Probability Mass Function.

Def'n. For a discrete random variable X, the probability mass function (pmf) is the function $p_X : \mathbb{R} \to [0,1]$ defined as

$$\boxed{p_X(x) = \mathbb{P}[X = x]} \quad \text{for all } x \in \mathbb{R}.$$

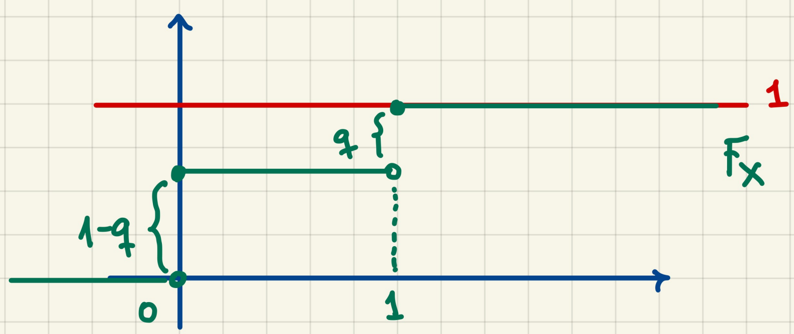
Note: On the support of the random variable X, the function p_X is strictly positive. Everywhere else, the function p_X is zero.

Note:

$$p_X(x) = F_X(x) - F_X(x-) \quad \text{for all } x$$

left limit

Example: Bernoulli trials w/ parameter q



$$p_X(0) = 1-q$$
$$p_X(1) = q$$

Special Case: In the special case where the support is (contained in) $\{0, 1, 2, \dots\}$, we say that the random variable is

$$\boxed{\mathbb{N}_0\text{-valued}}$$

Then, it's convenient to write the pmf as a sequence.

Problem. We model the number of accidents N in a particular year so that we assume:

$$\boxed{p_N(n+1) = \frac{1}{5} p_N(n)} \quad \text{for all } n \geq 0 .$$

What's the probability that there is @ least one accident in that year?

$\longrightarrow$: $\mathbb{P}[\text{@ least one accident}] = 1 - \mathbb{P}[\text{no accidents}]$

$$= 1 - \mathbb{P}[N = 0]$$

$$= 1 - p_N(0)$$

From our recursive property:

$$p_N(n+1) = \left(\frac{1}{5}\right)^1 p_N(n) = \frac{1}{5}\left(\frac{1}{5} p_N(n-1)\right) = \left(\frac{1}{5}\right)^2 p_N(n-1)$$

$$= \cdots = \left(\frac{1}{5}\right)^{n+1} p_N(0)$$

We know that the pmf sums up to one. So,

$$p_N(0) + p_N(1) + \cdots = 1$$

$$\sum_{n=0}^{+\infty} p_N(n) = 1$$

$$\sum_{n=0}^{+\infty} \left(\left(\frac{1}{5}\right)^n \cdot p_N(0)\right) = 1$$

$$\left.\begin{array}{l} p_N(0) \cdot \boxed{\sum_{n=0}^{+\infty} \left(\frac{1}{5}\right)^n} = 1 \\ = \dfrac{1}{1-\frac{1}{5}} = \dfrac{5}{4} \end{array}\right\} \Rightarrow \boxed{p_N(0) = \frac{4}{5}}$$

$$\boxed{\text{answer} = 1 - \frac{4}{5} = \frac{1}{5}}$$

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Def'n. Let X be a continuous random variable. Its probability density function (pdf) is defined as a function

$$f_X : \mathbb{R} \to \mathbb{R}_+ := [0,+\infty)$$

given by

$$\boxed{f_X(x) = F_X'(x)} = - S_X'(x)$$

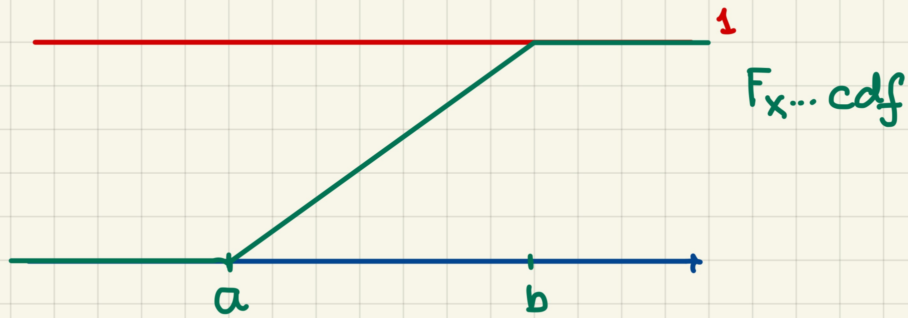
wherever the derivative exists.

Note: By the Fundamental Theorem of Calculus, we have

$$F_X(b) - F_X(a) = \int_a^b f_X(x)\,dx$$

$$= \mathbb{P}[a < X \leq b]$$

Example. Uniform dist'n. : $\boxed{U(a,b)}$



The probability density function:

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{for } x \in (a,b) \\ 0 & \text{otherwise} \end{cases}$$

Problem. Let T denote the lifetime of a particular device. The pdf of T is known to be proportional to

$(10+x)^{-2}$ on the interval $(0,40)$

and 0 otherwise.

What's the probability that the lifetime of the device exceeds 10?

$\longrightarrow$: Write: $f_X(x) = \mathcal{K} \cdot \frac{1}{(10+x)^2}$ for $x \in (0,40)$

$$\mathcal{K} \cdot \int_0^{40} (10+x)^{-2}\, dx = 1$$

$$\mathcal{K} \cdot (-1) \left[(10+x)^{-1} \right]_{x=0}^{40} = 1$$

$$\mathcal{K} (-1) \left(\frac{1}{50} - \frac{1}{10} \right) = 1$$

$$\mathcal{K} \cdot \frac{5-1}{50} = 1$$

$$\mathcal{K} \cdot \frac{2}{25} = 1 \implies \mathcal{K} = \frac{25}{2}$$

$$\mathbb{P}[T > 10] = 1 - \mathbb{P}[T \leq 10]$$

$$= 1 - \int_0^{10} f_X(x)\, dx = 1 - \frac{25}{2} \int_0^{10} (10+x)^{-2}\, dx$$

$$= 1 - \frac{25}{2} \left(\frac{1}{10} - \frac{1}{20} \right) = 1 - \frac{25}{2} \cdot \frac{2-1}{20} = \frac{15}{40} = \frac{3}{8} \quad \blacksquare$$