University of Texas at Austin

Problem set 5

Problem 5.1. Which one is greater: 23% of 61 or 61% of 23?

Solution: They are equal, of course.

Problem 5.2. (2 pts)
We define the minimum of two values in the usual way, i.e.,

$$\min(x,y) = \begin{cases} x & \text{if } x \leq y \\ y & \text{if } x \geq y \end{cases}$$

Then, for every x and y we have that

$$\min(x,y) = \min(x-y,0) + y$$

True or false?

Solution: TRUE

Problem 5.3. (2 pts)
We define the maximum of two values in the usual way, i.e.,

$$\max(x,y) = \begin{cases} x & \text{if } x \geq y \\ y & \text{if } x \leq y \end{cases}$$

Then, for every x and y we have that

$$-\max(x,y) = \max(x-y,0) - x$$

True or false?

Solution: FALSE

Problem 5.4. (2 pts)
We define the minimum of two values in the usual way, i.e.,

$$\min(x,y) = \begin{cases} x & \text{if } x \leq y \\ y & \text{if } x \geq y \end{cases}$$

We define the maximum of two values in the usual way, i.e.,

$$\max(x,y) = \begin{cases} x & \text{if } x \geq y \\ y & \text{if } x \leq y \end{cases}$$

Then, for every x and y we have that

$$\max(x,y) + \min(x,y) = x + y.$$

True or false?

Solution: TRUE

Problem 5.5. Twelve people enter the elevator on the ground floor of a ten-story builiding and start riding up. By the time the elevator reaches the top floor, all the people have exited the elevator. What is the probability that at least two people exited the elevator on the same floor?

Solution: 1.

Problem 5.6. (5 pts) Let $\Omega = \{a_1, a_2, a_3, a_4\}$ be an outcome space, and let $\mathbb{P}$ be a probability distribution on Ω. Assume that $\mathbb{P}[\{a_1\}] = 1/3, \mathbb{P}[\{a_2\}] = 1/6$ and $\mathbb{P}[\{a_3\}] = 1/9$. Then we have that $\mathbb{P}[\{a_4\}]$ equals the following value:

(a) 1/3
(b) 2/3
(c) 7/18
(d) 7/9
(e) None of the above

Solution: (c)
For any outcome space Ω, from the axioms of probability, we must have that $\mathbb{P}[\Omega] = 1$. In this case, $\Omega = \{a_1, a_2, a_3, a_4\}$, and so

$$\begin{aligned}
\mathbb{P}[\Omega] &= \mathbb{P}[\{a_1, a_2, a_3, a_4\}] \\
&= \mathbb{P}[\{a_1\}] + \mathbb{P}[\{a_2\}] + \mathbb{P}[\{a_3\}] + \mathbb{P}[\{a_4\}] \\
&= \frac{1}{3} + \frac{1}{6} + \frac{1}{9} + \mathbb{P}[\{a_4\}].
\end{aligned}$$

Hence,

$$\mathbb{P}[\{a_4\}] = 1 - \frac{11}{18} = \frac{7}{18}\,.$$

Problem 5.7. Two dice are rolled, the probability that the sum of the upturned faces equals 7 is 1/6. *True or false?*

Solution: TRUE

Problem 5.8. (2 pts) Two dice are rolled, the probability that the maximum (and **not** necessarily a strict maximum) of the upturned faces is achieved on the second die equals 1/2. *True or false?*

Solution: FALSE
Let X denote the outcome on the first die and let Y denote the outcome on the second die. Then, the probability we are considering is

$$\mathbb{P}[X \geq Y] = 1 - \mathbb{P}[X < Y] = 1 - \mathbb{P}[X \leq Y] + \mathbb{P}[X = Y].$$

Due to symmetry, it must be that $\mathbb{P}[X \leq Y] = \mathbb{P}[Y \leq X]$. So,

$$\mathbb{P}[X \geq Y] = 1 - \mathbb{P}[X < Y] = 1 - \mathbb{P}[X \geq Y] + \mathbb{P}[X = Y] \quad \Rightarrow \quad 2\mathbb{P}[X \geq Y] = 1 + \mathbb{P}[X = Y]$$

Thus,

$$\mathbb{P}[X \geq Y] = \frac{1}{2}\left(1 + \frac{1}{6}\right)\,.$$

Problem 5.9. (5 pts) A class has 12 boys and 4 girls. If three students are selected at random from this class, what is the probability that they are all boys?

(a) 1/4
(b) 5/9
(c) 11/28
(d) 17/36
(e) None of the above

Solution: (c)
Let A_i stand for the event of choosing a boy in the i^{th} selection with $i = 1, 2, 3$. The probability we are seeking is

$$\mathbb{P}[A_1 \cap A_2 \cap A_3].$$

By the multiplication rule,

$$\begin{aligned}\mathbb{P}[A_1 \cap A_2 \cap A_3] &= \mathbb{P}[A_1]\mathbb{P}[A_2|A_1]\mathbb{P}[A_3|A_2 \cap A_1] \\ &= \frac{12}{16} \cdot \frac{11}{15} \cdot \frac{10}{14} = \frac{11}{2 \cdot 14} = \frac{11}{28}\,.\end{aligned}$$

Problem 5.10. A pair of dice is thrown. Find the probability that the sum of the outcomes is 10 or greater **if** a 5 appears on at least one of the dice.

(a) 1/6
(b) 3/11
(c) 1/3
(d) 1/2
(e) None of the above

Solution: (b)
Let

$$\begin{aligned}E &:= \{\text{the sum of the outcomes is greater than or equal to } 10\} \\ &= \{(i,j) : 1 \leq i,j \leq 6 \,\text{and}\, i+j \geq 10\}, \\ A &:= \{\text{the outcome on the first die is equal to } 5\} \\ &= \{(i,j) : 1 \leq j \leq 6 \,\text{and}\, i = 5\}, \\ B &:= \{\text{the outcome on the second die is equal to } 5\} \\ &= \{(i,j) : 1 \leq i \leq 6 \,\text{and}\, j = 5\}.\end{aligned}$$

Then, we are looking for the following conditional probability:

$$\mathbb{P}[E|A \cup B] = \frac{\mathbb{P}[E \cap (A \cup B)]}{\mathbb{P}[A \cup B]}\,.$$

We have that

$$E \cap (A \cup B) = \{(5,5), (5,6), (6,5)\},$$

and so $\mathbb{P}[E \cap (A \cup B)] = 3/36 = 1/12$. On the other hand,

$$\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B] = \frac{1}{6} + \frac{1}{6} - \frac{1}{36} = \frac{11}{36}\,.$$

Finally, we are looking for the following conditional probability:

$$\mathbb{P}[E|A \cup B] = \frac{\mathbb{P}[E \cap (A \cup B)]}{\mathbb{P}[A \cup B]} = \frac{\frac{3}{36}}{\frac{11}{36}} = \frac{3}{11}\,.$$

Of course, it is more natural and easier to look at the restricted outcome space with 5 of at least on one of the dice. This outcome space has 11 elements. Out of those, three pairs:

$$(5,5), (5,6), (6,5)$$

have the sum greater than or equal to 10. So, the probability we are looking for is 3/11.

Problem 5.11. (5 pts) Find the probability of obtaining exactly two fives in six rolls of a fair die.

(a) $5^5/(2^3 \cdot 3^6)$
(b) $5^5/(2^6 \cdot 3^6)$
(c) $5^5/(2^6 \cdot 3^5)$
(d) 1/5

(e) None of the above

Solution: (c)
The number of fives in six rolls of a die has the binomial distribution with parameters $n = 6$ and $p = 1/6$. The probability of getting exactly two fives is

$$\binom{6}{2} \cdot \frac{1}{6^2} \cdot \frac{5^4}{6^4} = 3 \cdot 5 \cdot \frac{5^4}{6^6} = \frac{5^5}{2^6 \cdot 3^5}\,.$$

Problem 5.12. Four balls are drawn (without replacement) from a box which contains 4 black and 5 red balls. Given that the four drawn balls were *not* all of the same color, what is the probability that there were exactly two balls of each color among the four.

Solution: Let A denote the event that the colors of the balls drawn are not all the same, and let B denote the event that there are exactly two black balls and two red balls. We are looking for $\mathbb{P}[B|A]$. Since $B \subseteq A$, we have

$$\mathbb{P}[B|A] = \mathbb{P}[B \cap A]/\mathbb{P}[A] = \mathbb{P}[B]/\mathbb{P}[A].$$

To compute $\mathbb{P}[A]$, we note that the event A^c consists of only two elementary outcomes: either all balls are black or all balls are red. The probability of picking all red balls is

$$\frac{\binom{5}{4}}{\binom{9}{4}}$$

while the probability of picking all black balls is

$$\frac{\binom{4}{4}}{\binom{9}{4}} = \frac{1}{\binom{9}{4}}.$$

Putting these two together, we obtain

$$\mathbb{P}[A] = 1 - \frac{\binom{5}{4} + 1}{\binom{9}{4}}.$$

To compute $\mathbb{P}[B]$ we note that we can choose 2 red balls out of 5 in $\binom{5}{2}$ ways and, then, for each such choice, we have $\binom{4}{2}$ ways of choosing 2 black balls from the set of 4 black balls. Therefore,

$$\mathbb{P}[B] = \left(\binom{5}{2} \times \binom{4}{2}\right) \Big/ \binom{9}{4}.$$

Finally,

$$\mathbb{P}[B|A] = \frac{\binom{5}{2}\binom{4}{2}}{\binom{9}{4} - \binom{5}{4} - 1} = \frac{10 \times 6}{126 - 5 - 1} = \frac{1}{2}.$$

Problem 5.13. (5 pts) Roger deposits \$100 into an account at time 0.
For the following three years, he does not make any subsequent withdrawals or deposits and the account earns at a constant continuously compounded, risk-free interest rate r.

After 15 years and 6 months, the balance in his account equals \$133. Then,

(a) $0 \leq r < 0.0150$
(b) $0.0150 \leq r < 0.0250$
(c) $0.0250 \leq r < 0.0550$
(d) $0.0550 \leq r < 0.0650$
(e) None of the above

Solution: (b)
The unknown continuously compounded, risk-free interest rate r must satisfy

$$133 = 100e^{15.5r}.$$

So,

$$r = \ln(1.33)/15.5 \approx 0.018.$$

Problem 5.14. (5 pts) *Source: Sample FM Problem #26.*
A 5-year loan for 10,000 is charged a nominal interest rate of 12% compounded semiannually.
The loan is to be repaid so that interest is repaid at the end of every 6 month period as it accrues and the principal is repaid in total at the end of the 5 years.

Denote the total amount of interest paid on this loan by I. Then

(a) $I \approx 2,750$
(b) $I \approx 3,000$
(c) $I \approx 3,250$
(d) $I \approx 3,500$
(e) None of the above

Solution: (e)

$$10 \cdot \frac{0.12}{2} \cdot 10,000 = 6,000.$$

Problem 5.15. To plant and harvest 20,000 bushels of corn, Farmer Jayne incurs total aggregate costs totaling \$33,000. The current spot price of corn is \$1.80 per bushel. What is the profit if the spot price is \$1.90 per bushel when she harvests and sells her corn?

(a) About \$3,000 gain
(b) About \$3,000 loss
(c) About \$5,000 loss
(d) About \$5,000 gain
(e) None of the above

Solution: (d)

$$1.90 \cdot 20,000 - 33,000 = 5,000$$

Problem 5.16. Assume that you open a 100−share short position in a common stock S when the bid-ask is \$100.00-\$101.00. When you close your position the bid-ask prices are \$99.50-\$100.00. Assume that you pay a commission rate of 1.00%. Calculate your (roundtrip) gain or loss on this short investment (assume $r = 0$)?

(a) The investor breaks even; i.e., the gain/loss is 0.
(b) About \$200 loss
(c) About \$132.50 loss
(d) About \$200 gain
(e) None of the above

Solution: (b)
The commission needs to be paid for both transactions, so the total outcome for the short-seller is

$$100(100 \cdot 0.99 - 100 \cdot 1.01) = -200.$$