04/02/2018.

47. Several months ago, an investor sold 100 units of a one-year European call option on a nondividend-paying stock. She immediately delta-hedged the commitment with shares of the stock, but has not ever re-balanced her portfolio. She now decides to close out all positions.

You are given the following information:

(i) The risk-free interest rate is constant.

(ii)

	Several months ago time 0	Now time t
Stock price	\$40.00	\$50.00
Call option price	\$ 8.88	\$14.42
Put option price	\$ 1.63	\$ 0.26
Call option delta	0.794	

The put option in the table above is a European option on the same stock and with the same strike price and expiration date as the call option.

Calculate her profit.

(A) \$11

(B) \$24

(C) \$126

(D) \$217

(E) \$240

Init. Cost: $-100 \cdot v_c(S(0), 0) + 100 \cdot \Delta_c(S(0),0) \cdot S(0)$

$= 100(-8.88 + 0.794 \cdot 40)$

$= 2{,}288.00$

Payoff: $-100 \cdot v_c(S(t), t) + 100 \cdot \Delta_c(S(0),0) \cdot S(t)$

$= 100(-14.42 + 0.794 \cdot 50)$

$= 2{,}528.00$

Profit: $2{,}528 - FV_{0,t}(2{,}288) =$

$= 2{,}528 - 2{,}288\, e^{r \cdot t}$

Put·call Parity :

$$8.88 - 1.63 = \underbrace{F^{P}_{0,T}(S)}_{=S(0)} - Ke^{-r\cdot T}$$

$$14.42 - 0.26 = \underbrace{F^{P}_{t,T}(S)}_{=S(t)} - Ke^{-r(T-t)}$$

$$Ke^{-rT} = 40 - 7.25 = 32.75$$

$$Ke^{-r(T-t)} = 50 - 14.16 = 35.84$$

$$\frac{\cancel{K}e^{-r(\cancel{T}-t)}}{\cancel{Ke^{-rT}}} = \frac{35.84}{32.75}$$

$$\boxed{e^{r\cdot t} = 1.0944}$$

$\Rightarrow$ Profit $= 2528 - 2288 \cdot 1.0944 \approx 24.$

$\Rightarrow$ (B)

Delta-hedger's profit over a "small" time interval

Set up:

- An agent writes an option @ time-0. They proceed to delta hedge the commitment. So, the wealth, i.e., the total value of the portfolio:

$$w(S(0),0) = -\underbrace{v(S(0),0)}_{\text{value of the option}} + \underbrace{\Delta(S(0),0)}_{\text{time-0 }\Delta\text{ of the option}} \cdot S(0)$$

- Look @ the value of this portfolio at time dt (before the agent had the chance to rebalance). The total (exact) wealth:

$$w(S(dt),dt) = -\boxed{v(S(dt),dt)} + \Delta(S(0),0)\cdot S(dt)$$

For a small time interval, we can use the Δ-Γ-Θ approximation for the price of the option:

$$\begin{aligned} v(S(dt),dt) \cong\ & v(S(0),0) + \Delta(S(0),0)\cdot dS \\ & + \frac{1}{2}\Gamma(S(0),0)\,(dS)^2 \\ & + \Theta(S(0),0)\,dt \end{aligned}$$

$\Rightarrow$ The approximate time-dt wealth is:

$$\begin{aligned} w(S(dt),dt) = -\Big(& v(S(0),0) + \cancel{\Delta(S(0),0)\,dS} + \frac{1}{2}\Gamma(S(0),0)\,(dS)^2 \\ & + \Theta(S(0),0)\,dt\Big) + \underline{\Delta(S(0),0)}\,\big(S(0) \cancel{+dS}\big) \end{aligned}$$

9. Consider the Black-Scholes framework. A market-maker, who delta-hedges, sells a three-month at-the-money European call option on a nondividend-paying stock.

You are given:

(i) The continuously compounded risk-free interest rate is 10%.

(ii) The current stock price is 50.

(iii) The current call option delta is 0.61791.

(iv) There are 365 days in the year.

If, after one day, the market-maker has zero profit or loss, determine the stock price move over the day.

(A) 0.41

(B) 0.52

(C) 0.63

(D) 0.75

(E) 1.11

$$\boxed{ds = \underline{S(0)} \cdot \sigma\sqrt{h}}$$

w/ $h = 1\text{day} = \frac{1}{365}$

$\sigma = ?$

Given $\Delta_c(S(0),0) = 0.61791$

(nodiv) 1 $\cancel{e^{-\delta(T-0)}}$

$\cdot N(d_1(S(0),0)) = 0.61791$

$$d_1(S(0),0) = 0.3$$

$$\frac{1}{\sigma\sqrt{1/4}}\left[\cancel{\ln\left(\frac{S(0)}{K}\right)} + \left(0.10 + \frac{\sigma^2}{2}\right)\cdot\frac{1}{4}\right] = 0.3$$

at·the·money

$$\frac{0.10 + \frac{\sigma^2}{2}}{\sigma}\sqrt{1/4} = 0.3$$

$$\frac{1}{2}\left(0.10 + \frac{\sigma^2}{2}\right) = 0.3\cdot\sigma \qquad /\cdot 4$$

$$\sigma^2 - 1.2\cdot\sigma + 0.2 = 0$$

$\sigma = 1$ is a solution

$$(\sigma - 1)(\sigma - 0.2) = 0$$

Consider both sol'ns starting w/ the more likely one, i.e., $\sigma = 0.2$.

For $\sigma = 0.2$, we have

$$50 \cdot 0.2 \cdot \sqrt{\frac{1}{365}} = 10\sqrt{\frac{1}{365}} \approx 0.52$$

$\Rightarrow$ (B) ∎

Delta-Gamma Hedging:

The investor starts w/ a delta-neutral portfolio. So, if the value function of the investor's portfolio is $v(s,t)$, the investor maintains

$$\Delta(s,t) = \frac{\partial}{\partial s} v(s,t) = 0.$$

Then, the investor decides to Γ-hedge as well. They want to create a <u>Γ-neutral</u> portfolio.

<u>Recall</u>: the gamma of the stock is <u>0</u>. So, they need to trade in another option. Let the "new" option's price be $\tilde{v}(s,t)$ w/

$$\tilde{\Delta}(s,t) = \frac{\partial}{\partial s} \tilde{v}(s,t)$$

$$\text{and } \tilde{\Gamma}(s,t) = \frac{\partial^2}{\partial s^2} \tilde{v}(s,t)$$

Let $\tilde{n}$ be the # of these options to hold @ this particular time:

with $\Gamma(s,t) = \frac{\partial^2}{\partial s^2} v(s,t)$

we must have:

$$\Gamma(s,t) + \tilde{n} \cdot \tilde{\Gamma}(s,t) = 0$$

$\uparrow$ Γ-neutrality

$$\Rightarrow \tilde{n} = \frac{-\tilde{\Gamma}(s,t)}{\Gamma(s,t)}$$