Homework assignment #3: Solutions

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2021-09-19

More probability review

Problem #1.

A piggy bank contains coins of three different types: T_1, T_2 and T_3. There are twice as many type T_1 coins as type T_2 coins, and twice as % many type T_2 coins as type T_3 coins. The coins are indistinguishable to touch.

a. (5 points)

A coin is extracted from the piggy bank at random. Let the probability that the coin is of type T_i be denoted by p_i for $i = 1, 2, 3$. Find p_1, p_2 and p_3.

Solution: From the problem statement, we have that

$$p_1 = 2p_2 = 4p_3.$$

Since $p_1 + p_2 + p_3 = 1$, we have that $p_3 = 1/7$, $p_2 = 2/7$ and $p_1 = 4/7$.

b. (10 points)

Coins of type T_1 are fair, coins of type T_2 come up heads (H) when tossed with probability 3/10, and coins of type T_3 come up heads (H) when tossed with probability 1/10.

A coin is drawn from the piggy bank at random and tossed. What is the probability that the result of the coin toss was heads?

Solution: By the rule of average conditional probabilities, we get

$$\begin{aligned}\mathbb{P}[H] &= \mathbb{P}[T_1]\mathbb{P}[H \mid T_1] + \mathbb{P}[T_2]\mathbb{P}[H \mid T_2] + \mathbb{P}[T_3]\mathbb{P}[H \mid T_3] \\ &= \frac{4}{7} \cdot \frac{1}{2} + \frac{2}{7} \cdot \frac{3}{10} + \frac{1}{7} \cdot \frac{1}{10} = \frac{27}{70}\,.\end{aligned}$$

Problem 2. (15 points)

There are three variants of a genetic marker for *goosepox*: **immune, middling,** and **susceptible**. In the population, 10% are **immune**, 70% are **middling**, and 20% are **susceptible**. Within each category, here are the chances of contracting *goosepox*:

- for **immune** it is 0%,

- for **middling** it is 50%, and
- for **susceptible** it is 90%.

Say that you learn that a randomly chosen individual contracted *goosepox*. What is the probability that this individual was **susceptible**?

Solution: By the Bayes' Theorem,

$$\mathbb{P}[Susc \mid Goose] = \frac{\mathbb{P}[Goose \mid Susc]\mathbb{P}[Susc]}{\mathbb{P}[Goose]}.$$

By the Law of Total Probability, we have

$$\begin{aligned}\mathbb{P}[Goose] &= \mathbb{P}[Goose \mid Imm]\mathbb{P}[Imm] + \mathbb{P}[Goose \mid Mid]\mathbb{P}[Mid] + \mathbb{P}[Goose \mid Susc]\mathbb{P}[Susc] \\ &= 0(0.10) + 0.5(0.7) + 0.9(0.2) = 0.53.\end{aligned}$$

So,

$$\mathbb{P}[Susc \mid Goose] = \frac{0.9(0.2)}{0.53} = 0.3396226.$$

The binomial distribution

Problem 3. (5 points)

Using both R and analytic methods, find the probability that three independent tosses of a fair coin have exactly two successes.

Solution: This is exactly the probability that a random variable $X \sim Binomial(size = 3, p = 0.5)$ takes the value 2. We have

$$\mathbb{P}[X = 2] = \binom{3}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right) = \frac{3}{8} = 0.375.$$

In R, we have

```
dbinom(2,3,0.5)
## [1] 0.375
```

Problem 4. (10 points)

Using both R and analytic methods, find the probability that four independent tosses of a fair coin have at most two successes.

Solution: This is exactly the probability that a random variable $X \sim Binomial(size = 4, p = 0.5)$ takes the value less than or equal to 2. We have

$$\begin{aligned}\mathbb{P}[X \leq 2] &= \mathbb{P}[X = 0] + \mathbb{P}[X = 1] + \mathbb{P}[X = 2] \\ &= \binom{4}{0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^4 + \binom{4}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^3 + \binom{4}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 \\ &= (1 + 4 + 6)\left(\frac{1}{2}\right)^4 = \frac{11}{16} = 0.6875.\end{aligned}$$

In R, we have

```r
pbinom(2,4,0.5)
## [1] 0.6875
```

Problem 5. (5 points)

Consider a coin whose probability of landing on *heads* is 1/5. You encode *heads* as "success". Using both `R` and analytic methods, find the probability that five independent tosses of this coin have exactly four successes.

Solution: This is exactly the probability that a random variable $X \sim Binomial(size = 5, p = 0.2)$ takes the value equal to 4. We have

$$\mathbb{P}[X = 4] = \binom{5}{4}\left(\frac{1}{5}\right)^4\left(\frac{4}{5}\right)^1 = \frac{4}{5^4} = 0.0064.$$

In `R`, we have

```r
dbinom(4,5,0.2)
## [1] 0.0064
```