M378K: October 6th, 2025.

The CDF Method in 2D.

Goal: We want to find the density f_W of a r.v.

$$\boxed{W = g(Y_1, Y_2)}$$

where (Y_1, Y_2) are jointly continuous w/ pdf $\boxed{f_{Y_1,Y_2}}$

$\longrightarrow$: $F_W(w) = \mathbb{P}[\boxed{W} \leq w] = \mathbb{P}[g(Y_1,Y_2) \leq w] \underset{?}{=} \mathbb{P}[(Y_1,Y_2) \in A]$

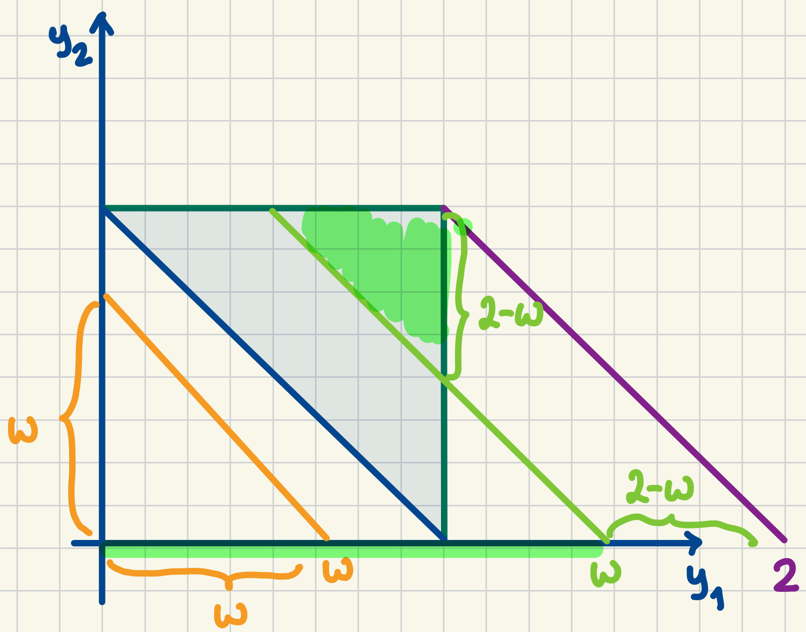
$$A = \{(y_1,y_2) \in \mathbb{R}^2 : g(y_1,y_2) \leq w\}$$

$$\boxed{F_W(w) = \iint_A f_{Y_1,Y_2}(y_1,y_2)\, dy_2\, dy_1}$$

Example. Say (Y_1,Y_2) represent points chosen @ random in a unit square $[0,1] \times [0,1] = [0,1]^2$

$$\boxed{f_{Y_1,Y_2}(y_1,y_2) = \mathbb{1}_{[0,1]^2}(y_1,y_2)}$$

Define $\boxed{W = Y_1 + Y_2}$ i.e., $\boxed{g(y_1,y_2) = y_1 + y_2}$



for $w<0$: $F_W(w)=0$

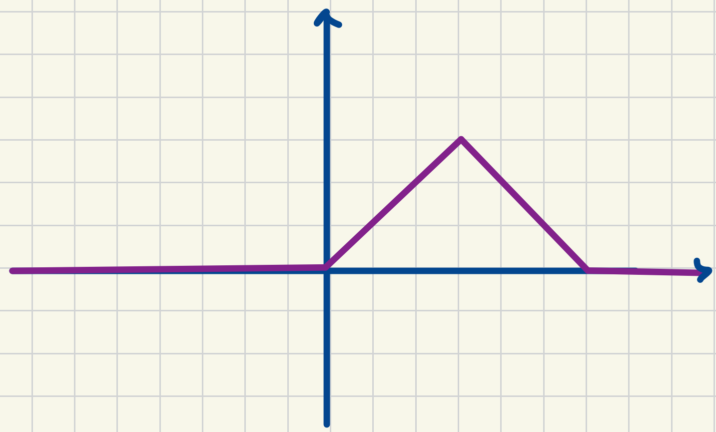
for $0\le w<1$: $F_W(w)=\frac{1}{2}w^2$

for $w=1$: $F_W(1)=\frac{1}{2}$

for $1<w\le 2$: $F_W(w)=1-\frac{(2-w)^2}{2} = -1+2w-\frac{1}{2}w^2$

for $w>2$: $F_W(w)=1$

$$f_W(w) = \begin{cases} 0 & w<0 \\ w & w\in[0,1) \\ 2-w & w\in[1,2] \\ 0 & w>2 \end{cases}$$

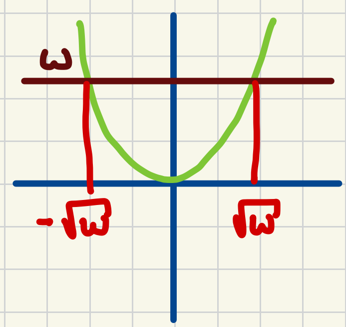


Example. Let $Y \sim N(0,1)$

Set $W=Y^2$, i.e., $W=g(Y)$ w/ $g(y)=y^2$

For all $w\le 0$: $F_W(w)=0$

For all $w>0$:



$$\begin{aligned} F_W(w) &= \mathbb{P}[W\le w] = \mathbb{P}[Y^2\le w] \\ &= \mathbb{P}[-\sqrt{w}\le Y\le \sqrt{w}] \\ &= F_Y(\sqrt{w}) - F_Y(-\sqrt{w}) \\ &= \Phi(\sqrt{w}) - \Phi(-\sqrt{w}) \\ &= 2\Phi(\sqrt{w})-1 \end{aligned}$$

By symmetry of $N(0,1)$

$\Phi(-\sqrt{w}) = 1-\Phi(\sqrt{w})$

for $w>0$:

$$f_W(w) = \frac{d}{dw} F_W(w)$$

$$= \frac{d}{dw}\left(2\Phi(\sqrt{w}) - 1\right)$$

$$= \cancel{2}\cdot \varphi(\sqrt{w}) \cdot \frac{1}{\cancel{2}\sqrt{w}} = \frac{1}{\sqrt{w}}\varphi(\sqrt{w})$$

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \quad \text{for all } z \in \mathbb{R}$$

$$f_W(w) = \frac{1}{\sqrt{w}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{w}{2}} \quad \text{for } w>0$$

$$\boxed{f_W(w) = \frac{1}{\sqrt{2\pi w}} e^{-\frac{w}{2}} \cdot \mathbb{1}_{(0,\infty)}(w)}$$

W is said to have the χ^2-distribution w/ 1 degree of freedom

$$\boxed{W \sim \chi^2(df=1)}$$

More generally, for $Y_1, Y_2, \dots, Y_K$ independent, standard normal r.v.s,

set $\boxed{W = Y_1^2 + Y_2^2 + \dots + Y_K^2}$

We say that $\boxed{W \sim \chi^2(df=K)}$