M339D: May 3rd, 2021.

The Forward Binomial Tree.

$$S(0) \nearrow S_u = u \cdot S(0)$$
$$S(0) \searrow S_d = d\, S(0)$$
$$\underbrace{\qquad\qquad}_{h = T}$$

The no-arbitrage condition:

$$d < e^{(r-\delta)h} < u$$

✓

$u, d = ?$

"Def'n." The volatility σ is the standard deviation of realized returns on a continuously compounded scale and $\boxed{\text{annualized}}$.

In M339W:

The realized return over a time interval $(0,h)$:

$$R(0,h) := \ln\left(\frac{S(h)}{S(0)}\right)$$

$$\Longleftrightarrow$$

$$S(h) = S(0)\, e^{R(0,h)}$$

In continuous time: $Var[R(0,1)] = \sigma^2$

Heuristics: $h = \frac{1}{m}$ years

Q: What is the volatility for my time period of length h?

σ_h

$$0 \quad \frac{1}{m} \quad \frac{2}{m} \quad \cdots \quad \frac{m-1}{m} \quad 1$$

$R(0,\frac{1}{m}), R(\frac{1}{m},\frac{2}{m}), \ldots, R(\frac{m-1}{m}, 1)$ ... realized returns for the periods of length $\frac{1}{m}$

$R\left(\frac{k-1}{m}, \frac{k}{m}\right)$ are all random variables.

We make the following <u>assumptions</u>:

- all the returns are identically distributed;
- the returns (over disjoint intervals) are independent.

$$\underline{\sigma^2} = Var[R(0,1)] = Var\left[R\left(0,\tfrac{1}{m}\right) + R\left(\tfrac{1}{m},\tfrac{2}{m}\right) + \dots + R\left(\tfrac{m-1}{m},1\right)\right]$$

$$= Var\left[R\left(0,\tfrac{1}{m}\right)\right] + \dots + Var\left[R\left(\tfrac{m-1}{m},1\right)\right]$$

independent

$$= m \cdot Var\left[R\left(0,\tfrac{1}{m}\right)\right] = \underline{m \cdot \sigma_h^2}$$

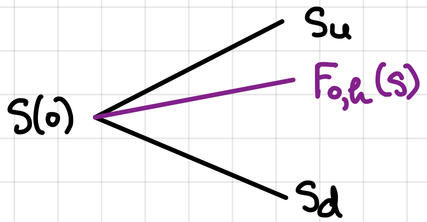
identically dist'd

$$h = \frac{1}{m}$$

$$\sigma^2 = \frac{1}{h} \cdot \sigma_h^2 \quad \Rightarrow \quad \sigma_h^2 = h \cdot \sigma^2$$

$$\Rightarrow \quad \boxed{\sigma_h = \sigma\sqrt{h}} \quad ✓$$

We generalize this equality to arbitrary lengths h.



<u>Recall:</u> $\underline{F_{0,h}(S)} = S(0)\,e^{-\delta h} \cdot e^{rh} = S(0)\,e^{(r-\delta)h}$

$$S_u = F_{0,h}(S) \cdot e^{\sigma\sqrt{h}} = S(0)\,e^{(r-\delta)h + \sigma\sqrt{h}} \quad =: u$$

$$S_d = F_{0,h}(S)\, e^{-\sigma\sqrt{h}} = S(0)\,e^{(r-\delta)h - \sigma\sqrt{h}} \quad =: d$$

2nd and 3rd

Q: What is S_u / S_d?

$$\rightarrow: \quad \frac{S_u}{S_d} = \frac{u}{d} = \frac{\cancel{e^{(r-\delta)h}} \cdot e^{\sigma\sqrt{h}}}{\cancel{e^{(r-\delta)h}} \cdot e^{-\sigma\sqrt{h}}} = e^{2\sigma\sqrt{h}}$$

Q: Risk-neutral Probability?

→ : $$p^* = \frac{e^{(r-\delta)h} - d}{u-d}$$

$$p^* = \frac{\cancel{e^{(r-\delta)h}}^{\,1} - e^{\cancel{(r-\delta)h} - \sigma\sqrt{h}}}{e^{\cancel{(r-\delta)h} + \sigma\sqrt{h}} - e^{\cancel{(r-\delta)h} - \sigma\sqrt{h}}}$$

$$p^* = \frac{1 - e^{-\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}}$$

$$p^* = \frac{\cancel{1 - e^{-\sigma\sqrt{h}}}^{\,1}}{e^{\sigma\sqrt{h}}\underbrace{\left(1 - e^{-2\sigma\sqrt{h}}\right)}_{\cancel{(1-e^{-\sigma\sqrt{h}})}(1+e^{-\sigma\sqrt{h}})}}$$

$$\boxed{p^* = \frac{1}{1+e^{\sigma\sqrt{h}}}}$$

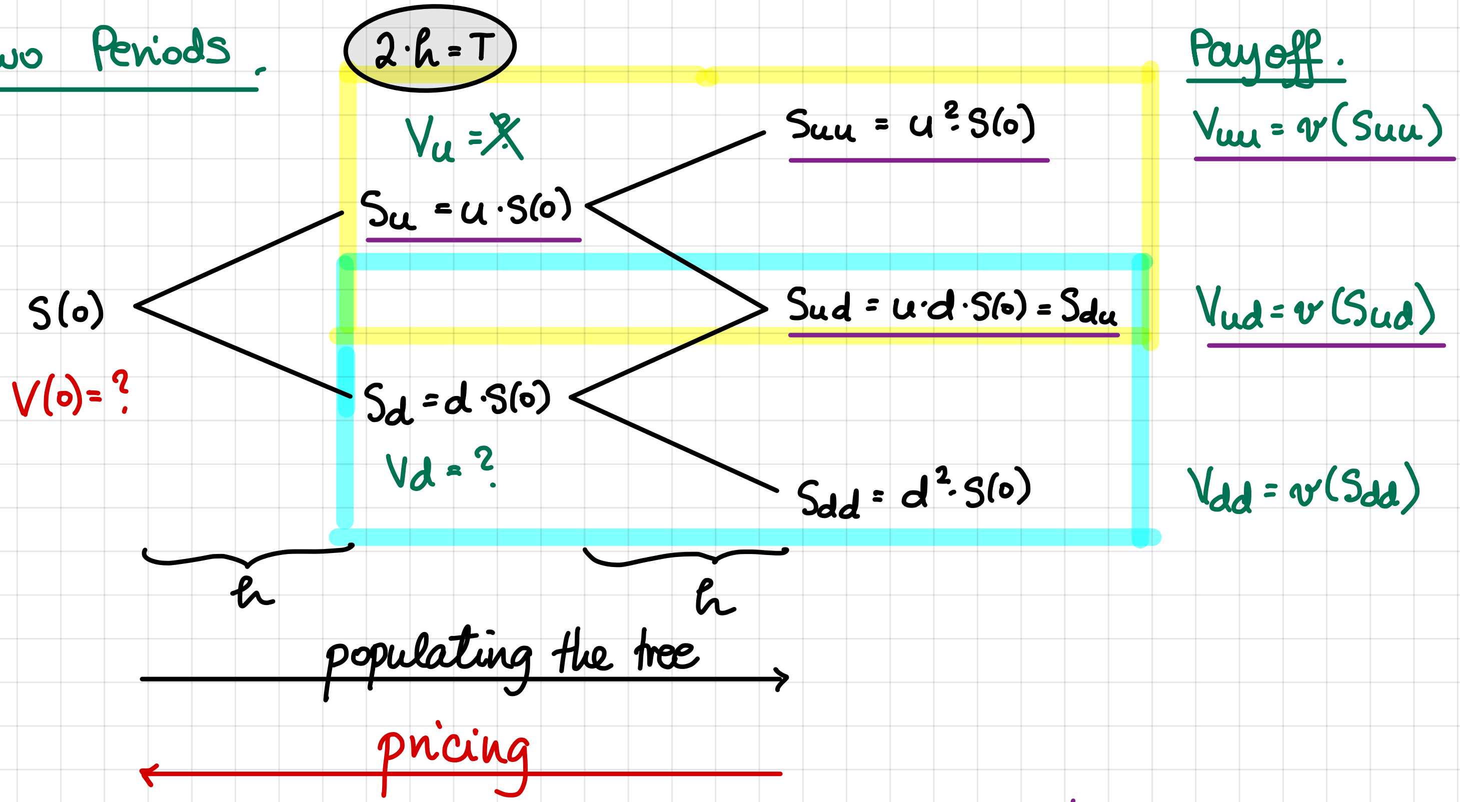
the Shortcut only for the forward tree

$$p^* \xrightarrow{h \to 0} \frac{1}{2}$$

Q: Is there a need to check the no-arbitrage condition?

→ : For $\sigma > 0$, no!

Two Periods.



- **up node:** replicating portfolio for the option @ the up node:

$$\begin{cases} \Delta_u = e^{-\delta \cdot h} \cdot \dfrac{V_{uu} - V_{ud}}{S_{uu} - S_{ud}} \\ B_u = e^{-r \cdot h} \cdot \dfrac{u \cdot V_{ud} - d \cdot V_{uu}}{u - d} \end{cases}$$

$\Rightarrow$ the option's value @ the up node:

$$V_u = \Delta_u \cdot S_u + B_u = e^{-rh} \cdot \left[p^* \cdot V_{uu} + (1-p^*) \cdot V_{ud} \right]$$

$$\text{w/ } p^* = \frac{e^{(r-\delta)h} - d}{u - d}$$

- **down node:** Think about the analogous calculation @ the down node :)