Fact:

$$\mathbb{E}\left[(Y-\hat{f}(X))^2 \mid X=x\right] \overset{?}{=} \underbrace{(f(x)-\hat{f}(x))^2}_{\text{Reducible}} + \underbrace{\mathrm{Var}[\varepsilon]}_{\text{Irreducible}}$$

$\longrightarrow$:

By our model:

$Y = f(X) + \varepsilon$ w/ ε independent from X
and $\mathbb{E}[\varepsilon]=0$

$$\mathbb{E}\left[(f(X)+\varepsilon-\hat{f}(X))^2 \mid X=x\right] = \text{(linearity of expectation)}$$

$$= \mathbb{E}\left[(f(X)-\hat{f}(X))^2 \mid X=x\right]$$

$$+ \boxed{2\,\mathbb{E}\left[(f(X)-\hat{f}(X))\cdot\varepsilon \mid X=x\right]} \quad (\varepsilon \text{ independent from } X)$$

$$+ \mathbb{E}\left[\varepsilon^2 \mid X=x\right]$$

$$= (f(x)-\hat{f}(x))^2$$

$$+ \boxed{\cancel{2\,\mathbb{E}\left[(f(X)-\hat{f}(X)) \mid X=x\right]\cdot\mathbb{E}[\varepsilon]}} = 0$$

$$+\mathrm{Var}[\varepsilon] \quad \square$$

In general, for any r.v. W

$$\mathrm{Var}[W] = \mathbb{E}[W^2] - (\mathbb{E}[W])^2$$

$$\mathbb{E}[W^2] = \boxed{\mathrm{Var}[W] + (\mathbb{E}[W])^2}$$

If we take $\varepsilon = W$ and note $\mathbb{E}[\varepsilon]=0$

$$\Rightarrow \quad \mathbb{E}[\varepsilon^2 \mid X=x] \underset{\substack{\uparrow \\ \text{independence}}}{=} \mathbb{E}[\varepsilon^2] = \mathrm{Var}[\varepsilon] + (\mathbb{E}[\varepsilon])^2 = \mathrm{Var}[\varepsilon]$$