M378K: September 19th, 2025.

Moments.

Def'n. For a r.v. Y w/ pdf f_Y and for $k=1,2,\dots$, we define the k^{th} (raw) moment μ_k as

$$\boxed{\mu_k = \mathbb{E}[Y^k] = \int_{-\infty}^{\infty} y^k f_Y(y)\,dy}$$

$$\mu = \mu_1 = \mathbb{E}[Y]$$

The k^{th} central moment is

$$\boxed{\mu_k^c = \mathbb{E}\left[(Y-\mu)^k\right] = \int_{-\infty}^{\infty} (y-\mu)^k f_Y(y)\,dy}$$

Q: $\mu_2^c = \cancel{?}\ \mathrm{Var}[Y]$

The Cumulative Distribution Function.

Def'n. The cumulative dist'n f'tion (cdf) of a r.v. Y is a function $F_Y : \mathbb{R} \longrightarrow \underline{[0,1]}$ defined as

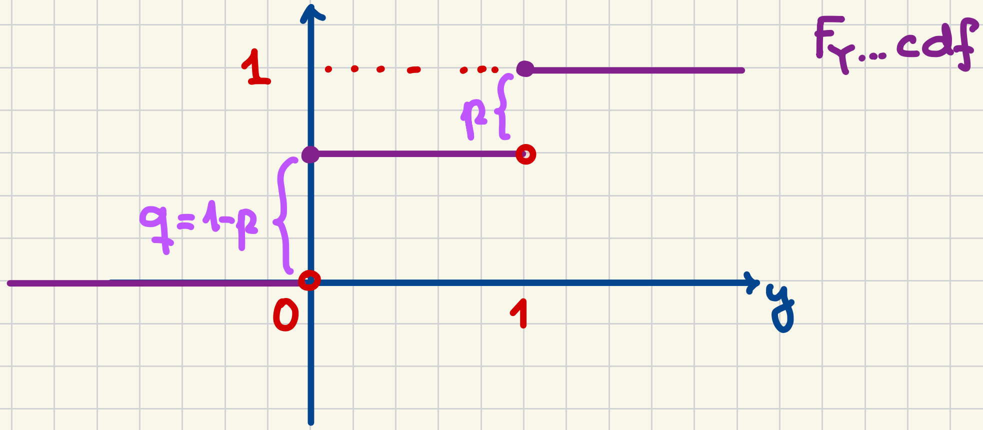
$$\boxed{F_Y(y) = \mathbb{P}[Y \leq y] \quad \text{for all } y \in \mathbb{R}}$$

Properties:

- $0 \leq F_Y(y) \leq 1$ for all y
- F_Y is non-decreasing
- $\lim_{y\to-\infty} F_Y(y) = \underline{0}$
- $\lim_{y\to+\infty} F_Y(y) = 1$

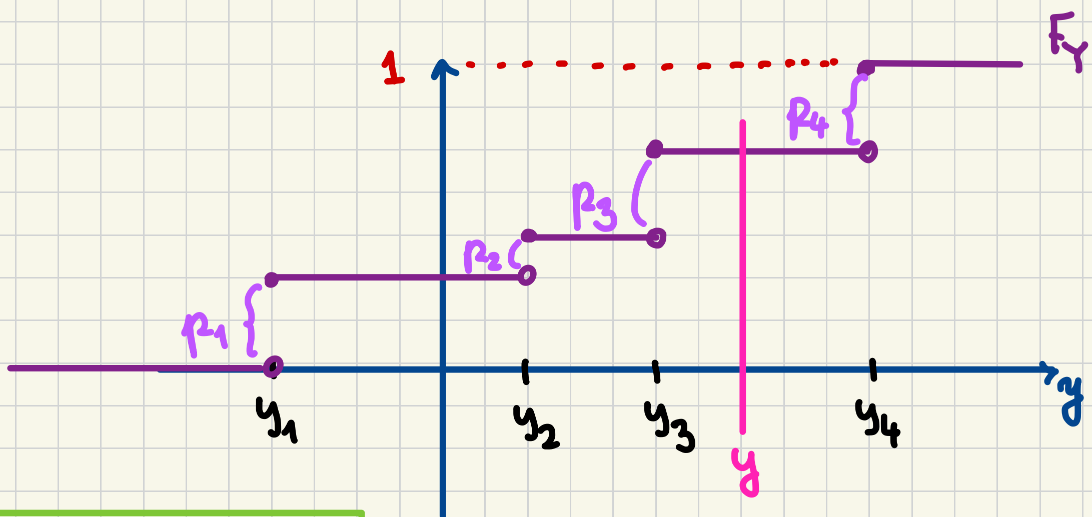
Example. Bernoulli Dist'n.

$\boxed{Y \sim B(p)}$



Example. Discrete w/ Finite Support.

y_1	y_2	$\cdots$	y_m
p_1	p_2	$\cdots$	p_m



The Discrete Case

Say that Y is discrete w/ pmf p_Y.

Then,

$$\boxed{F_Y(y) = \sum_{\substack{u \leq y \\ u \in S_Y}} p_Y(u)}$$

M378K Introduction to Mathematical Statistics
Problem Set #6
Cumulative distribution functions.

Problem 6.1. Source: Sample P exam, Problem #342.
Consider a Poisson distributed random variable X. As usual, let's denote its cumulative distribution function by F_X. You are given that

$$\boxed{\frac{F_X(2)}{F_X(1)} = 2.6}$$

Calculate the expected value of the random variable X.

→ : $X \sim P(\lambda)$

$\mathbb{E}[X] = \lambda$

pmf of X: $k = 0, 1, 2, \ldots$ $\boxed{p_X(k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}} = p_k$

$$\frac{\mathbb{P}[X \leq 2]}{\mathbb{P}[X \leq 1]} = 2.6$$

$$\frac{p_0 + p_1 + p_2}{p_0 + p_1} = 2.6$$

$$\frac{\cancel{e^{-\lambda}} + \cancel{e^{-\lambda}} \cdot \lambda + \cancel{e^{-\lambda}} \cdot \frac{\lambda^2}{2}}{\cancel{e^{-\lambda}} + \cancel{e^{-\lambda}} \cdot \lambda} = 2.6$$

$$\frac{1 + \lambda + \frac{\lambda^2}{2}}{1 + \lambda} = 2.6$$

$$1 + \lambda + \frac{\lambda^2}{2} = 2.6(1 + \lambda)$$

$$\frac{\lambda^2}{2} - 1.6\lambda - 1.6 = 0 \quad / \cdot 10$$

$$5\lambda^2 - 16\lambda + 16 = 0 \qquad \lambda_{1,2} = \frac{\quad}{\quad}$$

$$(\lambda + 20)(\lambda - 4) = 0$$

$$\boxed{\lambda = 4}$$

Keep positive solution!

The Continuous Case

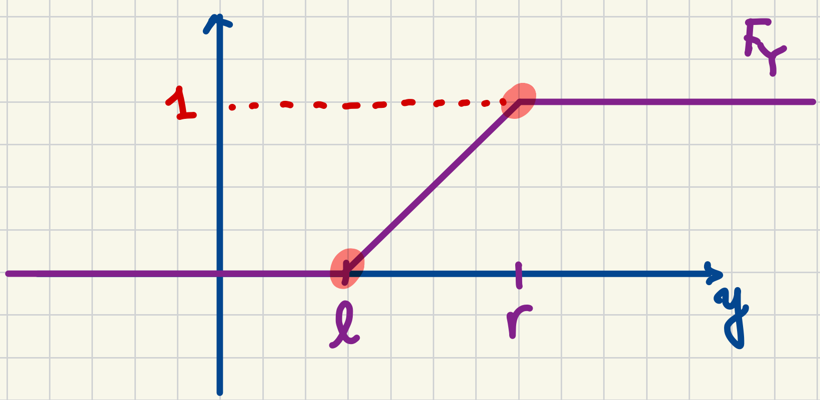
Let Y be a continuous r.v. w/ pdf f_Y.

Then,

$$F_Y(y) = \mathbb{P}[Y \le y] = \mathbb{P}[-\infty < Y \le y] = \int_{-\infty}^{y} f_Y(u)\,du$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = F_Y'(y) \qquad \text{wherever the derivative exists}$$

Example. Uniform $Y \sim U(l, r)$



Fact: The cdf of a continuous random variable is a continuous function w/ @ most countably many points @ which its not differentiable.

Problem 6.2. *Consider a random variable* Y *whose cumulative distribution function is given by*

$$F_Y(y) = \begin{cases} 0, & \textit{for } y < 0 \\ y^4, & \textit{for } 0 \le y < 1 \\ 1, & \textit{for } 1 \le y \end{cases}$$

Calculate the expectation of the random variable Y.

$\longrightarrow$:

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \begin{cases} 0 & y < 0 \\ 4y^3 & 0 \le y < 1 \\ 0 & y \geq 1 \end{cases}$$

$$\boxed{f_Y(y) = 4y^3 \cdot \mathbb{1}_{(0,1)}(y)}$$

$$\mathbb{E}[Y] = \int_{-\infty}^{\infty} y\, f_Y(y)\, dy = \int_0^1 y \cdot 4y^3\, dy =$$

$$= 4\int_0^1 y^4\, dy = 4 \cdot \left(\frac{y^5}{5}\right)_{y=0}^{1} = \frac{4}{5} \quad \square$$