M339 J: April 5^{th}, 2021.

Quiz #2: Problem #3.

$$X \sim \text{Loglogistic}(\gamma, \theta)$$

STAM TABLES

$$: \quad F_X(x) = \frac{\left(\frac{x}{\theta}\right)^{\gamma}}{1+\left(\frac{x}{\theta}\right)^{\gamma}}$$

$$X' = \frac{1}{X}$$

$$y>0: \quad F_{X'}(y) = ?$$

$$F_{X'}(y) = \mathbb{P}[X' \le y] = \mathbb{P}\left[\frac{1}{X} \le y\right] =$$

$$= \mathbb{P}[1 \le X \cdot y]$$

$$= \mathbb{P}\left[\frac{1}{y} \le X\right] = 1 - F_X\left(\frac{1}{y}\right)$$

$$= 1 - \frac{\left(\frac{1}{y\theta}\right)^{\gamma}}{1+\left(\frac{1}{y\theta}\right)^{\gamma}}$$

$$= \frac{1}{1+\left(\frac{1}{y\theta}\right)^{\gamma}} = \frac{(y\theta)^{\gamma}}{(y\theta)^{\gamma}+1} \qquad \theta' = \frac{1}{\theta}$$

$$= \frac{\left(\frac{y}{\theta'}\right)^{\gamma}}{1+\left(\frac{y}{\theta'}\right)^{\gamma}} \quad \Rightarrow \quad X' \sim \text{Loglogistic}\left(\theta' = \frac{1}{\theta}\right).$$

$\blacksquare$

Poisson-Gamma Mixing.

Let N has a mixture distribution w/ the mixing parameter Λ.

More precisely, let

$$\left.\begin{array}{l}\boxed{N \mid \Lambda \sim \text{Poisson}(\text{mean}=\Lambda)} \\ \Lambda \sim \text{Gamma}(\alpha, \theta)\end{array}\right\}$$

Q: What is the support of N?

$\longrightarrow$: $\text{Support}(N) = \mathbb{N}_0 = \{0, 1, 2, \dots\}$ ■

Focus on the pmf of N:

for $k = 0, 1, 2, \dots$:

$$p_N(k) = \mathbb{P}[N=k] = F_N(k) - F_N(k-1)$$

$$\overset{\text{mixing}}{=} \int F_{N|\Lambda}(k|\lambda) f_\Lambda(\lambda)\, d\lambda - \int F_{N|\Lambda}(k-1|\lambda) f_\Lambda(\lambda)\, d\lambda$$

$$= \int \left(F_{N|\Lambda}(k|\lambda) - F_{N|\Lambda}(k-1|\lambda)\right) f_\Lambda(\lambda)\, d\lambda$$

$$= \int \mathbb{P}[N=k \mid \Lambda=\lambda]\, f_\Lambda(\lambda)\, d\lambda$$

In general:

$$p_N(k) = \int p_{N|\Lambda}(k|\lambda) f_\Lambda(\lambda)\, d\lambda$$

In this case:

$$p_N(k) = \int_0^{+\infty} \underbrace{e^{-\lambda} \cdot \frac{\lambda^k}{k!}}_{\text{conditional pmf}} \underbrace{\frac{\left(\frac{\lambda}{\theta}\right)^\alpha e^{-\frac{\lambda}{\theta}}}{\lambda \cdot \Gamma(\alpha)}}_{\Gamma\text{-density}} d\lambda$$

$$= \frac{1}{k! \cdot \theta^{\alpha} \cdot \Gamma(\alpha)} \int_0^{+\infty} e^{-\lambda\underbrace{\left(1+\frac{1}{\theta}\right)}_{=:\frac{1}{\theta^*}}} \lambda^{\overbrace{k+\alpha}^{=:\alpha^*}-1} \, d\lambda \checkmark$$

Consider: $Y \sim \Gamma(\alpha^*, \theta^*)$

$$1 = \int_0^{+\infty} f_Y(\lambda)\, d\lambda$$

$$= \int_0^{+\infty} \frac{\lambda^{\alpha^*-1}}{(\theta^*)^{\alpha^*}} \cdot \frac{e^{-\frac{\lambda}{\theta^*}}}{\Gamma(\alpha^*)} \, d\lambda$$

$$= \frac{1}{(\theta^*)^{\alpha^*}} \cdot \frac{1}{\Gamma(\alpha^*)} \int_0^{+\infty} \lambda^{\alpha^*-1} \cdot e^{-\frac{\lambda}{\theta^*}} \, d\lambda$$

$$\Rightarrow p_N(k) = \frac{1}{k! \cdot \theta^{\alpha} \cdot \Gamma(\alpha)} \cdot (\theta^*)^{\alpha^*} \cdot \Gamma(\alpha^*)$$

$$= \frac{1}{k! \cdot \theta^{\alpha} \cdot \Gamma(\alpha)} \cdot \left(\frac{\theta}{1+\theta}\right)^{k+\alpha} \cdot \Gamma(k+\alpha)$$

$$= \frac{\Gamma(k+\alpha)}{k! \cdot \Gamma(\alpha)} \cdot \left(\frac{\theta}{1+\theta}\right)^{k} \cdot \left(\frac{1}{1+\theta}\right)^{\alpha}$$

$\Rightarrow$ N is NEGATIVE BINOMIAL

w/ $r = \alpha$ and $\beta = \theta$

Problem.

$$\left.\begin{array}{l} N \mid \Lambda \sim \text{Poisson}(\text{mean} = \Lambda) \\ \Lambda \sim \text{Gamma}\left(\alpha = \frac{5}{2}, \theta = 4\right) \end{array}\right\}$$

What is the (unconditional) probability $\mathbb{P}[N=3]$?

$\longrightarrow$: $N \sim \text{NegBinomial}\left(r = \frac{5}{2}, \beta = 4\right)$

$$\mathbb{P}[N=3] = \frac{\Gamma\left(3+\frac{5}{2}\right)}{\Gamma(3+1)\cdot\Gamma\left(\frac{5}{2}\right)} \cdot \frac{4^3}{(4+1)^{3+5/2}}$$

Recall:
- $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$
- $\Gamma(k) = (k-1)!$ — k integer
- $\Gamma(1/2) = \sqrt{\pi}$

$$\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\cdot\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right) = \frac{3}{4}\sqrt{\pi}$$

$$\Gamma\left(3+\frac{5}{2}\right) = \left(2+\frac{5}{2}\right)\cdot\Gamma\left(2+\frac{5}{2}\right) = \left(2+\frac{5}{2}\right)\left(1+\frac{5}{2}\right)\Gamma\left(1+\frac{5}{2}\right)$$

$$= \left(2+\frac{5}{2}\right)\left(1+\frac{5}{2}\right)\cdot\frac{5}{2}\cdot\Gamma\left(\frac{5}{2}\right) = \frac{9\cdot7\cdot5\cdot3}{2^3\cdot4}\sqrt{\pi}$$

$$\Rightarrow \mathbb{P}[N=3] = \frac{\frac{9\cdot7\cdot5\cdot3}{2^5}\cancel{\sqrt{\pi}}}{3!\cdot\frac{3}{4}\cdot\cancel{\sqrt{\pi}}} \cdot \frac{4^3}{5^{11/2}} \cong 0.0601 \blacksquare$$