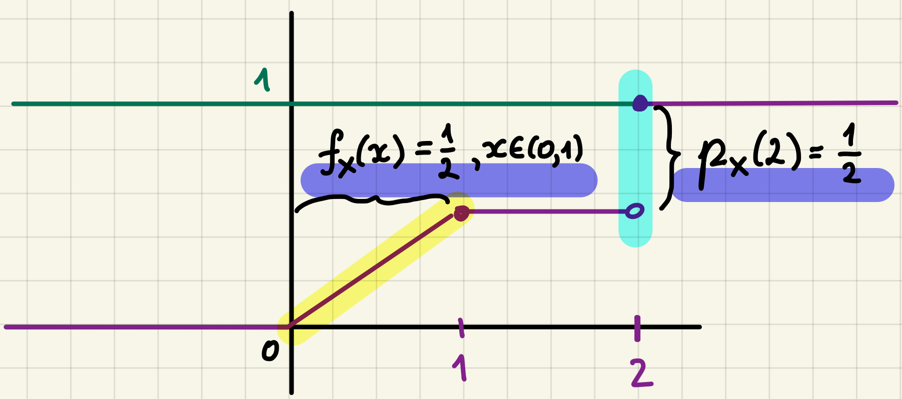
M339J: February 1st, 2021.

Expected Value

Def'n. The expected value or expectation or mean is given by:

- for discrete random variables as
$$\mathbb{E}[X] := \sum_x x \cdot p_X(x) \quad \text{if the sum exists}$$
- for continuous random variables as
$$\mathbb{E}[X] := \int_{-\infty}^{+\infty} x \cdot f_X(x)\,dx \quad \text{if the integral exists}$$
- for mixed random variable as
$$\mathbb{E}[X] = ?$$



This is a cdf a mixed r.v.

$$\boxed{\mathbb{E}[X] = \sum_x x \cdot p_X(x) + \int_{-\infty}^{\infty} x f_X(x)\,dx}$$

Problem. An insurance policy pays 100 per day for up to three days of hospitalization and 50 per day for each day thereafter.
The number of days N of hospitalization is a discrete r.v. with pmf :

$$p_N(k) = \frac{6-k}{15} \quad \text{for } k = 1, 2, 3, 4, 5$$
and 0 otherwise

Find the expected pmt per hospitalization under this policy.

$\longrightarrow$:

hospitalization length	probab.	pmt amount
1	1/3	100
2	4/15	200
3	1/5	300
4	2/15	350
5	1/15	400

The expected pmt per hospitalization:

$$\frac{1}{3} \times 100 + \frac{4}{15} \cdot 200 + \frac{1}{5} \cdot 300 + \frac{2}{15} \cdot 350 + \frac{1}{15} \cdot 400$$
$$= 220.$$

$\blacksquare$

Problem. Let X be a continuous r.v. w/ the pdf

$$f_X(x) = \begin{cases} \dfrac{p-1}{x^p}, & x > 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the value of p such that $\mathbb{E}[X] = 2$.

→: By def'n.

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx$$

$$= \int_{1}^{\infty} x\, (p-1) \cdot x^{-p}\, dx$$

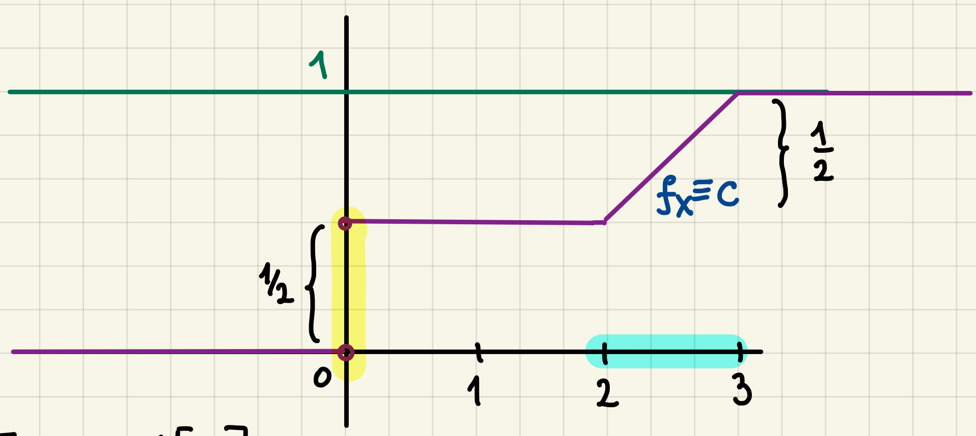
$$= (p-1) \int_{1}^{+\infty} x^{1-p}\, dx \qquad \text{for } p \neq 2$$

$$= (p-1) \cdot \frac{1}{2-p}\, x^{2-p} \Big|_{x=1}^{+\infty} \qquad \text{take } p > 2$$

$$= \frac{p-1}{2-p}(0-1) = \frac{p-1}{p-2} = 2$$

$$\Rightarrow \boxed{p = 3}$$

Problem. Consider this graph of a cdf of a r.v. X



Find $\mathbb{E}[X]$.

- jump @ zero, but this does not affect the expectation
- all the values $(-\infty,0)$, $(0,2)$, and $(3,\infty)$ are impossible
- between 2 and 3 the dist'n is uniform :

$$\int_2^3 c\,dx = \frac{1}{2} \quad \Rightarrow \quad c = \frac{1}{2}$$

$$\Rightarrow \quad \mathbb{E}[X] = \cancel{0 \cdot p_X(0)} + \int_2^3 x \cdot f_X(x)\,dx$$

$$= \int_2^3 x \cdot \frac{1}{2}\,dx = \frac{1}{2}\int_2^3 x\,dx =$$

$$= \frac{1}{2}\cdot\frac{1}{2}\cdot x^2\Big|_{x=2}^{3} = \frac{1}{4}\left(3^2-2^2\right) = \frac{1}{4}\cdot 5 = 1.25$$

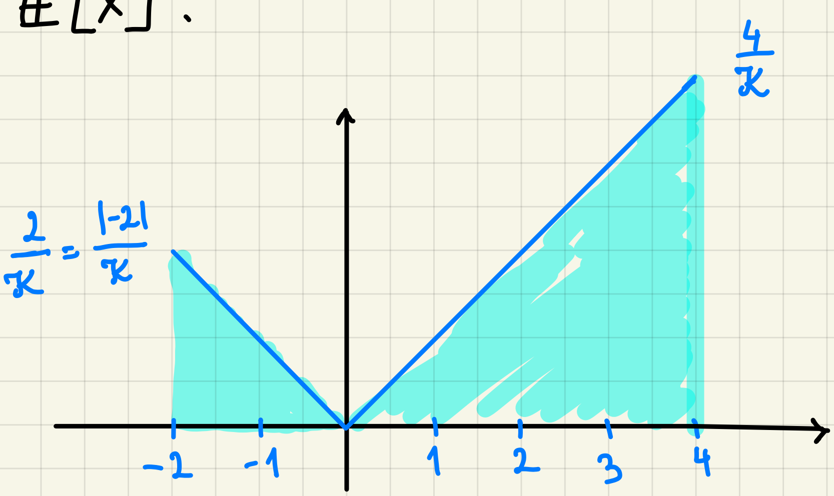
$\blacksquare$

Problem. Let the pdf of a continuous r.v. X be given by:

$$f_X(x) = \begin{cases} \dfrac{|x|}{K}, & -2 \le x \le 4 \\ 0, & \text{otherwise} \end{cases}$$

w/ K constant.

Find $\mathbb{E}[X]$.



$K = 10$

$\Uparrow$

$$\frac{\frac{2}{K}\cdot 2}{2} + \frac{\frac{4}{K}\cdot 4}{2} = 1 \Rightarrow \frac{2}{K} + \frac{8}{K} = 1$$