

## UNIVERSITY OF TEXAS AT AUSTIN

## HW Assignment 5

## Prerequisite material. Log-normal stock prices. Jensen's inequality.

Provide your **complete solution** to the following problems:

**Problem 5.1.** (10 points) A continuous-dividend-paying stock is valued at \$75.00 per share. Its dividend yield is 0.03. The time- $t$  realized (rate of) return is modeled as

$$R(0, t) \sim N(\text{mean} = 0.035t, \text{variance} = 0.09t)$$

An investor purchases a single share of stock at time-0 and continuously (and immediately) reinvests any dividends received in the same asset. What are the mean and median values of the investor's position at time-4?

**Solution:** The expected rate of return (per annum) is

$$0.035 + 0.03 + \frac{1}{2} \times 0.09 = 0.035 + 0.03 + 0.045 = 0.11.$$

The mean is

$$e^{4\delta} \mathbb{E}[S(T)] = 75e^{4\alpha} = 75e^{0.44} = 116.45.$$

Similarly, the median is

$$116.45 \times e^{-0.09 \times 4/2} = 97.27.$$

**Problem 5.2.** (10 points) A continuous-dividend-paying stock is valued at \$75.00 per share. Its dividend yield is 0.03. The time- $t$  realized return is modeled as

$$R(0, t) \sim N(\text{mean} = 0.035t, \text{variance} = 0.09t)$$

Find the probability that the time-4 stock price exceeds today's stock price.

**Solution:** We need to find

$$\mathbb{P}[S(4) > S(0)]$$

with

$$S(t) = S(0)e^{R(0,t)}.$$

Since  $R(0, t)$  follows the normal distribution with the above parameters, we have

$$\mathbb{P}[S(4) > S(0)] = \mathbb{P}[S(0)e^{R(0,4)} > S(0)] = \mathbb{P}[R(0, 4) > 0] = 1 - N\left(-\frac{0.035 \times 4}{0.3 \times 2}\right) = N(0.23) = 0.591.$$

**Problem 5.3.** (10 points) A non-dividend-paying stock is valued at \$75.00 per share. The annual expected (rate of) return is 16.0% and the standard deviation of annualized returns is given to be 0.30. If the stock price is modeled using the lognormal distribution (as discussed in class), what is the constant  $s_{1/2}^U$  such that

$$\mathbb{P}[S(1/2) > s_{1/2}^U] \leq 0.05.$$

**Solution:** Note that the 95<sup>th</sup> percentile of the standard normal distribution equals 1.645. So,

$$s_{1/2}^U = 75e^{(0.16 - \frac{1}{2} \times 0.3^2) \times \frac{1}{2} + 0.3 \times \frac{1}{\sqrt{2}} \times 1.645} = 112.61.$$

Provide your **final answer only** for the following problems:

**Problem 5.4.** (2 points) A time- $T$  exchange call with underlying **S** and strike asset **Q** is always worth strictly more than an exchange put option with underlying **Q** and strike asset **S**. *True or false?*

**Solution: FALSE**

**Problem 5.5.** (2 points) A bear spread is a long position with respect to the underlying asset. *True or false?*

**Solution: FALSE**

**Problem 5.6.** (2 points) If the random variable  $X$  has the distribution function  $F_X$ , then the distribution function of the random variable  $Y = |X|$  equals

$$F_Y(y) = 2F_X(y).$$

*True or false?*

**Solution: FALSE**

**Problem 5.7.** (2 points) Let  $X_1, \dots, X_n$  be random variables with finite expectations and let  $\alpha_1, \dots, \alpha_n$  be constants. Then, we always have that

$$\mathbb{E}[\alpha_1 X_1 + \dots + \alpha_n X_n] = \sum_{i=1}^n \alpha_i \mathbb{E}[X_i].$$

*True or false?*

**Solution: TRUE**

**Problem 5.8.** (2 points) Let the stock price be modeled by a lognormal distribution. Then, the expected payoff of a European put option with exercise date  $T$  and strike  $K$  greater than or equal to  $\max(0, K - \mathbb{E}[S(T)])$ . *True or false?*

**Solution: TRUE**

**Problem 5.9.** (5 points) The random vector  $(X_1, X_2)$  is jointly normal. Its marginal distributions are:

$$X_1 \sim N(\text{mean} = 0, \text{variance} = 4), \quad X_2 \sim N(\text{mean} = 1, \text{variance} = 1).$$

The correlation coefficient is given to be

$$\text{corr}[X_1, X_2] = -0.2.$$

What is the variance of the random variable  $X = 3X_1 - 2X_2$ ?

- (a) 32.8
- (b) 47.2
- (c) 54.4
- (d) 58.2
- (e) None of the above.

**Solution: (e)**

The variance of  $X$  is

$$\begin{aligned} \text{Var}[X] &= 9\text{Var}[X_1] + 4\text{Var}[X_2] - 2(3)(2)\text{Cov}[X_1, X_2] \\ &= 9(4) + 4(1) + 12(2)(1)(0.2) = 44.8. \end{aligned}$$

**Problem 5.10.** (5 points) Let the stochastic process  $S = \{S(t); t \geq 0\}$  denote the stock price. The stock's rate of appreciation is 10% while its volatility is 0.30. Then,

- (a)  $\text{Var}[\ln(S(t))] = 0.3t$

(b)  $Var[\ln(S(t))] = 0.09t^2$

(c)  $Var[\ln(S(t))] = 0.09t$

(d)  $Var[\ln(S(t))] = 0.09$

(e) None of the above.

**Solution: (c)**

The random variable  $S(t)$  is lognormal so that the random variable  $\ln(S(t))$  is normal with variance  $0.3^2t = 0.09t$ .