Multiple linear regression

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Here, I am adapting the lab associated with Chapter 3 of the textbook.

We load the `ISLR2` package, which includes the data sets associated with this book.

```r
library(ISLR2)
```

The `ISLR2` library contains the `Boston` data set, which records `medv` (median house value) for 506 census tracts in Boston. We will seek to predict `medv` using 12 predictors such as `rmvar` (average number of rooms per house), `age` (proportion of owner-occupied units built prior to 1940) and `lstat` (percent of households with low socioeconomic status).

```r
head(Boston)
```

```
##      crim zn indus chas   nox    rm  age    dis rad tax ptratio lstat medv
## 1 0.00632 18  2.31    0 0.538 6.575 65.2 4.0900   1 296    15.3  4.98 24.0
## 2 0.02731  0  7.07    0 0.469 6.421 78.9 4.9671   2 242    17.8  9.14 21.6
## 3 0.02729  0  7.07    0 0.469 7.185 61.1 4.9671   2 242    17.8  4.03 34.7
## 4 0.03237  0  2.18    0 0.458 6.998 45.8 6.0622   3 222    18.7  2.94 33.4
## 5 0.06905  0  2.18    0 0.458 7.147 54.2 6.0622   3 222    18.7  5.33 36.2
## 6 0.02985  0  2.18    0 0.458 6.430 58.7 6.0622   3 222    18.7  5.21 28.7
```

We should also investigate the variables therein and `attach` so that we don't have to use the `$` notation.

```r
names(Boston)
```

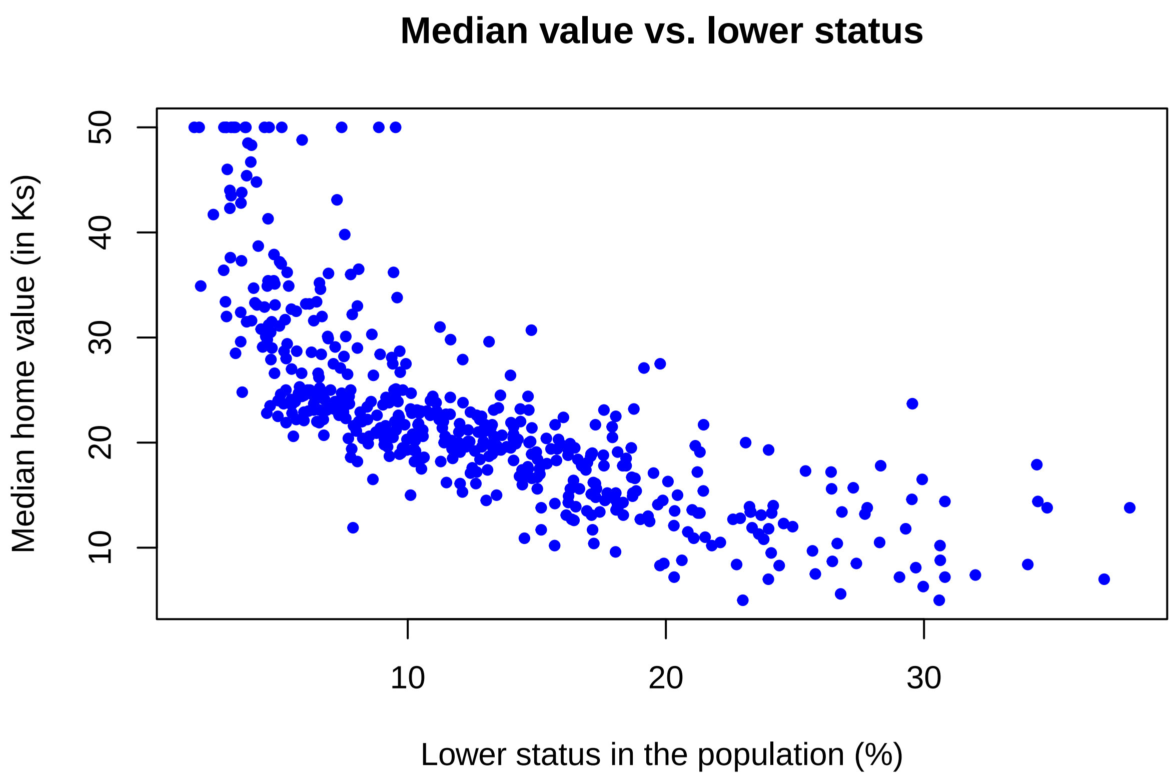
```
##  [1] "crim"    "zn"      "indus"   "chas"    "nox"     "rm"      "age"
##  [8] "dis"     "rad"     "tax"     "ptratio" "lstat"   "medv"
```

```r
attach(Boston)
```

To find out more about the data set, we can type `?Boston`.

We will start by fitting a simple linear regression model, with `medv` as the response and `lstat` as the predictor. Here is the scatterplot.

```r
plot(lstat, medv,
     main="Median value vs. lower status",
     xlab="Lower status in the population (%)",
     ylab="Median home value (in Ks)",
     pch=20, col="blue")
```

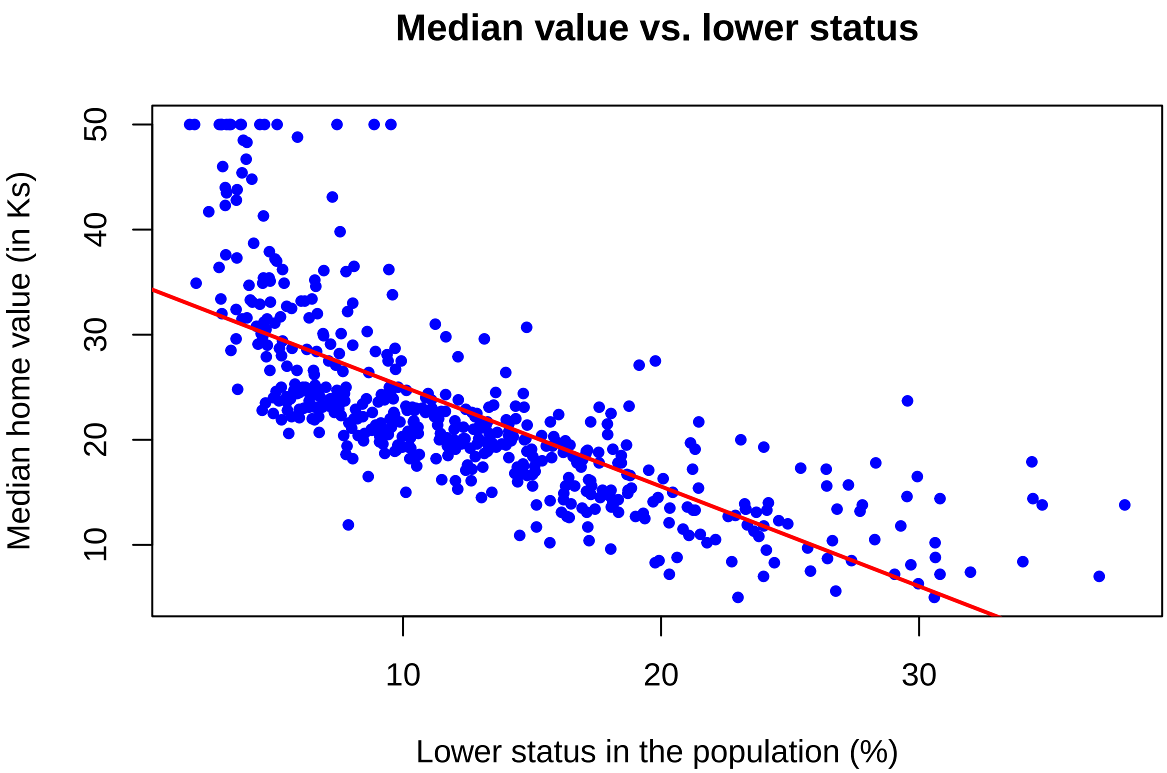


For the implementation of the linear model, the basic syntax is `lm(y ~ x, data)`, where `y` is the response, `x` is the predictor, and `data` is the data set in which these two variables are kept.

```
lm.fit <- lm(medv ~ lstat)
summary(lm.fit)
```

```
##
## Call:
## lm(formula = medv ~ lstat)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -15.168  -3.990  -1.318   2.034  24.500
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept) 34.55384    0.56263   61.41   <2e-16 ***
## lstat       -0.95005    0.03873  -24.53   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 6.216 on 504 degrees of freedom
## Multiple R-squared:  0.5441, Adjusted R-squared:  0.5432
## F-statistic: 601.6 on 1 and 504 DF,  p-value: < 2.2e-16
```

```
plot(lstat, medv,
     main="Median value vs. lower status",
     xlab="Lower status in the population (%)",
     ylab="Median home value (in Ks)",
     pch=20, col="blue")
abline(lm.fit, col="red", lwd=2)
```



We can use the `names()` function in order to find out what other pieces of information are stored in `lm.fit`. Although we can extract these quantities by name—e.g. `lm.fit$coefficients`—it is safer to use the extractor functions like `coef()` to access them.

```
names(lm.fit)
```

```
##  [1] "coefficients"  "residuals"     "effects"       "rank"
##  [5] "fitted.values" "assign"        "qr"            "df.residual"
##  [9] "xlevels"       "call"          "terms"         "model"
```

```
coef(lm.fit)
```

```
## (Intercept)       lstat
##  34.5538409  -0.9500494
```

In order to obtain a confidence interval for the coefficient estimates, we can use the `confint()` command.

```
confint(lm.fit)
```

```
##                 2.5 %     97.5 %
## (Intercept) 33.448457 35.6592247
## lstat       -1.026148 -0.8739505
```

The `predict()` function can be used to produce confidence intervals and prediction intervals for the prediction of `medv` for a given value of `lstat`.

```
predict(lm.fit, data.frame(lstat = (c(5, 10, 15))),
    interval = "confidence")
```

```
##        fit      lwr      upr
## 1 29.80359 29.00741 30.59978
## 2 25.05335 24.47413 25.63256
## 3 20.30310 19.73159 20.87461
```

```r
predict(lm.fit, data.frame(lstat = (c(5, 10, 15))),
    interval = "prediction")
```

```
##        fit       lwr      upr
## 1 29.80359 17.565675 42.04151
## 2 25.05335 12.827626 37.27907
## 3 20.30310  8.077742 32.52846
```

How about the full array of confidence and prediction intervals?

```r
newdata<-data.frame(lstat=seq(min(lstat),max(lstat),0.1))

conf<-predict(lm.fit,newdata,interval="confidence")
conf
```

```
##             fit          lwr        upr
## 1   32.91025550 31.917443200 33.9030678
## 2   32.81525056 31.828800866 33.8017003
## 3   32.72024563 31.740140752 33.7003505
## 4   32.62524069 31.651462508 33.5990189
## 5   32.53023576 31.562765779 33.4977057
## 6   32.43523082 31.474050201 33.3964114
## 7   32.34022589 31.385315402 33.2951364
## 8   32.24522095 31.296561001 33.1938809
## 9   32.15021601 31.207786608 33.0926454
## 10  32.05521108 31.118991823 32.9914303
## 11  31.96020614 31.030176238 32.8902360
## 12  31.86520121 30.941339436 32.7890630
## 13  31.77019627 30.852480987 32.6879116
## 14  31.67519134 30.763600455 32.5867822
## 15  31.58018640 30.674697391 32.4856754
## 16  31.48518147 30.585771336 32.3845916
## 17  31.39017653 30.496821822 32.2835312
## 18  31.29517160 30.407848368 32.1824948
## 19  31.20016666 30.318850482 32.0814828
## 20  31.10516173 30.229827662 31.9804958
## 21  31.01015679 30.140779392 31.8795342
## 22  30.91515185 30.051705147 31.7785986
## 23  30.82014692 29.962604387 31.6776895
## 24  30.72514198 29.873476561 31.5768074
## 25  30.63013705 29.784321104 31.4759530
## 26  30.53513211 29.695137439 31.3751268
## 27  30.44012718 29.605924976 31.2743294
## 28  30.34512224 29.516683111 31.1735614
## 29  30.25011731 29.427411226 31.0728234
## 30  30.15511237 29.338108688 30.9721161
## 31  30.06010744 29.248774852 30.8714400
## 32  29.96510250 29.159409056 30.7707959
## 33  29.87009757 29.070010626 30.6701845
##  [ reached 'max' / getOption("max.print") -- omitted 330 rows ]
```

```r
pred<-predict(lm.fit,newdata,interval="prediction")
pred
```

```
##             fit         lwr      upr
## 1   32.91025550  20.65797246 45.16254
```

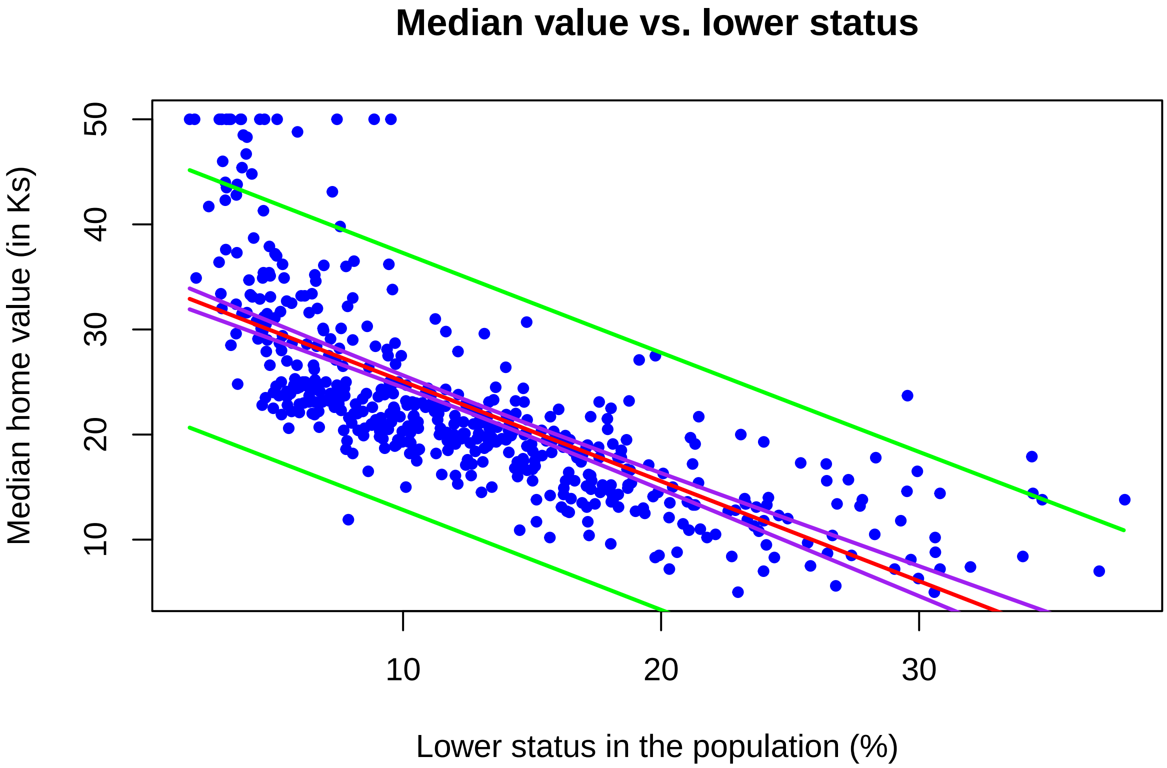
```
## 2   32.81525056  20.56348145 45.06702
## 3   32.72024563  20.46898573 44.97151
## 4   32.62524069  20.37448531 44.87600
## 5   32.53023576  20.27998018 44.78049
## 6   32.43523082  20.18547035 44.68499
## 7   32.34022589  20.09095581 44.58950
## 8   32.24522095  19.99643656 44.49401
## 9   32.15021601  19.90191260 44.39852
## 10  32.05521108  19.80738393 44.30304
## 11  31.96020614  19.71285055 44.20756
## 12  31.86520121  19.61831246 44.11209
## 13  31.77019627  19.52376966 44.01662
## 14  31.67519134  19.42922215 43.92116
## 15  31.58018640  19.33466993 43.82570
## 16  31.48518147  19.24011299 43.73025
## 17  31.39017653  19.14555134 43.63480
## 18  31.29517160  19.05098498 43.53936
## 19  31.20016666  18.95641390 43.44392
## 20  31.10516173  18.86183811 43.34849
## 21  31.01015679  18.76725760 43.25306
## 22  30.91515185  18.67267238 43.15763
## 23  30.82014692  18.57808244 43.06221
## 24  30.72514198  18.48348779 42.96680
## 25  30.63013705  18.38888842 42.87139
## 26  30.53513211  18.29428433 42.77598
## 27  30.44012718  18.19967552 42.68058
## 28  30.34512224  18.10506200 42.58518
## 29  30.25011731  18.01044375 42.48979
## 30  30.15511237  17.91582079 42.39440
## 31  30.06010744  17.82119311 42.29902
## 32  29.96510250  17.72656070 42.20364
## 33  29.87009757  17.63192358 42.10827
##  [ reached 'max' / getOption("max.print") -- omitted 330 rows ]
```

```
all.ints<-cbind(conf,pred[,-1])
all.ints
```

```
##             fit          lwr        upr         lwr      upr
## 1   32.91025550 31.917443200 33.9030678  20.65797246 45.16254
## 2   32.81525056 31.828800866 33.8017003  20.56348145 45.06702
## 3   32.72024563 31.740140752 33.7003505  20.46898573 44.97151
## 4   32.62524069 31.651462508 33.5990189  20.37448531 44.87600
## 5   32.53023576 31.562765779 33.4977057  20.27998018 44.78049
## 6   32.43523082 31.474050201 33.3964114  20.18547035 44.68499
## 7   32.34022589 31.385315402 33.2951364  20.09095581 44.58950
## 8   32.24522095 31.296561001 33.1938809  19.99643656 44.49401
## 9   32.15021601 31.207786608 33.0926454  19.90191260 44.39852
## 10  32.05521108 31.118991823 32.9914303  19.80738393 44.30304
## 11  31.96020614 31.030176238 32.8902360  19.71285055 44.20756
## 12  31.86520121 30.941339436 32.7890630  19.61831246 44.11209
## 13  31.77019627 30.852480987 32.6879116  19.52376966 44.01662
## 14  31.67519134 30.763600455 32.5867822  19.42922215 43.92116
## 15  31.58018640 30.674697391 32.4856754  19.33466993 43.82570
## 16  31.48518147 30.585771336 32.3845916  19.24011299 43.73025
## 17  31.39017653 30.496821822 32.2835312  19.14555134 43.63480
```

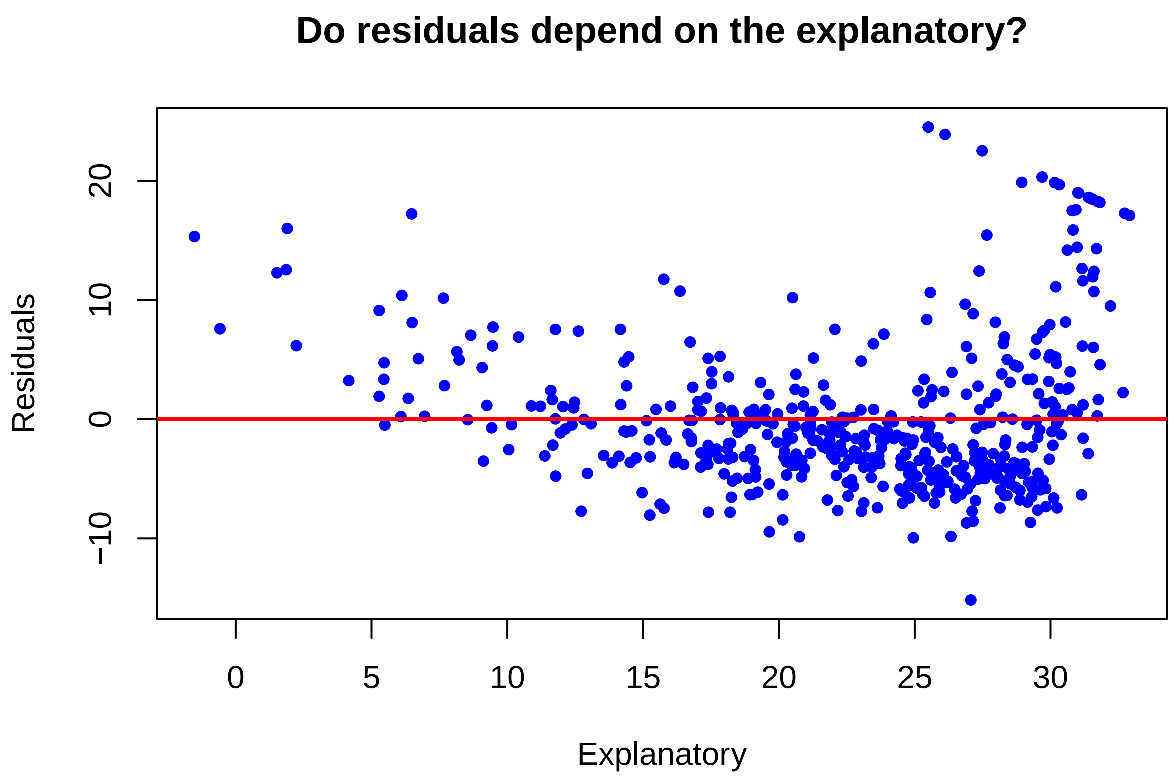
```
## 18  31.29517160 30.407848368 32.1824948  19.05098498 43.53936
## 19  31.20016666 30.318850482 32.0814828  18.95641390 43.44392
## 20  31.10516173 30.229827662 31.9804958  18.86183811 43.34849
##  [ reached 'max' / getOption("max.print") -- omitted 343 rows ]
```

```
plot(lstat, medv,
     main="Median value vs. lower status",
     xlab="Lower status in the population (%)",
     ylab="Median home value (in Ks)",
     pch=20, col="blue")
matplot(newdata, all.ints, type="l", lty=1, lwd=2, col=c("red", "purple", "purple","green","green"), ad
```

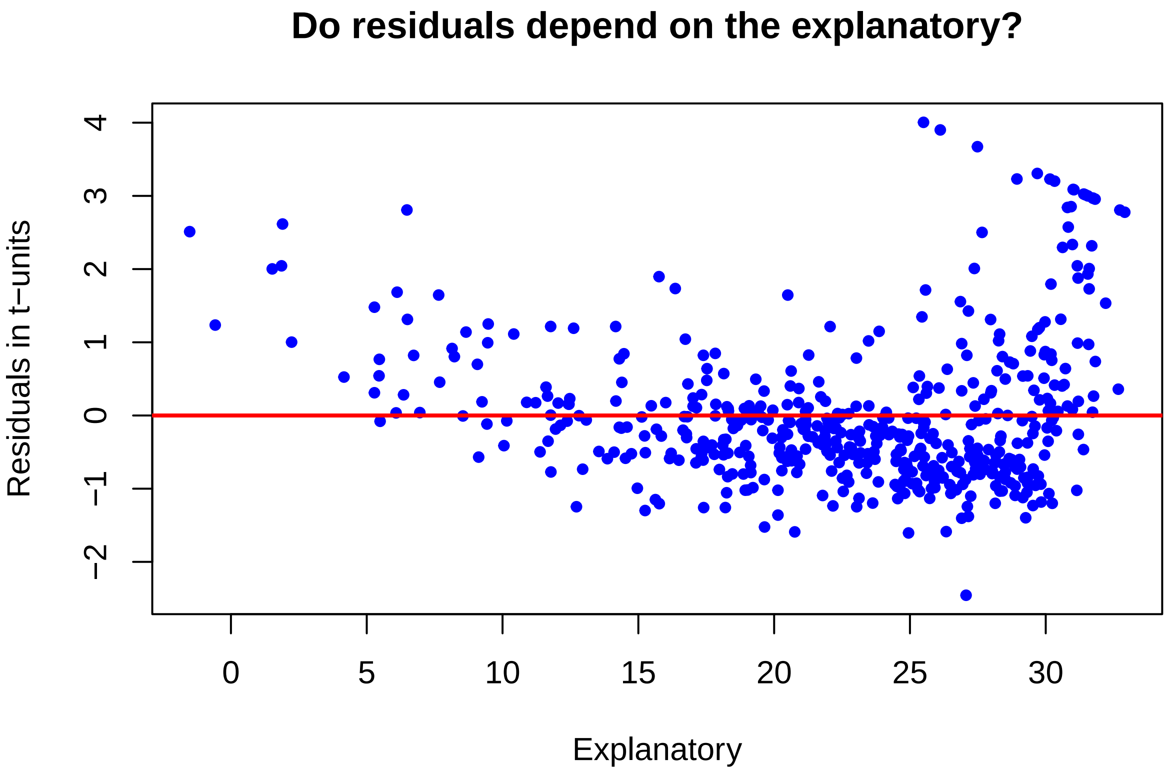


We can compute the residuals from a linear regression fit using the `residuals()` function. The function `rstudent()` will return the studentized residuals, and we can use this function to plot the residuals against the fitted values.

```
plot(predict(lm.fit), residuals(lm.fit),
     main="Do residuals depend on the explanatory?",
     xlab="Explanatory",
     ylab="Residuals",
     pch=20, col="blue")
abline(0,0, col="red", lwd=2)
```



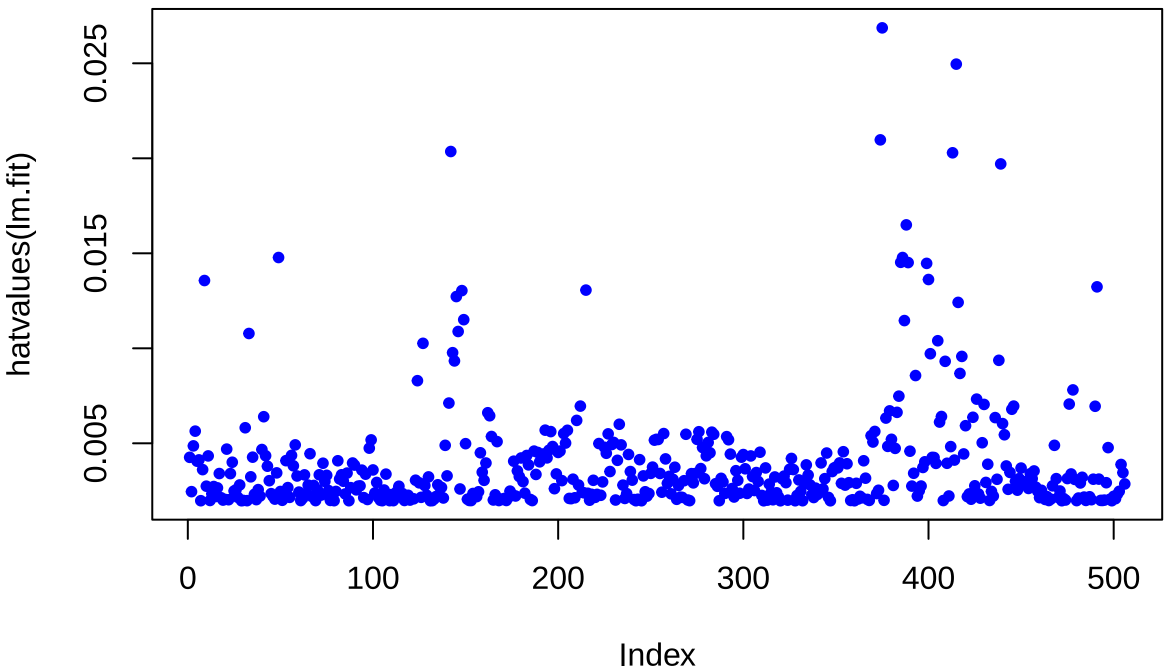

```r
plot(predict(lm.fit), rstudent(lm.fit),
     main="Do residuals depend on the explanatory?",
     xlab="Explanatory",
     ylab="Residuals in t-units",
     pch=20, col="blue")
abline(0,0, col="red", lwd=2)
```



On the basis of the residual plots, there is some evidence of non-linearity.

Leverage statistics can be computed for any number of predictors using the `hatvalues()` function.

```
plot(hatvalues(lm.fit),
     pch=20, col="blue")
```




```
which.max(hatvalues(lm.fit))
```

```
## 375
```

```
## 375
```

The `which.max()` function identifies the index of the largest element of a vector. In this case, it tells us which observation has the largest leverage statistic.

Multiple Linear Regression

In order to fit a multiple linear regression model using least squares, we again use the `lm()` function. The syntax `lm(y ~ x1 + x2 + x3)` is used to fit a model with three predictors, `x1`, `x2`, and `x3`. The `summary()` function now outputs the regression coefficients for all the predictors.

```r
lm.fit <- lm(medv ~ lstat + age, data = Boston)
summary(lm.fit)
```

```
##
## Call:
## lm(formula = medv ~ lstat + age, data = Boston)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -15.981  -3.978  -1.283   1.968  23.158
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept) 33.22276    0.73085  45.458  < 2e-16 ***
## lstat       -1.03207    0.04819 -21.416  < 2e-16 ***
## age          0.03454    0.01223   2.826  0.00491 **
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 6.173 on 503 degrees of freedom
## Multiple R-squared:  0.5513, Adjusted R-squared:  0.5495
## F-statistic:   309 on 2 and 503 DF,  p-value: < 2.2e-16
```

The `Boston` data set contains 12 variables, and so it would be cumbersome to have to type all of these in order to perform a regression using all of the predictors. Instead, we can use the following short-hand:

```r
lm.fit <- lm(medv ~ ., data = Boston)
summary(lm.fit)
```

```
##
## Call:
## lm(formula = medv ~ ., data = Boston)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -15.1304  -2.7673  -0.5814   1.9414  26.2526
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept)  41.617270   4.936039   8.431 3.79e-16 ***
## crim         -0.121389   0.033000  -3.678 0.000261 ***
## zn            0.046963   0.013879   3.384 0.000772 ***
## indus         0.013468   0.062145   0.217 0.828520
## chas          2.839993   0.870007   3.264 0.001173 **
## nox         -18.758022   3.851355  -4.870 1.50e-06 ***
## rm            3.658119   0.420246   8.705  < 2e-16 ***
```

```
## age           0.003611   0.013329   0.271 0.786595
## dis          -1.490754   0.201623  -7.394 6.17e-13 ***
## rad           0.289405   0.066908   4.325 1.84e-05 ***
## tax          -0.012682   0.003801  -3.337 0.000912 ***
## ptratio      -0.937533   0.132206  -7.091 4.63e-12 ***
## lstat        -0.552019   0.050659 -10.897  < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 4.798 on 493 degrees of freedom
## Multiple R-squared:  0.7343, Adjusted R-squared:  0.7278
## F-statistic: 113.5 on 12 and 493 DF,  p-value: < 2.2e-16
```

What if we would like to perform a regression using all of the variables but one? For example, in the above regression output, `age` has a high p-value. So we may wish to run a regression excluding this predictor. The following syntax results in a regression using all predictors except `age`.

```
lm.fit1 <- lm(medv ~ . - age, data = Boston)
summary(lm.fit1)
```

```
##
## Call:
## lm(formula = medv ~ . - age, data = Boston)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -15.1851  -2.7330  -0.6116   1.8555  26.3838
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept)  41.525128   4.919684   8.441 3.52e-16 ***
## crim         -0.121426   0.032969  -3.683 0.000256 ***
## zn            0.046512   0.013766   3.379 0.000785 ***
## indus         0.013451   0.062086   0.217 0.828577
## chas          2.852773   0.867912   3.287 0.001085 **
## nox         -18.485070   3.713714  -4.978 8.91e-07 ***
## rm            3.681070   0.411230   8.951  < 2e-16 ***
## dis          -1.506777   0.192570  -7.825 3.12e-14 ***
## rad           0.287940   0.066627   4.322 1.87e-05 ***
## tax          -0.012653   0.003796  -3.333 0.000923 ***
## ptratio      -0.934649   0.131653  -7.099 4.39e-12 ***
## lstat        -0.547409   0.047669 -11.483  < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 4.794 on 494 degrees of freedom
## Multiple R-squared:  0.7343, Adjusted R-squared:  0.7284
## F-statistic: 124.1 on 11 and 494 DF,  p-value: < 2.2e-16
```

Alternatively, the `update()` function can be used.

```
lm.fit1 <- update(lm.fit, ~ . - age)
```

Take away the next explanatory with a sizeable p−value, i.e., ***`indus`****.*

```
lm.fit2 <- lm(medv ~ . - age - indus, data = Boston)
summary(lm.fit2)
```

```
##
## Call:
## lm(formula = medv ~ . - age - indus, data = Boston)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -15.1814  -2.7625  -0.6243   1.8448  26.3920
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept)  41.451747   4.903283   8.454 3.18e-16 ***
## crim         -0.121665   0.032919  -3.696 0.000244 ***
## zn            0.046191   0.013673   3.378 0.000787 ***
## chas          2.871873   0.862591   3.329 0.000935 ***
## nox         -18.262427   3.565247  -5.122 4.33e-07 ***
## rm            3.672957   0.409127   8.978  < 2e-16 ***
## dis          -1.515951   0.187675  -8.078 5.08e-15 ***
## rad           0.283932   0.063945   4.440 1.11e-05 ***
## tax          -0.012292   0.003407  -3.608 0.000340 ***
## ptratio      -0.930961   0.130423  -7.138 3.39e-12 ***
## lstat        -0.546509   0.047442 -11.519  < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 4.789 on 495 degrees of freedom
## Multiple R-squared:  0.7342, Adjusted R-squared:  0.7289
## F-statistic: 136.8 on 10 and 495 DF,  p-value: < 2.2e-16
```

The above approach is called **backward selection**. *How would you design* **forward selection?**

Interaction Terms

It is easy to include interaction terms in a linear model using the `lm()` function. The syntax `lstat:age` tells `R` to include an interaction term between `lstat` and `age`.

```r
summary(lm(medv ~ lstat:age, data = Boston))
```

```
##
## Call:
## lm(formula = medv ~ lstat:age, data = Boston)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -13.347  -4.372  -1.534   1.914  27.193
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept) 30.1588631  0.4828240   62.46   <2e-16 ***
## lstat:age   -0.0077146  0.0003799  -20.31   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 6.827 on 504 degrees of freedom
## Multiple R-squared:  0.4501, Adjusted R-squared:  0.449
## F-statistic: 412.4 on 1 and 504 DF,  p-value: < 2.2e-16
```

```r
summary(lm(medv ~ lstat + age + lstat:age, data = Boston))
```

```
##
## Call:
## lm(formula = medv ~ lstat + age + lstat:age, data = Boston)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -15.806  -4.045  -1.333   2.085  27.552
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept) 36.0885359  1.4698355  24.553  < 2e-16 ***
## lstat       -1.3921168  0.1674555  -8.313 8.78e-16 ***
## age         -0.0007209  0.0198792  -0.036   0.9711
## lstat:age    0.0041560  0.0018518   2.244   0.0252 *
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 6.149 on 502 degrees of freedom
## Multiple R-squared:  0.5557, Adjusted R-squared:  0.5531
## F-statistic: 209.3 on 3 and 502 DF,  p-value: < 2.2e-16
```

```r
summary(lm(medv ~ lstat + lstat:age, data = Boston))
```

```
##
## Call:
## lm(formula = medv ~ lstat + lstat:age, data = Boston)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -15.815  -4.039  -1.335   2.086  27.491
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept) 36.041514   0.691334  52.133  < 2e-16 ***
## lstat       -1.388161   0.126911 -10.938  < 2e-16 ***
## lstat:age    0.004103   0.001133   3.621 0.000324 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 6.142 on 503 degrees of freedom
## Multiple R-squared:  0.5557, Adjusted R-squared:  0.554
## F-statistic: 314.6 on 2 and 503 DF,  p-value: < 2.2e-16
```

The syntax `lstat * age` simultaneously includes `lstat`, `age`, and the interaction term `lstat`×`age` as predictors; it is a shorthand for `lstat + age + lstat:age`. We can also pass in transformed versions of the predictors.

```r
summary(lm(medv ~ lstat * age, data = Boston))
```

```
##
## Call:
## lm(formula = medv ~ lstat * age, data = Boston)
##
## Residuals:
```

```
##     Min      1Q  Median      3Q     Max
## -15.806  -4.045  -1.333   2.085  27.552
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept) 36.0885359  1.4698355  24.553  < 2e-16 ***
## lstat       -1.3921168  0.1674555  -8.313 8.78e-16 ***
## age         -0.0007209  0.0198792  -0.036   0.9711
## lstat:age    0.0041560  0.0018518   2.244   0.0252 *
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 6.149 on 502 degrees of freedom
## Multiple R-squared:  0.5557, Adjusted R-squared:  0.5531
## F-statistic: 209.3 on 3 and 502 DF,  p-value: < 2.2e-16
```