Policy Limits.

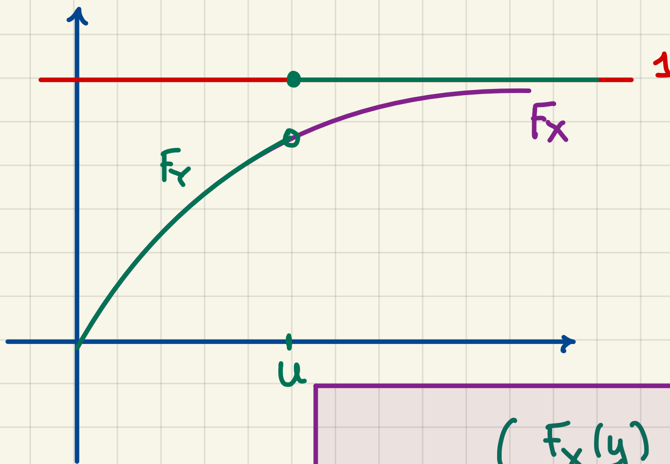
For an insurance policy w/ no deductible and a policy limit u, the insurer's payment will be

$$\boxed{Y = X \wedge u.}$$

In other words, Y is the right-censored random variable (also limited loss).

Q: Start w/ a continuous random variable X such that $\boxed{S_X(u) > 0}$.
What's the cumulative distribution function of Y?

$\longrightarrow$:



$$\boxed{F_Y(y) = \begin{cases} F_X(y) & \text{for } y < u \\ 1 & \text{for } y \geq u \end{cases}}$$

A mixed dist'n.

$$\begin{cases} f_Y(y) = f_X(y) & \text{for } y < u \\ p_Y(u) = S_X(u) & \end{cases}$$

Problem 2.3. *Source: Two old exams 3; I forgot to note the years.*
A jewelry store purchases two separate insurance policies that together provide full coverage. You are given:

- The expected ground-up loss is $\boxed{11,100}$.
- Policy A has an ordinary deductible of $5,000$ and **no** policy limit.
- Under policy A, the expected amount paid per loss is $6,500$.
- Under policy A, the expected amount paid per payment is $10,000$.
- Policy B has **no** deductible and has a policy limit of $5,000$.

i. Given that a loss has occurred, find the probability that the payment under policy B equals $5,000$.
ii. Given that a loss less than or equal to $5,000$ has occurred, what is the expected payment under policy B?

Think!

X... the random variable denoting the ground-up loss

Assume that X is continuous.

$$\mathbb{P}[X \geq 5000] = S_X(5000) = ?$$

$d = 5000$

$$\mathbb{E}[Y_A^L] = 6500 = \mathbb{E}[(X-d)_+]$$

$$\mathbb{E}[Y_A^P] = 10000 = \mathbb{E}[X-d \mid X>d] = \frac{\mathbb{E}[(X-d)_+]}{S_X(d)}$$

$$\boxed{S_X(5000) = \frac{6500}{10\,000} = 0.65}$$

ii.

$$\mathbb{E}[X \mid X \le 5000] = \cancel{?} \quad \text{by the def'n of conditional expectation}$$

$$= \frac{\boxed{\mathbb{E}[X \cdot \mathbb{I}_{[X \le 5000]}]}}{\mathbb{P}[X \le 5000]}$$

$= 1 - 0.65 = 0.35$ ✓

$$\mathbb{E}[X \cdot \mathbb{I}_{[X \le 5000]}] = \boxed{\mathbb{E}[X \wedge 5000]} - 5000 \underbrace{\mathbb{P}[X > 5000]}_{= 0.65}$$

$$\mathbb{E}[X \wedge 5000] = \mathbb{E}[X] - \mathbb{E}[(X - 5000)_+]$$

$$= 11{,}100 - 6{,}500 = 4{,}600$$

$$\text{answer} = \frac{4{,}600 - 5000 \cdot 0.65}{0.35} = 3{,}857.14 \quad \blacksquare$$

Problem 2.4. Let the ground-up loss X be exponentially distributed with mean \$500. An insurance policy has an ordinary deductible of \$50 and a policy limit of \$2000. Find the expected value of the amount paid (by the insurance company) per positive payment.

$\longrightarrow$: losses : $X \sim Exponential(\theta = 500)$

deductible : $d = 50$ ✓

the policy limit : $u - d = 2000$

$\boxed{u = 2050}$ ✓

Method I.

We need :

$\mathbb{E}[Y^P] = \mathbb{E}[Y^L \mid Y^L > 0] = \mathbb{E}[Y^L \mid X > d]$

$$Y^L = \begin{cases} (X-d)_+ & X < u \\ u-d & X \geq u \end{cases}$$

$$\boxed{Y^L = (X \wedge u - d)_+}$$

$Y = X - d \mid X > d \sim Exponential(\theta)$

memoryless property

$$\boxed{\mathbb{E}[Y^P] = \mathbb{E}[Y \wedge (u-d)]} = 500\left(1 - e^{-\frac{2000}{500}}\right) = 500(1 - e^{-4})$$

$= 490.84$

Method II.

$$\boxed{\mathbb{E}[Y^L] = \mathbb{E}[X \wedge u] - \mathbb{E}[X \wedge d]}$$ Thm.

In this problem,

$$\mathbb{E}[Y^L] = \mathbb{E}[X \wedge 2050] - \mathbb{E}[X \wedge 50]$$
$$= 500\left(\cancel{1} - e^{-\frac{2050}{500}}\right) - 500\left(\cancel{1} - e^{-\frac{50}{500}}\right)$$
$$= 500\, e^{-\frac{50}{500}}\left(1 - e^{-\frac{2000}{500}}\right)$$

$$\mathbb{E}[Y^P] = \frac{\mathbb{E}[Y^L]}{S_X(d)} = \frac{500\, \cancel{e^{-\frac{50}{500}}}\left(1 - e^{-\frac{2000}{500}}\right)}{\cancel{e^{-\frac{50}{500}}}}$$

$$\mathbb{E}[Y^P] = 500(1 - e^{-4}) = 490.84$$

Coinsurance.

If the insurance company pays a proportion α of the loss, while the policyholder covers the rest, and if this is the only modification, the insurance company pays $\boxed{Y = \alpha \cdot X}$

The General Situation.

X... the ground-up loss

The insurance policy:

- the ordinary deductible d
- the policy limit $\alpha(u-d)$
- coinsurance α
- inflation rate r

The per-loss random variable is

$$Y^L = \begin{cases} 0 & \text{if } (1+r)X < d \\ \alpha\cdot((1+r)X-d)_+ & \text{if } d \leq (1+r)X \leq u \\ \boxed{\alpha(u-d)} & \text{if } (1+r)X > u \end{cases}$$

u... maximum covered loss

The policy limit, i.e., maximum amount payable by the insurer

$$\boxed{\alpha(u-d)}$$ ✓

The per payment random variable is

$$Y^P = \begin{cases} \text{undefined} & \text{if } (1+r)X < d \\ Y^L & \text{if } d \leq (1+r)X \end{cases}$$

Thm.

$$\boxed{\begin{aligned} \mathbb{E}[Y^L] &= \alpha\left(\mathbb{E}[(1+r)X \wedge u] - \mathbb{E}[(1+r)X \wedge d]\right) \\ \mathbb{E}[Y^P] &= \mathbb{E}[Y^L \mid (1+r)X > d] = \frac{\mathbb{E}[Y^L]}{S_X\left(\frac{d}{1+r}\right)} \end{aligned}}$$