Name:

UTeid:

M378K Introduction to Mathematical Statistics
Fall 2025
University of Texas at Austin
In-Term Exam III
Instructor: Milica Čudina

Notes: This is a closed book and closed notes exam. The maximal score on the exam is 100 points.
Time: 50 minutes

All written work handed in by the student is considered to be
their own work, prepared without unauthorized assistance.

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3.1. **Formulas.** If Y has the binomial distribution with parameters n and p, then $p_Y(k) = \mathbb{P}[Y = k] = \binom{n}{k}p^k(1-p)^{n-k}$, for $k = 0, \ldots, n$, $\mathbb{E}[Y] = np$, $\mathrm{Var}[Y] = np(1-p)$. The binomial coefficients are defined as follows for integers $0 \le k \le n$, $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. The moment generating function of Y is given by $m_Y(t) = (pe^t + q)^n$.

If Y has a geometric distribution with parameter p, then $p_Y(k) = p(1-p)^k$ for $k = 0, 1, \ldots$, $\mathbb{E}[Y] = \frac{1-p}{p}$, $\mathrm{Var}[Y] = \frac{1-p}{p^2}$. Its mgf is $m_Y(t) = \frac{p}{1-qe^t}$ for t such that $qe^t < 1$.

If Y has a Poisson distribution with parameter λ, then $p_Y(k) = e^{-\lambda}\frac{\lambda^k}{k!}$ for $k = 0, 1, \ldots$, $\mathbb{E}[Y] = \mathrm{Var}[Y] = \lambda$. Its mgf is $m_Y(t) = e^{\lambda(e^t - 1)}$.

If Y has a uniform distribution on $[l, r]$, its density is

$$f_Y(y) = \frac{1}{r - l}\mathbf{1}_{(l,r)}(y),$$

its mean is $\frac{l+r}{2}$, and its variance is $\frac{(r-l)^2}{12}$. Let $U \sim U(0, 1)$. The mgf of U is $m_U(t) = \frac{1}{t}(e^t - 1)$.

If Y has the standard normal distribution, then its mean is zero, its variance is one, and its density equals

$$\varphi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2} \quad x \in \mathbb{R}.$$

Its mgf is $m_Y(t) = e^{\frac{t^2}{2}}$.

If Y has the exponential distribution with parameter τ, then its cumulative distribution function is $F_Y(y) = 1 - e^{-\frac{y}{\tau}}$ for $y \ge 0$, its probability density function is $f_Y(y) = \frac{1}{\tau}e^{-y/\tau}$ for $y \ge 0$. Also, $\mathbb{E}[Y] = SD[Y] = \tau$. Its mgf is $m_Y(t) = \frac{1}{1-\tau t}$.

The mgf of $Y \sim \Gamma(k, \tau)$ is

$$m_Y(t) = \frac{1}{(1 - \tau t)^k} \text{ for } t < 1/\tau.$$

Its expectation is $k\tau$ and its variance is $k\tau^2$. The χ^2-distribution with n degrees of freedom is the special case $\Gamma\left(\frac{n}{2}, 2\right)$

3.2. **DEFINITIONS.**

Problem 3.1. (10 points) Write down the definition of the **random sample** of size n from a distribution D.

Solution: A *random sample* of size n from a distribution D is a random vector

$$(Y_1, Y_2, \ldots, Y_n)$$

such that:

1. $Y_1, Y_2, \ldots, Y_n$ are **independent**, and
2. Y_i has the distribution D for every $i = 1, 2, \ldots, n$.

Problem 3.2. (10 points) Write down the definition of the **bias** of an estimator $\hat{\theta}$ of a parameter θ.

Solution:

$$\text{bias}(\hat{\theta}) = \mathbb{E}[\hat{\theta}] - \theta$$

Problem 3.3. (10 points) Write down the definition of the **mean squared error** of an estimator $\hat{\theta}$ of a parameter θ.

Solution:

$$\text{MSE}(\hat{\theta}) = \mathbb{E}[(\hat{\theta} - \theta)^2]$$

3.3. **TRUE/FALSE QUESTIONS.**

Problem 3.4. (5 points) Let Z be a standard normal random variable and let Q^2 have the χ^2-distribution with ν degrees of freedom. Assume that Z and Q^2 are independent. Set

$$T = \frac{Z}{\sqrt{\frac{Q^2}{\nu}}}.$$

Then, T has a $t-$distribution with ν degrees of freedom. *True or false?*

Solution: TRUE
From the definition of the $t-$distribution, we know that

$$T = \frac{Z}{\sqrt{\frac{Q^2}{\nu}}}$$

has the $t-$ distribution with ν degrees of freedom.

Problem 3.5. (5 points) Let $Y_1, Y_2, \ldots, Y_n$ be a random sample. Then,

$$\frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{(0,\infty)}(Y_i)$$

is a well-defined statistic. *True or false?*

Solution: TRUE

Problem 3.6. (5 points) Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from $N(\mu, \sigma)$ with μ **known** and σ **unknown**. Then,

$$\frac{1}{n} \sum_{i=1}^{n} (Y_i - \mu)^2$$

is a well-defined estimator for σ^2. *True or false?*

Solution: TRUE

3.4. **Free-response problems.**

Please, explain carefully all your statements and assumptions. Numerical results or single-word answers without an explanation (even if they're correct) are worth 0 points.

Problem 3.7. (15 points) Let $Y_1, \ldots, Y_n$ be a random sample from an exponential distribution with an unknown parameter τ. What is the maximum likelihood estimator for τ? Make sure that you prove your claim!

Solution: Let $y_1, \ldots, y_n$ represent a set of observations of the above random sample. Then, the likelihood function is

$$L(\tau; y_1, \ldots, y_n) = \prod_{i=1}^{n} \left(\frac{1}{\tau} e^{-\frac{y_i}{\tau}} \right) = \frac{1}{\tau^n} e^{-\frac{1}{\tau} \sum y_i} .$$

The log-likelihood function is

$$\ell(\tau; y_1, \ldots, y_n) = -n \ln(\tau) - \frac{1}{\tau} \sum_{i=1}^{n} y_i.$$

Differentiating with respect to τ, we get

$$\ell'(\tau; y_1, \ldots, y_n) = -\frac{n}{\tau} + \frac{\sum_{i=1}^{n} y_i}{\tau^2} .$$

Equating the above to 0 and solving for τ, we get

$$\hat{\tau}_{MLE} = \bar{Y}.$$

Problem 3.8. (20 points) Let the time it takes for a single elf to construct a toy engine be a continuous random variable with the following probability density function

$$f_Y(y) = e^{-(y-\theta)} \mathbf{1}_{(\theta,\infty)}(y). \tag{3.1}$$

The parameter θ is assumed positive and corresponds to the minimum required engine construction time. The group of n independent elves is competing about who will construct their toy engine first. What is the density of the time until the first toy engine is constructed?

Solution: Let $Y_1, \ldots, Y_n$ denote the individual times until each elf is done constructing their toy engine. The time at which the fastest elf is done is, then, the first order statistic

$$Y_{(1)} = \min(Y_1, \ldots, Y_n). \tag{3.2}$$

Let's start by figuring out its cumulative distribution function. For each $y \leq \theta$, the cumulative distribution function equals 0. For $y > \theta$, we have

$$\begin{aligned}
F_{(1)}(y) &= \mathbb{P}[Y_{(1)} \leq y] \\
&= \mathbb{P}[\min(Y_1, \ldots, Y_n) \leq y] \\
&= 1 - \mathbb{P}[\min(Y_1, \ldots, Y_n) > y] \\
&= 1 - \mathbb{P}[Y_1 > y, \ldots, Y_n > y].
\end{aligned} \tag{3.3}$$

Since the elves are independent, the above equals

$$\begin{aligned}
F_{(1)}(y) &= 1 - \mathbb{P}[Y_1 > y] \ldots \mathbb{P}[Y_n > y] \\
&= 1 - (1 - \mathbb{P}[Y_1 \leq y]) \ldots (1 - \mathbb{P}[Y_n \leq y]) \\
&= 1 - (1 - F_Y(y))^n.
\end{aligned} \tag{3.4}$$

So, the density of $Y_{(1)}$ is

$$g_{(1)}(y) = n f_Y(y)(1 - F_Y(y))^{n-1}. \tag{3.5}$$

Of course, some students would remember the above expression for the density of the first order statistic from class without needing to re-derive it.

Now, for each $y > \theta$, the cdf of Y can be obtained by integration as

$$F_Y(y) = \int_\theta^y f_Y(u) \, du = \int_\theta^y e^{-(u-\theta)} \, du = e^\theta \int_\theta^y e^{-u} \, du = e^\theta(e^{-\theta} - e^{-y}) = 1 - e^{-(y-\theta)}. \tag{3.6}$$

So, for $y > \theta$, we have

$$g_{(1)}(y) = n e^{-(y-\theta)}(e^{-(y-\theta)})^{n-1} = n e^{-n(y-\theta)}. \tag{3.7}$$

Alternatively: You could have noticed that an individual elf's work time could be expressed as

$$Y = \theta + Y' \tag{3.8}$$

where $Y' \sim E(1)$. Then, the minimum of $Y_1', \ldots, Y_n'$ is $E(1/n)$. So, the minimum work time would be

$$Y_{(1)} = \theta + Y_{(1)}' \tag{3.9}$$

with the density

$$g_{(1)}(y) = n e^{-n(y-\theta)}. \tag{3.10}$$

3.5. MULTIPLE CHOICE QUESTIONS.

Problem 3.9. (5 points) In a sample $Y_1, \ldots, Y_n$ from the uniform distribution on $(0, \theta)$ with parameter $\theta > 0$, $U = cY_{(n)}$ is a pivotal quantity if the value of the constant c is

(a) $\frac{1}{2}$

(b) $1/\theta$

(c) θ

(d) $\frac{n+1}{n}$

(e) none of the above

Solution: The correct answer is **(b)**.
The pivotal quantity

$$U = \frac{Y_{(n)}}{\theta} = \max(Y_1, \ldots, Y_n)/\theta$$

has the distribution with cdf y^n on $[0, 1]$.

Problem 3.10. (5 points) In a random sample of 100 voters 64 prefer candidate A and the rest prefer candidate B. The (approximate) $(1 - \alpha)$-confidence interval for the parameter p (the population proportion of A voters) is of the form

$$[0.64 - z_{\alpha/2} \times c, 0.64 + z_{\alpha/2} \times c],$$

where $z_{\alpha/2} = \texttt{qnorm}(1 - \alpha/2, 0, 1)$.
   The value of c is:

(a) 0.016

(b) 0.024

(c) 0.036

(d) 0.048

(e) ≥ 0.05

Solution: The correct answer is **(d)** since $c = \sqrt{\hat{p}(1 - \hat{p})/n} = \sqrt{0.64 \times 0.36/100} = 0.048$.

Problem 3.11. (5 points) A sample of size $n = 2$ from normal distribution with unknown μ and σ is collected and the data are
$$y_1 = 1 \text{ and } y_2 = 5.$$
The right end-point of a symmetric 95% confidence interval for σ^2 is

(a) 8/qchisq(0.975,2)

(b) 16/qchisq(0.025,1)

(c) 8/qchisq(0.025,1)

(d) 16/qchisq(0.975,2)

(e) **None of the above**

Solution: The correct answer is **(c)**.
The confidence interval is based on the pivotal quantity $(n-1)S^2/\sigma^2$ whose distribution is $\chi^2(n-1)$.
In this case, $n = 2$, $\bar{Y} = 3$ so that $(n-1)s^2 = (y_1 - \bar{y})^2 + (y_2 - \bar{y})^2 = 2^2 + 2^2 = 8$, which produces the interval $[8/\texttt{qchisq(0.975,1)}, 8/\texttt{qchisq(0.025,1)}]$.

Problem 3.12. (5 points)Let $Y_1, \ldots, Y_5$ be a random sample from the normal distribution $N(\mu, 2)$, with an <u>unknown</u> mean μ and the <u>known</u> standard deviation $\sigma = 2$. The collected data turn out to be
$$y_1 = 2, \ y_2 = 5, \ y_3 = 1, \ y_4 = 4, \ y_5 = 3.$$
The <u>right</u> end-point $\hat{\mu}_R$ of the symmetric 90%-confidence interval $[\hat{\mu}_L, \hat{\mu}_R]$ for μ is

(a) $3 + \frac{2}{\sqrt{5}}\texttt{qnorm}(0.95, 0, 1)$.

(b) $3 - \frac{2}{\sqrt{5}}\texttt{qnorm}(0.95, 0, 1)$.

(c) $3 - \frac{1}{\sqrt{5}}\texttt{qt}(0.95, 4)$.

(d) $3 + \frac{1}{5}\texttt{qnorm}(0.9, 5)$.

(e) none of the above

Solution: The correct answer is **(a)**.
The confidence interval in this case is based on the pivotal quantity $\sqrt{5}\left(\frac{\mu - \bar{Y}}{2}\right)$ which has the $N(0, 1)$ distribution. Therefore, for $a = \texttt{qnorm}(0.95, 0, 1)$ we have
$$\mathbb{P}\left[-a \leq \sqrt{5}\left(\frac{\mu - \bar{Y}}{2}\right) \leq a\right] = 0.90.$$
We solve for μ to obtain
$$\mathbb{P}[\bar{Y} - \frac{2}{\sqrt{5}}a \leq \mu \leq \bar{Y} + \frac{2}{\sqrt{5}}a] = 0.90.$$
For our data set $\bar{y} = 3$, so $\hat{\mu}_R = 3 + \frac{2}{\sqrt{5}}\texttt{qnorm}(0.95, 0, 1)$.

Problem 3.13. (5 points) You are monitoring a cash register at a store. A seemingly endless queue of customers is waiting. The times it takes for Cassie the Cashier to check out a single customer is exponential with mean 5 minutes for each customer. Moreover, the customer service times are independent. Cassie the cashier can go on a break after every batch of 10 customers leave. She just came in from a break. What is the distribution of her waiting time until the next break?

(a) $\Gamma(10, 5)$, i.e., $k = 10$, $\tau = 5$

(b) $\Gamma(50, 1)$, i.e., $k = 50$, $\tau = 1$

(c) $\chi^2(25)$

(d) $E(50)$

(e) $E(1/50)$

Solution: The correct answer is **(a)**.

We need the distribution of the sum of 10 independent exponential random variables $E(\tau)$ with parameter $\tau = 5$. Each $E(5)$ is a special case of the gamma distribution with parameters $k = 1$ and $\tau = 5$. Gammas are additive in the shape parameter, so the result is $\Gamma(10, 5)$.