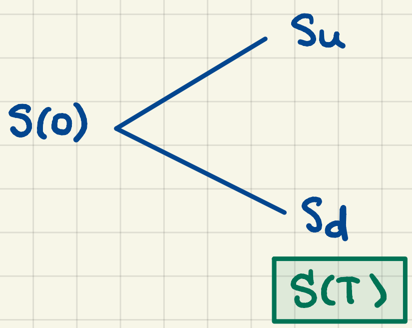
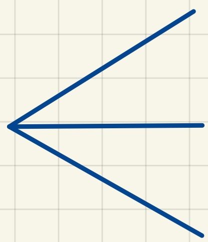
M339D: March 25th, 2024.

Inspiration.

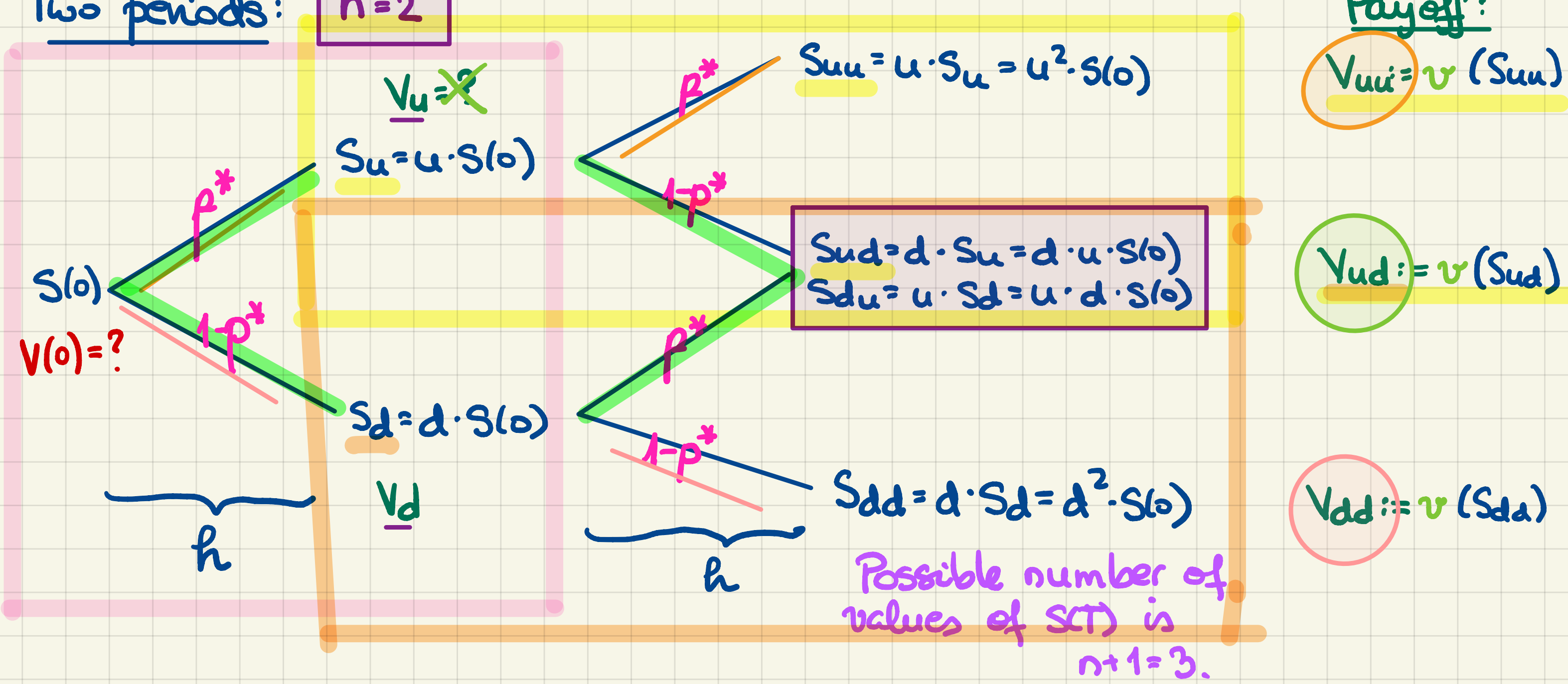


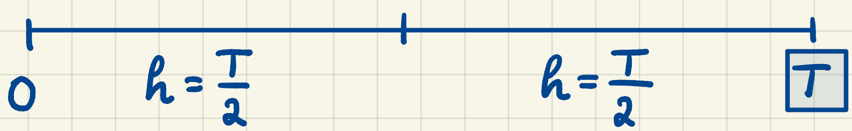
Q: How can we make the model for S(T) more complex?



k·nary trees

Two periods: $n=2$





populating the tree →

← pricing the option

- up node:

replicating portfolio for the option:

$$\Delta_u = \frac{V_{uu} - V_{ud}}{S_{uu} - S_{ud}}$$

$$B_u = e^{-rh} \cdot \frac{u \cdot V_{ud} - d \cdot V_{uu}}{u-d}$$

$\Rightarrow$ the option's value @ the up node:

$$V_u = \Delta_u \cdot S_u + B_u = e^{-rh} \cdot \left[p^* \cdot V_{uu} + (1-p^*) \cdot V_{ud} \right]$$

w/ $\boxed{p^* = \frac{e^{rh} - d}{u-d}}$

- down node: Δ_d, B_d

$\Rightarrow$ $V_d = \Delta_d \cdot S_d + B_d = e^{-rh} \left[p^* \cdot V_{ud} + (1-p^*) \cdot V_{dd} \right]$ w/

- ROOT node:

$$\Delta_0 = \frac{V_u - V_d}{S_u - S_d}$$

$$B_0 = e^{-rh} \cdot \frac{u \cdot V_d - d \cdot V_u}{u-d}$$

$\Rightarrow$ $\boxed{V(0) = \Delta_0 \cdot S(0) + B_0}$

From the risk-neutral "perspective":

$$V(0) = e^{-rh} \cdot \left[p^* \cdot V_u + (1-p^*) \cdot V_d \right]$$

$$= e^{-rh} \cdot \left[p^* \cdot e^{-rh} \left(p^* \cdot V_{uu} + (1-p^*) V_{ud} \right) + (1-p^*) e^{-rh} \left(p^* \cdot V_{ud} + (1-p^*) \cdot V_{dd} \right) \right]$$

$$= \underbrace{e^{-r(2h)}}_{\text{Discounting.}} \underbrace{\left[(p^*)^2 \cdot V_{uu} + 2p^*(1-p^*) \cdot V_{ud} + (1-p^*)^2 V_{dd} \right]}_{\text{Risk-Neutral Expectation of the Payoff.}}$$

Generally: $\boxed{V(0) = e^{-rT} \mathbb{E}^* \left[V(T) \right]}$

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Problem Set #10

Binomial option pricing: Two or more periods.

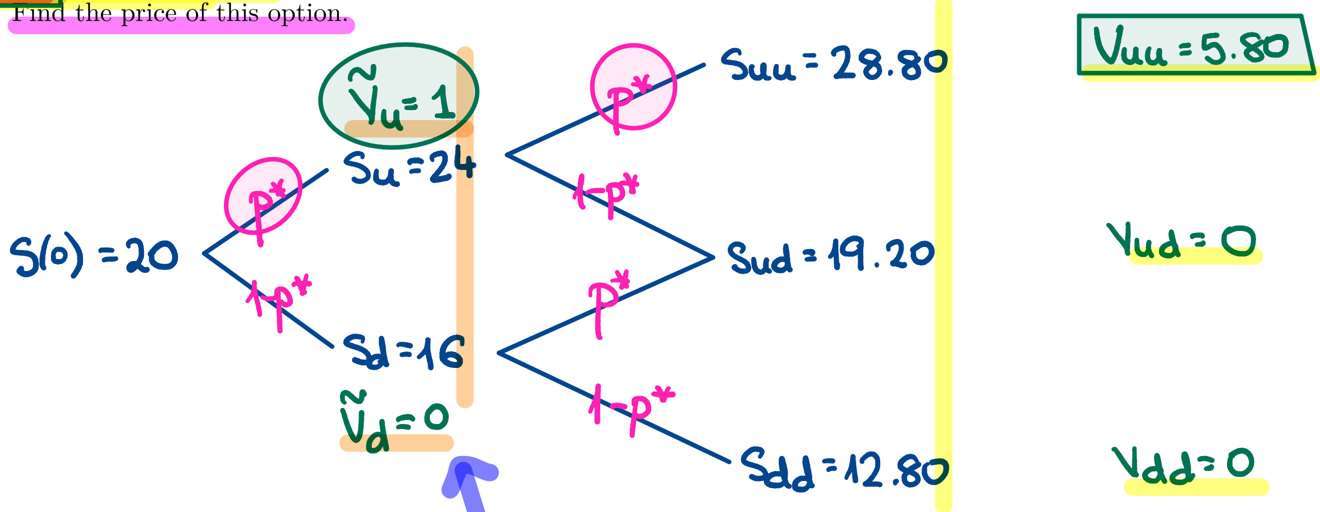
Problem 10.1. For a two-period binomial model, you are given that:

(1) each period is one year;
(2) the current price of a non-dividend-paying stock S is $S(0) = \$20$;
(3) $u = 1.2$, with u as in the standard notation for the binomial model;
(4) $d = 0.8$, with d as in the standard notation for the binomial model;
(5) the continuously compounded risk-free interest rate is $r = 0.04$.

Consider a **special** call option which pays the excess above the strike price $K = 23$ (if any!) at the end of **every** binomial period.

Find the price of this option.

:



Risk-neutral Probability:

$$p^* = \frac{e^{rh} - d}{u - d} = \frac{e^{0.04} - 0.8}{1.2 - 0.8} = 0.602$$

$$\tilde{V}(0) = e^{-0.04} \cdot p^* \cdot 1 = 0.5784$$

$$V(0) = e^{-0.04(2)} (p^*)^2 \cdot 5.80 = 1.9413$$

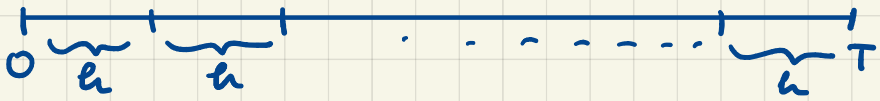
+

answer: the price of the special call is $\boxed{2.5197}$

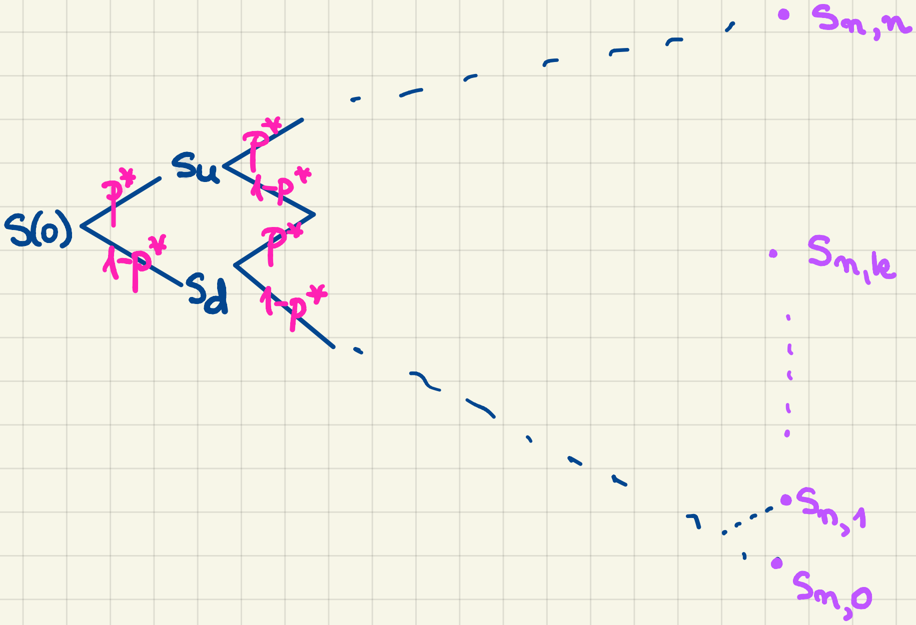
Multiple Binomial Periods.

T... exercise date of a European option
n... # of periods

the length of each period: $\boxed{h = \frac{T}{n}}$



Payoffs



$v_{n,n}$

$S_{n,k}$ w/ k ... # of steps up $v_{n,k}$

$v_{n,0}$

The $(n+1)$ values are the support of $S(T)$

$\Rightarrow$ for every $k = 0, 1, \ldots, n$:

$$S_{n,k} = S(0)u^k \cdot d^{n-k} = S(0)\left(\frac{u}{d}\right)^k \cdot d^n$$

k — # of upsteps