M339D: April 5th, 2023.

The Log-Normal Distribution.

Def'n. Let $X \sim Normal(mean = m, variance = \nu^2)$.

Define $\boxed{Y = e^X}$

We say that Y is **lognormally distributed**.

$$\mathbb{E}[Y] = \mathbb{E}[e^X] = \mathbb{E}[e^{X \cdot 1}] = M_X(1) = e^{m + \frac{\nu^2}{2}}$$

Consider: $\mathbb{E}[X] = m$

Caveat: $\boxed{\mathbb{E}[e^X] \geq e^{\mathbb{E}[X]}}$ ✓

This is a special case of Jensen's Inequality.

Theorem. Let X be a random variable, and g be a convex function such that

$g(X)$ is well-defined

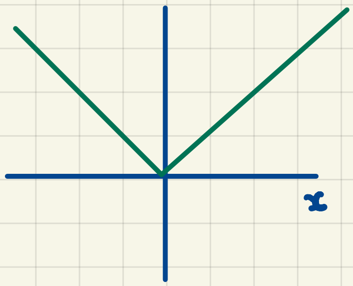
and

$\mathbb{E}[g(X)]$ exists.

Then, $\boxed{\mathbb{E}[g(X)] \geq g(\mathbb{E}[X])}$

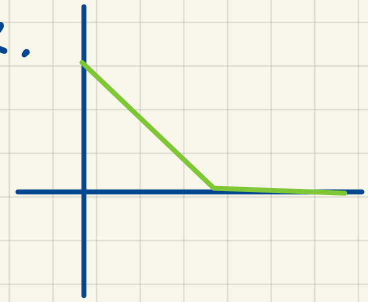
Examples. i. $g(x) = |x|$

$$\mathbb{E}[|X|] \geq |\mathbb{E}[X]|$$



ii. Look @ a European put w/ strike K.

Its payoff f'tion : $v_p(s) = (K-s)_+$



The expected payoff is:

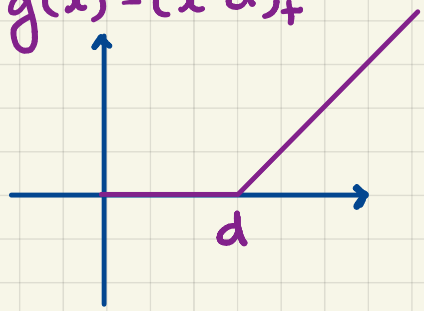
$$\mathbb{E}[v_p(S(T))] = \mathbb{E}[(K-S(T))_+]$$

By Jensen, its lower bound is $\geq (K - \mathbb{E}[S(T)])_+$

iii. In classical insurance :

$$\begin{cases} X \ldots \text{(ground-up) loss, i.e., the severity r.v.} \\ d \ldots \text{deductible} \end{cases}$$

The insurer pays $(X-d)_+$, i.e., $g(x) = (x-d)_+$



By Jensen's inequality:

$$\boxed{\mathbb{E}[(X-d)_+] \geq (\mathbb{E}[X]-d)_+}$$

The median = ?

Find $\pi_{0.5}$ such that $\mathbb{P}[Y \leq \pi_{0.5}] = 0.5$

$$\mathbb{P}[e^X \leq \pi_{0.5}] = 0.5$$

$$\mathbb{P}[X \leq \ln(\pi_{0.5})] = 0.5$$

$\uparrow$ mean of a normal $=$ median of the normal

median of X, i.e., m

$\Downarrow$

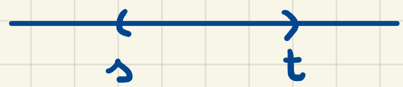
$$\boxed{\pi_{0.5} = e^m}$$

Task: How would you modify the above to any quantile of the lognormal distribution?

Log-Normal Stock Prices.

Temporarily fix a time-horizon T.

$S(t),\ t \in [0,T]$ ... time-t stock price



Define
$$R(s,t) := \ln\left(\frac{S(t)}{S(s)}\right)$$

In other words: $S(t) = S(s)e^{R(s,t)}$

In particular: $R(0,T)$... realized return over $(0,T)$

We model realized returns as normal

$R(0,T) \sim \text{Normal}(\text{mean} = m, \text{variance} = \nu^2)$

$\Rightarrow$ $S(T)$ is lognormal

and $\mathbb{E}^*[S(T)] = S(0)e^{m + \frac{\nu^2}{2}}$ ☆

Market model.

- Riskless Asset w/ ccrfir r
- Risky Asset : a non-dividend-paying stock
  - σ... volatility

Under the risk-neutral measure $\mathbb{E}^*[S(T)] = S(0)e^{rT}$ ☆☆

Equating: ☆ & ☆☆

$$\boxed{m + \frac{\nu^2}{2} = rT}$$

Consider: $Var[R(0,T)] = \nu^2 =$