Name:

M339D/M389D Introduction to Financial Mathematics for Actuaries
University of Texas at Austin
In-Term Three
Instructor: Milica Čudina

Notes: This is a closed book and closed notes exam. The maximum number of points on this exam is 80. The total number of points available is 95.
Time: 50 minutes

1.1. **Free-response problems.** Please, explain carefully all your statements and assumptions. Numerical results or single-word answers without an explanation (even if they're correct) are worth 0 points.

Problem 1.1. (10 points) You roll a fair tetrahedron whose sides are labeled by $1, 2, 3$, and 4 a total of 4000 times. What is the approximate probability that you see a 1 strictly more than 1025 times? There is no need to use the continuity correction.

Solution: The number of heads is $X \sim Binomial(n = 4000, p = 0.25)$. Evidently, we can use the normal approximation to the binomial. We have

$$\mu_X = \mathbb{E}[X] = 1000 \quad \text{and} \quad \sigma_X = 27.38613.$$

The probability we are seeking is

$$\mathbb{P}[X > 1025] \approx 1 - N\left(\frac{1025 - 1000}{27.38613}\right) \approx 1 - N(0.91) = 1 - 0.8186 = 0.1814.$$

Problem 1.2. (10 points) *Source: Open Course Intro to Statistics.*
Cholesterol levels for women aged 20 to 34 follow an approximately normal distribution with mean 185 milligrams per deciliter (mg/dl). Women with cholesterol levels above 220 mg/dl are considered to have high cholesterol and about 18.5% of women fall into this category. What is the standard deviation of the distribution of cholesterol levels for women aged 20 to 34?

Solution: Let X be the random variable denoting the cholesterol level. Then,

$$X \sim N(mean = 185, variance = \sigma^2).$$

We are given that

$$\mathbb{P}[X > 220] = 0.185 \quad \Rightarrow \quad \mathbb{P}[X \leq 220] = 1 - 0.185 = 0.815.$$

So,

$$220 = 185 + \sigma z_*$$

where z_* is the critical value such that $N(z_*) = 0.815$. The closest value in the standard normal tables is $z_* = 0.9$. Hence, our answers is

$$\sigma = \frac{220 - 185}{0.9} = 38.8889$$

Problem 1.3. (10 points) Suppose that the failure time (in seconds) of a certain component is modeled as lognormal random variable $Y = e^X$ such that the mean of X is -0.35 and its variance is 0.04.

What is the failure time t^* such that 95% of the components of the same type would still function after that time?

Solution: We are looking for the value t^* such that

$$\mathbb{P}[Y > t^*] = 0.95 \quad \Leftrightarrow \quad \mathbb{P}[Y \leq t^*] = 0.05.$$

The critical value z^* such that $N(z^*) = 0.05$ is -1.645. So,

$$t^* = e^{-0.35+0.2(-1.645)} = 0.5071.$$

Problem 1.4. (15 points) The current price of a non-dividend-paying stock is \$100 per share. Its volatility is given to be 0.25.

The continuously compounded risk-free interest rate equals 0.04.

Consider a \$95-strike European call option on the above stock with nine months to expiration. Using a three-period **forward** binomial tree, find the price of this put option. **I priced the call below. I graded both the call and the put for accuracy.**

Solution: The risk-neutral probability of the stock price moving up in a single period is

$$p^* = \frac{1}{1 + e^{0.25\sqrt{0.25}}} = 0.4687906.$$

The up and down factors in the above model are

$$u = e^{0.04\times 0.25+0.25\sqrt{0.25}} = 1.144537,$$
$$d = e^{0.04\times 0.25-0.25\sqrt{0.25}} = 0.8913661.$$

The relevant possible stock prices at the "leaves" of the binomial tree are

$$S_{uuu} = u^3 S(0) = 149.9303,$$
$$S_{uud} = u^2 d S(0) = 116.7658.$$

The remaining two final states of the world result in the call option being out-of-the-money at expiration.

So, the price of the European call option equals

$$V_C(0) = e^{-0.04(3/4)} \left[(149.9303 - 95)(p^*)^3 + (116.7658 - 95)(3)(p^*)^2(1 - p^*)\right] = 12.88946.$$

Problem 1.5. (10 points) Assume the Black-Scholes model. Under the risk-neutral probability, you expect the stock price in half a year to be \$86.45. The stock's volatility is 0.30. What is the median stock price in half a year according to that same model?

Solution: In our usual notation, we have that

$$\frac{\text{mean stock price}}{\text{median stock price}} = \frac{S(0)e^{rT}}{S(0)e^{(r-\frac{\sigma^2}{2})T}} = e^{\frac{\sigma^2 T}{2}}$$

So, in this problem,

$$\text{median stock price} = \mathbb{E}[S(T)]e^{-\frac{\sigma^2 T}{2}} = 86.45e^{-\frac{0.09(1/2)}{2}} = 84.52659.$$

Problem 1.6. (10 points) Assume the Black-Scholes framework. For an at-the-money, $T-$year European call option on a non-dividend-paying stock you are given that its delta equals 0.6772. What is the delta of an otherwise identical option with exercise date at time $4T$?

Solution: For the first option,

$$N(d_1) = 0.6772 \quad \Rightarrow \quad d_1 = 0.46.$$

On the other hand, by definition, and since the option is in-the-money

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T\right] = \frac{r + \frac{\sigma^2}{2}}{\sigma}\sqrt{T}$$

Using the same reasoning, for the $4T-$option, we have

$$\tilde{d}_1 = \frac{1}{\sigma\sqrt{4T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(4T)\right] = \frac{r + \frac{\sigma^2}{2}}{\sigma}\sqrt{4T}$$

Hence,

$$\tilde{d}_1 = d_1\sqrt{4} = 2(0.46) = 0.92.$$

So, the delta of the second option equals

$$N(0.92) = 0.8212.$$

Problem 1.7. (15 points) Assume the Black-Scholes setting. Let $S(0) = \$50, \sigma = 0.32, r = 0.04$. The stock pays no dividends. Consider a \$45-strike put option which expires in four months. What is the price of the put?

Solution: In our usual notation, the price is

$$V_P(0) = Ke^{-r \cdot T}N(-d_2) - S(0)N(-d_1)$$

with

$$d_1 = 0.7348253, \quad d_2 = 0.5500733.$$

So, $V_P(0) = 1.366387$.

Problem 1.8. (15 points) The current price of a non-dividend-paying stock is given to be \$92. The stock's volatility is 0.25.

The continuously compounded risk-free interest rate is 0.04.

Consider a \$90-strike European call option on the above stock with exercise date in a quarter-year. What is the Black-Scholes price of this call option?

Solution: In our usual notation,

$$d_1 = 0.3183313, d_2 = 0.1933313.$$

So,

$$V_C(0) = S(0)N(d_1) - Ke^{-rT}N(d_2) = 6.107129.$$