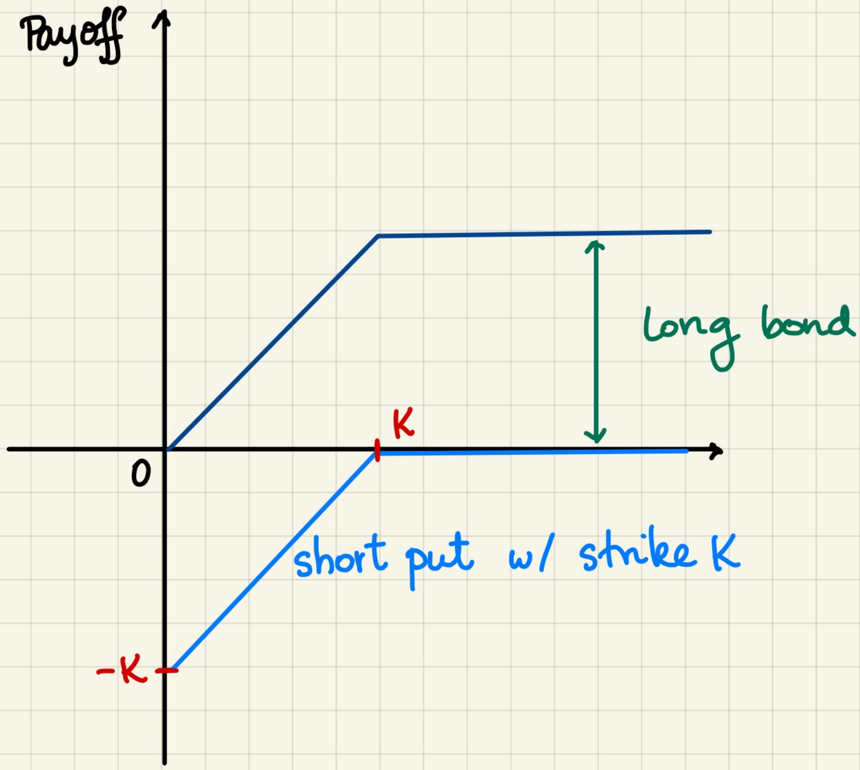
M339W: April 2nd, 2021.

Option Elasticity [review].

$$\Omega(s,t) := \frac{\Delta(s,t)\cdot s}{v(s,t)}$$



$T=1$

41. Assume the Black-Scholes framework. Consider a 1-year European contingent claim on a stock.

You are given:

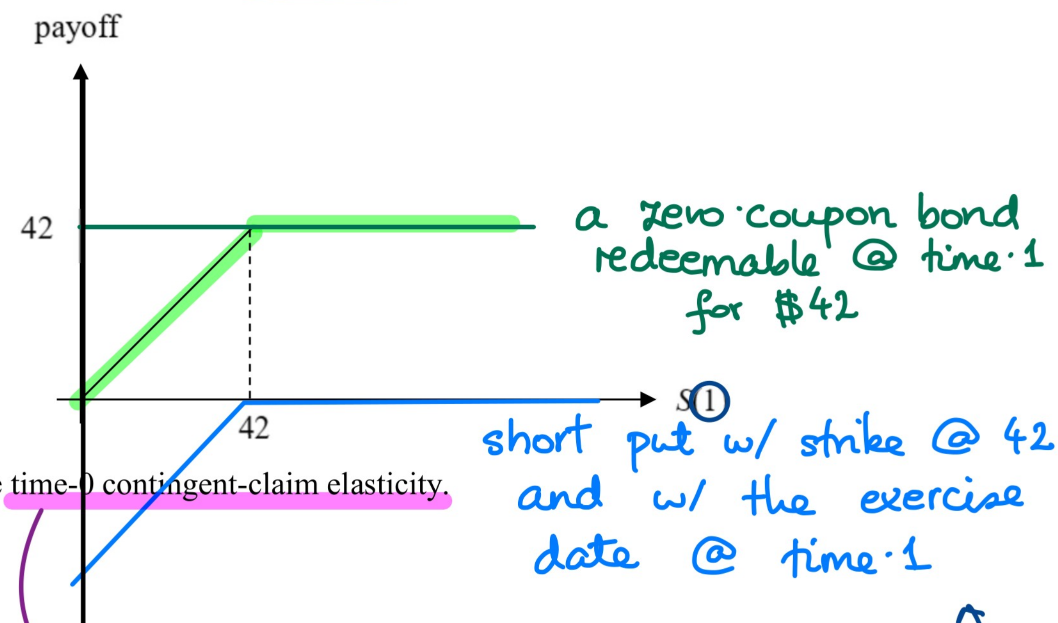
(i) The time-0 stock price is 45. $S(0)=45$

(ii) The stock's volatility is 25%. $\sigma=0.25$

(iii) The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 3%. $\delta=0.03$

(iv) The continuously compounded risk-free interest rate is 7%. $r=0.07$

(v) The time-1 payoff of the contingent claim is as follows:



Calculate the time-0 contingent-claim elasticity.

$\Omega(S(0),0)=?$

(A) 0.24

(B) 0.29

(C) 0.34

(D) 0.39

(E) 0.44

We designed our replicating portfolio

So, @ any (s,t):

the value function of our contingent claim is

$$v(s,t) = v_B(s,t) - v_P(s,t)$$

$$= \underbrace{42e^{-r(T-t)}} - v_P(s,t)$$

$$\frac{\partial}{\partial s}\Bigg| \quad \Delta(s,t) = 0 - \Delta_P(s,t) = e^{-\delta(T-t)}N(-d_1(s,t))$$

$$v(s,t) = 42e^{-r(T-t)} - \left(42e^{-r(T-t)} \cdot N(-d_2(s,t)) - se^{-\delta(T-t)} N(-d_1(s,t))\right)$$

$$= 42e^{-r(T-t)} N(d_2(s,t)) + se^{-\delta(T-t)} N(-d_1(s,t))$$

At time 0:

$$d_1(S(0),0) = \frac{1}{0.25\sqrt{1}}\left[\ln\left(\frac{45}{42}\right) + \left(0.07 - 0.03 + \frac{(0.25)^2}{2}\right)\cdot 1\right]$$

$$= 0.56097$$

$$d_2(S(0),0) = d_1(S(0),0) - \sigma\sqrt{T} = 0.31097$$

$$\Rightarrow N(-d_1(S(0),0)) = N(-0.56) = 1 - N(0.56) = \boxed{0.2877}$$

$$N(d_2(S(0),0)) = N(0.31) = \boxed{0.6217}$$

$$\Rightarrow v(S(0),0) = 42e^{-0.07}\cdot(0.6217) + 45e^{-0.03}\cdot(0.2877)$$

$$= 36.91$$

$$\Delta(S(0),0) = e^{-0.03}(0.2877) = 0.2797$$

$$\Rightarrow \text{Finally, } \Omega(S(0),0) = \frac{0.2797\cdot 45}{36.91} = 0.341 \quad \blacksquare$$

Q: What is the current volatility of this contingent claim?

$$\longrightarrow : \quad \sigma_{opt} = \sigma_S \cdot |\Omega|$$

$$\Rightarrow \text{At time } 0: \quad \sigma_{opt}(S(0),0) = 0.25 \cdot 0.341 = \underline{0.08525}$$

$\blacksquare$

The Gamma.

Γ... the second-order sensitivity of the portfolio price w/ respect to the perturbations in the price of the underlying asset, i.e.,

$$\Gamma(s,t) := \frac{\partial^2}{\partial s^2} v(s,t)$$

Example. [EUROPEAN CALL]

$$\Gamma_c(s,t) := \frac{\partial^2}{\partial s^2} v_c(s,t) = \frac{\partial}{\partial s}\Big(\underbrace{\frac{\partial}{\partial s} v_c(s,t)}_{\Delta_c(s,t)}\Big)$$

$$= \frac{\partial}{\partial s}\left(e^{-\delta(T-t)} \cdot N(d_1(s,t))\right)$$

$$= e^{-\delta(T-t)} \cdot \underbrace{\frac{\partial}{\partial s} N(d_1(s,t))}_{\text{use the chain rule}}$$

$$\underbrace{N'(d_1(s,t))}_{\varphi(d_1(s,t)) = f_Z(d_1(s,t))} \cdot \underbrace{\frac{\partial}{\partial s}(d_1(s,t))}_{?}$$

$$d_1(s,t) = \frac{1}{\sigma\sqrt{T-t}}\left[\ln(s) - \ln(K) + \left(r - \delta + \frac{\sigma^2}{2}\right)(T-t)\right]$$

$$\Rightarrow \frac{\partial}{\partial s} d_1(s,t) = \frac{1}{\sigma\sqrt{T-t}} \cdot \frac{1}{s}$$

$$\Rightarrow \Gamma_c(s,t) = e^{-\delta(T-t)} \cdot \varphi(d_1(s,t)) \cdot \frac{1}{s \cdot \sigma\sqrt{T-t}}$$

Q: What is the put's gamma?

$\longrightarrow$: Put-Call Parity.

$$v_c(s,t) - v_p(s,t) = s e^{-\delta(T-t)} - K e^{-r(T-t)}$$

$\frac{\partial}{\partial s}$

$$\Delta_c(s,t) - \Delta_p(s,t) = e^{-\delta(T-t)}$$

$\frac{\partial}{\partial s}$

$$\boxed{\Gamma_c(s,t) = \Gamma_p(s,t)}$$

Implied Volatility on the Wiki page.

Video demo on Option Greeks.