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Problem Set # 1

Conditional probability. Independence.

Problem 1.1. Let E and F be any two events. If

$$\mathbb{P}[E|F] > \mathbb{P}[E],$$

then

$$\mathbb{P}[F|E] > \mathbb{P}[F].$$

Solution: TRUE
The given inequality implies

$$\mathbb{P}[E|F] = \frac{\mathbb{P}[E \cap F]}{\mathbb{P}[F]} > \mathbb{P}[E].$$

So,

$$\frac{\mathbb{P}[E \cap F]}{\mathbb{P}[E]} > \mathbb{P}[F].$$

Using the definition of conditional probability, we get the desired inequality.

Problem 1.2. Let A and B be events such that $\mathbb{P}[A] = 1/2, \mathbb{P}[B] = 1/3$ and $\mathbb{P}[A \cap B] = 1/4$. Calculate the following probabilities:

(i) $\mathbb{P}[A \cup B]$
(ii) $\mathbb{P}[A|B]$
(iii) $\mathbb{P}[B|A]$
(iv) $\mathbb{P}[A^c|B^c]$

Solution:

(i)

$$\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B] = \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \frac{6+4-3}{12} = \frac{7}{12}\,.$$

(ii)

$$\mathbb{P}[A|B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]} = \frac{\frac{1}{4}}{\frac{1}{3}} = \frac{3}{4}.$$

(iii)

$$\mathbb{P}[B|A] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[A]} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{2}{4} = \frac{1}{2}\,.$$

(iv)

$$\mathbb{P}[A^c|B^c] = \frac{\mathbb{P}[A^c \cap B^c]}{\mathbb{P}[B^c]} = \frac{\mathbb{P}[(A \cup B)^c]}{\mathbb{P}[B^c]} = \frac{1 - \mathbb{P}[A \cup B]}{1 - \mathbb{P}[B]} = \frac{\frac{5}{12}}{\frac{2}{3}} = \frac{5}{8}\,.$$

Problem 1.3. Two cards are drawn at random from an ordinary deck of 52 cards. Find the probability that both are spades.

Solution: Let A_i denote the event that the i^{th} card drawn is a spade, for $i = 1, 2$. We need to find $\mathbb{P}[A_1 \cap A_2]$. By the multiplication formula,

$$\mathbb{P}[A_1 \cap A_2] = \mathbb{P}[A_1]\mathbb{P}[A_2|A_1] = \frac{13}{52} \cdot \frac{12}{51} = \frac{1}{17}\,.$$

Problem 1.4. If events E and F are independent, then they are necessarily mutually exclusive.

Solution: FALSE
Here is a counterexample: Consider two independent cointosses. Let E be the event that a toss of a fair coin comes up *Heads*. Let F be the event that a toss of the other fair coin comes up *Heads*. Then, we have

$$\frac{1}{4} = \mathbb{P}[HH] = \mathbb{P}[E \cap F] = \mathbb{P}[E]\mathbb{P}[F] = \mathbb{P}[H]\mathbb{P}[H] = \left(\frac{1}{2}\right)^2.$$

Obviously, since it's possible to have *Heads* come up on both coins, events E and F are **not** mutually exclusive.

Problem 1.5. The four standard blood types are distributed in a populations as follows:

$$\begin{aligned} A &- 42\% \\ O &- 33\% \\ B &- 18\% \\ AB &- 7\% \end{aligned}$$

Assuming that people choose their mates independently of their blood type, find the probability that a randomly chosen couple from this population has the same blood type.

Solution: Let us introduce the following events :

$$\begin{aligned} A &= \{\text{both have blood type } A\}, \\ B &= \{\text{both have blood type } B\}, \\ AB &= \{\text{both have blood type } AB\}, \\ O &= \{\text{both have blood type } O\}, \\ E &= \{\text{both have the same blood type}\}. \end{aligned}$$

Then,

$$\begin{aligned} \mathbb{P}[E] &= \mathbb{P}[A] + \mathbb{P}[B] + \mathbb{P}[AB] + \mathbb{P}[O] \\ &= \left(\frac{42}{100}\right)^2 + \left(\frac{33}{100}\right)^2 + \left(\frac{18}{100}\right)^2 + \left(\frac{7}{100}\right)^2 \\ &= 0.3226. \end{aligned}$$

Problem 1.6. *Source: Sample P exam problems.*
An actuary studying the insurance preferences of automobile owners makes the following conclusions:

(i) An automobile owner is twice as likely to purchase collision coverage as disability coverage.
(ii) The event that an automobile owner purchases collision coverage is independent of the event that he or she purchases disability coverage.
(iii) The probability that an automobile owner purchases both collision and disability coverages is 0.15.

Calculate the probability that an automobile owner purchases neither collision nor disability coverage.

Solution: Let

$$\begin{aligned} C &= \{\text{collision coverage is purchased}\} \\ D &= \{\text{disability coverage is purchased}\} \end{aligned}$$

We are seeking the probability $\mathbb{P}[C^c \cap D^c]$. Since C and D are assumed to be independent, we know that C^c and D^c are also independent. So, it suffices to calculate

$$\mathbb{P}[C^c \cap D^c] = \mathbb{P}[C^c]\mathbb{P}[D^c] = (1 - \mathbb{P}[C])(1 - \mathbb{P}[D]).$$

We are given that $\mathbb{P}[C] = 2\mathbb{P}[D]$. Moreover, since C and D are assumed to be **independent**, the following equality holds

$$\mathbb{P}[P \cap D] = \mathbb{P}[C]\mathbb{P}[D].$$

However, we are also told that $\mathbb{P}[P \cap D] = 0.15$. Hence,

$$0.15\mathbb{P}[P \cap D] = \mathbb{P}[C]\mathbb{P}[D] = 2\mathbb{P}[D]^2 \quad \Rightarrow \quad \mathbb{P}[D] = \sqrt{0.075}$$

Finally, our answer is

$\mathbb{P}[C^c \cap D^c] = (1 - \mathbb{P}[C])(1 - \mathbb{P}[D]) = (1 - 2\sqrt{0.075})(1 - \sqrt{0.075}) = 1 - 3\sqrt{0.075} + 2(0.075) = 1.15 - 3\sqrt{0.075}.$