UNIVERSITY OF TEXAS AT AUSTIN

Lecture 1

The cumulative distribution function and related concepts

1.1. Cumulative distribution function.

Definition 1.1. The **cumulative distribution function** (also called the **distribution function**) $F_X : \mathbb{R} \to [0, 1]$ of a random variable X is defined as

$$F_X(x) = \mathbb{P}[X \leq x], \quad \text{for } x \in \mathbb{R}$$

Conventions: We usually abbreviate "cumulative distribution function" to **cdf**.

It is customary to label (in the right subscript) the cdf by the random variable to which it "belongs"

Properties: Draw a graph!!!

- Codomain is $[0, 1]$
- Nondecreasing
- Right-continuous with left limits
- $$lim_{x\to-\infty} F_X(x) = 0$$
- $$lim_{x\to\infty} F_X(x) = 1$$

1.2. Survival function.

Definition 1.2. The **survival function** $S_X : \mathbb{R} \to [0, 1]$ of a random variable X is defined as

$$S_X(x) = 1 - F_X(x) = \mathbb{P}[X > x], \quad \text{for } x \in \mathbb{R}$$

Properties:

- Codomain is $[0, 1]$
- Nonincreasing
- Right-continuous
- $$lim_{x\to-\infty} S_X(x) = 1$$
- $$lim_{x\to\infty} S_X(x) = 0$$

Conventions. It is customary to label (in the right subscript) the survival function by the random variable to which it "belongs"

1.3. **Support.** This is not a proper definition of the *support* of a random variable, but it should serve our purposes.

Definition 1.3. The **support** of a random variable X is defined as the set of numbers that are possible values of the random variable.

If you look at X as a function from the set of elementary outcomes to the real numbers, then you can sort of imagine its support as the **image** of X.

1.4. **Quantiles.**

Definition 1.4. The $100p^{th}$ quantile/percentile of a random variable X is any value π_p such that

$$F_X(\pi_P-) \leq p \leq F_X(\pi_p).$$

In particular, the 50^{th} percentile is called the **median** of X.

Example 1.5. For distributions with a strictly increasing cumulative distribution function,

$$\pi_p = F_X^{-1}(p)$$

1.5. **Value at Risk.** Let p denote the probability of an adverse event that you - the insurance company - are "comfortable with", e.g., the probability with which you are "willing to" have a negative balance in the end.

Let X be a severity random variable, i.e., the random variable modeling the ground-up loss. Note that the adverse event for you - the insurance company - takes place when X is large.

One *risk measure* we can introduce is the following:

Definition 1.6. For a random variable X, the **value at risk** at the level p is the constant $VaR_p(X)$ such that

$$\mathbb{P}[X > VaR_p(X)] = p$$

In other words, the value at risk is just another term for the $100(1-p)^{th}$ quantile.

Note. If the random variable in question modeled, say, return on investment, we would be interested in the lower tail rather than the upper tail.