

M339D: October 14th, 2022.

Arbitrage Portfolios.

Def'n. An **arbitrage portfolio** is a portfolio whose **profit** is: ✓

- ① • non-negative in all states of the world
- and
- ② • strictly positive in at least one state of the world.

Unless it's specified otherwise in a particular problem/example, we assume **NO ARBITRAGE** exists.

Law of the Unique Price.

Assume that the payoffs of two static portfolios A and B, are equal, i.e.,

$$V_A(T) = V_B(T).$$

In general, two random variables, X and Y are said to be **equal** if

$$\mathbb{P}[X=Y] = 1$$

On a finite probability space, this means that they must take the exact same value on every elementary outcome.

Our claim:

$$V_A(0) = V_B(0)$$

→: Assume, to the contrary, that

$$V_A(0) \neq V_B(0). \quad \times$$

Without loss of generality, say

$$\underbrace{V_A(0)}_{\text{relatively cheap}} < \underbrace{V_B(0)}_{\text{relatively expensive}}$$

Diagnosis:

Proposal.

Propose an arbitrage portfolio:

- Long Portfolio A
 - short Portfolio B
- } Total Portfolio

Verification.

Verify:

- Payoff (Total Portfolio) = $V_A(T) - V_B(T) = 0$
- Initial Cost (Total Portfolio) = $V_A(0) - V_B(0) < 0$

↑
Inflow of money
@ time 0

$$\text{Profit} = \text{Payoff} - FV_{0,T}(\text{Initial Cost})$$

$$= 0 - FV_{0,T}(V_A(0) - V_B(0)) > 0$$

↑
Indeed, we created
an arbitrage portfolio!

$\Rightarrow < =$

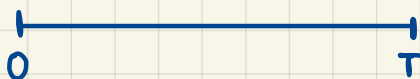
□

Replicating Portfolios.

Def'n. Consider a European-style derivative security.
A static portfolio w/ the same payoff as that of
the derivative security is called its replicating portfolio.

Note: The initial price of the derivative security must be equal
to the initial price of its replicating portfolio.

Example. Consider a forward contract on a non-dividend paying stock.
One possible replicating portfolio is: (?)



Forward contract:

$$S(T) - F$$

Replicating Portfolio:

- long 1 share of stock
- issue a bond w/ redemption amt F and delivery date T