Black-Scholes: Partial Expectation

The Model.

Under the risk-neutral probability measure $\mathbb{P}^*$

$$S(T) = S(0)\, e^{\left(r - \frac{\sigma^2}{2}\right)\cdot T + \sigma\sqrt{T}\cdot Z} \quad \text{w/ } Z \sim N(0,1)$$

The Motivation.

$$V_c(0) = e^{-rT}\,\mathbb{E}^*[Y_c(T)]$$

$$\ldots = e^{-rT}\,\boxed{\mathbb{E}^*[S(T)\cdot \mathbb{I}_{[S(T)\geq K]}]} - e^{-rT}\cdot K \cdot \boxed{\mathbb{P}^*[S(T)\geq K]}$$

$$\mathbb{P}^*[S(T)\geq K] = N(d_2)$$

w/ $d_2 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)\cdot T\right]$

$$\mathbb{E}^*[S(T)\cdot \mathbb{I}_{[S(T)\geq K]}] = ?$$

Method. Use the defining formula for the expectation of a f'tion of a r.v.

In this case, the r.v. is $Z \sim N(0,1)$

$$\{S(T)\geq K\} = \left\{S(0)\, e^{\left(r - \frac{\sigma^2}{2}\right)\cdot T + \sigma\sqrt{T}\cdot Z} \geq K\right\}$$

$$\vdots$$

$$= \{Z > -d_2\}$$

z ... our dummy variable within the integral; corresponds to Z,

i.e.,

$$g(z) = S(0)\, e^{\left(r - \frac{\sigma^2}{2}\right)\cdot T + \sigma\sqrt{T}\cdot z}\cdot \mathbb{I}_{[z \geq -d_2]}$$

so that $g(Z) = S(T)\cdot \mathbb{I}_{[S(T)\geq K]}$

$$\mathbb{E}^*[g(Z)] = \int_{-d_2}^{\infty} S(0)\, e^{\left(r - \frac{\sigma^2}{2}\right)\cdot T + \sigma\sqrt{T}\cdot z}\cdot f_Z(z)\,dz = \ldots$$

... lots of algebra / calculus

We get:

$$\mathbb{E}^*\left[S(T)\cdot \mathbb{I}_{[S(T)\geq K]}\right] = \boxed{S(0)e^{rT}}\cdot \boxed{N(d_1)}$$

$\mathbb{E}^*[S(T)]$

$$\text{w/ } d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T\right]$$

The expectation under $\mathbb{P}^*$ of the call payoff:

$$\mathbb{E}^*[V_c(T)] = S(0)e^{rT}N(d_1) - K\cdot N(d_2)$$

w/ d_1 as above and $\boxed{d_2 = d_1 - \sigma\sqrt{T}}$

$\Rightarrow$ The Black-Scholes call price

$$\boxed{V_c(0) = S(0)N(d_1) - Ke^{-rT}N(d_2)} \longleftarrow$$

$\Rightarrow$ The Black-Scholes put price

By put-call parity:

$$V_c(0) - V_p(0) = S(0) - Ke^{-rT}$$

$$\begin{aligned}
V_p(0) &= V_c(0) - S(0) + Ke^{-rT} \\
&= S(0)N(d_1) - Ke^{-rT}\cdot N(d_2) \\
&\quad - S(0) \qquad + Ke^{-rT} \\
&= S(0)\underbrace{(N(d_1)-1)}_{-N(-d_1)} + Ke^{-rT}\underbrace{(1-N(d_2))}_{N(-d_2)}
\end{aligned}$$

Symmetry of N(0,1)

$$\boxed{V_p(0) = Ke^{-rT}N(-d_2) - S(0)N(-d_1)}$$

Problem 14.3. (5 pts) Assume the Black-Scholes framework. Let the current price of a non-dividend-paying stock be equal to $S(0) = 95$ and let its volatility be equal to 0.35. Consider a European call on that stock with strike 100 and exercise date in 9 months. Let the risk-free continuously compounded interest rate be 6% per annum.

Denote the price of the call by $V_C(0)$. Then,

(a) $V_C(0) < \$5.20$

(b) $\$5.20 \leq V_C(0) < \$7.69$

(c) $\$7.69 \leq V_C(0) < \$9.04$

(d) $9.04 \leq V_C(0) < \$11.25$

(e) None of the above.

→: We'll use the Black·Scholes call price:

$$V_c(0) = S(0)\cdot N(d_1) - Ke^{-rT}\cdot N(d_2)$$

$$\text{w/ } d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)\cdot T\right]$$

$$\text{and } d_2 = d_1 - \sigma\sqrt{T}$$

✓ 1st Calculate d_1 and d_2.

✓ 2nd Use the standard normal table or 'R' (pnorm).

3rd Combine into the BS price.

$$d_1 = \frac{1}{0.35\sqrt{3/4}}\left[\ln\left(\frac{95}{100}\right) + \left(0.06 + \frac{0.35^2}{2}\right)\cdot\left(\frac{3}{4}\right)\right] = \underline{0.1307} \approx 0.13$$

$$d_2 = d_1 - \sigma\sqrt{T} = d_1 - 0.35\sqrt{3/4} = \underline{-0.1733} \approx -0.17$$

$$N(d_1) \approx N(0.13) = 0.5517$$

$$N(d_2) \approx N(-0.17) = 0.4325$$

$$V_c(0) = 95\cdot 0.5517 - 100e^{-0.06(3/4)}\cdot 0.4325 = \underline{11.06}$$

□

Problem 14.4. Assume the Black-Scholes setting. Let $S(0) = \$63.75$, $\sigma = 0.20$, $r = 0.055$. The stock pays no dividend and the option expires in 50 days (simplify the number of days in a year to 360).

What is the price of a \$60-strike European put?

(a) 0.66
(b) 0.55
(c) 0.44
(d) 0.37
(e) None of the above.

→ :

$$d_1 = \frac{1}{0.2\sqrt{\frac{50}{360}}}\left[\ln\left(\frac{63.75}{60}\right) + \left(0.055 + \frac{0.04}{2}\right)\cdot\left(\frac{50}{360}\right)\right]$$

$$d_1 = \underline{0.9531} \approx 0.95$$

$$d_2 = d_1 - \sigma\sqrt{T} = d_1 - 0.2\sqrt{\frac{50}{360}} = \underline{0.8786} \approx 0.88$$

$$N(-d_1) = N(-0.95) = 0.1711$$

$$N(-d_2) = N(-0.88) = 0.1894$$

$$V_P(0) = Ke^{-rT}\cdot N(-d_2) - S(0)N(-d_1)$$

$$= 60e^{-0.055\left(\frac{50}{360}\right)}\cdot 0.1894 - 63.75\cdot 0.1711$$

$$= \underline{0.37}$$

□