University of Texas at Austin

HW Assignment 4

Prerequisite material. Realized returns.

Provide your **complete solution** to the following problems. Final answers only, without appropriate justification, will receive zero points even if correct.

Problem 4.1. (5 points) *Source: Open Course Intro to Statistics.*
Cholesterol levels for women aged 20 to 34 follow an approximately normal distribution with mean 185 milligrams per deciliter (mg/dl). Women with cholesterol levels above 220 mg/dl are considered to have high cholesterol and about 18.5% of women fall into this category. What is the standard deviation of the distribution of cholesterol levels for women aged 20 to 34?

Solution: Let X be the random variable denoting the cholesterol level. Then,

$$X \sim N(mean = 185, variance = \sigma^2).$$

We are given that

$$\mathbb{P}[X > 220] = 0.185 \quad \Rightarrow \quad \mathbb{P}[X \leq 220] = 1 - 0.185 = 0.815.$$

So,

$$220 = 185 + \sigma z_*$$

where z_* is the critical value such that $N(z_*) = 0.815$. The closest value in the standard normal tables is $z_* = 0.9$. Hence, our answers is

$$\sigma = \frac{220 - 185}{0.9} = 38.8889$$

Problem 4.2. (5 points) A fair tetrahedron (a four-sided symmetric Platonic solid) with sides dyed pink, purple, mauve, and fuchsia will be rolled 2000 times. You intend to record the color of the side the tetrahedron fell on after every roll. What is the approximate probability that the outcome is mauve at most 510 times? There is no need to use the continuity correction.

Solution: The number of occurrences of mauve is

$$X \sim Binomial(2000, 1/4).$$

Since $2000(1/4) > 10$ and $2000(3/4) > 10$, we can use the normal approximation. We have

$$\mathbb{E}[X] = 2000(1/4) = 500 \quad \text{and} \quad SD[X] = \sqrt{2000(1/4)(3/4)} = 19.36492.$$

Hence,

$$\mathbb{P}[X \leq 510] = \mathbb{P}\left[\frac{X - 500}{19.36492} \leq \frac{510 - 500}{19.36492}\right] \approx N(0.52) = 0.6985.$$

Problem 4.3. (5 points) You roll a fair tetrahedron whose sides are labeled by 1, 2, 3, and 4 a total of 4000 times. What is the approximate probability that you see a 1 strictly more than 1025 times? There is no need to use the continuity correction.

Solution: The number of heads is $X \sim Binomial(n = 4000, p = 0.25)$. Evidently, we can use the normal approximation to the binomial. We have

$$\mu_X = \mathbb{E}[X] = 1000 \quad \text{and} \quad \sigma_X = 27.38613.$$

The probability we are seeking is

$$\mathbb{P}[X > 1025] \approx 1 - N\left(\frac{1025 - 1000}{27.38613}\right) \approx 1 - N(0.91) = 1 - 0.8186 = 0.1814.$$

Problem 4.4. (5 pts) For a stock price that was initially \$55.00, what is the price after 4 years if the continuously compounded returns for these 4 years are $4.5\%, 6.2\%, 8.9\%$, and -3.2%?

Solution:

$$55e^{0.045+0.062+0.089-0.032} \approx 64.80.$$

Problem 4.5. (5 pts) A non-dividend-paying stock is valued at \$55.00. The annual expected return is 12.0% and the standard deviation of annualized returns is 22.0%. If the stock is lognormally distributed, what is the expected stock price after 3 years?

Solution: Let us denote the stock price today by $S(0)$ and that in three years by $S(3)$. According to the work we did in class, we need to calculate

$$\mathbb{E}[S(3)] = S(0)e^{3\alpha}$$

with α equal to the expected continuously compounded rate of return on the stock S. We are given in the problem that $\alpha = 0.12$. So, the answer is $55e^{0.36} \approx 78.83$.

Problem 4.6. (5 pts) For a stock price that was initially \$55.00, what is the price after 4 years if the observed continuously compounded returns for these 4 years are $4.5\%, 6.2\%, 8.9\%$, and 3.2%?

Solution:

$$55e^{0.045+0.062+0.089+0.032} \approx 69.08.$$

Problem 4.7. (5 points) A continuous-dividend-paying stock is valued at \$100.00 per share. Its dividend yield is 0.02. The time$-t$ realized return is modeled as

$$R(0,t) \sim N(\text{mean} = 0.065t, \text{variance} = 0.16t)$$

The continuously-compounded, risk-free interest rate is 0.04.

Find the probability that the realized return in the first year exceeds the continuously compounded, risk-free interest rate.

Solution: We need to find

$$\mathbb{P}[R(0,1) > 0.04] = \mathbb{P}\left[0.065 + 0.4Z > 0.04\right]$$

where $Z \sim N(0,1)$. We get

$$\begin{aligned}\mathbb{P}[R(0,1) > 0.04] &= \mathbb{P}\left[Z > \frac{0.04 - 0.065}{0.4} = -0.0625\right] \\ &= \mathbb{P}[Z < 0.0625] = N(0.0625) \approx N(0.06) = 0.5239.\end{aligned}$$

Problem 4.8. (10 points) Assume the Black-Scholes model. Let $S(t), t \geq 0$, denote the stock price. You are given that the initial stock price equals \$36, the mean rate of appreciation equals 0.08, and the volatility is 0.20. Find the expected value of $\sqrt{S(2)}$.

Solution: The random variable $\sqrt{S(2)}$ is lognormal and it can be written as

$$\sqrt{S(2)} = (S(2))^{1/2} = (S(0))^{1/2}e^{\frac{1}{2}\left(0.08-\frac{0.04}{2}\right)(2)+\frac{1}{2}(0.2)\sqrt{2}Z} = (S(0))^{1/2}e^{0.06+0.1\sqrt{2}Z}$$

where $Z \sim N(0,1)$. So,

$$\mathbb{E}[\sqrt{S(2)}] = (36)^{1/2}\mathbb{E}[e^{0.06+0.1\sqrt{2}Z}] = 6e^{0.06}\mathbb{E}[e^{0.1\sqrt{2}Z}] = 6e^{0.06}e^{\frac{(0.1\sqrt{2})^2}{2}} = 6e^{0.07} = 6.435049.$$

Problem 4.9. (5 points) A certain common stock is priced at \$36.50 per share. The company just paid its \$0.50 quarterly dividend. The dividends will continue to be paid quarterly in the same amount. Assume that the continuously-compounded, risk-free interest rate equals $r = 6.0\%$. Consider a \$35 strike European call, maturing in 6 months which currently sells for \$3.20. What is the price of the corresponding $6-$month, \$35 strike put option? Assume that any dividends due are paid just prior to the options' exercise.

Solution: Due to put-call parity, we must have

$$\begin{aligned} V_P(0) &= V_C(0) + e^{-rT}K - S(0) + PV_{0,T}(Div) \\ &= 3.20 + e^{-0.03} \cdot 35 - 36.50 + e^{-0.06 \cdot 0.25} \cdot 0.50 + e^{-0.06 \cdot 0.5} \cdot 0.50 \\ &= 1.6433 \end{aligned}$$