University of Texas at Austin

Problem set 1

The cumulative distribution function.

Problem 1.1. The random variable X has the following cumulative distribution function:

$$F_X(x) = \begin{cases} \zeta & \text{for } x < 0 \\ \frac{1}{2} & \text{for } 0 \leq x < 1 \\ \kappa x + \nu & \text{for } 1 \leq x < 3 \\ \eta & \text{for } x \geq 3 \end{cases}$$

The function F_X is continuous at 1 and 3. How much are η, κ and ν? What is the probability that X is less than or equal to 2? What is the probability that X is equal to 1? What is the probability that X is equal to 0?

Solution: From the limiting behavior of any cdf, we know that $\zeta = 0$ and $\eta = 1$. From the continuity at 1 and 3, we have

$$\kappa + \nu = \frac{1}{2}$$
$$3\kappa + \nu = 1$$

So, $\kappa = \nu = \frac{1}{4}$. By definition,

$$\mathbb{P}[X \leq 2] = F_X(2) = \frac{1}{4}(2+1) = \frac{3}{4}\,.$$

Since F_X is continuous at 1, we have that $\mathbb{P}[X = 1] = 0$. There is a jump of size $\frac{1}{2}$ at 0, so $\mathbb{P}[X = 0] = \frac{1}{2}$.

Problem 1.2. The random variable X has the following cumulative distribution function:

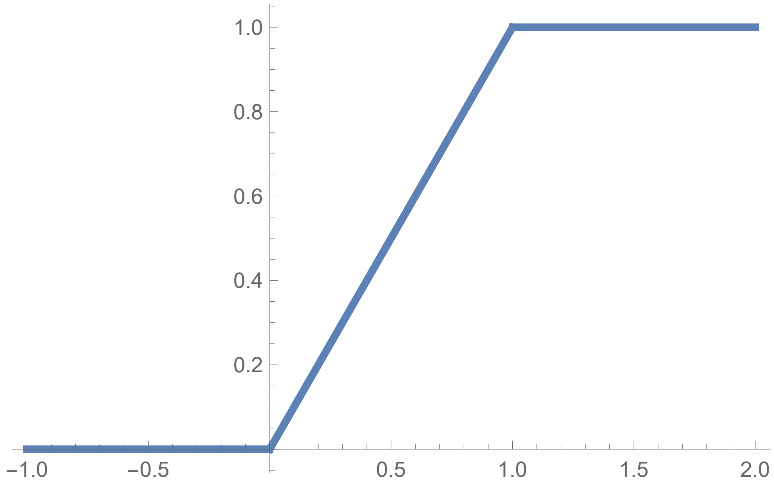
$$F_X(x) = x^3 \quad \text{for } x \in (0,1)$$

and is defined in the obvious way outside of the interval $(0,1)$. What is the probability that X exceeds $1/2$, **given** that it exceeds $1/4$?

Solution: We are supposed to calculate the probability

$$\mathbb{P}[X > \tfrac{1}{2} \,|\, X > \tfrac{1}{4}] = \frac{\mathbb{P}[X > \frac{1}{2}, X > \frac{1}{4}]}{\mathbb{P}[X > \frac{1}{4}]} = \frac{\mathbb{P}[X > \frac{1}{2}]}{\mathbb{P}[X > \frac{1}{4}]} = \frac{1-(\frac{1}{2})^3}{1-(\frac{1}{4})^3} = \frac{56}{63}\,.$$

Problem 1.3. The graph of the cumulative distribution function of the random variable X looks like this:



What is the support of the random variable X? What is the type of the random variable X?
Define the random variable Y as

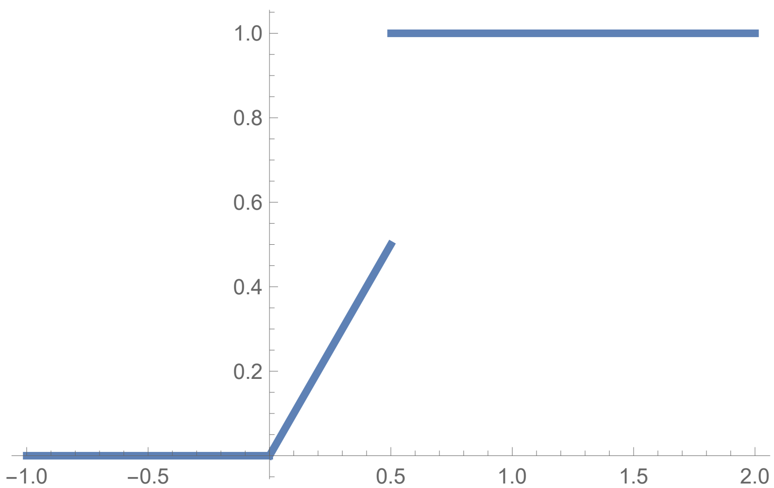
$$Y = \min(X, \tfrac{1}{2}).$$

What is the support of the random variable Y? Find the expression for the cumulative distribution function of Y. Sketch its graph. What is the type of the random variable Y?

Solution: Evidently, the random variable X is continuous and its support is the interval $[0, 1]$. The support of Y is $[0, \frac{1}{2}]$. Let F_Y denote the cumulative distribution function of Y. Then, $F_Y(x) = 0$ for all $x < 0$. Similarly, $F_Y(x) = 1$ for all $x \geq \frac{1}{2}$. For any $x \in [0, \frac{1}{2})$, we have

$$F_Y(x) = \mathbb{P}[Y \leq x] = \mathbb{P}[\min(X, \tfrac{1}{2}) \leq x] = \mathbb{P}[X \leq \tfrac{1}{2}] = F_X(x).$$

So, the graph of the cdf of Y looks like this:



The random variable Y is mixed.

Definition 1.1. Random variables X and Y with cumulative distribution functions F_X and F_Y (resp.) are said to be *independent* if

$$\mathbb{P}[X \le x, Y \le y] = F_X(x)F_Y(y) \quad \text{for all } x \text{ and } y.$$

Problem 1.4. Let T_1 and T_2 be two independent random variables with cumulative distributions functions denoted by F_1 and F_2, respectively. Define the random variables $T_\wedge$ and $T_\vee$ in the following fashion:

$$T_\wedge = \min(T_1, T_2), \quad T_\vee = \max(T_1, T_2).$$

Express the cumulative distribution functions of $T_\wedge$ and $T_\vee$ in terms of F_1 and F_2.

Solution: By definition, for all real t,

$$\begin{aligned} F_\wedge(t) &= \mathbb{P}[T_\wedge \le t] = \mathbb{P}[\min(T_1, T_2) \le t] \\ &= 1 - \mathbb{P}[\min(T_1, T_2) > t] = 1 - \mathbb{P}[T_1 > t, T_2 > t] \\ &= 1 - \mathbb{P}[T_1 > t]\mathbb{P}[T_2 > t] \\ &= 1 - (1 - \mathbb{P}[T_1 \le t])(1 - \mathbb{P}[T_2 \le t]) \\ &= 1 - (1 - F_1(t))(1 - F_2(t)). \end{aligned}$$

Similarly, for all real t,

$$\begin{aligned} F_\vee(t) &= \mathbb{P}[T_\vee \le t] = \mathbb{P}[\max(T_1, T_2) \le t] = \mathbb{P}[T_1 \le t, T_2 \le t] \\ &= \mathbb{P}[T_1 \le t]\mathbb{P}[T_2 \le t] = F_1(t)F_2(t). \end{aligned}$$