

**NAME:**

M339W/389W Financial Mathematics for Actuarial Applications  
University of Texas at Austin  
**In-Term Exam II**  
Instructor: Milica Čudina

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The maximum number of points on this exam is 100.

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**Problem 1.1.** (10 points) Assume the Black-Scholes model. Let the current price of a continuous-dividend-paying stock be \$40. The stock's volatility is 0.20 and its dividend yield is 0.02.

The continuously compounded, risk-free interest rate is 0.05.

Find the price of a \$42-strike, half-year European put option on the above stock.

**Solution:** In our usual notation, we have

$$d_1 = \frac{1}{0.2\sqrt{1/2}} \left[ \ln \left( \frac{40}{42} \right) + \left( 0.05 - 0.02 + \frac{(0.2)^2}{2} \right) \left( \frac{1}{2} \right) \right] = -0.1682 \approx -0.17,$$

$$d_2 = d_1 - \sigma\sqrt{T} = -0.3096 \approx -0.31.$$

Using the standard normal tables, we obtain

$$N(-d_1) = N(0.17) = 0.5675,$$

$$N(-d_2) = N(0.31) = 0.6217.$$

So, the Black-Scholes price of this call option is

$$\begin{aligned} v_P(S(0), 0) &= Ke^{-rT}N(-d_2) - S(0)e^{-\delta T}N(-d_1) \\ &= 42e^{-0.05/2}(0.6217) - 40e^{-0.02/2}(0.5675) = 2.99258. \end{aligned}$$

**Problem 1.2.** (5 points) Assume the Black-Scholes model. Let the current stock price be equal to \$90 per share. Its dividend yield is 0.02 and its volatility is 0.20.

The continuously compounded, risk-free interest rate is 0.03.

Consider a one-year European call option on the above stock. The delta of this call option is 0.49. What is the strike price of the call?

**Solution:** In our usual notation, the delta of the call is

$$\Delta_C(S(0), 0) = e^{-\delta T}N(d_1(S(0), 0)) = e^{-0.02}N(d_1(S(0), 0)) = 0.49.$$

So,

$$N(d_1(S(0), 0)) = 0.49e^{0.02} = 0.499899 \approx 0.5 \quad \Rightarrow \quad d_1(S(0), 0) = 0.$$

Hence,

$$\ln \left( \frac{S(0)}{K} \right) + \left( r - \delta + \frac{\sigma^2}{2} \right) = 0 \quad \Rightarrow \quad \ln \left( \frac{S(0)}{K} \right) = - \left( 0.03 - 0.02 + \frac{0.04}{2} \right) = -0.03.$$

Therefore,

$$\frac{S(0)}{K} = e^{-0.03} \quad \Rightarrow \quad K = S(0)e^{0.03} = 90e^{0.03} = 92.7409.$$

**Problem 1.3.** (10 points) The current price of a continuous-dividend-paying stock is given to be \$92. The stock's volatility is 0.35 and its dividend yield is 0.02.

The continuously-compounded, risk-free interest rate is 0.05.

Consider a \$90-strike European call option on the above stock with exercise date in a quarter-year. What is the Black-Scholes price of this call option?

**Solution:** In our usual notation,

$$d_1 = \frac{1}{0.35\sqrt{0.25}} \left[ \ln \left( \frac{92}{90} \right) + \left( 0.05 - 0.02 + \frac{(0.35)^2}{2} \right) \frac{1}{4} \right] = 0.255951,$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.255951 - 0.35(0.5) = 0.080951.$$

Using the standard normal tables, we get

$$N(d_1) = N(0.26) = 0.6026 \quad \text{and} \quad N(d_2) = N(0.08) = 0.5319.$$

Finally, the Black-Scholes call price equals

$$V_C(0) = S(0)e^{-\delta T}N(d_1) - Ke^{-rT}N(d_2) = 92e^{-0.02(0.25)}(0.6026) - 90e^{-0.05(0.25)}(0.5319) = 7.88636.$$

**Problem 1.4.** (5 points) Assume the Black-Scholes model. For a particular stock option, you are given that its price today equals \$11.84. You are given the following values of its Greeks today:

- the option's delta is 0.6122;
- the option's gamma is 0.0153;
- the option's theta is -0.0188 **per day**.

Approximately, what will this option's value be in a day should the stock price increase by 0.50?

**Solution:** By the delta-gamma-theta approximation, we have that

$$\begin{aligned} v(S(dt), dt) &\approx v(S(0), 0) + \Delta(S(0), 0)ds + \frac{1}{2}\Gamma(S(0), 0)(ds)^2 + \Theta(S(0), 0)dt \\ &= 11.84 + (0.6122)(0.5) + \frac{1}{2}(0.0153)(0.5)^2 + (-0.0188) = 12.1292. \end{aligned}$$

**Problem 1.5.** (10 points) Assume the Black-Scholes framework.

The goal is to delta-hedge a written one-year, (40, 60)-strangle on a non-dividend-paying stock whose current price is \$50. The stock's volatility is 0.20.

The continuously compounded, risk-free interest rate is 0.10.

What is the cost of delta-hedging the strangle using shares of the underlying stock?

**Solution:** The  $\Delta$  of the strangle equals

$$\Delta_P(S(0), 0; K_P = 40) + \Delta_C(S(0), 0; K_C = 60),$$

i.e., it is the sum of the delta of the call with strike 60 and the delta of the put with strike 40. We have

$$d_1(S(0), 0; K_P = 40) = \frac{1}{0.2} \left[ \ln \left( \frac{50}{40} \right) + \left( 0.10 + \frac{0.04}{2} \right) \right] = 1.715.$$

So, the put's delta is approximately

$$-N(-d_1(S(0), 0; K_P = 40)) = -N(-1.72) = N(1.72) - 1 = -0.0427.$$

Similarly, for the call, we have

$$d_1(S(0), 0; K_C = 60) = \frac{1}{0.2} \left[ \ln \left( \frac{50}{60} \right) + \left( 0.10 + \frac{0.04}{2} \right) \right] = -0.31.$$

So, the call's delta is approximately

$$N(d_1(S(0), 0; K_C = 60)) = N(-0.31) = 1 - N(0.31) = 0.3783.$$

Our answer is

$$50(0.3783 - 0.0427) = 16.78.$$

**Problem 1.6.** (5 points) Which of the following statements is always TRUE?

- (a) The call rho is greater than the put rho.
- (b) The put theta is always negative.
- (c) The call vega is the negative of the vega of the otherwise identical put.
- (d) The call psi is always positive.
- (e) None of the above.

**Solution:** (a)

**Problem 1.7.** (15 points) Assume the Black-Scholes model. The current stock price is \$60 per share. The stock's dividend yield is 0.02 and its volatility is 0.3.

Consider a \$70-strike, half-year European call option on this stock. Its price is \$2.40, and its delta is 0.3.

The continuously compounded, risk-free interest rate is 0.04.

What is the volatility of the otherwise identical put option?

**Solution:** The volatility of the put option is

$$\sigma_P = |\Omega_P|\sigma.$$

In our usual notation, we are given that

$$\Delta_C(S(0), 0) = e^{-0.02(0.5)}N(d_1(S(0), 0)) = 0.3$$

On the other hand, for the put, we have

$$\begin{aligned}\Delta_P(S(0), 0) &= -e^{-0.02(0.5)}N(-d_1(S(0), 0)) \\ &= -e^{-0.01}(1 - N(d_1(S(0), 0))) = -e^{-0.01} + \Delta_C(S(0), 0) = -0.69005.\end{aligned}$$

By put-call parity,

$$v_P(S(0), 0) = v_C(S(0), 0) - S(0)e^{-\delta T} + Ke^{-rT} = 2.40 - 60e^{-0.01} + 70e^{-0.02} = 11.61.$$

So, the put's elasticity is

$$\Omega_P(S(0), 0) = \frac{\Delta_P(S(0), 0)S(0)}{v_P(S(0), 0)} = \frac{-0.69005(60)}{11.61} = -3.56615.$$

Finally,  $\sigma_P = 3.56615(0.3) = 1.06984$ .

**Problem 1.8.** (15 points) Assume that the Black-Scholes setting holds. Let  $\mathbf{S} = \{S(t), t \geq 0\}$  denote the price of a continuous-dividend-paying stock. The stock price today equals \$80 and its dividend yield is 0.02.

Let  $r = 0.04$  be the continuously compounded risk-free interest rate.

Consider a European call option with exercise in three months and strike price  $K = 80e^{0.005}$ . You are given that its price today equals \$3.80.

What is the implied volatility of the stock  $S$ ?

**Solution:** According to the Black-Scholes formula, in our usual notation, the time-0 call price equals

$$v_C(S(0), 0) = v_C(0, S(0), K, r, \delta, \sigma, T) = S(0)e^{-\delta T}N(d_1) - Ke^{-rT}N(d_2)$$

with

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{T}}[\ln(S(0)/K) + (r - \delta + \tfrac{1}{2}\sigma^2)T], \\ d_2 &= d_1 - \sigma\sqrt{T}. \end{aligned}$$

Using the provided information, we obtain

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{\frac{1}{4}}} \left[ \ln(S(0)/S(0)e^{-0.005}) + (0.04 - 0.02 + \tfrac{1}{2}\sigma^2)\tfrac{1}{4} \right] \\ &= \frac{1}{\sigma\sqrt{\frac{1}{4}}} \left[ -0.005 + 0.005 + \tfrac{1}{2}\sigma^2 \cdot \tfrac{1}{4} \right] = \frac{\sigma}{4}, \\ d_2 &= d_1 - \frac{\sigma}{2} = \frac{\sigma}{4} = -d_1. \end{aligned}$$

Hence,

$$\begin{aligned} v_C(\dots, \sigma) &= S(0)e^{-\delta T}N(d_1) - S(0)e^{0.005}e^{-rT}N(d_2) \\ &= 80e^{-0.005}[N(d_1) - N(-d_1)] \\ &= 80e^{-0.005}[2N(\tfrac{\sigma}{4}) - 1]. \end{aligned}$$

From the problem, we know that

$$3.80 = 80e^{-0.005}[2N(\tfrac{\sigma}{4}) - 1].$$

So,

$$0.0475e^{0.005} = 2N(\tfrac{\sigma}{4}) - 1 \quad \Rightarrow \quad 2N(\tfrac{\sigma}{4}) = 1.0477381 \quad \Rightarrow \quad N(\tfrac{\sigma}{4}) = 0.523869$$

From the standard normal table, we get that

$$\tfrac{\sigma}{4} = 0.06 \Rightarrow \sigma = 0.24.$$

**Problem 1.9.** (5 points) The current price of a non-dividend-paying stock is \$40 per share. A market-maker writes a one-year European put option on this stock and proceeds to delta-hedge it.

The put premium is \$5.96, its delta is  $-0.5753$ , its gamma is  $0.0392$ , and its theta is  $0.01$  per day.

The continuously-compounded, risk-free interest rate is  $0.04$ .

Assuming that the stock price does not change, what is the **approximate** overnight profit for the market-maker?

**Solution:** The initial cost of the total delta-hedged portfolio is

$$-5.96 + (-0.5753)(40) = -28.972.$$

The approximate put price after one day is, according to the delta-gamma-theta approximation,

$$5.96 + 0.01 = 5.97.$$

So, the overnight profit is

$$-5.97 + (-0.5753)(40) + 28.972e^{0.04/365} = -0.00682481$$

**Problem 1.10.** (10 points) Assume the Black-Scholes model. The current price of a continuous-dividend-paying stock is denoted by  $S(0)$ . Its dividend yield is denoted by  $\delta$  and its volatility is  $\sigma = 0.20$ .

The continuously compounded, risk-free interest rate is equal to  $\delta$ .

Consider a one-year, at-the-money European put on the above stock. What is the elasticity of this put?

**Solution:** In our usual notation, since the option is at-the-money,

$$\begin{aligned} d_1(S(0), 0) &= \frac{1}{0.2} \left( r - \delta + \frac{(0.2)^2}{2} \right) = 0.1, \\ d_2(S(0), 0) &= d_1(S(0), 0) - \sigma = 0.1 - 0.2 = -0.1. \end{aligned}$$

So, the options delta is

$$\Delta_P(S(0), 0) = -e^{-\delta} N(-d_1(S(0), 0)) = -e^{-\delta} N(-0.1).$$

The put option's price is

$$\begin{aligned} v_P(S(0), 0) &= S(0)e^{-r} N(-d_2(S(0), 0)) - S(0)e^{-\delta} N(-d_1(S(0), 0)) \\ &= S(0)e^{-r} (N(0.1) - N(-0.1)) = S(0)e^{-r} (2N(0.1) - 1). \end{aligned}$$

Hence, the elasticity is

$$\Omega_P(S(0), 0) = \frac{\Delta_P(S(0), 0)S(0)}{v_P(S(0), 0)} = \frac{-e^{-\delta} N(-0.1)S(0)}{S(0)e^{-r} (2N(0.1) - 1)} = -\frac{1 - N(0.1)}{2N(0.1) - 1} = -5.78141.$$

**Problem 1.11.** (10 points) Assume the Black-Scholes model. Let the current price of a non-dividend-paying stock be \$100. Its volatility is given to be 0.25.

The continuously compounded, risk-free interest rate is 0.02.

Under the risk-neutral probability, the probability that a one-year, European call option is in the money at expiration is 0.484.

What is the current gamma of this call option?

**Solution:** We are given that, in our usual notation,

$$N(d_2(S(0), 0)) = 0.484 \quad \Rightarrow \quad d_2(S(0), 0) = -0.04.$$

So,  $d_1(S(0), 0) = -0.04 + 0.25 = 0.21$ . Using the IFM formula sheet, we get

$$\Gamma_C(S(0), 0) = \frac{N'(d_1(S(0), 0))}{S(0)\sigma} = \frac{\frac{1}{\sqrt{2\pi}}e^{-0.21^2/2}}{100(0.25)} = 0.0156097.$$