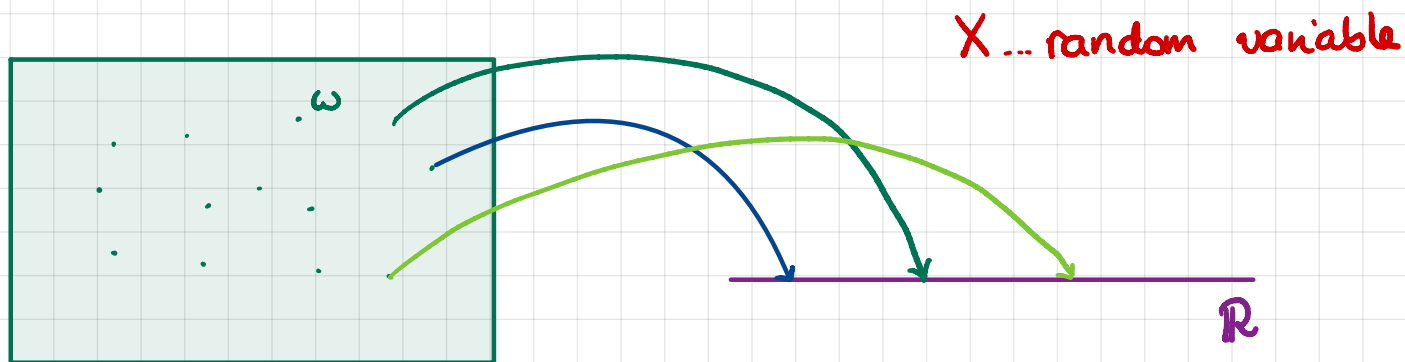


M3397: January 11th, 2023.

Probability Review.



Probability space Ω

(aka outcome space, sample space \mathcal{S})

$\omega \dots$ elementary outcomes

$E \dots$ event, i.e., a "nice" subset of the probability space

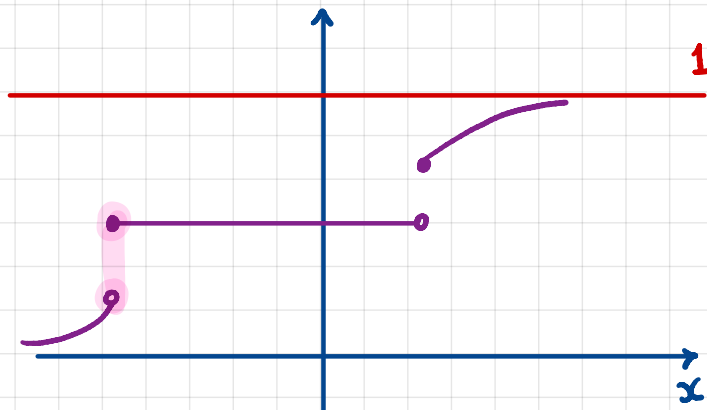
Def'n. The **cumulative distribution function (cdf)** of a random variable X is a function F_X such that

$$F_X: \mathbb{R} \rightarrow [0, 1]$$

given by

$$F_X(x) = \mathbb{P}[X \leq x] \quad \text{for all } x \in \mathbb{R}$$

Example.



Properties:

- Non-decreasing, i.e.,

$$x_1 < x_2 \Rightarrow F_X(x_1) \leq F_X(x_2)$$

- $\lim_{x \rightarrow -\infty} F_X(x) = \underline{0}$
- $\lim_{x \rightarrow +\infty} F_X(x) = \underline{1}$
- Right continuous w/ left limits

Def'n. The **survival function** of a random variable X is the function $S_X: \mathbb{R} \rightarrow [0, 1]$

given by $S_X(x) = 1 - F_X(x) = \mathbb{P}[X > x]$ for all $x \in \mathbb{R}$

Def'n. The **support** of a random variable X is the set of all the values it can take.

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Problem set 1The cumulative distribution function.**Problem 1.1.** The random variable X has the following cumulative distribution function:

$$F_X(x) = \begin{cases} \zeta & \text{for } x < 0 \\ \frac{1}{2} & \text{for } 0 \leq x < 1 \\ \kappa x + \nu & \text{for } 1 \leq x < 3 \\ \eta & \text{for } x \geq 3 \end{cases} \quad \Rightarrow \quad \begin{matrix} \zeta = 0 \\ \eta = 1 \end{matrix}$$

The function F_X is continuous at 1 and 3. How much are η , κ and ν ? What is the probability that X is less than or equal to 2? What is the probability that X is equal to 1? What is the probability that X is equal to 0?

$$\left. \begin{array}{l} \mathcal{K}(1) + \nu = \frac{1}{2} \\ - \mathcal{K}(3) + \nu = 1 \end{array} \right\} -$$

$$\hline -2\mathcal{K} = -\frac{1}{2} \Rightarrow \mathcal{K} = \frac{1}{4} \Rightarrow \nu = \frac{1}{4}$$

$$\mathbb{P}[X \leq 2] = F_X(2) = \mathcal{K}(2) + \nu = \frac{1}{4}(2) + \frac{1}{4} = \frac{3}{4}$$

$$\mathbb{P}[X=1] = F_X(1) - F_X(1^-) = 0$$

$$\mathbb{P}[X=0] = F_X(0) - F_X(0^-) = \frac{1}{2} - 0 = \frac{1}{2} \quad \square$$

Problem 1.2. The random variable X has the following cumulative distribution function:

$$F_X(x) = x^3 \quad \text{for } x \in (0, 1)$$

and is defined in the obvious way outside of the interval $(0, 1)$. What is the probability that X exceeds $1/2$, given that it exceeds $1/4$?

→:

E an event, $\mathbb{P}[E] > 0$

The conditional probability:

F an event

$$\mathbb{P}[F | E] := \frac{\mathbb{P}[E \cap F]}{\mathbb{P}[E]}$$

$$\begin{aligned} \mathbb{P}\left[X > \frac{1}{2} \mid X > \frac{1}{4}\right] &= \frac{\mathbb{P}\left[X > \frac{1}{2}, X > \frac{1}{4}\right]}{\mathbb{P}\left[X > \frac{1}{4}\right]} \\ &= \frac{\mathbb{P}\left[X > \frac{1}{2}\right]}{\mathbb{P}\left[X > \frac{1}{4}\right]} = \frac{1 - \left(\frac{1}{2}\right)^3}{1 - \left(\frac{1}{4}\right)^3} = \frac{\frac{7}{8}}{\frac{63}{64}} = \frac{56}{63} \quad \square \end{aligned}$$