M339W: March 9th, 2022.

41. Assume the Black-Scholes framework. Consider a 1-year European contingent claim on a stock.

You are given:

(i) The time-0 stock price is 45. $\boxed{S(0)=45}$ ✓

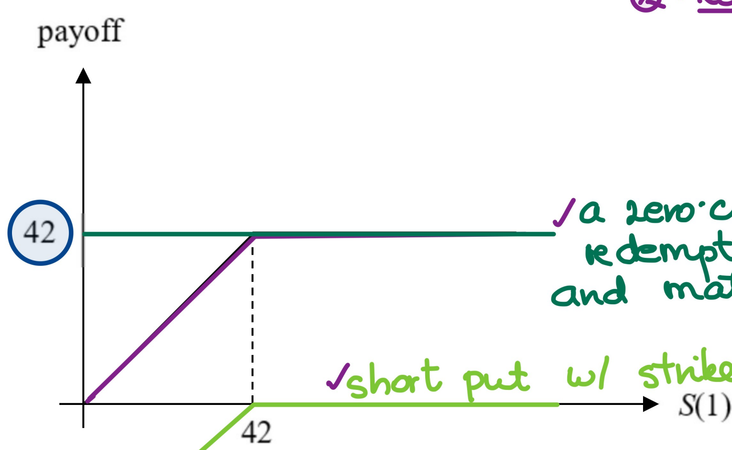
(ii) The stock's volatility is 25%. $\sigma = 0.25$

(iii) The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 3%. $\delta = 0.03$ ✓

(iv) The continuously compounded risk-free interest rate is 7%. $r = 0.07$ ✓

(v) The time-1 payoff of the contingent claim is as follows:

Q: Replicating Portfolio?



✓ a zero-coupon bond w/ redemption amt 42 and maturity date T=1

✓ short put w/ strike 42 and exercise date @ T=1

Calculate the time-0 contingent-claim elasticity.

(A) 0.24

(B) 0.29

(C) 0.34

(D) 0.39

(E) 0.44

The value function of my contingent claim $\boxed{v(s,t)}$

By definition, the option elasticity is

$$\boxed{\Omega(s,t) = \frac{\Delta(s,t)\cdot s}{v(s,t)}}$$

We need to calculate:

$$\Omega(S(0),0) = \frac{\boxed{\Delta(S(0),0)}\; \overbrace{S(0)}^{=45}}{\boxed{v(S(0),0)}}$$

The value f'tion of our contingent claim must satisfy for all (s,t):

$$v(s,t) = v_B(s,t) - v_p(s,t)$$
$$= \boxed{42e^{-r(T-t)} - v_p(s,t)}$$
$$= 42e^{-r(T-t)} - \left(42e^{-r(T-t)} \cdot N(-d_2(s,t)) - se^{-\delta(T-t)}N(-d_1(s,t))\right)$$
$$= 42e^{-r(T-t)}\underbrace{\left(1 - N(-d_2(s,t))\right)}_{N(d_2(s,t))} + se^{-\delta(T-t)}N(-d_1(s,t))$$

At time $\cdot$ 0:

$$d_1(S(0),0) = \frac{1}{0.25\sqrt{1}}\left[\ln\left(\frac{45}{42}\right) + \left(0.07 - 0.03 + \frac{(0.25)^2}{2}\right)\cdot 1\right]$$
$$= 0.56097$$

$$d_2(S(0),0) = d_1(S(0),0) - \sigma\sqrt{T} = 0.56097 - 0.25 = 0.31097$$

$$N(-d_1(S(0),0)) = \text{pnorm}(-0.56097) = \boxed{0.287409} \checkmark$$
$$N(d_2(S(0),0)) = \text{pnorm}(0.31097) = \boxed{0.6220883}$$

$$\Rightarrow v(S(0),0) = 42e^{-0.07}(0.6220883) + 45e^{-0.03}(0.287409)$$
$$= 36.91$$

$$\frac{\partial}{\partial s}\bigg| \; v(s,t) = 42e^{-r(T-t)} - v_p(s,t)$$

$$\Delta(s,t) = 0 - \Delta_p(s,t) = +\left(+e^{-\delta(T-t)}N(-d_1(s,t))\right) > 0$$

At time $\cdot$ 0:

$$\Delta(S(0),0) = e^{-0.03}\cdot(0.287409) = 0.2797$$

$$\Rightarrow \Omega(S(0),0) = \frac{0.2797 \cdot 45}{36.91} = 0.341$$

Q: What's the current volatility of this contingent claim?

$$\longrightarrow: \quad \sigma_{opt}(S(0),0) = \sigma_S \left| \Omega_{opt}(S(0),0) \right|$$
$$= 0.25 \cdot 0.341 = 0.08525 \quad \blacksquare$$

The Gamma.

Γ ... the second-order sensitivity to the perturbations of the price of the underlying asset, i.e.,

$$\boxed{\Gamma(s,t) := \frac{\partial^2}{\partial s^2} v(s,t)}$$

European call.

$$\Gamma_c(s,t) := \frac{\partial^2}{\partial s^2} v_c(s,t) = \frac{\partial}{\partial s}\Big(\underbrace{\frac{\partial}{\partial s} v_c(s,t)}_{\Delta_c(s,t)}\Big)$$
$$= \frac{\partial}{\partial s}\left(e^{-\delta(T-t)} \cdot N(d_1(s,t))\right)$$
$$= e^{-\delta(T-t)} \cdot \underbrace{\frac{\partial}{\partial s}\left(N(d_1(s,t))\right)}_{\text{Use the chain rule!}}$$

$$\underbrace{N'(d_1(s,t))}_{= f_Z(d_1(s,t))} \cdot \boxed{\underbrace{\frac{\partial}{\partial s} d_1(s,t)}_{= ?}}$$

$$d_1(s,t) = \frac{1}{\sigma\sqrt{T-t}}\Big[\underbrace{\ln\left(\frac{s}{K}\right)}_{\ln(s) - \ln(K)} + (r - \delta + \frac{\sigma^2}{2})(T-t)\Big]$$

$$\Rightarrow \boxed{\frac{\partial}{\partial s} d_1(s,t) = \frac{1}{\sigma\sqrt{T-t}} \cdot \frac{1}{s}}$$

$$\boxed{\Gamma(s,t) = e^{-\delta(T-t)} \cdot f_Z(d_1(s,t)) \cdot \frac{1}{\sigma\sqrt{T-t}} \cdot \frac{1}{s}}$$

European Put.

Put-Call Parity.

$$v_c(s,t) - v_p(s,t) = se^{-\delta(T-t)} - Ke^{-r(T-t)}$$

$\frac{\partial}{\partial s}$

$$\Delta_c(s,t) - \Delta_p(s,t) = e^{-\delta(T-t)}$$

$\frac{\partial}{\partial s}$

$$\boxed{\Gamma_c(s,t) = \Gamma_p(s,t)}$$