Name:

M339D=M389D Introduction to Actuarial Financial Mathematics
University of Texas at Austin
In-Term Exam II
Instructor: Milica Čudina

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Signature:

The maximum number of points on this exam is 100.

Provide your **complete solution** to the following problems. Final answers only, without appropriate justification, will receive zero points even if correct.

2.1. **DEFINITIONS.**

Problem 2.1. (10 points) Write the definition of an **arbitrage portfolio**.

Problem 2.2. (5 points) Write the definition of a **replicating portfolio** of a European option.

2.2. **TRUE/FALSE QUESTIONS.**

Problem 2.3. (2 points) Assume that the continuously compounded, risk-free interest rate is strictly positive. The forward price of a non-dividend-paying stock is always strictly increasing with respect to the delivery date. *True or false? Why?*

Solution: TRUE
The forward price is $F_{0,T} = S(0)e^{rT}$ as established in class.

Problem 2.4. (2 points) You are using a binomial asset-pricing model to model the evolution of the price of a particular stock. Then, the Δ in the replicating portfolio of a single call option on that stock never exceeds 1. *True or false? Why?*

Solution: TRUE
The call's Δ will always be between 0 and 1.

Problem 2.5. (2 points) You are using a binomial asset-pricing model to model the evolution of the price of a particular stock. Then, the Δ in the replicating portfolio of a single put option on that stock is between -1 and 0. *True or false? Why?*

Solution: TRUE
The puts's Δ will always be between -1 and 0.

Problem 2.6. (2 points) You are using a one-period binomial asset-pricing model to model the evolution of the price of a particular stock. Assume that, in our usual notation, $S_d < K < S_d$ for a European put option. Then, the risk-free component in the replicating portfolio of a single put option on that stock should be interpreted as lending. *True or false? Why?*

Solution: TRUE
The put's B will always be positive and should be intrpreted as lending.

Problem 2.7. (2 points) In the setting of the one-period binomial model, denote by i the **effective** interest rate **per period**. Let u denote the "up factor" and let d denote the "down factor" in the stock-price model.
If

$$d < u \leq 1 + i$$

then there certainly is no possibility for arbitrage. *True or false? Why?*

Solution: FALSE
The no-arbitrage condition is

$$d < 1 + i < u$$

Problem 2.8. (5 points) The payoff of a chooser option with the choice date coinciding with the exercise date T and with the strike K is given as $|S(T) - K|$. *True or false? Why?*

Solution: TRUE
The owner would choose whichever option is in-the-money on the exercise date. So, they would effectively get a **straddle**.

Problem 2.9. (5 points) In our usual notation, let $S(0) = 40, r = 0.08, \sigma = 0.3$. You need to construct a forward binomial tree with each period on length one year for the above stock. Then, $u > 1.31$. *True or false? Why?*

Solution: TRUE

$$u = \exp\{(0.08 - 0) \cdot 1 + 0.3\sqrt{1}\} \approx 1.46.$$

2.3. **FREE-RESPONSE PROBLEMS.**

Problem 2.10. (5 points) A portfolio consists of the following:

- two **short** one-year, $50-$strike call options with price equal to \$8.50,
- three **long** one-year, $60-$strike put option with price equal to \$6.75

All of the options are European and with the same underlying asset.

Assume that the continuously compounded, risk-free interest rate equals 0.02.

What is the portfolio's profit if the final price of the underlying asset equals \$55?

Solution:

$$-2(55-50)_+ + 3(60-55)_+ + (2(8.50) - 3(6.75))e^{0.02} = 1.684346$$

Problem 2.11. (10 points) A derivative security has the payoff function given by

$$v(s) = \begin{cases} 10 & \text{if } s < 90 \\ 0 & \text{if } 90 \leq s < 100 \\ 20 & \text{if } 100 \leq s \end{cases}$$

Its exercise date is in one year. You model the time-1 price of the underlying asset as

$$S(1) \sim \begin{cases} 85 & \text{with probability } 1/4 \\ 95 & \text{with probability } 1/2 \\ 105 & \text{with probability } 1/4 \end{cases}$$

What is the expected payoff of the above derivative security?

Solution:

$$10\left(\frac{1}{4}\right) + 20\left(\frac{1}{4}\right) = \frac{30}{4} = 7.5$$

Problem 2.12. (10 points) A non-dividend-paying stock is currently priced at \$100 per share. One-year, \$102-strike European call and put options on this stock have equal prices. What is the continuously-compounded annual risk-free rate of interest?

Solution: By put-call parity, we have

$$K = F_{0,T}(S) = S(0)e^{rT} \quad \Rightarrow \quad r = \frac{1}{T}\ln\left(\frac{K}{S(0)}\right).$$

So,

$$r = \frac{1}{T}\ln\left(\frac{K}{S(0)}\right) = \ln(102/100) = 0.01980263.$$

Problem 2.13. (15 points) *Source: Sample FM(DM) Problem #5.*
A market index has the following characteristics:

- One share of the PS index currently sells for 1,000.
- The PS index does not pay dividends.

Sam wants to lock in the ability to buy this index in one year for a price of $1,025. He can do this by buying or selling European put and call options with a strike price of $1,025. The annual effective risk-free interest rate is 5%.

Determine which of the following gives the hedging strategy that will achieve Sam's objective and also give the cost today of establishing this position.

(a) Buy the put and sell the call, receive 23.81.
(b) Buy the put and sell the call, spend 23.81.
(c) Buy the put and sell the call, no cost.
(d) Buy the call and sell the put, receive 23.81.
(e) Buy the call and sell the put, spend 23.81.

Solution: (e)
The reasoning for put-call parity applies.

Problem 2.14. (20 points) Consider a non-dividend-paying stock whose current price is \$80 per share. The stock has the volatility equal to 0.20.

Let the continuously-compounded, risk-free interest rate be equal to 0.04.

You model the evolution of this stock over the next quarter with a **forward** binomial tree.

What is the price of a \$75-strike, three-month call on the above stock consistent with this model?

Solution: In our usual notation, in the forward binomial tree, the risk-neutral probability is

$$p^* = \frac{1}{1+e^{\sigma\sqrt{h}}} = \frac{1}{1+e^{0.2\sqrt{1/4}}} = \frac{1}{1+e^{0.1}} = 0.4750208$$

The *up* and *down* factors are

$$\begin{aligned} u &= e^{rh+\sigma\sqrt{h}} = e^{0.04(1/4)+0.1} = e^{0.11}, \\ d &= e^{rh-\sigma\sqrt{h}} = e^{0.04(1/4)-0.1} = e^{-0.09}. \end{aligned}$$

Hence, the two possible stock prices at the end of the period are $S_u = 80e^{0.11} = 89.30225$ and $S_d = 80e^{-0.09} = 73.11449$. So, the option is in the money only in the *up* node where the payoff equals

$$V_u = (S_u - K)_+ = 14.30225.$$

By the risk neutral pricing formula, we have that

$$V_C(0) = e^{-0.04(1/4)}(0.4750208)(14.30225) = 6.726264.$$

Alternatively, the replicating portfolio has the following components

$$\begin{aligned} \Delta &= \frac{V_u - V_d}{S_u - S_d} = \frac{14.30225}{89.30225 - 73.1149} = 0.8835227, \\ B &= e^{-rh}\frac{uV_d - dV_u}{u-d} = -e^{0.01}\frac{e^{-0.09}(14.30225)}{e^{0.11} - e^{-0.09}} = 63.95555. \end{aligned}$$

So,

$$V_C(0) = \Delta S(0) + B = 0.8835227(80) + 63.95555 = 6.726264.$$

2.4. **MULTIPLE-CHOICE QUESTIONS.**

Problem 2.15. (5 points) Consider a one-year, \$40-strike European call option and a one-year, \$50-strike European put option on the same underlying asset. You observe that the time-0 stock price equals \$35 while the time-1 stock price equals \$55. Then,

(a) both of the options are out-of-the-money at expiration.

(b) both of the options are in-the-money at expiration.

(c) the call is out-of-the-money and the put is in-the-money at expiration.

(d) the put is out-of-the-money and the call is in-the-money at expiration.

(e) both options are at-the-money at expiration.

Solution: (d)