UNIVERSITY OF TEXAS AT AUSTIN

HW Assignment 7

Implied volatility. Hedging.

Provide your complete solution. Final answers only, even if correct, will receive zero credit. Thank you!

7.1. Implied volatility.

Problem 7.1. (10 points) Let $\mathbf{S} = \{S(t), t \geq 0\}$ denote the price of a non-dividend-paying stock. The stock price today equals \$100. Assume that the Black-Scholes setting holds.

Let r denote the continuously compounded risk-free interest rate.

Consider a European call option with exercise date $T = 10$ and strike price $K = S(0)e^{rT}$. You are given that its price today equals $V_C(0) = \$68.26$.

The goal of this problem is to obtain the implied volatility of the stock S.

(i) (5 pts) Write down the expression for the Black-Scholes price of the European call.

(ii) (3 pts) Simplify the expression you obtained in part (i) so that the call price depends only on the volatility σ.

(iii) (2 pts) Using the properties of the standard normal cumulative distribution function N, the standard normal table, the European call price given in the problem and your answer to part (ii), solve for σ.

Solution:

(i) According to the Black-Scholes formula,

$$V_C(0) = V_C(0, S(0), K, r, \delta, \sigma, T) = S(0)e^{-\delta T}N(d_1) - Ke^{-rT}N(d_2)$$

with

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{T}}[\ln(S(0)/K) + (r - \delta + \tfrac{1}{2}\sigma^2)T], \\ d_2 &= d_1 - \sigma\sqrt{T}. \end{aligned}$$

(ii) Using the provided information, we obtain

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{T}}[\ln(S(0)/S(0)e^{rT}) + (r + \tfrac{1}{2}\sigma^2)T] \\ &= \frac{1}{\sigma\sqrt{T}}[-rT + (r + \tfrac{1}{2}\sigma^2)T] \\ &= \frac{1}{2}\sigma\sqrt{T}, \\ d_2 &= -\frac{1}{2}\sigma\sqrt{T} = -d_1. \end{aligned}$$

Hence,

$$\begin{aligned} V_C(\ldots, \sigma) &= S(0)N(d_1) - S(0)e^{rT}e^{-rT}N(d_2) \\ &= S(0)[N(d_1) - N(-d_1)] \\ &= S(0)[2N(\tfrac{1}{2}\sigma\sqrt{T}) - 1] \\ &= 100[2N(\tfrac{\sqrt{10}}{2}\sigma) - 1]. \end{aligned}$$

(iii) From the problem, we know that

$$68.26 = 100[2N(\tfrac{\sqrt{10}}{2}\sigma) - 1].$$

So,

$$N(\tfrac{\sqrt{10}}{2}\sigma) = 0.8413.$$

From the standard normal table, we get that

$$\tfrac{\sqrt{10}}{2}\sigma = 1 \Rightarrow \sigma = \tfrac{2}{\sqrt{10}} \approx 0.6325.$$

Problem 7.2. (10 points) Let $\mathbf{S} = \{S(t),\, t \geq 0\}$ denote the price of a continuous-dividend-paying stock. The prepaid forward price for delivery of one share of this stock in one year equals \$98.02. Assume that the Black-Scholes model is used for the evolution of the stock price.

Consider a European call and a European put option both with exercise date in one year. They have the same strike price and the same Black-Scholes price equal to \$9.37. What is the implied volatility of the underlying stock?

Solution: Since the call and the put are both European and otherwise identical, put-call parity applies. We conclude that their strike price equals the forward price of the stock for delivery in one year. In other words, $PV_{0,1}(K) = F^P_{0,1}(S)$. Hence the Black-Scholes call price can be written as

$$v_C(S(0), 0) = F^P_{0,T}(S)(N(d_1) - N(d_2))$$

with

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{F^P_{0,T}(S)}{PV_{0,T}(K)}\right) + \frac{\sigma^2 T}{2}\right] = \frac{\sigma}{2}\,, \\ d_2 &= d_1 - \sigma\sqrt{T} = -\frac{\sigma}{2}\,. \end{aligned}$$

We obtain

$$v_C(S(0), 0) = F^P_{0,T}(2N\left(\frac{\sigma}{2}\right) - 1).$$

We are given the call price and the prepaid forward price, however, so that

$$9.37 = 98.02(2N\left(\frac{\sigma}{2}\right) - 1) \quad \Rightarrow \quad N\left(\frac{\sigma}{2}\right) = \frac{1}{2}\left(\frac{9.37}{98.02} + 1\right) = 0.5478.$$

Using the standard normal tables, we get $\frac{\sigma}{2} = 0.12$. Finally, the implied volatility is $\sigma = 0.24$.

Problem 7.3. (10 points) Let $\mathbf{S} = \{S(t),\, t \geq 0\}$ denote the price of a non-dividend-paying stock. The current stock price is \$50. Assume that the Black-Scholes model is used for the evolution of the stock price.

Let the continuously compounded, risk-free interest rate be equal to 0.05.

Consider a European call option on this stock with exercise date in one quarter-year and with the strike price equal to $K = 50e^{0.0125}$. The price of this option is observed to be \$3.98. What is the stock's implied volatility?

Solution: According to the given information, in our usual notation, we have

$$d_1 = \frac{1}{\sigma\sqrt{\frac{1}{4}}}\left[\ln\left(\frac{50}{50e^{0.0125}}\right) + \left(0.05 + \frac{\sigma^2}{2}\right)\cdot\frac{1}{4}\right] = \frac{\sigma}{2}\sqrt{\frac{1}{4}} = \frac{\sigma}{4}\,.$$

Hence,

$$d_2 = d_1 - \sigma\sqrt{\frac{1}{4}} = \frac{\sigma}{4} - \frac{\sigma}{2} = -\frac{\sigma}{4}\,.$$

Therefore, the call price can be written as

$$v_C(S(0), 0) = 50N(d_1) - 50e^{0.0125}e^{-0.05/4}N(d_2) = 50\left(N\left(\frac{\sigma}{4}\right) - N\left(-\frac{\sigma}{4}\right)\right) = 50\left(2N\left(\frac{\sigma}{4}\right) - 1\right)$$

We are given that the call price is observed as \$3.98. So,

$$50\left(2N\left(\frac{\sigma}{4}\right)-1\right)=3.98 \quad\Rightarrow\quad 2N\left(\frac{\sigma}{4}\right)-1=0.0796 \quad\Rightarrow\quad 2N\left(\frac{\sigma}{4}\right)=1.0796 \quad\Rightarrow\quad N\left(\frac{\sigma}{4}\right)=0.5398.$$

Perusing the standard normal tables, we can invert the standard normal cumulative distribution function and obtain that

$$\frac{\sigma}{4}=0.1 \quad\Rightarrow\quad \sigma=0.4$$

Problem 7.4. (10 points) Let $\mathbf{S}=\{S(t),\, t\geq 0\}$ denote the price of a continuous-dividend-paying stock. The current stock price is \$100 and its dividend yield is 0.01. Assume that the Black-Scholes model is used for the evolution of the stock price.

Let the continuously compounded, risk-free interest rate be equal to 0.025.

Consider a European call option on this stock with exercise date in nine months and with the strike price equal to $K=100e^{0.01125}$. The price of this option is observed to be \$10.26. What is the stock's implied volatility?

Solution: Again, let us start by calculating d_1. We have, using our usual notation,

$$\begin{aligned}
d_1&=\frac{1}{\sigma\sqrt{\frac{3}{4}}}\left[\ln\left(\frac{100}{100e^{0.01125}}\right)+\left(0.025-0.01+\frac{\sigma^2}{2}\right)\cdot\frac{3}{4}\right]\\
&=\frac{1}{\sigma\sqrt{\frac{3}{4}}}\left[-0.01125+0.015\cdot\frac{3}{4}+\frac{\sigma^2}{2}\cdot\frac{3}{4}\right]=\frac{\sigma\sqrt{3}}{4}.
\end{aligned}$$

Thus, we have that

$$d_2=d_1-\sigma\sqrt{T}=\frac{\sigma\sqrt{3}}{4}-\sigma\sqrt{\frac{3}{4}}=\frac{\sigma\sqrt{3}}{4}-\frac{\sigma\sqrt{3}}{2}=-\frac{\sigma\sqrt{3}}{4}=-d_1.$$

So, the Black-Scholes price of the European call option in the problem can be expressed as

$$\begin{aligned}
v_C(S(0),0)&=100e^{-0.01(0.75)}N\left(\frac{\sigma\sqrt{3}}{4}\right)-100e^{0.01125}e^{-0.025(0.75)}N\left(-\frac{\sigma\sqrt{3}}{4}\right)\\
&=100e^{-0.0075}N\left(\frac{\sigma\sqrt{3}}{4}\right)-100e^{-0.0075}N\left(-\frac{\sigma\sqrt{3}}{4}\right)\\
&=100e^{-0.0075}\left(N\left(\frac{\sigma\sqrt{3}}{4}\right)-N\left(-\frac{\sigma\sqrt{3}}{4}\right)\right)\\
&=100e^{-0.0075}\left(2N\left(\frac{\sigma\sqrt{3}}{4}\right)-1\right).
\end{aligned}$$

From the option price given in the problem, we obtain this condition on the implied volatility σ:

$$\begin{aligned}
10.26=100e^{-0.0075}\left(2N\left(\frac{\sigma\sqrt{3}}{4}\right)-1\right) \quad&\Rightarrow\quad 2N\left(\frac{\sigma\sqrt{3}}{4}\right)-1=0.103372\\
&\Rightarrow\quad N\left(\frac{\sigma\sqrt{3}}{4}\right)=0.5517.
\end{aligned}$$

Consulting the standard normal tables, we get that

$$\frac{\sigma\sqrt{3}}{4}=N^{-1}(0.5517)=0.13 \quad\Rightarrow\quad \sigma=\frac{0.13(4)}{\sqrt{3}}=0.3$$

7.2. **Delta-hedging.**

Problem 7.5. (2 points) An investor wants to delta-hedge a bull spread she bought. Then, she should short-sell shares of the underlying asset. *True or false? Why?*

Solution: TRUE
She owns a bull spread. The delta of the bull spread is positive. So, before she hedges, the delta of her portfolio is positive. Therefore, the delta of her stock investment needs to be negative. Hence, she needs to short sell shares of stock in order to create a delta-neutral portfolio.

Problem 7.6. (2 points) A market-maker writes a call option on a stock. To decrease the delta of this position, (s)he can **write** a call on the underlying stock. *True or false?*

Solution: TRUE

Problem 7.7. (2 points) Market makers usually do not need to rebalance their portfolios after the initial hedge is established. *True or false? Why?*

Solution: FALSE
The price of the underlying change; the valuation date changes; the inputs in the delta function change overall, so the value of the delta changes. Therefore, to maintain delta-neutrality, the market maker needs to continuously rebalance the portfolio.

Problem 7.8. (2 points) In order to **both** delta hedge and gamma hedge a position in a certain option, the market-maker must trade in another type of option (i.e., not only in the money-market and the underlying risky asset). *True or false? Why?*

Solution: TRUE
The gamma of the bond and the gamma of the stock are both equal to zero. So, trading in either of those does not affect the gamma of the portfolio.

Problem 7.9. (2 points) Consider an option whose payoff function is given by $v(s,T) = \min(s, 50)$. If a market-maker **writes** this option, they need to short sell shares of stock to create a delta-neutral portfolio. *True or false? Why?*

Solution: FALSE
The "special" option in the probem can be replicated using a put and a bond. More precisely,

$$v(s,T) = \min(s, 50) = 50 - \max(50 - s, 0) = 50 - v_P(s,T).$$

So, the value of the "special" option is equal to the value of the bond minus the put price at any point in time. Hence, the delta of the "special" option is

$$\Delta(s,t) = -\Delta_P(s,t) > 0.$$

Since the market maker writes the option, the original delta of their position is $\Delta_P(S(0), 0)$ which is a negative number. They need to **purchase** shares of stock to create a delta-neutral portfolio.