M339D: April 14^{th}, 2023.

Problem 12.2. (5 pts) Let the stochastic process $S = \{S(t); t \geq 0\}$ denote the stock price. The stock's rate of ~~appreciation~~ rate is 10% while its volatility is 0.30 Then,

(a) $Var[\ln(S(t))] = 0.3t$

(b) $Var[\ln(S(t))] = 0.09t^2$

(c) $Var[\ln(S(t))] = 0.09t$

(d) $Var[\ln(S(t))] = 0.09$

(e) None of the above.

→: In the Black-Scholes model:

$$S(t) = S(0)e^{(r-\frac{\sigma^2}{2})\cdot t + \sigma\sqrt{t}\cdot Z} \qquad Z \sim N(0,1)$$

deterministic

$$\ln(S(t)) = \boxed{\ln(S(0)) + (r-\frac{\sigma^2}{2})\cdot t} + \sigma\sqrt{t}\cdot Z$$

$$Var[\ln(S(t))] = Var[\sigma\sqrt{t}\cdot Z] = \sigma^2 \cdot t \underbrace{Var[Z]}_{=1} = \sigma^2 \cdot t \quad \square$$

Problem 12.3. (5 pts) Assume the Black-Scholes framework. Let the current price of a non-dividend-paying stock be equal to $S(0) = 95$ and let its volatility be equal to 0.35. Consider a European call on that stock with strike 100 and exercise date in 9 months. Let the risk-free continuously compounded interest rate be 6% per annum.

Denote the price of the call by $V_C(0)$. Then,

(a) $V_C(0) < \$5.20$

(b) $\$5.20 \leq V_C(0) < \$7.69$

(c) $\$7.69 \leq V_C(0) < \$9.04$

(d) $9.04 \leq V_C(0) < \$11.25$

(e) None of the above.

→ :

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)\cdot T\right]$$

$$d_1 = \frac{1}{0.35\sqrt{0.75}}\left[\ln\left(\frac{95}{100}\right) + \left(0.06 + \frac{(0.35)^2}{2}\right)\cdot 0.75\right]$$

$$d_1 = 0.13079 \approx 0.13 \qquad N(0.13) = \boxed{0.5517}$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.13079 - 0.35\sqrt{0.75} = -0.1723 \approx -0.17$$

$$N(-0.17) = \boxed{0.4325}$$

$$V_c(0) = S(0)\cdot N(d_1) - Ke^{-rT}\cdot N(d_2)$$

$$V_c(0) = 95\cdot 0.5517 - 100\,e^{-0.06(0.75)}\cdot 0.4325$$

$$V_c(0) = 11.06$$

Problem 12.4. Assume the Black-Scholes setting. Let $S(0) = \$63.75$, $\sigma = 0.20$, $r = 0.055$. The stock pays no dividend and the option expires in 50 days (simplify the number of days in a year to 360).

What is the price of a $\boxed{\$60\text{-strike}}$ European put?

(a) 0.66
(b) 0.55
(c) 0.44
(d) 0.37
(e) None of the above.

$$T = \frac{50}{360}$$

→ :

$$d_1 = \frac{1}{0.2\sqrt{\frac{50}{360}}}\left[\ln\left(\frac{63.75}{60}\right) + \left(0.055 + \frac{(0.2)^2}{2}\right)\cdot\frac{50}{360}\right]$$

$$d_1 = 0.9531189 \approx 0.95 \Rightarrow N(-0.95) = \boxed{0.1711}$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.8785833 \approx 0.88 \Rightarrow N(-0.88) = \boxed{0.1894}$$

$$V_P(0) = Ke^{-rT}\cdot N(-d_2) - S(0)\cdot N(-d_1)$$

$$V_P(0) = 60\,e^{-0.055\left(\frac{50}{360}\right)}\cdot 0.1894 - 63.75\cdot 0.1711$$

$$V_P(0) = 0.3699$$

□

8. Let $S(t)$ denote the price at time t of a stock that pays no dividends. The Black-Scholes framework holds. Consider a European call option with exercise date T, $T > 0$, and exercise price $S(0)e^{rT}$, where r is the continuously compounded risk-free interest rate.

$\boxed{K = S(0)e^{rT}}$

You are given:

(i) $S(0) = \$100$

(ii) $T = 10$

(iii) $\mathrm{Var}[\ln S(t)] = 0.4t, \quad t > 0.$ $\longrightarrow$ $\boxed{\sigma = \sqrt{0.4}}$

Determine the price of the call option.

(A) \$7.96

(B) \$24.82

(C) \$68.26

(D) \$95.44

(E) There is not enough information to solve the problem.

$$V_c(0) = S(0)N(d_1) - Ke^{-rT} \cdot N(d_2)$$

$$V_c(0) = S(0)N(d_1) - S(0)\cancel{e^{rT}} \cdot \cancel{e^{-rT}} \cdot N(d_2)$$

$$\boxed{V_c(0) = S(0)\left(N(d_1) - N(d_2)\right)}$$

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{\cancel{S(0)}}{\cancel{S(0)}e^{rT}}\right) + \left(r + \frac{\sigma^2}{2}\right) \cdot T\right]$$

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\cancel{-rT} + \cancel{rT} + \frac{\sigma^2}{2} \cdot T\right] = \frac{\sigma\sqrt{T}}{2}$$

$$d_2 = d_1 - \sigma\sqrt{T} = \frac{\sigma\sqrt{T}}{2} - \sigma\sqrt{T} = -\frac{\sigma\sqrt{T}}{2}$$

$$V_c(0) = S(0)\left(N\left(\frac{\sigma\sqrt{T}}{2}\right) - N\left(-\frac{\sigma\sqrt{T}}{2}\right)\right)$$

$$\boxed{V_c(0) = S(0)\left(2\cdot N\left(\frac{\sigma\sqrt{T}}{2}\right) - 1\right)}$$

$$V_c(0) = 100\left(2\cdot N\left(\underbrace{\frac{\sqrt{0.4}\cdot\sqrt{10}}{2}}_{1}\right) - 1\right)$$

$$V_c(0) = 100\left(2\cdot 0.8413 - 1\right) = \underline{68.26}$$

$\square$

Problem. Assume the Black-Scholes model.

For a European call option, the strike is $S(0)e^{rT}$ w/ T being the exercise date.

The price of a call option w/ one year to exercise is $0.6\cdot S(0)$.

Find the price of call option w/ three months to exercise in terms of $S(0)$.