Linear Discriminant Analysis (LDA)

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Here, I am adapting part of the lab associated with Chapter 4 of the textbook.

We re-examine the `Smarket` data, which is part of the `ISLR2` library. This data set consists of percentage returns for the S&P 500 stock index over $1,250$ days, from the beginning of 2001 until the end of 2005. For each date, we have recorded the percentage returns for each of the five previous trading days, `lagone` through `lagfive`. We have also recorded `volume` (the number of shares traded on the previous day, in billions), `Today` (the percentage return on the date in question) and `direction` (whether the market was `Up` or `Down` on this date). Our goal is to predict `direction` (a qualitative response) using the other features.

```r
library(ISLR2)
names(Smarket)
```

```
## [1] "Year"      "Lag1"      "Lag2"      "Lag3"      "Lag4"      "Lag5"
## [7] "Volume"    "Today"     "Direction"
```

```r
dim(Smarket)
```

```
## [1] 1250    9
```

```r
summary(Smarket)
```

```
##       Year           Lag1                Lag2                Lag3
##  Min.   :2001   Min.   :-4.922000   Min.   :-4.922000   Min.   :-4.922000
##  1st Qu.:2002   1st Qu.:-0.639500   1st Qu.:-0.639500   1st Qu.:-0.640000
##  Median :2003   Median : 0.039000   Median : 0.039000   Median : 0.038500
##  Mean   :2003   Mean   : 0.003834   Mean   : 0.003919   Mean   : 0.001716
##  3rd Qu.:2004   3rd Qu.: 0.596750   3rd Qu.: 0.596750   3rd Qu.: 0.596750
##  Max.   :2005   Max.   : 5.733000   Max.   : 5.733000   Max.   : 5.733000
##       Lag4                Lag5              Volume           Today
##  Min.   :-4.922000   Min.   :-4.92200   Min.   :0.3561   Min.   :-4.922000
##  1st Qu.:-0.640000   1st Qu.:-0.64000   1st Qu.:1.2574   1st Qu.:-0.639500
##  Median : 0.038500   Median : 0.03850   Median :1.4229   Median : 0.038500
##  Mean   : 0.001636   Mean   : 0.00561   Mean   :1.4783   Mean   : 0.003138
##  3rd Qu.: 0.596750   3rd Qu.: 0.59700   3rd Qu.:1.6417   3rd Qu.: 0.596750
##  Max.   : 5.733000   Max.   : 5.73300   Max.   :3.1525   Max.   : 5.733000
##  Direction
##  Down:602
##  Up  :648
##
##
##
##
```

```r
attach(Smarket)
```

We will again create a vector corresponding to the observations from 2001 through 2004. We will then use this vector to create a held out data set of observations from 2005.

```r
train <- (Year < 2005)
Smarket.2005 <- Smarket[!train, ]
dim(Smarket.2005)
```

```
## [1] 252   9
```

```r
Direction.2005 <- Direction[!train]
```

Linear Discriminant Analysis

Now we will perform LDA on the `Smarket` data. In `R`, we fit an LDA model using the `lda()` function, which is part of the `MASS` library. Notice that the syntax for the `lda()` function is identical to that of `lm()`, and to that of `glm()` except for the absence of the `family` option. We fit the model using only the observations before 2005.

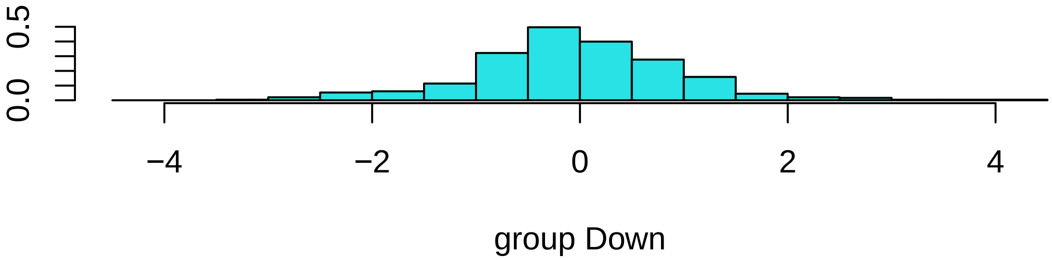
```r
library(MASS)
```

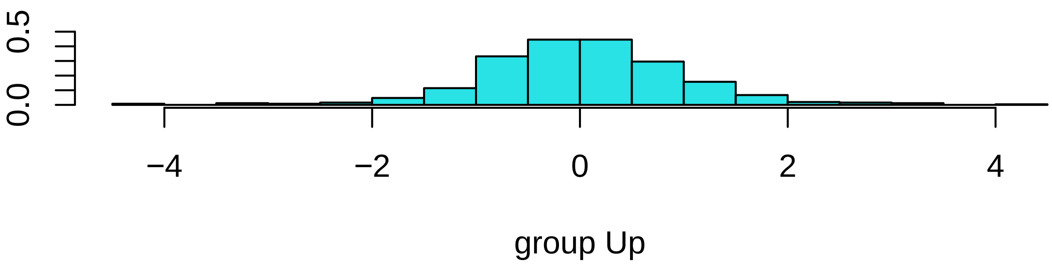
```
##
## Attaching package: 'MASS'
## The following object is masked from 'package:ISLR2':
##
##     Boston
```

```r
lda.fit <- lda(Direction ~ Lag1 + Lag2, data = Smarket,
    subset = train)
lda.fit
```

```
## Call:
## lda(Direction ~ Lag1 + Lag2, data = Smarket, subset = train)
##
## Prior probabilities of groups:
##     Down       Up
## 0.491984 0.508016
##
## Group means:
##             Lag1        Lag2
## Down  0.04279022  0.03389409
## Up   -0.03954635 -0.03132544
##
## Coefficients of linear discriminants:
##             LD1
## Lag1 -0.6420190
## Lag2 -0.5135293
```

```r
plot(lda.fit)
```





The LDA output indicates that $\hat{\pi}_1 = 0.492$ and $\hat{\pi}_2 = 0.508$; in other words, 49.2 % of the training observations correspond to days during which the market went down. It also provides the group means; these are the average of each predictor within each class, and are used by LDA as estimates of μ_k. These suggest that there is a tendency for the previous 2 days' returns to be negative on days when the market increases, and a tendency for the previous days' returns to be positive on days when the market declines. The *coefficients of linear discriminants* output provides the linear combination of `lagone` and `lagtwo` that are used to form the LDA decision rule. In other words, these are the multipliers of the elements of $X = x$ in

$$\delta_k(x) = x^T \mathbf{\Sigma}^{-1} \mu_k - \frac{1}{2} \mu_k^T \mathbf{\Sigma}^{-1} \mu_k + \log(\pi_k)$$

once the expression is simplified. If $-0.642 \times$ `‘lagone‘` $- 0.514 \times$ `‘lagtwo‘` is large, then the LDA classifier will predict a market increase, and if it is small, then the LDA classifier will predict a market decline.

The `plot()` function produces plots of the *linear discriminants*, obtained by computing $-0.642 \times$ `‘lagone‘` $-$ $0.514 \times$ `‘lagtwo‘` for each of the training observations. The `Up` and `Down` observations are displayed separately.

The `predict()` function returns a list with three elements. The first element, `class`, contains LDA's predictions about the movement of the market. The second element, `posterior`, is a matrix whose kth column contains the posterior probability that the corresponding observation belongs to the kth class, computed from

$$\mathbb{P}[Y = k \,|\, X = x] = \frac{\pi_k f_k(x)}{\sum_{j=1}^{K} \pi_j f_j(x)}$$

Finally, `x` contains the linear discriminants, described earlier.

```
lda.pred <- predict(lda.fit, Smarket.2005)
names(lda.pred)
```

```
## [1] "class"     "posterior" "x"
```

As we observed in Section 4.5, the LDA and logistic regression predictions are almost identical.

```
lda.class <- lda.pred$class
#lda.class
table(lda.class, Direction.2005)
```

```
##          Direction.2005
## lda.class Down  Up
##      Down   35  35
##      Up     76 106
```

```r
mean(lda.class == Direction.2005)
```

```
## [1] 0.5595238
```

Applying a 50 % threshold to the posterior probabilities allows us to recreate the predictions contained in `lda.pred$class`.

```r
sum(lda.pred$posterior[, 1] >= .5)
```

```
## [1] 70
```

```r
sum(lda.pred$posterior[, 1] < .5)
```

```
## [1] 182
```

Notice that the posterior probability output by the model corresponds to the probability that the market will *decrease*:

```r
lda.pred$posterior[1:20, 1]
```

```
##       999      1000      1001      1002      1003      1004      1005      1006
## 0.4901792 0.4792185 0.4668185 0.4740011 0.4927877 0.4938562 0.4951016 0.4872861
##      1007      1008      1009      1010      1011      1012      1013      1014
## 0.4907013 0.4844026 0.4906963 0.5119988 0.4895152 0.4706761 0.4744593 0.4799583
##      1015      1016      1017      1018
## 0.4935775 0.5030894 0.4978806 0.4886331
```

```r
lda.class[1:20]
```

```
##  [1] Up   Up   Up   Up   Up   Up   Up   Up   Up   Up   Up   Down Up   Up   Up  
## [16] Up   Up   Down Up   Up  
## Levels: Down Up
```

If we wanted to use a posterior probability threshold other than 50 % in order to make predictions, then we could easily do so. For instance, suppose that we wish to predict a market decrease only if we are very certain that the market will indeed decrease on that day—say, if the posterior probability is at least 90 %.

```r
sum(lda.pred$posterior[, 1] > .9)
```

```
## [1] 0
```

No days in 2005 meet that threshold! In fact, the greatest posterior probability of decrease in all of 2005 was 52.02 %.

Quadratic Discriminant Analysis

We will now fit a QDA model to the `Smarket` data. QDA is implemented in `R` using the `qda()` function, which is also part of the `MASS` library. The syntax is identical to that of `lda()`.

```r
qda.fit <- qda(Direction ~ Lag1 + Lag2, data = Smarket,
    subset = train)
qda.fit
```

```
## Call:
## qda(Direction ~ Lag1 + Lag2, data = Smarket, subset = train)
##
```

```
## Prior probabilities of groups:
##     Down       Up
## 0.491984 0.508016
##
## Group means:
##             Lag1        Lag2
## Down  0.04279022  0.03389409
## Up   -0.03954635 -0.03132544
```

The output contains the group means. But it does not contain the coefficients of the linear discriminants, because the QDA classifier involves a quadratic, rather than a linear, function of the predictors. The `predict()` function works in exactly the same fashion as for LDA.

```r
qda.class <- predict(qda.fit, Smarket.2005)$class
table(qda.class, Direction.2005)
```

```
##          Direction.2005
## qda.class Down  Up
##      Down   30  20
##      Up     81 121
```

```r
mean(qda.class == Direction.2005)
```

```
## [1] 0.5992063
```

Interestingly, the QDA predictions are accurate almost 60 % of the time, even though the 2005 data was not used to fit the model. This level of accuracy is quite impressive for stock market data, which is known to be quite hard to model accurately. This suggests that the quadratic form assumed by QDA may capture the true relationship more accurately than the linear forms assumed by LDA and logistic regression. However, we recommend evaluating this method's performance on a larger test set before betting that this approach will consistently beat the market!

Naive Bayes

Next, we fit a naive Bayes model to the `Smarket` data. Naive Bayes is implemented in `R` using the `naiveBayes()` function, which is part of the `e1071` library. The syntax is identical to that of `lda()` and `qda()`. By default, this implementation of the naive Bayes classifier models each quantitative feature using a Gaussian distribution. However, a kernel density method can also be used to estimate the distributions.

```r
library(e1071)
nb.fit <- naiveBayes(Direction ~ Lag1 + Lag2, data = Smarket,
    subset = train)
nb.fit
```

```
##
## Naive Bayes Classifier for Discrete Predictors
##
## Call:
## naiveBayes.default(x = X, y = Y, laplace = laplace)
##
## A-priori probabilities:
## Y
##     Down       Up
## 0.491984 0.508016
##
## Conditional probabilities:
##       Lag1
```

```
## Y              [,1]     [,2]
##   Down  0.04279022 1.227446
##   Up   -0.03954635 1.231668
##
##       Lag2
## Y              [,1]     [,2]
##   Down  0.03389409 1.239191
##   Up   -0.03132544 1.220765
```

The output contains the estimated mean and standard deviation for each variable in each class. For example, the mean for `lagone` is 0.0428 for `Direction=Down`, and the standard deviation is 1.23. We can easily verify this:

```r
mean(Lag1[train][Direction[train] == "Down"])
```

```
## [1] 0.04279022
```

```r
sd(Lag1[train][Direction[train] == "Down"])
```

```
## [1] 1.227446
```

The `predict()` function is straightforward.

```r
nb.class <- predict(nb.fit, Smarket.2005)
table(nb.class, Direction.2005)
```

```
##         Direction.2005
## nb.class Down  Up
##     Down   28  20
##     Up     83 121
```

```r
mean(nb.class == Direction.2005)
```

```
## [1] 0.5912698
```

Naive Bayes performs very well on this data, with accurate predictions over 59% of the time. This is slightly worse than QDA, but much better than LDA.

The `predict()` function can also generate estimates of the probability that each observation belongs to a particular class.

```r
nb.preds <- predict(nb.fit, Smarket.2005, type = "raw")
nb.preds[1:5, ]
```

```
##           Down        Up
## [1,] 0.4873164 0.5126836
## [2,] 0.4762492 0.5237508
## [3,] 0.4653377 0.5346623
## [4,] 0.4748652 0.5251348
## [5,] 0.4901890 0.5098110
```