

**Name:**

M339J: Probability models  
University of Texas at Austin

**Solution: Practice Problems for In-Term One**

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**Notes:** This is a closed book and closed notes exam. The maximal score on this exam is ?? points.

**Time:** 50 minutes

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**1.1. TRUE/FALSE QUESTIONS.** *Please, note your final answer on the front page of this exam.*

**Problem 1.1.** Let  $X$  denote the outcome of a roll of a fair, regular icosahedron (a polyhedron with 20 faces) with numbers  $1, 2, \dots, 20$  written on its sides. Then  $\mathbb{E}[X] = 15/2$ . *True or false? Why?*

**Solution: FALSE**

Straight from the definition of expectation, we have that

$$\mathbb{E}[X] = \frac{1}{20}(1 + 2 + \dots + 20) = \frac{1}{20} \cdot \frac{20 \cdot 21}{2} = \frac{21}{2}.$$

**Problem 1.2.** (2 pts) Let  $X$  be an exponential random variable. Then, its mean and its standard deviation are equal. *True or false?*

**Solution: TRUE**

Let  $X \sim \text{Exponential}(\theta)$ . From our tables, we get

$$\begin{aligned}\mathbb{E}[X] &= \theta, \\ \text{Var}[X] &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \theta^2 \cdot 2! - \theta^2 = \theta^2.\end{aligned}$$

Since the standard deviation is the square root of the variance, we are done!

**Problem 1.3.** (2 points) For a random variable  $X$  and for a positive constant  $d$ , in our usual notation, we have

$$(1.1) \quad \mathbb{E}[X] = e_X(d)S_X(d) + \mathbb{E}[X \wedge d].$$

*True or false?*

**Solution: TRUE**

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**1.2. Free-response problems.** Please, explain carefully all your statements and assumptions. Numerical results or single-word answers without an explanation (even if they're correct) are worth 0 points.

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**Problem 1.4.** Let the random variable  $X$  have a two-parameter Pareto distribution with  $\alpha = 2$  and  $\theta = 10$ . What is the 75<sup>th</sup> percentile of this distribution?

**Solution:** Let  $F_X$  denote the cumulative distribution function of  $X$ . We need to solve for  $x$  in

$$F_X(x) = \frac{3}{4}.$$

From the STAM tables, we learn that

$$F_X(x) = 1 - \left( \frac{\theta}{x + \theta} \right)^\alpha.$$

So, we solve for  $x$ :

$$\frac{3}{4} = 1 - \left( \frac{\theta}{x + \theta} \right)^\alpha \Leftrightarrow \frac{1}{4} = \left( \frac{10}{x + 10} \right)^2 \Leftrightarrow \frac{1}{2} = \frac{10}{x + 10} \Leftrightarrow x + 10 = 20 \Leftrightarrow x = 10$$

**Problem 1.5.** (10 points) A population of insureds consists of three types of people:  $\alpha$ ,  $\beta$  and  $\gamma$ . There is an equal number of Type  $\alpha$  and Type  $\beta$  people in the population. The number of Type  $\gamma$  people is equal to the total number of the remaining two types of people. The probability that a Type  $\alpha$  person makes at least one claim in a year is  $1/5$ . The probability that a Type  $\beta$  person makes at least one claim in a year is  $2/5$ . The probability that a Type  $\gamma$  person makes at least one claim in a year is  $3/5$ .

At the end of the year, a person is chosen at random from the population. It is observed that that person had at least one claim. What is the probability that the person was of Type  $\beta$ ?

**Solution:** From the given breakdown of the population, we conclude that

$$(1.2) \quad \mathbb{P}[\alpha] = \mathbb{P}[\beta] = 1/4, \quad \mathbb{P}[\gamma] = 1/2.$$

Let  $E$  denote the event that there was at least one claim. By Bayes' Theorem, we have that

$$(1.3) \quad \begin{aligned} \mathbb{P}[\beta | E] &= \frac{\mathbb{P}[E | \beta] \times \mathbb{P}[\beta]}{\mathbb{P}[E | \alpha] \times \mathbb{P}[\alpha] + \mathbb{P}[E | \beta] \times \mathbb{P}[\beta] + \mathbb{P}[E | \gamma] \times \mathbb{P}[\gamma]} \\ &= \frac{(2/5)(1/4)}{(1/5)(1/4) + (2/5)(1/4) + (3/5)(1/2)} = \frac{2}{9}. \end{aligned}$$

**Problem 1.6.** (15 points) Losses  $X$  follow a Pareto distribution with parameters  $\alpha > 1$  and  $\theta$  unspecified. For a positive constant  $c$ , determine the ratio of the mean excess loss function evaluated at  $c\theta$  to the mean excess loss function evaluated at  $\theta$ .

**Solution:** By definition, the *mean excess loss function* of the random variable  $X$  at a positive constant  $d$  such that  $\mathbb{P}[X > d] > 0$  is given by

$$(1.4) \quad e_X(d) = \mathbb{E}[X - d \mid X > d].$$

According to our class notes, we also have that

$$(1.5) \quad e_X(d) = \frac{\mathbb{E}[X] - \mathbb{E}[X \wedge d]}{S_X(d)}.$$

Using the STAM tables for the Pareto distribution, we get that in the present problem

$$(1.6) \quad e_X(d) = \frac{\frac{\theta}{\alpha-1} - \frac{\theta}{\alpha-1} \left[1 - \left(\frac{\theta}{d+\theta}\right)^{\alpha-1}\right]}{\left(\frac{\theta}{d+\theta}\right)^\alpha} = \frac{\frac{\theta}{\alpha-1}}{\frac{\theta}{d+\theta}} = \frac{d+\theta}{\alpha-1}.$$

Finally, our answer is

$$(1.7) \quad \frac{\frac{c\theta+\theta}{\alpha-1}}{\frac{\theta+\theta}{\alpha-1}} = \frac{c+1}{2}.$$

**Problem 1.7.** Let  $\{X_n, n \geq 1\}$  be a sequence of independent random variables. Assume that all the variables in the sequence have the two-parameter Pareto distribution with  $\theta = 10$  and  $\alpha = 3$ . For each  $n$ , define the random variable

$$Y_n = \frac{X_1^2 + X_2^2 + \cdots + X_n^2}{n}.$$

Does the limit of the sequence  $\{Y_n, n \geq 1\}$  as  $n \rightarrow \infty$  exist? If so, how much is it? If not, why not?

**Solution:** Since the random variables  $\{X_n, n \geq 1\}$  are independent and identically distributed, the random variables  $\{X_n^2, n \geq 1\}$  are also independent and identically distributed. If the mean of the random variable  $X_1^2$  exists and is finite, we can apply the Law of Large Numbers. Using the STAM tables, we get that

$$\mathbb{E}[X_1^2] = \frac{2\theta^2}{(\alpha-1)(\alpha-2)} = \frac{200}{2} = 100.$$

We see that the expected value is finite. So, not only does the Law of Large Numbers apply, but invoking it we can conclude that the limit of  $Y_n$  is 100.

**Problem 1.8.** (10 points) Let  $X \sim \text{Pareto}(\alpha = 3, \theta = 3000)$ . Assume that there is a deductible of  $d = 5000$ . Find the loss elimination ratio.

**Solution:** Using the tables, we get

$$\begin{aligned} \mathbb{E}[X] &= \frac{\theta}{\alpha-1} = \frac{3000}{3-1} = 1500, \\ \mathbb{E}[X \wedge d] &= \frac{\theta}{\alpha-1} \left[1 - \left(\frac{\theta}{d+\theta}\right)^{\alpha-1}\right] = \frac{3000}{3-1} \left[1 - \left(\frac{3000}{5000+3000}\right)^{3-1}\right] = 1500 \left[1 - \left(\frac{3}{8}\right)^2\right]. \end{aligned}$$

Finally, the loss elimination ratio is

$$\left[1 - \left(\frac{3}{8}\right)^2\right] = \frac{55}{64}.$$

**Problem 1.9.** (10 points) Let the ground-up loss  $X$  be exponentially distributed with mean \$800.

An insurance policy has an ordinary deductible of \$100 and the maximum amount payable per loss of \$2500.

Find the expected value of the amount paid (by the insurance company) per positive payment.

**Solution:** We are given  $X \sim \text{Exponential}(\theta = 800)$ , the deductible  $d = 100$  and the policy limit  $u - d = 2500$ . We need to calculate  $\mathbb{E}[Y^P]$  where  $Y^P = Y^L \mid Y^L > 0$  and

$$\begin{aligned} Y^L &= \begin{cases} (X - d)_+, & X < u, \\ u - d, & X \geq u \end{cases} \\ &= (X \wedge u - d)_+. \end{aligned}$$

By the memoryless property of the exponential distribution, we have that

$$Y = X - d \mid X > d$$

is also exponential with mean 800. So, using our tables, we get

$$\mathbb{E}[Y^P] = \mathbb{E}[Y \wedge (u - d)] = \mathbb{E}[Y \wedge 2500] = 800(1 - e^{-2500/800}) \approx 764.85.$$

**Problem 1.10.** (10 points) Losses in year  $y$  follow a two-parameter Pareto distribution with parameters  $\alpha = 4$  and  $\theta = 10$ .

Losses in year  $y + 1$  are uniformly 10% higher than those in year  $y$ .

An insurance covers each loss subject to a deductible  $d = 20$ . Calculate the **loss elimination ratio** for year  $y + 1$ .

**Solution:** Pareto is a scale distribution with the scale parameter  $\theta$ . So, the losses in year  $y + 1$ , denoted by  $X$  again have the two-parameter Pareto distribution with  $\alpha = 4$  and  $\theta = 11$ .

By definition, the loss elimination ratio is

$$\frac{\mathbb{E}[X \wedge d]}{\mathbb{E}[X]}.$$

In our case, using the provided tables, we get

$$\frac{\mathbb{E}[X \wedge d]}{\mathbb{E}[X]} = \frac{\frac{\theta}{\alpha-1} \left(1 - \left(\frac{\theta}{d+\theta}\right)^{\alpha-1}\right)}{\frac{\theta}{\alpha-1}} = 1 - \left(\frac{\theta}{d+\theta}\right)^{\alpha-1} = 1 - \left(\frac{11}{20+11}\right)^{4-1} = 1 - \left(\frac{11}{31}\right)^3 \approx 0.9553.$$

**Problem 1.11.** (10 points) Assume that the severity random variable  $X$  is uniform on the interval  $(0, 1000)$ . There is an insurance policy to cover this loss. The insurance policy has a deductible of 200 per loss and the maximum amount payable by the insurer per loss of 700.

Calculate the expected value of the per payment random variable  $Y^P$  under this policy.

**Solution:** Note that

$$X - 200 \mid X > 200 \sim U(0, 800).$$

So,

$$(X - 200) \wedge 700 \mid X > 200 \sim \begin{cases} U(0, 700) & \text{with probability } 7/8 \\ 700 & \text{with probability } 1/8 \end{cases}$$

$$\mathbb{E}[Y^P] = \frac{7}{8} \cdot 350 + \frac{1}{8} \cdot 700 = 393.75.$$

**Problem 1.12.** Assume that the severity random variable  $X$  is exponentially distributed with mean 1400. There is an insurance policy to cover this loss. This insurance policy has

- (1) a deductible of 200 per loss,
- (2) the coinsurance factor of  $\alpha = 0.25$ , and
- (3) the maximum amount payable by the insurer per loss of 700.

Calculate the expected value of the per payment random variable  $Y^P$  under this policy.

**Solution:** Due to the memoryless property of the exponential distribution, we have

$$X - 200 \mid X > 200 \sim \text{Exponential}(\theta = 1400).$$

Due to the fact that the exponential distribution is a scale distribution, when we introduce the coinsurance factor, we get

$$0.25(X - 200) \mid X > 200 \sim \text{Exponential}(\theta^* = 0.25 * 1400 = 350).$$

Hence, using our tables with  $Y \sim \text{Exponential}(\theta = 350)$ ,

$$\mathbb{E}[Y^P] = \mathbb{E}[Y \wedge 700] = 350(1 - e^{-700/350}) \approx 302.63.$$

### 1.3. MULTIPLE CHOICE QUESTIONS.

**Problem 1.13.** (5 pts) Let  $X$  be exponential with variance 225. Let  $a = \mathbb{E}[|20 - X|]$ . Then,

- (a)  $0 \leq a < 50$
- (b)  $50 \leq a < 150$
- (c)  $150 \leq a < 325$
- (d)  $325 \leq a < 550$
- (e) None of the above.

**Solution: (a)**

From the given information,  $X \sim \text{Exponential}(\theta = 15)$ . So,

$$\begin{aligned}
 a = \mathbb{E}[|20 - X|] &= \int_0^\infty |20 - x| f_X(x) dx \\
 &= \int_0^{20} (20 - x) f_X(x) dx + \int_{20}^\infty (-20 + x) f_X(x) dx \\
 &= \int_0^\infty (20 - x) f_X(x) dx + 2 \int_{20}^\infty (x - 20) f_X(x) dx \\
 &= 20 \int_0^\infty f_X(x) dx - \int_0^\infty x f_X(x) dx + 2 \int_0^\infty y f_X(y + 20) dy \\
 &= 20 - \theta + 2e^{-20/\theta} \int_0^\infty y f_X(y) dy \\
 &= 20 - \theta + 2e^{-20/\theta} \theta \\
 &= 20 - 15 + 30e^{-20/15} \approx 12.9079.
 \end{aligned}$$

**Problem 1.14.** Let  $E$  and  $F$  be two events on the same probability space. You know that  $\mathbb{P}[E \cup F] = 0.75$  and  $\mathbb{P}[E \cup F^c] = 0.85$ . What is the probability of the event  $E$ ?

- (a) 0.5
- (b) 0.6
- (c) 0.65
- (d) 0.7
- (e) None of the above.

**Solution: (b)**

By the basic properties of probability, we know that

$$\begin{aligned}
 0.75 &= \mathbb{P}[E \cup F] = \mathbb{P}[E] + \mathbb{P}[F] - \mathbb{P}[E \cap F], \\
 0.85 &= \mathbb{P}[E \cup F^c] = \mathbb{P}[E] + \mathbb{P}[F^c] - \mathbb{P}[E \cap F^c].
 \end{aligned}$$

We can sum up the above two equalities to get

$$1.6 = 2\mathbb{P}[E] + (\mathbb{P}[F] + \mathbb{P}[F^c]) - (\mathbb{P}[E \cap F] + \mathbb{P}[E \cap F^c]) = 2\mathbb{P}[E] + 1 - \mathbb{P}[E] = \mathbb{P}[E] + 1.$$

Finally,  $\mathbb{P}[E] = 0.6$ .

**Problem 1.15.** The time until the next bus arrives is a continuous random variable  $T$  with the density

$$f_T(t) = \begin{cases} \kappa(10 - t) & 0 < t < 10 \\ 0 & \text{otherwise} \end{cases}$$

for some constant  $\kappa$ . **Given** that you have already waited for 4 minutes, what is the probability that you will wait for at least another 4 minutes?

- (a)  $1/25$
- (b)  $1/9$
- (c)  $1/8$
- (d)  $1/3$
- (e) None of the above.

**Solution: (b)**

For any  $t$  between 0 and 10, the survival function of the random variable  $T$  is

$$S_T(t) = \frac{\kappa}{2}(10 - t)^2$$

So, the conditional probability in the problem is

$$\mathbb{P}[T > 8 \mid T > 4] = \frac{\mathbb{P}[T > 8, T > 4]}{\mathbb{P}[T > 4]} = \frac{\mathbb{P}[T > 8]}{\mathbb{P}[T > 4]} = \frac{\frac{\kappa}{2}(10 - 8)^2}{\frac{\kappa}{2}(10 - 4)^2} = \frac{4}{36} = \frac{1}{9}.$$

**Problem 1.16.** Let  $X_1$ ,  $X_2$ , and  $X_3$  be independent, identically distributed random variables with the probability mass function

$$p_X(x) = \begin{cases} \frac{1}{4} & x = 0 \\ \frac{1}{2} & x = 1 \\ \frac{1}{4} & x = 2 \end{cases}$$

Find  $\mathbb{P}[X_1 X_2 X_3 = 0]$ .

- (a)  $27/64$
- (b)  $1/8$
- (c)  $31/64$
- (d)  $37/64$
- (e) None of the above.

**Solution: (d)**

The product  $X_1 X_2 X_3$  equals zero if at least one of the random variables  $X_1$ ,  $X_2$  and  $X_3$  equals zero. So,

$$\mathbb{P}[X_1 X_2 X_3 = 0] = 1 - \mathbb{P}[X_1 \neq 0, X_2 \neq 0, X_3 \neq 0].$$

Since the random variables are independent, the above probability equals

$$1 - \mathbb{P}[X_1 \neq 0]\mathbb{P}[X_2 \neq 0]\mathbb{P}[X_3 \neq 0] = 1 - \left(\frac{3}{4}\right)^3 = 1 - \frac{27}{64} = \frac{37}{64}.$$

**Problem 1.17.** A recent study indicates that the annual cost of fertilizing a Japanese plum tree in Austin has a mean 100 with a variance of 20. A tax of 10% is introduced on fertilizer, i.e., fertilizer is made 10% more expensive. What is the variance of the new annual cost of fertilizing a Japanese plum tree in Austin after the tax is introduced?

- (a) 20
- (b) 22
- (c) 23.1
- (d) 24.2
- (e) None of the above.

**Solution: (d)**

Let  $X$  be the cost before the tax. Then, the cost after the tax equals  $1.1X$ . So, the new variance is

$$\text{Var}[1.1X] = 1.21\text{Var}[X] = 1.21(20) = 24.2.$$

**Problem 1.18.** Let the random variable  $X$  have the cumulative distribution function given by

$$F_X(x) = \begin{cases} 0 & \text{for } x < 1 \\ \frac{1}{2}(x^2 - 2x + 2) & \text{for } 1 \leq x < 2 \\ 1 & \text{for } 2 \leq x \end{cases}$$

What is the expectation of  $X$ ?

- (a)  $2/3$
- (b)  $5/6$
- (c)  $7/6$
- (d)  $4/3$
- (e) None of the above.

**Solution: (d)**

This is a mixed random variable with  $\mathbb{P}[X = 1] = 1/2$ . The pdf is  $f_X(x) = x - 1$  for  $1 < x < 2$ . So,

$$\begin{aligned} \mathbb{E}[X] &= \frac{1}{2}(1) + \int_1^2 x(x-1) dx = \frac{1}{2} + \left[ \frac{x^3}{3} - \frac{x^2}{2} \right]_{x=1}^2 \\ &= \frac{1}{2} + \frac{8}{3} - \frac{4}{2} - \frac{1}{3} + \frac{1}{2} = 4/3. \end{aligned}$$

**Problem 1.19.** Let the independent random variables  $X$  and  $Y$  have the same mean. You are given that coefficient of variation of  $X$  equals 2 and the coefficient of variation of  $Y$  equals 4. What is the coefficient of variation of the average of  $X$  and  $Y$ ?



- (a)  $3/2$
- (b)  $\frac{\sqrt{13}}{2}$
- (c)  $5/2$
- (d) There is not enough information to answer this problem.
- (e) None of the above.

**Solution: (b)**

Let  $\mu = \mathbb{E}[X] = \mathbb{E}[Y]$ . Then,  $\sigma_X = SD[X] = 2\mu$  and  $\sigma_Y = SD[Y] = 3\mu$ . The variance of the average of the two random variables is

$$\text{Var} \left[ \frac{1}{2}(X + Y) \right] = \frac{1}{4}(\text{Var}[X] + \text{Var}[Y]).$$

In terms of  $\mu$ , the variance of the average can be rewritten as

$$\text{Var} \left[ \frac{1}{2}(X + Y) \right] = \frac{1}{4}(4\mu^2 + 9\mu^2) = \frac{13\mu^2}{4}.$$

So, the standard deviation of the average can be expressed as  $\frac{\mu\sqrt{13}}{2}$ . Hence, the coefficient of variation of the average equals  $\frac{\sqrt{13}}{2}$ .

**Problem 1.20.** (5 points) Let  $X$  be the ground-up loss random variable. Assume that  $X$  has the two-parameter Pareto distribution with  $\theta = 4,000$  and  $\alpha = 3$ .

Let  $B$  denote the expected payment per loss on behalf of an insurer which wrote a policy with no deductible and with a policy limit of 5,000. Then,

- (a)  $B \approx 1,000$
- (b)  $B \approx 1,200$
- (c)  $B \approx 1,400$
- (d)  $B \approx 1,600$
- (e) None of the above

**Solution: (d)**

Using our tables,

$$B = \mathbb{E}[X \wedge 5000] = \frac{\theta}{\alpha - 1} \left[ 1 - \left( \frac{\theta}{5000 + \theta} \right)^{\alpha - 1} \right] = \frac{4000}{2} \left[ 1 - \left( \frac{4000}{9000} \right)^2 \right] \approx 1600.$$

**Problem 1.21.** (5 points) Let  $X$  be the ground-up loss random variable. Assume that  $X$  has the exponential distribution with mean 1,000.

Let  $B$  denote the expected payment per loss on behalf of an insurer who wrote a policy with a deductible of 1,500 and with the **maximum payment by the insurer** equal to 2,500. Then,

- (a)  $B \approx 714$
- (b)  $B \approx 816$
- (c)  $B \approx 918$

- (d)  $B \approx 1020$   
 (e) None of the above

**Solution: (e)**

Using our tables, we get

$$\begin{aligned} B &= \mathbb{E}[X \wedge 4000] - \mathbb{E}[X \wedge 1500] = 1000(1 - e^{-4000/1000}) - 1000(1 - e^{-1500/1000}) \\ &= 1000(e^{-1.5} - e^{-4}) \approx 204.8. \end{aligned}$$

**Problem 1.22.** (5 points) The ground-up loss  $X$  is modeled by a two-parameter Pareto distribution with parameters  $\alpha = 2$  and  $\theta = 200$ . For an insurance policy on the above loss, there is a **franchise** deductible of 200. Find the expected value of the per payment random variable.

- (a) 200  
 (b) 400  
 (c) 600  
 (d) 800  
 (e) None of the above.

**Solution: (c)**

In our usual notation, as we have shown in class,

$$\mathbb{E}[Y^P] = d + \frac{\mathbb{E}[X] - \mathbb{E}[X \wedge d]}{S_X(d)}.$$

In this problem, the ground-up loss is  $X \sim \text{Pareto}(\alpha = 2, \theta = 200)$ . So, using the STAM tables, we have

$$\begin{aligned} S_X(d) &= \left( \frac{\theta}{d + \theta} \right)^\alpha, \\ \mathbb{E}[X] &= \frac{\theta^1 \cdot 1!}{\alpha - 1} = \frac{\theta}{\alpha - 1}, \\ \mathbb{E}[X \wedge d] &= \frac{\theta}{\alpha - 1} \left( 1 - \left( \frac{\theta}{d + \theta} \right)^{\alpha-1} \right). \end{aligned}$$

So,

$$\mathbb{E}[Y^P] = d + \frac{\frac{\theta}{\alpha-1} - \frac{\theta}{\alpha-1} \left( 1 - \left( \frac{\theta}{d+\theta} \right)^{\alpha-1} \right)}{\left( \frac{\theta}{d+\theta} \right)^\alpha} = d + \frac{\frac{\theta}{\alpha-1} \left( \frac{\theta}{d+\theta} \right)^{\alpha-1}}{\left( \frac{\theta}{d+\theta} \right)^\alpha} = d + \frac{\frac{\theta}{\alpha-1}}{\frac{\theta}{d+\theta}} = d + \frac{d + \theta}{\alpha - 1}$$

Using the parameter values and the deductible from this problem, we get

$$\mathbb{E}[Y^P] = 200 + \frac{200 + 200}{2 - 1} = 600.$$