University of Texas at Austin

Extra-credit homework assignment 3

Probability. The normal distribution.

Please, provide your *final answer only* to the following questions:

Problem 3.1. (2 points) A linear combination of two normally distributed random variables is always also normally distributed. Assume that a constant is also considered normally distributed with variance zero. *True or false?*

Solution: TRUE

Problem 3.2. (2 points) If the random variable X has the distribution function F_X, then the distribution function of the random variable $Y = |X|$ equals

$$F_Y(y) = 2F_X(y).$$

True or false?

Solution: FALSE

Problem 3.3. (2 points) Let $X_1, \dots, X_n$ be random variables with finite expectations and let $\alpha_1, \dots, \alpha_n$ be constants. Then, we always have that

$$\mathbb{E}[\alpha_1 X_1 + \cdots + \alpha_n X_n] = \sum_{i=1}^{n} \alpha_i \mathbb{E}[X_i]\,.$$

True or false?

Solution: TRUE

Please, provide **your complete solution** to the following problems. Only the final answer without justification will receive zero credit.

Problem 3.4. (2 points) It is possible that a cumulative distribution function be even. *True or false? Why?*

Solution: FALSE
Any cumulative distribution function F must satisfy $F(-\infty) = 0$ and $F(+\infty) = 1$. Therefore, it cannot be even.

Problem 3.5. (2 pts) If the random variable X is standard normal, then the distribution function of the random variable $Y = |X|$ equals

$$F_Y(a) = 2\Phi(a) - 1 \text{ for every } a \geq 0.$$

True or false? Why?

Solution: TRUE
For every $a \geq 0$,

$$\begin{aligned} F_Y(a) &= \mathbb{P}[Y \leq a] \\ &= \mathbb{P}[|X| \leq a] \\ &= \mathbb{P}[-a \leq X \leq a] \\ &= \mathbb{P}[X \leq a] - \mathbb{P}[X < -a] \\ &= \mathbb{P}[X \leq a] - (1 - \mathbb{P}[X \geq -a]) \\ &= \mathbb{P}[X \leq a] - (1 - \mathbb{P}[X \leq a]) \\ &= 2\Phi(a) - 1. \end{aligned}$$

Problem 3.6. (10 pts) Let Z be a standard normal random variable. Find the following probabilities:

i. $\mathbb{P}[-1.33 < Z \leq 0.24]$
ii. $\mathbb{P}[0.49 < |Z|]$
iii. $\mathbb{P}[Z^4 < 0.0256]$
iv. $\mathbb{P}[e^{2Z} < 2.25]$
v. $\mathbb{P}\left[\frac{1}{Z} < 2\right]$

Solution:

i.
$$\begin{aligned} \mathbb{P}[-1.33 < Z \leq 0.24] &= \mathbb{P}[Z \leq 0.24] - \mathbb{P}[Z \leq -1.33] = \mathbb{P}[Z \leq 0.24] - (1 - \mathbb{P}[Z \leq 1.33]) \\ &= 0.5948 - 1 + 0.9082 = 0.503 \end{aligned}$$

ii.
$$\begin{aligned} \mathbb{P}[0.49 < |Z|] &= \mathbb{P}[Z < -0.49] + \mathbb{P}[0.49 < Z] = 2\mathbb{P}[Z > 0.49] \\ &= 2(1 - \mathbb{P}[Z \leq 0.49]) = 2(1 - 0.6879) = 0.6242 \end{aligned}$$

iii.
$$\begin{aligned} \mathbb{P}[Z^4 < 0.0256] &= \mathbb{P}[|Z| < \sqrt[4]{0.0256}] = \mathbb{P}[|Z| < 0.4] = \mathbb{P}[Z < 0.4] - \mathbb{P}[Z < -0.4] \\ &= 2\mathbb{P}[Z < 0.4] - 1 = 2(0.6554) - 1 = 0.3108 \end{aligned}$$

iv.
$$\mathbb{P}[e^{2Z} < 2.25] = \mathbb{P}[2Z < \ln(2.25)] = \mathbb{P}[Z < 0.5\ln(2.25)] \approx \mathbb{P}[Z \leq 0.41] = 0.6591$$

v.
$$\begin{aligned} \mathbb{P}\left[\tfrac{1}{Z} < 2\right] &= \mathbb{P}[\tfrac{1}{Z} < 0] + \mathbb{P}[0 < \tfrac{1}{Z} < 2] \\ &= \mathbb{P}[Z < 0] + \mathbb{P}[Z > 0.5] = 0.5 + (1 - \mathbb{P}[Z \leq 0.5]) = 0.5 + (1 - 0.6915) = 0.8085. \end{aligned}$$

Problem 3.7. (5 points) Let the density function of a random variable X be given as

$$f_X(x) = cx, \quad x \in [0,1],$$

for some constant c. Find $\mathbb{E}[X^3]$.

Solution: If you draw the graph of f_X, you will see that the integral of the density is the area of a triangle. You get $c = 2$. Then,

$$\mathbb{E}[X^3] = 2\int_0^1 x^4\,dx = 2/5.$$

Problem 3.8. (10 points) Two laser pointers are used to measure the length ℓ of a buidling. The error made by the less accurate laser pointer is normally distributed with mean 0 and standard deviation 0.0144ℓ. The error made by the more accurate laser pointer is normally distributed with mean 0 and standard deviation 0.0036ℓ. The errors from the two laser pointers are independent of each other. Calculate the probability that the average value of the two measurements is within 0.001ℓ of the true length ℓ of the building.

Solution: Let the error of the measurement from the first laser pointer be denoted by ε_1 and let the error of the measurement from the second laser pointer be denoted by ε_2. According to our modelling assumptions

$$\varepsilon_1 \sim Normal(mean = 0, sd = 0.0144\ell),$$
$$\varepsilon_2 \sim Normal(mean = 0, sd = 0.0036\ell).$$

Let the average error be $\varepsilon = \frac{1}{2}(\varepsilon_1 + \varepsilon_2)$. Then, ε is normally distributed with mean 0 and variance

$$Var[\varepsilon] = \frac{1}{4}(Var[\varepsilon_1] + Var[\varepsilon_2]) = \frac{1}{4}((0.0144\ell)^2 + (0.0036\ell)^2).$$

So,

$$\varepsilon \sim Normal(mean = 0, sd = 0.0074\ell)$$

We need to find the probability $\mathbb{P}[|\varepsilon| < 0.001\ell]$. We have

$$\begin{aligned}\mathbb{P}[|\varepsilon| < 0.001\ell] &= \mathbb{P}[-0.001\ell < \varepsilon < 0.001\ell]\\ &= \mathbb{P}\left[\frac{-0.001\ell - 0}{0.0074\ell} < \frac{\varepsilon - 0}{0.0074\ell} < \frac{0.001\ell - 0}{0.0074\ell}\right] = \mathbb{P}[-0.1351 < Z < 0.1351]\end{aligned}$$

where $Z \sim N(0,1)$. So,

$$\mathbb{P}[|\varepsilon| < 0.001\ell] = 2N(0.1351) - 1 \approx 2N(0.14) - 1 = 2(0.5557) - 1 = 0.1114.$$

Problem 3.9. (10 points) An astronomical instrument measures the distance d to a far-away planet. You know that the instrument is calibrated so that its measurement error is normally distributed, centered around zero, and with variance $(0.0001d)^2$. The different measurements using this same instrument are assumed to be independent. How many independent measurements would you have to perform so that your average is within $10^{-6}d$ with probability 99%?

Solution: For every $i \in \mathbb{N}$, let ε_i denote the error of the i^{th} measurement. Then, with n being the number of measurements, the average error $\varepsilon = \frac{1}{n}\sum_{i=1}^{n}\varepsilon_i$ is normally distributed with mean 0 and with variance

$$Var[\varepsilon] = Var\left[\frac{1}{n}(\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n)\right] = \frac{1}{n^2}Var[\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n].$$

Due to the fact that different measurements are independent, we get

$$Var[\varepsilon] = \frac{1}{n^2}(Var[\varepsilon_1] + Var[\varepsilon_2] + \cdots + Var[\varepsilon_n]) = \frac{1}{n^2}(n)(0.0001d)^2 = \frac{(0.0001d)^2}{n}.$$

The probability we are given is

$$\begin{aligned}\mathbb{P}[|\varepsilon| < 10^{-6}d] = 0.99 \quad &\Rightarrow \quad \mathbb{P}[-10^{-6}d < \varepsilon < 10^{-6}d] = 0.99\\ &\Rightarrow \quad \mathbb{P}\left[-\frac{10^{-6}d - 0}{\frac{0.0001d}{\sqrt{n}}} < \frac{\varepsilon - 0}{\frac{0.0001d}{\sqrt{n}}} < \frac{10^{-6}d - 0}{\frac{0.0001d}{\sqrt{n}}}\right] = 0.99\\ &\Rightarrow \quad \mathbb{P}\left[-\frac{10^{-6}\sqrt{n}}{10^{-4}} < Z < \frac{10^{-6}\sqrt{n}}{10^{-4}}\right] = 0.99\end{aligned}$$

where $Z \sim N(0,1)$. So, we have to find a condition for n in

$$\mathbb{P}\left[-\frac{\sqrt{n}}{100} < Z < \frac{\sqrt{n}}{100}\right] = 0.99 \quad \Rightarrow \quad 2N\left(\frac{\sqrt{n}}{100}\right) - 1 = 0.99 \quad \Rightarrow \quad N\left(\frac{\sqrt{n}}{100}\right) = 0.995.$$

From the standard normal tables, we get

$$\frac{\sqrt{n}}{100} = 2.575 \quad \Rightarrow \quad n \geq (257.5)^2 = 66306.25 \quad \Rightarrow \quad n \geq 66307.$$

Problem 3.10. (5 points)The profit of a certain company are modelled using a normal distribution with mean $1,000,000$ and standard deviation $400,000$. Given that the profit is positive, what is the probability that it is below $1,200,000$?

Solution: Let X denote the profit random variable. Then, we need to calculate

$$\mathbb{P}[X < 1200000 \mid X > 0] = \frac{\mathbb{P}[0 < X < 1200000]}{\mathbb{P}[X > 0]} = \frac{\mathbb{P}[X < 1200000] - \mathbb{P}[X \leq 0]}{\mathbb{P}[X > 0]}.$$

The given model for X is

$$X \sim Normal(mean = 1000000, sd = 400000).$$

First, let's calculate the probability that the profit is below zero

$$\mathbb{P}[X \leq 0] = \mathbb{P}\left[\frac{X - 1000000}{400000} \leq \frac{0 - 1000000}{400000}\right] = \mathbb{P}\left[Z \leq -2.5\right] = 0.0062.$$

Above, we denoted by Z a standard normal random variable. So, $\mathbb{P}[X > 0] = 0.9938$. Next, we calculate the probability that the profit is below $1,200,000$.

$$\mathbb{P}[X \leq 1200000] = \mathbb{P}\left[\frac{X - 1000000}{400000} \leq \frac{1200000 - 1000000}{400000}\right] = \mathbb{P}\left[Z \leq 0.5\right] = 0.6915.$$

Finally, our answer is

$$\mathbb{P}[X < 1200000 \mid X > 0] = \frac{0.6915 - 0.0062}{0.9938} = 0.6895754.$$