M362K Probability
University of Texas at Austin
Practice Problems for the Final Exam
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Notes: This is a closed book and closed notes exam. The maximal score on this exam is 100 points. **There are many ways in which any single problem can be solved. The solutions herein are just one possible way to tackle the given problems.**

All written work handed in by the student is considered to be
their own work, prepared without unauthorized assistance.

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4.1. **DEFINITIONS.**

Problem 4.1. (5 points) Complete the following definition:
Let X and Y be any two random variables on the same outcome space Ω, we say that X and Y are *independent* if ...

Solution: ...

$$\mathbb{P}[X \in A, Y \in B] = \mathbb{P}[X \in A]\mathbb{P}[Y \in B]$$

for all "nice" subsets A and B of $\mathbb{R}$.

or

...

$$F_{X,Y}(x,y) = F_X(x)F_Y(y) \quad \text{for all } (x,y) \in \mathbb{R}^2$$

Problem 4.2. (5 points) Complete the following definition:
Let X be a continuous random variable with the density function denoted by f_X. The *expected value* of X is defined as ...

Solution: ...

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

if the integral is absolutely convergent.

4.2. **TRUE/FALSE QUESTIONS.**

Problem 4.3. (3 points) We say that a function $g : \mathbb{R} \to \mathbb{R}$ is *even* if its graph is symmetric about the vertical axis, i.e., if $g(x) = g(-x)$ for all $x \in \mathbb{R}$.

It is possible that a cumulative distribution function be even. *True or false? Why?*

Solution: FALSE
For instance,

$$\lim_{x\to-\infty} F_X(x) = 0 \quad \text{while} \quad \lim_{x\to\infty} F_X(x) = 1.$$

Problem 4.4. (3 points) If the continuous random variable X has the distribution function F_X, then the distribution function of the random variable $Y = |X|$ equals

$$F_Y(y) = 2F_X(y).$$

True or false? Why?

Solution: FALSE
If the proposed statement were true, we would have

$$1 = \lim_{y\to\infty} F_Y(y) = \lim_{y\to\infty} 2F_X(y) = 2$$

which is nonsense.

Moreover,

$$\begin{aligned}F_Y(y) &= \mathbb{P}[Y \le y] = \mathbb{P}[|X| \le x]\\ &= \mathbb{P}[-x \le X \le x]\\ &= \mathbb{P}[X \le x] - \mathbb{P}[X \le -x]\\ &= F_X(x) - F_X(-x)\end{aligned}$$

Problem 4.5. (2 points) The minimum of two exponential random variables is also exponential. *True or false?*

Solution: TRUE
Check your notes! This was shown in class.

Problem 4.6. (2 points) Assume that **only** the marginal probability density functions f_X and f_Y are given for a random pair X, Y, then we can **always** calculate the joint probability density function $f_{X,Y}$ for the pair X, Y. *True or false?*

Solution: FALSE

4.3. **FREE RESPONSE PROBLEMS.** Please, explain carefully all your statements and assumptions. Numerical results or single-word answers without an explanation (even if they're correct) are worth 0 points.

Problem 4.7. (5 points) Let X and Y be random variables such that the random pair (X, Y) denotes the coordinates of a point uniformly chosen in a circle of radius 1 centered at the origin.

Write the expression for the joint density function of the pair (X, Y).

Solution:

$$f_{X,Y}(x, y) = \begin{cases}\frac{1}{\pi} & \text{for } x^2 + y^2 \le 1\\ 0 & \text{otherwise}\end{cases}$$

Problem 4.8. (6 points) Let $Z \sim N(0, 1)$. Find the following probabilities:

$$\mathbb{P}[Z \le 1.11] =$$

$$\mathbb{P}[1 \le Z \le 1.11] =$$

$$\mathbb{P}[Z \le -1.11] =$$

Solution: From the tables for the cdf of the standard normal distribution:

$$\mathbb{P}[X \leq 1.11] = \Phi(1.11) = 0.8665$$

$$\mathbb{P}[1 \leq X \leq 1.11] = \Phi(1.11) - \Phi(1) = 0.8665 - 0.8413 = 0.0252$$

$$\mathbb{P}[X \leq -1.11] = \mathbb{P}[X \geq 1.11] = 1 - \mathbb{P}[X < 1.11] = 1 - \Phi(1.11) = 1 - 0.8665 = 0.1335$$

Problem 4.9. Let X be an exponential random variable with parameter $\lambda > 0$. Compute the probability density function f_Y of the random variable $Y = \ln(X)$.

Solution: We start by computing the cdf F_Y of Y, keeping in mind that $F_X(x) = 1 - e^{-\lambda x}$, for $x \geq 0$. We also note that $f_Y(x) = 0$, for $x \leq 0$, since Y takes only nonnegative values. For $x \geq 0$ we get

$$\mathbb{P}[Y \leq x] = \mathbb{P}[\ln(X) \leq x] = \mathbb{P}[X \leq e^x] = 1 - e^{-\lambda e^x}.$$

Thus,

$$f_Y(x) = F_Y'(x) = \begin{cases} \lambda e^x e^{-\lambda e^x}, & x > 0 \\ 0, & x \leq 0 \end{cases} = \begin{cases} \lambda e^{x - \lambda e^x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

Problem 4.10. In a certain state, tax returns are audited for everyone whose income is in the top 15% of all incomes. Assume that the income in the state is modeled by a normally distributed random variable with mean $\mu = 54,000$ and standard deviation $\sigma = 15,000$.

Find the minimum income for which the tax-payer certainly gets audited in this state.

Solution: Let X stand for the random variable modeling the income in this state. Since $X \sim N(54,000, 15,000)$, we have that

$$\frac{X - 54,000}{15,000} \sim N(0,1). \tag{4.1}$$

Let x^* denote the minimum income for which a tax-payer certainly gets audited. From the conditions in the problem, we have that

$$\mathbb{P}[X > x^*] = 0.15.$$

Hence,

$$\mathbb{P}[\frac{X - 54,000}{15,000} > \frac{x^* - 54,000}{15,000}] = 0.15.$$

So, using (4.1), we have that

$$\Phi\left(\frac{x^* - 54,000}{15,000}\right) = 0.85.$$

From the tables for the function Φ, we conclude that

$$\frac{x^* - 54,000}{15,000} = 1.04 \quad \Rightarrow \quad x^* \approx 69,600.$$

Problem 4.11. An insurance company is handling claims from two categories of drivers: the good drivers and the bad drivers. The waiting time for the first claim from a **good** driver is modeled by an exponential random variable T_g with mean 6 (in years). The waiting time for the first claim from a **bad** driver is modeled by an exponential random variable T_b with mean 3 (in years). We assume that the random variables T_g and T_b are independent.

(i) (10 points) What is the distribution of the waiting time T until the first claim occurs (regardless of the type of driver this claim was filed by)?

(ii) (10 points) What is the probability that no claims from a good driver will be filed in the next 3 years **and** that the first claim from a bad driver will be filed within 2 years?

Solution: You can remember that the minimum of two independent exponential random variables is again exponential and what its parameter should be (we did this in class). Or, formally, we have that $T = \min\{T_b, T_g\}$, and the image of T is $[0, \infty)$. Let us calculate the cdf of the random variable T. For every $t \geq 0$, we have

$$F_T(t) = \mathbb{P}[T \leq t] = 1 - \mathbb{P}[T > t] = 1 - \mathbb{P}[\min\{T_b, T_g\} > t] = 1 - \mathbb{P}[T_b > t, T_g > t].$$

Due to the independence of T_b and T_g and the fact that $T_b \sim Exp(\lambda_b)$ and $T_g \sim Exp(\lambda_g)$, with $\lambda_g = 1/6$ and $\lambda_b = 1/3$, we can write

$$\mathbb{P}[T_b > t, T_g > t] = \mathbb{P}[T_b > t]\mathbb{P}[T_g > t] = e^{-\lambda_b t}e^{-\lambda_g t} = e^{-(\lambda_g+\lambda_b)t}.$$

So, $F_T(t) = 1 - e^{-(\lambda_g+\lambda_b)t}$, and $T \sim Exp(\lambda_g + \lambda_b)$, i.e., $T \sim Exp(1/2)$.

Using the independence of the two waiting times and their given distributions, we can simply calculate this probability as follows:

$$\begin{aligned}
\mathbb{P}[T_g > 3, T_b < 2] &= \mathbb{P}[T_g > 3]\mathbb{P}[T_b < 2] \\
&= e^{-3\lambda_g}(1 - e^{-2\lambda_b}) \\
&= e^{-3\cdot\frac{1}{6}}(1 - e^{-2\cdot\frac{1}{3}}) \\
&= e^{-1/2} - e^{-7/6}.
\end{aligned}$$

4.4. MULTIPLE CHOICE QUESTIONS.

Problem 4.12. (5 points) Let $X \sim U(0, 1)$. Calculate $\mathbb{E}[X^3]$

(a) 1/6
(b) 1/4
(c) 1/3
(d) 1/2
(e) None of the above

Solution: (b)

$$\mathbb{E}[X^3] = \int_{-\infty}^{+\infty} x^3 f_X(x)\, dx = \int_0^1 x^3\, dx = \frac{1}{4}\,.$$

Problem 4.13. (5 points) Let the density function of a random variable X be given as

$$f_X(x) = cxI_{[0,1]}(x),$$

for some constant c. Find $\mathbb{E}[X^3]$.

(a) 2/3
(b) 2/5
(c) 2/7
(d) 2/9
(e) None of the above

Solution: (b)
Since the density function must integrate up to 1, we get $c = 2$. Whence,

$$\mathbb{E}[X^3] = 2\int_0^1 x^4\,dx = \frac{2}{5}\,.$$

Problem 4.14. (5 points) Let X and Y be independent Poisson random variables with parameters $\lambda_1 = 1$ and $\lambda_2 = 3$, respectively. Define $Z = X + Y$. Find $\mathbb{E}[Z^2]$.

(a) 10
(b) 20
(c) 25
(d) 30
(e) None of the above

Solution: (b)
Method I: Since X and Y are independent Poisson r.v.s, their sum is a Poisson r.v. whose parameter is the sum of the two parameters of X and Y. So,

$$\mathbb{E}[Z^2] = Var[Z] + (\mathbb{E}[Z])^2 = 4 + 4^2 = 20.$$

Method II:

$$\mathbb{E}[Z^2] = \mathbb{E}[(X+Y)^2] = \mathbb{E}[X^2 + 2XY + Y^2] = \mathbb{E}[X^2] + 2\mathbb{E}[XY] + \mathbb{E}[Y^2].$$

Since X and Y are independent, $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y] = 3$. We get

$$\mathbb{E}[Z^2] = 1 + 1^2 + 2\cdot 3 + 3 + 3^2 = 20.$$

Problem 4.15. (5 points) Let X_1 and X_2 be independent normal random variables with a common mean μ and a common standard deviation σ. Then, the random variable $X = X_1 + X_2$ has the following distribution:

(a) $Normal(mean = \mu, sd = \sigma)$
(b) $Normal(mean = 2\mu, sd = 2\sigma)$
(c) $Normal(mean = 2\mu, sd = \sigma\sqrt{2})$
(d) $Normal(mean = \mu, sd = \sigma\sqrt{2})$
(e) None of the above.

Solution: (c)
A generalization of this statement was done in class.

Note: In addition to these problems, you should also work on the following: suggested problems from the textbook, past in-term exams, practice problem sets for the past in-term exams, homework problems, problems done in class, any other textbook problems.