The Inverse Transform Method.

Proposition.

(1) Let X be a continuous random variable, i.e., let X have the density function f_X.
Assume that $\boxed{f_X(x) > 0 \text{ for all } x.}$
Denote the cumulative dist'n function of X by F_X.
Set $\boxed{\tilde{X} := F_X(X)}$ ✓
Then, $\tilde{X} \sim \underline{U(0,1)}$

→: Support of $\tilde{X}$ will be contained in $[0,1]$.
Let $\underline{u \in [0,1]}$.

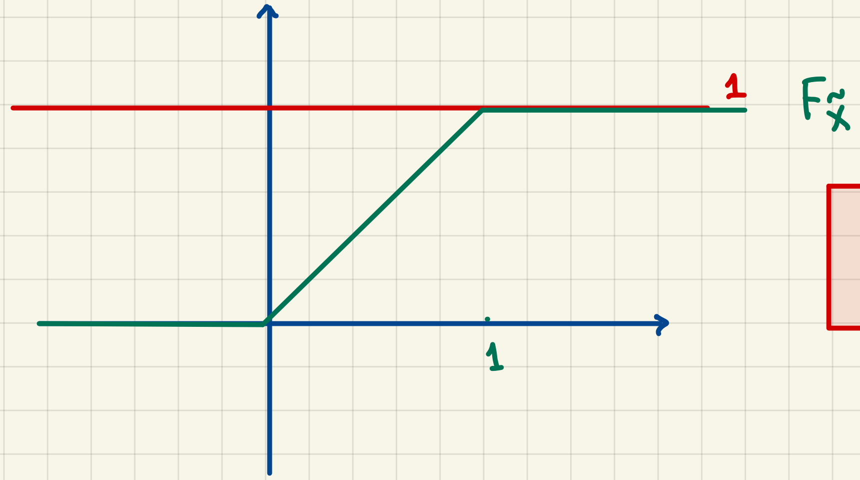
$$F_{\tilde{X}}(u) = \mathbb{P}[\tilde{X} \leq u]$$
$$= \mathbb{P}[F_X(X) \leq u] \ ✓$$

$\boxed{f_X(x) > 0}$ for all x

Recall: $F_X(a) = \int_{-\infty}^{a} f_X(x)\,dx$

⇒ the cdf F_X is strictly increasing
⇒ F_X is one-to-one
⇒ F_X^{-1} exists and is increasing

$$\underline{F_{\tilde{X}}(u)} = \mathbb{P}[X \leq F_X^{-1}(u)]$$
$$= \cancel{F_X}(\cancel{F_X^{-1}}(u)) = \underline{u} \quad \Rightarrow \quad \tilde{X} \sim \ ?$$



$$\boxed{\tilde{X} \sim U(0,1)}$$

(2) Let $U \sim U(0,1)$.

Let F be a cumulative dist'n function.

Set $\boxed{Y := F^{-1}(U)}$

Then, the cumulative dist'n f'tion of Y is F.

Implementation:

1. F... the cdf of the dist'n you want to draw the simulated values from
2. Find an "expression" for F^{-1}
3. Draw: $u_1, u_2, \dots, u_n \sim U(0,1)$ from your rng
4. Set $\boxed{x_i = F^{-1}(u_i), \quad i = 1, \dots, n}$

These are your simulated values from your desired dist'n.

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Problem Set # 1

The inverse transform method.

Problem 1.1. *Source: Course 3, November 1985, Problem #19.*
Your goal is to simulate four draws from a binomial distribution with two trials and the probability of success in every trial equal to 0.30. You intend to use the inverse transform method. Here are the four values produced by the random number generator:

$$\boxed{0.90}\quad 0.21 \quad \boxed{0.72} \quad 0.48.$$

Which values of the binomial were obtained from the above outputs of the random number generator?

$\rightarrow$: $X \sim Binomial(n=2, p=0.30)$

The pmf is

$$p_X(0) = (0.70)^2 = 0.49,$$

$$p_X(1) = 2(0.7)(0.3) = 0.42,$$

$$p_X(2) = (0.30)^2 = 0.09.$$

The cdf of X is:

$$F_X(x) = \begin{cases} 0, & \text{for } x<0 \\ 0.49, & \text{for } 0 \leq x < 1 \\ 0.91, & \text{for } 1 \leq x < 2 \\ 1, & \text{for } 2 \leq x \end{cases}$$

- 0.21 and 0.48 are both $< 0.49 \Rightarrow$ they map into 0
- 0.72 and 0.90 both map into 1.

Problem 1.2. Let the random variable X have the following density function:

$$f_X(x) = 3x^{-4}, \quad x > 1$$

You use the *inverse transform method* to simulate values from X. Let the simulated value of the unit uniform be equal to 0.25. What is the corresponding value of X?

Problem 1.3. *Source: Course 3, November 1980, Problem #33.*
The probability density function of the random variable X is

$$f_X(x) = \begin{cases} 0, & \text{for } x < -1, \\ \frac{3}{2}x^2, & \text{for } -1 \leq x \leq 1, \\ 0, & \text{for } x > 1. \end{cases}$$

Let u denote the simulated value from a unit uniform random number generator. Which transformation would you apply to u to generate simulated values of x?

Problem 1.4. Let the random variable X have the cumulative distribution function given by

$$F_X(x) = \begin{cases} 0, & \text{for } x < 0, \\ x^4, & \text{for } 0 \leq x \leq 1, \\ 1, & \text{for } x > 1. \end{cases}$$

The inverse transform method was applied to generate the following three simulated values of X:

$$0.09 \quad 0.64 \quad 0.81.$$

Which values of the random number generator where mapped into the above three draws from X?

Problem 1.5. Let the cumulative distribution function F_X of a random variable X satisfy the following conditions:

- $F_X(0) = 0$;
- $F_X(1) = 0.2$;
- $F_X(2) = 1$;
- F_X is linear on $(0, 1)$ and $(1, 2)$;
- F_X is continuous.

You use the inverse transform method to simulate values of X. The values given by the random number generator are

$$0.1 \quad 0.6 \quad 0.9.$$

Which simulated values from X were drawn based on the above three values?