NAME:

M339W/389W Financial Mathematics for Actuarial Applications
University of Texas at Austin
Mock In-Term Exam I
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Notes: This is a closed book and closed notes exam. The maximum number of points on this exam is 50.
Time: 50 minutes

Problem 2.1. Assume the Black-Scholes setting.
Today's price of a non-dividend paying stock is \$65, and its volatility is 0.20.
The continuously-compounded, risk-free interest rate is 0.055.
What is the price of a three-month, \$60-strike European put option on the above stock?

(a) 0.66
(b) 0.59
(c) 0.44
(d) 0.37
(e) None of the above.

Solution: **(b)**
In our usual notation, the price is

$$V_P(0) = Ke^{-r \cdot T} N(-d_2) - S(0)N(-d_1)$$

with

$$d_1 = \frac{1}{0.2\sqrt{1/4}} \left(\ln\left(\frac{65}{60}\right) + (0.055 + \frac{1}{2}\, 0.2^2) \left(\frac{1}{4}\right) \right) = 10(\ln(65/60) + (0.075)(0.25)) = 0.99,$$
$$d_2 = d_1 - 0.2\sqrt{0.25} = 0.89.$$

So,

$$V_P(0) = 60e^{-0.055 \cdot \frac{1}{4}} (1 - 0.8133) - 65 \cdot (1 - 0.8389) = 0.5922.$$

Problem 2.2. (5 points) A continuous-dividend-paying stock is valued at \$75.00 per share. Its dividend yield is 0.03. The time$-t$ realized return is modeled as

$$R(0,t) \sim N(\text{mean} = 0.035t, \text{variance} = 0.09t)$$

Find the probabilit y that the time-4 stock price exceeds today's stock price.

Solution: We need to find

$$\mathbb{P}[S(4) > S(0)]$$

with

$$S(t) = S(0)e^{R(0,t)}.$$

Since $R(0,t)$ follows the normal distribution with the above parameters, we have

$$\mathbb{P}[S(4) > S(0)] = \mathbb{P}[S(0)e^{R(0,4)} > S(0)] = \mathbb{P}[R(0,4) > 0] = 1 - N\left(-\frac{0.035 \times 4}{0.3 \times 2}\right) = N(0.23) = 0.591.$$

Problem 2.3. Assume the Black-Scholes model. According to your model, you expect the stock price in half a year to be \$84.10. The median stock price in half a year is \$83.26 according to that same model. What is the stock's volatility?

(a) 0.15
(b) 0.20
(c) 0.25
(d) Not enough information is given.
(e) None of the above.

Solution: (b)
In our usual notation, we have that

$$\frac{\text{mean stock price}}{\text{median stock price}} = \frac{S(0)e^{(\alpha-\delta)T}}{S(0)e^{(\alpha-\delta-\frac{\sigma^2}{2})T}} = e^{\frac{\sigma^2 T}{2}}$$

So, in this problem,

$$\frac{84.10}{83.26} = e^{\frac{\sigma^2}{4}} \quad \Rightarrow \quad \frac{\sigma^2}{4} = \ln\left(\frac{84.10}{83.26}\right) \quad \Rightarrow \quad \sigma = \sqrt{4\ln\left(\frac{84.10}{83.26}\right)} = 0.2004.$$

Problem 2.4. (5 points) The current stock price is given to be $S(0) = 30$. The stock has the rate of appreciation 0.12 and volatility 0.3

Find the probability that the stock price in three months is less than \$32.

(a) 0.5218
(b) 0.5412
(c) 0.5846
(d) 0.6217
(e) None of the above.

Solution: (d)
First, we calculate $\hat{d}_2$. We get

$$\hat{d}_2 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(\alpha - \delta - \frac{\sigma^2}{2}\right)T\right] = \frac{1}{0.3\sqrt{\frac{1}{4}}}\left[\ln\left(\frac{30}{32}\right) + \left(0.12 - \frac{0.09}{2}\right) \times \frac{1}{4}\right] = -0.30525.$$

Then, we use the standard normal tables to obtain

$$N(-\hat{d}_2) \approx N(0.31) = 0.6217 \tag{2.1}$$

Problem 2.5. (5 points) Assume the Black-Scholes model. Let the current price of a continuous-dividend-paying stock be \$80. Its dividend yield is 0.02 and its volatility is 0.25.

Let the continuously compounded, risk-free interest rate be 0.04.

Find the price of a 3−month, \$75-strike European call option on the above stock.

(a) 6.84
(b) 7
(c) 7.22
(d) 7.51
(e) None of the above.

Solution: (c)
In our usual notation,

$$d_1 = \frac{1}{0.25\sqrt{\frac{1}{4}}}\left[\ln\left(\frac{80}{75}\right) + \left(0.04 - 0.02 + \frac{(0.25)^2}{2}\right)\left(\frac{1}{4}\right)\right] = 0.6188 \approx 0.62,$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.6188 - 0.25\sqrt{\frac{1}{4}} = 0.4938 \approx 0.49.$$

Using the standard normal tables, we get

$$N(d_1) = N(0.62) = 0.7324 \quad \text{and} \quad N(d_2) = N(0.49) = 0.6879.$$

Finally, the Black-Scholes price of our call option is

$$\begin{aligned} V_C(0) &= S(0)e^{-\delta T}N(d_1) - Ke^{-rT}N(d_2) \\ &= 80e^{-0.02(0.25)}(0.7324) - 75e^{-0.04(0.25)}(0.6879) = 7.220625. \end{aligned}$$

Problem 2.6. Suppose that the failure time (in seconds) of a certain component is modeled as lognormal random variable $Y = e^X$ such that the mean of X is -0.35 and its variance is 0.04.

What is the failure time t^* such that 95% of the components of the same type would still function after that time?

(a) 0.306
(b) 0.402
(c) 0.507
(d) 0.701
(e) None of the above.

Solution: (c)
We are looking for the value t^* such that

$$\mathbb{P}[Y > t^*] = 0.95 \quad \Leftrightarrow \quad \mathbb{P}[Y \leq t^*] = 0.05.$$

The critical value z^* such that $N(z^*) = 0.05$ is -1.645. So,

$$t^* = e^{-0.35+0.2(-1.645)} = 0.5071.$$

Problem 2.7. (5 points) A stock is valued at \$55.00. The annual expected return is 12.0% and the standard deviation of annualized returns is 22.0%. If the stock is lognormally distributed, what is the expected stock price after 3 years?

(a) About \$78.83
(b) About \$88.83
(c) About \$98.83
(d) About \$108.83
(e) None of the above.

Solution: (a)
Let us denote the stock price today by $S(0)$ and that in three years by $S(3)$. According to the work we did in class, we need to calculate

$$\mathbb{E}[S(3)] = S(0)e^{3\alpha}$$

with α equal to the expected continuously compounded rate of return on the stock S. We are given in the problem that $\alpha = 0.12$. So, the answer is $55e^{0.36} \approx 78.83$.

Problem 2.8. (5 pts)
A non-dividend-paying stock is currently valued at \$100 per share. Its annual mean rate of return is given to be 12% while its volatility is given to be 30%.
Assuming the lognormal stock-price model, find

$$\mathbb{E}[S(2) \mid S(2) > 95].$$

(a) \$86.55
(b) \$101.60
(c) \$152.35
(d) \$159.07
(e) None of the above.

Solution: (c)
In our usual notation,

$$\mathbb{E}[S(T) \mid S(T) > K] = \frac{S(0)e^{(\alpha-\delta)T}N(\hat{d}_1)}{N(\hat{d}_2)}.$$

with

$$\hat{d}_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(\alpha - \delta + \frac{\sigma^2}{2}\right)T\right],$$
$$\hat{d}_2 = \hat{d}_1 - \sigma\sqrt{T}.$$

In the present probem,

$$\hat{d}_1 = \frac{1}{0.3\sqrt{2}}\left[\ln\left(\frac{100}{95}\right) + \left(0.12 + \frac{0.09}{2}\right) \times 2\right] = 0.8987,$$
$$\hat{d}_2 = 0.8987 - 0.3\sqrt{2} = 0.4745.$$

So, our answer is

$$\mathbb{E}[S(2) \mid S(2) > 95] = \frac{100e^{(0.12)\times 2}N(0.8987)}{N(0.4745)} = \frac{100e^{0.24} \times 0.8159}{0.6808} = 152.35.$$

Problem 2.9. Assume the Black-Scholes framework. For an at-the-money, T-year European call option on a non-dividend-paying stock you are given that its delta equals 0.5832. What is the delta of an otherwise identical option with exercise date at time $2T$?

(a) 0.62
(b) 0.66
(c) 0.70
(d) 0.74
(e) None of the above.

Solution: (a)

Problem 2.10. (5 points) Let the current price of a non-dividend-paying stock be denoted by $S(0)$. We model the time$-T$ stock price as lognormal. The mean rate of return on the stock is 0.12 and its volatility is 0.20. The continuously-compounded, risk-free interest rate is 0.04. You invest in one share of stock at time-0. and simultaneously deposit an amount $\varphi S(0)$ in a savings account. What should the proportion φ be so that the VaR at the level 0.05 of your total wealth at time-1 equals today's stock price $S(0)$?

(a) $\varphi = 0.1966$
(b) $\varphi = 0.5$
(c) $\varphi = 0.8034$
(d) $\varphi = 1$
(e) None of the above.

Solution: (a)
The total wealth at time-1 is equal to $S(1) + \varphi S(0)e^r$. So, our condition on the VaR is

$$\mathbb{P}[S(1) + \varphi S(0)e^r < S(0)] = 0.05.$$

The above is equivalent to

$$\mathbb{P}[S(0)e^{\alpha - \frac{\sigma^2}{2} + \sigma Z} + \varphi S(0)e^r < S(0)] = 0.05$$

with $Z \sim N(0,1)$. We have that the above is, in turn, equivalent to

$$\mathbb{P}[e^{\alpha - \frac{\sigma^2}{2} + \sigma Z} + \varphi e^r < 1] = 0.05.$$

The constant z^* such that $\mathbb{P}[Z < z^*] = 0.05$ equals -1.645. Therefore,

$$e^{\alpha - \frac{\sigma^2}{2} + \sigma(-1.645)} + \varphi e^r = 1.$$

So,

$$\varphi = e^{-r}\left(1 - e^{\alpha - \frac{\sigma^2}{2} + \sigma(-1.645)}\right) = e^{-0.04}\left(1 - e^{0.12 - \frac{(0.2)^2}{2} + 0.2(-1.645)}\right) = 0.196646.$$

Problem 2.11. (5 points) Assume the Black-Scholes model. The current price of a continuous-dividend-paying stock is \$63 per share. Its dividend yield is 0.01 and its volatility is 0.25. Its mean rate of return is 0.10.

Consider a three-month,\$65-strike call option on the above stock. What is the probability that the option is in-the-money at expiration?

(a) 0.4483
(b) 0.4325
(c) 0.3936
(d) 0.4207
(e) None of the above.

Solution: (a)
In our usual notation,

$$\mathbb{P}[S(1/4) > 65] = N(\hat{d}_2)$$

with

$$\hat{d}_2 = \frac{1}{0.25\sqrt{1/4}}\left[\ln\left(\frac{63}{65}\right) + \left(0.10 - 0.01 - \frac{(0.25)^2}{2}\right)\left(\frac{1}{4}\right)\right] = -0.1325203.$$

So, our answer is $N(-0.13) = 0.4483$.

Problem 2.12. Assume that the stock price follows the Black-Scholes model. You are given the following information:

- The current stock price is \$100.
- The mean rate of return on the stock is 0.15.
- The stock's dividend yield is 0.01.
- The stock's volatility is 0.35.
- The continuously-compounded, risk-free interest rate is 0.05.

Find

$$\mathbb{E}[S(1)\mathbb{I}_{[S(1)>80]}].$$

(a) \$102.02
(b) \$108.19
(c) \$115.03
(d) \$126.71

(e) None of the above.

Solution: (a)
Note that the given continuously-compounded, risk-free interest rate is not necessary to solve the problem!

In our usual notation, we have

$$\begin{aligned}\hat{d}_1 &= \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right)+\left(\alpha-\delta+\frac{\sigma^2}{2}\right)T\right] = \frac{1}{0.35\sqrt{1}}\left[\ln\left(\frac{100}{80}\right)+\left(0.15-0.01+\frac{0.35^2}{2}\right)\right]\\ &= 1.21255 \approx 1.21,\\ \hat{d}_2 &= \hat{d}_1-\sigma\sqrt{T} = 1.21-0.35 = 0.86255 \approx 0.86.\end{aligned}$$

Using the standard normal tables, we get

$$N(\hat{d}_1) = 0.8869 \quad \text{and} \quad N(\hat{d}_2) = 0.8051.$$

Finally, our answer is

$$\mathbb{E}[S(1)\mathbb{I}_{[S(1)>80]}] = 100e^{0.15-0.01}(0.8869) = 102.0178.$$