M378K: January 28th, 2026.

The Poisson Distribution.

The Poisson distribution is $\mathbb{N}_0$-valued
and its probability mass f'tion (pmf) is

$$\boxed{p_k := p_Y(k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}} \quad \text{for all } k \in \mathbb{N}_0$$

where λ is a positive parameter.

Problem 3.3. Source: Sample P exam, Problem #355.
The number of calls received by a certain emergency unit in a day is modeled by a Poisson distribution with parameter 2. *Calculate the probability that on a particular day the unit receives at least two calls.*

$$\Pr(\text{Calls} \geq 2) = 1 - \left(\Pr(\text{Calls} = 0) + \Pr(\text{Calls} = 1)\right)$$

$$= 1 - e^{-2} \cdot \frac{2^0}{0!} + e^{-2} \cdot \frac{2^1}{1!}$$

$$= 1 - e^{-2}\left(\frac{1}{1} + \frac{2}{1}\right)$$

$$= 1 - 3e^{-2}$$

$\square$

Expectation.

Def'n. For a discrete r.v. Y w/ support $S_Y \subseteq \mathbb{R}$ and pmf p_Y,

we define its **expectation/expected value/mean** as

$$\boxed{\mathbb{E}[Y] = \sum_{y \in S_Y} y \cdot p_Y(y)}$$ if the sum exists.

St. Petersburg Paradox.

Theorem. Let Y_1 and Y_2 be two r.v.s on the same Ω, both w/ finite expectations.

Let α and β be two constants.

Then, $\mathbb{E}[\alpha Y_1 + \beta Y_2]$ also exists, and

$$\boxed{\mathbb{E}[\alpha Y_1 + \beta Y_2] = \alpha\, \mathbb{E}[Y_1] + \beta\, \mathbb{E}[Y_2]}$$

Linearity of Expectation.

M378K Introduction to Mathematical Statistics

Problem Set #4

Expectation and variance: the discrete case.

Problem 4.1. Source: Sample P exam, Problem #481.
The number of days required for a damage control team to locate and repair a leak in the hull of a ship is modeled by a discrete random variable N. N is uniformly distributed on $\{1, 2, 3, 4, 5\}$.

The cost of locating and repairing a leak is $N^2 + N + 1$.

Calculate the expected cost of locating and repairing a leak in the hull of the ship.

$\longrightarrow$: $E[N^2+N+1] = E[N^2] + E[N] + E[1]$

$E[N^2] = \frac{1}{5}\sum_{n=1}^{5} n^2 = \frac{1}{5}(55) = 11$

$E[1] = 1$

$E[N] = \frac{1}{5}\sum_{n=1}^{5} n = \frac{1}{5}(15) = 3$

$1+3+11 = \boxed{15}$

$\square$

Example.

- **Bernoulli.** $Y \sim B(p)$

$\mathbb{E}[Y] = ?$

$$Y \sim \begin{cases} 1 & \text{w/ prob. } p \\ 0 & \text{w/ prob. } q = 1-p \end{cases}$$

$$\boxed{\mathbb{E}[Y] = 1 \cdot p + 0 \cdot (1-p) = p} \quad (\star)$$

- **Binomial.** $Y \sim b(n, p)$

$\mathbb{E}[Y] = ?$

Start w/ $I_j \sim B(p)$, $j = 1..n$, independent

$$Y = I_1 + I_2 + \cdots + I_n$$

$$\mathbb{E}[Y] = \mathbb{E}[I_1] + \mathbb{E}[I_2] + \cdots + \mathbb{E}[I_n] \quad (\text{linearity of } \mathbb{E})$$

$$\overset{(\star)}{=} p + p + \cdots + p = np$$

$$\boxed{\mathbb{E}[Y] = n \cdot p}$$

- **Geometric.** $Y \sim g(p)$

$\mathbb{E}[Y] = ?$

Idea #1. $\mathbb{E}[Y] = \sum_{k=0}^{\infty} k \, p_Y(k) = \sum_{k=0}^{\infty} k \cdot q^k \cdot p = p \cdot \sum_{k=0}^{\infty} k \cdot q^k$

NOT A GEOMETRIC SERIES!

Idea #2.

$$\sum_{k=\cancel{0}\,1}^{\infty} k\cdot p_k = p_1 + 2\cdot p_2 + \cdots + k\cdot p_k + \cdots$$

$$\begin{aligned}
= \; & p_1 + \\
& p_2 + p_2 + \\
& p_3 + p_3 + p_3 + \\
& \vdots \\
& p_k + p_k + p_k + p_k \cdots + p_k \\
& \vdots
\end{aligned}$$

$$= \boxed{\mathbb{P}[Y>0] + \mathbb{P}[Y>1] + \mathbb{P}[Y>2] + \cdots + \mathbb{P}[Y>k-1] + \cdots = \mathbb{E}[Y]}$$

The Tail Formula for Expectation

In the geometric case:

$$\mathbb{E}[Y] = ?$$

$$\mathbb{E}[Y] = q + q^2 + q^3 + \cdots + q^k + \cdots$$

$$= q(1 + q + q^2 + \cdots + q^{k-1} + \cdots)$$

$$= q\cdot\frac{1}{1-q} = \boxed{\frac{q}{p}}$$