Review:

W: April 17th, 2019.

$i = 1..n$ ... indices of individual components in the portfolio

R_i ... the realized return of component i

$$R_p = x_1 \cdot R_1 + \dots + x_n \cdot R_n = \sum_{i=1}^{n} x_i \cdot R_i$$

w/ $x_i = \dfrac{\text{value of investment } i}{\text{value of whole portfolio}}$

$$\Rightarrow \mathbb{E}[R_p] \underset{\substack{\uparrow \\ \text{linearity}}}{=} x_1 \cdot \mathbb{E}[R_1] + \dots + x_n \cdot \mathbb{E}[R_n]$$

$\operatorname{Var}[R_p] = ?$ We have to consider the correlation between individual returns.

In general:

$$\begin{aligned} \operatorname{Var}[R_p] &= \operatorname{Var}[x_1 \cdot R_1 + \dots + x_n R_n] \\ &= \sum_{i=1}^{n} x_i^2 \cdot \operatorname{Var}[R_i] + \sum_{i \neq j} x_i \cdot x_j \operatorname{Cov}[R_i, R_j] \\ &= \sum_{i=1}^{n} x_i^2 \cdot \operatorname{Var}[R_i] + 2 \cdot \sum_{i<j} x_i \cdot x_j \operatorname{Cov}[R_i, R_j] \end{aligned}$$

σ_i, σ_j ... volatilities ; $\rho_{i,j}$ ... correlation

$$\operatorname{Var}[R_p] = \sum_{i=1}^{n} x_i^2 \cdot \sigma_i^2 + 2 \cdot \sum_{i<j} x_i \cdot x_j \cdot \sigma_i \cdot \sigma_j \cdot \rho_{i,j}$$

11) You are given the following information about a portfolio that has two equally-weighted stocks, P and Q. $w_P = w_Q = \frac{1}{2}$

(i) The economy over the next year could be good or bad with equal probability.

$P_g = P_b = \frac{1}{2}$

(ii) The returns of the stocks can vary as shown in the table below:

Stock	Return when economy is good	Return when economy is bad
P	10%	–2%
Q	18%	–5%

Calculate the volatility of the portfolio return.

(A) 1.80%

(B) 6.90%

(C) 7.66%

(D) 8.75%

(E) 13.42%

R_T ... return of total portfolio

$R_T = \frac{1}{2}(R_P + R_Q)$

Q: What's the dist'n of R_T?

$$R_T \sim \begin{cases} 0.14, & \text{if good} \cdot \text{w/ probab. } 1/2 \\ -0.035, & \text{if bad} \cdot \text{w/ probab. } 1/2 \end{cases}$$

answer: $\sigma_T = \sqrt{Var[R_T]}$

- $\mathbb{E}[R_T] = \frac{1}{2}(0.14 + (-0.035)) = 0.0525$

- $\mathbb{E}[R_T^2] = \frac{1}{2}((0.14)^2 + (-0.035)^2) = 0.0104125$

$\Rightarrow Var[R_T] = \mathbb{E}[R_T^2] - (\mathbb{E}[R_T])^2 = 0.00765$

$\Rightarrow \sigma_T = \sqrt{0.00765} = 0.0875 \Rightarrow$ (D) ∎

Diversification w/ Equally Weighted Portfolios.

x_i ... the weight of i^{th} investment, $i = 1..n$

↳ the proportion of your wealth invested in investment i

For an equally weighted portfolio: $x_i = \frac{1}{n}$

$$\Rightarrow \underbrace{R_P}_{\text{Return of whole portfolio}} = \frac{1}{n}(R_1 + \cdots + R_n)$$

$$\Rightarrow Var[R_P] = \frac{1}{n^2} Var[R_1 + \cdots + R_n]$$

$$= \frac{1}{n^2}\left(\sum_{i=1}^{n} Var[R_i] + \sum_{i \neq j} Cov[R_i, R_j] \right)$$

$$= \frac{1}{n} \cdot \underbrace{\left(\frac{1}{n} \cdot \sum_{i=1}^{n} Var[R_i] \right)}_{} + \underbrace{\frac{1}{n^{\cancel{2}}} \cdot \cancel{n} \cdot (n-1)}_{} \cdot \underbrace{\left(\frac{1}{n(n-1)} \cdot \sum_{i \neq j} Cov[R_i, R_j] \right)}_{}$$

$$= \frac{1}{n} \cdot \left(\begin{matrix}\text{Average Variance} \\ \text{of Individual} \\ \text{Components}\end{matrix} \right) + \left(1 - \frac{1}{n}\right) \cdot \left(\begin{matrix}\text{Average} \\ \text{Covariance} \\ \text{between Stocks}\end{matrix} \right)$$

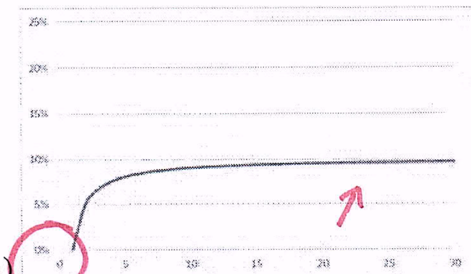
9) You are given the following information about an equally-weighted portfolio of n stocks:

(i) For each individual stock in the portfolio, the variance is 0.20.

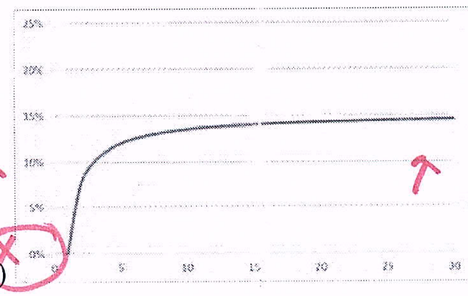
(ii) For each pair of distinct stocks in the portfolio, the covariance is 0.10.

Determine which graph displays the variance of the portfolio as a function of n.

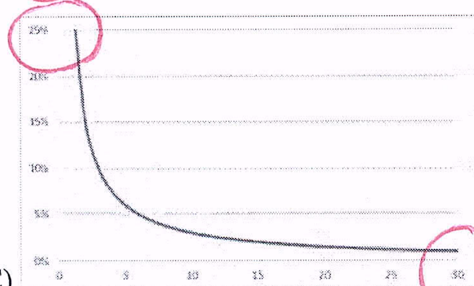
(A)



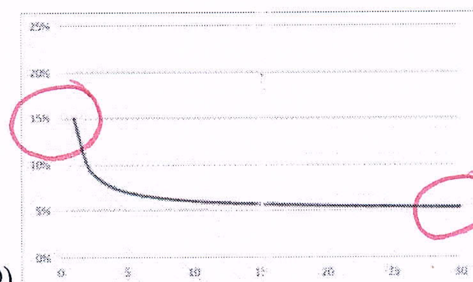
(B)



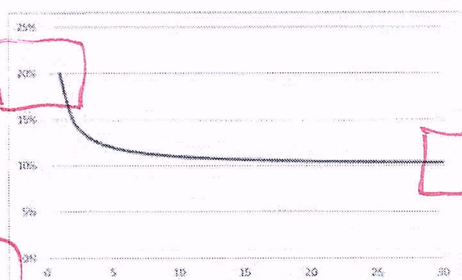
(C)



(D)



(E)



4.

Example. Volatility when Risks are Independent.

Independence $\Rightarrow$ **Uncorrelation**

$$\Rightarrow \operatorname{Var}[R_P] = \frac{1}{n}(\text{Avg Variance})$$

$$\Rightarrow \sigma_P = SD[R_P] = \frac{\sqrt{\text{Avg Variance}}}{\sqrt{n}}$$

Q: What if the risks have identical volatilities?

$$\Rightarrow \sigma_P = \frac{\sqrt{\frac{1}{n}\sum_{i=1}^{n}\operatorname{Var}[R_i]}}{\sqrt{n}} = \frac{\sqrt{\frac{1}{n}\cdot n \cdot \sigma_1^2}}{\sqrt{n}} = \frac{\sigma_1}{\sqrt{n}} \xrightarrow[n\to\infty]{} 0$$

Diversification w/ a General Portfolio.

x_i ... weights of individual components

Assume: $x_i \geq 0$

Recall: $\operatorname{Var}[R_P] = \sum_{i=1}^{n} x_i \cdot \operatorname{Cov}[R_i, R_P]$

$$= \sum_{i=1}^{n} x_i \cdot SD[R_i] \cdot SD[R_P] \cdot \operatorname{corr}[R_i, R_P]$$

$/ : SD[R_P]$

$$\Rightarrow SD[R_P] = \sum_{i=1}^{n} x_i \cdot SD[R_i] \cdot \underbrace{\operatorname{corr}[R_i, R_P]}_{\leq 1}$$

$$SD[R_P] \leq \sum_{i=1}^{n} x_i \cdot SD[R_i]$$

Equality happens only if all investments are perfectly correlated.