Problem. Assume the Black-Scholes model.
Let the current stock price be \$100.
You are given:

(i) $\mathbb{P}^*[S(1/4) < 95] = 0.2358$

(ii) $\mathbb{P}^*[S(1/2) < 110] = 0.6026$.

What's the expected time-1 stock price under $\mathbb{P}^*$?

→ :

$$\mathbb{E}^*[S(T)] = S(0)e^{rT}$$

In this problem : $\mathbb{E}^*[S(1)] = 100e^{r}$

In the B-S model:

$$S(T) = S(0)e^{\underbrace{(r-\frac{\sigma^2}{2})}_{=:\mu}\cdot T + \sigma\sqrt{T}\, Z}$$

$$\boxed{\mathbb{E}^*[S(1)] = 100\,e^{\mu+\frac{\sigma^2}{2}}}$$

(i): 95 is $\boxed{23.58^{th}}$ quantile of $S(1/4)$

The 23.58^{th} quantile of $N(0,1)$: standard normal tables: -0.72

or qnorm(0.2358) = -0.72

$$95 = 100e^{\mu\cdot(1/4) + \sigma\sqrt{1/4}\cdot(-0.72)} \quad /:100$$

$$0.95 = e^{\mu\cdot(1/4) + \sigma(1/2)(-0.72)}$$

$$\ln(0.95) = \frac{1}{4}\cdot\mu - 0.36\cdot\sigma$$

$$0.25\mu - 0.36\sigma = \ln(0.95) \qquad (i)$$

(ii): 110 is the 60.26^{th} quantile of $S(1/2)$

The 60.26^{th} quantile of $N(0,1)$: std normal tables: 0.26

or qnorm(0.6026) = 0.26

$$110 = 100\,e^{\mu(1/2) + \sigma\sqrt{1/2}\cdot(0.26)}$$

$$1.1 = e^{\mu(1/2) + \sigma\sqrt{1/2}\,(0.26)}$$

$$0.5\mu + 0.26\sqrt{\tfrac{1}{2}}\cdot\sigma = \ln(1.1) \qquad (ii)$$

We solve the system of two equations w/ two unknowns:

$$\mu = \underline{\quad ? \quad} \quad \text{and} \quad \sigma = \underline{\quad ? \quad}$$

2·(i): $-\cancel{0.5\mu} \underset{+}{-} 0.72\sigma = -2\ln(0.95)$

(ii): $\cancel{0.5\mu} + 0.26\sqrt{\tfrac{1}{2}}\cdot\sigma = \ln(1.1)$

$$\left(0.26\sqrt{\tfrac{1}{2}} + 0.72\right)\cdot\sigma = \ln(1.1) - 2\ln(0.95)$$

$$\sigma = \underline{0.2189492}$$

4·(i): $\mu = 1.44(0.2189492) + 4\ln(0.95) = \underline{0.11011}$

Finally: $100e^{\mu+\frac{\sigma^2}{2}} = 100e^{0.11011+\frac{(0.2189)^2}{2}} = \underline{114.3488}.$ $\square$

M339D: April 12th, 2023.

Black-Scholes: Partial Expectations.

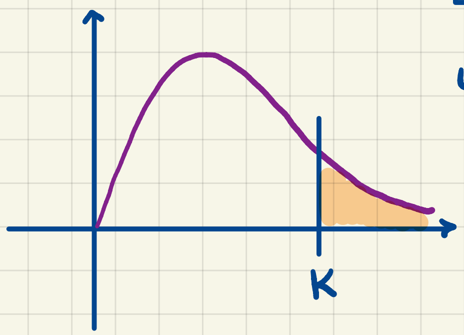
The Model.

Under the risk-neutral measure $\mathbb{P}^*$:

$$S(T) = S(0)e^{(r-\frac{\sigma^2}{2})\cdot T + \sigma\sqrt{T}\cdot Z} \quad \text{w/} \quad Z \sim N(0,1)$$

Tail Probabilities.

$$\mathbb{P}[S(T) > K] = \boxed{N(d_2)}$$



w/ $$\boxed{d_2 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)\cdot T\right]}$$

Motivation.

Get a formula for the price of European call and put options on a stock modeled in the Black-Scholes framework.

Idea: RISK-NEUTRAL PRICING

$$\boxed{V(0) = e^{-rT}\mathbb{E}^*[\underbrace{V(T)}_{\text{Payoff of a European Option}}]}$$

Implementation:

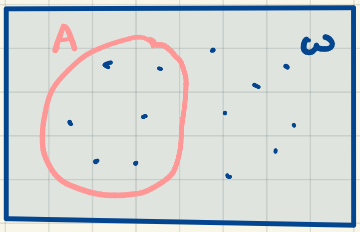
Temporarily, focus on a time-T, strike-K European call option.

The Payoff: $\boxed{V_c(T) = (S(T)-K)_+}$

Under $\mathbb{P}^*$:

$$\mathbb{E}^*[V_c(T)] = \mathbb{E}^*\left[(S(T)-K)_+\right]$$

$$= \mathbb{E}^*\left[(S(T)-K)\,\boxed{\mathbb{I}_{[S(T)\geq K]}}\right]$$



Ω

A is an event

$$\mathbb{I}_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}$$

$$\mathbb{I}_A = \begin{cases} 1 & \text{if } A \text{ happened} \\ 0 & \text{if } A \text{ did not happen} \end{cases}$$

$$\mathbb{E}[\mathbb{I}_A] = 1\cdot\mathbb{P}[A] + 0\cdot\mathbb{P}[A^c] = \mathbb{P}[A]$$

$$\mathbb{E}^*[V_c(T)] = \mathbb{E}^*\left[S(T)\cdot\mathbb{I}_{[S(T)\geq K]} - K\cdot\mathbb{I}_{[S(T)\geq K]}\right]$$

$$= \boxed{\mathbb{E}^*\left[S(T)\cdot\mathbb{I}_{[S(T)\geq K]}\right]} - \boxed{K\cdot\boxed{\mathbb{P}^*[S(T)\geq K]}}$$

$$\mathbb{P}^*[S(T)\geq K] = N(d_2)$$

The Partial Expectation from Title

$$\boxed{\mathbb{E}^*\left[S(T)\cdot\mathbb{I}_{\boxed{[S(T)\geq K]}}\right]} = ?$$

Method. Use the defining formula for the expectation of a function of a r.v.
In this case, that r.v. is $Z \sim N(0,1)$.

$$\{S(T)\geq K\} = \left\{S(0)e^{(r-\frac{\sigma^2}{2})\cdot T + \sigma\sqrt{T}\cdot Z} \geq K\right\}$$

$$= \{Z \geq -d_2\}$$

z... our dummy variable within the integral;
it corresponds to Z

i.e., $g(z) = S(0)e^{(r-\frac{\sigma^2}{2})\cdot T + \sigma\sqrt{T}\cdot z}$ (so that $g(Z) = S(T)$).

$$\mathbb{E}^*\left[S(T)\cdot \boxed{\mathbb{I}_{[S(T)\geq K]}}\right] = \mathbb{E}^*\left[g(Z)\cdot \mathbb{I}_{[Z\geq -d_2]}\right]$$

$$= \int_{-d_2}^{+\infty} g(z)\cdot f_Z(z)\,dz$$ (lots of algebra)

$$= \boxed{S(0)e^{rT}}\cdot \boxed{N(d_1)}$$

where $d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)\cdot T\right]$

The expectation under $\mathbb{P}^*$ of the call payoff:

$$\boxed{\begin{array}{l} \mathbb{E}^*\left[V_c(T)\right] = S(0)e^{rT}\cdot N(d_1) - K\cdot N(d_2) \\ \text{w/ } d_1 \text{ as above and } \boxed{d_2 = d_1 - \sigma\sqrt{T}} \end{array}}$$

$\Rightarrow$ The Black-Scholes call price:

$$V_c(0) = e^{-rT}\mathbb{E}^*\left[V_c(T)\right]$$

$$V_c(0) = e^{-rT}\left(S(0)e^{rT}N(d_1) - K\cdot N(d_2)\right)$$

$$\boxed{V_c(0) = S(0)N(d_1)\boxed{-Ke^{-rT}\cdot N(d_2)}}$$

The Black-Scholes put price:

By put-call parity:

$$V_c(0) - \boxed{V_p(0)} = S(0) - Ke^{-rT}$$

$$V_p(0) = V_c(0) - S(0) + Ke^{-rT} = S(0)N(d_1) - Ke^{-rT}N(d_2)$$

$$-S(0) \quad + Ke^{-rT}$$

$$= S(0)(\underbrace{N(d_1) - 1}_{-N(-d_1)}) + Ke^{-rT}(\underbrace{1 - N(d_2)}_{N(-d_2)})$$ symmetry of $N(0,1)$

$$\boxed{V_p(0) = \boxed{Ke^{-rT}N(-d_2)} - S(0)N(-d_1)}$$

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Problem Set 12

Black-Scholes pricing.

Problem 12.1. Let the stock prices be modeled using the lognormal distribution. Under the risk-neutral probability measure, the mean stock price at time−1 equals 120 and the median stock price 115. What is the risk-neutral probability that the time−1 stock price exceeds 100?

→ :

$$\mathbb{P}^*[S(1) > 100] = ?$$

$$\mathbb{P}^*\Big[\underbrace{\boxed{S(0)e^{(r-\frac{\sigma^2}{2})\cdot 1}}}_{\text{median of } S(1)}+\sigma\sqrt{1}\cdot Z > 100\Big] = ?$$

$$\mathbb{P}^*[115e^{\sigma \cdot Z} > 100] = \mathbb{P}^*\left[\sigma\cdot Z > \ln\left(\frac{100}{115}\right)\right] = 0.6844$$

$$\frac{\text{mean}}{\text{median}} = \frac{\cancel{S(0)}\cancel{e^{rT}}}{\cancel{S(0)}e^{(\cancel{r}-\frac{\sigma^2}{2})\cdot T}} = e^{\frac{\sigma^2}{2}\cdot T}$$

$$\frac{120}{115} = e^{\frac{\sigma^2}{2}} \Rightarrow \dots \Rightarrow \sigma = 0.2918$$

$$V(0) = e^{-rT}\mathbb{E}^*[V(T)] \qquad / \cdot e^{rT}$$

$$\boxed{\mathbb{E}^*[V(T)] = V(0)e^{rT}}$$

The rate of return under probability $\mathbb{P}$, was by def'n the constant α which satisfied:

$$\boxed{\mathbb{E}[W(T)] = W(0)\cdot e^{\alpha \cdot T}}$$

M339D: April 14^{th}, 2023.

Problem 12.2. (5 pts) Let the stochastic process $S = \{S(t); t \geq 0\}$ denote the stock price. The stock's rate of ~~appreciation~~ rate is 10% while its volatility is 0.30 Then,

(a) $Var[\ln(S(t))] = 0.3t$

(b) $Var[\ln(S(t))] = 0.09t^2$

(c) $Var[\ln(S(t))] = 0.09t$

(d) $Var[\ln(S(t))] = 0.09$

(e) None of the above.

$\rightarrow$: In the Black-Scholes model:

$$S(t) = S(0)e^{(r-\frac{\sigma^2}{2})\cdot t + \sigma\sqrt{t}\cdot Z} \qquad Z \sim N(0,1)$$

$$\ln(S(t)) = \overset{\text{deterministic}}{\boxed{\ln(S(0)) + \left(r-\frac{\sigma^2}{2}\right)\cdot t}} + \sigma\sqrt{t}\cdot Z$$

$$Var[\ln(S(t))] = Var[\sigma\sqrt{t}\cdot Z] = \sigma^2\cdot t\,\underbrace{Var[Z]}_{=1} = \sigma^2\cdot t \quad \square$$