
Note: You **must** show all your work. Numerical answers without a proper explanation or a clearly written down path to the solution will be assigned zero points.

Problem 9.1. (10 points) In the compound model for aggregate claims, let the frequency random variable N have the probability (mass) function

$$p_N(0) = 0.5, p_N(1) = 0.3, p_N(2) = 0.2.$$

Moreover, let the common distribution of the i.i.d. severity random variables $\{X_j; j = 1, 2, \dots\}$ be given by the probability (mass) function $p_X(1) = 0.3$ and $p_X(2) = 0.7$.
Let our usual assumptions hold, i.e., let N be independent of $\{X_j; j = 1, 2, \dots\}$.
Define the aggregate loss as $S = \sum_{j=1}^{N} X_j$.
Calculate $\mathbb{E}[(S-2)_+]$.

Solution: We use the equality

$$\mathbb{E}[(S-2)_+] = \mathbb{E}[S] - \mathbb{E}[S \wedge 2].$$

Using

$$\begin{aligned}
\mathbb{E}[N] &= 0.5 \cdot 0 + 0.3 \cdot 1 + 0.2 \cdot 2 = 0.3 + 0.4 = 0.7,\\
\mathbb{E}[X] &= 0.3 \cdot 1 + 0.7 \cdot 2 = 0.3 + 1.4 = 1.7,
\end{aligned}$$

we get

$$\mathbb{E}[S] = \mathbb{E}[N]\mathbb{E}[X] = 0.7 \cdot 1.7 = 1.19.$$

On the other hand,

$$\begin{aligned}
\mathbb{E}[S \wedge 2] &= \mathbb{P}[S > 0] + \mathbb{P}[S > 1]\\
&= (1 - F_S(0)) + (1 - F_S(1)).
\end{aligned}$$

From the problem statement, we conclude that

$$\begin{aligned}
F_S(0) &= \mathbb{P}[S \le 0] = \mathbb{P}[S = 0] = \mathbb{P}[N = 0] = 0.5\\
F_S(1) &= \mathbb{P}[S \le 1] = \mathbb{P}[S = 0] + \mathbb{P}[S = 1] = \mathbb{P}[N = 0] + \mathbb{P}[N = 1, X_1 = 1]\\
&= 0.5 + 0.3 \cdot 0.3 = 0.59.
\end{aligned}$$

Finally,

$$\mathbb{E}[S \wedge 2] = 0.5 + 0.41 = 0.91$$

and

$$\mathbb{E}[(S-2)_+] = 1.19 - 0.91 = 0.28.$$

Problem 9.2. (10 points) In the compound model for aggregate claims, let the frequency random variable N have the Poisson distribution with mean 1.

Let the common distribution of the i.i.d. severity random variables $\{X_j; j = 1, 2, \dots\}$ be given by the following p.m.f.

$$p_X(100) = 1/2, p_X(200) = 3/10, p_X(300) = 1/5.$$

Let our usual assumptions hold, i.e., let N be independent of $\{X_j; j = 1, 2, \dots\}$.
Define the aggregate loss as $S = \sum_{j=1}^{N} X_j$.
Find the expected value of the **policyholder's** payment for a stop-loss insurance policy with an ordinary deductible of 200, i.e., calculate $\mathbb{E}[S \wedge 200]$.

Solution: Note that S has the support of the form $\{0, 100, 200, 300, \dots\}$. So,

$$\mathbb{E}[S \wedge 200] = 100\mathbb{P}[S = 100] + 200\mathbb{P}[S \geq 200].$$

Next,

$$\begin{aligned}
\mathbb{P}[S = 0] &= \mathbb{P}[N = 0] = e^{-1},\\
\mathbb{P}[S = 100] &= \mathbb{P}[N = 1, X_1 = 100] = 0.5e^{-1},\\
\mathbb{P}[S \geq 200] &= 1 - \mathbb{P}[S = 0] - \mathbb{P}[S = 100] = 1 - 1.5e^{-1}.
\end{aligned}$$

So,

$$\mathbb{E}[S \wedge 200] = 100 \cdot 0.5e^{-1} + 200(1 - 1.5e^{-1}) = 200 - 250e^{-1} \approx 108.03.$$

Problem 9.3. (15 points) *Source: Based on Problem #165 from sample STAM Exam.*
Consider the following collective risk model:
(i) The claim count random variable N is Poisson with mean 3.
(ii) The severity random variable has the following probability (mass) function:

$$p_X(1) = 0.6,\ p_X(2) = 0.4.$$

(iii) As usual, individual loss random variables are mutually independent and independent of N.

Assume that an insurance covers **aggregate losses** subject to a deductible $d = 3$.

Find the expected value of aggregate payments for this insurance.

Solution:
Method I. Total aggregate losses are given by

$$S = X_1 + X_2 + \cdots + X_N.$$

So, the expected value of aggregate payments for this insurance equals

$$\mathbb{E}[(S - 3)_+] = \mathbb{E}[S] - \mathbb{E}[S \wedge 3].$$

Wald's identity gives us

$$\mathbb{E}[S] = \mathbb{E}[N]\mathbb{E}[X] = 3(0.6(1) + 0.4(2)) = 4.2.$$

On the other hand, the distribution of the random variable $S \wedge 3$ is given by

$$S \wedge 3 \sim \begin{cases} 0 & \text{if } N = 0, \\ 1 & \text{if } N = 1 \text{ and } X_1 = 1, \\ 2 & \text{if } \{N = 1 \text{ and } X_1 = 2\} \text{ or } \{N = 2 \text{ and } X_1 = X_2 = 1\} \\ 3 & \text{otherwise.} \end{cases}$$

So, we have that

$\mathbb{P}[S \wedge 3 = 0] = \mathbb{P}[N = 0] = e^{-3}$,

$\mathbb{P}[S \wedge 3 = 1] = \mathbb{P}[N = 1]\mathbb{P}[X = 1] = 3e^{-3}(0.6) = 1.8e^{-3}$,

$\mathbb{P}[S \wedge 3 = 2] = \mathbb{P}[N = 1]\mathbb{P}[X = 2] + \mathbb{P}[N = 2](\mathbb{P}[X = 1])^2 = 3e^{-3}(0.4) + \dfrac{3^2}{2}e^{-3}(0.6)^2 = 2.82e^{-3}$,

$\mathbb{P}[S \wedge 3 = 3] = 1 - 5.62e^{-3}$

Therefore,

$$\mathbb{E}[S \wedge 3] = 1.8e^{-3} + 2(2.82)e^{-3} + 3(1 - 5.62e^{-3}) = 2.53101.$$

So, our answer is $\mathbb{E}[(S-3)_+] = 4.2 - 2.53101 = 1.66899$.

Method II. We are supposed to calculate $\mathbb{E}[(S-3)_+]$. We wish to use the formula

$$\mathbb{E}[(S-3)_+] = \mathbb{E}[S] - \mathbb{E}[S \wedge 3].$$

We have

$$\begin{aligned} \mathbb{E}[X] &= 1 \cdot 0.6 + 2 \cdot 0.4 = 1.4, \\ \mathbb{E}[S] &= \mathbb{E}[N]\,\mathbb{E}[X] = 3 \cdot 1.4 = 4.2. \end{aligned}$$

Also,

$$\begin{aligned} \mathbb{E}[S \wedge 3] &= \mathbb{P}[S > 0] + \mathbb{P}[S > 1] + \mathbb{P}[S > 2] \\ &= 3 - (\mathbb{P}[S \le 0] + \mathbb{P}[S \le 1] + \mathbb{P}[S \le 2]) \\ &= 3 - (3\mathbb{P}[S = 0] + 2\mathbb{P}[S = 1] + \mathbb{P}[S = 2]). \end{aligned}$$

Calculating the above probabilities, using the provided distributions of N and X and their independence, we get

$\mathbb{P}[S = 0] = \mathbb{P}[N = 0] = e^{-3}$,

$\mathbb{P}[S = 1] = \mathbb{P}[N = 1, X_1 = 1] = 3e^{-3} \cdot 0.6 = 1.8e^{-3}$,

$\mathbb{P}[S = 2] = \mathbb{P}[N = 1, X_1 = 2] + \mathbb{P}[N = 2, X_1 = 1, X_2 = 1] = 3e^{-3} \cdot 0.4 + \dfrac{9}{2}e^{-3} \cdot 0.6 \cdot 0.6 = 2.82e^{-3}$.

So,

$$\begin{aligned} \mathbb{E}[S \wedge 3] &= 3 - (3\mathbb{P}[S = 0] + 2\mathbb{P}[S = 1] + \mathbb{P}[S = 2]) \\ &= 3 - (3e^{-3} + 2 \cdot 1.8e^{-3} + 2.82e^{-3}) \\ &= 3 - 9.42e^{-3}. \end{aligned}$$

Finally,

$$\mathbb{E}[(S-3)_+] = 4.2 - (3 - 9.42e^{-3}) = 1.2 + 9.42e^{-3} \approx 1.669.$$

Problem 9.4. (10 pts) In the compound model for aggregate claims, let the frequency random variable N have the geometric distribution with mean 4.

Moreover, let the individual losses have the distribution

$$p_X(0) = 1/2, p_X(100) = 1/2.$$

Define the aggregate loss as $S = \sum_{j=1}^{N} X_j$. How much is $\mathbb{E}[(S-100)_+]$?

Solution: As usual, we start with

$$\mathbb{E}[(S-100)_+] = \mathbb{E}[S] - \mathbb{E}[S \wedge 100].$$

We have

$$\mathbb{E}[S] = \mathbb{E}[N]\mathbb{E}[X] = 4 \cdot 50 = 200.$$

On the other hand, since the possible values of S are $\{0, 100, 200, \dots\}$,

$$\mathbb{E}[S \wedge 100] = 0 \cdot \mathbb{P}[S=0] + 100 \cdot \mathbb{P}[S \geq 100] = 100(1 - \mathbb{P}[S < 100]) = 100(1 - \mathbb{P}[S=0]).$$

Note that, due to the usual independence assumptions,

$$\begin{aligned}
\mathbb{P}[S=0] &= \mathbb{P}[N=0] + \mathbb{P}[N=1, X_1=0] + \cdots + \mathbb{P}[N=k, X_1 = X_2 = \cdots = X_k = 0] + \dots \\
&= \mathbb{P}[N=0] + \mathbb{P}[N=1]\,\mathbb{P}[X=0] + \cdots + \mathbb{P}[N=1]\,(\mathbb{P}[X=0])^k + \dots \\
&= \frac{1}{1+\beta} + \frac{\beta}{(1+\beta)^2} \cdot \frac{1}{2} + \cdots + \frac{\beta^k}{(1+\beta)^{k+1}} \cdot \frac{1}{2^k} + \cdots \\
&= \frac{1}{1+\beta}[1 + \frac{\beta}{1+\beta} \cdot \frac{1}{2} + \cdots + \frac{\beta^k}{(1+\beta)^k} \cdot \frac{1}{2^k} + \cdots] \\
&= \frac{1}{1+\beta} \cdot \frac{1}{1 - \frac{\beta}{2(1+\beta)}} \\
&= \frac{\beta}{2+\beta} \\
&= \frac{1}{3}.
\end{aligned}$$

So,

$$\mathbb{E}[(S-100)_+] = 200 - \frac{200}{3} = \frac{400}{3} \approx 133.33.$$

Problem 9.5. (5 points) An insurer pays aggregate claims in excess of the deductible d. In return, they receive a stop-loss premium $\mathbb{E}[(S-d)_+]$. You model the aggregate losses S using a continuous distribution. Moreover, you are given the following information about the aggregate losses S:

(i) $\mathbb{E}[(S-100)_+] = 15$,
(ii) $\mathbb{E}[(S-120)_+] = 10$,
(iii) $\mathbb{P}[80 < S \leq 120] = 0$.

Find the probability that the aggregate claim amounts are less than or equal to 80.

Solution: From the given fact (i), we know that

$$\mathbb{E}[(S-100)_+] = 15 = \int_{100}^{\infty} S_S(x)\, dx$$

where S_S denotes the survival function of the random variable S. Similarly, From the given fact (ii), we know that

$$\mathbb{E}[(S-120)_+] = 10 = \int_{120}^{\infty} S_S(x)\, dx$$

Therefore,

$$\int_{100}^{120} S_S(x)\, dx = 5.$$

From the given fact (iii), we know that the survival function is constant over the interval $[80, 120]$. In particular, we can write $S_S(x) = S_S(80)$ for all $x \in [100, 120]$. Substituting this finding into the equality above, we get

$$20 S_S(80) = 5 \quad \Rightarrow \quad S_S(80) = \frac{1}{4} \quad \Rightarrow \quad F_S(80) = \frac{3}{4}$$

where F_S denotes the cumulative distribution function of the aggregate losses S.