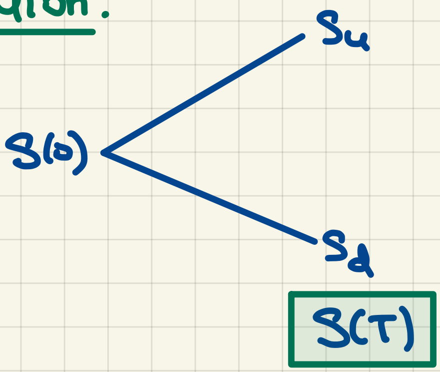
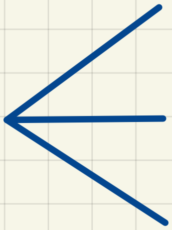
Motivation.

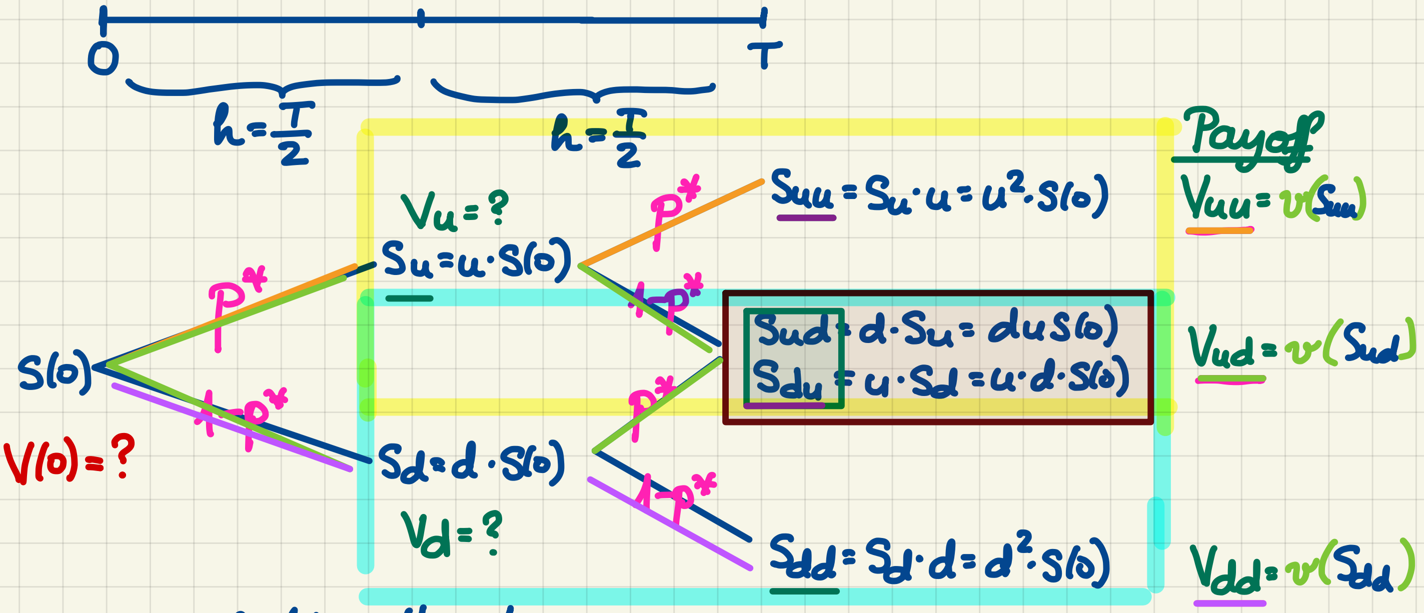


Q: How can we make the model for S(T) richer, but still interpretable?



k-nary tree

Two Periods. $\boxed{n=2}$



populating the tree →

← pricing the option

- **up node:** replicating portfolio for the option:

$$\Delta_u = \frac{V_{uu} - V_{ud}}{S_{uu} - S_{ud}}$$

$$B_u = e^{-rh} \cdot \frac{u \cdot V_{ud} - d \cdot V_{uu}}{u-d}$$

$\Rightarrow$ the option's value @ the up node:

$$V_u = \Delta_u \cdot S_u + B_u = e^{-rh}\left[p^* \cdot V_{uu} + (1-p^*) \cdot V_{ud}\right]$$

w/ $\boxed{p^* = \frac{e^{rh} - d}{u-d}}$

- **down node:** Δ_d, B_d

$$\Rightarrow \quad V_d = \Delta_d \cdot S_d + B_d = \underline{e^{-rh}\left[p^* \cdot V_{ud} + (1-p^*) \cdot V_{dd}\right]} \quad \text{w/}$$

- **ROOT node:**

$$\Delta_0 = \frac{V_u - V_d}{S_u - S_d}$$

$$B_0 = e^{-rh} \cdot \frac{u \cdot V_d - d \cdot V_u}{u-d}$$

$$\Rightarrow \quad \boxed{V(0) = \Delta_0 \cdot S(0) + B_0}$$

From the "risk-neutral perspective":

$$V(0) = e^{-rh} \cdot \left[p^* \cdot V_u + (1-p^*) \cdot V_d\right]$$

$$= e^{-rh} \cdot \left[p^* e^{-rh}\left(p^* \cdot V_{uu} + (1-p^*) \cdot V_{ud}\right) + (1-p^*) e^{-rh}\left(p^* \cdot V_{ud} + (1-p^*) \cdot V_{dd}\right)\right]$$

$$= e^{-r\overset{T}{(2h)}}\left[\underbrace{(p^*)^2 \cdot V_{uu} + 2 \cdot p^*(1-p^*) \cdot V_{ud} + (1-p^*)^2 \cdot V_{dd}}_{\text{Risk-neutral Expectation of the Payoff}}\right]$$

Generally: $\boxed{V(0) = e^{-rT} \, \mathbb{E}^*[V(T)]}$

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Problem Set #10

Binomial option pricing: Two or more periods.

Problem 10.1. For a two-period binomial model, you are given that:

(1) each period is one year;
(2) the current price of a non-dividend-paying stock S is $S(0) = \$20$;
(3) $u = 1.2$, with u as in the standard notation for the binomial model;
(4) $d = 0.8$, with d as in the standard notation for the binomial model;
(5) the continuously compounded risk-free interest rate is $r = 0.04$.

Consider a **special** call option which pays the excess above the strike price $K = 23$ (if any!) at the end of **every** binomial period.

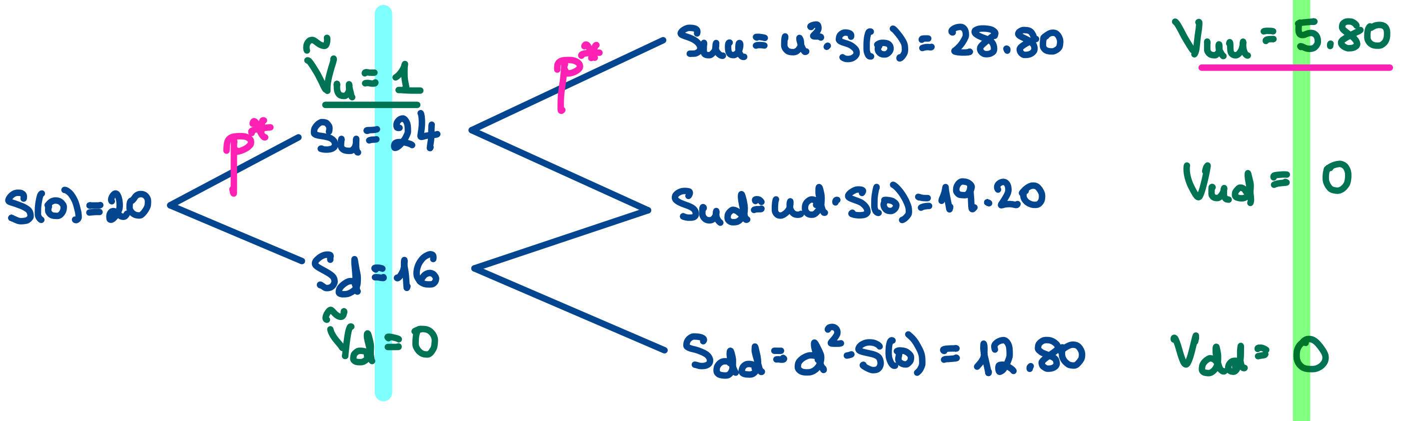
Find the price of this option.

→ : 1st $p^* = ?$ ✓

2nd Tree + Payoffs ✓ ✓

3rd Risk-Neutral Pricing

Risk-Neutral Probability: $p^* = \frac{e^{rh}-d}{u-d} = \frac{e^{0.04}-0.8}{1.2-0.8} = 0.602$



$\tilde{V}(0) = e^{-0.04} \cdot p^* \cdot 1 = 0.5784$

$V(0) = e^{-0.04(2)} \cdot (p^*)^2 \cdot 5.80 = 1.9413$ } +

answer: the price of the special call: 2.5197

$\square$