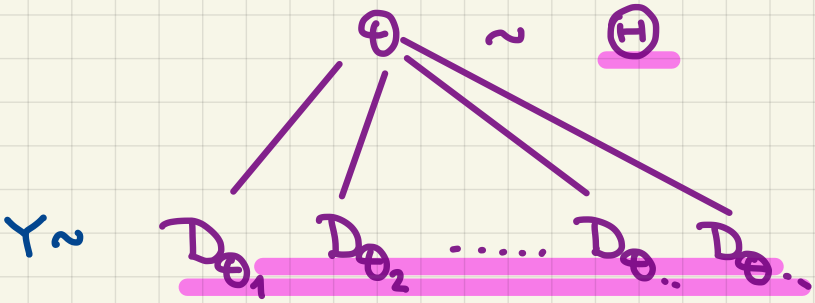
M378K: 04/25/2025.

The Bayesian Approach.

The Set Up.



The Simplest Version

Θ can be any one value from $\theta_1, \theta_2, \ldots, \theta_K$

Then, we see one value y of the random variable Y.

By the Bayes rule, we have for $k=1,\ldots,K$,

$$\boxed{\mathbb{P}[\Theta = \theta_k \mid Y=y]} = \frac{\mathbb{P}[\Theta=\theta_k]\cdot\mathbb{P}[Y=y \mid \Theta=\theta_k]}{\sum_{i=1}^{K} \boxed{\mathbb{P}[\Theta=\theta_i]}\cdot\mathbb{P}[Y=y \mid \Theta=\theta_i]}$$

posterior probabilities

prior probabilities

This procedure is called Bayesian updating.

When no prior information is available, we use the uninformed prior.

The Continuous Case.

Here, we assume that Θ admits a pdf denoted by $p(\theta)$

We denote the posterior density by

$$p(\theta \mid y_1, \ldots, y_n)$$

As usual, $L(\theta; y_1, \ldots, y_n)$ is the likelihood f'tion.

So, the posterior density becomes

$$p(\Theta \,|\, y_1,\dots,y_n) = \frac{p(\Theta)\cdot L(\Theta; y_1,\dots,y_n)}{\int p(\tilde{\Theta})\cdot L(\tilde{\Theta}; y_1,\dots,y_n)\, d\tilde{\Theta}}$$

Note: • By default, the integral is from $-\infty$ to $+\infty$.

• $\tilde{\Theta}$ is the "dummy" variable of integration.

Example. $Y_1,\dots,Y_n$ is a random sample from $N(\mu, \sigma=1)$ where the prior distribution for μ is $N(0,1)$.

$$p(\mu) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\mu^2}{2}}$$

and

$$L(\mu; y_1,\dots,y_n) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \exp\left(-\frac{1}{2}\sum_{i=1}^n (y_i-\mu)^2\right)$$

$$\Rightarrow \quad p(\mu \,|\, y_1,\dots,y_n) = \frac{\cancel{\frac{1}{\sqrt{2\pi}}} e^{-\frac{\mu^2}{2}} \cdot \cancel{\left(\frac{1}{\sqrt{2\pi}}\right)^n} \exp\left(-\frac{1}{2}\sum_{i=1}^n (y_i-\mu)^2\right)}{\boxed{\int_{-\infty}^{\infty} \cancel{(2\pi)^{-\frac{n+1}{2}}}\, e^{-\frac{\tilde{\mu}^2}{2}} \exp\left(-\frac{1}{2}\sum_{i=1}^n (y_i-\tilde{\mu})^2\right) d\tilde{\mu}}}$$

$$c :=$$

$$p(\mu \,|\, y_1,\dots,y_n) = c \cdot \exp\left(-\frac{1}{2}\left(\mu^2 + \sum (y_i-\mu)^2\right)\right)$$

$$= c \cdot \exp\left(-\frac{1}{2}\left(\mu^2 + \sum y_i^2 - 2\mu \sum y_i + n\cdot\mu^2\right)\right)$$

$$= c \cdot \exp\left(-\frac{1}{2}\left((n+1)\mu^2 - 2\mu \sum y_i + \sum y_i^2\right)\right)$$

$$= c \cdot \exp\left(-\frac{1}{\frac{2}{n+1}}\left(\mu^2 - 2\cdot\mu\cdot\frac{\sum y_i}{n+1} + \frac{\sum y_i^2}{n+1}\right)\right)$$

$$= \boxed{c} \cdot \exp\left(-\frac{1}{\frac{2}{n+1}}\left(\mu^2 - 2\mu\cdot\frac{\sum y_i}{n+1} + \left(\frac{\sum y_i}{n+1}\right)^2\right)\right)$$

$$\boxed{\exp\left(-\frac{1}{\frac{2}{n+1}}\left(\frac{\sum y_i}{n+1}\right)^2 - \frac{1}{2}\sum y_i^2\right)}$$

$$= c' \boxed{\exp\left(-\frac{1}{2 \cdot \frac{1}{n+1}} \left(\mu - \frac{\sum y_i}{n+1}\right)^2\right)}$$

$c' = ?$

$$\mu \sim \text{Normal}\left(\frac{\sum y_i}{n+1}, sd = \frac{1}{\sqrt{n+1}}\right)$$

$$c' = \frac{1}{SD\sqrt{2\pi}} = \frac{1}{\frac{\sqrt{2\pi}}{n+1}}$$