M362K Probability
University of Texas at Austin
Practice Problems for In-Term Exam III
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Notes: This is a closed book and closed notes exam. The maximal score on this exam is 100 points. **There are many ways in which any single problem can be solved. The solutions herein are just one possible way to tackle the given problems.**

All written work handed in by the student is considered to be
their own work, prepared without unauthorized assistance.

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"I agree that I have complied with the UT Honor Code during my completion of this exam."

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3.1. DEFINITIONS.

Problem 3.1. (5 points) Complete the definition of a *cumulative distribution function* below:

Let X be a random variable on an outcome space Ω. The *cumulative distribution function* of X is ...

Solution: ...the function $F_X : \mathbb{R} \to [0, 1]$ given by

$$F_X(x) = \mathbb{P}[X \leq x] \quad \text{for all } x \in \mathbb{R}.$$

Problem 3.2. (5 points) Complete the definition of the *probability mass function* of a random variable X on a finite outcome space Ω.

Let X be a random variable on a finite outcome space Ω. The *probability mass function* of X is ...

Solution: ...the function $p_X : Support(X) \to [0, 1]$ given by

$$p_X(x) = \mathbb{P}[X = x] \quad \text{for all } x \in Support(X).$$

3.2. TRUE/FALSE QUESTIONS.

Problem 3.3. (3 points) Let X denote the outcome of a roll of a fair, regular dodecahedron (a polyhedron with 12 faces) with numbers $1, 2, \cdots, 12$ written on its sides. Then $\mathbb{E}[X] = 13/2$. *True or false? Why?*

Solution: TRUE
Since each outcome is equally likely, by the definition of the expected value

$$\mathbb{E}[X] = \frac{1}{12} \cdot 1 + \frac{1}{12} \cdot 2 + \cdots + \frac{1}{12} \cdot 12 = \frac{1}{12}(1 + 2 + \cdots + 12) = \frac{1}{12} \cdot \frac{12 \cdot 13}{2} = \frac{13}{2}.$$

Problem 3.4. (3 points) If $Var[X] = 0$, then $\mathbb{P}[X = \mathbb{E}[X]] = 0$. *True or false? Why?*

Solution: FALSE
By Chebyshev's inequality, it should be $\mathbb{P}[X = \mathbb{E}[X]] = 1$.

3.3. FREE RESPONSE PROBLEMS.

Please, explain carefully all your statements and assumptions. Numerical results or single-word answers without an explanation (even if they're correct) are worth 0 points.

Problem 3.5. (10 points) Assume that the number of claims filed by a policyholder in a given year is modeled by a Poisson random variable with parameter λ.

An actuary has discovered that policyholders are three times as likely to file two claims in a single year as they are to file four claims.

(i) (6 points) Find the unknown parameter λ.
(ii) (4 points) Find the variance of the number of claims filed in a given year.

Solution:

(i) Let the number of claims filed in a single year be denoted by N. Then, $N \sim Poisson(\lambda)$, and the actuary's discovery can be expressed as

$$\mathbb{P}[N = 2] = 3\mathbb{P}[N = 4].$$

Using the expression for the pmf of the Poisson distribution, we get

$$e^{-\lambda}\frac{\lambda^2}{2!} = 3e^{-\lambda}\frac{\lambda^4}{4!}.$$

Hence,

$$\lambda^2 = 4 \Rightarrow \lambda = 2.$$

(ii) In general, we have that for $N \sim Poisson(\lambda)$

$$\mathbb{E}[N] = Var[N] = \lambda.$$

To convince yourselves that this is true, you can either develop the formula yourselves (it is an easy exercise in summation), or you can go over the calculations on p. 163 in the textbook.

Problem 3.6. (20 points) Let X and Y be independent random variables with distribution functions F_X and F_Y. Define $U = \min(X, Y)$ and $V = \max(X, Y)$ and denote their distribution functions by F_U and F_V, respectively. Express F_U and F_V in terms of F_X and F_Y.

Solution: For U, we have

$$\begin{aligned} F_U(a) &= \mathbb{P}[U \leq a] \\ &= 1 - \mathbb{P}[U > a] \\ &= 1 - \mathbb{P}[\min(X, Y) > a] \\ &= 1 - \mathbb{P}[X > a,\ Y > a]. \end{aligned}$$

Using independence of X and Y, we get

$$\begin{aligned} F_U(a) &= 1 - \mathbb{P}[X > a]\mathbb{P}[Y > a] \\ &= 1 - (1 - \mathbb{P}[X \leq a])(1 - \mathbb{P}[Y \leq a]) \\ &= 1 - (1 - F_X(a))(1 - F_Y(a)). \end{aligned}$$

For V, the story is even simpler

$$F_V(a) = \mathbb{P}[\max(X, Y) \leq a] = \mathbb{P}[X \leq a,\ Y \leq a].$$

Using the independence of X and Y, we obtain

$$F_V(a) = \mathbb{P}[X \leq a]\mathbb{P}[Y \leq a] = F_X(a)F_Y(a).$$

Problem 3.7. If 20 fair octahedra whose sides are labeled with numbers 1 through 8 are rolled, find the approximate probability p that the sum of the obtained numbers falls between 30 and 40 inclusive.

Note: You will be working on this problem at home with calculators and such readily accessible; if a problem such as this one appears on the actual exams, the numbers will be chosen so that it is possible to perform the calculations by hand and in a reasonable amount of time.

Solution: Let X_i denote the outcome of the i^{th} roll for $i = 1, 2, \ldots 20$. The octahedra are assumed to be fair, so all the outcomes are equally likely and have the probability 1/8. Hence, for every i,

$$\begin{aligned}
\mathbb{E}[X_i] &= \frac{1}{8}[1 + 2 + \cdots + 8] = \frac{1}{8} \cdot \frac{8 \cdot 9}{2} = \frac{9}{2}\,, \\
\mathbb{E}[X_i^2] &= \frac{1}{8}[1 + 4 + \cdots + 64] = \frac{1}{8} \cdot \frac{8 \cdot 9 \cdot (2 \cdot 8 + 1)}{6} = \frac{9 \cdot 17}{6} = \frac{51}{2}\,, \\
Var[X_i] &= \mathbb{E}[X_i^2] - (\mathbb{E}[X_i])^2 = \frac{51}{2} - \frac{81}{4} = \frac{102 - 81}{4} = \frac{21}{4}\,.
\end{aligned}$$

Define $S = \sum_{i=1}^{20} X_i$. By the Central Limit Theorem,

$$\begin{aligned}
\mathbb{P}[29.5 \leq S \leq 40.5] &= \mathbb{P}\left[\frac{29.5 - 20 \cdot \mathbb{E}[X_i]}{\sqrt{20 Var[X_i]}} \leq \frac{S - 20 \cdot \mathbb{E}[X_i]}{\sqrt{20 Var[X_i]}} \leq \frac{40.5 - 20 \cdot \mathbb{E}[X_i]}{\sqrt{20 Var[X_i]}}\right] \\
&= \mathbb{P}\left[\frac{29.5 - 20 \cdot \frac{9}{2}}{\sqrt{20 \cdot \frac{21}{4}}} \leq \frac{S - 20 \cdot \frac{9}{2}}{\sqrt{20 \cdot \frac{21}{4}}} \leq \frac{40.5 - 20 \cdot \frac{9}{2}}{\sqrt{20 \cdot \frac{21}{4}}}\right] \\
&= \mathbb{P}\left[\frac{29.5 - 90}{\sqrt{105}} \leq \frac{S - 90}{\sqrt{105}} \leq \frac{40.5 - 90}{\sqrt{105}}\right] \\
&\approx \mathbb{P}\left[-5.90 \leq \frac{S - 90}{\sqrt{105}} \leq -4.83\right] \\
&\approx \Phi(-4.83) - \Phi(-5.90) \approx 0.
\end{aligned}$$

3.4. MULTIPLE CHOICE QUESTIONS.

Problem 3.8. (5 points) Let X denote the number of $1's$ in 100 throws of a fair die. Find $\mathbb{E}[X^2]$.

(a) 125/9
(b) 50/3
(c) 875/3
(d) 1585/9
(e) None of the above

Solution: (c)
Evidently, $X \sim b(100, 1/6)$. So,

$$\mathbb{E}[X^2] = Var[X] + (\mathbb{E}[X])^2 = 100 \cdot \frac{1}{6} \cdot \frac{5}{6} + (100 \cdot \frac{1}{6})^2 = \frac{500 + 10000}{36} = \frac{875}{3}\,.$$