M358K : October 29^{th}, 2021.

UNIVERSITY OF TEXAS AT AUSTIN

Problem Set # 14

Power of Test.

Provide your **complete solution** for the following problems.

Problem 14.1. As the sample size increases, the power of a test will increase. *True or false? Why?*

→ : As a proof of concept: The left-sided alternative.

$H_0: \mu = \mu_0$ against $\boxed{H_a: \mu < \mu_0}$ ✓

The RR is of the form: $(-\infty, ?]$

$$? = \mu_0 + z_\alpha\left(\frac{\sigma}{\sqrt{n}}\right)$$

w/ $z_\alpha = \Phi^{-1}(\alpha) =$ qnorm(α)

Note: $z_\alpha < 0$

Let μ_a be a value from the alternative, i.e., $\boxed{\mu_a < \mu_0}$. Under that alternative, the sample mean has the dist'n:

$$\bar{X} \sim \text{Normal}\left(\text{mean} = \mu_a, \text{sd} = \frac{\sigma}{\sqrt{n}}\right)$$

The power of the test @ μ_a:

$$\mathbb{P}_{\mu_a}\left[\bar{X} \le \mu_0 + z_\alpha \cdot \left(\frac{\sigma}{\sqrt{n}}\right)\right] =$$

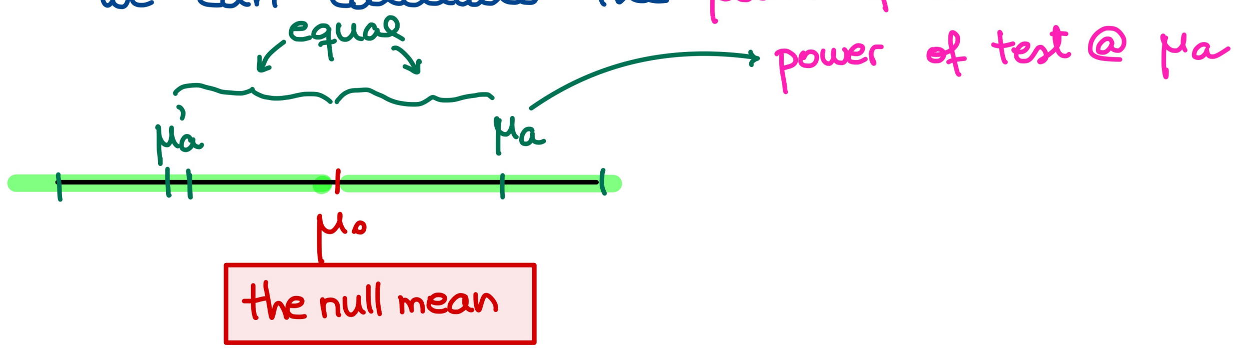
$$= \mathbb{P}_{\mu_a}\left[\frac{\bar{X} - \mu_a}{\frac{\sigma}{\sqrt{n}}} \le \frac{\mu_0 + z_\alpha\left(\frac{\sigma}{\sqrt{n}}\right) - \mu_a}{\frac{\sigma}{\sqrt{n}}}\right]$$

$\frac{\bar{X} - \mu_a}{\sigma/\sqrt{n}} \sim N(0,1)$!! Z

$$= \mathbb{P}\left[Z \le \underbrace{\overbrace{(\mu_0 - \mu_a)}^{>0} \cdot \frac{\sqrt{n}}{\sigma}}_{\uparrow \text{ w/ } n} + z_\alpha\right] \quad \uparrow \text{ w/ } n$$

Problem 14.2. (2 points) Consider a two-sided hypothesis test for the population mean of a normal population. Then, the power of the test is symmetric with respect to the null mean. *True or false? Why?*

$\longrightarrow$: For every μ_a from the alternative,
we can "calculate" the power of the test at that μ_a.



The power of the test can be understood as a function whose domain are all the values from the alternative.

Problem 14.3. (2 points) Let μ denote the population mean μ of a normally distributed population model with a known σ. At a given significance level α, we are testing

$$H_0 : \mu = \mu_0 \quad vs. \quad \boxed{H_a : \mu < \mu_0.}$$

Let μ_a and μ_a' be two values in the alternative such that $\mu_a < \mu_a'$. Then, the power of the test at the alternative μ_a exceeds the power of the test at the alternative μ_a'. *True or false? Why?*

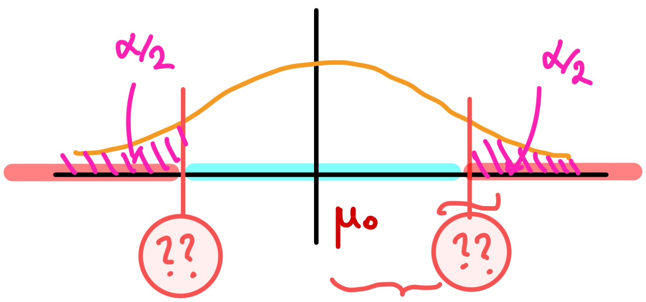
Problem 14.4. The time needed for college students to complete a certain mirror-symmetry puzzle is modeled using a normal distribution with a mean of 30 seconds and a standard deviation of 3 seconds. You wish to see if the population mean time μ is changed by vigorous exercise, so you have a group of nine college students exercise vigorously for 30 minutes and then complete the puzzle.

i. What are your null and alternative hypotheses?
ii. What is the rejection region at the significance level 0.01?
iii. What is the power of your test at $\mu = 28$ seconds?

i. $H_0: \mu = 30$ vs. $H_a: \mu \neq 30$

ii. $\alpha = 0.01$

The form of the RR is:



$$?? = \mu_0 + z_{\frac{\alpha}{2}}\left(\frac{\sigma}{\sqrt{n}}\right) = 30 + (-2.576)\cdot\frac{3}{\sqrt{9}} = 30 - 2.576 = 27.424$$

$$z_{\frac{\alpha}{2}} = \Phi^{-1}\left(\frac{\alpha}{2}\right) = \text{qnorm}\left(\frac{\alpha}{2}\right)$$

$$\text{qnorm}(0.005) = -2.575829$$

$$\widetilde{??} = \mu_0 + \left|z_{\frac{\alpha}{2}}\right|\cdot\left(\frac{\sigma}{\sqrt{n}}\right) = 30 + 2.576 = 32.576$$

$$\boxed{RR: (\cancel{-\infty}\,0, 27.424] \cup [32.576, +\infty)}$$

iii. $\mu_a = 28$ power of test

"fail to reject region": $(27.424, 32.576)$

$\bar{X} \sim$ Normal(mean $= 28$, sd $= 1$)

$\beta =$ pnorm$(32.576, 28, 1) -$ pnorm$(27.424, 28, 1) = 0.71769$

power of test: $1 - \beta \cong 0.28$