M339D: April 30th, 2021.

Binomial Option Pricing [cont'd].

$$\underbrace{\qquad}_{h=T}$$

		Payoff
$S(0)$	$S_u = u \cdot S(0)$	$V_u = v(S_u)$
$V(0) = ?$	$S_d = d \cdot S(0)$	$V_d = v(S_d)$

Replicating Portfolio:
- Δ shares of stock
- B @ ccrfir

$$\Delta = e^{-\delta h} \frac{V_u - V_d}{S_u - S_d}$$

$$B = e^{-rh} \cdot \frac{u \cdot V_d - d \cdot V_u}{u-d}$$

$$\Rightarrow \boxed{V(0) = \Delta \cdot S(0) + B}$$

Pricing by Replication

$$V(0) = e^{-\delta \cdot h} \cdot \underbrace{\frac{V_u - V_d}{S_u - S_d}}_{\cancel{S(0)}(u-d)} \cdot \cancel{S(0)} + e^{-r \cdot h} \cdot \frac{u \cdot V_d - d \cdot V_u}{u-d}$$

$$V(0) = \frac{1}{u-d}\left[e^{-\delta \cdot h} (V_u - V_d) + e^{-rh}(u \cdot V_d - d \cdot V_u) \right]$$

$$V(0) = e^{-r \cdot h} \cdot \frac{1}{u-d}\left[e^{(r-\delta)h} (V_u - V_d) + u \cdot V_d - d \cdot V_u \right]$$

$$V(0) = e^{-r \cdot h} \cdot \frac{1}{u-d}\left[V_u \left(e^{(r-\delta)h} - d \right) + V_d \left(u - e^{(r-\delta)h} \right) \right]$$

$$V(0) = e^{-rh}\left[V_u \cdot \frac{e^{(r-\delta)h} - d}{u-d} + V_d \cdot \frac{u - e^{(r-\delta)h}}{u-d} \right]$$

Add up to 1!
Both positive! Due to the no-arbitrage cond.

$\overbrace{}^{h=T}$

$$
\begin{tikzcd}[row sep=small]
 & S_u \quad V_u \\
S(0) \arrow[ur] \arrow[dr] & \\
 & S_d \quad V_d
\end{tikzcd}
\Bigg\} \text{ The two possible states of the world!}
$$

Choose to interpret the two weights above as probabilities.

We define the RISK-NEUTRAL PROBABILITY of the stock price going up in a single period:

$$p^* := \frac{e^{(r-\delta)\cdot h} - d}{u-d}$$

$\Rightarrow$ THE RISK-NEUTRAL PRICING FORMULA:

$$\boxed{V(0) = e^{-rT}\left[V_u \cdot p^* + V_d(1-p^*)\right]}$$

We can (and do) generalize this concept:

$$\boxed{V(0) = e^{-rT}\,\mathbb{E}^*\left[V(T)\right]}$$

Example. Consider a stock paying dividends continuously w/ the dividend yield δ

Q: How much should I be charged today in order to receive one share @ time T?

$$\boxed{F^P_{0,T}(S) = S(0)e^{-\delta \cdot T}} \quad \checkmark$$

Say that we model the time-T stock price w/ a one-period binomial tree:

$$
\begin{tikzcd}[row sep=small]
 & S_u & & \text{Payoff} \\
 & & & V_u = S_u \\
S(0) \arrow[uur, "p^*"] \arrow[dr, "1-p^*"'] & & & \\
 & S_d & & V_d = S_d
\end{tikzcd}
$$

By risk-neutral pricing:

$$V(0) = e^{-rT}\left[V_u \cdot p^* + V_d \cdot (1-p^*) \right]$$

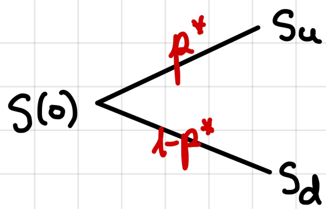
$$= e^{-rT}\left[\underbrace{S_u}_{S(0)\cdot u} \cdot \frac{e^{(r-\delta)h} - d}{u-d} + \underbrace{S_d}_{S(0)\cdot d} \cdot \frac{u - e^{(r-\delta)h}}{u-d} \right]$$

$$= e^{-r\cdot T} \frac{S(0)}{u-d}\left[u \cdot e^{(r-\delta)h} - \cancel{u \cdot d} + \cancel{d \cdot u} - d \cdot e^{(r-\delta)h} \right] \qquad h = T$$

$$= \cancel{e^{-rT}} \cdot \frac{S(0)}{\cancel{u-d}} e^{(\cancel{r}-\delta)\cdot T} \cdot \cancel{(u-d)} = S(0)\, e^{-\delta \cdot T}$$

Consistency!

Example. Consider a European call w/ exercise date @ the end of the period and the strike price K such that $S_d < K < S_u$



Payoff

$V_u = S_u - K$

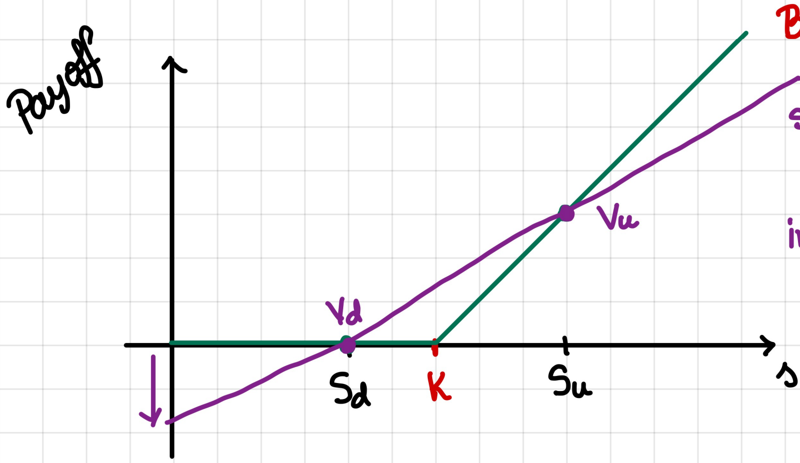
$V_d = 0$

In the replicating portfolio:

$$\Delta_c = e^{-\delta \cdot h} \cdot \frac{V_u}{S_u - S_d} = e^{-\delta \cdot h} \cdot \frac{S_u - K}{S_u - S_d} \in (0,1)$$

$$B_c = e^{-r\cdot h} \cdot \frac{\cancel{u V_d} - d V_u}{u-d} = - e^{-r \cdot h} \cdot \frac{d V_u}{u-d}$$

BORROWING!



slope: $\frac{V_u - V_d}{S_u - S_d}$

intercept: bellow zero

$\Rightarrow$ borrowing

$$\boxed{V_c(0) = e^{-r\cdot T} \cdot p^* \cdot V_u}$$