M378K: September 12th, 2025.

Even more about Variance.

Def'n. If Y_1 and Y_2 are two random variables, then, we say they're *independent* if

$$\mathbb{P}[Y_1 \in B_1, Y_2 \in B_2] = \mathbb{P}[Y_1 \in B_1] \cdot \mathbb{P}[Y_2 \in B_2]$$

for any B_1, B_2 "$\subseteq$" $\mathbb{R}$

Theorem. If Y_1 and Y_2 are *independent*, then

$$\boxed{Var[Y_1 + Y_2] = Var[Y_1] + Var[Y_2]}$$

Example. Binomial

$Y \sim b(n,p)$

$\mathbb{E}[Y] = n \cdot p$

$Var[Y] = npq$

Idea 1. $Var[Y] = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2$

$$\mathbb{E}[Y^2] = \sum_{y=0}^{n} \left(y^2 \cdot \binom{n}{y} p^y \cdot (1-p)^{n-y} \right)$$

:(

Idea 2.

$$\boxed{\begin{aligned} \tilde{Y} &\sim \begin{cases} 1 & \text{w/ prob } p \\ 0 & \text{w/ prob } q \end{cases} \\ \tilde{Y}^2 &\sim \begin{cases} 1^2 & \text{w/ prob } p \\ 0^2 & \text{w/ prob } q \end{cases} \end{aligned}}$$

Bernoulli

$\tilde{Y} \sim B(p)$

$$Var[\tilde{Y}] = \mathbb{E}[(\tilde{Y})^2] - (\mathbb{E}[\tilde{Y}])^2$$

($\mathbb{E}[\tilde{Y}] = p$)

$$= \boxed{\mathbb{E}[\tilde{Y}]} - p^2$$

$$= p - p^2 = p(1-p) = pq$$

Introduce:

$$I_j \sim \begin{cases} 1 & \text{w/ prob } p \\ 0 & \text{w/ prob } q = 1-p \end{cases}$$

independent

$$Y = I_1 + I_2 + \dots + I_n$$

$$Var[Y] = Var[I_1 + I_2 + \dots + I_n] =$$

independence

$$= Var[I_1] + Var[I_2] + \dots + Var[I_n]$$

$$= npq$$

Geometric. $Y \sim g(p)$

$$\mathbb{E}[Y] = \frac{q}{p^2}$$

$$Var[Y] = \frac{q}{p^2} \implies SD[Y] = \frac{\sqrt{q}}{p}$$

Poisson. $Y \sim P(\lambda)$

$$\mathbb{E}[Y] = Var[Y] = \lambda$$

Problem 4.2. Source: Sample P exam, Problem #458.

An investor wants to purchase a total of ten units of two assets, A and B, with annual payoffs **per unit** *purchased of X and Y, respectively. Each asset has the same purchase price per unit. The payoffs are independent random variables with equal expected values and with $\mathrm{Var}(X) = 30$ and $\mathrm{Var}(Y) = 20$.*

Calculate the number of units of asset A the investor should purchase to minimize the variance of the total payoff.

→ : n... # of units of asset A that is bought

10−n ... # of units of B bought

$$\mathrm{Var}[n \cdot X + (10-n) \cdot Y] \xrightarrow[n]{} \min$$

indep.

$$n^2 \cdot \mathrm{Var}[X] + (10-n)^2 \cdot \mathrm{Var}[Y] \xrightarrow[n]{} \min$$

$$30n^2 + 20(10-n)^2 \longrightarrow \min$$

Idea#1.

$$30 \cdot 2n + 20 \cdot 2 \cdot (-1)(10-n) = 0$$

$$60n - 400 + 40n = 0$$

$$100n = 400$$

$$\boxed{n = 4}$$

Idea#2.

$$50n^2 - 400n + 2000 \longrightarrow \min$$

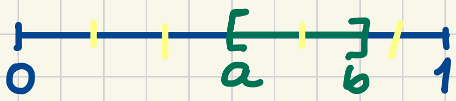
$$n^2 - 8n + 40 \longrightarrow \min$$

$$\boxed{n^* = -\frac{-8}{2 \cdot 1} = 4}$$

□

Continuous Distributions.

Example. The Uniform Distribution.



Imagine a r.v. Y on $[0,1]$.

The probability of Y landing between a and b where $0 \leq a \leq b \leq 1$ is

$$\mathbb{P}[a \leq Y \leq b] = \mathbb{P}[Y \in [a,b]] = b-a$$

Note: $\mathbb{P}[Y=y] = \mathbb{P}[y \leq Y \leq y] = y-y = 0$ for all $y \in [0,1]$

Def'n. A r.v. Y is said to be continuous if there exists a function

$$f_Y : \mathbb{R} \longrightarrow [0,\infty)$$

such that

$$\mathbb{P}[Y \in [a,b]] = \int_a^b f_Y(y)\,dy \quad \text{for all } a \leq b$$

The function f_Y is called the probability density function (pdf) of Y.

Properties.

- $f_Y(y) \geq 0$ for all $y \in \mathbb{R}$.
- $\int_{-\infty}^{\infty} f_Y(y)\,dy = 1$

Note:
- For a pmf p_Y, we have $p_Y(y) \leq 1$ for all $y \in S_Y$.
- For a pdf f_Y, it's possible to have $f_Y(y) > 1$ for some y.