M339D: October 13th, 2025.

The Binomial Asset Pricing Model.

$S(0)$... the observeable initial asset price

0 ——— T time horizon (e.g., the exercise date of an option)

$$
\begin{tikzcd}
 & \boxed{S_u = u \cdot S(0)} \text{ up node} \\
\boxed{S(0)} \arrow[ur] \arrow[dr] & \\
 & \boxed{S_d = d \cdot S(0)} \text{ down node}
\end{tikzcd}
$$

ROOT NODE

$\underbrace{\qquad\qquad}_{h \text{... length of a single period}}$

u... up factor
d... down factor

By convention: $\boxed{u > d}$

one period: $\boxed{S(T) = S(h)}$ .. a random variable denoting the time-T stock price w/ two possible values S_u and S_d

As a random variable:

- SIMPLE RETURN: $\dfrac{S(T) - S(0)}{S(0)}$

up node: $\dfrac{S_u - S(0)}{S(0)} = \dfrac{S_u}{S(0)} - 1 = u - 1$

down node: $\dfrac{S_d - S(0)}{S(0)} = \dfrac{S_d}{S(0)} - 1 = d - 1$

- **Continuously Compounded:** $\ln\left(\frac{S(T)}{S(0)}\right) \leftarrow$

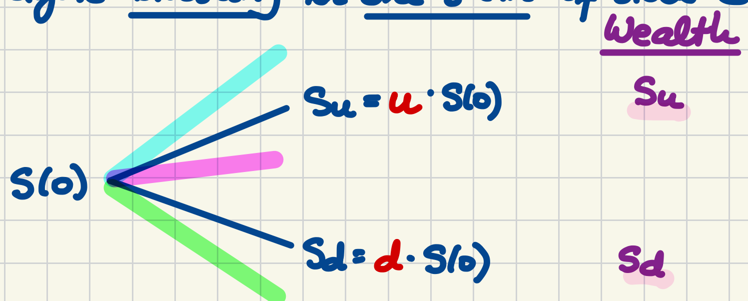
  up node: $\ln(u)$

  down node: $\ln(d)$

Market Model.

- **riskless asset:** @ the ccrfir r
- **risky asset:** non-dividend-paying stock

Imagine investing in one share of stock @ time 0:



At the risk-free interest rate, the amount S(0) accumulates to $S(0)e^{rh}$ in the same time period.

The No-Arbitrage Condition.

$$S_d < S(0)e^{rh} < S_u$$

$$d \cdot \cancel{S(0)} < \cancel{S(0)}e^{rh} < u \cdot \cancel{S(0)}$$

$$\boxed{d < e^{rh} < u}$$

Half·a·Proof.

Say, to the contrary, that $\boxed{e^{rh} \leq d < u}$. ←

Propose. Long the stock.

Verify. Profit = Payoff − $FV_{0,T}$(Initial Cost)

$$= S(h) - S(0)e^{rh}$$

down node: $S_d - S(0)e^{rh} = d \cdot S(0) - S(0)e^{rh}$

$$= S(0)\left(d - e^{rh}\right) \geq 0$$

up node: $S_u - S(0)e^{rh} = u \cdot S(0) - S(0)e^{rh}$

$$= S(0)\left(u - e^{rh}\right) > 0$$

Indeed, this is an arbitrage portfolio.

The Other Half of Proof : $d < u \leq e^{rh}$

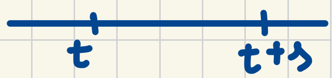
⋮

Task!

Forward Binomial Tree.

"Def'n". The volatility $\boxed{\sigma}$ is the standard deviation of realized returns on a continuously compounded scale and annualized.

Realized Returns.



$$R(t,t+s) := \ln\left(\frac{S(t+s)}{S(t)}\right)$$

It satisfies:

$$\boxed{S(t+s) = S(t)e^{R(t,t+s)}}$$

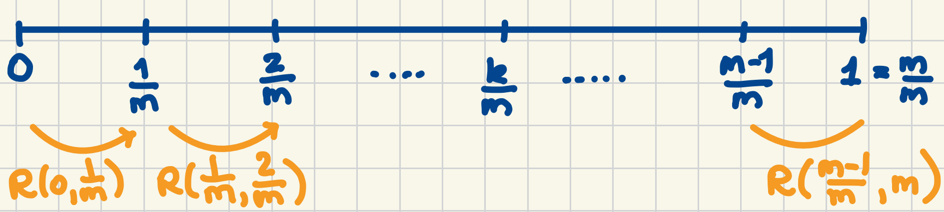
Heuristics.

$T=1$. Q: $Var[R(0,1)] = \underline{\sigma^2}$

$h_m = \frac{1}{m}$ (of a year)

Q: What is the volatility for the time period of length h_m?

$$\boxed{\sigma_{h_m} = ?}$$



$\boxed{R\left(\frac{k-1}{m}, \frac{k}{m}\right) \text{ for } k=1,\dots,m \text{ are all RANDOM VARIABLES.}}$