M339D: March 18^{th}, 2024.

More on the Forward Trees.

$$u = e^{rh + \sigma\sqrt{h}}$$
$$d = e^{rh - \sigma\sqrt{h}}$$

Q: Do u and d satisfy the no-arbitrage condition?

$\longrightarrow$:
$$d < e^{rh} < u$$
$$\Leftrightarrow \qquad \text{for the forward tree}$$
$$e^{rh-\sigma\sqrt{h}} < e^{rh} < e^{rh+\sigma\sqrt{h}}$$
$$\Leftrightarrow$$
$$\cancel{e^{rh}} \cdot e^{-\sigma\sqrt{h}} < \underbrace{\cancel{e^{rh}}}_{1} < \cancel{e^{rh}} \cdot e^{\sigma\sqrt{h}}$$
$$\Leftrightarrow$$
$$e^{-\sigma\sqrt{h}} < 1 < e^{\sigma\sqrt{h}} \qquad \sigma > 0$$

✓

Q: What is S_u / S_d?

$\longrightarrow$:
$$\frac{S_u}{S_d} = \frac{u \cdot \cancel{S(0)}}{d \cdot \cancel{S(0)}} = \frac{\cancel{e^{rh}} \cdot e^{\sigma\sqrt{h}}}{\cancel{e^{rh}} \cdot e^{-\sigma\sqrt{h}}} = \boxed{e^{2\sigma\sqrt{h}}}$$ ✓

Example. Consider this one-period tree w/ the time-horizon of one-quarter year.

$S_u = 105$

$100 = S(0)$

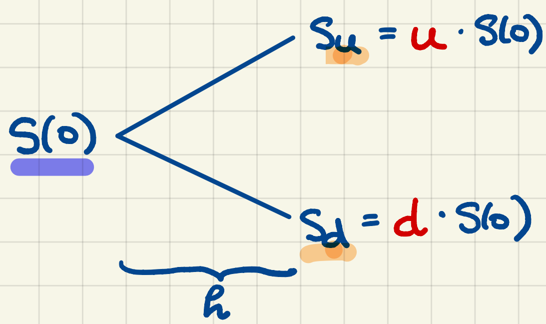
$S_d = 98$

Q: If this is a forward tree, what is the volatility?

$\longrightarrow$:
$$\frac{S_u}{S_d} = e^{2\sigma\sqrt{h}}$$
$$\frac{105}{98} = e^{\cancel{2}\sigma\cancel{\sqrt{1/4}}} \quad \Rightarrow \quad \sigma = \ln\left(\frac{105}{98}\right)$$ □

Binomial Option Pricing.

Stock Price Tree.



populating the tree

Goal: Pricing a European-style derivative security w/ exercise date @ the end of the tree, i.e., $T=h$.

It is completely determined by its payoff f'tion: $v(\cdot)$

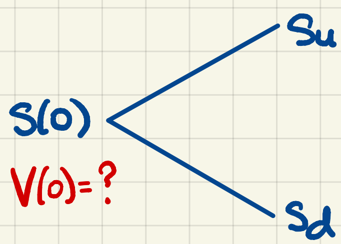
e.g., for a call: $v_c(s) = (s-K)_+$,

or for a put: $v_p(s) = (K-s)_+$,

or something a bit different: $\boxed{v(s) = (s^2-K)_+}$

The payoff of the derivative security is a random variable

$$\boxed{V(T) := v(S(T))}$$



	PAYOFF:		REPLICATING PORTFOLIO
S_u	$V_u := v(S_u)$	=	$\Delta \cdot S_u + Be^{rh}$
S_d	$V_d := v(S_d)$	=	$\Delta \cdot S_d + Be^{rh}$

population →

← pricing

In the binomial model, any derivative security can be **REPLICATED** w/ a portfolio of this form:

@ time 0

- Δ shares of stock
  - $\Delta > 0$ buying
  - $\Delta = 0$ "nothing"
  - $\Delta < 0$ short-selling
- and B @ the ccrfir r
  - $B > 0$ lending (buying a bond)
  - $B = 0$ "nothing"
  - $B < 0$ borrowing (issuing a bond)

If we can find Δ and B, then $\boxed{V(0) = \Delta \cdot S(0) + B}$

We get a system of two equations w/ two unknowns:

$$\left.\begin{aligned} \Delta \cdot S_u + \cancel{Be^{rh}} &= V_u \\ -\Delta \cdot S_d - \cancel{Be^{rh}} &= -V_d \end{aligned}\right\} -$$

$$\Delta\,(S_u - S_d) = V_u - V_d$$

$$\boxed{\Delta = \frac{V_u - V_d}{S_u - S_d}}$$

unitless

$$\frac{V_u - V_d}{S_u - S_d} \cdot S_u + Be^{rh} = V_u$$

$$Be^{rh} = V_u - \frac{V_u - V_d}{\cancel{S(0)}(u-d)} \cdot \cancel{S(0)} \cdot u = \frac{\cancel{u \cdot V_u} - d \cdot V_u - \cancel{u \cdot V_u} + u \cdot V_d}{u-d}$$

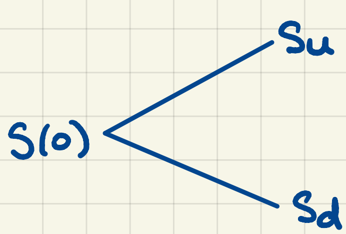
$$\boxed{B = e^{-rh} \cdot \frac{u \cdot V_d - d \cdot V_u}{u-d}}$$

cash

Graphical Interpretation.

Consider a European call w/ exercise date @ the end of the period and strike price K such that

$$\boxed{S_d < K < S_u}$$



PAYOFF:

$V_u = S_u - K$

$V_d = 0$

Recall: $v_c(s) = (s-K)_+$

In the replicating portfolio:

- $\Delta_c = \dfrac{V_u - V_d}{S_u - S_d} = \dfrac{S_u - K}{S_u - S_d} \in (0,1)$ Buy a fraction of a stock!
- $B_c = e^{-rh} \cdot \dfrac{u \cdot V_d - d V_u}{u-d} = -e^{-rh} \cdot \dfrac{d \cdot V_u}{u-d} < 0$ Borrowing!