Name:

M339D=M389D Introduction to Actuarial Financial Mathematics
University of Texas at Austin
In-Term One
Instructor: Milica Čudina

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The maximum number of points on this exam is 100.

Problem 1.1. (15 points) The primary ingredient in the production of a certain instrument is one ounce of platinum. The platinum will be bought to complete the final phase of production in exactly one quarter year. The prices of other ingredients and cumulative labor costs aggregate to \$400 per unit. Upon completion, each unit can be sold for \$2000.

The market price of platinum in three months is modeled as follows:

$$\text{Platinum price per ounce} \sim \begin{cases} \$1,433.00 & \text{with probability } 0.1 \\ \$1,533.00 & \text{with probability } 0.6 \\ \$1,633.00 & \text{with probability } 0.3 \end{cases}$$

The buyer hedges the price of platinum by buying a three-month call option with an exercise price of \$1,500 per ounce. The option costs \$78 per ounce now.

The continuously compounded risk-free interest rate is 5%.

Calculate the expected profit per unit produced of the **hedged** position.

Solution: If $S(T)$ denotes the price of one ounce of platinum in three months, then the profit of the hedged portfolio can be expressed as

$$2000 - 400 - \min(S(T), 1500) - 78e^{0.05/4} = 1521.02 - \min(S(T), 1500).$$

So, the expected profit equals

$$1521.02 - (1433 \times 0.1 + 1500 \times 0.9) = 27.72$$

Problem 1.2. (5 points)The probability mass function p_X of a discrete random variable X is given by

$$p_X(x) = \begin{cases} 1/4, & \text{for } x = 10 \\ 2/3, & \text{for } x = 20 \\ 1/12, & \text{for } x = 40 \end{cases}$$

Find $\mathbb{E}[\max(X - 15, 0)]$.

Solution: The random variable $\max(X - 15, 0)$ has the following distribution

$$\max(X - 15, 0) \sim \begin{cases} 0 & \text{with probability } 1/4 \\ 5 & \text{with probability } 2/3 \\ 25 & \text{with probability } 1/12 \end{cases}$$

So, its expectation is

$$\mathbb{E}[\max(X - 15, 0)] = 5\left(\frac{2}{3}\right) + 25\left(\frac{1}{12}\right) = \frac{10}{3} + \frac{25}{12} = \frac{40 + 25}{12} = \frac{65}{12}.$$

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Problem 1.3. (5 points) Here is some information about two forward contracts with delivery dates in one year:

	Current price of underlying	Forward price
Forward I	100	105
Forward II	90	92

Alfur enters a long position in Forward I and a short position in Forward II. It turns out that the final price of the underlying asset for Forward I equals \$102, while the final price of the underlying asset for Forward II equals \$89.

Let the continuously compounded, risk-free interest rate be 0.03.

What is Alfur's profit?

Solution: The initial cost of any forward contract is zero, so the profit and the payoff are equal. For the long Forward I, Alfur's payoff is

$$102 - 105 = -3.$$

For the short Forward II, Alfur's payoff is

$$92 - 89 = 3.$$

Alfur's overall profit is zero.

Problem 1.4. (5 points) The current spot price of corn is \$3.60 per bushel. There is a forward contract on corn for delivery in six months with the forward price of \$3.65. In order to hedge, farmer Brown shorts a 1000-bushel forward contract.

At the delivery date, it turns out that the spot price of corn is \$3.80.

You know that farmer Brown's total aggregate costs of production for 1000 bushels of corn equal \$3,450.

What is farmer Brown's profit?

Solution:

$$1000(3.65) - 3450 = 200$$

Problem 1.5. (5 points) The current price of zinc is \$2.74 kilograms. David needs to buy zinc in three months for the purposes of galvanization of a component his company is making. Per kilogram of zinc, the total revenue from the sale of the finished component will be \$10.23, while the total aggregate costs of non-zinc inputs equal \$5.12.

To hedge, David enters a forward contract for delivery of zinc in three months with the forward price equal to \$2.80.

The market price of zinc turns out to be \$2.78 in three months.

What is David's total profit (per kilo of zinc used)?

Solution:

$$10.23 - 5.12 - 2.80 = 2.31$$

Problem 1.6. (5 points) The **owner** of a call option has ...

(a) an obligation to sell the underlying asset at the strike price.

(b) a right, but **not** an obligation, to sell the underlying asset at the strike price.

(c) an obligation to buy the underlying asset at the strike price.
(d) a right, but **not** an obligation, to buy the underlying asset at the strike price.
(e) None of the above.

Explain your choice!

Solution: (d)

Problem 1.7. (5 points) Let the function f be given by

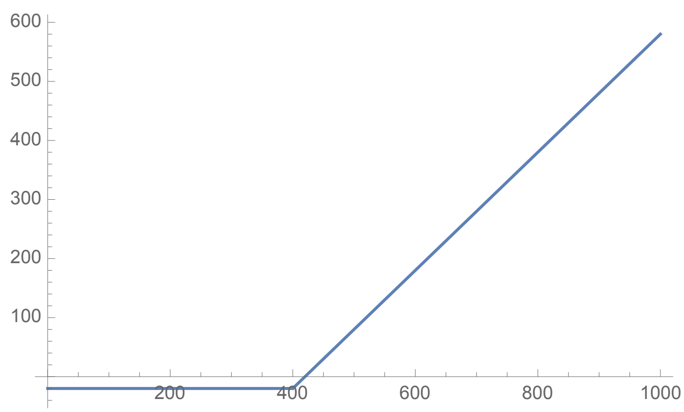
$$f(x) = \begin{cases} x - 400 & \text{for } x \geq 400 \\ 0 & \text{otherwise} \end{cases}$$

Draw the graph of the function g defined as

$$g(x) = f(x) - 20.$$

Carefully label your axes.

Solution:



Problem 1.8. (10 points) Let the function f be defined as

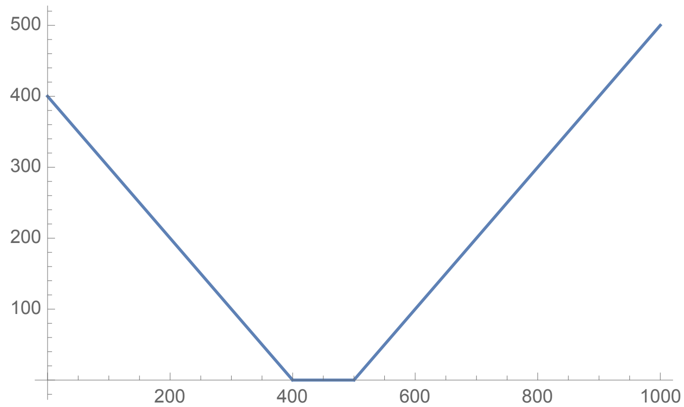
$$f(x) = \begin{cases} 400 - x & \text{for } x < 400 \\ 0 & \text{for } x \geq 400 \end{cases}$$

Let the function g be defined as

$$g(x) = \begin{cases} 0 & \text{for } x < 500 \\ x - 500 & \text{for } x \geq 500 \end{cases}$$

Draw the graph of the function $f + g$. Label your axes carefully.

Solution:



Problem 1.9. (5 points) Let the function $f : \mathbb{R}_+ \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} 10 - x, & \text{for } x < 12, \\ 0, & \text{for } x \geq 12. \end{cases}$$

Let the function $g : \mathbb{R}_+ \to \mathbb{R}$ be defined as

$$g(x) = \begin{cases} 2x, & \text{for } x < 20, \\ x - 20, & \text{for } x \geq 20. \end{cases}$$

Calculate $f(g(31))$.

Solution:

$$f(g(31)) = f(31 - 20) = f(11) = 10 - 11 = -1.$$

Problem 1.10. (5 points) Let the accumulation function be given by

$$a(t) = (1 + 0.03)^{t^2} (1 + 0.02)^{2t}.$$

What can you say about the continuously compounded, risk-free interest rate r associated with the above accumulation function?

Solution:

$$r = \frac{d}{dt}\ln(a(t)) = \frac{d}{dt}\ln[1.03^{t^2}1.02^{2t}] = \frac{d}{dt}\left[t^2\ln(1.03) + 2t\ln(1.02)\right] = 2t\ln(1.03) + 2\ln(1.02).$$

Problem 1.11. (10 points) Let the current price of a market index be \$80. Consider a European six-month, at-the-money call option on this market index.

We model the price of the market index in half a year as follows:

$$S(1/2) \sim \begin{cases} 78 & \text{with probability } 1/6 \\ 82 & \text{with probability } 1/2 \\ 84 & \text{with probability } 1/3 \end{cases}$$

What is the expected payoff of this call option?

Solution: Since the option is at-the-money, the strike price is \$80. We have

$$V_C(1/2) = (S(1/2) - 80)_+ \sim \begin{cases} 0 & \text{with probability } 1/6 \\ 2 & \text{with probability } 1/2 \\ 4 & \text{with probability } 1/3 \end{cases}$$

So,

$$\mathbb{E}[V_C(T)] = 2\left(\frac{1}{2}\right) + 4\left(\frac{1}{3}\right) = \frac{7}{3}.$$

Problem 1.12. (10 points) Let the current price of a non-dividend-paying stock be \$40. A market maker writes a \$38-strike, three-month call option on this stock. The option's price is \$2.72. The market-maker simultaneously buys one share of the underlying stock.

The continuously compounded, risk-free interest rate is 0.04.

For which final value of the stock price will the market maker break even?

Solution: The initial cost of the portfolio is $40 - 2.72 = 37.28$. This is a covered call, so the expression for the payoff is, in our usual notation,

$$-(S(T) - K)_+ + S(T) = \min(S(T), K).$$

In this problem, the payoff function for the portfolio is, therefore, $v(s) = min(s, 38)$. We need to solve for s in

$$\min(s, 38) - 37.28e^{0.04/4} = 0 \quad \Rightarrow \quad \min(s, 38) = 37.65467 \quad \Rightarrow \quad s = 37.65467.$$

Problem 1.13. (15 points) The following gamble is proposed:

First, you toss a fair coin. If the coin comes up *Heads*, then you roll a regular fair six-sided die. Your winnings are the number you get on the die. If the coin comes up *Tails*, they you toss a fair tetrahedron whose sides are numbered $\{1, 2, 3, 4\}$. Your winnings are the number displayed on the side of the tetrahedron which falls down.

(i) (5 points) Define a probability space Ω which is sufficient to formalize the above two-phase procedure.

(ii) (10 points) What are the expected winnings?

Solution:

(i)

$$\Omega = \{H\} \times \{1,2,3,4,5,6\} \cup \{T\} \times \{1,2,3,4\}$$

(ii)

$$\begin{aligned} 1\left(\frac{1}{2}\cdot\frac{1}{6}+\frac{1}{2}\cdot\frac{1}{4}\right)+2\left(\frac{1}{2}\cdot\frac{1}{6}+\frac{1}{2}\cdot\frac{1}{4}\right)+3\left(\frac{1}{2}\cdot\frac{1}{6}+\frac{1}{2}\cdot\frac{1}{4}\right)+4\left(\frac{1}{2}\cdot\frac{1}{6}+\frac{1}{2}\cdot\frac{1}{4}\right)& \\ +5\left(\frac{1}{2}\cdot\frac{1}{6}\right)+6\left(\frac{1}{2}\cdot\frac{1}{6}\right)& \\ =\frac{5}{24}+2\left(\frac{5}{24}\right)+3\left(\frac{5}{24}\right)+4\left(\frac{5}{24}\right)+5\left(\frac{1}{12}\right)+6\left(\frac{1}{12}\right)&=3 \end{aligned}$$