Name:

M339J/M389J: Probability Models with Actuarial Applications
The University of Texas at Austin
Practice Problems for In-Term Exam II
Instructor: Milica Čudina

Notes: This is a closed book and closed notes exam. The maximal score on this exam is 100 points.

2.1. **TRUE/FALSE QUESTIONS.**

Problem 2.1. (2 points) Any negative binomial distribution has the memoryless property. *True or false?*

Solution: FALSE
This is correct only for the geometric distribution.

2.2. **Free-response problems.** Please, explain carefully all your statements and assumptions. Numerical results or single-word answers without an explanation (even if they're correct) are worth 0 points.

Problem 2.2. (10 points) Let $X \sim Pareto(\alpha = 3, \theta = 3000)$.
Assume that there is a deductible of $d = 5000$.
Find the loss elimination ratio.

Solution: Using the tables, we get

$$\mathbb{E}[X] = \frac{\theta}{\alpha - 1} = \frac{3000}{3-1} = 1500,$$
$$\mathbb{E}[X \wedge d] = \frac{\theta}{\alpha - 1}\left[1 - (\tfrac{\theta}{d+\theta})^{\alpha-1}\right] = \frac{3000}{3-1}\left[1 - (\tfrac{3000}{5000+3000})^{3-1}\right] = 1500[1 - (\tfrac{3}{8})^2].$$

Finally, the loss elimination ratio is

$$[1 - (\tfrac{3}{8})^2] = \tfrac{55}{64}\,.$$

Problem 2.3. (10 points) Assume that conditional on the random variable Λ, N has the Poisson distribution with variance Λ.
Let $\Lambda \sim U(0, 2)$.
Find the (unconditional) mean and variance of N.

Solution: From the given information, we conclude that

$$N \mid \Lambda = \lambda \sim Poisson(\lambda).$$

So,

$$\mathbb{E}[N|\Lambda] = \Lambda,$$

and

$$\mathbb{E}[N] = \mathbb{E}[\mathbb{E}[N|\Lambda]] = \mathbb{E}[\Lambda] = 1.$$

On the other hand,

$$Var[N|\Lambda] = \Lambda.$$

To get the variance of N, we use

$$\begin{aligned} Var[N] &= \mathbb{E}[Var[N|\Lambda]] + Var[\mathbb{E}[N|\Lambda]] \\ &= \mathbb{E}[\Lambda] + Var[\Lambda] \\ &= \mathbb{E}[Var[N|\Lambda]] + Var[\mathbb{E}[N|\Lambda]] \\ &= 1 + 4 \cdot \frac{1}{12} \\ &= 4/3. \end{aligned}$$

Problem 2.4. (10 points) Let the ground-up loss X be exponentially distributed with mean \$800.
An insurance policy has an ordinary deductible of \$100 and the maximum amount payable per loss of \$2500.
Find the expected value of the amount paid (by the insurance company) per positive payment.

Solution: We are given $X \sim Exponential(\theta = 800)$, the deductible $d = 100$ and the policy limit $u - d = 2500$. We need to calculate $\mathbb{E}[Y^P]$ where $Y^P = Y^L \mid Y^L > 0$ and

$$\begin{aligned} Y^L &= \begin{cases} (X-d)_+, & X < u, \\ u - d, & X \geq u \end{cases} \\ &= (X \wedge u - d)_+. \end{aligned}$$

By the memoryless property of the exponential distribution, we have that

$$Y = X - d \mid X > d$$

is also exponential with mean 800. So, using our tables, we get

$$\mathbb{E}[Y^P] = \mathbb{E}[Y \wedge (u-d)] = \mathbb{E}[Y \wedge 2500] = 800(1 - e^{-2500/800}) \approx 764.85.$$

Problem 2.5. (10 points) Losses in year y follow a two-parameter Pareto distribution with parameters $\alpha = 4$ and $\theta = 10$.
Losses in year $y+1$ are uniformly 10% higher than those in year y.
An insurance covers each loss subject to a deductible $d = 20$.
Calculate the **loss elimination ratio** for year $y+1$.

Solution: Pareto is a scale distribution with the scale parameter θ. So, the losses in year $y+1$, denoted by X again have the two-parameter Pareto distribution with $\alpha = 4$ and $\theta = 11$.
By definition, the loss elimination ratio is

$$\frac{\mathbb{E}[X \wedge d]}{\mathbb{E}[X]}.$$

In our case, using the provided tables, we get

$$\begin{aligned}\frac{\mathbb{E}[X \wedge d]}{\mathbb{E}[X]} &= \frac{\frac{\theta}{\alpha-1}(1-(\frac{\theta}{d+\theta})^{\alpha-1})}{\frac{\theta}{\alpha-1}} = 1-(\frac{\theta}{d+\theta})^{\alpha-1}\\ &= 1-(\frac{11}{20+11})^{4-1} = 1-(\frac{11}{31})^3 \approx 0.9553.\end{aligned}$$

Problem 2.6. (10 points) Assume that the severity random variable X is uniform on the interval $(0, 1000)$. There is an insurance policy to cover this loss. The insurance policy has a deductible of 200 per loss and the maximum amount payable by the insurer per loss of 700.

Calculate the expected value of the per payment random variable Y^P under this policy.

Solution: Note that

$$X - 200 \,|\, X > 200 \quad \sim \quad U(0, 800).$$

So,

$$(X-200) \wedge 700 \,|\, X > 200 \quad \sim \quad \begin{cases} U(0,700) & \text{with probability } 7/8 \\ 700 & \text{with probability } 1/8 \end{cases}$$

$$\mathbb{E}[Y^P] = \frac{7}{8} \cdot 350 + \frac{1}{8} \cdot 700 = 393.75.$$

Problem 2.7. Assume that the severity random variable X is exponentially distributed with mean 1400. There is an insurance policy to cover this loss. This insurance policy has

(1) a deductible of 200 per loss,
(2) the coinsurance factor of $\alpha = 0.25$, and
(3) the maximum amount payable by the insurer per loss of 700.

Calculate the expected value of the per payment random variable Y^P under this policy.

Solution: Due to the memoryless property of the exponential distribution, we have

$$X - 200 \,|\, X > 200 \sim Exponential(\theta = 1400).$$

Due to the fact that the exponential distribution is a scale distribution, when we introduce the coinsurance factor, we get

$$0.25(X - 200) \,|\, X > 200 \sim Exponential(\theta^* = 0.25 * 1400 = 350).$$

Hence, using our tables with $Y \sim Exponential(\theta = 350)$,

$$\mathbb{E}[Y^P] = \mathbb{E}[Y \wedge 700] = 350(1 - e^{-700/350}) \approx 302.63.$$

Problem 2.8. (10 points) Let X have the exponential distribution with mean 4 and define $Y = \sqrt{X}$. Find $f_Y(3)$.

Solution: We are given that $X \sim Exponential(\theta = 4)$.
For every $y > 0$,

$$\begin{aligned} F_Y(y) &= \mathbb{P}[Y \leq y] \\ &= \mathbb{P}[\sqrt{X} \leq y] \\ &= \mathbb{P}[X \leq y^2] \\ &= F_X(y^2). \end{aligned}$$

So, for $y > 0$,

$$\begin{aligned} f_Y(y) &= 2y f_X(y^2) \\ &= 2y \cdot \frac{1}{\theta} e^{-y^2/\theta} \\ &= 2y \cdot \frac{1}{4} e^{-y^2/4}. \end{aligned}$$

In particular, $f_Y(3) = 1.5e^{-9/4} \approx 0.1581$.

2.3. **MULTIPLE CHOICE QUESTIONS.**

Problem 2.9. (5 points) Let X be the ground-up loss random variable. Assume that X has the two-parameter Pareto distribution with $\theta = 4,000$ and $\alpha = 3$.

Let B denote the expected payment per loss on behalf of an insurer which wrote a policy with no deductible and with a policy limit of $5,000$. Then,

(a) $B \approx 1,000$
(b) $B \approx 1,200$
(c) $B \approx 1,400$
(d) $B \approx 1,600$
(e) None of the above

Solution: (d)
Using our tables,

$$B = \mathbb{E}[X \wedge 5000] = \frac{\theta}{\alpha - 1}[1 - (\frac{\theta}{5000 + \theta})^{\alpha-1}] = \frac{4000}{2}[1 - (\frac{4000}{9000})^2] \approx 1600.$$

Problem 2.10. (5 points) Let X be the ground-up loss random variable. Assume that X has the exponential distribution with mean $1,000$.
Let B denote the expected payment per loss on behalf of an insurer who wrote a policy with a deductible of $1,500$ and with the **maximum payment by the insurer** equal to $2,500$. Then,

(a) $B \approx 714$
(b) $B \approx 816$
(c) $B \approx 918$
(d) $B \approx 1020$
(e) None of the above

Solution: (e)
Using our tables, we get

$$\begin{aligned} B = \mathbb{E}[X \wedge 4000] - \mathbb{E}[X \wedge 1500] &= 1000(1 - e^{-4000/1000}) - 1000(1 - e^{-1500/1000}) \\ &= 1000(e^{-1.5} - e^{-4}) \approx 204.8. \end{aligned}$$

Problem 2.11. (5 points) Assume that X is a mixture distribution with the mixing variable Λ. More precisely, let $X \mid \Lambda = \lambda$ be uniform on the interval $(0, \lambda)$ and let Λ be uniform on an interval of the form $(0, c)$ for some constant c with mean 5. Let $\pi = \mathbb{P}[X \leq 3]$. Then, we have that

(a) $0 \leq p < 1/4$
(b) $1/4 \leq p < 1/2$
(c) $1/2 \leq p < 5/8$
(d) $5/8 \leq p < 9/10$
(e) None of the above

Solution: (d)
We are given that

$$\begin{aligned} X \mid \Lambda = \lambda &\sim U(0, \lambda) \\ \Lambda &\sim U(0, 10). \end{aligned}$$

So,

$$f_\Lambda(\lambda) = \frac{1}{10} \qquad x \in (0, 10),$$

$$F_{X \mid \Lambda}(x \mid \lambda) = \begin{cases} 0, & x \leq 0, \\ \frac{x}{\lambda} & x \in (0, \lambda), \\ 1, & x \geq \lambda. \end{cases}$$

Therefore, the support of X is $(0,10)$ and for every $x \in (0,10)$,

$$\begin{aligned}\mathbb{P}[X \leq 3] &= \int_0^{10} F_{X\,|\,\Lambda}(3\,|\,\lambda) f_\Lambda(\lambda)\,d\lambda \\ &= \int_0^3 1 \cdot \frac{1}{10}\,d\lambda + \int_3^{10} \frac{3}{\lambda} \cdot \frac{1}{10}\,d\lambda \\ &= \frac{3}{10}[1 + \ln(\lambda)|_{\lambda=3}^{10}] \\ &= \frac{3}{10}[1 + (\ln(10) - \ln(3))] \\ &\approx 0.6612.\end{aligned}$$

Problem 2.12. (5 points) A model for the arrival time T for a particular event is initially an exponential distribution with mean 2 years. Upon reconsideration, this distribution is replaced with a spliced model whose density function:

(i) is uniform over $[0,1]$,
(ii) is proportional to the initial modeled density function after 1 year,
(iii) is continuous.

Calculate the probability of failure in the first year under the revised distribution.

(a) 1/4
(b) 1/3
(c) 1/2
(d) 3/5
(e) None of the above.

Solution: (b)
For the "old" model, the density function is

$$f_T(x) = \frac{1}{2}e^{-\frac{x}{2}} \quad \text{for } x > 0.$$

Let $\tilde{f}$ be the new density function. Then, it satisfies:

(i) $\tilde{f}(x) = \kappa$ for some constant κ for $x \in [0,1]$,
(ii) $\tilde{f}(x) = ce^{-\frac{x}{2}}$ for $x > 1$ for some constant c,
(iii) $\tilde{f}(1) = \kappa = ce^{-\frac{1}{2}}$

Since $\tilde{f}$ is a density function, it must integrate to 1. Thus,

$$1 = \kappa + c\int_1^\infty e^{-x/2}\,dx = \kappa + c(2)e^{-1/2}.$$

Using (iii) above, we can substitute $\kappa = ce^{-\frac{1}{2}}$ in the last equation to obtain

$$1 = \kappa + 2\kappa \quad \Rightarrow \quad \kappa = \frac{1}{3}.$$

The answer is $\kappa(1) = 1/3$.

Problem 2.13. (5 points) Let the ground-up loss X be modeled by a two-parameter Pareto distribution with parameters $\alpha = 2$ and $\theta = 200$. For an insurance policy on the above loss, there is a **franchise** deductible of 200. Find the expected value of the *per payment* random variable.

(a) 200
(b) 400
(c) 600
(d) 800
(e) None of the above.

Solution: (c)
In our usual notation, as we have shown in class,

$$\mathbb{E}[Y^P] = d + \frac{\mathbb{E}[X] - \mathbb{E}[X \wedge d]}{S_X(d)}.$$

In this problem, the ground-up loss is $X \sim Pareto(\alpha = 2, \theta = 200)$. So, using the STAM tables, we have

$$S_X(d) = \left(\frac{\theta}{d+\theta}\right)^{\alpha},$$
$$\mathbb{E}[X] = \frac{\theta^1 \cdot 1!}{\alpha - 1} = \frac{\theta}{\alpha - 1},$$
$$\mathbb{E}[X \wedge d] = \frac{\theta}{\alpha - 1}\left(1 - \left(\frac{\theta}{d+\theta}\right)^{\alpha-1}\right).$$

So,

$$\mathbb{E}[Y^P] = d + \frac{\frac{\theta}{\alpha-1} - \frac{\theta}{\alpha-1}\left(1 - \left(\frac{\theta}{d+\theta}\right)^{\alpha-1}\right)}{\left(\frac{\theta}{d+\theta}\right)^{\alpha}} = d + \frac{\frac{\theta}{\alpha-1}\left(\frac{\theta}{d+\theta}\right)^{\alpha-1}}{\left(\frac{\theta}{d+\theta}\right)^{\alpha}} = d + \frac{\frac{\theta}{\alpha-1}}{\frac{\theta}{d+\theta}} = d + \frac{d+\theta}{\alpha-1}$$

Using the parameter values and the deductible from this problem, we get

$$\mathbb{E}[Y^P] = 200 + \frac{200+200}{2-1} = 600.$$

Problem 2.14. (5 points) *Source: Sample P exam, Problem #262.*
The number of traffic accidents occurring on any given day in Coralville is Poisson distributed with mean 5. The probability that any such accident involves an uninsured driver is 0.25, independent of all other such accidents. Calculate the probability that on a given day in Coralville there are no traffic accidents that involve an uninsured driver.

(a) 0.007

(b) 0.010

(c) 0.124

(d) 0.237

(e) 0.287

Solution: (e)
By the "thinning" theorem, the number of accidents involving an uninsured driver is again Poisson with mean $5(0.25) = 1.25$. The probability that this random variable has the value of zero is

$$e^{-1.25} = 0.2865.$$

Problem 2.15. (5 points) A group dental policy has a negative binomial claim count distribution with mean 20 and variance 100. What is the probability that there is at most 1 claim?

(a) 0.0016

(b) 0.0024

(c) 0.0032

(d) 0.0048

(e) None of the above.

Solution: (a)
According to our tables, with $N \sim NegBinomial(r, \beta)$, the mean and the variance are

$$\mathbb{E}[N] = r\beta = 20 \quad \text{and} \quad Var[N] = r\beta(1+\beta) = 100.$$

Hence, $1 + \beta = 5$, so that $\beta = 4$ and $r = 5$.
We are looking for

$$F_N(1) = p_N(0) + p_N(1) = \frac{1}{5^5} + \frac{5 \cdot 4^1}{5^6} = 0.0016.$$

Problem 2.16. (5 points) The distribution of a loss X is a two-point mixture:

(i) With probability 3/4, X has a two-parameter Pareto distribution with parameters $\alpha = 4$ and $\theta = 100$.

(ii) With probability 1/4, X has a two-parameter Pareto distribution with parameters $\alpha = 3$ and $\theta = 1000$.

What is the variance of X?

(a) 229375

(b) 194167

(c) 51875

(d) 26179

(e) None of the above.

Solution: (a)
We are given that

$$X = \begin{cases} X_1 \sim Pareto(\alpha_1 = 4, \theta_1 = 200) & \text{with probability } a_1 = 3/4 \\ X_2 \sim Pareto(\alpha_2 = 3, \theta_2 = 1000) & \text{with probability } a_2 = 1/4 \end{cases}$$

According to the STAM tables, the first and the second moments of our two-parameter Pareto random variables $X_i, i = 1, 2$, can be expressed in terms of its parameters as

$$\mathbb{E}[X_i] = \frac{\theta_i}{\alpha_i - 1} \quad \text{and} \quad \mathbb{E}[X_i^2] = \frac{2\theta_i^2}{(\alpha_i - 1)(\alpha_i - 2)}.$$

So,

$$\mathbb{E}[X] = \frac{3}{4}\mathbb{E}[X_1] + \frac{1}{4}\mathbb{E}[X_2] = \frac{3}{4}\left(\frac{\theta_1}{\alpha_1 - 1}\right) + \frac{1}{4}\left(\frac{\theta_2}{\alpha_2 - 1}\right) = \frac{3}{4}\left(\frac{200}{4-1}\right) + \frac{1}{4}\left(\frac{1000}{3-1}\right) = 175,$$

$$\begin{aligned}\mathbb{E}[X^2] &= \frac{3}{4}\mathbb{E}[X_1^2] + \frac{1}{4}\mathbb{E}[X_2^2] = \frac{3}{4}\left(\frac{2\theta_1^2}{(\alpha_1 - 1)(\alpha_1 - 2)}\right) + \frac{1}{4}\left(\frac{2\theta_2^2}{(\alpha_2 - 1)(\alpha_2 - 2)}\right) \\ &= \frac{3}{4}\left(\frac{2(200)^2}{(4-1)(4-2)}\right) + \frac{1}{4}\left(\frac{2(1000)^2}{(3-1)(3-2)}\right) = 260000.\end{aligned}$$

Finally, the variance of X is

$$Var[X] = 260000 - 175^2 = 229375.$$

2.4. **More free-response problems.**

Problem 2.17. Let the random variable X have a two-parameter Pareto distribution with $\alpha = 2$ and $\theta = 10$. What is the 75^{th} percentile of this distribution?

Solution: Let F_X denote the cumulative distribution function of X. We need to solve for x in

$$F_X(x) = \frac{3}{4}.$$

From the STAM tables, we learn that

$$F_X(x) = 1 - \left(\frac{\theta}{x+\theta}\right)^{\alpha}.$$

So, we solve for x:

$$\frac{3}{4} = 1 - \left(\frac{\theta}{x+\theta}\right)^{\alpha} \quad \Leftrightarrow \quad \frac{1}{4} = \left(\frac{10}{x+10}\right)^2 \quad \Leftrightarrow \quad \frac{1}{2} = \frac{10}{x+10} \quad \Leftrightarrow \quad x + 10 = 20 \quad \Leftrightarrow \quad x = 10$$

Problem 2.18. Let the independent random variables X_1, X_2 and X_3 all have the following probability mass function:

$$p_{X_1}(-1) = 1/4, \quad p_{X_1}(0) = 1/2, \quad p_{X_1}(1) = 1/4.$$

Let $X = 1 + X_1X_2X_3$. What is the probability generating function of X?

Solution: The random variable X has the following probability mass function:

$$p_X(0) = \frac{1}{16}, \quad p_X(1) = \frac{7}{8}, \quad p_X(2) = \frac{1}{16}.$$

So, the probability generating fucntion of X equals

$$P_X(s) = \mathbb{E}[s^X] = s^0\left(\frac{1}{16}\right) + s^1\left(\frac{7}{8}\right) + s^2\left(\frac{1}{16}\right) = \frac{1}{16} + \frac{7}{8}s + \frac{1}{16}s^2.$$

Problem 2.19. Let the random variable X have the moment generating function given by

$$M_X(t) = (1-2t)^{-3}$$

Define the random variable $\tilde{X}$ as $\tilde{X} = 5X + 1$. What is the moment generating function $M_{\tilde{X}}$ of $\tilde{X}$?

Solution: In general, for $\tilde{X} = aX + b$, we have that

$$M_{\tilde{X}}(t) = \mathbb{E}[e^{t\tilde{X}}] = \mathbb{E}[e^{t(aX+b)}] = e^{tb}\mathbb{E}[e^{atX}] = e^{tb}M_X(at).$$

So, in our problem, we have that

$$M_{\tilde{X}}(t) = e^{tb}M_X(at) = e^{tb}(1-2(at))^{-3} = e^t(1-10t)^{-3} = \frac{e^t}{(1-10t)^3}.$$