M378K: January 27th, 2025.

Expectation [cont'd].

Def'n.

$$\boxed{\mathbb{E}[Y] = \sum_{y \in S_Y} y \cdot p_Y(y)}$$

Example. $Y \sim g(p)$

$$\mathbb{E}[Y] = \sum_{k=0}^{\infty} k \cdot \boxed{p_Y(k)} = \sum_{k=1}^{\infty} k \cdot q^k \cdot p = p \sum_{k=1}^{\infty} k \cdot q^k$$

Not a geometric series.

$$\sum_{k=1}^{\infty} k \cdot p_k = p_1 + 2p_2 + \cdots + k \cdot p_k + \cdots$$

$$\begin{aligned}
= & \; p_1 + \\
& + p_2 + p_2 \\
& + p_3 + p_3 + p_3 \\
& \;\vdots \\
& + p_k + p_k + p_k + \cdots + p_k \\
& \;\vdots
\end{aligned}$$

$$= \mathbb{P}[Y>0] + \mathbb{P}[Y>1] + \mathbb{P}[Y>2] + \cdots + \mathbb{P}[Y>k-1] + \cdots$$

$$= q + q^2 + q^3 + \cdots + q^k + \cdots$$

$$= q\,(1 + q + q^2 + \cdots + q^{k-1} + \cdots)$$

$$= q \cdot \frac{1}{1-q} = \frac{q}{p}$$

Variance.

Def'n. The variance of a random variable Y is defined as

$$Var[Y] := \mathbb{E}\left[(Y-\mathbb{E}[Y])^2\right] \qquad \text{if "finite"}$$

The standard deviation of Y is

$$SD[Y] := \sqrt{Var[Y]}.$$

Formula:

$$Var[Y] = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2$$

$\longrightarrow$: $\mu_Y := \mathbb{E}[Y]$

$$Var[Y] = \mathbb{E}[(Y-\mu_Y)^2]$$

$$= \mathbb{E}[Y^2 - 2\mu_Y \cdot Y + \mu_Y^2] \quad \text{linearity of } \mathbb{E}$$

$$= \mathbb{E}[Y^2] - 2\mu_Y \underbrace{\mathbb{E}[Y]}_{\mu_Y} + \mu_Y^2$$

$$= \mathbb{E}[Y^2] - \mu_Y^2 \quad \square$$

Theorem. Say that Y_1 and Y_2 are rvs w/ finite variances and that α is a real constant. Then,

- $Var[\alpha Y_1] = \alpha^2 Var[Y_1]$
- when, additionally, Y_1 and Y_2 are $\boxed{\text{independent}}$

$$\boxed{Var[Y_1 + Y_2] = Var[Y_1] + Var[Y_2]}$$

Problem 4.2. Source: Sample P exam, Problem #458.

An investor wants to purchase a total of ten units of two assets, A and B, with annual payoffs **per unit** *purchased of X and Y, respectively. Each asset has the same purchase price per unit. The payoffs are independent random variables with equal expected values and with* $\mathrm{Var}(X) = 30$ *and* $\mathrm{Var}(Y) = 20$.

Calculate the number of units of asset A the investor should purchase to minimize the variance of the total payoff.

$\longrightarrow$: n ... # of units of A that is bought

$10-n$ ... # of units of B bought

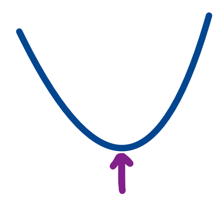
$$\mathrm{Var}[n\cdot X + (10-n)\cdot Y] \longrightarrow \min$$

$$n^2\cdot \mathrm{Var}[X] + (10-n)^2\cdot \mathrm{Var}[Y] \longrightarrow \min$$

$$30n^2 + 20(10-n)^2 \longrightarrow \min$$

$$50n^2 - 400n + 2000 \longrightarrow \min$$

$$n^2 - 8n + 40 \longrightarrow \min$$



$$n^* = -\frac{-8}{2\cdot 1} = 4$$

$\square$

Example. • Bernoulli. $Y \sim B(p)$

$$\mathbb{E}[Y] = 0 \cdot q + 1 \cdot p = p$$

$$\mathrm{Var}[Y] = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2$$

$$\mathbb{E}[Y^2] = 0^2 \cdot q + 1^2 \cdot p = p$$

$$\mathrm{Var}[Y] = p - p^2 = p(1-p) = p \cdot q$$

• Binomial. $Y \sim b(n, p)$

$$\mathbb{E}[Y] = np$$

$$Y = I_1 + I_2 + \dots + I_n$$

$I_j \sim B(p)$, $j = 1..n$, independent

$$\mathbb{E}[Y] = \mathbb{E}[I_1] + \dots + \mathbb{E}[I_n]$$
$$= p + \dots + p = np$$

$$\mathrm{Var}[Y] = npq$$

$$\mathrm{Var}[Y] = \mathrm{Var}[I_1 + \dots + I_n]$$ $\boxed{\text{independence}}$
$$= \mathrm{Var}[I_1] + \dots + \mathrm{Var}[I_n]$$
$$= n \cdot \mathrm{Var}[I_1] = n \cdot p \cdot q$$

• Geometric. $Y \sim g(p)$

$$\mathbb{E}[Y] = p \cdot 0 + q(1 + \mathbb{E}[Y]) = q + q\,\mathbb{E}[Y]$$

$$\mathbb{E}[Y] = \frac{q}{1-q} = \frac{q}{p}$$

$$\mathrm{Var}[Y] = \frac{q}{p^2} \implies SD[Y] = \frac{\sqrt{q}}{p}$$

- **Poisson.** $Y \sim P(\lambda)$

$$\mathbb{E}[Y] = \lambda = Var[Y]$$