M378K: January 31st, 2025.

Problem 5.2. Source: Sample P exam, Problem #419.
A customer purchases a lawnmower with a two-year warranty. The number of years before the lawnmower needs a repair is uniformly distributed on $[0, 5]$. *Calculate the probability that the lawnmower needs no repairs within* 4.5 *years after the purchase, given that the lawnmower needs no repairs within the warranty period*

→: T... the lifetime of the lawn mower

$T \sim U(0,5)$

$$\mathbb{P}[T>4.5 \mid T>2] = \frac{\mathbb{P}[T>4.5, T>2]}{\mathbb{P}[T>2]}$$

$$= \frac{\mathbb{P}[T>4.5]}{\mathbb{P}[T>2]} = \frac{\frac{5-4.5}{\cancel{5-0}}}{\frac{5-2}{\cancel{5-0}}} = \frac{1}{6} \quad \square$$

Remark: $Y \sim B(p)$

$Y \sim b(n,p)$

Example. $Y \sim N(\mu, \sigma)$ where $\mu \in \mathbb{R}$ and $\sigma > 0$ is normally distributed w/ mean μ and standard deviation σ

if

$$f_Y(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-\mu)^2}{2\sigma^2}} \quad \text{for all } y \in \mathbb{R}$$

If $\mu = 0$ and $\sigma = 1$, we say that Y is standard normal and we write $Y \sim N(0,1)$.

Its pdf is $f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$ for all $y \in \mathbb{R}$.

Q: Let $Y \sim N(0,1)$.
Let α and β be two real constants

$$\alpha Y + \beta \sim \text{Normal}(\mu = \beta, \sigma = |\alpha|)$$

Q: Let $Y \sim N(\mu, \sigma)$.

$$\frac{Y-\mu}{\sigma} \sim N(0,1)$$

Example. We say that Y is exponential w/ parameter τ if it has this pdf

$$\boxed{f_Y(y) = \frac{1}{\tau} e^{-\frac{y}{\tau}} \cdot \mathbb{1}_{[0,\infty)}(y)}$$

Alternative parametrization: $\lambda = \frac{1}{\tau}$

Expectations. Variances. Standard deviations.

In the discrete case:

$$\mathbb{E}[Y] = \sum_{y \in S_Y} y \cdot p_Y(y)$$

Google: Cauchy distribution

Def'n. For a continuous r.v. Y w/ pdf f_Y, we define its expectation as

$$\boxed{\mathbb{E}[Y] := \int_{-\infty}^{\infty} y\, f_Y(y)\, dy}$$

if the integral exist

Problem 5.3. *Consider a continuous random variable Y whose probability density function is given by*

$$f_Y(y) = 2y\mathbf{1}_{[0,1]}(y)$$

What is the expected value of this random variable?

$\longrightarrow$:

$$\mathbb{E}[Y] = \int_{-\infty}^{\infty} y\, f_Y(y)\, dy$$

$$= \int_{-\infty}^{\infty} y \cdot (2y) \cdot \mathbb{1}_{[0,1]}(y)$$

$$= 2\int_0^1 y^2\, dy = 2 \cdot \left(\frac{y^3}{3}\right)_{y=0}^{1} = \frac{2}{3} \quad \square$$

Def'n. $\mathrm{Var}[Y] = \mathbb{E}\left[(Y-\mu_Y)^2\right]$ w/ $\mu_Y = \mathbb{E}[Y]$

$\mathrm{SD}[Y] = \sqrt{\mathrm{Var}[Y]}$

Example. $\boxed{Y \sim U(\ell, r)}$

$\mathbb{E}[Y] = \dfrac{\ell + r}{2}$

$\mathrm{Var}[Y] = ?$

$Y - \ell \sim U(0, r-\ell)$

$U := \dfrac{Y-\ell}{r-\ell} \sim U(0,1)$

$$\mathrm{Var}[U] = ? = \boxed{\mathbb{E}[U^2]} - (\boxed{\mathbb{E}[U]})^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

$\mathbb{E}[U^2] \overset{?}{=}$, $\mathbb{E}[U] = \frac{1}{2}$

$$\mathbb{E}[g(U)] = \int_{-\infty}^{\infty} g(u) f_U(u)\,du$$

$$\mathbb{E}[U^2] = \int_0^1 u^2\,du = \frac{u^3}{3}\Big|_{u=0}^{1} = \frac{1}{3}$$

$Y = (r-\ell)U + \ell$

$$\Rightarrow \mathrm{Var}[Y] = \mathrm{Var}[(r-\ell)\cdot U + \ell] = (r-\ell)^2 \cdot \mathrm{Var}[U] = \frac{(r-\ell)^2}{12} \quad \square$$

Example. $Y \sim N(\mu, \sigma)$

μ: $\mathbb{E}[Y]$, σ: $\mathrm{SD}[Y]$

Example. $Y \sim E(\tau)$, i.e., Y is exponential w/ parameter τ

$$\mathbb{E}[Y] = \int_{-\infty}^{\infty} y f_Y(y)\,dy$$

$$= \int_{-\infty}^{\infty} y \cdot \left(\frac{1}{\tau}\right) e^{-\frac{y}{\tau}}\, \mathbb{1}_{[0,\infty)}(y)\,dy$$

$$= \int_0^\infty \left(\frac{y}{\tau}\right) e^{-\frac{y}{\tau}} dy$$

$$u = \frac{y}{\tau} \qquad du = \frac{dy}{\tau}$$

$$= \tau \int_0^\infty u\, e^{-u}\, du$$

$$u = u \qquad du = du$$

$$dv = e^{-u} du \qquad v = -e^{-u} du$$

$$= \tau \left(\underbrace{-u e^{-u} \Big|_{u=0}^{\infty}}_{0} + \underbrace{\int_0^\infty (+e^{-u})\, du}_{=1} \right) = \tau \qquad \square$$