Name:

M339J/M389J: Probability models for actuarial applications
Spring 2022
University of Texas at Austin
In-Term Exam II
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The maximum number of points on this exam is 100.

Problem 2.1. (5 points) Consider a severity random variable X which is modelled as a two-parameter Pareto with $\alpha = 2$ and an unknown value of the parameter θ. You are given that

$$\mathbb{E}[X - 10|X > 10] = \frac{3}{2}\,\mathbb{E}[X - 5|X > 5].$$

Find $\mathbb{E}[X - 15|X > 15]$.

Solution: For the Pareto distribution of X, we know that

$$\mathbb{E}[X - d|X > d] = \frac{d + \theta}{\alpha - 1}.$$

Since in this problem $\alpha = 2$, we know that

$$10 + \theta = \frac{3}{2}(5 + \theta) \quad \Rightarrow \quad 10 + 2\theta = 15 + 3\theta \quad \Rightarrow \quad \theta = 5.$$

So, our answer is $5 + 15 = 20$.

Problem 2.2. (5 points) Let the severity random variable X be modelled as exponential with mean 1000. There is an insurance policy on this loss with the deductible of 400. What is the expected value of the per-loss random variable under this policy?

Solution: The per-loss random variable is $Y^L = (X - 400)_+$. We have

$$\mathbb{E}[Y^L] = \mathbb{E}[(X - 400)_+] = \mathbb{E}[X] - \mathbb{E}[X \wedge 400].$$

Using the STAM tables, we obtain, with $\theta = 1000$,

$$\mathbb{E}[Y^L] = \theta - \theta(1 - e^{-\frac{400}{\theta}}) = \theta e^{-\frac{400}{\theta}} = 1000e^{-\frac{400}{1000}} = 1000e^{-\frac{2}{5}}.$$

Problem 2.3. (5 points) You will replace your hair dryer at failure or in two years, whichever occurs first. The dryer's age at failure is a random variable T which is modelled as uniform over the interval $[0, 4]$. What is the expected time at which the dryer is replaced?

Solution: The density of the random variable T is

$$f_T(t) = \begin{cases} \frac{1}{4} & 0 < x < 4 \\ 0 & \text{otherwise} \end{cases}$$

We are calculating $\mathbb{E}[T \wedge 2]$ with T uniform over $[0, 4]$. We have

$$\mathbb{E}[T \wedge 2] = \int_0^2 \frac{x}{4}\,dx + 2\left(\frac{1}{2}\right) = \left[\frac{x^2}{8}\right]_{x=0}^{2} + 1 = \frac{1}{2} + 1 = 3/2.$$

Problem 2.4. (5 points) Let T denote the time in minutes for a customer service representative to respond to a telephone inquiry. T is uniformly distributed on the interval with endpoints 8 minutes and 12 minutes. Let R denote the average rate, in customers per minute, at which the representative responds to inquiries. Which one of the following is a density function for R on the interval $(1/12, 1/8)$?

Solution: First of all, note that $R = 1/T$. So, the support of R is $(1/12, 1/8)$. For x within this interval, we have

$$F_R(x) = \mathbb{P}[R \leq x] = \mathbb{P}[1/T \leq x] = \mathbb{P}[\frac{1}{x} \leq T] = \int_{x^{-1}}^{12} \frac{1}{4}\,dt = \frac{1}{4}(12 - x^{-1}).$$

So,

$$f_R(x) = F_R'(x) = -\frac{1}{4}(-1)x^{-2} = \frac{1}{4x^2}.$$

Problem 2.5. (5 points) Let X be a two-point mixture. More precisely, let X be

- a two-parameter Pareto with mean equal to 10 and variance equal to 200 with probability 3/4;
- gamma distributed with parameters $\alpha = 4$ and $\theta = 10$ with probability 1/4.

What is the variance of the random variable X?

Solution: We have that

$$\begin{aligned}\mathbb{E}[X] &= \frac{3}{4}(10) + \frac{1}{4}(10)(4) = 17.5,\\ \mathbb{E}[X^2] &= \frac{3}{4}(200 + 100) + \frac{1}{4}(10)^2(4+1)(4) = 725.\end{aligned}$$

So,

$$Var[X] = 725 - 17.5^2 = 418.75$$

Problem 2.6. (10 points) Assume that X has a mixing distribution such that

$$X \mid \Lambda \sim Exponential(mean = \Lambda)$$

with $\Lambda \sim U(100, 200)$. Find the (unconditional) coefficient of variation of X.

Solution: In our usual notation, the coefficient of variation is

$$\frac{\sigma_X}{\mu_X} = \frac{SD[X]}{\mathbb{E}[X]}\,.$$

First, we focus on the mean of X. We know that

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X \mid \Lambda]] = \mathbb{E}[\Lambda] = 150.$$

Next, we focus on the variance of X. We have

$$\begin{aligned}Var[X] &= Var[\mathbb{E}[X \mid \Lambda]] + \mathbb{E}[Var[X \mid \Lambda]] = Var[\Lambda] + \mathbb{E}[\Lambda^2]\\ &= Var[\Lambda] + Var[\Lambda] + (\mathbb{E}[\Lambda])^2 = 2Var[\Lambda] + (\mathbb{E}[\Lambda])^2\end{aligned}$$

We are given that $\Lambda \sim U(100, 200)$. So,

$$Var[X] = 2(100)^2 \cdot \frac{1}{12} + (150)^2 = \frac{72500}{3}\,.$$

Finally, the coefficient of variation is

$$\frac{\sigma_X}{\mu_X} = \frac{\sqrt{\frac{72500}{3}}}{150} = 1.0364.$$

Problem 2.7. (5 points) The distribution of the random variable X is a spliced distribution with a continuous probability density function f_X. It is assumed that:

- the pdf is constant on $[0, 100]$, and
- the pdf is proportional to the pdf of an exponential distribution with mean 50 on $(100, \infty)$.

What is the probability that the random variable X is less than 50?

Solution: We know that

$$f_X(x) = \begin{cases} c & \text{for } 0 \le x \le 100 \\ \kappa e^{-x/50} & \text{for } 100 < x < \infty \end{cases}$$

with $c = \kappa e^{-100/50} = \kappa e^{-2}$ due to continuity. Since f_X is a density function, it must integrate to 1. Hence,

$$\begin{aligned} 1 &= \int_0^\infty f_X(x)\,dx = 100c + \kappa \int_{100}^\infty e^{-x/50}\,dx = 100c + \kappa\,(-50)\,e^{-x/50}\big|_{x=100}^{\infty} \\ &= 100c - 50\kappa(0 - e^{-2}) = 100c + 50\kappa e^{-2} = 100c + 50c = 150c. \end{aligned}$$

So, $c = 1/150$. The probability we were looking for is

$$\mathbb{P}[X \le 50] = 50c = \frac{50}{150} = \frac{1}{3}.$$

Problem 2.8. (10 points) Let X be the ground-up loss random variable. Assume that X has the two-parameter Pareto distribution with mean 100 and variance $15,000$.

Let B denote the expected payment **per loss** on behalf of an insurer which wrote a policy with an ordinary deductible of 500 and with no policy limit. How much is B?

Solution: Let $X \sim Pareto(\alpha, \theta)$. Then, according to our STAM tables,

$$\mathbb{E}[X] = \frac{\theta}{\alpha - 1} \quad \text{and} \quad \mathbb{E}[X^2] = \frac{2\theta^2}{(\alpha-1)(\alpha-2)} = \frac{\theta}{\alpha-1} \cdot \frac{2\theta}{\alpha-2}.$$

Using the mean given in the problem, we conclude that

$$\mathbb{E}[X] = \frac{\theta}{\alpha-1} = 100 \quad \Rightarrow \quad \theta = 100(\alpha - 1). \tag{2.1}$$

On the other hand,

$$\mathbb{E}[X^2] = Var[X] + (\mathbb{E}[X])^2 = 15000 + 100^2 = 25000.$$

Combining the expression for the second moment of the Pareto with the mean given in the problem and the value we obtained for the second raw moment, we get

$$100 \cdot \frac{2(100)(\alpha-1)}{\alpha-2} = 25000 \quad \Rightarrow \quad 200(\alpha-1) = 250(\alpha-2) \quad \Rightarrow \quad \alpha = 6.$$

Reusing the expression in (2.1), we get $\theta = 100(6-1) = 500$. Now, we consult the STAM tables to obtain

$$\begin{aligned}
\mathbb{E}[X \wedge 500] &= \frac{\theta}{\alpha-1}\left[1-\left(\frac{\theta}{500+\theta}\right)^{\alpha-1}\right] = 100\left[1-\left(\frac{500}{500+500}\right)^{6-1}\right] \\
&= 100\left[1-\left(\frac{1}{2}\right)^{5}\right] = 100\left[1-\frac{1}{32}\right] = 100\left(\frac{31}{32}\right) = 96.875.
\end{aligned}$$

Finally,

$$\mathbb{E}[Y^L] = \mathbb{E}[(X-500)_+] = 100 - 96.875 = 3.125.$$

Problem 2.9. (5 points) Let the ground-up loss X be modeled by a two-parameter Pareto distribution with parameters $\alpha = 2$ and $\theta = 400$. For an insurance policy on the above loss, there is a franchise deductible of 200. Find the expected value of the **per loss** random variable.

Solution: The expected value of the **per loss** random variable in the franchise deductible case is

$$\mathbb{E}[Y^L] = \mathbb{E}[(X-d)_+] + dS_X(d) = \mathbb{E}[X] - \mathbb{E}[X \wedge d] + dS_X(d).$$

In this problem, the ground-up loss is $X \sim \mathit{Pareto}(\alpha = 2, \theta = 400)$. So, using the STAM tables, we have

$$\begin{aligned}
S_X(d) &= \left(\frac{\theta}{d+\theta}\right)^{\alpha}, \\
\mathbb{E}[X] &= \frac{\theta^1 \cdot 1!}{\alpha-1} = \frac{\theta}{\alpha-1}, \\
\mathbb{E}[X \wedge d] &= \frac{\theta}{\alpha-1}\left(1-\left(\frac{\theta}{d+\theta}\right)^{\alpha-1}\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathbb{E}[Y^L] &= \frac{\theta}{\alpha-1} - \frac{\theta}{\alpha-1}\left(1-\left(\frac{\theta}{d+\theta}\right)^{\alpha-1}\right) + d\left(\frac{\theta}{d+\theta}\right)^{\alpha} \\
&= \frac{\theta}{\alpha-1}\left(\frac{\theta}{d+\theta}\right)^{\alpha-1} + d\left(\frac{\theta}{d+\theta}\right)^{\alpha} \\
&= \frac{400}{2-1}\left(\frac{400}{200+400}\right)^{2-1} + 200\left(\frac{400}{200+400}\right)^{2} \\
&= 400\left(\frac{2}{3}\right) + 200\left(\frac{2}{3}\right)^{2} = 355.556.
\end{aligned}$$

Problem 2.10. (5 points) Losses in a particular year follow a two-parameter Pareto distribution with parameters $\alpha = 4$ and $\theta = 11$.An insurance covers each loss subject to a deductible $d = 20$.

Calculate the **loss elimination ratio** for that year.

Solution: By definition, the loss elimination ratio is

$$\frac{\mathbb{E}[X \wedge d]}{\mathbb{E}[X]}.$$

In our case, using the provided tables, we get

$$\frac{\mathbb{E}[X \wedge d]}{\mathbb{E}[X]} = \frac{\frac{\theta}{\alpha-1}(1-(\frac{\theta}{d+\theta})^{\alpha-1})}{\frac{\theta}{\alpha-1}} = 1-\left(\frac{\theta}{d+\theta}\right)^{\alpha-1} = 1-\left(\frac{11}{20+11}\right)^{4-1} = 1-\left(\frac{11}{31}\right)^{3} \approx 0.9553.$$

Problem 2.11. (5 points) Let the ground-up loss X be exponentially distributed with mean \$400. An insurance policy has an ordinary deductible of \$200 and the maximum amount payable per loss of \$2200. Find the expected value of the amount paid (by the insurance company) **per positive payment**.

Solution: We are given $X \sim \mathit{Exponential}(\theta = 400)$, the deductible $d = 200$ and the policy limit $u - d = 2200$. We need to calculate $\mathbb{E}[Y^P]$ where $Y^P = Y^L \,|\, , Y^L > 0$ and

$$\begin{aligned}
Y^L &= \begin{cases} (X-d)_+, & X < u, \\ u-d, & X \geq u \end{cases} \\
&= (X \wedge u - d)_+.
\end{aligned}$$

By the memoryless property of the exponential distribution, we have that

$$Y = X - d \,|\, X > d$$

is also exponential with mean 400. So, using our STAM tables, we get

$$\mathbb{E}[Y^P] = \mathbb{E}[Y \wedge (u-d)] = \mathbb{E}[Y \wedge 2200] = 400(1-e^{-2200/400}) = 400(1-e^{-5.5}) \approx 398.3652914.$$

Problem 2.12. (10 points) Let the loss amounts X have the distribution function given by

$$F_X(x) = \begin{cases} 0, & x < 0 \\ x/100, & 0 \le x \le 100 \\ 1, & 100 < x. \end{cases}$$

An insurance policy has the following properties:
(i) there is an ordinary deductible of 20,
(ii) the maximum payment per loss that the insurer pays equals 60,
(iii) the insurance pays 80% of the amount of the loss in excess of the deductible and subject to the above maximum payment.

Let Y^P denote the per-payment random variable, i.e., the amount paid by the insurer given that a payment was made. Find $\mathbb{E}[Y^P]$.

Solution: Since this insurance pays $\alpha = 80\%$ of the loss amount above the deductible $d = 20$ and up to the amount of 60, the maximum loss amount u for which there is a payment satisfies

$$\alpha(u - d) = 60 \Rightarrow u = 95.$$

The expected value of the per-loss random variable is

$$\alpha(\mathbb{E}[X \wedge u] - \mathbb{E}[X \wedge d]).$$

One can recognize X as a uniform random variable on $(0, 100)$. For any a:

$$\mathbb{E}[X \wedge a] = \int_0^a S_X(x)\,dx = \int_0^a (1 - \frac{x}{100})\,dx = a - \frac{a^2}{200}\,.$$

Hence,

$$\begin{aligned}\alpha(\mathbb{E}[X \wedge u] - \mathbb{E}[X \wedge d]) &= \alpha(u - \frac{u^2}{200} - (d - \frac{d^2}{200})) = \alpha((u-d) - \frac{1}{200}(u^2 - d^2)) \\ &= \alpha(u-d)(1 - \frac{1}{200}(u+d)) = 0.8 \cdot 75 \left(1 - \frac{115}{200}\right) \\ &= 0.8 \cdot 75 \cdot \frac{85}{200} = \frac{51}{2} = 25.5.\end{aligned}$$

So, the per payment random variable has the expected value of

$$\mathbb{E}[Y^P] = \frac{25.5}{0.8} = 31.875.$$

Problem 2.13. (5 points) Let N represent the number of customers arriving during the morning hours and let N' represent the number of customers arriving during the afternoon hours at a diner. You are given:

(i) N and N' are Poisson distributed.
(ii) The first moment of N is less than the first moment of N' by 8.
(iii) The second raw moment of N is 60% of the second raw moment of N'.

Calculate the variance of N'.

Solution: We are given that

$$N \sim Poisson(\lambda) \quad \text{and} \quad N' \sim Poisson(\lambda').$$

Moreover,

$$\lambda = \lambda' - 8 \quad \text{and} \quad \lambda + \lambda^2 = 0.6(\lambda' + (\lambda')^2).$$

So,

$$\begin{aligned} \lambda + \lambda^2 = 0.6(\lambda + 8 + (\lambda+8)^2) \quad &\Rightarrow \quad \lambda + \lambda^2 = 0.6(\lambda + 8 + \lambda^2 + 16\lambda + 64) \\ &\Rightarrow \quad \lambda + \lambda^2 = 0.6(\lambda^2 + 17\lambda + 72) \\ &\Rightarrow \quad 0.4\lambda^2 - 9.2\lambda - 43.2 = 0. \end{aligned}$$

The only acceptable solution is $\lambda = 27$. So, $Var[N'] = \lambda' = 27 + 8 = 35$.

Problem 2.14. (5 points) The number of cars one sees passing by the local playground in an afternoon is modeled using a Poisson distribution with mean 25. The proportion of black cars in the stream is 1/5. The color of the cars is independent of the number of cars that drive by. What is the probability that exactly 5 black cars and exactly 10 non-black cars drive by in a particular afternoon?

Solution: Let N_1 denote the number of black cars and let N_2 denote the number of non-black cars. By the thinning theorem,

$$N_1 \sim Poisson(\lambda_1 = 5) \quad \text{and} \quad N_2 \sim Poisson(\lambda_2 = 20)$$

with N_1 and N_2 independent. Hence, the probability we are looking for is

$$\mathbb{P}[N_1 = 5, N_2 = 10] = \mathbb{P}[N_1 = 5]\mathbb{P}[N_2 = 10] = \left(e^{-5} \cdot \frac{5^5}{5!}\right)\left(e^{-20} \cdot \frac{20^{10}}{10!}\right) = 0.00102.$$

Problem 2.15. (5 points) The Clampetts conduct a geological study seeking sites for oil wells. Past date indicate that each exploratory oil well should have a 20% chance of striking oil independent of all other wells. The Clampetts are modest folk and they will stop the study when they strike oil for the third time. What is the probability that they drill exactly ten exploratory oil wells total before stopping the study?

Solution: This is a case where the negative binomial distribution is appropriate. The parameters r and β are chosen so that r counts the required number of successes 3 (i.e., the wells where oil is struck) and β satisfies that $\frac{1}{1+\beta}$ is the probability of success $20\% = 1/5$ (i.e., the probability that oil is struck in an exploratory well). So, the total number of failures before the 3^{rd} success in encountered is

$$N \sim NegBinomial(r = 3, \beta = 4).$$

Since we are looking for the probability that the total number of exploratory wells is equal to 10, this means that we are looking for the probability that the number of failures is 7. Using our STAM tables, we get

$$\mathbb{P}[N=7]=\binom{9}{2}\left(\frac{1}{5}\right)^3\left(\frac{4}{5}\right)^7=\frac{36\cdot 4^7}{5^{10}}=0.060397978.$$

Problem 2.16. (5 points) Let the independent random variables X_1, X_2 and X_3 all have the following probability mass function:

$$p_{X_1}(0)=1/4, \quad p_{X_1}(1)=1/2, \quad p_{X_1}(2)=1/4.$$

Let $X=X_1X_2X_3$. What is the probability generating function of X?

Solution: The random variable X has the following probability mass function:

$$\begin{aligned}
p_X(0) &= 1-\left(\frac{3}{4}\right)^3=0.578125,\\
p_X(1) &= \left(\frac{1}{2}\right)^3=0.125,\\
p_X(2) &= 3\left(\frac{1}{2}\right)^2\left(\frac{1}{4}\right)=0.1875,\\
p_X(4) &= 3\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)^2=0.09375,\\
p_X(8) &= \left(\frac{1}{4}\right)^3=0.015625.
\end{aligned}$$

So, the probability generating function of X equals

$$P_X(s)=\mathbb{E}[s^X]=0.578125+0.125s+0.1875s^2+0.09375s^4+0.015625s^8.$$

Problem 2.17. (5 points) Let the random variable X have a Weibull distribution with parameters $\tau=2$ and $\theta=10$. What is the 75^{th} percentile of this distribution?

Solution: Let F_X denote the cumulative distribution function of X. We need to solve for x in

$$F_X(x)=\frac{3}{4}.$$

From the STAM tables, we learn that

$$F_X(x)=1-e^{-\left(\frac{x}{10}\right)^2}.$$

So, we solve for x:

$$\begin{aligned}
\frac{3}{4}=1-e^{-\left(\frac{x}{10}\right)^2} \quad &\Leftrightarrow \quad \frac{1}{4}=e^{-\left(\frac{x}{10}\right)^2} \quad \Leftrightarrow \quad \ln(0.25)=-\left(\tfrac{x}{10}\right)^2\\
&\Leftrightarrow \quad \sqrt{\ln(4)}=\tfrac{x}{10} \quad \Leftrightarrow \quad x=10\sqrt{(\ln(4))}=11.7741
\end{aligned}$$