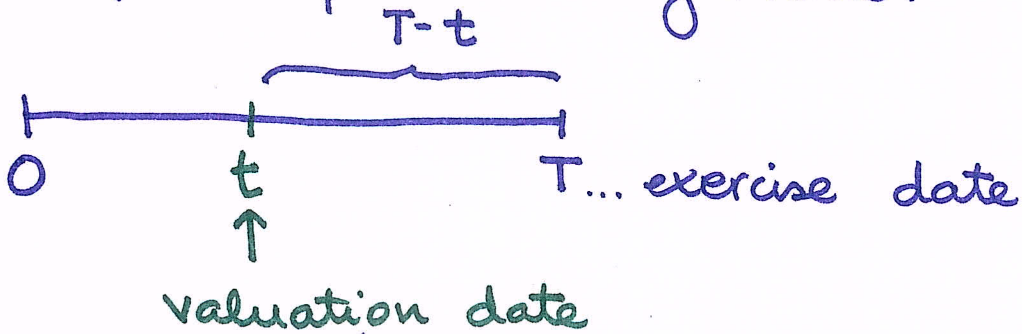
Option Greeks

W: March 8^{th}, 2019.

Value of a portfolio as it depends on a set of independent arguments.



The underlying asset's price will be modeled in the Black-Scholes framework.

Notation: $\{S(t), t \geq 0\}$ ... this is a stochastic process

We introduce:

s ... independent argument which stands for the current asset price

$\Rightarrow$ When we look @ the pair (t, s), we are considering the valuation time t and the stock price @ that time t equals s

We, so far, had our prices depend also on:

r, σ, δ

K ... strike price

For any of our portfolios, its value @ time t can be written as

$$v(s, t, r, \delta, \sigma)$$

value function

<u>Def'n.</u>

- $\frac{\partial}{\partial s} v(\ldots) =: \Delta(\ldots)$ <u>delta</u>
- $\frac{\partial^2}{\partial s^2} v(\ldots) =: \Gamma(\ldots)$ <u>gamma</u>
- $\frac{\partial}{\partial t} v(\ldots) =: \Theta(\ldots)$ <u>theta</u>
- $\frac{\partial}{\partial r} v(\ldots) =: \rho(\ldots)$ <u>rho</u>
- $\frac{\partial}{\partial \delta} v(\ldots) =: \psi(\ldots)$ <u>psi</u>
- $\frac{\partial}{\partial \sigma} v(\ldots) = \text{vega}(\ldots)$ <u>vega</u>

Example. Consider a bond w/ redemption amount of \$1 and maturity date @ time T

$\Rightarrow$ The value of the portfolio @ time t:

$$v(s,t,r,\delta,\sigma) = e^{-r(T-t)}$$

$$\Rightarrow \Delta(\dots) = 0 \quad \Rightarrow \Gamma(\dots) = 0$$

$$\Theta(\dots) = \frac{\partial}{\partial t} e^{-r(T-t)} = re^{-r(T-t)} > 0$$

$$\rho(\dots) = \frac{\partial}{\partial r} e^{-r(T-t)} = -(T-t)e^{-r(T-t)} < 0$$

Example. Outright purchase of a non-dividend-paying stock.

$$\Rightarrow v(s,t,r,\delta,\sigma) = s$$

$$\Rightarrow \Delta(\dots) = 1 \ ; \ \Gamma(\dots) = 0$$

$$\Rightarrow \Theta(\dots) = 0$$

Example. Continuous-dividend-paying stock

We have a prepaid forward on the stock.

$$\Rightarrow v(s,t,r,\delta,\sigma) = se^{-\delta(T-t)}$$

$$\Rightarrow \Delta(\dots) = e^{-\delta(T-t)} \quad \Rightarrow \Gamma(\dots) = 0$$

$$\Theta(\dots) = \delta \cdot se^{-\delta(T-t)}$$

Example. <u>A European time-T, strike-K call</u>

The Black-Scholes price is

$$v_c(s,t,r,\delta,\sigma) = se^{-\delta(T-t)} \cdot N(\underline{\underline{d_1}}) - Ke^{-r(T-t)} \cdot N(\underline{\underline{d_2}})$$

w/ $$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{s}{K}\right) + \left(r-\delta+\frac{\sigma^2}{2}\right)(T-t)\right]$$

and $$d_2 = d_1 - \sigma\sqrt{T-t}$$

$$\Rightarrow \Delta_c(\dots) = \frac{\partial}{\partial s} v_c(s,t,r,\delta,\sigma)$$

After the chain rule & product rule:

$$\Delta_c(s,t,r,\delta,\sigma) = e^{-\delta(T-t)} \cdot N(d_1) > 0$$

Example. <u>A European time-T, strike-K put</u>

<u>Put-call Parity</u>

$$v_c(\dots) - v_p(\dots) = se^{-\delta(T-t)} - Ke^{-r(T-t)} \quad \Big/ \frac{\partial}{\partial s}$$

$$\Delta_c(\dots) - \Delta_p(\dots) = e^{-\delta(T-t)}$$

$$\Rightarrow \Delta_p(\dots) = \Delta_c(\dots) - e^{-\delta(T-t)}$$

$$= e^{-\delta(T-t)} \cdot N(d_1) - e^{-\delta(T-t)}$$

$$= -e^{-\delta(T-t)} N(-d_1) < 0$$

FORMULAS FOR OPTION GREEKS:

Delta (Δ)

Call: $e^{-\delta(T-t)}N(d_1)$,

Put: $-e^{-\delta(T-t)}N(-d_1)$

Gamma (Γ)

Call and Put: $\dfrac{e^{-\delta(T-t)}N'(d_1)}{S\sigma\sqrt{T-t}}$

Theta (θ)

Call: $\delta Se^{-\delta(T-t)}N(d_1)-rKe^{-r(T-t)}N(d_2)-\dfrac{Ke^{-r(T-t)}N'(d_2)\sigma}{2\sqrt{T-t}}$,

Put: Call Theta + $rKe^{-r(T-t)}-\delta Se^{-\delta(T-t)}$

Vega

Call and Put: $Se^{-\delta(T-t)}N'(d_1)\sqrt{T-t}$

Rho (ρ)

Call: $(T-t)Ke^{-r(T-t)}N(d_2)$,

Put: $-(T-t)Ke^{-r(T-t)}N(-d_2)$

Psi (ψ)

Call: $-(T-t)Se^{-\delta(T-t)}N(d_1)$,

Put: $(T-t)Se^{-\delta(T-t)}N(-d_1)$

(A) 7.32 million

(B) 7.42 million

(C) 7.52 million

(D) 7.62 million

(E) 7.72 million

$T = 1/4$ $K = 41.50$

8. You are considering the purchase of a 3-month 41.5-strike American call option on a nondividend-paying stock. → Same as European ⇒ B-S works!

You are given:

(i) The Black-Scholes framework holds.

(ii) The stock is currently selling for 40. $S(0) = 40$

(iii) The stock's volatility is 30%. $\sigma = 0.3$

(iv) The current call option delta is 0.5. $\Delta_C(S(0), 0) = 0.5$

Determine the current price of the option.

(A) $20 - 20.453 \int_{-\infty}^{0.15} e^{-x^2/2} dx$

(B) $20 - 16.138 \int_{-\infty}^{0.15} e^{-x^2/2} dx$

(C) $20 - 40.453 \int_{-\infty}^{0.15} e^{-x^2/2} dx$

(D) $16.138 \int_{-\infty}^{0.15} e^{-x^2/2} dx - 20.453$

(E) $40.453 \int_{-\infty}^{0.15} e^{-x^2/2} dx - 20.453$

6.

Problem # 8.

The current B·S call price:

$$v_c(S(0),0) = \underbrace{S(0)}_{\checkmark} \cdot \underbrace{N(d_1)}_{\checkmark} - \underbrace{K}_{\checkmark} \underbrace{e^{-r\cdot T}}_{?} \cdot N(d_2)$$

r = ?

Use (iv) to get the ccrfir (r):

$$\Delta_c(S(0),0) = 0.5 = N(d_1)$$

$$\Rightarrow \boxed{d_1 = 0}$$

$$\Rightarrow \frac{1}{\sigma\sqrt{T}} \underbrace{\left[\ln\left(\frac{40}{41.5}\right) + \left(r + \frac{0.09}{2}\right)\cdot\frac{1}{4}\right]}_{=0} = 0$$

$$\Rightarrow r + 0.045 = 4\ln\left(\frac{41.5}{40}\right)$$

$$\Rightarrow \boxed{r = 0.1023}$$

$$\Rightarrow d_2 = d_1 - \sigma\sqrt{T} = -0.15$$

$$\Rightarrow v_c(S(0),0) = 40\cdot 0.5 - 41.5\, e^{-0.1023(1/4)} \cdot \underbrace{N(-0.15)}$$

Recall:

$$N(x) = \int_{-\infty}^{x} \varphi(z)\,dz = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\,dz$$