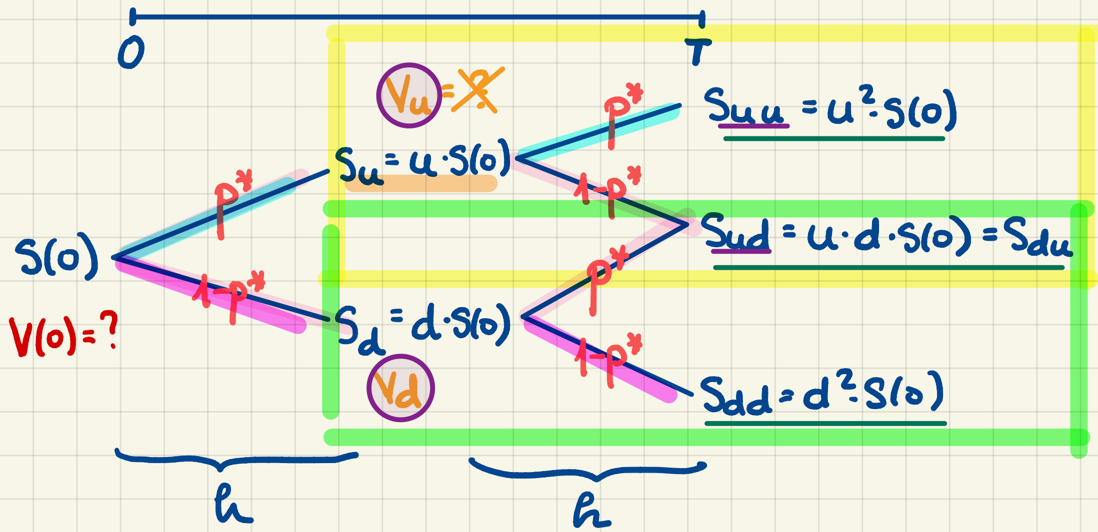
M339D: March 24th, 2023.

Two Binomial Periods.

$\boxed{2h = T}$



Payoff:

$V_{uu} = v(S_{uu})$

$V_{ud} = v(S_{ud})$

$V_{dd} = v(S_{dd})$

populating the tree $\longrightarrow$

$\longleftarrow$ pricing the option

- **up node**:

replicating portfolio for the option:

$$\begin{cases} \Delta_u = \dfrac{V_{uu} - V_{ud}}{S_{uu} - S_{ud}} \\ B_u = e^{-rh} \cdot \dfrac{u \cdot V_{ud} - d \cdot V_{uu}}{u - d} \end{cases}$$

$\Rightarrow$ the option's value @ the up node:

$$V_u = \Delta_u \cdot S_u + B_u = e^{-rh}\left[p^* \cdot V_{uu} + (1-p^*) \cdot V_{ud}\right] \longleftarrow$$

w/ $\boxed{p^* = \dfrac{e^{rh} - d}{u - d}}$

- **down node**: Δ_d, B_d

$$\Rightarrow \quad V_d = \Delta_d \cdot S_d + B_d = e^{-rh}\left[p^* \cdot V_{ud} + (1-p^*) \cdot V_{dd}\right]$$

- **ROOT node:**

$$\Delta_0 = \frac{V_u - V_d}{S_u - S_d}$$

$$B_0 = e^{-rh} \cdot \frac{u \cdot V_d - d \cdot V_u}{u - d}$$

$$\Rightarrow \boxed{V(0) = \Delta_0 \cdot S(0) + B_0}$$

From the risk·neutral "perspective":

$$V(0) = e^{-rh}\left[p^* \cdot V_u + (1-p^*) \cdot V_d\right]$$

$$V(0) = e^{-rh}\Big[p^* e^{-rh}\left(p^* \cdot V_{uu} + (1-p^*) \cdot V_{ud}\right) + (1-p^*) e^{-rh}\left(p^* \cdot V_{ud} + (1-p^*) \cdot V_{dd}\right)\Big]$$

$$V(0) = e^{-r\overset{T}{(2h)}} \underbrace{\left[(p^*)^2 \cdot V_{uu} + 2p^*(1-p^*) V_{ud} + (1-p^*)^2 \cdot V_{dd}\right]}_{\text{Risk·Neutral Expectation of the Payoff}}$$

Generally:

$$\boxed{V(0) = e^{-rT}\, \mathbb{E}^*\left[V(T)\right]}$$

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Problem Set #9

Binomial option pricing: Two or more periods.

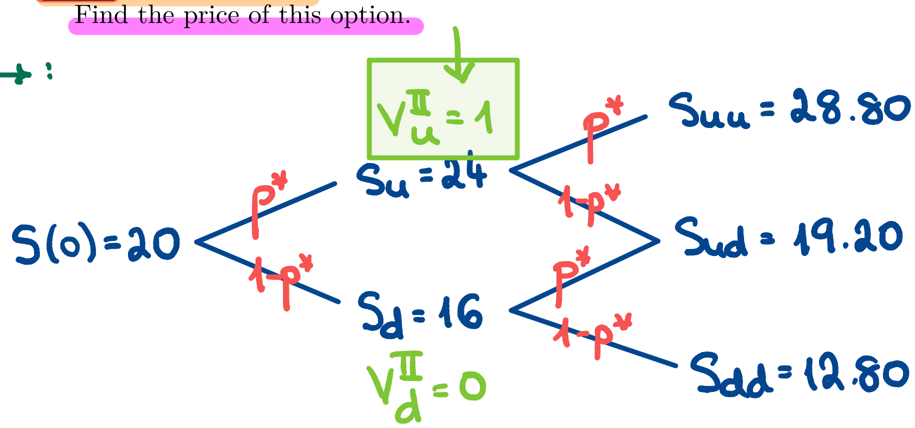
Problem 9.1. For a two-period binomial model, you are given that:

(1) each period is one year;
(2) the current price of a non-dividend-paying stock S is $S(0) = \$20$;
(3) $u = 1.2$, with u as in the standard notation for the binomial model;
(4) $d = 0.8$, with d as in the standard notation for the binomial model;
(5) the continuously compounded risk-free interest rate is $r = 0.04$.

Consider a **special** call option which pays the excess above the strike price $K = 23$ (if any!) at the end of **every** binomial period.

Find the price of this option.

→ :



$V_{uu}^{I} = 5.80$

$V_{ud}^{I} = 0$

$V_{dd}^{I} = 0$

The risk-neutral Probability:

$$p^* = \frac{e^{rh} - d}{u-d} = \frac{e^{0.04} - 0.8}{1.2 - 0.8} = \underline{0.602}$$

$$V^{I}(0) = e^{-0.04(2)} \cdot (p^*)^2 (5.8) = \underline{1.941}$$

$$V^{II}(0) = e^{-0.04} \cdot p^* \cdot 1 = \underline{0.5784}$$

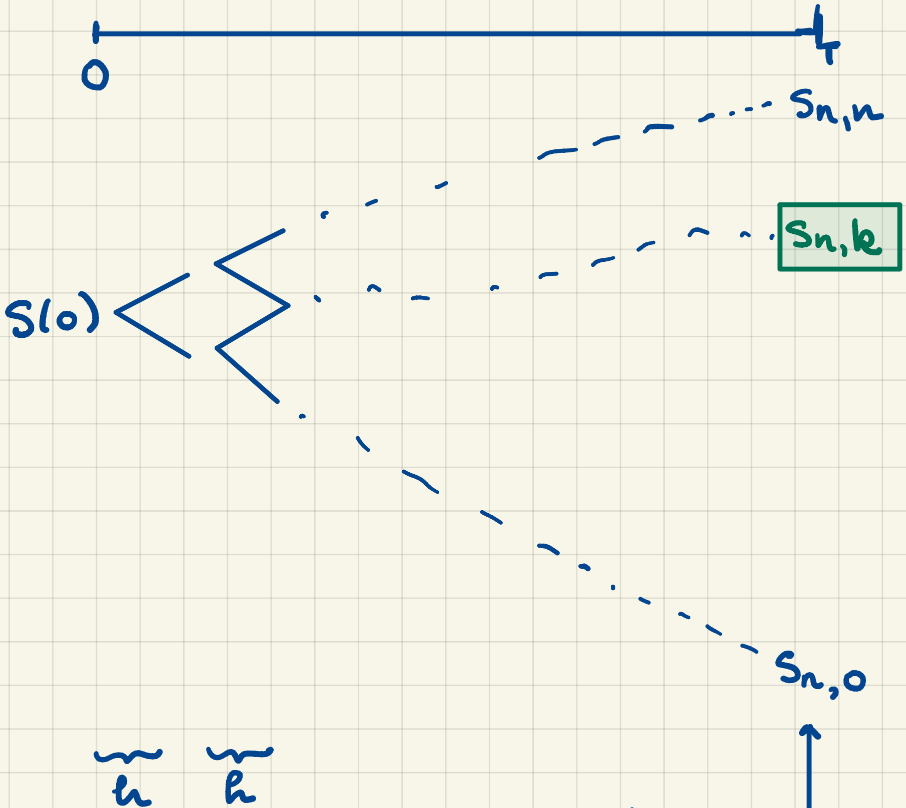
$+ \implies V(0) \approx 2.5$

$\square$

Multiple Binomial Periods.

T ... exercise date of a European option
n ... # of periods

the length of each period $\boxed{h = \frac{T}{n}}$



k .... # of steps up

These (n+1) values are the support of S(T).

⇒ for every $k = 0, 1, \dots, n$:

$$S_{n,k} = S(0) \cdot u^k \cdot d^{n-k} = S(0) \cdot \left(\frac{u}{d}\right)^{k} \cdot d^n$$

# of upsteps