UNIVERSITY OF TEXAS AT AUSTIN

Lecture 5

Moment generating functions and probability generating functions

Definition 5.1. Let X be a random variable. The **moment generating function** of X, denoted by M_X, is defined by

$$M_X(t) = \mathbb{E}[e^{tX}]$$

for all $t \in \mathbb{R}$ for which the above expectation exists.

Remark 5.2. For all X, at least $t = 0$ is in the domain.

Definition 5.3. Let X be a random variable. The **probability generating function** of X, denoted by P_X, is defined by

$$P_X(z) = \mathbb{E}[z^X]$$

for all $z > 0$ for which the above expectation exists.

Remark 5.4. For all X, at least $z = 1$ is in the domain.

The domains of the mgf and the pgf will depend on the distribution of X (neither is ever empty, though).

Remark 5.5. Consider the correspondence $s \leftrightarrow e^t$. We, thus, have

$$P_X(z) = M_X(\ln(z)) \qquad M_X(t) = P_X(e^t)$$

Theorem 5.6. *Let $\{X_k; k = 1, 2, \dots\}$ be independent random variables. Then, in the usual notation, for every $t \in \mathbb{R}$ for which all the mgf's are well-defined,*

$$M_{S_k}(t) = \prod_{j=1}^{k} M_{X_j}(t) \quad \textit{for every } k$$

Also, in the usual notation, for every $z > 0$ for which all the pgf's are well-defined,

$$P_{S_k}(z) = \prod_{j=1}^{k} P_{X_j}(z) \quad \textit{for every } k$$

Example 5.7. Let $X_i \sim Normal(mean = \mu_i, variance = \sigma_i^2), i = 1, \dots, n$. Assume that $X_i, i = 1, \dots, n$ are independent.

The moment generating function for every X_i can be written as

$$M_{X_i}(t) = e^{\mu_i t + \frac{\sigma_i^2 t}{2}} \quad \text{for all } i.$$

Let $S = X_1 + X_2 + \cdots + X_n$. By our theorem,

$$M_S(t) = \prod_{i=1}^{n} M_{X_i}(t) = \prod_{i=1}^{n} e^{\mu_i t + \frac{\sigma_i^2 t^2}{2}} = \exp\left(\sum_{i=1}^{n} \mu_i t + \frac{\sigma_i^2 t^2}{2}\right) = \exp\left(\left(\sum_{i=1}^{n} \mu_i\right) t + \frac{(\sum_{i=1}^{n} \sigma_i^2) t^2}{2}\right)$$

We set

$$\mu = \sum_{i=1}^{n} \mu_i \quad \text{and} \quad \sigma^2 = \sum_{i=1}^{n} \sigma_i^2$$

With these parameters, we can write

$$M_S(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

and we conclude that $S \sim Normal(\mu, \sigma^2)$.

Example 5.8. Let $X_i \sim Gamma(\alpha_i, \theta), i = 1, \dots, n$ be independent random variables. Using our tables, we have that

$$M_{X_i}(t) = (1 - \theta t)^{-\alpha_i} \quad \text{for } t < \frac{1}{\theta}$$

Set

$$S = X_1 + X_2 + \cdots + X_n$$

Then, by our theorem,

$$M_S(t) = \prod_{i=1}^{n} M_{X_i}(t) = \prod_{i=1}^{n} (1 - \theta t)^{-\alpha_i} = (1 - \theta t)^{\sum_{i=1}^{n} \alpha_i}$$

We can conclude that

$$S \sim \Gamma\left(\alpha = \sum_{i=1}^{n} \alpha_i, \theta\right)$$