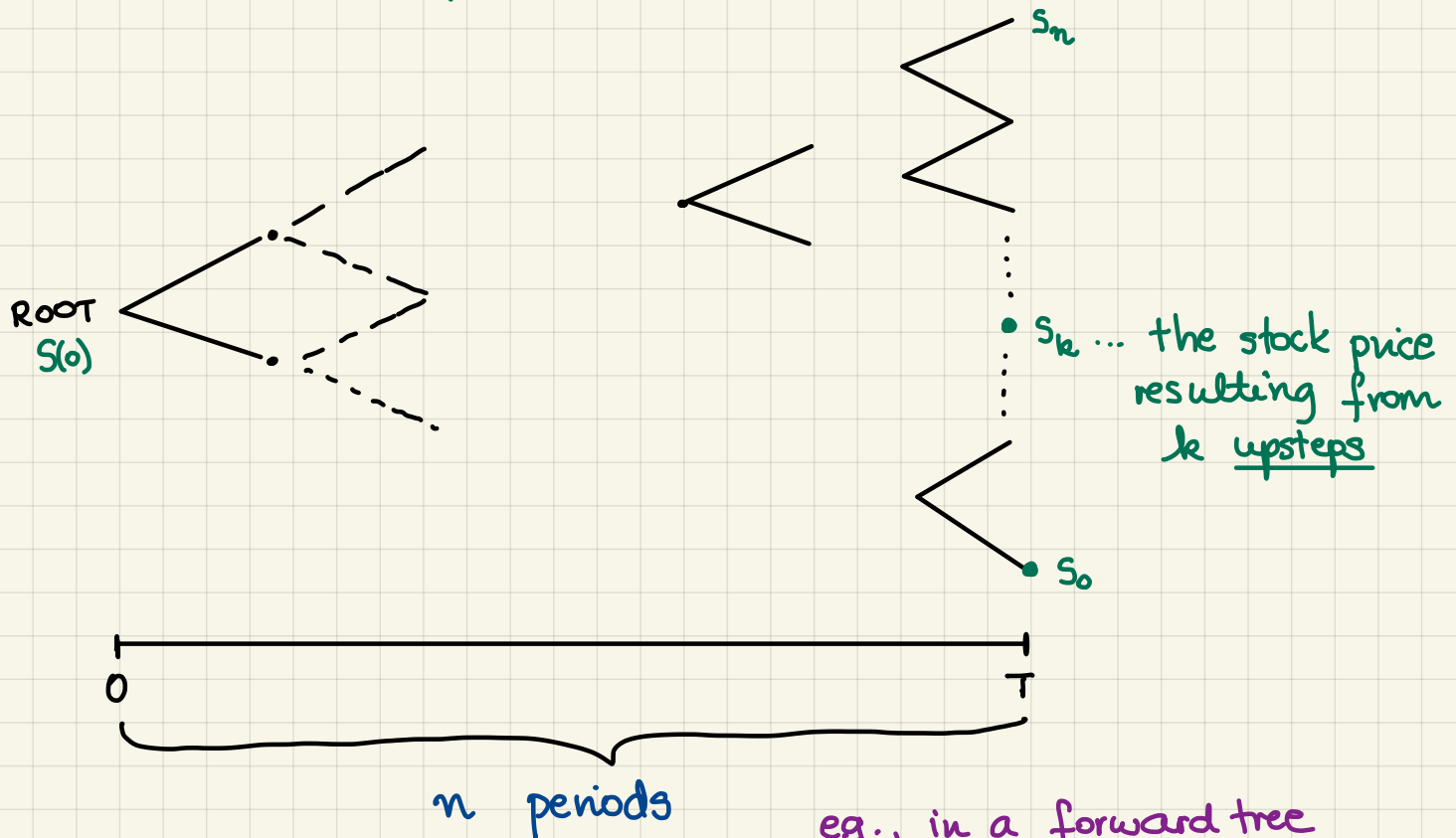


M339 W: February 5th, 2021.

The Pre-Limit: n-period Binomial Trees



u_n ... up factor

d_n ... down factor

eg., in a forward tree

$$u_n = e^{(r-s)(T/n) + \sigma \sqrt{T/n}}$$

$$d_n = e^{(r-s)(T/n) - \sigma \sqrt{T/n}}$$

$$\Rightarrow S_k = S(0) u_n^k \cdot d_n^{n-k} = S(0) \left(\frac{u_n}{d_n} \right)^k \cdot d_n^n$$

k corresponds to a realization of the binomial distribution w/ n trials and

p_n as the probability of success in every trial

\Rightarrow Say, X_n ... # of upsteps in the n periods.

Then, $X_n \sim \text{Binomial}(\text{\# of trials} = n,$

prob. of success = p_n)

$$S(T) = S(0) \left(\frac{u_n}{d_n} \right)^{X_n} \cdot d_n^n$$

Note: • p_n can generally be a subjective probability of an upstep

• If we're pricing options, we use $p_n = p_n^*$,
e.g., in a forward tree $p_n^* := \frac{1}{1 + e^{\sigma \sqrt{T_n}}}$

Q: In the past, when you were looking for a limiting dist'n of a sequence of binomial r.v.s, which theorem did you use?

→: Normal Approximation to the Binomial
(de Moivre-Laplace).

Consider a sequence of binomial random variables

$Y_n \sim \text{Binomial}(n = \# \text{ of trials}, p = \text{prob. of success})$

Set: $E[Y_n] = n \cdot p$

$\text{Var}[Y_n] = np(1-p) \Rightarrow \text{SD}[Y_n] = \sqrt{np(1-p)}$

$$\frac{Y_n - np}{\sqrt{np(1-p)}} \xrightarrow{D} N(0,1)$$

Usage: • Look @ "large" n (rule of thumb:

$np \geq 10$ and $n(1-p) \geq 10$)

$$\mathbb{P}[a < Y_n \leq b] =$$

$$= \mathbb{P}\left[\frac{a - np}{\sqrt{np(1-p)}} < \underbrace{\frac{Y_n - np}{\sqrt{np(1-p)}}}_{\approx N(0,1)} \leq \frac{b - np}{\sqrt{np(1-p)}} \right]$$

N ... cumulative
dist'n f'n of $N(0,1)$

i.e.,

$$N(z) = \mathbb{P}[Z \leq z]$$

$$\approx \mathbb{P}\left[\frac{a - np}{\sqrt{np(1-p)}} < Z \leq \frac{b - np}{\sqrt{np(1-p)}} \right]$$

$$= N\left(\frac{b - np}{\sqrt{np(1-p)}}\right) - N\left(\frac{a - np}{\sqrt{np(1-p)}}\right)$$

- In statistics, we use this theorem like this:

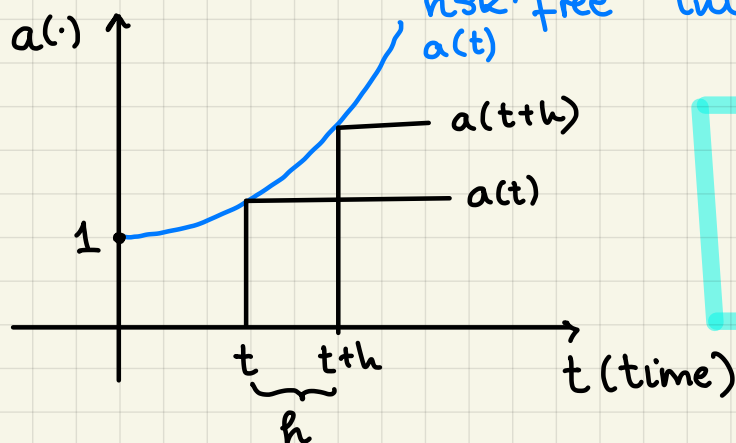
$$Y_n \sim \text{Normal}(\text{mean} = np, \text{sd} = \sqrt{np(1-p)})$$

In our model, the probability of success p_n depends on n .

Realized Returns.

Inspiration.

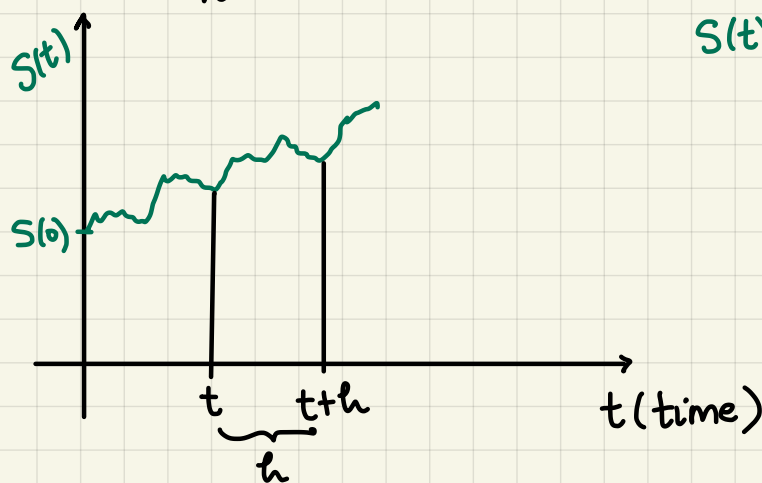
Consider an accumulation function in the compound interest case. Let r ... continuously compounded, risk-free interest rate.



$$a(t+h) = a(t) \cdot e^{r \cdot h}$$

$$r \cdot h = \ln\left(\frac{a(t+h)}{a(t)}\right)$$

Definition.



$S(t), t \geq 0$... time t stock price