M339W/389W Financial Mathematics for Actuarial Applications
University of Texas at Austin
Mock In-Term Exam 2
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Notes: This is a closed book and closed notes exam. The maximum number of points is 50.
Time: 50 minutes

Problem 2.1. (5 points) Assume the Black-Scholes model. Let the current price of a continuous-dividend-paying stock be \$80. Its dividend yield is 0.02 and its volatility is 0.25.

Let the continuously compounded, risk-free interest rate be 0.04.

Find the price of a 3−month, \$75-strike European call option on the above stock.

(a) 6.84
(b) 7
(c) 7.22
(d) 7.51
(e) None of the above.

Solution: (c)
In our usual notation,

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right)T\right] \\ &= \frac{1}{0.25\sqrt{\frac{1}{4}}}\left[\ln\left(\frac{80}{75}\right) + \left(0.04 - 0.02 + \frac{(0.25)^2}{2}\right)\left(\frac{1}{4}\right)\right] = 0.6188 \approx 0.62, \\ d_2 &= d_1 - \sigma\sqrt{T} = 0.6188 - 0.25\sqrt{\frac{1}{4}} = 0.4938 \approx 0.49. \end{aligned}$$

Using the standard normal tables, we get

$$N(d_1) = N(0.62) = 0.7324 \quad \text{and} \quad N(d_2) = N(0.49) = 0.6879.$$

Finally, the Black-Scholes price of our call option is

$$\begin{aligned} V_C(0) &= S(0)e^{-\delta T}N(d_1) - Ke^{-rT}N(d_2) \\ &= 80e^{-0.02(0.25)}(0.7324) - 75e^{-0.04(0.25)}(0.6879) = 7.220625. \end{aligned}$$

Problem 2.2. (5 points) Assume the Black-Scholes framework for the evolution of a stock price. The stock pays no dividends. Consider a one-year European call on this stock. You are given the following:

- the call's delta is 0.6591,

- under the risk-neutral probability measure the probability that the option is in the money at expiration is 0.3409.

What is the volatility of this call option?

(a) 0.3409
(b) 1.6985
(c) 2.0713
(d) 3.0257
(e) None of the above.

Solution: (b)
The volatility of the call option is

$$\sigma_C = \Omega_c \sigma.$$

We are given that

$$N(d_1) = 0.6591 \quad \text{and} \quad N(d_2) = 0.3409.$$

So, using the standard normal table, we get

$$d_1 = -d_2 = 0.41$$

Hence,

$$\sigma = d_1 - d_2 = 0.82.$$

Next, we need to calculate the elasticity of the call option. So,

$$\Omega_C = \frac{S(0)\Delta_C}{V_C(0)} = \frac{S(0)N(d_1)}{S(0)N(d_1) - Ke^{-r}N(d_2)} = \frac{1}{1 - \frac{Ke^{-r}}{S(0)} \times \frac{N(d_2)}{N(d_1)}}$$

Reusing the given value of the delta, we get

$$d_1 = 0.41 = \frac{1}{0.82}\left[\ln\left(\frac{S(0)}{Ke^{-r}}\right) + \frac{(0.82)^2}{2}\right] \quad \Rightarrow \quad \ln\left(\frac{S(0)}{Ke^{-r}}\right) = 0.$$

So,

$$\Omega_C = \frac{1}{1 - \left(\frac{0.3409}{0.6591}\right)} = 2.07134.$$

Finally, $\sigma_C = 2.07134(0.82) = 1.6985$.

Problem 2.3. (5 points) The current price of a non-dividend-paying stock is \$25 per share. A market-maker writes a three-month European put option on this stock and proceeds to delta-hedge it. The put premium is \$2.50, its delta is -0.30, its gamma is 0.04, and its theta is -0.01 per day.

The continuously compounded risk-free interest rate is 0.04.

Assuming that the stock price does not change, what is the **approximate** overnight profit for the market-maker?

(a) 0.011096

(b) 0.018026

(c) 0.021064

(d) -0.014062

(e) None of the above.

Solution: (a)
The initial cost of the total delta-hedged portfolio is

$$-2.50 + (-0.3)(25) = -10.$$

The approximate put price after one day is, according to the delta-gamma-theta approximation,

$$2.50 - 0.01 = 2.49.$$

So, the overnight profit is

$$-2.49 + (-0.3)(25) + 10e^{0.04/365} = -9.99 + 10e^{0.04/365} = 0.011096.$$

Problem 2.4. A market-maker sells option I for \$10. This option's delta is 0.6557 and its gamma is 0.02. The market maker proceeds to delta-gamma hedge this commitment by trading in the underlying and also in option II on the same stock. The latter option's price is \$4.70, its delta is 0.5794 and its gamma is 0.04.

What is the market-maker's resulting position in option II?

(a) Buy 0.5 of option II.
(b) Write 0.5 of option II.
(c) Buy 2 of option II.
(d) Write 2 of option II.
(e) None of the above.

Solution: (a)
With n_{II} denoting the position in option II, to achieve gamma-neutrality we need

$$-0.02 + n_{II}(0.04) = 0 \quad \Rightarrow \quad n_{II} = 0.5.$$

Problem 2.5. Assume the Black-Scholes model. The current price of a particular stock is \$100 per share.

Here is some information about the current prices and Greeks for a pair of European call options on this stock:

Strike price	80	90
Price	4.32	2.15
Δ	0.34	0.24

What is the current elasticity of the $(80, 90)-$bull spread constructed using the above options?

(a) 4.61

(b) 5.28

(c) 6.34
(d) 7.16
(e) None of the above.

Solution: (a)
The above call bull spread consists of a long 80−strike call and a short 90−strike call. So, its price and its delta are

$$v(S(0),0) = 4.32 - 2.15 = 2.17 \quad \text{and} \quad \Delta(S(0),0) = 0.34 - 0.24 = 0.1.$$

So, the elasticity of the call bull spread equals

$$\Omega(S(0),0) = \frac{\Delta(S(0),0)S(0)}{v(S(0),0)} = \frac{0.1(100)}{2.17} = 4.60829.$$

Problem 2.6. (5 points) Which of the following statements is **FALSE**?

(a) The call theta is negative.
(b) The put gamma is positive.
(c) The vegas of otherwise identical calls and puts are equal.
(d) The put rho is negative.
(e) None of the above.

Solution: (e)

Problem 2.7. Assume the Black-Scholes model. Let the current price of a continuous-dividend-paying stock be \$80. Its dividend yield is 0.03.

Let the continuously compounded risk-free interest rate be 0.05.

Consider a European call on the above stock with a quarter-year to exercise and with the strike price equal to $\$80e^{0.01}$. The current delta of this call option is 0.496264. What is the volatility of the stock?

(a) 0.15
(b) 0.2
(c) 0.25
(d) 0.3
(e) None of the above.

Solution: (b)
The Black-Scholes call delta at time−0 is

$$e^{-\delta T}N(d_1(S(0),0)) = 0.496264 \quad \Rightarrow \quad N(d_1(S(0),0)) = e^{0.03(0.25)}(0.496264) = 0.5.$$

Hence, $d_1(S(0),0) = 0$. By the definition of d_1, we have that

$$\ln\left(\frac{80}{80e^{0.01}}\right) + \left(0.05 - 0.03 + \frac{\sigma^2}{2}\right) \cdot \frac{1}{4} = 0.$$

So,

$$-0.01 + \frac{0.02}{4} + \frac{\sigma^2}{8} = 0 \quad \Rightarrow \quad \frac{\sigma^2}{8} = 0.005 \quad \Rightarrow \quad \sigma^2 = 0.04.$$

Finally, we conclude that $\sigma = 0.2$.

Problem 2.8. Assume the Black-Scholes model. The current price of a continuous-dividend-paying stock is \$100. Its volatility is 0.25.

The continuously compounded risk-free interest rate is 0.04.

Consider an investor who buys a share of the above stock. Now, they want to create a $\Delta-$neutral portfolio by writing European call options on this stock. You are given the following information about the call options:

- the time to exercise is T,
- in our usual notation, $\delta T = 0.02$,
- in our usual notation, $d_1(S(0), 0) = 0.25$.

What is the number of call options that need to be written at time-0 to create a $\Delta-$neutral portfolio?

(a) 11.9043

(b) 13.2603

(c) 15.0457

(d) 17.0403

(e) None of the above.

Solution: (e)
The Δ of the call option at time-0 is

$$\Delta_C(S(0), 0) = e^{-\delta T} N(d_1(S(0), 0)) = e^{-0.02} N(0.25) = e^{-0.02}(0.5987) = 0.586845.$$

Let $n_C(S(0), 0)$ denote the number of call options needed for the initial delta-hedge. Then,

$$1 + n_C(S(0), 0)(0.586845) = 0 \quad \Rightarrow \quad n_C(S(0), 0) = -\frac{1}{0.586845} = -1.70403.$$

Problem 2.9. Assume the Black-Scholes model. Let the current price of a non-dividend-paying stock be \$50. The stock's volatility is 0.25.

The continuously compounded risk-free interest rate is 0.03.

An investor writes a European call option with the following properties:

- the option is at the money at time-0,
- the option's time to exercise is four months,
- the option's premium is \$3.03,
- the option's delta is 0.5557.

The investor then, delta-hedges the written call by buying the underlying stock. The delta-hedge is not rebalanced for a month.

After one month, the investor realizes that the option is again at-the-money, and decides to liquidate the portfolio. What is their **exact** profit?

(a) -0.07

(b) 0.39

(c) 1.24

(d) 3.57

(e) None of the above.

Solution: (b)
The initial cost of the portfolio is

$$v(S(0), 0) = -3.03 + 0.5557(50) = 24.755.$$

After one month, the call option has a quarter-year to exercise. Our next task is to calculate the call's price. We start with d_1 and d_2, as usual.

$$d_1(S(1/12), 1/12) = \frac{1}{0.25\sqrt{1/4}}\left[\ln\left(\frac{50}{50}\right) + \left(0.03 + \frac{(0.25)^2}{2}\right) \cdot \frac{1}{4}\right] = \frac{0.03 + 0.03125}{0.25}\sqrt{\frac{1}{4}} = 0.1225,$$

$$d_2(S(1/12), 1/12) = d_1(S(1/12), 1/12) - 0.25\sqrt{\frac{1}{4}} = 0.1225 - \frac{0.25}{2} = -0.0025.$$

Using the standard normal tables, we get

$$N(d_1(S(1/12), 1/12)) = 0.5478 \quad \text{and} \quad N(d_2(S(1/12), 1/12)) = 0.5.$$

So, the call's price after one month equals

$$v_C(S(1/12), 1/12) = 50(0.5478 - e^{-0.03(0.25)}(0.5)) = 2.5768.$$

Therefore, the investor's profit is

$$-2.5768 + 0.5557(50) - 24.755e^{0.03/12} = 0.391235$$

Problem 2.10. Assume the Black-Scholes framework. For an at-the-money, T-year European call option on a non-dividend-paying stock you are given that its delta equals 0.5832. What is the delta of an otherwise identical option with exercise date at time $2T$?

(a) 0.62

(b) 0.66

(c) 0.70

(d) 0.74

(e) None of the above.

Solution: (a)
See `https://gordanz.github.io/cudina/M339W/exams/m339w-quiz-seven-delta-solns.pdf`

Problem 2.11. (5 points) Assume the Black-Scholes model. Bertie Wooster was looking at stock-price and option data from yesterday. He decides to pose his friend Tuppy Glossop a riddle. Bertie tells Tuppy the following about yesterday's price of a stock and information on an option on this stock:

- the stock price yesterday was greater than \$77;
- the option's price was \$2.45;

- the option's delta was -0.1814;
- the option's gamma was 0.04;
- the option's theta was 0.01 **per day**.

Tuppy is allowed to see today's stock price and today's option price. They turn out to be \$80 and \$2.20, respectively. What is Tuppy going to guess to be yesterday's stock price?

(a) \$77.08
(b) \$77.27
(c) \$78.63
(d) \$78.22
(e) None of the above.

Solution: (d)
Using the delta-gamma-theta approximation, we get that

$$2.20 = 2.45 + (-0.1814)ds + \frac{1}{2}(0.04)(ds)^2 + 0.01.$$

Simplifying the above quadratic equation, we obtain

$$0.02(ds)^2 - 0.1814ds + 0.26 = 0 \quad \Leftrightarrow \quad (ds)^2 - 9.07ds + 13 = 0.$$

Solving for ds, we get

$$ds_{1,2} = \frac{9.07 \pm \sqrt{9.07^2 - 4(13)}}{2} = \frac{9.07 \pm 5.50135}{2}.$$

The two solutions are

$$ds_1 = 1.7843 \quad \text{and} \quad ds_2 = 7.2857.$$

We conclude that yesterday's stock price was \$78.22.

Problem 2.12. (5 points) Assume the Black-Scholes model. Let the current price of a non-dividend-paying stock be equal to \$80 per share. Its volatility is 0.20.

The continuously compounded risk-free interest rate is 0.04.

Consider a one-year, at-the-money European call option on the above stock. The current delta of the call option is 0.6179. What is the current gamma of the call option?

(a) 0
(b) 0.01256
(c) 0.0238
(d) 0.03862
(e) None of the above.

Solution: (c)
Using our IFM tables, or differentiating, we know that

$$\Gamma_C(S(0), 0) = \frac{e^{-\delta T} N'(d_1(S(0), 0))}{S(0)\sigma\sqrt{T}} = \frac{N'(d_1(S(0), 0))}{S(0)\sigma}$$

From the given value of the call's delta, we have that

$$N(d_1(S(0),0)) = 0.6179 \quad \Rightarrow \quad d_1(S(0),0) = 0.3.$$

So, the gamma of the call equals

$$\Gamma_C(S(0),0) = \frac{N'(0.3)}{80(0.2)} = \frac{\frac{1}{\sqrt{2\pi}}e^{-\frac{0.09}{2}}}{16} = 0.0238367.$$