Name:

M339J/M389J: Probability Models with Actuarial Applications
The University of Texas at Austin
Practice Problems for In-Term Exam II
Instructor: Milica Čudina

Notes: This is a closed book and closed notes exam. The maximal score on this exam is 100 points.

2.1. **Prior material.** Please, solve problems **1.8, 1.9, 1.10, 1.11, 1.12, 1.21, and 1.22** from the practice problem set posted for the first in-term exam on the course website.

2.2. **TRUE/FALSE QUESTIONS.**

Problem 2.1. (2 points) Any negative binomial distribution has the memoryless property. *True or false?*

Solution: FALSE
This is correct only for the geometric distribution.

Problem 2.2. (2 points) With our usual assumptions in place, assume that the claim count follows the Poisson distribution with parameter $\lambda = 10$.
The severities of individual claims are such that on average half of the claims are expected to have claim amounts equal to 50 and the other half are expected to have the claim amounts equal to 100.
Then, the total losses have the expected value 750. *True or false?*

Solution: TRUE

Problem 2.3. (2 points) In the compound aggregate loss model, with our usual notation and with the usual assumptions in place, we have that the claim count distribution is binomial with $m = 40$ and $q = 1/4$, and that the p.m.f. of the individual losses is

$$p_X(10) = p_X(20) = 0.3, p_X(50) = 0.4.$$

Then, $\mathbb{E}[S] \leq 290$. *True or false?*

Solution: TRUE
In fact,

$$\mathbb{E}[S] = \mathbb{E}[N]\mathbb{E}[X]$$

with $N \sim bin(m = 40, q = 1/4)$ and X as in the problem statement. We have

$$\mathbb{E}[N] = mq = 40(1/4) = 10$$
$$\mathbb{E}[X] = 10 \cdot 0.3 + 20 \cdot 0.3 + 50 \cdot 0.4 = 29.$$

So,

$$\mathbb{E}[S] = \mathbb{E}[N]\mathbb{E}[X] = 10 \cdot 29 = 290 \leq 290.$$

2.3. **MULTIPLE CHOICE QUESTIONS.**

Problem 2.4. (5 points) *Source: Sample P exam, Problem #262.*
The number of traffic accidents occurring on any given day in Coralville is Poisson distributed with mean 5. The probability that any such accident involves an uninsured driver is 0.25, independent of all other such accidents. Calculate the probability that on a given day in Coralville there are no traffic accidents that involve an uninsured driver.

(a) 0.007
(b) 0.010
(c) 0.124
(d) 0.237
(e) 0.287

Solution: (e)
By the "thinning" theorem, the number of accidents involving an uninsured driver is again Poisson with mean $5(0.25) = 1.25$. The probability that this random variable has the value of zero is

$$e^{-1.25} = 0.2865.$$

Problem 2.5. (5 points) A group dental policy has a negative binomial claim count distribution with mean 20 and variance 100. What is the probability that there is at most 1 claim?

(a) 0.0016
(b) 0.0024
(c) 0.0032
(d) 0.0048
(e) None of the above.

Solution: (a)
According to our tables, with $N \sim NegBinomial(r, \beta)$, the mean and the variance are

$$\mathbb{E}[N] = r\beta = 20 \quad \text{and} \quad Var[N] = r\beta(1+\beta) = 100.$$

Hence, $1 + \beta = 5$, so that $\beta = 4$ and $r = 5$.
We are looking for

$$F_N(1) = p_N(0) + p_N(1) = \frac{1}{5^5} + \frac{5 \cdot 4^1}{5^6} = 0.0016.$$

Problem 2.6. (5 points) Let the number of typos in the new edition of the *"Lord of the Rings"* trilogy be denoted by N and modeled using the Poisson distribution with mean 80. Some typos involve elves' names and others do not involve the elves' names. The number of typos and the type of the typo are assumed to be independent.

Assume that the probability that an observed typo involves an elf's name equals $1/5$.

Find the probability that the number of typos involving elves' names is 2, given that the **total** number of typos in the trilogy equals 5.

(a) 0.0406
(b) 0.0842
(c) 0.1632
(d) 0.2048
(e) None of the above.

Solution: (d)
Let N denote the total number of typos and let N_e denote the r.v. which stands for the number of typos involving elves' names. According to our extension of the *"Thinning" theorem*,

$$N_e \,|\, N = 5 \sim \mathit{Binomial}(m = 5, q = 1/5).$$

We have that

$$\mathbb{P}[N_e = 2 \,|\, N = 5] = \binom{5}{2}\left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)^3 = 10\left(\frac{4^3}{5^5}\right) = 0.2048$$

Problem 2.7. (8 points) Let N be an $\mathbb{N}_0-$valued random variable from the $(a, b, 0)$ class. You are given that

$$p_0 = 0.049787, \quad p_1 = 0.149361, \quad \text{and} \quad p_2 = 0.224042.$$

Find p_6.

(a) 0.04
(b) 0.05
(c) 0.06
(d) 0.07
(e) None of the above.

Solution: (b)
From the given information, we conclude that

$$\frac{p_1}{p_0} = 3 = a + \frac{b}{1} = a + b, \quad \text{and} \quad \frac{p_2}{p_1} = 1.5 = a + \frac{b}{2}\,.$$

So, $b = 3$ and $a = 0$. We conclude that the distribution of N is Poisson with parameter $\lambda = 3$. Finally,

$$p_6 = e^{-3} \times \frac{3^6}{6!} = 0.050409.$$

2.4. **Free-response problems.** Please, explain carefully all your statements and assumptions. Numerical results or single-word answers without an explanation (even if they're correct) are worth 0 points.

Problem 2.8. Let the number of losses be negative binomial with mean 6 and variance 24. The losses have the Pareto distribution with $\theta = 100$ and $\alpha = 6$. The number of losses and loss amounts are assumed to be independent. There is an ordinary deductible of 25 on individual losses. What is the variance of the number of claims?

Solution: We are given that $N^L \sim NegBinomial(r, \beta)$. We are also given that $r\beta = 6$ and $r\beta(1+\beta) = 24$. Whence, we obtain that $r = 2$ and $\beta = 3$. As we derived in class, the number of claims will also be negative binomial with parameters $r^P = r$ and $\beta^P = \beta v$ where v denotes the probability that an individual loss meets the deductible. For the Pareto distribution with the given parameter values, we have

$$\mathbb{P}[X > 25] = \left(\frac{100}{100+25}\right)^6 = (0.8)^6 = 0.262144.$$

So, the variance of the number of claims is

$$2(3(0.262144))(1 + 3(0.262144)) = 2.809815.$$

Problem 2.9. Let the independent random variables X_1, X_2 and X_3 all have the following probability mass function:

$$p_{X_1}(-1) = 1/4, \quad p_{X_1}(0) = 1/2, \quad p_{X_1}(1) = 1/4.$$

Let $X = 1 + X_1X_2X_3$. What is the probability generating function of X?

Solution: The random variable X has the following probability mass function:

$$p_X(0) = \frac{1}{16}, \quad p_X(1) = \frac{7}{8}, \quad p_X(2) = \frac{1}{16}.$$

So, the probability generating fucntion of X equals

$$P_X(s) = \mathbb{E}[s^X] = s^0\left(\frac{1}{16}\right) + s^1\left(\frac{7}{8}\right) + s^2\left(\frac{1}{16}\right) = \frac{1}{16} + \frac{7}{8}s + \frac{1}{16}s^2.$$

Problem 2.10. Let the random variable X have the moment generating function given by

$$M_X(t) = (1-2t)^{-3}$$

Define the random variable $\tilde{X}$ as $\tilde{X} = 5X + 1$. What is the moment generating function $M_{\tilde{X}}$ of $\tilde{X}$?

Solution: In general, for $\tilde{X} = aX + b$, we have that

$$M_{\tilde{X}}(t) = \mathbb{E}[e^{t\tilde{X}}] = \mathbb{E}[e^{t(aX+b)}] = e^{tb}\mathbb{E}[e^{atX}] = e^{tb}M_X(at).$$

So, in our problem, we have that

$$M_{\tilde{X}}(t) = e^{tb}M_X(at) = e^{tb}(1-2(at))^{-3} = e^t(1-10t)^{-3} = \frac{e^t}{(1-10t)^3}\,.$$

Problem 2.11. *Source: "Probability" by Jim Pitman.*

Consider the average

$$\bar{X}_n = \frac{1}{n}(X_1 + X_2 + \cdots + X_n)$$

of n independent random variables, with each $X_i, i \in \mathbb{N}$ uniformly distributed over $[0,1]$. Find the n such that $\mathbb{P}[\bar{X}_n < 0.51]$ is approximately 90%.

Solution: By the Central Limit Theorem, $\bar{X}_n$ is approximately normal with mean $\frac{1}{2}$ and variance $\frac{1}{12n}$. Hence, we need to figure out an n such that

$$0.90 \approx \mathbb{P}[\bar{X}_n < 0.51] \approx \Phi\left(\frac{0.51 - 0.5}{\sqrt{\frac{1}{12n}}}\right).$$

We can use 'R' to get

$$0.01\sqrt{12n} \approx \Phi^{-1}(0.90) \approx 1.281552.$$

So,

$$\sqrt{12n} \approx 128.1552 \quad \Leftrightarrow \quad 12n \approx (128.1552)^2 \quad \Leftrightarrow \quad n \approx 1369.$$

Problem 2.12. (10 points) In the compound model for aggregate claims, let the frequency random variable N have the Poisson distribution with mean 20.

Moreover, let the common distribution of the i.i.d. severity random variables $\{X_j; j = 1,2,\dots\}$ be the Gamma distribution with parameters $\alpha = 2.5$ and $\theta = 3,000$.

Let our usual assumptions hold, i.e., let N be independent of $\{X_j; j = 1,2,\dots\}$.
Define the aggregate loss as $S = \sum_{j=1}^{N} X_j$.

Using the normal approximation, find the amount of the premium π such that the total aggregate losses S exceed the premium with the probability of at most 1%.

Solution: Let $\mu_S = \mathbb{E}[S]$ and $\sigma_S = \sqrt{Var[S]}$. Then, using the normal approximation, we have

$$0.01 = \mathbb{P}[S > \pi] = \mathbb{P}\left[\frac{S-\mu_S}{\sigma_S} > \frac{\pi - \mu_S}{\sigma_S}\right] \approx 1 - \Phi\left(\frac{\pi-\mu_S}{\sigma_S}\right)$$

where Φ denotes the c.d.f. of the standard normal distribution. From the tables for Φ, we get

$$\pi = \mu_S + 2.33\sigma_S.$$

From the given information on the severity r.v.s, we obtain

$$\begin{aligned}\mathbb{E}[X] &= \theta\alpha = 7,500\\ \mathbb{E}[X^2] &= \theta^2\alpha(\alpha+1) = 3000^2 \cdot 2.5 \cdot 3.5 = 7.875 \cdot 10^7,\\ \mathbb{E}[N] &= Var[N] = 20.\end{aligned}$$

So,

$$\begin{aligned}\mu_S &= \mathbb{E}[S] = \mathbb{E}[X]\mathbb{E}[N] = 20 \cdot 7500 = 150,000,\\ \sigma_S^2 &= Var[S] = Var[X]\mathbb{E}[N] + Var[N]\mathbb{E}[X]^2 = \lambda \cdot \mathbb{E}[X^2] = 20 \cdot 7.875 \cdot 10^7 = 15.75 \cdot 10^8.\end{aligned}$$

Hence,

$$\pi = 150000 + 2.33 \cdot 3.96863 \cdot 10^4 \approx 242469.$$