M339D: March 28th, 2025.

Normal Distributions.

Review

$Z \sim N(0,1)$

$$f_Z(z) = \varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \quad \text{for all } z \in \mathbb{R}$$

We can completely specify any normal dist'n by providing its mean, μ ✓ and its standard deviation σ (or its variance σ^2)!

We write: $X \sim Normal(mean = \mu, variance = \sigma^2)$

X can always be written as a linear transform of a standard normal Z, i.e.,

$$\boxed{X = \mu + \sigma \cdot Z} \longleftrightarrow \frac{X - \mu}{\sigma} = Z \sim N(0,1)$$

We can check:

- $\mathbb{E}[X] = \mathbb{E}[\mu + \sigma \cdot Z] = \mu + \sigma \cdot \underbrace{\mathbb{E}[Z]}_{=0} = \mu$

  linearity of expectation

- $Var[X] = Var[\mu + \sigma \cdot Z]$

  a deterministic shift which doesn't affect the variance

  $= Var[\sigma \cdot Z] = \sigma^2 \cdot \underbrace{Var[Z]}_{=1} = \sigma^2$

Fact. (X_1, X_2) are jointly normal, then

$$\alpha_1 X_1 + \alpha_2 X_2 \sim Normal(\_\_, \_\_)$$

In particular, if X_1 and X_2 are independent, then

$$\alpha_1 X_1 + \alpha_2 X_2 \sim Normal(\_\_, \_\_)$$

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim \text{Normal}\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \boxed{\Sigma = \begin{bmatrix} \sigma_1^2 & Cov \\ Cov & \sigma_2^2 \end{bmatrix}} \right)$$

The Normal Approximation to the Binomial

(de Moivre-Laplace Thm).

Consider a sequence of binomial random variables

$Y_n \sim \text{Binomial}(n = \#\text{ of trials}, p = \text{success probability})$

Then,

- $\mathbb{E}[Y_n] = np$
- $\text{Var}[Y_n] = np(1-p) \Rightarrow SD[Y_n] = \sqrt{np(1-p)}$

$$\boxed{\boxed{\frac{Y_n - np}{\sqrt{np(1-p)}}} \xrightarrow[n\to\infty]{\mathcal{D}} N(0,1)}$$

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Usage:

- "Rule of Thumb" : n is "large", i.e.,

  $np \geq 10$ and $n(1-p) \geq 10$

- $\mathbb{P}[a \leq Y_n \leq b] =$

$$= \mathbb{P}\left[\frac{a-np}{\sqrt{np(1-p)}} \leq \boxed{\frac{Y_n - np}{\sqrt{np(1-p)}}} \leq \frac{b-np}{\sqrt{np(1-p)}} \right]$$

$\approx N(0,1) \sim Z$

$$\approx \mathbb{P}\left[\frac{a-np}{\sqrt{np(1-p)}} \leq Z \leq \frac{b-np}{\sqrt{np(1-p)}} \right]$$

$$= N\left(\frac{b-np}{\sqrt{np(1-p)}} \right) - N\left(\frac{a-np}{\sqrt{np(1-p)}} \right)$$

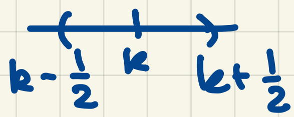
$\boxed{N\ldots \text{ cdf of } N(0,1)}$

- In statistics: $Y_n \approx N(\text{mean} = np, \text{var} = np(1-p))$

- **In M362K:**

$$\mathbb{P}[Y_n = k] = \mathbb{P}[k \le Y_n \le k] = 0$$

Continuity Correction!!!



$$\mathbb{P}[Y_n = k] = \mathbb{P}\left[k-\tfrac{1}{2} \le Y_n \le k+\tfrac{1}{2}\right] = \cdots$$