

## M378K Introduction to Mathematical Statistics

### Problem Set #3

#### Continuous distributions.

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**Problem 3.1.** Source: Sample P exam, Problem #33.

*The lifetime of a machine part has a continuous distribution on the interval  $(0, 40)$  with probability density function  $f_X$ , where*

$$f_X(x) \propto \frac{1}{(10+x)^2}$$

*on the interval.*

*Calculate the probability that the lifetime of the machine part is less than 6.*

**Solution:** Using the indicator function, we can write our probability density function as

$$f_X(x) = \kappa(10+x)^{-2}\mathbf{1}_{(0,40)}(x)$$

where  $\kappa$  is the proportionality constant. Evidently, we need to figure out  $\kappa$  first. To accomplish this, we use the fact that the area under the density must equal 1, i.e.,

$$\kappa \int_0^{40} (10+x)^{-2} dx = 1.$$

We have

$$\int_0^{40} (10+x)^{-2} dx = \frac{(10+x)^{-1}}{-1} \Big|_{x=0}^{40} = \frac{1}{10} - \frac{1}{50} = \frac{5-1}{50} = \frac{2}{25}.$$

So,  $\kappa = \frac{25}{2}$ . The probability we seek is

$$\frac{25}{2} \int_0^6 (10+x)^{-2} dx = \frac{25}{2} \left( \frac{1}{10} - \frac{1}{16} \right) = \frac{25}{2} \cdot \frac{8-5}{80} = \frac{15}{32}.$$

**Problem 3.2.** Source: Sample P exam, Problem #419.

*A customer purchases a lawnmower with a two-year warranty. The number of years before the lawnmower needs a repair is uniformly distributed on  $[0, 5]$ . Calculate the probability that the lawnmower needs no repairs within 4.5 years after the purchase, given that the lawnmower needs no repairs within the warranty period*

**Solution:** Let the lifetime of the lawnmower be  $T \sim U(0, 5)$ . We have

$$\mathbb{P}[T > 4.5 \mid T > 2] = \frac{\mathbb{P}[T > 4.5, T > 2]}{\mathbb{P}[T > 2]} = \frac{\mathbb{P}[T > 4.5]}{\mathbb{P}[T > 2]} = \frac{5-4.5}{5-2} = \frac{1}{6}.$$

**Problem 3.3.** Consider a continuous random variable  $Y$  whose probability density function is given by

$$f_Y(y) = 2y\mathbf{1}_{[0,1]}(y)$$

What is the expected value of this random variable?

**Solution:** Straight from the definition, we get

$$\mathbb{E}[Y] = \int_0^1 y(2y) dy = 2 \int_0^1 y^2 dy = 2 \left( \frac{1^3}{3} - \frac{0^3}{3} \right) = \frac{2}{3}.$$