Poisson-Gamma Mixing.

Let N has a mixture distribution w/ the mixing parameter Λ.

More precisely, let

$$\boxed{N \mid \Lambda \sim \text{Poisson}(\text{mean} = \Lambda)}$$
$$\Lambda \sim \text{Gamma}(\alpha, \theta)$$

Q: What is the support of N?

$\longrightarrow$: $\text{Support}(N) = \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ $\blacksquare$

Focus on the pmf of N:

for $k = 0, 1, 2, \ldots$:

$$p_N(k) = \mathbb{P}[N = k] = F_N(k) - F_N(k-1)$$

$$= \int F_{N|\Lambda}(k|\lambda) f_\Lambda(\lambda)\, d\lambda - \int F_{N|\Lambda}(k-1|\lambda) f_\Lambda(\lambda)\, d\lambda$$

mixing

$$= \int \left(F_{N|\Lambda}(k|\lambda) - F_{N|\Lambda}(k-1|\lambda) \right) f_\Lambda(\lambda)\, d\lambda$$

$$= \int \mathbb{P}[N = k \mid \Lambda = \lambda]\, f_\Lambda(\lambda)\, d\lambda$$

In general:

$$p_N(k) = \int p_{N|\Lambda}(k|\lambda) f_\Lambda(\lambda)\, d\lambda$$

In this case:

$$p_N(k) = \int_0^{+\infty} \underbrace{e^{-\lambda} \cdot \frac{\lambda^k}{k!}}_{\text{conditional pmf}} \; \underbrace{\frac{\left(\frac{\lambda}{\theta}\right)^\alpha e^{-\frac{\lambda}{\theta}}}{\lambda \cdot \Gamma(\alpha)}}_{\Gamma\text{-density}} \, d\lambda$$

$$= \frac{1}{k! \cdot \theta^{\alpha} \cdot \Gamma(\alpha)} \int_0^{+\infty} e^{-\lambda \underbrace{\left(1+\frac{1}{\theta}\right)}_{=:\frac{1}{\theta^*}}} \lambda^{\overbrace{k+\alpha}^{=:\alpha^*} - 1} \, d\lambda \quad ✓$$

Consider: $Y \sim \Gamma(\alpha^*, \theta^*)$

$$1 = \int_0^{+\infty} f_Y(\lambda)\, d\lambda$$

$$= \int_0^{+\infty} \frac{\lambda^{\alpha^*-1}}{(\theta^*)^{\alpha^*}} \cdot \frac{e^{-\frac{\lambda}{\theta^*}}}{\Gamma(\alpha^*)} \, d\lambda$$

$$= \frac{1}{(\theta^*)^{\alpha^*}} \cdot \frac{1}{\Gamma(\alpha^*)} \int_0^{+\infty} \lambda^{\alpha^*-1} \cdot e^{-\frac{\lambda}{\theta^*}} \, d\lambda$$

$$\Rightarrow p_N(k) = \frac{1}{k! \cdot \theta^{\alpha} \cdot \Gamma(\alpha)} \cdot (\theta^*)^{\alpha^*} \cdot \Gamma(\alpha^*)$$

$$= \frac{1}{k! \cdot \theta^{\alpha} \cdot \Gamma(\alpha)} \cdot \left(\frac{\theta}{1+\theta}\right)^{k+\alpha} \cdot \Gamma(k+\alpha)$$

$$= \frac{\Gamma(k+\alpha)}{k! \cdot \Gamma(\alpha)} \cdot \left(\frac{\theta}{1+\theta}\right)^{k} \cdot \left(\frac{1}{1+\theta}\right)^{\alpha}$$

$\Rightarrow$ N is NEGATIVE BINOMIAL

w/ $r = \alpha$ and $\beta = \theta$