M378K Introduction to Mathematical Statistics
Problem Set #16
Confidence intervals.

Problem 16.1. *Suppose that the thumb sizes of the US males are following a normal distribution with an unknown mean μ and standard deviation $\sigma = 20$ on the LDI - scale (*Lauretski's Digital Index - LDI - from 50 to 280*). The US Department of Thumbs and Toes (DTT) reports that the mean thumb size in the country is $\mu = 150$. Being the chairman of the Faculty of Thumbs of the local university you see an excellent opportunity here and decide to conduct your own study of the size of the average American thumb.*

(i) *After carefully collecting a random sample of 100 American thumbs you obtain the following sample mean: $\bar{x} = 153$. This result doesn't seem to be compatible with the DTT report so you decide to construct a 95%-confidence interval for the unknown parameter μ based on your study. What is your confidence interval?*

(ii) *Now, you dream about achieving fame and fortune by being the first person ever to estimate the mean thumb size up to ± 0.1. How large a sample size do you need for that?*

Solution:

(i) In this case, we are dealing with **exactly** the normal distribution. The confidence interval is

$$\bar{x} \pm \frac{z^*\sigma}{\sqrt{n}} = 153 \pm \frac{1.96(20)}{\sqrt{100}} = 153 \pm 3.92 \text{ i.e. } (149.08, 156.92)$$

.

(ii) With the given *margin of error* of $m = 0.1$, we get

$$n = \left(\frac{z^*\sigma}{m}\right)^2 = \left(\frac{1.96(20)}{0.1}\right)^2 = 153664.$$

Problem 16.2. Gallup's inaugural measure of global loneliness shows over one in five people worldwide (23%) said they felt loneliness "a lot of the day yesterday."[1] *However, there were considerable variations between countries. For instance, out of* 1000 *individuals polled in Taiwan, 11% reported having felt loneliness "a lot of the day" before. What 90%-confidence would you report for the population proportion of Taiwanese who had felt lonely the day before?*

Solution: Evidently, the point estimate is $\hat{p} = 0.11$. With the 1000 sample size, our standard error will be

$$\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \sqrt{\frac{0.11(0.89)}{1000}} = 0.009894443.$$

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[1] https://news.gallup.com/poll/646718/people-worldwide-feel-lonely-lot.aspx

The given confidence level is 90%. So, the associated critical value equals

$$z^* = \Phi^{-1}(0.95) = 1.645.$$

The margin of error is, hence,

$$1.645(0.009894443) = 0.01627636$$

Our confidence interval is

$$p = 0.11 \pm 0.01627636$$

Problem 16.3. *What is the unbiased estimator for σ^2.*

Solution:

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y}_n)^2$$

Problem 16.4. *Assume a random sample $Y_1, Y_2, \dots, Y_n$ from a normal distribution with mean μ and standard deviation σ - both unknown. What's the distribution of*

$$\frac{\sum_{i=1}^{n}(Y_i - \bar{Y}_n)^2}{\sigma^2} = \frac{(n-1)S^2}{\sigma^2}\ ?$$

Solution: It's $\chi^2(n-1)$.

Problem 16.5. *Assume that you are assigned a* confidence level $1-\alpha$. *What does it mean to find a* confidence interval *for S^2?*

Solution: We need to find $\hat{\chi}^2_L$ and $\hat{\chi}^2_U$ such that

$$\mathbb{P}[\hat{\chi}^2_L \leq \frac{(n-1)S^2}{\sigma^2} \leq \hat{\chi}^2_U] = 1-\alpha$$

Problem 16.6. *Are $\hat{\chi}^2_L$ and $\hat{\chi}^2_U$ as above uniquely defined?*

Solution: Nope!

So, we really seek $\hat{\chi}^2_L = \hat{\chi}^2_{1-(\alpha/2)}$ and $\hat{\chi}^2_U = \hat{\chi}^2_{(\alpha/2)}$ such that

$$\mathbb{P}\left[\frac{(n-1)S^2}{\sigma^2} \leq \hat{\chi}^2_{1-(\alpha/2)}\right] = \mathbb{P}\left[\frac{(n-1)S^2}{\sigma^2} \geq \hat{\chi}^2_{(\alpha/2)}\right] = \alpha/2\,.$$

Problem 16.7. *What's the form of the confidence interval, then?*

Solution:

$$\left(\frac{(n-1)S^2}{\hat{\chi}^2_{(\alpha/2)}}, \frac{(n-1)S^2}{\hat{\chi}^2_{1-(\alpha/2)}} \right)$$

Problem 16.8. *Assume the above setting. Let the random sample be of size $n = 9$. You do the arithmetic and arrive at the estimate $s^2 = 7.93$ (based on the data set). Using the above procedure, find the $90\%-$confidence interval for σ^2.*

Solution: Here, $\alpha = 1 - 0.90 = 0.10$. For $n = 9$, there are $n - 1 = 8$ degrees of the freedom to the χ^2 distribution. We look at the χ^2-table and find:

$$\hat{\chi}^2_{(\alpha/2)} = \hat{\chi}^2_{(0.05)} = 15.5, \quad \hat{\chi}^2_{1-(\alpha/2)} = \hat{\chi}^2_{(0.95)} = 2.73.$$

The confidence interval is

$$\left(\frac{8 \times 7.93}{15.5}, \frac{8 \times 7.93}{2.73} \right) = (4.09, 23.24)\,.$$

Problem 16.9. *(20 points)*
A random sample of size 10 is drawn from a normal distribution with **both** *mean and standard deviation* **unknown.** *The generated sample has the sample mean $\bar{y}_{10} = 14$ and the (unbiased) estimate of the variance $s^2 = 25$.*

(i) (10 points) Construct a (symmetric) 90%-confidence interval for μ.

(ii) (10 points) Construct a (symmetric) 90%-confidence interval for σ^2.
Hint: *Remember that you know the distribution of $(n-1)S^2/\sigma^2$.*

Solution:

(i) We have learned in class that

$$\frac{\bar{Y}_n - \mu}{S/\sqrt{n}} \sim t(df = n-1).$$

So, in this problem

$$\frac{\bar{Y}_{10} - \mu}{S/\sqrt{n}} \sim t(df = 9).$$

Looking at the $t-$distribution table for 9 degrees of freedom, we see that

$$t_{\alpha/2} = t_{0.05} = 1.833.$$

So, the desired confidence interval can be expressed as

$$14 \pm 1.833 \times \frac{5}{\sqrt{10}} = 14 \pm 2.8982.$$

(ii) We have learned in class that

$$(n-1)S^2/\sigma^2 \sim \chi^2(df = n-1).$$

In our problem, we have that $n = 10$, and so the χ^2-distribution has 9 degrees of freedom. From the χ^2-tables, we conclude that

$$\chi^2_{0.05} = 16.9190 \quad \text{and} \quad \chi^2_{0.95} = 3.32511.$$

The required confidence interval is

$$\left(\frac{9 \times 25}{16.919}, \frac{9 \times 25}{3.32511}\right) = (13.2987, 67.6669) = (13.32, 67.67).$$