M339W: October 15th, 2021.

Implied Volatility.

- Assume that we can observe call/put prices in the market.
- Assume the Black-Scholes model

$\Rightarrow$ we have nice formulae for Black-Scholes prices of European calls/puts w/ arguments $(s, t, r, \delta, \sigma)$

Say the value of an option at a particular time t is:

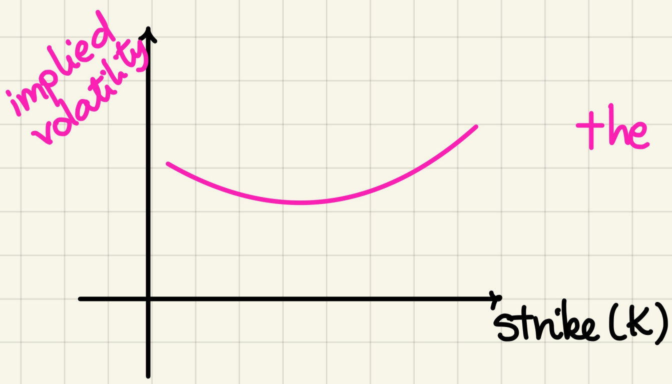
$$\boxed{v(\underbrace{s, t, r, \delta,}_{\text{Assume they are given/observed.}} \sigma)}$$

Consider the Black-Scholes pricing formula as a function of σ.

We invert the Black-Scholes pricing formula to obtain the σ.
The solution σ is called implied volatility.

Theoretically. If all of the above assumptions are true, the observed call prices for different strikes K should give us the same value for the implied volatility.

Practically.



the effect is called the VOLATILITY SMILE

17. Assume the Black-Scholes framework. Consider a one-year at-the-money European put option on a nondividend-paying stock.

$\boxed{T=1}$ $\boxed{S(0)=K}$ $\delta=0$

You are given:

(i) The ratio of the put option price to the stock price is less than 5%. $\boxed{\frac{v_p(S(0),0)}{S(0)} < 0.05}$

(ii) Delta of the put option is –0.4364. $\boxed{\Delta_p(S(0),0) = -0.4364}$

(iii) The continuously compounded risk-free interest rate is 1.2%. $\boxed{r = 0.012}$

Determine the stock's volatility. $\sigma = ?$

(A) 12%

(B) 14%

(C) 16%

(D) 18%

(E) 20%

$$(ii) \Rightarrow + \cancel{e^{-\delta T}} \cdot \boxed{N(-d_1(S(0),0)) = +0.4364}$$

$$N(d_1(S(0),0)) = 1 - 0.4364 = 0.5636$$

$$d_1(S(0),0) = N^{-1}(0.5636) = 0.16$$

$$\frac{1}{\sigma\sqrt{1}}\left[\underbrace{\cancel{\ln\left(\frac{S(0)}{K}\right)}}_{\text{at the money}} + \left(0.012 + \frac{\sigma^2}{2}\right)\cdot 1\right] = 0.16$$

$$0.5\sigma^2 - 0.16\sigma + 0.012 = 0$$

$$\sigma^2 - 0.32\sigma + 0.024 = 0$$

$$\Rightarrow \sigma_1 = 0.12 \text{ and } \sigma_2 = 0.20$$

$$(i) \Rightarrow \cancel{K}e^{-rT}N(-d_2(S(0),0)) - \cancel{S(0)}\cdot \underbrace{N(-d_1(S(0),0))}_{0.4364} < 0.05\cdot\cancel{S(0)}$$

$$e^{-0.012}\cdot N(-d_2(S(0),0)) - 0.4364 < 0.05$$

$$N(-d_2(S(0),0)) < 0.4864\cdot e^{0.012} = 0.49227$$

We know:

$$d_2 = d_1 - \sigma\sqrt{T} \Rightarrow d_2(S(0),0) = 0.16 - \sigma$$

Finally : choose $\sigma = 0.12$.

Example. Consider:

- an at-the-money call/put with $r=\delta$ ✓

or

- a call/put on a non-dividend-paying stock w/ strike $K=S(0)e^{rT}$ ✓

or

- any choice of parameter values such that
$$F^P_{0,T}(S) = PV_{0,T}(K)$$

If this is the case:

$$d_1(S(0),0) = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(r-\delta+\frac{\sigma^2}{2}\right)\cdot T\right]$$
$$= \frac{1}{\sigma\sqrt{T}}\left[\cancel{\ln\left(\frac{S(0)e^{-\delta\cdot T}}{Ke^{-rT}}\right)} + \frac{\sigma^2}{2}\cdot T\right] = \frac{\sigma\sqrt{T}}{2}$$
$$\ln\left(\frac{S(0)e^{-\delta\cdot T}}{Ke^{-rT}}\right) = 0$$

$$\Rightarrow d_2(S(0),0) = d_1(S(0),0) - \sigma\sqrt{T} = \frac{\sigma\sqrt{T}}{2} - \sigma\sqrt{T} = -\frac{\sigma\sqrt{T}}{2}$$

$\Rightarrow$ The call price is

$$v_c(S(0),0) = S(0)e^{-\delta\cdot T}\cdot N(d_1(S(0),0)) - Ke^{-rT}\cdot N(d_2(S(0),0))$$
$$= S(0)e^{-\delta\cdot T}\left(N\left(\frac{\sigma\sqrt{T}}{2}\right) - N\left(-\frac{\sigma\sqrt{T}}{2}\right)\right)$$
$$= S(0)e^{-\delta T}\left(2N\left(\frac{\sigma\sqrt{T}}{2}\right) - 1\right)$$

Given the price of the call, we can invert the price function to get the implied volatility. ∎

Ponder the Taylor expansion!