University of Texas at Austin, Department of Mathematics
M358K - Applied Statistics

The Practice In-term One

**Problem 1.1.** (5 points) Write down the definition of *independence* of two *events*.

**Solution:** Two events $A$ and $B$ are said to be *independent* if

$$\mathbb{P}[A \cap B] = \mathbb{P}[A] \cdot \mathbb{P}[B]$$

**Problem 1.2.** (5 points) Write down the definition of the *cumulative distribution function* of a random variable.

**Solution:** Let $X$ be a random variable. Its *cumulative distribution function* is a function $F_X : \mathbb{R} \to [0,1]$ defined by

$$F_X(x) = \mathbb{P}[X \leq x], \quad \text{for every } x \in \mathbb{R}.$$

**Problem 1.3.** (20 points) Four balls are drawn (without replacement) from a box which contains 4 black and 5 red balls. Given that the four drawn balls were *not* all of the same color, what is the probability that there were exactly two balls of each color among the four.

**Solution:** Let $A$ denote the event that the colors of the balls drawn are not all the same, and let $B$ denote the event that there are exactly two black balls and two red balls. We are looking for $\mathbb{P}[B|A]$. Since $B \subseteq A$, we have

$$\mathbb{P}[B|A] = \mathbb{P}[B \cap A]/\mathbb{P}[A] = \mathbb{P}[B]/\mathbb{P}[A].$$

To compute $\mathbb{P}[A]$, we note that the event $A^c$ consists of only two elementary outcomes: either all balls are black or all balls are red. The probability of picking all red balls is

$$\frac{\binom{5}{4}}{\binom{9}{4}}$$

while the probability of picking all black balls is

$$\frac{\binom{4}{4}}{\binom{9}{4}} = \frac{1}{\binom{9}{4}}.$$

Putting these two together, we obtain

$$\mathbb{P}[A] = 1 - \frac{\binom{5}{4} + 1}{\binom{9}{4}}.$$

To compute $\mathbb{P}[B]$ we note that we can choose 2 red balls out of 5 in $\binom{5}{2}$ ways and, then, for each such choice, we have $\binom{4}{2}$ ways of choosing 2 black balls from the set of 4 black balls. Therefore,

$$\mathbb{P}[B] = \left(\binom{5}{2} \times \binom{4}{2}\right) \Big/ \binom{9}{4}.$$

Finally,

$$\mathbb{P}[B|A] = \frac{\binom{5}{2}\binom{4}{2}}{\binom{9}{4} - \binom{5}{4} - 1} = \frac{10 \times 6}{126 - 5 - 1} = \frac{1}{2}.$$

**Problem 1.4.** (15 points)Consider a set-up in which a transmitter is transmitting either a 0 or a 1 and the receiver indicates that it received either a 0 or a 1. Denote the events that $i = 0, 1$ was transmitted by $T_i$, and the events that $i = 0, 1$ was indicated as received by $R_i$.
It is possible to have transmission errors. In fact, you are given the following data on accuracy and the frequency of transmitted signals:

$$\mathbb{P}[R_0 \,|\, T_0] = 0.99,\ \mathbb{P}[R_1 \,|\, T_1] = 0.98,$$

and

$$\mathbb{P}[T_0] = 0.75.$$

(a) (10pts) Given that the receiver indicated 1, what is the probability that there was an error in the transmission?

(b) (5pts) What is the overall probability that there was an error in transmission?

**Solution:**

(1) We need $\mathbb{P}[T_0|R_1]$. By the Bayes formula,

$$\begin{aligned}\mathbb{P}[T_0|R_1] &= \frac{\mathbb{P}[R_1|T_0]\mathbb{P}[T_0]}{\mathbb{P}[R_1|T_0]\mathbb{P}[T_0] + \mathbb{P}[R_1|T_1]\mathbb{P}[T_1]}\\ &= \frac{(1-0.99)\times 0.75}{(1-0.99)\times 0.75 + 0.98\times 0.25}\\ &= \frac{3}{101} \cong 0.030.\end{aligned}$$

(2) An error will happen if $T_0 \cap R_1$ or $T_1 \cap R_0$ occur, i.e.,

$$\begin{aligned}\mathbb{P}[\text{error}] &= \mathbb{P}[T_0 \cap R_1] + \mathbb{P}[T_1 \cap R_0]\\ &= \mathbb{P}[R_1|T_0] \times \mathbb{P}[T_0] + \mathbb{P}[R_0|T_1] \times \mathbb{P}[T_1]\\ &= (1 - \mathbb{P}[R_0|T_0]) \times \mathbb{P}[T_0]\\ &\quad + (1 - \mathbb{P}[R_1|T_1]) \times (1 - \mathbb{P}[T_0])\\ &= 0.01 \times 0.75 + 0.02 \times 0.25\\ &= \tfrac{1}{80} \cong 0.013\end{aligned}$$

**Problem 1.5.** (25 points) *Source: "Probability" by Pitman.*
A final exam consists of multiple choice problems – each problem with 5 offered answers only one of which is correct. Before the final exam the diligent student is given a practice set of multiple choice problems. Knowing that exactly

70% of the final exam will be out of the practice set, the student works out the entire practice set and gets the correct answer to each question.

When he takes the final exam, the student proceeds to answer the known questions correctly. However, for the remaining questions, he panics and chooses the answers completely at random.

(i) (5 points) What is the probability that the student answers a randomly chosen question correctly?

(ii) (5 points) **Given** that the student answered a particular question correctly, what is the probability that he was guessing at random when he was answering that question?

Let the total number of questions in the exam be 20. Let the random variable $N$ represent the total number of questions the student answered correctly.

(iii) (5 points) What is the distribution of the random variable $N$?

(iv) (5 points) What is the expected value of $N$?

(v) (5 points) What is the standard deviation of $N$?

**Solution:**

(i)

$$0.7(1) + 0.3(0.2) = 0.76$$

(ii)

$$\frac{0.3(0.2)}{0.76} = 0.0789$$

(iii) The random variable $N$ can be written as

$$N = 14 + X,$$

where $X \sim Binomial(n = 6, p = 1/5)$.

(iv)

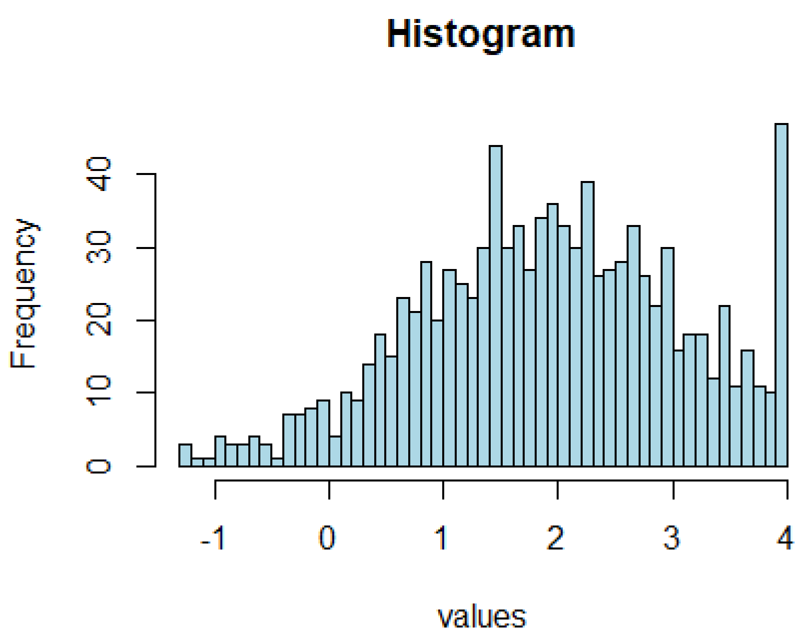
$$\mathbb{E}[N] = 14 + 6 * 0.2 = 15.2$$

(v)

$$Var[N] = Var[X] = 6 * 0.2 * 0.8 = 0.96 \quad \Rightarrow \quad SD[N] = \sqrt{0.96} = 0.979796.$$

---

**Problem 1.6.** (5 points) Consider the following histogram:




The histogram is ...

(a) ...unimodal.

(b) ...left-skewed.

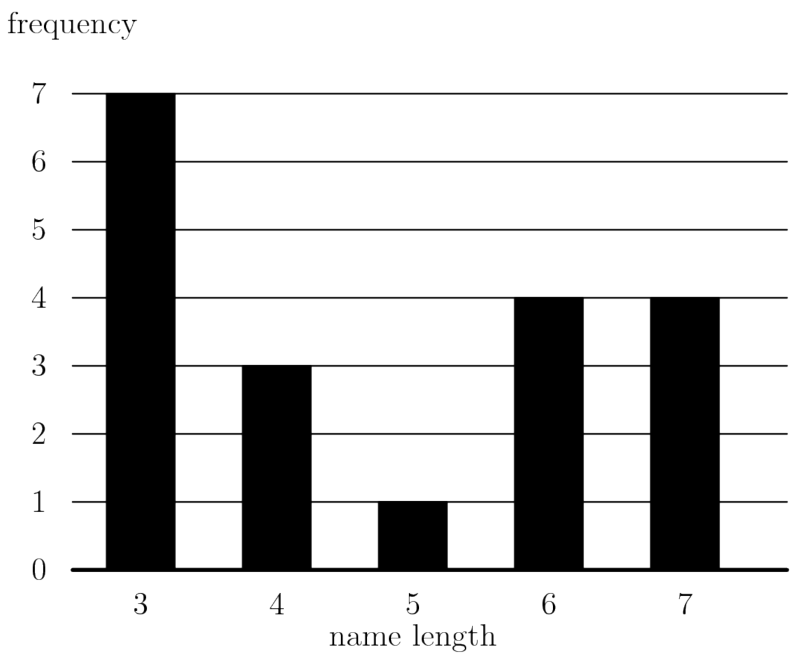
(c) ...right-skewed.

(d) ...symmetric.

(e) None of the above.

**Solution: (b)**
The "hump" is on the right; see how the values "max-out" at 4. So, it's left-skewed.

**Problem 1.7.** (5 points) *Source: AMC8, 2016.* The following bar graph represents the length (in letters) of the names of 19 people. What is the median length of these names?




(a) 3
(b) 4
(c) 5
(d) 6
(e) None of the above.

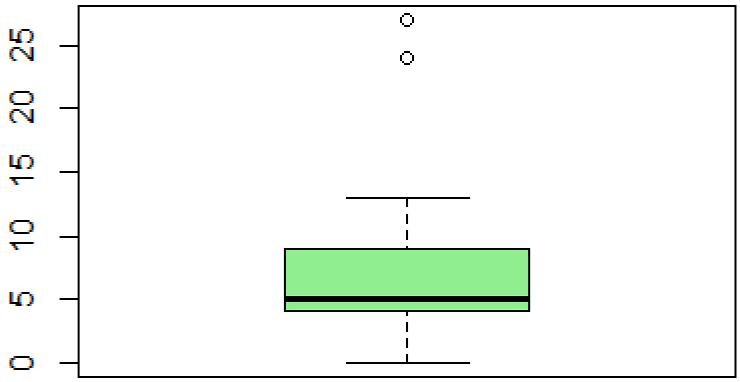
**Solution: (b)**
According to the bar graph, there is a total of $7+3+1+4+4=19$ names. The median name length is, thus, in the $10^{th}$ spot. This means it's of length 4.

**Problem 1.8.** (5 points) Which one of the following statements is **false**?

(a) The observed sample variance is more sensitive than the sample average to a few observations with extreme values.
(b) The observed sample variance is equal to zero if and only if all of the observations are identical.
(c) When all observations are multiplied by a constant $\kappa \neq 0$, the IQR increases by $|\kappa|$.
(d) When all the observations are increased by 5, the IQR increases by 5.
(e) None of the above.

**Solution: d**

**Problem 1.9.** (5 points) Consider the following box plot:




Which summary statistics does it correspond to?

(a) Min=0.000, Q1=0.000, median=5.000, mean=7.103, Q3=9.000, Max=27.000

(b) Min=0.000, Q1=0.000, median=5.000, mean=7.103, Q3=12.500, Max=27.000

(c) Min=0.000, Q1=4.000, median=7.103, mean=5.000, Q3=9.000, Max=27.000

(d) Min=0.000, Q1=4.000, median=5.000, mean=7.103, Q3=9.000, Max=27.000

(e) None of the above.

**Solution: (d)**

**Problem 1.10.** (5 points) A class has 12 boys and 4 girls. If three students are selected at random from this class, what is the probability that they are all boys?

(a) 1/4

(b) 5/9

(c) 11/28

(d) 17/36

(e) None of the above

**Solution: (c)**
Let $A_i$ stand for the event of choosing a boy in the $i^{th}$ selection with $i = 1, 2, 3$. The probability we are seeking is

$$\mathbb{P}[A_1 \cap A_2 \cap A_3].$$

By the multiplication rule,

$$\begin{aligned}\mathbb{P}[A_1 \cap A_2 \cap A_3] &= \mathbb{P}[A_1]\mathbb{P}[A_2|A_1]\mathbb{P}[A_3|A_2 \cap A_1] \\ &= \frac{12}{16} \cdot \frac{11}{15} \cdot \frac{10}{14} = \frac{11}{2 \cdot 14} = \frac{11}{28}.\end{aligned}$$

**Problem 1.11.** (5 points) A coin is tossed, and, independently, a 6-sided die is rolled. Let

$$\begin{aligned}A &= \{4 \text{ is obtained on the die}\} \text{ and}\\ B &= \{\textit{Heads} \text{ is obtained on the coin and}\\ &\qquad \text{an even number is obtained on the die}\}.\end{aligned}$$

Then

(a) $A$ and $B$ are mutually exclusive

(b) $A$ and $B$ are independent

(c) $A \subseteq B$

(d) $A \cap B = B$

(e) none of the above

**Solution:** The correct answer is (e).

**Problem 1.12.** (5 pts) *Source: Sample P exam problem set.*

An insurance company pays hospital claims.

The number of claims that include emergency room or operating room charges is 85% of the total number of hospital claims.

The number of claims that do not include emergency room charges is 25% of the total number of claims.

The occurrences of emergency room charges and operating room charges are independent.

Calculate the probability that a claim submitted to the insurance company includes operating room charges.

(a) 0.1

(b) 0.2

(c) 0.25

(d) 0.4

(e) None of the above

**Solution: (d)**

Let us denote by $E$ the event that a given claim includes emergency room charges and by $O$ the event that it includes operating room charges. Then, we can formalize the data provided in the problem as

$$\begin{aligned}\mathbb{P}[E \cup O] &= 0.85\\ \mathbb{P}[E^c] &= 0.25 \Rightarrow \mathbb{P}[E] = 0.75\\ \mathbb{P}[E \cap O] &= \mathbb{P}[E]\mathbb{P}[O].\end{aligned}$$

We need to find $\mathbb{P}[O]$.

From the first equality above, we get

$$0.85 = \mathbb{P}[E \cup O] = \mathbb{P}[E] + \mathbb{P}[O] - \mathbb{P}[E \cap O].$$

Then, using the second qnd third equalities,

$$0.85 = \mathbb{P}[E \cup O] = 0.75 + \mathbb{P}[O] - 0.75\mathbb{P}[O].$$

So, $\mathbb{P}[O] = 0.1/0.25 = 0.4$.

**Problem 1.13.** (5 pts) A class contains 20 men and 10 women. You know that half the men and half the women have brown eyes. What is the probability that a person chosen at random from this class is a woman or has brown eyes?

(a) 1/3

(b) 2/3

(c) 7/18

(d) 7/9

(e) None of the above

**Solution: (b)**
Let

$$W := \{\text{the chosen person is a woman}\},$$
$$B := \{\text{the chosen person has brown eyes}\}.$$

Then, we need to calculate the probability

$$\mathbb{P}[W \cup B] = \mathbb{P}[W] + \mathbb{P}[B] - \mathbb{P}[W \cap B] = \frac{1}{3} + \frac{1}{2} - \frac{1}{6} = \frac{2+3-1}{6} = \frac{4}{6} = \frac{2}{3}.$$

**Problem 1.14.** (5 points)

If a fair coin is flipped 1600 times, what is the probability of getting more than 800 heads?

**a.:** Less than 0.01.

**b.:** More than 0.01, but less than 0.05.

**c.:** More than 0.05, but less than 0.10.

**d.:** More than 0.10, but less than 0.25.

**e.:** None of the above

**Solution: e.**