UNIVERSITY OF TEXAS AT AUSTIN

Problem Set # 7

The inverse transform method.

Problem 7.1. *Source: Course 3, November 1985, Problem #19.*
Your goal is to simulate four draws from a binomial distribution with two trials and the probability of success in every trial equal to 0.30. You intend to use the inverse transform method. Here are the four values produced by the random number generator:

$$0.90 \quad 0.21 \quad 0.72 \quad 0.48.$$

Which values of the binomial were obtained from the above outputs of the random number generator?

Solution: Let X denote the random variable in the problem. The distribution of X is given to be binomial. More precisely, in the usual notation, we have

$$X \sim Binomial(n = 2, p = 0.30).$$

The probability mass function of X is

$$p_X(0) = (0.7)^2 = 0.49, \quad p_X(1) = 2(0.7)(0.3) = 0.42, \quad p_X(2) = (0.3)^2 = 0.09.$$

So, the cumulative distribution function of X is

$$F_X(x) = \begin{cases} 0, & \text{for } x < 0, \\ 0.49, & \text{for } 0 \leq x < 1, \\ 0.91, & \text{for } 1 \leq x < 2, \\ 1, & \text{for } x \geq 2. \end{cases}$$

If the drawn value of the random number generator is **less**[1] than 0.49, that value maps into a simulated value of 0 for X. So, in the present problem, both 0.21 and 0.48 will generate a 0.

If the drawn value of the random number generator is greater than or equal to 0.49 and less than 0.91, that value maps into a simulated value of 1 for X. So, in our problem, 0.78 and 0.90 both map to 1.

Problem 7.2. Let the random variable X have the following density function:

$$f_X(x) = 3x^{-4}, \quad x > 1$$

You use the *inverse transform method* to simulate values from X. Let the simulated value of the unit uniform be equal to 0.25. What is the corresponding value of X?

Solution: The cumulative distribution function of X is

$$F_X(x) = \begin{cases} 0, & \text{for } x \leq 1, \\ 1 - x^{-3}, & \text{for } x > 1. \end{cases}$$

Now, we find the inverse of the cumulative distribution function.

$$y = 1 - x^{-3} \quad \Leftrightarrow \quad x^{-3} = 1 - y \quad \Leftrightarrow \quad x^3 = \frac{1}{1-y} \quad \Leftrightarrow \quad x = \frac{1}{\sqrt[3]{1-y}}$$

So, our answer is

$$x = \frac{1}{\sqrt[3]{1-0.25}} = 1.100642.$$

[1] This is the convention that the IFM study note uses; other sources might have different rules for the boundary cases.

Problem 7.3. *Source: Course 3, November 1980, Problem #33.*
The probability density function of the random variable X is

$$f_X(x) = \begin{cases} 0, & \text{for } x < -1, \\ \frac{3}{2}x^2, & \text{for } -1 \leq x \leq 1, \\ 0, & \text{for } x > 1. \end{cases}$$

Let u denote the simulated value from a unit uniform random number generator. Which transformation would you apply to u to generate simulated values of x?

Solution: First, we figure out the cumulative distribution function of X.

$$F_X(x) = \frac{3}{2}\int_{-1}^{x} \xi^2 d\xi = \frac{\xi^3}{2}\Big]_{\xi=-1}^{x} = \frac{x^3+1}{2}.$$

Now, we invert the cumulative distribution function of X.

$$y = \frac{x^3+1}{2} \quad \Leftrightarrow \quad 2y - 1 = x^3 \quad \Leftrightarrow \quad \sqrt[3]{2y-1} = x.$$

So, we obtain simulated values of X through the following transform $x = \sqrt[3]{2u-1}$.

Problem 7.4. Let the random variable X have the cumulative distribution function given by

$$F_X(x) = \begin{cases} 0, & \text{for } x < 0, \\ x^4, & \text{for } 0 \leq x \leq 1, \\ 1, & \text{for } x > 1. \end{cases}$$

The inverse transform method was applied to generate the following three simulated values of X:

$$0.09 \quad 0.64 \quad 0.81.$$

Which values of the random number generator where mapped into the above three draws from X?

Solution: To undo the inverse transform, we simply apply F_X to the three given values. We get

$$(0.09)^4 \quad (0.64)^4 \quad (0.81)^4.$$

Problem 7.5. Let the cumulative distribution function F_X of a random variable X satisfy the following conditions:

- $F_X(0) = 0$;
- $F_X(1) = 0.2$;
- $F_X(2) = 1$;
- F_X is linear on $(0, 1)$ and $(1, 2)$;
- F_X is continuous.

You use the inverse transform method to simulate values of X. The values given by the random number generator are

$$0.1 \quad 0.6 \quad 0.9.$$

Which simulated values from X were drawn based on the above three values?

Solution: From the given information about F_X, we can conclude that $F_X(0.5) = 0.1$. Therefore, the value of the random number generator equal to 0.1 maps into the simulated value 0.5 of the random variable X.

Similarly, $F_X(1.5) = 0.6$. So, the value of the random number generator equal to 0.6 maps into the simulated value 1.5 of the random variable X.

The last mapping is tricky. Between 1 and 2, the function F_X satisfies

$$F_X(x) = 0.2 + 0.8(x-1).$$

Now, we need to solve for x in

$$F_X(x) = 0.9 \quad \Leftrightarrow \quad 0.2 + 0.8(x-1) = 0.9 \quad \Leftrightarrow \quad 0.8(x-1) = 0.7 \quad \Leftrightarrow \quad x - 1 = 0.875 \quad \Leftrightarrow \quad x = 1.875$$