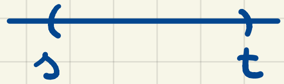
M339D: November 6th, 2023.

Log-Normal Stock Prices.

Temporarily fix a time-horizon T.

$S(t), t \in [0,T]$ ... time-t stock price



Define:

$$R(s,t) := \ln\left(\frac{S(t)}{S(s)}\right)$$

Equivalently:

$$\boxed{S(t) = S(s)e^{R(s,t)}}$$

In particular: $R(0,T)$ ... realized return over $(0,T)$

We model realized returns as normal.

$$R(0,T) \sim Normal(mean = m, variance = \nu^2) \longleftarrow$$

$\Rightarrow$ $S(T)$ is lognormal

and $\mathbb{E}^*[S(T)] = S(0)e^{m+\frac{\nu^2}{2}}$ ☆

Market model.

- Riskless Asset w/ ccrfir r
- Risky Asset : a non-dividend-paying stock

$\boxed{\sigma\text{... volatility}}$ ✓

Under the risk-neutral measure $\mathbb{E}^*[S(T)] = \boxed{S(0)e^{rT}}$ ☆☆

Equating ☆ to ☆☆

$$\boxed{m + \frac{\nu^2}{2} = rT}$$

Consider: $\mathrm{Var}[R(0,T)] = \nu^2 = \,?$ $\longrightarrow$ $\boxed{\nu^2 = \sigma^2 \cdot T}$

Recall: $\mathrm{Var}[R(0,1)] = \sigma^2$, i.e., $SD[R(0,1)] = \sigma$

$$\mathrm{Var}[R(0,2)] = \mathrm{Var}[R(0,1)] + \mathrm{Var}[R(1,2)] = 2\mathrm{Var}[R(0,1)] = 2\sigma^2$$

$$\boxed{m = rT - \frac{\sigma^2 \cdot T}{2} = \left(r - \frac{\sigma^2}{2}\right)\cdot T}$$

Finally:

$$\boxed{R(0,T) \sim \mathrm{Normal}\left(\text{mean} = \left(r - \frac{\sigma^2}{2}\right)\cdot T,\ \text{variance} = \sigma^2 \cdot T\right)}$$

Say that $\boxed{Z \sim N(0,1)}$

Then, we can express $R(0,T)$ as

$$\boxed{R(0,T) = \left(r - \frac{\sigma^2}{2}\right)\cdot T + \sigma\sqrt{T}\cdot Z}$$

Thus,

$$\boxed{S(T) = S(0)e^{\left(r - \frac{\sigma^2}{2}\right)\cdot T + \sigma\sqrt{T}\cdot Z}}$$

Q: What is the median of $S(T)$ under $\mathbb{P}^*$?

$\longrightarrow$: $\boxed{S(0)e^{\left(r - \frac{\sigma^2}{2}\right)\cdot T}}$ ✓

Note:

$$\frac{\text{mean}}{\text{median}} = \frac{\cancel{S(0)}e^{\cancel{rT}}}{\cancel{S(0)}e^{\left(\cancel{r} - \frac{\sigma^2}{2}\right)\cdot T}} = e^{\frac{\sigma^2 \cdot T}{2}}$$

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Problem Set # 100

Mean and median of the log-normal stock prices.

Problem 100.1. The current price of a non-dividend-paying stock is \$80 per share. Under the risk-neutral probability measure, its mean rate of return is 12% and its volatility is 30%.

Let $R(0,t)$ denote the realized return of this stock over the time period $[0,t]$ for any $t > 0$. Calculate $\mathbb{E}^*[R(0,2)]$.

→:

$r = 0.12$

$\sigma = 0.30$

$R(0,2) \sim Normal\left(mean = \boxed{\left(r - \frac{\sigma^2}{2}\right) \cdot 2}, var = \sigma^2 \cdot 2\right)$

$\mathbb{E}^*[R(0,2)] = \left(0.12 - \frac{0.09}{2}\right) \cdot 2 = 0.15$ □

Problem 100.2. A stock is valued at \$75.00. The continuously compounded, risk-free interest rate is 10.0% and the standard deviation of annualized returns is $\boxed{25.0\%}$. If the stock is lognormally distributed, what is the expected stock price after 2 years under the risk-neutral probability measure?

→: $\mathbb{E}^*[S(2)] = S(0)e^{2r} = 75e^{0.20} \approx 91.605$ □

Problem 100.3. A non-dividend-paying stock is valued at \$55.00 per share. Its standard deviation of annualized returns is given to be $\boxed{22.0\%}$. The continuously compounded risk-free interest rate is 12%. If the stock price is modeled using the lognormal distribution (as discussed in class), what is the median of the stock price in 3 years under the risk-neutral probability measure?

→: $S(0)e^{\left(r - \frac{\sigma^2}{2}\right) \cdot T} = 55e^{\left(0.12 - \frac{0.22^2}{2}\right) \cdot 3} = \underline{73.313}$ □

Problem 100.4. Assume that the stock price is modeled using the lognormal distribution. The stock pays no dividends. Under $\mathbb{P}^*$, the annual mean rate of return on the stock is given to be 12%. Also under $\mathbb{P}^*$, the median time$-t$ stock price is evaluated to be $S(0)e^{0.1t}$. What is the volatility parameter of this stock price?

→ :
$$S(0)e^{(r-\frac{\sigma^2}{2})\cdot t} = S(0)e^{0.1t}$$

$$r - \frac{\sigma^2}{2} = 0.1$$

$$0.12 - \frac{\sigma^2}{2} = 0.1$$

$$\frac{\sigma^2}{2} = 0.02 \quad \Rightarrow \quad \sigma^2 = 0.04 \quad \Rightarrow \quad \boxed{\sigma = 0.20}$$

Problem 100.5. The current stock price is \$100 per share. The stock price at any time $t > 0$ is modeled using the lognormal distribution. Assume that the continuously compounded risk-free interest rate equals 8%. Let its volatility equal 20%.

Find the value t^* at which the median stock price equals \$120, under the risk-neutral probability measure.

→ : median time$\cdot$t stock price $= S(0)e^{(r-\frac{\sigma^2}{2})\cdot t}$

$$\cancel{120}\ 1.2 = 1\cancel{00}e^{(0.08-\frac{0.04}{2})\cdot t^*}$$

$$t^* = \frac{\ln(1.2)}{0.06} = \underline{3.039}$$

□

Problem 100.6. The volatility of the price of a non-dividend-paying stock is 0.2. The stock price is modeled using a log-normal distribution. Under $\mathbb{P}^*$, the expected time-2 stock price is \$120. What is the median of the time-2 stock price under $\mathbb{P}^*$?

→ :
$$\mathbb{E}^*[S(2)]e^{-\frac{2\sigma^2}{2}} = 120e^{-0.04} = 115.295$$

□