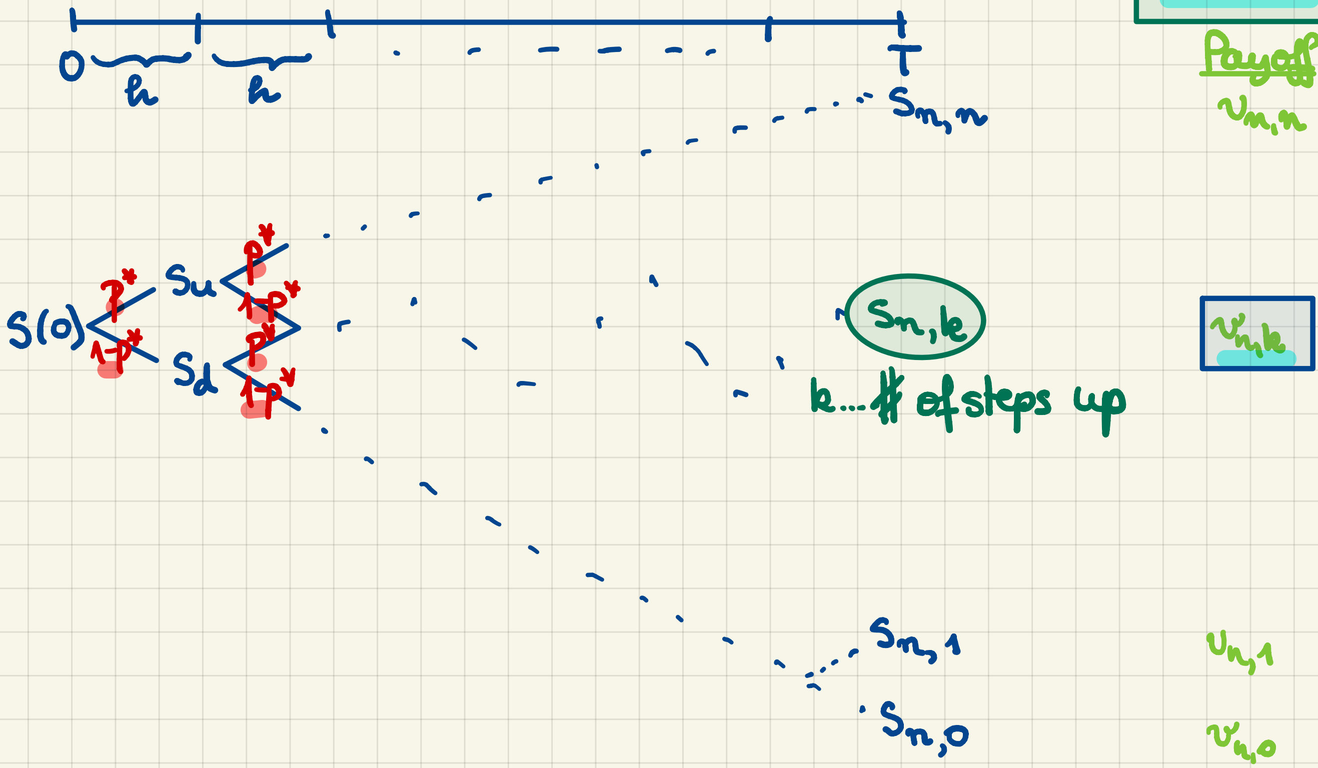
M339D: October 20th, 2023.

Multiple Binomial Periods.

T... exercise date of a European option
n... # of periods

the length of each period: $\boxed{h = \frac{T}{n}}$



The $(n+1)$ values are the support of $S(T)$.

$\Rightarrow$ for every $k = 0, 1, \dots, n$:

$$S_{n,k} = S(0) \cdot u^k \cdot d^{n-k} = S(0) \left(\frac{u}{d}\right)^{k} \cdot d^n$$

k ... # of upsteps

Consider a European option w/ payoff function $v(\cdot)$.
Then, the possible payoff values will be

$$\boxed{v_{n,k} := v(S_{n,k})}$$

Recall: Risk-Neutral Pricing: $\boxed{V(0) = e^{-rT}\,\mathbb{E}^*[V(T)]}$

p^*... the risk-neutral probability of a single upstep, i.e.,

$$\boxed{p^* = \frac{e^{rh} - d}{u - d}}$$

=> The risk-neutral probability of attaining the payoff $v_{n,k}$:

$$\boxed{\binom{n}{k}(p^*)^k(1-p^*)^{n-k}}$$

The risk-neutral option price:

$$\boxed{V(0) = e^{-rT} \cdot \sum_{k=0}^{n} \left(\binom{n}{k} (p^*)^k (1-p^*)^{n-k} \cdot v_{n,k} \right)}$$

Problem 9.2. Let the continuously compounded risk-free interest rate be 0.10. Let the initial price of a non-dividend-paying stock be $100 per share. You use a five-period binomial tree to model the stock price over the next year. Let $u = 1.04$ and $\boxed{d = 0.96}$.

$T=1$ $K=S(0)=100$ $n=5$

What is the price of a one-year, $\boxed{\text{at-the-money}}$ European call option on the above stock?

→: The risk-neutral probability:

$$p^* = \frac{e^{rh}-d}{u-d} = \frac{e^{0.10(0.2)}-0.96}{1.04-0.96} \approx 0.7525$$

The relevant stock prices in our tree:

$$S_{5,5} = S(0)u^5 = 100(1.04)^5 = 121.67 \quad \Rightarrow \quad v_{5,5} = 21.67$$

$$S_{5,4} = S(0)u^4 \cdot d = 100(1.04)^4(0.96) = 112.31 \quad \Rightarrow \quad v_{5,4} = 12.31$$

$$S_{5,3} = S(0)u^3 \cdot d^2 = 100(1.04)^3(0.96)^2 = 103.67 \quad \Rightarrow \quad v_{5,3} = 3.67$$

The remaining terminal nodes are all out-o-money.

⇒

$$V_C(0) = e^{-0.10}\Big(21.67(p^*)^5 + 12.31 \cdot 5 \cdot (p^*)^4(1-p^*) + 3.67 \cdot 10 \cdot (p^*)^3(1-p^*)^2\Big) = 10.002$$

$$\binom{5}{2} = \frac{5 \cdot 4}{2} = 10$$

□