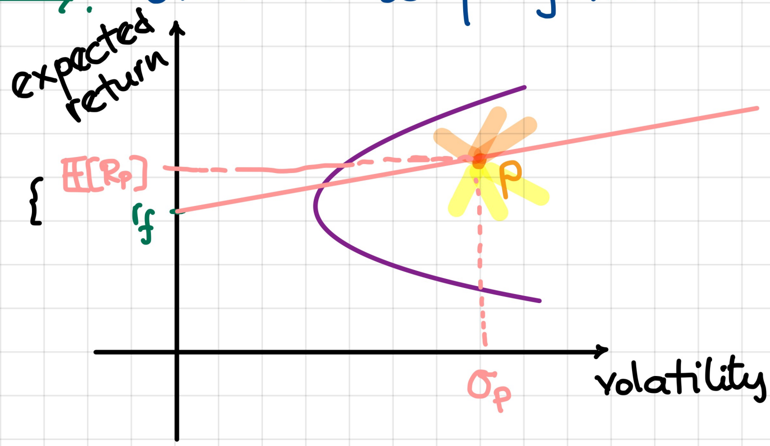
M339W: November 5th, 2021.

Required Returns.

Objective: To figure out if we can improve a portfolio by "adding" (more of) a particular security.

Q: What is the condition for the improvement of the portfolio and what consequences does this condition have on the desired expected return of the security?

$\longrightarrow$: Start w/ a portfolio P.



slope = Sharpe ratio of P

$$\boxed{\eta_P := \frac{\mathbb{E}[R_P] - r_f}{\sigma_P}}$$

Consider an investment I.

Construct P':

- keep P
- borrow $\boxed{x \cdot (\text{Value of } P)}$ @ the rate r_f
- invest $\boxed{x \cdot (\text{Value of } P)}$ in the investment I

Assume the weight x is small!

The new return: $R_{P'} = R_P - x \cdot r_f + x \cdot R_I$

$\Rightarrow$ The risk premium of P':

$$\mathbb{E}[R_{P'}] - r_f = \mathbb{E}[R_P] - x \cdot r_f + x \cdot \mathbb{E}[R_I] - r_f$$

$$= \underbrace{\left(\mathbb{E}[R_P] - r_f\right)}_{\text{the risk premium of P}} + \boxed{x \cdot \underbrace{\left(\mathbb{E}[R_I] - r_f\right)}_{\text{the risk premium of I}}} \checkmark$$

The variance of $R_{P'}$:

deterministic

$$\sqrt{Var[R_{P'}]} = Var[R_P - x \cdot r_f + x \cdot R_I] =$$

$$= Var[R_P + x \cdot R_I] =$$

$$= \sqrt{\underbrace{Var[R_P]}_{y_0} + \underbrace{2 \cdot x \cdot Cov[R_P, R_I]}_{dy}} + \Big(\underbrace{x^2 \cdot Var[R_I]}_{\text{lower order term}}\Big)$$

negligible

lower order term, i.e., ~~diversified~~ since x is small

$$f(y) = \sqrt{y} = y^{1/2}$$

$$f'(y) = \frac{1}{2} \cdot y^{-\frac{1}{2}} = \frac{1}{2} \cdot \frac{1}{\sqrt{y}}$$

$$f(y_0 + dy) = f(y_0) + f'(y_0)dy + \text{lower order terms}$$

$$= f(y_0) + \frac{1}{2} \cdot \frac{1}{\sqrt{y_0}} dy + \ldots$$

$$\sqrt{Var[R_{P'}]} = \sqrt{Var[R_P]} + \frac{1}{\cancel{2}} \cdot \frac{1}{\cancel{\sqrt{Var[R_P]}}} \cdot \cancel{2} \cdot x \cdot \overbrace{Cov[R_P, R_I]}^{corr[R_P,R_I] \cdot \cancel{SD[R_P]} \cdot SD[R_I]}$$

$$SD[R_{P'}] = SD[R_P] + \boxed{x \cdot SD[R_I] \cdot corr[R_P, R_I]}$$

"incremental" risk added to the portfolio by "adding" I.

Our criterion is:

$$\cancel{x}(\mathbb{E}[R_I] - r_f) > \underbrace{\cancel{x} \cdot SD[R_I] \cdot corr[R_P, R_I] \cdot \eta_P}_{}$$

the effect of staying on the line through P w/ the same "incremental" risk added

$$\mathbb{E}[R_I] - r_f > SD[R_I] \cdot corr[R_P, R_I] \cdot \frac{\mathbb{E}[R_P] - r_f}{SD[R_P]}$$

$$\mathbb{E}[R_I] > r_f + \boxed{\frac{\sigma_I}{\sigma_P} \cdot \rho_{P,I}} (\mathbb{E}[R_P] - r_f)$$

!!

β_I^P ... the beta of the investment I w/ the portfolio P

<u>Def'n</u>. The required return of investment I given portfolio P:

$$\boxed{r_I := r_f + \beta_I^P (\mathbb{E}[R_P] - r_f)}$$

<u>Important Consequence:</u>

<u>Recall</u>: A portfolio P^* is efficient if no other portfolio outperforms it (in the sense of the Sharpe ratio).

Imagine there is an investment I such that

$$\mathbb{E}[R_I] > r_I = r_f + \beta_I^{P^*} (\mathbb{E}[R_P] - r_f)$$

$\Rightarrow$ Portfolio P^* can be improved by investing in I.

$\Rightarrow\Leftarrow$ Contradicts the fact that P^* is efficient.

$\Rightarrow$ For any security I:

$$\boxed{\mathbb{E}[R_I] = r_f + \beta_I^* (\mathbb{E}[R_{P^*}] - r_f)}$$

beta of investment I w/ the efficient portfolio P^*.

16) You are given the following information about Stock X and the ~~market~~ P:

(i) The annual effective risk-free rate is 5%.

(ii) The expected return and volatility for Stock X and the ~~market~~ P are shown in the table below:

	Expected Return	Volatility
Stock X	5%	40%
~~Market~~ P	8%	25%

(iii) The correlation between the returns of stock X and the ~~market~~ P is –0.25.

~~Assume the Capital Asset Pricing Model holds~~. Calculate the required return for Stock X and determine if the investor should invest in Stock X.

(A) The required return is 1.8%, and the investor should invest in Stock X.

(B) The required return is 3.8%, and the investor should NOT invest in stock X.

(C) The required return is 3.8%, and the investor should invest in stock X.

(D) The required return is 6.2%, and the investor should NOT invest in Stock X.

(E) The required return is 6.2%, and the investor should invest in stock X.

$$r_X = r_f + \beta_X^P \left(\mathbb{E}[R_P] - r_f\right)$$

w/ $\beta_X^P = \frac{\sigma_X}{\sigma_P} \cdot \rho_{X,P} = \frac{0.4}{0.25}(-0.25) = -0.4$

$$r_X = 0.05 + (-0.4)(0.08 - 0.05) = \boxed{0.038}$$

$\mathbb{E}[R_X] = 0.05 > 0.038 = r_X$ $\Rightarrow$ (C)

14) You are given the following information about Stock X, Stock Y, and the ~~market:~~ P

(i) The annual effective risk-free rate is 4%.

(ii) The expected return and volatility for Stock X, Stock Y, and the market are shown in the table below:

	Expected Return	Volatility
Stock X	5.5%	40%
Stock Y	4.5%	35%
~~Market~~ P	6.0%	25%

(iii) The correlation between the returns of stock X and the ~~market~~ P is –0.25.

(iv) The correlation between the returns of stock Y and the ~~market~~ P is 0.30.

~~Assume the Capital Asset Pricing Model holds.~~ Calculate the required returns for Stock X and Stock Y, and determine which of the two stocks an investor should choose.

(A) The required return for Stock X is 3.20%, the required return for Stock Y is 4.84%, and the investor should choose Stock X.

(B) The required return for Stock X is 3.20%, the required return for Stock Y is 4.84%, and the investor should choose Stock Y.

(C) The required return for Stock X is 4.80%, the required return for Stock Y is 4.84%, and the investor should choose Stock X.

(D) The required return for Stock X is 6.40%, the required return for Stock Y is 3.16%, and the investor should choose Stock Y.

(E) The required return for Stock X is 3.50%, the required return for Stock Y is 3.16%, and the investor should choose both Stock X and Stock Y.