Homework assignment #6: Solutions

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Textbook exercises

Problem 1. $(1 + 1 + 1 + 2 = 5$ points)

Solve **Problem 4.2** from the textbook.

Solution:

```r
knitr::include_graphics("oc-p4-2.png")
```

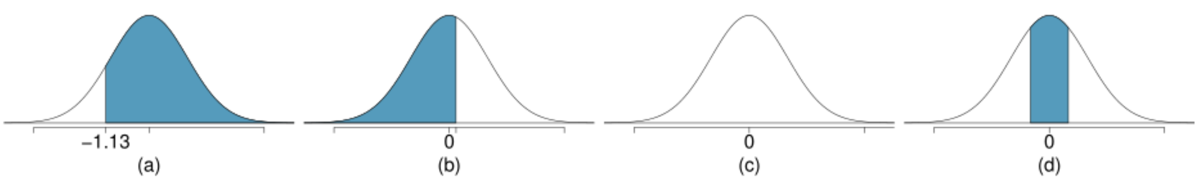
4.2

(a) $P(Z > -1.13) = 1 - 0.1292 = 0.8708 \rightarrow 87\%$

(b) $P(Z < 0.18) = 0.5714 \rightarrow 57\%$

(c) $P(Z > 8) \approx 0 \rightarrow 0\%$

(d) $P(|Z| < 0.5) = P(-0.5 < Z < 0.5) = P(Z < 0.5) - P(Z < -0.5)$
$= 0.6915 - 0.3085 = 0.3830 \rightarrow 38\%$



Problem 2. $(1 + 3 + 2 + 2 + 2 + 3 = 13$ points)

Solve **Problem 4.4** from the textbook.

Solution:

```r
knitr::include_graphics("oc-p4-4.png")
```

(a) Let X denote the finishing times of *Men, Ages 30 - 34* and Y denote the finishing times of emphWomen, Ages 25 - 29. Then,

$$X \sim N(\mu = 4313, \sigma = 583)$$
$$Y \sim N(\mu = 5261, \sigma = 807)$$

(b) The Z scores can be calculated as follows:

$$Z_{Leo} = \frac{x-\mu}{\sigma} = \frac{4948-4313}{583} = 1.09$$
$$Z_{Mary} = \frac{y-\mu}{\sigma} = \frac{5513-5261}{807} = 0.31$$

Leo finished 1.09 standard deviations above the mean of his group's finishing time and Mary finished 0.31 standard deviations above the mean of her group's finishing time.

(c) Mary ranked better since she she has a lower Z score indicating that her finishing time is relatively shorter.

(d) Leo:

$$\begin{aligned} P(Z > 1.09) &= 1 - P(Z < 1.09) \\ &= 1 - 0.8621 \\ &= 0.1379 \ \rightarrow \ 13.79\% \end{aligned}$$

(e) Mary:

$$\begin{aligned} P(Z > 0.31) &= 1 - P(Z < 0.31) \\ &= 1 - 0.6217 \\ &= 0.3783 \ \rightarrow \ 37.83\% \end{aligned}$$

(f) Answer to part (b) would not change as Z scores can be calculated for distributions that are not normal. However, we could not answer parts (c)-(e) since we cannot use the Z table to calculate probabilities and percentiles without a normal model.

Problem 3. $(3 + 3 = 6$ points)

Solve **Problem 4.6** from the textbook.

Solution:

```
knitr::include_graphics("oc-p4-6.png")
```

(a) The fastest 5% are in the 5^{th} percentile of the distribution. The Z score corresponding to the 5^{th} percentile of the normal distribution is approximately -1.64. Then,

$$Z = -1.65 = \frac{x - 4313}{583} \rightarrow x = -1.65 \times 583 + 4313 = 3351 \ sec$$

The fastest 5% of males in this age group finished in less than 56 minutes.

(b) The slowest 10% are in the 90^{th} percentile of the distribution. The Z score corresponding to the 90^{th} percentile of the normal distribution is approximately 1.28. Then,

$$Z = 1.28 = \frac{y - 5261}{807} \rightarrow y = 1.28 \times 807 + 5261 = 6294 \ sec$$

The slowest 10% of females in this age group took 1 hour, 45 minutes or longer to finish.

Problem 4. ($3 + 3 = 6$ points)

Solve **Problem 4.8** from the textbook.

Solution:

```r
knitr::include_graphics("oc-p4-8.png")
```

(a) Let X denote returns on this portfolio, then $X \sim N(\mu = 14.7, \sigma = 33)$.

$$P(X < 0) = P\left(Z < \frac{0 - 14.7}{33}\right) = P(Z < -0.45) = 0.3264 \rightarrow 32.64\%$$

(b) The Z score corresponding to the top 15% (or 85^{th} percentile) is 1.04.

$$Z = 1.04 = \frac{x - 14.7}{33} \rightarrow x = 1.04 \times 33 + 14.7 = 49.02$$

Problem 5. (5 points)

Solve **Problem 4.10** from the textbook.

Solution:

```r
knitr::include_graphics("oc-p4-10.png")
```

4.10 The Z score corresponding to the top 18.5% (or the 81.5^{th} percentile) is approximately 0.90.

$$Z = 0.90 = \frac{220 - 185}{\sigma} \rightarrow \sigma = \frac{220 - 185}{0.90} = 38.9 \ mg/dl$$

Additional problems

Problem 6. ($3 \times 2 = 6$ points)

Let Z be a standard normal random variable. Using the standard normal tables, calculate the following probabilities:

(i) $\mathbb{P}[-1.23 < Z < 2.37]$

(ii) $\mathbb{P}[1/Z < 1]$

(iii) $\mathbb{P}[Z^2 > 2.56]$

Solution: (i) $\mathbb{P}[-1.23 < Z < 2.37] = \mathbb{P}[Z < 2.37] - \mathbb{P}[Z \leq -1.23]$

```
pnorm(2.37) - pnorm(-1.23)
## [1] 0.8817574
```

(ii) $\mathbb{P}[1/Z < 1] = \mathbb{P}[Z < 0] + \mathbb{P}[Z > 1] = \mathbb{P}[Z < 0] + 1 - \mathbb{P}[Z \leq 1]$

```
pnorm(0) + 1 - pnorm(1)
## [1] 0.6586553
```

(iii) $\mathbb{P}[Z^2 > 2.56] = \mathbb{P}[|Z| > 1.6] = \mathbb{P}[Z < -1.6] + \mathbb{P}[Z > 1.6] = 2\mathbb{P}[Z < -1.6]$

```
2 * pnorm(-1.6)
## [1] 0.1095986
```

Problem 7. ($4 + 5 = 9$ points)

Source: Problem #139 from Moore-McCabe-Craig.

The interquartile range (IQR) of a distribution is defined as the distance between the first and the third quartiles.

(i) (4 points) What is the IQR for the standard normal distribution?

(ii) (5 points) What is the IQR for a normal distribution with mean μ and variance σ^2?

Solution: (i) The value $z*$ of the third quartile can be obtained as

$$z^* = \Phi^{-1}(0.75)$$

```
qnorm(0.75)
## [1] 0.6744898
```

By the symmetry of the standard normal distribution, we have

$$-z^* = \Phi^{-1}(0.25)$$

```r
qnorm(0.25)
## [1] -0.6744898
```

Therefore the IQR for the standard normal distribution is

```r
qnorm(0.75) - qnorm(0.25)
## [1] 1.34898
```

(ii) Any normal random variable $X \sim Normal(mean = \mu, variance = \sigma^2)$ can be represented as a linear transformation of the standard normal random variable Z. Namely, we have

$$X = \mu + \sigma Z.$$

So, the interquartile range is about 1.34898σ.