Subjective Probability.

When pricing, we use the risk-neutral measure.

$$\underbrace{S(0) \begin{cases} \xrightarrow{p^*} S_u = u \cdot S(0) \\ \xrightarrow{1-p^*} S_d = d \cdot S(0) \end{cases}}_{h=T}$$

$$\boxed{p^* = \frac{e^{rh} - d}{u - d}}$$

Q: If we invest in one share of non-dividend-paying stock @ time·0, what is our expected wealth @ time·T, under the risk-neutral probability measure?

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$$\longrightarrow: \mathbb{E}^*[S(T)] = S_u \cdot p^* + S_d \cdot (1 - p^*)$$

$$= S_u \cdot \frac{e^{rh} - d}{u - d} + S_d \frac{u - e^{rh}}{u - d} = \cdots$$

$$\cdots \underset{h=T}{=} S(0)e^{rT}$$

In Contrast:

There can be a subjective probability measure $\mathbb{P}$.
We can think about the quality of our investment under that probability, e.g.,

$$\mathbb{E}[S(T)] = S(0)e^{\alpha \cdot T}.$$

We usually refer to α as the mean rate of return.

In a binomial tree, we can talk about the "true" probability of a step up

$$\boxed{p = \frac{e^{\alpha h} - d}{u - d}}$$

Moment Generating Functions.

For any random variable Y,
and for independent arguments denoted by t,
we define the **moment generating function (mgf)** of Y
as this function of t:

$$\boxed{M_Y(t) := \boxed{\mathbb{E}\left[e^{Y\cdot t}\right]}}$$

for all t such that the expectation exists, i.e., if it's finite

Note: • $M_Y(0) = \underline{1}$ $\Rightarrow$ @ least $t=0$ is in the domain of M_Y

Goal: To understand e^X w/ $X \sim Normal(mean = m, var = \nu^2)$

Recall. In terms of $Z \sim N(0,1)$,

$$\boxed{\nu = nu}$$

$$\frac{X-m}{\nu} = Z \iff \boxed{X = m + \nu \cdot Z} \checkmark$$

Fact. $\boxed{M_Z(t) = e^{\frac{t^2}{2}}} \checkmark$ for all $t \in \mathbb{R}$

$\Rightarrow$ For any normal X:

$$M_X(t) = \mathbb{E}\left[e^{X\cdot t}\right] = \mathbb{E}\left[e^{(m+\nu Z)t}\right] =$$

$$= \mathbb{E}\left[e^{m\cdot t} \cdot e^{\nu t Z}\right] = e^{mt}\,\mathbb{E}\left[e^{\nu t Z}\right]$$

$$= e^{mt} M_Z(\nu t)$$

$$= e^{mt}\, e^{\frac{\nu^2 t^2}{2}} = e^{mt + \frac{\nu^2 t^2}{2}}$$

The Log-Normal Distribution.

Def'n. Let $X \sim \text{Normal}(\text{mean} = m, \text{variance} = \nu^2)$.

Define $\boxed{Y = e^X}$

We say that Y is lognormally distributed.

$$\mathbb{E}[Y] = \mathbb{E}[e^X] = \mathbb{E}[e^{1 \cdot X}] = M_X(1) = e^{m + \frac{\nu^2}{2}}$$

Consider: $\mathbb{E}[X] = m$

Caveat: $\boxed{\mathbb{E}[e^X] \geq e^{\mathbb{E}[X]}}$

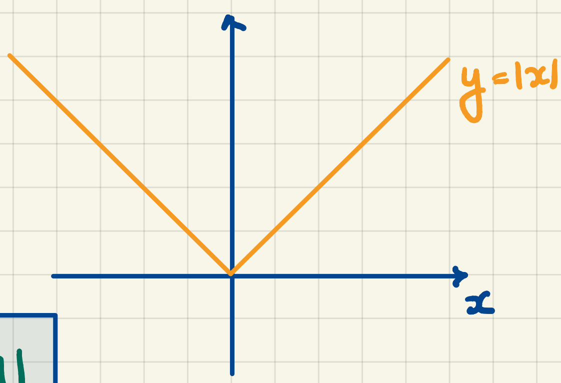
This is a special case of Jensen's Inequality.

Theorem. Let X be a random variable, and g be a convex function such that

$g(X)$ is well-defined and $\mathbb{E}[g(X)]$ exists.

Then, $\boxed{\mathbb{E}[g(X)] \geq g(\mathbb{E}[X])}$

Examples. i. $\boxed{g(x) = |x|}$



$$\boxed{\mathbb{E}[|X|] \geq |\mathbb{E}[X]|}$$

ii. Look @ a European put w/ strike K.

Its payoff f'tion: $v_p(s) = (K-s)_+$

The expected payoff is

$$\mathbb{E}[v_p(S(T))] = \boxed{\mathbb{E}[(K-S(T))_+]}$$

$$\geq$$

By Jensen, its lower bound is $(K - \mathbb{E}[S(T)])_+$

iii. In classical insurance:

$$\begin{cases} X \ldots \text{(ground-up) loss, i.e., the severity r.v.} \\ d \ldots \text{deductible} \end{cases}$$