M378K: October 1st, 2025.

Marginal Distributions & Independence.

Theorem. Say that $(Y_1, Y_2, \dots, Y_n)$ has the joint pdf $f_{Y_1,\dots,Y_n}$.

Then, for every $i = 1, \dots, n$, the random variable Y_i is also continuous with its marginal density

$$f_{Y_i}(y) := \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{n-1} f_{Y_1,\dots,Y_n}(y_1, \dots, y_{i-1}, y, y_{i+1}, \dots, y_n)\, dy_n \cdots dy_{i+1}\, dy_{i-1} \cdots dy_1$$

Example. [cont'd]

(Y_1, Y_2) has the joint pdf

$$f_{Y_1,Y_2}(y_1, y_2) = 6y_1 \mathbb{1}_{[0 \le y_1 \le y_2 \le 1]}$$

Marginal of Y_1:

$$f_{Y_1}(y) = \int_{-\infty}^{\infty} f_{Y_1,Y_2}(y, y_2)\, dy_2$$

$$= \int_{-\infty}^{\infty} 6y \mathbb{1}_{[0 \le y \le y_2 \le 1]}\, dy_2$$

$$= 6y \int_{y}^{1} dy_2 = 6y(1-y) \cdot \mathbb{1}_{[0,1]}(y)$$

Marginal of Y_2:

$$f_{Y_2}(y) = \int_{-\infty}^{\infty} f_{Y_1,Y_2}(y_1, y)\, dy_1$$

$$= \int_{-\infty}^{\infty} 6y_1 \mathbb{1}_{[0 \leq y_1 \leq y \leq 1]}\, dy_1$$

$$= 6 \int_0^y y_1\, dy_1 \cdot \mathbb{1}_{[0,1]}(y)$$

$$= 6 \cdot \frac{y^2}{2} \mathbb{1}_{[0,1]}(y) = 3y^2 \mathbb{1}_{[0,1]}(y)$$

Def'n. The random variables $Y_1, \dots, Y_n$ are independent iff the events

$$\{Y_i \in [a_i, b_i]\} \quad i = 1..n$$

are independent events for all a_i, b_i, $i = 1..n$.

Theorem. The Factorization Criterion.

Jointly continuous r.v.s $Y_1, \dots, Y_n$ are

independent

iff

$$f_{Y_1,\dots,Y_n}(y_1, \dots, y_n) = f_{Y_1}(y_1) \cdots f_{Y_n}(y_n) \text{ for all } y_1, \dots, y_n \in \mathbb{R}$$

Corollary. If

$$f_{Y_1,\dots,Y_n}(y_1, \dots, y_n) = g_1(y_1) \cdot g_2(y_2) \cdots g_n(y_n)$$

for all $y_1, \dots, y_n$

and some functions $g_1, \dots, g_n$,

then r.v.s $Y_1, \dots, Y_n$ are independent.

Theorem. Let $Y_1, \dots, Y_n$ be independent r.v.s and let $g_1, \dots, g_n$ be functions such that

$g_i(Y_i)$, $i = 1..n$

are all well-defined

Then, if all the expectations are finite

$$\boxed{\mathbb{E}[g_1(Y_1) \cdot g_2(Y_2) \cdots g_n(Y_n)] = \mathbb{E}[g_1(Y_1)] \cdot \mathbb{E}[g_2(Y_2)] \cdots \mathbb{E}[g_n(Y_n)]}$$

e.g., Y_1, Y_2 independent

$g_1(y) = g_2(y) = e^y$ for all $y \in \mathbb{R}$

$$\mathbb{E}[\exp(Y_1 + Y_2)] = \mathbb{E}[e^{Y_1} \cdot e^{Y_2}] \underset{\text{independence}}{=} \mathbb{E}[e^{Y_1}] \cdot \mathbb{E}[e^{Y_2}]$$

If Y_1 and Y_2 are also identically distributed,

i.e., if $F_{Y_1} = F_{Y_2}$,

then, $\mathbb{E}[\exp(Y_1 + Y_2)] = \left(\mathbb{E}[e^{Y_1}]\right)^2$

M378K Introduction to Mathematical Statistics
Problem Set #8
Transformations of Random Variables.

Problem 8.1. *Let X be a continuous random variable with the cumulative distribution function denoted by F_X and the probability density function denoted by f_X.*

Let the random variable $\boxed{Y = 2X}$ have the p.d.f. denoted by f_Y. Then,

(a) $f_Y(x) = 2f_X(2x)$

(b) $f_Y(x) = \frac{1}{2}f_X\left(\frac{x}{2}\right)$

(c) $f_Y(x) = f_X(2x)$

(d) $f_Y(x) = f_X\left(\frac{x}{2}\right)$

(e) *None of the above*

The CDF Method

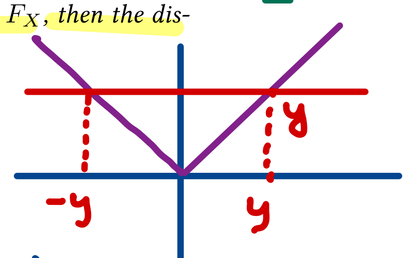
$\rightarrow$: $y \in \mathbb{R}$: $F_Y(y) = \mathbb{P}[Y \leq y] = \mathbb{P}[2X \leq y] =$

$= \mathbb{P}[X \leq \frac{y}{2}] = F_X\left(\frac{y}{2}\right)$

$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}F_X\left(\frac{y}{2}\right) = \frac{1}{2}f_X\left(\frac{y}{2}\right)$ □

Problem 8.2. *If the continuous random variable X has the distribution function F_X, then the distribution function of the random variable $Y = |X|$ equals*

$$F_Y(y) = ?$$



$\rightarrow$: $F_Y(y) = \mathbb{P}[Y \leq y] = \mathbb{P}[|X| \leq y] =$

$= \mathbb{P}[-y \leq X \leq y]$

$= \mathbb{P}[X \leq y] - \mathbb{P}[X < -y] = F_X(y) - F_X(-y)$ □

$f_Y(y) = (f_X(y) + f_X(-y)) \cdot \mathbb{1}_{[0,\infty)}(y)$

Remark 8.1. *The goal is to figure out the distribution of the random variable*

$$X = g(Y_1, Y_2, \ldots, Y_n)$$

where $Y_i, i = 1, \ldots, n$ are a **random sample** *with a common density f_Y.*