M339D: October 21st, 2024.

Forward Binomial Tree.

"Def'n". The **volatility σ** is the standard deviation of realized returns on a continuously compound scale and annualized.

Heuristics. $T = 1$

$h_m = \frac{1}{m}$ (of a year)

Q: What is the volatility for the time period of length h_m?
Call this volatility $\boxed{\sigma_{h_m}}$

Realized Returns.

$t \qquad t+s$

$R(t, t+s)$ satisfies

$$\boxed{S(t+s) = S(t)e^{R(t,t+s)}}$$

or, equivalently,

$$\boxed{R(t, t+s) = \ln\left(\frac{S(t+s)}{S(t)}\right)}$$

Compare to Simple Returns:

$$\frac{S(t+s) - S(t)}{S(t)} = \frac{S(t+s)}{S(t)} - 1$$

$0 \quad \frac{1}{m} \quad \frac{2}{m} \quad \cdots \quad \frac{k}{m} \quad \cdots\cdots \quad \frac{m-1}{m} \quad 1 = \frac{m}{m}$

$R(0,\frac{1}{m}) \quad R(\frac{1}{m},\frac{2}{m}) \quad \cdots \quad R(\frac{m-1}{m}, 1)$

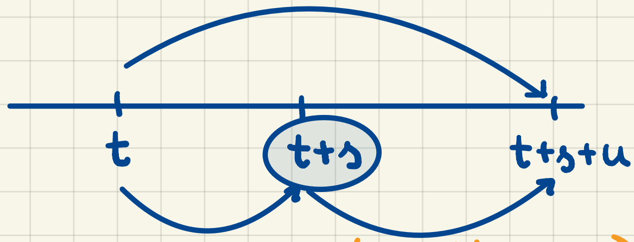
Note: $\boxed{R\left(\frac{k-1}{m}, \frac{k}{m}\right) \text{ for } k = 1..m \text{ are all Random Variables.}}$

We make the following assumptions:

- all the returns are $\boxed{\text{identically distributed}}$,
- the returns over disjoint intervals are independent.

Since their time intervals are equal

We also know that realized returns defined as above are additive, i.e.,



$$R(t, t+s+u) = \ln\left(\frac{S(t+s+u)}{S(t)}\right)$$

$$= \ln\left(\frac{S(t+s)}{S(t)} \cdot \frac{S(t+s+u)}{S(t+s)}\right)$$

$$= \boxed{\ln\left(\frac{S(t+s)}{S(t)}\right)} + \boxed{\ln\left(\frac{S(t+s+u)}{S(t+s)}\right)}$$

$$= R(t, t+s) + R(t+s, t+s+u)$$

$$\boxed{R\left(0, \tfrac{1}{m}\right) + R\left(\tfrac{1}{m}, \tfrac{2}{m}\right) + \cdots + R\left(\tfrac{m-1}{m}, 1\right) = R(0,1)}$$

Q: $\mathrm{Var}[R(0,1)] = \sigma^2$

$$\Rightarrow \sigma^2 = \mathrm{Var}\left[R\left(0, \tfrac{1}{m}\right) + \cdots + R\left(\tfrac{m-1}{m}, 1\right)\right] = \text{(independence)}$$

$$= \mathrm{Var}\left[R\left(0, \tfrac{1}{m}\right)\right] + \cdots + \mathrm{Var}\left[R\left(\tfrac{m-1}{m}, 1\right)\right] = \text{(identically dist'd)}$$

$$= m \cdot \mathrm{Var}\left[R\left(0, \tfrac{1}{m}\right)\right] = m \cdot \sigma_{h_m}^2$$

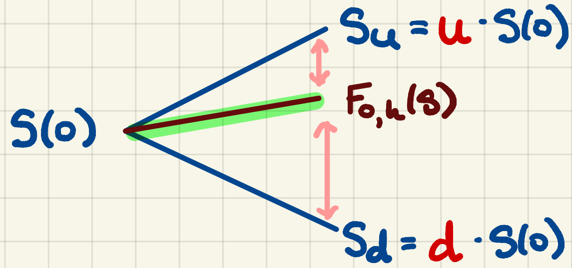
$$\Rightarrow \sigma_{h_m}^2 = \frac{1}{m}\sigma^2$$

$$\boxed{\sigma_{h_m} = \sigma\sqrt{\frac{1}{m}} = \sigma\sqrt{h_m}}$$

We generalize this identity to arbitrary lengths h:

$$\sigma_h = \sigma\sqrt{h}$$

Back to the forward binomial tree:



Recall: $\boxed{F_{0,h}(S) = S(0)e^{rh}}$

$$S_u := F_{0,h}(S)e^{\sigma\sqrt{h}} = S(0)e^{rh} \cdot e^{\sigma\sqrt{h}} = S(0)\underbrace{\boxed{e^{rh+\sigma\sqrt{h}}}}_{=u}$$

$$S_d := F_{0,h}(S)\, e^{-\sigma\sqrt{h}} = S(0)e^{rh} \cdot e^{-\sigma\sqrt{h}} = S(0)\underbrace{\boxed{e^{rh-\sigma\sqrt{h}}}}_{=d}$$

Q: Do u and d satisfy the no-arbitrage condition?

$\longrightarrow$:

$$d < e^{rh} < u$$

$$\cancel{e^{rh}} \cdot e^{-\sigma\sqrt{h}} < \cancel{e^{rh}} < \cancel{e^{rh}} \cdot e^{\sigma\sqrt{h}}$$

$$\Longleftrightarrow$$

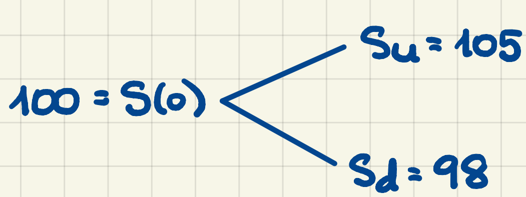
$$\boxed{e^{-\sigma\sqrt{h}} < 1 < e^{\sigma\sqrt{h}}} \qquad \sigma > 0$$

Q: What is S_u/S_d?

$\longrightarrow$:

$$\frac{S_u}{S_d} = \frac{\cancel{S(0)}e^{\cancel{rh}+\sigma\sqrt{h}}}{\cancel{S(0)}e^{\cancel{rh}-\sigma\sqrt{h}}} = \boxed{e^{2\sigma\sqrt{h}}}$$

Example. Consider this one-period forward binomial tree w/ the time horizon of one quarter-year.



Q: What is the volatility?

$\longrightarrow$: $\frac{S_u}{S_d} = e^{2\sigma\sqrt{h}} \Rightarrow \ln\left(\frac{105}{98}\right) = 2\sigma\sqrt{\frac{1}{4}} = \sigma$ $\square$