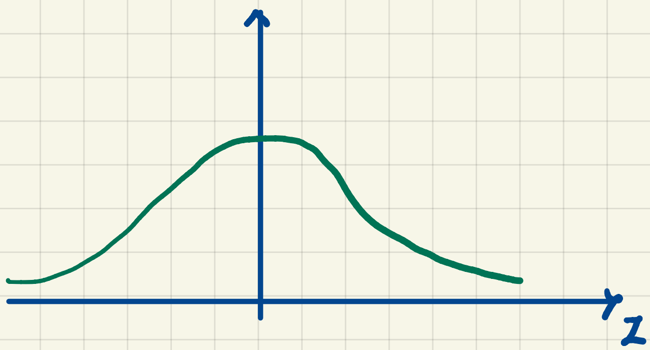
M358K: September 27th, 2023.

Standard Normal Distribution.

We say that a random variable Z has the **standard normal distribution** if its probability density f'tion equals

$$\boxed{\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \quad \text{for all } z \in \mathbb{R}}$$



- even, i.e., $\varphi(z) = \varphi(-z)$

$\Downarrow$

- median of $Z = 0$

The **cdf** of Z is:

$$\boxed{\Phi(x) = \mathbb{P}[Z \leq x] = \int_{-\infty}^{x} \varphi(z)\,dz = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\,dz}$$

No analytic form! $\Rightarrow$ Standard Normal Tables or R.

- $\mathbb{E}[Z] = ?$

Let Y be a continuous r.v. w/ density f_Y

$$\mathbb{E}[Y] := \int_{-\infty}^{+\infty} y f_Y(y)\,dy \quad \text{if the integral exists}$$

$$\mathbb{E}[Z] = \int_{-\infty}^{+\infty} \underbrace{z \underbrace{\varphi(z)}_{\text{even}}}_{\text{odd}}\,dz = 0 \checkmark$$

φ ... phi

- $Var[Z] = 1 \implies SD[Z] = 1$

We write: $\boxed{Z \sim Normal(mean = 0, var = 1) \sim N(0,1)}$

Normal Distribution.

We can completely specify any normal distribution by its mean and its variance (or standard deviation).

$$\boxed{X \sim Normal(mean = \mu_X, variance = \sigma_X^2)}$$

which means that X can be written as a linear transform of a standard normal Z, i.e.,

$$\boxed{X = \mu_X + \sigma_X \cdot Z} \iff \boxed{\frac{X - \mu_X}{\sigma_X} \sim N(0,1)}$$

We can check:

- $\mathbb{E}[X] \stackrel{?}{=} \mu_X$

$$\mathbb{E}[X] = \mathbb{E}[\mu_X + \sigma_X \cdot Z] = \text{(linearity of expectation)}$$

$$= \mu_X + \sigma_X \cdot \underbrace{\mathbb{E}[Z]}_{=0} = \mu_X$$

- $Var[X] \stackrel{?}{=} \sigma_X^2$

$$Var[X] = Var[\mu_X + \sigma_X \cdot Z] =$$

deterministic (added, does not affect the variance)

$$= Var[\sigma_X \cdot Z] = \sigma_X^2 \cdot \overbrace{Var[Z]}^{=1} = \sigma_X^2$$

multiplicative const.