M339J: Probability models with actuarial applications
University of Texas at Austin
HW VI
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Note: You **must** show all your work. Numerical answers without a proper explanation or a clearly written down path to the solution will be assigned zero points.

Problem 6.1. (10 points) The aggregate loss random variable S has a compound Poisson claims distribution, i.e., let the frequency random variable N have the Poisson distribution. You are given that

i. Individual claim amounts may only be equal to 1, 2, or 3.
ii. $\mathbb{E}[S] = 56$
iii. $Var[S] = 126$
iv. The rate of the Poisson claim count random variable is $\lambda = 29$.

Determine the probability mass function of the claim amounts.

Solution: We are given that, in our usual notation,

$$N \sim Poisson(\lambda = 29),$$

and that the support of the severity r.v. X is $\{1, 2, 3\}$.

Moreover,

$$56 = \mathbb{E}[S] = \mathbb{E}[N]\mathbb{E}[X] = 29(p_X(1) + 2p_X(2) + 3p_X(3)),$$

and

$$126 = Var[S] = 29\mathbb{E}[X^2] = 29(p_X(1) + 4p_X(2) + 9p_X(3)).$$

Together with the law of total one, i.e.,

$$p_X(1) + p_X(2) + p_X(3) = 1,$$

we get three equations with three unknowns. The solution is

$$p_X(1) = 10/29, p_X(2) = 11/29, p_X(3) = 8/29.$$

Problem 6.2. (15 points) In the compound model for aggregate claims, let the frequency random variable N be Negative Binomial with parameters $r = 2$ and $\beta = 4$, and let the common distribution of the i.i.d. severity random variables $\{X_j; j = 1, 2, \dots\}$ be given by the probability (mass) function $p_X(1) = 0.3$ and $p_X(2) = 0.7$.
Let our usual assumptions hold, i.e., let N be independent of $\{X_j; j = 1, 2, \dots\}$.
Define the aggregate loss as $S = \sum_{j=1}^{N} X_j$.
Calculate $\mathbb{P}[S \leq 3]$.

Solution: Evidently,

$$\mathbb{P}[S \leq 3] = \mathbb{P}[S = 0] + \mathbb{P}[S = 1] + \mathbb{P}[S = 2] + \mathbb{P}[S = 3]$$

We simplify one probability at a time, using independence of N and $\{X_j; j \geq 1\}$:

$$\begin{aligned}
\mathbb{P}[S = 0] &= \mathbb{P}[N = 0] = (1+\beta)^{-r} = 5^{-2} = 1/25 = 0.04,\\
\mathbb{P}[S = 1] &= \mathbb{P}[N = 1, X_1 = 1] = \mathbb{P}[N = 1]\mathbb{P}[X_1 = 1] = p_N(1)\, p_X(1),\\
\mathbb{P}[S = 2] &= \mathbb{P}[N = 1, X_1 = 2] + \mathbb{P}[N = 2, X_1 = 1, X_2 = 1]\\
&= p_N(1)\, p_X(2) + p_N(2)\, p_X(1)^2,\\
\mathbb{P}[S = 3] &= \mathbb{P}[N = 2, X_1 = 2, X_2 = 1] + \mathbb{P}[N = 2, X_1 = 1, X_2 = 2] + \mathbb{P}[N = 3, X_1 = X_2 = X_3 = 1]\\
&= 2p_N(2)\, p_X(2)\, p_X(1) + p_N(3)\, p_X(1)^3.
\end{aligned}$$

So,

$$\begin{aligned}
\mathbb{P}[S \leq 3] &= 0.04 + p_N(1)\, p_X(1) + p_N(1)\, p_X(2) + p_N(2)\, p_X(1)^2 + 2p_N(2)\, p_X(2)\, p_X(1) + p_N(3)\, p_X(1)^3\\
&= 0.04 + p_N(1) + p_N(2)[p_X(1)^2 + 2p_X(2)\, p_X(1)] + p_N(3)\, p_X(1)^3\\
&= 0.04 + p_N(1) + p_N(2)[(p_x(1) + p_X(2))^2 - p_X(2)^2] + p_N(3)\, p_X(1)^3\\
&= 0.04 + p_N(1) + p_N(2)[1 - p_X(2)^2] + p_N(3)\, p_X(1)^3.
\end{aligned}$$

Since

$$\begin{aligned}
p_N(1) &= r\beta^1(1+\beta)^{-(r+1)} = 2 \cdot 4 \cdot 5^{-3} = 0.064,\\
p_N(2) &= \frac{1}{2}\, r(r+1)\beta^2(1+\beta)^{-(r+2)} = \frac{1}{2} \cdot 2 \cdot 3 \cdot 4^2 \cdot 5^{-4} = 0.0768\\
p_N(3) &= \frac{1}{2 \cdot 3}\, r(r+1)(r+2)\beta^3(1+\beta)^{-(r+3)} = \frac{1}{2 \cdot 3} \cdot 2 \cdot 3 \cdot 4 \cdot 4^3 \cdot 5^{-5} = 4^4/5^5 = 0.08192,
\end{aligned}$$

we get

$$\mathbb{P}[S \leq 3] = 0.04 + 0.064 + 0.0768(1 - 0.7^2) + 0.08192 \cdot 0.3^3 = 0.14537984.$$

Problem 6.3. (10 points) In the compound model for aggregate claims, let the frequency random variable N have the probability (mass) function

$$p_N(0) = 0.5, p_N(1) = 0.3, p_N(2) = 0.2.$$

Moreover, let the common distribution of the i.i.d. severity random variables $\{X_j; j = 1, 2, \dots\}$ be given by the probability (mass) function $p_X(1) = 0.3$ and $p_X(2) = 0.7$.
Let our usual assumptions hold, i.e., let N be independent of $\{X_j; j = 1, 2, \dots\}$.
Define the aggregate loss as $S = \sum_{j=1}^{N} X_j$.
Calculate $\mathbb{E}[(S-2)_+]$.

Solution: We use the equality

$$\mathbb{E}[(S-2)_+] = \mathbb{E}[S] - \mathbb{E}[S \wedge 2].$$

Using

$$\begin{aligned}\mathbb{E}[N] &= 0.5 \cdot 0 + 0.3 \cdot 1 + 0.2 \cdot 2 = 0.3 + 0.4 = 0.7, \\ \mathbb{E}[X] &= 0.3 \cdot 1 + 0.7 \cdot 2 = 0.3 + 1.4 = 1.7,\end{aligned}$$

we get

$$\mathbb{E}[S] = \mathbb{E}[N]\mathbb{E}[X] = 0.7 \cdot 1.7 = 1.19.$$

On the other hand,

$$\begin{aligned}\mathbb{E}[S \wedge 2] &= \mathbb{P}[S > 0] + \mathbb{P}[S > 1] \\ &= (1 - F_S(0)) + (1 - F_S(1)).\end{aligned}$$

From the problem statement, we conclude that

$$\begin{aligned}F_S(0) &= \mathbb{P}[S \leq 0] = \mathbb{P}[S = 0] = \mathbb{P}[N = 0] = 0.5 \\ F_S(1) &= \mathbb{P}[S \leq 1] = \mathbb{P}[S = 0] + \mathbb{P}[S = 1] = \mathbb{P}[N = 0] + \mathbb{P}[N = 1, X_1 = 1] \\ &= 0.5 + 0.3 \cdot 0.3 = 0.59.\end{aligned}$$

Finally,

$$\mathbb{E}[S \wedge 2] = 0.5 + 0.41 = 0.91$$

and

$$\mathbb{E}[(S-2)_+] = 1.19 - 0.91 = 0.28.$$

Problem 6.4. (10 points) In the compound model for aggregate claims, let the frequency random variable N have the Poisson distribution with mean 1.

Let the common distribution of the i.i.d. severity random variables $\{X_j; j = 1, 2, \dots\}$ be given by the following p.m.f.

$$p_X(100) = 1/2, p_X(200) = 3/10, p_X(300) = 1/5.$$

Let our usual assumptions hold, i.e., let N be independent of $\{X_j; j = 1, 2, \dots\}$.
Define the aggregate loss as $S = \sum_{j=1}^{N} X_j$.
Find the expected value of the **policyholder's** payment for a stop-loss insurance policy with an ordinary deductible of 200, i.e., calculate $\mathbb{E}[S \wedge 200]$.

Solution: Note that S has the support of the form $\{0, 100, 200, 300, \dots\}$. So,

$$\mathbb{E}[S \wedge 200] = 100\mathbb{P}[S = 100] + 200\mathbb{P}[S \geq 200].$$

Next,

$$\begin{aligned}
\mathbb{P}[S = 0] &= \mathbb{P}[N = 0] = e^{-1}, \\
\mathbb{P}[S = 100] &= \mathbb{P}[N = 1, X_1 = 100] = 0.5e^{-1}, \\
\mathbb{P}[S \geq 200] &= 1 - \mathbb{P}[S = 0] - \mathbb{P}[S = 100] = 1 - 1.5e^{-1}.
\end{aligned}$$

So,

$$\mathbb{E}[S \wedge 200] = 100 \cdot 0.5e^{-1} + 200(1 - 1.5e^{-1}) = 200 - 250e^{-1} \approx 108.03.$$

Problem 6.5. (5 pts) In the compound model for aggregate claims, let the frequency random variable N have the Poisson distribution with mean 5. Moreover, let the common distribution of the i.i.d. severity random variables $\{X_j; j = 1, 2, \dots\}$ be the two-parameter Pareto with parameters $\alpha = 3$ and $\theta = 10$. Let our usual assumptions hold, i.e., let N be independent of $\{X_j; j = 1, 2, \dots\}$.
Define the aggregate loss as $S = \sum_{j=1}^{N} X_j$.
What is the variance of S?

Solution: We will use the formula

$$Var[S] = \mathbb{E}[N]Var[X] + Var[N]\mathbb{E}[X]^2.$$

We are given that

$$\mathbb{E}[N] = Var[N] = 5.$$

So,

$$Var[S] = 5(Var[X] + \mathbb{E}[X]^2) = 5(\mathbb{E}[X^2] - \mathbb{E}[X]^2 + \mathbb{E}[X]^2) = 5\mathbb{E}[X^2].$$

Using our tables, we get

$$\mathbb{E}[X^2] = \frac{\theta^2 \cdot 2!}{(\alpha - 1)(\alpha - 2)} = \frac{10^2 \cdot 2}{(3 - 1)(3 - 2)} = 100.$$

Finally, $Var[S] = 5 \cdot 100 = 500$.