M378K: September 18th, 2024.

Joint Distributions: The Discrete Case

Example 5.2.1. We independently throw two dice and record the results as Y_1 and Y_2, resp.

joint pmf: $p_{ij} = p_{Y_1,Y_2}(i,j) = \mathbb{P}[Y_1 = i, Y_2 = j]$

In this example. $p_{ij} = \frac{1}{36}$ for $1 \leq i,j \leq 6$

Our joint distribution table.

	1	2	3	4	5	6
1	$\frac{1}{36}$					
2						
3						
4						
5						
6						

Define $\boxed{Z = Y_1 + Y_2}$

Q: What's the joint distribution table for $(\underline{Y_1}, Z)$?

Y_1 \ Z	2	3	4	5	6	7	8	9	10	11	12
1	$\frac{1}{36}$						0	0	0	0	0
2	0							0	0	0	0
3	0	0							0	0	0
4	0	0	0							0	0
5	0	0	0	0							0
6	0	0	0	0	0						

$Y_1 \backslash Z$	2	3	4	5	6	7	8	9	10	11	12
1	$\frac{1}{36}$						0	0	0	0	0
2	0							0	0	0	0
3	0	0							0	0	0
4	0	0	0							0	0
5	0	0	0	0							0
6	0	0	0	0	0						
	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Joint Distributions: The continuous Case.

Recall that for a continuous random variable Y w/ pdf f_Y, we can can calculate probabilities using

$$\mathbb{P}[Y \in [a,b]] = \mathbb{P}[a \leq Y \leq b]$$
$$= \int_a^b f_Y(y)\,dy$$

In multiple dimensions:

Say that the random vector $(Y_1, Y_2, \ldots, Y_n)$ is jointly continuous w/ density $f_{Y_1,\ldots,Y_n}$.

Then,

$$\mathbb{P}[Y_1 \in [a_1,b_1], Y_2 \in [a_2,b_2], \ldots, Y_n \in [a_n,b_n]] =$$
$$= \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_n}^{b_n} f_{Y_1,\ldots,Y_n}(y_1, y_2, \ldots, y_n)\,dy_n \cdots dy_2\,dy_1$$

For "any" region $A \subseteq \mathbb{R}^n$,

$$\mathbb{P}[(Y_1, \dots, Y_n) \in A] = \int_A \cdots \int f_{Y_1, \dots, Y_n}(y_1, \dots, y_n)\, dy_n \cdots dy_1$$

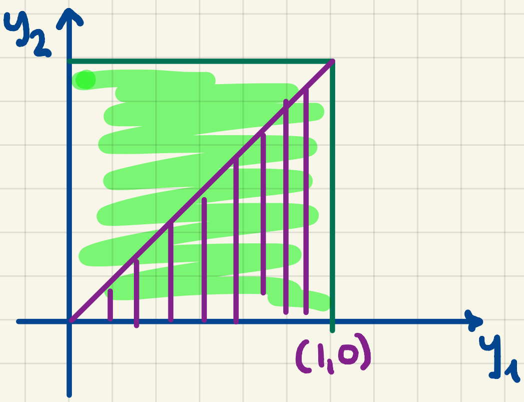
<u>Example.</u> (Y_1, Y_2) ... represents a point chosen @ random in the unit square $[0,1]^2$

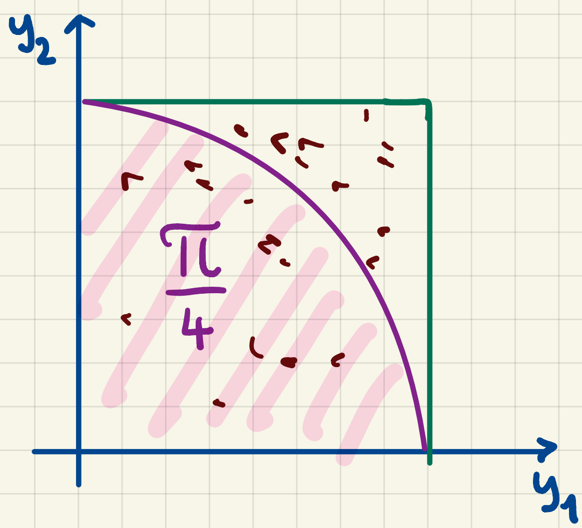
$$f_{Y_1,Y_2}(y_1,y_2) = \begin{cases} 1 & \text{if } (y_1,y_2) \in [0,1]^2 \\ 0 & \text{otherwise} \end{cases}$$

OR

$$f_{Y_1,Y_2}(y_1,y_2) = 1 \cdot \mathbb{1}_{[0,1]\times[0,1]}(y_1,y_2)$$

$$\mathbb{P}[\boxed{Y_1 > Y_2}] = \cancel{?} = \frac{1}{2}$$





$$A = \left\{ (y_1,y_2) \in [0,1]^2 : \sqrt{y_1^2 + y_2^2} \leq 1 \right\}$$

$$\mathbb{P}[(Y_1, Y_2) \in A] = \iint_A f_{Y_1 Y_2}(y_1, y_2)\, dy_2\, dy_1 = \iint_A dy_2\, dy_1$$

$$= \text{area}(A) = \frac{\pi}{4}$$

<u>Example.</u> Let (Y_1, Y_2) be jointly continuous w/ pdf

$$f_{Y_1, Y_2}(y_1, y_2) = 6y_1 \quad , \quad 0 \leq y_1 \leq y_2 \leq 1$$

and 0 otherwise

OR

$$f_{Y_1, Y_2}(y_1, y_2) = 6y_1 \cdot \mathbb{1}_{[0 \leq y_1 \leq y_2 \leq 1]}$$

$$\mathbb{P}[Y_1 > \tfrac{1}{2}, Y_2 > \tfrac{1}{2}] =$$

$$= \int_{\frac{1}{2}}^{1} \int_{\frac{1}{2}}^{1} 6y_1 \cdot \mathbb{1}_{[0 \leq y_1 \leq y_2 \leq 1]}\, dy_2\, dy_1$$

$$= \int_{\frac{1}{2}}^{1} 6y_1 \underbrace{\int_{\cancel{\frac{1}{2}}\, y_1}^{1} dy_2}_{= 1-y_1}\, dy_1 = \int_{\frac{1}{2}}^{1} 6y_1(1-y_1)\, dy_1$$

$$= 6\left(\int_{\frac{1}{2}}^{1} y_1\, dy_1 - \int_{\frac{1}{2}}^{1} y_1^2\, dy_1 \right) = 6\left(\frac{y_1^2}{2} \Big|_{y_1 = \frac{1}{2}}^{1} - \frac{y_1^3}{3} \Big|_{y_1 = \frac{1}{2}}^{1} \right)$$

$$= 6\left(\frac{1}{2} - \frac{1}{8} - \left(\frac{1}{3} - \frac{1}{24}\right)\right) =$$

$$= \cancel{6} \cdot \frac{12 - 3 - 8 + 1}{\underset{4}{\cancel{24}}} = \frac{1}{2}$$

$\square$