M358K: October 2nd, 2020

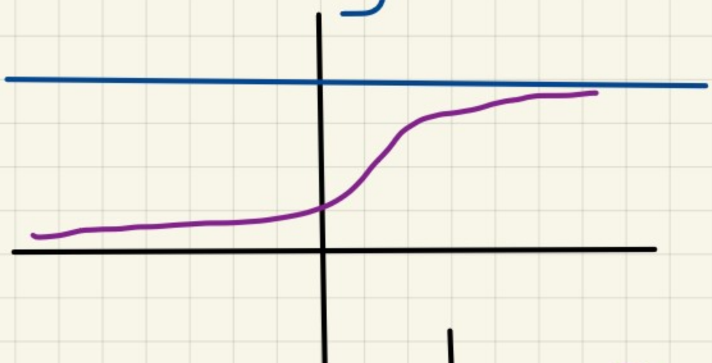
Continuous Dist'ns.

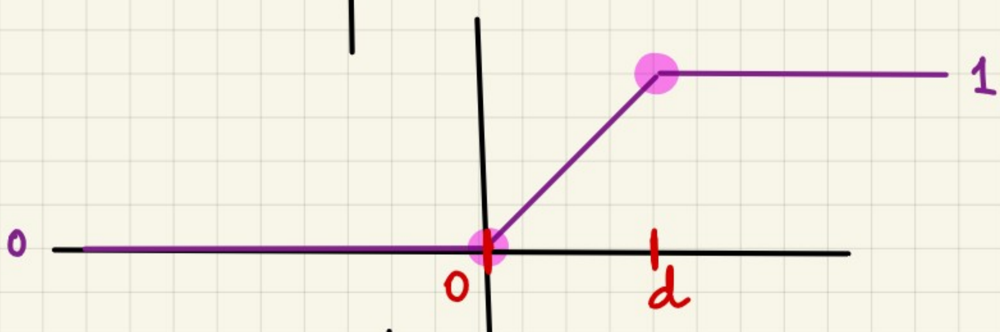
For every random variable X, we can define its cumulative distribution f'tion $F_X : \mathbb{R} \to [0,1]$.

If X is such that F_X is:

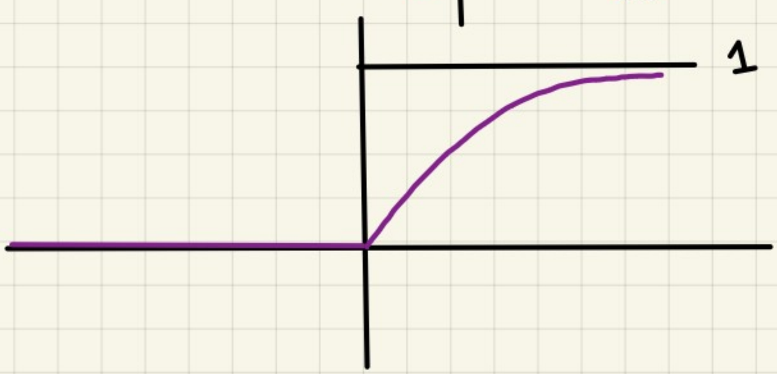
- everywhere continuous
- everywhere differentiable except at at most countably many points,

then we say that X is a continuous r.v.





uniform



exponential

The derivative of the cdf F_X exists everywhere but at at most countably many points. We define

$$f_X(x) = F_X'(x) \quad \text{for all } x \text{ where it exists}$$

Any resulting f'tion f_X is the

probability density function

of the r.v. X.

Note: for any $a<b$:

$$\mathbb{P}[a < X \leq b] = \mathbb{P}[X \leq b] - \mathbb{P}[X \leq a]$$

$$= F_X(b) - F_X(a)$$

$$= \int_a^b f_X(x)\,dx$$

$\Rightarrow$ Knowing the pdf is sufficient to know all you want to know about the dist'n of the r.v. X.

Standard Normal Distribution.

We say that the random variable Z has the standard normal dist'n if it has the pdf of the form:

$$\boxed{\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \quad \text{for all } z \in \mathbb{R}}$$

The cdf of the standard normal:

$$\Phi(x) = \mathbb{P}[Z \le x] = \int_{-\infty}^{x} \varphi(z)\,dz$$

$$= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\,dz$$

There is no analytic form, but we do have <u>standard normal tables.</u>

$$\mathbb{E}[Z] = \int_{-\infty}^{+\infty} \underbrace{z \cdot \underbrace{\varphi(z)}_{\text{even}}}_{\text{odd}}\,dz = 0$$

$$\mathrm{Var}[Z] = 1$$

Mean & Variance completely determine the normal dist'n:

We write: $\boxed{Z \sim N(\text{mean} = 0, \ \text{var} = 1)}$

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Problem Set # 4

Normal distribution.

Problem 4.1. Let Z be a standard normal random variable. Find the following probabilities:

i. $\mathbb{P}[-1.33 < Z \leq 0.24]$
ii. $\mathbb{P}[0.49 < |Z|]$
iii. $\mathbb{P}[Z^4 < 0.0256]$
iv. $\mathbb{P}[e^{2Z} < 2.25]$
v. $\mathbb{P}\left[\frac{1}{Z} < 2\right]$

Problem 4.2. (10 points)

At the *Hogwarts School of Witchcraft and Wizardry* the *Ordinary Wizarding Level (OWL)* exam is typically taken at the end of the fifth year. Based on hystorical data, we model the *OWL* scores as roughly normal with mean 100 and standard deviation of 16.

(a) (5 points)
What is the range of scores for the bottom 15% of the *OWL* takers?

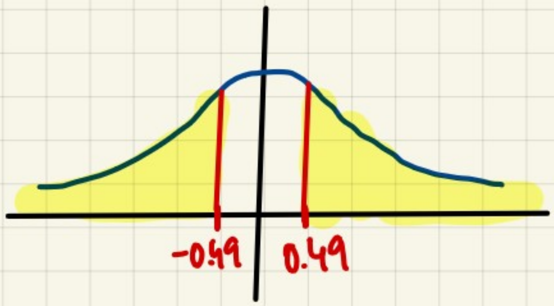
(b) (5 points)
What is the probability that a randomly chosen *OWL* taker has a score higher than 125?

(i) $\mathbb{P}[-1.33 < Z < 0.24] =$

$$= \Phi(0.24) - \Phi(-1.33) =$$

$$= 0.5948 - 0.0918 = 0.5030$$

(ii) $\mathbb{P}[0.49 < |Z|] = \mathbb{P}[Z > 0.49] + \mathbb{P}[Z < -0.49]$



$$= 2 \cdot \mathbb{P}[Z > 0.49]$$

$$= 2(1 - \mathbb{P}[Z \leq 0.49])$$

$$= 2(1 - \Phi(0.49))$$

$$= 0.6241$$