Lines. Planes. Hyperplanes.

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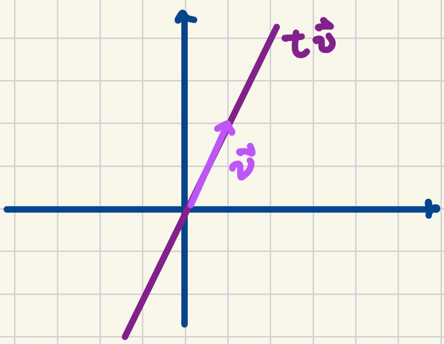
Lines in $\mathbb{R}^n$.

Start w/ $\vec{v}$, a non-zero vector in $\mathbb{R}^n$, i.e.,

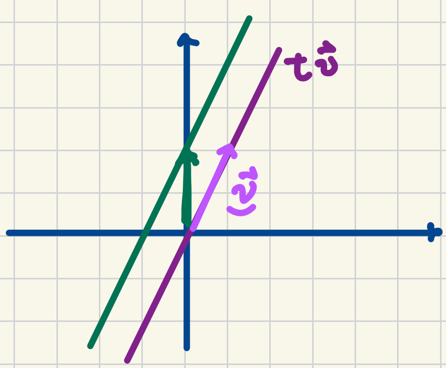
$$\vec{v} = (v_1, v_2, \ldots, v_n).$$

For any scalar $t \in \mathbb{R}$, the vector $\boxed{t\vec{v}}$ will have
the same direction as $\vec{v}$ if $t>0$,
the opposite direction when $t<0$,
be $\vec{0}$ when $t=0$.



If we add a vector, say $\vec{p} \neq 0$,
then we get a line shifted from the origin

$$\{\, t\vec{v} + \vec{p}\ ,\ -\infty < t < \infty \,\} \text{ is a line in } \mathbb{R}^n$$



VECTOR EQUATION

Can be expressed as

PARAMETRIC EQ'NS

$$y_1 = t \cdot v_1 + p_1$$
$$y_2 = t \cdot v_2 + p_2$$
$$\vdots$$
$$y_n = t \cdot v_n + p_n$$

Hyperplanes.

Focus temporarily on $\mathbb{R}^2$.
Consider the set of all points in $\mathbb{R}^2$ which satisfy

$$\boxed{a \cdot x + b \cdot y + d = 0}$$

w/ a, b, and d all scalars and
@ least one of a and b is $\neq 0$

$$\boxed{a^2 + b^2 > 0}$$

Say, $b \neq 0$.
Then, we can rewrite the above as

$$\boxed{y = -\frac{a}{b} x - \frac{d}{b}}$$

The eq'n we remember from childhood

The vector form is obtained via $t \leftrightarrow x$

$$(x, y) = \left(t, -\frac{a}{b} \cdot t - \frac{d}{b}\right)$$

$$= t\underbrace{\left(1, -\frac{a}{b}\right)}_{\vec{v}} + \underbrace{\left(0, -\frac{d}{b}\right)}_{\vec{p}}$$

Return to: $ax + by + d = 0$

Define $\vec{n} = (a, b)$

We can now write, w/ $\vec{x} = (x, y)$,

$$\boxed{\vec{n} \cdot \vec{x} + d = 0}$$

Say that $\vec{p} = (p_1, p_2)$ is a point on this line

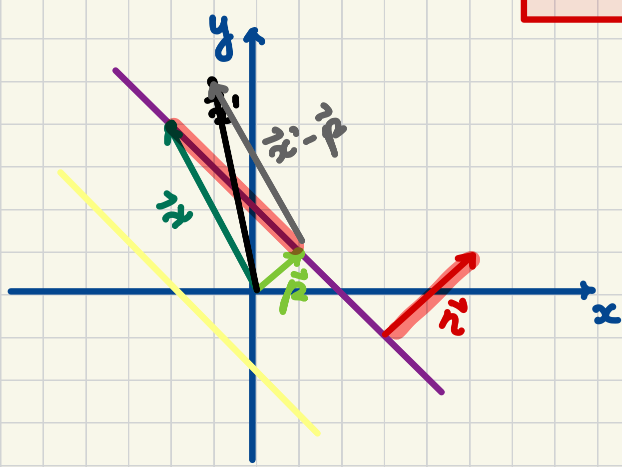
$$\Rightarrow \vec{n} \cdot \vec{p} + d = 0 \quad \Rightarrow \quad \boxed{d = -\vec{n} \cdot \vec{p}}$$

$\Rightarrow \vec{n}\cdot\vec{x} - \vec{n}\cdot\vec{p} = 0$

$\Rightarrow \boxed{\vec{n}\cdot(\vec{x}-\vec{p}) = 0}$ $\Rightarrow$ An equivalent condition for $\vec{x}$ being on the line is $\boxed{\vec{n} \perp \vec{x}-\vec{p}}$

THE NORMAL EQ'N

$\vec{n}$ ... normal vector



The hyperplane is the set of all points $\vec{x} \in \mathbb{R}^2$ which satisfy the normal equation.

Now, we generalize to $\mathbb{R}^n$.

Def'n. Let $\vec{n}$ and $\vec{p}$ be vectors in $\mathbb{R}^n$ w/ $\vec{n} \neq \vec{0}$. The set of all vectors $\vec{x}$ in $\mathbb{R}^n$ which satisfy the normal equation

$$\boxed{\vec{n}\cdot(\vec{x}-\vec{p}) = 0}$$

is called a hyperplane through the point $\vec{p}$ normal to the vector $\vec{n}$.

Example. Suppose that L is a line in $\mathbb{R}^2$ w/ the equation

$$2x+3y=1$$

Then, a normal vector for L is $\boxed{\vec{n}=(2,3)}$.

We can easily find points on L; say $x=2 \Rightarrow y=-1$, i.e., the point

$$\vec{p}=(2,-1) \text{ is on } L$$

As a normal equation, all the points (x,y) on L must satisfy

$$\vec{n}\cdot(\vec{x}-\vec{p})=0$$

$$(2,3)\cdot\big((x,y)-(2,-1)\big)=0$$

$$\Longleftrightarrow$$

$$(2,3)\cdot\big((x-2),(y+1)\big)=0$$

Let's find another point on L; say $\vec{q}=(q_1,q_2)$
Pick

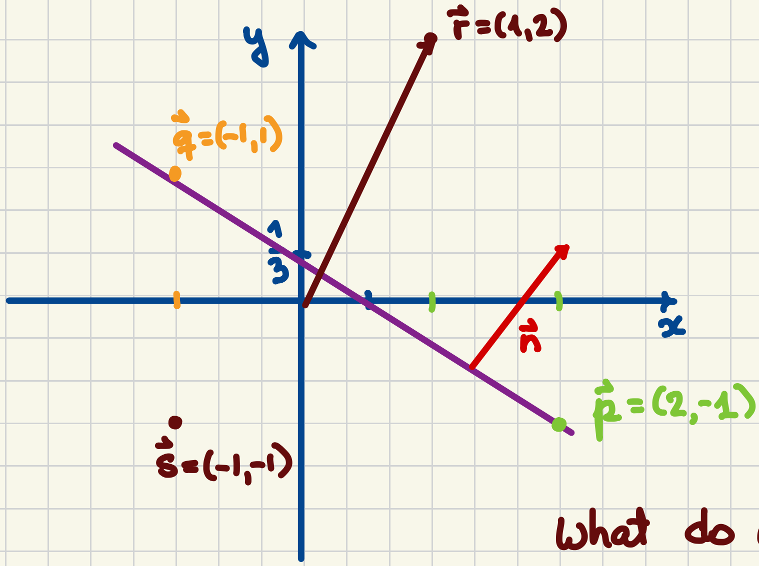
$$q_1=-1 \Rightarrow q_2=1$$

We can check the normal eq'n

$$(2,3)\cdot(-1-2,1+1) \overset{?}{=} 0$$

$$(2,3)\cdot(-3,2) \overset{?}{=} 0$$

$$2(-3)+3(2) \overset{?}{=} 0$$



$$2x+3y=1$$
$$3y=-2x+1$$
$$y=-\frac{2}{3}x+\frac{1}{3}$$

What do we get for $\vec{r}=(1,2)$?

$$2(1)+3(2)-1=2+6-1=7>0$$

What about $\vec{s}=(-1,-1)$?

$$2(-1)+3(-1)-1=-2-3-1=-6<0$$