So, the UNBIASED estimator for σ^2 is

$$\boxed{S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y})^2}$$

Mean Squared Error.

Def'n. Let $\hat{\theta}$ be an estimator for the parameter θ:

1. the error of $\hat{\theta}$ is $\hat{\theta} - \theta$
2. the absolute error of $\hat{\theta}$ is $|\hat{\theta} - \theta|$;
3. the squared error of $\hat{\theta}$ is $(\hat{\theta} - \theta)^2$;
4. the mean squared error of $\hat{\theta}$ is

$$\boxed{MSE(\hat{\theta}) = \mathbb{E}[(\hat{\theta} - \theta)^2]}$$

Proposition. $\boxed{MSE(\hat{\theta}) = (bias(\hat{\theta}))^2 + Var[\hat{\theta}]}$

$$\begin{aligned}
\longrightarrow: MSE(\hat{\theta}) &= \mathbb{E}[(\hat{\theta} - \theta)^2] \\
&= \mathbb{E}[((\hat{\theta} - \mathbb{E}[\hat{\theta}]) + (\mathbb{E}[\hat{\theta}] - \theta))^2] \\
&= \boxed{\mathbb{E}[(\hat{\theta} - \mathbb{E}[\hat{\theta}])^2]} \\
&\quad + 2\boxed{\mathbb{E}[(\hat{\theta} - \mathbb{E}[\hat{\theta}])(\mathbb{E}[\hat{\theta}] - \theta)]} \\
&\quad + \boxed{\mathbb{E}[(\mathbb{E}[\hat{\theta}] - \theta)^2]} \\
&= Var[\hat{\theta}] + (bias(\hat{\theta}))^2 \\
&\quad \cancel{+ 2\mathbb{E}[(\hat{\theta} - \mathbb{E}[\hat{\theta}])(\mathbb{E}[\hat{\theta}] - \theta)]}
\end{aligned}$$

Focus on:

$$\mathbb{E}[(\hat{\theta} - \mathbb{E}[\hat{\theta}])(\mathbb{E}[\hat{\theta}] - \theta)] = (\mathbb{E}[\hat{\theta}] - \theta)\underbrace{\mathbb{E}[\hat{\theta} - \mathbb{E}[\hat{\theta}]]}_{\underbrace{\mathbb{E}[\hat{\theta}] - \mathbb{E}[\hat{\theta}]}_{=0}}$$

Def'n. The standard error of $\hat{\theta}$ is

$$\boxed{SE(\hat{\theta}) = \sqrt{Var[\hat{\theta}]}}$$

M378K Introduction to Mathematical Statistics
Problem Set #15
Bias. MSE.

Problem 15.1. Source: "Mathematical Statistics with Applications" by Wackerly, Mendenhall, Scheaffer.
Let $Y_1, , Y_2, Y_3$ be a random sample from $E(\tau)$. Consider the following five estimators of τ:

$$\hat{\theta}_1 = Y_1, \quad \hat{\theta}_2 = \frac{Y_1 + Y_2}{2}, \quad \hat{\theta}_3 = \frac{Y_1 + 2Y_2}{3}, \quad \hat{\theta}_4 = 3Y_{(1)}, \quad \hat{\theta}_5 = \bar{Y}.$$

Which ones of these estimators are unbiased? Among the unbiased ones which one has the smallest variance?

→ :

$\mathbb{E}[\hat{\theta}_1] = \mathbb{E}[Y_1] = \tau$ ✓

$\mathbb{E}[\hat{\theta}_2] = \mathbb{E}\left[\frac{Y_1+Y_2}{2}\right] = \tau$ ✓

$\mathbb{E}[\hat{\theta}_3] = \mathbb{E}\left[\frac{Y_1+2Y_2}{3}\right] = \tau$ ✓

$Y_{(1)} \sim E(\tau/3)$

$\mathbb{E}[\hat{\theta}_4] = \mathbb{E}[3 \cdot Y_{(1)}] = 3\boxed{\mathbb{E}[Y_{(1)}]} = 3 \cdot \frac{\tau}{3} = \tau$ ✓

$\mathbb{E}[\hat{\theta}_5] = \mathbb{E}[\bar{Y}] = \tau$ ✓

$Var[\hat{\theta}_1] = Var[Y_1] = \tau^2$

$Var[\hat{\theta}_2] = Var\left[\frac{Y_1+Y_2}{2}\right] = \frac{\tau^2}{2}$

$Var[\hat{\theta}_3] = Var\left[\frac{Y_1+2Y_2}{3}\right] = \frac{1}{9}\left(\overbrace{Var[Y_1]}^{\tau^2} + 4\overbrace{Var[Y_2]}^{\tau^2}\right) = \frac{5}{9}\tau^2$

$Var[\hat{\theta}_4] = Var[3Y_{(1)}] = 9 \cdot Var[Y_{(1)}] = 9 \cdot \frac{\tau^2}{9} = \tau^2$

$Var[\hat{\theta}_5] = Var[\bar{Y}] = \boxed{\frac{\tau^2}{3}}$

Remark: When we want to estimate the mean,

$\mathbb{E}[\bar{Y}] =$ mean, i.e., unbiased

$\Rightarrow MSE(\bar{Y}) = Var[\bar{Y}] = \frac{Var[Y_1]}{n}$

$SE[\bar{Y}] = \frac{SD[Y_1]}{\sqrt{n}}$

Problem 15.2. *Suppose that the two estimators $\hat{\theta}_1$ and $\hat{\theta}_2$ are unbiased. We know that* $\mathrm{Var}[\hat{\theta}_1] = \sigma_1^2$ *and* $\mathrm{Var}[\hat{\theta}_2] = \sigma_2^2$.

Consider the estimator all the estimators that can be obtained as convex combinations of $\hat{\theta}_1$ and $\hat{\theta}_2$, i.e., all the estimators of the form

$$\hat{\theta} = \alpha\hat{\theta}_1 + (1-\alpha)\hat{\theta}_2.$$

What can you say about the bias *of estimators $\hat{\theta}$ of the form above? Assuming that $\hat{\theta}_1$ and $\hat{\theta}_2$ are* **independent**, *for which weight α is the variance minimal?*

$\longrightarrow$:

$$\mathbb{E}[\hat{\theta}] = \mathbb{E}[\alpha\hat{\theta}_1 + (1-\alpha)\hat{\theta}_2] \overset{\uparrow}{=} \alpha\overbrace{\mathbb{E}[\hat{\theta}_1]}^{\theta} + (1-\alpha)\overbrace{\mathbb{E}[\hat{\theta}_2]}^{\theta} = \theta$$

linearity

$\Rightarrow \hat{\theta}$ is unbiased for all α

$$\mathrm{Var}[\hat{\theta}] \xrightarrow[\alpha]{} \min$$

$$\mathrm{Var}[\alpha\hat{\theta}_1 + (1-\alpha)\hat{\theta}_2] \xrightarrow[\alpha]{} \min$$

independence

$$\alpha^2 \cdot \sigma_1^2 + (1-\alpha)^2 \cdot \sigma_2^2 \xrightarrow[\alpha]{} \min$$

$$\cancel{2}\alpha\sigma_1^2 + \cancel{2}(1-\alpha)(-1)\sigma_2^2 = 0$$

$$\alpha\sigma_1^2 + \alpha\sigma_2^2 = \sigma_2^2$$

$$\boxed{\alpha^* = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}}$$

$\square$