The University of Texas at Austin
Solutions to In-Term Exam 1

M378K Introduction to Mathematical Statistics

February 11, 2026

Instructions: This is a closed book and closed notes exam. The maximal score on the exam is 100 points.
Time: 50 minutes

Formulas.

If Y has the binomial distribution with parameters n and p, then $p_Y(k) = \mathbb{P}[Y = k] = \binom{n}{k} p^k (1-p)^{n-k}$, for $k = 0, ..., n$, $\mathbb{E}[Y] = np$, Var $[Y] = np(1-p)$. The binomial coefficients are defined as follows for integers $0 \leq k \leq n$, $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

If Y has a geometric distribution with parameter p, then $p_Y(k) = p(1-p)^k$ for $k = 0, 1, ..., \mathbb{E}[Y] = \frac{1-p}{p}$, Var $[Y] = \frac{1-p}{p^2}$.

If Y has a Poisson distribution with parameter λ, then $p_Y(k) = e^{-\lambda} \frac{\lambda^k}{k!}$ for $k = 0, 1, ..., \mathbb{E}[Y] =$ Var $[Y] = \lambda$.

If Y has a uniform distribution on $[l, r]$, its density is

$$f_Y(y) = \frac{1}{r - l} 1_{(l,r)}(y),$$

its mean is $\frac{l+r}{2}$, and its variance is $\frac{(r-l)^2}{12}$.

If Y has the standard normal distribution, then its mean is zero, its variance is one, and its density equals

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad x \in \mathbb{R}.$$

If Y has the exponential distribution with parameter τ, then its cumulative distribution function is $F_Y(y) = 1 - e^{-\frac{y}{\tau}}$ for $y \geq 0$, its probability density function is $f_Y(y) = \frac{1}{\tau} e^{-y/\tau}$ for $y \geq 0$. Also, $\mathbb{E}[Y] = SD[Y] = \tau$.

DEFINITIONS

Problem 1.1. (10 points) Write down the definition of the f cumulative distribution function of a random variable Y.

Solution.

$$F_Y(x) = \mathbb{P}[Y \leq x] \quad \text{for} \quad x \in \mathbb{R}.$$

Problem 1.2. (10 points) Let Y be a continuous random variable with the probability density function denoted by f_Y. Let g be a function taking real values such that $g(Y)$ is well defined. How is $\mathbb{E}[g(Y)]$ evaluated using f_Y, if it exists?

Solution. We have that

$$\mathbb{E}[g(Y)] = \int_{-\infty}^{\infty} g(y) f_Y(y)\, \mathrm{d}y$$

if the above integral is absolutely convergent.

TRUE/FALSE QUESTIONS

Problem 1.3. (5 points) The pdf (probability density function) of the random variable Y is

$$f_Y(y) = c\mathbf{1}_{()[0,4]}(y), \quad \text{for} \quad y \in \mathbb{R}.$$

The constant c is 4. *True or false? Why?*

Solution. FALSE
We can recognize Y as uniform on the interval $[0,4]$ which implies that $c = \frac{1}{4}$.

Alternatively, we have

$$1 = \int_0^4 c\,\mathrm{d}y = 4c \quad \Rightarrow \quad c = \frac{1}{4}$$

.

Problem 1.4. (5 points) Let Y be a binomial random variable with mean 4 and variance 3.2. Then, its success probability is $1/5$. *True or false? Why?*

Solution. TRUE
Since Y is binomial, we know that, in our usual notation,

$$\mathbb{E}[Y] = np = 4 \quad \text{and} \quad \mathrm{Var}\,[Y] = np(1-p) = 3.2.$$

Hence,

$$\frac{\mathrm{Var}\,[Y]}{\mathbb{E}[Y]} = \frac{np(1-p)}{np} = 1 - p = \frac{3.2}{4} = \frac{4}{5}\,.$$

Thus, $p = \frac{1}{5}$.

Problem 1.5. (5 points) Let Y be a continuous random variable. Then, $\mathbb{P}[Y = y] = 0$ for every $y \in \mathbb{R}$. *True or false? Why?*

Solution. TRUE
For every y, we have that, in our usual notation,

$$\mathbb{P}[Y = y] = \mathbb{P}[y \le Y \le y] = \int_y^y f_Y(u)\,\mathrm{d}u = 0.$$

Problem 1.6. (5 points) The second (raw) moment of a Poisson random variable is 90. Then, its mean is 10. *True or false? Why?*

Solution. FALSE
Since Y is Poisson, we know that, in our usual notation,

$$\mathbb{E}[Y] = \mathrm{Var}\,[Y] = \lambda.$$

Hence,

$$90 = \mathbb{E}[Y^2] = \mathrm{Var}\,[Y] + (\mathbb{E}[Y])^2 = \lambda + \lambda^2 = \lambda(\lambda + 1).$$

Thus, $\lambda = 9$.

Free-response problems.

Please, explain carefully all your statements and assumptions. Numerical results or single-word answers without an explanation (even if they're correct) are worth 0 points.

Problem 1.7. (15 points) A random variable Y has the normal distribution with mean 3 and standard deviation 5. Then, the probability that Y exceeds 4 given that it exceeds 2 equals

$$\frac{1}{\Phi(0.2)} - 1$$

where Φ stands for the standard normal cumulative distribution function. *True or false? Why?*

Solution. Since $Y \sim N(\mu = 3, \sigma = 5)$, we know that Y can be expressed as

$$Y = \mu + \sigma Z = 3 + 5Z$$

where Z is standard normal.

We are asked to calculate

$$\begin{aligned}\mathbb{P}[Y > 4 \,|\, Y > 2] &= \frac{\mathbb{P}[Y > 4, Y > 2]}{\mathbb{P}[Y > 2]} = \frac{\mathbb{P}[Y > 4]}{\mathbb{P}[Y > 2]} = \frac{\mathbb{P}[3 + 5Z > 4]}{\mathbb{P}[3 + 5Z > 2]} \\ &= \frac{\mathbb{P}[Z > 0.2]}{\mathbb{P}[Z > -0.2]} = \frac{1 - \Phi(0.2)}{\Phi(0.2)} = \frac{1}{\Phi(0.2)} - 1.\end{aligned}$$

So, **TRUE**.

Problem 1.8. (10 points) Assume that the lifetime T of a lightbulb is exponential with mean 10 hours. The lighbulb has already been burning for 20 hours. What is the probability that its f total lifetime will **exceed** 30 hours? *Note: You can leave your response in the form that uses the exponential function, but you must **simplify** it as much as possible!*

Solution. By the memoryless property, the probability equals

$$\mathbb{P}[T > 10] = e^{-\frac{10}{10}} = e^{-1}\,.$$

Problem 1.9. (15 points) Let Y be binomial with 10 trials and probability of success equal to $1/4$. What is the probability of at most 2 successes? *Note: Leave your answer in the form of a fraction containing only integers **without** any binomial coefficients.*

Solution. With $Y \sim b(10, 1/4)$, we have to calculate

$$
\begin{aligned}
\mathbb{P}[Y \le 2] &= \mathbb{P}[Y=0] + \mathbb{P}[Y=1] + \mathbb{P}[Y=2] \\
&= \binom{10}{0}\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^{10} + \binom{10}{1}\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^9 + \binom{10}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^8 \\
&= \frac{3^{10}}{4^{10}} + 10 \times \frac{3^9}{4^{10}} + 45 \times \frac{3^8}{4^{10}} \\
&= \frac{3^{10} + 10 \times 3^9 + 45 \times 3^8}{4^{10}} \\
&= \frac{3^{10} + 10 \times 3^9 + 15 \times 3^9}{4^{10}} \\
&= \frac{3^{10} + 25 \times 3^9}{4^{10}} \\
&= \frac{28 \times 3^9}{4^{10}} \\
&= \frac{28 \times 3^9}{2^{20}} = \frac{7 \times 3^9}{2^{18}} = \frac{137781}{262144}.
\end{aligned}
$$

Problem 1.10. (10 points) A sensor records the number of particle hits per minute, and the count is modeled as a Poisson random variable. It is known that exactly one hit in a minute is three times as likely as exactly three hits in a minute. What is the probability that zero hits occur in a minute? *Note: You can leave your response in the form that uses the exponential function, but you must **simplify** it as much as possible!*

Solution. Let the number of particle hits be $Y \sim \text{Poisson}\ (\lambda)$. Then,

$$\mathbb{P}[Y = k] = \frac{\lambda^k e^{-\lambda}}{k!} \quad \text{for all} \quad k \ge 0.$$

We are given that

$$\mathbb{P}[Y = 1] = 3\mathbb{P}[Y = 3].$$

Now, we substitute the expression for the Poisson probability mass function into the above equation and obtain

$$\frac{\lambda e^{-\lambda}}{1!} = 3 \cdot \frac{\lambda^3 e^{-\lambda}}{3!}.$$

Of course, we can cancel $e^{-\lambda} > 0$ which simplifies the above equation to

$$\lambda = 3 \times \frac{\lambda^3}{6} = \frac{\lambda^3}{2}.$$

If we multiply both sides by 2, we arrive at

$$2\lambda = \lambda^3.$$

The parameter λ cannot be zero. So, we get

$$\lambda^2 = 2 \quad \Rightarrow \quad \lambda = \sqrt{2}.$$

Finally, the probability of zero events is

$$\mathbb{P}[Y = 0] = \frac{\lambda^0 e^{-\lambda}}{0!} = e^{-\lambda} = e^{-\sqrt{2}}.$$

MULTIPLE CHOICE QUESTIONS

Problem 1.11. (5 points) There are 10 green balls and 20 red balls in a box, well mixed together. We pick a ball at random, and then roll a die. If the ball was green, we write down the outcome of the die, but if the ball is red, we write down the number 3.

If the number 3 was written down, the probability that the ball was green is:

a. $\frac{1}{13}$.
b. $\frac{1}{12}$.
c. $\frac{5}{6}$.
d. 1.
e. **None of the above**.

Solution. The correct answer is **(a)**.
Let R denote the event when the ball drawn was red, and $G = R^c$ the event corresponding to drawing a green ball, so that $\mathbb{P}[R] = 2/3$ and $\mathbb{P}[G] = 1/3$. If X denotes the number written down, we have

$$\mathbb{P}[X = 3 \mid G] = 1/6 \quad \text{and} \quad \mathbb{P}[X = 3 \mid R] = 1.$$

Using Bayes formula,

$$\begin{aligned}\mathbb{P}[G \mid X = 3] &= \frac{\mathbb{P}[X = 3 \mid G] \times \mathbb{P}[G]}{\mathbb{P}[X = 3 \mid G] \times \mathbb{P}[G] + \mathbb{P}[X = 3 \mid R] \times \mathbb{P}[R]} \\ &= \frac{1/6 \times 1/3}{1/6 \times 1/3 + 1 \times 2/3} = \frac{1}{13}.\end{aligned}$$

Problem 1.12. (5 points) Let the density function of a random variable X be given as

$$f_X(x) = cxI_{[0,2]}(x),$$

for some constant c. Find $\mathbb{E}[X^3]$.

a. $\frac{1}{2}$.
b. $\frac{2}{5}$.
c. $\frac{16}{5}$.
d. 4.
e. **None of the above**.

Solution. The correct answer is **(c)**.
Since the density function must integrate up to 1, it must be that

$$\int_0^2 cx \,\mathrm{d}x = 1 \quad \Rightarrow \quad c\left(\frac{x^2}{2}\right)\Bigg|_{x=0}^{2} = 1 \quad \Rightarrow \quad 2c = 1 \quad \Rightarrow \quad c = \frac{1}{2}.$$

Hence,

$$\mathbb{E}[X^3] = \frac{1}{2}\int_0^2 x^4 \,\mathrm{d}x = \frac{1}{2}\left(\frac{x^5}{5}\right)\Bigg|_{x=0}^{2} = \frac{2^5}{2 \times 5} = \frac{2^4}{5} = \frac{16}{5}.$$