University of Texas at Austin, Department of Mathematics
M358K - Applied Statistics

The In-term One Problem Bank

1.1. Experimental design.

Problem 1.1. A researcher would like to study the effect of eating breakfast on a cognitive function. Volunteers are recruited through the study by posting flyers on campus. He plans to randomly assign subjects to two groups, one told to eat before participating in the study and one asked to eat breakfast following the study. However, he suspects whether or not the person typically eats breakfast affects the suspected relationship. In order to address this, what should he do prior to assigning subjects to experimental groups?

(a) Cluster on typical breakfast habits.
(b) Randomly assign subjects to typical breakfast habits.
(c) Sample from each strata, typical breakfast eater and not.
(d) Block on typical breakfast habits.
(e) There is no way he can address his suspicions in his experimental design.

Solution: (d)

Problem 1.2. An attendance officer would like to study the effect of drinking coffee in the morning on class attendance. Volunteers are recruited to the study by posting flyers on her high school campus. She plans to randomly assign subjects to two groups, one told to drink coffee every morning for a fortnight and the other told not to drink coffee for a fortnight. However, she suspects that whether or not the person is already a caffeine addict affects the suspected relationship. In order to address this, what should she do prior to assigning subjects to experimental groups?

(a) Construct a stratified sample of caffeine addicts and non-caffeine addicts.
(b) Randomly assign subjects to pre-existing coffee addiction.
(c) Block by caffeine addiction.
(d) Just take note of whether they habitually drink coffee or not and proceed with random assignment on the entire sample.
(e) There is no way she can address these suspicions in his experimental design.

Solution: (c)

Problem 1.3. (5 points) A new headache remedy is given to a group of 250 patients who suffer severe headaches. Of these patients, 200 report that the remedy is very helpful in treating their headaches. From this information you conclude the following:

a.: The remedy is effective for the treatment of headaches.
b.: Nothing, because the sample size is too small.
c.: The new treatment is better than aspirin.
d.: Nothing, because there is no control group.
e.: None of the above.

Solution: d.

Problem 1.4. Scientists want to study whether a new type of contact lenses is less irritating than the existing type. They collect a random sample of 200 nearsighted people who never wore contact lenses before. They had a suspicion that the people with gray or blue eyes are already more sensitive. So, they divided their random sample into a group consisting of subjects with gray or blue eyes and another group consisting of everyone else. Then, they randomly split each group in half. One half each group wears the existing type of contact lenses while the other half wears the new type for a month. For all subjects, the contact lenses are put in and taken out by a specialist. The subjects are unaware of the type of contact lenses they receive. The specialists are unaware of the type of contact lenses they administer. Which of these statements is correct?

(a) This is an observational study.

(b) This is a double-blind experiment with blocking.

(c) This is a double-blind experiment without blocking.

(d) This is a double-blind experiment with a stratified sample.

(e) None of the above are correct.

Solution: (b)

Problem 1.5. For your honors thesis, you want to study the effect of eating Lindt Truffles on the score on the General Applied Statistics Proficiency (GASP) exam. You recruit 120 students and you split them at random into two groups of 60. You feed Lindt Truffles to the lucky half and you do not feed the Lindt Truffles to the other half. Then, you administer the GASP exam to all the 120 students, record, and analyse their scores. Which of these statements is correct?

(a) This is an observational study.

(b) This is a double-blind experiment with blocking.

(c) This is a blind experiment with blocking.

(d) This is an experiment without blocking.

(e) None of the above.

Solution: (d)

1.2. **The standard normal distribution.**

Problem 1.6. Let $Z \sim N(0,1)$. Given that Z is at least 2, what is the probability that Z is less than 3?

(a) 0.0215

(b) 0.9430

(c) 0.9772

(d) 0.9987

(e) None of the above.

Solution: (b)
We need to calculate

$$\mathbb{P}[Z < 3 \,|\, Z > 2] = \frac{\mathbb{P}[2 < Z < 3]}{\mathbb{P}[Z > 2]} = \frac{\mathbb{P}[Z < 3] - \mathbb{P}[Z \leq 2]}{\mathbb{P}[Z > 2]}\,.$$

From the standard normal tables, we obtain

$$\begin{aligned}&\mathbb{P}[Z \leq 2] = \Phi(2) = 0.9772 \quad \Rightarrow \quad \mathbb{P}[Z > 2] = 1 - 0.9772 = 0.0228, \quad \text{and}\\ &\mathbb{P}[Z < 3] = \Phi(3) = 0.9987.\end{aligned}$$

So, our final answer is

$$\mathbb{P}[Z < 3 \,|\, Z > 2] = \frac{\mathbb{P}[Z < 3] - \mathbb{P}[Z \leq 2]}{\mathbb{P}[Z > 2]} = \frac{0.9987 - 0.9772}{0.0228} = 0.9429825.$$

Problem 1.7. Let $Z \sim N(0,1)$. Given that Z is at least 0.5, what is the probability that Z is at least 1?

(a) 0.1587

(b) 0.3174

(c) 0.5144

(d) 1

(e) None of the above.

Solution: (c)
By the definition of conditional probability, we are looking for

$$\mathbb{P}[Z \geq 1 \,|\, Z \geq 0.5] = \frac{\mathbb{P}[Z \geq 1, Z \geq 0.5]}{\mathbb{P}[Z \geq 0.5]} = \frac{\mathbb{P}[Z \geq 1]}{\mathbb{P}[Z \geq 0.5]}\,.$$

Using the standard normal tables, we get

$$\begin{aligned}\mathbb{P}[Z \geq 0.5] &= 1 - \mathbb{P}[Z < 0.5] = 1 - \Phi(0.5) = 1 - 0.6915 = 0.3085,\\ \mathbb{P}[Z \geq 1] &= 1 - \mathbb{P}[Z < 1] = 1 - \Phi(1) = 1 - 0.8413 = 0.1587.\end{aligned}$$

So, our answer is

$$\mathbb{P}[Z \geq 1 \,|\, Z \geq 0.5] = \frac{\mathbb{P}[Z \geq 1]}{\mathbb{P}[Z \geq 0.5]} = \frac{0.1587}{0.3085} = 0.5144.$$

Problem 1.8. Let $Z \sim N(0,1)$. Given that Z is at most 2, what is the probability that Z is at least 0.5?

(a) 0.2777

(b) 0.2842

(c) 0.6915

(d) 0.7158

(e) None of the above.

Solution: (b) or (e)
By the definition of conditional probability, we need to find

$$\mathbb{P}[Z \geq 0.5 \,|\, Z \leq 2] = \frac{\mathbb{P}[0.5 \leq Z \leq 2]}{\mathbb{P}[Z \leq 2]} = \frac{\mathbb{P}[Z \leq 2] - \mathbb{P}[Z < 0.5]}{\mathbb{P}[Z \leq 2]} = 1 - \frac{\mathbb{P}[Z < 0.5]}{\mathbb{P}[Z \leq 2]}\,.$$

From the standard normal tables, we get

$$\mathbb{P}[Z < 0.5] = \Phi(0.5) = 0.6915 \quad \text{and} \quad \mathbb{P}[Z \leq 2] = \Phi(2) = 0.9772.$$

So, our final answer is

$$\mathbb{P}[Z \geq 0.5 \,|\, Z \leq 2] = 1 - \frac{\mathbb{P}[Z < 0.5]}{\mathbb{P}[Z \leq 2]} = 1 - \frac{0.6915}{0.9772} = 0.2924.$$

Problem 1.9. Let $Z \sim N(0,1)$. Given that Z is at most 3, what is the probability that Z is at most 2.5?

(a) 0.0049

(b) 0.9938

(c) 0.9951

(d) 0.9987

(e) None of the above.

Solution: (c)
From the definition of conditional probability, we are looking for

$$\mathbb{P}[Z \leq 2.5 \,|\, Z \leq 3] = \frac{\mathbb{P}[Z \leq 3, Z \leq 2.5]}{\mathbb{P}[Z \leq 3]} = \frac{\mathbb{P}[Z \leq 2.5]}{\mathbb{P}[Z \leq 3]}.$$

From the standard normal tables, we obtain

$$\mathbb{P}[Z \leq 2.5] = \Phi(2.5) = 0.9938 \quad \text{and} \quad \mathbb{P}[Z \leq 3] = \Phi(3) = 0.9987.$$

So, our answer is

$$\mathbb{P}[Z \leq 2.5 \,|\, Z \leq 3] = \frac{\mathbb{P}[Z \leq 2.5]}{\mathbb{P}[Z \leq 3]} = \frac{0.9938}{0.9987} = 0.9951.$$

Problem 1.10. Let $Z \sim N(0,1)$. Given that Z is at most 0.3, what is the probability that Z is at most -2.5?

(a) 0.0062

(b) 0.0100

(c) 0.6117

(d) 0.6179

(e) None of the above.

Solution: (b)
From the definition of conditional probability, we are looking for

$$\mathbb{P}[Z \leq -2.5 \,|\, Z \leq 0.3] = \frac{\mathbb{P}[Z \leq 0.3, Z \leq -2.5]}{\mathbb{P}[Z \leq 0.3]} = \frac{\mathbb{P}[Z \leq -2.5]}{\mathbb{P}[Z \leq 0.3]}.$$

From the standard normal tables, we obtain

$$\mathbb{P}[Z \leq -2.5] = \Phi(-2.5) = 0.0062 \quad \text{and} \quad \mathbb{P}[Z \leq 0.3] = \Phi(0.3) = 0.6179.$$

So, our answer is

$$\mathbb{P}[Z \leq -2.5 \,|\, Z \leq 0.3] = \frac{\mathbb{P}[Z \leq -2.5]}{\mathbb{P}[Z \leq 0.3]} = \frac{0.0062}{0.6179} = 0.0100.$$

1.3. **The normal distribution.**

Problem 1.11. The distribution of lengths of adult bass in Cumberland Lake is modeled as normal with mean 32" and standard deviation 6".

At the annual Cumberland Lake bass fishing competition, you win a blue ribbon if you catch a bass that is over 38" in length. If you catch a bass over 42" in length you also win a gold medallion.

Assume that an angler (the person fishing) is only allowed to catch the first fish s/he reels in.

Which of the following is the closest to the probability that an angler has not won a gold medallion **given** that s/he has won a blue ribbon?

(a) 0.1112

(b) 0.1684

(c) 0.3012
(d) 0.7007
(e) None of the above.

Solution: (d)
Let X denote the length of a randomly chosen adult bass in Cumberland Lake. We are given that

$$X \sim Normal(mean = 32, sd = 6).$$

We are looking for

$$\mathbb{P}[X < 42 \,|\, X > 38] = \frac{\mathbb{P}[38 < X < 42]}{\mathbb{P}[X > 38]} = \frac{\mathbb{P}[X < 42] - \mathbb{P}[X \leq 38]}{\mathbb{P}[X > 38]}.$$

Transitioning to standard units and using the standard normal table, we get

$$\mathbb{P}[X \leq 38] = \mathbb{P}\left[\frac{X-32}{6} \leq \frac{38-32}{6}\right] = \mathbb{P}[Z \leq 1] = \Phi(1) = 0.8413 \quad \Rightarrow \quad \mathbb{P}[X > 38] = 0.1587,$$

$$\mathbb{P}[X < 42] = \mathbb{P}\left[\frac{X-32}{6} \leq \frac{42-32}{6}\right] \approx \mathbb{P}[Z \leq 1.67] = \Phi(1.67) = 0.9525.$$

Finally, our answer is

$$\mathbb{P}[X < 42 \,|\, X > 38] = \frac{0.9525 - 0.8413}{0.1587} = 0.7007$$

Problem 1.12. Let Z_1, Z_2, and Z_3 be independent, standard normal random variables. What is the probability that the sum of Z_1 and Z_2 exceeds $1.2Z_3 + 1$?

(a) 0.2946
(b) 0.3694
(c) 0.3773
(d) 0.3856
(e) None of the above.

Solution: (a)
We are looking for the probability

$$\mathbb{P}[Z_1 + Z_2 > 1.2Z_3 + 1] = \mathbb{P}[Z_1 + Z_2 - 1.2Z_3 > 1].$$

The random variable $X = Z_1 + Z_2 - 1.2Z_3$ is a linear combination of independent normal random variables, so that it has the following distribution

$$X \sim Normal(mean = 0, var = 1 + 1 + (1.2)^2 = 3.44).$$

So, rewriting X in standard units, the probability we are looking for becomes

$$\mathbb{P}[X > 1] = \mathbb{P}\left[\frac{X-0}{\sqrt{3.44}} > \frac{1-0}{\sqrt{3.44}}\right] = \mathbb{P}\left[Z > \frac{1}{\sqrt{3.44}} = 0.54\right]$$

where $Z \sim N(0,1)$. Using the standard normal tables, we obtain

$$\mathbb{P}[Z > 0.54] = 1 - \Phi(0.54) = 1 - 0.7054 = 0.2946.$$

1.4. **The sample mean: The normal case.**

Problem 1.13. Consider a normal population with mean 10 and standard deviation 1. What is the probability that a sample mean of a sample of size 16 falls into the interval $[9.525, 9.9]$?

(a) 0.0548
(b) 0.1428
(c) 0.2816
(d) 0.3159
(e) None of the above.

Solution: (d)
The sample mean of a sample of size 16 has the distribution

$$\bar{X} \sim Normal\left(mean = 10, sd = \frac{1}{\sqrt{16}} = \frac{1}{4}\right).$$

We are looking for the probability

$$\mathbb{P}[\bar{X} \in [9.525, 9.9]] = \mathbb{P}[9.525 \leq \bar{X} \leq 9.9] = \mathbb{P}[\bar{X} \leq 9.9] - \mathbb{P}[\bar{X} < 9.525].$$

Rewriting $\bar{X}$ in standard units (with $Z \sim N(0,1)$), we get

$$\mathbb{P}[\bar{X} < 9.525] = \mathbb{P}\left[\frac{\bar{X}-10}{\frac{1}{4}} < \frac{9.525-10}{\frac{1}{4}}\right] = \mathbb{P}[Z < -1.9],$$
$$\mathbb{P}[\bar{X} < 9.9] = \mathbb{P}\left[\frac{\bar{X}-10}{\frac{1}{4}} < \frac{9.9-10}{\frac{1}{4}}\right] = \mathbb{P}[Z < -0.4].$$

Using the standard normal tables, we get our answer:

$$\mathbb{P}[Z < -0.4] - \mathbb{P}[Z < -1.9] = 0.3446 - 0.0287 = 0.3159.$$

Problem 1.14. Consider a normal population distribution for a large population. You know that the standard deviation of the sampling distribution of the sample mean for samples of size 36 is 2. How large should a sample from the same population be so that the standard deviation of the sample mean becomes 1.2?

(a) 10
(b) 51
(c) 52
(d) 100
(e) None of the above.

Solution: (d)
Let σ denote the population standard deviation. Then, the standard deviation of the sample mean $\bar{X}_{36}$ of a sample of size 36 is

$$SD[\bar{X}_{36}] = \frac{\sigma}{\sqrt{36}} = \frac{\sigma}{6} = 2 \quad \Rightarrow \quad \sigma = 12.$$

Let n denote the unknown sample size for the sample whose standard deviation for the sample mean $\bar{X}_n$ needs to be 1.2. Then,

$$SD[\bar{X}_n] = \frac{\sigma}{\sqrt{n}} = \frac{12}{\sqrt{n}} = 1.2 \quad \Rightarrow \quad \sqrt{n} = \frac{12}{1.2} = 10 \quad \Rightarrow \quad n = 100.$$

Problem 1.15. A particular type of wool for clothes manufacturing has to have a specific tensile strength in order to be used in weaving machines without breaking. We model its tensile strength as normally distributed with standard deviation 0.4 MPa. How is the variance of the sample mean changed when the sample size increases from 64 to 196?

(a) The variance does not change.
(b) It decreases from 0.0025 to 0.0008.
(c) It decreases from 0.02 to 0.0114.
(d) Not enough information is given.
(e) None of the above.

Solution: (b)
The variance of the sample mean $\bar{X}_{64}$ for the sample of size 64 is

$$Var[\bar{X}_{64}] = \frac{(0.4)^2}{64} = 0.0025.$$

The variance of the sample mean $\bar{X}_{196}$ for the sample of size 196 is

$$Var[\bar{X}_{196}] = \frac{(0.4)^2}{196} = 0.0008.$$

Problem 1.16. A particular type of wool for clothes manufacturing has to have a specific tensile strength in order to be used in weaving machines without breaking. We model its tensile strength as normally distributed with standard deviation 0.4 MPa. How is the variance of the sample mean changed when the sample size decreases from 64 to 36?

(a) The variance does not change.
(b) It increases from 0.0025 to 0.0044.
(c) It increases from 0.02 to 0.0267.
(d) Not enough information is given.
(e) None of the above.

Solution: (b)
The variance of the sample mean $\bar{X}_{64}$ for the sample of size 64 is

$$Var[\bar{X}_{64}] = \frac{(0.4)^2}{64} = 0.0025.$$

The variance of the sample mean $\bar{X}_{36}$ for the sample of size 36 is

$$Var[\bar{X}_{36}] = \frac{(0.4)^2}{36} = 0.0044.$$

1.5. **Point estimates and sampling variability.**

Problem 1.17. (5 points) Dr. Theodore Gauss conducts survey. He draws a random sample of 100 citizens of Whoburgh. He finds that **winter** is the absolute favorite season for 88% of the surveyed citizens. In this situation, 88% is ...

(a) ... the point estimate.
(b) ... the population parameter.
(c) ... the measure of spread.
(d) ... the margin of error.
(e) None of the above.

Solution: (a)

1.6. **Confidence intervals: General principles.**

Problem 1.18. Consider the following statement:

"At the 95% confidence level, we estimate that the true population mean of the tail length of rhesus macaques is between 20.7 cm and 22.9 cm. If we were to construct a confidence interval at the 99% confidence level, the resulting interval would be narrower."

Are there any errors in the above two statements?

(a) This interpretation is correct. There are no errors.
(b) In the first sentence, the phrase "true population mean" should be replaced with "sample mean."
(c) In the second sentence, "narrower" should be replaced with "wider."
(d) In the first sentence, the word "estimate" should be replaced with "know for certain."
(e) None of the above.

Solution: (c)

Problem 1.19. The Chicago Tribune and the Los Angeles Times conducted separate national polls where they asked full-time employees how many hours they work per week, and reported confidence intervals at the 95% confidence level. The Chicago Tribune surveyed 500 people, and the Los Angeles Times surveyed 300 people. Which paper reported a larger margin of error? Assume the standard deviations of the two populations were equal.

(a) The Chicago Tribune
(b) The Los Angeles Times
(c) The margins of error are the same
(d) The margins of error will depend on the means of the samples
(e) There is not enough information to answer this question

Solution: (b)

1.7. **Confidence intervals for the mean: The normal case with a known standard deviation.**

Problem 1.20. *For your own RT, go to:* `https://humanbenchmark.com/tests/reactiontime`. You will also see that the distribution is not really well-modelled as normal.

You want to study the reaction time (RT) of undergraduate students. From past research, you are comfortable modelling the RT using the normal distribution with a standard deviation of 30 ms. You gather a simple random sample of 225 undergraduates. Their sample average is 215 ms. What is the 80%-confidence interval?

(a) 215 ± 1.28
(b) 215 ± 1.96
(c) 215 ± 2.56
(d) 215 ± 3.29
(e) None of the above.

Solution: (c)
The critical value z^* corresponding to the confidence level of 80% is $z^* = \Phi^{-1}(0.90) = 1.28$. The standard deviation of the sample mean with the given $\sigma = 30$ and the sample size $n = 225$ is

$$\frac{\sigma}{\sqrt{n}} = \frac{30}{\sqrt{225}} = \frac{30}{15} = 2.$$

So, the margin of error equals

$$z^*\left(\frac{\sigma}{\sqrt{n}}\right) = 1.28(2) = 2.56.$$

1.8. **Hypothesis testing: General principles.**

Problem 1.21. Public policy researchers are studying whether a new school lunch program reduces obesity amongst elementary school children. The authors compute the p-value for their sample to be 0.10. Which of the following interpretations of the p-value is correct?

(a) The probability that the policy is effective.
(b) The probability that the policy is *not* effective.
(c) The probability of determining the policy is not effective when it actually is.
(d) The probability of getting results as extreme or more extreme than the ones in the study if the policy is actually effective.
(e) The probability of getting results as extreme or more extreme than the ones in the study if the policy is actually *not* effective.

Solution: (e)

1.9. **Hypothesis testing: The normal case with an unknown mean and known standard deviation (left-sided).**

Problem 1.22. Every jar of *Nocciolata* is labeled to contain 270 mg of delicious hazelnut spread. You suspect that the mean contents of every jar is actually less. You do some research and you are comfortable assuming that the amount of spread per jar is normally distributed with a known standard deviation of 10 mg. Your plan is to randomly collect 64 jars and weigh their contents. What is the form of your rejection region with the significance level of 1%?

(a) $[0, 267.0875]$
(b) $[272.9125, \infty)$
(c) $[0, 267.55]$
(d) $[272.055, \infty)$
(e) None of the above.

Solution: (a)
We are testing

$$H_0 : \mu = \mu_0 = 270 \quad vs. \quad H_a : \mu < \mu_0 = 270.$$

So, remembering that the amount of spread in the jar cannot be negative, the form of our rejection region is, in our usual notation,

$$RR = \left[0, \mu_0 + z_\alpha\left(\frac{\sigma}{\sqrt{n}}\right)\right]$$

where $z_\alpha = \Phi^{-1}(0.01) = -2.33$. We conclude that

$$RR = \left[0, 270 - 2.33\left(\frac{10}{\sqrt{64}}\right)\right] = [0, 270 - 2.9125] = [0, 267.0875].$$

1.10. **Hypothesis testing: The normal case with an unknown mean and known standard deviation (right-sided).**

Problem 1.23. The manufacturer of the *Fortunate Trinket* cereal claims that the total sugars in their cereal amount to at most 30 g per 100 g serving. You have witnessed your hyperactive nephew after he eats the aforementioned cereal and you suspect that the manufacturer's claim is erroneous. Your plan is to test your hypotheses at the 4% significance level. Your model for the amount of sugars in a single serving of cereal is normal with the standard deviation of 4 g. The sample size you plan to use is 64 servings. What will your rejection region look like (in real units)?

(a) It will have a lower bound at 30.875.

(b) It will have a lower bound at 37.

(c) It will have an upper bound at 30.875.

(d) It will have an upper bound at 37.

(e) None of the above.

Solution: (a)
Let the mean sugar content be denoted by μ. We are testing

$$H_0 : \mu = \mu_0 = 30 \quad vs. \quad H_a : \mu > \mu_0 = 30.$$

This is a right-tailed test, so the form of the rejection region is, in our usual notation,

$$RR = \left[\mu_0 + z_{1-\alpha}\left(\frac{\sigma}{\sqrt{n}}\right), \infty\right)$$

with $z_{1-\alpha} = \Phi^{-1}(0.96) = 1.75$. For the lower bound of our rejection region, we have

$$\mu_0 + z_{1-\alpha}\left(\frac{\sigma}{\sqrt{n}}\right) = 30 + 1.75\left(\frac{4}{\sqrt{64}}\right) = 30 + 1.75\left(\frac{4}{8}\right) = 30.875.$$

1.11. **Hypothesis testing: The normal case with an unknown mean and known standard deviation (two-sided).**

Problem 1.24. An manufacturing process produces cylindrical components parts. It is important that the process produce parts having the diameter of 10 mm. A study is planned to see whether the machine involved is correctly calibrated. One hundred parts are randomly selected and measured exactly. It is found that the sample average of the diameter is 9.975 mm. Assuming that the diameters are normally distributed with the known standard deviation of 0.1 mm, what is the $p-$value associated with the observed sample average?

(a) 0.0062

(b) 0.0124

(c) 0.0401

(d) 0.0802

(e) None of the above.

Solution: (b)
We ar testing

$$H_0 : \mu = \mu_0 = 10 \quad vs. \quad H_a : \mu \neq \mu_0 = 10.$$

Under the null hypothesis, the $z-$score associated with the observed sample average is

$$z = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{9.975 - 10}{\frac{0.1}{\sqrt{100}}} = -2.5$$

So, with $Z \sim N(0,1)$, the $p-$value is

$$\mathbb{P}[Z < -2.5] + \mathbb{P}[Z > 2.5] = 2\mathbb{P}[Z < -2.5] = 2(0.0062) = 0.0124$$

where we used the standard normal tables to evaluate the probability.

Problem 1.25. A medicine dispensing machine is supposed to be calibrated to dispense 20 ml of medication. Of course, the amount dispensed is not exact. You model the amount actually dispensed using the normal distribution with standard deviation 1.5 ml. Periodically, the machine is tested to see if it's correctly calibrated. Each time, a sample of 9 doses is taken and measured carefully. With a particular significance level, the rejection region is the complement of the interval $(19.1, 20.9)$. What was the significance level used?

(a) 0.0359
(b) 0.05
(c) 0.0718
(d) 0.0968
(e) None of the above.

Solution: (c)
We are testing

$$H_0 : \mu = 20 \quad vs. H_a : \mu \neq 20.$$

Under the null $\mu = \mu_0 = 20$ and the distribution of the sample mean is

$$\bar{X} \sim Normal\left(mean = 20, sd = \frac{1.5}{\sqrt{9}} = \frac{1.5}{3} = 0.5\right).$$

The probability of falling outside the rejection region is

$$\begin{aligned}\beta = \mathbb{P}[19.1 < \bar{X} < 20.9] &= \mathbb{P}\left[\frac{19.1 - 20}{0.5} < \frac{\bar{X} - 20}{0.5} < \frac{20.9 - 20}{0.5}\right] \\ &= \mathbb{P}[-1.8 < Z < 1.8] = 2\mathbb{P}[Z < 1.8] - 1 = 2\Phi(1.8) - 1 = 2(0.9641) - 1 = 0.9282.\end{aligned}$$

So, the significance level of the test is $\alpha = 1 - 0.9282 = 0.0718$.

1.12. **The power of the test.**

Problem 1.26. A medicine dispensing machine is supposed to be calibrated to dispense 20 ml of medication. Of course, the amount dispensed is not exact. You model the amount actually dispensed using the normal distribution with standard deviation 1.5 ml. Periodically, the machine is tested to see if it's correctly calibrated. Each time, a sample of 9 doses is taken and measured carefully. With a particular significance level, the rejection region is the complement of the interval $(19.1, 20.9)$. What is the power of the test at the alternative mean $\mu_a = 21.5$?

(a) 0.8849
(b) 0.9289

(c) 0.9332

(d) 0.9641

(e) None of the above.

Solution: (a)
Under the alternative $\mu_a = 21.5$, the distribution of the sample mean is

$$\bar{X} \sim Normal\left(mean = 21.5, sd = \frac{1.5}{\sqrt{9}} = \frac{1.5}{3} = 0.5\right).$$

The probability of making a Type II Error is

$$\begin{aligned}\beta = \mathbb{P}[19.1 < \bar{X} < 20.9] &= \mathbb{P}\left[\frac{19.1 - 21.5}{0.5} < \frac{\bar{X} - 21.5}{0.5} < \frac{20.9 - 21.5}{0.5}\right] \\ &= \mathbb{P}[-4.8 < Z < -1.2] = \mathbb{P}[Z < -1.2] - \mathbb{P}[Z \leq -4.8] = \Phi(-1.2) - 0 = 0.1151.\end{aligned}$$

So, the power of the test is $1 - \beta = 1 - 0.1151 = 0.8849$.