M339D: November 10th, 2025.

The lognormal distribution.

Definition 1.1. Let $X \sim Normal(mean = m, variance = \nu^2)$. Define the random variable $Y = e^X$. We say that the random variable Y is *lognormally distributed*.

1.1. **First properties.**

- The expected value of the lognormally distributed random variable Y can be obtained as follows:

$$\boxed{\mathbb{E}[Y] = \mathbb{E}[e^X] = M_X(1) = e^{m+\frac{\nu^2}{2}}.}$$

- Let Y be a lognormal and let $a \neq 0$. Then, the random variable Y^a is also lognormal. *Note:* For $a = 0$, we get a degenerate random variable at 1 which can, technically, be interpreted as lognormal, but is not fun.
- Let Y_1 and Y_2 be independent and lognormally distributed. Then, Y_1Y_2 is also lognormal.

1.2. **Quantiles.**

Definition 1.2. For p such that $0 < p < 1$, we define the *$100p^{th}$ quantile* of a random variable X as any value π_p such that

$$F_X(\pi_p-) \leq p \leq F_X(\pi_p).$$

In particular, the 50^{th} quantile is referred to as the *median*.

Note: When the random variable X is continuous, we can obtain the $100p^{th}$ quantile by simply solving for π_p in

$$F_X(\pi_p) = p.$$

Consider a probability p. Let z_p be the $100p^{th}$ quantile of the standard normal distribution. Let Y be lognormally distributed as above. My claim is that the value

$$\boxed{y_p = e^{m+\nu z_p}}$$

is the $100p^{th}$ quantile of Y. Let us simply verify this claim by calculating $F_Y(y_p)$. We have, with $Z \sim N(0,1)$,

$$F_Y(y_p) = \mathbb{P}[Y \leq y_p] = \mathbb{P}[e^X \leq y_p] = \mathbb{P}[e^{m+\nu Z} \leq e^{m+\nu z_p}].$$

Since the logarithmic function is increasing, we have that the above equals

$$F_Y(y_p) = \mathbb{P}[m + \nu Z \leq m + \nu z_p] = \mathbb{P}[Z \leq z_p] = p.$$

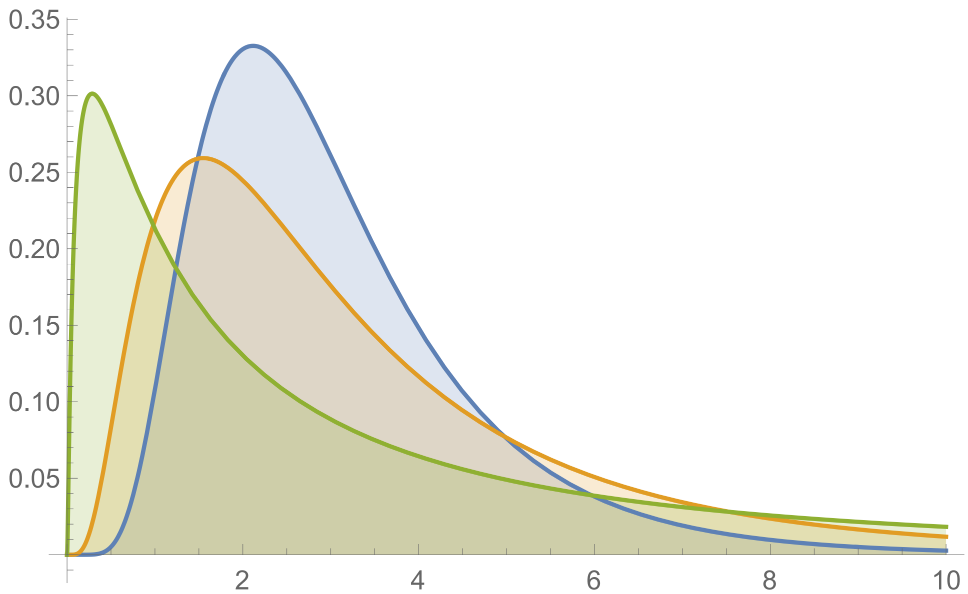
The above concludes our proof.

In particular, since the median of the standard normal distribution equals 0, the median of the lognormal distribution will be e^m.

Note: Since

$$\boxed{e^m < e^{m+\frac{\nu^2}{2}},} \tag{1.1}$$

i.e., since the mean of a lognormal distribution always exceeds the median, we say that it's *right-skewed*. In fact, this is what its probability density function looks like.


0.35
0.30
0.25
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10

Jensen's Inequality.

Caveat: $\boxed{\mathbb{E}[e^X] \geq e^{\mathbb{E}[X]}}$

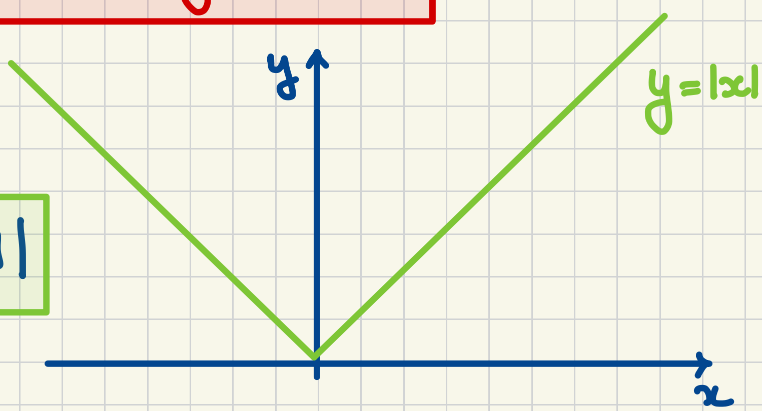
Theorem. Let X be a random variable, and let g be a convex function such that $g(X)$ is well defined and $\mathbb{E}[g(X)]$ exists.

Then,

$$\boxed{\mathbb{E}[g(X)] \geq g(\mathbb{E}[X])}.$$

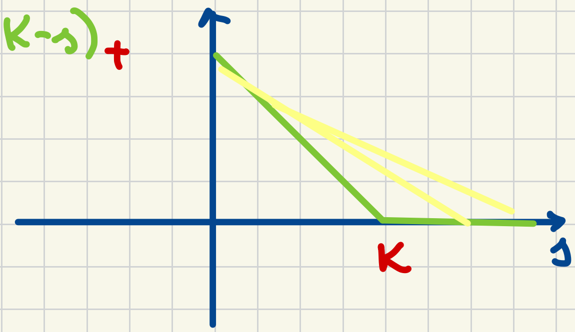
Example. i. $\boxed{g(x) = |x|}$

$$\boxed{\mathbb{E}[|X|] \geq |\mathbb{E}[X]|}$$



ii. Consider a European put w/ strike K.

Its payoff function: $v_p(s) = (K-s)_+$



The expected payoff

$$\mathbb{E}[v_p(S(T))] = \mathbb{E}[(K-S(T))_+]$$

By Jensen's Inequality it's

$$\geq (K - \mathbb{E}[S(T)])_+$$

iii. In classical insurance:

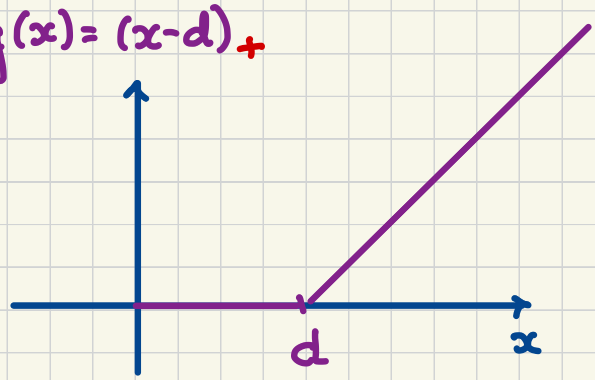
$$\begin{cases} X \ldots \text{(ground-up) loss, i.e., severity r.v.} \\ d \ldots \text{deductible} \end{cases}$$

The insurer pays

$$(X-d)_+, \text{ i.e., } \quad g(x) = (x-d)_+$$

By Jensen's inequality

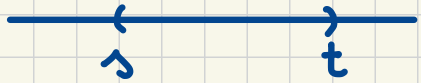
$$\mathbb{E}[(X-d)_+] \geq (\mathbb{E}[X]-d)_+$$



Log-Normal Stock Price.

Temporarily fix a time horizon T.

$S(t)$, $t \in \boxed{[0,T]}$ .... time-t stock price



Recall: $R(s,t) = \ln\left(\frac{S(t)}{S(s)}\right) \iff S(t) = S(s)\,e^{R(s,t)}$

In particular: $R(0,T)$ ... realized return for $[0,T]$

$\boxed{\text{We model realized returns as normal.}}$

$R(0,T) \sim \text{Normal}(\text{mean} = m, \text{Var} = \nu^2)$

$\Rightarrow$ $S(T)$ is lognormal

$\Rightarrow$ $\mathbb{E}^*[S(T)] = \mathbb{E}[S(0)e^{R(0,T)}] = \boxed{S(0)\,e^{m+\frac{\nu^2}{2}}}$ ★

Market model:

- Riskless asset w/ ccrfir r
- Risky asset : a non-dividend paying stock
  w/ volatility σ

Q: Under the risk-neutral measure $\mathbb{P}^*$, we have... ?

$$\boxed{\mathbb{E}^*[S(T)] = S(0)e^{rT}} \quad \star\star$$

Equating $\star$ & $\star\star$, we get $\boxed{m + \frac{\nu^2}{2} = rT}$

Q: $\mathrm{Var}[R(0,1)] = \cancel{?}\ \sigma^2$

$$\Rightarrow \quad \mathrm{Var}[R(0,T)] = \cancel{?}\ \boxed{\sigma^2 \cdot T = \nu^2}$$

$$\underbrace{m} = rT - \frac{\nu^2}{2} = rT - \frac{\sigma^2}{2}\cdot T = \underbrace{\left(r - \frac{\sigma^2}{2}\right)\cdot T}$$

$$\Rightarrow \boxed{R(0,T) \sim \mathrm{Normal}\left(\text{mean} = \left(r - \frac{\sigma^2}{2}\right)\cdot T,\ \text{var} = \sigma^2 \cdot T\right)}$$