e) None of the above.

Solution: (d)
The βs of stocks X and Y are

$$\beta_X = \frac{0.4(-0.2)}{0.25} = -0.32,$$
$$\beta_Y = \frac{0.4(0.25)}{0.25} = 0.4.$$

So, the required return of stock X must satisfy

$$\begin{aligned} 0.08 = r_X = r_f + (-0.32)(0.10 - r_f) \quad &\Rightarrow \quad 0.08 = r_f - 0.032 + 0.32 r_f \\ &\Rightarrow \quad 1.32 r_f = 0.112 \quad \Rightarrow \quad r_f = 0.0848. \end{aligned}$$

Finally, the required return of stock Y equals

$$r_Y = 0.0848 + 0.4(0.10 - 0.0848) = 0.09088.$$

Problem 3.15. (5 points) For a certain stock, you are given that its expected return equals 0.0944 and that its β equals 1.24. For another stock, you are given that its expected return equals 0.068 and that its β equals 0.8. Both stocks lie on the **Security Market Line (SML)**. What is the risk-free interest rate r_f?

(a) About 0.02
(b) About 0.025
(c) About 0.03
(d) About 0.035
(e) None of the above.

Solution: (a)
Denote the expected return of the market portfolio by r_m. Then,

$$0.0944 = r_f + 1.24(r_m - r_f),$$
$$0.068 = r_f + 0.8(r_m - r_f).$$

Subtracting the second equation from the first one, we get

$$0.0264 = 0.44(r_m - r_f) \quad \Rightarrow \quad r_m - r_f = \frac{0.0264}{0.44} = 0.06.$$

Substituting the obtained risk premium of the market portfolio into the first equation above, we obtain

$$r_f = 0.0944 - 1.24(0.06) = 0.02.$$

Problem 3.16. (5 points) In a market, the risk-free interest rate is given to be 0.04.

Consider an investment I in this market, whose Sharpe ratio is 0.42. You construct an equally weighted portfolio consisting of the investment I and the risk-free asset. The expected return of this portfolio is 0.10.

You decide to rebalance your portfolio so that one quarter of your wealth gets invested in the investment I and the remainder is invested in the risk-free asset. What is the volatility of this new portfolio?

(a) 0.0625
(b) 0.0714
(c) 0.1225
(d) 0.1625
(e) None of the above.

Solution: (b)
Let's denote the volatility of investment I by σ_I and the volatility of the new portfolio by $\sigma_{P'}$. Then, $\sigma_{P'} = 0.25\sigma_I$.

The expected return of the old portfolio is

$$\mathbb{E}[R_P] = \frac{1}{2}\mathbb{E}[R_I] + \frac{1}{2}r_f\,.$$

So,

$$\mathbb{E}[R_I] = 2\mathbb{E}[R_P] - r_f = 2(0.10) - 0.04 = 0.16.$$

We are given the Sharpe ratio of the investment I, and so we can calculate

$$\sigma_I = \frac{0.16 - 0.04}{0.42} = 0.2857143.$$

Finally, the new portfolio's volatility is

$$\sigma_{P'} = 0.25(0.2857143) = 0.07142857.$$

Problem 3.17. (5 points) According to your model, the economy over the next year could be *good* or *bad*. You are a pessimist and believe that the economy is twice as likely to be *bad* than *good*.

Consider two assets, X and Y, existing in this market. If the economy is *good* the return on asset X is 0.12, and the return on asset Y is 0.11. If the economy is *bad* the return on asset X is -0.03 and the return on asset Y is -0.01.

You construct a portfolio P using assets X and Y so that the portfolio's expected return equals 0.025.

Calculate the volatility of this portfolio's return.

(a) 0.0458

(b) 0.0512

(c) 0.0584

(d) 0.0637

(e) None of the above.

Solution: (d)
First, we need to figure out the weight w_X the asset X is given in portfolio P. The expected returns of assets X and Y are

$$\mathbb{E}[R_X] = \frac{1}{3}(0.12) + \frac{2}{3}(-0.03) = \frac{0.12 - 0.06}{3} = 0.02,$$
$$\mathbb{E}[R_Y] = \frac{1}{3}(0.11) + \frac{2}{3}(-0.01) = \frac{0.11 - 0.02}{3} = 0.03.$$

So, the portfolio P must be equally weighted, i.e., $w_X = 0.5$. Hence, the distribution of the portfolio's return can be described as

$$R_P \sim \begin{cases} 0.115, & \text{with probability } 1/3 \\ -0.02, & \text{with probability } 2/3 \end{cases}$$

The second moment of the portfolio's return is

$$\mathbb{E}[R_P^2] = (0.115) \times \frac{1}{3} + (-0.02)^2 \times \frac{2}{3} = 0.004675.$$

The variance of the portfolio's return is

$$Var[R_P] = 0.004675 - 0.025^2 = 0.00405.$$

Finally, its volatility equals $\sigma = \sqrt{0.00405} = 0.0636396$.

Problem 3.18. (5 points) Consider two assets X and Y such that:

- their expected returns are $\mathbb{E}[R_X] = 0.10$ and $\mathbb{E}[R_Y] = 0.08$;
- their volatilities are $\sigma_X = 0.25$ and $\sigma_Y = 0.35$;
- the correlation coefficient of their returns is $\rho_{X,Y} = -1$.

You are tasked with constructing a portfolio consisting of shares of X and Y with a risk-free return. What should the weight w_Y given to asset Y be?

(a) 5/12

(b) 1/2
(c) 7/12
(d) Such a weight does not exist.
(e) None of the above.

Solution: (a)

$$w_Y = \frac{\sigma_X}{\sigma_X + \sigma_Y} = \frac{0.25}{0.25 + 0.35} = \frac{5}{12}\,.$$

Problem 3.19. (5 points) For stock S_1, you are given that its expected return equals 0.08 and its β is 1.22. For stock S_2, you are given that its expected return equals 0.05 and its β is 0.56. Both of these stocks lie on the *Security Market Line.* For stock S_3, you are given that its expected return equals 0.07 and its β is 0.7. What is the α of stock S_3?

(a) 0
(b) 0.0137
(c) 0.0245
(d) 0.0455
(e) None of the above.

Solution: (b)
Since both S_1 and S_2 are on the **SML**, we know that

$$0.08 = r_f + 1.22(r_m - r_f),$$
$$0.05 = r_f + 0.56(r_m - r_f),$$

where r_f stands for the risk-free interest rate and r_m stands for the expected return of the market. Subtracting the second equation from the first one, we get

$$0.03 = 0.66(r_m - r_f) \quad \Rightarrow \quad r_m - r_f = \frac{0.03}{0.66} = 0.0455.$$

The risk-free interest rate r_f can, then, be calculated as

$$r_f = 0.08 - 1.22(0.0455) = 0.02449.$$

Hence, the α of stock S_3 is

$$0.07 - 0.02449 - 0.7(0.0455) = 0.01366.$$