t-Distribution.

Def'n. A Student t-distribution w/ k degrees of freedom is the dist'n of the random variable

$$T = \frac{Z}{\sqrt{Q^2/k}}$$

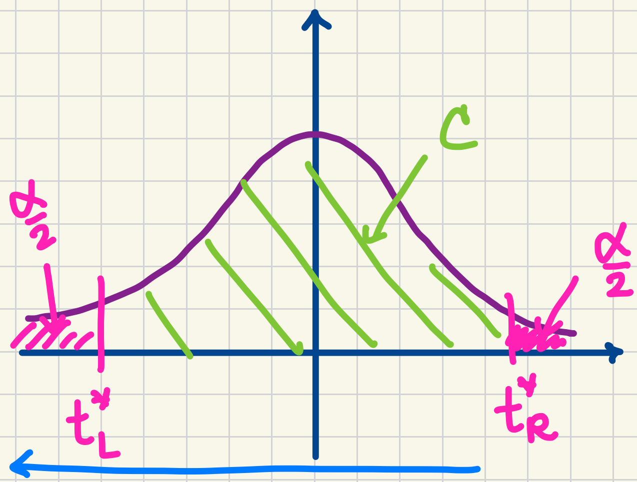
w/

- $Z \sim N(0,1)$
- $Q^2 \sim \chi^2(df=k)$
- Z and Q^2 are independent.

We write $\boxed{T \sim t(df=k)}$

More on t-Confidence Intervals.

To construct a confidence interval w/ the confidence level $C = 1-\alpha$, we do



$$-t^*_L = t^*_R =: t^*$$

$$\left.\begin{aligned} t^* &= qt(1-\alpha/2,\ df=n-1) \\ t^* &= qt((1+C)/2,\ df=n-1) \end{aligned}\right\}$$

$$\mathbb{P}\left[-t^* \le \frac{\bar{Y}-\mu}{S/\sqrt{n}} \le t^*\right] = C = 1-\alpha$$

$$\mathbb{P}\left[-t^* \cdot \frac{S}{\sqrt{n}} \le \bar{Y} - \mu \le t^* \cdot \frac{S}{\sqrt{n}}\right] = 1-\alpha$$

$$\mathbb{P}\left[\underbrace{\boxed{\bar{Y} - t^* \cdot \frac{S}{\sqrt{n}}}}_{\hat{\theta}_L} \le \mu \le \underbrace{\boxed{\bar{Y} + t^* \cdot \frac{S}{\sqrt{n}}}}_{\hat{\theta}_R}\right] = 1-\alpha$$

$$\boxed{\mu = \underbrace{\bar{Y}}_{\text{pt. estimator}} \pm \underbrace{t^*}_{\text{critical value}} \underbrace{\frac{S}{\sqrt{n}}}_{\text{std error}}}$$

Problem 16.9. *(20 points)*

A random sample of size 10 *is drawn from a normal distribution with* **both** *mean and standard deviation* **unknown.** *The generated sample has the sample mean* $\bar{y}_{10} = 14$ *and the (unbiased) estimate of the variance* $s^2 = 25$.

(i) (10 points) *Construct a (symmetric) 90%-confidence interval for* μ. ✓

(ii) (10 points) *Construct a (symmetric) 90%-confidence interval for* σ^2.
Hint: *Remember that you know the distribution of* $\boxed{(n-1)S^2/\sigma^2}$.

Critical value t^* of the t-dist'n w/ df = 10-1=9

$$t^* = qt(0.95, df=9) = 1.833$$

$$\mu = 14 \pm 1.833 \cdot \frac{5}{\sqrt{10}}$$

□

$$Q^2 = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2(df=9)$$

$$\mathbb{P}\left[\chi_L^2 \le \frac{(n-1)S^2}{\sigma^2} \le \chi_R^2\right] = 0.90$$

$$\mathbb{P}\left[\frac{S^2(n-1)}{\chi_R^2} \le \sigma^2 \le \frac{S^2 \cdot (n-1)}{\chi_L^2}\right] = 0.90$$

$$\chi_L^2 = qchisq(0.05, df=9)$$

$$\chi_R^2 = qchisq(0.95, df=9)$$

□

Q: $\mathbb{E}\left[(Y-a)^2\right] \xrightarrow[a]{} \min$ $\boxed{a^* = \mathbb{E}[Y]}$

Maximum Likelihood Estimation.

Likelihood.

Def'n. Given a random sample $Y_1, Y_2, \ldots, Y_n$ from a discrete dist'n D w/ an unknown parameter θ, the likelihood f'tion is defined as

$$L(\theta; y_1, y_2, \ldots, y_n) \overset{\checkmark}{=} p^{\theta}_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n)$$

$$= p^{\theta}_{Y_1}(y_1) \cdot p^{\theta}_{Y_2}(y_2) \cdots p^{\theta}_{Y_n}(y_n)$$

$$= p^{\theta}(y_1) \cdot p^{\theta}(y_2) \cdots p^{\theta}(y_n)$$

where p^{θ} is the pmf of D.