M378K: December 5^{th}, 2025.

Tests of Significance.

Set α... significance level

Typically: $\alpha = 0.05, 0.01, 0.001$

Decision Process:

If p-value $\leq \alpha$, we REJECT the null hypothesis.

If p-value $> \alpha$, we FAIL TO REJECT the null hypothesis.

Note: The p-value corresponding to an observed value of the test statistic is the LOWEST significance level @ which the null hypothesis would still be REJECTED.

M378K Introduction to Mathematical Statistics

Problem Set #19

Hypothesis testing.

Problem 19.1. *An instructor of a massive online course claims that students solve at most* 20 *problems per week (on average). To verify this conviction, the instructor intends to conduct a hypothesis test.*

What are the null and alternative hypotheses in this case?

$$H_0: \mu = \mu_0 = 20$$

$$H_a: \mu < \mu_0 = 20$$

With a sample size of 256*, what is the test statistics appropriate the test the above claim? What is its (approximate) distribution under the null?*

$$TS := \frac{\bar{Y} - \mu_0}{\frac{\sigma}{\sqrt{n}}} \approx N(0,1)$$

Say that the sample average equals 19.7 *and that the sample variance equals* 9*. What is the* $p-$*value associated with these data?*

$$p\cdot\text{value} = \mathbb{P}\left[TS < \frac{19.7-20}{\frac{3}{\sqrt{256}}}\right] = \mathbb{P}\left[Z < -\frac{0.3}{\frac{3}{16}}\right] =$$

$$= \Phi(-1.6) = 0.0548$$

Assume that the given significance level is 5%*. What would the decision be?*

Fail to reject the null hypothesis. $\square$

Problem 19.2. *A candy-cane twisting machine is considered defective it at least 10% of the candy canes crack or break in the twisting process. A random sample of 100 candy canes was collected and it was found that it contained 12 cracked candy canes. You believe that the machine is defective. Formulate and conduct the relevant hypothesis test with a 2% significance level.*

$\longrightarrow$:

$H_0: p = p_0 = \underline{0.10}$

vs.

$H_a: p > p_0 = 0.10$

$\hat{P}$ ... sample proportion (a random variable)

$$TS := \frac{\hat{P} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} \approx N(0,1)$$

under the null hypothesis

$$\Rightarrow \quad p\text{-value} = \mathbb{P}\left[TS > \frac{0.12 - 0.10}{\sqrt{\frac{0.1 \cdot 0.9}{100}}}\right] \approx \mathbb{P}\left[Z > \frac{0.02}{0.03}\right]$$

$$= \mathbb{P}\left[Z > \frac{2}{3}\right]$$

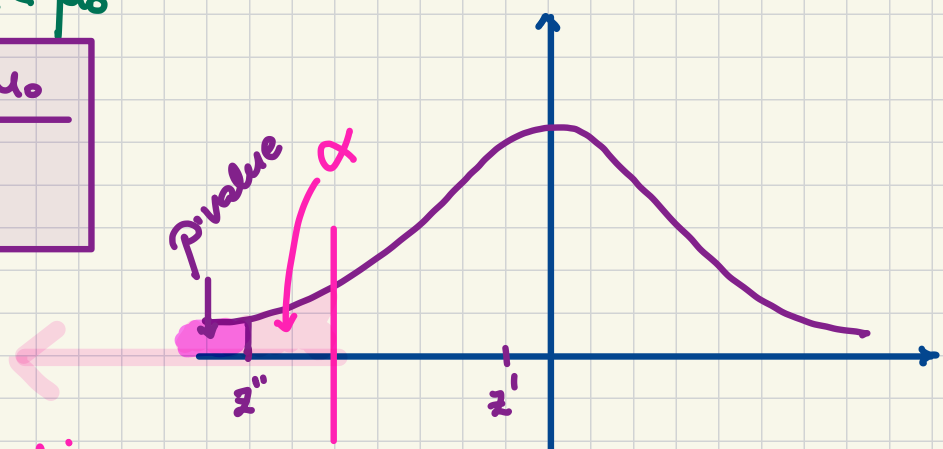
Fail to Reject. $\square$

Rejection Regions. (α... significance level)

The left-sided alternative.

$H_a: \mu < \mu_0$

$$z = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}}$$



Rejection Region (RR) z^*_α

$\hookrightarrow \Phi^{-1}(\alpha) = \text{qnorm}(\alpha)$

$RR = (-\infty, z^*_\alpha]$ **In Standard Units**

$$z \leq z^*_\alpha$$

$$\frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}} \leq z^*_\alpha$$

In Raw Units

$$\bar{x} \leq \boxed{\mu_0 + \frac{\sigma}{\sqrt{n}} \cdot z^*_\alpha}$$

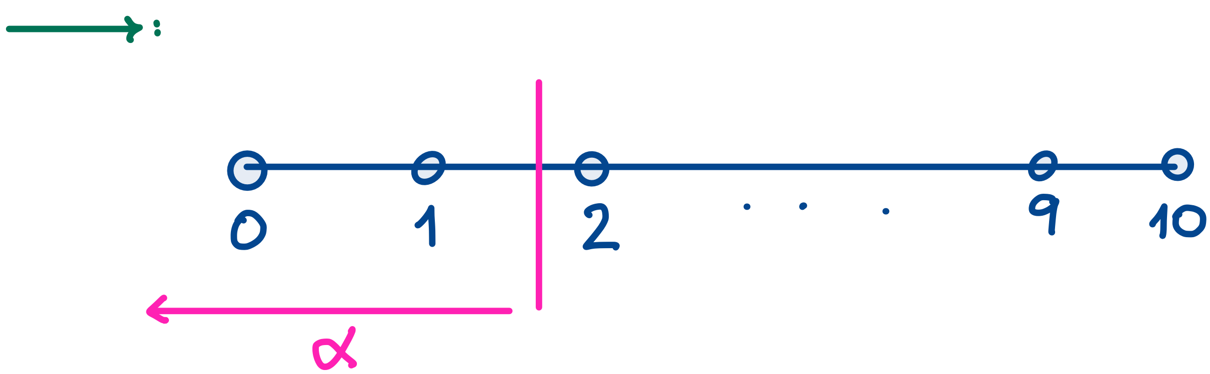
upper bound of the RR in raw units

$$\boxed{RR = \left(-\infty, \mu_0 + \frac{\sigma}{\sqrt{n}} z^*_\alpha\right]}$$

Problem 19.3. *Consider a poll ahead of an election with two candidates: A and B. Let p denote the population proportion of voters who will vote for A. We want to conduct a hypothesis test on whether candidate A will win, i.e., our hypotheses are*

$$H_0 : p = 0.5 \quad \textit{vs.} \quad H_a : p < 0.5$$

Let our significance level be 5%. What is the rejection region (RR) for a sample size of 10?



pbinom(0,10,0.5)

pbinom(1,10,0.5)