M378K Introduction to Mathematical Statistics
Problem Set #8
Transformations of Random Variables.

Problem 8.1. *Let X be a continuous random variable with the cumulative distribution function denoted by F_X and the probability density function denoted by f_X.*

Let the random variable $Y = 2X$ have the p.d.f. denoted by f_Y. Then,

(a) $f_Y(x) = 2f_X(2x)$

(b) $f_Y(x) = \frac{1}{2}f_X\left(\frac{x}{2}\right)$

(c) $f_Y(x) = f_X(2x)$

(d) $f_Y(x) = f_X\left(\frac{x}{2}\right)$

(e) None of the above

Problem 8.2. *If the continuous random variable X has the distribution function F_X, then the distribution function of the random variable $Y = |X|$ equals*

$$F_Y(y) = ?$$

Remark 8.1. *The goal is to figure out the distribution of the random variable*

$$X = g(Y_1, Y_2, \dots, Y_n)$$

where $Y_i, i = 1, \dots, n$ are a **random sample** *with a common density f_Y.*

1. *Identify the objective: We want* f_X.
2. *Realize:* $f_X = F'_X$
3. *Recall the definition:* $F_X(x) = \mathbb{P}[X \le x] = \mathbb{P}[g(Y_1, \dots, Y_n) \le x]$
4. *Identify the region* A_x *in* $\mathbb{R}^n$ *where*

$$g(y_1, \dots, y_n) \le x$$

for every x, *i.e., express*

$$A_x = \{(y_1, \dots, y_n) : g(y_1, \dots, y_n) \le x\}$$

5. *Calculate*

$$F_X(x) = \int \cdots \int_{\mathbb{R}^n} \mathbf{1}_{A_x}(y_1, \dots, y_n) f_Y(y_1) \dots f_Y(y_n) dy_1 \dots dy_n.$$

6. *Differentiate:* $f_X = F'_X$.
7. *Pat yourself on the back!*

Problem 8.3. One-to-one transformations: Step-by-step *Let* Y *be a random variable with density* f_Y. *Let* $g : \mathbb{R} \to \mathbb{R}$ *be a strictly increasing differentiable function. Define* $\tilde{Y} = g(Y)$. *What is the density function* $f_{\tilde{Y}}$ *of* $\tilde{Y}$ *expressed in terms of* f_Y *and* g*?*

1. Identify the objective: We want $f_{\tilde{Y}}$.
2. Realize: $f_{\tilde{Y}} = F'_{\tilde{Y}}$
3. Recall the definition:

$$F_{\tilde{Y}}(x) =$$

4. The function g is assumed to be **strictly increasing**. In which way can you modify the inequality in the probability you obtained above to *separate* the random variable Y from the transformation g?

5. Express your result from above in terms of the c.d.f. F_Y of the r.v. Y.

6. Differentiate: $f_{\tilde{Y}} = F'_{\tilde{Y}}$.

Problem 8.4. *The time T that a manufacturing distribution system is out of operation is modeled by a distribution with the following c.d.f.*

$$F_T(t) = (1 - (2/t)^2)\mathbf{1}_{(2,\infty)}(t)$$

The resulting cost to the company is $Y = T^2$. Find the probability density function f_Y of the r.v. Y.

Problem 8.5. *What if h is strictly decreasing?*

Problem 8.6. *The unifying formula?*

Do not forget: it always makes sense to simply attack a problem without giving it a "label" .... Just look at the following problem:

Problem 8.7. *Let T_1 and T_2 be independent shifted geometric random variables with parameters $p_1 = 1/2$ and $p_2 = 1/3$. Compute $\mathbb{E}[\min(T_1, T_2)]$.*