UNIVERSITY OF TEXAS AT AUSTIN

HW Assignment 12

The Delta.

Please, provide your **complete solutions** to the following problems. Final answers only, even if correct will earn zero points for those problems.

Problem 12.1. (2 points) The Black-Scholes delta of a European call option is always between 0 and 1. *True or false? Why?*

Solution: TRUE
The expression for the call delta is

$$\Delta_C(s,t) = N(d_1(s,t)).$$

Both terms in the product on the right-hand side are non-negative and smaller than 1.

Problem 12.2. (2 points) The Black-Scholes delta of a European put option is always between -1 and 0. *True or false? Why?*

Solution: TRUE
The expression for the call delta is

$$\Delta_P(s,t) = -N(-d_1(s,t)).$$

Both terms in the product on the right-hand side are non-negative and smaller than 1. However, their product is preceded by a -1 which makes the entire expression always have a value between -1 and 0.

Problem 12.3. (10 points) *Source: Sample MFE Problem #8.*
Consider a non-dividend-paying stock whose price $\mathbf{S} = \{S(t), t \geq 0\}$ is modeled using the Black-Scholes model. Suppose that the current stock price equals \$40 and that its volatility is given to be 0.30.

Consider a three-month, \$41.5-strike European call option on the above stock. You learn that the current call delta equals 0.5.

What is the Black-Scholes price of this call option?

Solution: 2.18521.

Problem 12.4. (4 points) An investor wants to delta-hedge a portfolio consisting of a long K_1- strike call option and an otherwise identical K_2-strike call option with $K_1 < K_2$. Then, she should short-sell shares of the underlying asset. *True or false? Why?*

Solution: TRUE
The portfolio she owns is called a **bull spread**. The delta of the bull spread is positive since it is a long position with respect to the underlying (if in doubt, plot the payoff diagram). So, before she hedges, the delta of her portfolio is positive. Therefore, the delta of her stock investment needs to be negative. Hence, she needs to short sell shares of stock in order to create a delta-neutral portfolio.

Problem 12.5. (2 points) A market-maker writes a call option on a stock. To decrease the delta of this position, (s)he can **write** a call on the underlying stock. *True or false? Why?*

Solution: TRUE
The expression for the call delta is

$$\Delta_C(s,t) = N(d_1(s,t)).$$

Problem 12.6. (6 points) Consider an option whose payoff function is given by $v(s,T) = \min(s, 50)$. If a market-maker **writes** this option, they need to short sell shares of stock to create a delta-neutral portfolio. *True or false? Why?*

Solution: FALSE
The "special" option in the probem can be replicated using a put and a bond. More precisely,

$$v(s,T) = \min(s,50) = 50 - \max(50-s,0) = 50 - v_P(s,T).$$

So, the value of the "special" option is equal to the value of the bond minus the put price at any point in time. Hence, the delta of the "special" option is

$$\Delta(s,t) = -\Delta_P(s,t) > 0.$$

Since the market maker writes the option, the original delta of their position is $\Delta_P(S(0),0)$ which is a negative number. They need to **purchase** shares of stock to create a delta-neutral portfolio.

Problem 12.7. (14 points) Assume the Black-Scholes framework.

Let $K_P < K_C$. A $(K_P, K_C)-$strangle consists of a long K_P-strike put and an otherwise identical long K_C-strike call.

The goal is to delta-hedge a written one-year, $(40,60)$-strangle on a non-dividend-paying stock whose current price is \$50. The stock's volatility is 0.20.

The continuously compounded, risk-free interest rate is 0.10.

What is the cost of delta-hedging the strangle using shares of the underlying stock?

Solution: The Δ of the strangle equals

$$\Delta_P(S(0),0;K_P=40) + \Delta_C(S(0),0;K_C=60),$$

i.e., it is the sum of the delta of the call with strike 60 and the delta of the put with strike 40. We have

$$d_1(S(0),0;K_P=40) = \frac{1}{0.2}\left[\ln\left(\frac{50}{40}\right) + \left(0.10 + \frac{0.04}{2}\right)\right] = 1.715.$$

So, the put's delta is approximately

$$-N(-d_1(S(0),0;K_P=40)) = -N(-1.72) = N(1.72) - 1 = -0.0427.$$

Similarly, for the call, we have

$$d_1(S(0),0;K_C=60) = \frac{1}{0.2}\left[\ln\left(\frac{50}{60}\right) + \left(0.10 + \frac{0.04}{2}\right)\right] = -0.31.$$

So, the call's delta is approximately

$$N(d_1(S(0),0;K_C=60)) = N(-0.31) = 1 - N(0.31) = 0.3783.$$

Our answer is

$$50(0.3783 - 0.0427) = 16.78.$$

Problem 12.8. (10 points) Assume the Black-Scholes model. Let the current stock price be equal to \$90 per share. Its volatility is 0.20.

The continuously compounded, risk-free interest rate is 0.01.

Consider a one-year European call option on the above stock. The delta of this call option is 0.50. What is the strike price of the call?

Solution: In our usual notation, the delta of the call is

$$\Delta_C(S(0),0) = N(d_1(S(0),0)) = N(d_1(S(0),0)) = 0.50.$$

So,

$$d_1(S(0),0) = 0.$$

Hence,

$$\ln\left(\frac{S(0)}{K}\right) + \left(r + \frac{\sigma^2}{2}\right) = 0 \quad \Rightarrow \quad \ln\left(\frac{S(0)}{K}\right) = -\left(0.01 + \frac{0.04}{2}\right) = -0.03.$$

Therefore,

$$\frac{S(0)}{K} = e^{-0.03} \quad \Rightarrow \quad K = S(0)e^{0.03} = 90e^{0.03} = 92.7409.$$