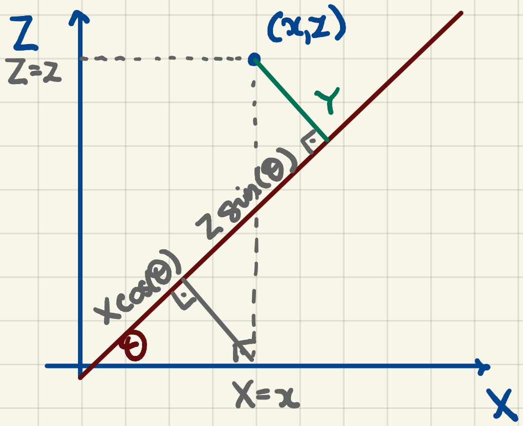
M339G: March 20th, 2024.

Bivariate Normal [cont'd].

Standard.

Start w/ a pair of independent, standard normal random variables. Say, X and Z.



$$\Rightarrow \boxed{Y = X\cdot\cos\theta + Z\cdot\sin\theta}$$

We know: Y is normally dist'd.

$$\mathbb{E}[Y] = \mathbb{E}[X\cdot\cos\theta + Z\cdot\sin\theta]$$

(linearity of $\mathbb{E}$)
$$= \underbrace{\mathbb{E}[X]}_{=0}\cdot\cos\theta + \underbrace{\mathbb{E}[Z]}_{=0}\sin\theta$$

$$= 0$$

$$Var[Y] = Var[X\cdot\cos\theta + Z\sin\theta] = \quad (X \text{ and } Z \text{ independent})$$

$$= \cos^2(\theta)\cdot\underbrace{Var[X]}_{=1} + \sin^2(\theta)\cdot\underbrace{Var[Z]}_{=1} = 1$$

$\Rightarrow \quad Y \sim N(0,1)$

Q: What's the correlation coefficient between X and Y?

$$\longrightarrow: \quad \rho(X,Y) = \frac{Cov[X,Y]}{\underbrace{SD[X]}_{=1}\cdot\underbrace{SD[Y]}_{=1}} = Cov[X,Y]$$

$$= \mathbb{E}[(X-\mu_X)(Y-\mu_Y)] = \mathbb{E}[XY] - \underbrace{\mu_X\cdot\mu_Y}_{=0}$$ (std normal X and Y)

$$= \mathbb{E}[XY] =$$

$$= \mathbb{E}[X(X\cos\theta + Z\sin\theta)] =$$

$$= \cos\theta\cdot\underbrace{\mathbb{E}[X^2]}_{\substack{= Var[X] + (\mathbb{E}[X])^2 \\ = 1}} + \sin\theta\,\underbrace{\mathbb{E}[XZ]}_{\substack{= \text{(independence)} \\ = \mathbb{E}[X]\cdot\mathbb{E}[Z] = 0}}$$

$$\boxed{\rho(X,Y) = \cos\theta}$$

$\square$

Special Cases: $\theta = 0 \Rightarrow X = Y$

$\theta = \frac{\pi}{2} \Rightarrow Y = Z$ (and X and Y are independent)

$\theta = \pi \Rightarrow Y = -X$

In general: For each correlation coefficient $-1 < \rho < 1$, there exists an angle $\boxed{\theta = \arccos(\rho)}$

such that X and Y as above have the correlation ρ.

Alternatively,
$$\boxed{Y = \rho \cdot X + \sqrt{1-\rho^2} \cdot Z}$$

w/ X and Z independent and $N(0,1)$.

Joint Density:
$$f_{X,Y}(x,y) = \frac{1}{(1-\rho^2)(2\pi)} \exp\left(-\frac{1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2)\right)$$

Marginal Dist'ns. $X \sim N(0,1)$, $Y \sim N(0,1)$

Conditional Dist'ns: Given $X = x$, $Y \sim \text{Normal}(\rho x, 1-\rho^2)$

Given $Y = y$, $X \sim \text{Normal}(\rho y, 1-\rho^2)$

Independence: For (X,Y) w/ a standard bivariate normal w/ correlation ρ, X and Y are independent iff $\rho = 0$.

Any Bivariate Normal.

Random variables U and V are said to have the bivariate normal distribution w/ parameters $\mu_U, \mu_V, \sigma_U, \sigma_V$ and ρ if

$$\left(X = \frac{U - \mu_U}{\sigma_U},\ Y = \frac{V - \mu_V}{\sigma_V} \right)$$

has the standard bivariate normal dist'n w/ correlation ρ.

Note: • $\rho(U, V) = ?$

By def'n:

$$\rho(U, V) = \frac{Cov[U, V]}{SD[U] \cdot SD[V]} = \cdots$$