M339D: February 26^{th}, 2024.

The Inverse Transform Method.

Proposition.

(1) Let X be a continuous random variable, i.e., let X have a density function f_X.
Assume that $\boxed{f_X(x) > 0 \text{ for all } x}$

Denote the cumulative distribution f'tion of X by F_X.
Set: $\boxed{\tilde{X} := F_X(X)}$ ✓

Then, $\tilde{X} \sim U(0,1)$ ✓

$\longrightarrow$: Support of $\tilde{X}$ will be contained in $[0,1]$.

$F_{\tilde{X}}(u) = \underline{0}$ for $u < 0$

$F_{\tilde{X}}(u) = \underline{1}$ for $u \geq 1$

Let $u \in [0,1]$:

$$F_{\tilde{X}}(u) = \mathbb{P}[\tilde{X} \leq u] = \mathbb{P}[F_X(X) \leq u] = \dots$$

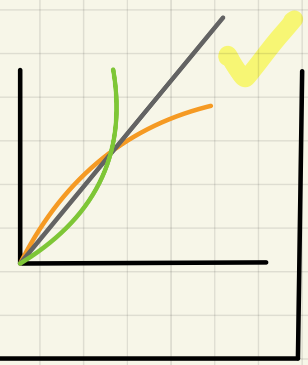
$f_X(x) > 0$ for all x

Recall: $F_X(a) = \int_{-\infty}^{a} f_X(x)\,dx$

$\Rightarrow$ the cdf F_X is strictly increasing

$\Rightarrow$ F_X is one-to-one

$\Rightarrow$ F_X^{-1} exists and is increasing



$$F_{\tilde{X}}(u) = \mathbb{P}[X \leq \boxed{F_X^{-1}(u)}] = \cancel{F_X}(\boxed{\cancel{F_X^{-1}}(u)}) = u \quad \Rightarrow \quad \tilde{X} \sim U(0,1)$$

$\square$

(2) Let $U \sim U(0,1)$.

Let F be a cumulative distribution f'tion.

Set $\boxed{Y := F^{-1}(U)}$

Then, the cumulative dist'n f'tion of Y is F.

Implementation:

1. F... the cdf of the dist'n you want to draw the simulated values from
2. Find an "expression" for F^{-1}
3. Draw: $u_1, u_2, \ldots, u_n \sim U(0,1)$ from your random number generator (rng)
4. Set $\boxed{x_i = F^{-1}(u_i), \quad i = 1, \ldots, n}$

The are your simulated values from the target distribution.

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Problem Set # 7

The inverse transform method.

Problem 7.1. *Source: Course 3, November 1985, Problem #19.*
Your goal is to simulate four draws from a binomial distribution with two trials and the probability of success in every trial equal to 0.30. You intend to use the inverse transform method. Here are the four values produced by the random number generator:

$$\boxed{0.90 \quad 0.21 \quad 0.72 \quad 0.48.}$$

0.91 ?

Which values of the binomial were obtained from the above outputs of the random number generator?

→: $X \sim Binomial(n=2, p=0.3)$

The probability mass function is

$$p_X(0) = (0.7)^2 = 0.49$$

$$p_X(1) = 2(0.7)(0.3) = 0.42$$

$$p_X(2) = (0.3)^2 = 0.09$$

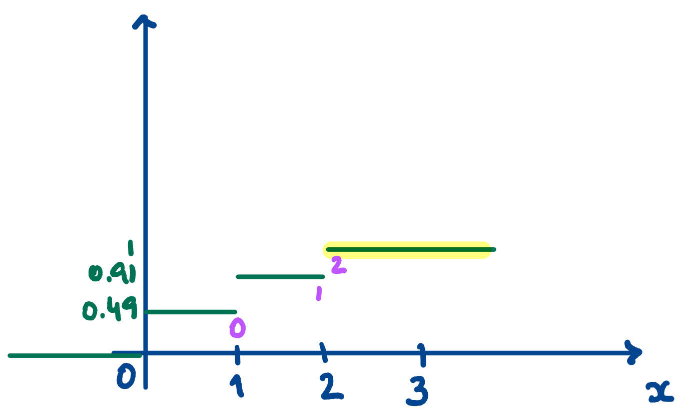
The cdf of X is:

$$F_X(x) = \begin{cases} 0 & \text{for } x<0 \\ 0.49 & \text{for } 0 \le x < 1 \\ 0.91 & \text{for } 1 \le x < 2 \\ 1 & \text{for } x \ge 2 \end{cases}$$

0.21 and 0.48 ⇒ They both map into 0.

0.90 and 0.72 ⇒ They both map into 1.

□



Problem 7.2. Let the random variable X have the following density function:

$$\boxed{f_X(x) = 3x^{-4}, \quad x > 1}$$

You use the *inverse transform method* to simulate values from X. Let the simulated value of the unit uniform be equal to $\boxed{0.25}$. What is the corresponding value of X?

→: The cumulative dist'n f'tion is

$$F_X(x) = \begin{cases} 0 & \text{for } x < 1 \\ \boxed{\int_1^x f_X(u)\,du} & \text{for } x \geq 1 \end{cases}$$

$$\int_1^x 3u^{-4}\,du = \cancel{3} \cdot \frac{u^{-3}}{(\cancel{-3})}\Bigg|_{u=1}^{x} = 1 - x^{-3}$$

The inverse of the cumulative dist'n f'tion (aka the quantile function)?

$$y = 1 - x^{-3} \iff 1 - y = x^{-3} \iff x^3 = \frac{1}{1-y}$$

$$\iff \boxed{x = \frac{1}{\sqrt[3]{1-y}}}$$

So, our answer is

$$x = \frac{1}{\sqrt[3]{1-0.25}} \approx \underline{1.100642}$$

Problem 7.4. Let the random variable X have the cumulative distribution function given by

$$F_X(x) = \begin{cases} 0, & \text{for } x < 0, \\ x^4, & \text{for } 0 \leq x \leq 1, \\ 1, & \text{for } x > 1. \end{cases}$$

The inverse transform method was applied to generate the following three simulated values of X:

$$0.09 \quad 0.64 \quad 0.81.$$

Which values of the random number generator where mapped into the above three draws from X?

:

$(0.09)^4, (0.64)^4, (0.81)^4$

$\square$