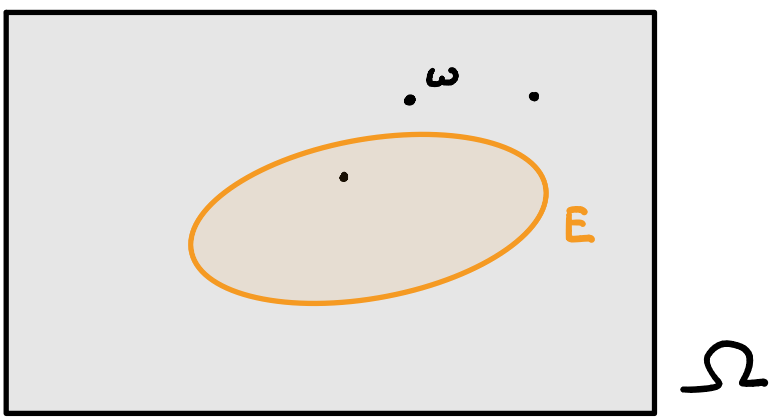
M378K: January 15th, 2025.

M378K Introduction to Mathematical Statistics
Problem Set #1
Probability spaces.

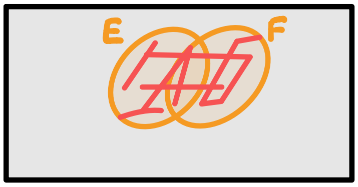
1.1. Probability distributions. Consider an **outcome space** (also known as a **sample space**) Ω. In statistics, it's convenient to imagine it as containing your entire target population. The individual elements $\omega \in \Omega$ are known in probability as **elementary outcomes**; in statistics, they can be understood as individuals in your target population.

We are usually not interested that much in individual ω, but want to consider **events** E in Ω. In full mathematical generality, the set Ω can be complicated so that the family of all of its subsets is also a complicated object. Due to mathematical difficulties, it is sometimes impossible to measure in any meaningful way absolutely all subsets of Ω [1]. For the purposes of this course, we will not have to dwell on these matters. So, we will refer to any (nice) subset of Ω as an **event**.

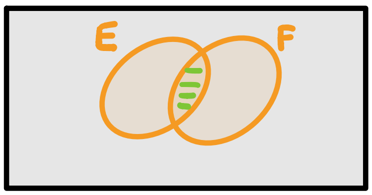


[1] See https://en.wikipedia.org/wiki/Banach\T1\textendashTarski_paradox

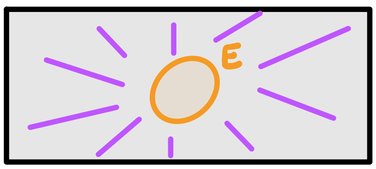
We will inherit all the notation and rules from set theory while adding some more vocabulary and context. Most notably, we will consider *intersections, unions*, and *complements* of events. These are best understood via Venn diagrams.



$E \cup F$ ... @ least one happened



$E \cap F$ ... both happened



E^c ... did not happen

Moreover, in a probabilistic setting, we have the following definition:

Definition 1.1. *Let E and F be two events on the same Ω such that*

$$E \cap F = \emptyset.$$

Then, we say that E and F are mutually exclusive *(or* disjoint*).*

Now, we are ready for the following (crucial!) definition:

Definition 1.2. *Consider a mapping $\mathbb{P}$ from the set of all events on Ω to $\mathbb{R}$. We say that $\mathbb{P}$ is a* probability (distribution, measure) *if it satisfies the following three conditions:*

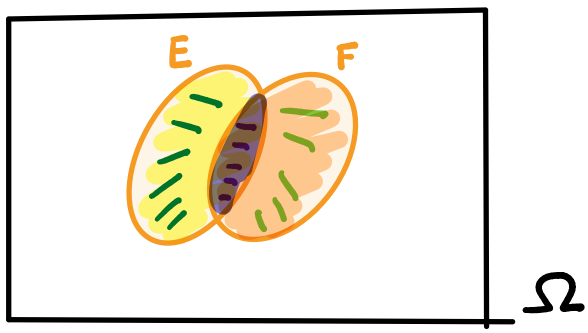
- $\mathbb{P}[E] \geq 0$ *for all events E;*
- $\mathbb{P}[\Omega] = 1$;
- *for all* **pairwise disjoint** *sequences of events $\{E_j : j = 1, 2, \ldots\}$, we have that*

$$\mathbb{P}\left[\bigcup_{j=1}^{\infty} E_j\right] = \sum_{j=1}^{\infty} \mathbb{P}[E_j].$$

One immediately useful consequence is the **inclusion-exclusion formula**. This is its simplest version.

Proposition 1.3. *Let E and F be two events on Ω. Then,*

$$\mathbb{P}[E \cup F] = \mathbb{P}[E] + \mathbb{P}[F] - \mathbb{P}[E \cap F].$$

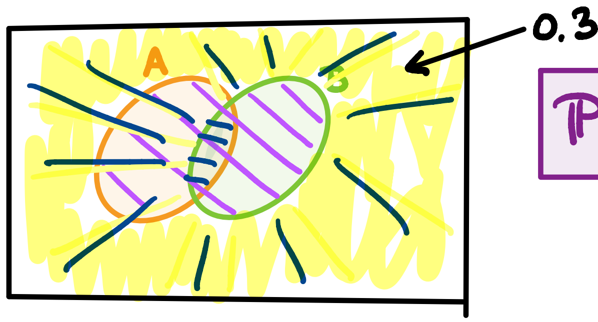


Of course, the above formula can be generalized to arbitrary unions of finitely many events. *Try to figure it out!*

Problem 1.1. Source: An old P exam problem.
For two events A and B, you are given that

$$\mathbb{P}[A \cup B] = 0.7 \quad \textit{and} \quad \mathbb{P}[A \cup B^c] = 0.9.$$

Calculate $\mathbb{P}[A]$.



$$\boxed{\mathbb{P}[A] = 0.9 - 0.3 = 0.6}$$

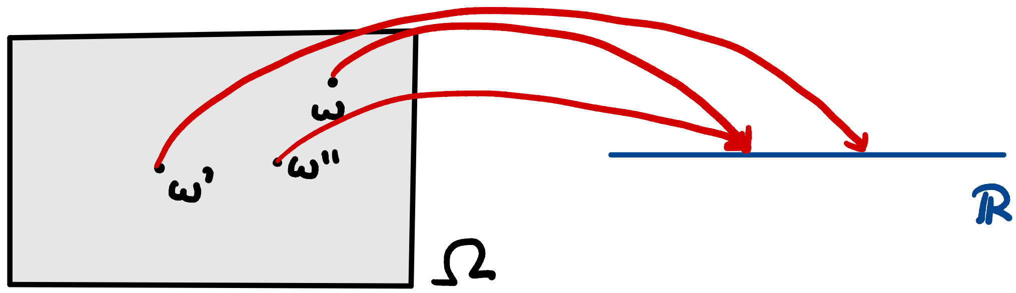
$$\left.\begin{aligned} \mathbb{P}[A\cup B] &= \boxed{\mathbb{P}[A]} + \boxed{\mathbb{P}[B]} - \boxed{\mathbb{P}[A\cap B]} = 0.7 \\ \mathbb{P}[A\cup B^c] &= \boxed{\mathbb{P}[A]} + \boxed{\mathbb{P}[B^c]} - \boxed{\mathbb{P}[A\cap B^c]} = 0.9 \end{aligned}\right\} +$$

$$2\mathbb{P}[A] + 1 - \mathbb{P}[A] = 1.6$$

$$\boxed{\mathbb{P}[A] = 0.6.}$$

$\square$

1.2. **Random variables.** Informally speaking, any "nice" mapping/function from Ω to a target set $\mathcal{S}$ is a *random element* [2]. When $\mathcal{S}$ is $\mathbb{R}$, we like to use the term *random variable*. When $\mathcal{S}$ is $\mathbb{R}^n$ for some n, we like to use the term *random vector*.



Let's consider a classroom of students as our Ω and give examples of a

- random element

  Major

  Eye Color

- random variable

  GPA

- random vector

  (height, weight)

To keep track of what values a random variable is "allowed" we use the following terminology[3]:

Definition 1.4. *Given a set B, we say that a random variable Y is $B-$valued if*

$$\mathbb{P}[Y \in B] = 1.$$

[2] In practice, people like to use the term *random variable* even in more general context when there is no source of confusion. We will habitually do this.

[3] Read your lecture notes: https://web.ma.utexas.edu/users/gordanz/notes/discrete_probability_color.pdf