M339G: January 26th, 2024.

Fact.

$$\mathbb{E}\left[(Y-\hat{f}(X))^2 \mid X=x\right] = \underbrace{(f(x)-\hat{f}(x))^2}_{\text{Reducible}} + \underbrace{Var[\varepsilon]}_{\text{Irreducible}}$$

→ :

By the model:

$\boxed{Y = f(X) + \varepsilon}$ w/ ε is independent from X and $\boxed{\mathbb{E}[\varepsilon]=0}$

$$\mathbb{E}\left[(f(X)+\varepsilon-\hat{f}(X))^2 \mid X=x\right] = \quad \text{(linearity of expectation)}$$

$$= \mathbb{E}\left[(f(X)-\hat{f}(X))^2 \mid X=x\right]$$

$$+ 2\,\mathbb{E}\left[(f(X)-\hat{f}(X))\cdot\varepsilon \mid X=x\right] \quad (\varepsilon \text{ indep. of } X)$$

$$+ \boxed{\mathbb{E}\left[\varepsilon^2 \mid X=x\right]}$$

$$= (f(x)-\hat{f}(x))^2 + Var[\varepsilon]$$

□

In general: for any r.v. W

$$Var[W] = \mathbb{E}[W^2] - (\mathbb{E}[W])^2$$

$$\mathbb{E}[W^2] = Var[W] + (\mathbb{E}[W])^2$$

We take $W = \varepsilon$ and note $\mathbb{E}[\varepsilon]=0$

$$\Rightarrow \quad \boxed{\mathbb{E}[\varepsilon^2] = Var[\varepsilon]}$$