M378K: November 12th, 2025.

Confidence Intervals for the Variance.

Consider a normal model w/ both parameters unknown, i.e., a random sample $(Y_1, \dots, Y_n)$ from $N(\mu, \sigma)$ both unknown.

A $\boxed{\text{good}}$ unbiased point estimator for σ^2 is

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y})^2$$

Theorem. Consider the above set-up. Set

$$Q^2 = \frac{1}{\sigma^2}\sum_{i=1}^{n}(Y_i - \bar{Y})^2$$

Then,
- Q^2 is a pivotal quantity for σ^2; in fact, $\boxed{Q^2 \sim \chi^2(df = n-1)}$
- $\boxed{\bar{Y} \text{ and } Q^2 \text{ are INDEPENDENT}}$

Problem 16.3. *What is the unbiased estimator for σ^2.*

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (Y_i - \bar{Y})^2$$

Problem 16.4. *Assume a random sample $Y_1, Y_2, \ldots, Y_n$ from a normal distribution with mean μ and standard deviation σ - both unknown. What's the distribution of*

$$\boxed{Q^2} = \frac{\sum_{i=1}^{n}(Y_i - \bar{Y}_n)^2}{\sigma^2} = \frac{(n-1)S^2}{\sigma^2}$$

$\chi^2(df = n-1)$

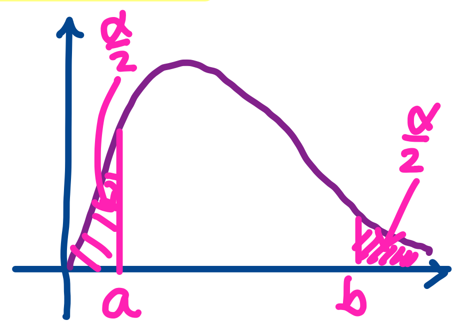
A PIVOTAL QUANTITY for σ^2

Problem 16.5. *Assume that you are assigned a* confidence level $1-\alpha$. *What does it mean to find a* confidence interval *for* $\cancel{\sigma}^2$? σ^2

$$\mathbb{P}[a \le Q^2 \le b] = 1-\alpha$$

$$\mathbb{P}\left[a \le \frac{(n-1)S^2}{\sigma^2} \le b\right] = 1-\alpha$$

a: χ^2_L $\qquad$ b: χ^2_U



Problem 16.6. *Are χ^2_L and χ^2_U as above uniquely defined?*

No. ☺

We can choose a symmetric confidence interval via

$a = \chi^2_L = \text{qchisq}(\alpha/2, df=n-1)$

and

$b = \chi^2_U = \text{qchisq}(1-\alpha/2, df=n-1)$

Problem 16.7. *What's the form of the confidence interval, then?*

$$\mathbb{P}\left[\chi^2_L \leq \frac{(n-1)s^2}{\sigma^2} \leq \chi^2_U\right] = 1-\alpha$$

$$\mathbb{P}\left[\frac{1}{\chi^2_L} \geq \frac{\sigma^2}{(n-1)s^2} \geq \frac{1}{\chi^2_U}\right] = 1-\alpha$$

$$\mathbb{P}\left[\underbrace{\boxed{\frac{(n-1)s^2}{\chi^2_U}}}_{\hat{\theta}_L} \leq \sigma^2 \leq \underbrace{\boxed{\frac{(n-1)s^2}{\chi^2_L}}}_{\hat{\theta}_R}\right] = 1-\alpha$$

Problem 16.8. *Assume the above setting. Let the random sample be of size $n = 9$. You do the arithmetic and arrive at the estimate $s^2 = 7.93$ (based on the data set). Using the above procedure, find the 90%−confidence interval for σ^2.*

$\longrightarrow$: $n=9 \Rightarrow df = n-1 = 8$

$\alpha = 0.10$

$s^2 = 7.93$

Our confidence interval:

$$\left(\frac{8 \cdot 7.93}{\chi^2_U}, \frac{8 \cdot 7.93}{\chi^2_L}\right) = ?$$

$\chi^2_L = \text{qchisq}(0.05, df=8) = 2.732637$

$\chi^2_U = \text{qchisq}(0.95, df=8) = 15.50731$

$(4.090973, 23.21567)$

$\square$

Confidence intervals for the μ w/ variance unknown.

Focus on the normal model $N(\mu, \sigma)$ w/ both parameters unknown, but, w/ target parameter μ.

Theorem.
- $Q^2 = \frac{1}{\sigma^2} \sum_{i=1}^{n} (Y_i - \bar{Y})^2 \sim \chi^2(df = n-1)$
- $\bar{Y}$ and Q^2 are independent.

Goal: Confidence interval for μ.

Idea: $\bar{Y} \sim Normal(mean = \mu, sd = \frac{\sigma}{\sqrt{n}})$

Use $\boxed{U = \frac{\bar{Y} - \mu}{\frac{S}{\sqrt{n}}}}$ w/ $S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (Y_i - \bar{Y})^2$ as a pivotal quantity.

$$\frac{\frac{\bar{Y}-\mu}{\frac{\sigma}{\sqrt{n}}}}{\frac{S}{\sigma}} = \frac{Z}{\sqrt{\frac{S^2}{\sigma^2}}} = \frac{Z}{\sqrt{\frac{Q^2}{n-1}}}$$

$\sim N(0,1) \sim Z$