M339J *Probability models*
University of Texas at Austin
HW #10
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Note: You **must** show all your work. Numerical answers without a proper explanation or a clearly written down path to the solution will be assigned zero points.

Problem 10.1. (5 points) *Source: Sample STAM Problem #213.*
For an insurance portfolio, you are given that:

(i) The number of claims N has the probability mass function:

$$p_N(0) = 0.1, \quad p_N(1) = 0.4, \quad p_N(2) = 0.3, \quad \text{and} \quad p_N(3) = 0.2.$$

(ii) Each claim amount has a Poisson distribution with mean 3.
(iii) The number of claims and claim amounts are mutually independent.

Calculate the variance of aggregate claims.

Solution: In our usual notation, the total aggregate claims S satisfy

$$Var[S] = \mathbb{E}[N]Var[X] + Var[N](\mathbb{E}[X])^2$$

where X stands for the representative random variable of the common loss distribution. In our case, $X \sim Poisson(\lambda = 3)$, and N has the provided distribution, so that

$$\begin{aligned} \mathbb{E}[N] &= 0.4(1) + 0.3(2) + 0.2(3) = 1.6, \\ \mathbb{E}[N^2] &= 0.4(1) + 0.3(2)^2 + 0.2(3)^2 = 3.4 \\ &\Rightarrow \quad Var[N] = 3.4 - 1.6^2 = 0.84, \\ \mathbb{E}[X] &= Var[X] = 3. \end{aligned}$$

Therefore,

$$Var[S] = 1.6(3) + 0.84(3)^2 = 12.36.$$

Problem 10.2. (5 points) *Source: Sample STAM Problem #287.*
For an aggregate loss random variable S, you are given that

(i) The number of claims N has a negative binomial distribution with parameters $r = 16$ and $\beta = 6$.
(ii) The claim amounts $X_j, j \geq 1$, are uniformly distributed on the interval $(0, 8)$.
(iii) The number of claims and claim amounts are mutually independent.

Using the normal approximation for aggregate losses, calculate the premium π such that the probability that aggregate losses will exceed the premium is 5%.

Solution: Consulting the standard normal tables, we see that the critical z^* corresponding to the upper tail probability of 5% equals 1.645. So, the value π we are looking for is

$$\mathbb{E}[S] + 1.645\sqrt{Var[S]}$$

where S, as usual, denotes the total aggregate loss. We re going to need the first couple of moments of the random variables X and N. Consulting the STAM tables, we get

$$\begin{aligned}
\mathbb{E}[N] &= r\beta = 16(6) = 96,\\
Var[N] &= r\beta(\beta+1) = 16(6)(7) = 672,\\
\mathbb{E}[X] &= 4,\\
Var[X] &= \frac{64}{12} = \frac{16}{3}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathbb{E}[S] &= \mathbb{E}[X]\mathbb{E}[N] = 96(4) = 384,\\
Var[S] &= \mathbb{E}[N]Var[X] + Var[N](\mathbb{E}[X])^2 = 96 \times \frac{16}{3} + 672(4)^2 = 11264.
\end{aligned}$$

Finally, our premium is

$$\pi = 384 + 1.645\sqrt{11264} = 558.587.$$

Problem 10.3. (10 points) *Source: Based on Problem #165 from sample C Exam.*
Consider the following collective risk model:
(i) The claim count random variable N is Poisson with mean 3.
(ii) The severity random variable has the following probability (mass) function:

$$p_X(1) = 0.6,\ p_X(2) = 0.4.$$

(iii) As usual, individual loss random variables are mutually independent and independent of N.

Assume that an insurance covers **aggregate losses** subject to a deductible $d = 3$.
Find the expected value of aggregate payments for this insurance.

Solution: Total aggregate losses are given by

$$S = X_1 + X_2 + \cdots + X_N.$$

So, the expected value of aggregate payments for this insurance equals

$$\mathbb{E}[(S-3)_+] = \mathbb{E}[S] - \mathbb{E}[S \wedge 3].$$

Wald's identity gives us

$$\mathbb{E}[S] = \mathbb{E}[N]\mathbb{E}[X] = 3(0.6(1) + 0.4(2)) = 4.2.$$

On the other hand, the distribution of the random variable $S \wedge 3$ is given by

$$S \wedge 3 \sim \begin{cases} 0 & \text{if } N = 0, \\ 1 & \text{if } N = 1 \text{ and } X_1 = 1, \\ 2 & \text{if } \{N = 1 \text{ and } X_1 = 2\} \text{ or } \{N = 2 \text{ and } X_1 = X_2 = 1\} \\ 3 & \text{otherwise.} \end{cases}$$

So,

$$\begin{aligned} \mathbb{E}[S \wedge 3] &= p_N(1)p_X(1) + 2(p_N(1)p_X(2) + p_N(2)(p_X(1))^2) \\ &\quad + 3(1 - p_N(0) - p_N(1)p_X(1) - 2(p_N(1)p_X(2) + p_N(2)(p_X(1))^2)) \\ &= 3e^{-3}(0.6) + 2(3e^{-3}(0.4) + e^{-3}\frac{3^2}{2}(0.6)^2) + 3(0.720197) \\ &= 0.0896167 + 2(0.1404) + 3(0.720197) \\ &= 2.53101 \end{aligned}$$

So, our answer is $\mathbb{E}[(S-3)_+] = 4.2 - 2.53101 = 1.66899$.

Problem 10.4. (10 points) *Source: Sample STAM Problem #280.*
A compound Poisson claim distribution has the parameter λ equal to 5 and individual claim amounts X distributed as follows:

$$p_X(5) = 0.6 \quad \text{and} \quad p_X(9) = 0.4.$$

What is the expected cost of an aggregate stop-loss insurance subject to a deductible of 5?

Solution: With the individual claim amounts $X_j, j \geq 1$ distributed as above, the aggregate claims are

$$S = X_1 + X_2 + \cdots + X_N$$

where $N \sim Poisson(\lambda = 5)$. We are looking for $\mathbb{E}[(S-5)_+]$. The most straightforward approach is to use the following relationship

$$\mathbb{E}[(S-5)_+] = \mathbb{E}[S] - \mathbb{E}[S \wedge 5].$$

By Wald's identity, we know that $\mathbb{E}[S] = \mathbb{E}[N]\mathbb{E}[X]$. We are given that $\mathbb{E}[N] = 5$ and we can calculate

$$\mathbb{E}[X] = 5(0.6) + 9(0.4) = 6.6.$$

Focusing on the random variable $S \wedge 5$ and taking into account the support of the claim amounts, we conclude that

$$S \wedge 5 \sim \begin{cases} 0 & \text{if } N = 0, \\ 5 & \text{oterwise.} \end{cases}$$

Hence,

$$\mathbb{E}[S \wedge 5] = 5(1 - p_N(0)) = 5(1 - e^{-5}).$$

Pooling our findings together, we get

$$\mathbb{E}[(S-5)_+] = 5(6.6) - 5(1-e^{-5}) = 28 + 5e^{-5} = 28.034.$$

Problem 10.5. (5 points) Aggregate losses for a portfolio of policies are modeled as follows:

(i) The number of losses before any coverage modifications follows a geometric distribution with mean β.
(ii) The severity of each loss before any coverage modifications is uniformly distributed between 0 and ω.

The insurer would like to model the effect of imposing an ordinary deductible d such that $0 < d < \omega$ on each loss and reimbursing only a percentage α, such that $0 < \alpha < 1$ of each loss in excess of the deductible.

It is assumed that the coverage modifications will not affect the loss distribution. The insurer models its claims with modified frequency and severity distributions.

What is the mean of the modified frequency distribution?

Solution: The probability of exceeding the deductible is $\frac{\omega-d}{\omega}$, so the mean number of claims is $\frac{\beta(\omega-d)}{\omega}$.

Problem 10.6. (5 points) A group insurance policy has a negative binomial claim count distribution with mean 200 and variance 600.

The severity random variable X has the following probability mass function:

$$p_X(60) = p_X(120) = p_X(160) = p_X(200) = 1/4.$$

There is a per-loss deductible of 100. Calculate the expected total claim payment.

Solution: The per-loss random variable has the following distribution

$$Y^L \sim \begin{cases} 0 & \text{with probability } 1/4 \\ 20 & \text{with probability } 1/4 \\ 60 & \text{with probability } 1/4 \\ 100 & \text{with probability } 1/4 \end{cases}$$

Its expectation is

$$\mathbb{E}[Y^L] = 20\left(\frac{1}{4}\right) + 60\left(\frac{1}{4}\right) + 100\left(\frac{1}{4}\right) = 45.$$

So,

$$\mathbb{E}[S] = 200(45) = 9000.$$

Problem 10.7. (5 points) Let the loss count random variable have the Poisson distribution with parameter λ. The losses are assumed to be uniform on $(0, a)$. The losses are all mutually independent and independent from the loss count random variable.

There is a per-loss deductible of d such that $d < a$. What is the variance of aggregate claim payments? Express your answer in terms of λ, a and d.

Solution: The payment count random variable is Poisson with parameter $\lambda^P = \lambda\left(\frac{a-d}{a}\right)$.

The per payment random variable Y^P is uniform on $(0, a-d)$.

The aggregate claims S are compound Poisson and their variance equals

$$\mathrm{Var}[S] = \lambda^P \mathbb{E}[(Y^P)^2] = \lambda^P(\mathrm{Var}[Y^P] + (\mathbb{E}[Y^P])^2) = \lambda\left(\frac{a-d}{a}\right)\left(\frac{(a-d)^2}{12} + \left(\frac{a-d}{2}\right)^2\right) = \frac{\lambda(a-d)^3}{3a}.$$