M339D: November 11th, 2024.

Moment Generating Functions.

For a random variable Y,
and for an independent argument denoted by t,
we define the moment generating function (mgf) of Y
as this function of t:

$$\boxed{M_Y(t) := \mathbb{E}\left[e^{Yt}\right]}$$

for all t such that the expectation exists

Note: • $M_Y(0) = \underline{1}$ $\Rightarrow$ @ least $t=0$ is in the domain of M_Y

Goal: To understand e^X w/ $X \sim Normal(mean = m, var = \nu^2)$

Recall: In terms of $Z \sim N(0,1)$,

$$\boxed{\nu = nu}$$

$$\boxed{X = m + \nu \cdot Z}$$

Fact. $\boxed{M_Z(t) = e^{\frac{t^2}{2}}}$ for all $t \in \mathbb{R}$

$\Rightarrow$ For any normal X:

$$M_X(t) = \mathbb{E}\left[e^{X \cdot t}\right] = \mathbb{E}\left[e^{(m+\nu Z)t}\right]$$

$$= \mathbb{E}\left[e^{mt} e^{\nu t \cdot Z}\right] = e^{mt}\,\mathbb{E}\left[e^{\nu t Z}\right]$$

$$= e^{mt} M_Z(\nu t)$$

$$= e^{mt} e^{\frac{\nu^2 t^2}{2}} = e^{mt + \frac{\nu^2 t^2}{2}}$$

The lognormal distribution.

Definition 1.1. Let $X \sim Normal(mean = m, variance = \nu^2)$. Define the random variable $Y = e^X$. We say that the random variable Y is *lognormally distributed.*

1.1. First properties.

- The expected value of the lognormally distributed random variable Y can be obtained as follows:

$= \mathbb{E}[e^{X\cdot 1}] =$

$$\mathbb{E}[Y] = \mathbb{E}[e^X] = M_X(1) = e^{m+\frac{\nu^2}{2}}.$$

- Let Y be a lognormal and let $a \neq 0$. Then, the random variable Y^a is also lognormal. *Note:* For $a = 0$, we get a degenerate random variable at 1 which can, technically, be interpreted as lognormal, but is not fun.
- Let Y_1 and Y_2 be independent and lognormally distributed. Then, Y_1Y_2 is also lognormal.

1.2. Quantiles.

Definition 1.2. For p such that $0 < p < 1$, we define the $100p^{th}$ *quantile* of a random variable X as any value π_p such that

$$F_X(\pi_p-) \leq p \leq F_X(\pi_p).$$

In particular, the 50^{th} quantile is referred to as the *median.*

Note: When the random variable X is continuous, we can obtain the $100p^{th}$ quantile by simply solving for π_p in

$$F_X(\pi_p) = p.$$

Consider a probability p. Let z_p be the $100p^{th}$ quantile of the standard normal distribution. Let Y be lognormally distributed as above. My claim is that the value

$$\boxed{y_p = e^{m+\nu z_p}}$$

is the $100p^{th}$ quantile of Y. Let us simply verify this claim by calculating $F_Y(y_p)$. We have, with $Z \sim N(0,1)$,

$$F_Y(y_p) = \mathbb{P}[Y \leq y_p] = \mathbb{P}[e^X \leq y_p] = \mathbb{P}[e^{m+\nu Z} \leq e^{m+\nu z_p}].$$

Since the logarithmic function is increasing, we have that the above equals

$$F_Y(y_p) = \mathbb{P}[m + \nu Z \leq m + \nu z_p] = \mathbb{P}[Z \leq z_p] = p.$$

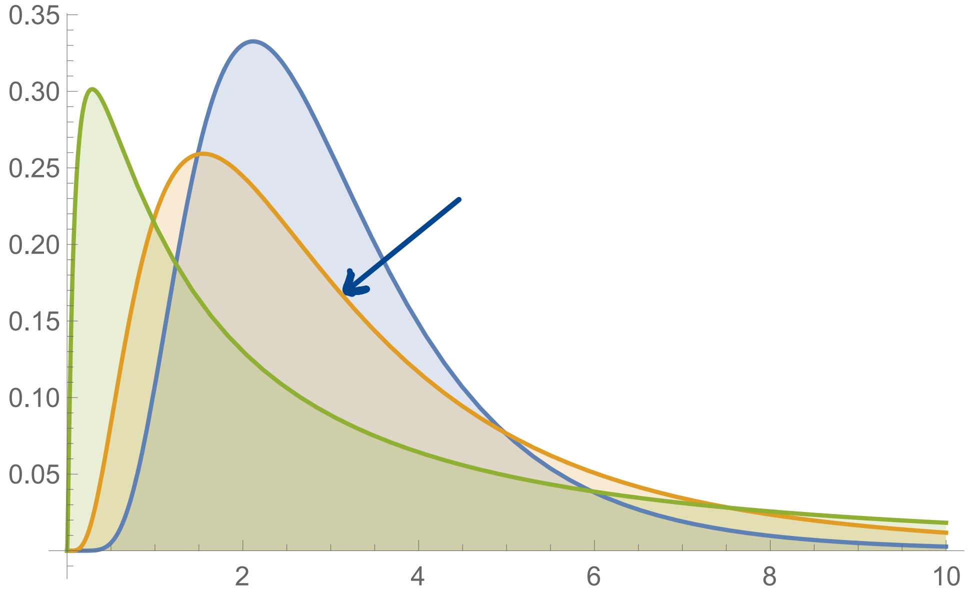
The above concludes our proof.

In particular, since the median of the standard normal distribution equals 0, the median of the lognormal distribution will be e^m.

Note: Since

$$e^m < e^{m+\frac{\nu^2}{2}}, \tag{1.1}$$

i.e., since the mean of a lognormal distribution always exceeds the median, we say that it's *right-skewed.* In fact, this is what its probability density function looks like.


0.35
0.30
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2
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10

The Log-Normal Dist'n.

Def'n. Let $X \sim \text{Normal}(\text{mean} = m, \text{variance} = \nu^2)$

Define $Y = e^X$

We say that Y is **lognormally distributed**.

$$\mathbb{E}[Y] = \mathbb{E}[e^X] = \mathbb{E}[e^{X \cdot 1}] = M_X(1) = e^{m + \frac{\nu^2}{2}}$$

Consider: $\mathbb{E}[X] = m$

Caveat: $\boxed{\mathbb{E}[e^X] \geq e^{\mathbb{E}[X]}}$

This is a special case of **Jensen's Inequality**.

Theorem. Let X be a random variable, and let g be a convex function such that

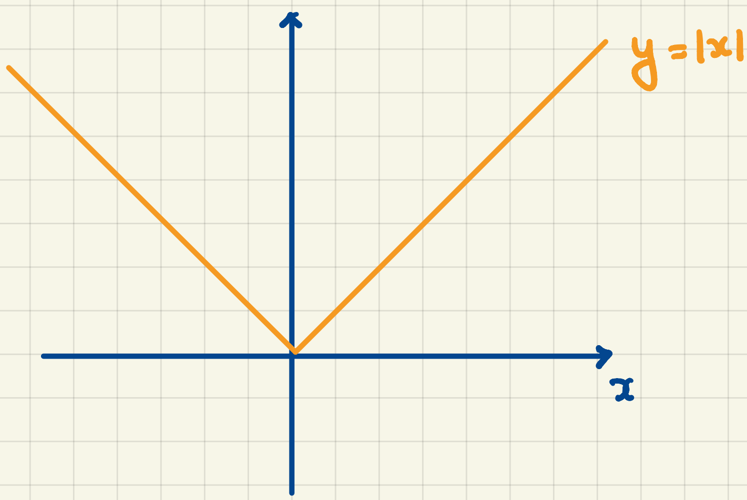
$g(X)$ is well-defined

and

$\mathbb{E}[g(X)]$ exists.

Then, $\boxed{\mathbb{E}[g(X)] \geq g(\mathbb{E}[X])}$

Examples. i. $\boxed{g(x) = |x|}$



$\boxed{\mathbb{E}[\,|X|\,] \geq |\mathbb{E}[X]|}$

ii. Look @ a European put w/ strike K.

Its payoff f'tion: $v_p(s) = (K-s)_+$

The expected payoff

$$\mathbb{E}[v_p(S(T))] = \boxed{\mathbb{E}\left[(K-S(T))_+\right]}$$

$$\geq$$

By Jensen, its lower bound is $(K - \mathbb{E}[S(T)])_+$