M378K Introduction to Mathematical Statistics
Problem Set #16
Consistency.

Definition 16.1. *$\hat{\theta}_n$ is said to be a* consistent estimator *of θ if*

$$\hat{\theta}_n \to \theta \quad \textit{in probability as } n \to \infty,$$

i.e., if for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left[|\hat{\theta}_n - \theta| > \varepsilon\right] = 0.$$

Theorem 16.2. *Let $\hat{\theta}_n$ be* unbiased *and such that*

$$\boxed{\mathrm{Var}\left[\hat{\theta}_n\right] \xrightarrow{n\to\infty} 0.}$$

Then, $\hat{\theta}_n$ is a **consistent estimator**.

Problem 16.1. *Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from any distribution with finite first and second moments. Propose a consistent estimator for the population mean μ and* **prove** *that it is, indeed, consistent.*

→ :

$$\bar{Y} = \frac{1}{n}(Y_1 + Y_2 + \cdots + Y_n)$$

unbiased ✓

$$\mathrm{Var}[\bar{Y}] = \frac{\mathrm{Var}[Y_1]}{n} \xrightarrow[n\to\infty]{} 0$$

□

Problem 16.2. *Consider a random sample $Y_1, Y_2, \ldots, Y_n$ from a power distribution with the density of the form*

$$f_Y(y) = \theta y^{\theta-1}\mathbf{1}_{(0,1)}(y).$$

What is a consistent estimator for $\frac{\theta}{\theta+1}$? **Prove** *that your choice is indeed consistent.*

$\longrightarrow$:

$$\mathbb{E}[Y_1] = \int_0^1 y\,\theta y^{\theta-1}\,dy = \theta\int_0^1 y^{\theta}\,dy = \theta\cdot\frac{y^{\theta+1}}{\theta+1}\Big|_{y=0}^{1} = \boxed{\frac{\theta}{\theta+1}}$$

We propose $\bar{Y}_n$:

- unbiased ✓
- $\mathrm{Var}[\bar{Y}_n] \longrightarrow 0$
- $\mathrm{Var}[Y_1] = \boxed{\mathbb{E}[Y_1^2]} - \underbrace{(\mathbb{E}[Y_1])^2}$

$$\mathbb{E}[Y_1^2] = \int_0^1 y^2\,\theta y^{\theta-1}\,dy = \theta\cdot\int_0^1 y^{\theta+1}\,dy = \frac{\theta}{\theta+2}$$

Maximum $\boxed{\text{Likelihood}}$ Estimation.

Likelihood.

Def'n. Given a random sample $Y_1, Y_2, \dots, Y_n$ from a discrete dist'n D w/ an unknown parameter θ, the likelihood f'tion is defined as

$$L(\theta; y_1, y_2, \dots, y_n) = P^{\theta}_{Y_1,\dots,Y_n}(y_1,\dots,y_n) = p^{\theta}(y_1)\,p^{\theta}(y_2)\cdots p^{\theta}(y_n)$$

where p^{θ} is a pmf of D.

If $Y_1,\dots,Y_n$ come from a continuous dist'n D w/ pdf f^{θ}, we have this definition:

$$L(\theta; y_1,\dots,y_n) = f^{\theta}_{Y_1,\dots,Y_n}(y_1,\dots,y_n) = f^{\theta}(y_1)\cdot f^{\theta}(y_2)\cdots f^{\theta}(y_n)$$

Example.

Bernoulli. $Y_1,\dots,Y_n \sim B(p)$ $\boxed{p \longleftrightarrow \theta}$

$$p(y) = \begin{cases} p & y=1 \\ 1-p & y=0 \end{cases}$$

$$= p^{y}(1-p)^{1-y} \quad \text{for } y=0,1$$

$$L(p; y_1, y_2, \dots, y_n) = p^{y_1}(1-p)^{1-y_1}\cdot p^{y_2}(1-p)^{1-y_2}\cdot \dots \cdot p^{y_n}(1-p)^{1-y_n}$$
$$= p^{\sum y_i}(1-p)^{n-\sum y_i}$$