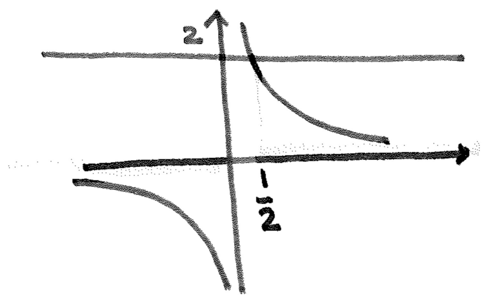
M339W: February 7th, 2020.

$\boxed{*}$ $Z \sim N(0,1)$

$$\mathbb{P}\left[\frac{1}{Z} < 2\right] = \cancel{?} \ \mathbb{P}[Z<0] + \mathbb{P}\left[Z > \frac{1}{2}\right]$$



$= \dots$

Moment Generating Functions.

For any random variable Y,
for independent arguments denoted by t,
we set

$$M_Y(t) := \mathbb{E}\left[e^{t \cdot Y}\right]$$

whenever the expectation exists, i.e., for all t where it is finite

Note: $\boxed{M_Y(0) = 1}$

$\Rightarrow$ @ least $t=0$ is in the domain.

Goal: To understand normal rnd variables.
Any normal rnd variable

$$X \sim \text{Normal}(\text{mean} = m, \text{var} = v^2)$$

can be expressed as

$$X \overset{(d)}{=} m + v \cdot Z \quad \text{w/} \quad Z \sim N(0,1)$$

In general: Take constants a and b, and define $\tilde{Y} = a \cdot Y + b$ w/ Y any r.v.

By def'n
$\Rightarrow$

$$M_{\tilde{Y}}(t) = \mathbb{E}\left[e^{t \cdot \tilde{Y}}\right]$$
$$= \mathbb{E}\left[e^{t(a \cdot Y + b)}\right]$$
$$= \mathbb{E}\left[e^{taY} e^{t \cdot b}\right]$$
$$= e^{t \cdot b} \cdot \mathbb{E}\left[e^{taY}\right] = e^{bt} \cdot M_Y(at)$$

In particular: For our normal X:

$$M_X(t) = e^{m \cdot t} \cdot M_Z(v \cdot t) = e^{mt + \frac{v^2 \cdot t^2}{2}} \quad \star$$

$$M_Z(t) = e^{t^2/2}$$

Recall our motivation :)

We model our realized returns as normal, i.e.,

$$R(0,t) \sim \text{Normal}(\text{mean} = m, \text{var} = v^2)$$

$\Rightarrow$ the time $\cdot t$ stock price is of the form

$$S(t) = S(0)e^{R(0,t)}$$

LogNormal Distribution.

Def'n. A random variable Y is said to be lognormally dist'd if it is of the form

$$Y = e^X \quad \text{w/ } X \sim \text{Normal}$$

Note:

(i) $\mathbb{E}[Y] \underset{\uparrow}{=} \mathbb{E}[e^X] \underset{\uparrow}{=} \mathbb{E}[e^{m+Z\cdot v}]$ w/ $Z \sim N(0,1)$

$X \sim \text{Normal}(\text{mean} = m, \text{var} = v^2)$

$$= e^m \cdot \mathbb{E}[e^{Z\cdot v}] = e^m \cdot M_Z(v)$$

$$= e^{m+\frac{v^2}{2}}$$

CAVEAT: $\mathbb{E}[e^X] \geq e^{\mathbb{E}[X]}$

This is a special case of $\boxed{\text{Jensen's Inequality}}$.

Thm. If X is a random variable and g is a CONVEX f'tion such that $g(X)$ is well-defined and $\mathbb{E}[g(X)]$ is well-defined,

then

$$\mathbb{E}[g(X)] \geq g(\mathbb{E}[X])$$

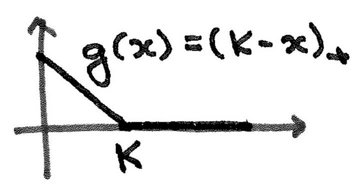
Examples:

(i) $g(x) = |x|$

$$\mathbb{E}[|X|] \geq |\mathbb{E}[X]|$$

(ii) Consider a put option w/ strike K.
The expected payoff of the put is

$$\mathbb{E}[(K - S(T))_+]$$



$$\geq$$

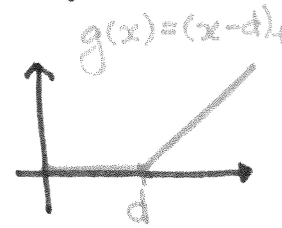
$$(K - \mathbb{E}[S(T)])_+$$

(iii) In classical insurance:

X... ground-up loss, i.e., the severity random variable

d... deductible

→ the insurer pays $(X-d)_+$



$$\mathbb{E}[(X-d)_+] \geq (\mathbb{E}[X] - d)_+$$

Note:

(ii) The median of $Y = ?$

Looking for

$a^* = ?$ such that $\boxed{F_Y(a^*) = \frac{1}{2}}$

$\rightarrow$: $\frac{1}{2} = F_Y(a^*) \underset{\substack{\uparrow \\ \text{by def'n} \\ \text{of a cdf}}}{=} \mathbb{P}[Y \leq a^*] \underset{\substack{\uparrow \\ \text{by def'n} \\ \text{of } Y}}{=}$

$$= \mathbb{P}\left[e^{m+v \cdot Z} \leq a^*\right] \quad \text{w/ } Z \sim N(0,1)$$

$$= \mathbb{P}\left[m + v \cdot Z \leq \ln(a^*)\right] \qquad v > 0$$

$$= \mathbb{P}\left[Z \leq \underbrace{\frac{1}{v}(\ln(a^*) - m)}_{= 0}\right]$$

$$\Rightarrow a^* = e^m$$

Mean of lognormal $\geq$ median of lognormal