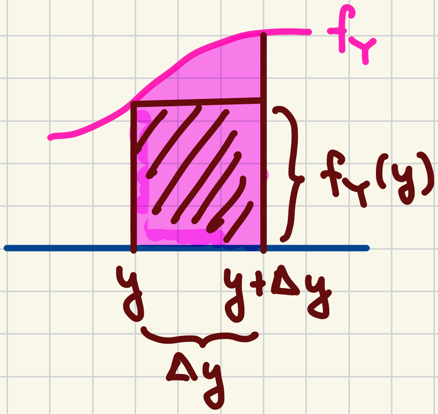
M378K: September 15th, 2025.

More on Continuous Distribution.



$$\mathbb{P}[Y \in [y, y+\Delta y]] = \quad \approx f_Y(y)\,\Delta y \approx f_Y(y)\,dy$$

i.e., $\mathbb{P}[Y \in [a,b]] = \int_a^b f_Y(y)\,dy$

Caveat: There are r.v.s that are neither discrete nor continuous!

Example. Y is uniformly distributed between ℓ and r.

$$\boxed{Y \sim U(\ell, r)}$$

$$f_Y(y) = \begin{cases} \frac{1}{r-\ell} & \text{for } \boxed{y \in [\ell, r]} \\ 0 & \text{otherwise} \end{cases}$$

We introduce, for any subset $A \subseteq \mathbb{R}$

$$\mathbb{1}_A : \mathbb{R} \to \boxed{\mathbb{R}}$$

$$\mathbb{1}_A(y) = \begin{cases} 1 & \text{if } y \in A \\ 0 & \text{if } y \notin A \end{cases}$$

This function is called the indicator function.

$$\boxed{f_Y(y) = \frac{1}{r-\ell} \cdot \mathbb{1}_{[\ell, r]}(y)}$$

M378K Introduction to Mathematical Statistics
Problem Set #5
Continuous distributions.

Problem 5.1. Source: Sample P exam, Problem #33.
The lifetime of a machine part has a continuous distribution on the interval $(0, 40)$ *with probability density function* f_X*, where*

$$f_X(x) \propto \frac{1}{(10+x)^2}$$

is proportional to

on the interval.

Calculate the probability that the lifetime of the machine part is less than 6.

$\longrightarrow$: There is a constant κ such that

$$f_X(x) = \kappa \cdot \frac{1}{(10+x)^2} \cdot \mathbb{1}_{[0,40]}(x)$$

We know: $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

So, $\kappa \cdot \int_0^{40} (10+x)^{-2}\,dx = 1$

$$\int_0^{40} (10+x)^{-2}\,dx = \frac{(10+x)^{-1}}{-1}\Big|_{x=0}^{40} = -\frac{1}{10+x}\Big|_{x=0}^{40}$$

$$= -\frac{1}{50} - \left(-\frac{1}{10}\right) = \frac{5-1}{50} = \frac{2}{25} \Rightarrow \kappa = \frac{25}{2}$$

$$\mathbb{P}[X \leq 6] = \mathbb{P}[X \in [0,6]]$$

$$= \int_0^6 f_X(x)\,dx = \frac{25}{2}\int_0^6 (10+x)^{-2}\,dx$$

$$= \frac{25}{2}\left(-\frac{1}{10+x}\right)\Big|_{x=0}^{6} = \frac{25}{2}\left(-\frac{1}{16}+\frac{1}{10}\right) = \frac{25}{2}\cdot\frac{8-5}{80}$$

$$= \frac{15}{32} \quad \square$$

Problem 5.2. Source: Sample P exam, Problem #419.

A customer purchases a lawnmower with a two-year warranty. The number of years before the lawnmower needs a repair is uniformly distributed on $[0, 5]$. *Calculate the probability that the lawnmower needs no repairs within* 4.5 *years after the purchase, given that the lawnmower needs no repairs within the warranty period*

$\longrightarrow$: T... the lifetime of the lawnmower

$T \sim U(0,5)$

$$\mathbb{P}[T > 4.5 \mid T > 2] = \frac{\mathbb{P}[T > 4.5, T > 2]}{\mathbb{P}[T > 2]}$$

$$= \frac{\mathbb{P}[T > 4.5]}{\mathbb{P}[T > 2]} = \frac{\frac{5-4.5}{\cancel{5-0}}}{\frac{5-2}{\cancel{5-0}}} = \frac{0.5}{3} = \frac{1}{6}$$

$\square$

Example. $Y \sim N(\mu, \sigma^2)$ where $\mu \in \mathbb{R}$ and $\sigma > 0$ is said to be normally distributed w/ mean μ and standard deviation σ

If $$f_Y(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-\mu)^2}{2\sigma^2}} \quad \text{for all } y \in \mathbb{R}$$

If $\mu = 0$ and $\sigma = 1$, we say that Y is standard normal.

Its pdf is $$\boxed{f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \quad \text{for all } y \in \mathbb{R}}$$

Q: Let $Y \sim N(\mu, \sigma^2)$.

$$\frac{Y-\mu}{\sigma} \sim N(0,1)$$

Q: Let $Y \sim N(0,1)$.
Let α and β are two real constants

$$\alpha \cdot Y + \beta \sim \underline{\text{Normal}}(\beta, \alpha^2)$$

Example. We say that Y is exponential w/ parameter τ if its pdf is

$$\boxed{f_Y(y) = \frac{1}{\tau} e^{-\frac{y}{\tau}} \cdot \mathbb{1}_{[0,\infty)}(y)}$$