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- In this setting, we need to be careful to avoid incorrectly rejecting too many null hypotheses, i.e. having too many false positives.

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- If we reject all null hypotheses for which the p-value falls below 0.01, then how many Type I errors will we make?

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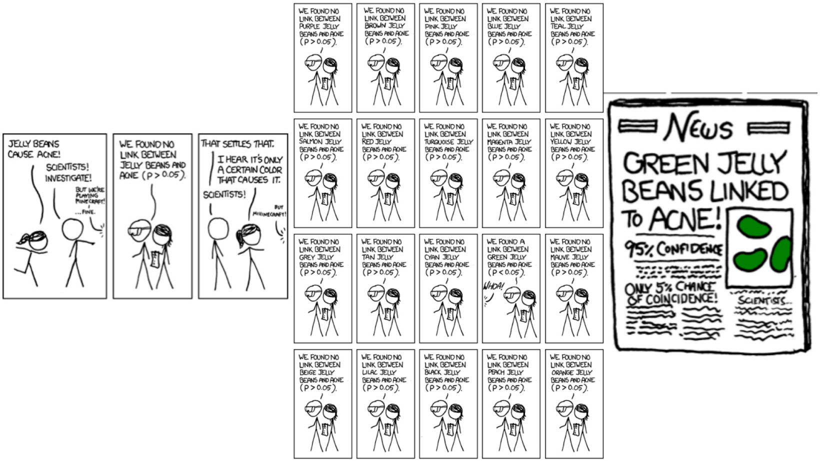
  - So we would conclude it is not fair, i.e. we *reject H_0*, even though it's a fair coin.
- If we test a lot of hypotheses, we are almost certain to get one very small p-value by chance!

Multiple Testing: Even XKCD Weighs In



`https://xkcd.com/882/`

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- *That's a lot of Type I errors, i.e. false positives!*

The Family-Wise Error Rate

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- $\text{FWER} = \Pr(V \geq 1)$

	H_0 is True	H_0 is False	Total
Reject H_0	V	S	R
Do Not Reject H_0	U	W	$m - R$
Total	m_0	$m - m_0$	m

Challenges in Controlling the Family-Wise Error Rate

$$\begin{aligned}
\text{FWER} &= 1 - \Pr(\text{do not falsely reject any null hypotheses}) \\
&= 1 - \Pr\left(\bigcap_{j=1}^{m} \{\text{do not falsely reject } H_{0j}\}\right).
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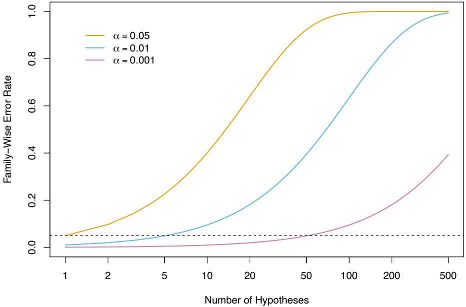

The Bonferroni Correction

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- If we only reject hypotheses when the p-value is less than α/m, then

$$\text{FWER} \leq \sum_{j=1}^{m} \Pr(A_j) \leq \sum_{j=1}^{m} \frac{\alpha}{m} = m \times \frac{\alpha}{m} = \alpha,$$

  because $\Pr(A_j) \leq \alpha/m$.

- This is the *Bonferroni Correction*: to control FWER at level α, reject any null hypothesis with p-value below α/m.

Fund Manager Data

Manager	Mean, $\bar{x}$	s	t-statistic	p-value
One	3.0	7.4	2.86	0.006
Two	-0.1	6.9	-0.10	0.918
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- *H_{0j}: the jth manager's expected excess return equals zero*.
- If we reject H_{0j} if the p-value is less than $\alpha = 0.05$, then we will conclude that the *first* and *third* managers have significantly non-zero excess returns.

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- If we reject H_{0j} if the p-value is less than $\alpha = 0.05$, then we will conclude that the *first* and *third* managers have significantly non-zero excess returns.
- However, we have tested multiple hypotheses, so the FWER is *greater* than 0.05.

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- Consequently, we will reject the null hypothesis only for the *first* manager.
- Now the FWER is at most 0.05.

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- Holm's method controls the FWER at level α.

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- The Holm procedure rejects the first two null hypotheses, because
  - $p_{(1)} = 0.006 < 0.05/(5+1-1) = 0.0100$
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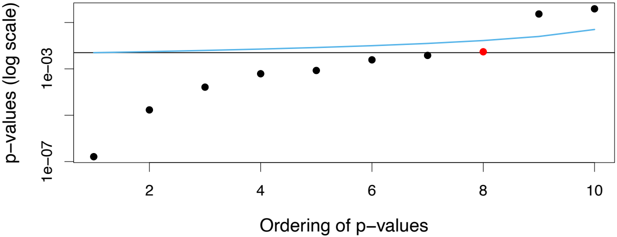
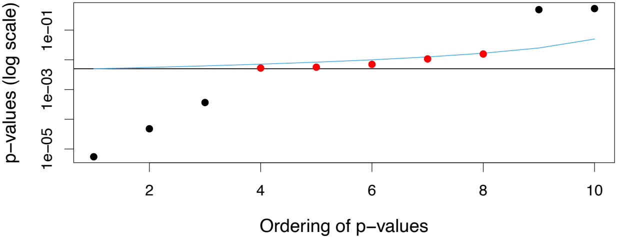
  - $p_{(3)} = 0.601 > 0.05/(5 + 1 - 3) = 0.0167$.
- Holm rejects H_0 for the *first* and *third* managers, but Bonferroni only rejects H_0 for the *first* manager.

A Comparison with $m = 10$ p-values



- Aim to control FWER at 0.05.
- p-values below the black horizontal line are rejected by Bonferroni.
- p-values below the blue line are rejected by Holm.
- Holm and Bonferroni make the same conclusion on the black points, but only Holm rejects for the red point.

A More Extreme Example



- Now five hypotheses are rejected by Holm but not by Bonferroni ....
- .... even though both control FWER at 0.05.

Holm or Bonferroni?

- Bonferroni is simple ... reject any null hypothesis with a p-value below α/m.
- Holm is slightly more complicated, but it will lead to more rejections while controlling FWER!!
- So, *Holm is a better choice*!