M339J: February 3rd, 2021.

Tail Formula for Expectation.

Let Y be a nonnegative continuous r.v.

Then, we have that $\mathbb{E}[Y] = \int_0^{+\infty} S_Y(y)\,dy$. ⊛

→: $\mathbb{E}[Y] \underset{\text{by def'n}}{=} \int_0^{+\infty} y f_Y(y)\,dy$

Start from the right-hand side in ⊛

$$\int_0^{+\infty} S_Y(y)\,dy \underset{\text{by def'n of } S_X}{=} \int_0^{+\infty} \mathbb{P}[Y > y]\,dy$$

$$= \int_0^{+\infty} \int_y^{+\infty} f_Y(u)\,du\,dy$$

Now, we swich the integrals!

$$= \int_0^{+\infty} \int_0^{u} f_Y(u)\,dy\,du$$

$$= \int_0^{+\infty} f_Y(u) \underbrace{\left(\int_0^{u} dy\right)}_{= u} du = \int_0^{+\infty} f_Y(u)\cdot u\,du = \mathbb{E}[Y].$$

■

Task: Figure out the analogous formula for discrete non-negative r.v.s :)

Problem. Let the lifetime of a generator be modelled by a r.v. X which has the Weibull dist'n w/ parameters $\nu \geq 0$, $\alpha > 0$, $\beta > 0$, i.e.,

$$F_X(x) = \begin{cases} 0 & \text{for } x \leq \nu \\ 1 - \exp\left(-\left(\frac{x-\nu}{\alpha}\right)^\beta\right) & \text{for } x > \nu. \end{cases}$$

Q: Say that $\nu = 0$, $\beta = 1$, $\alpha > 0$.
Which dist'n do you get?

$\longrightarrow$: $F_X(x) = 1 - \exp^{-\frac{x}{\alpha}}$ for $x > 0$

$\Rightarrow X \sim \text{Exponential}(\theta = \alpha)$

Choose $\nu = 0$, $\underline{\alpha = 1}$, $\beta = 2$.
Find the expected lifetime of the generator.

$\longrightarrow$:

$$\mathbb{E}[X] = \int_0^{+\infty} S_X(x)\,dx$$

$$= \int_0^{+\infty} \exp\left(-\left(\frac{x-0}{1}\right)^2\right) dx$$

$$= \int_0^{+\infty} e^{-x^2}\,dx$$

Resembles the density of a standard normal:

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad x \in \mathbb{R}$$

$$u = x\sqrt{2} \Rightarrow \begin{cases} du = \sqrt{2}\,dx \\ x = \frac{u}{\sqrt{2}} \end{cases}$$

$$= \int_0^{+\infty} e^{-\frac{u^2}{2}} \cdot \frac{1}{\sqrt{2}}\,du$$

$$= \frac{\sqrt{\pi}}{\sqrt{\pi}} \underbrace{\frac{1}{\sqrt{2}} \int_0^{+\infty} e^{-\frac{u^2}{2}} \, du}_{= \frac{1}{2}} = \frac{\sqrt{\pi}}{2} \quad \blacksquare$$

Raw Moments.

Def'n. The k^{th} raw moment of a r.v. X is given by

$$\mu_k' := \mathbb{E}\left[X^k\right]$$

Note: The 1^{st} raw moment is actually the mean which we frequently denote by $\mu \equiv \mu_X$

Central Moments.

Def'n. The k^{th} central moment of a r.v. X is

$$\mu_k := \mathbb{E}\left[(X-\mu)^k\right]$$

Q: What is the 2^{nd} central moment of a r.v. X?

$\rightarrow$: $\mu_2 = Var[X] = \mathbb{E}[(X-\mu)^2]$

The computational formula for the variance:

$$Var[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

Having fun w/ the notation:

$$\mu_2 = \mu_2' - \mu^2$$

Problem. Let X be a two-parameter Pareto r.v. w/ $\alpha = 3$ and $\theta = 10$. Find $Var[X]$.

$\rightarrow$: Attempt this problem!