

UNIVERSITY OF TEXAS AT AUSTIN

Homework Assignment #1

Prerequisite material.

Provide your **complete solution** to the following problems. Final answers only, without appropriate justification, will receive zero points even if correct.

**Problem 1.1.** (5 points) Roger initially deposits \$4,000 in an investment fund which pays him \$2,000 at time 1 and \$4,000 at time 2.

Sally gets \$2,000 at time 0 and \$4,000 at time 1, and deposits \$5,460 at time 2 in return.

Both investments are governed by compound interest with the same annual effective interest rate  $i$  and they have the same net present values.

Find  $i$ .

- (a) About 9%
- (b) About 10.0%
- (c) About 11.5%
- (d) About 12%
- (e) None of the above

**Solution:** (b)

**Problem 1.2.** (5 pts) Find the total amount of interest that would be paid on a \$1,000 loan over a 10–year period, if the effective interest rate is 0.09 per annum under the following repayment method:

The entire loan plus entire accumulated interest is paid as one lump-sum at the end of the loan term.

**Solution:** Using compound interest, the accumulated value at the end of the 10 years is

$$1000 \cdot 1.09^{10} \approx 2367.36.$$

The total amount of interest is

$$2367.36 - 1000 = 1367.36.$$

**Problem 1.3.** (5 points) Let  $\Omega = \{a_1, a_2, a_3, a_4\}$  be an outcome space, and let  $\mathbb{P}$  be a probability distribution on  $\Omega$ . Assume that  $\mathbb{P}[\{a_1, a_2\}] = 1/3$ ,  $\mathbb{P}[\{a_2, a_3\}] = 1/4$  and  $\mathbb{P}[\{a_1, a_3\}] = 1/9$ . How much is  $\mathbb{P}[\{a_4\}]$ ?

**Solution:** For any outcome space  $\Omega$ , from the axioms of probability, we must have that  $\mathbb{P}[\Omega] = 1$ . In this case,  $\Omega = \{a_1, a_2, a_3, a_4\}$ , and so

$$\mathbb{P}[\Omega] = \mathbb{P}[\{a_1, a_2, a_3, a_4\}] = \mathbb{P}[\{a_1, a_2, a_3\}] + \mathbb{P}[\{a_4\}] = \frac{1}{2} \left( \frac{1}{3} + \frac{1}{4} + \frac{1}{9} \right) + \mathbb{P}[\{a_4\}].$$

Hence,

$$\mathbb{P}[\{a_4\}] = 1 - \frac{25}{72} = \frac{47}{72}.$$

**Problem 1.4.** (10 points) Let  $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$  be a probability space. We denote by  $p_k$  the probability of the elementary outcome  $\omega_k$ , i.e.,  $p_k = \mathbb{P}[\{\omega_k\}]$  for  $k = 1, \dots, 5$ . You are given that  $p_k/p_{k-1}$  is constant for  $k = 2, 3, 4, 5$ . You are also given that  $p_1 = 16/31$ . Find  $p_5$ .

- (a) 1/31
- (b) 2/31
- (c) 4/31
- (d) Not enough information is given.

(e) None of the above.

**Solution: (a)**

From the given recursive property, we know that, for some constant  $\kappa$ ,

$$p_2 = \kappa p_1, \quad p_3 = \kappa^2 p_1, \quad p_4 = \kappa^3 p_1, \quad p_5 = \kappa^4 p_1.$$

We also know that

$$p_1 + p_2 + p_3 + p_4 + p_5 = 1.$$

So,

$$p_1(1 + \kappa + \kappa^2 + \kappa^3 + \kappa^4) = 1 \quad \Rightarrow \quad \frac{16}{31} \left( \frac{1 - \kappa^5}{1 - \kappa} \right) = 1 \quad \Rightarrow \quad \frac{1 - \kappa^5}{1 - \kappa} = \frac{31}{16} \quad \Rightarrow \quad \kappa = 1/2.$$

Finally,  $p_5 = \frac{1}{2^4} \left( \frac{16}{31} \right) = \frac{1}{31}$ .

**Problem 1.5.** (5 points) Emmanuel entered an extra special kind of game with his friend Fischer. First, they toss a fair coin. If the coin comes up heads, Emmanuel gives \$5,000 to Fischer. If the coin comes up tails, Fischer gives \$2,000 to Emmanuel. Then, regardless of the outcome of the first cointoss, they toss the same fair coin again. If it comes up heads, Emmanuel gives Fischer \$4,000. If the coin comes up tails, Fischer gives \$3,000 to Emmanuel. What is the expected cashflow, i.e., what is the expected amount of money that changes hands and who gives it to whom?

**Solution:** Let  $X_i$  be the cashflow from Emmanuel's perspective after the  $i^{th}$  cointoss for  $i = 1, 2$ . Then, we are looking for

$$\begin{aligned} \mathbb{E}[X_1 + X_2] &= \mathbb{E}[X_1] + \mathbb{E}[X_2] \\ &= \frac{1}{2}(-5,000 + 2,000) + \frac{1}{2}(-4,000 + 3,000) = -2,000. \end{aligned}$$

So, the expected cashflow is \$2,000 from Emmanuel to Fischer.

**Problem 1.6.** (5 points) The random variables  $(R_1, R_2)$  have the following moments:

$$\mathbb{E}[R_1] = 0.08, \quad SD[R_1] = 0.2,$$

$$\mathbb{E}[R_2] = 0.10, \quad SD[R_2] = 0.25.$$

The correlation coefficient between  $R_1$  and  $R_2$  is given to be 0.2. What is the standard deviation of the random variable  $R = \frac{1}{2}(R_1 + R_2)$ ?

**Solution:** The variance of  $R$  is

$$\begin{aligned} Var[R] &= \frac{1}{4}Var[R_1 + R_2] = \frac{1}{4}(Var[R_1] + Var[R_2] + 2Cov[R_1, R_2]) \\ &= \frac{1}{4}(Var[R_1] + Var[R_2] + 2SD[R_1]SD[R_2]corr[R_1, R_2]) \\ &= \frac{1}{4}((0.2)^2 + (0.25)^2 + 2(0.2)(0.25)(0.2)) = 0.030625. \end{aligned}$$

So, the standard deviation of  $R$  is  $\sigma = \sqrt{0.030625} = 0.175$ .