M339J: March 12th, 2021.

Continuous Mixing [Review]. [Section 5.3.6]

$$\begin{cases} X \mid \Lambda = \lambda \text{ has the pdf } f_{X|\Lambda}(x|\lambda) \\ \qquad\qquad \text{and the cdf } F_{X|\Lambda}(x|\lambda) \\ \text{w/ } \Lambda \text{ w/ pdf } f_\Lambda(\cdot) \end{cases}$$

Then, unconditionally

- $f_X(x) = \int f_{X|\Lambda}(x|\lambda) f_\Lambda(\lambda)\, d\lambda$
- $F_X(x) = \int F_{X|\Lambda}(x|\lambda) f_\Lambda(\lambda)\, d\lambda$
- $\boxed{\mathbb{E}[\mathbb{E}[X^k \mid \Lambda]] = \mathbb{E}[X^k]}$
- $\boxed{Var[X] = \mathbb{E}[Var[X \mid \Lambda]] + Var[\mathbb{E}[X \mid \Lambda]]}$

Problem. Assume that

$$\begin{cases} X \mid \Lambda = \lambda \sim \text{Exponential}(\text{mean} = \theta = \lambda) \\ \text{w/ } \Lambda \sim U(50, 100) \end{cases}$$

Find the (unconditional) coefficient of variation of the r.v. X.

→: The coefficient of variation is $\dfrac{\sigma_X}{\mu_X}$.

✓ Focus on μ_X.

We know: $\mathbb{E}[X] = \mathbb{E}[\underbrace{\boxed{\mathbb{E}[X \mid \Lambda]}}_{\text{Always a FUNCTION of } \Lambda}]$

In this problem:

$$\boxed{\mathbb{E}[X \mid \Lambda] = \Lambda}$$

$X|\Lambda \sim$ Exponential (mean $= \Lambda$) ✓

$\Rightarrow \mathbb{E}[X] = \mathbb{E}[\Lambda] = 75$.

✓ Focus on $\sigma_X^2 = Var[X]$.

We know:

$$Var[X] = \mathbb{E}[\underbrace{Var[X|\Lambda]}_{\Lambda^2}] + Var[\underbrace{\mathbb{E}[X|\Lambda]}_{\Lambda}]$$

$$= \mathbb{E}[\Lambda^2] + Var[\Lambda]$$

$$= Var[\Lambda] + (\mathbb{E}[\Lambda])^2 + Var[\Lambda]$$

$$= 2 \cdot Var[\Lambda] + (\mathbb{E}[\Lambda])^2$$

$Var[\Lambda] = ? = \frac{50^2}{12}$

$\Lambda \sim U(50, 100)$

$\Lambda = 50 + \tilde{\Lambda}$ w/ $\tilde{\Lambda} \sim U(0, 50)$

$Var[\Lambda] = Var[\tilde{\Lambda}]$

$\tilde{\Lambda} \sim 50 \cdot U(0,1) = 50 \cdot U$

$Var[\tilde{\Lambda}] = Var[50 \cdot U] = 50^2 \cdot \frac{1}{12}$

$$Var[X] = 2 \cdot \frac{50^2}{12} + (75)^2 = 6{,}041.67$$

$\Rightarrow \sigma_X = 77.73$

$\Rightarrow$ answer: $\frac{77.73}{75} = 1.04$

$U \sim U(0,1)$

$\mathbb{E}[U^2] = \int_0^1 u^2 \cdot 1 \, du = \frac{u^3}{3} \Big|_{u=0}^{1} = \frac{1}{3}$

$Var[U] = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$

Problem. Let X have a mixture dist'n w/ the mixing variable Λ. More precisely,

$$\begin{cases} X \mid \Lambda = \lambda \sim \text{Exponential}\left(\text{mean} = \frac{1}{\lambda}\right) \\ \Lambda \sim \text{Exponential}(\text{mean} = 5) \end{cases}$$

Find the (unconditional) $\mathbb{P}[X \le 3] = F_X(3)$.

→: Since X is a continuous mixture, its cdf can be expressed as

for $x > 0$: $F_X(x) = \int_0^{+\infty} F_{X|\Lambda}(x|\lambda) f_\Lambda(\lambda)\, d\lambda$

The specific distributions in this problem give us:

$$\begin{cases} \cdot\ f_\Lambda(\lambda) = \frac{1}{\theta} e^{-\frac{\lambda}{\theta}} & \lambda > 0 \quad (\theta = 5) \\ \cdot\ F_{X|\Lambda}(x|\lambda) = 1 - e^{-\frac{x}{\frac{1}{\lambda}}} = \boxed{1 - e^{-\lambda \cdot x}} \end{cases}$$

$$F_X(x) = \int_0^{+\infty} (1 - e^{-\lambda \cdot x}) \cdot \frac{1}{\theta} e^{-\frac{\lambda}{\theta}}\, d\lambda$$

$$= \underbrace{\int_0^{+\infty} \frac{1}{\theta} e^{-\frac{\lambda}{\theta}}\, d\lambda}_{1} - \int_0^{+\infty} e^{-\lambda \cdot x} \cdot \frac{1}{\theta} e^{-\frac{\lambda}{\theta}}\, d\lambda$$

$$= 1 - \frac{1}{\theta} \int_0^{+\infty} e^{-\lambda\left(x + \frac{1}{\theta}\right)}\, d\lambda$$

$$= 1 + \frac{1}{\theta}\left(+\frac{1}{x + \frac{1}{\theta}}\right) e^{-\lambda\left(x + \frac{1}{\theta}\right)} \Big]_{\lambda=0}^{+\infty}$$

$$= 1 + \frac{1}{\theta\left(x + \frac{1}{\theta}\right)} (0 - 1)$$

$$\Rightarrow \quad F_X(x) = 1 - \frac{1}{\theta x + 1}$$

In this problem : $F_X(3) = 1 - \frac{1}{5\cdot 3+1} = 1 - \frac{1}{16} = \frac{15}{16}$

$$\boxed{F_X(x) = 1 - \frac{\frac{1}{\theta}}{x + \frac{1}{\theta}}} \quad \Rightarrow \quad \boxed{X \sim \text{Pareto}\left(\alpha^* = 1,\ \theta^* = \frac{1}{\theta}\right)}$$

Consult your STAM tables at home and convince yourselves :)