

## More Review: Continuous Distributions.

Example.



Imagine a r.v.  $Y$  on  $[0, 1]$

The probability of it landing between  $a$  and  $b$  where  $0 \leq a < b \leq 1$  is

$$\text{P}[Y \in [a, b]] = b - a$$

Note:  $\text{P}[Y = y] = 0$  for all  $y$

Def'n. A r.v.  $Y$  is said to be **continuous** if there exists a function

$$f_Y : \mathbb{R} \rightarrow [0, \infty)$$

such that

$$\text{P}[Y \in [a, b]] = \int_a^b f_Y(y) dy \text{ for all } a < b.$$

The function  $f_Y$  is called the **probability density function (pdf)** of  $Y$ .

- Properties:
- $f_Y(y) \geq 0$  for all  $y$
  - $\int_{-\infty}^{\infty} f_Y(y) dy = 1$

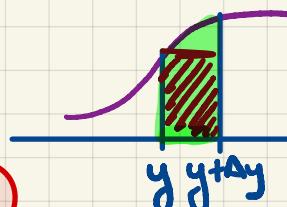
Q: Is it possible that  $p_Y(k) > 1$  where  $p_Y$  is a pmf?

Q: Is it possible that  $f_Y(y) > 1$  at some  $y$ ?

Note:



$$\text{P}[Y \in [y, y + \Delta y]] \approx f_Y(y) \Delta y$$



Caveat: There are r.v.s that are neither discrete, nor continuous.

Example.  $Y \sim U(0, 1/4)$ , i.e.,  $Y$  is uniformly distributed between 0 and  $1/4$

$$f_Y(y) = C \cdot \mathbb{1}_{[0, 1/4]}(y)$$

where  $\mathbb{1}_A : \mathbb{R} \rightarrow \mathbb{R}$

$$\mathbb{1}_A(y) = \begin{cases} 1 & y \in A \\ 0 & \text{otherwise} \end{cases}$$

is the indicator function

$$\int_{-\infty}^{\infty} C \cdot \mathbb{1}_{[0, 1/4]}(y) dy = 1$$

$$C \int_0^{1/4} dy = 1$$

$$C \cdot \left(\frac{1}{4}\right) = 1 \Rightarrow C = 4$$

□

M378K Introduction to Mathematical Statistics  
Problem Set #3  
Continuous distributions.

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**Problem 3.1.** Source: Sample P exam, Problem #33.

The lifetime of a machine part has a continuous distribution on the interval  $(0, 40)$  with probability density function  $f_X$ , where

$$f_X(x) \propto \frac{1}{(10+x)^2}$$

on the interval.

Calculate the probability that the lifetime of the machine part is less than 6.

→ :  $f_X(x) = K(10+x)^{-2} \mathbb{1}_{[0,40]}(x)$

$$K \int_0^{40} (10+x)^{-2} dx = 1$$

$$\int_0^{40} (10+x)^{-2} dx = \frac{(10+x)^{-1}}{-1} \Big|_{x=0}^{40} = -\frac{1}{10+x} \Big|_{x=0}^{40}$$

$$= \frac{1}{10} - \frac{1}{50} = \frac{4}{50} = \frac{2}{25} \Rightarrow K = \frac{25}{2}$$

$$\begin{aligned} \mathbb{P}[Y \leq 6] &= \mathbb{P}[Y \in [0, 6]] \\ &= \frac{25}{2} \int_0^6 (10+x)^{-2} dx \\ &= \frac{25}{2} \left( \frac{1}{10} - \frac{1}{16} \right) = \frac{25}{2} \cdot \frac{8-5}{80} = \frac{15}{32} \end{aligned}$$



Example.  $Y \sim U(l, r)$

$$P[Y \in [a, b]] = \frac{b-a}{r-l} \quad \text{for } l \leq a < b \leq r$$

$$f_Y(y) = \frac{1}{r-l} \mathbb{1}_{[l,r]}(y)$$

**Problem 3.2.** Source: Sample P exam, Problem #419.

A customer purchases a lawnmower with a two-year warranty. The number of years before the lawnmower needs a repair is uniformly distributed on  $[0, 5]$ . Calculate the probability that the lawnmower needs no repairs within 4.5 years after the purchase, given that the lawnmower needs no repairs within the warranty period.

→ : T... lifetime of lawnmower

$$T \sim U(0, 5)$$

$$\begin{aligned} P[T > 4.5 \mid T > 2] &= \frac{P[T > 4.5, T > 2]}{P[T > 2]} \\ &= \frac{P[T > 4.5]}{P[T > 2]} = \frac{\frac{5-4.5}{5-0}}{\frac{5-2}{5-0}} = \frac{1}{6} \quad \square \end{aligned}$$

Example.  $Y \sim N(\mu, \sigma)$  where  $\mu \in \mathbb{R}$  and  $\sigma > 0$   
 is normally distributed w/ mean  $\mu$   
 and standard deviation  $\sigma$

If  $f_Y(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-\mu)^2}{2\sigma^2}}$  for all  $y \in \mathbb{R}$

If  $\mu=0$  and  $\sigma=1$ , we say that  $Y$  is  
 standard normal and write  $Y \sim N(0,1)$

w/ pdf  $f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$  for all  $y \in \mathbb{R}$

Q: Let  $Y \sim N(0,1)$

Let  $\alpha$  and  $\beta$  be two real constants

$$\alpha Y + \beta \sim \text{Normal}(\mu=\beta, \sigma=|\alpha|)$$

Q: Let  $Y \sim N(\mu, \sigma)$

$$\frac{Y-\mu}{\sigma} \sim N(0,1)$$

**Problem 3.3.** Consider a continuous random variable  $Y$  whose probability density function is given by

$$f_Y(y) = 2y\mathbf{1}_{[0,1]}(y)$$

What is the expected value of this random variable?