M339J: February 14th, 2022.

Per payment and per loss random variables.

Review:

Def'n. Let X be a loss random variable.
Let d be a positive constant such that

$$\mathbb{P}[X > d] > 0.$$

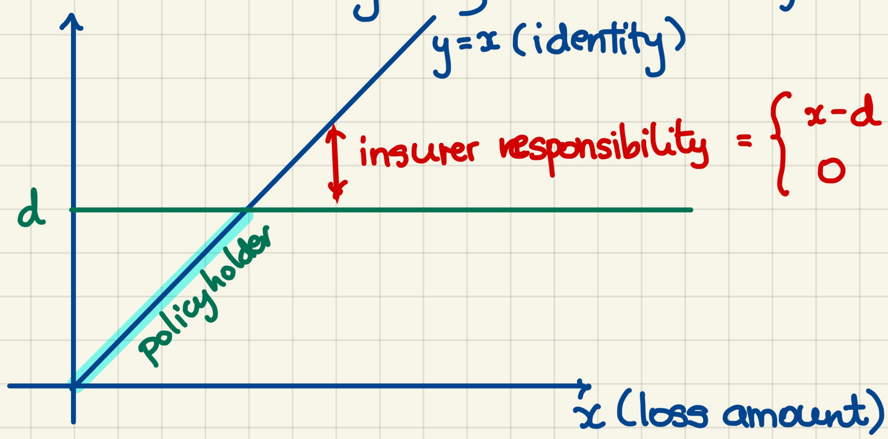
The excess loss (random) variable, usually denoted by Y^P, is defined as:

$$\boxed{Y^P = X - d \mid X > d}$$

The actuarial context:

X ... (ground-up) loss; a random variable w/ a nonnegative support

d ... deductible, i.e., the amount up to which the policyholder is responsible for the losses; the insurance company is responsible for everything in excess of the (ordinary) deductible



$$\text{insurer responsibility} = \begin{cases} x-d & \text{if } x > d \\ 0 & \text{if } x \le d \end{cases}$$

We refer to the random variable Y^P as the per payment (random) variable.

Example. $X \sim \text{Exponential}(\theta)$

What's the distribution of the per payment random variable Y^P?

$\longrightarrow$: Support of Y^P is $\boxed{[0,+\infty)}$.

Let's figure out its survival function.

For $y>0$:

$$S_{Y^P}(y) = \mathbb{P}[Y^P > y]$$
$$= \mathbb{P}[X-d > y \mid X>d] =$$
$$= \mathbb{P}[X > d+y \mid X>d] = \text{(memoryless property)}$$
$$= \mathbb{P}[X>y] = S_X(y)$$

$$\Rightarrow \quad Y^P \sim \text{Exponential}(\theta) \qquad \blacksquare$$

Recall: The mean excess loss function is $\boxed{e_X(d) = \mathbb{E}[Y^P]}$

Def'n. The left censored and shifted random variable, usually denoted by Y^L, is defined by

$$\boxed{Y^L = \begin{cases} X-d & \text{if } X>d \\ 0 & \text{if } X \le d \end{cases}}$$

Using the indicator random variables; we can write

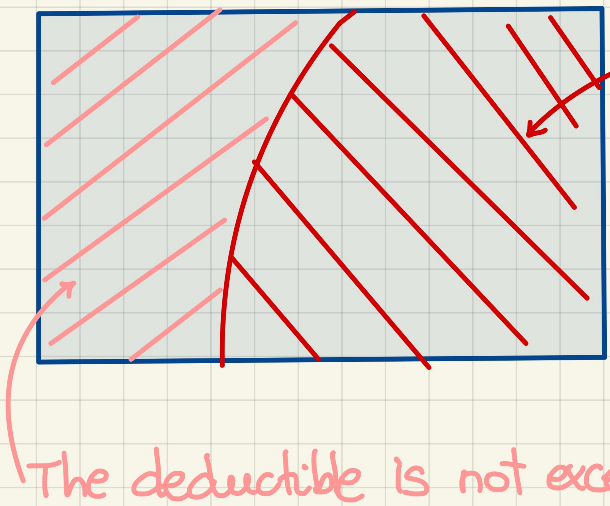
$$\boxed{Y^L = (X-d)\cdot \mathbb{I}_{[X>d]}}$$

Introduce: the positive part function

$$\xi \longmapsto (\xi)_+ := \begin{cases} \xi & \text{if } \xi>0 \\ 0 & \text{if } \xi \le 0 \end{cases}$$

We can write $\boxed{Y^L = (X-d)_+}$

Y^L is usually called the per loss (random) variable.



The deductible is exceeded: Y^P is only defined here.

Ω... all possible scenarios

The deductible is not exceeded: Y^L equals zero here.

Example. $X \sim \text{Exponential}(\theta)$

What's the dist'n of the per loss random variable Y^L?

$\longrightarrow$: What's the support of Y^L?

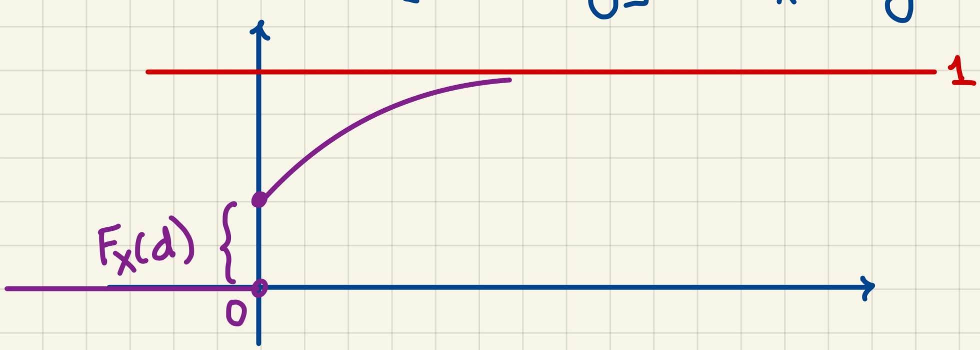
$[0, +\infty)$

Q: What is the probability $\mathbb{P}[Y^L = 0]$?

$\longrightarrow$:
$$\mathbb{P}[Y^L = 0] = \mathbb{P}[X \leq d] = F_X(d) = \boxed{1 - e^{-\frac{d}{\theta}}} > 0$$

Consider $y > 0$:

$$F_{Y^L}(y) = \mathbb{P}[Y^L \leq y] \quad \text{(law of total probability)}$$
$$= \mathbb{P}[Y^L \leq y, X \leq d] + \mathbb{P}[Y^L \leq y, X > d]$$
$$= \mathbb{P}[0 \leq y, X \leq d] + \mathbb{P}[X - d \leq y, X > d]$$
$$= \mathbb{P}[X \leq d] + \mathbb{P}[d < X \leq d + y]$$
$$= \mathbb{P}[X \leq d + y] = F_X(d+y) = 1 - e^{-\frac{d+y}{\theta}}$$



Q: Is it important that X is exponential?

→: Not much :)

Q: Assume that X is continuous. What type of a random variable is Y^P?

→: continuous.

But: Y^L is mixed.