M339D: April 12th, 2023.

Black-Scholes: Partial Expectations.

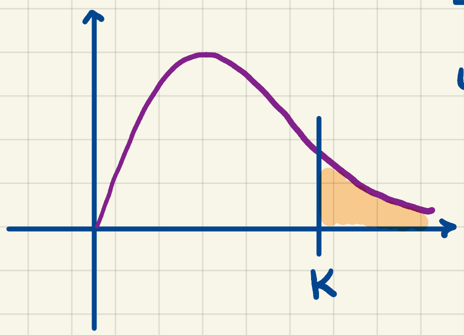
The Model.

Under the risk-neutral measure $\mathbb{P}^*$:

$$S(T) = S(0)e^{(r-\frac{\sigma^2}{2})\cdot T + \sigma\sqrt{T}\cdot Z} \quad w/ \quad Z \sim N(0,1)$$

Tail Probabilities.

$$\mathbb{P}[S(T) > K] = \boxed{N(d_2)}$$



w/ $\boxed{d_2 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)\cdot T\right]}$

Motivation.

Get a formula for the price of European call and put options on a stock modeled in the Black-Scholes framework.

Idea: $\boxed{\text{RISK-NEUTRAL PRICING}}$

$$V(0) = e^{-rT}\mathbb{E}^*[V(T)]$$

Payoff of a European Option

Implementation:

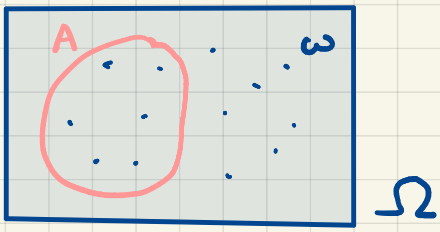
Temporarily, focus on a time-T, strike-K European call option.

The Payoff: $\boxed{V_c(T) = (S(T)-K)_+}$

Under $\mathbb{P}^*$:

$$\mathbb{E}^*[V_c(T)] = \mathbb{E}^*[(S(T)-K)_+]$$

$$= \mathbb{E}^*\left[(S(T)-K)\,\boxed{\mathbb{I}_{[S(T)\geq K]}}\right]$$



A is an event

$$\mathbb{I}_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}$$

$$\mathbb{I}_A = \begin{cases} 1 & \text{if } A \text{ happened} \\ 0 & \text{if } A \text{ did not happen} \end{cases}$$

$$\mathbb{E}[\mathbb{I}_A] = 1\cdot\mathbb{P}[A] + 0\cdot\mathbb{P}[A^c] = \mathbb{P}[A]$$

$$\mathbb{E}^*[V_c(T)] = \mathbb{E}^*\left[S(T)\cdot\mathbb{I}_{[S(T)\geq K]} - K\cdot\mathbb{I}_{[S(T)\geq K]}\right]$$

$$= \boxed{\mathbb{E}^*\left[S(T)\cdot\mathbb{I}_{[S(T)\geq K]}\right]} - \boxed{K\cdot\boxed{\mathbb{P}^*[S(T)\geq K]}}$$

$$\mathbb{P}^*[S(T)\geq K] = N(d_2)$$

The Partial Expectation from Title

$$\boxed{\mathbb{E}^*\left[S(T)\cdot\mathbb{I}_{\boxed{[S(T)\geq K]}}\right]} = ?$$

<u>Method</u>. Use the defining formula for the expectation of a function of a r.v.
In this case, that r.v. is $Z \sim N(0,1)$.

$$\{S(T)\geq K\} = \left\{S(0)e^{(r-\frac{\sigma^2}{2})\cdot T + \sigma\sqrt{T}\cdot Z} \geq K\right\}$$

$$= \{Z \geq -d_2\}$$

z... our dummy variable within the integral;
it corresponds to Z

i.e., $g(z) = S(0)e^{(r-\frac{\sigma^2}{2})\cdot T + \sigma\sqrt{T}\cdot z}$ (so that $g(Z) = S(T)$).

$$\mathbb{E}^*\left[S(T)\cdot \boxed{\mathbb{I}_{[S(T)\geq K]}}\right] = \mathbb{E}^*\left[g(Z)\cdot \mathbb{I}_{[Z\geq -d_2]}\right]$$

$$= \int_{-d_2}^{+\infty} g(z)\cdot f_Z(z)\,dz \qquad \text{(lots of algebra)}$$

$$= \boxed{S(0)e^{rT}}\cdot \boxed{N(d_1)}$$

where $d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(r+\frac{\sigma^2}{2}\right)\cdot T\right]$

The expectation under $\mathbb{P}^*$ of the call payoff:

$$\boxed{\mathbb{E}^*\left[V_c(T)\right] = S(0)e^{rT}\cdot N(d_1) - K\cdot N(d_2)}$$

w/ d_1 as above and $\boxed{d_2 = d_1 - \sigma\sqrt{T}}$

$\Rightarrow$ The Black-Scholes call price:

$$V_c(0) = e^{-rT}\,\mathbb{E}^*\left[V_c(T)\right]$$

$$V_c(0) = e^{-rT}\left(S(0)e^{rT}N(d_1) - K\cdot N(d_2)\right)$$

$$\boxed{V_c(0) = S(0)N(d_1) - Ke^{-rT}\cdot N(d_2)}$$

The Black-Scholes put price:

By put-call parity:

$$V_c(0) - \boxed{V_p(0)} = S(0) - Ke^{-rT}$$

$$V_p(0) = V_c(0) - S(0) + Ke^{-rT} = S(0)N(d_1) - Ke^{-rT}N(d_2) \quad -S(0) \quad + Ke^{-rT}$$

$$= S(0)(\underbrace{N(d_1)-1}_{-N(-d_1)}) + Ke^{-rT}(\underbrace{1-N(d_2)}_{N(-d_2)})$$

symmetry of $N(0,1)$

$$\boxed{V_p(0) = Ke^{-rT}N(-d_2) - S(0)N(-d_1)}$$

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Problem Set 12

Black-Scholes pricing.

Problem 12.1. Let the stock prices be modeled using the lognormal distribution. Under the risk-neutral probability measure, the mean stock price at time−1 equals 120 and the median stock price 115. What is the risk-neutral probability that the time−1 stock price exceeds 100?

→ :

$$\mathbb{P}^*[S(1) > 100] = ?$$

$$\mathbb{P}^*\Big[\underbrace{\boxed{S(0)e^{(r-\frac{\sigma^2}{2})\cdot 1}}}_{\text{median of } S(1)}+\sigma\sqrt{1}\cdot Z > 100\Big] = ?$$

$$\mathbb{P}^*[115\, e^{\sigma\cdot Z} > 100] = \mathbb{P}^*\left[\sigma\cdot Z > \ln\left(\frac{100}{115}\right)\right] = 0.6844$$

$$\frac{\text{mean}}{\text{median}} = \frac{\cancel{S(0)}\cancel{e^{rT}}}{\cancel{S(0)}e^{(\cancel{r}-\frac{\sigma^2}{2})\cdot T}} = e^{\frac{\sigma^2}{2}\cdot T}$$

$$\frac{120}{115} = e^{\frac{\sigma^2}{2}} \Rightarrow \cdots \Rightarrow \quad \sigma = 0.2918$$