M339W: February 21st, 2022.

The Black-Scholes Model [Review].

$$S(T) = S(0)e^{(\alpha - \delta - \frac{\sigma^2}{2})T + \sigma\sqrt{T}Z} \quad \text{w/ } Z \sim N(0,1)$$

Under the risk-neutral measure $\mathbb{P}^*$:

$$S(T) = S(0)e^{(r - \delta - \frac{\sigma^2}{2})\cdot T + \sigma\sqrt{T}Z^*} \quad \text{w/ } Z^* \sim N(0,1)$$

$$\boxed{\mathbb{P}[S(T) > K] = N(\hat{d}_2) \quad \text{w/ } \hat{d}_2 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(\alpha - \delta - \frac{\sigma^2}{2}\right)\cdot T\right]}$$

Under the risk-neutral probability measure $\mathbb{P}^*$:

$$\boxed{\mathbb{P}^*[S(T) > K] = N(d_2) \quad \text{w/ } d_2 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(r - \delta - \frac{\sigma^2}{2}\right)\cdot T\right]}$$

Partial and Conditional Expectations.

- Motivation I: $TVaR_p(S(T))$
- Motivation II: PRICING

Goal: Get a formula for the price of European options on a stock modeled using the Black-Scholes framework.

Idea: RISK-NEUTRAL PRICING

$$\boxed{V(0) = e^{-rT}\mathbb{E}^*[\underbrace{V(T)}_{\text{payoff of a European option}}]}$$

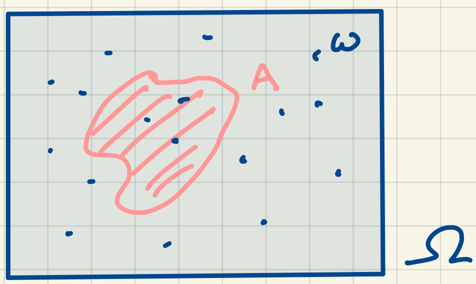
Implementation:

Temporarily, focus on a time$\cdot T$, strike$\cdot K$ European call option.

The payoff : $V_c(T) = (S(T)-K)_+$

Under any measure $\mathbb{P}$:

$$\boxed{\mathbb{E}[V_c(T)]} = \mathbb{E}\left[(S(T)-K)_+\right]$$
$$= \mathbb{E}\left[(S(T)-K)\cdot \mathbb{I}_{[S(T)>K]}\right]$$
$$= \mathbb{E}\left[S(T)\cdot \mathbb{I}_{[S(T)>K]}\right] - K\cdot \boxed{\underbrace{\mathbb{E}\left[\mathbb{I}_{\boxed{[S(T)>K]}}\right]}_{\overset{\shortparallel}{?}}} \checkmark$$



A is an event

$$\mathbb{I}_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}$$

$$\mathbb{I}_A = \begin{cases} 1 & \text{if } A \text{ happened} \\ 0 & \text{if } A \text{ did not happen} \end{cases}$$

$$\mathbb{E}[\mathbb{I}_A] = 1\cdot \mathbb{P}[A] + 0\cdot \mathbb{P}[A^c] = \mathbb{P}[A]$$

$$\boxed{\mathbb{E}\left[\mathbb{I}_{[S(T)>K]}\right] = \mathbb{P}[S(T)>K] = N(\hat{d}_2)}\ \checkmark$$

w/ $\hat{d}_2$ as above

Focus on partial expectation:

$$\mathbb{E}\left[S(T)\cdot \mathbb{I}_{\boxed{[S(T)>K]}}\right] = ?$$

Method: Use the defining formula for the expectation of a f'tion of a r.v.
In this case, that r.v. is $Z \sim N(0,1)$.

$$\{S(T) > K\} =$$
$$= \left\{S(0)e^{(\alpha-\delta-\frac{\sigma^2}{2})\cdot T + \sigma\sqrt{T}\cdot Z} > K\right\}$$
$$= \left\{Z > -\hat{d}_2\right\}$$

z... my dummy variable within the integral; it corresponds to Z

i.e.,
$$g(z) = S(0)e^{(\alpha-\delta-\frac{\sigma^2}{2})\cdot T + \sigma\sqrt{T}\cdot z}$$
(so that $g(Z) = S(T)$)

$$\mathbb{E}\left[g(Z)\cdot \mathbb{I}_{[Z>-\hat{d}_2]}\right] = \int_{-\hat{d}_2}^{+\infty} g(z)\cdot\varphi(z)\,dz \qquad \text{(lots of algebra)}$$

$$= S(0)e^{(\alpha-\delta)T}\cdot N(\hat{d}_1)$$

w/ $\hat{d}_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(\alpha-\delta+\frac{\sigma^2}{2}\right)\cdot T\right]$

$$\boxed{\mathbb{E}\left[S(T)\cdot \mathbb{I}_{[S(T)>K]}\right] = \underbrace{S(0)e^{(\alpha-\delta)T}}_{\mathbb{E}[S(T)]}\cdot N(\hat{d}_1)}$$

✓

The expectation of a call payoff:

$$\boxed{\begin{aligned}&\mathbb{E}[V_C(T)] = S(0)e^{(\alpha-\delta)\cdot T}\cdot N(\hat{d}_1) - K\cdot N(\hat{d}_2)\\ &\text{w/ } \hat{d}_1 \text{ as above and } \hat{d}_2 = \hat{d}_1 - \sigma\sqrt{T}\end{aligned}}$$

For the otherwise identical put option:

$$\mathbb{E}[V_p(T)] = \mathbb{E}[(K-S(T))_+]$$
$$= \mathbb{E}[(K-S(T))\cdot \mathbb{I}_{[S(T)<K]}]$$
$$= K\cdot \underbrace{\mathbb{P}[S(T)<K]}_{\substack{\parallel \\ N(-\hat{d}_2)}} - \boxed{\mathbb{E}[S(T)\cdot\mathbb{I}_{[S(T)<K]}]}$$

$$\mathbb{E}[S(T)] - \mathbb{E}[S(T)\cdot\mathbb{I}_{[S(T)\geq K]}] =$$
$$= S(0)e^{(\alpha-\delta)T} - S(0)e^{(\alpha-\delta)\cdot T}\cdot N(\hat{d}_1)$$
$$= S(0)e^{(\alpha-\delta)\cdot T}(1-N(\hat{d}_1))$$
$$= S(0)e^{(\alpha-\delta)\cdot T}\cdot N(-\hat{d}_1)$$

Conditional Expectation.

Let X be a r.v.

Let A be an event such that $\mathbb{P}[A]>0$.

Then, $$\boxed{\mathbb{E}[X \mid A] := \frac{\mathbb{E}[X\cdot\mathbb{I}_A]}{\mathbb{P}[A]}}$$

$$\mathbb{E}[S(T) \mid S(T)\geq K] = ?$$

$$\mathbb{E}[S(T) \mid S(T)< K] = ?$$

Fill in the formulae @ home, please.