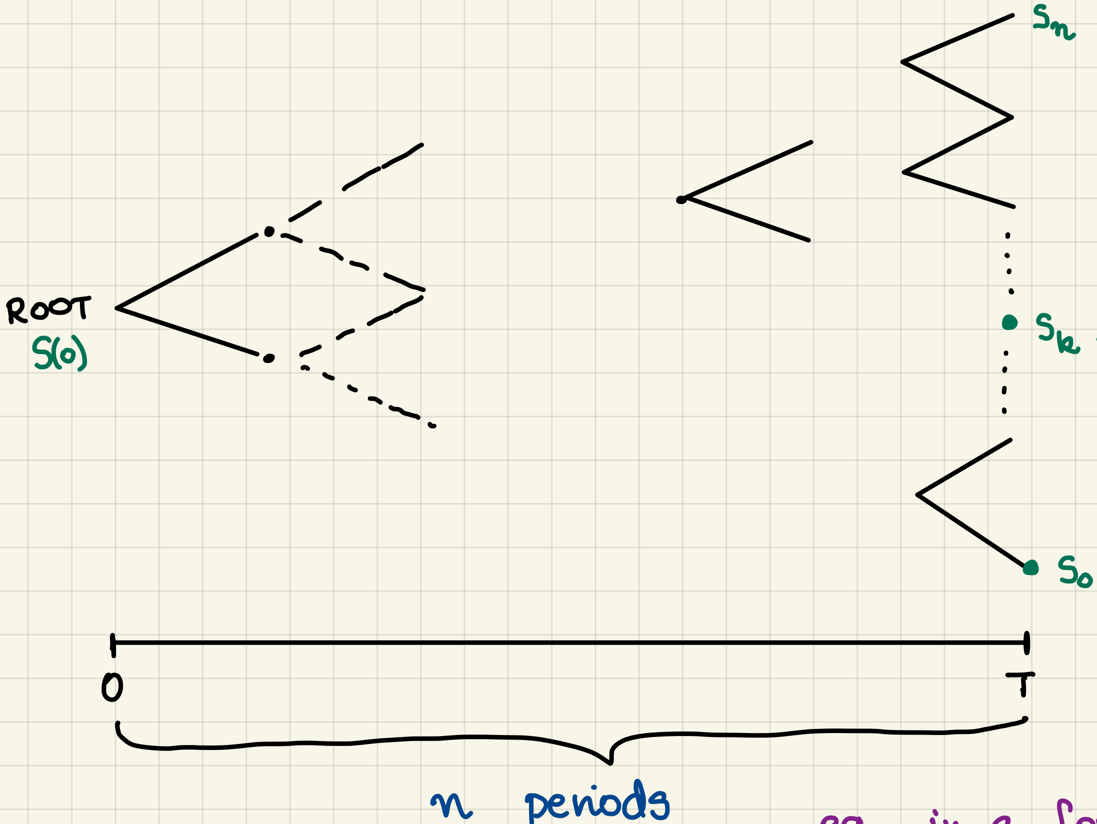
M339W: September 10th, 2021.

The Pre-Limit : n-period Binomial Trees



S_k ... the stock price resulting from k upsteps

u_n ... up factor

d_n ... down factor

e.g., in a forward tree

$u_n = e^{(r-\delta)(T/n) + \sigma\sqrt{T/n}}$

$d_n = e^{(r-\delta)(T/n) - \sigma\sqrt{T/n}}$

$$\Rightarrow \quad S_k = S(0)\, u_n^k \cdot d_n^{n-k} = S(0)\left(\frac{u_n}{d_n}\right)^k \cdot d_n^n$$

k corresponds to a realization of the binomial distribution w/ n trials and p_n as the probability of success in every trial

$\Rightarrow$ Say, X_n ... # of upsteps in the n periods.

Then, $X_n \sim$ Binomial(# of trials $= n$, prob. of success $= p_n$)

$$S(T) = S(0)\left(\frac{u_n}{d_n}\right)^{X_n} \cdot d_n^n$$

Note: • p_n can generally be a subjective probability of an upstep

• If we're pricing options, we use $\boxed{p_n = p_n^*}$,

e.g., in a forward tree $\boxed{p_n^* := \frac{1}{1+e^{\sigma\sqrt{T/n}}}}$

Q: In the past, when you were looking for a limiting dist'n of a sequence of binomial r.v.s, which theorem did you use?

$\longrightarrow$: Normal Approximation to the Binomial (de Moivre·Laplace).

Consider a sequence of binomial random variables

$$Y_n \sim \text{Binomial}(n = \#\text{ of trials}, p = \text{prob. of success})$$

Set: $\mathbb{E}[Y_n] = n \cdot p$

$\text{Var}[Y_n] = np(1-p) \Rightarrow SD[Y_n] = \sqrt{np(1-p)}$

$$\frac{Y_n - np}{\sqrt{np(1-p)}} \xRightarrow{\mathcal{D}} N(0,1)$$

Usage: • Look @ "large" n (rule of thumb: $np \geq 10$ and $n(1-p) \geq 10$)

$$\mathbb{P}[a < Y_n \leq b] =$$

$$= \mathbb{P}\left[\frac{a-np}{\sqrt{np(1-p)}} < \underbrace{\frac{Y_n - np}{\sqrt{np(1-p)}}}_{\approx N(0,1)} \leq \frac{b-np}{\sqrt{np(1-p)}}\right]$$

$$\approx \mathbb{P}\left[\frac{a-np}{\sqrt{np(1-p)}} < \underbrace{Z}_{Z \sim N(0,1)} \leq \frac{b-np}{\sqrt{np(1-p)}}\right]$$

$$= N\left(\frac{b-np}{\sqrt{np(1-p)}}\right) - N\left(\frac{a-np}{\sqrt{np(1-p)}}\right)$$

N… cumulative dist'n ftion of N(0,1)

i.e., $N(z) = \mathbb{P}[Z \leq z]$

- In statistics, we use this theorem like this:

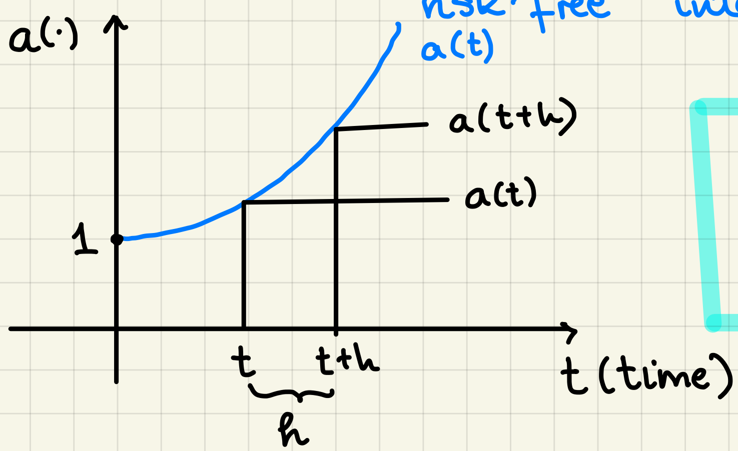
$$Y_n \text{ "}\sim\text{" } \text{Normal}\left(\text{mean} = np,\ \text{sd} = \sqrt{np(1-p)}\right)$$

In our model, the probability of success p_n depends on n.

Realized Returns.

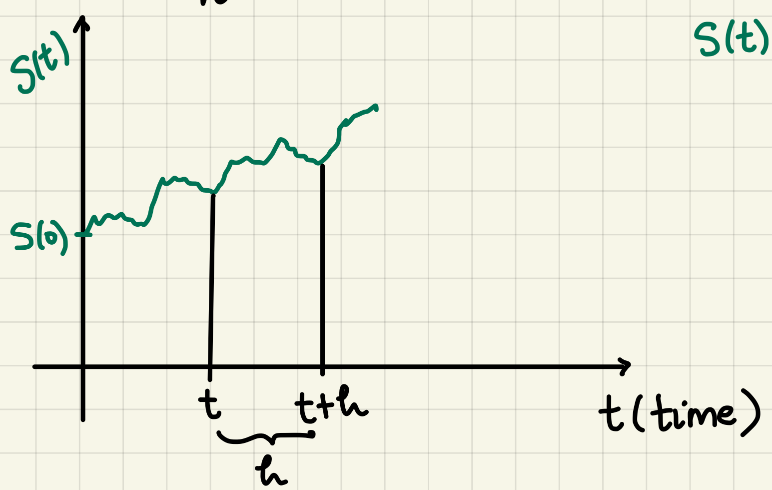
Inspiration.

Consider an accumulation f'tion in the compound interest case. Let r... continuously compounded, risk-free interest rate.



$$\boxed{\begin{aligned} a(t+h) &= a(t)\cdot e^{r\cdot h} \\ r\cdot h &= \ln\left(\frac{a(t+h)}{a(t)}\right) \end{aligned}}$$

Definition.



$S(t), t \geq 0$... time-t stock price