M339D: October 30^{th}, 2024.

UNIVERSITY OF TEXAS AT AUSTIN

Problem Set #10

Binomial option pricing: Two or more periods.

Problem 10.1. For a two-period binomial model, you are given that:

(1) each period is one year; $h=1$

(2) the current price of a non-dividend-paying stock S is $S(0) = \$20$;

(3) $u = 1.2$ with u as in the standard notation for the binomial model;

(4) $d = 0.8$, with d as in the standard notation for the binomial model;

(5) the continuously compounded risk-free interest rate is $r = 0.04$.

Consider a **special** call option which pays the excess above the strike price $K = 23$ (if any!) at the end of **every** binomial period.

Find the price of this option.

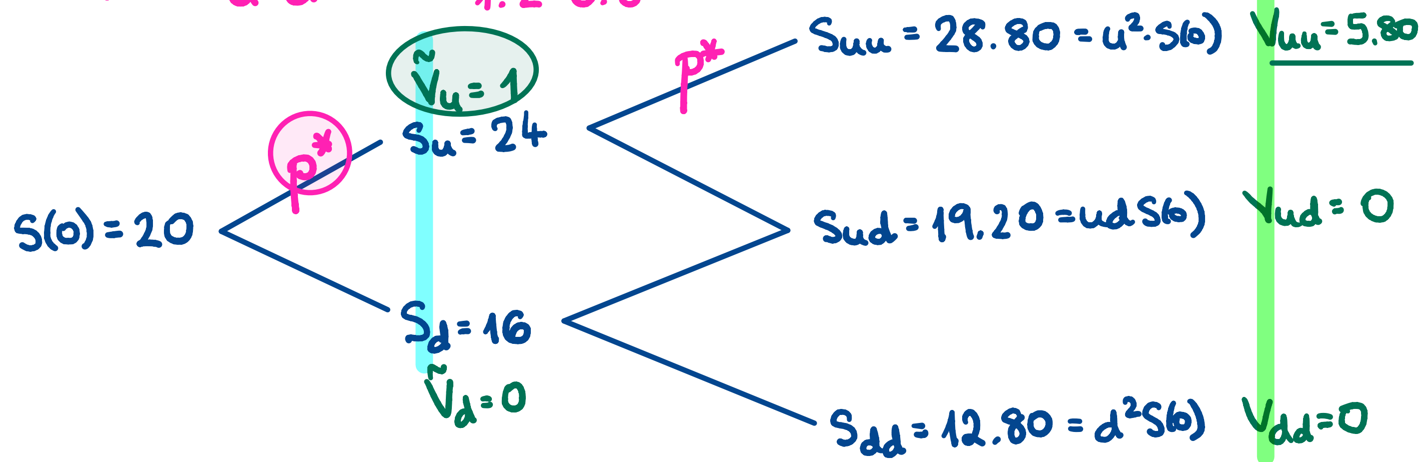
1st $p^* = ?$

2nd Tree + Payoffs

3rd Risk-Neutral Pricing.

Risk-Neutral Probability:

$$p^* = \frac{e^{rh} - d}{u-d} = \frac{e^{0.04} - 0.8}{1.2 - 0.8} = 0.602$$



$$\tilde{V}(0) = e^{-0.04} \cdot p^* \cdot 1 = 0.5784$$

$$V(0) = e^{-0.04(2)} (p^*)^2 \cdot 5.80 = 1.9413$$

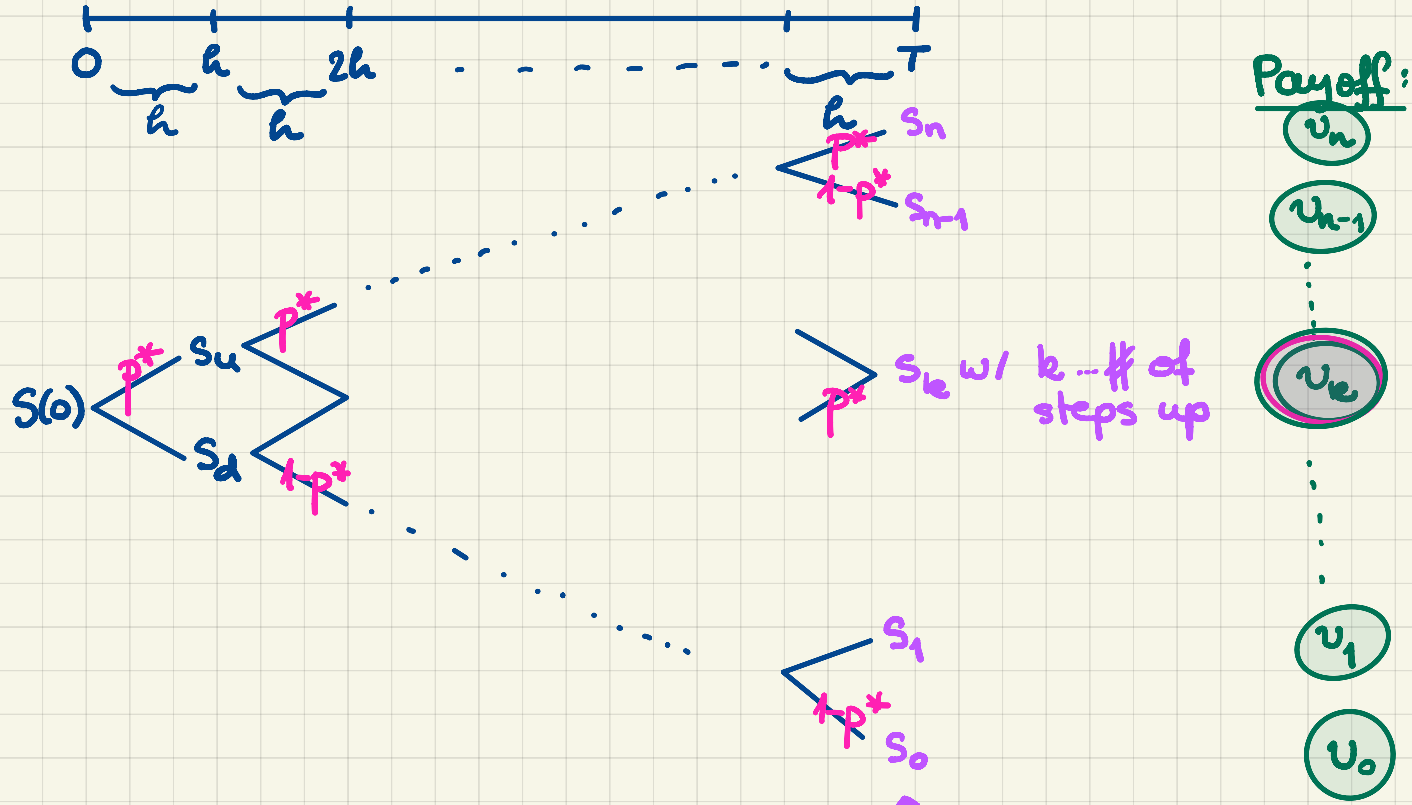
$+$

answer: the price of the special call is $\boxed{2.5197}$

Multiple Binomial Periods.

T... exercise date of a European option
n... # of periods

the length of each period: $\boxed{h = \frac{T}{n}}$



The $n+1$ values are the support of $S(T)$

$\Rightarrow$ for every $k = 0, 1, \ldots, n$:

$$S_k = S(0)\, u^k \cdot d^{n-k} = S(0) \left(\frac{u}{d}\right)^{k} \cdot d^n$$

# of upsteps

Consider a European option w/ payoff function $v(\cdot)$.
Then, the possible payoff values will be

$$\boxed{v_k := v(S_k)}$$

Risk-Neutral Pricing:

$$V(0) = e^{-rT}\, \mathbb{E}^*[V(T)] \quad \text{w/} \quad \boxed{p^* = \frac{e^{rh} - d}{u - d}}$$

⇒ The risk-neutral probability of reaching the payoff v_k is

$$\boxed{\binom{n}{k}(p^*)^k(1-p^*)^{n-k}}$$

⇒ The risk-neutral option price:

$$\boxed{V(0) = e^{-rT}\sum_{k=0}^{n}\left(\binom{n}{k}(p^*)^k(1-p^*)^{n-k}\cdot v_k\right)}$$

Problem 10.2. Let the continuously compounded risk-free interest rate be 0.10. Let the initial price of a non-dividend-paying stock be $100 per share. You use a five-period binomial tree to model the stock price over the next year. Let $u = 1.04$ and $d = 0.96$.

$n=5 \Rightarrow h=\frac{1}{5}$

$T=1$ What is the price of a one-year, at-the-money European call option on the above stock?

$K=100$

→: The risk-neutral probability:

$$p^* = \frac{e^{rh}-d}{u-d} = \frac{e^{0.1(1/5)}-0.96}{1.04-0.96} = \underline{0.7525}$$

The relevant stock prices in our tree:

$S_5 = S(0)\cdot u^5 = 100(1.04)^5 = \underline{121.67}$ $\Rightarrow v_5 = 21.67$

$S_4 = S(0)\cdot u^4\cdot d = 100(1.04)^4\cdot d = \underline{112.31}$ $\Rightarrow v_4 = 12.31$

$S_3 = S(0)\cdot u^3\cdot d^2 = 100(1.04)^3\cdot(0.96)^2 = \underline{103.67}$ $\Rightarrow v_3 = 3.67$

The remaining terminal nodes are all out-of-money

$\Rightarrow$

$$V(0) = e^{-0.10}\Big(21.67\cdot(p^*)^5 + 12.31\cdot 5\cdot(p^*)^4\cdot(1-p^*) + 3.67\underbrace{\binom{5}{2}}_{10}\cdot(p^*)^3(1-p^*)^2\Big) = \underline{10.01821}$$

$\square$