M339D: March 11th, 2022.

Exchange Options.

... calls/puts where BOTH the underlying asset & the strike are RISKY ASSETS

Notation: S and Q will denote our two risky assets

For now: $\begin{cases} \bullet\ S \text{... the underlying asset} \\ \bullet\ Q \text{... the strike asset} \end{cases}$

Exchange Call.

At time T, the owner of the exchange call has the right, but not an obligation to:

RECEIVE 1 share of S

and GIVE UP 1 share of Q

$\Rightarrow$ The payoff:

$$V_{EC}(T, S, Q) = (S(T) - Q(T))_+$$

S: the underlying asset

Q: the strike asset

Exchange Put.

At time T, the owner of the exchange put has the right, but not an obligation to:

GIVE UP 1 share of S

and RECEIVE 1 share of Q

$\Rightarrow$ The payoff:

$$V_{EP}(T, S, Q) = (Q(T) - S(T))_+$$

$\Rightarrow$ A special symmetry: $\boxed{V_{EC}(T, S, Q) = V_{EP}(T, Q, S)}$

$\Rightarrow$ For all $t \in [0, T]$: $\boxed{V_{EC}(t, S, Q) = V_{EP}(t, Q, S)}$

Maximum Options.

$\{S(t), t \geq 0\}$ and $\{Q(t), t \geq 0\}$ ... prices of the two risky assets

Set the payoff of the maximum option:

$$\boxed{V_{MAX}(T) := \max(S(T), Q(T))}$$

Q: Think of a financial implementation for this option!

$\longrightarrow$: The owner of the maximum option receives: either one share of S or one share of Q.

Q: Bounds on the price of the maximum option?

$\longrightarrow$: The lower bounds on the payoff:

$$V_{MAX}(T) \geq \begin{cases} S(T) \\ Q(T) \end{cases}$$

$$\underset{\text{no arbitrage}}{\Rightarrow} \quad V_{MAX}(0) \geq \begin{cases} F^P_{0,T}(S) \\ F^P_{0,T}(Q) \end{cases}$$

$$\Rightarrow \boxed{V_{MAX}(0) \geq \max(F^P_{0,T}(S), F^P_{0,T}(Q))}$$

Q: Can we construct a replicating portfolio for the maximum option?

$\longrightarrow$: $V_{MAX}(T) = \max(S(T), Q(T))$

$$\checkmark \quad = S(T) + \boxed{\max(0, Q(T) - S(T))}$$

$$= Q(T) + \boxed{\max(S(T) - Q(T), 0)}$$

Payoffs of Exchange Options

$\Rightarrow$ One possibility for the replicating portfolio:

- one LONG prepaid forward on S ✓

and

- one Long exchange put w/ underlying S and strike asset Q

$\Rightarrow$ no arbitrage

$$V_{MAX}(0) = F^P_{0,T}(S) + V_{EP}(0, S, Q)$$

$$\ldots = F^P_{0,T}(S) + V_{EC}(0, Q, S)$$

$$\ldots = F^P_{0,T}(Q) + V_{EC}(0, S, Q)$$

$$\ldots = F^P_{0,T}(Q) + V_{EP}(0, Q, S)$$

$$\Rightarrow \quad F^P_{0,T}(S) + V_{EP}(0, S, Q) = F^P_{0,T}(Q) + V_{EC}(0, S, Q)$$

$$\Rightarrow \quad \boxed{V_{EC}(0, S, Q) - V_{EP}(0, S, Q) = F^P_{0,T}(S) - F^P_{0,T}(Q)}$$

Generalized Put·Call Parity

6. Consider a model with two stocks. Each stock pays dividends continuously at a rate proportional to its price.

$S_j(t)$ denotes the price of one share of stock *j* at time *t*.

Consider a claim maturing at time 3. The payoff of the claim is

$$\text{Maximum}\left(S_1(3), S_2(3)\right).$$

You are given:

(i) $S_1(0) = \$100$

(ii) $S_2(0) = \$200$

$\delta_1 = 0.05$

(iii) Stock 1 pays dividends of amount $(0.05)S_1(t)dt$ between time *t* and time $t + dt$.

$\delta_2 = 0.10$

(iv) Stock 2 pays dividends of amount $(0.1)S_2(t)dt$ between time *t* and time $t + dt$.

(v) The price of ~~a European option to exchange Stock 2 for Stock 1 at time 3~~ is $10.

An exchange call w/ underlying $\underline{S_1}$ and the strike asset $\underline{S_2}$.

Calculate the price of the claim.

(A) $96

(B) $145

(C) $158

(D) $200

(E) $234

$$V_{MAX}(0) = V_{EC}(0, S_1, S_2) + F^P_{0,3}(S_2)$$

$$= 10 + 200e^{(-0.1(3))} = 158.16$$