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Problem Set 3

Payoff. Profit.

3.1. **Static portfolios.**

Step #1. Remember the **bottom-line approach** from *theory of interest.* Decide who your **protagonist** is!

Step #2. Set up the **timeline** (on paper or mentally):

static, i.e., no intermediate cashflows

"today"

Cashflow from the perspective of our agent.

!!

Initial Cost

TIME HORIZON

Cashflow/Wealth from the perspective of our agent.

!!

Payoff

$$\boxed{\text{Profit} := \text{Payoff} - FV_{0,T}(\text{Initial Cost})}$$

This is how we will talk about **profit**:

- If Profit > 0, then we call it a gain.
- If Profit < 0, then we call it a loss.
- If Profit $= 0$, then we say that we break even.

3.2. **Riskless assets.**

Example 3.1. Investing in a zero-coupon bond

$\boxed{+C}$ ... redemption amount

0 — T

0: bond bought

T: maturity date

$\boxed{r \ldots \text{ccrfir}}$

Initial Cost: Ce^{-rT}

Payoff: C

$$\text{Profit} = \text{Payoff} - FV_{0,T}(\text{Initial Cost}) = C - FV(Ce^{-rT}) = 0$$

Example 3.2. Taking a loan

L ... loan amt

0 — T

0: loan taken

T: loan repaid

Initial Cost: $-L$

Payoff: $-Le^{rT}$

$$\text{Profit} = -Le^{rT} + FV_{0,T}(+L) = 0$$

3.3. **Risky assets.**

Example 3.3. Outright purchase of a stock

$S(t), t \geq 0$ ... time-t Stock Price

0 — T

stock bought — stock sold

Initial Cost: $S(0)$

Payoff: $S(T)$ ... a random variable

$$\text{Profit} = S(T) - FV_{0,T}(S(0))$$
$$= S(T) - S(0)e^{rT}$$

Problem 3.1. Let the current price of a non-dividend-paying stock be \$40. The continuously compounded, risk-free interest rate is 0.04. You model the distribution of the time−1 price of the above stock as follows:

$$S(1) \sim \begin{cases} 45, & \text{with probability } 1/4, \\ 42, & \text{with probability } 1/2, \\ 38, & \text{with probability } 1/4. \end{cases}$$

What is your expected profit under the above model, if you invest in one share of stock at time−0 and liquidate your investment at time−1?

Solution: The initial cost is $S(0)$ and the payoff is $S(T)$ with $T = 1$. So, the profit equals

$$S(T) - S(0)e^{rT}.$$

Thus, the expected profit equals

$$\mathbb{E}[S(T)] - S(0)e^{rT}.$$

According to the given model for the stock price, we have

$$\mathbb{E}[S(T)] = 45\left(\frac{1}{4}\right) + 42\left(\frac{1}{2}\right) + 38\left(\frac{1}{4}\right) = 41.75.$$

Finally, the expected profit is

$$41.75 - 40e^{0.04} = 0.117569.$$

$\longrightarrow$:

$$\boxed{\mathbb{E}[\text{Profit}] = \mathbb{E}[\text{Payoff}] - FV_{0,T}(\text{Initial Cost})}$$

In this problem:

$$\mathbb{E}[\text{Profit}] = \mathbb{E}[S(T)] - S(0)e^{rT}$$

$$\mathbb{E}[S(T)] = 45 \cdot \left(\tfrac{1}{4}\right) + 42 \cdot \left(\tfrac{1}{2}\right) + 38 \cdot \left(\tfrac{1}{4}\right)$$

$$= 41.75$$

$$\mathbb{E}[\text{Profit}] = 41.75 - 40e^{0.04} = \underline{0.1176}$$

$\square$

Goal. To study the payoff and the profit as **functions** of the **final asset price**.
Introduce s . . . an independent **argument** taking values in $[0, \infty)$ which will stand for the **final asset price**, i.e., it will be a "placeholder" for the random variable $S(T)$

Now, we can define the PAYOFF FUNCTION which describes the dependence of the payoff amount on the independent argument s.

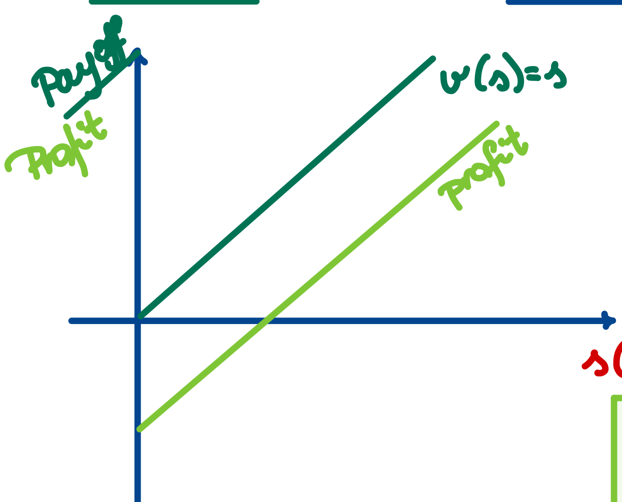
Notation: v ... payoff f'tion.

$$v : [0, \infty) \longrightarrow \mathbb{R}$$

$v(s)$ ... the agent's payoff if the final asset price equals s

Example. For the outright purchase: $\boxed{v(s) = s}$

identity function



When we plot the payoff f'tion, we get the payoff curve/diagram

In general: the profit function is $v(s) - FV_{0,T}(\text{Initial Cost})$