M339J : March 23rd, 2022.

Poisson Thinning.

Theorem. Let $\boxed{N \sim Poisson(\lambda)}$ be a counting random variable for some events of interest.
Suppose that independently from N, each of these events falls into a particular category indexed by $i = 1, 2, \dots, m$ w/ probability p_i $(i = 1, 2, \dots, m)$ $(p_1 + p_2 + \dots + p_m = 1)$.

Let N_i be the count of events from category i, $i = 1..m$.

Then:

- $N_i \sim \underline{Poisson}(\lambda_i = p_i \cdot \lambda)$ for all $i = 1..m$
- $N_1, N_2, \dots, N_m$ are <u>independent</u> random variables.

$N \sim \text{Poisson}(\lambda = 12)$

111. The number of accidents follows a Poisson distribution with mean 12. Each accident generates 1, 2, or 3 claimants with probabilities 1/2, 1/3, and 1/6, respectively.

Assume INDEPENDENCE between N and the # of claimants.

Calculate the variance of the total number of claimants.

S... the total # of claimants

$\mathrm{Var}[S] = ?$

- for $i = 1, 2, 3$, N_i... # of accidents w/ i claimants

$N_i \sim \text{Poisson}(\lambda_i = \lambda \cdot p_i)$

$\Rightarrow N_1 \sim \text{Poisson}(\lambda_1 = 6)$ ✓

$N_2 \sim \text{Poisson}(\lambda_2 = 4)$ ✓

$N_3 \sim \text{Poisson}(\lambda_3 = 2)$ ✓

(A) 20

(B) 25

(C) 30

(D) 35

(E) 40

112. In a clinic, physicians volunteer their time on a daily basis to provide care to those who are not eligible to obtain care otherwise. The number of physicians who volunteer in any day is uniformly distributed on the integers 1 through 5. The number of patients that can be served by a given physician has a Poisson distribution with mean 30.

Determine the probability that 120 or more patients can be served in a day at the clinic, using the normal approximation with continuity correction.

(A) $1-\Phi(0.68)$

(B) $1-\Phi(0.72)$

(C) $1-\Phi(0.93)$

(D) $1-\Phi(3.13)$

(E) $1-\Phi(3.16)$

$$S = 1 \cdot N_1 + 2 \cdot N_2 + 3 \cdot N_3$$

$$\mathrm{Var}[S] = \mathrm{Var}[N_1 + 2N_2 + 3N_3] =$$

N_1, N_2, N_3 are independent due to the "thinning" theorem

$$= \mathrm{Var}[N_1] + 4\,\mathrm{Var}[N_2] + 9\,\mathrm{Var}[N_3]$$

$$= \lambda_1 + 4\,\lambda_2 + 9 \cdot \lambda_3$$

$$= 6 + 4 \cdot 4 + 9 \cdot 2 = 40. \quad \square$$

Negative Binomial Distrubution.

Binomial Coefficients.

For $n, k \in \mathbb{N}_0$ w/ $n \geq k$:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

We can generalize to $x \in \mathbb{R}_+$ and $k \in \mathbb{N}_0$ such that

$$\boxed{x > k-1}$$

$$\binom{x}{k} = \frac{\Gamma(x+1)}{\Gamma(k+1) \cdot \Gamma(x-k+1)}$$

Inspiration: A Modelling Problem.

Consider a sequence of independent, identically dist'd Bernoulli trials.

Say: the probability of success in every trial is p.

The repetition of the Bernoulli trials continues until a total of r successes is achieved.

N... # of failures before the r^{th} success.

Q: What is the support of N? $\mathbb{N}_0 = \{0, 1, 2, \dots\}$

For $k = 0, 1, 2, \ldots$

The last trial must be a success!

$$p_N(k) = \underbrace{\mathbb{P}[N = k]}_{} = \binom{r + k - 1}{k} p^r (1-p)^k$$

What is the probability of seeing k failures before the r^{th} success?

In this class, and the STAM exam, we use this parametrisation:

$$\boxed{p = \frac{1}{1+\beta}}$$ for some $\beta > 0$

With this parametrisation:

$$p_N(k) = \binom{r + k - 1}{k} \left(\frac{1}{1+\beta}\right)^r \left(\frac{\beta}{1+\beta}\right)^k$$

Return to our generalization of binomial coefficients and have $\boxed{r > 0}$

We write: $N \sim NegBinomial(r, \beta)$

- $\mathbb{E}[N] = r \cdot \beta$
- $Var[N] = r \cdot \beta(1+\beta) > \mathbb{E}[N]$
- $\boxed{P_N(z) = (1 - \beta(z-1))^{-r}}$

Q: What do we get in the special case where $r = 1$?

$\longrightarrow$: We have $N \sim Geometric(\beta)$.

The geometric distribution has the memoryless property.