M378K: March 26^{th}, 2025.

M378K Introduction to Mathematical Statistics
Problem Set #16
Confidence intervals.

Problem 16.1. *Suppose that the thumb sizes of the US males are following a normal distribution with an unknown mean μ and standard deviation $\boxed{\sigma = 20}$ on the LDI - scale* (Lauretski's Digital Index - LDI - from 50 to 280). *The US Department of Thumbs and Toes (DTT) reports that the mean thumb size in the country is $\mu = 150$. Being the chairman of the Faculty of Thumbs of the local university you see an excellent opportunity here and decide to conduct your own study of the size of the average American thumb.*

(i) *After carefully collecting a random sample of $\boxed{100}$ American thumbs you obtain the following sample mean: $\boxed{\bar{x} = 153.}$ This result doesn't seem to be compatible with the DTT report so you decide to construct a 95%-confidence interval for the unknown parameter μ based on your study. What is your confidence interval?*

(ii) *Now, you dream about achieving fame and fortune by being the first person ever to estimate the mean thumb size up to ± 0.1. How large a sample size do you need for that?*

i. point estimate $\pm$ margin of error

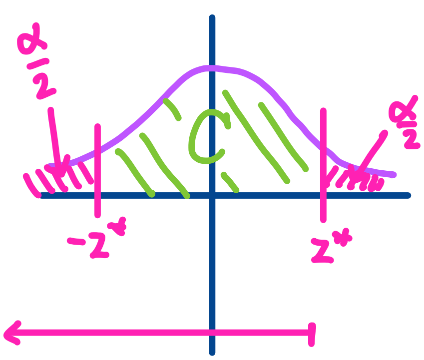
$$\bar{x} \pm z^* \frac{\sigma}{\sqrt{n}}$$

$\boxed{\sigma = 20}$ ✓

$\boxed{n = 100}$

$\alpha = 1 - C$

$\boxed{z^* = 1.96}$



$$\frac{1-C}{2} + C = \frac{1+C}{2}$$

$$z^* = \Phi^{-1}\left(\frac{1+C}{2}\right) = \text{qnorm}\left((1+C)/2\right)$$

$$\mu = \bar{x} \pm z^* \cdot \left(\frac{\sigma}{\sqrt{n}}\right) = 153 \pm 1.96 \cdot \left(\frac{20}{10}\right)$$

$$= 153 \pm 3.92 \quad \square$$

Choosing the Sample Size.

By def'n., the margin of error is $\frac{1}{2}(\hat{\theta}_R - \hat{\theta}_L)$

We can prescribe a margin of error m and a confidence level $C (= 1-\alpha)$

We seek a sample size n so that the margin of error is @ most m.

Example. $n = ?$

Let $Y_1, \dots, Y_n$ is a random sample from $N(\mu, \sigma)$? known

The form of the confidence interval for μ is

$$\bar{Y} \pm \boxed{z^* \cdot \frac{\sigma}{\sqrt{n}}}$$

$\leq m$

$$z^* \cdot \frac{\sigma}{\sqrt{n}} \leq m$$

$$z^* \cdot \frac{\sigma}{m} \leq \sqrt{n} \quad /^2$$

$$\boxed{\left(z^* \cdot \frac{\sigma}{m}\right)^2 \leq n}$$

Always round up!

Back to the problem.

ii.

$$n \geq \left(1.96 \cdot \frac{20}{0.1}\right)^2 = (1.96 \cdot 200)^2$$

Example. $Y_1, Y_2, \ldots, Y_n$ is a random sample from $E(\tau)$ w/ τ unknown.

$$\bar{Y} = \frac{1}{n}(Y_1 + Y_2 + \cdots + Y_n) \sim \,?$$

Each $Y_i \sim \Gamma(k=1, \tau)$ and they're independent.

We recall $\quad Y_1 + \cdots + Y_n \sim \Gamma(n, \tau)$ — shape: n, scale: τ

$$\Rightarrow \boxed{\bar{Y} \text{ is NOT A PIVOTAL QUANTITY!}}$$

The second parameter of a Γ dist'n is a scale parameter, as we know from

$$\boxed{m_Y(t) = (1 - \tau t)^{-k}} \qquad \text{for } t < \frac{1}{\tau}$$

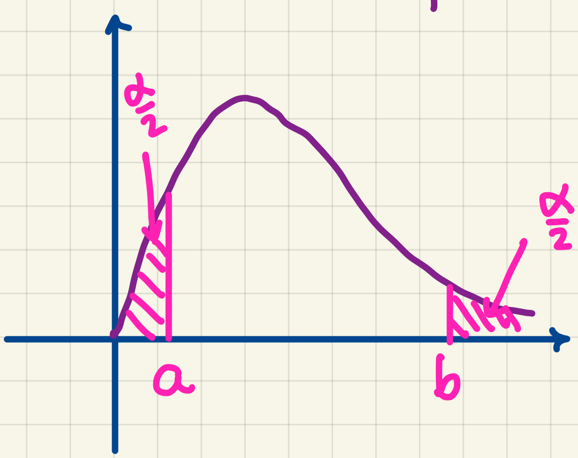
for $Y \sim \Gamma(k, \tau)$

$$m_{aX}(t) = \mathbb{E}\left[e^{aX \cdot t}\right] = \mathbb{E}\left[e^{at \cdot X}\right] = m_X(at)$$

$$\Rightarrow \boxed{U = \frac{1}{\tau}\bar{Y}} = \frac{1}{\tau} \cdot \frac{1}{n} \overbrace{(Y_1 + \cdots + Y_n)}^{\Gamma(n,\tau)}$$

$$\sim \Gamma\left(n, \frac{\cancel{\tau}}{n\cancel{\tau}}\right) = \Gamma\left(n, \frac{1}{n}\right)$$

The dist'n doesn't depend on τ, so we do have a pivotal quatity.



Pick a confidence level. Say $C = 0.90$, i.e., $\alpha = 0.10$.

a = qgamma (0.05, shape=n, scale=1/n) ←

b = qgamma (0.95, shape=n, scale=1/n) ←

We know that

$$\mathbb{P}[a \le U \le b] = 0.90$$

$$\mathbb{P}\left[a \le \frac{1}{\tau}\cdot\bar{Y} \le b\right] = 0.90$$

$$\mathbb{P}\left[\frac{a}{\bar{Y}} \le \boxed{\frac{1}{\tau}} \le \frac{b}{\bar{Y}}\right] = 0.90$$

$$\mathbb{P}\left[\underbrace{\boxed{\frac{\bar{Y}}{b}}}_{\hat{\theta}_L} \le \tau \le \underbrace{\boxed{\frac{\bar{Y}}{a}}}_{\hat{\theta}_R}\right] = 0.90$$

$\square$