M378K: February 7^{th}, 2025.

Problem 7.2. Source: Sample P exam, Problem #267.
The lifetime of a certain electronic device has an exponential distribution with mean 0.50. *Calculate the probability that the lifetime of the device is greater than* 0.70, $\boxed{\textit{given}}$ *that it is greater than* 0.40.

→ : Y... lifetime

$Y \sim E(\tau = 0.5)$

$$\mathbb{P}[Y > 0.7 \mid Y > 0.4] = \frac{\boxed{\mathbb{P}[Y>0.7, Y>0.4]}}{\mathbb{P}[Y>0.4]}$$

$$= \frac{\mathbb{P}[Y>0.7]}{\mathbb{P}[Y>0.4]}$$

$$= \frac{e^{-\frac{0.7}{\tau}}}{e^{-\frac{0.4}{\tau}}} = e^{-\frac{0.3}{\tau}}$$

$$= e^{-0.6}$$

$\square$

This is a special case of the memoryless property, i.e.,

$$\boxed{\mathbb{P}[Y > t+s \mid Y > t] = \mathbb{P}[Y > s]}$$

Random Vector.

Say, we are interested in two (or more) r.v.s as a PAIR (or VECTOR), i.e.,
we look @ $\boxed{(Y_1, Y_2)}$

Then, we must not only look @ their "individual" dist'ns, but also @ how they're associated.

Example. Y_i... cointoss for $i=1,2$ of fair coins

independence

$\{Y_1=H, Y_2=H\}$ $\{Y_1=T, Y_2=H\}$
$\{Y_1=H, Y_2=T\}$ $\{Y_1=T, Y_2=T\}$

complete dependence.

$\{Y_1=H, Y_2=H\}$ ✗
✗ $\{Y_1=T, Y_2=T\}$

Discrete 2D Environment.

The Joint Distribution table.

X \ Y	y_1	y_2	....	y_j	.....	y_ℓ	The Marginal Dist'n of X
x_1							$p_X(x_1)=\sum_{j=1}^{\ell} p_{1j}$
x_2							$\vdots$
$\vdots$							
x_i				p_{ij}			$p_X(x_i)=\sum_{j=1}^{\ell} p_{ij}$
$\vdots$							$\vdots$
x_m							
	The Marginal Dist'n of Y						

$p_{ij} = \mathbb{P}[X=x_i, Y=y_j]$, i.e., the joint pmf

X and Y are independent iff $\boxed{p_{ij} = p_X(x_i)\cdot p_Y(y_j) \quad \forall i,j}$

Example. $\boxed{5.2.1.}$

We independently throw two dice and record the results as Y_1 and Y_2, resp.

joint pmf: $\quad p_{ij} = \frac{1}{36}$ for all $1 \le i, j \le 6$.

Define: $\boxed{Z = Y_1 + Y_2}$

Q: What is the joint dist'n table for (Y_1, Z)?

$Y_1 \backslash Z$	2	3	4	5	6	7	8	9	10	11	12
1	$\frac{1}{36}$	$\frac{1}{36}$					0	0	0	0	0
2	0							0	0	0	0
3	0	0							0	0	0
4	0	0	0							0	0
5	0	0	0	0							0
6	0	0	0	0	0						
	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Joint Distributions: The Continuous Case.

Recall: For a continuous r.v. Y w/ a pdf f_Y, we can calculate probabilities using

$$\mathbb{P}[Y \in [a,b]] = \mathbb{P}[a \le Y \le b] =$$

$$= \int_a^b f_Y(y)\,dy \qquad \text{for all } a \le b$$

In multiple dimensions:

Say that the random vector $(Y_1, Y_2, \ldots, Y_n)$ is jointly continuous w/ density $f_{Y_1,\ldots,Y_n}$.

Then,

$$\mathbb{P}[Y_1 \in [a_1,b_1], Y_2 \in [a_2,b_2], \ldots, Y_n \in [a_n,b_n]] =$$

$$= \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_n}^{b_n} f_{Y_1,\ldots,Y_n}(y_1, y_2, \ldots, y_n)\, dy_n \ldots dy_2\, dy_1$$

For "any $\boxed{\text{nice}}$" region $A \subseteq \mathbb{R}^n$,

$$\mathbb{P}[(Y_1, Y_2, \ldots, Y_n) \in A] = \underbrace{\int \cdots \int}_{A} f_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n)\, dy_n \ldots dy_1$$

Example. (Y_1, Y_2) .... represents a point chosen @ random in a unit square $[0,1]^2$

$$\boxed{f_{Y_1,Y_2}(y_1, y_2) = 1 \cdot \mathbb{1}_{[0,1]\times[0,1]}(y_1, y_2)}$$

$$\mathbb{P}[Y_1 > Y_2] = ?$$