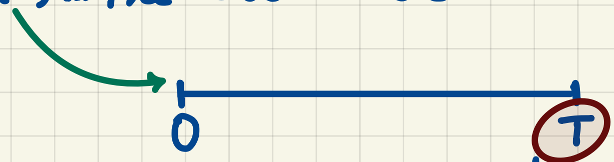
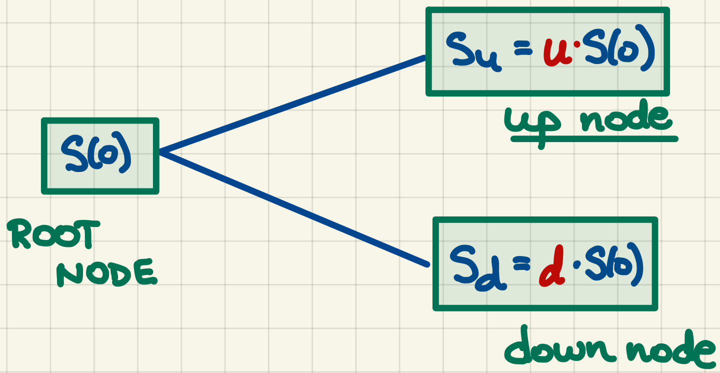
M339D: March 3rd, 2025.

The Binomial Asset Pricing Model.

$S(0)$ ... the observeable initial asset price



time horizon (e.g., the exercise date of an option)



u ... up factor
d ... down factor

By convention:

$\boxed{u > d}$

h ... length of a single period

one period: $\boxed{S(T) = S(h)}$ ... a random variable denoting the time-T stock price w/ two possible values S_u and S_d

As a random variable: SIMPLE RETURN $\boxed{\dfrac{S(T) - S(0)}{S(0)}}$ ←

up node: $\dfrac{S_u - S(0)}{S(0)} = \dfrac{S_u}{S(0)} - 1 = u - 1$

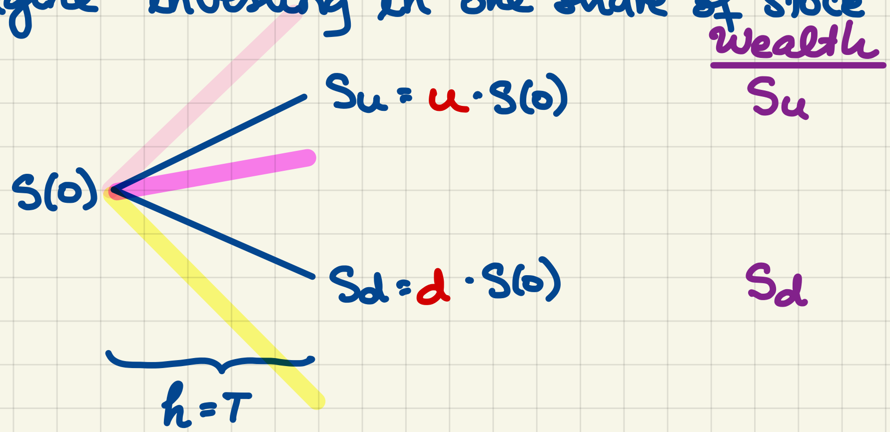
down node: $\dfrac{S_d - S(0)}{S(0)} = \dfrac{S_d}{S(0)} - 1 = d - 1$

No-Arbitrage Condition.

Market Model.

- <u>riskless asset</u>: @ the ccrfir r
- <u>risky asset</u>: non-dividend-paying stock

Imagine investing in one share of stock @ time 0:



At the risk-free rate, the amount $S(0)$ accumulates to $S(0)e^{rh}$ in the same time period.

The No-Arbitrage Condition.

$$S_d < S(0)e^{rh} < S_u$$

$$d \cdot \cancel{S(0)} < \cancel{S(0)}e^{rh} < u \cdot \cancel{S(0)}$$

$$\boxed{d < e^{rh} < u}$$

Half-a-Proof.

Say, to the contrary, $\boxed{e^{rh}} \leq \boxed{d < u}$

Propose. Long one share of stock.

Verify. Profit = Payoff − $FV_{0,T}$(Initial Cost)

$= S(h) - S(0)e^{rh}$

down node: $S_d - S(0)e^{rh} = d \cdot S(0) - S(0)e^{rh} = S(0)(d - e^{rh}) \geq 0$

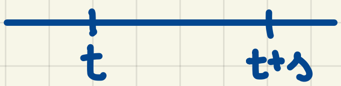
up node: $S_u - S(0)e^{rh} = u\,S(0) - S(0)e^{rh} = S(0)(u - e^{rh}) > 0$

Indeed, this is an arbitrage portfolio.

Forward Binomial Tree.

"Def'n". The volatility $\boxed{\sigma}$ is the standard deviation of realized returns on a continuously compounded scale and annualized.

Realized Return.



$$R(t, t+s) := \ln\left(\frac{S(t+s)}{S(t)}\right)$$

It satisfies: $\boxed{S(t+s) = S(t)e^{R(t,t+s)}}$

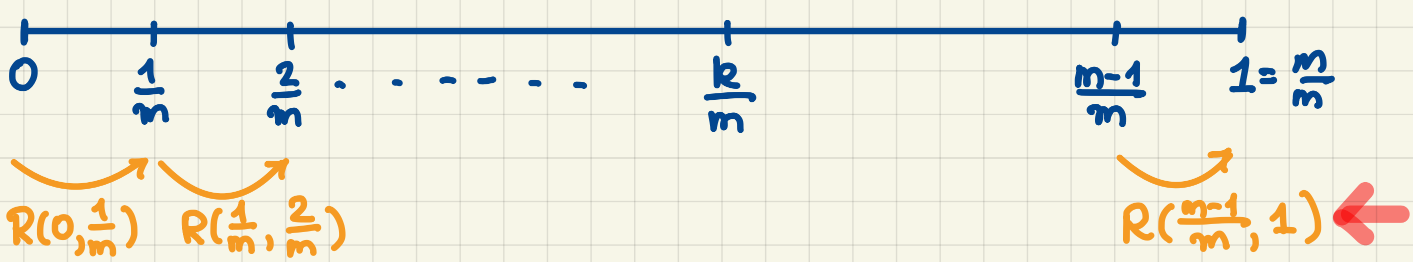
Heuristics.

$T = 1$

$h_m = \frac{1}{m}$ (of a year)

Q: What is the volatility for the time period of length h_m?

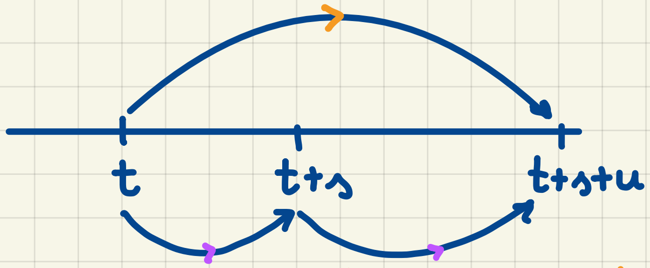
$$\boxed{\sigma_{h_m} = ?}$$



Note: $\boxed{R\left(\frac{k-1}{m}, \frac{k}{m}\right) \text{ for } k = 1 \dots m \text{ are all RANDOM VARIABLES.}}$

We make the following assumptions:

- the returns over disjoint intervals are independent,
- all the returns over the same-length intervals are identically dist'd.



$$R(t,t+s+u) = \ln\left(\frac{S(t+s+u)}{S(t)}\right)$$
$$= \ln\left(\frac{S(t+s)}{S(t)} \cdot \frac{S(t+s+u)}{S(t+s)}\right)$$
$$= \boxed{\ln\left(\frac{S(t+s)}{S(t)}\right)} + \boxed{\ln\left(\frac{S(t+s+u)}{S(t+s)}\right)}$$
$$= R(t,t+s) + R(t+s,t+s+u)$$

Hence, realized returns are additive.

$$\boxed{R\left(0,\tfrac{1}{m}\right) + R\left(\tfrac{1}{m},\tfrac{2}{m}\right) + \cdots + R\left(\tfrac{m-1}{m},1\right) = R(0,1)}$$

Q: $Var[R(0,1)] = \sigma^2$

$$\Rightarrow \sigma^2 = Var\left[R\left(0,\tfrac{1}{m}\right) + \cdots + R\left(\tfrac{m-1}{m},1\right)\right] = \quad \text{(independence)}$$
$$= Var\left[R\left(0,\tfrac{1}{m}\right)\right] + \cdots + Var\left[R\left(\tfrac{m-1}{m},1\right)\right] = \quad \text{(identically dist'd)}$$
$$= m \cdot Var\left[R\left(0,\tfrac{1}{m}\right)\right] = m \cdot \sigma_{h_m}^2$$

$$\sigma_{h_m}^2 = \frac{1}{m}\sigma^2 \qquad \boxed{\sigma_{h_m} = \sigma\sqrt{\frac{1}{m}} = \sigma\sqrt{h_m}}$$

We generalize this identity to arbitrary lengths h:

$$\boxed{\sigma_h = \sigma\sqrt{h}}$$