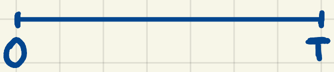
M339D: February 19th, 2024.

Replicating Portfolio.

Def'n. Consider a European-style derivative security. A static portfolio w/ the same payoff as that of the derivative security is called its **replicating portfolio**.

Note. The initial price of the derivative security is equal to the initial price of its replicating portfolio.

Example. Consider a forward contract on a non-dividend-paying stock.



Forward contract: $S(T) - F$

Replicating portfolio:
- long 1 share of stock
- issue a bond w/ redemption amount F and maturity date T

$$\text{Payoff (Portfolio)} = S(T) - F$$

$\Rightarrow$ The forward contract and its replicating portfolio must have the same initial cost, i.e.,

$$0 = \underbrace{S(0)}_{\text{long stock}} - \underbrace{PV_{0,T}(F)}_{\text{short bond}}$$

$$\Rightarrow \quad PV_{0,T}(F) = S(0)$$

$$\Rightarrow \quad \boxed{F = S(0)e^{rT}}$$

$\square$

3. An insurance company sells single premium deferred annuity contracts with return linked to a stock index, the time-t value of one unit of which is denoted by $S(t)$. The contracts offer a minimum guarantee return rate of $g\%$. At time 0, a single premium of amount π is paid by the policyholder, and $\pi \times y\%$ is deducted by the insurance company. Thus, at the contract maturity date, T, the insurance company will pay the policyholder

$$\boxed{\pi \times (1 - y\%) \times \text{Max}[S(T)/S(0), (1 + g\%)^T].}$$

You are given the following information:

(i) The contract will mature in one year. $\boxed{T=1}$

(ii) The minimum guarantee rate of return, $g\%$, is 3%. $g=0.03$

(iii) ~~Dividends are incorporated in the stock index. That is, the stock index is constructed with all stock dividends reinvested.~~ no dividends

(iv) $S(0) = 100$.

(v) The price of a one-year European $\boxed{\text{put}}$ option, with strike price of \$103, on the stock index is $\boxed{\$15.21}$.

Determine $y\%$, so that the insurance company does not make or lose money on this contract.

$\longrightarrow$:

(A) 12.8%.

(B) 13.0%

(C) 13.2%

(D) 13.4%

(E) 13.6%.

The insurance company's liability:

$$\underbrace{\boxed{\pi(1-y)}}_{\text{constant}} \times \text{Max}\left[\frac{S(T)}{S(0)}, (1+g)^T\right]$$

$$\underbrace{\frac{1}{S(0)}}_{\text{Constant}} \boxed{\text{Max}[S(T), S(0)(1+g)^T]}$$

$$= \text{Max}[S(T), 100(1+0.03)^1]$$

$$= \boxed{\text{Max}[S(T), 103]} \checkmark$$

a, b

$\max(a,b) = a + \max(0, b-a) = a + (b-a)_+$

$= b + \max(a-b, 0) = b + (a-b)_+$

Payoff of Put: $(K - S(T))_+$

$$\max[S(T), 103] = \boxed{S(T)} + \boxed{(103 - S(T))_+}$$

Long Stock Index

Payoff of the put w/ strike 103 and exercise date @ T=1.

The insurance company can <u>perfectly hedge</u> by:

- <u>longing/buying</u> $\frac{\pi(1-y)}{S(0)}$ units of the stock index

and

- buying $\frac{\pi(1-y)}{S(0)}$ European puts w/ K=103 and T=1.

> The amount they receive @ time 0 is **EQUAL** to the cost of the hedge.

$$\cancel{\pi} = \frac{\cancel{\pi}(1-y)}{S(0)}\left(S(0) + V_p(0)\right)$$

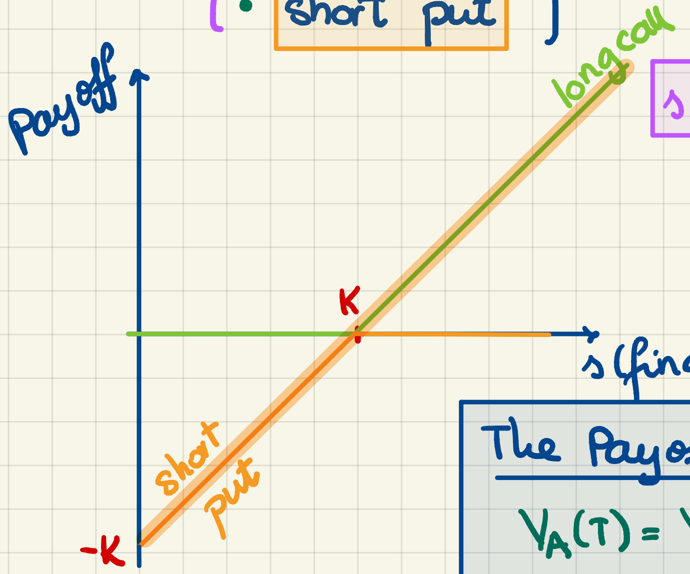
$$100 = (1-y)(100 + 15.21)$$

$$1-y = \frac{100}{115.21} \implies \boxed{y = \frac{15.21}{115.21} = 0.132}$$

Put-Call Parity.

Portfolio A:
- long call
- short put

both European & otherwise identical



$\boxed{s-K}$ payoff function of portfolio A

The Payoff:

$$V_A(T) = V_c(T) - V_p(T) = S(T) - K$$

Portfolio B:
- long non-dividend-paying stock
- borrow $PV_{0,T}(K)$ @ the risk-free interest r to be repaid @ time T

$$\Rightarrow \boxed{V_B(T) = S(T) - K}$$

$\Rightarrow$ NO ARBITRAGE!

$$V_A(0) = V_B(0)$$

$$\Rightarrow \boxed{V_c(0) - V_p(0) = S(0) - PV_{0,T}(K)}$$

Put-Call Parity