M378K: February 27th, 2026.

Corollary 9.18. *Let N_1 and N_2 be independent and Poisson distributed. Define $N = N_1 + N_2$. Then, the distribution of N is . . .*

$$\boxed{N \sim \text{Poisson}(\lambda = \lambda_1 + \lambda_2)}$$

Proof. □

$$m_N(t) = \cancel{?}\ \mathbb{E}[e^{t \cdot N}] = \mathbb{E}[e^{t \cdot (N_1 + N_2)}]$$

$$= \mathbb{E}[e^{tN_1} \cdot e^{tN_2}] = \underbrace{\mathbb{E}[e^{tN_1}]} \cdot \underbrace{\mathbb{E}[e^{tN_2}]}$$

$$= m_{N_1}(t) \cdot m_{N_2}(t)$$

$$= e^{\lambda_1(e^t - 1)} \cdot e^{\lambda_2(e^t - 1)} = e^{\overset{\lambda}{(\lambda_1 + \lambda_2)}(e^t - 1)}$$

M378K Introduction to Mathematical Statistics
Problem Set #10
The Normal Distribution.

Definition 10.1. *The* moment-generating function (mgf) m_Y *for a random variable* Y *is defined as*

$$m_Y(t) = \mathbb{E}[e^{tY}]$$

for all t *for which the above expectation exists. In fact, we say that the moment-generating function* **exists** *there exists a positive number* b *such that* $m_Y(t)$ *is finite for all* t *such that* $|t| \leq b$.

Proposition 10.2.
1. *If* m_Y *exists for a certain probability distribution, then it is unique.*
2. *If* m_{Y_1} *and* m_{Y_2} *are equal on an interval, then* $Y_1 \stackrel{(d)}{=} Y_2$.

Corollary 10.3. *Let* Y_1 *and* Y_2 *be independent and normally distributed. Define* $Y = Y_1 + Y_2$. *Then, the distribution of* [redacted] Y *is* …

Proof. Note that $Y_i \sim N(\mu = mu_i, \sigma_i)$ for $i = 1, 2$. Now, let's look at the mgf of Y. Then, since Y_1 and Y_2 are independent, we have

$$m_Y(t) = m_{Y_1}(t)m_{Y_2}(t).$$

We can now use the fact that for any $X \sim N(\mu, \sigma)$,

$$m_X(t) = e^{\mu t}m_Z(\sigma t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

Hence,

$$m_Y(t) = e^{\mu_1 t + \frac{\sigma_1^2 t^2}{2}} \times e^{\mu_2 t + \frac{\sigma_2^2 t^2}{2}} = e^{(\mu_1 + \mu_2)t + \frac{(\sigma_1^2 + \sigma_2^2)t^2}{2}}$$

We can conclude that $Y \sim N(\mu_1 + \mu_2, \sqrt{\sigma_1^2 + \sigma_2^2})$. $\square$

Problem 10.1. *Two scales are used to measure the mass m of a precious stone. The first scale makes an error in measurement which we model by a normally distributed random variable X_1 with mean $\mu_1 = 0$ and standard deviation $\sigma_1 = 0.04m$. The second scale is more accurate. We model its error by a normal random variable X_2 with mean $\mu_2 = 0$ and standard deviation $\sigma_2 = 0.03m$.*

We assume that the measurements made using the two different scales are independent, i.e., that the random variables X_1 and X_2 are independent.

To get our final estimate of the mass of the stone, we take the average of the two results from the two different scales, i.e., we define $Y = \frac{X_1+X_2}{2}$.

(i) *What is the distribution of the random variable Y? State the* **name** *of its distribution and the* **values** *of the parameters.*

(ii) *What is the probability that the error Y we get is within $0.005m$ of the actual mass of the stone? Namely, calculate*

$$\boxed{\mathbb{P}[|Y| < 0.005m].}$$

$\longrightarrow$: $Y \sim$ Normal $(\mu_Y = 0,\ \sigma^2 = ?)$

$$\mathrm{Var}[Y] = \mathrm{Var}\left[\frac{1}{2}(X_1+X_2)\right] = \frac{1}{4}\mathrm{Var}[X_1+X_2]$$

(independence)

$$= \frac{1}{4}\left(\mathrm{Var}[X_1] + \mathrm{Var}[X_2]\right)$$

$$= \frac{1}{4}\left(0.04^2 m^2 + 0.03^2 m^2\right) = \frac{1}{4}\cdot 0.05^2\ m^2$$

$$\Rightarrow \boxed{\sigma = 0.025m}$$

$Y \sim$ Normal $(\mu_Y = 0,\ \sigma = 0.025m)$

$$\mathbb{P}[\,|Y| < 0.005m] = \cancel{?}$$

$$= \mathbb{P}[-0.005m < Y < 0.005m]$$

$$= \mathbb{P}\left[\frac{-0.005\cancel{m}}{0.025\cancel{m}} < \frac{Y-0}{0.025m} < \frac{0.005\cancel{m}}{0.025\cancel{m}}\right]$$

$\sim N(0,1) \sim Z$

$$= \mathbb{P}[-0.2 < Z < 0.2]$$

$$= \Phi(0.2) - \Phi(-0.2) = 2\Phi(0.2) - 1$$

$$= \text{pnorm}(0.2) - \text{pnorm}(-0.2) \quad \square$$

Corollary 10.4. *Let $Y_1, \dots, Y_n$ be independent and identically distributed. Assume that $Y_1 \sim N(\mu, \sigma)$. Define*

$$\boxed{S = Y_1 + Y_2 + \cdots + Y_n}$$

Then, the distribution of S is …

$S \sim$ Normal (mean $= n\mu$, var $= n \cdot \sigma^2$)

Proof. $\square$

Note: Define $\bar{Y} = \dfrac{Y_1 + \cdots + Y_n}{n} = \dfrac{S}{n}$

$\bar{Y} \sim$ Normal (mean $= \mu$, var $= \dfrac{\sigma^2}{n}$)

M378K Introduction to Mathematical Statistics
Problem Set #11
De Moivre-Laplace.

Problem 11.1. *You are given a TRUE/FALSE exam with 30 questions. Suppose that you need to answer 21 questions correctly in order to pass. You have no idea what the class is about and decide to toss a fair coin to answer all the questions; you circle TRUE if the outcome is tails and you circle FALSE if the outcome is heads. What is your approximation of the probability p that you manage to pass the exam using this strategy?*

For $Y \sim Binomial(n, p)$ we know that its probability mass function is:

$$p_Y(k) = \binom{n}{k} p^k (1-p)^{n-k} \quad \text{for } k = 0, 1, \dots, n$$

Moreover, its expectation and its variance are

$$\mathbb{E}[Y] = np \quad \text{and} \quad Var[Y] = np(1-p).$$

Now, consider a sequence of binomial random variables $Y_n \sim Binomial(n, p)$. Note that, while the number of trials n varies, the probability of success in every trial p remains the same for all n. The *normal approximation to the binomial* is a theorem which states that

$$\frac{Y_n - np}{\sqrt{np(1-p)}} \stackrel{\mathcal{D}}{\Rightarrow} N(0,1) \quad \text{as } n \to \infty.$$

Practically, this means that Y_n is "approximately" normal with mean np and variance $np(1-p)$ for "large" n. The usual rule of thumb is that both $np > 10$ and $n(1-p) > 10$.

Another practical adjustment needs to be made due to the fact that discrete distributions of Y_n are approximated by a continuous (normal) distribution. This adjustment is usually referred to as the **continuity correction**. More specifically, provided that the conditions above are satisfied, for every integer $a \leq b$, we have that

$$\begin{aligned}
\boxed{\mathbb{P}[a \leq Y_n \leq b]} &= \mathbb{P}\left[a - \tfrac{1}{2} < Y_n < b + \tfrac{1}{2}\right] \\
&= \mathbb{P}\left[\frac{a - \frac{1}{2} - np}{\sqrt{np(1-p)}} < \frac{Y_n - np}{\sqrt{np(1-p)}} < \frac{b + \frac{1}{2} - np}{\sqrt{np(1-p)}}\right] \\
&\approx \Phi\left(\frac{b + \frac{1}{2} - np}{\sqrt{np(1-p)}}\right) - \Phi\left(\frac{a - \frac{1}{2} - np}{\sqrt{np(1-p)}}\right)
\end{aligned}$$

where Φ, as usual, stands for the cummulative distribution function of the standard normal distribution.

For more about the history of the theorem and ideas for its proof, go to:
Wikipedia: de Moivre-Laplace.

$Y_n \sim b(1000, 0.5)$

$\mathbb{P}[Y_n = 500] = \binom{1000}{500} (0.5)^{1000}$

$\mathbb{P}[500 \leq Y_n \leq 500] \approxeq 0$

$1 = \sum_{k=0}^{1000} \mathbb{P}[Y_n = k] \approxeq 0$

$\longrightarrow$:

Y... # of questions answered correctly

$Y \sim b(\text{\# of trials} = 30, \text{ succ. prob} = 0.5)$

$$\mathbb{P}[Y \geq 21] = \sum_{k=21}^{30} \left(\binom{30}{k} (0.5)^{30} \right) \quad \text{exact}$$

Using the normal approximation:

$$\mathbb{P}[Y \geq 21] = \mathbb{P}[Y > 20.5]$$

$$\mathbb{E}[Y] = 30 \cdot 0.5 = 15$$

$$\text{Var}[Y] = 30 \cdot 0.5 \cdot 0.5 = 7.5 \Rightarrow SD[Y] = \sqrt{7.5}$$

$$\mathbb{P}[Y > 20.5] = \mathbb{P}\left[\underbrace{\frac{Y - 15}{\sqrt{7.5}}}_{\approx N(0,1)} > \frac{20.5 - 15}{\sqrt{7.5}} \right]$$

$$\approx \mathbb{P}[Z > 2] = 1 - \Phi(2)$$

$$= 1 - \text{pnorm}(2) \quad \square$$