The University of Texas at Austin
Homework Assignment 3

February 07, 2026

Instructions: Provide your <u>complete solution</u> to the following problems. Final answers only, without appropriate justification, will receive zero points even if correct.

Expectation

Problem 3.1. (5 points) Let the density function of a random variable X be given as

$$f_X(x) = cxI_{[0,1]}(x),$$

for some constant c. Find $\mathbb{E}[X^3]$.

Solution. Since the density function must integrate up to 1, we get $c = 2$. Whence,

$$\mathbb{E}[X^3] = 2\int_0^1 x^4 \, \mathrm{d}x = \frac{2}{5}.$$

Problem 3.2. (5 points) Let X denote the outcome of a roll of a fair, regular dodecahedron (a polyhedron with 12 faces) with numbers $1, 2, \ldots, 12$ written on its sides. Find $\mathbb{E}[X]$.

Solution. Since the dodecahedron is fair, we have $\mathbb{P}[X = n] = \frac{1}{12}$ for $n = 1, 2, \ldots, 12$. Therefore,

$$\mathbb{E}[X] = \sum_{n=1}^{12} n\left(\frac{1}{12}\right) = \frac{13}{2}.$$

Problem 3.3. (5 points) Let X be a random variable with mean $\mu = 2$ and standard deviation equal to $\sigma = 1$. Find $\mathbb{E}[X^2]$.

Solution. We have

$$\mathbb{E}[X^2] = \mathrm{Var}\ (X) + (\mathbb{E}[X])^2 = \sigma^2 + \mu^2 = 1 + 4 = 5.$$

Problem 3.4. (5 points) Let X denote the number of $1's$ in 100 throws of a fair die. Find $\mathbb{E}[X^2]$.

Solution. We have

$$\mathbb{E}[X^2] = \mathrm{Var}\ (X) + (\mathbb{E}[X])^2 = 100 \times \frac{1}{6} \times \frac{5}{6} + \left(100 \times \frac{1}{6}\right)^2 = \frac{875}{3}.$$

Problem 3.5. (10 points) Let the random variable Y have the following cumulative distribution function

$$F_Y(y) = \begin{cases} 0 & \text{for } \ y < 0 \\ \frac{y}{2} & \text{for } \ 0 \leq y < 1 \\ \frac{y^2}{\alpha} & \text{for } \ 1 \leq y < \beta \\ 1 & \text{for } \ \beta \leq y \end{cases}$$

- (3 points) Find the constants α and β such that the random variable Y is continuous.

- (7 points) Calculate the expectation of the random variable Y for the κ you obtained in the previous part of the problem.

Solution.

- In order for the random variable Y to be continuous, its cumulative distribution function must be continuous. So,

$$\frac{1}{2} = \frac{1^2}{\alpha} \quad \text{and} \quad \frac{\beta^2}{\alpha} = 1.$$

  So, $\alpha = 2$ and $\beta = \sqrt{2}$.
- The probability density function of the random variable Y is

$$f_Y(y) = \begin{cases} 0 & \text{for} \quad y < 0 \\ \frac{1}{2} & \text{for} \quad 0 \leq y < 1 \\ y & \text{for} \quad 1 \leq y < \sqrt{2} \\ 0 & \text{for} \quad \sqrt{2} \leq y \end{cases}.$$

  So, the expectation of Y is

$$\mathbb{E}[Y] = \int_0^1 \frac{y}{2}\,\mathrm{d}y + \int_1^{\{\sqrt{2}\}} y^2\,\mathrm{d}y = \frac{1}{4} + \frac{2\sqrt{2}}{3} - \frac{1}{3} = \frac{2\sqrt{2}}{3} - \frac{1}{12} = \frac{8\sqrt{2}-1}{12}$$

Problem 3.6. (20 points) Let X be a discrete random variable with the support $\mathcal{S}_X = \mathbb{N}$, such that $\mathbb{P}[X = n] = C\frac{1}{n^2}$, for $n \in \mathbb{N}$, where C is a constant chosen so that $\sum_n \mathbb{P}[X = n] = 1$. The distribution table of X is, therefore, given by

1	2	3	...
$C\frac{1}{1^2}$	$C\frac{1}{2^2}$	$C\frac{1}{3^2}$	...

1. (10 points) Show that $\mathbb{E}[X]$ does not exist.
2. (10 points) Construct a distribution of a similar random variable whose expectation does exist, but the variance does not. (*Hint:* Use the same support $\mathbb{N}$, but tweak the probabilities so that the sum for $\mathbb{E}[X]$ converges, while the sum for $\mathbb{E}[X^2]$ does not.)

Solution.

1. The expression for $\mathbb{E}[X]$ is

$$\mathbb{E}[X] = \sum_{n=1}^{\infty} n\mathbb{P}[X = n] = C\sum_{n=1}^{\infty} \frac{1}{n} = +\infty,$$

   because the *harmonic series* $1 + 1/2 + 1/3 + \ldots$ diverges.
2. The distribution of Y we need to construct should have the following properties

$$\sum_{n=1}^{\infty} n\mathbb{P}[Y = n] < \infty \quad \text{but} \quad \sum_{n=1}^{\infty} n^2\mathbb{P}[Y = n] = \infty.$$

   We can try to achieve this by taking $\mathbb{P}[Y = n] = C'\frac{1}{n^3}$, where, as above, C' is simply a constant that ensures that $\sum_n \mathbb{P}[Y = n] = 1$. Indeed, in this case, we have

$$\mathbb{E}[X] = C'\sum_n \frac{1}{n^2} \quad \text{while} \quad \mathbb{E}[X^2] = C'\sum_n \frac{1}{n}.$$

The first sum converges, but the second one diverges.