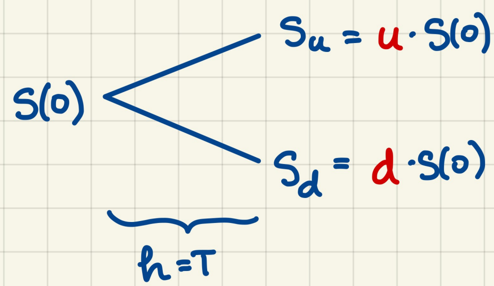
M339W: April 15th, 2022.

Forward Binomial Trees.



The no-arbitrage condition:

$$d < e^{(r-\delta)h} < u$$

$$\boxed{u, d = ?}$$

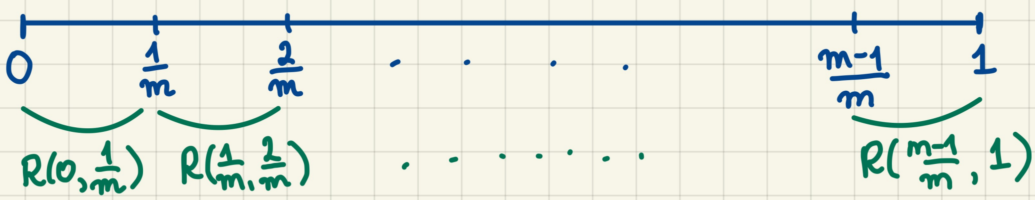
"Def'n". The volatility σ is the standard deviation of the realized returns on a continuously compounded scale and $\boxed{\text{annualized}}$.

Heuristics: $T = 1$

$h = \frac{1}{m}$ (year)

Q: What is the volatility for a time period of length h?

Call this volatility: σ_h.



Note: $R(\frac{k-1}{m}, \frac{k}{m})$ for $k = 1, 2, \dots, m$

are all random variable

We make the following assumptions:

- all the returns above are identically distributed;
- the returns over disjoint intervals are independent.

We also know that returns defined as above are additive, i.e.,

$$\boxed{R(0, \tfrac{1}{m}) + R(\tfrac{1}{m}, \tfrac{2}{m}) + \dots + R(\tfrac{m-1}{m}, 1) = R(0, 1)}$$

$$\sigma^2 = Var[R(0,1)] = Var\left[R(0,\tfrac{1}{m}) + \cdots + R(\tfrac{m-1}{m},1)\right]$$

$$\overset{\text{independence}}{=} Var[R(0,\tfrac{1}{m})] + \cdots + Var[R(\tfrac{m-1}{m},1)]$$

$$\overset{\text{identically dist'd}}{=} m \cdot Var[R(0,\tfrac{1}{m})] = m \cdot \sigma_h^2$$

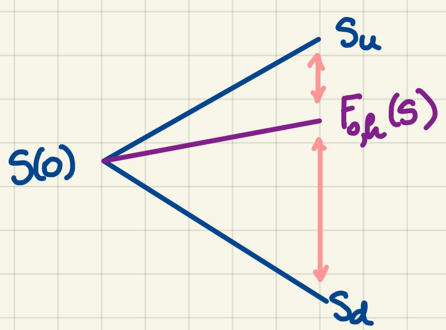
$$h = \frac{1}{m}$$

$$\sigma^2 = m \cdot \sigma_h^2$$

$$\Rightarrow \quad \sigma_h^2 = h \cdot \sigma^2$$

$$\Rightarrow \quad \boxed{\sigma_h = \sigma\sqrt{h}}$$

We generalize this identity to arbitrary lengths h.



Recall: $\boxed{F_{0,h}(S) = S(0)e^{(r-\delta)h}}$

$$S_u = F_{0,h}(S)e^{\sigma\sqrt{h}} = S(0)e^{(r-\delta)h}\cdot e^{\sigma\sqrt{h}} = S(0)\boxed{e^{(r-\delta)h+\sigma\sqrt{h}}} = u$$

$$S_d = F_{0,h}(S)e^{-\sigma\sqrt{h}} = S(0)e^{(r-\delta)h}\cdot e^{-\sigma\sqrt{h}} = S(0)\boxed{e^{(r-\delta)h-\sigma\sqrt{h}}} = d$$

Note: u and d immediately satisfy the no-arbitrage condition.

Q: What is S_u/S_d?

$$\longrightarrow: \frac{S_u}{S_d} = \frac{u\cdot \cancel{S(0)}}{d\cdot\cancel{S(0)}} = \frac{\cancel{e^{(r-\delta)h}}\cdot e^{\sigma\sqrt{h}}}{\cancel{e^{(r-\delta)h}}\cdot e^{-\sigma\sqrt{h}}} = \boxed{e^{2\sigma\sqrt{h}}}$$

□

Q: Risk-neutral probability?

$$\longrightarrow: \quad p^* = \frac{e^{(r-\delta)h} - d}{u - d} = \frac{\cancel{e^{(r-\delta)h}} - \cancel{e^{(r-\delta)h}} \cdot e^{-\sigma\sqrt{h}}}{\cancel{e^{(r-\delta)h}} \cdot e^{\sigma\sqrt{h}} - \cancel{e^{(r-\delta)h}} \cdot e^{-\sigma\sqrt{h}}}$$

$$p^* = \frac{1 - e^{-\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}} = \frac{\cancel{1 - e^{-\sigma\sqrt{h}}}}{e^{\sigma\sqrt{h}} \underbrace{\left(1 - e^{-2\sigma\sqrt{h}}\right)}_{\cancel{(1-e^{-\sigma\sqrt{h}})}(1+e^{-\sigma\sqrt{h}})}}$$

$$\boxed{p^* = \frac{1}{1 + e^{\sigma\sqrt{h}}}}$$

the shortcut only for the forward binomial tree

$$h \longrightarrow 0: \quad \frac{1}{2}$$

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Problem Set #10

Binomial option pricing: Forward trees. Two periods.

Problem 10.1. (5 points) Assume that the a stock price is modeled using a one-period forward binomial tree with the length of a single period equal to three months. According to this model, the stock price can take either the value of $50 or the value of $40 in exactly three months. Calculate the volatility of the stock price.

$h = \frac{1}{4}$

(a) About 0.22
(b) About 0.28
(c) About 0.30
(d) About 0.32
(e) None of the above.

$$\frac{S_u}{S_d} = e^{2\sigma\sqrt{h}}$$

$$2\sigma\sqrt{h} = \ln\left(\frac{S_u}{S_d}\right)$$

$$\cancel{2}\cdot\sigma\cdot\sqrt{\cancel{\tfrac{1}{4}}} = \ln\left(\frac{50}{40}\right) = \ln(1.25) \cong 0.223$$

Problem 10.2. The current price of a continuous-dividend-paying stock is \$100 per share. Its volatility is given to be 0.30 and its dividend yield is 0.03.

The continuously compounded risk-free interest rate equals 0.06.

Consider a \$95-strike European put option on the above stock with three months to expiration. Using a one-period forward binomial tree, find the price of this put option.

(a) \$3.97
(b) \$4.52
(c) \$4.70
(d) \$4.97
(e) None of the above.

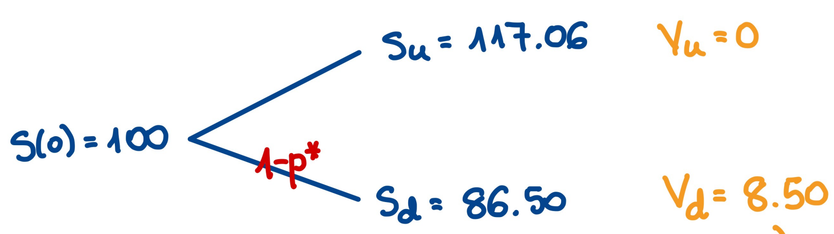
$$p^* = \frac{1}{1+e^{\sigma\sqrt{h}}} = \frac{1}{1+e^{0.3\sqrt{1/4}}} = \frac{1}{1+e^{0.15}} = 0.46257$$

forward tree

$$u = e^{(r-\delta)h+\sigma\sqrt{h}} = e^{(0.06-0.03)(0.25)+0.15} = e^{0.155} = 1.17058$$

$$d = e^{(r-\delta)h-\sigma\sqrt{h}} = e^{0.005-0.15} = e^{-0.145} = 0.865$$

$\boxed{K=95}$



$$V(0) = e^{-rT}(1-p^*)\cdot V_d = e^{-0.06(0.25)}(1-p^*)\cdot 8.50 \approx 4.50$$