M378K: September 13^{th}, 2024.

Cumulative Distribution Function.

Def'n. The cumulative distribution function (cdf) of a r.v. Y is a function

$$F_Y : \mathbb{R} \longrightarrow [0,1]$$

defined as

$$F_Y(y) = \mathbb{P}[Y \leq y] \text{ for all } y \in \mathbb{R}$$

Discrete Case. Say Y is discrete w/ pmf p_Y.

Then,

$$F_Y(y) = \sum_{u \in S_Y, u \leq y} p_Y(y)$$

M378K Introduction to Mathematical Statistics
Problem Set #4
Cumulative distribution functions.

Problem 4.1. Source: Sample P exam, Problem #342.
Consider a Poisson distributed random variable X. As usual, let's denote its cumulative distribution function by F_X. You are given that

$$\boxed{\frac{F_X(2)}{F_X(1)} = 2.6}$$

Calculate the expected value of the random variable X.

→: $\mathbb{E}[X] = \lambda$

pmf of X: $\boxed{p_X(k) = e^{-\lambda}\cdot\frac{\lambda^k}{k!}}$ for $k = 0, 1, 2, \ldots$

$$\frac{p_X(0)+p_X(1)+p_X(2)}{p_X(0)+p_X(1)} = 2.6$$

$$\frac{\overset{1}{\cancel{e^{-\lambda}}} + \cancel{e^{-\lambda}}\cdot\lambda + \cancel{e^{-\lambda}}\cdot\frac{\lambda^2}{2}}{\overset{1}{\cancel{e^{-\lambda}}} + \cancel{e^{-\lambda}}\cdot\lambda} = 2.6$$

$$1 + \lambda + \frac{\lambda^2}{2} = 2.6(1+\lambda)$$

$$5\lambda^2 - 16\lambda - 16 = 0$$

$$\lambda_{1,2} = \frac{16 \pm \sqrt{256+320}}{10} = \frac{16 \pm \sqrt{576}}{10}$$

$$\lambda_{1,2} = \frac{16 \pm 24}{10}$$

Only keep the positive solution: $\boxed{\lambda = 4}$ $\square$

Continuous Case.

Y is a continuous r.v. w/ pdf f_Y

$$F_Y(y) = \mathbb{P}[Y \leq y] = \int_{-\infty}^{y} f_Y(u)\,du$$

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$ wherever the derivative exists

Problem 4.2. *Consider a random variable Y whose cumulative distribution function is given by*

$$F_Y(y) = \begin{cases} 0, & \text{for } y < 0 \\ y^4, & \text{for } 0 \leq y < 1 \\ 1, & \text{for } 1 \leq y \end{cases}$$

Calculate the expectation of the random variable Y.

$\to$: $\mathbb{E}[Y] = \int_{-\infty}^{\infty} y \cdot f_Y(y)\, dy$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \begin{cases} 0 & y<0 \\ 4y^3 & 0<y<1 \\ 0 & y>1 \end{cases}$$

$$\boxed{f_Y(y) = 4y^3 \mathbb{1}_{(0,1)}(y)}$$

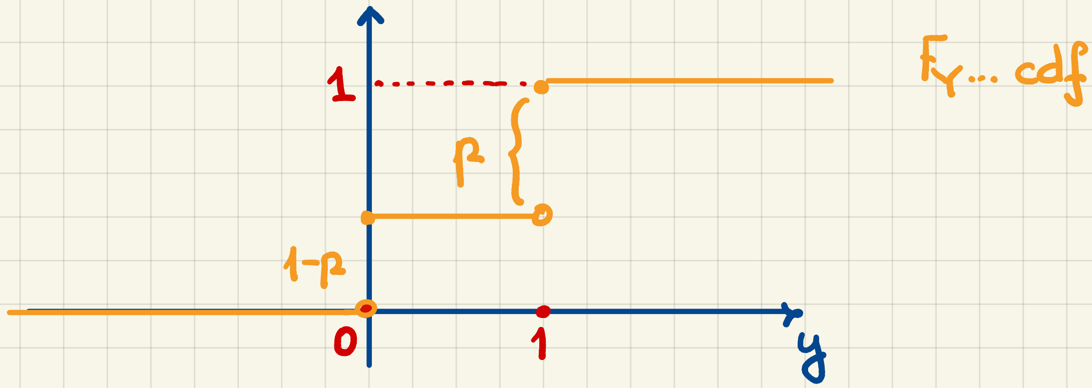
$$\mathbb{E}[Y] = \int_0^1 y(4y^3)\, dy = 4\int_0^1 y^4\, dy = 4 \cdot \left(\frac{y^5}{5}\right)_{y=0}^{1} = \frac{4}{5} \quad \square$$

Properties of cdf:

- $0 \leq F_Y(y) \leq 1$
- F_Y is nondecreasing
- $\lim_{y \to -\infty} F_Y(y) = 0$; $\lim_{y \to \infty} F_Y(y) = 1$

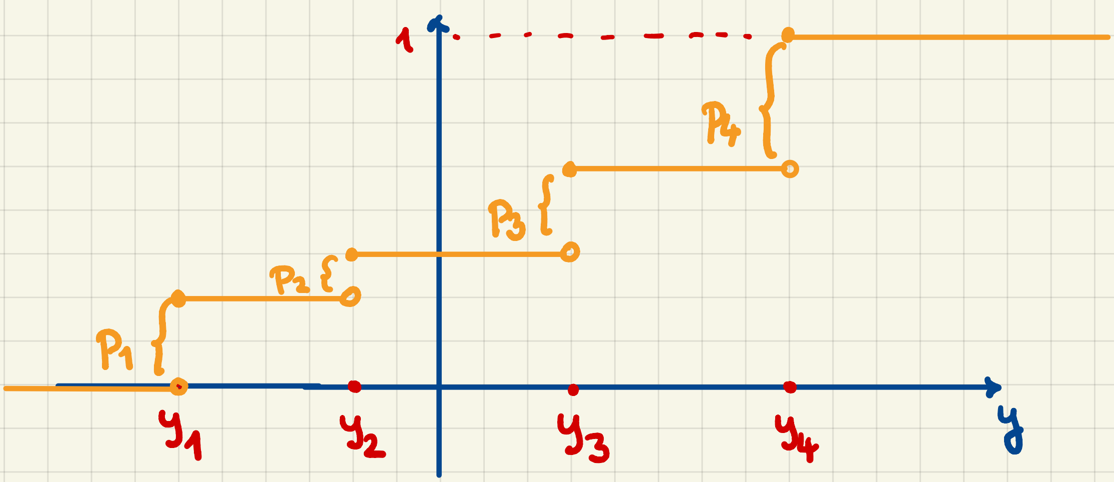
Example. Bernoulli Dist'n.

$$\boxed{Y \sim B(p)}$$

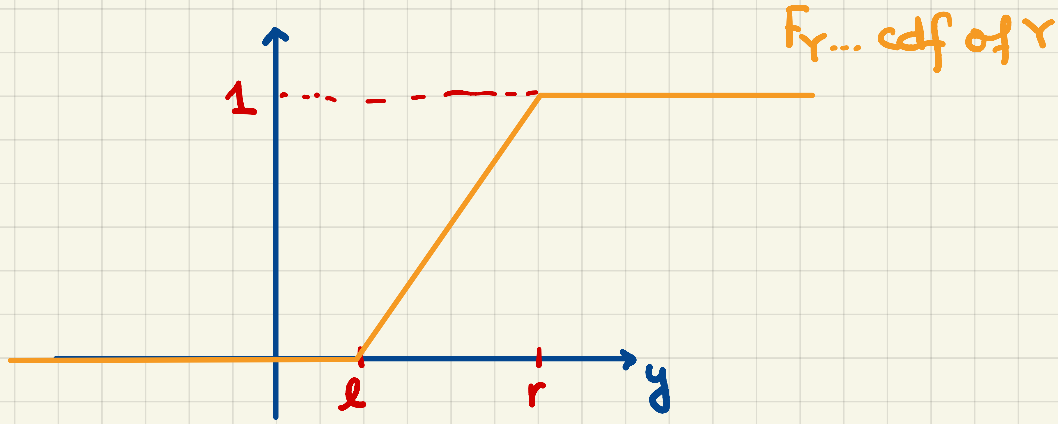


Discrete r.v. w/ a Finite Support

	y_1	y_2	......	y_m
	p_1	p_2	......	p_m



Uniform Dist'n. $Y \sim U(\ell, r)$



Normal Dist'n. $Y \sim N(\mu, \sigma)$

M378K Introduction to Mathematical Statistics

Problem Set #5

Cumulative distribution functions: Named continuous distributions.

Problem 5.1. Source: Sample P exam, Problem #126.
A company's annual profit is normally distributed with mean 100 and variance 400. Then, we can express the probability that the company's profit in a year is at most 60, given that the profit in the year is positive in terms of the standard normal cumulative distribution function denoted by Φ as

$$\boxed{1-\frac{\Phi(2)}{\Phi(5)}}$$

$\rightarrow$: $Y \sim N(\mu = 100, \sigma = 20)$

$$Z = \frac{Y-100}{20} \sim N(0,1)$$

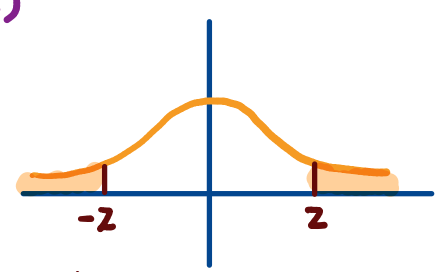
$$\mathbb{P}[Y \le 60 \mid Y > 0] = \frac{\mathbb{P}[0 < Y \le 60]}{\mathbb{P}[Y>0]}$$

$$= \frac{\mathbb{P}[0 < 100 + 20Z \le 60]}{\mathbb{P}[100 + 20Z > 0]}$$

$$= \frac{\mathbb{P}\left[\frac{0-100}{20} < Z \le \frac{60-100}{20}\right]}{\mathbb{P}\left[Z > \frac{0-100}{20}\right]} =$$

$$= \frac{\mathbb{P}[-5 < Z \le -2]}{\mathbb{P}[Z > -5]}$$

$$= \frac{\Phi(-2) - \Phi(-5)}{1 - \Phi(-5)} = ?$$



$$\Phi(-z) = \mathbb{P}[Z > z]$$
$$= 1 - \mathbb{P}[Z \le z]$$
$$= 1 - \Phi(z)$$