M339J *Probability Models with Actuarial Applications* University of Texas at Austin
HW VII
Instructor: Milica Čudina

Note: You **must** show all your work. Numerical answers without a proper explanation or a clearly written down path to the solution will be assigned zero points.

Problem 7.1. (15 points) *Source: Based on Problem #165 from sample STAM Exam.*
Consider the following collective risk model:
(i) The claim count random variable N is Poisson with mean 3.
(ii) The severity random variable has the following probability (mass) function:

$$p_X(1) = 0.6,\ p_X(2) = 0.4.$$

(iii) As usual, individual loss random variables are mutually independent and independent of N.

Assume that an insurance covers **aggregate losses** subject to a deductible $d = 3$.

Find the expected value of aggregate payments for this insurance.

Solution:
Method I. Total aggregate losses are given by

$$S = X_1 + X_2 + \cdots + X_N.$$

So, the expected value of aggregate payments for this insurance equals

$$\mathbb{E}[(S-3)_+] = \mathbb{E}[S] - \mathbb{E}[S \wedge 3].$$

Wald's identity gives us

$$\mathbb{E}[S] = \mathbb{E}[N]\mathbb{E}[X] = 3(0.6(1) + 0.4(2)) = 4.2.$$

On the other hand, the distribution of the random variable $S \wedge 3$ is given by

$$S \wedge 3 \sim \begin{cases} 0 & \text{if } N = 0, \\ 1 & \text{if } N = 1 \text{ and } X_1 = 1, \\ 2 & \text{if } \{N = 1 \text{ and } X_1 = 2\} \text{ or } \{N = 2 \text{ and } X_1 = X_2 = 1\} \\ 3 & \text{otherwise.} \end{cases}$$

So, we have that
$\mathbb{P}[S \wedge 3 = 0] = \mathbb{P}[N = 0] = e^{-3}$,
$\mathbb{P}[S \wedge 3 = 1] = \mathbb{P}[N = 1]\mathbb{P}[X = 1] = 3e^{-3}(0.6) = 1.8e^{-3}$,
$\mathbb{P}[S \wedge 3 = 2] = \mathbb{P}[N = 1]\mathbb{P}[X = 2] + \mathbb{P}[N = 2](\mathbb{P}[X = 1])^2 = 3e^{-3}(0.4) + \dfrac{3^2}{2}e^{-3}(0.6)^2 = 2.82e^{-3}$,
$\mathbb{P}[S \wedge 3 = 3] = \mathbb{P}[N = 0] = 1 - 5.62e^{-3}$

Therefore,

$$\mathbb{E}[S \wedge 3] = 1.8e^{-3} + 2(2.82)e^{-3} + 3(1 - 5.62e^{-3}) = 2.53101.$$

So, our answer is $\mathbb{E}[(S-3)_+] = 4.2 - 2.53101 = 1.66899$.

Method II. We are supposed to calculate $\mathbb{E}[(S-3)_+]$. We wish to use the formula

$$\mathbb{E}[(S-3)_+] = \mathbb{E}[S] - \mathbb{E}[S \wedge 3].$$

We have

$$\begin{aligned} \mathbb{E}[X] &= 1 \cdot 0.6 + 2 \cdot 0.4 = 1.4, \\ \mathbb{E}[S] &= \mathbb{E}[N]\,\mathbb{E}[X] = 3 \cdot 1.4 = 4.2. \end{aligned}$$

Also,

$$\begin{aligned} \mathbb{E}[S \wedge 3] &= \mathbb{P}[S > 0] + \mathbb{P}[S > 1] + \mathbb{P}[S > 2] \\ &= 3 - (\mathbb{P}[S \le 0] + \mathbb{P}[S \le 1] + \mathbb{P}[S \le 2]) \\ &= 3 - (3\mathbb{P}[S = 0] + 2\mathbb{P}[S = 1] + \mathbb{P}[S = 2]). \end{aligned}$$

Calculating the above probabilities, using the provided distributions of N and X and their independence, we get

$\mathbb{P}[S = 0] = \mathbb{P}[N = 0] = e^{-3}$,

$\mathbb{P}[S = 1] = \mathbb{P}[N = 1, X_1 = 1] = 3e^{-3} \cdot 0.6 = 1.8e^{-3}$,

$\mathbb{P}[S = 2] = \mathbb{P}[N = 1, X_1 = 2] + \mathbb{P}[N = 2, X_1 = 1, X_2 = 1] = 3e^{-3} \cdot 0.4 + \dfrac{9}{2}e^{-3} \cdot 0.6 \cdot 0.6 = 2.82e^{-3}$.

So,

$$\begin{aligned} \mathbb{E}[S \wedge 3] &= 3 - (3\mathbb{P}[S = 0] + 2\mathbb{P}[S = 1] + \mathbb{P}[S = 2]) \\ &= 3 - (3e^{-3} + 2 \cdot 1.8e^{-3} + 2.82e^{-3}) \\ &= 3 - 9.42e^{-3}. \end{aligned}$$

Finally,

$$\mathbb{E}[(S-3)_+] = 4.2 - (3 - 9.42e^{-3}) = 1.2 + 9.42e^{-3} \approx 1.669.$$

Problem 7.2. (10 pts) We are using the aggregate loss model and our usual notation. The frequency random variable N is assumed to be Poisson distributed with mean equal to 1. The severity random variable is assumed to have the following probability mass function:

$$p_X(100) = 3/5, \quad p_X(200) = 3/10, \quad p_X(300) = 1/10.$$

Find the probability that the total aggregate loss **exactly** equals 300.

Solution: If we focus on the event that $\{S = 300\}$, we know that the number of losses must be $1, 2$ or 3.

$$\begin{aligned}
\mathbb{P}[S = 300] &= \mathbb{P}[S = 300 \mid N = 1]\mathbb{P}[N = 1] + \mathbb{P}[S = 300 \mid N = 2]\mathbb{P}[N = 2] + \mathbb{P}[S = 300 \mid N = 3]\mathbb{P}[N = 3] \\
&= \mathbb{P}[X_1 = 300 \mid N = 1]\mathbb{P}[N = 1] + \mathbb{P}[X_1 + X_2 = 300 \mid N = 2]\mathbb{P}[N = 2] \\
&\quad + \mathbb{P}[X_1 + X_2 + X_3 = 300 \mid N = 3]\mathbb{P}[N = 3] \\
&= p_X(300)p_N(1) + 2p_X(100)p_X(200)p_N(2) + (p_X(100))^3 p_N(3) \\
&= \frac{1}{10}e^{-1} + 2\left(\frac{3}{5}\right)\left(\frac{3}{10}\right)e^{-1}\left(\frac{1}{2}\right) + \left(\frac{3}{5}\right)^3 e^{-1}\left(\frac{1}{6}\right) = 0.316e^{-1} = 0.11625.
\end{aligned}$$

Problem 7.3. (10 pts) In the compound model for aggregate claims, let the frequency random variable N have the geometric distribution with mean 4.

Moreover, let the individual losses have the distribution

$$p_X(0) = 1/2, p_X(100) = 1/2.$$

Define the aggregate loss as $S = \sum_{j=1}^{N} X_j$. How much is $\mathbb{E}[(S - 100)_+]$?

Solution: As usual, we start with

$$\mathbb{E}[(S - 100)_+] = \mathbb{E}[S] - \mathbb{E}[S \wedge 100].$$

We have

$$\mathbb{E}[S] = \mathbb{E}[N]\mathbb{E}[X] = 4 \cdot 50 = 200.$$

On the other hand, since the possible values of S are $\{0, 100, 200, \dots\}$,

$$\mathbb{E}[S \wedge 100] = 0 \cdot \mathbb{P}[S = 0] + 100 \cdot \mathbb{P}[S \geq 100] = 100(1 - \mathbb{P}[S < 100]) = 100(1 - \mathbb{P}[S = 0]).$$

Note that, due to the usual independence assumptions,

$$\begin{aligned}
\mathbb{P}[S = 0] &= \mathbb{P}[N = 0] + \mathbb{P}[N = 1, X_1 = 0] + \dots + \mathbb{P}[N = k, X_1 = X_2 = \dots = X_k = 0] + \dots \\
&= \mathbb{P}[N = 0] + \mathbb{P}[N = 1]\,\mathbb{P}[X = 0] + \dots + \mathbb{P}[N = 1]\,(\mathbb{P}[X = 0])^k + \dots \\
&= \frac{1}{1+\beta} + \frac{\beta}{(1+\beta)^2} \cdot \frac{1}{2} + \dots + \frac{\beta^k}{(1+\beta)^{k+1}} \cdot \frac{1}{2^k} + \cdots \\
&= \frac{1}{1+\beta}[1 + \frac{\beta}{1+\beta} \cdot \frac{1}{2} + \dots + \frac{\beta^k}{(1+\beta)^k} \cdot \frac{1}{2^k} + \cdots] \\
&= \frac{1}{1+\beta} \cdot \frac{1}{1 - \frac{\beta}{2(1+\beta)}} \\
&= \frac{\beta}{2+\beta} \\
&= \frac{1}{3}.
\end{aligned}$$

So,

$$\mathbb{E}[(S-100)_+] = 200 - \frac{200}{3} = \frac{400}{3} \approx 133.33.$$

Problem 7.4. (10 points) In the compound model for aggregate claims, let the frequency random variable N be negative binomial with parameters $r = 15$ and $\beta = 5$.

Moreover, let the common distribution of the i.i.d. severity random variables $\{X_j; j = 1, 2, \dots\}$ be the two-parameter Pareto with $\alpha = 3$ and $\theta = 10$.

Let our usual assumptions hold, i.e., let N be independent of $\{X_j; j = 1, 2, \dots\}$. The insurer is interested in finding the total premium π such that the aggregate losses exceed it with the probability less than or equal to 5%. Using the normal approximation, find π such that

$$\mathbb{P}[S > \pi] = 0.05.$$

Solution: Let $\mu_S = \mathbb{E}[S]$ and $\sigma_S = \sqrt{Var[S]}$. Then, using the normal approximation, we have

$$0.05 = \mathbb{P}[S > \pi] = \mathbb{P}\left[\frac{S-\mu_S}{\sigma_S} > \frac{\pi-\mu_S}{\sigma_S}\right] \approx 1 - \Phi\left(\frac{\pi-\mu_S}{\sigma_S}\right)$$

where Φ denotes the c.d.f. of the standard normal distribution. From the tables for Φ, we get

$$\pi = \mu_S + 1.645\sigma_S.$$

From the given information on the severity r.v.s, we obtain

$$\begin{aligned}
\mathbb{E}[X] &= \frac{\theta}{\alpha-1} = \frac{10}{3-1} = 5,\\
Var[X] &= \frac{\theta^2 \cdot 2}{(\alpha-1)(\alpha-2)} - (\frac{\theta}{\alpha-1})^2 = \frac{\theta^2 \cdot \alpha}{(\alpha-1)^2(\alpha-2)} = \frac{10^2 \cdot 3}{(3-1)^2(3-2)} = 75,\\
\mathbb{E}[N] &= r\beta = 75,\\
Var[N] &= r\beta(1+\beta) = 450.
\end{aligned}$$

So,

$$\begin{aligned}
\mu_S &= \mathbb{E}[S] = \mathbb{E}[X]\mathbb{E}[N] = 5 \cdot 75 = 375,\\
\sigma_S &= Var[S] = Var[X]\mathbb{E}[N] + Var[N]\mathbb{E}[X]^2 = 75 \cdot 75 + 450 \cdot 5^2 = 16,875.
\end{aligned}$$

Hence,

$$\pi = 375 + 1.645 \cdot \sqrt{16875} \approx 588.692.$$

Problem 7.5. (5 points) An insurer pays aggregate claims in excess of the deductible d. In return, they receive a stop-loss premium $\mathbb{E}[(S-d)_+]$. You model the aggregate losses S using a continuous distribution. Moreover, you are given the following information about the aggregate losses S:

(i) $\mathbb{E}[(S-100)_+] = 15$,
(ii) $\mathbb{E}[(S-120)_+] = 10$,

(iii) $\mathbb{P}[80 < S \leq 120] = 0$.

Find the probability that the aggregate claim amounts are less than or equal to 80.

Solution: From the given fact (i), we know that

$$\mathbb{E}[(S-100)_+] = 15 = \int_{100}^{\infty} S_S(x)\,dx$$

where S_S denotes the survival function of the random variable S. Similarly, From the given fact (ii), we know that

$$\mathbb{E}[(S-120)_+] = 10 = \int_{120}^{\infty} S_S(x)\,dx$$

Therefore,

$$\int_{100}^{120} S_S(x)\,dx = 5.$$

From the given fact (iii), we know that the survival function is constant over the interval $[80, 120]$. In particular, we can write $S_S(x) = S_S(80)$ for all $x \in [100, 120]$. Substituting this finding into the equality above, we get

$$20 S_S(80) = 5 \quad \Rightarrow \quad S_S(80) = \frac{1}{4} \quad \Rightarrow \quad F_S(80) = \frac{3}{4}$$

where F_S denotes the cumulative distribution function of the aggregate losses S.