M378K Introduction to Mathematical Statistics
Problem Set #2
Expectation and variance: the discrete case.

Problem 2.1. Source: Sample P exam, Problem #481.
The number of days required for a damage control team to locate and repair a leak in the hull of a ship is modeled by a discrete random variable N. N is uniformly distributed on $\{1, 2, 3, 4, 5\}$.

The cost of locating and repairing a leak is $N^2 + N + 1$.

Calculate the expected cost of locating and repairing a leak in the hull of the ship.

Solution: We need to calculate

$$\mathbb{E}[N^2 + N + 1] = \mathbb{E}[N^2] + \mathbb{E}[N] + 1.$$

By definition

$$\mathbb{E}[N] = 1 \cdot \frac{1}{5} + \cdots + 5 \cdot \frac{1}{5} = \frac{1}{5} \cdot \frac{5 \cdot 6}{2} = 3,$$
$$\mathbb{E}[N^2] = 1^2 \cdot \frac{1}{5} + \cdots + 5^2 \cdot \frac{1}{5} = \frac{1}{5} \cdot \frac{5 \cdot 6 \cdot 11}{6} = 11.$$

Our final answer is $3 + 11 + 1 = 15$.

Problem 2.2. Source: Sample P exam, Problem #458.
An investor wants to purchase a total of ten units of two assets, A and B, with annual payoffs **per unit** *purchased of X and Y, respectively. Each asset has the same purchase price per unit. The payoffs are independent random variables with equal expected values and with $\mathrm{Var}(X) = 30$ and $\mathrm{Var}(Y) = 20$.*

Calculate the number of units of asset A the investor should purchase to minimize the variance of the total payoff.

Solution: Let the number of units of asset A be denoted by n. Them, the total payoff will be equal to

$$nX + (10 - n)Y.$$

The variance of the total payoff will be

$$\mathrm{Var}[nX + (10 - n)Y].$$

Due to independence of X and Y, using the properties of variance, we get that the above equals

$$n^2 \,\mathrm{Var}[X] + (10 - n)^2 \,\mathrm{Var}[Y] = 30n^2 + 20(10 - n)^2 = 50n^2 - 400n + 2000 = 50(n^2 - 8n + 40).$$

We need to minimize the above with respect to n. This is easily done by noticing that $n^2 - 8n + 40$ is a parabola facing up. So, all we need to do is to figure out the position of its "bottom". In the notation you might recall from quadratic equations, it is

$$n^* = -\frac{B}{2A} = -\frac{-8}{2 \cdot 1} = 4.$$

Note: Yes, you could have also differentiated to find the minimum.