Example 10.1.1. Consider the following two data sets, both consisting of measurements of the same quantity (say the distance to Proxima Centauri) and in the same units, but made with two different methods.

method 1: 4.51, 4.52, 4.48, 4.49, 4.47, 4.53
method 2: 14.12, 1.30, 0.40, 2.50, 1.00, 3.18

If we use the sample mean $\bar{Y}$ as the (point) estimator for the "true" mean μ, both of these data sets yield the same result, namely $\bar{Y} = 4.5$. It is clear, however, that the first method is more accurate and that, in general, one should trust the results produced by method 1 more than those obtained by method 2.

M378K : March 24th, 2025.

Point vs. Interval Estimators.

An interval estimator is a pair

$$\hat{\theta}_L \leq \hat{\theta}_R$$

of point estimators.

Good traits:

- being narrow
- containing the true parameter θ "w/ a high probability"

First: Pick a confidence level $C = 0.95, 0.99$, another probability close to 1

or pick a significance level $\alpha = 0.05, 0.01$, another probability close to 0

Convention: $\boxed{C = 1-\alpha}$

The purpose:

$$\boxed{\mathbb{P}\left[\hat{\theta}_L \leq \theta \leq \hat{\theta}_R\right] = 1-\alpha = C}$$

Def'n. Consider a random sample $(Y_1, Y_2, \ldots, Y_n)$ from a distribution D which is parameterized by an unknown parameter θ.
A pivotal quantity is a function of the data $(Y_1, Y_2, \ldots, Y_n)$ and the parameter θ whose distribution does not depend on θ.

Example. Say that $Y_i \sim N(\mu, 1)$, $i = 1..n$, is our random sample

$\bar{Y}$ is a great estimator for μ.

$\bar{Y} \sim \text{Normal}(\mu, \frac{1}{\sqrt{n}}) \Rightarrow \bar{Y}$ is NOT a PIVOTAL QUANTITY

BUT $\boxed{\bar{Y} - \mu}$ is a PIVOTAL QUANTITY!

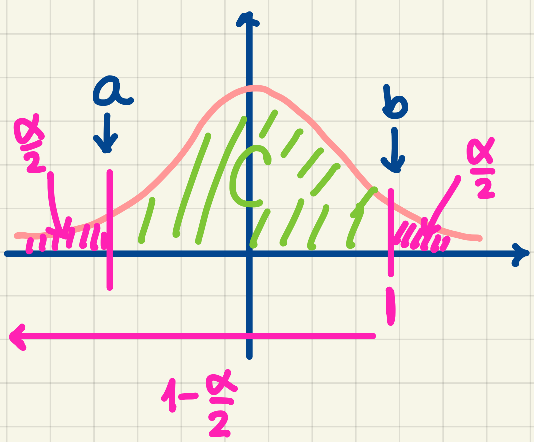
$$\boxed{\bar{Y} - \mu \sim \text{Normal}(0, \tfrac{1}{\sqrt{n}})}$$

Example. Consider a random sample $(Y_1, \dots, Y_n)$ from $\text{Normal}(\mu, \sigma)$ w/ both parameters unknown

$$\boxed{\frac{\bar{Y} - \mu}{\frac{\sigma}{\sqrt{n}}}} \sim \text{Normal}(0,1)$$

is a PIVOTAL QUANTITY for $\theta = (\mu, \sigma)$

Recipe.

1. $\boxed{\text{Consider}}$ a pivotal quantity U
2. 

Define $a = q_U(\alpha/2)$, $b = q_U(1-\alpha/2)$
w/ q_U the quantile function of U

Because: $\mathbb{P}[a \leq \boxed{U} \leq b] = C = 1-\alpha$

Has θ in its expression!

3. **Algebra:**

"Undo" the "rule" for U so that the final expression looks like

$$\hat{\theta}_L \leq \theta \leq \hat{\theta}_R$$

$\boxed{\text{Works for 1D parameters!}}$

Example. $Y_1, \dots, Y_n$ a random sample from $N(\mu, \sigma)$ w/ μ unknown and σ known

① Propose a pivotal quantity:

$$\boxed{U = \frac{\bar{Y} - \mu}{\frac{\sigma}{\sqrt{n}}}} \sim N(0,1)$$

② Confidence level : $C = 0.95$

Significance level: $\alpha = 0.05$

$$a = q_U(\alpha/2) = -1.96 = \text{qnorm}(0.025) =: -z^*$$

$$b = q_U(1-\alpha/2) = 1.96 = \text{qnorm}(0.975) =: z^*$$

③

$$\mathbb{P}[a \le U \le b] = 0.95$$

$$\mathbb{P}\left[a \le \frac{\bar{Y}-\mu}{\frac{\sigma}{\sqrt{n}}} \le b\right] = 0.95$$

$$\mathbb{P}\left[-z^* \cdot \frac{\sigma}{\sqrt{n}} \le \bar{Y} - \mu \le z^* \cdot \frac{\sigma}{\sqrt{n}}\right] = 0.95$$

$$\mathbb{P}\left[-z^* \cdot \frac{\sigma}{\sqrt{n}} - \bar{Y} \le -\mu \le z^* \cdot \frac{\sigma}{\sqrt{n}} - \bar{Y}\right] = 0.95$$

$$\mathbb{P}\left[\underbrace{\bar{Y} - \boxed{z^* \cdot \frac{\sigma}{\sqrt{n}}}}_{\hat{\theta}_L} \le \mu \le \underbrace{\bar{Y} + z^* \cdot \frac{\sigma}{\sqrt{n}}}_{\hat{\theta}_R}\right] = 0.95$$

The anatomy of a normal confidence interval for mean:

pt. estimate $\pm$ margin of error

$z^* \cdot \frac{\sigma}{\sqrt{n}}$ σ ... population, $\sqrt{n}$ ... sample size

z^* $\Updownarrow$ choice of C