M378K: September 10^{th}, 2025.

Expectation.

Def'n. For a discrete random variable Y w/ support S_Y and pmf p_Y, its **expectation** is:

$$\boxed{\mathbb{E}[Y] = \sum_{y \in S_Y} y \cdot p_Y(y)}$$

Example.

- **Bernoulli** $Y \sim B(p)$

$\mathbb{E}[Y] = ?$

$$Y \sim \begin{cases} 1 & \text{w/ probab. } p \\ 0 & \text{w/ probab. } q = 1-p \end{cases}$$

$$\boxed{\mathbb{E}[Y] = 1 \cdot p + 0 \cdot q = p}$$

- **Binomial** $Y \sim b(n, p)$

$\mathbb{E}[Y] = ?$

Start w/ $I_j \sim B(p)$, $j = 1..n$, independent

$$Y = I_1 + I_2 + \cdots + I_n$$

$$\mathbb{E}[Y] = \mathbb{E}[I_1] + \cdots + \mathbb{E}[I_n] \quad \text{linearity}$$

$$= p + \cdots + p = np$$

$$\boxed{\mathbb{E}[Y] = np}$$

- Geometric

$$Y \sim g(p)$$

$$\mathbb{E}[Y] = ?$$

$$\mathbb{E}[Y] = \sum_{k=0}^{\infty} k \cdot p_Y(k) = \sum_{k=0}^{\infty} k \cdot q^k \cdot p = p \cdot \sum_{k=\cancel{0}\,1}^{\infty} k \cdot q^k$$

Not a geometric series.

$$\sum_{k=1}^{\infty} k \cdot p_k = p_1 + 2 \cdot p_2 + \dots + k \cdot p_k + \dots$$

$$\begin{aligned} = &\; p_1 + \\ & p_2 + p_2 + \\ & p_3 + p_3 + p_3 \\ & \vdots \\ & p_k + p_k + p_k + \dots + p_k \\ & \vdots \end{aligned}$$

$$= \mathbb{P}[Y>0] + \mathbb{P}[Y>1] + \mathbb{P}[Y>2] + \dots + \mathbb{P}[Y>k-1] + \dots$$

Tail Formula for Expectation

In the geometric case:

$$\mathbb{E}[Y] = q + q^2 + q^3 + \dots + q^k + \dots$$

$$= q\left(1 + q + q^2 + \dots + q^{k-1} + \dots\right)$$

geometric!

$$= q \cdot \frac{1}{1-q} = \frac{q}{p}$$

$\square$

Problem. A game is played by tossing a loaded coin independently until the first Heads. Assume that the probability of Tails is 1/3. What's the expected number of Tails until the first Heads?

$\rightarrow$:

$$q = \frac{1}{3} \qquad Y \sim g\left(p = \frac{2}{3}\right)$$

$$\mathbb{E}[Y] = \frac{\frac{1}{\cancel{3}}}{\frac{2}{\cancel{3}}} = \frac{1}{2}$$

$\square$

Variance.

Def'n. The variance of the random variable Y is defined as

$$\boxed{\mathrm{Var}[Y] := \mathbb{E}\left[(Y - \mathbb{E}[Y])^2\right]}$$ if "finite"

The standard deviation of Y is

$$\boxed{SD[Y] := \sqrt{\mathrm{Var}[Y]}}$$

Formula: $\mathrm{Var}[Y] = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2$

$\rightarrow$: $\mu_Y := \mathbb{E}[Y]$

$$\begin{aligned}
\mathrm{Var}[Y] &= \mathbb{E}\left[(Y - \mathbb{E}[Y])^2\right] \\
&= \mathbb{E}\left[Y^2 - 2\mu_Y \cdot Y + \mu_Y^2\right] \\
&= \mathbb{E}[Y^2] - 2\mu_Y \cdot \underbrace{\mathbb{E}[Y]}_{\mu_Y} + \mu_Y^2 \quad \text{linearity} \\
&= \mathbb{E}[Y^2] - \mu_Y^2 \quad \square
\end{aligned}$$

Theorem. Say that Y_1 and Y_2 are r.v.s w/ finite variances and that α is a real constant. Then,

- $\mathrm{Var}[\alpha Y_1] = \alpha^2 \cdot \mathrm{Var}[Y_1]$
- $\mathrm{Var}[Y_1 + Y_2] = \mathrm{Var}[Y_1] + \mathrm{Var}[Y_2] + 2\,\mathrm{Cov}[Y_1, Y_2]$

w/

$$\mathrm{Cov}[Y_1, Y_2] = \mathbb{E}\left[(Y_1 - \mathbb{E}[Y_1]) \cdot (Y_2 - \mathbb{E}[Y_2])\right]$$

The correlation coefficient is

$$\mathrm{corr}[Y_1, Y_2] = \frac{\mathrm{Cov}[Y_1, Y_2]}{SD[Y_1] \cdot SD[Y_2]}$$

Two r.v.s are uncorrelated if $\mathrm{corr}[Y_1, Y_2] = 0$

- If, in addition, Y_1 and Y_2 are uncorrelated, then, $\boxed{\mathrm{Var}[Y_1 + Y_2] = \mathrm{Var}[Y_1] + \mathrm{Var}[Y_2]}$

Theorem. If Y_1 and Y_2 are independent, they are also uncorrelated.