M339D : November 20^{th}, 2024.

Problem 14.2. (5 pts) Let the stochastic process $S = \{S(t); t \geq 0\}$ denote the stock price. The stock's rate of ~~appreciation~~ return is 10% while its volatility is 0.30. Then,

(a) $Var[\ln(S(t))] = 0.3t$
(b) $Var[\ln(S(t))] = 0.09t^2$
(c) $Var[\ln(S(t))] = 0.09t$
(d) $Var[\ln(S(t))] = 0.09$
(e) None of the above.

→ : In the Black-Scholes model:

$$S(t) = S(0)e^{(r-\frac{\sigma^2}{2})\cdot t + \sigma\sqrt{t}\cdot Z} \quad \text{w/ } Z \sim N(0,1)$$

$$\ln(S(t)) = \underbrace{\boxed{\ln(S(0)) + (r-\frac{\sigma^2}{2})\cdot t}}_{\text{deterministic}} + \sigma\sqrt{t}\cdot Z$$

$$Var[\ln(S(t))] = Var[\sigma\sqrt{t}\cdot Z] = \sigma^2\cdot t\cdot Var[Z] = \sigma^2\cdot t$$

$\square$

Problem 14.3. (5 pts) Assume the Black-Scholes framework. Let the current price of a non-dividend-paying stock be equal to $S(0) = 95$ and let its volatility be equal to 0.35. Consider a European call on that stock with strike 100 and exercise date in 9 months. Let the risk-free continuously compounded interest rate be 6% per annum.

$T = 3/4$

Denote the price of the call by $V_C(0)$. Then,

(a) $V_C(0) < \$5.20$

(b) $\$5.20 \leq V_C(0) < \$7.69$

(c) $\$7.69 \leq V_C(0) < \$9.04$

(d) $9.04 \leq V_C(0) < \$11.25$

(e) None of the above.

→: We'll use the Black·Scholes call·price:

$$V_c(0) = S(0)N(d_1) - Ke^{-rT} \cdot N(d_2)$$

w/ $d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)\cdot T\right]$

and $d_2 = d_1 - \sigma\sqrt{T}$

1st Calculate d_1 and d_2 ✓

2nd Use the standard normal tables or 'R' ✓

3rd Combine into the BS pricing formula.

$$d_1 = \frac{1}{0.35\sqrt{3/4}}\left[\ln\left(\frac{95}{100}\right) + \left(0.06 + \frac{0.35^2}{2}\right)\cdot\left(\frac{3}{4}\right)\right] = 0.1307 \approx 0.13$$

$$d_2 = d_1 - \sigma\sqrt{T} = d_1 - 0.35\sqrt{3/4} = -0.1733 \approx -0.17$$

$N(d_1) \approx N(0.13) = 0.5517$

$N(d_2) \approx N(-0.17) = 0.4325$

$$V_c(0) = 95 \cdot 0.5517 - 100e^{-0.06(3/4)} \cdot 0.4325 = 11.06$$

□

Problem 14.4. Assume the Black-Scholes setting. Let $S(0) = \$63.75$, $\sigma = 0.20$, $r = 0.055$. The stock pays no dividend and the option expires in 50 days (simplify the number of days in a year to 360).

What is the price of a \$60-strike European put?

(a) 0.66
(b) 0.55
(c) 0.44
(d) 0.37
(e) None of the above.

$\rightarrow$: The Black·Scholes put price is

$$V_P(0) = Ke^{-rT} \cdot N(-d_2) - S(0)N(-d_1)$$

$$d_1 = \frac{1}{0.2\sqrt{\frac{50}{360}}}\left[\ln\left(\frac{63.75}{60}\right) + \left(0.055 + \frac{0.04}{2}\right)\cdot\left(\frac{50}{360}\right)\right]$$

$d_1 = \underline{0.9531} \approx 0.95$

$d_2 = d_1 - \sigma\sqrt{T} = \underline{0.8786} \approx 0.88$

$N(-d_1) = N(-0.95) = \underline{0.1711}$

$N(-d_2) = N(-0.88) = \underline{0.1894}$

$$V_P(0) = 60e^{-0.055\cdot\left(\frac{50}{360}\right)} \cdot 0.1894 - 63.75 \cdot 0.1711$$

$V_P(0) = \underline{0.37}$ □

8. Let $S(t)$ denote the price at time t of a stock that pays no dividends. The Black-Scholes framework holds. Consider a European call option with exercise date T, $T > 0$, and exercise price $\boxed{S(0)e^{rT}}$, where r is the continuously compounded risk-free interest rate.

K

You are given:

(i) $S(0) = \$100$

(ii) $\boxed{T = 10}$

(iii) $\text{Var}[\ln S(t)] = 0.4t, \quad t > 0.$ $\longrightarrow$ $\boxed{\sigma = \sqrt{0.4}}$ ✓

Determine the price of the call option.

(A) $7.96

(B) $24.82

(C) $68.26

(D) $95.44

(E) There is not enough information to solve the problem.

$$V_c(0) = S(0)N(d_1) - Ke^{-rT}N(d_2)$$

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{\cancel{S(0)}}{\cancel{S(0)}e^{rT}}\right) + \left(r + \frac{\sigma^2}{2}\right)\cdot T\right]$$

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\cancel{-rT} + \cancel{rT} + \frac{\sigma^2}{2}\cdot T\right] = \boxed{\frac{\sigma\sqrt{T}}{2}}$$

$$d_2 = d_1 - \sigma\sqrt{T} = -\frac{\sigma\sqrt{T}}{2}$$

$$\Rightarrow d_1 = \frac{\sqrt{0.4}\sqrt{10}}{2} = 1 \text{ and } d_2 = -1$$

$$V_c(0) = S(0)N(d_1) - S(0)\cancel{e^{rT}}\cdot\cancel{e^{-rT}}N(d_2) = S(0)(N(1) - N(-1))$$

$$V_c(0) = 100(2\cdot N(1) - 1) = 100(2\cdot 0.8413 - 1) = 68.26$$

Problem. Assume the Black-Scholes model.

For a European call, the strike is $S(0)e^{rT}$ where T is the exercise date.

The price of a call w/ one year to exercise is $0.6 \cdot S(0)$

Find the price of such a call option w/ three months to exercise in terms of $S(0)$.

→: For any T, from the previous problem, we know

$$V_c(0,T) = S(0)\left(2 \cdot N\left(\frac{\sigma\sqrt{T}}{2}\right) - 1\right)$$