M339D : April 27th, 2022.

Properties of American Option Prices.

Fact 1.

$T < \tilde{T}$

Expiration dates for a pair of otherwise identical American options.

$V^A(t, \cdot)$

the expiration date

the valuation date

$$V^A(t, T) \leq V^A(t, \tilde{T})$$

The longer the time to expiration, the more early exercise dates are admissible.

In words: Longer-lived American options are worth @ least as much as otherwise identical shorter-lived American option.

Fact 2.

$$\boxed{V^A(t) \geq V(t)} \quad \text{for } t \leq T$$

American option price @ time t

European option price @ time t

otherwise identical

Fact 3. Bounds on American call prices.

$$V_C^A(t) \geq V_C(t) \geq \max(0, F_{t,T}^P(S) - PV_{t,T}(K))$$

$$V_C^A(t) \geq S(t) - K$$

due to the possibility of early exercise

$$\boxed{V_C^A(t) \geq \max(0, F_{t,T}^P(S) - PV_{t,T}(K), S(t) - K)}$$

Lower Bound

$$\boxed{V_C^A(t) \leq S(t)}$$ Upper Bound

Fact 4. Bounds on American put prices.

$$\boxed{K \geq V_P^A(t) \geq \max\left(0,\ PV_{t,T}(K) - F_{t,T}^P(S),\ K - S(t)\right)}$$

Fact 5. American calls on non-dividend-paying stock.

Assume a nonnegative ccrfir. It is never optimal to exercise an American call on a non-dividend-paying stock early.

$\rightarrow$: Say, to the contrary, that there is a random, non-anticipating time t^* @ which it's optimal to exercise this American call early, i.e., $t^* < T$.

Then,

$$V_C^A(t^*) = S(t^*) - K > CV(t^*)$$

$$\Rightarrow \quad \cancel{S(t^*)} - K > V_C(t^*) \geq F_{t^*,T}^P(S) - Ke^{-r(T-t^*)}$$

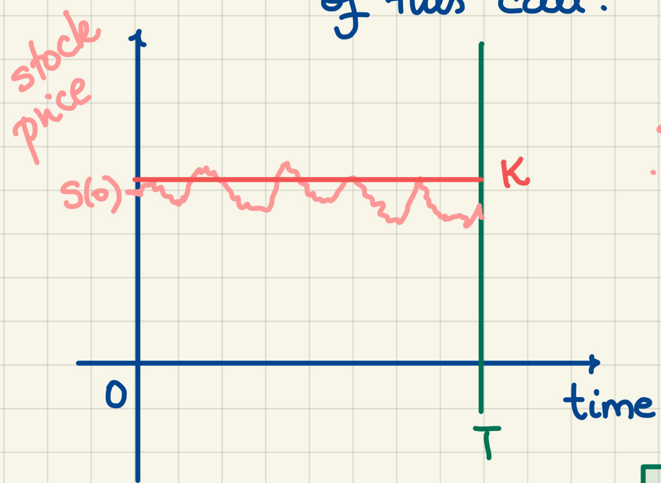
$$F_{t^*,T}^P(S) \overset{\text{no div}}{=} \cancel{S(t^*)}$$

$$\Rightarrow \quad Ke^{-r(T-t^*)} > K \qquad \Rightarrow\Leftarrow \qquad \square$$

Asian Option.

... are a type of exotic option.

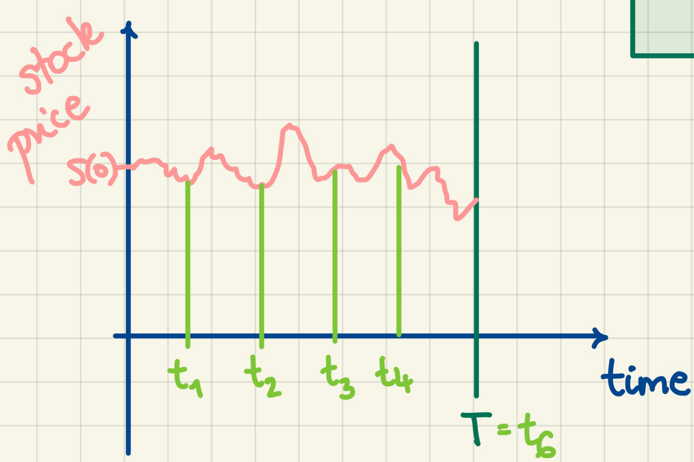
Inspiration: Consider a vanilla call option.
Its payoff will be $V_c(T) = (S(T)-K)_+$.
Imagine you own this call.
Look @ the stock-price evolution during the life of this call.



A large investor could manipulate the stock price @ the end of the time-horizon to affect the payoff of the option.

Idea: Construct an option which takes into account the $\boxed{\text{average}}$ of the stock prices in the payoff.

$\boxed{\text{sampled during the life of the option}}$



$$\boxed{(S(t_k),\ k = 1..n)}$$

$$\downarrow$$

average stock price

Set:

$$\boxed{\begin{array}{l} \bullet\ A(T) = \frac{1}{n}\left(S(t_1) + S(t_2) + \cdots + S(t_n)\right) \\ \bullet\ G(T) = \left(S(t_1)\cdot S(t_2) \cdots S(t_n)\right)^{1/n} \end{array}}$$

arithmetic average

geometric average

Fact: $\boxed{A(T) \geq G(T)}$ ☆

The basis for the payoff < call / put

and

the stock-price average can be used as < the underlying / the strike price.

Payoffs:

		Geometric		Arithmetic
Strike Price	Call:	$(S(T)-G(T))_+$	$\geq$	$(S(T)-A(T))_+$
	Put:	$(G(T)-S(T))_+$	$\leq$	$(A(T)-S(T))_+$
Underlying asset (w/ a given strike K)	Call:	$(G(T)-K)_+$	$\leq$	$(A(T)-K)_+$
	Put:	$(K-G(T))_+$	$\geq$	$(K-A(T))_+$

$\Downarrow$

Analogous inequalities hold for option prices.