M378K: February 23rd, 2026.

In. Term One Aftermath.

- $$F_Y(y) = \underbrace{\int_{-\infty}^{y} f_Y(u)\,du}_{} \quad \times$$

  Domain! $y \in \mathbb{R}$

  $$\underbrace{\sum_{k \leq y} p_Y(k)}_{} \quad \times$$

  $$\boxed{F_Y(y) = \mathbb{P}[Y \leq y] \quad , \text{ for all } y \in \mathbb{R}}$$

- Y cont. w/ $f_Y(y)$

  $$\mathbb{E}[g(Y)] = \int_{-\infty}^{\infty} g(y) f_Y(y)\,dy$$

- $$\int_{-\infty}^{\boxed{y}} f_Y(y)\,dy \quad \leftarrow$$

  $$\int_0^x g(x-s)\,ds$$

χ^2-Distribution.

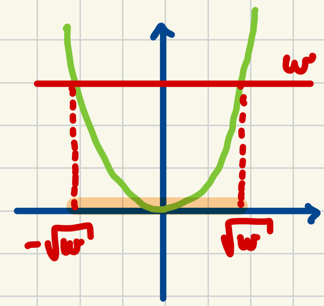
Let $Y \sim N(0,1)$.

Set $W = Y^2$, i.e., $W = g(Y)$ w/ $g(y) = y^2$

For all $w \le 0$: $F_W(w) = \underline{0}$

For all $w > 0$:

$$\begin{aligned} F_W(w) &= \mathbb{P}[W \le w] = \mathbb{P}[Y^2 \le w] \\ &= \mathbb{P}[-\sqrt{w} \le Y \le \sqrt{w}] \\ &= F_Y(\sqrt{w}) - F_Y(-\sqrt{w}) \\ &= \Phi(\sqrt{w}) - \Phi(-\sqrt{w}) \\ &= \Phi(\sqrt{w}) - (1 - \Phi(\sqrt{w})) \\ &= 2\Phi(\sqrt{w}) - 1 \end{aligned}$$



By symmetry of $N(0,1)$:

$$\Phi(-z) = 1 - \Phi(z)$$

for $w > 0$:

$$\begin{aligned} f_W(w) &= \frac{d}{dw} F_W(w) = \\ &= \frac{d}{dw}\left(2\Phi(\sqrt{w}) - 1\right) \\ &= \cancel{2}\, \varphi(\sqrt{w}) \cdot \frac{1}{\cancel{2}\sqrt{w}} \end{aligned}$$

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \quad \text{for all } z \in \mathbb{R}$$

$$f_W(w) = \frac{1}{\sqrt{w}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{w}{2}} \quad \text{for } w > 0$$

$$\boxed{f_W(w) = \frac{1}{\sqrt{2\pi w}} e^{-\frac{w}{2}} \mathbb{1}_{(0,\infty)}(w)}$$

W is said to have the χ^2-distribution w/ 1 degree of freedom

$$\boxed{W \sim \chi^2(df=1)}$$

More generally, for $Y_1, \dots, Y_K$ independent standard normal r.v.s, set

$$\boxed{W = Y_1^2 + Y_2^2 + \dots + Y_K^2}$$

We say that W has the χ^2-dist'n w/ K degrees of freedom.

$$\boxed{W \sim \chi^2(df=K)}$$

The F-Distribution.

Let Y_1 and Y_2 be two independent, χ^2-distributed r.v.s w/ df=1.

For both Y_1 and Y_2, the pdf is

$$f_Y(y) = \frac{1}{\sqrt{2\pi y}} e^{-\frac{y}{2}} \mathbb{1}_{(0,\infty)}(y)$$

Define $\boxed{W = \frac{Y_1}{Y_2}}$, i.e., $W = g(Y_1, Y_2)$ w/ $g(y_1, y_2) = \frac{y_1}{y_2}$

Goal: Density of W, i.e., f_W!

Start by figuring out the cdf F_W.

- $w \leq 0$: $F_W(w) = 0$
- $w > 0$:

$$F_W(w) = \mathbb{P}[W \leq w] = \mathbb{P}\left[\frac{Y_1}{Y_2} \leq w\right] = \mathbb{P}[Y_1 \leq w \cdot Y_2]$$

$$= \int_0^\infty \int_0^{w \cdot y_2} f_{Y_1, Y_2}(y_1, y_2)\, dy_1\, dy_2$$

$$= \int_0^\infty \int_0^{w \cdot y_2} \frac{1}{\sqrt{2\pi y_1}} e^{-\frac{y_1}{2}} \cdot \frac{1}{\sqrt{2\pi y_2}} e^{-\frac{y_2}{2}}\, dy_1\, dy_2$$

$$= \int_0^\infty \frac{1}{\sqrt{2\pi y_2}} e^{-\frac{y_2}{2}} \boxed{\int_0^{w \cdot y_2} \frac{1}{\sqrt{2\pi y_1}} e^{-\frac{y_1}{2}}\, dy_1}\, dy_2$$

$F_{Y_1}(w \cdot y_2)$

$$F_W(w) = \int_0^\infty \frac{1}{\sqrt{2\pi y_2}} e^{-\frac{y_2}{2}} \cdot F_{Y_1}(w \cdot y_2)\, dy_2$$

$$f_W(w) = \frac{d}{dw} \int_0^\infty \frac{1}{\sqrt{2\pi y_2}} e^{-\frac{y_2}{2}} F_{Y_1}(w \cdot y_2)\, dy_2$$

$$f_W(w) = \int_0^\infty \frac{1}{\sqrt{2\pi y_2}} e^{-\frac{y_2}{2}} f_{Y_1}(w\, y_2) \cdot y_2\, dy_2$$

$$f_W(w) = \int_0^\infty \frac{1}{\sqrt{2\pi \cancel{y_2}}} e^{-\frac{y_2}{2}} \cdot \frac{1}{\sqrt{2\pi w \cancel{y_2}}} e^{-\frac{w y_2}{2}} \cdot \cancel{y_2}\, dy_2$$

$$f_W(w) = \int_0^\infty \frac{1}{2\pi} \cdot \frac{1}{\sqrt{w}} e^{-\frac{y_2}{2}(1+w)}\, dy_2$$

$$f_W(w) = \frac{1}{2\pi\sqrt{w}} \boxed{\int_0^\infty e^{-\frac{1+w}{2} y_2}\, dy_2}$$

$$= -\frac{2}{1+w} e^{-\frac{1+w}{2} y_2} \Big|_{y_2=0}^{\infty} = \frac{2}{1+w}$$

$$f_W(w) = \frac{1}{\cancel{2}\pi\sqrt{w}} \cdot \frac{\cancel{2}}{1+w} = \frac{1}{\pi\sqrt{w}(1+w)} \mathbb{1}_{(0,\infty)}(w)$$

is the density of $F(1,1)$, i.e., the F-distribution w/ 1 numerator df and 1 denominator df

In general:

$$F(\nu_1, \nu_2) = \frac{\frac{\chi^2(\nu_1)}{\nu_1}}{\frac{\chi^2(\nu_2)}{\nu_2}}$$