Name:

M339D=M389D Introduction to Actuarial Financial Mathematics
University of Texas at Austin
Solution: Practice Problems for In-Term Exam I
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Notes: This is a closed book and closed notes exam. The maximal score on this exam is 80 points.
Time: 50 minutes

Problem 2.1. (2 points) Put-call parity applies only to European-style options.

Solution: TRUE

Problem 2.2. (2 points) The strike price at which the European call and the otherwise identical European put have the same premiums is the future value (on the exercise date) of the intial price of the underlying of the two options. *True or false?*

Solution: TRUE

Problem 2.3. The following nine-month European put options are available in the market:

- a \$120-strike put with the premium of \$12,
- a \$127-strike put with the premium of \$10,

The continuously compounded, risk-free interest rate is 0.04.

You construct a portfolio by buying the \$127-strike put and writing the \$120-strike put. Which of the following statements is correct?

(a) The minimum **profit** of this portfolio is -9.06.
(b) The minimum **profit** of this portfolio is -2.06.
(c) The minimum **profit** of this portfolio is -7.
(d) This is an arbitrage portfolio.
(e) None of the above.

Solution: (d)
The initial cost of this portfolio is $10 - 12 = -2$. The minimum payoff of this portfolio happens for the final asset price below 120. It is equal to 7. So, the minimal gain of this portfolio is

$$7 - (-2)e^{0.03} = 9.06.$$

Problem 2.4. (5 points) The current price of stock a certain type of stock is \$80. The premium for a $6-$month, at-the-money call option is \$5.84. Let the continuously compounded, risk-free interest rate be 0.04. What is the break-even point of this call option?

(a) \$80
(b) \$85.72
(c) \$85.84
(d) \$85.96
(e) None of the above.

Solution: (d)
The break-even point is

$$80 + 5.84e^{0.04/2} = 85.958$$

Problem 2.5. Let the current price of a non-dividend-paying stock equal 100. The forward price for delivery of this stock in 3 months equals \$101.26

Consider a \$90-strike, six-month put option on this stock whose premium today equals \$2.22.

What will the profit of this long put option be if the stock price at expiration equals \$96?

(a) About \$2.28 loss.
(b) About \$2.22 loss.
(c) About \$2.28 gain.
(d) About \$2.22 gain.
(e) None of the above.

Solution: (a)
The option is out-of-the money at expiration, so its owner suffers a loss of the future value of its premium

$$2.22 \times \left(\frac{101.26}{100}\right)^2 = 2.2763.$$

Problem 2.6. (5 points) A derivative security has the payoff function given by

$$v(s) = (s^2 - 100)_+$$

Its exercise date is in one year. You model the time-1 price of the underlying asset as

$$S(1) \sim \begin{cases} 9.5 & \text{with probability } 1/4 \\ 10 & \text{with probability } 1/2 \\ 11 & \text{with probability } 1/4 \end{cases}$$

The continuously compounded, risk-free interest rate is 10%. What is the expected **payoff** of the above derivative security?

(a) 5.25
(b) 2.81
(c) 0.31
(d) 1.42
(e) None of the above.

Solution: (a)

$$(9.5^2 - 100)_+ \left(\frac{1}{4}\right) + (10^2 - 100)_+ \left(\frac{1}{2}\right) + (11^2 - 100)_+ \left(\frac{1}{4}\right) = 21 \left(\frac{1}{4}\right) = 5.25$$

Problem 2.7. (5 points) The current stock price is 40 per share. The price at the end of a three-month period is modeled with a one-period binomial tree so that the stock price can either increase by \$5, or decrease by \$4.

The continuously compounded risk-free interest rate is 0.06.

What is the price of a \$40-strike European **straddle** on the above stock?

(a) 4.40
(b) 3.30
(c) 2.20
(d) 1.10
(e) None of the above.

Solution: **(a)**
The risk-neutral probability is

$$p^* = \frac{e^{rh} - d}{u - d} = \frac{S(0)e^{(0.06)(0.25)} - S_d}{S_u - S_d} = \frac{40e^{(0.06)(0.25)} - 36}{45 - 36} = 0.5116136.$$

The possible payoffs are $V_u = 5$ and $V_d = 4$. So,

$$V(0) = e^{-0.06/4}[5p^* + 4(1 - p^*)] = 4.444444.$$

Problem 2.8. Consider a non-dividend-paying stock with the current price of \$50.

The continuously compounded risk-free interest rate is 0.03.

You are using a one-period binomial tree to model the stock price at the end of the next quarter. You assume that the stock price can either increase by 0.04 or decrease by 0.02. What is the risk-neutral probability associated with this tree?

Solution: By defition, in our usual notation, we have

$$p^* = \frac{e^{rh} - d}{u - d} = \frac{e^{(0.03)(0.25)} - 0.98}{1.04 - 0.98} = 0.4588.$$

Problem 2.9. (2 points) A **long** straddle has a non-negative payoff function. *True or false?*

Solution: **TRUE**

Problem 2.10. (5 points) Let the continuously compounded risk-free interest rate be equal to 0.04. Consider a one-period binomial tree with every period of length one year used to model the stock price of a stock whose current price is \$100 per share. In the model, it is assumed that the stock price can either go up by 3% or down by 4%.

You use the binomial tree to construct a replicating portfolio for a at-the-money, one-year European call on the above stock. What is the stock investment in the replicating portfolio?

Solution: The two possible stock prices are $S_u = 103$ and $S_d = 96$. So, the possible payoffs of the call are $V_u = 3$ and $V_d = 0$. The Δ of the call, thus, equals

$$\Delta = \frac{V_u - V_d}{S_u - S_d} = \frac{3 - 0}{103 - 96} = \frac{3}{7} = 0.4285714. \tag{2.1}$$

Problem 2.11. (5 points) Let the continuously compounded risk-free interest rate be equal to 0.04. Consider a one-period binomial tree with every period of length one year used to model the stock price of a non-dividend-paying stock whose current price is \$50 per share. In the model, it is assumed that the stock price can either go up by 5% or down by 10%.

You use the binomial tree to construct a replicating portfolio for a \$48-strike, one-year European put on the above stock. What is the risk-free investment in the replicating portfolio? Explicitly state whether one should be borrowing or lending.

Solution: The two possible stock prices are $S_u = 52.5$ and $S_d = 45$. So, the possible payoffs of the put are $V_u = 0$ and $V_d = 3$. The risk-free investment B in the replicating portoflio of our put, thus, equals

$$B = e^{-rh}\frac{uV_d - dV_u}{u-d} = e^{-0.04}\frac{1.05(3) - 0.9(0)}{1.05 - 0.9} = 20.17658. \tag{2.2}$$

This is lending!

Problem 2.12. (10 pts) For a two-period binomial model, you are given that:

(1) each period is one year;
(2) the current price of a non-dividend-paying stock S is $S(0) = \$20$;
(3) $u = 1.2$, with u as in the standard notation for the binomial model;
(4) $d = 0.8$, with d as in the standard notation for the binomial model;
(5) the continuously compounded risk-free interest rate is $r = 0.04$.

Consider a **chooser** option such that its owner can decide after one year whether the option becomes a put or a call option with exercise date at time-2 and strike equal to \$20.

Find the price of the chooser option.

Solution: The risk-neutral probability is

$$p^* = \frac{e^{0.04} - 0.8}{1.2 - 0.8} = 0.6020.$$

When one constructs the two-period binomial tree, one gets

$$S_u = 24,\ S_d = 16,$$
$$S_{uu} = 28.80,\ S_{ud} = S_{dd} = 19.2,\ S_{dd} = 12.8$$

The call will be worth more than the put in the *up* node while the put will be worth more than the call in the *down* node. This means that the chooser option's owner will choose the call in the *up* node and will choose the put in the *down* node.

The possible payoffs of the call at the end of the second period are

$$V_{uu} = 8.80 \quad \text{and} \quad V_{ud} = 0.$$

So, taking the discounted expected value at the *up* node of the payoff with respect to the risk-neutral probability, we get that the price of this call (and, hence, the price of the chooser option) at the *up* node equals

$$V_u^{CH} = e^{-0.04} \times 8.80 \times 0.602 = 5.0899.$$

The possible payoffs of the put at the end of the second period are

$$V_{ud} = 0.80 \quad \text{and} \quad V_{dd} = 7.20.$$

So, taking the discounted expected value at the *down* node of the payoff with respect to the risk-neutral probability, we get that the price of this put (and, hence, the price of the chooser option) at the *down* node equals

$$V_d^{CH} = e^{-0.04}\left[0.80 \times 0.602 + 7.20 \times 0.398\right] = 3.21595.$$

Finally, the time-0 price of the chooser option equals

$$V_{CH}(0) = e^{-0.04}\left[5.0899 \times 0.602 + 3.21595 \times 0.398\right] = 4.1737. \tag{2.3}$$

Problem 2.13. (10 points) Today's price of a non-dividend-paying stock is observed to be \$80. Its volatility is 0.2 and its dividend yield is 0.01. The evolution of this stock price over the following year is modelled using a three-period binomial tree such that the stock price can either go up by 2% or down by 1% at the end of every period. The continuously compounded risk-free interest rate is 0.03.

What is the price of an \$82-strike European put option on the above stock?

Solution: The up factor is given to be $u = 1.02$ while the down factor equals $d = 0.99$. The possible stock prices at the end of the year are

$$S_{uuu} = S(0)u^3 = 84.8966, \quad S_{uud} = S(0)u^2d = 82.3997,$$
$$S_{udd} = S(0)ud^2 = 79.9762, \quad \text{and} \quad S_{ddd} = S(0)d^3 = 77.6239.$$

Therefore, the possible payoffs of our European put are

$$V_{uuu} = V_{uud} = 0, \quad V_{udd} = 82 - 79.9762 = 2.0238, \quad \text{and} \quad V_{ddd} = 82 - 77.6239 = 4.37608.$$

The risk-neutral probability of the stock price going up in a single period is

$$p^* = \frac{e^{rh} - d}{u - d} = \frac{e^{0.03(1/3)} - 0.99}{1.02 - 0.99} = 0.668339.$$

Hence, the put's price equals

$$V_P(0) = e^{-0.03}[2.0238(3)(0.668339)(1 - 0.668339)^2 + 4.37608(1 - 0.668339)^3] = 0.5880888.$$