M358K: November 11th, 2020.

χ^2-distribution.

The following "definition" of the χ^2-random variable can be extended, but for our purposes:

If $Z_1, Z_2, \ldots, Z_n$ are independent, standard normal random variables, then we say that

$$X = Z_1^2 + Z_2^2 + \cdots + Z_n^2$$

is χ^2 (chi squared) distributed w/ n degrees of freedom

and we write $X \sim \chi^2(df = n)$

df: degrees of freedom (parameter)

(aka r aka ν)

ν: nu

Example. Let $X \sim \chi^2(df = 5)$.

Find $\mathbb{P}[1.145 \leq X \leq 12.83] = ?$

1st Tables.

$$\begin{aligned}
\mathbb{P}[1.145 \leq X \leq 12.83] &= \\
&= \mathbb{P}[X \leq 12.83] - \mathbb{P}[X \leq 1.145] \\
&= F_X(12.83) - F_X(1.145) \\
&= 0.975 - 0.050 = 0.925 \checkmark
\end{aligned}$$

2^{nd} R.

pchisq(12.83, df=5) - pchisq(1.145, df=5) =

= 0.9250188. ∎

Example. Let $X \sim \chi^2(df=7)$.

Find a and b such that

$\mathbb{P}[X < a] = 0.025$

and $\mathbb{P}[X < b] = 0.975$.

1^{st} Tables.

$a = \chi^2_{0.975}(df=7) = 1.69$,

$b = \chi^2_{0.025}(df=7) = 16.01$.

2^{nd} R.

a = qchisq(0.025, df=7) = 1.689869,

b = qchisq(0.975, df=7) = 16.01276 ∎

χ^2 connections to normal samples.

Fact: Let $X_1, X_2, \dots, X_n$ are independent,

Normal(mean $= \mu$, var $= \sigma^2$).

Set, for all $i = 1..n$,

$$\boxed{Z_i = \frac{X_i - \mu}{\sigma}}$$

Note: $Z_1, Z_2, \dots, Z_n$ are independent and standard normal.

$$\Rightarrow \quad Z_1^2 + Z_2^2 + \dots + Z_n^2 \sim \chi^2(df = n)$$

Fact: Let $X_1, X_2, \dots, X_n$ be independent and $N(\text{mean} = \mu, \text{var} = \sigma^2)$

unknown

Set $\bar{X} = \frac{1}{n}(X_1 + \dots + X_n)$... the sample mean

Then:

$$\boxed{\sum_{i=1}^{n} \left(\frac{X_i - \bar{X}}{\sigma} \right)^2 \sim \chi^2(df = n-1)}$$

$$\Rightarrow \quad \frac{1}{\sigma^2} \boxed{\sum_{i=1}^{n} (X_i - \bar{X})^2} \sim \chi^2(df = n-1)$$

S^2... sample variance

$$\boxed{\frac{(n-1) \cdot S^2}{\sigma^2} \sim \chi^2(df = n-1)}$$

$$\boxed{S^2 := \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2}$$