M339G: March 12th, 2025.

Bivariate Normal Random Variables.

(Based on Pitman's "Probability")

Recall: In 1-D, the standard normal density is

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, \text{ for } z \in \mathbb{R}$$

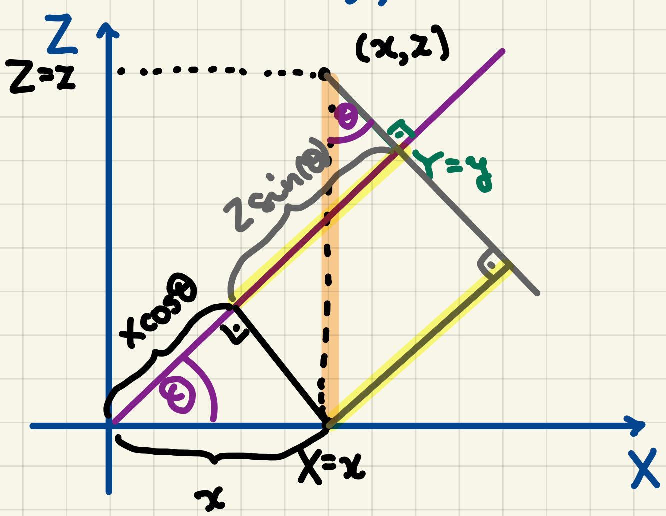
In 2-D, we start w/ X and Y that are independent and both are standard normal, i.e., $N(0,1)$.

Then, their joint density, i.e., the density of the pair (X,Y) is

$$\boxed{f(x,y) = \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)} \quad \text{for all } (x,y) \in \mathbb{R}^2}$$

Standard.

Start w/ a pair of independent, standard normal random variables. Say, X and Z.



$$\Rightarrow \quad \boxed{Y = X\cdot\cos\theta + Z\cdot\sin\theta}$$

We know: Y is normal

$$\mathbb{E}[Y] = \mathbb{E}[X\cdot\cos\theta + Z\cdot\sin\theta]$$

$$\begin{pmatrix}\text{linearity}\\ \text{of}\\ \mathbb{E}\end{pmatrix} = \underbrace{\mathbb{E}[X]}_{=0}\cdot\cos\theta + \underbrace{\mathbb{E}[Z]}_{=0}\cdot\sin\theta$$

$$= 0$$

$$Var[Y] = Var[X\cdot\cos\theta + Z\cdot\sin\theta] \quad (X \text{ and } Z \text{ independent})$$

$$= \underbrace{Var[X]}_{1}\cdot\cos^2\theta + \underbrace{Var[Z]}_{1}\cdot\sin^2\theta = 1$$

Q: What is the correlation coefficient between X and Y?

$\longrightarrow$:

$$\rho(X,Y) = \frac{Cov[X,Y]}{\underbrace{SD[X]}_{=1} \cdot \underbrace{SD[Y]}_{=1}} = Cov[X,Y]$$

$$= \mathbb{E}\left[(X-\mu_X)(Y-\mu_Y)\right] =$$

$$= \mathbb{E}[XY - X\mu_Y - \mu_X Y + \mu_X\mu_Y]$$

$$= \mathbb{E}[XY] - \mathbb{E}[X]\cdot\mu_Y - \cancel{\mu_X \mathbb{E}[Y]} + \cancel{\mu_X\mu_Y}$$

$$= \mathbb{E}[XY] - \mu_X\cdot\mu_Y$$

$\mu_X = \mu_Y = 0$

$$= \mathbb{E}[XY]$$

$$= \mathbb{E}[X(X\cdot\cos\theta + Z\cdot\sin\theta)] =$$

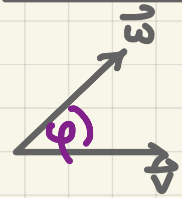
$$= \cos\theta\cdot\underbrace{\mathbb{E}[X^2]}_{\underbrace{Var[X]+(\mathbb{E}[X])^2}_{=1}} + \sin\theta\,\underbrace{\mathbb{E}[X\cdot Z]}_{= \mathbb{E}[X]\cdot\mathbb{E}[Z] = 0}$$

independent

$$\boxed{\rho(X,Y) = \cos\theta}$$

$\square$

Linear Algebra



$$\vec{v}\cdot\vec{w} = \|v\|\cdot\|w\|\cdot\cos(\varphi)$$

$$\cos(\varphi) = \frac{\vec{v}\cdot\vec{w}}{\|v\|\|w\|}$$

Special Cases:

$\theta = 0 \Rightarrow Y = X$

$\theta = \frac{\pi}{2} \Rightarrow Y = Z$ (So, X and Y are independent.)

$\theta = \pi \Rightarrow Y = -X$

In general: For each correlation coefficient $-1 \le \rho \le 1$, there exists an angle $\boxed{\theta = \arccos(\rho)}$ such that X and Y as above have the correlation coefficient ρ.

Alternatively, $\boxed{Y = \rho \cdot X + \sqrt{1-\rho^2} \cdot Z}$

w/ X and Z independent and $N(0,1)$.

Joint Density:

$$\boxed{f_{X,Y}(x,y) = \frac{1}{2\pi(1-\rho^2)} \exp\left(-\frac{1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2)\right)}$$

Marginal Dist'ns: $X \sim N(0,1)$, $Y \sim N(0,1)$

Conditional Dist'n: Given $X = x$, $Y \sim \text{Normal}(\rho x, \text{var} = 1-\rho^2)$

Given $Y = y$, $X \sim \text{Normal}(\rho y, \text{var} = 1-\rho^2)$

Independence.

X and Y are independent

iff

$\rho(X,Y) = 0$