M339D : November 18th, 2022.

Problem 10.4. Assume the Black-Scholes setting. Let $S(0) = \$63.75$, $\sigma = 0.20$, $r = 0.055$. The stock pays no dividend and the option expires in 50 days (simplify the number of days in a year to 360).

What is the price of a \$60-strike European put?

(a) 0.66
(b) 0.55
(c) 0.44
(d) 0.37
(e) None of the above.

$\longrightarrow$:

$$d_1 = \frac{1}{0.2\sqrt{\frac{50}{360}}}\left[\ln\left(\frac{63.75}{60}\right) + \left(0.055 + \frac{(0.2)^2}{2}\right)\left(\frac{50}{360}\right)\right]$$

$d_1 = 0.9531189 \approx 0.95$ $\quad N(-0.95) = \underline{0.1711}$

$d_2 = d_1 - \sigma\sqrt{T} = 0.8785833 \approx 0.88$ $\quad N(-0.88) = \underline{0.1894}$

$$V_p(0) = Ke^{-rT}\boxed{N(-d_2)} - S(0)\boxed{N(-d_1)}$$

$$V_p(0) = 60\,e^{-0.055\left(\frac{50}{360}\right)} \cdot 0.1894 - 63.75 \cdot 0.1711$$

$$V_p(0) = 0.3698974$$

8. Let $S(t)$ denote the price at time t of a stock that pays no dividends. The Black-Scholes framework holds. Consider a European call option with exercise date T, $T > 0$, and exercise price $S(0)e^{rT}$, where r is the continuously compounded risk-free interest rate.

$\boxed{K = S(0)e^{rT}}$ ✓

You are given:

(i) $S(0) = \$100$

(ii) $T = 10$

(iii) $\mathrm{Var}[\ln S(t)] = \boxed{0.4t}$, $t > 0$. $\longrightarrow$

$$S(t) = S(0)e^{(r - \frac{\sigma^2}{2})\cdot t + \sigma\sqrt{t}\cdot Z}$$

$$\ln(S(t)) = \ln(S(0)) + (r - \frac{\sigma^2}{2})\cdot t + \sigma\sqrt{t}\cdot Z$$

$$\mathrm{Var}[\ln(S(t))] = \mathrm{Var}[\sigma\sqrt{t}\cdot Z]$$

$$= \sigma^2 \cdot t = 0.4\cdot t$$

$$\sigma^2 = 0.4$$

$$\boxed{\sigma = \sqrt{0.4}}$$

Determine the price of the call option.

(A) $7.96

(B) $24.82

(C) $68.26

(D) $95.44

(E) There is not enough information to solve the problem.

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{\cancel{S(0)}}{\cancel{S(0)}e^{rT}}\right) + (r + \frac{\sigma^2}{2})\cdot T\right]$$

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\cancel{-rT} + \cancel{rT} + \frac{\sigma^2}{2}\cdot T\right] = \boxed{\frac{\sigma\sqrt{T}}{2}}$$ ✓

$$d_2 = d_1 - \sigma\sqrt{T} = \frac{\sigma\sqrt{T}}{2} - \sigma\sqrt{T} = \boxed{-\frac{\sigma\sqrt{T}}{2}}$$ ✓

$$V_c(0) = S(0)N(d_1) - \boxed{K}e^{-rT}\cdot N(d_2)$$

$$V_c(0) = S(0)N\left(\frac{\sigma\sqrt{T}}{2}\right) - S(0)\cancel{e^{rT}}\cdot\cancel{e^{-rT}}\cdot N\left(-\frac{\sigma\sqrt{T}}{2}\right)$$

$$V_c(0) = S(0)\left(N\left(\frac{\sigma\sqrt{T}}{2}\right) - \underbrace{N\left(-\frac{\sigma\sqrt{T}}{2}\right)}_{= 1 - N\left(\frac{\sigma\sqrt{T}}{2}\right)}\right)$$

symmetry of $N(0,1)$

$$\boxed{V_c(0) = S(0)\left(2\cdot N\left(\frac{\sigma\sqrt{T}}{2}\right) - 1\right)}$$

$$\frac{\sigma\sqrt{T}}{2} = \frac{\sqrt{0.4}\cdot\sqrt{10}}{2} = 1$$

$$V_c(0) = 100\left(2\cdot N(1) - 1\right)$$

$$V_c(0) = 100\left(2\cdot 0.8413 - 1\right) = \underline{68.27} \quad \square$$

Problem. For European call options on a non-dividend-paying stock. (BS Model.)

(i) The time-t stock price is $S(t)$.

(ii) The strike price of the option is $S(0)e^{rT}$ where T is the exercise date.

(iii) The price of a call option w/ one year to exercise equals $\underline{0.6\cdot S(0)}$.

Find the price of the call w/ exercise in three months in terms of $S(0)$.

$\longrightarrow$: For any T:

$$d_1 = \frac{\sigma\sqrt{T}}{2} = -d_2$$

$$V_c(0,T) = S(0)\cdot\left(2\cdot N\left(\frac{\sigma\sqrt{T}}{2}\right) - 1\right)$$

For $T=1$.

$$V_c(0,T=1) = 0.6\cdot \cancel{S(0)} = \cancel{S(0)}\left(2\cdot N\left(\frac{\sigma}{2}\right) - 1\right)$$

$$2\cdot N\left(\frac{\sigma}{2}\right) - 1 = 0.6$$

$$N\left(\frac{\sigma}{2}\right) = 0.8 \quad \Rightarrow \quad \frac{\sigma}{2} = 0.84$$

For $T=\frac{1}{4}$.

$$V_c\left(0,T=\tfrac{1}{4}\right) = S(0)\left(2\cdot N\left(\frac{\sigma\sqrt{1/4}}{2}\right) - 1\right)$$

$$= S(0)\left(2\cdot N\left(\frac{\sigma}{4}\right) - 1\right)$$

$$= S(0)\left(2\cdot N(0.42) - 1\right)$$

$$= S(0)\left(2\cdot 0.6628 - 1\right)$$

$$= S(0)\cdot \underline{0.3256} \quad \square$$