M378K Introduction to Mathematical Statistics
Homework assignment #8

Please, provide your **final answer only** to the following problems.

Problem 8.1. *(5 points) Which of the following estimators is* ***not*** *unbiased for μ if $Y_1, \dots, Y_n$ is a random sample from the normal distribution $N(\mu, \sigma)$:*

(a) Y_n

(b) $\frac{1}{2}(Y_1 + Y_2)$

(c) $Y_1 - Y_2 + Y_3$

(d) $\bar{Y}$

(e) **All of the above are unbiased.**

Problem 8.2. *(5 points) Let $Y_1, \dots, Y_n$ be a random sample of size $n \geq 2$, from $N(\mu, \sigma)$ and let the estimators $\hat{\mu}_1$, $\hat{\mu}_2$ and $\hat{\mu}_3$, for μ, be given by*

$$\hat{\mu}_1 = Y_1, \hat{\mu}_2 = \frac{1}{2}(Y_1 + Y_2) \text{ and } \hat{\mu}_3 = \bar{Y}.$$

Then, no matter what μ and σ are, we always have

(a) $MSE(\hat{\mu}_1) \leq MSE(\hat{\mu}_2) \leq MSE(\hat{\mu}_3)$

(b) $MSE(\hat{\mu}_3) \leq MSE(\hat{\mu}_2) \leq MSE(\hat{\mu}_1)$

(c) $MSE(\hat{\mu}_3) \leq MSE(\hat{\mu}_1) \leq MSE(\hat{\mu}_2)$

(d) $MSE(\hat{\mu}_1) \leq MSE(\hat{\mu}_3) \leq MSE(\hat{\mu}_2)$

(e) **None of the above.**

Please, provide your **complete solutions** to the following problems. Final answers only, even if correct will earn zero points for those problems.

Problem 8.3. *(30 points) Let (Y_1, Y_2) be a random sample (of size $n = 2$) from the uniform distribution $U(0, \theta)$, with $\theta > 0$ unknown.*

1. *($2 + 3 + 10 = 15$ points) Find constants c_1, c_2 and c_3 such that the following estimators*

$$\hat{\theta}_1 = c_1 Y_1,\ \hat{\theta} = c_2 Y_2 \text{ and } \hat{\theta}_3 = c_3 \max(Y_1, Y_2),$$

   are unbiased. (Hint: For $\hat{\theta}_3$, integrate the function $\max(y_1, y_2)$ multiplied by the joint density of Y_1, Y_2. Split the integral over $[0, \theta] \times [0, \theta]$ into two parts - one where $y_1 \geq y_2$ and the other where $y_1 < y_2$ and note that $\max(y_1, y_2) = y_1 1_{\{y_1 \geq y_2\}} + y_2 1_{\{y_1 < y_2\}}$.)

2. *($2 + 3 + 10 = 15$ points) With values c_1, c_2 and c_3 as above, compute mean-squared errors MSE($\hat{\theta}_1$), MSE($\hat{\theta}_2$) and MSE($\hat{\theta}_3$) of $\hat{\theta}_1$, $\hat{\theta}_2$ and $\hat{\theta}_3$.*

3. *(10 points) Sketch the graphs of MSE($\hat{\theta}_1$), MSE($\hat{\theta}_2$) and MSE($\hat{\theta}_3$) as functions of θ. Is one of the three clearly better (in the mean-square sense) than the others?*