

The University of Texas at Austin
IN-CLASS WORK 9

M378K Introduction to Mathematical Statistics

February 28, 2026

MOMENT-GENERATING FUNCTIONS.

DEFINITION 9.1: *The k^{th} moment of a random variable Y taken about the origin is defined as $\mathbb{E}[Y^k]$ provided that the expectation exists. We write*

$$\mu_k = \mathbb{E}[Y^k]$$

when there is no ambiguity about the random variable in question.

Remark. μ_k is also referred to as the k^{th} raw moment.

Remark. In particular, $\mu_1 = \mu$ happens to be the mean of the random variable Y .

DEFINITION 9.2: *The k^{th} central moment of a random variable Y is defined as $\mathbb{E}[(Y - \mu)^k]$ provided that the expectation exists. We write*

$$\mu_k^c = \mathbb{E}[(Y - \mu)^k]$$

when there is no ambiguity about the random variable in question.

Remark. μ_k^c is also referred to as the k^{th} moment of a random variable Y taken about its mean.

DEFINITION 9.3: *The moment-generating function (mgf) m_Y for a random variable Y is defined as*

$$m_Y(t) = \mathbb{E}[e^{tY}]$$

for all t for which the above expectation exists. In fact, we say that the moment-generating function exists if there exists a positive number b such that $m_Y(t)$ is finite for all t such that $|t| \leq b$.

Problem 9.1. How much is $m_Y(0)$?

Solution.

$$m_Y(0) = 1.$$

Remark. On the choice of terminology ...

Step 1.

$$\frac{d}{dt}m_Y(t) = ?$$

Solution.

$$\mathbb{E}[Ye^{tY}]$$

Step 2.

$$m'_Y(0) = ?$$

Solution.

$$\mu_Y = \mathbb{E}[Y]$$

Step 3.

$$\frac{d^2}{dt^2}m_Y(t) = ?$$

Solution.

$$\mathbb{E}[Y^2e^{tY}]$$

Step 4.

$$m''_Y(0) = ?$$

Solution.

$$\mu_2 = \mathbb{E}[Y^2]$$

Step 5. What do you suspect the **generalization** of the above would be?

THEOREM 9.4: If m_Y exists, then for $k \in \mathbb{N}$, we have

$$m_Y^{(k)}(0) = \mu_k.$$

EXAMPLE 9.5: Let $Y \sim b(n = 1, p)$, i.e., let Y model a Bernoulli trial with the probability of success denoted by p . Find m_Y .

Solution.

$$m_Y(t) = \mathbb{E}[e^{tY}] = (1-p)e^0 + pe^t = (1-p) + pe^t, \quad t \in \mathbb{R}.$$

PROPOSITION 9.6: Let Y_1 and Y_2 be independent random variables with m.g.f.s denoted by m_{Y_1} and m_{Y_2} . Define $Y = Y_1 + Y_2$. Then, for every t for which both m_{Y_1} and m_{Y_2} are well defined, we have

$$m_Y(t) =$$

Proof: By definition:

$$m_Y(t) =$$

Solution.

$$\mathbb{E}[e^{tY}]$$

Using $Y = Y_1 + Y_2$, we can substitute $Y_1 + Y_2$ for Y in the expression above. So,

$$m_Y(t) =$$

Solution.

$$\mathbb{E}[e^{t(Y_1+Y_2)}]$$

One of the properties of the exponential function is that $e^{A+B} = e^A \times e^B$. Thus, the above becomes:

$$m_Y(t) =$$

Solution.

$$\mathbb{E}[e^{tY_1}] \times \mathbb{E}[e^{tY_2}]$$

Recall that Y_1 and Y_2 are assumed to be independent random variables. With this in mind, we get:

$$m_Y(t) =$$

Solution.

$$\mathbb{E}[e^{tY_1}] \times \mathbb{E}[e^{tY_2}]$$

Finally, using the definition of a moment generating function, we have

$$m_Y(t) =$$

Solution.

$$m_{Y_1}(t)m_{Y_2}(t)$$

■

EXAMPLE 9.7: Let $Y \sim b(n, p)$. What is the moment generating function of Y ?

Solution.

$$Y \stackrel{(d)}{=} X_1 + \dots + X_n$$

with $X_i \sim b(1, p)$, $i = 1, \dots, n$ independent random variables. Then,

$$m_Y(t) = m_{X_1}(t) \times \dots \times m_{X_n}(t) = (m_{X_1}(t))^n = (1 - p + pe^t)^n.$$

EXAMPLE 9.8: Let $N \sim \text{Poisson}(\lambda)$. What is the moment generating function m_N of N ?

Solution.

$$\begin{aligned} m_N(t) &= \mathbb{E}[e^{tN}] \\ &= \sum_{n=0}^{\infty} e^{tn} p_N(n) \\ &= \sum_{n=0}^{\infty} e^{tn} e^{-\lambda} \frac{\lambda^n}{n!} \\ &= e^{-\lambda} \sum_{n=0}^{\infty} \frac{(e^t \lambda)^n}{n!} = e^{-\lambda} \times e^{\lambda e^t} = e^{\lambda(e^t - 1)}. \end{aligned}$$

EXAMPLE 9.9: Let $Z \sim N(0, 1)$. What is the moment generating function m_Z of Z ?

Solution.

$$m_Z(t) = e^{t^2/2} \quad t \in \mathbb{R}.$$

EXAMPLE 9.10: Let the random variable Y have the mgf m_Y . Define $X = aY + b$ for some constants a and b . Express the mgf m_X of X in terms of m_Y , a and b .

Solution.

$$m_X(t) = e^{bt} m_Y(at)$$

EXAMPLE 9.11: Let $X \sim N(\mu, \sigma^2)$. What is the moment generating function m_X of X ?

Solution. Since X can be expressed as a linear transform of $Z \sim N(0, 1)$ in the following way

$$X = \mu + \sigma Z,$$

we get that

$$m_X(t) = e^{\mu t} m_Z(\sigma t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

Problem 9.2. A random variable Y is said to be *lognormal* if there exists a normally distributed random variable $X \sim N(\mu, \sigma^2)$ such that $Y \stackrel{(d)}{=} e^X$. Express the mean and the variance of the lognormal r.v. Y in terms of the parameters μ and σ .

Solution.

$$\mathbb{E}[Y] = \mathbb{E}[e^X] = \mathbb{E}[e^{1 \times X}] = m_X(1) = \exp\left\{\frac{1}{2}\sigma^2 + \mu\right\}.$$

$$\mathbb{E}[Y^2] = \mathbb{E}\left[(e^X)^2\right] = \mathbb{E}[e^{2 \times X}] = m_X(2) = \exp\left\{\frac{1}{2}\sigma^2 \times 4 + \mu \times 2\right\} = \exp\{2(\sigma^2 + \mu)\}.$$

$$Var[Y] = \mathbb{E}\left[(e^X)^2\right] - (\mathbb{E}[e^X])^2 = \exp\{2(\sigma^2 + \mu)\} - \exp\{\sigma^2 + 2\mu\}.$$

PROPOSITION 9.12:

- If m_Y exists for a certain probability distribution, then it is unique.
- If m_{Y_1} and m_{Y_2} are equal on an interval, then $Y_1 \stackrel{(d)}{=} Y_2$.

COROLLARY 9.13: Let Y_1 and Y_2 be independent and normally distributed. Define $Y = Y_1 + Y_2$. Then, the distribution of Y is ...

Proof:

Solution. Note that $Y_i \sim N(\mu = mu_i, \sigma_i)$ for $i = 1, 2$. Now, let's look at the mgf of Y . Then, since Y_1 and Y_2 are independent, we have

$$m_Y(t) = m_{Y_1}(t)m_{Y_2}(t).$$

We can now use the fact that for any $X \sim N(\mu, \sigma)$,

$$m_X(t) = e^{\mu t} m_Z(\sigma t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

Hence,

$$m_Y(t) = e^{\mu_1 t + \frac{\sigma_1^2 t^2}{2}} \times e^{\mu_2 t + \frac{\sigma_2^2 t^2}{2}} = e^{(\mu_1 + \mu_2)t + \frac{(\sigma_1^2 + \sigma_2^2)t^2}{2}}$$

We can conclude that $Y \sim N(\mu_1 + \mu_2, \sqrt{\sigma_1^2 + \sigma_2^2})$. ■

COROLLARY 9.14: Let N_1 and N_2 be independent and Poisson distributed. Define $N = N_1 + N_2$. Then, the distribution of N is ...

Proof:

Solution. We are given that $N_i \sim Poisson(\lambda_i)$ for $i = 1, 2$. We saw in a previous example that

$$m_{N_i}(t) = e^{\lambda_i(e^t - 1)}.$$

Hence,

$$m_N(t) = m_{N_1}(t)m_{N_2}(t) = e^{\lambda_1(e^t - 1)} \times e^{\lambda_2(e^t - 1)} = e^{(\lambda_1 + \lambda_2)(e^t - 1)}$$

We can conclude that $N \sim Poisson(\lambda_1 + \lambda_2)$. ■