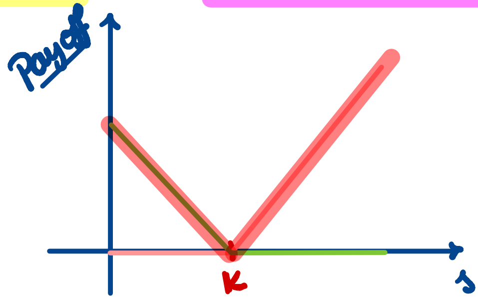
M339D: March 20th, 2023.

Problem 8.5. Consider a non-dividend paying stock whose current price is \$95 per share. You model the evolution of this stock price over the following year using a one-period binomial tree under the assumption that the stock price can be either $\boxed{\$120}$, or $\boxed{\$75}$ in one year.

The continuously compounded risk-free interest rate is 0.06. $r = 0.06$

A **straddle** consists of a long call and a $\boxed{\text{long otherwise identical put}}$. Consider a $\boxed{\$100}$ strike, one-year European **straddle** on the above stock. What is the straddle's price consistent with the above stock-price model?

(a) About \$10
(b) About \$10.83
(c) About \$15.45
(d) About \$20.84
(e) None of the above.



Payoff f'tion of straddle:

$$\boxed{v(s) = |s-K|}$$

$K = 100$

$S(0) = 95$ $\quad S_u = 120$, $V_u = 20$ $\quad S_d = 75$, $V_d = 25$

$$\begin{cases} \Delta = \dfrac{V_u - V_d}{S_u - S_d} = \dfrac{20-25}{120-75} = -\dfrac{5}{45} = -\dfrac{1}{9} \\ B = \dfrac{u V_d - d \cdot V_u}{u-d} e^{-rh} = e^{-0.06(1)} \cdot \dfrac{\frac{120}{95}\cdot 25 - \frac{75}{95}\cdot 20}{\frac{120}{95} - \frac{75}{95}} \end{cases}$$

$$= e^{-0.06} \cdot \frac{120(25) - 75(20)}{120-75} = \underline{31.392}$$

$$V(0) = \Delta \cdot S(0) + B = -\frac{1}{9}(95) + 31.392 = \underline{20.83} \quad \square$$

Risk-Neutral Probability.

Start w/

$$V(0) = \Delta \cdot S(0) + B$$

$$V(0) = \frac{V_u - V_d}{S_u - S_d} \cdot S(0) + e^{-rh} \cdot \frac{u \cdot V_d - d \cdot V_u}{u-d}$$

⋮ algebra

$$V(0) = e^{-rh}\left[V_u \cdot \frac{e^{rh}-d}{u-d} + V_d \cdot \frac{u-e^{rh}}{u-d}\right]$$

Add up to 1!

Both positive! Due to the no-arbitrage condition.

We choose to interpret these two quantities as probabilities!

We define the risk-neutral probability of the stock price going up in a single period as

$$\boxed{p^* := \frac{e^{rh}-d}{u-d}}$$

=> The risk-neutral pricing formula

$$\boxed{V(0) = e^{-rT}\left[V_u \cdot p^* + V_d \cdot (1-p^*)\right]}$$

We will generalize this principle:

$$\boxed{V(0) = e^{-rT}\,\mathbb{E}^*[V(T)]}$$

8.5 w/ this pricing approach:

$$p^* = \frac{e^{rh}-d}{u-d} = \frac{S(0)e^{rh}-S_d}{S_u-S_d} = \frac{95e^{0.06}-75}{120-75} = 0.5749$$

$$V(0) = e^{-0.06}\left(20 \cdot 0.5749 + 25 \cdot (1-0.5749)\right) = 20.84 \quad \square$$

More on the Forward Binomial Tree.

σ ... volatility

$u := e^{rh + \sigma\sqrt{h}}$

$d := e^{rh - \sigma\sqrt{h}}$

The risk-neutral probability in this special case.

$$p^* = \frac{e^{rh} - d}{u - d} = \frac{e^{rh} - e^{rh - \sigma\sqrt{h}}}{e^{rh+\sigma\sqrt{h}} - e^{rh-\sigma\sqrt{h}}}$$

$$p^* = \frac{\cancel{e^{rh}} - \cancel{e^{rh}} \cdot e^{-\sigma\sqrt{h}}}{\cancel{e^{rh}} \cdot e^{\sigma\sqrt{h}} - \cancel{e^{rh}} \cdot e^{-\sigma\sqrt{h}}} = \frac{1 - e^{-\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}}$$

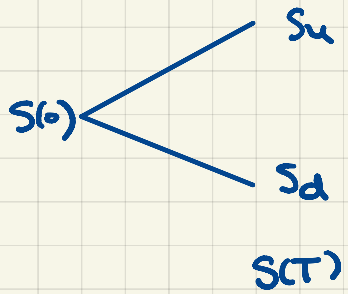
$$p^* = \frac{\cancel{1 - e^{-\sigma\sqrt{h}}}}{e^{\sigma\sqrt{h}} \underbrace{\left(1 - e^{-2\sigma\sqrt{h}}\right)}_{(\cancel{1 - e^{-\sigma\sqrt{h}}})(1 + e^{-\sigma\sqrt{h}})}}$$

$$\boxed{p^* = \frac{1}{1 + e^{\sigma\sqrt{h}}}}$$

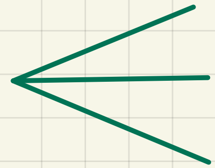
The shortcut only for the FORWARD BINOMIAL TREE.

$$\Big\downarrow \; h \to 0$$

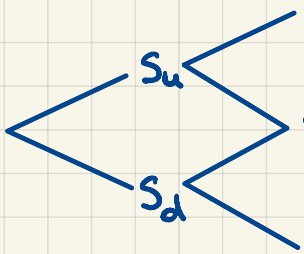
$$\frac{1}{2}$$



Q: How could you make the model for $S(T)$ more complex?



Two-Periods:



$S_{uu} = u^2 \cdot S(0)$

$S_{ud} = u \cdot d \cdot S(0) = S_{du}$

$S_{dd} = d^2 \cdot S(0)$