M339J: January 18th, 2023.

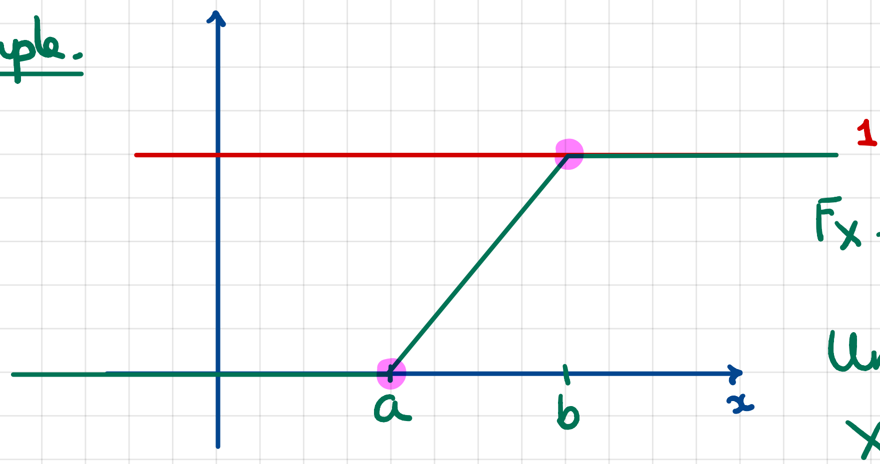
Let X be a continuous r.v.

$$\mathbb{P}[X=x] = \underline{\ 0\ } \quad \text{for } x \in \mathbb{R}$$

Let f_X be its pdf.

$$\left.\begin{array}{l} \bullet\ f_X(x) \geq 0 \\ \bullet\ \displaystyle\int_{-\infty}^{+\infty} f_X(x)\,dx = \underline{\ 1\ } \end{array}\right\}$$

Example.

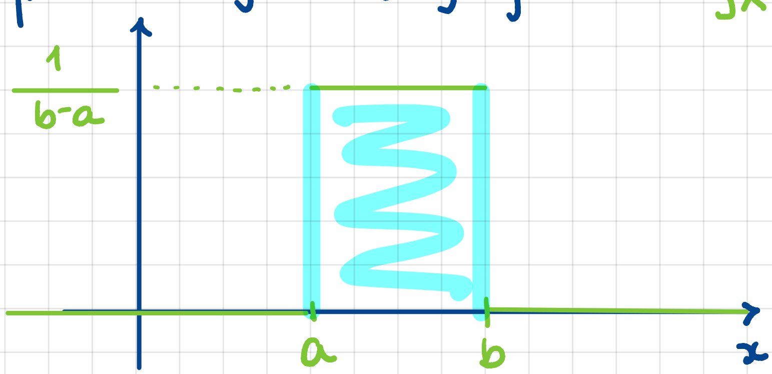


F_X ... cdf of a random variable

Uniform on (a,b)

$X \sim U(a,b)$

The probability density function f_X:



$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & \text{for } x \in (a,b) \\ 0 & \text{otherwise} \end{cases}$$

Problem. Let T denote the lifetime of a particular device. The pdf of T is known to be proportional to

$(10+x)^{-2}$ on the interval $(0,40)$

and 0 otherwise.

What's the probability that the lifetime of the device exceeds 10?

→: Write

$$f_X(x) = \mathcal{K} \cdot (10+x)^{-2} \quad \text{for} \quad x \in (0,40)$$

$$\mathcal{K} \int_0^{40} (10+x)^{-2} dx = 1$$

$$\mathcal{K} \left(\frac{1}{-1}\right) (10+x)^{-1} \Big|_{x=0}^{40} = 1$$

$$\mathcal{K} (-1) \left(\frac{1}{50} - \frac{1}{10}\right) = 1$$

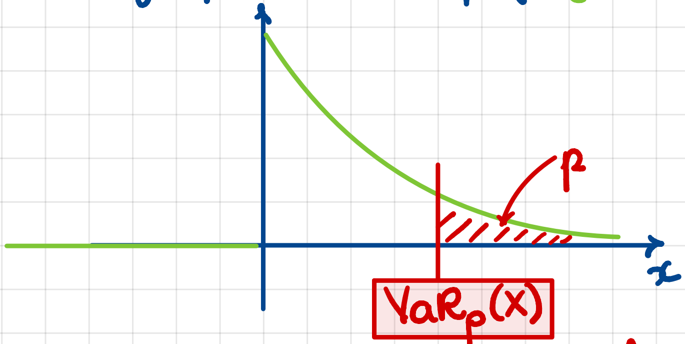
$$\mathcal{K} \cdot \frac{5-1}{50} = 1 \implies \boxed{\mathcal{K} = \frac{50}{4} = \frac{25}{2} = 12.5}$$

$$P[T > 10] = 1 - P[T \leq 10]$$

$$= 1 - \frac{25}{2} \int_0^{10} (10+x)^{-2} dx = 1 - \frac{25}{2} \left(\frac{1}{10} - \frac{1}{20}\right) =$$

$$= 1 - \frac{\cancel{25}^{5}}{2} \cdot \frac{2-1}{\cancel{20}_{4}} = \frac{3}{8} = 0.375 \qquad \square$$

Example. Consider a loss random variable X which is continuous. Let the graph of its pdf f_X look like this:



value at risk

Generally, with continuous r.v.s X, we solve for

$VaR_p(X)$ in $\boxed{S_X(VaR_p(X)) = p}$

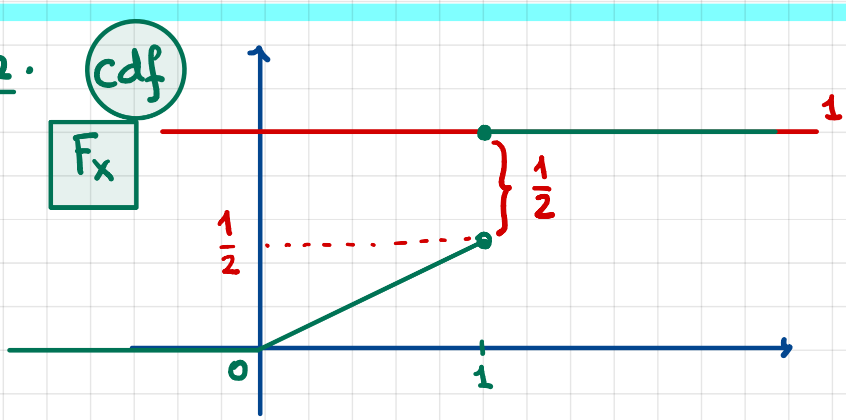
or

$$\boxed{F_X(VaR_p(X)) = 1-p}$$

or

$$\boxed{VaR_p(X) = F_X^{-1}(1-p)}$$

Example.



$f_X(x) = F_X'(x)$

$$f_X(x) = \begin{cases} \frac{1}{2} & \text{for } x \in (0,1) \\ \frac{1}{2} & \text{for } x = 1 \end{cases}$$

2.3. Mixed random variables.

Definition 2.7. A random variable is called **mixed** if

(a) it is **not** discrete, and
(b) its cumulative distribution function is continuous everywhere except for at least one and at most countably many points, and
(c) its cumulative distribution function is differentiable everywhere except for at most countably many points.

Example 2.8. Benefit payments

Let X represent the total dollars paid on a policy in one year.
Clearly, its support is contained in $[0, \infty)$
One possible model is to set

$$F_X(x) = F_4(x) = \begin{cases} 0 & x < 0 \\ 1 - 0.3e^{-10^{-5}x} & x \geq 0 \end{cases}$$

This is a **mixed** r.v.
The event $\{X = 0\}$ has a positive probability:

$$\mathbb{P}[X = 0] = F_4(0) - F_4(0-) = 0.7 - 0 = 0.7$$

At all $x \neq 0$, F_4 is differentiable, and we have

$$f_4(x) = F_4'(x) = \begin{cases} 0 & x < 0 \\ 3 \times 10^{-6}\, e^{-10^{-5}x} & x > 0 \end{cases}$$

Throughout this course, we will abuse the notation slightly and write

$$f_4(x) = \begin{cases} 0.7 & x = 0 \\ 3 \times 10^{-6}\, e^{-10^{-5}x} & x > 0 \end{cases}$$

understanding that $f_4(x) = 0$ otherwise and that the value assigned to $f_4(0)$ is not really the density

2.4. The Hazard Rate.

Definition 2.9. The bf hazard rate (also known as the **force of mortality** and the **failure rate**) of a r.v. X is a function $h_X : I \to \mathbb{R}_+$ is defined as

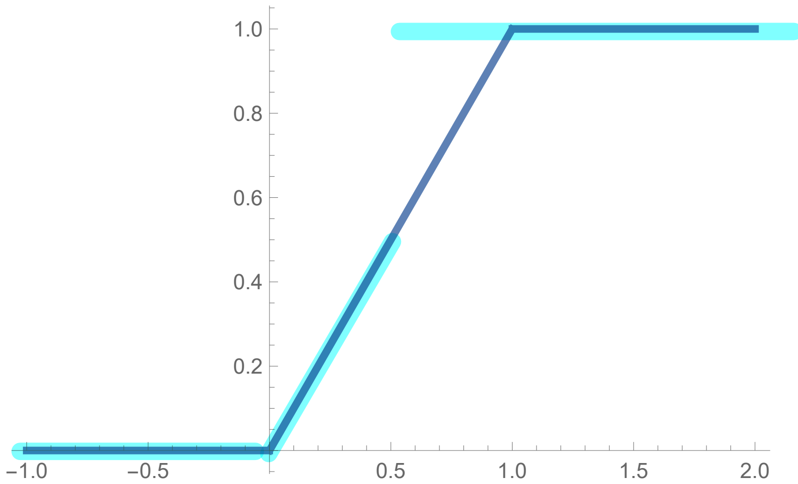
$$h_X(x) = \frac{f_X(x)}{S_X(x)} = -\frac{S_X'(x)}{S_x(x)} = -d[\ln(S_X(x))]$$

for all $x \in I$, where I is the set of all real numbers at which the density f_X is defined and $S_X \neq 0$.

The terminology depends on the context.
To recover the survival function from the hazard rate we use

$$S_X(x) = \exp\{-\int_0^b h(x)\, dx\}$$

Problem 1.3. The graph of the cumulative distribution function of the random variable X looks like this:



What is the support of the random variable X? What is the type of the random variable X?
Define the random variable Y as

$$Y = \min(X, \tfrac{1}{2}).$$

What is the support of the random variable Y? Find the expression for the cumulative distribution function of Y. Sketch its graph. What is the type of the random variable Y?

$\longrightarrow$: $\begin{cases} X \sim U(0,1) \\ \text{Support}(X) = [0,1] \\ \text{Continuous} \end{cases}$ Mixed

$\text{Support}(Y) = [0, \frac{1}{2}]$

for $x \in [0, \frac{1}{2})$: $F_Y(x) = \mathbb{P}[Y \le x] = \mathbb{P}[\min(X, \frac{1}{2}) \le x] = \mathbb{P}[X \le \frac{1}{2}] = F_X(x)$