M339J: May 6th, 2022.

MLE: Bernoulli.

$X \sim \text{Bernoulli}(q)$ ← the probab. of "success"

$\text{Support}(X) = \{0, 1\}$

$$X \sim \begin{cases} 0 & \text{w/ probab. } 1-q \\ 1 & \text{w/ probab } q \end{cases}$$

Let $\underline{x_1, x_2, \dots, x_n}$ be the observations from Bernoulli(q).
They will all be 0 or 1.

We write the pmf as:

$$f(x;q) = \begin{cases} q & \text{if } x=1 \\ 1-q & \text{if } x=0 \end{cases}$$

$$\boxed{f(x;q) = q^{x}(1-q)^{1-x}}$$

⇒ The likelihood ftion:

$$L(q) = \prod_{j=1}^{n} f(x_j;q) = \prod_{j=1}^{n}\left(q^{x_j}(1-q)^{1-x_j}\right)$$

$$L(q) = \prod_{j=1}^{n} q^{x_j} \cdot \prod_{j=1}^{n}(1-q)^{1-x_j} = q^{\sum x_j}(1-q)^{n-\sum x_j}$$

⇒ The log-likelihood:

$$\ell(q) = \left(\sum x_j\right)\cdot \ln(q) + \left(n - \sum x_j\right)\cdot \ln(1-q)$$

Differentiating w.r.t. q, we get

$$\ell'(q) = \left(\sum x_j\right)\cdot \frac{1}{q} + \left(n - \sum x_j\right)(-1)\frac{1}{1-q} = 0$$

Solve for q:

$$\left(\sum x_j\right)(1-\cancel{q}) = nq - \cancel{\left(\sum x_j\right)\cdot q}$$

$$\boxed{\hat{q}_{MLE} = \bar{x}}$$ ... sample mean

MLE: Poisson.

$X \sim \text{Poisson}(\text{mean} = \lambda)$

pmf: $f_X(x; \lambda) = e^{-\lambda} \dfrac{\lambda^x}{x!}$; $x = 0, 1, 2, \dots$

Let $x_1, x_2, \dots, x_n$ be the observations from Poisson(λ).

$\Rightarrow$ The likelihood ftion is:

$$L(\lambda) = \prod_{j=1}^{n} f(x_j; \lambda) = \prod_{j=1}^{n} \left(e^{-\lambda} \frac{\lambda^{x_j}}{x_j!} \right)$$

$$L(\lambda) = e^{-n \cdot \lambda} \cdot \prod_{j=1}^{n} \lambda^{x_j} \cdot \frac{1}{\prod_{j=1}^{n} x_j!} \propto e^{-n \cdot \lambda} \cdot \lambda^{\sum x_j}$$

$\Rightarrow$ The log likelihood is up to a constant:

$$\ell(\lambda) = -n \cdot \lambda + \left(\sum x_j\right) \ln(\lambda)$$

$$\ell'(\lambda) = -n + \left(\sum x_j\right) \cdot \frac{1}{\lambda} = 0$$

$$\Rightarrow \boxed{\hat{\lambda}_{MLE} = \bar{x}}$$

Problem.

$X \sim \text{Poisson}(\text{mean} = \lambda)$

Observations other than 0 and 1 have been deleted from the data.

The data contain an equal number of 0s and 1s.

$\frac{n}{2}$ zeros and $\frac{n}{2}$ ones

Find the MLE for λ.

$\longrightarrow$: Condition on all the remaining data points being ≤ 1.

$$L(\lambda) = \boxed{\frac{1}{(\underbrace{\mathbb{P}[X \leq 1; \lambda]})^n}} \underbrace{\left(e^{-\lambda} \cdot \frac{\lambda^0}{0!}\right)}_{e^{-\lambda}}^{\frac{n}{2}} \underbrace{\left(e^{-\lambda} \frac{\lambda^1}{1!}\right)}_{e^{-\lambda} \cdot \lambda}^{\frac{n}{2}}$$

$$L(\lambda) = \frac{1}{(\cancel{e^{-\lambda}}(1+\lambda))^n} \cdot \cancel{e^{-\lambda \cdot \frac{n}{2}}} \cdot \cancel{e^{-\lambda \cdot \frac{n}{2}}} \cdot \lambda^{\frac{n}{2}}$$

$$L(\lambda) = \frac{\lambda^{\frac{n}{2}}}{(1+\lambda)^n}$$

$$\Rightarrow \quad \ell(\lambda) = \frac{n}{2} \cdot \ln(\lambda) - n \cdot \ln(1+\lambda)$$

$$\Rightarrow \quad \ell'(\lambda) = \frac{n}{2} \cdot \frac{1}{\lambda} - n \cdot \frac{1}{1+\lambda} = 0 \quad / :n$$

$$\frac{1}{2\lambda} = \frac{1}{1+\lambda} \quad \Rightarrow \quad \boxed{\hat{\lambda}_{MLE} = 1}$$

$\square$