Quiz #6: Solutions

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Problem 1. (3 points)

Let X denote the number of $1's$ in 100 throws of a fair die. Find $\mathbb{E}[X^2]$.

Solution: Evidently, $X \sim Binomial(n = 100, p = 1/2)$. So,

$$\mathbb{E}[X^2] = Var[X] + (\mathbb{E}[X])^2 = 100 \cdot \frac{1}{6} \cdot \frac{5}{6} + \left(100 \cdot \frac{1}{6}\right)^2 = \frac{500 + 10000}{36} = \frac{875}{3}.$$

Problem 2. (2 points)

If $Var[X] = 0$, then $\mathbb{P}[X = \mathbb{E}[X]] = 1$. *True or false? You do not have to justify your response.*

Solution: The statement is **TRUE.** If it were false, then we would have that $\mathbb{P}[(X - \mathbb{E}[X])^2 > 0] > 0$. So, we would have $Var[X] = \mathbb{E}[(X - \mathbb{E}[X])^2] > 0$.

Problem 3. (2 points)

Let X denote the outcome of a roll of a fair, regular dodecahedron (a polyhedron with 12 faces) with numbers $1, 2, \cdots, 12$ written on its sides. Find $\mathbb{E}[X]$.

Solution: Since each outcome is equally likely, by the definition of the expected value

$$\mathbb{E}[X] = \frac{1}{12} \cdot 1 + \frac{1}{12} \cdot 2 + \cdots + \frac{1}{12} \cdot 12 = \frac{1}{12}(1 + 2 + \cdots + 12) = \frac{1}{12} \cdot \frac{12 \cdot 13}{2} = \frac{13}{2}.$$

Problem 4. (8 points)

Your sample consists of 50 sixth-graders from Kealing Middle School (KMS) and 60 sixth-graders from Murchison Middle school (MMS). The measures of center and spread of the students' heights are:

$KMS:$ the mean of 60 inches with the standard deviation of 1 inch, and

$MMS:$ the mean of 60 inches with the standard deviation of 2 inches.

What are the measures of center and spread for the pooled sample of 110 sixth-graders?

Solution: Let the KMS mean and standard deviation be denoted by $\bar{x}_K$ and SD_K and let the MMS mean and standard deviation be denoted by $\bar{x}_M$ and SD_M. Also, let the pooled mean and standard deviation be denoted by $\bar{x}$ and SD.

Then, the overall mean will be

$$\bar{x} = \frac{50\bar{x}_K + 60\bar{x}_M}{110} = 60.$$

As for the standard deviation, we have (with measurements from Kealing denoted by $\{x_i^K\}$ and measurements from Murchison denoted by $\{x_i^M\}$)

$$SD_K^2 = \frac{1}{50-1}\sum_{i=1}^{50}(x_i^K - \bar{x})^2;$$

$$SD_M^2 = \frac{1}{60-1}\sum_{i=1}^{60}(x_i^M - \bar{x})^2;$$

$$SD^2 = \frac{1}{50+60-1}\left(\sum_{i=1}^{50}(x_i^K - \bar{x})^2 + \sum_{i=1}^{60}(x_i^M - \bar{x})^2\right)$$

The key is to notice that $\bar{x}_K = \bar{x}_M = \bar{x} = 60$. Therefore, we have

$$\begin{aligned} SD^2 &= \frac{1}{50+60-1}\left(\sum_{i=1}^{50}(x_i^K - \bar{x})^2 + \sum_{i=1}^{60}(x_i^M - \bar{x})^2\right) \\ &= \frac{1}{109}\left(49(1)^2 + 59(2)^2\right) = \frac{285}{109}. \end{aligned}$$

So, the standard deviation is $\sqrt{\frac{285}{109}}$ which is about 1.616997.

An example: Consider the following:

```
mms=c(rep(50,46),50-3.5,53.5,50-3.5,53.5)
mean(mms)
## [1] 50
sd(mms)
## [1] 1
kms=c(rep(50,56),50-sqrt(59),50+sqrt(59),50-sqrt(59),50+sqrt(59))
mean(kms)
## [1] 50
sd(kms)
## [1] 2
ms=c(kms,mms)
mean(ms)
## [1] 50
sd(ms)
## [1] 1.616997
```