M339W/389W Financial Mathematics for Actuarial Applications
University of Texas at Austin
Practice Problems for In-Term Exam 2
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Notes: This is a closed book and closed notes exam.
Time: 50 minutes

TRUE/FALSE

1 (2)	TRUE	FALSE
2 (2)	TRUE	FALSE
3 (2)	TRUE	FALSE
4 (2)	TRUE	FALSE
5 (2)	TRUE	FALSE

MULTIPLE CHOICE

1 (5)	a	b	c	d	e
2 (5)	a	b	c	d	e
3 (5)	a	b	c	d	e
4 (5)	a	b	c	d	e
5 (5)	a	b	c	d	e
6 (5)	a	b	c	d	e

FOR GRADER'S USE ONLY:

T/F	1.	2.	M.C.	Σ

2.1. **TRUE/FALSE QUESTIONS.** *Please note your answers on the front page.*

2.2. **FREE-RESPONSE PROBLEMS.** Please, explain carefully all your statements and assumptions. Numerical results or single-word answers without an explanation (even if they're correct) are worth 0 points.

Problem 2.1. (10 points) Assume that $Y_1 = e^X$ where X is a standard normal random variable.

(i) (2 points) What is the probability that Y_1 exceeds 5?

(ii) $(3+5$ points) Find the mean and the variance of Y_1.

Hint: It helps if you use the expression for the moment generating function of a standard normal random variable.

Solution:

(i)

$$\mathbb{P}[Y_1 > 5] = \mathbb{P}[e^X > 5] = \mathbb{P}[X > \ln(5)] = 1 - N(\ln(5)) \approx 1 - N(1.61) = 1 - 0.9463 = 0.0537.$$

(ii)

$$\mathbb{E}[Y_1] = \mathbb{E}[e^X] = \mathbb{E}[e^{1 \cdot X}] = M_X(1)$$

where M_X denotes the moment generating function of X. In class, we recalled the following expression for M_X:

$$M_X(t) = e^{t^2/2}.$$

So, $\mathbb{E}[Y_1] = e^{1/2} = \sqrt{e}$.

The second moment of Y_1 is obtained similarly as

$$\mathbb{E}[Y_1^2] = \mathbb{E}[e^{2 \cdot X}] = M_X(2) = e^2.$$

So,

$$Var[Y_1] = \mathbb{E}[Y_1^2] - (\mathbb{E}[Y_1])^2 = e^2 - e = e(e-1).$$

Problem 2.2. (10 points) Let the stock prices be modeled using the lognormal distribution. The mean stock price at time-1 equals 120 and the median stock price 115. What is the probability that the time-1 stock price exceeds 100?

Solution: The stock price at time-1 is lognormally distributed. In fact, using our usual parameters, we can rewrite it as

$$S(1) = S(0)e^{(\alpha-\delta-\frac{1}{2}\sigma^2)+\sigma Z(1)}.$$

Recall that the median of $S(1)$ equals $S(0)e^{(\alpha-\delta-\frac{1}{2}\sigma^2)}$. So, the required probability can be expressed as

$$\begin{aligned}\mathbb{P}[S(1) > 100] = \mathbb{P}[115e^{\sigma Z(1)} > 100] &= \mathbb{P}\left[Z(1) > \frac{1}{\sigma}\ln\left(\frac{100}{115}\right)\right] \\ &= \mathbb{P}\left[Z(1) < \frac{1}{\sigma}\ln\left(\frac{115}{100}\right)\right] = N\left(\frac{1}{\sigma}\ln\left(\frac{115}{100}\right)\right).\end{aligned}$$

Since the mean of $S(1)$ equals $S(0)e^{(\alpha-\delta)}$, we have

$$e^{\frac{1}{2}\sigma^2} = \frac{120}{115} \quad \Rightarrow \quad \sigma = \sqrt{2\ln(1.04348)} = 0.2918.$$

So, our final answer is

$$\mathbb{P}[S(1) > 100] = N(0.48) = 0.6844.$$

Problem 2.3. (5 points) Assume the Black-Scholes model. The initial price of a continuous-dividend-paying stock is \$100. Its dividend yield is 0.03 and its volatility is 0.15. According to your model, the mean rate of return is 0.08.

The continuously compounded risk-free interest rate is 0.04.

Calculate the probability that the realized return for the time period $[0, 2]$ exceeds 0.06.

Solution: In our usual notation, the realized returns are normally distributed as

$$R(0,t) \sim Normal(mean = (\alpha - \delta - \tfrac{\sigma^2}{2})t, variance = \sigma^2 t).$$

In the present problem, we are focused on

$$R(0,2) \sim Normal(mean = (0.08 - 0.03 - \tfrac{(0.15)^2}{2})(2) = 0.0775, variance = (0.15)^2(2) = 0.045).$$

Finally, we calculate

$$\begin{aligned}\mathbb{P}[R(0,2) > 0.06] &= \mathbb{P}\left[\frac{R(0,2) - 0.0775}{\sqrt{0.045}} > \frac{0.06 - 0.0775}{\sqrt{0.045}}\right] \\ &= \mathbb{P}[Z > -0.08] = N(0.08) = 0.5319.\end{aligned}$$

Problem 2.4. (5 points) Assume the Black-Scholes model. Consider a non-dividend-paying stock with the intial stock price of \$100 and volatility equal to 0.30. According to your model, the stock's mean rate of return is 0.10. Find

$$\mathbb{E}[S(1)\mathbb{I}_{[S(1) \geq 105]}].$$

Solution: According to the work done in class,

$$\mathbb{E}[S(1)\mathbb{I}_{[S(1) \geq 105]}] = \mathbb{E}[S(1)]N(\hat{d}_1)$$

where

$$\hat{d}_1 = \frac{1}{\sigma\sqrt{1}}\left[\ln\left(\frac{100}{105}\right) + \left(0.10 + \frac{(0.3)^2}{2}\right)(1)\right] \approx 0.32.$$

So,

$$\mathbb{E}[S(1)\mathbb{I}_{[S(1)\geq 105]}] = 100e^{0.10}N(0.32) = 69.12844.$$

2.3. **MULTIPLE CHOICE QUESTIONS.**

Problem 2.5. (5 points) The current price of a continuous-dividend-paying stock is given to be \$92. The stock's volatility is 0.35 and its dividend yield is 0.02.

The continuously compounded risk-free interest rate is 0.05.

Consider a \$90-strike European call option on the above stock with exercise date in a quarter-year.

What is the Black-Scholes price of this call option?

(a) 5.05
(b) 7.66
(c) 7.71
(d) 7.89
(e) None of the above.

Solution: (d)

$$d_1 = 0.26, d_2 = 0.08.$$

So,

$$V_C(0) = 92e^{-0.02/4} \times 0.6026 - 90e^{-0.05/4} \times 0.5319 \approx 7.89.$$

Problem 2.6. (5 pts) Assume the Black-Scholes framework. Let the current price of a non-dividend-paying stock be equal to $S(0) = 95$ and let its volatility be equal to 0.35. Consider a European call on that stock with strike 100 and exercise date in 9 months. Let the risk-free continuously compounded interest rate be 6% per annum.

Denote the price of the call by $V_C(0)$. Then,

(a) $V_C(0) < \$5.20$
(b) $\$5.20 \leq V_C(0) < \$7.69$
(c) $\$7.69 \leq V_C(0) < \$9.04$
(d) $9.04 \leq V_C(0) < \$11.25$
(e) None of the above.

Solution: (d)
Using the Black-Scholes formula one gets the price of about 11.06.

Problem 2.7. Assume the Black-Scholes setting. Let $S(0) = \$63.75, \sigma = 0.20, r = 0.055$. The stock pays no dividend and the option expires in 50 days (simplify the number of days in a year to 360).

What is the price of a \$60-strike European put?

(a) 0.66

(b) 0.55
(c) 0.44
(d) 0.37
(e) None of the above.

Solution: (d)
In our usual notation, the price is

$$V_P(0) = Ke^{-r\cdot T}N(-d_2) - S(0)N(-d_1)$$

with

$$d_1 = \frac{1}{0.2\sqrt{5/36}}\left(\ln\left(\frac{63.75}{60}\right) + (0.055 + \frac{1}{2}0.2^2)\left(\frac{5}{36}\right)\right) = 0.95,$$
$$d_2 = d_1 - 0.25\sqrt{0.125} = 0.88.$$

So,

$$V_P(0) = 60e^{-0.055\cdot\frac{5}{36}}(1 - 0.8106) - 63.75\cdot(1 - 0.8289) = 0.37.$$

Problem 2.8. Assume the Black-Scholes setting. Assume $S(0) = \$28.50, \sigma = 0.32, r = 0.04$. The stock pays a 1.0% continuous dividend and the option expires in 110 days (simplify the number of days in a year to 360).
What is the price of a \$30-strike put?

(a) 2.75
(b) 2.10
(c) 1.80
(d) 1.20
(e) None of the above.

Solution: (a)
In our usual notation, the price is

$$V_P(0) = Ke^{-r\cdot T}N(-d_2) - S(0)e^{-\delta\cdot T}N(-d_1)$$

with

$$d_1 = -0.15, \quad d_2 = -0.33.$$

So, $V_P(0) = 2.75$.

Problem 2.9. (5 points) Let the current price of a continuous-dividend-paying stock be denoted by $S(0)$. We model the time$-T$ stock price as lognormal. The mean rate of return on the stock is 0.10, its dividend yield is 0.01, and its volatility is 0.20. The continuously compounded risk-free interest rate is 0.03. You invest in one share of stock at time-0. and simultaneously deposit an amount $\varphi S(0)$ in a savings account. Assume continuous and immediate reinvestment of all dividends in the same stock. What should the proportion φ be so that the VaR at the level 0.05 of your total wealth at time-1 equals today's stock price $S(0)$?

(a) $\varphi = 0.0573$
(b) $\varphi = 0.1966$
(c) $\varphi = 0.2139$
(d) $\varphi = 0.5$
(e) None of the above.

Solution: (c)
The total wealth at time-1 is equal to $e^{\delta}S(1) + \varphi S(0)e^{r}$. So, our condition on the VaR is

$$\mathbb{P}[e^{\delta}S(1) + \varphi S(0)e^{r} < S(0)] = 0.05.$$

The above is equivalent to

$$\mathbb{P}[S(0)e^{\alpha - \frac{\sigma^2}{2} + \sigma Z} + \varphi S(0)e^{r} < S(0)] = 0.05$$

with $Z \sim N(0,1)$. We have that the above is, in turn, equivalent to

$$\mathbb{P}[e^{\alpha - \frac{\sigma^2}{2} + \sigma Z} + \varphi e^{r} < 1] = 0.05.$$

The constant z^* such that $\mathbb{P}[Z < z^*] = 0.05$ equals -1.645. Therefore,

$$e^{\alpha - \frac{\sigma^2}{2} + \sigma(-1.645)} + \varphi e^{r} = 1.$$

So,

$$\varphi = e^{-r}\left(1 - e^{\alpha - \frac{\sigma^2}{2} + \sigma(-1.645)}\right) = e^{-0.03}\left(1 - e^{0.10 - \frac{(0.2)^2}{2} + 0.2(-1.645)}\right) = 0.2139.$$

Problem 2.10. (5 points) Assume that the stock price of a certain non-dividend-paying stock is modeled using the lognormal distribution, i.e., the Black-Scholes framework.

The time-0 delta of an at-the-money, time$-T$ European call option is 0.5557. What is the time-0 delta of an otherwise identical call option with exercise date $4T$?

(a) 0.3011
(b) 0.4145
(c) 0.5255
(d) 0.6103
(e) None of the above.

Solution: (d)

The delta of a time$-T$ European call option in the Black-Scholes setting can be expressed as

$$\Delta_C(s,t) = e^{-\delta(T-t)}N(d_1(s,t))$$

with

$$d_1(s,t) = \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{s}{K}\right) + \left(r - \delta + \frac{1}{2}\sigma^2\right)(T-t)\right].$$

In the present problem, we get

$$d_1(s,0) = N^{-1}(0.5557) = 0.14.$$

On the other hand, for an at-the money call option with exercise at time$-T$, we have

$$d_1(s,0) = \frac{r - \delta + \frac{1}{2}\sigma^2}{\sigma}\sqrt{T}$$

Similarly, for an at-the-money call with exercise date at $4T$, we get

$$\tilde{d}_1(s,0) = \frac{r - \delta + \frac{1}{2}\sigma^2}{\sigma}\sqrt{4T} = 2d_1(s,0) = 2 \times 0.14 = 0.28 \approx 0.20.$$

So, the Δ we are looking for equals $N(0.28) = 0.6103$.

Problem 2.11. (5 points) Assume the Black-Scholes framework. The current price of a certain stock is \$50 per share. Its dividend yield is 0.04 and its volatility is 0.14.

The continuously compounded risk-free interest rate is 0.02.

What is the current delta of a European, \$43.75-strike, six-year put on the above stock?

(a) -0.13
(b) -0.23
(c) -0.33
(d) -0.45
(e) None of the above.

Solution: (c)
In our usual notation,

$$d_1(S(0),0) = \frac{1}{0.14\sqrt{6}}\left[\ln\left(\frac{50}{43.75}\right) + \left(0.02 - 0.04 + \frac{0.14^2}{2}\right)(6)\right] = 0.21.$$

The put's current delta is

$$\Delta_P(S(0),0) = -e^{-0.04(6)}N(-0.21) = -e^{-0.24}(0.4168) = -0.327866.$$

Problem 2.12. (5 pts) Let the stochastic process $S = \{S(t); t \geq 0\}$ denote the stock price. The stock's rate of appreciation is 10% while its volatility is 0.30 Then,

(a) $Var[\ln(S(t))] = 0.3t$
(b) $Var[\ln(S(t))] = 0.09t^2$
(c) $Var[\ln(S(t))] = 0.09t$
(d) $Var[\ln(S(t))] = 0.09$
(e) None of the above.

Solution: (c)
The random variable $S(t)$ is lognormal so that the random variable $\ln(S(t))$ is normal with variance $0.3^2 t = 0.09t$.