M339D: February 16th, 2024.

Finite Probability Space.

... serve as environments for the possible paths that the asset price can take.

e.g.,

$$S(T) \sim \begin{cases} \underline{120} & \text{w/ probab. } 1/6 \\ 80 & \text{w/ probab. } 1/2 \\ 50 & \text{w/ probab. } 1/3 \end{cases}$$

Q: What is the expected payoff of a 105-strike put on S?

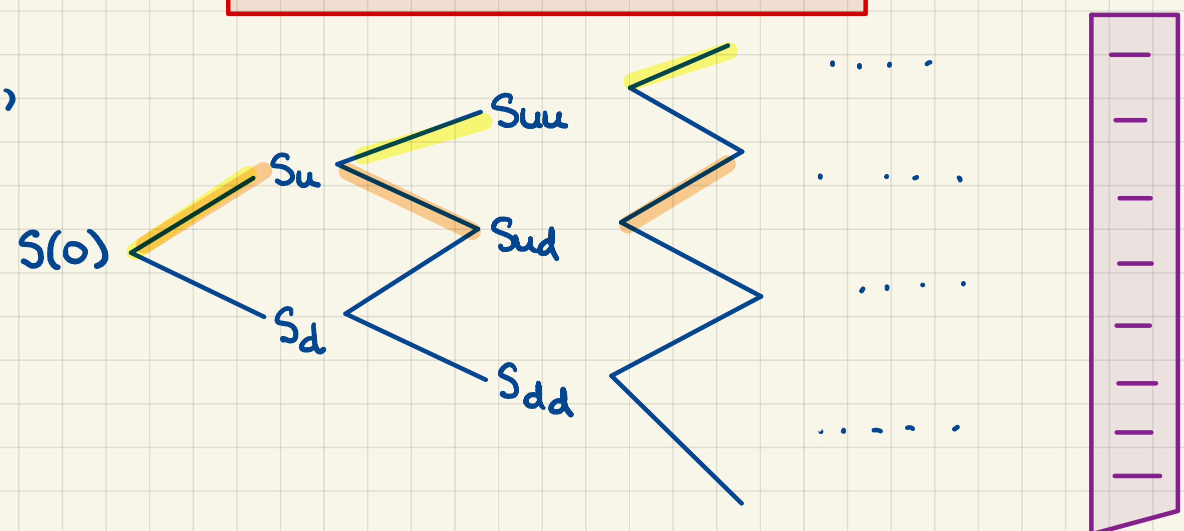
$\longrightarrow$:

$$V_P(T) \sim \begin{cases} 0 & \text{w/ probab. } 1/6 \\ 25 & \text{w/ probab. } 1/2 \\ 55 & \text{w/ probab. } 1/3 \end{cases}$$

$$\mathbb{E}[V_P(T)] = 25\left(\frac{1}{2}\right) + 55\left(\frac{1}{3}\right) = \frac{185}{6} \quad \square$$

In general: $\boxed{\mathbb{E}[g(X)] \neq g(\mathbb{E}[X])}$

e.g.,



All the finitely many scenarios are called states of the world.

We assume that:

- each can happen, i.e., it has $\boxed{\text{probab} > 0}$

and

- they exhaust all possibilities, i.e., $\boxed{\sum \text{probab} = 1}$

Arbitrage Portfolios.

Def'n. An arbitrage portfolio is a portfolio whose profit is:

- nonnegative in all states of the world

and

- strictly positive in at least one state of the world.

> Unless it's specified otherwise in a particular problem/example, we assume NO ARBITRAGE.

Law of the Unique Price.

Assume that the payoffs of two static portfolios A and B are equal, i.e.,

$$\boxed{V_A(T) = V_B(T)} \quad ✓$$

Claim.

$$\boxed{V_A(0) = V_B(0)}$$

Proof. Assume, to the contrary, that

$$V_A(0) \neq V_B(0)$$

Without loss of generality, say,

$$\boxed{\underbrace{V_A(0)}_{\text{relatively cheap}} < \underbrace{V_B(0)}_{\text{relatively expensive}}}$$

Diagnosis.

Propose an arbitrage portfolio:

- Long Portfolio A
- Short Portfolio B

} Total Portfolio

Verify:

- Payoff (Total Portfolio) = $V_A(T) - V_B(T) = 0$
- Initial Cost (Total Portfolio) = $V_A(0) - V_B(0) < 0$

Inflow of money @ time 0.

$$\text{Profit} = \text{Payoff} - FV_{0,T}(\text{Initial Cost})$$

$$= 0 - FV_{0,T}\big(\underbrace{V_A(0) - V_B(0)}_{<0}\big) > 0$$

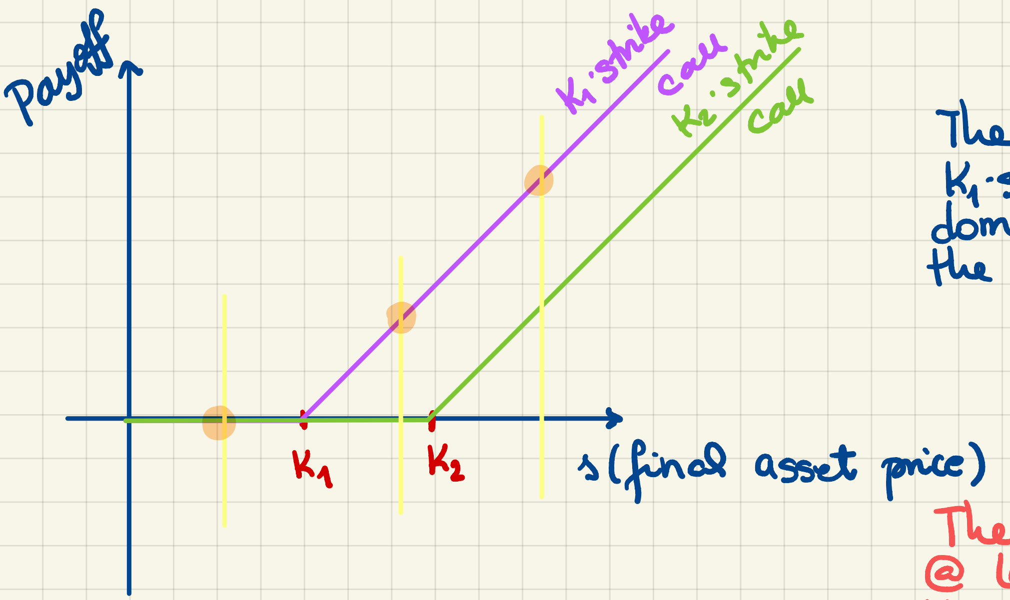
Indeed, this is an arbitrage portfolio!

$\Rightarrow\Leftarrow$ $\square$

Remark: If $V_A(T) \geq V_B(T)$, then $V_A(0) \geq V_B(0)$.

Example. $\boxed{K_1 < K_2}$

A: one $\boxed{\text{long } K_1\text{-strike call}}$
B: one $\boxed{\text{long } K_2\text{-strike call}}$ } w/ the same underlying asset and exercise date



The payoff of the K_1-strike call dominates the payoff of the K_2-strike call.

The K_1-strike call costs @ least as much as the K_2-strike call.

$\boxed{\text{In Math:}}$ $\boxed{K_1 < K_2 \Rightarrow V_C(0, K_1) \geq V_C(0, K_2)}$

As a function of the strike price, the call price is decreasing