Q: $\mathbb{E}\left[(Y-a)^2\right] \xrightarrow[a]{} \min$

$\boxed{a^* = \mathbb{E}[Y]}$

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Maximum Likelihood Estimation.

Likelihood.

Def'n. Given a random sample $Y_1, Y_2, \ldots, Y_n$ from a discrete dist'n D w/ an unknown parameter θ, the likelihood f'tion is defined as

$$L(\theta; y_1, y_2, \ldots, y_n) = p^{\theta}_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n)$$

$$= p^{\theta}_{Y_1}(y_1) \cdot p^{\theta}_{Y_2}(y_2) \cdots p^{\theta}_{Y_n}(y_n)$$

$$= p^{\theta}(y_1) \cdot p^{\theta}(y_2) \cdots p^{\theta}(y_n)$$

where p^{θ} is the pmf of D.

Example. Bernoulli.

$Y_1, Y_2, \ldots, Y_n \sim B(p)$ $\boxed{p \longleftrightarrow \theta}$

pmf of $B(p)$: $p(y) = \begin{cases} p & \text{for } y=1 \\ 1-p & \text{for } y=0 \end{cases}$

$\boxed{p(y) = p^y (1-p)^{1-y}}$ for $y = 0, 1$

$$L(p; y_1, y_2, \ldots, y_n) = p^{y_1}(1-p)^{1-y_1}\, p^{y_2}(1-p)^{1-y_2} \cdots p^{y_n}(1-p)^{1-y_n}$$

$$= \prod_{i=1}^{n} p^{y_i} \cdot \prod_{i=1}^{n} (1-p)^{1-y_i}$$

$$= p^{\sum y_i} \cdot (1-p)^{n-\sum y_i}$$

For computational reasons, take the $\ln(\cdot)$ increasing and get the log-likelihood, i.e.,

$$\ell(p; y_1, \ldots, y_n) = \sum_{i=1}^{n} y_i \cdot \ln(p) + \left(n - \sum_{i=1}^{n} y_i\right) \ln(1-p) \xrightarrow[p]{} \max$$

Next, we differentiate with respect to p

$$\ell'(p; y_1, \dots, y_n) = \left(\sum_{i=1}^{n} y_i\right)\frac{1}{p} + \left(n - \sum_{i=1}^{n} y_i\right)(-1)\cdot\frac{1}{1-p}$$

We equate this derivative to zero and solve for p.

$$\left(\sum_i y_i\right)\cdot\frac{1}{p} - \left(n - \sum_i y_i\right)\frac{1}{1-p} = 0 \quad / \cdot p(1-p)$$

$$\left(\sum_i y_i\right)(1-p) - \left(n - \sum_i y_i\right)p = 0$$

$$\sum_i y_i - \cancel{p\cdot\sum_i y_i} - np + \cancel{p\cdot\sum_i y_i} = 0$$

$$p = \frac{\sum y_i}{n} = \bar{y}$$

$$\boxed{\hat{p}_{MLE} = \bar{Y}}$$

$\square$

<u>Def'n.</u> If $Y_1, \dots, Y_n$ come from a continuous dist'n D w/ pdf f^θ, then the likelihood f'tion is

$$L(\theta; y_1, \dots, y_n) = f^\theta_{Y_1, \dots, Y_n}(y_1, \dots, y_n) = f^\theta(y_1)\cdot f^\theta(y_2)\cdots f^\theta(y_n)$$

<u>Example</u>. <u>Exponential</u>.

$$Y_1, \dots, Y_n \sim E(\theta)$$

the pdf is $f^\theta(y) = \frac{1}{\theta} e^{-\frac{y}{\theta}} \quad y > 0$

$$\Rightarrow L(\theta; y_1, \dots, y_n) = \frac{1}{\theta} e^{-\frac{y_1}{\theta}} \cdot \frac{1}{\theta} \cdot e^{-\frac{y_2}{\theta}} \cdot \dots \cdot \frac{1}{\theta} e^{-\frac{y_n}{\theta}}$$

$$= \left(\frac{1}{\theta}\right)^n e^{-\frac{1}{\theta}\sum y_i}$$

$$\Rightarrow \ell(\theta; y_1, \dots, y_n) = -n\cdot\ln(\theta) + \left(-\frac{1}{\theta}\sum_{i=1}^{n} y_i\right)$$

$$\Rightarrow \ell'(\theta; y_1, \dots, y_n) = -n\cdot\frac{1}{\theta} + (+1)\cdot\frac{1}{\theta^2}\sum_{i=1}^{n} y_i = 0 \quad / \cdot \theta^2$$

$$-n\cdot\theta + \sum_{i=1}^{n} y_i = 0$$

$$\theta = \bar{y} \quad \Rightarrow \boxed{\hat{\theta}_{MLE} = \bar{Y}}$$

Example. **Normal**

$Y_1, \dots, Y_n \sim N(\mu, \sigma)$

the pdf: $f(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}$ for all $y \in \mathbb{R}$

$$L(\mu, \sigma; y_1, \dots, y_n) = \prod_{i=1}^{n} \left(\frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\frac{(y_i-\mu)^2}{\sigma^2}} \right)$$

unknown

$$= (2\pi)^{-\frac{n}{2}} \cdot \sigma^{-n} \cdot e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu)^2}$$

$$\propto e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu)^2}$$

$$\ell(\mu; y_1, \dots, y_n) = \text{const} - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu)^2$$

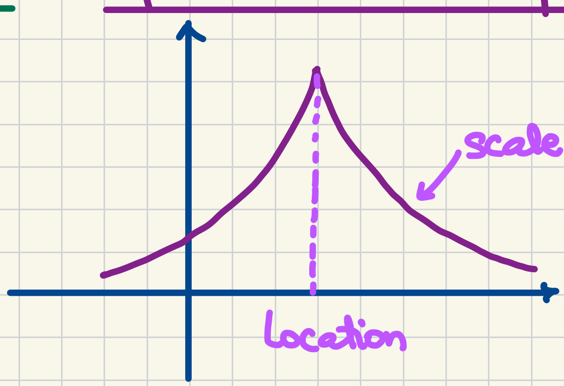
$$\ell'(\mu; y_1, \dots, y_n) = -\frac{1}{\cancel{2}\cancel{\sigma^2}}\sum_{i=1}^{n}\cancel{2}(y_i-\mu)(-1) = 0$$

$$\sum_{i=1}^{n}(y_i-\mu) = 0$$

$$\sum_{i=1}^{n} y_i - n\cdot\mu = 0$$

$$\boxed{\hat{\mu}_{MLE} = \bar{Y}}$$

Example. **Laplace / Double exponential.**



Look up on Wikipedia