M339J: April 21st, 2021.

Collective Risk Model.

independent

- N ... frequency random variable : $\mathbb{N}_0$-valued w/ $P_N(\cdot)$ as its pgf
- $\{X_1, X_2, \dots\}$ ... severity i.i.d. r.v.s. w/ $M_X(\cdot)$ as their mgf (if continuous) or $P_X(\cdot)$ as their pgf (if discrete)

For the aggregate loss $\boxed{S = X_1 + \cdots + X_N}$

we have:

- $P_S(z) = P_N(P_X(z))$ if X discrete

  or

  $M_S(z) = P_N(M_X(z))$ if X continuous

- $\mathbb{E}[S]$ = Wald's Identity = $\mathbb{E}[N] \cdot \mathbb{E}[X]$
- $\mathrm{Var}[S] = \mathbb{E}[N] \cdot \mathrm{Var}[X] + \mathrm{Var}[N] \cdot (\mathbb{E}[X])^2$

32. For an individual over 65:

$N \sim \mathbb{P}\text{oisson}(\lambda=25)$

(i) The number of pharmacy claims is a Poisson random variable with mean 25.

$X \sim U(5,95)$

(ii) The amount of each pharmacy claim is uniformly distributed between 5 and 95.

(iii) The amounts of the claims and the number of claims are mutually independent.

Determine the probability that aggregate claims (S) for this individual will exceed 2000 using the
normal approximation.

(A) $1-\Phi(1.33)$

(B) $1-\Phi(1.66)$

(C) $1-\Phi(2.33)$

(D) $1-\Phi(2.66)$

(E) $1-\Phi(3.33)$

$S = X_1 + X_2 + \cdots + X_N$

$$\mathbb{P}[S > 2000] = \mathbb{P}\left[\underbrace{\frac{S-\mu_S}{\sigma_S}}_{\approx N(0,1)\sim Z} > \frac{2000-\mu_S}{\sigma_S}\right]$$

$$\approx \mathbb{P}\left[Z > \frac{2000-\mu_S}{\sigma_S}\right]$$

$$= 1 - \Phi\left(\frac{2000-\mu_S}{\sigma_S}\right)$$

- $\mu_S = \mathbb{E}[N]\cdot\mathbb{E}[X] = 25\cdot\left(\frac{5+95}{2}\right) = 25\cdot 50 = 1250$

- $\mathrm{Var}[S] = \mathbb{E}[N]\cdot\boxed{\mathrm{Var}[X]} + \mathrm{Var}[N]\cdot(\mathbb{E}[X])^2$

$$\mathrm{Var}[X] = \frac{(95-5)^2}{12} = 675$$

$$\mathrm{Var}[S] = 25(675 + 50^2) = 25(675+2500) = 79375$$

$$\Rightarrow \quad \sigma_S = 281.7357$$

$$\frac{2000-1250}{281.7357} = 2.6621 \Rightarrow (D)$$

"Def'n": Insurance on the aggregate losses, subject to a deductible is called stop·loss insurance.

The expected cost of this insurance is called the net stop·loss premium.

Note:

- $\mathbb{E}[(S-d)_+]$ is the expression for the net stop·loss premium.
- Using the tail formula for the expectation:

$$\mathbb{E}[(S-d)_+] = \int_d^{+\infty} (1-F_S(x))\,dx \quad ✓$$

- For S continuous:

$$\mathbb{E}[(S-d)_+] = \int_d^{+\infty} (x-d)\, f_S(x)\,dx \quad ✓$$

- For S discrete:

$$\mathbb{E}[(S-d)_+] = \sum_{x>d} (x-d)\, p_S(x) \quad ✓$$

Useful in problems:

- $\mathbb{E}[(S-d)_+] = \mathbb{E}[S] - \mathbb{E}[S \wedge d]$
- combinatorics ☺

11. A company provides insurance to a concert hall for losses due to power failure. You are given:

(i) The number of power failures in a year has a Poisson distribution with mean 1.

$N \sim \text{Poisson}(\lambda = 1)$

(ii) The distribution of ground up losses due to a single power failure is:

x	Probability of x
10	0.3
20	0.3
50	0.4

pmf of X: $p_X(\cdot)$

$S = X_1 + \cdots + X_N$

(iii) The number of power failures and the amounts of losses are independent.

(iv) There is an annual deductible of 30. $d = 30$

Calculate the expected amount of claims paid by the insurer in one year.

$$\mathbb{E}[(S-30)_+] = \mathbb{E}[S] - \mathbb{E}[S \wedge 30]$$

(A) 5

(B) 8

(C) 10

(D) 12

(E) 14

$$\mathbb{E}[S] = \underbrace{\mathbb{E}[N]}_{=1} \cdot \mathbb{E}[X] = 29$$

w/ $\mathbb{E}[X] = 0.3(10) + 0.3(20) + 0.4(50) = 29$

$\mathbb{E}[S \wedge 30] = ?$

Q: What is the support of S?

$\{0, 10, 20, 30, \dots\}$

Q: What is the support of $S \wedge 30$?

$\{0, 10, 20, 30\}$

$p_{S\wedge 30}(0) = p_S(0) = p_N(0) = e^{-1}$

$p_{S\wedge 30}(10) = p_S(10) = \mathbb{P}[N=1, X_1 = 10] = p_N(1) \cdot p_X(10) = e^{-1} \cdot \underline{0.3}$ ✓

independence

$$p_{S\wedge 30}(20) = p_S(20) = \mathbb{P}[N=1,\ X_1=20] + \mathbb{P}[N=2,\ X_1=X_2=10]$$

$$= p_N(1)\cdot p_X(20) + p_N(2)\cdot\left(p_X(10)\right)^2$$

independence

$$= e^{-1}\cdot 0.3 + e^{-1}\cdot\frac{1}{2}\cdot(0.3)^2$$

$$= e^{-1}\cdot 0.3\cdot 1.15 = e^{-1}\cdot 0.345$$

Easiest way to get $p_{S\wedge 30}(30) = ?$