M378K: February 28th, 2025.

The $\chi^2(n)$-distribution.

If $Y \sim N(0,1)$, then $\boxed{W = Y^2 \sim \chi^2(df=1)}$

Its density is $\boxed{f_W(w) = \frac{1}{\sqrt{2\pi w}} e^{-w/2} \mathbb{1}_{(0,\infty)}(w)}$

Its mgf is $\boxed{m_W(t) = (1-2t)^{-1/2} = \frac{1}{\sqrt{1-2t}}}$ $\longleftarrow$

Example. $Y_1 \sim N(0,1)$, $Y_2 \sim N(0,1)$ independent

Set $\boxed{W = Y_1^2 + Y_2^2} \sim \chi^2(df=2)$

Let's get its mgf.

$\longrightarrow$: $m_W(t) = m_{Y_1^2}(t) \cdot m_{Y_2^2}(t)$

$$= \frac{1}{\sqrt{1-2t}} \cdot \frac{1}{\sqrt{1-2t}} = \boxed{\frac{1}{1-2t}} = \frac{\frac{1}{2}}{\frac{1}{2} - t}$$

From the HW, we know that $\boxed{W \sim E(\tau=2)}$.

$\boxed{\text{So, } \chi^2(df=2) \text{ is the same a } E(\tau=2).}$

Def'n. The χ^2-dist'n w/ n degrees of freedom is the dist'n of the sum

$$W = Y_1^2 + Y_2^2 + \cdots + Y_n^2$$

where $Y_i \sim N(0,1)$ for $i=1..n$ and independent.

We write $\boxed{W \sim \chi^2(n) = \chi^2(df=n)}$

Note: $m_W(t) = \left(\frac{1}{\sqrt{1-2t}}\right)^n = (1-2t)^{-\frac{n}{2}} = \left(\frac{1}{1-2t}\right)^{\frac{n}{2}}$
$\uparrow$

Problem. Let $Y \sim \chi^2(df=5)$.

Find $\mathbb{P}[1.145 \leq Y \leq 12.83] = ?$

$\longrightarrow$: **Tables:** $\mathbb{P}[Y \leq 12.83] - \mathbb{P}[Y \leq 1.145] =$

$= 0.975 - 0.05 = 0.925$

R. pchisq(12.83, df=5) - pchisq(1.145, df=5) =

$= 0.9250188$ $\square$

The Gamma Distribution.

Def'n. A random variable Y is said to have the gamma distribution w/ parameters $k>0$ and $\tau>0$, if its mgf is of this form

$$m_Y(t) = \left(\frac{1}{1-\tau t}\right)^k.$$

We write $\boxed{Y \sim \Gamma(k, \tau)}$

Note:
$$\boxed{\begin{aligned} \mathbb{E}[Y] &= k \cdot \tau \\ Var[Y] &= k \cdot \tau^2 \end{aligned}}$$

Q: Say that $Y \sim \Gamma(1, \tau)$. Do you know another name for it?

$Y \sim E(\tau)$ ☺

Q: Say that $Y \sim \Gamma(\frac{n}{2}, 2)$. What's another name for it?

$Y \sim \chi^2(n)$ ☺

Q: $Y_1 \sim \Gamma(k_1, \tau)$, $Y_2 \sim \Gamma(k_2, \tau)$ independent

$$\boxed{Y_1 + Y_2 \sim \Gamma(k_1 + k_2, \tau)}$$