NAME:

M339W/389W Financial Mathematics for Actuarial Applications
University of Texas at Austin
In-Term Exam I
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The maximum number of points on this exam is 100.

Problem 1.1. (5 points) Provide a real-life example of a real option. How are real options different from, say, options we covered in M339D? If you provide the example equivalent to the computational problem from this exam, you will get zero points.

Solution: Solutions vary.

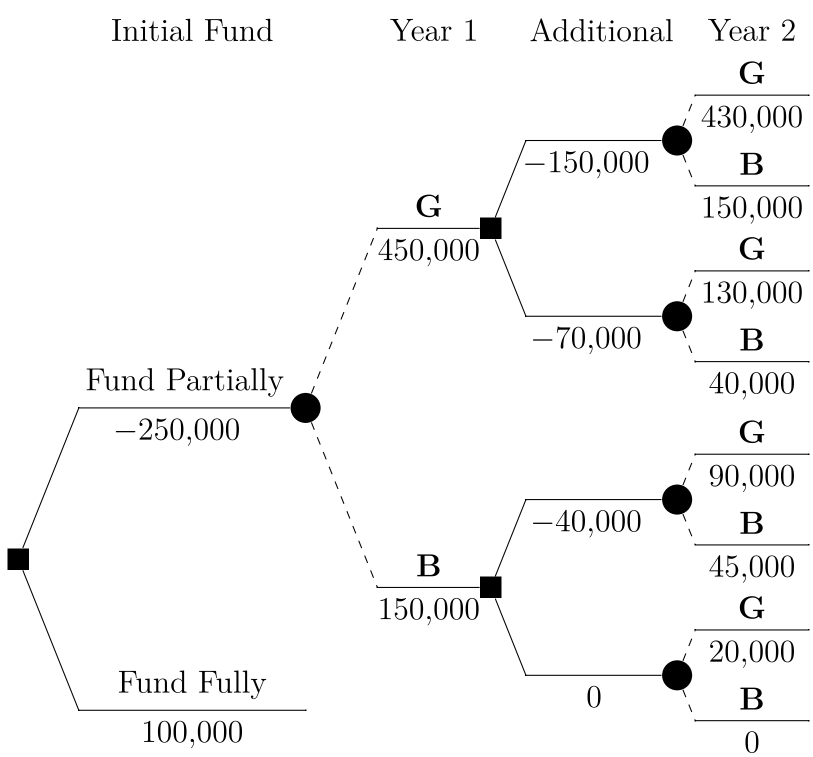
Problem 1.2. (5 points) A publisher is considering a young-adult duology: *The Willing Warlock* and *The Successful Sorcerer*. When the publication of the duology is fully funded at time−0 the project has a net present value of 100,000.

The decision tree below shows the cash flows when the marketing at the beginning of the Year 1 (i.e., at $t = 0$) is only partial with an option to provide different amounts of funding at the beginning of Year 2 (i.e., at $t = 1$) depending on how well *The Willing Warlock* did.

This tree reflects two possible receptions of the two tomes at each information node (**G** = good, **B** = bad). As with most young-adult literature, it's a cointoss, i.e., the probability of the first book being a success is given to be 1/2.

Assume the interest rate is 0%.

Find the **initial** (i.e., at $t = 0$) value of the option to fund partially.



Solution: As usual, when pricing options, we are moving backwards through the tree.

- In the *uppermost final* information node, the possible cashflows are 430,000 with probability 1/2 and 150,000 with probability 1/2. So, the value of the project at that node equals

$$430000\left(\frac{1}{2}\right)+150000\left(\frac{1}{2}\right)=290000.$$

- In the *second-by-height final* information node, the possible cashflows are 130,000 with probability 1/2 and 40,000 with probability 1/2. So, the value of the project at that node equals

$$130000\left(\frac{1}{2}\right)+40000\left(\frac{1}{2}\right)=85000.$$

- In the *third-by-height final* information node, the possible cashflows are 90,000 with probability 1/2 and 45,000 with probability 1/2. So, the value of the project at that node equals

$$90000\left(\frac{1}{2}\right)+45000\left(\frac{1}{2}\right)=67500.$$

- In the *lowest final* information node, the possible cashflows are 20,000 with probability 1/2 and 0 with probability 1/2. So, the value of the project at that node equals

$$20000\left(\frac{1}{2}\right)=10000.$$

We continue working backwards, at the **upper decision** node at the end of Year 1, we can go "up" or "down" in the tree.

- We go "up" by investing 150,000; combining this cashflow with the average revenue at the *uppermost final* node, we get the total effect of going "up" to be

$$290000-150000=140000.$$

- We go "down" by investing 70,000; combining this cashflow with the average revenue at the *second-by-height final* node, we get the total effect of going "down" to be

$$85000-70000=15000.$$

Comparing the two values we obtained, we conclude that the **optimal** decision at this node is to go "up" and we keep the value of this project at this node to be

$$140000+450000=590000.$$

Here, we took into account that the first book was a success resulting in 450,000 in revenue in Year 1.

Similarly, at the **lower decision** node at the end of Year 1, we can go "up" or "down" in the tree.

- We go "up" by investing 40,000; combining this cashflow with the average revenue at the *third-by-height final* node, we get the total effect of going "up" to be

$$67500-40000=27500.$$

- We go "down" by investing nothing; so, the total effect of going "down" is 10000.

Comparing the two values we obtained, we conclude that the **optimal** decision at this node is to go "up" and we keep the value of this project at this node to be

$$27500 + 150000 = 177500.$$

Here, we took into account that the first book was "just readable" resulting in 150,000 in revenue in Year 1.

Altogether, at the information node corresponding to Year 1, we have that the expected value of the project is

$$590000\left(\frac{1}{2}\right) + 177500\left(\frac{1}{2}\right) = 383750.$$

Now, we take into account that we funded the duology partially with 250,000. So, the total expected present value of the cashflows we get should we decide to fund partially is

$$383750 - 250000 = 133750.$$

The total value of the option is

$$133750 - 100000 = 33750.$$

Problem 1.3. (5 points) Describe the mechanism of a prepaid forward contract and a forward contract. Give a real-life example of a situation which can be understood as a prepaid forward contract or a forward contract.

Solution: See your M339D notes.

Problem 1.4. (5 points) A discrete-dividend-paying stock sells today for \$90 per share. The continuously compounded, risk-free interest rate is 0.05. The first dividend will be paid at in three months in the amount of \$2.50. The remaining dividends will be equal to \$2 and continue to be paid out quarterly for three more years. What is the **prepaid forward price** of this stock for delivery in eight months?

Solution:

$$F^P_{0,7/12}(S) = 90 - 2.50e^{-0.0125} - 2e^{-0.025} = 85.58044.$$

Problem 1.5. (10 points) What are the three properties of realized returns in the Black-Scholes model? *Caveat: Nobody is asking you to state the actual model; you need to actually state the three properties.*

Solution:

- **i. independence:** realized returns over non-overlapping time intervals are **independent**;
- **ii. time homogeneity:** realized returns over time intervals of the same length are **identically distributed**;
- **iii. additivity:** for $t, u, s \geq 0$, we have

$$R(t, t+u+s) = R(t, t+u) + R(t+u, t+u+s)$$

Problem 1.6. (10 points) A continuous-dividend-paying stock is currently valued at \$80 per share. Its annual mean rate of return is given to be 10% while its dividend yield equals 2% and its volatility is given to be 30%.

Assuming the lognormal stock-price model, find

$$\mathbb{E}[S(4) \,|\, S(4) > 90].$$

Solution: In our usual notation,

$$\mathbb{E}[S(T) \,|\, S(T) > K] = \frac{S(0)e^{(\alpha-\delta)T}N(\hat{d}_1)}{N(\hat{d}_2)}.$$

with

$$\hat{d}_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(\alpha - \delta + \frac{\sigma^2}{2}\right)T\right],$$
$$\hat{d}_2 = \hat{d}_1 - \sigma\sqrt{T}.$$

In the present probem,

$$\hat{d}_1 = \frac{1}{0.3\sqrt{4}}\left[\ln\left(\frac{80}{90}\right) + \left(0.10 - 0.02 + \frac{0.09}{2}\right) \times 4\right] = 0.6370283,$$
$$\hat{d}_2 = 0.6370283 - 0.3\sqrt{4} = 0.03702827.$$

So, our answer is

$$\mathbb{E}[S(4) \,|\, S(4) > 90] = \frac{80e^{(0.10-0.02)\times 4}N(0.6370283)}{N(0.03702827)} = 157.9345.$$

Problem 1.7. (5 points) In the setting of the one-period binomial model, denote by i the **effective** interest rate **per period**. Let u denote the "up factor" and let d denote the "down factor" in the stock-price model. Which of the following statements is the correct no-arbitrage condition for the binomial asset-pricing model?

(a) $d < 1 + i < u$

(b) $d < 1 < u$

(c) $d < e^i < u$

(d) $d = \frac{i}{1+i}$

(e) None of the above.

Solution: (a)

Problem 1.8. (5 pts) Consider a non-dividend-paying stock currently priced at \$100 per share.

The price of this stock in one year is modeled using a one-period binomial tree under the assumption that the stock price can either go up to 110 or down to 90.

Let the continuously-compounded, risk-free interest rate equal 0.04. What is the risk-neutral probability of the stock price going up?

(a) About 0.2969
(b) About 0.3039
(c) About 0.5000
(d) About 0.7041
(e) None of the above.

Solution: (d)

$$p^* = \frac{100e^{0.04} - 90}{110 - 90} = 0.7041.$$

Problem 1.9. (5 points) You roll a fair tetrahedron whose sides are labeled by $1, 2, 3$, and 4 a total of 4000 times. What is the approximate probability that you see a 1 strictly more than 1025 times? There is no need to use the continuity correction.

Solution: The number of heads is $X \sim Binomial(n = 4000, p = 0.25)$. Evidently, we can use the normal approximation to the binomial. We have

$$\mu_X = \mathbb{E}[X] = 1000 \quad \text{and} \quad \sigma_X = 27.38613.$$

The probability we are seeking is

$$\mathbb{P}[X > 1025] \approx 1 - N\left(\frac{1025 - 1000}{27.38613}\right) \approx 1 - N(0.91) = 1 - 0.8186 = 0.1814.$$

Problem 1.10. (5 points) The **writer** of a call option has ...

(a) an obligation to sell the underlying asset at the strike price.
(b) a right, but **not** an obligation, to sell the underlying asset at the strike price.
(c) an obligation to buy the underlying asset at the strike price.
(d) a right, but **not** an obligation, to buy the underlying asset at the strike price.
(e) None of the above.

Solution: (a)

Problem 1.11. (5 points) The current price of a non-dividend-paying stock is \$50 per share. You observe that the price of a three-month, at-the-money American call option on this stock equals \$3.50.

The continuously-compounded, risk-free interest rate is 0.04.

Find the premium of the European three-month, at-the-money put option on the same underlying asset.

Solution: Recall that the price of an American call on a non-dividend-paying stock equals the price of the otherwise identical European call option. So, put-call parity yields

$$V_P(0) = V_C(0) + Ke^{-rT} - S(0) = 3.50 - 50(e^{-0.01} - 1) = 3.0025.$$

Problem 1.12. (10 points) Let the current price of a continuous-dividend-paying stock be denoted by $S(0)$. We model the time$-T$ stock price as lognormal. The mean rate of return on the stock is 0.10, its dividend yield is 0.01, and its volatility is 0.25. The continuously-compounded, risk-free interest rate is 0.02. You invest in one share of stock at time-0 and simultaneously deposit an amount $\varphi S(0)$ in a savings account. What should the proportion φ be so that the VaR at the level 0.01 of your total wealth at time-1 equals today's stock price $S(0)$?

Solution: The total wealth at time-1 is equal to $e^{\delta}S(1) + \varphi S(0)e^r$. So, our condition on the VaR is

$$\mathbb{P}[e^{\delta}S(1) + \varphi S(0)e^r < S(0)] = 0.01.$$

The above is equivalent to

$$\mathbb{P}[e^{\delta}S(0)e^{\alpha-\delta-\frac{\sigma^2}{2}+\sigma Z} + \varphi S(0)e^r < S(0)] = \mathbb{P}[S(0)e^{\alpha-\frac{\sigma^2}{2}+\sigma Z} + \varphi S(0)e^r < S(0)] = 0.01$$

with $Z \sim N(0,1)$. We have that the above is, in turn, equivalent to

$$\mathbb{P}[e^{\alpha-\frac{\sigma^2}{2}+\sigma Z} + \varphi e^r < 1] = 0.01.$$

The constant z^* such that $\mathbb{P}[Z < z^*] = 0.01$ equals -2.326348. Therefore,

$$e^{\alpha-\frac{\sigma^2}{2}+\sigma(-2.326348)} + \varphi e^r = 1.$$

So,

$$\varphi = e^{-r}\left(1 - e^{\alpha-\frac{\sigma^2}{2}+\sigma(-2.326348)}\right) = e^{-0.02}\left(1 - e^{0.10-\frac{(0.25)^2}{2}+0.25(-2.326348)}\right) = 0.355407.$$

Problem 1.13. (10 points) Your goal is to price a call option on a futures contract. The movements of the futures price are modeled by a binomial tree. You are given:

(i) Each period is three months.
(ii) $u_F/d_F = 5/4$, where u_F is one plus the rate of gain on the futures price if it goes up, and d_F is one plus the rate of loss if it goes down.
(iii) The risk-neutral probability of an up move is $1/2$.
(iv) The initial futures price is 80.
(v) The continuously compounded risk-free interest rate is 5%.

Find the price of a half-year, 85-strike American call option on the futures contract.

Solution: We are given that

$$\frac{1}{2} = \frac{1-d_F}{u_F-d_F} = \frac{d_F^{-1}-1}{\frac{u_F}{d_F}-1} = \frac{d_F^{-1}-1}{\frac{5}{4}-1} \quad \Rightarrow \quad d_F^{-1} = \frac{9}{8} \quad \Rightarrow \quad d_F = \frac{8}{9} \quad \Rightarrow \quad u_F = \frac{10}{9}.$$

The prices in the futures-price tree are, thus,

$$\begin{array}{ccc} & & F_{uu} = 98.77 \\ & F_u = 88.89 & \\ F_0 = 80 & & F_{ud} = 79.01 \\ & F_d < 85 & \\ & & F_{dd} < 85 \end{array}$$

At the *up* node, we have that

$$\begin{aligned} CV_u &= e^{-0.0125}(13.77)(0.5) = 6.799473, \\ IE_u &= 3.89. \end{aligned}$$

So, we conclude that it's optimal to hold onto the option at this node and $V_A^u = 6.799473$. At the *down* node, the option is out of the money. So, it will not be exercised early. Moreover, the option is also out of the money in the *up-down* and *down-down* nodes. So, overall, we get that $V_d^A = 0$. The option's price is

$$V_C(0) = e^{-0.0125}(0.5)(6.799473) = 3.357504.$$

Problem 1.14. (10 points) The current exchange rate is \$1.13 per euro. Let the continuously copmounded, risk-free interest rate for the USD be 0.03 and let the continuously compounded, risk-free interest rate for the euro be 0.04. The volatility of the exchange rate is 0.10. You model the evolution of the exchange rate over the following nine months using a three-period forward binomial tree. What is the price of a nine-month, \$1.25-strike European put option on the euro?

Solution: First, we calculate the risk-neutral probability. In our usual notation, recalling that we are using a **forward** binomial tree, we have

$$p^* = \frac{1}{1+e^{\sigma\sqrt{h}}} = \frac{1}{1+e^{0.1\sqrt{1/4}}} = \frac{1}{1+e^{0.05}} = 0.4875026.$$

Next, let's find the up and down factors so that we can populate the tree. Taking into account that the domestic currency is the USD and that the foreign currency is the euro, we have

$$\begin{aligned} u &= e^{(r_D-r_F)h+\sigma\sqrt{h}} = e^{(0.03-0.04)(0.25)+0.05} = e^{0.0475}, \\ d &= e^{(r_D-r_F)h-\sigma\sqrt{h}} = e^{(0.03-0.04)(0.25)-0.05} = e^{-0.0525}. \end{aligned}$$

Since we are pricing a European option, we are only interested in the possible final asset prices. We have

$$\begin{aligned} x_{uuu} &= u^3x(0) = 1.303063, \quad x_{uud} = u^2dx(0) = 1.17906, \\ x_{udd} &= ud^2x(0) = 1.066858, \quad x_{ddd} = d^3x(0) = 0.9653328. \end{aligned}$$

The option's price is

$$\begin{aligned} V_P(0) = e^{-0.03(0.75)}&[3(p^*)^2(1-p^*)(1.25-1.17906) \\ &+ 3p^*(1-p^*)^2(1.25-1.066858) + (1-p^*)^3(1.25-0.9653328)] = 0.1315966. \end{aligned}$$

Problem 1.15. (5 points) The current price of a continuous-dividend-paying stock is \$100 per share. Its dividend yield is 0.02. According to your model, the expected value of the stock price in two years is \$110 per share. You are also given:

The risk-free interest rate is strictly greater than 0.02.

The two-year forward price on a share of this stock is denoted by F. At this price you are willing to short the forward contract. What is the smallest range of values F can take according to the above information?

Solution: Using the fact that the investor is willing to enter a **short** forward contract, we conclude that the **short** forward contract's profit is positive. So,

$$\mathbb{E}[S(T)] < F \quad \Rightarrow \quad 110 < F$$

On the other hand, we know that

$$F = S(0)e^{(r-\delta)T} = 100e^{2(r-\delta)} > 100.$$

So, the most we can say about F is that $F > 110$.