M339G: April 17th, 2024.

More on Hyperplanes.

Example.

Suppose that L is line in $\mathbb{R}^2$ w/ the equation

$$\boxed{2x+3y=1.} \checkmark$$

Then, a normal vector for L is $\vec{n}=(2,3)$.

We can easily find points on L: Say that $x=2 \Rightarrow y=-1$, i.e., the point $\boxed{\vec{p}=(2,-1)}$ is on L.

As a normal equation, all the points (x,y) on L must satisfy

$$\boxed{\vec{n}\cdot(\vec{x}-\vec{p})=0} \qquad \left.\begin{aligned} &(2,3)\cdot((x,y)-(2,-1))=0 \\ &\Longleftrightarrow \\ &(2,3)\cdot(x-2,y+1)=0 \end{aligned}\right\}$$

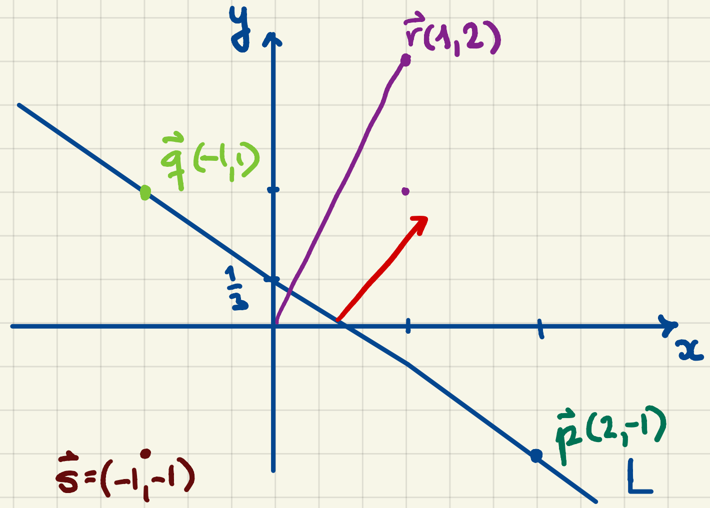
Let's find another point on L. Say, we denote it by $\boxed{\vec{q}=(q_1,q_2)}$

Pick $\boxed{q_1=-1 \quad \Rightarrow \quad q_2=1}$

We can check the normal equation: $(2,3)\cdot(-1-2,1+1) \overset{?}{=} 0$

$(2,3)\cdot(-3,2) \overset{?}{=} 0$

$2(-3)+3\cdot 2 \overset{\checkmark}{=} 0$



$2x+3y=1$

$3y=-2x+1$

$y=-\frac{2}{3}x+\frac{1}{3}$

What do we get for $\vec{r}=(1,2)$?

$2(1)+3(2)-1=2+6-1=7>0$

What do we get for $\vec{s}=(-1,-1)$?

$2(-1)+3(-1)-1=-2-3-1=-6<0$

A Short Outline of Lagrange Multipliers.

Example. Find a point $\vec{p}$ on the plane $x+y-2z=6$ which lies closest to the origin.

→: Q: why is this a constrained optimization problem?

→: Function that we want to minimize:

$$\tilde{D}(x,y,z) = x^2+y^2+z^2$$

subject to: $\boxed{x+y-2z=6}$

In general, $f(x,y,z) \longrightarrow \text{min/max}$

subject to the constraint $F(x,y,z)=0$

First, we construct the "Lagrangian function"

$$L(x,y,z,\lambda) = f(x,y,z) + \lambda F(x,y,z)$$

Then, we optimize the function L as a function of 4 variables (x,y,z,λ)

Back to the example:

$$\tilde{D}(x,y,z) = x^2+y^2+z^2 \longrightarrow \text{min}$$

Subject to $F(x,y,z) = x+y-2z-6 = 0$

$$\Rightarrow L(x,y,z,\lambda) = \tilde{D}(x,y,z) + \lambda F(x,y,z)$$

$$= x^2+y^2+z^2+\lambda(x+y-2z-6)$$

$$\frac{\partial L}{\partial x} = 2x+\lambda = 0 \quad \Rightarrow \quad x = -\frac{\lambda}{2}$$

$$\frac{\partial L}{\partial y} = 2y+\lambda = 0 \quad \Rightarrow \quad y = -\frac{\lambda}{2}$$

$$\frac{\partial L}{\partial z} = 2z-2\lambda = 0 \quad \Rightarrow \quad z = \lambda$$

$$\frac{\partial L}{\partial \lambda} = x+y-2z-6 = 0 \quad \Rightarrow \quad x+y-2z-6=0$$

$$-\frac{\lambda}{2}-\frac{\lambda}{2}-2\lambda-6=0$$

$$3\lambda=-6 \quad \Rightarrow \boxed{\lambda=-2} \quad \Rightarrow \boxed{\vec{p}=(1,1,-2)}$$

$\square$