Maximum Likelihood for Individual, Unmodified Data.

Let X_j, $j=1..n$, be continuous random variables.
Denote by f_{X_j} the pdf of X_j for all $j=1..n$.
All of the f_{X_j} must depend on the same parameter θ.

Say that our data set consists of singletons, i.e.,

$$x_1, x_2, \ldots, x_n$$

The likelihood function is: $L(\theta) = \prod_{j=1}^{n} f_{X_j}(x_j; \theta)$

Goal: Maximize the likelihood function across all θ.

Introduce: $\ell(\theta) = \ln(L(\theta)) = \sum_{j=1}^{n} \ln(f_{X_j}(x_j;\theta))$

Now, we maximize the log-likelihood f'tion; typically, we differentiate.

Example. $X_j \sim$ Exponential(mean $=\theta$), $j=1..n$

Data set: $x_1, x_2, \ldots, x_n$

For every $j=1..n$, the pdf is

$$f_{X_j}(x_j;\theta) = \frac{1}{\theta} e^{-\frac{x}{\theta}}, \quad x>0$$

The likelihood function is:

$$L(\theta) = \prod_{j=1}^{n} \left(\frac{1}{\theta} e^{-\frac{x_j}{\theta}}\right) = \left(\frac{1}{\theta}\right)^n \cdot \prod_{j=1}^{n} e^{-\frac{x_j}{\theta}}$$

$$L(\theta) = \left(\frac{1}{\theta}\right)^n \cdot e^{-\sum_{j=1}^{n} \frac{x_j}{\theta}}$$

$$L(\theta) = \left(\frac{1}{\theta}\right)^n \cdot e^{-\frac{1}{\theta}\sum_{j=1}^{n} x_j}$$

The log-likelihood will be:

$$\ell(\theta) = \ln(L(\theta)) = \underbrace{n \cdot \ln\left(\frac{1}{\theta}\right)}_{-n\cdot\ln(\theta)} - \frac{1}{\theta}\sum_{j=1}^{n} x_j$$

We are seeking the maximum, so we differentiate:

$$l'(\theta) = -n\cdot\frac{1}{\theta} + (+1)\frac{1}{\theta^2}\sum_{j=1}^{n} x_j = 0$$

$$0 \neq \frac{1}{\theta^2}\underbrace{\left(-n\cdot\theta + \sum_{j=1}^{n} x_j\right)}_{=0} = 0$$

$$\Rightarrow \boxed{\hat{\theta}_{MLE} = \frac{1}{n}\sum_{j=1}^{n} x_j = \bar{x}}$$

Example. $X_j \sim \text{Pareto}(\alpha, \theta)$, $j = 1, \dots, n$

Data set: $x_1, x_2, \dots, x_n$

We assume that θ is known and we're estimating α.

The likelihood function:

$$L(\alpha, \theta) = \prod_{j=1}^{n} f(x_j; \alpha, \theta)$$

$$= \prod_{j=1}^{n} \frac{\alpha\cdot\theta^\alpha}{(x_j+\theta)^{\alpha+1}}$$

$$L(\alpha, \theta) = \frac{\alpha^n\cdot\theta^{\alpha n}}{[(x_1+\theta)(x_2+\theta)\cdot\ldots\cdot(x_n+\theta)]^{\alpha+1}}$$

The log-likelihood function:

$$l(\alpha, \theta) = n\cdot\ln(\alpha) + \alpha n\cdot\ln(\theta) - (\alpha+1)\boxed{\sum_{j=1}^{n}\ln(x_j+\theta)}$$

Q: What if α is given and we're looking for θ?

Omit the known θ from the notation.

Differentiate:

$$l'(\alpha) = n\cdot\frac{1}{\alpha} + n\ln(\theta) - \sum_{j=1}^{n}\ln(x_j+\theta) = 0$$

$$\frac{n}{\alpha} = \sum_{j=1}^{n}\ln(x_j+\theta) - n\cdot\ln(\theta)$$

$$\boxed{\hat{\alpha}_{MLE} = \frac{n}{\sum_{j=1}^{n}\ln(x_j+\theta) - n\cdot\ln(\theta)}}$$

35. You are given the following information about a credibility model:

First Observation	Unconditional Probability	Bayesian Estimate of Second Observation
1	1/3	1.50
2	1/3	1.50
3	1/3	3.00

Calculate the Bühlmann credibility estimate of the second observation, given that the first observation is 1.

(A) 0.75

(B) 1.00

(C) 1.25

(D) 1.50

(E) 1.75

36. DELETED

37. A random sample of three claims from a dental insurance plan is given below:

225 525 950

Claims are assumed to follow a Pareto distribution with parameters $\theta = 150$ and α.

Calculate the maximum likelihood estimate of α.

$$\hat{\alpha}_{MLE} = \frac{3}{\ln(375) + \ln(675) + \ln(1100) - 3\ln(150)}$$

(A) Less than 0.6

(B) At least 0.6, but less than 0.7

(C) At least 0.7, but less than 0.8

(D) At least 0.8, but less than 0.9

(E) At least 0.9

$$\hat{\alpha}_{MLE} = 0.6798$$

Grouped Data.

The data are grouped into "bins":

$$c_0 < c_1 < \cdots < c_k$$

We are given the number of observations in each interval

$$(c_{j-1}, c_j].$$

Denote this number of observations by n_j for every $j = 1..k$.

Then, $n_1 + n_2 + \cdots + n_k = n$, i.e., the sample size.

For every "bin": $\left(\mathbb{P}(c_{j-1} < X \leq c_j)\right)^{n_j} =$

$$= \left(\mathbb{P}[X \leq c_j] - \mathbb{P}[X \leq c_{j-1}]\right)^{n_j}$$

$$= \left(F_X(c_j;\theta) - F_X(c_{j-1};\theta)\right)^{n_j}$$

The overall likelihood function is:

$$\boxed{L(\theta) = \prod_{j=1}^{k} \left(F_X(c_j;\theta) - F_X(c_{j-1};\theta)\right)^{n_j}}$$

Then, the log-likelihood f'tion is:

$$\boxed{\ell(\theta) = \sum_{j=1}^{k} n_j \ln\left(F_X(c_j;\theta) - F_X(c_{j-1};\theta)\right)}$$

43. You are given:

(i) The prior distribution of the parameter Θ has probability density function:

$$\pi(\theta) = \frac{1}{\theta^2}, \quad 1 < \theta < \infty$$

(ii) Given $\Theta = \theta$, claim sizes follow a Pareto distribution with parameters $\alpha = 2$ and θ.

A claim of 3 is observed.

Calculate the posterior probability that Θ exceeds 2.

(A) 0.33

(B) 0.42

(C) 0.50

(D) 0.58

(E) 0.64

44. You are given:

(i) Losses follow an exponential distribution with mean θ.

(ii) A random sample of 20 losses is distributed as follows:

Loss Range	Frequency
[0, 1000]	7
(1000, 2000]	6
(2000, ∞)	7

Calculate the maximum likelihood estimate of θ.

$\longrightarrow$: $F_X(x;\theta) = 1 - e^{-\frac{x}{\theta}}$

(A) Less than 1950

(B) At least 1950, but less than 2100

(C) At least 2100, but less than 2250

(D) At least 2250, but less than 2400

(E) At least 2400

My likelihood:

$$L(\theta) = \left(1-e^{-\frac{1000}{\theta}}\right)^7 \left[\left(\cancel{1}-e^{-\frac{2000}{\theta}}\right) - \left(\cancel{1}-e^{-\frac{1000}{\theta}}\right)\right]^6 \left(e^{-\frac{2000}{\theta}}\right)^7$$

$$L(\theta) = \left(1-e^{-\frac{1000}{\theta}}\right)^7 \left(e^{-\frac{1000}{\theta}} - e^{-\frac{2000}{\theta}}\right)^6 \left(e^{-\frac{2000}{\theta}}\right)^7$$

Substitute $y = e^{-\frac{1000}{\theta}}$ (a strictly increasing transform)

$$L(y) = (1-y)^7 (y-y^2)^6 (y^2)^7 = (1-y)^7 y^6 (1-y)^6 y^{14}$$

$$L(y) = y^{20}(1-y)^{13}$$

$\Rightarrow$ The log likelihood: $\ell(y) = 20\ln(y) + 13\ln(1-y)$

$$\ell'(y) = \frac{20}{y} - \frac{13}{1-y} = 0$$

$$20(1-y) - 13y = 0$$

$$20 - 20y - 13y = 0$$

$$\boxed{y = \frac{20}{33}}$$

Last:

$$\frac{20}{33} = e^{-\frac{1000}{\theta}}$$

$$\ln\left(\frac{33}{20}\right) = \frac{1000}{\theta}$$

$$\hat{\theta}_{MLE} = \frac{1000}{\ln\left(\frac{33}{20}\right)} = \underline{1996.9} \quad \square$$