UNIVERSITY OF TEXAS AT AUSTIN

Quiz #4

Log-normal stock prices: Tail probabilities.

Problem 4.1. (5 points)

The current stock price is given to be $S(0) = 30$. The stock has the rate of appreciation 0.12 and volatility 0.3

Let a denote the probability that the stock price in three months is less than \$32, i.e., set $a = \mathbb{P}[S(1/4) < 32]$.

Then,

(a) $0 \leq a < 0.35$
(b) $0.35 \leq a < 0.45$
(c) $0.45 \leq a < 0.50$
(d) $0.50 \leq a < 0.64$
(e) None of the above.

Solution: (d)

$$a = N(-d_2) = N(0.16) = 0.5636$$

Problem 4.2. (10 points)

Let $\mathbf{S} = \{S(t), t \geq 0\}$ denote the stock-price process. For any time$-t$, the stock price is modeled as lognormal. The mean stock price at time-2 equals 140 and the median stock price at time-2 equals 130. What is the probability that the time-2 stock price exceeds 140?

Solution: Since $\mathbf{S} = \{S(t), t \geq 0\}$ is the stock-price process modeled by a geometric Brownian motion, the stock price at time-1 is lognormally distributed. In fact, using our usual parameters, we can rewrite it as

$$S(2) = S(0)e^{(\alpha-\delta-\frac{1}{2}\sigma^2)2+\sigma Z(2)}.$$

Hence $S(2)$ is log-normally distributed, and the median of $S(2)$ equals $S(0)e^{(\alpha-\delta-\frac{1}{2}\sigma^2)2}$. So, the required probability can be expressed as

$$\begin{aligned}\mathbb{P}[S(2) > 140] = \mathbb{P}[130e^{\sigma Z(2)} > 140] &= \mathbb{P}\left[Z(2) > \frac{1}{\sigma}\ln\left(\frac{140}{130}\right)\right] \\ &= \mathbb{P}\left[Z(2) < \frac{1}{\sigma}\ln\left(\frac{130}{140}\right)\right] = N\left(\frac{1}{\sqrt{2}\sigma}\ln\left(\frac{130}{140}\right)\right).\end{aligned}$$

Since the mean of $S(2)$ equals $S(0)e^{(\alpha-\delta)2}$, we have

$$e^{\sigma^2} = \frac{140}{130} \quad \Rightarrow \quad \sigma = \sqrt{\ln(140/130)}.$$

So, our final answer is

$$\begin{aligned}\mathbb{P}[S(2) > 140] &= N\left(\frac{1}{\sqrt{2} \times \sqrt{\ln(140/130)}}\ln\left(\frac{130}{140}\right)\right) \\ &= N\left(-\sqrt{\frac{\ln(14/13)}{2}}\right) = N(-0.1925) = 1 - N(0.19) = 1 - 0.5753 = 0.4247.\end{aligned}$$