M378K: September 11th, 2024.

More review.

Example. We say that Y is exponential w/ parameter $\tau>0$ if it has the following pdf

$$\boxed{f_Y(y) = \frac{1}{\tau} e^{-\frac{y}{\tau}} \mathbb{1}_{[0,\infty)}(y)}$$

Alternative parameterization : $\lambda = \frac{1}{\tau}$

Expecations and Standard Deviations.

In the discrete case: $\mathbb{E}[Y] = \sum_{y \in S_Y} y \, p_Y(y)$

Def'n. For a continuous r.v. Y w/ pdf f_Y, we define the expectation $\mathbb{E}[Y]$ as

$$\boxed{\mathbb{E}[Y] := \int_{-\infty}^{+\infty} y \cdot f_Y(y)\,dy}$$

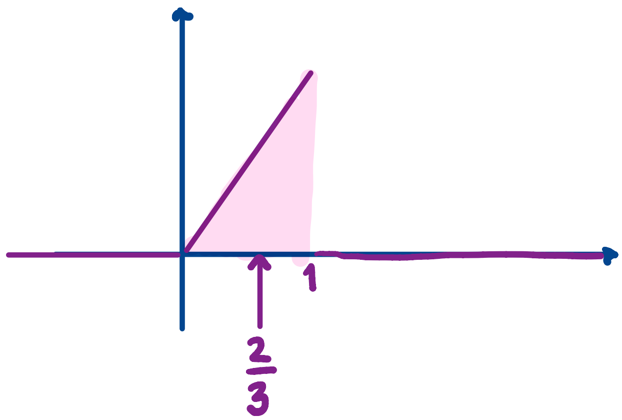
if the integral exists.

Problem 3.3. *Consider a continuous random variable Y whose probability density function is given by*

$$f_Y(y) = 2y\mathbf{1}_{[0,1]}(y)$$

What is the expected value of this random variable?

$\longrightarrow$: $\mathbb{E}[Y] = \int_0^1 y(2y)\,dy = 2\int_0^1 y^2\,dy = 2\,\frac{y^3}{3}\Big|_{y=0}^{1} = \frac{2}{3}$ $\square$



Example. $Y \sim U(a,b)$

$$\mathbb{E}[Y] = \frac{a+b}{2}$$

$$\mathrm{Var}[Y] = ?$$

$$\boxed{\begin{aligned}\mathrm{Var}[Y] &= \mathbb{E}[(Y-\mu_Y)^2] \\ &= \mathbb{E}[Y^2] - \mu_Y^2\end{aligned}}$$

$$Y - a \sim U(0, b-a)$$

$$U := \frac{Y-a}{b-a} \sim U(0,1)$$

$$\boxed{Y = a + (b-a)\cdot U}$$

$$\mathrm{Var}[Y] = \mathrm{Var}[\underbrace{a}_{\text{deterministic}} + (b-a)\cdot U] = (b-a)^2 \cdot \boxed{\mathrm{Var}[U]}$$

$$\mathrm{Var}[U] = \boxed{\mathbb{E}[U^2]} - (\underbrace{\mathbb{E}[U]}_{=\frac{1}{2}})^2$$

$$\mathbb{E}[U^2] = \int_0^1 u^2 \underbrace{f_U(u)}_{\text{pdf of } U \sim U(0,1)} du = \int_0^1 u^2 du = \frac{u^3}{3}\Big|_{u=0}^{1} = \frac{1}{3}$$

$$\mathrm{Var}[U] = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12}$$

$$\mathrm{Var}[Y] = \frac{(b-a)^2}{12}$$

$\square$

Example. $Y \sim N(\mu, \sigma)$

μ: $\mathbb{E}[Y]$ σ: $SD[Y]$

Example. $Y \sim E(\tau)$, Y is exponential w/ parameter τ

$$\mathbb{E}[Y] = \cancel{?} = \int_{-\infty}^{+\infty} y\, f_Y(y)\, dy$$

$$= \int_{-\infty}^{+\infty} y \cdot \frac{1}{\tau} e^{-\frac{y}{\tau}} \mathbb{1}_{[0,\infty)}(y)\, dy$$

$$= \int_{0}^{+\infty} \left(\frac{y}{\tau}\right) e^{-\left(\frac{y}{\tau}\right)} dy$$

$$u = \frac{y}{\tau} \qquad du = \frac{dy}{\tau} \qquad dy = \tau \cdot du$$

$$= \tau \underbrace{\int_{0}^{+\infty} u\, e^{-u}\, du}$$

$$u = u \qquad du = du$$

$$dv = e^{-u} du \qquad v = -e^{-u} du$$

$$= \tau \left(\underbrace{-u e^{-u} \Big|_{u=0}^{+\infty}}_{0} + \underbrace{\int_{0}^{+\infty} (+e^{-u}) du}_{-e^{-u} \Big|_{u=0}^{+\infty} = -(0-1) = 1} \right) = \tau$$

Def'n. For a r.v. Y w/ pdf f_Y and $k = 1, 2, \ldots$, we define the k^{th} (raw) moment μ_k by

$$\mu_k = \mathbb{E}[Y^k] = \int_{-\infty}^{\infty} y^k f_Y(y)\, dy \;;$$

$$\boxed{\mu_1 = \mathbb{E}[Y]}$$

the k^{th} central moment μ_k^c is defined by

$$\mu_k^c = \mathbb{E}\left[(Y-\mu_Y)^k\right] = \int_{-\infty}^{\infty} (y-\mu_Y)^k f_Y(y)\,dy$$

w/ $\mu_Y = \mu_1$;

the skewness is

$$\mathbb{E}\left[\left(\frac{Y-\mu_Y}{\sigma_Y}\right)^3\right] \quad \text{w/} \quad \sigma_Y^2 = \mathbb{E}\left[(Y-\mu_Y)^2\right];$$

the kurtosis is

$$\mathbb{E}\left[\left(\frac{Y-\mu_Y}{\sigma_Y}\right)^4\right] = \frac{\mathbb{E}\left[(Y-\mu_Y)^4\right]}{\sigma_Y^4}$$

Cumulative Distribution Function.

Defn. The cumulative distribution function (cdf) of a r.v. Y is a function

$$F_Y : \mathbb{R} \longrightarrow [0,1]$$

defined as $F_Y(y) = \mathbb{P}[Y \leq y]$ for all $y \in \mathbb{R}$