UNIVERSITY OF TEXAS AT AUSTIN

Quiz #5

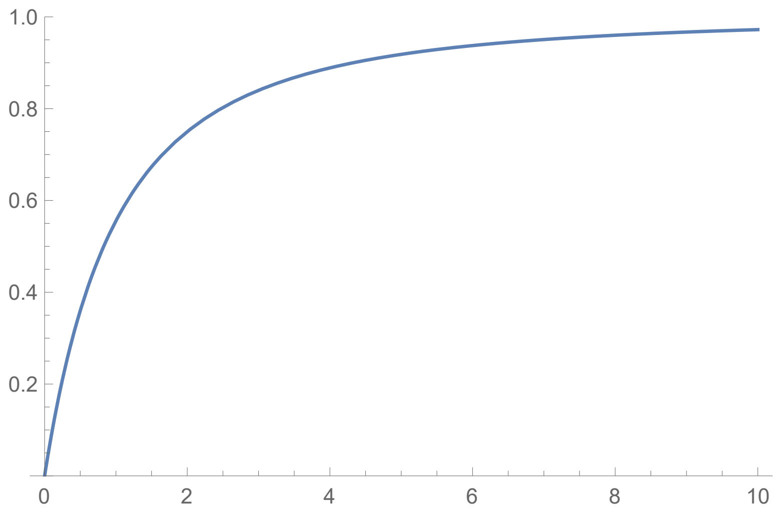
Inverse transform.

Provide your **complete solution** to the following problems. Final answers only, without appropriate justification, will receive zero points even if correct.

Problem 5.1. (15 points) The two-parameter Pareto distribution is a named distribution with two parameters usually denoted by θ and α. You can find this distribution in the STAM tables. Note that it is **different** both from the generalized Pareto and the single-parameter Pareto. In fact, its cumulative distribution function is defined as follows on the positive half-line (on the negative half-line it's 0):

$$F_X(x) = 1 - \left(\frac{\theta}{x+\theta}\right)^{\alpha}.$$

This is what the graph of the cdf looks like for $\theta = \alpha = 2$ on $\mathbb{R}_+$:



You want to use the *inverse transform method* to simulate draws from the Pareto distribution (not just for the choice of θ and α used in the graph above, but in general). Which transformation are you going to apply to the draws of the unit uniform random number generator to create simulated values from the two-parameter Pareto?

Solution: We need to apply the inverse (on $\mathbb{R}_+$) of the cumulative distribution function F_X to the draws from the unif uniform. The function F_X is strictly increasing on $\mathbb{R}_+$, so the inverse exists in that region. What remains is a simple calculation

$$y = 1 - \left(\frac{\theta}{x+\theta}\right)^{\alpha} \quad \Leftrightarrow \quad (1-y)(x+\theta)^{\alpha} = \theta^{\alpha} \quad \Leftrightarrow \quad (x+\theta)^{\alpha} = \frac{\theta^{\alpha}}{1-y} \quad \Leftrightarrow \quad x+\theta = \frac{\theta}{(1-y)^{1/\alpha}}$$

$$\Leftrightarrow \quad x = \theta\left(\frac{1}{(1-y)^{1/\alpha}} - 1\right)$$

So, for $y \in (0,1)$,

$$F_X^{-1}(y) = \theta\left(\frac{1}{(1-y)^{1/\alpha}} - 1\right).$$

For every value u simulated from the unit uniform, we obtain a value simulated from the two-parameter Pareto as

$$\theta\left(\frac{1}{(1-u)^{1/\alpha}} - 1\right).$$