M378K Introduction to Mathematical Statistics
Problem Set #9
Moment generating functions.

Definition 9.1. *The* k^{th} moment of a random variable Y taken about the origin *is defined as* $\mathbb{E}[Y^k]$ *provided that the expectation exists. We write*

$$\mu_k = \mathbb{E}[Y^k]$$

when there is no ambiguity about the random variable in question.

Remark 9.2. μ_k *is also referred to as the* k^{th} raw moment.

Remark 9.3. *In particular,* $\mu_1 = \mu$ *happens to be the* **mean** *of the random variable* Y.

Definition 9.4. *The* k^{th} central moment *of a random variable* Y *is defined as* $\mathbb{E}[(Y-\mu)^k]$ *provided that the expectation exists. We write*

$$\mu_k^c = \mathbb{E}[(Y-\mu)^k]$$

when there is no ambiguity about the random variable in question.

Remark 9.5. μ_k *is also referred to as the* k^{th} moment of a random variable Y taken about its mean.

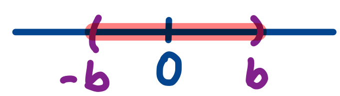
Definition 9.6. *The* moment-generating function (mgf) m_Y *for a random variable* Y *is defined as*

$$m_Y(t) = \mathbb{E}[e^{tY}]$$

for all t *for which the above expectation exists. In fact, we say that the moment-generating function* **exists** *if there exists a positive number* b *such that* $m_Y(t)$ *is finite for all* t *such that* $|t| \leq b$.

Problem 9.1. *How much is* $m_Y(0)$*?*

$$m_Y(0) = \mathbb{E}[e^{0 \cdot Y}] = 1$$



Remark 9.7. *On the choice of terminology...*

Step 1.

$$\frac{d}{dt} m_Y(t) = ?$$

$$\frac{d}{dt} m_Y(t) = \frac{d}{dt} \mathbb{E}[e^{tY}] = \mathbb{E}\left[\frac{d}{dt} e^{t \cdot Y}\right]$$

$$= \mathbb{E}[Y e^{t \cdot Y}]$$

Step 2.

$$m'_Y(0) = ?$$

$$m'_Y(0) = \mathbb{E}[Y\underbrace{e^{0\cdot Y}}_{=1}] = \mathbb{E}[Y] = \mu_Y$$

Step 3.

$$\frac{d^2}{dt^2} m_Y(t) = ?$$

$$\frac{d}{dt}\left(\frac{d}{dt} m_Y(t)\right) = \frac{d}{dt}\mathbb{E}[Ye^{t\cdot Y}] = \mathbb{E}[Y^2 e^{tY}]$$

Step 4.

$$m''_Y(0) = ?$$

$m''_Y(0) = \mathbb{E}[Y^2] = \mu_2$, i.e., the second moment

Step 5. *What do you suspect the* ***generalization*** *of the above would be?*

$$m_Y^{(k)}(0) = \mathbb{E}[Y^k] = \mu_k$$

Theorem 9.8. *If m_Y exists, then for $k \in \mathbb{N}$, we have*

$$m_Y^{(k)}(0) = \mu_k.$$

Example 9.9. *Let $Y \sim b(n=1, p)$, i.e., let Y model a Bernoulli trial with the probability of success denoted by p. Find m_Y.*

→: $m_Y(t) = \mathbb{E}[e^{t \cdot Y}] = e^{t \cdot 0} \cdot (1-p) + e^{t \cdot 1} \cdot p$
$= \boxed{(1-p) + pe^t} \quad t \in \mathbb{R}$

Proposition 9.10. *Let Y_1 and Y_2 be independent random variables with m.g.f.s denoted by m_{Y_1} and m_{Y_2}. Define $Y = Y_1 + Y_2$. Then, for every t for which both m_{Y_1} and m_{Y_2} are well defined, we have*

$$m_Y(t) = ?$$

Proof. By definition:

$$m_Y(t) = \mathbb{E}[e^{t \cdot Y}]$$

Using $Y = Y_1 + Y_2$, we can subsitute $Y_1 + Y_2$ for Y in the expression above. So,

$$m_Y(t) = \mathbb{E}[e^{t \cdot (Y_1 + Y_2)}]$$

One of the properties of the exponential function is that $e^{A+B} = e^A \times e^B$. Thus, the above becomes:

$$m_Y(t) = \mathbb{E}[e^{t \cdot Y_1} \cdot e^{t \cdot Y_2}]$$

Recall that Y_1 and Y_2 are assumed to be independent random variables. With this in mind, we get:

$$m_Y(t) = \underbrace{\mathbb{E}[e^{t\cdot Y_1}]}\cdot\underbrace{\mathbb{E}[e^{t\cdot Y_2}]}$$

Finally, using the definition of a m.g.f., we have

$$\boxed{m_Y(t) = m_{Y_1}(t)\cdot m_{Y_2}(t)}$$

Example 9.11. *Let $Y \sim b(n,p)$. What is the moment generating function of Y?*

$\longrightarrow$: $m_Y(t) = ?$

$Y = I_1 + I_2 + \dots + I_n$ w/ $I_j, j=1..n$ are all $B(p)$ and independent

$$m_Y(t) = m_{I_1}(t)\cdots m_{I_n}(t) \underset{\uparrow}{=} (m_{I_1}(t))^n = (1-p+pe^t)^n$$

identically dist'd

□

Example 9.12. *Let $N \sim Poisson(\lambda)$. What is the moment generating function m_N of N?*

$$\longrightarrow: m_N(t) = \mathbb{E}[e^{t\cdot N}] = \sum_{n=0}^{\infty} e^{t\cdot n}\cdot p_N(n) = \sum_{n=0}^{\infty} e^{t\cdot n}\, e^{-\lambda}\, \frac{\lambda^n}{n!} =$$

$$= e^{-\lambda}\sum_{n=0}^{\infty}\frac{e^{t\cdot n}\lambda^n}{n!} = e^{-\lambda}\cdot\sum_{n=0}^{\infty}\frac{(e^t\cdot\lambda)^n}{n!} = e^{-\lambda}\cdot e^{e^t\cdot\lambda} = e^{\lambda(e^t-1)}$$

□

Example 9.13. *Let $Z \sim N(0,1)$. What is the moment generating function m_Z of Z?*

$$\longrightarrow: m_Z(t) = \mathbb{E}[e^{t\cdot Z}] = \int_{-\infty}^{\infty} e^{tz}\cdot\varphi(z)\,dz = \int_{-\infty}^{\infty} e^{tz}\cdot\frac{1}{\sqrt{2\pi}}\, e^{-\frac{z^2}{2}}\,dz$$

$$= \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\, e^{-\frac{z^2}{2} + t\cdot z - \frac{t^2}{2}}\cdot e^{\frac{t^2}{2}}\,dz = e^{\frac{t^2}{2}}\underbrace{\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}(z-t)^2}\,dz}_{=1}$$

density $N(t,\sigma=1)$

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$$\boxed{m_Z(t) = e^{\frac{t^2}{2}}}$$

□

Example 9.14. *Let the random variable Y have the mgf m_Y. Define $\boxed{X = aY + b}$ for some constants a and b. Express the mgf m_X of X in terms of m_Y, a and b.*

$$\to: \; m_X(t) = \mathbb{E}[e^{t\cdot X}] = \mathbb{E}[e^{t(aY+b)}] = \mathbb{E}[e^{taY+tb}]$$
$$= \mathbb{E}[e^{taY}\cdot e^{tb}] = e^{tb}\,\mathbb{E}[e^{taY}] = e^{tb}\cdot m_Y(ta)$$

Example 9.15. *Let $X \sim N(\mu, \sigma^2)$. What is the moment generating function m_X of X?*

$$\to: \; X = \mu + \sigma\cdot Z \quad \text{w/} \quad Z \sim N(0,1)$$
$$\mu \leftrightarrow b, \quad \sigma \leftrightarrow a$$
$$m_X(t) = e^{t\cdot\mu}\cdot m_Z(t\cdot\sigma) = e^{t\mu}\cdot e^{\frac{t^2\sigma^2}{2}} = \underline{e^{t\mu + \frac{t^2\sigma^2}{2}}} \qquad \square$$

Problem 9.2. *A random variable Y is said to be lognormal if there exists a normally distributed random variable $X \sim N(\mu, \sigma^2)$ such that $Y \stackrel{(d)}{=} e^X$. Express the mean and the variance of the lognormal r.v. Y in terms of the parameters μ and σ.*

$$\to: \; \mathbb{E}[Y] = \mathbb{E}[e^X] = \mathbb{E}[e^{1\cdot X}] = m_X(1) = e^{\mu + \frac{\sigma^2}{2}}$$
$$\mathbb{E}[Y^2] = \mathbb{E}[(e^X)^2] = \mathbb{E}[e^{2\cdot X}] = m_X(2) = e^{2\mu + \frac{2^2\cdot\sigma^2}{2}} = e^{2(\mu+\sigma^2)}$$
$$\mathrm{Var}[Y] = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2 = e^{2(\mu+\sigma^2)} - e^{2\mu+\sigma^2} = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1) \qquad \square$$

Proposition 9.16.
1. *If m_Y exists for a certain probability distribution, then it is unique.*
2. *If m_{Y_1} and m_{Y_2} are equal on an interval, then $Y_1 \stackrel{(d)}{=} Y_2$.*

Corollary 9.17. *Let Y_1 and Y_2 be independent and normally distributed. Define $Y = Y_1 + Y_2$. Then, the distribution of Y is ...*

Proof. $\square$