M339J: April 7th, 2021.

Practice 2: Problem 2.11.

$$\boxed{X \mid \Lambda \sim U(0, \Lambda)}$$

$$\Lambda \sim U(0, c) \qquad c = 10$$

$$\mathbb{P}[X \leq 3] = ?$$

$$\mathbb{P}[X \leq 3] = F_X(3)$$

For a mixing dist'n:

$$F_X(x) = \int_0^{10} F_{X|\Lambda}(x|\lambda) \underbrace{f_\Lambda(\lambda)}_{=\frac{1}{10}} d\lambda$$

$$F_{X|\Lambda}(x|\lambda) = \begin{cases} 0 & \text{for } x \leq 0 \\ \boxed{\frac{x}{\lambda} \quad \text{for } 0 < x < \lambda} \\ 1 & \text{for } x \geq \lambda \end{cases}$$

$$F_X(3) = \int_0^{10} F_{X|\Lambda}(3|\lambda) f_\Lambda(\lambda) d\lambda$$

$\lambda < 3 \Rightarrow F_{X|\Lambda}(3|\lambda) = 1$ $\qquad$ $3 < \lambda \Rightarrow F_{X|\Lambda}(3|\lambda) = \frac{3}{\lambda}$

$$= \int_0^3 1 \cdot f_\Lambda(\lambda) d\lambda + \int_3^{10} \left(\frac{3}{\lambda}\right) \cdot f_\Lambda(\lambda) d\lambda$$

$$= \int_0^3 \frac{1}{10} d\lambda + \int_3^{10} \frac{3}{\lambda} \cdot \frac{1}{10} d\lambda$$

$$= \frac{3}{10} + \frac{3}{10} \int_3^{10} \frac{1}{\lambda} d\lambda = \frac{3}{10} + \frac{3}{10} \ln\left(\frac{10}{3}\right) = \ldots$$

Binomial Distribution.

Consider m independent, identically dist'd risks w/ probability q of making a claim.

Formally, for $j = 1, 2, \dots m$, we set

$$I_j = \begin{cases} 1 & \text{if risk } j \text{ makes a claim} \\ 0 & \text{if not} \end{cases}$$

Then, for every $j = 1..m$, $I_j \sim \text{Bernoulli}(q)$

and $\{I_j, j = 1..m\}$ are independent.

N.. the number of claims made

$$\Rightarrow \quad N = I_1 + I_2 + \dots + I_m = \sum_{j=1}^{m} I_j$$

Q: What is the pmf of N?

Note: There are many ways to answer this question; I chose this approach to illustrate a particular method.

→: We want to get and then use the pgf of N!

Start w/ $\{I_j, j=1..m\}$ which are independent, identically dist'd.

$$P_N(z) = \prod_{j=1}^{m} P_{I_j}(z) = \left(P_{I_1}(z)\right)^m$$

independence

$I_j, j = 1..m$ are identically dist'd

For a single Bernoulli trial:

$$P_{I_1}(z) = \mathbb{E}\left[z^{I_1}\right] = \overbrace{p_{I_1}(0)}^{=1-q} z^0 + \overbrace{p_{I_1}(1)}^{=q} z^1 = (1-q) + q \cdot z$$

by the def'n of pgf

$$\Rightarrow \quad \boxed{P_{I_1}(z) = 1 + q(z-1)}$$

$$\Rightarrow \quad \boxed{P_N(z) = \left(1 + q(z-1)\right)^m}$$

$$P_N(z) = \left((1-q) + q\cdot z\right)^m$$

Use the binomial formula $\Big\}$ $p_N(k) = \binom{m}{k}(1-q)^{m-k}\cdot q^k, \quad k = 0,1,\dots,m$ ∎

Q: $\mathbb{E}[N] = ?$

$\longrightarrow$: $\mathbb{E}[N] = \mathbb{E}[I_1 + I_2 + \dots + I_m] \overset{\text{linearity of } \mathbb{E}}{=}$

$= \mathbb{E}[I_1] + \mathbb{E}[I_2] + \dots + \mathbb{E}[I_m] = m\cdot q$ ∎

Q: $\mathrm{Var}[N] = ?$

$\longrightarrow$: $\mathrm{Var}[N] = \mathrm{Var}[I_1 + \dots + I_m] \overset{\text{Independence}}{=}$

$= \mathrm{Var}[I_1] + \dots + \mathrm{Var}[I_m] = m\cdot q(1-q)$ ∎

w/ $\mathrm{Var}[I_1] = \mathbb{E}[I_1^2] - (\mathbb{E}[I_1])^2 = q - q^2 = q(1-q)$

Note on the "counting" distributions:

	mean		variance
Poisson(λ)	λ	$=$	λ
NegBinomial(r, β)	$r\beta$	$<$	$r\beta(1+\beta)$
Binomial(m, q)	mq	$>$	$mq(1-q)$

Example. Let $N \sim$ Poisson(λ) be our frequency r.v.
Every loss is from:

- Category 1 w/ probab. p_1
- Category 2 w/ probab. p_2 w/ $p_1 + p_2 = 1$

Let N_i be the number of losses from Category i, $i = 1,2$.

From our "thinning" theorem, we know that

$$N_i \sim \text{Poisson}(\lambda_i := p_i \cdot \lambda) \qquad i = 1,2.$$

We also know that N_1 and N_2 are independent.

Q: Given that $N = m$, what is the probability that $N_1 = k$, for $k = 0, 1, \dots, m$?

$\longrightarrow$:

$$\mathbb{P}[N_1 = k \mid N = m] =$$ (by the def'n of conditional probability)

$$= \frac{\mathbb{P}[N_1 = k, N = m]}{\mathbb{P}[N = m]} = \frac{\mathbb{P}[N_1 = k, N_1 + N_2 = m]}{\mathbb{P}[N = m]}$$

$$= \frac{\mathbb{P}[N_1 = k, N_2 = m - k]}{\mathbb{P}[N = m]}$$

(we know that N_1 & N_2 are independent)

Please, complete the calculation and recognize the conditional dist'n of N_1 given that $N = m$. :)