M339J: March 31st, 2021.

Poisson Distribution: Practice.

N... frequency r.v.

$N \sim \text{Poisson}(\lambda)$

- <u>pmf.</u> $p_k = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$, $k = 0, 1, 2, \dots$
- <u>pgf.</u> $\boxed{P_N(z) = e^{\lambda(z-1)}}$
- $\mathbb{E}[N] = Var[N] = \lambda$

284. A risk has a loss amount that has a Poisson distribution with mean 3.

atypical

$X \sim \text{Poisson}(\text{mean} = \lambda = 3)$

An insurance covers the risk with an ordinary deductible of 2. An alternative insurance replaces the deductible with coinsurance α, which is the proportion of the loss paid by the insurance, so that the expected insurance cost remains the same.

$d = 2$

$\alpha \ldots$ coinsurance

Calculate α.

(A) 0.22

(B) 0.27

(C) 0.32

(D) 0.37

(E) 0.42

285. You are the producer for the television show Actuarial Idol. Each year, 1000 actuarial clubs audition for the show. The probability of a club being accepted is 0.20.

The number of members of an accepted club has a distribution with mean 20 and variance 20. Club acceptances and the numbers of club members are mutually independent.

Your annual budget for persons appearing on the show equals 10 times the expected number of persons plus 10 times the standard deviation of the number of persons.

Calculate your annual budget for persons appearing on the show.

(A) 42,600

(B) 44,200

(C) 45,800

(D) 47,400

(E) 49,000

$\longrightarrow$: With an ordinary deductible $d=2$, the expected cost:

$$\mathbb{E}[(X-2)_+] = \underbrace{\mathbb{E}[X]}_{=3} - \mathbb{E}[X \wedge 2]$$

$$X \wedge 2 \sim \begin{cases} 0 & \text{w/ probab. } p_X(0) = p_0 = e^{-3} \\ 1 & \text{w/ probab. } p_X(1) = p_1 = e^{-3} \cdot \frac{3}{1!} = 3e^{-3} \\ 2 & \text{w/ probab. } \mathbb{P}[X \geq 2] = 1 - p_0 - p_1 \\ & \qquad\qquad\qquad\;\; = 1 - 4e^{-3} \end{cases}$$

$$\Rightarrow \mathbb{E}[X \wedge 2] = 1 \cdot (3e^{-3}) + 2 \cdot (1 - 4e^{-3}) =$$

$$= 2 - 5e^{-3} \approx 1.751$$

$$\Rightarrow \mathbb{E}[(X-2)_+] = 3 - 1.75 = 1.25$$

- The expected cost under the second insurance policy:

$$\mathbb{E}[\alpha \cdot X] = \alpha \cdot \underbrace{\mathbb{E}[X]}_{3}$$

$$\Rightarrow \alpha = \frac{1.25}{3} \approx 0.42.$$

130. Bob is a carnival operator of a game in which a player receives a prize worth $W = 2^N$ if the player has N successes, $N = 0, 1, 2, 3, \ldots$ Bob models the probability of success for a player as follows:

(i) N has a Poisson distribution with mean Λ.

(ii) Λ has a uniform distribution on the interval (0, 4).

} A MIXING DIST'N.

Calculate $E[W]$.

(A) 5

(B) 7

(C) 9

(D) 11

(E) 13

$$N \mid \Lambda = \lambda \sim \text{Poisson}(\lambda)$$

$$\Lambda \sim U(0,4)$$

$$\mathbb{E}[W] = \mathbb{E}[2^N] = \mathbb{E}\Big[\underbrace{\mathbb{E}[2^N \mid \Lambda]}_{\text{a function of } \Lambda}\Big]$$

131. DELETED

Focus on $\underbrace{\mathbb{E}[2^N \mid \Lambda]}_{\text{the pgf of } N \mid \Lambda}$.

132. DELETED

$$\mathbb{E}[2^N \mid \Lambda] = e^{\Lambda(2-1)} = e^{\Lambda}$$

$$\Rightarrow \mathbb{E}[2^N] = \mathbb{E}[e^{\Lambda}] \underset{\Lambda \sim U(0,4)}{=} \int_0^4 e^{\lambda} \cdot \frac{1}{4}\, d\lambda = \frac{1}{4}(e^4 - 1) = 13.4$$

Theorem. Let $N_1, N_2, \dots, N_\ell$ be independent Poisson r.v.s w/ parameters $\lambda_1, \lambda_2, \dots, \lambda_\ell$, resp.

Set $N := N_1 + N_2 + \dots + N_\ell$.

Then: $N \sim \text{Poisson}(\lambda = \lambda_1 + \lambda_2 + \dots + \lambda_\ell)$.

$\rightarrow$: Focus on the pgf of N.

$$
\begin{aligned}
P_N(z) = \mathbb{E}[z^N] &= \mathbb{E}[z^{N_1+N_2+\dots+N_\ell}] \\
&= \mathbb{E}[z^{N_1} \cdot z^{N_2} \cdot \dots \cdot z^{N_\ell}] && \text{independence} \\
&= \mathbb{E}[z^{N_1}] \cdot \mathbb{E}[z^{N_2}] \cdots \mathbb{E}[z^{N_\ell}] \\
&= P_{N_1}(z) \cdot P_{N_2}(z) \cdots P_{N_\ell}(z) && \text{Poisson} \\
&= e^{\lambda_1(z-1)} \cdot e^{\lambda_2(z-1)} \cdots e^{\lambda_\ell(z-1)} \\
&= e^{(\lambda_1+\lambda_2+\dots+\lambda_\ell)(z-1)}
\end{aligned}
$$

172. A policyholder has probability 0.7 of having no claims, 0.2 of having exactly one claim, and 0.1 of having exactly two claims. Claim amounts are uniformly distributed on the interval [0, 60] and are independent. The insurer covers 100% of each claim.

Calculate the probability that the total benefit paid to the policyholder is 48 or less.

(A) 0.320
(B) 0.400
(C) 0.800
(D) 0.892
(E) 0.924

Sample P

173. In a given region, the number of tornadoes in a one-week period is modeled by a Poisson distribution with mean 2. The numbers of tornadoes in different weeks are mutually independent.

Calculate the probability that fewer than four tornadoes occur in a three-week period.

(A) 0.13
(B) 0.15
(C) 0.29
(D) 0.43
(E) 0.86

→ : N... total # of tornadoes in a 3-week period

$\mathbb{P}[N < 4] = ?$

$N \sim \mathrm{Poisson}(\lambda = 2 \cdot 3 = 6)$

$\mathbb{P}[N \leq 3] = p_0 + p_1 + p_2 + p_3 =$

$= e^{-6} + e^{-6} \cdot 6 + e^{-6} \cdot 18 + e^{-6} \cdot 36 = e^{-6} \cdot 61 = 0.1512.$

174. An electronic system contains three cooling components that operate independently. The probability of each component's failure is 0.05. The system will overheat if and only if at least two components fail.

Calculate the probability that the system will overheat.

(A) 0.007
(B) 0.045
(C) 0.098
(D) 0.135
(E) 0.143

Think about this:

You have a model for a total count of events coming from several distinct categories. The model is Poisson. Imagine that you know the proportion of occurrencies in different categories. What is the model for the count in one individual category?