M339J: March 25th, 2022. Poisson·Gamma Mixing.

Let N have the following mixture distribution w/ mixing parameter Λ:

$$\left.\begin{array}{l} N \mid \Lambda \sim \text{Poisson}(\text{mean} = \Lambda) \\ \Lambda \sim \text{Gamma}(\alpha, \theta) \end{array}\right\}$$

Q: What is the support of N?

$\longrightarrow$: $\mathbb{N}_0 = \{0, 1, 2, \dots\}$

$$\boxed{N \sim \text{NegBinomial}(r = \underline{\alpha}, \beta = \underline{\theta})}$$ ✓

Problem.

$$\left.\begin{array}{l} N \mid \Lambda \sim \text{Poisson}(\text{mean} = \Lambda) \\ \Lambda \sim \text{Gamma}(\alpha = \frac{5}{2}, \theta = 4) \end{array}\right\}$$

What is the (unconditional) probability $\mathbb{P}[N=3]$?

$\longrightarrow$: $N \sim \text{NegBin}(r = \frac{5}{2}, \beta = 4)$

$$\mathbb{P}[N=3] = \binom{\frac{5}{2}+3-1}{3} \boxed{\left(\frac{1}{1+\beta}\right)^{5/2}\left(\frac{\beta}{1+\beta}\right)^{3}}$$ w/ $\beta = 4$ ✓

by our generalization from last class

$$\frac{\Gamma(3+\frac{5}{2})}{3! \cdot \Gamma(\frac{5}{2})}$$

Hint:
- $\Gamma(x) = (x-1)\cdot\Gamma(x-1)$ *
- $\Gamma(k) = (k-1)!$ for $k \in \mathbb{N}$ **
- $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ ***

$$\Gamma\left(3+\frac{5}{2}\right) = \left(2+\frac{5}{2}\right)\cdot\Gamma\left(2+\frac{5}{2}\right) = \left(2+\frac{5}{2}\right)\cdot\left(1+\frac{5}{2}\right)\Gamma\left(1+\frac{5}{2}\right)$$
$$= \left(2+\frac{5}{2}\right)\left(1+\frac{5}{2}\right)\left(\frac{5}{2}\right)\Gamma\left(\frac{5}{2}\right)$$

$$\boxed{\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}}$$

$$\frac{\Gamma\left(3+\frac{5}{2}\right)}{3!\cdot\Gamma\left(\frac{5}{2}\right)} = \frac{\left(2+\frac{5}{2}\right)\left(1+\frac{5}{2}\right)\left(\frac{5}{2}\right)\cancel{\Gamma\left(\frac{5}{2}\right)}}{3!\cdot\cancel{\Gamma\left(\frac{5}{2}\right)}} = \frac{\overset{3}{\cancel{\frac{9}{2}}}\cdot\frac{7}{2}\cdot\frac{5}{\cancel{2}}}{\cancel{6}_{2}} = \frac{105}{16}$$

$$\mathbb{P}[N=3] = \frac{105}{\cancel{16}}\cdot\boxed{\frac{\cancel{4^3}\,4}{5^{3+\frac{5}{2}}}} = 0.0601 \quad \square$$

Binomial Distribution.

Consider m independent, identically distributed $\boxed{\text{risks}}$ w/ probability of making a claim denoted by q.

Formally, for $j = 1, \dots, m$, we set

$$I_j = \begin{cases} 1 & \text{if risk } j \text{ makes a claim} \\ 0 & \text{if risk } j \text{ does not make claim} \end{cases}$$

Then, for all j,
$$I_j \sim \text{Bernoulli}(q)$$
and $\{I_j : j = 1..m\}$ are independent.

N... the total number of claims made

$$\boxed{N = I_1 + I_2 + \dots + I_m}$$

Q: What is the pmf of N?

Note: There are many approaches to this. I'm choosing this one so that we can practice a particular technique!

$\longrightarrow$: First, we figure out the pgf of N.

$$P_N(z) = ?$$

$$P_N(z) = \underbrace{P_{I_1}(z) \cdot P_{I_2}(z) \cdots P_{I_m}(z)}_{} = \left(P_{I_1}(z)\right)^m \quad ✓$$

independence

identically dist'd

For a single Bernoulli trial w/ probability of success q, ✓

$$P_{I_1}(z) = \mathbb{E}\left[z^{I_1}\right] = \overbrace{p_{I_1}(0)}^{=1-q} \cdot z^0 + \overbrace{p_{I_1}(1)}^{=q} \cdot z^1 = 1-q + q \cdot z$$

by def'n of pgf

$$= 1 + q(z-1)$$

$$\boxed{P_{I_1}(z) = 1 + q(z-1)}$$

$$\boxed{P_N(z) = \left(1 + q(z-1)\right)^m}$$

Using the binomial formula on $P_N(z) = \left((1-q) + q \cdot z\right)^m$, we get, for $k = 0, 1, \ldots, m$,

$$\boxed{p_N(k) = \binom{m}{k} (1-q)^{m-k} \cdot q^k}$$