M339J: January 23rd, 2023.

Problem 2.3. Find the ratio of the 90^{th} percentile to the median of the exponential distribution with parameter θ.

→:

$X \sim Exponential(\theta)$

$F_X(x) = 1 - e^{-\frac{x}{\theta}} \qquad x > 0$

Let $p \in (0,1)$.

$\pi_p = ?$

$F_X(\pi_p) = p$

$1 - e^{-\frac{\pi_p}{\theta}} = p$

$e^{-\frac{\pi_p}{\theta}} = 1 - p$

$-\frac{\pi_p}{\theta} = \ln(1-p)$

$$\boxed{\pi_p = -\theta \ln(1-p)}$$

$$\frac{\pi_{0.9}}{\pi_{0.5}} = \frac{-\cancel{\theta}\ln(1-0.9)}{-\cancel{\theta}\ln(1-0.5)} = \frac{\ln(0.1)}{\ln(0.5)} = \underline{3.3219} \quad \square$$

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Lecture 4

The Inverse Transformation (Simulation) Method

Proposition 4.1. *Let X be a continuous random variable with the cumulative distribution function F_X and probability density function f_X.*
Assume that $\boxed{f(x) > 0}$ for all positive x and zero elsewhere.
Define $Y = \boxed{F_X}(X)$.
Then, $Y \sim U(0,1)$.

$\longrightarrow$: Support$(Y) \subseteq [0,1]$

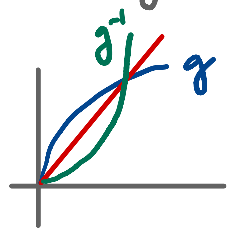
for $y \in (0,1)$:

$$F_Y(y) = \mathbb{P}[Y \leq y] = \mathbb{P}[F_X(X) \leq y]$$

$$\left.\begin{array}{l} f_X(x) > 0 \\ F_X(a) = \int_0^a f_X(x)\,dx \end{array}\right\} \Rightarrow F_X \text{ is strictly increasing}$$

$\Rightarrow F_X$ is one-to-one on $\mathbb{R}_+$

$\Rightarrow F_X^{-1}$ exists and increasing



$$F_Y(y) = \mathbb{P}[\cancel{F_X^{-1}}(\cancel{F_X}(X)) \leq F_X^{-1}(y)]$$
$$= \mathbb{P}[X \leq F_X^{-1}(y)]$$
$$= \cancel{F_X}(\cancel{F_X^{-1}}(y)) = y \quad \square$$

Proposition 4.2. *Let $U \sim U(0,1)$ and let F be a cumulative distribution function.*
Define $X = F^{-1}(U)$.
Then, the random variable X has the cumulative distribution function F.

Proof in the book.

An Informal Implementation.

1. Set F to be the cdf of the distribution from which we want to simulate values. "Figure out" F^{-1}; this can be analytic or numerical.
2. Draw the simulated values from the unit uniform $U(0,1)$:
$$u_1, u_2, \dots, u_n$$
3. Apply F^{-1} to the simulated values to obtain
$$x_1 = F^{-1}(u_1), x_2 = F^{-1}(u_2), \dots, x_n = F^{-1}(u_n)$$

The $x_1, x_2, \dots, x_n$ are the simulated values from your target distribution.

Example 4.3. In the exponential case $X \sim Exponential(\theta)$, we have already obtained the analytic expression for the quantile function F_X^{-1}. It is
$$F_X^{-1}(y) = -\theta \ln(1-y)$$
So, with $\{u_i, i = 1, \dots, n\}$ generated from the unit uniform, the x_i defined as
$$-\theta \ln(1-u_i) \quad \text{for } i = 1, \dots, n$$
will be simulated values from the exponential distribution with parameter θ.