M339D: October 14th, 2022.

Arbitrage Portfolios.

✓

Def'n. An arbitrage portfolio is a portfolio whose $\boxed{\text{profit}}$ is:

? • non-negative in all states of the world

and

?? • strictly positive in at least one state of the world.

> Unless it's specified otherwise in a particular problem/example, we assume NO ARBITRAGE exists.

Law of the Unique Price.

Assume that the payoffs of two static portfolios A and B, are equal, i.e.,

$$\boxed{\boxed{V_A(T)} = V_B(T).}$$

In general, two random variables, X and Y are said to be equal if

$$\boxed{\mathbb{P}[X=Y]=1}$$

On a finite probability space, this means that they must take the exact same value on every elementary outcome.

Our claim: $\boxed{V_A(0) = V_B(0)}$

$\longrightarrow$: Assume, to the contrary, that

$$V_A(0) \neq V_B(0).$$

X

Without loss of generality, say

$$\boxed{\underbrace{V_A(0)}_{\substack{\text{relatively}\\ \text{cheap}}} < \underbrace{V_B(0)}_{\substack{\text{relatively}\\ \text{expensive}}}}$$

Diagnosis.

Proposal.

Propose an arbitrage portfolio:

- Long Portfolio A
- short Portfolio B

} Total Portfolio

Verification.

Verify:

- Payoff (Total Portfolio) $= V_A(T) - V_B(T) = 0$
- Initial Cost (Total Portfolio) $= V_A(0) - V_B(0) < 0$

Inflow of money @ time 0

$$\text{Profit} = \text{Payoff} - FV_{0,T}(\text{Initial Cost})$$

$$= 0 - FV_{0,T}\left(V_A(0) - V_B(0)\right) > 0$$

Indeed, we created an arbitrage portfolio!

$\Rightarrow\Leftarrow$

$\square$

Replicating Portfolios.

Def'n. Consider a European-style derivative security. A static portfolio w/ the same payoff as that of the derivative security is called its replicating portfolio.

Note: The initial price of the derivative security must be equal to the initial price of its replicating portfolio.

Example. Consider a $\boxed{\text{forward contract}}$ on a non-dividend paying stock. One possible replicating portfolio is: ?

0 T

Forward contract: $S(T) - F$

Replicating Portfolio:

- long 1 share of stock
- issue a bond w/ redemption amt F and delivery date T