M378K : November 8^{th}, 2024.

Problem 14.3. *What is the unbiased estimator for σ^2.*

$$\boxed{S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y})^2}$$

Problem 14.4. *Assume a random sample $Y_1, Y_2, \ldots, Y_n$ from a normal distribution with mean μ and standard deviation σ - both unknown. What's the distribution of*

$$Q^2 = \boxed{\frac{\sum_{i=1}^{n}(Y_i - \bar{Y}_n)^2}{\sigma^2}} = \boxed{\frac{(n-1)S^2}{\sigma^2}}?$$

$$\sim \chi^2(df = n-1)$$

$$\boxed{\text{PIVOTAL QUANTITY}} = U$$

Problem 14.5. *Assume that you are assigned a* confidence level $\boxed{1 - \alpha}$. *What does it mean to find a* confidence interval *for S^2?*

$$\mathbb{P}[a \leq U \leq b] = 1-\alpha$$

$$\mathbb{P}\Big[\chi^2_L \leq \underbrace{\boxed{\frac{(n-1)S^2}{\sigma^2}}}_{Q^2} \leq \chi^2_R\Big] = 1-\alpha$$

Problem 14.6. *Are $\hat{\chi}^2_L$ and $\hat{\chi}^2_U$ as above uniquely defined?*

No :)

We choose:

$\chi^2_L = \text{qchisq}(\alpha/2)$, $\chi^2_R = \text{qchisq}(1-\alpha/2)$

Problem 14.7. *What's the form of the confidence interval, then?*

$$\mathbb{P}\left[\chi^2_L \le \frac{(n-1)S^2}{\sigma^2} \le \chi^2_R\right] = 1-\alpha$$

$$\mathbb{P}\left[\frac{1}{\chi^2_L} \ge \frac{\sigma^2}{(n-1)S^2} \ge \frac{1}{\chi^2_R}\right] = 1-\alpha$$

$$\mathbb{P}\left[\underbrace{\boxed{\frac{(n-1)S^2}{\chi^2_R}}}_{=\hat{\Theta}_L} \le \sigma^2 \le \underbrace{\boxed{\frac{(n-1)S^2}{\chi^2_L}}}_{=\hat{\Theta}_R}\right] = 1-\alpha$$

Problem 14.8. *Assume the above setting. Let the random sample be of size* $n = 9$. *You do the arithmetic and arrive at the estimate* $\boxed{s^2 = 7.93}$ *(based on the data set). Using the above procedure, find the* 90%*–confidence interval for* σ^2.

→ : $n = 9 \Rightarrow df = n-1 = 8$

$\alpha = 0.10$

$\chi^2_L = 2.733 \qquad \chi^2_R = 15.51$

Our CI :

$$\left(\frac{8 \cdot 7.93}{15.51}, \frac{8 \cdot 7.93}{2.733}\right) = (4.09, 23.24)$$

□

t-Distribution.

Def'n. A Student t-distribution w/ k degrees of freedom is the dist'n of random variable given by

$$\boxed{T = \frac{Z}{\sqrt{Q^2/k}}}$$

w/
- $Z \sim N(0,1)$
- $Q^2 \sim \chi^2(k)$
- Z and Q^2 are independent

t-Procedures.

Consider a random sample $Y_1, \ldots, Y_n$ from $N(\mu, \sigma)$ w/ both parameters unknown.

Goal: Confidence intervals for μ

Idea: Use

$$\boxed{\frac{\bar{Y} - \mu}{S/\sqrt{n}}}$$

as a pivotal quantity.

$$\frac{\bar{Y} - \mu}{S/\sqrt{n}} = \frac{\frac{\bar{Y}-\mu}{\sigma/\sqrt{n}}}{S/\sigma} = \frac{\frac{\bar{Y}-\mu}{\sigma/\sqrt{n}}}{\sqrt{\frac{S^2}{\sigma^2}}} = \frac{\overbrace{\frac{\bar{Y}-\mu}{\sigma/\sqrt{n}}}^{\sim N(0,1) \sim Z}}{\sqrt{\frac{Q^2}{n-1}}}$$

$$\sim t(df = n-1)$$

To construct a confidence interval w/ the confidence level $1-\alpha$

$$t_L^* = qt(\alpha/2)$$
$$t_R^* = qt(1-\alpha/2)$$

$$\mathbb{P}\left[t_L^* \leq \frac{\bar{Y}-\mu}{S/\sqrt{n}} \leq t_R^* \right] = 1-\alpha$$

$$\mathbb{P}\left[\bar{Y} + t_L^* \frac{S}{\sqrt{n}} \leq \mu \leq \bar{Y} + t_R^* \frac{S}{\sqrt{n}} \right] = 1-\alpha$$

$$\boxed{-t_L^* = t_R^* = t^*}$$