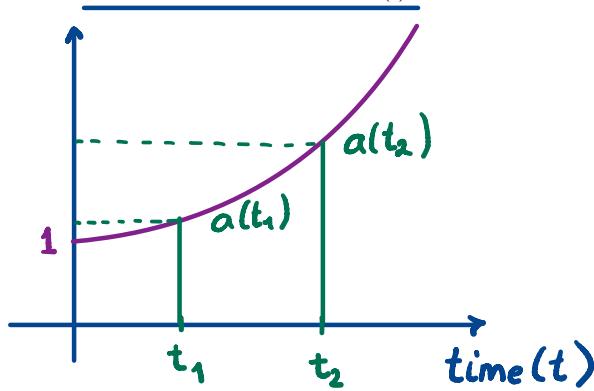


UNIVERSITY OF TEXAS AT AUSTIN

Problem set #2

Prerequisites. Conventions.

Problem 2.1. Write down the definition of the effective interest rate for the time period $[t_1, t_2]$ in terms of the accumulation function $a(\cdot)$.



$$i_{[t_1, t_2]} := \frac{a(t_2) - a(t_1)}{a(t_1)}$$

... annual effective
 $a(t) = (1+i)^t$

Problem 2.2. A 5-year loan for 10,000 is charged an effective interest rate of 6% per half-year period. The loan is to be repaid so that interest is repaid at the end of every 6 month period as it accrues and the principal is repaid in total at the end of the 5 years.

Denote the total amount of interest paid on this loan by I . Then,

- (a) $I \approx 2,750$
- (b) $I \approx 3,000$
- (c) $I \approx 3,250$
- (d) $I \approx 3,500$
- (e) None of the above

OLB_k ... outstanding loan balance after k^{th} period after the payment is made

we keep current on the interest

$$\Rightarrow OLB_k = L = 10000$$

$$\Rightarrow \text{amt of every interest pmt: } (0.06)(10,000) = 600) \times$$

$$\text{total # of pmts} = 5 \cdot 2 = 10$$

answer: 6,000

Problem 2.3. Source: Exam FM/2, May 2005, Problem #7.

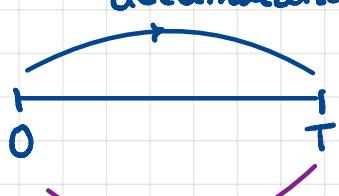
Mike receives cash flows of 100 today, 200 in one year, and 100 in two years. The net present value of these cash flows is 364.46 at an annual effective interest rate i .

Calculate i .

- (a) About 10%
- (b) About 11%
- (c) About 12%
- (d) About 13%
- (e) None of the above

M339D: January 18th, 2023.

i ... effective annual interest rate
 $a(t) = \underline{(1+i)^t}$ accumulation function



$$v(t) = \frac{1}{a(t)} = (1+i)^{-t}$$

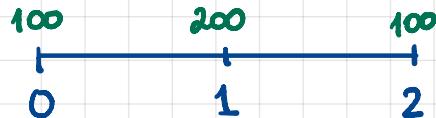
discounting

$$v = \frac{1}{1+i}$$

discount factor

$$v(t) = v^t$$

Problem 2.3.



$$\begin{aligned} NPV &= 364.46 = PV_{0,0}(100) + PV_{0,1}(200) + PV_{0,2}(100) \\ &= 100 + 200 \cdot v + 100 \cdot v^2 \\ &= 100 (1 + 2v + v^2) \quad /:100 \end{aligned}$$

$$(v+1)^2 = 3.6446$$

$$v+1 = \sqrt{3.6446}$$

$$\Rightarrow v = 0.9090$$

$$\Rightarrow i = \frac{1}{v} - 1 = 0.10$$

Problem 2.4. Write down the definition of the (time-varying) **force of interest** in terms of the accumulation function $a(\cdot)$.

$$\delta_t := \frac{d}{dt} [\ln(a(t))] = \frac{a'(t)}{a(t)}$$

Definition 2.1. In the context of financial markets, the constant force of interest is usually referred to as the **continuously compounded, risk-free interest rate** and denoted by r .

Problem 2.5. Let the accumulation function be given by

$$a(t) = (1 + 0.05)^{3t}(1 + 0.02)^{t/2}.$$

Then, we can say the following about the **continuously compounded, risk-free interest rate r** associated with the above accumulation function:

- (a) $r = 0.07$
- (b) $r = \ln(1.05) + \ln(1.02)$
- (c) $r = 3\ln(1.05) + 0.5\ln(1.02)$
- (d) The continuously compounded, risk-free interest rate is not constant.
- (e) None of the above

$$\ln(x \cdot y) = \ln(x) + \ln(y)$$

$$\begin{aligned} \rightarrow r &= \frac{d}{dt} [\ln(a(t))] = \frac{d}{dt} [3t \ln(1.05) + \frac{t}{2} \ln(1.02)] \\ &= 3\ln(1.05) + 0.5\ln(1.02) \end{aligned}$$

Problem 2.6. Let r be the **continuously compounded, risk-free interest rate**. What is the expression for the accumulation function $a(\cdot)$ in terms of r ?

$$a(t) = ?$$

$$\frac{d}{dt} [\ln(a(t))] = r$$

$$d[\ln(a(t))] = r dt \quad | \int$$

$$\ln[a(t)] = rt + c$$

$$\begin{aligned} a(t) &= e^{rt+c} \\ a(0) &= e^c = 1 \end{aligned}$$

$$a(t) = e^{rt}$$

Problem 2.7. Roger deposits \$100 into an account at time 0.

~~For the following three years, he does not make any withdrawals or deposits and the account earns at a continuously compounded, risk-free interest rate r .~~

After 15 years and 6 months, the balance in his account equals \$133. Then,

- (a) $0 \leq r < 0.0150$
- (b) $0.0150 \leq r < 0.0250$
- (c) $0.0250 \leq r < 0.0550$
- (d) $0.0550 \leq r < 0.0650$
- (e) None of the above

$$\begin{aligned}100e^{r(15.5)} &= 133 \\ e^{15.5r} &= 1.33 \\ \ln(1.33) &= 15.5r \\ r &= \frac{\ln(1.33)}{15.5} = 0.0182\end{aligned}$$

Notation . Assumptions . Conventions .

Assumptions .

- There is a single prevailing interest rate:
 - the same for borrowing and lending;
 - the same for everyone;
 - deterministic and constant.
- There is an infinite line of credit .
- Money is continuously valued.

Example. Say that the ccrfir equals 0.02.

You invest \$100 @ time 0.

You make no subsequent deposits or withdrawals.

Your balance in three months :

$$\frac{100 e^{(0.02)(0.25)}}{ } = 100 e^{0.005}$$

□

Note. Borrowing & Lending via savings accounts , loans, bonds
are all

RISKLESS INVESTMENT.

Riskless Asset:

governed by a fixed, deterministic
continuously compounded, risk-free interest rate ;
usually denoted by r

Risky Asset:

///	///	///
///	///	///

We will allow for continuously valued investment in the shares of stock, e.g., $\frac{11}{23}$ shares
or $e^{-0.03}$ shares.

The structure of returns

dividend pmts

stock appreciation

Dividends

NO DIVIDENDS ? (This class :)

YES DIVIDENDS

DISCRETE

CONTINUOUS