UNIVERSITY OF TEXAS AT AUSTIN

Problem set 3

The tail formula for expectation.

The Weibull distribution. The nonnegative random variable X is said to have the *Weibull distribution* is its cumulative distribution function is of the form

$$F_X(x) = 1 - e^{-\left(\frac{x}{\theta}\right)^{\tau}}$$

Remark 3.1. In the special case that $\tau = 1$, we get the *exponential distribution*.

Problem 3.1. Let $X \sim Weibull(\theta = 1, \tau = 2)$. What is the expectation of X?

Solution: By the tail formula for the expectation, we have

$$\mathbb{E}[X] = \int_0^\infty S_X(x)\, dx = \int_0^\infty e^{-x^2}\, dx.$$

Note that the integrand resembles the density of a normal distribution. In fact, for a standard normal distribution, the density is

$$\varphi(z) = \frac{1}{\sqrt{\pi}} e^{-\frac{z^2}{2}} \quad \text{for } z \in \mathbb{R}.$$

Let us use the $u-$substitution to modify the integral we are trying to calculate to the one we see in the standard normal density.

$$\int_0^\infty e^{-x^2}\, dx = \left\{ \begin{array}{c} u = x\sqrt{2} \\ du = \sqrt{2}\, dx \end{array} \right\} = \int_0^\infty e^{-\frac{u^2}{2}} \left(\frac{1}{\sqrt{2}}\right) du = \frac{1}{\sqrt{2}} \int_0^\infty e^{-\frac{u^2}{2}}\, du = \frac{\sqrt{\pi}}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{u^2}{2}}\, du.$$

Using the fact that

$$\int_{-\infty}^\infty \varphi(z)\, dz = 1,$$

we obtain that $\mathbb{E}[X] = \frac{\sqrt{\pi}}{2}$.