M339D: March 6th, 2023.

The Binomial Asset Pricing Model.

$S(0)$ .... the observable initial asset price

$$0 \longrightarrow T$$

T ... time horizon (i.e., the exercise date of an option)

$$\begin{tikzcd} & \boxed{S_u = u \cdot S(0)} \\ S(0) \arrow[ur] \arrow[dr] & \\ & \boxed{S_d = d \cdot S(0)} \end{tikzcd}$$

ROOT NODE

up node

down node

By convention: $\boxed{u > d}$

u ... up factor
d ... down factor

h ... length of a single period

one period $\Rightarrow$ $\boxed{S(T) = S(h)}$ ... a r.v. denoting the time-T stock price w/ two possible values: S_u and S_d

Returns:

As a random variable: simple return $= \dfrac{S(T) - S(0)}{S(0)}$

- $\dfrac{S_u - S(0)}{S(0)} = \dfrac{u \cdot S(0) - S(0)}{S(0)} = u - 1$
- $\dfrac{S_d - S(0)}{S(0)} = \dfrac{d \cdot S(0) - S(0)}{S(0)} = d - 1$

Market Model.

- riskless asset: @ the ccrfir r
- risky asset: non-dividend paying stock

Imagine investing in one share of stock @ time·0.

$$\begin{array}{lll} & & & \underline{\text{Wealth}} \\ & \nearrow & S_u = u\cdot S(0) & S_u \\ S(0) & & & \\ & \searrow & S_d = d\cdot S(0) & S_d \end{array}$$

At the risk·free rate $S(0)$ accumulates to $S(0)e^{rh}$ @ time $T=h$

The No·Arbitrage Condition.

$$d\cdot \cancel{S(0)} < \cancel{S(0)}e^{rh} < u\cdot \cancel{S(0)}$$

$$\boxed{d < e^{rh} < u}$$

Half·a·Proof.

Say, to the contrary, $e^{rh} \leq d < u$

Propose: Long one share of stock.

Verify: Profit $= \underline{S(T) - S(0)e^{rh}}$

In the up·node: $S(0)\cdot u - S(0)e^{rh} > 0$

In the down·node: $S(0)\cdot d - S(0)e^{rh} \geq 0$

Indeed, an arbitrage portfolio.

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Problem Set #8

Binomial option pricing.

Problem 8.1. In the setting of the one-period binomial model, denote by i the **effective** interest rate **per period**. Let u denote the "up factor" and let d denote the "down factor" in the stock-price model.
If

$$d < u \leq 1 + i$$

then there certainly is no possibility for arbitrage.

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Fixed statement: $\boxed{d < 1+i < u}$

Problem 8.2. In our usual notation, does this parameter choice create a binomial model with an arbitrage opportunity?

$u = 1.18, \quad d = 0.87, \quad r = 0.05, \quad \delta = 0, \quad h = 1/4$

No

$$d = 0.87 \overset{?}{<} \underbrace{e^{rh} = e^{0.05(0.25)}}_{1.0125} \overset{?}{<} 1.18 = u$$

$\tilde{d} = 1.01$