

M378K: September 22nd, 2025.

More on the Cumulative Distribution Function.

Example. The Normal Distribution.

$$Y \sim N(\mu, \sigma)$$

The pdf is: $f_Y(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-\mu)^2}{2\sigma^2}}$ for all $y \in \mathbb{R}$

The cdf is:

$$\begin{aligned} F_Y(y) &= \int_{-\infty}^y f_Y(u) du \\ &= \int_{-\infty}^y \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(u-\mu)^2}{2\sigma^2}} du \end{aligned}$$

No ANALYTIC CDF!

Fact. $\frac{Y - \mu_Y}{\sigma_Y} \sim N(0,1) \sim Z$

$$Y = \mu_Y + \sigma_Y \cdot Z$$

$$\begin{aligned} F_Y(y) &= \mathbb{P}[Y \leq y] = \mathbb{P}\left[\frac{Y - \mu_Y}{\sigma_Y} \leq \frac{y - \mu_Y}{\sigma_Y}\right] \\ &= \mathbb{P}\left[Z \leq \frac{y - \mu_Y}{\sigma_Y}\right] = \Phi\left(\frac{y - \mu_Y}{\sigma_Y}\right) \end{aligned}$$

CDF of $N(0,1)$... Φ

Def'n. Let Y be a random variable w/ the cdf F_Y .
 For $\alpha \in (0,1)$ the α -quantile of the dist'n of the
 r.v. Y is defined as the number

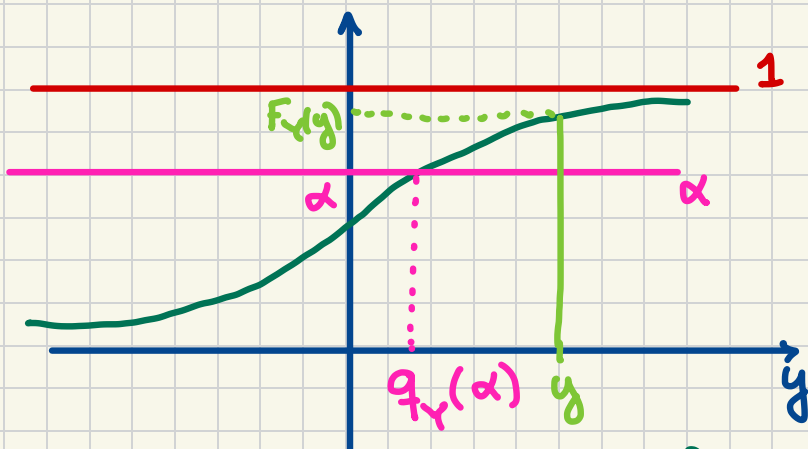
$$q_Y(\alpha) \in \mathbb{R}$$

which satisfies

$$P[Y \leq q_Y(\alpha)] = \alpha$$



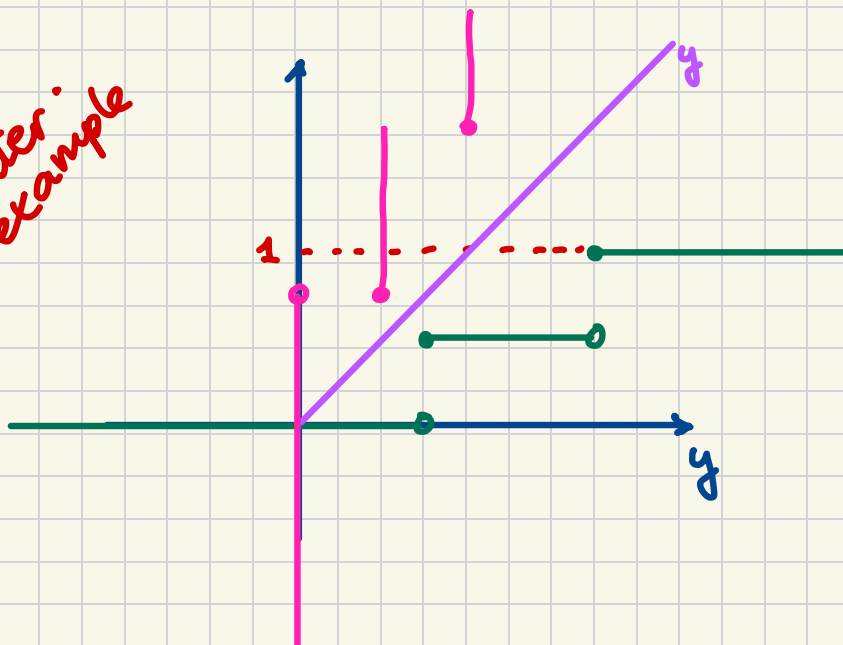
$$F_Y(q_Y(\alpha)) = \alpha$$



If F_Y^{-1} inverse function of the cdf exists, then $q_Y(\alpha) = F_Y^{-1}(\alpha)$.

This is, for example, the case w/ the normal.

Counter example



M378K Introduction to Mathematical Statistics

Problem Set #7

Cumulative distribution functions: Named continuous distributions.

Problem 7.1. Source: Sample P exam, Problem #126.

A company's annual profit is normally distributed with mean 100 and variance 400. Then, we can express the probability that the company's profit in a year is at most 60, given that the profit in the year is positive in terms of the standard normal cumulative distribution function denoted by Φ as

$$1 - \frac{\Phi(2)}{\Phi(5)}$$

True or false? Why?

→: $Y \sim N(\mu=100, \text{sd}=\sigma=20)$

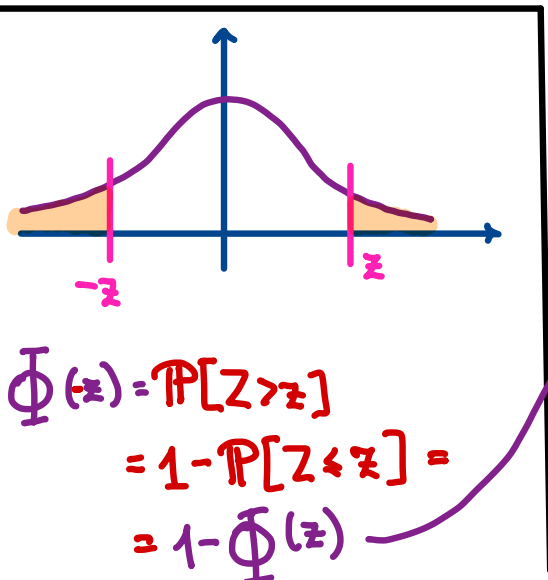
$$\begin{aligned} \mathbb{P}[Y \leq 60 | Y > 0] &= \frac{\mathbb{P}[0 < Y \leq 60]}{\mathbb{P}[Y > 0]} \\ &= \frac{\mathbb{P}\left[\frac{0-100}{20} < \frac{Y-100}{20} \leq \frac{60-100}{20}\right]}{\mathbb{P}\left[\frac{Y-100}{20} > \frac{0-100}{20}\right]} = \end{aligned}$$

$$= \frac{\mathbb{P}[-5 < Z \leq -2]}{\mathbb{P}[-5 < Z]}$$

$$= \frac{\Phi(-2) - \Phi(-5)}{1 - \Phi(-5)}$$

$$= \frac{\cancel{1 - \Phi(2)} - \cancel{(1 - \Phi(5))}}{\Phi(5)}$$

$$= \frac{\Phi(5) - \Phi(2)}{\Phi(5)} = 1 - \frac{\Phi(2)}{\Phi(5)}$$



Example. The Exponential Dist'n.

$$Y \sim E(\tau)$$

pdf... $f_Y(y) = \frac{1}{\tau} e^{-\frac{y}{\tau}} \cdot \mathbb{1}_{[0, \infty)}(y)$

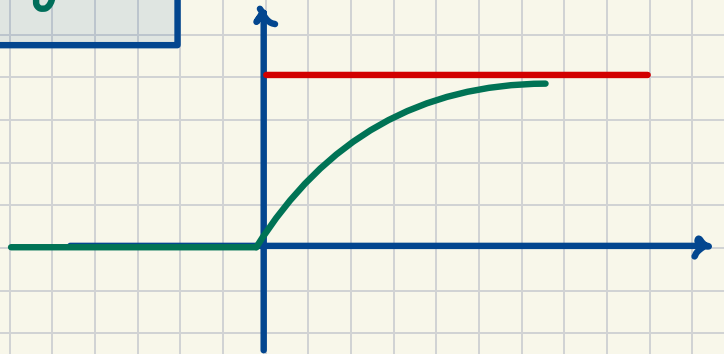
cdf... $F_Y(y) = ?$

Evidently, $F_Y(y) = \underline{0}$ for $y < 0$

For $y \geq 0$:

$$\begin{aligned} F_Y(y) &= \mathbb{P}[Y \leq y] = \int_0^y \frac{1}{\tau} e^{-\frac{u}{\tau}} du \\ &= \frac{1}{\tau} (-\cancel{\tau}) e^{-\frac{u}{\tau}} \Big|_{u=0}^y \\ &= - \left(e^{-\frac{y}{\tau}} - 1 \right) \end{aligned}$$

$$F_Y(y) = 1 - e^{-\frac{y}{\tau}} \quad y \geq 0$$



Problem 7.2. Source: Sample P exam, Problem #267.

The lifetime of a certain electronic device has an exponential distribution with mean 0.50. Calculate the probability that the lifetime of the device is greater than 0.70, given that it is greater than 0.40.

→ : Y... lifetime

$$Y \sim E(\tau = 0.5)$$

$$\begin{aligned} \mathbb{P}[Y > 0.7 \mid Y > 0.4] &= \frac{\mathbb{P}[Y > 0.7, Y > 0.4]}{\mathbb{P}[Y > 0.4]} \\ &= \frac{\mathbb{P}[Y > 0.7]}{\mathbb{P}[Y > 0.4]} \\ &= \frac{e^{-\frac{0.7}{\tau}}}{e^{-\frac{0.4}{\tau}}} = e^{-\frac{0.3}{\tau}} \\ &= e^{-0.6} \end{aligned}$$



This is a special case of the **memoryless property**, i.e.,

$$\mathbb{P}[Y > t+s \mid Y > t] = \mathbb{P}[Y > s]$$