# M378K: September $6^{th}$, 2024.

**Problem 1.3.** Source: Sample P exam, Problem #355.
*The number of calls received by a certain emergency unit in a day is modeled by a Poisson distribution with parameter* 2. *Calculate the probability that on a particular day the unit receives at least two calls.*

$\longrightarrow$: $N$... # of calls

$N \sim \text{Poisson}(\lambda = 2) \sim P(\lambda = 2)$

Support is $\mathbb{N}_0 = \{0, 1, 2, \dots\}$

pmf for $k = 0, 1, \dots$

$$\boxed{p_Y(k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}}$$

$$\mathbb{P}[N \geq 2] = 1 - \mathbb{P}[N = 0] - \mathbb{P}[N = 1]$$
$$= 1 - e^{-\lambda} \cdot \frac{\lambda^0}{0!} - e^{-\lambda} \cdot \frac{\lambda^1}{1!}$$
$$= 1 - e^{-2} - e^{-2} \cdot 2 = 1 - 3e^{-2}$$

$\square$

**Def'n.** For a discrete r.v. $Y$ w/ support $S_Y \subseteq \mathbb{R}$, we define its expectation as

$$\boxed{\mathbb{E}[Y] = \sum_{y \in S_Y} y \cdot p_Y(y)}$$

If the sum exists

**Theorem.** If $Y_1$ and $Y_2$ both have expectations $\mathbb{E}[Y_1]$ and $\mathbb{E}[Y_2]$, resp, then for any two constants $\alpha$ and $\beta$ $\mathbb{E}[\alpha Y_1 + \beta Y_2]$ also exists and

$$\boxed{\mathbb{E}[\alpha Y_1 + \beta Y_2] = \alpha \mathbb{E}[Y_1] + \beta \mathbb{E}[Y_2]}$$

M378K Introduction to Mathematical Statistics

Problem Set #2

Expectation and variance: the discrete case.

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**Problem 2.1.** Source: Sample P exam, Problem #481.

*The number of days required for a damage control team to locate and repair a leak in the hull of a ship is modeled by a discrete random variable $N$. $N$ is uniformly distributed on $\{1,2,3,4,5\}$.*

*The cost of locating and repairing a leak is $N^2 + N + 1$.*

*Calculate the expected cost of locating and repairing a leak in the hull of the ship.*

$\longrightarrow$: $\mathbb{E}[N^2+N+1] = \mathbb{E}[N^2] + \mathbb{E}[N] + \underline{1}$

linearity of $\mathbb{E}$

$$\mathbb{E}[N] = 1\cdot\frac{1}{5} + 2\cdot\frac{1}{5} + \cdots + 5\cdot\frac{1}{5} = \underline{3}$$

$$\mathbb{E}[N^2] = 1\cdot\frac{1}{5} + 4\cdot\frac{1}{5} + \cdots + 25\cdot\frac{1}{5}$$

$$= \frac{1}{5}(1+2^2+\cdots+5^2) = \frac{1}{\cancel{5}}\cdot\frac{\cancel{5}\cdot\cancel{6}\cdot(2\cdot 5+1)}{\cancel{6}} = \underline{11}$$

<u>answer</u>: $11+3+1 = 15$

$\square$

**Def'n.** The variance of the random variable $Y$ is defined as

$$\mathrm{Var}[Y] := \mathbb{E}\left[(Y-\mu_Y)^2\right] \qquad \text{w/ } \mu_Y = \mathbb{E}[Y]$$

The standard deviation of $Y$ is

$$SD[Y] = \sqrt{\mathrm{Var}[Y]}$$

**Theorem.** **Pf.**

$$\mathrm{Var}[Y] = \mathbb{E}\left[(Y-\mu_Y)^2\right]$$

$$= \mathbb{E}\left[Y^2 - 2\mu_Y \cdot Y + \mu_Y^2\right] \quad \text{linearity of } \mathbb{E}$$

$$= \mathbb{E}[Y^2] - 2\mu_Y \underbrace{\boxed{\mathbb{E}[Y]}}_{=\mu_Y} + \mu_Y^2$$

$$= \mathbb{E}[Y^2] - \mu_Y^2 \qquad \square$$

**Theorem.** Say that $Y_1$ and $Y_2$ are r.v.s w/ finite variances and that $\alpha$ is a real constant. Then,

- $\boxed{\mathrm{Var}[\alpha \cdot Y_1] = \alpha^2 \cdot \mathrm{Var}[Y_1]}$
- when, additionally, $Y_1$ and $Y_2$ are independent,

$$\boxed{\mathrm{Var}[Y_1 + Y_2] = \mathrm{Var}[Y_1] + \mathrm{Var}[Y_2]}$$

**Problem 2.2.** Source: Sample P exam, Problem #458.

*An investor wants to purchase a total of ten units of two assets, $A$ and $B$, with annual payoffs* **per unit** *purchased of $X$ and $Y$, respectively. Each asset has the same purchase price per unit. The payoffs are independent random variables with equal expected values and with* $\mathrm{Var}(X) = 30$ *and* $\mathrm{Var}(Y) = 20$.

*Calculate the number of units of asset $A$ the investor should purchase to minimize the variance of the total payoff.*

$\longrightarrow$ : $n$ ... # of units of A that is bought

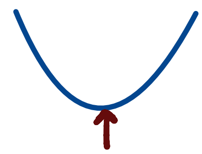
$$n \cdot X + (10-n) \cdot Y$$

$$\mathrm{Var}[n \cdot X + (10-n) \cdot Y] \longrightarrow \min$$

$$n^2 \cdot \mathrm{Var}[X] + (10-n)^2 \cdot \mathrm{Var}[Y] \longrightarrow \min$$

$$30n^2 + 20(10-n)^2 = 50n^2 - 400n + 2000 \longrightarrow \min$$

$$n^2 - 8n + 40 \longrightarrow \min$$



$$n^* = -\frac{-8}{2} = 4 \quad \square$$

## Example.

- **Bernoulli** $Y \sim B(p)$

$$\mathbb{E}[Y] = 0 \cdot q + 1 \cdot p = p$$

$$Var[Y] = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2$$

$$\mathbb{E}[Y^2] = 0^2 \cdot q + 1^2 \cdot p = p$$

$$Var[Y] = p - p^2 = p(1-p) = \underline{pq}$$

- **Binomial** $Y \sim B(n, p)$

$$\mathbb{E}[Y] = np$$

$$Var[Y] = n \cdot p q$$

- **Geometric** $Y \sim g(p)$

$$\mathbb{E}[Y] = p \cdot 0 + q \cdot (1 + \mathbb{E}[Y])$$

$$= q + q\,\mathbb{E}[Y]$$

$$\mathbb{E}[Y](1-q) = q$$

$$\boxed{\mathbb{E}[Y] = \frac{q}{p}}$$

$$Var[Y] = \frac{q}{p^2} \iff SD[Y] = \frac{\sqrt{q}}{p}$$

- **Poisson** $Y \sim P(\lambda)$

$$\mathbb{E}[Y] = Var[Y] = \lambda$$