Elementary Probability Review
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TRUE/FALSE

1	TRUE	FALSE
2	TRUE	FALSE

MULTIPLE CHOICE

1 (5)	a	b	c	d	e
2 (5)	a	b	c	d	e

FOR GRADER'S USE ONLY:

T/F	1.	2.	3.	4.	5.	6.	7.	8.	Σ

Part I. **DEFINITIONS**

1. (5 points) Write down the definition of *independence* of two *events*.
Solution: Two events A and B are said to be *independent* if
$$\mathbb{P}[A \cap B] = \mathbb{P}[A] \cdot \mathbb{P}[B]$$
2. (5 points) Write down the definition of a *cumulative distribution function* of a random variable.
Solution: Let X be a random variable. Its *cumulative distribution function* is a function $F_X : \mathbb{R} \to [0,1]$ defined by
$$F_X(x) = \mathbb{P}[X \leq x], \quad \text{for every } x \in \mathbb{R}.$$

Part II. **TRUE/FALSE QUESTIONS**

1. (2 pts) Assume that **only** the marginal p.m.f.s p_X and p_Y are given for a random pair (X, Y), then we can **always** calculate the joint p.m.f. $p_{X,Y}$ for the pair X, Y.

   Solution: FALSE

2. (2 pts) If X and Y are independent random variables, then the cumulative distribution functions satisfy, for every a,
$$F_{X+Y}(a) = F_X(a) \cdot F_Y(a).$$
   Solution: FALSE

Part III. **FREE RESPONSE PROBLEMS**

Please, explain carefully all your statements and assumptions. Numerical results or single-word answers without an explanation (even if they're correct) are worth 0 points.

1. (10 points)Let $\Omega = \{a_1, a_2, a_3, a_4, a_5\}$ be an outcome space, and let $\mathbb{P}$ be a probability distribution on Ω. Assume that $\mathbb{P}[A] = 0.5$, $\mathbb{P}[B] = 0.4$, $\mathbb{P}[C] = 0.4$, and $\mathbb{P}[D] = 0.2$, where
$$A = \{a_1, a_2, a_3\},\ B = \{a_2, a_3, a_4\},$$
$$C = \{a_3, a_5\} \text{ and } D = \{a_4\}.$$
Are the events A and B independent?

   Solution: We need to check whether $\mathbb{P}[A \cap B] = \mathbb{P}[A] \times \mathbb{P}[B]$. Since
$$\begin{aligned}\mathbb{P}[A \cap B] &= \mathbb{P}[\{a_2, a_3\}] \\ &= \mathbb{P}[\{a_2, a_3, a_4\}] - \mathbb{P}[\{a_4\}] \\ &= \mathbb{P}[B] - \mathbb{P}[D] = 0.2\end{aligned}$$
and $\mathbb{P}[A] \times \mathbb{P}[B] = 0.5 \times 0.4 = 0.2$, we conclude that A and B *are* independent.

2. (20 points) Four balls are drawn (without replacement) from a box which contains 4 black and 5 red balls. Given that the four drawn balls were *not* all of the same color, what is the probability that there were exactly two balls of each color among the four.

Solution. Let A denote the event that the colors of the balls drawn are not all the same, and let B denote the event that there are exactly two black balls and two red balls. We are looking for $\mathbb{P}[B|A]$. Since $B \subseteq A$, we have

$$\mathbb{P}[B|A] = \mathbb{P}[B \cap A]/\mathbb{P}[A] = \mathbb{P}[B]/\mathbb{P}[A].$$

To compute $\mathbb{P}[A]$, we note that the event A^c consists of only two elementary outcomes: either all balls are black or all balls are red. The probability of picking all red balls is

$$\frac{\binom{5}{4}}{\binom{9}{4}}$$

while the probability of picking all black balls is

$$\frac{\binom{4}{4}}{\binom{9}{4}} = \frac{1}{\binom{9}{4}}.$$

Putting these two together, we obtain

$$\mathbb{P}[A] = 1 - \frac{\binom{5}{4} + 1}{\binom{9}{4}}.$$

To compute $\mathbb{P}[B]$ we note that we can choose 2 red balls out of 5 in $\binom{5}{2}$ ways and, then, for each such choice, we have $\binom{4}{2}$ ways of choosing 2 black balls from the set of 4 black balls. Therefore,

$$\mathbb{P}[B] = \left(\binom{5}{2} \times \binom{4}{2} \right) \Big/ \binom{9}{4}.$$

Finally,

$$\mathbb{P}[B|A] = \frac{\binom{5}{2}\binom{4}{2}}{\binom{9}{4} - \binom{5}{4} - 1} = \frac{10 \times 6}{126 - 5 - 1} = \frac{1}{2}.$$

3. (15 points)Consider a set-up in which a transmitter is transmitting either a 0 or a 1 and the receiver indicates that it received either a 0 or a 1. Denote the events that $i = 0, 1$ was transmitted by T_i, and the events that $i = 0, 1$ was indicated as received by R_i.
   It is possible to have transmission errors. In fact, you are given the following data on accuracy and the frequency of transmitted signals:

$$\mathbb{P}[R_0 \,|\, T_0] = 0.99, \;\; \mathbb{P}[R_1 \,|\, T_1] = 0.98,$$

and

$$\mathbb{P}[T_0] = 0.75.$$

   (a) (10pts) Given that the receiver indicated 1, what is the probability that there was an error in the transmission?

   (b) (5pts) What is the overall probability that there was an error in transmission?

Solution:

(1) We need $\mathbb{P}[T_0|R_1]$. By the Bayes formula,

$$\begin{aligned}
\mathbb{P}[T_0|R_1] &= \frac{\mathbb{P}[R_1|T_0]\mathbb{P}[T_0]}{\mathbb{P}[R_1|T_0]\mathbb{P}[T_0] + \mathbb{P}[R_1|T_1]\mathbb{P}[T_1]} \\
&= \frac{(1-0.99)\times 0.75}{(1-0.99)\times 0.75 + 0.98\times 0.25} \\
&= \frac{3}{101} \cong 0.030.
\end{aligned}$$

(2) An error will happen if $T_0 \cap R_1$ or $T_1 \cap R_0$ occur, i.e.,

$$\begin{aligned}
\mathbb{P}[\text{error}] &= \mathbb{P}[T_0 \cap R_1] + \mathbb{P}[T_1 \cap R_0] \\
&= \mathbb{P}[R_1|T_0] \times \mathbb{P}[T_0] + \mathbb{P}[R_0|T_1] \times \mathbb{P}[T_1] \\
&= (1 - \mathbb{P}[R_0|T_0]) \times \mathbb{P}[T_0] \\
&\qquad + (1 - \mathbb{P}[R_1|T_1]) \times (1 - \mathbb{P}[T_0]) \\
&= 0.01 \times 0.75 + 0.02 \times 0.25 \\
&= \tfrac{1}{80} \cong 0.013
\end{aligned}$$

4. (20 points) A fair coin is tossed 3 times. Let the random variable X stand for the number of heads (H) in the *first* two of the three coin tosses, and let Y stand for the number of tails (T) in the *last* two of the three coin tosses.
   (a) (4pts) what is the outcome space associated with the above procedure?

   (b) (4pts) Write down the joint-distribution table of the random pair (X, Y)

(c) (4pts) Find the marginal distribution of Y.

(d) (4pts) Determine the conditional distribution of X, given $Y = 1$.

(e) (4pts) Find the distribution of $Z = X + Y$.

Solution:

(a) The outcome space is

$$\Omega = \{TTT, TTH, THT, HTT, \ldots, HHH\},$$

i.e., the 8-element set consisting of all "three-letter words", where each letter is either T or H.

(b)

	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	
2	$\frac{1}{8}$	$\frac{1}{8}$	0	$\frac{1}{4}$
1	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{2}$
0	0	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$
Y / X	**0**	**1**	**2**	

(c)

k	**0**	**1**	**2**
$\mathbb{P}[Y = k]$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

(d)

k	**0**	**1**	**2**
$\mathbb{P}[X = k \mid Y = 1]$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

(e)

k	**0**	**1**	**2**	**3**	**4**
$\mathbb{P}[Z = k]$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{8}$

Part IV. **Multiple choice questions**

1. (5 points) Let X be a continuous random variable with the cumulative distribution function denoted by F_X and the probability density function denoted by f_X.
   Let the random variable $Y = \frac{1}{2}X$ have the p.d.f. denoted by f_Y. Then,
   (a) $f_Y(x) = 2f_X(2x)$
   (b) $f_Y(x) = \frac{1}{2}f_X\left(\frac{x}{2}\right)$
   (c) $f_Y(x) = f_X(2x)$
   (d) $f_Y(x) = f_X\left(\frac{x}{2}\right)$
   (e) None of the above

   Solution: (a)
   For every $x \in \mathbb{R}$, the cumulative distribution function is

$$F_Y(x) = \mathbb{P}[Y \leq x] = \mathbb{P}[\frac{1}{2}X \leq x] = \mathbb{P}[X \leq 2x] = F_X(2x).$$

As for the probability density function, we have that for all x,

$$f_Y(x) = F_Y'(x) = 2f_X(2x).$$

Problem 0.1. (5 points) A class has 12 boys and 4 girls. If three students are selected at random from this class, what is the probability that they are all boys?

(a) 1/4
(b) 5/9
(c) 11/28
(d) 17/36
(e) None of the above

Solution: (c)
Let A_i stand for the event of choosing a boy in the i^{th} selection with $i = 1, 2, 3$. The probability we are seeking is

$$\mathbb{P}[A_1 \cap A_2 \cap A_3].$$

By the multiplication rule,

$$\begin{aligned}\mathbb{P}[A_1 \cap A_2 \cap A_3] &= \mathbb{P}[A_1]\mathbb{P}[A_2|A_1]\mathbb{P}[A_3|A_2 \cap A_1] \\ &= \frac{12}{16} \cdot \frac{11}{15} \cdot \frac{10}{14} = \frac{11}{2 \cdot 14} = \frac{11}{28}\,.\end{aligned}$$