UNIVERSITY OF TEXAS AT AUSTIN

Problem Set # 7
The Central Limit Theorem.

Let $\{X_n, n = 1, 2, 3, \dots\}$ be a sequence of independent, identically distributed random variables such that $\mu_X = \mathbb{E}[X_1] < \infty$ and $Var[X] = \sigma_X^2 < \infty$. For every $n = 1, 2, \dots$ define

$$\bar{X}_n = \frac{X_1 + X_2 + \cdots + X_n}{n}.$$

Problem 7.1. Find the expected value of $\bar{X}_n$ for every n.

Problem 7.2. Find the variance and standard deviation of $\bar{X}_n$ for every n.

Theorem 7.1. The Central Limit Theorem (CLT). *If the above conditions are satisfied, we have that*

$$\frac{\bar{X}_n - \mu_X}{\frac{\sigma_X}{\sqrt{n}}} \overset{\mathcal{D}}{\Rightarrow} N(0, 1) \quad as\ n \to \infty.$$

Practically, for "large enough" n, $\bar{X}_n$ is approximately normal with mean μ_X and variance $\frac{\sigma_X^2}{n}$. The rule of thumb is that we use the theorem for $n \geq 30$. If that is the case, we have that for any real $a < b$,

$$\mathbb{P}[a < \bar{X}_n \leq b] = \mathbb{P}\left[\frac{a - \mu_X}{\frac{\sigma_X}{\sqrt{n}}} < \frac{\bar{X}_n - \mu_X}{\frac{\sigma_X}{\sqrt{n}}} \leq \frac{b - \mu_X}{\frac{\sigma_X}{\sqrt{n}}}\right] \approx \Phi\left(\frac{b - \mu_X}{\frac{\sigma_X}{\sqrt{n}}}\right) - \Phi\left(\frac{a - \mu_X}{\frac{\sigma_X}{\sqrt{n}}}\right).$$

Problem 7.3. Travel time by sled between Whoville and Whoburgh takes on average 36 minutes with a standard deviation of 6 minutes. Over a particular weekend, 64 sled trips take place. What is the (approximate) probability that the average sled trip took more than 38 minutes?

Problem 7.4. The amount of time your friendly taquero at *Torchy's Tacos* spends to assemble any one tasty taco is a random variable with mean 3 minutes and 15 seconds and standard deviation of thirty seconds. You and your 31 friends from *Applied Statistics* celebrate by ordering two tacos each. What is the probability that the average taco-assembly time is:

- less than 2 minutes and 30 seconds;
- more than 3 minutes and 15 seconds;
- at least 3 minutes but at most 3 minutes and 30 seconds?