M339D: November 4^{th}, 2024.

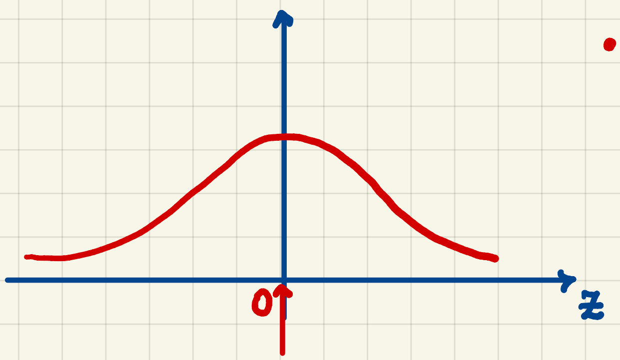
Standard Normal Distribution.

We say that a random variable Z has the

standard normal distribution

if its probability density function (pdf) has this form

$$f_Z(z) = \boxed{\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}} \quad \text{for all } z \in \mathbb{R}$$



- symmetric about the vertical axis, i.e.,

  $\varphi(z) = \varphi(-z)$, i.e., even

- mean/median/mode $= 0$

The cumulative distribution function (cdf) of the std normal is

$$N(z) = \Phi(z) = \mathbb{P}[Z \le z]$$

$$= \int_{-\infty}^{z} f_Z(u)\,du = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\,du$$

No Analytic Form!

There are <u>standard normal tables</u>!

We can use the built-in `pnorm` and `qnorm` in `R`.

$$\underbrace{\text{pnorm}}_{N} \qquad \underbrace{\text{qnorm}}_{N^{-1}}$$

We write $\boxed{Z \sim N(0,1)}$

The Normal Distributions.

We completely specify any normal distribution by providing its mean μ_X and variance σ_X^2 (or its std deviation σ_X).

We write:

$$\boxed{X \sim \text{Normal}(\text{mean} = \mu_X,\ \text{variance} = \sigma_X^2)}$$

X can be written as a linear transform of a standard normal Z:

$$\boxed{X = \mu_X + \sigma_X \cdot Z}$$

We can check:

- $\mathbb{E}[X] = \mathbb{E}[\mu_X + \sigma_X \cdot Z] \overset{\text{linearity of expectation}}{=} \mu_X + \sigma_X \cdot \underbrace{\mathbb{E}[Z]}_{=0} = \mu_X$

- $\text{Var}[X] = \text{Var}[\mu_X + \sigma_X \cdot Z]$

a deterministic shift which doesn't affect the variance

$$= \text{Var}[\sigma_X Z] = \sigma_X^2 \underbrace{\text{Var}[Z]}_{=1} = \sigma_X^2$$