University of Texas at Austin

Quiz # 14

The χ^2−distribution.

Provide your **complete solution** to the following problems.

Problem 14.1. (5 points) Let the random sample $X_1, \dots, X_6$ be drawn from a normal distribution with mean 4 and variance 1. Define

$$Y = \sum_{i=1}^{6} (X_i - 4)^2.$$

Using **R**, find the constant q such that

$$\mathbb{P}[Y \geq q] = 0.07.$$

Solution: In class, we learned that for $X_1, X_2, \dots, X_n$ independent, normally distributed all with mean μ and variance σ^2, the random variable

$$Y = \sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma}\right)^2$$

has the χ^2−distribution with n degrees of freedom. So, in the present problem

$$Y \sim \chi^2(df = 6).$$

Using the following command in **R**:

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qchisq(0.93, df=6)
```

we get $q = 11.65992$.

Problem 14.2. (10 points) Let $X_1, X_2, \dots, X_{11}$ be a simple random sample from a normal distribution with an **unknown** mean μ and a known variance of 2. Let S^2 denote the sample variance.

Using the χ^2−distribution tables, find the constants a and b such that

$$\mathbb{P}[S^2 \leq a] = 0.025 \quad \text{and} \quad \mathbb{P}[S^2 \leq b] = 0.975.$$

Solution: In class, we learned that for $X_1, X_2, \dots, X_n$ independent and normally distributed with an unknown mean μ and a known variance σ^2 we have

$$\sum_{i=1}^{n} \left(\frac{X_i - \bar{X}}{\sigma}\right)^2 \sim \chi^2(df = n-1).$$

Since, by definition, the sample variance equals

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2,$$

we can conclude that

$$\frac{(n-1)S^2}{\sigma} \sim \chi^2(df = n-1).$$

So, in the present problem,

$$\frac{(11-1)S^2}{2} \sim \chi^2(df = 11-1 = 10) \quad \Rightarrow \quad 5S^2 \sim \chi^2(df = 10).$$

For the first probability, we have

$$\mathbb{P}[S^2 \leq a] = 0.025 \quad \Leftrightarrow \quad \mathbb{P}[5S^2 \leq 5a] = 0.025 \quad \Rightarrow \quad 5a = \chi^2_{0.975}(df = 10) = 3.247. \quad \Rightarrow \quad a = 0.6494.$$

Similarly, for the second probability, we have

$$\mathbb{P}[S^2 \leq b] = 0.975 \quad \Leftrightarrow \quad \mathbb{P}[5S^2 \leq 5b] = 0.975 \quad \Rightarrow \quad 5b = \chi^2_{0.025}(df = 10) = 20.48. \quad \Rightarrow \quad b = 4.096.$$