M378K Introduction to Mathematical Statistics
Problem Set #13
Bias. MSE.

Problem 13.1. Source: "Mathematical Statistics with Applications" by Wackerly, Mendenhall, Scheaffer.
Let $Y_1, , Y_2, Y_3$ be a random sample from $E(\tau)$. Consider the following five estimators of τ:

$$\hat{\theta}_1 = Y_1, \quad \hat{\theta}_2 = \frac{Y_1 + Y_2}{2}, \quad \hat{\theta}_3 = \frac{Y_1 + 2Y_2}{3}, \quad \hat{\theta}_4 = 3Y_{(1)}, \quad \hat{\theta}_5 = \bar{Y}.$$

Which ones of these estimators are unbiased? Among the unbiased ones which one has the smallest variance?

Solution: First, let us check for unbiasedness.

$$\begin{aligned}
\mathbb{E}[\hat{\theta}_1] &= \mathbb{E}[Y_1] = \tau \\
\mathbb{E}[\hat{\theta}_2] &= \mathbb{E}\left[\frac{Y_1 + Y_2}{2}\right] = \tau \\
\mathbb{E}[\hat{\theta}_3] &= \mathbb{E}\left[\frac{Y_1 + 2Y_2}{3}\right] = \tau \\
\mathbb{E}[\hat{\theta}_4] &= \mathbb{E}\left[3Y_{(1)}\right] = 3 \cdot \frac{\tau}{3} = \tau \\
\mathbb{E}[\hat{\theta}_5] &= \mathbb{E}\left[\frac{Y_1 + Y_2 + Y_3}{3}\right] = \tau
\end{aligned}$$

So, all of the offered estimators are unbiased.

Now, for the variance, we have

$$\begin{aligned}
\mathrm{Var}[\hat{\theta}_1] &= \mathrm{Var}[Y_1] = \tau^2 \\
\mathrm{Var}[\hat{\theta}_2] &= \mathrm{Var}\left[\frac{Y_1 + Y_2}{2}\right] = \frac{2\tau^2}{4} = \frac{\tau^2}{2} \\
\mathrm{Var}[\hat{\theta}_3] &= \mathrm{Var}\left[\frac{Y_1 + 2Y_2}{3}\right] = \frac{1}{9}(\tau^2 + 4\tau^2) = \frac{5\tau^2}{9} \\
\mathrm{Var}[\hat{\theta}_4] &= \mathrm{Var}\left[3Y_{(1)}\right] = 9 \cdot \left(\frac{\tau}{3}\right)^2 = \tau^2 \\
\mathrm{Var}[\hat{\theta}_5] &= \mathrm{Var}\left[\frac{Y_1 + Y_2 + Y_3}{3}\right] = \frac{\tau^2}{3}
\end{aligned}$$

The sample mean has the minimum variance.

Problem 13.2. *Suppose that the two estimators $\hat{\theta}_1$ and $\hat{\theta}_2$ are unbiased. We know that* $\mathrm{Var}[\hat{\theta}_1] = \sigma_1^2$ *and* $\mathrm{Var}[\hat{\theta}_2] = \sigma_2^2$.

Consider the estimator all the estimators that can be obtained as convex combinations of $\hat{\theta}_1$ and $\hat{\theta}_2$, i.e., all the estimators of the form

$$\hat{\theta} = \alpha\hat{\theta}_1 + (1-\alpha)\hat{\theta}_2.$$

What can you say about the bias *of estimators $\hat{\theta}$ of the form above? Assuming that $\hat{\theta}_1$ and $\hat{\theta}_2$ are* **independent**, *for which weight α is the variance minimal?*

Solution: Due to the linearity of expectation, we have

$$\mathbb{E}[\hat{\theta}] = \alpha\mathbb{E}[\hat{\theta}_1] + (1-\alpha)\mathbb{E}[\hat{\theta}_2] = \theta.$$

So, each estimator of the above form is *unbiased.* Now, let's consider the variance. We want to solve this optimization problem

$$\begin{gathered}\mathrm{Var}[\hat{\theta}] \to \min \\ \Leftrightarrow \\ \mathrm{Var}[\alpha\hat{\theta}_1 + (1-\alpha)\hat{\theta}_2] \to \min\end{gathered}$$

Due to **independence** of $\hat{\theta}_1$ and $\hat{\theta}_2$, the above is equivalent to

$$\begin{gathered}\alpha^2\,\mathrm{Var}[\hat{\theta}_1] + (1-\alpha)^2\,\mathrm{Var}[\hat{\theta}_2] \to \min \Leftrightarrow \\ \alpha^2\sigma_1^2 + (1-\alpha)^2\sigma_2^2 \to \min\end{gathered}$$

This is an upward facing parabola in α. So, if we differentiate with respect to α, set our derivative equal to zero, and solve for α we will figure out the minimum.

$$2\alpha\sigma_1^2 - 2(1-\alpha)\sigma_2^2 = 0 \quad\Leftrightarrow\quad \alpha(\sigma_1^2+\sigma_2^2) = \sigma_2^2 \quad\Leftrightarrow\quad \alpha = \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}\,.$$

Problem 13.3. *Let $Y_1, Y_2, \dots, Y_n$ be a random sample from a continuous distribution with probability density function*

$$f_Y(y) = \frac{\alpha y^{\alpha-1}}{\theta^\alpha}\mathbf{1}_{[0,\theta]}(y)$$

with a known parameter $\alpha > 0$ and an unknown parameter $\theta > 0$. We propose the estimator $\hat{\theta} = \max(Y_1,\dots,Y_n)$. Is this estimator unbiased*? If not, how would you modify it to create an* unbiased *estimator? What is the* **mean-squared error** *of the* unbiased *estimator you obtained?*

Solution: The proposed estimator is the n^{th} order statistic. So, let's start by figuring out its density; it will be, for $y \in [0,\theta]$,

$$g_{(n)}(y) = nf_Y(y)(F_Y(y))^{n-1}$$

where F_Y stands for the cumulative distribution function of Y. In fact, for $y \in [0, \theta]$, we have that

$$F_Y(y) = \int_0^y f_Y(u)\, du = \frac{\alpha}{\theta^\alpha} \int_0^y u^{\alpha-1}\, du = \frac{\alpha}{\theta^\alpha} \cdot \frac{y^\alpha}{\alpha} = \left(\frac{y}{\theta}\right)^\alpha$$

This is the reason that distributions from this family are called **power distributions**. Now, we have that, for $y \in [0, \theta]$,

$$g_{(n)}(y) = n\frac{\alpha y^{\alpha-1}}{\theta^\alpha} \left(\frac{y}{\theta}\right)^{\alpha(n-1)} = \frac{n\alpha}{\theta^{\alpha n}} y^{\alpha n-1}$$

Hence,

$$\mathbb{E}[\hat{\theta}] = \int_0^\theta y g_{(n)}(y)\, dy = \int_0^\theta y \frac{n\alpha}{\theta^{\alpha n}} y^{\alpha n-1}\, dy = \frac{n\alpha}{\theta^{\alpha n}} \int_0^\theta y^{\alpha n}\, dy = \frac{n\alpha}{\theta^{\alpha n}} \cdot \frac{\theta^{\alpha n+1}}{\alpha n+1} = \frac{\alpha n}{\alpha n+1}\theta \neq \theta.$$

So, our estimator is *biased*, but if we instead consider

$$\hat{\theta}' = \frac{\alpha n+1}{\alpha n}\hat{\theta},$$

this estimator **will** be *unbiased.* By the proposition from class, we know that

$$MSE[\hat{\theta}'] = \mathrm{Var}(\hat{\theta}') = \left(\frac{\alpha n+1}{\alpha n}\right)^2 \mathrm{Var}[\hat{\theta}].$$

Focusing on the second moment of $\hat{\theta}$, we obtain

$$\mathbb{E}[\hat{\theta}^2] = \int_0^\theta y^2 g_{(n)}(y)\, dy = \int_0^\theta y^2 \frac{n\alpha}{\theta^{\alpha n}} y^{\alpha n-1}\, dy = \frac{n\alpha}{\theta^{\alpha n}} \int_0^\theta y^{\alpha n+1}\, dy = \frac{n\alpha}{\theta^{\alpha n}} \frac{\theta^{\alpha n+2}}{\alpha n+2} = \frac{\alpha n}{\alpha n+2}\theta^2.$$

Finally,

$$\mathrm{Var}[\hat{\theta}] = \frac{\alpha n}{\alpha n+2}\theta^2 - \left(\frac{\alpha n}{\alpha n+1}\right)^2 \theta^2$$

and

$$\mathrm{Var}[\hat{\theta}'] = \left(\frac{\alpha n+1}{\alpha n}\right)^2 \left(\frac{\alpha n}{\alpha n+2}\theta^2 - \left(\frac{\alpha n}{\alpha n+1}\right)^2 \theta^2\right) = \left(\frac{(\alpha n+1)^2}{\alpha n(\alpha n+2)} - 1\right)\theta^2 = \frac{1}{\alpha n(\alpha n+2)}\theta^2.$$