The Monte Carlo Method.

1.1. **The Strong Law of Large Numbers.** Let $\{X_k : k = 1, 2, \dots\}$ be a sequence of **independent**, **identically distributed** random variables with a finite first moment

$$\mu_X := \mathbb{E}[X_1]$$

Then,

$$\frac{X_1 + X_2 + \cdots + X_n}{n} \xrightarrow{n\to\infty} \mu_X$$

Moreover, let g be a function such that $g(X_1)$ is well defined. *What does this mean?* Assume that $\mathbb{E}[g(X_1)]$ is finite.
Then,

$$\frac{g(X_1) + g(X_2) + \cdots + g(X_n)}{n} \xrightarrow{n\to\infty} \mathbb{E}[g(X_1)]$$

1.2. **Monte Carlo.** The idea of Monte Carlo is to "marry" our ability to simulate pseudo-random numbers using software with the *Strong Law of Large Numbers.*

1.2.1. *Recipe.* Here is the algorithm for estimating "complicated" expectations.

- ☐ Draw simulated values from a specific distribution.
- ☐ Apply the function to the simulated values.
- ☐ Calculate the arithmetic average of the obtained quantities.

The result is a value **close to** the theoretical expectation.

1.2.2. **Precision.** Notice that our estimates are merely simulated realizations of a random variable (the **sample mean** in the language of statistics). The measure of bow **close** we expect our estimate to be to the theoretical mean is its **variance** or the **standard deviation**. We have

$$\begin{aligned}
\mathrm{Var}\left[\frac{X_1 + X_2 + \cdots + X_n}{n}\right] &= \frac{1}{n^2}\,\mathrm{Var}\left[X_1 + X_2 + \cdots + X_n\right] \\
&= \textbf{(independence)}\ \frac{1}{n^2}\left(\mathrm{Var}[X_1] + \mathrm{Var}[X_2] + \cdots + \mathrm{Var}[X_n]\right) \\
&= \textbf{(identical distribution)}\ \frac{1}{n^2}(n\,\mathrm{Var}[X_1]) \\
&= \frac{\mathrm{Var}[X_1]}{n}.
\end{aligned}$$

Ergo,

$$SD\left[\frac{X_1 + X_2 + \cdots + X_n}{n}\right] = \frac{SD[X_1]}{\sqrt{n}}.$$

In words: **to increase our precision by a factor of η, we must increase our number of variates (draws) by a factor of η^2.**