M339D : November 8th, 2023.

Log Normal Tail Probabilities.

Example. Consider a non-dividend-paying stock. What is the probability that the stock outperforms a risk-free investment under the risk-neutral probability measure?

→ : The initially invested amount: $S(0)$.

- If it's the risk-free investment, the balance @ time T is: $\boxed{S(0)e^{rT}}$ ←
- If it's the stock investment, the wealth @ time T is $S(T)$

$$\boxed{\mathbb{P}^*[S(T) > S(0)e^{rT}] = ?}$$

This question is equivalent to the question of whether the profit is positive under $\mathbb{P}^*$.

In the Black-Scholes model:

$$S(T) = S(0)e^{(r-\frac{\sigma^2}{2})\cdot T + \sigma\sqrt{T}Z} \quad \text{w/ } Z \sim N(0,1)$$

$$\mathbb{P}^*[\cancel{S(0)}e^{(\cancel{r}-\frac{\sigma^2}{2})\cdot T + \sigma\sqrt{T}\cdot Z} > \cancel{S(0)}e^{\cancel{rT}}] = \text{(ln}(\cdot)\text{ is increasing)}$$

$$= \mathbb{P}^*\left[-\frac{\sigma^2}{2}\cdot T + \sigma\sqrt{T}\cdot Z > 0\right]$$

$$= \mathbb{P}^*\left[\sigma\sqrt{T}\cdot Z > \frac{\sigma^2}{2}\cdot T\right] = \mathbb{P}^*\left[Z > \frac{\sigma\sqrt{T}}{2}\right] \quad \text{(symmetry of } N(0,1))$$

$$= N\left(-\frac{\sigma\sqrt{T}}{2}\right) \xrightarrow[T\to\infty]{} 0 \quad \square$$

$\updownarrow$

Φ

Motivation.

Consider a European call option w/ strike K and exercise date T. Under the risk-neutral probability measure $\mathbb{P}^*$, what is the probability that the option is in the money on the exercise date?

$\longrightarrow$: In the Black-Scholes model:

$$\boxed{S(T) = S(0)e^{\left(r-\frac{\sigma^2}{2}\right)\cdot T + \sigma\sqrt{T}\cdot Z}} \qquad \text{w/ } Z \sim N(0,1)$$

We are calculating:

$$\mathbb{P}^*\left[\boxed{S(T)} > K\right] =$$

$$= \mathbb{P}^*\left[S(0)e^{\left(r-\frac{\sigma^2}{2}\right)\cdot T + \sigma\sqrt{T}\cdot Z} > K\right]$$

$$= \mathbb{P}^*\left[e^{\left(r-\frac{\sigma^2}{2}\right)\cdot T + \sigma\sqrt{T}\cdot Z} > \frac{K}{S(0)}\right] \qquad (\ln(\cdot) \text{ is increasing})$$

$$= \mathbb{P}^*\left[\left(r-\frac{\sigma^2}{2}\right)\cdot T + \sigma\sqrt{T}\cdot Z > \ln\left(\frac{K}{S(0)}\right)\right]$$

$$= \mathbb{P}^*\left[\sigma\sqrt{T}\cdot Z > \ln\left(\frac{K}{S(0)}\right) - \left(r-\frac{\sigma^2}{2}\right)\cdot T\right]$$

$$= \mathbb{P}^*\left[Z > \boxed{\frac{1}{\sigma\sqrt{T}}\left(\ln\left(\frac{K}{S(0)}\right) - \left(r-\frac{\sigma^2}{2}\right)\cdot T\right)}\right] \qquad (\text{symmetry of } N(0,1))$$

$$= \mathbb{P}^*\left[Z < \underbrace{\boxed{\frac{1}{\sigma\sqrt{T}}\left(\ln\left(\frac{S(0)}{K}\right) + \left(r-\frac{\sigma^2}{2}\right)\cdot T\right)}}_{=: d_2}\right]$$

$$\boxed{\mathbb{P}^*[S(T) > K] = N(d_2)}$$

Consequently: The probability that the otherwise identical put is in the money is

$$\mathbb{P}^*[S(T) < K] = 1 - N(d_2) = N(-d_2)$$

Why?

Goal: Price calls and puts in the Black-Scholes model.

By risk-neutral pricing:

$$V_c(0) = e^{-rT}\boxed{\mathbb{E}^*[V_c(T)]}$$

$$\mathbb{E}^*[V_c(T)] = \mathbb{E}^*[(S(T)-K)_+]$$

$$= \mathbb{E}^*[(S(T)-K)\cdot \mathbb{I}_{[S(T)\geq K]}]$$

$$= \mathbb{E}^*[S(T)\cdot \mathbb{I}_{[S(T)\geq K]}] - K\cdot \boxed{\mathbb{E}^*[\mathbb{I}_{[S(T)\geq K]}]}$$

$$\mathbb{E}^*[\mathbb{I}_{[S(T)\geq K]}] = \mathbb{P}^*[S(T)\geq K]$$