NAME:

M339W/389W Financial Mathematics for Actuarial Applications
University of Texas at Austin
Practice Problems for In-Term Three
Instructor: Milica Čudina

3.1. **TRUE/FALSE QUESTIONS.**

Problem 3.1. (2 points) According to the weak formulation of the efficient market hypothesis, one cannot consistently make gains by trading based on the information contained in past prices. *True or false?*

Solution: TRUE

Problem 3.2. (2 points) Under the **CAPM**, the expected return and the required return of the market portfolio are equal. *True or false?*

Solution: TRUE

Problem 3.3. (2 points) Assume the assumptions of CAPM. Then, the **capital market line (CML)** is the tangent line of the feasible set going through the market portfolio. *True or false?*

Solution: TRUE

Problem 3.4. (2 points) The variability of an investment portfolio that is balanced evenly between the stocks it contains is lower than the average variability of the individual stocks it contains. *True or false?*

Solution: TRUE

Problem 3.5. (2 points) Consider the feasible set for two stocks. The higher the correlation of the two stocks' returns, the flatter the curve of the feasible set. *True or false?*

Solution: TRUE

3.2. MULTIPLE CHOICE QUESTIONS.

Problem 3.6. You are given the following information about the return of a security, using a three-factor model:

Factor	Beta	Expected Return
T	0.10	12%
U	0.15	15%
V	0.20	10%

The expected return of this security using the given three-factor model is equal to 0.09. What is the annual effective risk-free rate of return?

(a) About 0.0375
(b) About 0.0415
(c) About 0.0485
(d) About 0.06455
(e) None of the above.

Solution: (d)
By our three-factor model, we have that the expected return of our security S satisfies

$$\begin{aligned}\mathbb{E}[R_S] &= r_f + \beta^T(\mathbb{E}[R_T] - r_f) + \beta^U(\mathbb{E}[R_U] - r_f) + \beta^V(\mathbb{E}[R_V] - r_f) \\ &= \beta_T\mathbb{E}[R_T] + \beta_U\mathbb{E}[R_U] + \beta_V\mathbb{E}[R_V] + r_f(1 - \beta_T - \beta_U - \beta_V).\end{aligned} \tag{3.1}$$

So,

$$\begin{aligned}r_f &= \frac{\mathbb{E}[R_S] - \beta_T\mathbb{E}[R_T] - \beta_U\mathbb{E}[R_U] - \beta_V\mathbb{E}[R_V]}{1 - \beta_T - \beta_U - \beta_V} \\ &= \frac{0.09 - 0.10(0.12) - 0.15(0.15) - 0.2(0.1)}{1 - 0.10 - 0.15 - 0.2} = 0.06454545.\end{aligned}$$

Problem 3.7. (5 points) Consider a two-year project. There are only three cash flows for this project:

- The first occurs at $t = 0$, and is -80.
- The second occurs at $t = 1$, and is 40.
- The third occurs at $t = 2$, and is 44.30.

Determine r, the cost of capital, that leads to the project breaking even.

(a) 0.035
(b) 0.04
(c) 0.045
(d) 0.05
(e) None of the above.

Solution: (a)
The break-even value of the cost of capital must satisfy

$$-80(1+r)^2 + 40(1+r) + 44.30 = 0 \quad \Leftrightarrow \quad (1+r)^2 - 0.5(1+r) - 0.55375 = 0.$$

Solving the quadratic equation, we obtain

$$(1+r)_{1,2} = \frac{0.5 \pm \sqrt{0.5^2 + 4(0.55375)}}{2} = \frac{0.5 \pm \sqrt{2.465}}{2} = \frac{0.5 \pm 1.57}{2}.$$

Our acceptable solution is $1 + r = 1.035$, i.e., $r = 0.035$.

Problem 3.8. (5 points) Assume the **Capital Asset Pricing Model** holds.

You are given the following information about stock X, stock Y, and the market:

- The required return and volatility for the market portfolio are 0.10 and 0.25, respectively.
- The required return and volatility for the stock X are 0.08 and 0.4, respectively.
- The correlation between the returns of stock X and the market is -0.2.
- The volatility of stock Y is 0.25.
- The correlation between the returns of stock Y and the market is 0.4.

Calculate the required return for stock Y.

(a) About 0.075.
(b) About 0.08.
(c) About 0.085.
(d) About 0.09.
(e) None of the above.

Solution: (d)

The βs of stocks X and Y are

$$\beta_X = \frac{0.4(-0.2)}{0.25} = -0.32,$$
$$\beta_Y = \frac{0.4(0.25)}{0.25} = 0.4.$$

So, the required return of stock X must satisfy

$$\begin{aligned} 0.08 = r_X = r_f + (-0.32)(0.10 - r_f) \quad &\Rightarrow \quad 0.08 = r_f - 0.032 + 0.32 r_f \\ &\Rightarrow \quad 1.32 r_f = 0.112 \quad \Rightarrow \quad r_f = 0.0848. \end{aligned}$$

Finally, the required return of stock Y equals

$$r_Y = 0.0848 + 0.4(0.10 - 0.0848) = 0.09088.$$

Problem 3.9. (5 points) For a certain stock, you are given that its expected return equals 0.0944 and that its β equals 1.24. For another stock, you are given that its expected return equals 0.068 and that its β equals 0.8. Both stocks lie on the **Security Market Line (SML)**. What is the risk-free interest rate r_f?

(a) About 0.02
(b) About 0.025
(c) About 0.03
(d) About 0.035
(e) None of the above.

Solution: (a)
Denote the expected return of the market portfolio by r_m. Then,

$$0.0944 = r_f + 1.24(r_m - r_f),$$
$$0.068 = r_f + 0.8(r_m - r_f).$$

Subtracting the second equation from the first one, we get

$$0.0264 = 0.44(r_m - r_f) \quad \Rightarrow \quad r_m - r_f = \frac{0.0264}{0.44} = 0.06.$$

Substituting the obtained risk premium of the market portfolio into the first equation above, we obtain

$$r_f = 0.0944 - 1.24(0.06) = 0.02.$$

Problem 3.10. (5 points) There are two stocks present in our market: **S** and **Q**. Their current prices are $S(0) = 50$ and $Q(0) = 55$. Both stocks pay dividends continuously. The dividend yield for **S** is 0.02 while the dividend yield for **Q** equals 0.03.

You are given that for $t \geq 0$

$$\mathrm{Var}[\ln(S(t)/Q(t))] = 0.09t.$$

What is the Black-Scholes price of a one-year **exchange call** with underlying **S** and the strike asset **Q**?

(a) \$2.89
(b) \$3.01
(c) \$3.57
(d) \$4.36
(e) None of the above.

Solution: (d)
In our usual notation, the volatility of the difference of the stocks' realized returns is $\sigma = 0.3$. So,

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right)T\right] = \frac{1}{0.3}\left[\ln\left(\frac{50}{55}\right) + \left(0.03 - 0.02 + \frac{0.09}{2}\right)\right] \approx -0.13,$$
$$d_2 = -0.13 - 0.3 = -0.43.$$

So,

$$N(d_1) = 1 - N(0.13) = 0.4483, \quad N(d_2) = 1 - N(0.44) = 0.33.$$

Finally,

$$V_{EC}(0, \mathbf{S}, \mathbf{Q}) = 50e^{-0.02}(0.4483) - 55e^{-0.03}(0.33) = 4.357567.$$

Problem 3.11. (5 points) In a market, the risk-free interest rate is given to be 0.04.

Consider an investment I in this market, whose Sharpe ratio is 0.42. You construct an equally weighted portfolio consisting of the investment I and the risk-free asset. The expected return of this portfolio is 0.10.

You decide to rebalance your portfolio so that one quarter of your wealth gets invested in the investment I and the remainder is invested in the risk-free asset. What is the volatility of this new portfolio?

(a) 0.0625
(b) 0.0714
(c) 0.1225
(d) 0.1625
(e) None of the above.

Solution: (b)
Let's denote the volatility of investment I by σ_I and the volatility of the new portfolio by $\sigma_{P'}$. Then, $\sigma_{P'} = 0.25\sigma_I$.

The expected return of the old portfolio is

$$\mathbb{E}[R_P] = \frac{1}{2}\mathbb{E}[R_I] + \frac{1}{2}r_f\,.$$

So,

$$\mathbb{E}[R_I] = 2\mathbb{E}[R_P] - r_f = 2(0.10) - 0.04 = 0.16.$$

We are given the Sharpe ratio of the investment I, and so we can calculate

$$\sigma_I = \frac{0.16 - 0.04}{0.42} = 0.2857143.$$

Finally, the new portfolio's volatility is

$$\sigma_{P'} = 0.25(0.2857143) = 0.07142857.$$

Problem 3.12. (5 points) According to your model, the economy over the next year could be *good* or *bad*. You are a pessimist and believe that the economy is twice as likely to be *bad* than *good*.

Consider two assets, X and Y, existing in this market. If the economy is *good* the return on asset X is 0.12, and the return on asset Y is 0.11. If the economy is *bad* the return on asset X is -0.03 and the return on asset Y is -0.01.

You construct a portfolio P using assets X and Y so that the portfolio's expected return equals 0.025.

Calculate the volatility of this portfolio's return.

(a) 0.0458
(b) 0.0512
(c) 0.0584
(d) 0.0637
(e) None of the above.

Solution: (d)
First, we need to figure out the weight w_X the asset X is given in portfolio P. The expected returns of

assets X and Y are

$$\mathbb{E}[R_X] = \frac{1}{3}(0.12) + \frac{2}{3}(-0.03) = \frac{0.12 - 0.06}{3} = 0.02,$$
$$\mathbb{E}[R_Y] = \frac{1}{3}(0.11) + \frac{2}{3}(-0.01) = \frac{0.11 - 0.02}{3} = 0.03.$$

So, the portfolio P must be equally weighted, i.e., $w_X = 0.5$. Hence, the distribution of the portfolio's return can be described as

$$R_P \sim \begin{cases} 0.115, & \text{with probability } 1/3 \\ -0.02, & \text{with probability } 2/3 \end{cases}$$

The second moment of the portfolio's return is

$$\mathbb{E}[R_P^2] = (0.115) \times \frac{1}{3} + (-0.02)^2 \times \frac{2}{3} = 0.004675.$$

The variance of the portfolio's return is

$$Var[R_P] = 0.004675 - 0.025^2 = 0.00405.$$

Finally, its volatility equals $\sigma = \sqrt{0.00405} = 0.0636396$.

Problem 3.13. (5 points) Consider two assets X and Y such that:

- their expected returns are $\mathbb{E}[R_X] = 0.10$ and $\mathbb{E}[R_Y] = 0.08$;
- their volatilities are $\sigma_X = 0.25$ and $\sigma_Y = 0.35$;
- the correlation coefficient of their returns is $\rho_{X,Y} = -1$.

You are tasked with constructing a portfolio consisting of shares of X and Y with a risk-free return. What should the weight w_Y given to asset Y be?

(a) 5/12
(b) 1/2
(c) 7/12
(d) Such a weight does not exist.
(e) None of the above.

Solution: (a)

$$w_Y = \frac{\sigma_X}{\sigma_X + \sigma_Y} = \frac{0.25}{0.25 + 0.35} = \frac{5}{12}.$$

Problem 3.14. (5 points) For stock S_1, you are given that its expected return equals 0.08 and its β is 1.22. For stock S_2, you are given that its expected return equals 0.05 and its β is 0.56. Both of these stocks lie on the *Security Market Line*. For stock S_3, you are given that its expected return equals 0.07 and its β is 0.7. What is the α of stock S_3?

(a) 0
(b) 0.0137
(c) 0.0245
(d) 0.0455
(e) None of the above.

Solution: (b)
Since both S_1 and S_2 are on the **SML**, we know that

$$0.08 = r_f + 1.22(r_m - r_f),$$
$$0.05 = r_f + 0.56(r_m - r_f),$$

where r_f stands for the risk-free interest rate and r_m stands for the expected return of the market. Subtracting the second equation from the first one, we get

$$0.03 = 0.66(r_m - r_f) \quad \Rightarrow \quad r_m - r_f = \frac{0.03}{0.66} = 0.0455.$$

The risk-free interest rate r_f can, then, be calculated as

$$r_f = 0.08 - 1.22(0.0455) = 0.02449.$$

Hence, the α of stock S_3 is

$$0.07 - 0.02449 - 0.7(0.0455) = 0.01366.$$