M339D: April 19th, 2024.

Black-Scholes Practice.

Problem. Assume the Black-Scholes model.

For a European call option, the strike is $\boxed{S(0)e^{rT}}$ ✓ w/ T being the exercise date.

The price of a call option w/ one year to exercise is $\boxed{0.6 \cdot S(0)}$

Find the price of such a call option w/ three months to exercise in terms of $S(0)$.

→: For any T:

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)\cdot T\right]$$

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\underbrace{\ln\left(\frac{\cancel{S(0)}}{\cancel{S(0)}e^{rT}}\right)}_{\cancel{-rT}} + \left(\cancel{r} + \frac{\sigma^2}{2}\right)\cdot T\right] = \frac{\sigma\sqrt{T}}{2}$$

$$d_2 = d_1 - \sigma\sqrt{T} = \frac{\sigma\sqrt{T}}{2} - \sigma\sqrt{T} = -\frac{\sigma\sqrt{T}}{2}$$

Compare to the previous problem ☺

$$V_c(0,T) = S(0)\,N\left(\frac{\sigma\sqrt{T}}{2}\right) - S(0)\cancel{e^{rT}}\cdot\cancel{e^{-rT}}\,N\left(-\frac{\sigma\sqrt{T}}{2}\right)$$

$$\boxed{V_c(0,T) = S(0)\left(2N\left(\frac{\sigma\sqrt{T}}{2}\right) - 1\right)}$$

For $T=1$:

$$V_c(0,T=1) = S(0)\left(2N\left(\frac{\sigma}{2}\right) - 1\right) = 0.6\cdot S(0)$$

$$\Rightarrow \quad 2N\left(\frac{\sigma}{2}\right) - 1 = 0.6$$

$$\Rightarrow \quad N\left(\frac{\sigma}{2}\right) = 0.8$$

$$\Rightarrow \quad \frac{\sigma}{2} = 0.84 \quad \Rightarrow \quad \boxed{\sigma = 1.68}$$

For $T=\frac{1}{4}$:

$$V_c\left(0,T=\tfrac{1}{4}\right) = S(0)\left(2\cdot N\left(\frac{\sigma\sqrt{\frac{1}{4}}}{2}\right) - 1\right) = S(0)\cdot(2\cdot 0.6628 - 1)$$

0.42

$$= S(0)\cdot 0.3256 \quad \square$$

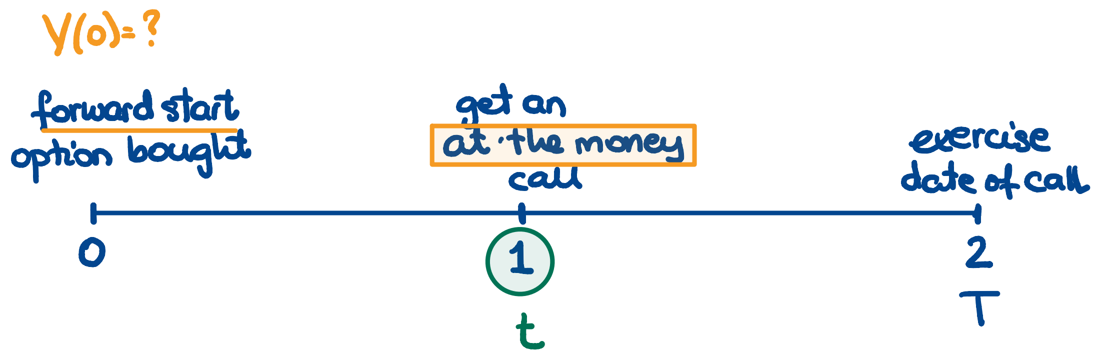
(A) 586

(B) 594

(C) 684

(D) 692

(E) 797



19. Consider a *forward start option* which, 1 year from today, will give its owner a 1-year European call option with a strike price equal to the stock price at that time.

You are given:

(i) The European call option is on a stock that pays no dividends.

(ii) The stock's volatility is 30%. $\boxed{\sigma = 0.3}$

(iii) The forward price for delivery of 1 share of the stock 1 year from today is 100.

$\boxed{F_{0,1}(S) = 100} = S(0)e^{r \cdot 1}$

(iv) The continuously compounded risk-free interest rate is 8%.

Under the Black-Scholes framework, determine the price today of the forward start option.

(A) 11.90

(B) 13.10

(C) 14.50

(D) 15.70

(E) 16.80

At $t < T$:

$$V_c(t) = S(t) \cdot N(d_1(t)) - Ke^{-r(T-t)} \cdot N(d_2(t))$$

w/

$$d_1(t) = \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S(t)}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right]$$

and

$$d_2(t) = d_1(t) - \sigma\sqrt{T-t}$$

In this problem: $\boxed{t=1}$

$$V_c(1) = S(1) \cdot N(d_1(1)) - Ke^{-r(2-1)} \cdot N(d_2(1))$$

$$V_c(1) = S(1) \cdot \boxed{\left(N(d_1(1)) - e^{-r} \cdot N(d_2(1))\right)} \leftarrow$$

w/ $d_1(1) = \frac{1}{0.3\sqrt{2-1}}\left[\cancel{\ln\left(\frac{S(1)}{S(1)}\right)} + \left(0.08 + \frac{0.09}{2}\right)(2-1)\right]$

@ the money

$$d_1(1) = \frac{0.08+0.045}{0.3} = \frac{0.125}{0.3} = 0.42$$

$$d_2(1) = d_1(1) - 0.3\sqrt{2-1} = 0.12$$

$$N(d_1(1)) = N(0.42) = 0.6628$$

$$N(d_2(1)) = N(0.12) = 0.5478$$

$$V_c(1) = S(1)\boxed{\left(0.6628 - e^{-0.08}\cdot 0.5478\right)} = S(1)\cdot 0.1571$$

At time 0, our forward start option is worth:

$$\boxed{S(0)\cdot 0.1571}$$

Our answer: $0.1571 \cdot F_{0,1}(S)\cdot e^{-0.08} = 0.1571\cdot 100\cdot e^{-0.08}$

$$= 14.50$$

□

33. You own one share of a nondividend-paying stock. Because you worry that its price may drop over the next year, you decide to employ a *rolling insurance strategy*, which entails obtaining one 3-month European put option on the stock every three months, with the first one being bought immediately.

You are given:

(i) The continuously compounded risk-free interest rate is 8%.

(ii) The stock's volatility is 30%.

(iii) The current stock price is 45.

(iv) The strike price for each option is 90% of the then-current stock price.

Your broker will sell you the four options but will charge you for their total cost now.

Under the Black-Scholes framework, how much do you now pay your broker?

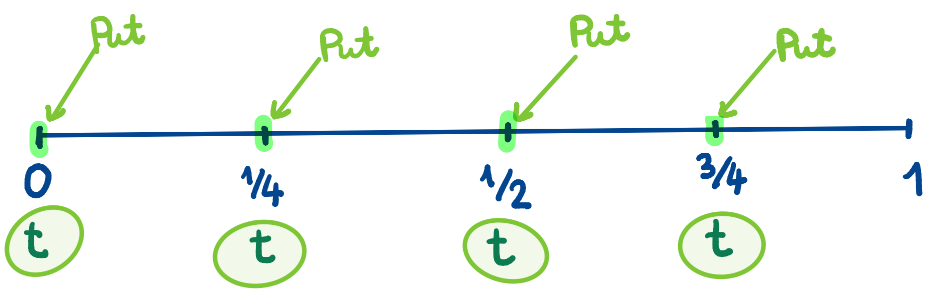
(A) 1.59

(B) 2.24

(C) 2.86

(D) .48

(E) 3.61



t... represents the valuation date

34-39. DELETED

For each of the four puts in the rolling insurance strategy:

- one quarter year to exercise
- $\boxed{K_t = 0.9 \cdot S(t)}$

For every t @ which a put option is received:

$$d_1(\cancel{t}) = \frac{1}{\sigma\sqrt{\frac{1}{4}}}\left[\ln\left(\frac{\cancel{S(t)}}{0.9\cdot \cancel{S(t)}}\right) + \left(r + \frac{\sigma^2}{2}\right)\left(\frac{1}{4}\right)\right]$$

$$d_1(\cancel{t}) = \frac{1}{0.3(1/2)}\left[-\ln(0.9) + \left(0.08 + \frac{0.09}{2}\right)\left(\frac{1}{4}\right)\right] = 0.9107$$

$$d_2(\cancel{t}) = 0.7607$$

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In general,

$$V_P(t) = K e^{-r(T-t)} \cdot N(-d_2) - S(t) N(-d_1)$$

$$N(-0.9107) = 0.1812267$$

$$N(-0.7607) = 0.2234181$$

$$V_P(t) = 0.9\, S(t) \cdot e^{-0.08(0.25)} \cdot 0.2234181 - S(t) \cdot 0.1812267$$

$$V_P(t) = S(t) \cdot 0.01586801$$

=> Note that for every "put delivery" date, i.e., $t = 0, 1/4, 1/2, 3/4$, the value of the put delivered @ that time is

$$\boxed{0.01586801 \cdot S(t)}$$

In order to perfectly replicate, we should buy 0.01586801 shares of stock today.
This costs:

$$0.01586801 \cdot \underset{\sim}{\overset{45}{S(0)}}$$

Since there are 4 puts, each w/ the same cost of replicating today, the total cost of the rolling insurance policy is

$$4 \cdot 0.01586801 \cdot 45 = \underline{2.856242}$$

□