M339D: March 8^{th}, 2024.

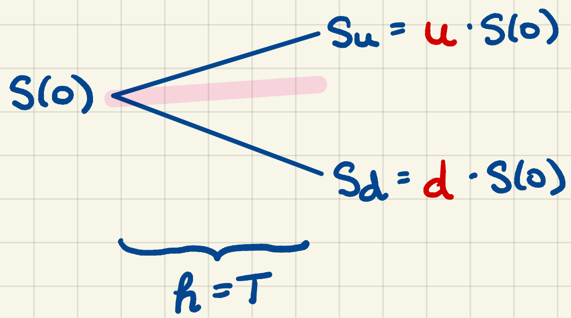
The Binomial Asset Pricing Model.

The No-Arbitrage Condition.

Market Model.

- riskless asset: @ the ccrfir r
- risky asset: non-dividend-paying stock

Imagine investing in one share of this stock @ time·0:



Wealth.

S_u

S_d

At the risk-free interest rate $S(0)$ accumulates to $S(0)e^{rh}$ at time h.

The no-arbitrage condition

$$S_d < S(0)e^{rh} < S_u$$

$$d \cdot \cancel{S(0)} < \cancel{S(0)}e^{rh} < u \cdot \cancel{S(0)}$$

$$\boxed{d < e^{rh} < u}$$

Half-a-Proof.

Say, to the contrary, $e^{rh} \le d < u$

Propose: Long one share of stock

Verify: Profit = Payoff − FV(Initial Cost) = $S(h) - S(0)e^{rh}$

In the down node: $S_d - S(0)e^{rh} = S(0) \cdot d - S(0)e^{rh} = S(0)(d - e^{rh}) \ge 0$

In the up node: $S_u - S(0)e^{rh} = S(0) \cdot u - S(0)e^{rh} = S(0)(u - e^{rh}) > 0$

Indeed, this is an arbitrage portfolio.

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Problem Set #9

Binomial option pricing.

Problem 9.1. In the setting of the one-period binomial model, denote by i the **effective** interest rate **per period**. Let u denote the "up factor" and let d denote the "down factor" in the stock-price model.
If

$$d < u \leq 1 + i$$

then there certainly is no possibility for arbitrage.

F

Fixed statement: $\boxed{d < 1+i < u}$

Problem 9.2. In our usual notation, does this parameter choice create a binomial model with an arbitrage opportunity?

$u = 1.18, \quad d = 0.87, \quad r = 0.05, \quad \cancel{\delta = 0}, \quad h = 1/4$

$\rightarrow$: $d = 0.87 \overset{?}{<} e^{rh} = \underbrace{e^{0.05(0.25)}}_{\approx}\overset{?}{<} 1.18 = u$

1.0125

Taylor expansion of e^x

$$\boxed{e^x = 1 + x} + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

Q: What if $\tilde{d} = 1.01$?

It still works.

Forward Binomial Tree.

$$
\begin{tikzcd}
 & S_u = u \cdot S(0) \\
S(0) \arrow[ur] \arrow[dr] & \\
 & S_d = d \cdot S(0)
\end{tikzcd}
$$

$\underbrace{\qquad}_{h=T}$

The no-arbitrage condition:

$$d < e^{rh} < u$$

$\boxed{u, d = ?}$

"Def'n". The volatility σ is the standard deviation of realized returns on a continuously compounded scale and annualized.

Heuristics:

$T=1$

$h_m = \frac{1}{m}$ (of a year)

Q: What is the volatility for the time period of length h_m? Call this volatility σ_{h_m} ✓

Realized Return: $R(t, t+s)$ satisfies

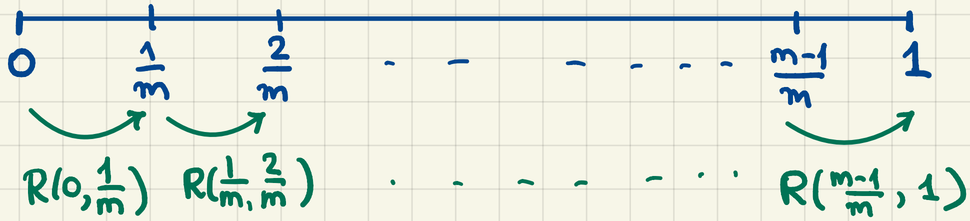
$$S(t+s) = S(t)e^{R(t,t+s)},$$

or, equivalently,

$$\boxed{R(t,t+s) = \ln\left(\frac{S(t+s)}{S(t)}\right)}$$

Compare to simple returns:

$$\frac{S(t+s) - S(t)}{S(t)} = \frac{S(t+s)}{S(t)} - 1$$



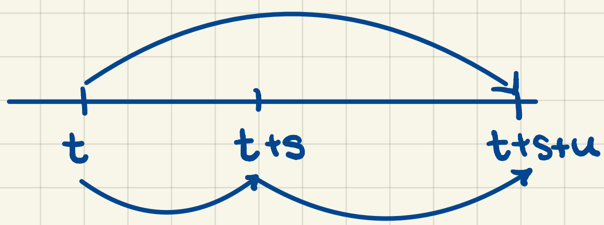
<u>Note</u>:

$$\boxed{\begin{array}{c} R\left(\frac{k-1}{m},\frac{k}{m}\right) \quad \text{for} \quad k=1,2,\dots,m \\ \text{are all random variables} \end{array}}$$

<u>We make the following assumptions:</u>

- all the returns are identically distributed;
- the returns over disjoint intervals are independent

We also know that realized returns defined as above are additive, i.e.,



$$R(t,t+s+u) = \ln\left(\frac{S(t+s+u)}{S(t)}\right) =$$

$$= \ln\left(\frac{S(t+s+u)}{S(t+s)}\cdot\frac{S(t+s)}{S(t)}\right)$$

$$= \underbrace{\boxed{\ln\left(\frac{S(t+s+u)}{S(t+s)}\right)}}_{R(t+s,t+s+u)} + \underbrace{\boxed{\ln\left(\frac{S(t+s)}{S(t)}\right)}}_{R(t,t+s)}$$

$$= R(t,t+s) + R(t+s,t+s+u)$$

$$\Rightarrow \boxed{R\left(0,\tfrac{1}{m}\right) + R\left(\tfrac{1}{m},\tfrac{2}{m}\right) + \dots + R\left(\tfrac{m-1}{m},1\right) = R(0,1)}$$

Q: $\mathrm{Var}[R(0,1)] = \cancel{X} = \sigma^2$

$$\Rightarrow \sigma^2 = \mathrm{Var}[R(0,1)] = \mathrm{Var}\left[R(0,\tfrac{1}{m}) + R(\tfrac{1}{m},\tfrac{2}{m}) + \dots + R(\tfrac{m-1}{m},1)\right]$$

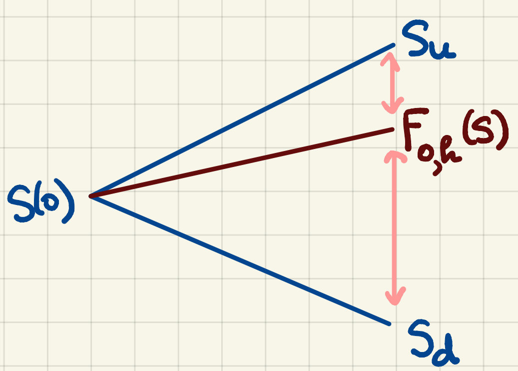
$$\underset{\text{independence}}{=} \mathrm{Var}[R(0,\tfrac{1}{m})] + \mathrm{Var}[R(\tfrac{1}{m},\tfrac{2}{m})] + \dots + \mathrm{Var}[R(\tfrac{m-1}{m},1)]$$

$$\underset{\text{identically dist'd}}{=} m \cdot \mathrm{Var}[R(0,\tfrac{1}{m})] = m \cdot \sigma_{h_m}^2$$

$$\sigma^2 = m \cdot \sigma_{h_m}^2$$

$$\Rightarrow \boxed{\sigma_{h_m} = \sigma\sqrt{\tfrac{1}{m}} = \sigma\sqrt{h_m}}$$

We generalize this identity to arbitrary lengths h: $\boxed{\sigma_h = \sigma\sqrt{h}}$



Recall: $\boxed{F_{0,h}(S) = S(0)e^{rh}}$

$$S_u := F_{0,h}(S)\, e^{\sigma\sqrt{h}} = S(0)\, e^{rh} \cdot e^{\sigma\sqrt{h}} = S(0)\, \boxed{e^{rh+\sigma\sqrt{h}}} = S(0)\, u$$

$$S_d := F_{0,h}(S)\, e^{-\sigma\sqrt{h}} = S(0)\, e^{rh} \cdot e^{-\sigma\sqrt{h}} = S(0)\, \boxed{e^{rh-\sigma\sqrt{h}}} = S(0)\, d$$