UNIVERSITY OF TEXAS AT AUSTIN

Quiz #4

Poisson-gamma mixture. Poisson "thinning". $(a, b, 0)$ class.

Please, provide your **complete solutions** to the following questions:

Problem 4.1. (5 points) Let us denote the claim count r.v. by N. We are given that N is a mixture random variable such that

$$N \mid \Lambda = \lambda \sim Poisson(\lambda)$$

while Λ is Gamma distributed with mean 4 and variance 8. Calculate $F_N(1)$.

Solution: The distribution of N is negative binomial. Let us find its parameters r and β from the provided mean and variance of the gamma dsitribution. Using the fact that $N \mid \Lambda$ is Poisson, we get

$$r\beta = \mathbb{E}[N] = \mathbb{E}[\mathbb{E}[N \mid \Lambda]] = \mathbb{E}[\Lambda] = 4,$$
$$r\beta(1+\beta) = Var[N] = \mathbb{E}[Var[N \mid \Lambda]] + Var[\mathbb{E}[N \mid \Lambda]] = \mathbb{E}[\Lambda] + Var[\Lambda] = 12.$$

So, $\beta = 2$ and $r = 2$. Finally, using our tables, we get

$$F_N(1) = p_N(0) + p_N(1) = \frac{1}{3^2} + \frac{2 \cdot 2}{3^3} = \frac{7}{27}.$$

Problem 4.2. (5 points) Let the random variable N be in the $(a, b, 0)$ class with $a = 0$ and $b = 8$. Find $\mathbb{P}[N = 10]$.

Solution: From the given values of a and b, we conclude that $N \sim Poisson(\lambda = 8)$. So,

$$\mathbb{P}[N = 10] = e^{-8} \frac{8^{10}}{10!} \approx 0.0993.$$

Problem 4.3. (5 points) Let the number of floods in a calendar year be denoted by N and modeled using the Poisson distribution with mean 5. We say that a flood is "minor" if the damages associated with it do not exceed \$1,000,000. Otherwise, a flood is designated as "major". The number of floods and the damages caused by individual floods are assumed to be independent.

Assume that the probability that an observed flood is "major" equals 1/5.

Find the probability that the number of "major" floods is 2, given that the **total** number of floods in that year equals 5.

Solution: Let N_1 denote the r.v. which stands for the number of "major"floods, and let N_2 be the number of "minor" floods. According to the *"Thinning" theorem*, N_1 and N_2 are independent and

$$N_1 \sim Poisson(\frac{1}{5} \cdot 5 = 1),$$
$$N_2 \sim Poisson(\frac{4}{5} \cdot 5 = 4).$$

We are ready to calculate the conditional probability

$$\begin{aligned}\mathbb{P}[N_1 = 2 \mid N = 5] &= \frac{\mathbb{P}[N_1 = 2, N = 5]}{\mathbb{P}[N = 5]} \\ &= \frac{\mathbb{P}[N_1 = 2, N_1 + N_2 = 5]}{\mathbb{P}[N = 5]} \\ &= \frac{\mathbb{P}[N_1 = 2, N_2 = 3]}{\mathbb{P}[N = 5]}.\end{aligned}$$

Since N_1 and N_2 are independent, this probability equals

$$\begin{aligned}
\frac{\mathbb{P}[N_1 = 2]\,\mathbb{P}[N_2 = 3]}{\mathbb{P}[N = 5]} &= \frac{e^{-1}\,\frac{1^2}{2!}\cdot e^{-4}\,\frac{4^3}{3!}}{e^{-5}\,\frac{5^5}{5!}} \\
&= \frac{\frac{1^2}{2!}\cdot\frac{4^3}{3!}}{\frac{5^5}{5!}} \\
&= \frac{4^3\cdot 5!}{5^5\cdot 2!\cdot 3!} = \frac{2^7}{5^4} = 0.2048.
\end{aligned}$$

Of course, we obtained the binomial conditional distribution above. This is a fact we have shown in class and you could have just used it directly.