Delta-hedger's profit over a "small" time period.

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Set up:

1st An agent writes an option @ time 0.
2nd Then, they Δ-hedge @ time 0 by trading in the shares of the underlying asset.

$\Rightarrow$ At time 0, the agent's wealth is:

$$\boxed{w(S(0),0) = -\underbrace{v(S(0),0)}_{\text{value ftion of the option they wrote}} + \underbrace{\Delta(S(0),0)}_{\text{delta of the option}} \cdot S(0)}$$

Consider the value of their portfolio @ time dt (before the portfolio is rebalanced to re-establish the Δ-hedge).

$$w(S(dt),dt) = -v(S(dt),dt) + \Delta(S(0),0)\, S(dt)$$

For a "small" time dt, we use the Δ-Γ-Θ approximation to get:

$dS = S(dt) - S(0)$

$$\begin{aligned}
w(S(dt),dt) \approx -\Big(v(S(0),0) &+ \cancel{\Delta(S(0),0)\,dS} \\
&+ \frac{1}{2}\Gamma(S(0),0)(dS)^2 \\
&+ \Theta(S(0),0)\,dt\Big) \\
&+ \Delta(S(0),0)\underbrace{S(dt)}_{(S(0)+\cancel{dS})}
\end{aligned}$$

$$\begin{aligned}
= \boxed{-v(S(0),0) + \Delta(S(0),0)\cdot S(0)} \\
- \frac{1}{2}\Gamma(S(0),0)(dS)^2 \\
- \boxed{\Theta(S(0),0)\,dt}
\end{aligned}$$

To calculate the approximate profit:

$$w(S(dt),dt) - w(S(0),0)e^{r\,dt} \approx$$

$$\approx w(S(0),0)\left(1-e^{r\,dt}\right) - \frac{1}{2}\Gamma(S(0),0)(dS)^2 - \Theta(S(0),0)\,dt$$

Delta-Gamma Hedging.

Let's start w/ a Δ-neutral portfolio whose value f'tion is $v(s,t)$, i.e., we maintain $\Delta(s,t)=0$. This is accomplished by trading in the shares of the underlying asset.

The investor decides to Γ-hedge as well, i.e., they want to maintain a Γ-neutral portfolio.

Q: Can they accomplish this by trading in the shares of the underlying?

$\longrightarrow$: Recall that the Γ of the stock is 0.

So: NO!

We need another option on the same underlying w/ a non-zero gamma.

Denote the value f'tion of this option by $\tilde{v}(s,t)$; its delta is $\tilde{\Delta}(s,t)$ and its gamma is $\tilde{\Gamma}(s,t)$.

Let $\tilde{N}(s,t)$ is the # of the new options which needs to be held at time t w/ the stock price s. To have Γ-neutrality:

$$\Gamma(s,t) + \tilde{N}(s,t)\cdot\tilde{\Gamma}(s,t) = 0$$

$$\Rightarrow \boxed{\tilde{N}(s,t) = -\frac{\Gamma(s,t)}{\tilde{\Gamma}(s,t)}}$$