The University of Texas at Austin
HOMEWORK ASSIGNMENT 6

Introduction to Mathematical Statistics

February 28, 2026

Instructions: Provide your complete solution to the following problems. Final answers only, without appropriate justification, will receive zero points even if correct.

MOMENT-GENERATING FUNCTIONS.

Problem 6.1. (5 points) Let $Z_1 \sim N(1,1)$, $Z_2 \sim N(2,2)$ and $Z_3 \sim N(3,3)$ be independent random variables. The distribution of the random variable $W = Z_1 + \frac{1}{2}Z_2 + \frac{1}{3}Z_3$ is ...

a. $N(5/3, 7/6)$
b. $N(3,3)$
c. $N\left(3, \sqrt{3}\right)$
d. $N\left(3, \sqrt{5/3}\right)$
e. None of the above

(*Note:* In our notation $N(\mu, \sigma)$ means normal with mean μ and *standard deviation* σ.)

Solution. The correct answer is **(c)**.
As a linear combination of independent normals, the random variable W is normally distributed itself. To compute its parameters, we compute its mean and its variance:

$$\mathbb{E}[W] = \mathbb{E}[Z_1] + \frac{1}{2}\mathbb{E}[Z_2] + \frac{1}{3}\mathbb{E}[Z_3] = 1 + \frac{1}{2} \times 2 + \frac{1}{3} \times 3 = 3,$$
$$\text{Var }[W] = \text{Var }[Z_1] + \frac{1}{4}Var[Z_2] + \frac{1}{9}\text{ Var }[Z_3] = 1 + \frac{1}{4} \times 4 + \frac{1}{9} \times 9 = 3.$$

Therefore $W \sim N\left(3, \sqrt{3}\right)$.

Problem 6.2. (5 points) Let $Y_1, ..., Y_{100}$ be independent random variables with the Bernoulli $B(p)$ distribution, with $p = 0.2$ The best approximation to $\bar{Y} = \frac{1}{n}(Y_1 + ... + Y_n)$ (among the offered answers) is

a. $N(0,1)$
b. $N(100, 20)$
c. $N(0.2, 0.04)$
d. $N(20, 4)$
e. $N(20, 20)$

(*Note:* In our notation $N(\mu, \sigma)$ means normal with mean μ and *standard deviation* σ.)

Solution. The correct answer is **(c)**.
The sum $W = Y_1 + ... + Y_n$ is binomially distributed with mean $np = 20$ and variance $np(1-p) = 16$, i.e., standard deviation 4. It is well approximated by a normal $N(20, 4)$. Since $\bar{Y} = \frac{1}{n}W$, its best approximation will a normal with mean $\frac{1}{100}20 = 0.2$ and standard deviation $\sigma = \frac{1}{100}4 = 0.04$.

Problem 6.3. (5 points) Use the uniqueness of moment-generating functions to give the distribution of a random variable Y with moment-generating function $m_Y(t) = \left(0.7e^t + 0.3\right)^3$.

a. $Y \sim b(3, 0.7)$
b. $Y \sim b(3, 0.3)$
c. $Y \sim B(0.7)$
d. $Y \sim P(0.7)$

+None of the above.

Solution. The correct answer is **(a)**.
See *Example 6.2.2* from the lecture notes.

Problem 6.4. (10 points) The moment generating function of a certain random variable Y is given to be equal to

$$m_Y(t) = (1 - 2500t)^{-4}.$$

Calculate the standard deviation of the random variable Y.

Solution. The mean of the given distribution is

$$\mathbb{E}[Y] = m'_Y(0)$$

with

$$m'_Y(t) = -4(1 - 2500t)^{-5} \times (-2500) = 10000(1 - 2500t)^{-5}\,.$$

So, $\mathbb{E}[Y] = 10,000$.

The second raw moment of the given distribution is

$$\mathbb{E}[Y^2] = m''_Y(0)$$

with

$$m''_Y(t) = 10000 \times (-5)(1 - 2500t)^{-6}(-2500) = 125 \times 10^6(1 - 2500t)^{-6}\,.$$

So, $\mathbb{E}[Y^2] = 125 \times 10^6$.

Hence,

$$Var[Y] = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2 = 125 \times 10^6 - 100 \times 10^6 = 25 \times 10^6 \quad \Rightarrow \quad \sigma_Y = 5,000.$$

Problem 6.5. (10 points) Let Y be a geometric random variable with parameter p. What is its moment generating function m_Y? Do not forget to explicitly state the domain of m_Y!

Solution. See *Example 6.1.23* from the lecture notes.

Problem 6.6. (15 points) Let $Y \sim E(\tau)$. Find the moment generating function on Y not forgetting to explicitly state the domain. Using the moment generating function, recalculate the mean and the variance of the random variable Y.

Solution. We are given that $Y \sim E(\tau)$. Then,

$$
\begin{aligned}
m_Y(t) = \mathbb{E}[e^{tY}] &= \int_0^\infty e^{ty}\frac{1}{\tau}e^{-y/\tau}\,\mathrm{d}y \\
&= \frac{1}{\tau}\int_0^\infty e^{(t-\frac{1}{\tau})y}\,\mathrm{d}y.
\end{aligned}
$$

The indefinite integral converges only if $t < \frac{1}{\tau}$. In that case, we get

$$
m_Y(t) == \frac{1}{\tau} \times \frac{1}{t-\frac{1}{\tau}} e^{(t-\frac{1}{\tau})y}[]]_{y=0}^{\infty} = \frac{1}{1-t\tau}\,.
$$

As for the moments, we have

$$
m'_Y(t) = -\frac{-\tau}{(1-t\tau)^2} \quad \Rightarrow \quad \mathbb{E}[Y] = m'_Y(0) = \tau
$$

$$
m''_Y(t) = -\frac{-2\tau^2}{(1-t\tau)^2} \quad \Rightarrow \quad \mathbb{E}[Y^2] = m''_Y(0) = 2\tau^2 \quad \Rightarrow \quad Var[Y] = \tau^2
$$