University of Texas at Austin

HW Assignment 1

Provide your **complete solution** to the following problems. Final answers only, without appropriate justification, will receive zero points even if correct.

Problem 1.1. (5 points) Let $\Omega = \{a_1, a_2, a_3, a_4, a_5\}$ be an outcome space, and let $\mathbb{P}$ be a probability distribution on Ω. Assume that $\mathbb{P}[A] = 0.5$, $\mathbb{P}[B] = 0.4$, $\mathbb{P}[C] = 0.4$, and $\mathbb{P}[D] = 0.2$, where

$$A = \{a_1, a_2, a_3\}, \ B = \{a_2, a_3, a_4\},$$
$$C = \{a_3, a_5\} \text{ and } D = \{a_4\}.$$

Are the events A and B independent?

Solution: We need to check whether $\mathbb{P}[A \cap B] = \mathbb{P}[A] \times \mathbb{P}[B]$. Since

$$\begin{aligned}\mathbb{P}[A \cap B] &= \mathbb{P}[\{a_2, a_3\}] \\ &= \mathbb{P}[\{a_2, a_3, a_4\}] - \mathbb{P}[\{a_4\}] \\ &= \mathbb{P}[B] - \mathbb{P}[D] = 0.2\end{aligned}$$

Since $\mathbb{P}[A] \times \mathbb{P}[B] = 0.5 \times 0.4 = 0.2$, we conclude that A and B *are* independent.

Problem 1.2. (10 points) Consider a set-up in which a transmitter is transmitting either a 0 or a 1 and the receiver indicates that it received either a 0 or a 1. Denote the events that $i = 0, 1$ was transmitted by T_i, and the events that $i = 0, 1$ was indicated as received by R_i.
It is possible to have transmission errors. In fact, you are given the following data on accuracy and the frequency of transmitted signals:

$$\mathbb{P}[R_0 \mid T_0] = 0.99, \ \mathbb{P}[R_1 \mid T_1] = 0.98,$$

and

$$\mathbb{P}[T_0] = 0.75.$$

(a) Given that the receiver indicated 1, what is the probability that there was an error in the transmission?
(b) What is the overall probability that there was an error in transmission?

Solution:

(1) We need $\mathbb{P}[T_0|R_1]$. By the Bayes formula,

$$\begin{aligned}\mathbb{P}[T_0|R_1] &= \frac{\mathbb{P}[R_1|T_0]\mathbb{P}[T_0]}{\mathbb{P}[R_1|T_0]\mathbb{P}[T_0] + \mathbb{P}[R_1|T_1]\mathbb{P}[T_1]} \\ &= \frac{(1-0.99) \times 0.75}{(1-0.99) \times 0.75 + 0.98 \times 0.25} \\ &= \frac{3}{101} \cong 0.030.\end{aligned}$$

(2) An error will happen if $T_0 \cap R_1$ or $T_1 \cap R_0$ occur, i.e.,

$$\begin{aligned}\mathbb{P}[\text{error}] &= \mathbb{P}[T_0 \cap R_1] + \mathbb{P}[T_1 \cap R_0] \\ &= \mathbb{P}[R_1|T_0] \times \mathbb{P}[T_0] + \mathbb{P}[R_0|T_1] \times \mathbb{P}[T_1] \\ &= (1 - \mathbb{P}[R_0|T_0]) \times \mathbb{P}[T_0] \\ &\quad + (1 - \mathbb{P}[R_1|T_1]) \times (1 - \mathbb{P}[T_0]) \\ &= 0.01 \times 0.75 + 0.02 \times 0.25 \\ &= \tfrac{1}{80} \cong 0.013\end{aligned}$$

Problem 1.3. (15 points) Two people are picked at random from a group of 50 and given \$10 each. After that, independently of what happened before, three people are picked from the same group - one or more people could have been picked both times - and given \$10 each. What is the probability that at least one person received \$20?

Solution: Define

$$A = \{\text{no person picked the first time was also picked the second time}\},$$

so that the probability that at least one person received \$20 is given by $\mathbb{P}[A^c] = 1 - \mathbb{P}[A]$. In order to compute $\mathbb{P}[A]$, we note that we can write

$$A = \bigcup_{1 \le i < j \le 50} A_{ij} \cap B_{ij},$$

where

$$A_{ij} = \{ \text{ the first two people picked are } i \text{ and } j \text{ (not necessarily in that order)}\}, \text{ and}$$
$$B_{ij} = \{ \ i \text{ and } j \text{ are not among the next three people picked}\}.$$

The sets $A_{ij} \cap B_{ij}$ and $A_{i'j'} \cap B_{i'j'}$ are mutually exclusive whenever $i \neq i'$ or $j \neq j'$, so we have

$$\mathbb{P}[A] = \sum_{1 \le i < j \le 50} \mathbb{P}[A_{ij} \cap B_{ij}].$$

Furthermore, A_{ij} and B_{ij} are independent by the assumption so $\mathbb{P}[A_{ij} \cap B_{ij}] = \mathbb{P}[A_{ij}]\mathbb{P}[B_{ij}]$.

Clearly, $\mathbb{P}[A_{ij}] = \frac{1}{\binom{50}{2}}$, since there are $\binom{50}{2}$ equally likely ways to choose 2 people out of 50, and only one of these corresponds to the choice (i, j). Similarly, $\mathbb{P}[B_{ij}] = \frac{\binom{48}{3}}{\binom{50}{3}}$, because there are $\binom{50}{3}$ ways to choose 3 people out of 50, and $\binom{48}{3}$ of those do not involve i or j. Therefore,

$$\mathbb{P}[A] = \sum_{1 \le i < j \le 50} \frac{1}{\binom{50}{2}} \frac{\binom{48}{3}}{\binom{50}{3}}.$$

The terms inside the sum are all equal and there are $\binom{50}{2}$ of them, so

$$\mathbb{P}[A] = \binom{50}{2} \frac{1}{\binom{50}{2}} \frac{\binom{48}{3}}{\binom{50}{3}} = \frac{\binom{48}{3}}{\binom{50}{3}},$$

and the required probability is

$$1 - \frac{\binom{48}{3}}{\binom{50}{3}}.$$

Problem 1.4. (10 points) A simple experiment consists of drawing a single ball at random from each of two urns containing red and blue marbles. The first urn contains 4 red and 6 blue marbles. A second urn contains 16 red marbles and an unknown number of blue marbles. You are told that the probability that both marbles are the same color equals 0.44. Calculate the number of blue marbles in the second urn.

Solution: Let b denote the number of blue marbles in the second urn. Then, the probability of both extracted marbles being the same color equals

$$\begin{aligned} 0.44 &= \mathbb{P}[\text{both marbles of the same color}] \\ &= \mathbb{P}[\text{both marbles are red}] + \mathbb{P}[\text{both marbles are blue}] \\ &= \left(\frac{4}{10}\right)\left(\frac{16}{16+b}\right) + \left(\frac{6}{10}\right)\left(\frac{b}{16+b}\right). \end{aligned}$$

Tidying up the expression above, we get

$$4.4(16 + b) = 64 + 6b \quad \Rightarrow \quad 70.4 + 4.4b = 64 + 6b \quad \Rightarrow \quad 1.6b = 6.4 \quad \Rightarrow \quad b = 4.$$

Problem 1.5. (5 points) A particular disease is known to afflict 1% of the population at any given time. There is a blood test for the disease. We know that:

- the test shows positive for people who actually have the disease 95% of the time, and
- the test shows positive for people who actually don't have the disease 0.5% of the time.

Find the probability that a randomly chosen person whose test shows positive actually does have the disease.

Solution:

$$\begin{aligned}\mathbb{P}[D\,|\,+] &= \frac{\mathbb{P}[D\cap +]}{\mathbb{P}[+]} = \frac{\mathbb{P}[D\cap +]}{\mathbb{P}[D\cap +]+\mathbb{P}[nD\cap +]}\\ &= \frac{(0.01)(0.95)}{(0.01)(0.95)+(0.99)(0.005)} = 0.6574394.\end{aligned}$$

Problem 1.6. (5 points) Event A happens with probability 0.8 and event B happens with probability 0.6. The probability that at least one of the events happens is 0.9. Find the probability of event B, given that event A happened.

Solution: We are given $\mathbb{P}[A] = 0.8, \mathbb{P}[B] = 0.6$, and $\mathbb{P}[A\cup B] = 0.9$. By definition,

$$\mathbb{P}[B|A] = \frac{\mathbb{P}[A\cap B]}{\mathbb{P}[A]} = \frac{\mathbb{P}[A]+\mathbb{P}[B]-\mathbb{P}[A\cup B]}{\mathbb{P}[A]} = \frac{0.8+0.6-0.9}{0.8} = \frac{0.5}{0.8} = \frac{5}{8}\,.$$