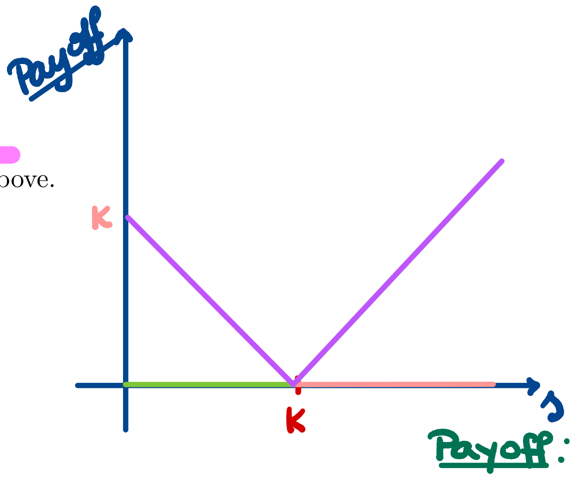
M339D: October 25th, 2024.

Problem 9.5. Consider a non-dividend paying stock whose current price is \$95 per share. You model the evolution of this stock price over the following year using a one-period binomial tree under the assumption that the stock price can be either $\boxed{\$120}$ or $\boxed{\$75}$ in one year.

The continuously compounded risk-free interest rate is 0.06.

A **straddle** consists of a $\boxed{\text{long call}}$ and a long otherwise identical $\boxed{\text{put}}$. Consider a \$100-strike, one-year European **straddle** on the above stock. What is the straddle's price consistent with the above stock-price model?

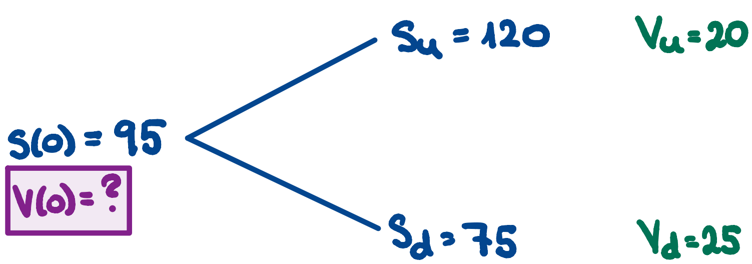
(a) About \$10
(b) About \$10.83
(c) About \$15.45
(d) About \$20.84
(e) None of the above.



Payoff f'tion of a straddle:

$$\boxed{v(s) = |s-K|}$$

In this problem $\boxed{K=100}$



$$\Delta = \frac{V_u - V_d}{S_u - S_d} = \frac{20-25}{120-75} = -\frac{5}{45} = -\frac{1}{9}$$

$$B = e^{-rh} \cdot \frac{u \cdot V_d - d \cdot V_u}{u-d} =$$

$$= e^{-0.06} \cdot \frac{\frac{120}{\cancel{95}} \cdot 25 - \frac{75}{\cancel{95}} \cdot 20}{\frac{120}{\cancel{95}} - \frac{75}{\cancel{95}}} = e^{-0.06} \cdot \frac{120(25) - 75(20)}{45}$$

$$= \underline{31.392}$$

$$V(0) = \Delta \cdot S(0) + B = -\frac{1}{9}(95) + 31.392 = \underline{20.84} \quad \square$$

Risk·Neutral Pricing.

Start w/ $V(0) = \Delta \cdot S(0) + B$

$$V(0) = \frac{V_u - V_d}{\underbrace{S_u - S_d}_{\cancel{S(0)}(u-d)}} \cdot \cancel{S(0)} + e^{-rh} \cdot \frac{u \cdot V_d - d \cdot V_u}{u-d}$$

$$V(0) = \frac{1}{u-d}\left[(V_u - V_d) + e^{-rh}(u V_d - d \cdot V_u)\right]$$

$$V(0) = e^{-rh} \cdot \frac{1}{u-d}\left[e^{rh} \cdot V_u - e^{rh} \cdot V_d + u \cdot V_d - d \cdot V_u\right]$$

$$V(0) = e^{-rh} \frac{1}{u-d}\left[V_u(e^{rh} - d) + V_d(u - e^{rh})\right]$$

$$V(0) = e^{-rh}\left[V_u \cdot \underbrace{\boxed{\frac{e^{rh}-d}{u-d}}}_{=: p^*} + V_d \cdot \underbrace{\boxed{\frac{u-e^{rh}}{u-d}}}_{= 1-p^*}\right]$$

Both Positive (due to the no·arbitrage condition!)

Add up to 1!

We choose to interpret the two fractions as probabilities!

We define the risk·neutral probability of the stock price going up in a single period as

$$\boxed{p^* := \frac{e^{rh}-d}{u-d}}$$

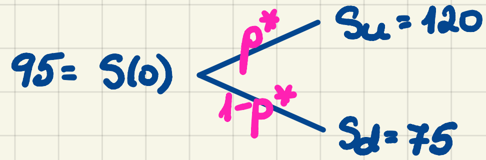
$\Rightarrow$ The risk·neutral pricing formula:

$$\boxed{V(0) = \underbrace{e^{-rT}}_{\text{discounting}} \underbrace{\left[V_u \cdot p^* + V_d \cdot (1-p^*)\right]}_{\text{expected payoff under the risk·neutral probability}}}$$

We can generalize this principle:

$$\boxed{V(0) = e^{-rT} \mathbb{E}^*[V(T)]}$$

Problem 9.5. revisited:



$V_u = 20$

$V_d = 25$

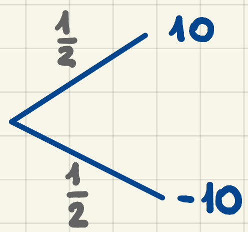
w/ $p^* = \dfrac{e^{rh} - d}{u - d} = \dfrac{S(0)e^{rh} - S_d}{S_u - S_d} = \dfrac{95e^{0.06} - 75}{120 - 75} = \underline{0.5749}$

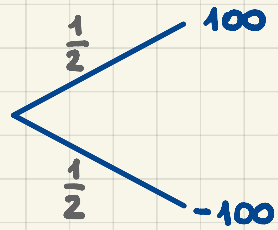
$$V(0) = e^{-rT}\left[V_u \cdot p^* + V_d \cdot (1 - p^*)\right]$$
$$= e^{-0.06}\left[20 \cdot p^* + 25 \cdot (1 - p^*)\right] = \underline{20.84}$$

$\square$

Q: Why "risk-neutral"?

Imagine bets:



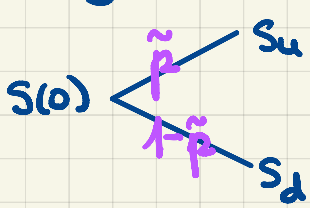


Consider a risk-neutral investor. They are indifferent to risk and only care about the expectation.

(*) What is the probability $\tilde{p}$ such that, for a specific stock-price tree, this investor is indifferent between investing in the stock and the risk-free investment? (*)

Say, they start w/ S(0). If they invest @ the ccrfir r, then, their balance @ time h will be $\underline{S(0)e^{rh}}$.

If they invest in the stock:



$$\mathbb{E}[\text{Wealth}] = \mathbb{E}[S(h)]$$
$$= \tilde{p} \cdot S_u + (1 - \tilde{p}) S_d =$$
$$= \underline{\tilde{p} \cdot S(0) \cdot u + (1 - \tilde{p}) \cdot S(0) \cdot d}$$

$\tilde{p} = ?$ w/ $\cancel{S(0)} e^{rh} = \tilde{p} \cdot \cancel{S(0)} \cdot u + (1-\tilde{p}) \cdot \cancel{S(0)} \cdot d$

$$\tilde{p} \cdot u + (1-\tilde{p}) d = e^{rh}$$

$$\tilde{p}(u-d) = e^{rh} - d$$

$$\boxed{\tilde{p} = \frac{e^{rh} - d}{u-d} = p^*}$$