M358K : September 29th, 2023.

UNIVERSITY OF TEXAS AT AUSTIN

Problem Set # 5

Normal distribution.

Problem 5.1. Let Z be a standard normal random variable. Find the following probabilities:

i. $\mathbb{P}[-1.33 < Z \leq 0.24]$
ii. $\mathbb{P}[0.49 < |Z|]$
iii. $\mathbb{P}[Z^4 < 0.0256]$
iv. $\mathbb{P}[e^{2Z} < 2.25]$
v. $\mathbb{P}\left[\frac{1}{Z} < 2\right]$

i. $\mathbb{P}[-1.33 < Z \leq 0.24] = \mathbb{P}[Z \leq 0.24] - \mathbb{P}[Z \leq -1.33]$

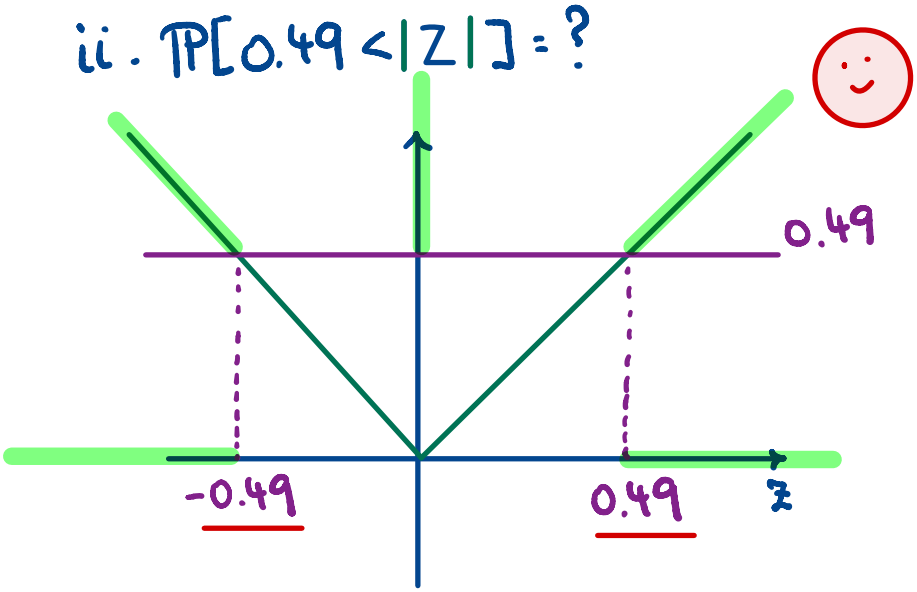
$= \Phi(0.24) - \Phi(-1.33)$

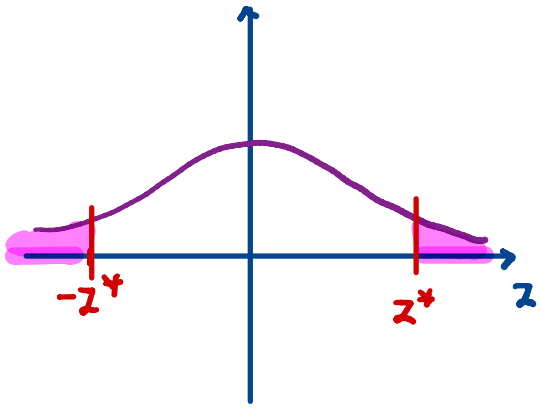
standard normal table

$= 0.5948 - 0.0918 = 0.5030$

In R: pnorm(0.24) - pnorm(-1.33) = 0.5030757

ii. $\mathbb{P}[0.49 < |Z|] = ?$





$\mathbb{P}[Z < -0.49] + \mathbb{P}[Z > 0.49] = 2 \cdot \mathbb{P}[Z < -0.49]$

$= 2 * \text{pnorm}(-0.49) = 0.6241339$

std normal tables: $2 \cdot 0.3121 = 0.6242$

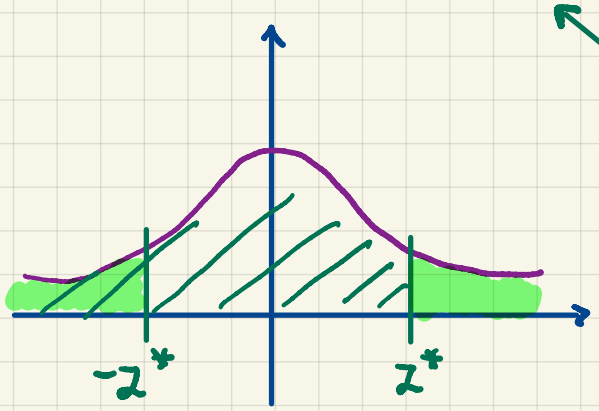
iii. $\mathbb{P}[Z^4 < 0.0256] = \mathbb{P}[|Z| < \sqrt[4]{0.0256} = 0.4]$

$= \mathbb{P}[-0.4 < Z < 0.4]$

$= \mathbb{P}[Z \leq 0.4] - \mathbb{P}[Z \leq -0.4]$

$\mathbb{P}[Z < -z^*] = \mathbb{P}[Z > z^*]$

$= 1 - \mathbb{P}[Z \leq z^*]$



$= \mathbb{P}[Z \leq 0.4] - (1 - \mathbb{P}[Z \leq 0.4])$

$= 2 \cdot \mathbb{P}[Z \leq 0.4] - 1$

$= 2 * \text{pnorm}(0.4) - 1 = 0.3108435$

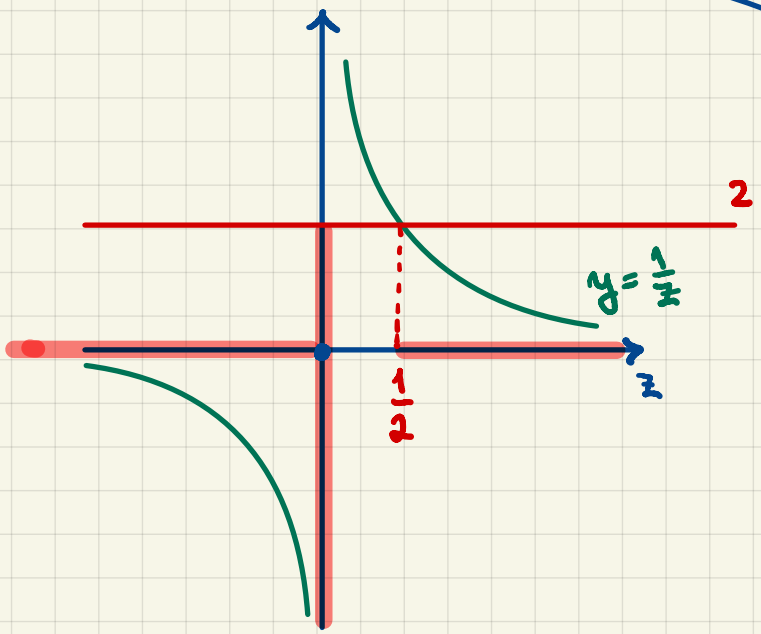
iv. $\mathbb{P}[e^{2Z} < 2.25] = \mathbb{P}[2Z < \ln(2.25)] = \mathbb{P}[Z < 0.5 \cdot \ln(2.25)]$

ln is strictly increasing

$= \text{pnorm}(0.5 * \log(2.25))$

$= 0.6574322$

v. $\mathbb{P}[\frac{1}{Z} < 2] = $ ✗



$\mathbb{P}[Z < 0] + \mathbb{P}[Z > \frac{1}{2}] =$

$= 0.5 + (1 - \mathbb{P}[Z \leq \frac{1}{2}])$

$= 1.5 - \mathbb{P}[Z \leq 0.5]$

$= 1.5 - \text{pnorm}(0.5)$

$= 0.8085375$

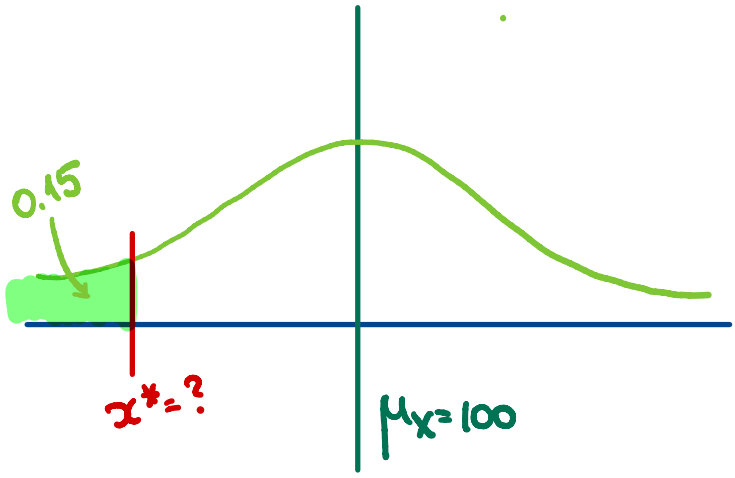
Problem 5.2. (10 points)

At the *Hogwarts School of Witchcraft and Wizardry* the *Ordinary Wizarding Level (OWL)* exam is typically taken at the end of the fifth year. Based on hystorical data, we model the *OWL* scores as roughly normal with mean 100 and standard deviation of 16.

$X \sim \text{Normal}(\text{mean} = 100, \text{sd} = 16)$

(a) (5 points)

What is the range of scores for the bottom 15% of the *OWL* takers?



$\text{qnorm}(0.15, \text{mean} = 100, \text{sd} = 16) = 83.41707$

OR

std normal tables: $z^*_{0.15} = -1.04$

$\downarrow$

$x^* = 100 + 16 \cdot (-1.04) = 83.36$

(b) (5 points)
What is the probability that a randomly chosen *OWL* taker has a score higher than 125?

The raw score of 125 corresponds to their score in standard units:

$$z = \frac{125 - 100}{16} = \frac{25}{16} = 1.5625$$

In the std normal tables: $\Phi(1.56) = 0.9406$

$\Rightarrow$ answer: $1 - 0.9406 = 0.0594$

OR

pnorm(25/16) = 0.05908512

OR

pnorm(125, mean=100, sd=16, lower.tail=FALSE)
= 0.05908512