M378K Introduction to Mathematical Statistics
Problem Set #17
Relative efficiency.

Definition 17.1. *Given two unbiased estimators $\hat{\theta}_1$ and $\hat{\theta}_2$, the efficiency of $\hat{\theta}_1$ relative to $\hat{\theta}_2$ is defined as*

$$\mathit{eff}(\hat{\theta}_1, \hat{\theta}_2) = \frac{\mathrm{Var}[\hat{\theta}_2]}{\mathrm{Var}[\hat{\theta}_1]}\,.$$

Problem 17.1. *Let Y_1, Y_2 be a random sample from the exponential distribution with the unknown parameter θ.*

(i) The estimator $\hat{\theta}_1 = (Y_1 + Y_2)/2$ for θ is proposed. What is its variance?

(ii) The estimator $\hat{\theta}_2 = cY_{(1)}$ for θ is proposed. Find the constant c such that $\hat{\theta}_2$ is an unbiased estimator of θ. What is its variance?

(iii) Calculate the efficiency of $\hat{\theta}_1$ relative to $\hat{\theta}_2$.

Solution:

(i)

$$\mathrm{Var}[\hat{\theta}_1] = \mathrm{Var}\left[\frac{Y_1 + Y_2}{2}\right] = \frac{1}{4}\left(\mathrm{Var}[Y_1] + \mathrm{Var}[Y_2]\right) = \frac{2\tau^2}{4} = \frac{\tau^2}{2}\,.$$

(ii) By definition, $Y_{(1)} = \min(Y_1, Y_2)$. As we have proved earlier in the class

$$Y_{(1)} \sim E\left(\frac{\tau}{2}\right)$$

So, $\mathbb{E}[Y_{(1)}] = \frac{\tau}{2}$. In order to have $\hat{\theta}_2$ be unbiased, we must set $c = 2$. Since, $\mathrm{Var}[Y_{(1)}] = \frac{\tau^2}{4}$, we have that

$$\mathrm{Var}[\hat{\theta}_2] = \mathrm{Var}[2Y_{(1)}] = 4\left(\frac{\tau^2}{4}\right) = \tau^2.$$

(iii)

$$\mathrm{eff}(\hat{\theta}_1, \hat{\theta}_2) = \frac{\mathrm{Var}[\hat{\theta}_2]}{\mathrm{Var}[\hat{\theta}_1]} = \frac{\tau^2}{\frac{\tau^2}{2}} = 2.$$