

Poisson-Gamma Mixing.

Let N has a mixture distribution w/
the mixing parameter Δ .

More precisely, let

$$\boxed{\begin{aligned} N | \Delta &\sim \text{Poisson}(\text{mean} = \Delta) \\ \Delta &\sim \text{Gamma}(\alpha, \theta) \end{aligned}} \quad \boxed{\quad}$$

Q: What is the support of N ?

$$\rightarrow: \text{Support}(N) = \mathbb{N}_0 = \{0, 1, 2, \dots\}$$

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Focus on the pmf of N :

for $k = 0, 1, 2, \dots$:

$$\begin{aligned} \underline{p_N(k)} &= \mathbb{P}[N=k] = \underline{F_N(k)} - \underline{F_N(k-1)} \\ &= \int \underline{F_{N|\Delta}(k|\lambda)} f_\Delta(\lambda) d\lambda \\ &\quad - \int \underline{F_{N|\Delta}(k-1|\lambda)} f_\Delta(\lambda) d\lambda \\ &= \int (\underline{F_{N|\Delta}(k|\lambda)} - \underline{F_{N|\Delta}(k-1|\lambda)}) f_\Delta(\lambda) d\lambda \\ &= \int \mathbb{P}[N=k \mid \Delta=\lambda] f_\Delta(\lambda) d\lambda \end{aligned}$$

In general:

$$\underline{p_N(k)} = \int p_{N|\Delta}(k|\lambda) f_\Delta(\lambda) d\lambda$$

In this case:

$$p_N(k) = \int_0^\infty e^{-\lambda} \cdot \frac{\lambda^k}{k!} \cdot \frac{(\frac{\lambda}{\theta})^\alpha e^{-\frac{\lambda}{\theta}}}{\lambda \cdot \Gamma(\alpha)} d\lambda$$

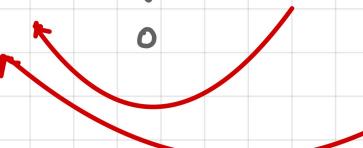
conditional pmf Γ-density

$$= \frac{1}{k! \cdot \theta^\alpha \cdot \Gamma(\alpha)} \int_0^{+\infty} e^{-\lambda} \lambda^{k+\alpha-1} d\lambda$$

\$e^{-\lambda(1+\frac{1}{\theta})}\$ \$\frac{1}{\theta^*}\$ \$\alpha - 1\$ \$\alpha^*\$
\$e^{-\lambda}\$ \$\lambda\$ \$k + \alpha - 1\$

Consider : $Y \sim \Gamma(\alpha^*, \theta^*)$

$$1 = \int_0^{+\infty} f_Y(\lambda) d\lambda$$

$$= \int_0^{+\infty} \frac{\lambda^{\alpha^*-1}}{(\theta^*)^{\alpha^*}} \cdot \frac{e^{-\frac{\lambda}{\theta^*}}}{\Gamma(\alpha^*)} d\lambda$$


$$= \frac{1}{(\theta^*)^{\alpha^*}} \cdot \frac{1}{\Gamma(\alpha^*)} \int_0^{+\infty} \lambda^{\alpha^*-1} \cdot e^{-\frac{\lambda}{\theta^*}} d\lambda$$

$$\Rightarrow P_N(k) = \frac{1}{k! \cdot \theta^\alpha \cdot \Gamma(\alpha)} \cdot (\theta^*)^{\alpha^*} \cdot \Gamma(\alpha^*)$$

$$= \frac{1}{k! \cdot \theta^\alpha \cdot \Gamma(\alpha)} \cdot \left(\frac{\theta}{1+\theta} \right)^{k+\alpha} \cdot \Gamma(k+\alpha)$$

$$= \frac{\Gamma(k+\alpha)}{k! \cdot \Gamma(\alpha)} \cdot \left(\frac{\theta}{1+\theta} \right)^k \cdot \left(\frac{1}{1+\theta} \right)^\alpha$$

\Rightarrow N is NEGATIVE BINOMIAL

w/ $r = \alpha$ and $\beta = \theta$