M378K Introduction to Mathematical Statistics

Homework assignment #2

Please, provide your **final answer only** to the following problems.

Problem 2.1. *(5 points) Let X be a binomial random variable with parameters $n = 10$ and $p = 4/5$. Then*

(a) *If $Y = 2X$ then $\mathcal{S}_Y = \{0, 1, 2, 3, 4, \ldots, 20\}$.*

(b) *If $Y = -X$ then Y is also binomial, but with parameters $n = 10$ and $p = 1 - \frac{4}{5} = \frac{1}{5}$.*

(c) *The support $\mathcal{S}_X$ of X is $10 \times \frac{4}{5} = 8$.*

(d) *$\mathbb{P}[X] = 10 \times \frac{4}{5} = 8$.*

(e) *None of the above.*

Solution: The answer is **(e)**.

Problem 2.2. *(5 points) n people vote in a general election, with only two candidates running. The vote of person i is denoted by Y_i and it can take values 0 and 1, depending which candidate they voted for (we encode one of them as 0 and the other as 1). We assume that votes are independent of each other and that each person votes for candidate 1 with probability p. If the total number of votes for candidate 1 is denoted by Y, then*

(a) *Y is a geometric random variable*

(b) *Y^2 is a binomial random variable*

(c) *Y is uniform on $\{0, 1, \ldots, n\}$*

(d) *$\text{Var}[Y] \leq \mathbb{E}[Y]$*

(e) *None of the above.*

Solution: The correct answer is **(d)**.
Y is a binomial random variable with parameters n and p, and, so, $\mathbb{E}[Y] = np$, $\text{Var}[Y] = np(1 - p) \leq np$. The Y is clearly not geometric, as it takes only finitely many possible values. Y^2 is not binomial, because the set of possible values is not contiguous (some values are skipped), and Y is not uniform since the binomial probabilities are not all the same.

Please, provide your **complete solutions** to the following problems. Final answers only, even if correct will earn zero points for those problems.

Problem 2.3. *(5 points) Let Y be a random variable such that*

$$\mathbb{P}[Y = 2] = 1/2, \mathbb{P}[Y = 3] = 1/3 \text{ and } \mathbb{P}[Y = 6] = 1/6.$$

How much is $\mathbb{E}[Y^2]$?

Solution:

$$\mathbb{E}[Y^2] = 2^2 \cdot \frac{1}{2} + 3^2 \cdot \frac{1}{3} + 6^2 \cdot \frac{1}{6} = 2 + 3 + 6 = 11$$

Problem 2.4. *(5 points) Let Y be a random variable such that*

$$\mathbb{P}[Y = 1] = 1/2, \mathbb{P}[Y = 3] = 1/3 \text{ and } \mathbb{P}[Y = 6] = 1/6.$$

With $|\cdot|$ denoting the absolute value, find $\mathbb{E}[|Y - 2|]$.

Solution: $\mathbb{E}[|Y - 2|] = |1 - 2| \times \frac{1}{2} + |3 - 2| \times \frac{1}{3} + |6 - 2| \times \frac{1}{6} = \frac{1}{2} + \frac{1}{3} + \frac{4}{6} = 3/2.$

Problem 2.5. *(5 points) Let X be a Poisson random variable with parameter $\lambda > 0$. Express $\mathbb{P}[X \geq 3]$ in terms of λ.*

Solution: $\mathbb{P}[X \geq 3] = 1 - \mathbb{P}[X = 0] - \mathbb{P}[X = 1] - \mathbb{P}[X = 2] = 1 - e^{-\lambda}(1 + \lambda + \frac{1}{2}\lambda^2)$

Problem 2.6. *(10 points) The probability that Janet makes a free throw is 0.6. What is the probability that she will make at least 16 out of 23 (independent) throws? Write down the answer as a sum - no need to evaluate it.*

Solution: Let Y denote the number of free throws Janet makes. It is a binomial random variable with parameters $n = 23$ and $p = 0.6$, i.e. $Y \sim b(23, 0.6)$. The probability we are interested in is $\mathbb{P}[Y \geq 16]$. We split this into the following sum

$$\mathbb{P}[Y \geq 16] = \sum_{k=16}^{23} \mathbb{P}[Y = k] = \sum_{k=16}^{23} \binom{23}{k} (0.6)^k (0.4)^{23-k}.$$

(*Note:* If you do evaluate this sum numerically, you get about 0.24.)

Problem 2.7. *(15 points) A mail lady has $l \in \mathbb{N}$ letters in her bag when she starts her shift and is scheduled to visit $n \in \mathbb{N}$ different households during her round. If each letter is equally likely to be addressed to any one of the n households, what is the expected number of households that will receive no letters?*

Note: It is quite possible that some households will receive more than 1 letter.

Solution: Let A_i, $i = 1, \ldots, n$ denote the event where the i-th household does not receive any letters. The sum $X = 1_{A_1} + \cdots + 1_{A_n}$ of the indicators of A_i equals the total number of households that receive no letters. Therefore, by linearity of expectation and the fact that $\mathbb{E}[1_{A_i}] = \mathbb{P}[A_i]$, we get

$$\begin{aligned} \mathbb{E}[X] &= \mathbb{E}[1_{A_1} + \cdots + 1_{A_n}] \\ &= \mathbb{P}[A_1] + \cdots + \mathbb{P}[A_n]. \end{aligned}$$

It remains to compute $\mathbb{P}[A_i]$, for $i = 1, \ldots, n$. For this, we note that the household i will receive no letters if each of the l letters get delivered to another household. This probability, for the individual letter, is $\frac{n-1}{n}$. By independence, thus, we have

$$\mathbb{P}[A_i] = \left(\tfrac{n-1}{n}\right)^l.$$

Finally,

$$\mathbb{E}[X] = n\left(\tfrac{n-1}{n}\right)^l = \tfrac{(n-1)^l}{n^{l-1}}.$$

Note: Compare to Problems 3.2.6, 3.2.13(d) or 3.2.14 from Pitman's "Probability".