M378K Introduction to Mathematical Statistics
Homework assignment #3

Please, provide your **complete solutions** to the following problems. Final answers only, even if correct will earn zero points for those problems.

Problem 3.1. *(5 points) Let the density function of a random variable X be given as*

$$f_X(x) = cxI_{[0,1]}(x),$$

for some constant c. Find $\mathbb{E}[X^3]$.

Solution: Since the density function must integrate up to 1, we get $c = 2$. Whence,

$$\mathbb{E}[X^3] = 2\int_0^1 x^4\,dx = \frac{2}{5}\,.$$

Problem 3.2. *(5 points) Let X denote the outcome of a roll of a fair, regular dodecahedron (a polyhedron with 12 faces) with numbers $1, 2, \cdots, 12$ written on its sides. Find $\mathbb{E}[X]$.*

Solution:

$$\mathbb{E}[X] = \frac{1}{12}(1 + \cdots + 12) = \frac{13}{2}\,.$$

Problem 3.3. *(5 points) Let X be a random variable with mean $\mu = 2$ and standard deviation equal to $\sigma = 1$. Find $\mathbb{E}[X^2]$.*

Solution:

$$\mathbb{E}[X^2] = Var[X] + (\mathbb{E}[X])^2 = 1 + 2^2 = 5.$$

Problem 3.4. *(5 points) Let X denote the number of $1's$ in 100 throws of a fair die. Find $\mathbb{E}[X^2]$.*

Solution: Evidently, $X \sim b(100, 1/6)$. So,

$$\mathbb{E}[X^2] = Var[X] + (\mathbb{E}[X])^2 = 100 \cdot \frac{1}{6} \cdot \frac{5}{6} + (100 \cdot \frac{1}{6})^2 = \frac{500 + 10000}{36} = \frac{875}{3}\,.$$

Problem 3.5. *(10 points) Let the random variable Y have the following cumulative distribution function*

$$F_Y(y) = \begin{cases} 0 & \textit{for } y < 0 \\ \frac{y}{2} & \textit{for } 0 \leq y < 1 \\ \frac{y^2}{\alpha} & \textit{for } 1 \leq y < \beta \\ 1 & \textit{for } \beta \leq y \end{cases}$$

(i) (3 points) Find the constants α and β such that the random variable Y is continuous.

(ii) (7 points) Calculate the expectation of the random variable Y for the κ you obtained in the previous part of the problem.

Solution:

(i) In order for the random variable Y to be continuous, its cumulative distribution function must be continuous. So,

$$\frac{1}{2} = \frac{1^2}{\alpha} \quad \text{and} \quad \frac{\beta^2}{\alpha} = 1.$$

So, $\alpha = 2$ and $\beta = \sqrt{2}$.

(ii) The probability density function of the random variable Y is

$$f_Y(y) = \begin{cases} 0 & \text{for } y < 0 \\ \frac{1}{2} & \text{for } 0 \leq y < 1 \\ y & \text{for } 1 \leq y < \sqrt{2} \\ 0 & \text{for } \sqrt{2} \leq y \end{cases}$$

So, the expectation of Y is

$$\mathbb{E}[Y] = \int_0^1 \frac{y}{2}\, dy + \int_1^{\sqrt{2}} y^2\, dy = \frac{1}{4} + \frac{2\sqrt{2}}{3} - \frac{1}{3} = \frac{2\sqrt{2}}{3} - \frac{1}{12} = \frac{8\sqrt{2} - 1}{12}\,.$$

Problem 3.6. *(20 points) Let X be a discrete random variable with the support $\mathcal{S}_X = \mathbb{N}$, such that $\mathbb{P}[X = n] = C\frac{1}{n^2}$, for $n \in \mathbb{N}$, where C is a constant chosen so that $\sum_n \mathbb{P}[X = n] = 1$. The distribution table of X is, therefore, given by*

1	2	3	...
$C\frac{1}{1^2}$	$C\frac{1}{2^2}$	$C\frac{1}{3^2}$	...

1. *(10 points) Show that $\mathbb{E}[X]$ does not exist.*
2. *(10 points) Construct a distribution of a similar random variable whose expectation does exist, but the variance does not. (*Hint: *Use the same support $\mathbb{N}$, but tweak the probabilities so that the sum for $\mathbb{E}[X]$ converges, while the sum for $\mathbb{E}[X^2]$ does not.)*

Solution:

1. The expression for $\mathbb{E}[X]$ is

$$\mathbb{E}[X] = \sum_{n=1}^{\infty} n\mathbb{P}[X = n] = C \sum_{n=1}^{\infty} \frac{1}{n} = +\infty,$$

because the *harmonic series* $1 + 1/2 + 1/3 + \dots$ diverges.

2. The distribution of Y we need to construct should have the following properties

$$\sum_{n=1}^{\infty} n\mathbb{P}[Y = n] < \infty \text{ but } \sum_{n=1}^{\infty} n^2\mathbb{P}[Y = n] = \infty.$$

We can try to achieve this by taking $\mathbb{P}[Y = n] = C'\frac{1}{n^3}$, where, as above, C' is simply a constant that ensures that $\sum_n \mathbb{P}[Y = n] = 1$. Indeed, in this case, we have

$$\mathbb{E}[X] = C' \sum_n \frac{1}{n^2} \text{ while } \mathbb{E}[X^2] = C' \sum_n \frac{1}{n}.$$

The first sum converges, but the second one diverges.