M378K Introduction to Mathematical Statistics
Problem Set #6
Moment generating functions.

Definition 6.1. *The* k^{th} moment of a random variable Y taken about the origin *is defined as* $\mathbb{E}[Y^k]$ *provided that the expectation exists. We write*

$$\mu_k = \mathbb{E}[Y^k]$$

when there is no ambiguity about the random variable in question.

Remark 6.2. μ_k *is also referred to as the* k^{th} raw moment.

Remark 6.3. *In particular,* $\mu_1 = \mu$ *happens to be the* **mean** *of the random variable* Y.

Definition 6.4. *The* k^{th} central moment *of a random variable* Y *is defined as* $\mathbb{E}[(Y-\mu)^k]$ *provided that the expectation exists. We write*

$$\mu_k^c = \mathbb{E}[(Y-\mu)^k]$$

when there is no ambiguity about the random variable in question.

Remark 6.5. μ_k *is also referred to as the* k^{th} moment of a random variable Y taken about its mean.

Definition 6.6. *The* moment-generating function (mgf) m_Y *for a random variable* Y *is defined as*

$$m_Y(t) = \mathbb{E}[e^{tY}]$$

for all t *for which the above expectation exists. In fact, we say that the moment-generating function* **exists** *if there exists a positive number* b *such that* $m_Y(t)$ *is finite for all* t *such that* $|t| \leq b$.

Problem 6.1. *How much is* $m_Y(0)$?

Remark 6.7. *On the choice of terminology ...*

Step 1.

$$\frac{d}{dt} m_Y(t) = ?$$

Step 2.

$$m_Y'(0) = ?$$

Step 3.

$$\frac{d^2}{dt^2} m_Y(t) = ?$$

Step 4.

$$m_Y''(0) = ?$$

Step 5. *What do you suspect the* **generalization** *of the above would be?*

Theorem 6.8. *If m_Y exists, then for $k \in \mathbb{N}$, we have*

$$m_Y^{(k)}(0) = \mu_k.$$

Example 6.9. *Let $Y \sim b(n = 1, p)$, i.e., let Y model a Bernoulli trial with the probability of success denoted by p. Find m_Y.*

Proposition 6.10. *Let Y_1 and Y_2 be independent random variables with m.g.f.s denoted by m_{Y_1} and m_{Y_2}. Define $Y = Y_1 + Y_2$. Then, for every t for which both m_{Y_1} and m_{Y_2} are well defined, we have*

$$m_Y(t) =$$

Proof. By definition:

$$m_Y(t) =$$

Using $Y = Y_1 + Y_2$, we can subsitute $Y_1 + Y_2$ for Y in the expression above. So,

$$m_Y(t) =$$

One of the properties of the exponential function is that $e^{A+B} = e^A \times e^B$. Thus, the above becomes:

$$m_Y(t) =$$

Recall that Y_1 and Y_2 are assumed to be independent random variables. With this in mind, we get:

$$m_Y(t) =$$

Finally, using the definition of a m.g.f., we have

$$m_Y(t) =$$

$\square$

Example 6.11. *Let $Y \sim b(n, p)$. What is the moment generating function of Y?*

Example 6.12. *Let $N \sim Poisson(\lambda)$. What is the moment generating function m_N of N?*

Example 6.13. *Let $Z \sim N(0, 1)$. What is the moment generating function m_Z of Z?*

Example 6.14. *Let the random variable Y have the mgf m_Y. Define $X = aY + b$ for some constants a and b. Express the mgf m_X of X in terms of m_Y, a and b.*

Example 6.15. *Let $X \sim N(\mu, \sigma^2)$. What is the moment generating function m_X of X?*

Problem 6.2. *A random variable Y is said to be* lognormal *if there exists a normally distributed random variable $X \sim N(\mu, \sigma^2)$ such that $Y \overset{(d)}{=} e^X$. Express the mean and the variance of the lognormal r.v. Y in terms of the parameters μ and σ.*

Proposition 6.16.

1. *If m_Y exists for a certain probability distribution, then it is unique.*
2. *If m_{Y_1} and m_{Y_2} are equal on an interval, then $Y_1 \overset{(d)}{=} Y_2$.*

Corollary 6.17. *Let X_1 and X_2 be independent and normally distributed. Define $X = X_1 + X_2$. Then, the distribution of X is ...*

Proof. $X_i \sim N(\mu = \check{\mu}_i, \sigma_i^2)$ for $i = 1, 2$

□

Corollary 6.18. *Let N_1 and N_2 be independent and Poisson distributed. Define $N = N_1 + N_2$. Then, the distribution of N is ...*

Proof. $N_i \sim Poisson(\lambda_i)$ for $i = 1, 2$

□