Name:

M339D=M389D Introduction to Actuarial Financial Mathematics
University of Texas at Austin
Solution: In-Term One
Instructor: Milica Čudina

All written work handed in by the student is considered to be
their own work, prepared without unauthorized assistance.

The University Code of Conduct

"The core values of The University of Texas at Austin are learning, discovery, freedom, leadership, individual opportunity, and responsibility. Each member of the university is expected to uphold these values through integrity, honesty, trust, fairness, and respect toward peers and community. As a student of The University of Texas at Austin, I shall abide by the core values of the University and uphold academic integrity."

"I agree that I have complied with the UT Honor Code during my completion of this exam."

Signature:

The maximum number of points on this exam is 100.

Problem 1.1. (5 points)The probability mass function p_X of a discrete random variable X is given by

$$p_X(x) = \begin{cases} 1/4, & \text{for } x = 10 \\ 2/3, & \text{for } x = 20 \\ 1/12, & \text{for } x = 40 \end{cases}$$

Find $\mathbb{E}[\max(X-15,0)]$.

Solution: The random variable $\max(X-15,0)$ has the following distribution

$$\max(X-15,0) \sim \begin{cases} 0 & \text{with probability } 1/4 \\ 5 & \text{with probability } 2/3 \\ 25 & \text{with probability } 1/12 \end{cases}$$

So, its expectation is

$$\mathbb{E}[\max(X-15,0)] = 5\left(\frac{2}{3}\right) + 25\left(\frac{1}{12}\right) = \frac{10}{3} + \frac{25}{12} = \frac{40+25}{12} = \frac{65}{12}.$$

.

Problem 1.2. (5 points) The current exchange rate is that 1 USD can be exchanged for 0.93 Swiss Francs. Hilda lives in Lausanne and receives her salary in Swiss Francs. She decides to spend 1000 Swiss Francs to buy USD and let the proceeds of the exchange accrue interest at the USD continuously compounded, risk-free interest rate. She will withdraw the balance in six months and exchange it back to Swiss Francs. Assume that there were no intermediate deposits or withdrawals. You know the following:

- The USD continuously compounded, risk-free interest rate is equal to $r_\$ = 0.05$.
- The Swiss Franc continuously compounded, risk-free interest rate is equal to $r_{SF} = 0.02$.

Given that the exchange rate at the end of the half equals 0.90 Swiss Francs per USD, how much (in Swiss Francs) does Hilda receive?

Solution: Hilda spends 1000 Swiss Francs. So, she receives $1000/0.93 = 1075.269$ USD. The balance in her USD savings account at the end of the six-month period is

$$1075.269e^{0.05/2} = 1102.489.$$

Taking into account the final exchange rate, her payoff is

$$1102.489(0.90) = 992.2401.$$

Problem 1.3. (5 points) Let the current price of a non-dividend-paying stock S be \$40 and let the current price of a continuous-dividend-paying stock Q be \$40, as well. The dividend yield for stock Q is 0.02.

You sell short one share of stock S at time-0 and intend to close the short sale at time-1. You use the proceeds to purchase one share of stock Q at time-0. You make no subsequent trades for a year.

The continuously compounded, risk-free interest rate is 0.04.

At time-1, you close the short sale and sell stock Q. If the time-1 stock price of S equals \$38 per share and the time-1 stock price for Q equals \$42 per share, what is your profit?

Solution: The initial cost is 0. So the profit will be equal to the payoff, i.e., it will be equal to

$$-S(T) + e^{0.02}Q(T) = -38 + e^{0.02}(42) = 4.848456.$$

Problem 1.4. (5 points) Provide the definition of the *forward contract.* Carefully describe all the transactions involved.

Solution: See the lecture notes.

Problem 1.5. (5 points) Here is some information about two forward contracts with delivery dates in one year:

	Current price of underlying	Forward price
Forward I	100	105
Forward II	90	92

Alfur enters a long position in Forward I and a short position in Forward II. It turns out that the final price of the underlying asset for Forward I equals \$102, while the final price of the underlying asset for Forward II equals \$89.

Let the continuously compounded, risk-free interest rate be 0.03.

What is Alfur's profit?

Solution: The initial cost of any forward contract is zero, so the profit and the payoff are equal. For the long Forward I, Alfur's payoff is

$$102 - 105 = -3.$$

For the short Forward II, Alfur's payoff is

$$92 - 89 = 3.$$

Alfur's overall profit is zero.

Problem 1.6. (5 points) The current spot price of corn is \$3.60 per bushel. There is a forward contract on corn for delivery in six months with the forward price of \$3.65. In order to hedge, farmer Brown shorts a 1000-bushel forward contract.

At the delivery date, it turns out that the spot price of corn is \$3.80.

You know that farmer Brown's total aggregate costs of production for 1000 bushels of corn equal \$3,450.

What is farmer Brown's profit?

Solution:

$$1000(3.65) - 3450 = 200$$

Problem 1.7. (5 points) The current price of zinc is \$2.74 kilograms. David needs to buy zinc in three months for the purposes of galvanization of a component his company is making. Per kilogram of zinc, the total revenue from the sale of the finished component will be \$10.23, while the total aggregate costs of non-zinc inputs equal \$5.12.

To hedge, David enters a forward contract for delivery of zinc in three months with the forward price equal to \$2.80.

The market price of zinc turns out to be \$2.78 in three months.

What is David's total profit (per kilo of zinc used)?

Solution:

$$10.23 - 5.12 - 2.80 = 2.31$$

Problem 1.8. (5 points) Provide the definition of the *European call option*. Carefully describe all the transactions involved.

Solution: See the lecture notes.

Problem 1.9. (5 points) The **owner** of a call option has ...

(a) an obligation to sell the underlying asset at the strike price.

(b) a right, but **not** an obligation, to sell the underlying asset at the strike price.

(c) an obligation to buy the underlying asset at the strike price.

(d) a right, but **not** an obligation, to buy the underlying asset at the strike price.

(e) None of the above.

Solution: (d)

Problem 1.10. (5 points) Let the function f be given by

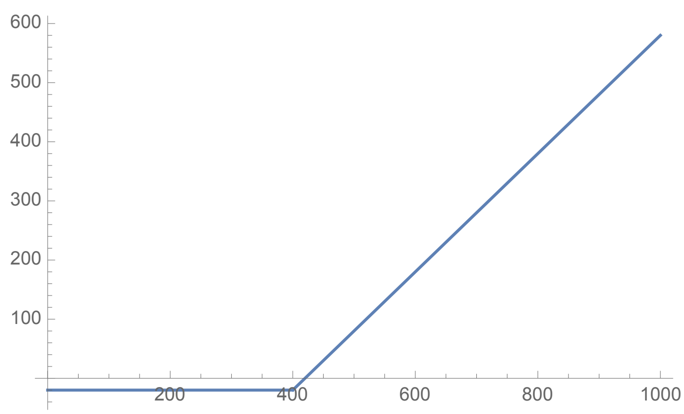
$$f(x) = \begin{cases} x - 400 & \text{for } x \geq 400 \\ 0 & \text{otherwise} \end{cases}$$

Draw the graph of the function g defined as

$$g(x) = f(x) - 20.$$

Carefully label your axes.

Solution:



Problem 1.11. (10 points) Let the function f be defined as

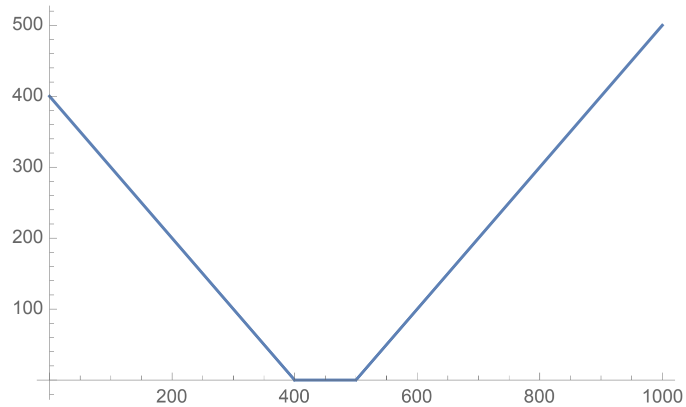
$$f(x) = \begin{cases} 400 - x & \text{for } x < 400 \\ 0 & \text{for } x \geq 400 \end{cases}$$

Let the function g be defined as

$$g(x) = \begin{cases} 0 & \text{for } x < 500 \\ x - 500 & \text{for } x \geq 500 \end{cases}$$

Draw the graph of the function $f + g$. Label your axes carefully.

Solution:



Problem 1.12. (5 points) Let the function $f : \mathbb{R}_+ \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} 10 - x, & \text{for } x < 12, \\ 0, & \text{for } x \geq 12. \end{cases}$$

Let the function $g : \mathbb{R}_+ \to \mathbb{R}$ be defined as

$$g(x) = \begin{cases} 2x, & \text{for } x < 20, \\ x - 20, & \text{for } x \geq 20. \end{cases}$$

Calculate $f(g(31))$.

Solution:

$$f(g(31)) = f(31 - 20) = f(11) = 10 - 11 = -1.$$

Problem 1.13. (10 points) Let the accumulation function be given by

$$a(t) = (1 + 0.03)^{t^2}(1 + 0.02)^{2t}.$$

What can you say about the continuously compounded, risk-free interest rate r associated with the above accumulation function?

Solution:

$$r = \frac{d}{dt}\ln(a(t)) = \frac{d}{dt}\ln[1.03^{t^2}1.02^{2t}] = \frac{d}{dt}\left[t^2\ln(1.03) + 2t\ln(1.02)\right] = 2t\ln(1.03) + 2\ln(1.02).$$

Problem 1.14. (20 points) In Country X, the latest census has revealed the following:

- 40% of the population exercise regularly,
- 30% own a dog,
- 20% like cauliflower,
- 60% of all dog owners exercise regularly,
- 10% own a dog and like cauliflower,
- 4% exercise regularly, own a dog and like cauliflower.

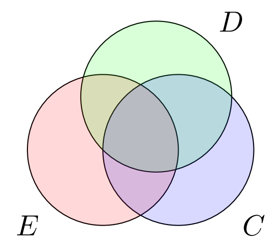
(1) (10pts) A person is selected at random. Compute the probability that they are a dog owner who does not exercise regularly.

(2) (10pts) If it is known that the randomly selected person either likes cauliflower or owns a dog (or both), what is the probability that he/she exercises regularly, owns a dog and likes cauliflower?

Solution:

Let E denote the event that the person chosen exercises regularly, D that he/she owns a dog and C that he/she likes cauliflower. The problem states that

$$\mathbb{P}[E|D] = 0.6, \quad \mathbb{P}[C \cap D] = 0.1, \quad \mathbb{P}[C \cap D \cap E] = 0.04,$$
$$\mathbb{P}[E] = 0.4, \quad \mathbb{P}[D] = 0.3, \quad \mathbb{P}[C] = 0.2.$$



It follows that $\mathbb{P}[E \cap D] = \mathbb{P}[E|D] \times \mathbb{P}[D] = 0.6 \times 0.3 = 0.18$, so $\mathbb{P}[D \cap E^c] = \mathbb{P}[D] - \mathbb{P}[E \cap D] = 0.12$.

Next, we are looking for $\mathbb{P}[E \cap C \cap D | C \cup D] = \mathbb{P}[E \cap C \cap D]/\mathbb{P}[C \cup D]$. We know that $\mathbb{P}[C \cup D] = \mathbb{P}[C] + \mathbb{P}[D] - \mathbb{P}[C \cap D] = 0.2 + 0.3 - 0.1 = 0.4$, so the required probability is $0.04/0.4 = 0.1$.

Problem 1.15. (5 points) You buy one share of continuous-dividend-paying stock today with the intention of holding onto this investment for exactly six months. The stock price today is \$60 per share and its dividend yield is 0.02. Let the continuously compounded risk-free interest rate be equal to 0.04. Assume continuous and immediate reinvestment of all dividends in the same stock. What is the profit of your investment if the stock price at the end of the six-month period equals \$62?

Solution:

$$62e^{0.02(0.5)} - 60e^{0.04(0.5)} = 62e^{0.01} - 60e^{0.02} = 1.41103.$$