M339W: March 22^{nd}, 2021.

Black-Scholes Model

$$\boxed{S(T) = S(0)e^{(\alpha-\delta-\frac{\sigma^2}{2})\cdot T + \sigma\sqrt{T}\cdot Z}}$$ w/ $Z \sim N(0,1)$

Under the risk-neutral measure $\mathbb{P}^*$:

$$S(T) = S(0)e^{(r-\delta-\frac{\sigma^2}{2})\cdot T + \sigma\sqrt{T}\cdot Z}$$ w/ $Z \sim N(0,1)$

$$\mathbb{P}[S(T) > K] = N(\hat{d}_2)$$

w/ $$\hat{d}_2 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + (\alpha-\delta-\frac{\sigma^2}{2})\cdot T\right]$$

Under the risk-neutral measure $\mathbb{P}^*$:

$$\mathbb{P}^*[S(T) > K] = N(d_2)$$

w/ $$d_2 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + (r-\delta-\frac{\sigma^2}{2})\cdot T\right]$$

Partial & Conditional Expectations.

- Motivation I: Tail Value @ Risk $\boxed{TVaR_p(S(T))}$
- Motivation II: $\boxed{\text{PRICING}}$

Goal: Get a formula for prices of European options on a stock modeled using the Black-Scholes framework.

Idea: $\boxed{\text{RISK-NEUTRAL PRICING}}$

$$\boxed{V(0) = e^{-rT}\mathbb{E}^*[\underbrace{V(T)}_{\text{payoff of a European option}}]}$$

Implementation: Temporarily, focus on a European time-T, strike-K call option.

The payoff:

$$V_C(T) = (S(T)-K)_+$$

$\Rightarrow$ Under any measure $\mathbb{P}$:

$$\mathbb{E}[V_C(T)] = \mathbb{E}[(S(T)-K)_+]$$
$$= \mathbb{E}[(S(T)-K)\cdot \mathbb{I}_{[S(T)>K]}]$$
$$= \mathbb{E}[S(T)\cdot \mathbb{I}_{[S(T)>K]}] - K\cdot \underbrace{\mathbb{E}[\mathbb{I}_{[S(T)>K]}]}_{?}$$

A is an event.

$$\mathbb{I}_A = \begin{cases} 1 & \text{if } A \text{ happened} \\ 0 & \text{if } A \text{ did not happen} \end{cases}$$

$$\mathbb{E}[\mathbb{I}_A] = 1\cdot \mathbb{P}[A] = \mathbb{P}[A]$$

$$\mathbb{E}[\mathbb{I}_{[S(T)>K]}] = \mathbb{P}[S(T)>K] = N(\hat{d}_2)$$

w/ $\hat{d}_2$ as above

Focus on the partial expectation:

$$\mathbb{E}[S(T)\cdot \mathbb{I}_{[S(T)>K]}] = ?$$

Idea: Use the formula for the expectation of a function of a random variable Z.

Note: $\{S(T) \geq K\} =$

$$= \left\{ S(0) e^{(\alpha - \delta - \frac{\sigma^2}{2}) \cdot T + \sigma\sqrt{T} \cdot Z} \geq K \right\}$$

$$= \{ Z \geq -\hat{d}_2 \}$$

z ... dummy variable in the integration corresponding to Z

$$\Rightarrow \quad g(z) = S(0) e^{(\alpha - \delta - \frac{\sigma^2}{2}) \cdot T + \sigma\sqrt{T} \cdot z}$$

(so that $g(Z) = S(T)$).

$\Rightarrow$ Our partial expectation is:

$$\mathbb{E}[g(Z) \cdot \mathbb{I}_{[Z \geq -\hat{d}_2]}] =$$

$$= \int_{-\hat{d}_2}^{+\infty} g(z) \cdot \varphi(z)\, dz$$

$\boxed{\text{(fill in the algebra)}}$

$$= S(0) e^{(\alpha - \delta) \cdot T} \cdot N(\hat{d}_1)$$

w/ $\hat{d}_1 = \frac{1}{\sigma\sqrt{T}} \left[\ln\left(\frac{S(0)}{K}\right) + (\alpha - \delta + \frac{\sigma^2}{2}) \cdot T \right]$

$$\mathbb{E}[S(T) \cdot \mathbb{I}_{[S(T) > K]}] = \boxed{\underbrace{S(0) e^{(\alpha - \delta) \cdot T}}_{\mathbb{E}[S(T)]}} \cdot \boxed{N(\hat{d}_1)}$$

$\Rightarrow$ Our expected payoff of the call:

$$\mathbb{E}[V_C(T)] = \mathbb{E}[S(T)] \cdot N(\hat{d}_1) - K \cdot N(\hat{d}_2)$$

w/ $\hat{d}_1 = \frac{1}{\sigma\sqrt{T}} \left[\ln\left(\frac{S(0)}{K}\right) + (\alpha - \delta + \frac{\sigma^2}{2}) \cdot T \right]$ and $\hat{d}_2 = \hat{d}_1 - \sigma\sqrt{T}$

For the put option:

$$\mathbb{E}[V_P(T)] = \mathbb{E}[(K-S(T))_+]$$
$$= \mathbb{E}[(K-S(T))\cdot \mathbb{I}_{[K>S(T)]}]$$
$$= K\cdot \underbrace{\mathbb{P}[K>S(T)]}_{N(-\hat{d}_2)} - \boxed{\mathbb{E}[S(T)\cdot \mathbb{I}_{[K>S(T)]}]}$$

$$\mathbb{E}[S(T)\cdot \mathbb{I}_{[K>S(T)]}] = \mathbb{E}[S(T)] - \mathbb{E}[S(T)\cdot \mathbb{I}_{[S(T)\geq K]}]$$
$$= \mathbb{E}[S(T)] - \mathbb{E}[S(T)]\cdot N(\hat{d}_1)$$
$$= \mathbb{E}[S(T)]\,(\underbrace{1-N(\hat{d}_1)})$$
$$= \mathbb{E}[S(T)]\,N(-\hat{d}_1)$$

Conditional Expectations.

Let X be a r.v.

Let A be an event s.t. $\mathbb{P}[A]>0$.

Then, $\mathbb{E}[X\,|\,A] := \dfrac{\mathbb{E}[X\cdot \mathbb{I}_A]}{\mathbb{P}[A]}$.

At home:

- $\mathbb{E}[S(T)\,|\,S(T)\geq K] = ?$
- $\mathbb{E}[S(T)\,|\,S(T)<K] = ?$