M339D: September 27th, 2024.

Finite Probability Spaces.

... serve as environments for the possible paths that asset prices can take.

e.g.,

$$S(T) \sim \begin{cases} 120 & \text{w/ probab. } 1/6 \\ 80 & \text{w/ probab. } 1/2 \\ 50 & \text{w/ probab. } 1/3 \end{cases}$$

Q: What is the expected put payoff w/ strike equal to 105?

$$V_p(T) \sim \begin{cases} 0 & \text{w/ probab. } 1/6 \\ 25 & \text{w/ probab. } 1/2 \\ 55 & \text{w/ probab. } 1/3 \end{cases}$$

$$V_p(T) = (K - S(T))_+$$

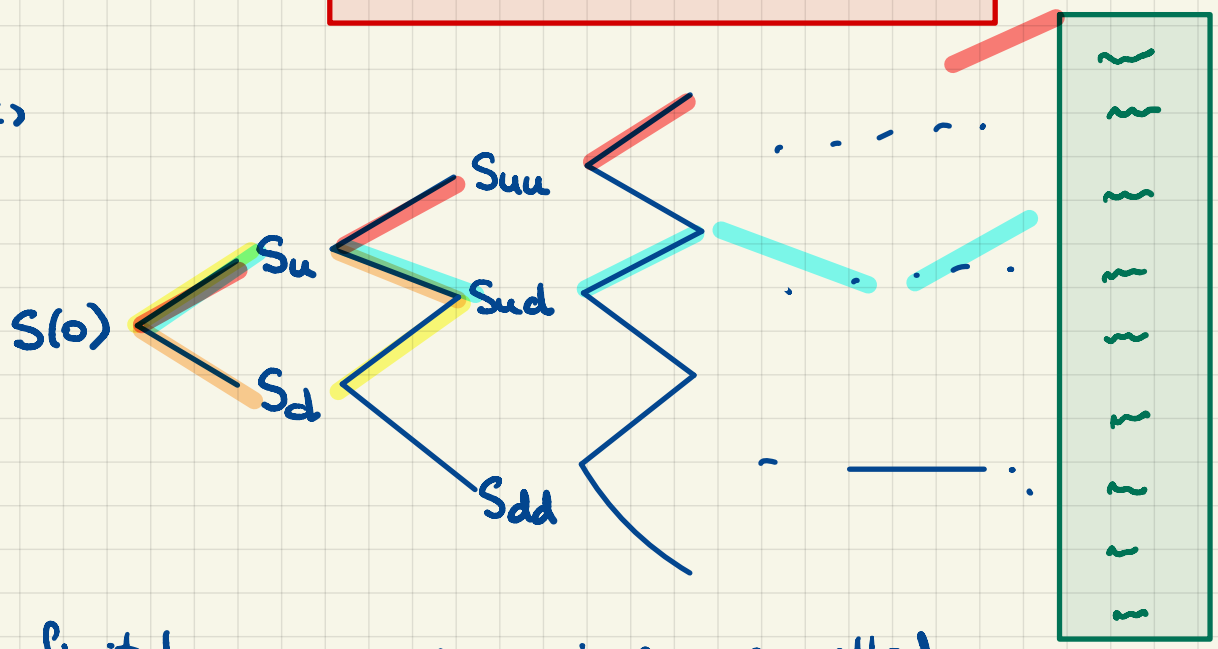
$$\mathbb{E}[V_p(T)] = 25 \cdot \left(\frac{1}{2}\right) + 55\left(\frac{1}{3}\right) = \frac{?}{6}$$

□

In general:

$$\mathbb{E}[g(X)] \neq g(\mathbb{E}[X])$$

e.g.,



All the finitely many scenarios are called

states of the world.

We assume that:

- each can happen, i.e., probab > 0

and

- they exhaust all possibilities, i.e., $\sum$ probab = 1

Arbitrage Portfolio.

Def'n. An arbitrage portfolio is a portfolio whose profit is:

- nonnegative in (ALL) states of the world, i.e., w/ probab. 1;

and

- strictly positive in (AT LEAST ONE) state of the world w/ probab. > 0

Unless it's specified otherwise in a specific problem/example, we assume NO ARBITRAGE.

Law of the Unique Price.

Assume that the payoffs of two static portfolios A and B are equal, i.e.,

$V_A(T) = V_B(T)$

($V_A(T)$: random variable; $V_B(T)$: random variable)

T... time horizon (temporarily fixed)

We say that random variables X and Y are equal if $\mathbb{P}[X=Y] = 1$

We say that r.v.s X and Y are identically distributed if $F_X(x) = F_Y(x)$ for all $x \in \mathbb{R}$.

Claim. $\boxed{V_A(0) = V_B(0)}$

Proof. Assume, to the contrary, that

$$V_A(0) \neq V_B(0)$$

Without loss of generality,

$$\underbrace{V_A(0)}_{\text{relatively cheap}} < \underbrace{V_B(0)}_{\text{relatively expensive}}$$

Propose an arbitrage portfolio:

- Long Portfolio A
- Short Portfolio B

} Total Portfolio

Verify that this is, indeed, an arbitrage portfolio.