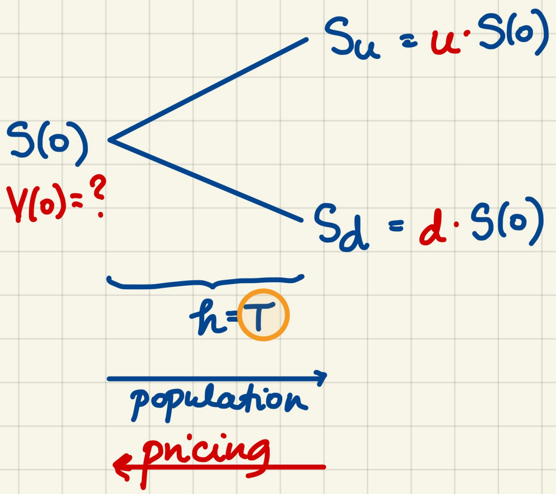
M339D: April 11th, 2022.

Binomial Option Pricing [cont'd].



PAYOFF / REPLICATING PORTFOLIO

$$\boxed{\begin{aligned} V_u = v(S_u) &= \Delta e^{\delta h} \cdot S_u + Be^{rh} \\ V_d = v(S_d) &= \Delta e^{\delta h} \cdot S_d + Be^{rh} \end{aligned}}$$

European-style derivative security w/ the payoff f'tion $v(\cdot)$

In this simple model, we can replicate our derivative security w/ a portfolio w/ the following structure:

- ✓ • Δ shares of stock
- ✓ • B invested @ the risk-free interest rate

We get a system of two eq'ns w/ two unknowns:

$$\begin{aligned} \Delta e^{\delta \cdot h} \cdot S_u + \cancel{Be^{rh}} &= V_u \\ - \quad \Delta e^{\delta \cdot h} \cdot S_d + \cancel{Be^{rh}} &= V_d \end{aligned}$$

$$\Delta e^{\delta h}(S_u - S_d) = V_u - V_d$$

$$\boxed{\Delta = e^{-\delta \cdot h} \frac{V_u - V_d}{S_u - S_d}}$$

unitless

$$\frac{V_u - V_d}{S_u - S_d} \cdot S_u + Be^{rh} = V_u$$

$$Be^{rh} = V_u - \frac{V_u - V_d}{\cancel{S(0)}(u-d)} \cdot u \cdot \cancel{S(0)} = \frac{\cancel{u \cdot V_u} - d \cdot V_u - \cancel{u \cdot V_u} + u \cdot V_d}{u - d}$$

$$\boxed{B = e^{-rh} \cdot \frac{u \cdot V_d - d \cdot V_u}{u-d}}$$

(domestic) currency

By the Law of the Unique Price:

$$\boxed{V(0) = \Delta \cdot S(0) + B}$$

Pricing by Replication

$h=1$

Problem 9.3. Let the continuously compounded risk-free interest rate be equal to 0.04. Consider a one-period binomial tree with every period of length one year used to model the stock price of a stock whose current price is \$80 per share. The stock pays dividends continuously with a dividend yield of 0.02. In the model, it is assumed that the stock price can either go up by \$5 or down by \$4.

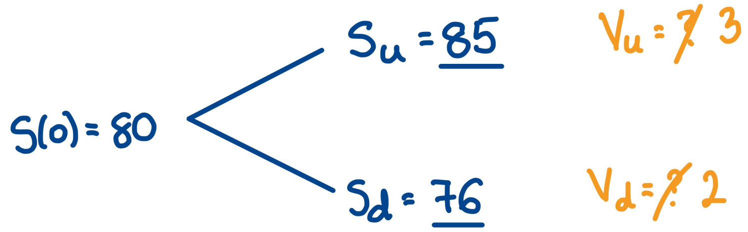
You use the binomial tree to construct a replicating portfolio for a $(78, 82)-$strangle on the above stock. What is the stock investment in the replicating portfolio?

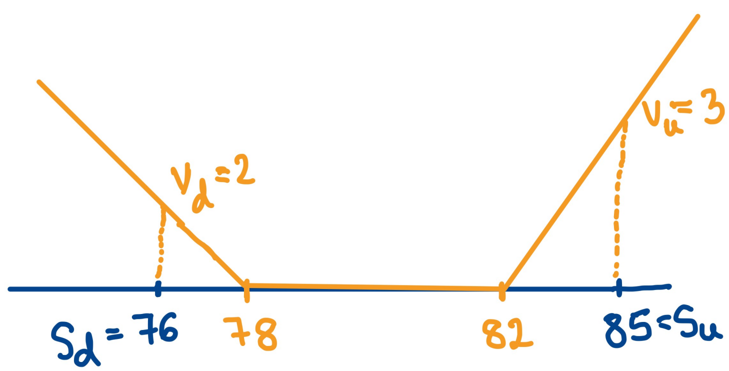
(a) Long 0.1089 shares.
(b) Long 0.33 shares.
(c) Short 0.1089 shares.
(d) Short 0.33 shares.
(e) None of the above.

$\Delta = ?$

$$\Delta = e^{-\delta \cdot h} \cdot \frac{V_u - V_d}{S_u - S_d} = e^{-0.02(1)} \cdot \frac{3-2}{9} = 0.1089$$

✓





$h=1$ $\boxed{r=0.04}$

Problem 9.4. Let the continuously compounded risk-free interest rate be equal to 0.04. Consider a one-period binomial tree with every period of length one year used to model the stock price of a non-dividend-paying stock whose current price is \$50 per share. In the model, it is assumed that the stock price can either go up by 5% or down by 10%.

You use the binomial tree to construct a replicating portfolio for a (45, 55)−call bull spread on the above stock. What is the risk-free investment in the replicating portfolio?

(a) Borrow \$45
(b) Borrow \$43.24
(c) Lend \$45
(d) Lend \$43.24
(e) None of the above.

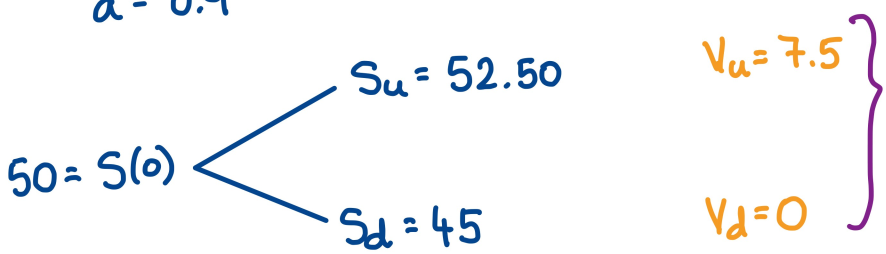
$B = ?$

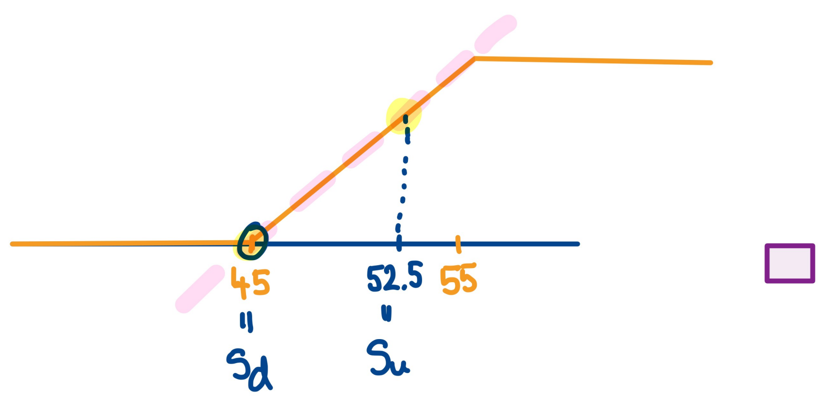
$$B = e^{-rh} \cdot \frac{u \cdot V_d - d \cdot V_u}{u-d} = e^{-0.04} \cdot \frac{(1.05)(0) - 0.9(7.5)}{1.05 - 0.9} = -43.24$$

borrowing

$u = 1.05$

$d = 0.9$





For laughs:

$$\Delta = e^{-\delta \cdot h} \cdot \frac{V_u - V_d}{S_u - S_d} = \frac{7.5 - 0}{52.50 - 45} = 1$$

The initial spread price:

$$V(0) = 1 \cdot 50 - 43.24 = 6.76$$

Start w/ : $V(0) = \Delta \cdot S(0) + B$

⋮ algebra

$$V(0) = e^{-rh}\left[V_u \cdot \frac{e^{(r-\delta)h} - d}{u-d} + V_d \cdot \frac{u - e^{(r-\delta)h}}{u-d}\right]$$

Add up to 1!

Both positive!

No-arbitrage condition:

$d < e^{(r-\delta)h} < u$

We choose to interpret these two quantities as probabilities!

We define the risk-neutral probability of the stock price going up in a single period as:

$$\boxed{p^* := \frac{e^{(r-\delta)h} - d}{u-d}}$$

$\Rightarrow$ The Risk-Neutral Pricing Formula:

$$\boxed{V(0) = e^{-rT}\left[V_u \cdot p^* + V_d(1-p^*)\right]}$$

We will generalize this principle:

$$\boxed{V(0) = e^{-rT}\,\mathbb{E}^*[V(T)]}$$