M378K: January 24th, 2025.

M378K Introduction to Mathematical Statistics
Problem Set #3
Named discrete random variables.

Problem 3.1. Source: Sample P exam, Problem #125.
An electronic system contains three cooling components that operate independently. The probability of each component's failure is 0.05. The system will overheat if and only if at least two components fail. Calculate the probability that the system will overheat.

→: Y ... # of components that overheated

$\mathbb{P}[Y \geq 2] = ?$

$Y \sim b(n=3, p=0.05)$

$$\mathbb{P}[Y \geq 2] = \mathbb{P}[Y=2] + \mathbb{P}[Y=3]$$
$$= \binom{3}{2}(0.05)^2(0.95) + (0.05)^3$$
$$= \cdots = 0.00725$$

$\square$

Example. Say we repeat independently Bernoulli trials w/ the same success probability p until the first success. The random variable Y which denotes the number of failures until the first success is called geometric.

We write $\boxed{Y \sim g(p)}$

Set $q = 1-p$.

y	0	1	2	3	.....	k	.....
$p_Y(y)$	p	$q \cdot p$	$q^2 \cdot p$	$q^3 \cdot p$	....	$q^k \cdot p$	.....

Q:
$$
\begin{aligned}
\mathbb{P}[Y > 2] &= \underbrace{1-p} - qp - q^2 \cdot p \\
&= q - qp - q^2 \cdot p \\
&= q(\underbrace{1-p} - qp) \\
&= q(q - qp) \\
&= q \cdot q \underbrace{(1-p)}_{q} = q^3
\end{aligned}
$$

$\square$

Problem 3.2. Source: Sample P exam, Problem #462.
Each person in a large population independently has probability p *of testing positive for diabetes where* $0 < p < 1$*. People are tested for diabetes, one person at a time, until a test is positive. Individual tests are independent. Determine the probability that* m *or fewer people are tested, given that* n *or fewer people are tested, where* $1 \leq m \leq n$*.*

→: Y'... total # of people test

SHIFTED geometric r.v. w/ parameter p,

i.e., $\boxed{Y = Y' - 1 \sim g(p)}$

$$\mathbb{P}[Y' \leq m \mid Y' \leq n] = \mathbb{P}[Y+1 \leq m \mid Y+1 \leq n]$$

$$= \mathbb{P}[Y \leq m-1 \mid Y \leq n-1]$$

$$= \frac{\mathbb{P}[Y \leq m-1, Y \leq n-1]}{\mathbb{P}[Y \leq n-1]}$$

$$= \frac{\mathbb{P}[Y \leq m-1]}{\mathbb{P}[Y \leq n-1]} =$$

$$= \frac{1-\mathbb{P}[Y > m-1]}{1-\mathbb{P}[Y > n-1]} = \frac{1-q^m}{1-q^n}$$

□

Example. The Poisson distribution is $\mathbb{N}_0$-valued and has the pmf:

$$p_k = p_Y(k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!} \quad \text{for } k \in \mathbb{N}_0$$

where λ is a positive parameter.

Problem 3.3. Source: Sample P exam, Problem #355.
The number of calls received by a certain emergency unit in a day is modeled by a Poisson distribution with parameter 2*. Calculate the probability that on a particular day the unit receives at least two calls.*

$\longrightarrow$: Y... # of calls

$Y \sim \text{Poisson}(\lambda = 2) \sim \mathbb{P}(2)$

$$\mathbb{P}[Y \geq 2] = 1 - \mathbb{P}[Y=0] - \mathbb{P}[Y=1]$$

$$= 1 - e^{-\lambda} \cdot \frac{\lambda^0}{0!} - e^{-\lambda} \cdot \frac{\lambda^1}{1!}$$

$$= 1 - e^{-2} - e^{-2} \cdot 2 = 1 - 3e^{-2}$$

$\square$

Expectation.

Def'n. For a discrete r.v. Y w/ support $S_Y \subseteq \mathbb{R}$, we define its expectation as

$$\boxed{\mathbb{E}[Y] = \sum_{y \in S_Y} y \cdot p_Y(y)}$$

if the sum exists.

Theorem. Linearity. Let Y_1 and Y_2 be two r.v.s on the same Ω, both w/ finite expectations. Let α and β be two constants.

Then, $\mathbb{E}[\alpha Y_1 + \beta Y_2]$ also exists and

$$\boxed{\mathbb{E}[\alpha Y_1 + \beta Y_2] = \alpha \mathbb{E}[Y_1] + \beta \mathbb{E}[Y_2]}$$

M378K Introduction to Mathematical Statistics
Problem Set #4
Expectation and variance: the discrete case.

Problem 4.1. Source: Sample P exam, Problem #481.
The number of days required for a damage control team to locate and repair a leak in the hull of a ship is modeled by a discrete random variable N. N is uniformly distributed on $\{1, 2, 3, 4, 5\}$.

The cost of locating and repairing a leak is $N^2 + N + 1$.

Calculate the expected cost of locating and repairing a leak in the hull of the ship.

$\rightarrow$:
$$\mathbb{E}[N^2 + N + 1] \underset{\uparrow}{=} \mathbb{E}[N^2] + \mathbb{E}[N] + 1$$

linearity

$$\mathbb{E}[N] = \frac{1}{5} \cdot 1 + \cdots + \frac{1}{5} \cdot 5 = 3$$

$$\mathbb{E}[N^2] = \frac{1}{5} \cdot 1^2 + \frac{1}{5} \cdot 2^2 + \cdots + \frac{1}{5} \cdot 5^2$$

$$= \frac{1}{5}(1 + 2^2 + 3^2 + 4^2 + 5^2) = \frac{1}{\cancel{5}} \cdot \frac{\cancel{5} \cdot \cancel{6} \cdot (2 \cdot 5 + 1)}{\cancel{6}} = 11$$

answer: $11 + 3 + 1 = 15$ $\square$