UNIVERSITY OF TEXAS AT AUSTIN

Homework assignment 11

The log-normal distribution. The Black-Scholes model.

Problem 11.1. (15 points) You roll a fair tetrahedron whose sides are labeled by $1, 2, 3$, and 4 a total of 4000 times. What is the approximate probability that you see a 1 strictly more than 1025 times? There is no need to use the continuity correction.

Solution: The number of heads is $X \sim Binomial(n = 4000, p = 0.25)$. Evidently, we can use the normal approximation to the binomial. We have

$$\mu_X = \mathbb{E}[X] = 1000 \quad \text{and} \quad \sigma_X = 27.38613.$$

The probability we are seeking is

$$\mathbb{P}[X > 1025] \approx 1 - N\left(\frac{1025 - 1000}{27.38613}\right) \approx 1 - N(0.91) = 1 - 0.8186 = 0.1814.$$

Problem 11.2. (15 points) *Source: Open Course Intro to Statistics.*
Cholesterol levels for women aged 20 to 34 follow an approximately normal distribution with mean 185 milligrams per deciliter (mg/dl). Women with cholesterol levels above 220 mg/dl are considered to have high cholesterol and about 18.5% of women fall into this category. What is the standard deviation of the distribution of cholesterol levels for women aged 20 to 34?

Solution: Let X be the random variable denoting the cholesterol level. Then,

$$X \sim N(mean = 185, variance = \sigma^2).$$

We are given that

$$\mathbb{P}[X > 220] = 0.185 \quad \Rightarrow \quad \mathbb{P}[X \leq 220] = 1 - 0.185 = 0.815.$$

So,

$$220 = 185 + \sigma z_*$$

where z_* is the critical value such that $N(z_*) = 0.815$. The closest value in the standard normal tables is $z_* = 0.9$. Hence, our answers is

$$\sigma = \frac{220 - 185}{0.9} = 38.8889$$

Problem 11.3. (10 points) Suppose that the failure time (in seconds) of a certain component is modeled as lognormal random variable $Y = e^X$ such that the mean of X is -0.35 and its variance is 0.04.

What is the failure time t^* such that 95% of the components of the same type would still function after that time?

Solution: We are looking for the value t^* such that

$$\mathbb{P}[Y > t^*] = 0.95 \quad \Leftrightarrow \quad \mathbb{P}[Y \leq t^*] = 0.05.$$

The critical value z^* such that $N(z^*) = 0.05$ is -1.645. So,

$$t^* = e^{-0.35 + 0.2(-1.645)} = 0.5071.$$

Problem 11.4. (10 points) Assume the Black-Scholes model. Under the risk-neutral probability, you expect the stock price in half a year to be \$86.45. The stock's volatility is 0.30. What is the median stock price in half a year according to that same model?

Solution: In our usual notation, we have that

$$\frac{\text{mean stock price}}{\text{median stock price}} = \frac{S(0)e^{rT}}{S(0)e^{(r-\frac{\sigma^2}{2})T}} = e^{\frac{\sigma^2 T}{2}}$$

So, in this problem,

$$\text{median stock price} = \mathbb{E}[S(T)]e^{-\frac{\sigma^2 T}{2}} = 86.45e^{-\frac{0.09(1/2)}{2}} = 84.52659.$$