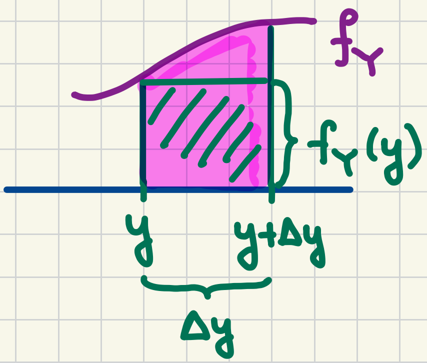
M378K: February 2nd, 2026.

More on Continuous Distributions.



$$\mathbb{P}[Y \in [y, y+\Delta y]] = \quad \approx f_Y(y)\Delta y \approx \boxed{f_Y(y)\,dy}, \text{ i.e.,}$$

$$\mathbb{P}[Y \in [a,b]] = \int_a^b f_Y(y)\,dy$$

Caveat: There are r.v.s that are neither discrete nor continuous!

Example. Y is uniformly distributed between l and r.

$$\boxed{Y \sim U(l,r)}$$

$$f_Y(y) = \begin{cases} \frac{1}{r-l} & \text{for } \boxed{y \in [l,r]} \\ 0 & \text{otherwise} \end{cases}$$

We introduce, for any subset $A \subseteq \mathbb{R}$,

$$\mathbb{1}_A : \mathbb{R} \longrightarrow \mathbb{R}$$

$$\mathbb{1}_A(y) = \begin{cases} 1 & \text{if } y \in A \\ 0 & \text{if } y \notin A \end{cases}$$

This f'tion is called the indicator function.

$$\boxed{f_Y(y) = \frac{1}{r-l}\mathbb{1}_{[l,r]}(y)}$$

<u>M378K Introduction to Mathematical Statistics</u>
<u>Problem Set #5</u>
<u>Continuous distributions.</u>

Problem 5.1. Source: Sample P exam, Problem #33.
The lifetime of a machine part has a continuous distribution on the interval $(0, 40)$ *with probability density function* f_X*, where*

is proportional to

$$f_X(x) \propto \frac{1}{(10+x)^2}$$

on the interval.

Calculate the probability that the lifetime of the machine part is less than 6.

→ :

$$\int_{-\infty}^{\infty} \frac{K}{(10+x)^2}\,dx = \int_0^{40} \frac{K}{(10+x)^2}\,dx = 1$$

$$K \int_0^{40} \frac{1}{(10+x)^2}\,dx = 1$$

$$\int_0^{40} \frac{1}{(10+x)^2}\,dx = \frac{-1}{10+x}\Bigg|_0^{40}$$

$$= \frac{-1}{10+40} + \frac{1}{10}$$

$$= \frac{4}{50}K = 1$$

$$K = \frac{50}{4} = \frac{25}{2}$$

$$\mathbb{P}[X<6] = \int_0^6 f_X(x)\,dx = \frac{25}{2}\int_0^6 (10+x)^{-2}\,dx$$

$$= \frac{25}{2}\cdot\left(-\frac{1}{10+x}\right)\Bigg|_{x=0}^{6} = \frac{25}{2}\left(-\frac{1}{16}+\frac{1}{10}\right) = \frac{25}{2}\cdot\frac{8-5}{80} = \frac{15}{32}$$

□

Example. We say that Y is exponential w/ parameter τ if its pdf is

$$\boxed{f_Y(y) = \frac{1}{\tau} e^{-\frac{y}{\tau}} \cdot \mathbb{1}_{[0,\infty)}(y)}$$

τ... scale parameter

Problem 5.2. Source: Sample P exam, Problem #419.

A customer purchases a lawnmower with a two-year warranty. The number of years before the lawnmower needs a repair is uniformly distributed on $[0, 5]$*. Calculate the probability that the lawnmower needs no repairs within* 4.5 *years after the purchase, given that the lawnmower needs no repairs within the warranty period*

$\longrightarrow$: $X \sim U(0, 5)$ $\quad Pr(X > K) = \frac{5-K}{5-0} = \frac{5-K}{5}$

$$Pr(X > 4.5 \mid X > 2) = \frac{Pr(X > 4.5 \cap X > 2)}{Pr(X > 2)}$$

$$= \frac{Pr(X > 4.5)}{Pr(X > 2)} \qquad Pr(X > 2) = \frac{5-2}{5} = \frac{3}{5}$$

$$Pr(X > 4.5) = \frac{5-4.5}{5} = \frac{0.5}{5}$$

$$\frac{0.5/5}{3/5} = \frac{0.5}{3} = \boxed{\frac{1}{6}}$$

$\square$

Example. $\boxed{Y \sim E(\tau)}$

$t, s > 0$

$$\boxed{\mathbb{P}[Y > t+s \mid Y > t] = ?}$$

In general. $y > 0$

$$\mathbb{P}[Y > u] = \int_u^\infty f_Y(y)\,dy = \int_u^\infty \frac{1}{\tau} e^{-\frac{y}{\tau}}\,dy = \frac{1}{\tau}\int_u^\infty e^{-\frac{y}{\tau}}\,dy =$$

$x = -\frac{y}{\tau}$

$dx = -\frac{1}{\tau}\,dy \Rightarrow -\tau\,dx = dy$

$$= \frac{1}{\cancel{\tau}} \int_{-\frac{u}{\tau}}^{-\infty} e^{x} (-\cancel{\tau})\,dx = -\int_{-\frac{u}{\tau}}^{-\infty} e^{x}\,dx = -\left(e^{x}\right)_{x=-\frac{u}{\tau}}^{-\infty} = e^{-\frac{u}{\tau}}$$

$$\mathbb{P}[Y > t+s \mid Y > t] = \frac{\mathbb{P}[Y > t+s, Y > t]}{\mathbb{P}[Y > t]} = \frac{\mathbb{P}[Y > t+s]}{\mathbb{P}[Y > t]} = \frac{e^{-\frac{1}{\tau}(t+s)}}{e^{-\frac{1}{\tau}t}} = e^{-\frac{s}{\tau}} = \mathbb{P}[Y > s]$$

Memoryless property.