M339D : April 7th, 2025.

Under the risk·neutral probability measure $\mathbb{P}^*$, we have

$$\boxed{\mathbb{E}^*[S(T)] = \underline{S(0)e^{rT}}} \quad (\star\star)$$

Equating $(\star)$ & $(\star\star)$, we get $\boxed{m + \frac{\nu^2}{2} = rT}$

Recall: $Var[R(0,1)] = \sigma^2$, i.e., $SD[R(0,1)] = \sigma$

$$\Rightarrow Var[R(0,T)] = \boxed{\underline{\sigma^2 \cdot T = \nu^2}}$$

$$\underline{m} = rT - \frac{\nu^2}{2} = rT - \frac{\sigma^2 \cdot T}{2} = \boxed{\underline{\left(r - \frac{\sigma^2}{2}\right) \cdot T}}$$

Finally,

$$\boxed{R(0,T) \sim Normal\left(mean = \boxed{\left(r - \frac{\sigma^2}{2}\right)T}, var = \sigma^2 \cdot T\right)}$$

Say, $\underline{Z \sim N(0,1)}$

Then, we can express $R(0,T)$ as

$$\boxed{R(0,T) = \left(r - \frac{\sigma^2}{2}\right) \cdot T + \sigma\sqrt{T} \cdot Z} \quad ✓$$

Thus,

$$\boxed{S(T) = S(0)e^{R(0,T)} = S(0)e^{\left(r - \frac{\sigma^2}{2}\right) \cdot T + \sigma\sqrt{T} \cdot Z}}$$

Q: What is the median of $S(T)$ under the risk·neutral probability measure $\mathbb{P}^*$?

$$\longrightarrow: \quad \boxed{S(0)e^{\left(r - \frac{\sigma^2}{2}\right) \cdot T}}$$

Note:

$$\frac{mean}{median} = \frac{\cancel{S(0)}\cancel{e^{rT}}}{\cancel{S(0)}e^{(\cancel{r} - \frac{\sigma^2}{2}) \cdot T}} = e^{\frac{\sigma^2 \cdot T}{2}}$$

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Problem Set # 13

Mean and median of the log-normal stock prices.

Problem 13.1. The current price of a non-dividend-paying stock is \$80 per share. Under the risk-neutral probability measure, its mean rate of return is 12% and its volatility is 30%.

Let $R(0,t)$ denote the realized return of this stock over the time period $[0,t]$ for any $t > 0$. Calculate $\mathbb{E}^*[R(0,2)]$.

$$\mathbb{E}^*[R(0,T)] = \left(r - \frac{\sigma^2}{2}\right)\cdot T = \left(0.12 - \frac{0.09}{2}\right)\cdot 2$$

$$= (0.12 - 0.045)\cdot 2$$

$$= 0.15$$

Problem 13.2. A stock is valued at $\boxed{\$75.00}$. The continuously compounded, risk-free interest rate is $\boxed{10.0\%}$ and the standard deviation of annualized returns is 25.0%. If the stock is lognormally distributed, what is the expected stock price after 2 years under the risk-neutral probability measure?

→ :

$$\mathbb{E}^*[S(2)] = S(0)e^{r\cdot 2} = 75\cdot e^{0.1\cdot 2} = 75e^{0.2} = \dots$$

Problem 13.3. A non-dividend-paying stock is valued at \$55.00 per share. Its standard deviation of annualized returns is given to be 22.0%. The continuously compounded risk-free interest rate is $\boxed{12\%}$. If the stock price is modeled using the lognormal distribution (as discussed in class), what is the median of the stock price in $\boxed{3}$ years under the risk-neutral probability measure?

→ :

$$\text{median} = S(0)e^{\left(r - \frac{\sigma^2}{2}\right)\cdot T}$$

$$= 55\, e^{\left(0.12 - \frac{0.22^2}{2}\right)\cdot 3} = \dots\dots$$

Problem 13.4. Assume that the stock price is modeled using the lognormal distribution. The stock pays no dividends. Under $\mathbb{P}^*$, the annual mean rate of return on the stock is given to be 12%. Also under $\mathbb{P}^*$, the median time$-t$ stock price is evaluated to be $S(0)e^{0.1t}$. What is the volatility parameter of this stock price?

→:

$$\cancel{S(0)}e^{(r-\frac{\sigma^2}{2})\cdot \cancel{t}} = \cancel{S(0)}e^{0.1\cancel{t}}$$

$$r-\frac{\sigma^2}{2}=0.1$$

$$0.12-\frac{\sigma^2}{2}=0.1$$

$$\frac{\sigma^2}{2}=0.02$$

$$\sigma^2=0.04$$

$$\boxed{\sigma=0.2}$$

Problem 13.5. The current stock price is $\boxed{\$100}$ per share. The stock price at any time $t>0$ is modeled using the lognormal distribution. Assume that the continuously compounded risk-free interest rate equals 8%. Let its volatility equal 20%.

Find the value t^* at which the median stock price equals \$120, under the risk-neutral probability measure.

→: median time$\cdot$t stock price $= S(0)e^{(r-\frac{\sigma^2}{2})\cdot t}$

$$12\cancel{0} = 1\cancel{00}\, e^{(0.08-\frac{0.04}{2})\cdot t^*}$$

$$\ln(1.2) = 0.06t^*$$

$$t^* = \frac{\ln(1.2)}{0.06} = \underline{3.039}$$

Problem 13.6. The volatility of the price of a non-dividend-paying stock is 0.2. The stock price is modeled using a log-normal distribution. Under $\mathbb{P}^*$, the expected time-2 stock price is $\boxed{\$120}$. What is the median of the time-2 stock price under $\mathbb{P}^*$?

→: median $= S(0)e^{(r-\frac{\sigma^2}{2})\cdot T} = \underbrace{\boxed{S(0)e^{rT}}}_{120}\cdot e^{-\frac{\sigma^2}{2}\cdot T}$

$$= 120\,e^{-\frac{0.04}{2}\cdot 2}$$

$$= 120\,e^{-0.04} = \underline{115.29}$$

Motivation.

Consider a European call option w/ strike K and exercise date T.

By our risk-neutral pricing

$$V_c(0) = e^{-rT}\mathbb{E}^*\left[\boxed{V_c(T)}\right]$$
$$= e^{-rT}\mathbb{E}^*\left[(S(T)-K)_+\right]$$
$$= e^{-rT}\mathbb{E}^*\left[(S(T)-K)\cdot\mathbb{I}_{[S(T)\geq K]}\right]$$
$$= e^{-rT}\mathbb{E}^*\left[S(T)\cdot\mathbb{I}_{[S(T)\geq K]}\right] - e^{-rT}\boxed{\mathbb{E}^*\left[K\cdot\mathbb{I}_{[S(T)\geq K]}\right]}$$

$$\mathbb{E}^*\left[K\cdot\mathbb{I}_{[S(T)\geq K]}\right] = K\,\mathbb{E}^*\left[\mathbb{I}_{[S(T)\geq K]}\right] = K\cdot\mathbb{P}^*\left[S(T)\geq K\right]$$

Log Normal Tail Probabilities.

Example. Consider a non-dividend-paying stock. What is the probability that the stock outperforms a risk-free investment under the risk-neutral probability measure?

→: The initially invested amount is: $S(0)$

- If it's a risk-free investment, the balance @ time-T is
  $S(0)e^{rT}$
- If it's a stock investment, the wealth @ time-T is
  $S(T)$

$$\mathbb{P}^*\left[S(T) > S(0)e^{rT}\right] = ?$$

This question is equivalent to the one of whether the profit for the stock investment is positive under $\mathbb{P}^*$.