Moment Generating Functions

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For any random variable Y:
for independent arguments t

$$M_Y(t) := \mathbb{E}\left[e^{t \cdot Y}\right]$$

wherever it exists, i.e, for whichever t it is finite

<u>Note:</u> $\boxed{M_Y(0) = 1}$ So, at least $t=0$ is in the domain

For $Z \sim N(0,1)$, we have $M_Z(t) = e^{t^2/2}$ for all $t \in \mathbb{R}$

If we want to understand
$X \sim \text{Normal}(\text{mean} = m, \text{var} = \tau^2)$,
it's convenient to look at

$$\boxed{X \overset{(d)}{=} m + \tau \cdot Z}$$

Returning to any rnd variable Y, define

$$\tilde{Y} = a \cdot Y + b$$

$$M_{\tilde{Y}}(t) \underset{\substack{\uparrow \\ \text{by def'n} \\ \text{of m.g.f.}}}{=} \mathbb{E}\left[e^{\tilde{Y} t}\right] \underset{\substack{\uparrow \\ \text{by def'n} \\ \text{of } \tilde{Y}}}{=} \mathbb{E}\left[e^{(aY+b) \cdot t}\right]$$

$$= \mathbb{E}\left[e^{aY\cdot t + bt}\right] = \mathbb{E}\left[e^{aY\cdot t}\cdot e^{bt}\right]$$

$$= e^{bt}\,\mathbb{E}\left[e^{aY\cdot t}\right] = e^{bt}\cdot\mathbb{E}\left[e^{(at)Y}\right]$$ rnd var

$$= e^{bt} M_Y(at)$$

$\Rightarrow$ In particular, for the normally dist'd X, we have: $M_X(t) = e^{m\cdot t}\cdot M_Z(\tau\cdot t) = e^{mt + \frac{\tau^2\cdot t^2}{2}}$ ☆

<u>Motivation</u>: We will model realized returns as normally dist'd

$\Rightarrow$ the stock prices will be of the form

$$S(t) = S(0)\, e^{\overbrace{R(0,t)}^{\sim \text{Normal}}}$$

<u>LogNormal Distribution</u>

<u>Def'n</u>. A random variable Y is said to be lognormally distributed (or lognormal) if there exists a normal rnd variable X such that

$$Y = e^X$$

$$\Longleftrightarrow$$

$$\ln(Y) = X$$

Mean / Expectation of Y = ?

$\mathbb{E}[Y] = ?$

If Y is lognormal, then $Y = e^X$ for some

$X \sim \text{Normal}(\text{mean}=m, \text{var}=\tau^2)$

$$\mathbb{E}[Y] = \mathbb{E}[e^X] = M_X(1) \underset{\star}{=} e^{m \cdot 1 + \frac{\tau^2 \cdot 1}{2}}$$

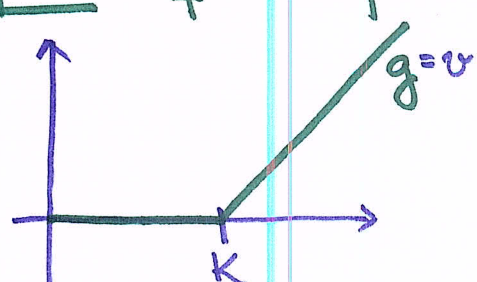
$$\boxed{\mathbb{E}[Y] = e^{m + \frac{\tau^2}{2}}}$$

Caveat: $\mathbb{E}[e^X] \geq e^{\mathbb{E}[X]}$

This is a special case of <u>Jensen's Inequality</u>:

If X is a random variable,
and g is a $\boxed{\text{convex}}$ function
such that $g(X)$ is well-defined
and $\mathbb{E}[g(X)]$ is well-defined,
then
$$\boxed{\mathbb{E}[g(X)] \geq g(\mathbb{E}[X])}$$

<u>Examples</u> of useful functions g:

- 

Payoff of a call option:

$v(s) = (s-K)_+$

<u>We have:</u>

$$\mathbb{E}[(S(T)-K)_+] \geq (\mathbb{E}[S(T)]-K)_+$$

• in short-term insurance (M339J):

$\begin{cases} X \ldots \text{severity ind variable} \\ d \ldots \text{deductible} \end{cases}$

→ the insurer has to pay: $(X-d)_+$

the expected value: $\mathbb{E}[(X-d)_+]$

$$\mathbb{E}[(X-d)_+] \geq (\mathbb{E}[X]-d)_+$$

Median of $Y = ?$

$a^* = ?$ such that $\boxed{F_Y(a^*) = \frac{1}{2}}$

$$\frac{1}{2} = F_Y(a^*) \underset{\substack{\uparrow \\ \text{by def'n} \\ \text{of cdf}}}{=} \mathbb{P}[Y \leq a^*] \underset{\substack{\uparrow \\ \text{by def'n} \\ \text{of } Y}}{=} \mathbb{P}[e^{m+\tau \cdot Z} \leq a^*]$$

$Z \sim N(0,1)$

$$\frac{1}{2} = \mathbb{P}[m + \tau \cdot Z \leq \ln(a^*)] = \mathbb{P}[\tau \cdot Z \leq \ln(a^*) - m]$$

since $\ln(\cdot)$ is increasing

$$= \mathbb{P}[Z \leq \underbrace{\tfrac{1}{\tau}(\ln(a^*) - m)}_{=0}]$$

$$\Downarrow$$

$$\boxed{a^* = e^m}$$

Q: What if you're looking for quatiles of the lognormal?

You can generalize the above: find the quantiles of the std normal (z^*) → the quantile of $Y = e^{m+\tau \cdot z^*}$

For the lognormal :

$$\underline{\text{mean}} \geq \underline{\text{median}}$$

$$e^{m+\frac{\tau^2}{2}} \geq e^{m}$$