M339W: March 8th, 2021.

Risk Measures.

The Variance.

For any random variable X, we denote the expected value of X as

$$\mu_X := \mathbb{E}[X] \qquad \text{(if it exists)}$$

We define the variance of X as follows

$$\boxed{Var[X] := \mathbb{E}\left[(X-\mu_X)^2\right] = \mathbb{E}[X^2] - \mu_X^2 \qquad \text{(if it exists)}}$$

We define the standard deviation of X as

$$\boxed{\sigma_X = SD[X] = \sqrt{Var[X]}}$$

Usage: (i) $X \longleftrightarrow R$... return on an investment

(ii) X... severity r.v. / the loss amount

The Semi-Variance

I will just define the version which is relevant when X is interpreted as a return on an investment.

$$\boxed{\sigma_{SV}^2 := \mathbb{E}\left[\left(\min(0, X-\mu_X)\right)^2\right]}$$

Value @ Risk.

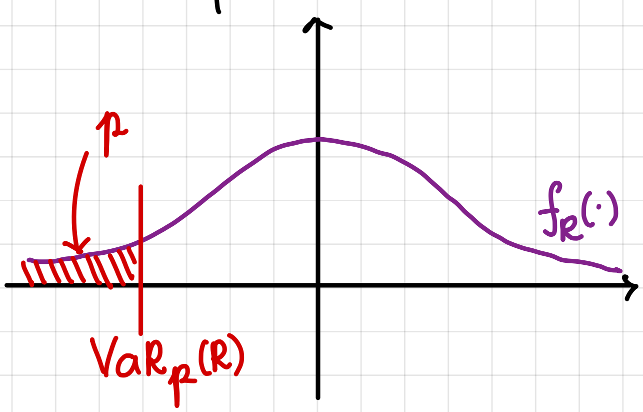
p ... probability of an adverse event you're still willing to live with (e.g., the probability of experiencing a loss)

R ... return random variable (i.e., we benefit if the value of R is high an we have an adverse effect if it's low)

Define $VaR_p(R)$ as the value such that

$$\mathbb{P}[R \leq \underbrace{VaR_p(R)}_{\equiv \pi_p \text{ ... } p^{th} \text{ percentile}}] = p$$

Example. Temporarily, assume that R is a continuous random variable; then we can graph its probability density function



In particular: Consider an R such that it has a pdf and that density is always positive (think normal).

Then, for any $a \in \mathbb{R}$:

$$F_R(a) = \mathbb{P}[R \le a] = \int_{-\infty}^{a} \underbrace{f_R(x)}_{>0}\, dx$$

$\boxed{\text{cdf of } R}$

$\Longrightarrow$ F_R is strictly increasing

$\Longrightarrow$ F_R is one-to-one

$\Longrightarrow$ F_R^{-1} exists

$\Longrightarrow$ $\boxed{VaR_p(R) = F_R^{-1}(p)}$

In particular: we can use the std normal tables or the Prometric calculator for normal returns

Note: In case we are worried about upper-tail probabilities (say, for losses in classical insurance), we will look at $VaR_{1-p}(X)$ w/ X denoting the loss severity.

SAMPLE IFM : Part II.

34) Let X be the random gain from operations of a company. You are given:

↖ Our PROFIT random variable.

(i) X is normally distributed with mean 42 and variance 6400.

(ii) p is the probability that X is negative.

(iii) K is the amount of capital such that the Value-at-Risk (VaR) at the 5th percentile for $X + K$ is zero.

Calculate p and K.

(A) $p = 0.7$; $K = 157$

(B) $p = 0.7$; $K = 131$

(C) $p = 0.5$; $K = 115$

(D) $p = 0.3$; $K = 115$ *

(E) $p = 0.3$; $K = 90$ *

(i) $\Rightarrow X \sim \text{Normal}(\text{mean} = 42, \sigma = 80)$.

(ii) $p = \mathbb{P}[X < 0] =$ (rewrite in std units)

$$= \mathbb{P}\left[\frac{X-42}{80} < \frac{0-42}{80}\right] =$$

$$= \mathbb{P}[Z < -0.525] = N(-0.525) \approx 0.30$$

(iii) $VaR_{0.05}(X+K) = 0$

$$\mathbb{P}[X + K < 0] = 0.05$$

$$\mathbb{P}[X < -K] = 0.05$$

this the 5th percentile of X

On the other hand, we can express X as

$$X = 42 + 80 \cdot Z \quad \text{w/} \quad Z \sim N(0,1)$$

$z^*_{0.05}$ ... the 5th percentile of $N(0,1)$

$$z^*_{0.05} = -1.645$$

$$\Rightarrow -K = 42 + 80(-1.645) \cong -90 \Rightarrow K = 90$$

$\Rightarrow$ (E)