

Notes: This is a closed book and closed notes exam. The maximum number of points is 50.

Time: 50 minutes

Problem 3.1. (5 points) Assume the Black-Scholes model. Let the current price of a non-dividend-paying stock be equal to \$80 per share. Its volatility is 0.20.

The continuously compounded risk-free interest rate is 0.04.

Consider a one-year, at-the-money European call option on the above stock. The current delta of the call option is 0.6179. What is the current gamma of the call option?

- (a) 0
- (b) 0.01256
- (c) 0.0238
- (d) 0.03862
- (e) None of the above.

Solution: (c)

Using our IFM tables, or differentiating, we know that

$$\Gamma_C(S(0), 0) = \frac{e^{-\delta T} N'(d_1(S(0), 0))}{S(0)\sigma\sqrt{T}} = \frac{N'(d_1(S(0), 0))}{S(0)\sigma}$$

From the given value of the call's delta, we have that

$$N(d_1(S(0), 0)) = 0.6179 \quad \Rightarrow \quad d_1(S(0), 0) = 0.3.$$

So, the gamma of the call equals

$$\Gamma_C(S(0), 0) = \frac{N'(0.3)}{80(0.2)} = \frac{\frac{1}{\sqrt{2\pi}}e^{-\frac{0.09}{2}}}{16} = 0.0238367.$$

Problem 3.2. A market-maker sells option *I* for \$10. This option's delta is 0.6557 and its gamma is 0.02. The market maker proceeds to delta-gamma hedge this commitment by trading in the underlying and also in option *II* on the same stock. The latter option's price is \$4.70, its delta is 0.5794 and its gamma is 0.04.

What is the market-maker's resulting position in option *II*?

- (a) Buy 0.5 of option *II*.
- (b) Write 0.5 of option *II*.
- (c) Buy 2 of option *II*.
- (d) Write 2 of option *II*.

(e) None of the above.

Solution: (a)

With n_{II} denoting the position in option II , to achieve gamma-neutrality we need

$$-0.02 + n_{II}(0.04) = 0 \quad \Rightarrow \quad n_{II} = 0.5.$$

Problem 3.3. Assume the Black-Scholes model. The current price of a particular stock is \$100 per share.

Here is some information about the current prices and Greeks for a pair of European call options on this stock:

Strike price	80	90
Price	4.32	2.15
Δ	0.34	0.24

What is the current elasticity of the (80, 90)–bull spread constructed using the above options?

- (a) 4.61
- (b) 5.28
- (c) 6.34
- (d) 7.16
- (e) None of the above.

Solution: (a)

The above call bull spread consists of a long 80–strike call and a short 90–strike call. So, its price and its delta are

$$v(S(0), 0) = 4.32 - 2.15 = 2.17 \quad \text{and} \quad \Delta(S(0), 0) = 0.34 - 0.24 = 0.1.$$

So, the elasticity of the call bull spread equals

$$\Omega(S(0), 0) = \frac{\Delta(S(0), 0)S(0)}{v(S(0), 0)} = \frac{0.1(100)}{2.17} = 4.60829.$$

Problem 3.4. (5 points) Assume the Black-Scholes framework for the evolution of a stock price. The stock pays no dividends. Consider a one-year European call on this stock. You are given the following:

- the call's delta is 0.6591,
- under the risk-neutral probability measure the probability that the option is in the money at expiration is 0.3409.

What is the volatility of this call option?

- (a) 0.3409
- (b) 1.6985
- (c) 2.0713
- (d) 3.0257

(e) None of the above.

Solution: (b)

The volatility of the call option is

$$\sigma_C = \Omega_c \sigma.$$

We are given that

$$N(d_1) = 0.6591 \quad \text{and} \quad N(d_2) = 0.3409.$$

So, using the standard normal table, we get

$$d_1 = -d_2 = 0.41$$

Hence,

$$\sigma = d_1 - d_2 = 0.82.$$

Next, we need to calculate the elasticity of the call option. So,

$$\Omega_C = \frac{S(0)\Delta_C}{V_C(0)} = \frac{S(0)N(d_1)}{S(0)N(d_1) - Ke^{-r}N(d_2)} = \frac{1}{1 - \frac{Ke^{-r}}{S(0)} \times \frac{N(d_2)}{N(d_1)}}$$

Reusing the given value of the delta, we get

$$d_1 = 0.41 = \frac{1}{0.82} \left[\ln \left(\frac{S(0)}{Ke^{-r}} \right) + \frac{(0.82)^2}{2} \right] \Rightarrow \ln \left(\frac{S(0)}{Ke^{-r}} \right) = 0.$$

So,

$$\Omega_C = \frac{1}{1 - \left(\frac{0.3409}{0.6591} \right)} = 2.07134.$$

Finally, $\sigma_C = 2.07134(0.82) = 1.6985$.

Problem 3.5. (5 points) Assume the Black-Scholes framework for the pair of stocks **S** and **Q**.

For the stock **S**, you are given that

- the current stock price is \$40 per share;
- the stock pays dividends in the amount $0.05S(t) dt$ during the time period $(t, t + dt)$;
- the stock's volatility is 0.2.

For the stock **Q**, you are given that

- the current stock price is \$40 per share;
- the stock pays no dividends;
- the stock's volatility is 0.4.

The correlation between the returns of **S** and **Q** is -0.4 .

The continuously compounded, risk-free interest rate is 0.055.

What is the price of the exchange call option on **S** with the strike asset **Q** with exercise date in a quarter year?

- (a) 3.87
- (b) 9.13

- (c) 18.26
- (d) 31.17
- (e) None of the above.

Solution: (a)

In order to price the exchange call, we first need to find the “relative” volatility between **S** and **Q**. We get

$$\begin{aligned}\sigma^2 &= \sigma_S^2 + \sigma_Q^2 - 2\sigma_S\sigma_Q\rho \\ &= 0.04 + 0.16 - 2(0.2)(0.4)(-0.4) = 0.264 \quad \Rightarrow \quad \sigma = 0.5138.\end{aligned}$$

Next, we calculate the terms in the Black-Scholes price of the exchange call. We obtain

$$\begin{aligned}d_1 &= \frac{1}{0.5138\sqrt{1/4}} \left[\ln\left(\frac{40}{40}\right) + \left(0 - 0.05 + \frac{0.5138^2}{2}\right) \left(\frac{1}{4}\right) \right] = 0.07979614 = 0.08, \\ d_2 &= 0.07979614 - 0.5138\sqrt{\frac{1}{4}} = -0.1771085 = -0.18.\end{aligned}$$

From the standard normal tables, we get

$$N(d_1) = N(0.08) = 0.5319, \quad N(d_2) = 1 - N(0.18) = 1 - 0.5714 = 0.4286.$$

So,

$$V_{EC}(0, \mathbf{S}, \mathbf{Q}) = 40e^{-0.05(1/4)}(0.5319) - 40(0.4286) = 3.87.$$

Problem 3.6. Assume the Black-Scholes model. Let the current price of a non-dividend-paying stock be \$50. The stock’s volatility is 0.25.

The continuously compounded risk-free interest rate is 0.03.

An investor writes a European call option with the following properties:

- the option is at the money at time=0,
- the option’s time to exercise is four months,
- the option’s premium is \$3.03,
- the option’s delta is 0.5557.

The investor then, delta-hedges the written call by buying the underlying stock. The delta-hedge is not rebalanced for a month.

After one month, the investor realizes that the option is again at-the-money, and decides to liquidate the portfolio. What is their **exact** profit?

- (a) -0.07
- (b) 0.39
- (c) 1.24
- (d) 3.57
- (e) None of the above.

Solution: (b)

The initial cost of the portfolio is

$$v(S(0), 0) = -3.03 + 0.5557(50) = 24.755.$$

After one month, the call option has a quarter-year to exercise. Our next task is to calculate the call's price. We start with d_1 and d_2 , as usual.

$$d_1(S(1/12), 1/12) = \frac{1}{0.25\sqrt{1/4}} \left[\ln\left(\frac{50}{50}\right) + \left(0.03 + \frac{(0.25)^2}{2}\right) \cdot \frac{1}{4} \right] = \frac{0.03 + 0.03125}{0.25} \sqrt{\frac{1}{4}} = 0.1225,$$

$$d_2(S(1/12), 1/12) = d_1(S(1/12), 1/12) - 0.25\sqrt{\frac{1}{4}} = 0.1225 - \frac{0.25}{2} = -0.0025.$$

Using the standard normal tables, we get

$$N(d_1(S(1/12), 1/12)) = 0.5478 \quad \text{and} \quad N(d_2(S(1/12), 1/12)) = 0.5.$$

So, the call's price after one month equals

$$v_C(S(1/12), 1/12) = 50(0.5478 - e^{-0.03(0.25)}(0.5)) = 2.5768.$$

Therefore, the investor's profit is

$$-2.5768 + 0.5557(50) - 24.755e^{0.03/12} = 0.391235$$

Problem 3.7. (5 points) Which of the following statements is **FALSE**?

- (a) The call theta is negative.
- (b) The put gamma is positive.
- (c) The vegas of otherwise identical calls and puts are equal.
- (d) The put rho is negative.
- (e) None of the above.

Solution: (e)

Problem 3.8. Assume the Black-Scholes model. The current price of a continuous-dividend-paying stock is \$100. Its volatility is 0.25.

The continuously compounded risk-free interest rate is 0.04.

Consider an investor who buys a share of the above stock. Now, they want to create a Δ -neutral portfolio by writing European call options on this stock. You are given the following information about the call options:

- the time to exercise is T ,
- in our usual notation, $\delta T = 0.02$,
- in our usual notation, $d_1(S(0), 0) = 0.25$.

What is the number of call options that need to be written at time-0 to create a Δ -neutral portfolio?

- (a) 11.9043
- (b) 13.2603
- (c) 15.0457
- (d) 17.0403
- (e) None of the above.

Solution: (e)

The Δ of the call option at time-0 is

$$\Delta_C(S(0), 0) = e^{-\delta T} N(d_1(S(0), 0)) = e^{-0.02} N(0.25) = e^{-0.02}(0.5987) = 0.586845.$$

Let $n_C(S(0), 0)$ denote the number of call options needed for the initial delta-hedge. Then,

$$1 + n_C(S(0), 0)(0.586845) = 0 \quad \Rightarrow \quad n_C(S(0), 0) = -\frac{1}{0.586845} = -1.70403.$$

Problem 3.9. (5 points) Consider a two-year project. There are only three cash flows for this project:

- The first occurs at $t = 0$, and is -50 .
- The second occurs at $t = 1$, and is 40 .
- The third occurs at $t = 2$, and is 11.50 .

Determine r , the cost of capital, that leads to the project breaking even.

- (a) 0.0245
- (b) 0.0345
- (c) 0.045
- (d) 0.05
- (e) None of the above.

Solution: (a)

The break-even value of the cost of capital must satisfy

$$-50(1+r)^2 + 40(1+r) + 11.50 = 0 \quad \Leftrightarrow \quad (1+r)^2 - 0.8(1+r) - 0.23 = 0.$$

Solving the quadratic equation, we obtain

$$(1+r)_{1,2} = \frac{0.8 \pm \sqrt{0.8^2 + 4(0.23)}}{2} = \frac{0.8 \pm \sqrt{1.56}}{2} = \frac{0.8 \pm 1.249}{2}.$$

Our acceptable solution is $1+r = 1.0245$, i.e., $r = 0.0245$.

Problem 3.10. (5 points) The current price of a non-dividend-paying stock is \$25 per share. A market-maker writes a three-month European put option on this stock and proceeds to delta-hedge it. The put premium is \$2.50, its delta is -0.30 , its gamma is 0.04 , and its theta is -0.01 per day.

The continuously compounded risk-free interest rate is 0.04 .

Assuming that the stock price does not change, what is the **approximate** overnight profit for the market-maker?

- (a) 0.011096

- (b) 0.018026
- (c) 0.021064
- (d) -0.014062
- (e) None of the above.

Solution: (a)

The initial cost of the total delta-hedged portfolio is

$$-2.50 + (-0.3)(25) = -10.$$

The approximate put price after one day is, according to the delta-gamma-theta approximation,

$$2.50 - 0.01 = 2.49.$$

So, the overnight profit is

$$-2.49 + (-0.3)(25) + 10e^{0.04/365} = -9.99 + 10e^{0.04/365} = 0.011096.$$

Problem 3.11. (5 points) Assume the Black-Scholes model. Bertie Wooster was looking at stock-price and option data from yesterday. He decides to pose his friend Tuppy Glossop a riddle. Bertie tells Tuppy the following about yesterday's price of a stock and information on an option on this stock:

- the stock price yesterday was greater than \$77;
- the option's price was \$2.45;
- the option's delta was -0.1814 ;
- the option's gamma was 0.04 ;
- the option's theta was 0.01 **per day**.

Tuppy is allowed to see today's stock price and today's option price. They turn out to be \$80 and \$2.20, respectively. What is Tuppy going to guess to be yesterday's stock price?

- (a) \$77.08
- (b) \$77.27
- (c) \$78.63
- (d) \$78.22
- (e) None of the above.

Solution: (d)

Using the delta-gamma-theta approximation, we get that

$$2.20 = 2.45 + (-0.1814)ds + \frac{1}{2}(0.04)(ds)^2 + 0.01.$$

Simplifying the above quadratic equation, we obtain

$$0.02(ds)^2 - 0.1814ds + 0.26 = 0 \quad \Leftrightarrow \quad (ds)^2 - 9.07ds + 13 = 0.$$

Solving for ds , we get

$$ds_{1,2} = \frac{9.07 \pm \sqrt{9.07^2 - 4(13)}}{2} = \frac{9.07 \pm 5.50135}{2}.$$

The two solutions are

$$ds_1 = 1.7843 \quad \text{and} \quad ds_2 = 7.2857.$$

We conclude that yesterday's stock price was \$78.22.

Problem 3.12. Assume the Black-Scholes model. Let the current price of a continuous-dividend-paying stock be \$80. Its dividend yield is 0.03.

Let the continuously compounded risk-free interest rate be 0.05.

Consider a European call on the above stock with a quarter-year to exercise and with the strike price equal to $\$80e^{0.01}$. The current delta of this call option is 0.496264. What is the volatility of the stock?

- (a) 0.15
- (b) 0.2
- (c) 0.25
- (d) 0.3
- (e) None of the above.

Solution: (b)

The Black-Scholes call delta at time-0 is

$$e^{-\delta T} N(d_1(S(0), 0)) = 0.496264 \quad \Rightarrow \quad N(d_1(S(0), 0)) = e^{0.03(0.25)}(0.496264) = 0.5.$$

Hence, $d_1(S(0), 0) = 0$. By the definition of d_1 , we have that

$$\ln\left(\frac{80}{80e^{0.01}}\right) + \left(0.05 - 0.03 + \frac{\sigma^2}{2}\right) \cdot \frac{1}{4} = 0.$$

So,

$$-0.01 + \frac{0.02}{4} + \frac{\sigma^2}{8} = 0 \quad \Rightarrow \quad \frac{\sigma^2}{8} = 0.005 \quad \Rightarrow \quad \sigma^2 = 0.04.$$

Finally, we conclude that $\sigma = 0.2$.
