NAME:

M339W/389W Financial Mathematics for Actuarial Applications
University of Texas at Austin
In-Term Exam II
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The maximum number of points on this exam is 100.

Problem 1.1. (10 points) Assume the Black-Scholes model. Let the current price of a continuous-dividend-paying stock be \$40. The stock's volatility is 0.20 and its dividend yield is 0.02.

The continuously compounded, risk-free interest rate is 0.05.

Find the price of a \$42-strike, half-year European put option on the above stock.

Solution: In our usual notation, we have

$$d_1 = \frac{1}{0.2\sqrt{1/2}}\left[\ln\left(\frac{40}{42}\right) + \left(0.05 - 0.02 + \frac{(0.2)^2}{2}\right)\left(\frac{1}{2}\right)\right] = -0.1682 \approx -0.17,$$
$$d_2 = d_1 - \sigma\sqrt{T} = -0.3096 \approx -0.31.$$

Using the standard normal tables, we obtain

$$N(-d_1) = N(0.17) = 0.5675,$$
$$N(-d_2) = N(0.31) = 0.6217.$$

So, the Black-Scholes price of this call option is

$$\begin{aligned} v_P(S(0),0) &= Ke^{-rT}N(-d_2) - S(0)e^{-\delta T}N(-d_1) \\ &= 42e^{-0.05/2}(0.6217) - 40e^{-0.02/2}(0.5675) = 2.99258. \end{aligned}$$

Problem 1.2. (5 points) Assume the Black-Scholes model. Let the current stock price be equal to \$90 per share. Its dividend yield is 0.02 and its volatility is 0.20.

The continuously compounded, risk-free interest rate is 0.03.

Consider a one-year European call option on the above stock. The delta of this call option is 0.49. What is the strike price of the call?

Solution: In our usual notation, the delta of the call is

$$\Delta_C(S(0),0) = e^{-\delta T}N(d_1(S(0),0)) = e^{-0.02}N(d_1(S(0),0)) = 0.49.$$

So,

$$N(d_1(S(0),0)) = 0.49e^{0.02} = 0.499899 \approx 0.5 \quad \Rightarrow \quad d_1(S(0),0) = 0.$$

Hence,

$$\ln\left(\frac{S(0)}{K}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right) = 0 \quad \Rightarrow \quad \ln\left(\frac{S(0)}{K}\right) = -\left(0.03 - 0.02 + \frac{0.04}{2}\right) = -0.03.$$

Therefore,

$$\frac{S(0)}{K} = e^{-0.03} \quad \Rightarrow \quad K = S(0)e^{0.03} = 90e^{0.03} = 92.7409.$$

Problem 1.3. (10 points) The current price of a continuous-dividend-paying stock is given to be \$92. The stock's volatility is 0.35 and its dividend yield is 0.02.

The continuously-compounded, risk-free interest rate is 0.05.

Consider a \$90-strike European call option on the above stock with exercise date in a quarter-year. What is the Black-Scholes price of this call option?

Solution: In our usual notation,

$$d_1 = \frac{1}{0.35\sqrt{0.25}}\left[\ln\left(\frac{92}{90}\right) + \left(0.05 - 0.02 + \frac{(0.35)^2}{2}\right)\frac{1}{4}\right] = 0.255951,$$
$$d_2 = d_1 - \sigma\sqrt{T} = 0.255951 - 0.35(0.5) = 0.080951.$$

Using the standard normal tables, we get

$$N(d_1) = N(0.26) = 0.6026 \quad \text{and} \quad N(d_2) = N(0.08) = 0.5319.$$

Finally, the Black-Scholes call price equals

$$V_C(0) = S(0)e^{-\delta T}N(d_1) - Ke^{-rT}N(d_2) = 92e^{-0.02(0.25)}(0.6026) - 90e^{-0.05(0.25)}(0.5319) = 7.88636.$$

Problem 1.4. (5 points) Assume the Black-Scholes model. For a particular stock option, you are given that its price today equals \$11.84. You are given the following values of its Greeks today:

- the option's delta is 0.6122;
- the option's gamma is 0.0153;
- the option's theta is -0.0188 **per day**.

Approximately, what will this option's value be in a day should the stock price increase by 0.50?

Solution: By the delta-gamma-theta approximation, we have that

$$v(S(dt), dt) \approx v(S(0), 0) + \Delta(S(0), 0)ds + \frac{1}{2}\Gamma(S(0), 0)(ds)^2 + \Theta(S(0), 0)dt$$
$$= 11.84 + (0.6122)(0.5) + \frac{1}{2}(0.0153)(0.5)^2 + (-0.0188) = 12.1292.$$

Problem 1.5. (10 points) Assume the Black-Scholes framework.

The goal is to delta-hedge a written one-year, $(40, 60)$-strangle on a non-dividend-paying stock whose current price is \$50. The stock's volatility is 0.20.

The continuously compounded, risk-free interest rate is 0.10.

What is the cost of delta-hedging the strangle using shares of the underlying stock?

Solution: The Δ of the strangle equals

$$\Delta_P(S(0), 0; K_P = 40) + \Delta_C(S(0), 0; K_C = 60),$$

i.e., it is the sum of the delta of the call with strike 60 and the delta of the put with strike 40. We have

$$d_1(S(0), 0; K_P = 40) = \frac{1}{0.2}\left[\ln\left(\frac{50}{40}\right) + \left(0.10 + \frac{0.04}{2}\right)\right] = 1.715.$$

So, the put's delta is approximately

$$-N(-d_1(S(0), 0; K_P = 40)) = -N(-1.72) = N(1.72) - 1 = -0.0427.$$

Similarly, for the call, we have

$$d_1(S(0), 0; K_C = 60) = \frac{1}{0.2}\left[\ln\left(\frac{50}{60}\right) + \left(0.10 + \frac{0.04}{2}\right)\right] = -0.31.$$

So, the call's delta is approximately

$$N(d_1(S(0), 0; K_C = 60)) = N(-0.31) = 1 - N(0.31) = 0.3783.$$

Our answer is

$$50(0.3783 - 0.0427) = 16.78.$$

Problem 1.6. (5 points) Which of the following statements is always TRUE?

(a) The call rho is greater than the put rho.
(b) The put theta is always negative.
(c) The call vega is the negative of the vega of the otherwise identical put.
(d) The call psi is always positive.
(e) None of the above.

Solution: (a)

Problem 1.7. (15 points) Assume the Black-Scholes model. The current stock price is \$60 per share. The stock's dividend yield is 0.02 and its volatility is 0.3.

Consider a \$70-strike, half-year European call option on this stock. Its price is \$2.40, and its delta is 0.3.

The continuously compounded, risk-free interest rate is 0.04.

What is the volatility of the otherwise identical put option?

Solution: The volatility of the put option is

$$\sigma_P = |\Omega_P|\sigma.$$

In our usual notation, we are given that

$$\Delta_C(S(0), 0) = e^{-0.02(0.5)}N(d_1(S(0), 0)) = 0.3$$

On the other hand, for the put, we have

$$\begin{aligned}\Delta_P(S(0), 0) &= -e^{-0.02(0.5)}N(-d_1(S(0), 0)) \\ &= -e^{-0.01}(1 - N(d_1(S(0), 0))) = -e^{-0.01} + \Delta_C(S(0), 0) = -0.69005.\end{aligned}$$

By put-call parity,

$$v_P(S(0), 0) = v_C(S(0), 0) - S(0)e^{-\delta T} + Ke^{-rT} = 2.40 - 60e^{-0.01} + 70e^{-0.02} = 11.61.$$

So, the put's elasticity is

$$\Omega_P(S(0), 0) = \frac{\Delta_P(S(0), 0)S(0)}{v_P(S(0), 0)} = \frac{-0.69005(60))}{11.61} = -3.56615.$$

Finally, $\sigma_P = 3.56615(0.3) = 1.06984$.

Problem 1.8. (15 points) Assume that the Black-Scholes setting holds. Let $\mathbf{S} = \{S(t),\, t \geq 0\}$ denote the price of a continuous-dividend-paying stock. The stock price today equals \$80 and its dividend yield is 0.02.

Let $r = 0.04$ be the continuously compounded risk-free interest rate.

Consider a European call option with exercise in three months and strike price $K = 80e^{0.005}$. You are given that its price today equals \$3.80.

What is the implied volatility of the stock S?

Solution: According to the Black-Scholes formula, in our usual notation, the time-0 call price equals

$$v_C(S(0), 0) = v_C(0, S(0), K, r, \delta, \sigma, T) = S(0)e^{-\delta T}N(d_1) - Ke^{-rT}N(d_2)$$

with

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{T}}[\ln(S(0)/K) + (r - \delta + \tfrac{1}{2}\sigma^2)T], \\ d_2 &= d_1 - \sigma\sqrt{T}. \end{aligned}$$

Using the provided information, we obtain

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{\frac{1}{4}}}\left[\ln(S(0)/S(0)e^{-0.005}) + (0.04 - 0.02 + \tfrac{1}{2}\sigma^2)\frac{1}{4}\right] \\ &= \frac{1}{\sigma\sqrt{\frac{1}{4}}}\left[-0.005 + 0.005 + \tfrac{1}{2}\sigma^2 \cdot \frac{1}{4}\right] = \frac{\sigma}{4}\,, \\ d_2 &= d_1 - \frac{\sigma}{2} = \frac{\sigma}{4} = -d_1. \end{aligned}$$

Hence,

$$\begin{aligned} v_C(\ldots, \sigma) &= S(0)e^{-\delta T}N(d_1) - S(0)e^{0.005}e^{-rT}N(d_2) \\ &= 80e^{-0.005}[N(d_1) - N(-d_1)] \\ &= 80e^{-0.005}[2N(\tfrac{\sigma}{4}) - 1]. \end{aligned}$$

From the problem, we know that

$$3.80 = 80e^{-0.005}[2N(\tfrac{\sigma}{4}) - 1].$$

So,

$$0.0475e^{0.005} = 2N(\tfrac{\sigma}{4}) - 1 \quad \Rightarrow \quad 2N(\tfrac{\sigma}{4}) = 1.0477381 \quad \Rightarrow \quad N(\tfrac{\sigma}{4}) = 0.523869$$

From the standard normal table, we get that

$$\tfrac{\sigma}{4} = 0.06 \;\Rightarrow\; \sigma = 0.24.$$

Problem 1.9. (5 points) The current price of a non-dividend-paying stock is \$40 per share. A market-maker writes a one-year European put option on this stock and proceeds to delta-hedge it.

The put premium is \$5.96, its delta is -0.5753, its gamma is 0.0392, and its theta is 0.01 per day.

The continuously-compounded, risk-free interest rate is 0.04.

Assuming that the stock price does not change, what is the **approximate** overnight profit for the market-maker?

Solution: The initial cost of the total delta-hedged portfolio is

$$-5.96 + (-0.5753)(40) = -28.972.$$

The approximate put price after one day is, according to the delta-gamma-theta approximation,

$$5.96 + 0.01 = 5.97.$$

So, the overnight profit is

$$-5.97 + (-0.5753)(40) + 28.972e^{0.04/365} = -0.00682481$$

Problem 1.10. (10 points) Assume the Black-Scholes model. The current price of a continuous-dividend-paying stock is denoted by $S(0)$. Its dividend yield is denoted by δ and its volatility is $\sigma = 0.20$.

The continuously compounded, risk-free interest rate is equal to δ.

Consider a one-year, at-the-money European put on the above stock. What is the elasticity of this put?

Solution: In our usual notation, since the option is at-the-money,

$$\begin{aligned} d_1(S(0),0) &= \frac{1}{0.2}\left(r - \delta + \frac{(0.2)^2}{2}\right) = 0.1, \\ d_2(S(0),0) &= d_1(S(0),0) - \sigma = 0.1 - 0.2 = -0.1. \end{aligned}$$

So, the options delta is

$$\Delta_P(S(0),0) = -e^{-\delta}N(-d_1(S(0),0)) = -e^{-\delta}N(-0.1).$$

The put option's price is

$$\begin{aligned} v_P(S(0),0) &= S(0)e^{-r}N(-d_2(S(0),0)) - S(0)e^{-\delta}N(-d_1(S(0),0)) \\ &= S(0)e^{-r}(N(0.1) - N(-0.1)) = S(0)e^{-r}(2N(0.1) - 1). \end{aligned}$$

Hence, the elasticity is

$$\Omega_P(S(0),0) = \frac{\Delta_P(S(0),0)S(0)}{v_C(S(0),0)} = \frac{-e^{-\delta}N(-0.1)S(0)}{S(0)e^{-r}(2N(0.1)-1)} = -\frac{1-N(0.1)}{2N(0.1)-1} = -5.78141.$$

Problem 1.11. (10 points) Assume the Black-Scholes model. Let the current price of a non-dividend-paying stock be \$100. Its volatility is given to be 0.25.

The continuously compounded, risk-free interest rate is 0.02.

Under the risk-neutral probability, the probability that a one-year, European call option is in the money at expiration is 0.484.

What is the current gamma of this call option?

Solution: We are given that, in our usual notation,

$$N(d_2(S(0),0)) = 0.484 \quad \Rightarrow \quad d_2(S(0),0) = -0.04.$$

So, $d_1(S(0),0) = -0.04 + 0.25 = 0.21$. Using the IFM formula sheet, we get

$$\Gamma_C(S(0),0) = \frac{N'(d_1(S(0),0))}{S(0)\sigma} = \frac{\frac{1}{\sqrt{2\pi}}e^{-0.21^2/2}}{100(0.25)} = 0.0156097.$$