M339G: October 25^{th}, 2024.

Any Bivariate Normal.

Random variables U and V are said to be bivariate normal w/ parameters $\mu_U, \mu_V, \sigma_U, \sigma_V$, and ρ if

$$\boxed{\left(X := \frac{U-\mu_U}{\sigma_U}, Y := \frac{V-\mu_V}{\sigma_V}\right)}$$

has the standard bivariate normal dist'n w/ correlation ρ.

Note: • $\rho(U,V) = ?$

By def'n:

$$\boxed{\begin{aligned} U &= \mu_U + \sigma_U \cdot X \\ V &= \mu_V + \sigma_V \cdot Y \end{aligned}}$$

$$\rho(U,V) = \frac{Cov[U,V]}{SD[U] \cdot SD[V]}$$

$$= \frac{Cov[\cancel{\mu_U +} \sigma_U X, \cancel{\mu_V +} \sigma_V Y]}{SD[\cancel{\mu_U +} \sigma_U X] \cdot SD[\cancel{\mu_V +} \sigma_V \cdot Y]}$$

μ_U and μ_V are deterministic

$$= \frac{Cov[\sigma_U \cdot X, \sigma_V \cdot Y]}{SD[\sigma_U \cdot X] \cdot SD[\sigma_V \cdot Y]}$$

$$= \frac{\cancel{\sigma_U} \cdot \cancel{\sigma_V} Cov[X,Y]}{\cancel{\sigma_U} \cdot SD[X] \cdot \cancel{\sigma_V} \cdot SD[Y]} = \rho(X,Y) = \rho$$

• U and V are independent

$$\iff$$

$$\rho = 0$$

Example. Midterm and Final.

Midterm and final scores in a large class have an (approximately) bivariate normal dist'n w/ parameters:

	mean	sd
midterm scores:	65	18
final scores:	60	20

correlation: 0.76

Q: What is the "estimated" mean final score of the students who were above the mean on the midterm?

→: Let U be the midterm score, and let V be the final score.

Let X and Y be U and V in std units, resp.

Our task is to find:

$$\mathbb{E}[V \mid U > \mu_U] = ?$$

Our ancillary task is to find:

$$\mathbb{E}[Y \mid X>0] = \int_{\cancel{-\infty}\; 0}^{\infty} \underbrace{\mathbb{E}[Y \mid X=x]}_{\rho x} \cdot f_X(x \mid X>0)\, dx$$

$Y \mid X=x \sim \text{Normal}(\rho x,$ var $= 1-\rho^2)$

The Law of Total Probability

and for $x>0$:

$$f_X(x \mid X>0)\,dx = \mathbb{P}[X \in (x, x+dx) \mid X>0]$$

$$= \frac{\mathbb{P}[X \in (x,x+dx) \text{ and } X>0]}{\mathbb{P}[X>0]} =$$

$$= \frac{\mathbb{P}[X \in (x,x+dx)]}{\frac{1}{2}} = 2 \cdot f_X(x)\,dx$$

$$= 2 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx$$

$$\mathbb{E}[Y \mid X>0] = \int_0^\infty \rho x \cdot 2 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= \frac{2\rho}{\sqrt{2\pi}} \int_0^\infty x \cdot e^{-\frac{x^2}{2}} dx = \boxed{\begin{array}{l} u = \frac{x^2}{2} \\ du = x\,dx \end{array}}$$

$$= \frac{2\rho}{\sqrt{2\pi}} \int_0^\infty e^{-u} du = \frac{2\rho}{\sqrt{2\pi}} \left[-e^{-u}\right]_{u=0}^{\infty} = \frac{2\rho}{\sqrt{2\pi}}$$

In this problem: $\frac{1.52}{\sqrt{2\pi}} = 0.6063923$

$\Rightarrow$ Our answer is $60 + 20(0.6063923) = 72.12785$ $\square$

Matrix Notation.

In two dimensions, we can place the means in a vector $\begin{pmatrix} \mu_U \\ \mu_V \end{pmatrix}$ and the variances/covariance in a matrix

$$\boxed{\Sigma = \begin{bmatrix} \sigma_U^2 & \sigma_U \sigma_V \rho \\ \sigma_U \sigma_V \rho & \sigma_V^2 \end{bmatrix}}$$

(positive definite)

In 1D: $\boxed{U \sim N(\mu, \sigma)}$ $f_U(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$