M339W: March 10th, 2021.

Tail Probabilities.

Example. You are considering an investment in a continuous-dividend-paying stock and you are comparing it to a risk-free investment.
The dividend yield is δ.
The ccrfir is r.

Q: What is the probability that the stock outperforms the risk-free account @ time T?

$\rightarrow$: The invested amount: $S(0)$

- If it's a risk-free investment, the balance @ time T will be $S(0)e^{rT}$.
- If it's a stock investment, the number of shares owned @ time T will be $e^{\delta \cdot T}$.

$\Rightarrow$ The wealth will be $e^{\delta \cdot T} \cdot S(T)$.

We are interested in:

$$\mathbb{P}[e^{\delta \cdot T} \cdot S(T) > S(0)e^{rT}] = ?$$

This question is equivalent to asking if the profit of a purchase of stock is positive.

In the Black-Scholes model:

$$S(T) = S(0)e^{(\alpha - \delta - \frac{\sigma^2}{2}) \cdot T + \sigma\sqrt{T} \cdot Z}$$

w/ $Z \sim N(0,1)$

Recall: α ... (mean) rate of return.

The probability we're looking for is:

$$\mathbb{P}\left[\cancel{e^{\delta T}} \cdot \cancel{S(0)} e^{(\alpha - \cancel{\delta} - \frac{\sigma^2}{2})\cdot T + \sigma\sqrt{T}\cdot Z} > \cancel{S(0)} e^{rT}\right] =$$

$$= \mathbb{P}\left[e^{(\alpha - \frac{\sigma^2}{2})\cdot T + \sigma\sqrt{T}\cdot Z} > e^{rT}\right]$$

$$= \mathbb{P}\left[(\alpha - \frac{\sigma^2}{2})\cdot T + \sigma\sqrt{T}\cdot Z > rT\right]$$

$$= \mathbb{P}\left[\sigma\sqrt{T}\cdot Z > r\cdot T - (\alpha - \frac{\sigma^2}{2})\cdot T = (r - \alpha + \frac{\sigma^2}{2})\cdot T\right]$$

$$= \mathbb{P}\left[Z > \frac{1}{\sigma\cancel{\sqrt{T}}}(r - \alpha + \frac{\sigma^2}{2})\cdot \cancel{T}^{\sqrt{T}}\right]$$

$$= \mathbb{P}\left[Z > \frac{(r - \alpha + \frac{\sigma^2}{2})\cdot\sqrt{T}}{\sigma}\right]$$ symmetry of $N(0,1)$

$$= \mathbb{P}\left[Z < -\frac{(r - \alpha + \frac{\sigma^2}{2})\cdot\sqrt{T}}{\sigma} = \frac{(\alpha - r - \frac{\sigma^2}{2})\cdot\sqrt{T}}{\sigma}\right]$$

$$= N\left(\frac{(\alpha - r - \frac{\sigma^2}{2})\cdot\sqrt{T}}{\sigma}\right)$$

Q: What if we look @ this example under $\mathbb{P}^*$?

→: Under $\mathbb{P}^*$, we have $\alpha = r$.

So, we get

$$\mathbb{P}^*\left[e^{\delta\cdot T}\cdot S(T) > S(0)e^{r\cdot T}\right] = N\left(-\frac{\frac{\sigma^{\cancel{2}}}{2}\cdot\sqrt{T}}{\cancel{\sigma}}\right)$$

$$= N\left(-\frac{\sigma\sqrt{T}}{2}\right)$$

∎

Motivation: Given a particular exercise date T and a strike price K, what's the probability that a European call option will be exercised?

→: In our Black-Scholes model,

$$S(T) = S(0)\, e^{(\alpha - \delta - \frac{\sigma^2}{2})\cdot T + \sigma\sqrt{T}\cdot Z}$$ w/ $Z \sim N(0,1)$.

We need:

$$\mathbb{P}[S(T)>K] =$$

$$= \mathbb{P}\left[S(0)e^{(\alpha-\delta-\frac{\sigma^2}{2})\cdot T+\sigma\sqrt{T}\cdot Z} > K\right]$$

$$= \mathbb{P}\left[e^{(\alpha-\delta-\frac{\sigma^2}{2})\cdot T+\sigma\sqrt{T}\cdot Z} > \frac{K}{S(0)}\right]$$

$$= \mathbb{P}\left[(\alpha-\delta-\frac{\sigma^2}{2})\cdot T+\sigma\sqrt{T}\cdot Z > \ln\left(\frac{K}{S(0)}\right)\right]$$

$$= \mathbb{P}\left[\sigma\sqrt{T}\cdot Z > \ln\left(\frac{K}{S(0)}\right) - (\alpha-\delta-\frac{\sigma^2}{2})\cdot T\right]$$

$$= \mathbb{P}\left[Z > \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{K}{S(0)}\right) - (\alpha-\delta-\frac{\sigma^2}{2})\cdot T\right]\right]$$

symmetry of $N(0,1)$

$$= \mathbb{P}\left[Z < \underbrace{-\frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{K}{S(0)}\right) - (\alpha-\delta-\frac{\sigma^2}{2})\cdot T\right]}_{}\right]$$

$$\boxed{\frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + (\alpha-\delta-\frac{\sigma^2}{2})\cdot T\right]}$$

!! shorthand

$$\hat{d}_2$$

$$\boxed{\mathbb{P}[S(T)>K] = N(\hat{d}_2)}$$

Consequently: $\mathbb{P}[S(T)<K] = 1-\mathbb{P}[S(T)>K]$

$$= 1-N(\hat{d}_2) = N(-\hat{d}_2)$$

$$\boxed{\mathbb{P}[S(T)<K] = N(-\hat{d}_2)}$$