M378K: September 8th, 2025.

Named Discrete Distributions [cont'd].

Geometric Dist'n

Say, we repeat independently Bernoulli trials w/ the same success probability p until the first success.
The random variable Y which denotes the number of failures until the first success is called geometric.

We write $\boxed{Y \sim g(p)}$

Set $q = 1-p$.

y	0	1	2	$\cdots\cdots$	k	$\cdots\cdots$
$p_Y(y)$	p	$q \cdot p$	$q^2 \cdot p$	$\cdots\cdots$	$q^k \cdot p$	$\cdots\cdots$

Q: $\mathbb{P}[Y > 2] = 1 - \mathbb{P}[Y \leq 2]$

$$= 1 - \mathbb{P}[Y=0] - \mathbb{P}[Y=1] - \mathbb{P}[Y=2]$$

$$= \underbrace{1 - p}_{} \quad - q \cdot p \quad - q^2 p$$

$$= q \quad - qp \quad - q^2 p$$

$$= q(\underbrace{1-p}_{q} - qp)$$

$$= q \cdot (q - qp) = q \cdot q \cdot (1-p) = q^3$$

$\square$

Problem 3.2. Source: Sample P exam, Problem #462.
Each person in a large population independently has probability p of testing positive for diabetes where $0 < p < 1$. People are tested for diabetes, one person at a time, until a test is positive. Individual tests are independent. Determine the probability that m or fewer people are tested, given that n or fewer people are tested, where $1 \leq m \leq n$.

→: Y'... total # of people tested

↳ SHIFTED geometric w/ parameter p

i.e., $\boxed{Y = Y' - 1 \sim g(p)}$

$$\mathbb{P}[Y' \leq m \mid Y' \leq n] = \mathbb{P}[Y+1 \leq m \mid Y+1 \leq n]$$

$$= \mathbb{P}[Y \leq m-1 \mid Y \leq n-1]$$

$$= \frac{\mathbb{P}[Y \leq m-1, Y \leq n-1]}{\mathbb{P}[Y \leq n-1]}$$

$$= \frac{\mathbb{P}[Y \leq m-1]}{\mathbb{P}[Y \leq n-1]}$$

$$= \frac{1-\mathbb{P}[Y > m-1]}{1-\mathbb{P}[Y > n-1]} = \frac{1-q^m}{1-q^n}$$

□

Task: Google the memoryless property of the geometric.

Poisson Distribution.

The Poisson distribution is $\mathbb{N}_0$-valued and has the pmf:

$$\boxed{p_k := p_Y(k) = e^{-\lambda}\frac{\lambda^k}{k!}} \quad \text{for } k \in \mathbb{N}_0$$

where λ is a positive parameter.

Problem 3.3. Source: Sample P exam, Problem #355.
The number of calls received by a certain emergency unit in a day is modeled by a Poisson distribution with parameter 2*. Calculate the probability that on a particular day the unit receives at least two calls.*

→ : Y… # of calls

$Y \sim \text{Poisson}(\lambda = 2) \sim \mathbb{P}(2)$

$$\boxed{p_k := p_Y(k) = e^{-\lambda}\frac{\lambda^k}{k!}}$$

$$\begin{aligned}\mathbb{P}[Y \geq 2] &= 1 - \mathbb{P}[Y \leq 1] \\ &= 1 - \mathbb{P}[Y=0] - \mathbb{P}[Y=1] \\ &= 1 - e^{-\lambda} - e^{-\lambda}\cdot\lambda \\ &= 1 - e^{-2} - e^{-2}\cdot 2 \\ &= 1 - 3e^{-2}\end{aligned}$$

□

Expectation.

Def'n. For a discrete r.v. Y w/ support $S_Y \subseteq \mathbb{R}$ and w/ pmf p_Y, we define its expectation (or expected value, or mean) as

$$\boxed{\mathbb{E}[Y] = \sum_{y \in S_Y} y \cdot p_Y(y)}$$

if the sum exists

Theorem. Let Y_1 and Y_2 be two r.v.s on the same Ω, both w/ finite expectations.

Let α and β be two constants.

Then, $\mathbb{E}[\alpha Y_1 + \beta Y_2]$ also exists, and

$$\boxed{\mathbb{E}[\alpha Y_1 + \beta Y_2] = \alpha \mathbb{E}[Y_1] + \beta \mathbb{E}[Y_2]}$$

Linearity of Expection.

M378K Introduction to Mathematical Statistics
Problem Set #4
Expectation and variance: the discrete case.

Problem 4.1. Source: Sample P exam, Problem #481.
The number of days required for a damage control team to locate and repair a leak in the hull of a ship is modeled by a discrete random variable N. N is uniformly distributed on $\{1, 2, 3, 4, 5\}$.

The cost of locating and repairing a leak is $N^2 + N + 1$.

Calculate the expected cost of locating and repairing a leak in the hull of the ship.

→ :

$$\mathbb{E}[N^2+N+1] \underset{\text{linearity}}{=} \mathbb{E}[N^2] + \mathbb{E}[N] + 1$$

$$\mathbb{E}[N] = \frac{1}{5}\cdot 1 + \frac{1}{5}\cdot 2 + \frac{1}{5}\cdot 3 + \frac{1}{5}\cdot 4 + \frac{1}{5}\cdot 5 = \frac{1}{5}\cdot 15 = 3 \checkmark$$

$$\mathbb{E}[N^2] = \frac{1}{5}\cdot 1^2 + \frac{1}{5}\cdot 2^2 + \frac{1}{5}\cdot 3^2 + \frac{1}{5}\cdot 4^2 + \frac{1}{5}\cdot 5^2 =$$

$$= \frac{1}{5}(1 + 2^2 + 3^2 + 4^2 + 5^2) = \frac{1}{\cancel{5}} \cdot \frac{\cancel{5}\cdot\cancel{6}\cdot(2\cdot 5+1)}{\cancel{6}} = 11$$

answer: $11 + 3 + 1 = 15$

$\square$

Task:

$$\mathrm{Var}[X] = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right]$$

$$\mathbb{E}\left[(X-a)^2\right] \xrightarrow[a]{} \min$$