Motivation.

Get a formula for the price of European calls and puts on a stock modeled in the Black-Scholes framework.

Idea.

$\boxed{\text{RISK-NEUTRAL PRICING}}$

$$\boxed{V(0) = e^{-rT}\,\mathbb{E}^*[V(T)]}$$

Payoff of a European option

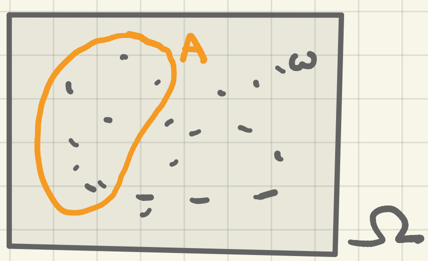
Implementation.

Temporarily, focus on a time-T, strike-K European call option.

The Payoff: $\boxed{V_c(T) = (S(T)-K)_+}$

Under $\mathbb{P}^*$:

$$\mathbb{E}^*[V_c(T)] = \mathbb{E}^*[(S(T)-K)_+]$$
$$= \mathbb{E}^*\left[(S(T)-K)\,\boxed{\mathbb{I}_{[S(T)\geq K]}}\right]$$



A is an event

$$\mathbb{I}_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}$$

$$\mathbb{I}_A = \begin{cases} 1 & \text{if } A \text{ happened} \\ 0 & \text{if } A \text{ did not happen} \end{cases}$$

$$\mathbb{E}[\mathbb{I}_A] = 1\cdot\mathbb{P}[A] + 0\cdot\mathbb{P}[A^c] = \mathbb{P}[A]$$

$$\mathbb{E}^*[V_c(T)] = \boxed{\mathbb{E}^*\left[S(T)\cdot\mathbb{I}_{[S(T)\geq K]}\right]} - K\cdot\boxed{\mathbb{P}^*[S(T)\geq K]}$$

$\mathbb{E}^*\left[S(T)\cdot\mathbb{I}_{[S(T)\geq K]}\right] \overset{?}{=}$ The partial expectation from the title

$\mathbb{P}^*[S(T)\geq K] = N(d_2)$

$$\text{w/ } d_2 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)\cdot T\right]$$

$$\mathbb{E}^*\left[\boxed{S(T)}\,\mathbb{I}_{\boxed{[S(T)\geq K]}}\right] = ?$$

$$\mathbb{E}^*\left[S(T)\cdot\mathbb{I}_{\boxed{[S(T)<K]}}\right] = ?$$

Method. Use the defining formula for the expectation of a function of a r.v.

In this case, the r.v. is $\boxed{Z \sim N(0,1)}$

$$\{S(T) \geq K\} = \left\{S(0)e^{(r-\frac{\sigma^2}{2})\cdot T + \sigma\sqrt{T}\cdot Z} \geq K\right\} = \{Z \geq -d_2\}$$

z ... our dummy variable within the integral which corresponds to Z, i.e.,

we set

$$g(z) = S(0)e^{(r-\frac{\sigma^2}{2})\cdot T + \sigma\sqrt{T}\cdot z} \quad \left(\text{so that } g(Z) = S(T)\right)$$

$$\mathbb{E}^*\left[\boxed{S(T)}\cdot\mathbb{I}_{[S(T)\geq K]}\right] = \mathbb{E}^*\left[g(Z)\cdot\mathbb{I}_{[Z\geq -d_2]}\right]$$

$$= \int_{-d_2}^{\infty} g(z)\cdot f_Z(z)\,dz \quad \text{.... lots of algebra ...}$$

$$= \boxed{S(0)e^{rT}}\cdot\boxed{N(d_1)}$$

where $d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)\cdot T\right]$

The expectation under $\mathbb{P}^*$ of the call payoff:

$$\mathbb{E}^*[V_c(T)] = S(0)e^{rT}\cdot N(d_1) - K\cdot N(d_2)$$

w/ d_1 as above

and $\boxed{d_2 = d_1 - \sigma\sqrt{T}}$

$\Rightarrow$ **The Black-Scholes call price:**

$$V_c(0) = e^{-rT}\,\mathbb{E}^*[V_c(T)]$$

$$V_c(0) = e^{-rT}\left(S(0)e^{rT}\cdot N(d_1) - K\cdot N(d_2)\right)$$

$$\boxed{V_c(0) = S(0)\cdot N(d_1) - Ke^{-rT}\cdot N(d_2)}$$

$\Rightarrow$ **The Black-Scholes Put Price:**

By put-call parity:

$$V_c(0) - \boxed{V_p(0)} = S(0) - Ke^{-rT}$$

$$V_p(0) = V_c(0) - S(0) + Ke^{-rT} = S(0)N(d_1) - Ke^{-rT}N(d_2) - S(0) + Ke^{-rT}$$

$$= S(0)\big(\underbrace{N(d_1)-1}_{-N(-d_1)}\big) + Ke^{-rT}\big(\underbrace{1-N(d_2)}_{N(-d_2)}\big)$$

symmetry of $N(0,1)$

$$\boxed{V_p(0) = Ke^{-rT}N(-d_2) - S(0)N(-d_1)}$$

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Problem Set 12

$\boxed{\text{Black-Scholes pricing.}}$

Problem 12.1. Let the stock prices be modeled using the lognormal distribution. Under the risk-neutral probability measure, the mean stock price at time−1 equals 120 and the median stock price 115. What is the risk-neutral probability that the time−1 stock price exceeds 100?

$\rightarrow$:

$$\frac{\text{mean}}{\text{median}} = \frac{\cancel{S(0)}e^{\cancel{rT}}}{\cancel{S(0)}e^{(\cancel{r}-\frac{\sigma^2}{2})\cdot T}} = e^{\frac{\sigma^2}{2}\cdot T}$$

$$\frac{120}{115} = e^{\sigma^2/2} \Rightarrow \frac{\sigma^2}{2} = \ln\left(\frac{120}{115}\right) \Rightarrow \sigma^2 = 2\ln\left(\frac{120}{115}\right)$$

$$\Rightarrow \sigma = \sqrt{2\ln\left(\frac{120}{115}\right)} = \underline{0.2918}$$

$$\mathbb{P}^*[S(1) > 100] = ?$$

$$\mathbb{P}^*\left[\boxed{S(0)e^{(r-\frac{\sigma^2}{2})\cdot 1}}\,e^{+\sigma\sqrt{1}\cdot Z} > 100\right]$$

median of $S(1)$ = 115

$$\mathbb{P}^*[115e^{\sigma\cdot Z} > 100] = \mathbb{P}^*\left[\sigma\cdot Z > \ln\left(\frac{100}{115}\right)\right] =$$

$$= \mathbb{P}^*[Z > -0.479] \cong 0.6844 \quad \square$$

Problem 12.2. (5 pts) Let the stochastic process $S = \{S(t); t \geq 0\}$ denote the stock price. The stock's rate of appreciation is 10% while its volatility is 0.30 Then,

(a) $Var[\ln(S(t))] = 0.3t$

(b) $Var[\ln(S(t))] = 0.09t^2$

(c) $Var[\ln(S(t))] = 0.09t$

(d) $Var[\ln(S(t))] = 0.09$

(e) None of the above.

$\longrightarrow$: In the Black-Scholes model:

$$S(t) = S(0)e^{(r-\frac{\sigma^2}{2})\cdot t + \sigma\sqrt{t}\cdot Z} \quad \text{w/ } Z \sim N(0,1)$$

$$\ln(S(t)) = \underbrace{\boxed{\ln(S(0)) + (r-\frac{\sigma^2}{2})\cdot t}}_{\text{deterministic}} + \sigma\sqrt{t}\cdot Z$$

$$Var[\ln(S(t))] = Var[\sigma\sqrt{t}\cdot Z] = \sigma^2\cdot t\cdot \underbrace{Var[Z]}_{1} = \sigma^2\cdot t \quad \square$$

Problem 12.3. (5 pts) Assume the Black-Scholes framework. Let the current price of a non-dividend-paying stock be equal to $S(0) = 95$ and let its volatility be equal to 0.35. Consider a European call on that stock with strike 100 and exercise date in $\boxed{\text{9 months.}}$ Let the risk-free continuously compounded interest rate be 6% per annum.

Denote the price of the call by $V_C(0)$. Then,

(a) $V_C(0) < \$5.20$

(b) $\$5.20 \leq V_C(0) < \$7.69$

(c) $\$7.69 \leq V_C(0) < \$9.04$

(d) $9.04 \leq V_C(0) < \$11.25$

(e) None of the above.

$\longrightarrow$: $\underline{\underline{1^{st}}}$ Calculate d_1 and d_2.

$\underline{\underline{2^{nd}}}$ Use the standard normal tables or R.

$\underline{\underline{3^{rd}}}$ Final pricing formula.

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)\cdot T\right]$$

$$d_1 = \frac{1}{0.35\sqrt{3/4}}\left[\ln\left(\frac{95}{100}\right) + \left(0.06 + \frac{(0.35)^2}{2}\right)\cdot 0.75\right] = \_\_\_\_\_\_$$