M339D: April 9th, 2025.

Example [cont'd]. In the Black-Scholes model:

$$S(T) = S(0)e^{\left(r-\frac{\sigma^2}{2}\right)\cdot T + \sigma\sqrt{T}\cdot Z} \quad \text{w/ } Z \sim N(0,1)$$

$$\mathbb{P}^*[S(T) > S(0)e^{rT}] =$$

$$= \mathbb{P}^*\left[\cancel{S(0)}\,e^{\left(r-\frac{\sigma^2}{2}\right)\cdot T + \sigma\sqrt{T}\cdot Z} > \cancel{S(0)}\,e^{rT}\right]$$

$$= \mathbb{P}^*\left[\cancel{r\cdot T} - \frac{\sigma^2}{2}\cdot T + \sigma\sqrt{T}\cdot Z > \cancel{rT}\right]$$

$$= \mathbb{P}^*\left[\sigma\sqrt{T}\cdot Z > \frac{\sigma^2}{2}\cdot T\right]$$

$$= \mathbb{P}^*\left[Z > \frac{\sigma\sqrt{T}}{2}\right] \qquad \text{(symmetry of } N(0,1))$$

$$= \mathbb{P}^*\left[Z < -\frac{\sigma\sqrt{T}}{2}\right] = N\left(-\frac{\sigma\sqrt{T}}{2}\right) \xrightarrow[T\to\infty]{} 0$$

Example. Consider a European call option w/ strike K and exercise date T. Under the risk-neutral probability measure, what is the probability that the call is in-the-money @ expiration?

$$\longrightarrow: \mathbb{P}^*[S(T) > K] =$$

$$= \mathbb{P}^*\left[S(0)e^{\left(r-\frac{\sigma^2}{2}\right)\cdot T + \sigma\sqrt{T}\cdot Z} > K\right]$$

$$= \mathbb{P}^*\left[e^{\left(r-\frac{\sigma^2}{2}\right)\cdot T + \sigma\sqrt{T}\cdot Z} > \frac{K}{S(0)}\right] \quad (\ln(\cdot) \text{ is increasing})$$

$$= \mathbb{P}^*\left[\left(r-\frac{\sigma^2}{2}\right)\cdot T + \sigma\sqrt{T}\cdot Z > \ln\left(\frac{K}{S(0)}\right)\right]$$

$$= \mathbb{P}^*\left[\sigma\sqrt{T}\cdot Z > \ln\left(\frac{K}{S(0)}\right) - \left(r-\frac{\sigma^2}{2}\right)\cdot T\right]$$

$$= \mathbb{P}^*\left[Z > \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{K}{S(0)}\right) - \left(r-\frac{\sigma^2}{2}\right)\cdot T\right]\right]$$

$$= \mathbb{P}^*\left[Z < \boxed{\frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(r-\frac{\sigma^2}{2}\right)\cdot T\right]}\right]$$

$$=: d_2$$

$$\boxed{\mathbb{P}^*[S(T) > K] = N(d_2)}$$

$\square$

Consequently. The probability that the otherwise identical put is in-the-money @ expiration is

$$\mathbb{P}^*[S(T) < K] = 1 - N(d_2) = N(-d_2)$$

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Problem Set 14

Black-Scholes pricing.

Problem 14.1. Let the stock prices be modeled using the lognormal distribution. Under the risk-neutral probability measure, the mean stock price at time−1 equals 120 and the median stock price 115. What is the risk-neutral probability that the time−1 stock price exceeds 100?

$\rightarrow$:

$$\mathbb{P}^*[S(1) > 100] = ?$$

1st Figure out σ.

$$\frac{\text{mean}}{\text{median}} = \frac{S(0)e^{rT}}{S(0)e^{(r-\frac{\sigma^2}{2})\cdot T}} = e^{\frac{\sigma^2}{2}\cdot T}$$

$$\frac{120}{115} = e^{\frac{\sigma^2}{2}\cdot T}$$

$$\ln\left(\frac{120}{115}\right) = \frac{\sigma^2}{2}\cdot T = \frac{\sigma^2}{2}$$

$$\sigma = \sqrt{2\cdot\ln\left(\frac{120}{115}\right)} = 0.2918$$

2nd $\mathbb{P}^*[\underbrace{\boxed{S(0)e^{(r-\frac{\sigma^2}{2})\cdot T}}}_{\text{median of } S(1) \; 115} + \sigma\sqrt{T}\cdot Z > 100]$

$$\mathbb{P}^*[115 e^{\sigma\cdot Z} > 100] = \mathbb{P}^*\left[e^{\sigma\cdot Z} > \frac{100}{115}\right]$$

$$= \mathbb{P}^*\left[Z > \frac{1}{\sigma}\ln\left(\frac{100}{115}\right)\right] = \cdots = 0.6844$$

$\square$

Problem 14.2. (5 pts) Let the stochastic process $S = \{S(t); t \geq 0\}$ denote the stock price. The stock's rate of appreciation is 10% while its volatility is 0.30. Then,

(a) $Var[\ln(S(t))] = 0.3t$

(b) $Var[\ln(S(t))] = 0.09t^2$

(c) $Var[\ln(S(t))] = 0.09t$ ←

(d) $Var[\ln(S(t))] = 0.09$

(e) None of the above.

→: In the Black·Scholes model:

$$S(t) = S(0)e^{(r-\frac{\sigma^2}{2})\cdot t + \sigma\sqrt{t}\cdot Z} \qquad Z \sim N(0,1)$$

deterministic

$$\ln(S(t)) = \boxed{\ln(S(0)) + (r-\frac{\sigma^2}{2})\cdot t} + \sigma\sqrt{t}\cdot Z$$

$$Var[\ln(S(t))] = Var[\sigma\sqrt{t}\cdot Z] = \sigma^2\cdot t \qquad \square$$

LogNormal Stock Prices: Tail Probabilities [cont'd].

Problem. Assume the Black-Scholes model.
Let the current stock price be \$100.

You are given:

(i) $\mathbb{P}^*[S(1/4) < 95] = 0.2358$

(ii) $\mathbb{P}^*[S(1/2) < 110] = 0.6026$

What's the expected time-1 stock price under $\mathbb{P}^*$?

→:

$$\boxed{\mathbb{E}^*[S(T)] = S(0)e^{rT}}$$

In this problem: $\boxed{\mathbb{E}^*[S(1)] = S(0)e^{r}}$

In the Black-Scholes model:

$$S(T) = S(0)\, e^{\left(r - \frac{\sigma^2}{2}\right)\cdot T + \sigma\sqrt{T}\cdot Z} \quad \text{w/ } Z \sim N(0,1)$$

μ

$$\boxed{\mathbb{E}^*[S(1)] = 100\, e^{\mu + \frac{\sigma^2}{2}}}$$

(i) 95 is the 23.58^{th} quantile of $S(1/4)$

The 23.58^{th} quantile of $N(0,1)$: standard normal tables: -0.72

$$95 = 100\, e^{\mu(1/4) + \sigma\sqrt{1/4}\cdot(-0.72)} \quad / :100$$

$$0.95 = e^{\mu/4 + \sigma(-0.36)}$$

$$\boxed{\ln(0.95) = 0.25\cdot\mu - 0.36\,\sigma} \quad \text{(i)}$$

(ii) 110 is the 60.26^{th} quantile of $S(1/2)$

The 60.26^{th} quantile of $N(0,1)$: 0.26

$$110 = 100\, e^{\mu(1/2) + \sigma\sqrt{1/2}\cdot(0.26)} \quad / :100$$

$$1.1 = e^{\mu/2 + \sigma\sqrt{1/2}\,(0.26)}$$

$$\boxed{\ln(1.1) = 0.5\mu + \sigma\sqrt{1/2}\,(0.26)} \quad \text{(ii)}$$

We solve this system of two eq'ns w/ two unknowns:

.... $\sigma = 0.21895$

$\Rightarrow$ $\mu = 0.11011$

Finally, $100e^{\mu+\frac{\sigma^2}{2}} = 100e^{0.11011+\frac{(0.21895)^2}{2}} = \underline{114.3488}$

$\square$