M378K : January 22nd, 2025.

3. Independent events

What if knowing that an event happened in fact does **not** give any information about the probability of another event?

Definition 3.1. *We say that events E and F on Ω are independent if*

$$\mathbb{P}[E \cap F] = \mathbb{P}[E]\mathbb{P}[F].$$

In the case when E or F have a positive probability, it's possible to rewrite the above condition in a different (illustrative!) way. *How?*

$$\mathbb{P}[F \mid E] = \frac{\mathbb{P}[E \cap F]}{\mathbb{P}[E]} = \mathbb{P}[F]$$

Now that we know the notion of **independence**, we can construct random variables in many creative ways.

Example 3.2. *A fair coin is tossed repeatedly and* **independently** *until the first* Heads*. Let the random variable Y represent the total number of* Tails *observed by the end of the procedure.*

What is the support of the random variable Y?

$$S_Y = \{0, 1, 2, \dots\} = \mathbb{N}_0$$

What is the **probability mass function** *of the random variable Y?*

for $y \in S_Y$

$$p_Y(y) = \left(\frac{1}{2}\right)^{y+1}$$

Moreover, now that we remember the definition of **conditional probability**, we can solve interesting problems such as this one:

Problem 3.1. *The number of pieces of gossip that break out in a particular high school in a week is modeled by a random variable Y with the following probability mass function:*

$$p_n := p_Y(n) = \frac{1}{(n+1)(n+2)} \quad \textit{for all } n \in \mathbb{N}_0.$$

(i) Is the above a well-defined probability mass function?

(ii) Calculate the probability that at least one piece of gossip occurred in a week **given** *that at most four pieces of gossip occurred.*

$\longrightarrow$: (i) • $p_Y(n) \geq 0$ for all n ✓

• $\sum_{n=0}^{\infty} p_Y(n) \overset{?}{=} 1$

$$\sum_{n=0}^{N} p_Y(n) = \sum_{n=0}^{N} \frac{1}{(n+1)(n+2)}$$

$$= \sum_{n=0}^{N} \boxed{\left(\frac{1}{n+1} - \frac{1}{n+2}\right)}$$

$$\frac{1}{n+1} - \frac{1}{n+2} = \frac{n+2-(n+1)}{(n+1)(n+2)} = \frac{1}{(n+1)(n+2)}$$

$$= 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\cdots} + \cancel{\frac{1}{N+1}} - \frac{1}{N+2}$$

$$= 1 - \frac{1}{N+2} \xrightarrow[N\to\infty]{} 1 \quad ✓$$

(ii)

$$\mathbb{P}[Y \geq 1 \mid Y \leq 4] = \frac{\mathbb{P}[1 \leq Y \leq 4]}{\mathbb{P}[Y \leq 4]} =$$

$$= \frac{p_1 + p_2 + p_3 + p_4}{p_0 + p_1 + p_2 + p_3 + p_4}$$

$$= \frac{\frac{1}{2} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{4}} - \cancel{\frac{1}{5}} + \cancel{\frac{1}{5}} - \frac{1}{6}}{1 - \frac{1}{6}} = \frac{\frac{1}{3}}{\frac{5}{6}} = \frac{2}{5} \quad \square$$

Named Discrete Distributions.

Def'n. Bernoulli trials have two possible outcomes.
They are also known as indicators (or indicator r.v.s).
Usually, the outcomes are encoded as

$$\begin{cases} 1 & \text{for "success"} \\ 0 & \text{for "failure"} \end{cases}$$

Example. Y_i ... result of a throw of a die $(i=1,2)$

$$S_{Y_1} = S_{Y_2} = \{1,2,3,4,5,6\}$$

Define $\boxed{W = Y_1 + Y_2}$

$$S_W = \{2, \dots, 12\}$$

Define
$$\boxed{I = \begin{cases} 1 & \text{if } W \geq 9 \\ 0 & \text{if } W < 9 \end{cases}}$$

Example.
- Quality control: Say, whether a lightbulb is deficient or not.
- Insurance: An indicator of whether a deductible is met.

Example. Bernoulli Dist'n. $Y \sim B(p)$ w/ $p \in (0,1)$

y	0	1
$p_Y(y)$	$1-p$	p

Example. Say that we repeat independently the Bernoulli trials w/ the same p a fixed number of times n.
Then, we count the total number of successes Y.
Its dist'n is the binomial distribution
We write

$$\boxed{Y \sim b(n,p)}$$

$$S_Y = \{0, 1, \dots, n\}$$

for all $k = 0, \dots, n$:

$$p_Y(k) = \mathbb{P}[Y=k] = \boxed{\binom{n}{k}} p^k (1-p)^{n-k}$$

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$