NAME:

M339W/389W Financial Mathematics for Actuarial Applications
University of Texas at Austin
Mock In-Term Exam I
Instructor: Milica Čudina

Notes: This is a closed book and closed notes exam. The maximum number of points on this exam is 50.
Time: 50 minutes

Problem 1.1. (5 points) A continuous-dividend-paying stock is valued at \$75.00 per share. Its dividend yield is 0.03. The time$-t$ realized return is modeled as

$$R(0,t) \sim N(\text{mean} = 0.035t, \text{variance} = 0.09t)$$

Find the probability that the time-4 stock price exceeds today's stock price.

Solution: We need to find

$$\mathbb{P}[S(4) > S(0)]$$

with

$$S(t) = S(0)e^{R(0,t)}.$$

Since $R(0,t)$ follows the normal distribution with the above parameters, we have

$$\mathbb{P}[S(4) > S(0)] = \mathbb{P}[S(0)e^{R(0,4)} > S(0)] = \mathbb{P}[R(0,4) > 0] = 1 - N\left(-\frac{0.035 \times 4}{0.3 \times 2}\right) = N(0.23) = 0.591.$$

Problem 1.2. (5 points) The current price of a continuous-dividend-paying stock is \$80 per share. The stock's dividend yield is 0.02. According to your model, the expected value of the stock price in two years is \$90 per share. You are also given:

The risk-free interest rate exceeds the dividend yield.

The two-year forward price on a share of this stock is denoted by F. At this price you are willing to enter into the forward. What is the smallest range of values F can take according to the above information?

(a) $F < 77$
(b) $77 < F < 80$
(c) $80 < F < 90$
(d) $F > 90$
(e) None of the above.

Solution: (c)
Using the fact that the investor is willing to enter a forward contract, we conclude that the forward contract's profit is positive. So,

$$\mathbb{E}[S(T)] > F \quad \Rightarrow \quad 90 > F$$

On the other hand, we know that

$$F = S(0)e^{(r-\delta)T} = 80e^{2(r-0.02)} > 80.$$

So, the most we can say about F is that $80 < F < 90$.

Problem 1.3. (5 points) Let the current price of a non-dividend-paying stock be denoted by $S(0)$. We model the time$-T$ stock price as lognormal. The mean rate of return on the stock is 0.12 and its volatility is 0.20. The continuously-compounded, risk-free interest rate is 0.04. You invest in one share of stock at time-0. and simultaneously deposit an amount $\varphi S(0)$ in a savings account. What should the proportion φ be so that the VaR at the level 0.05 of your total wealth at time-1 equals today's stock price $S(0)$?

(a) $\varphi = 0.1966$
(b) $\varphi = 0.5$
(c) $\varphi = 0.8034$
(d) $\varphi = 1$
(e) None of the above.

Solution: (a)
The total wealth at time-1 is equal to $S(1) + \varphi S(0)e^r$. So, our condition on the VaR is

$$\mathbb{P}[S(1) + \varphi S(0)e^r < S(0)] = 0.05.$$

The above is equivalent to

$$\mathbb{P}[S(0)e^{\alpha - \frac{\sigma^2}{2} + \sigma Z} + \varphi S(0)e^r < S(0)] = 0.05$$

with $Z \sim N(0,1)$. We have that the above is, in turn, equivalent to

$$\mathbb{P}[e^{\alpha - \frac{\sigma^2}{2} + \sigma Z} + \varphi e^r < 1] = 0.05.$$

The constant z^* such that $\mathbb{P}[Z < z^*] = 0.05$ equals -1.645. Therefore,

$$e^{\alpha - \frac{\sigma^2}{2} + \sigma(-1.645)} + \varphi e^r = 1.$$

So,

$$\varphi = e^{-r}\left(1 - e^{\alpha - \frac{\sigma^2}{2} + \sigma(-1.645)}\right) = e^{-0.04}\left(1 - e^{0.12 - \frac{(0.2)^2}{2} + 0.2(-1.645)}\right) = 0.196646.$$

Problem 1.4. Consider a non-dividend-paying stock whose current price is \$95 per share. You model the evolution of this stock price over the following year using a one-period binomial tree under the assumption that the stock price can be either \$120, or \$75 in one year.

The continuously-compounded, risk-free interest rate is 0.06.

Consider a \$100-strike, one-year European **straddle** on the above stock. What is the straddle's price consistent with the above stock-price model?

(a) About \$10
(b) About \$10.83
(c) About \$15.45
(d) About \$20.84

(e) None of the above.

Solution: (d)
The risk-neutral probability of an up movement is

$$p^* = \frac{95e^{0.06} - 75}{120 - 75} = 0.575.$$

So, the price of our straddle is

$$V_C(0) = e^{-0.06}[0.575 \times (120 - 100) + (1 - 0.575) * (100 - 75)] = 20.8366.$$

Problem 1.5. The current exchange rate is given to be \$1.25 per Euro and its volatility is given to be 0.15.

The continuously-compounded, risk-free interest rate for the US dollar is 0.03, while the continuously-compounded, risk-free interest rate for the Euro equals 0.06.

The evolution of the exchange rate over the following nine-month period is modeled using a three-period forward binomial tree.

What is the value of the so-called down factor in the above tree?

(a) $d \approx 0.8586$
(b) $d \approx 0.8982$
(c) $d \approx 0.9208$
(d) $d \approx 0.9347$
(e) None of the above.

Solution: (c)
In the forward binomial tree, the up and down factors are given as

$$u = e^{(0.03-0.06)\times 0.25 + 0.15\times\sqrt{0.25}} = 1.0698$$
$$d = e^{(0.03-0.06)\times 0.25 - 0.15\times\sqrt{0.25}} = 0.9208.$$

Problem 1.6. The evolution of a market index over the following year is modeled using a four-period binomial tree. We are given that the current value of the market index equals \$144, that its volatility equals 0.25, and that it pays dividends continuously.

You are tasked with constructing a four-period forward tree for the evolution over the following year of the forward price of the above market index with delivery at time-2.

What is the down factor d_F in the forward price tree for the futures prices on the stock?

(a) 0.7788
(b) 0.8825
(c) 0.9914
(d) There is not enough information given.
(e) None of the above.

Solution: (b)
In our usual notation,

$$d_F = de^{-(r-\delta)h} = e^{-\sigma\sqrt{h}} = e^{-0.25\sqrt{1/4}} = 0.8825.$$

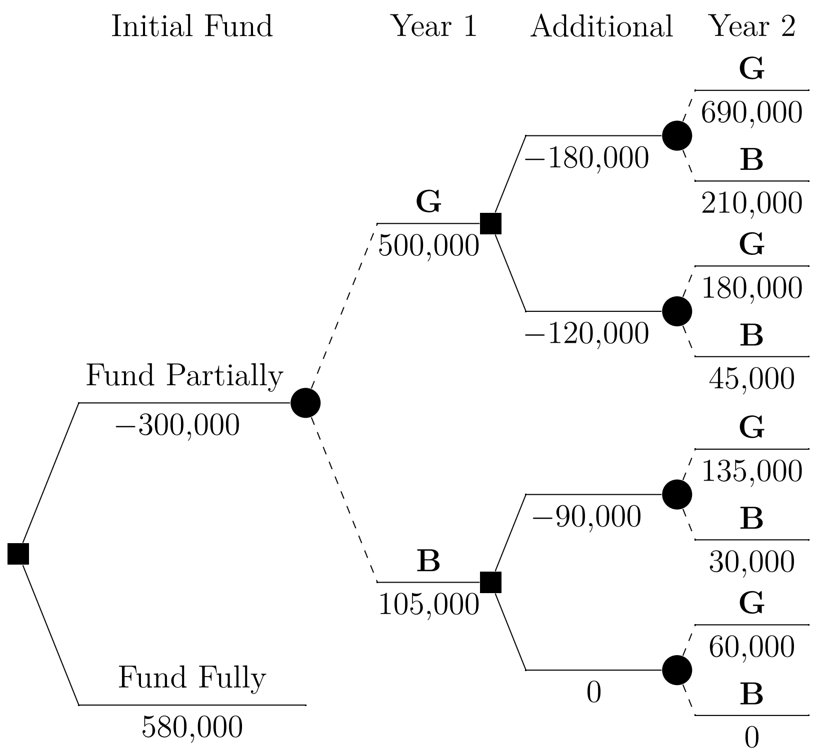
Problem 1.7. (5 points) Netflix is considering a cartoon series. When the production of two seasons is fully funded at time−0 the project has a net present value of 580,000.

The decision tree below shows the cash flows of the series when the promotion at the beginning of the Year 1 (i.e., at $t = 0$) is only partial with an option to provide different amounts of funding at the beginning of Year 2 (i.e., at $t = 1$) depending on how well the first season did.

This tree reflects two possible receptions of the two seasons at each information node (**G** = good, **B** = bad). The probability of the series being a success is given to be 1/2 and the probability of it being merely watchable is 1/2.

Assume the interest rate is 0%.

Find the **initial** (i.e., at $t = 0$) value of the option to fund partially.



(a) 15000

(b) 20000

(c) 25000

(d) 30000

(e) None of the above.

Solution: **(e)** As usual, when pricing options, we are moving backwards through the tree.

- In the *uppermost final* information node, the possible cashflows are 690,000 with probability 1/2 and 210,000 with probability 1/2. So, the value of the project at that node equals

$$690000\left(\frac{1}{2}\right) + 210000\left(\frac{1}{2}\right) = 450000.$$

- In the *second-by-height final* information node, the possible cashflows are 180,000 with probability 1/2 and 45,000 with probability 1/2. So, the value of the project at that node equals

$$180000\left(\frac{1}{2}\right) + 45000\left(\frac{1}{2}\right) = 112500.$$

- In the *third-by-height final* information node, the possible cashflows are 135,000 with probability 1/2 and 30,000 with probability 1/2. So, the value of the project at that node equals

$$135000\left(\frac{1}{2}\right) + 30000\left(\frac{1}{2}\right) = 82500.$$

- In the *lowest final* information node, the possible cashflows are 60,000 with probability 1/2 and 0 with probability 1/2. So, the value of the project at that node equals

$$60000\left(\frac{1}{2}\right) = 30000.$$

We continue working backwards, at the **upper decision** node at the end of Year 1, we can go "up" or "down" in the tree.

- We go "up" by investing 180,000; combining this cashflow with the average revenue at the *uppermost final* node, we get the total effect of going "up" to be

$$450000 - 180000 = 270000.$$

- We go "down" by investing 120,000; combining this cashflow with the average revenue at the *second-by-height final* node, we get the total effect of going "down" to be

$$112500 - 120000 = -7500.$$

Comparing the two values we obtained, we conclude that the **optimal** decision at this node is to go "up" and we keep the value of this project at this node to be

$$270000 + 500000 = 770000.$$

Here, we took into account that the first season was a success resulting in 500,000 in revenue in Year 1.

Similarly, at the **lower decision** node at the end of Year 1, we can go "up" or "down" in the tree.

- We go "up" by investing 90,000; combining this cashflow with the average revenue at the *third-by-height final* node, we get the total effect of going "up" to be

$$82500 - 90000 = -7500.$$

- We go "down" by investing nothing; so, the total effect of going "down" is 30000. Comparing the two values we obtained, we conclude that the **optimal** decision at this node is to go "down" and we keep the value of this project at this node to be

$$30000 + 105000 = 135000.$$

Here, we took into account that the first season was "meh" resulting in 105,000 in revenue in Year 1.

Altogether, at the information node corresponding to Year 1, we have that the expected value of the project is

$$770000\left(\frac{1}{2}\right) + 135000\left(\frac{1}{2}\right) = 425500.$$

Now, we take into account that we funded the series partially with 30,000. So, the total expected present value of the cashflows we get should we decide to fund partially is

$$425500 - 30000 = 395500$$

The total value of the option is

$$395500 - 580000 = -184500.$$

Problem 1.8. *Source: Open Course Intro to Statistics.*
Cholesterol levels for women aged 20 to 34 follow an approximately normal distribution with mean 185 milligrams per deciliter (mg/dl). Women with cholesterol levels above 220 mg/dl are considered to have high cholesterol and about 18.5% of women fall into this category. What is the standard deviation of the distribution of cholesterol levels for women aged 20 to 34?

(a) 38.9
(b) 41.3
(c) 43.7
(d) 45.1
(e) None of the above.

Solution: (a)
Let X be the random variable denoting the cholesterol level. Then,

$$X \sim N(mean = 185, variance = \sigma^2).$$

We are given that

$$\mathbb{P}[X > 220] = 0.185 \quad \Rightarrow \quad \mathbb{P}[X \leq 220] = 1 - 0.185 = 0.815.$$

So,

$$220 = 185 + \sigma z_*$$

where z_* is the critical value such that $N(z_*) = 0.815$. The closest value in the standard normal tables is $z_* = 0.9$. Hence, our answers is

$$\sigma = \frac{220 - 185}{0.9} = 38.8889$$

Problem 1.9. Suppose that the failure time (in seconds) of a certain component is modeled as lognormal random variable $Y = e^X$ such that the mean of X is -0.35 and its variance is 0.04.

What is the failure time t^* such that 95% of the components of the same type would still function after that time?

(a) 0.306
(b) 0.402
(c) 0.507
(d) 0.701
(e) None of the above.

Solution: (c)
We are looking for the value t^* such that

$$\mathbb{P}[Y > t^*] = 0.95 \quad \Leftrightarrow \quad \mathbb{P}[Y \le t^*] = 0.05.$$

The critical value z^* such that $N(z^*) = 0.05$ is -1.645. So,

$$t^* = e^{-0.35+0.2(-1.645)} = 0.5071.$$

Problem 1.10. Assume that the stock price follows the Black-Scholes model. You are given the following information:

- The current stock price is \$100.
- The mean rate of return on the stock is 0.15.
- The stock's dividend yield is 0.01.
- The stock's volatility is 0.35.
- The continuously-compounded, risk-free interest rate is 0.05.

Find

$$\mathbb{E}[S(1)\mathbb{I}_{[S(1)>80]}].$$

(a) \$102.02
(b) \$108.19
(c) \$115.03
(d) \$126.71
(e) None of the above.

Solution: (a)
Note that the given continuously-compounded, risk-free interest rate is not necessary to solve the problem!

In our usual notation, we have

$$\begin{aligned}\hat{d}_1 &= \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(\alpha - \delta + \frac{\sigma^2}{2}\right)T\right] = \frac{1}{0.35\sqrt{1}}\left[\ln\left(\frac{100}{80}\right) + \left(0.15 - 0.01 + \frac{0.35^2}{2}\right)\right] \\ &= 1.21255 \approx 1.21, \\ \hat{d}_2 &= \hat{d}_1 - \sigma\sqrt{T} = 1.21 - 0.35 = 0.86255 \approx 0.86.\end{aligned}$$

Using the standard normal tables, we get

$$N(\hat{d}_1) = 0.8869 \quad \text{and} \quad N(\hat{d}_2) = 0.8051.$$

Finally, our answer is

$$\mathbb{E}[S(1)\mathbb{I}_{[S(1)>80]}] = 100e^{0.15-0.01}(0.8869) = 102.0178.$$

Problem 1.11. (5 points) The current stock price is given to be $S(0) = 30$. The stock has the rate of appreciation 0.12 and volatility 0.3

Find the probability that the stock price in three months is less than \$32.

(a) 0.5218
(b) 0.5412
(c) 0.5846
(d) 0.6217
(e) None of the above.

Solution: (d)
First, we calculate $\hat{d}_2$. We get

$$\hat{d}_2 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(\alpha - \delta - \frac{\sigma^2}{2}\right)T\right] = \frac{1}{0.3\sqrt{\frac{1}{4}}}\left[\ln\left(\frac{30}{32}\right) + \left(0.12 - \frac{0.09}{2}\right) \times \frac{1}{4}\right] = -0.30525.$$

Then, we use the standard normal tables to obtain

$$a = N(-\hat{d}_2) \approx N(0.31) = 0.6217 \tag{1.1}$$

Problem 1.12. Assume the Black-Scholes model. According to your model, you expect the stock price in half a year to be \$84.10. The median stock price in half a year is \$83.26 according to that same model. What is the stock's volatility?

(a) 0.15
(b) 0.20
(c) 0.25
(d) Not enough information is given.
(e) None of the above.

Solution: (b)
In our usual notation, we have that

$$\frac{\text{mean stock price}}{\text{median stock price}} = \frac{S(0)e^{(\alpha-\delta)T}}{S(0)e^{(\alpha-\delta-\frac{\sigma^2}{2})T}} = e^{\frac{\sigma^2 T}{2}}$$

So, in this problem,

$$\frac{84.10}{83.26} = e^{\frac{\sigma^2}{4}} \quad \Rightarrow \quad \frac{\sigma^2}{4} = \ln\left(\frac{84.10}{83.26}\right) \quad \Rightarrow \quad \sigma = \sqrt{4\ln\left(\frac{84.10}{83.26}\right)} = 0.2004.$$