M339D: March 31st, 2025.

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Problem Set 12

The normal approximation to the binomial.

Problem 12.1. According to the Pew research center, 64% of Americans say that social media have a mostly negative effect on things (see https://pewrsr.ch/3dsV7uR). You take a sample of 1000 randomly chosen Americans. What is the approximate probability that at least 650 of them say that social media have a mostly negative effect on things?

→: Y... a r.v. denoting the number of the surveyed people who say that social media bad

$Y \sim \text{Binomial}(n=1000, p=0.64)$

$\mathbb{P}[Y \geq 650] = ?$

Just for laughs:

$$\mathbb{P}[Y \geq 650] = \sum_{k=650}^{1000} \mathbb{P}[Y=k] = \sum_{k=650}^{1000} \binom{1000}{k} (0.64)^k (0.36)^{1000-k}$$

$$\mathbb{P}[Y \geq 650] = 1 - \mathbb{P}[Y \leq 649]$$
$$= 1 - \text{pbinom}(649, size=1000, prob=0.64)$$
$$= 0.2663257$$

$\boxed{n \cdot p = 640} \geq 10$ ✓ $n \cdot (1-p) = 360 \geq 10$ ✓

$$\mu_Y = \mathbb{E}[Y] = n \cdot p = 640$$
$$\sigma_Y = \sqrt{Var[Y]} = \sqrt{np(1-p)} = \sqrt{1000(0.64)(0.36)} = 15.17893$$

$$\mathbb{P}[Y \geq 650] = \mathbb{P}\Big[\underbrace{\boxed{\frac{Y-640}{15.17893}}}_{\approx N(0,1) \sim Z} \geq \frac{650-640}{15.17893}\Big] = \mathbb{P}[Z \geq 0.66]$$
$$= 1 - N(0.66)$$
$$= 1 - 0.7454$$
$$= 0.2546$$

Problem 12.2. According to a Gallup survey, only 22% of American young adults rate their mental health as *excellent*:
https://www.gallup.com/education/328961/why-higher-education-lead-wellbeing-revolution.aspx.

You sample 6000 randomly chosen American young adults. What is the approximate probability that at most 1400 of them rate their mental health as *excellent*?

→: Y... a r.v. denoting the # of sampled people who are "excellent"

$Y \sim Binomial(\#\text{ of trials} = 6000, \text{ success prob} = 0.22)$

$\mathbb{P}[Y \leq 1400] = ?$

Check the condition for the normal approximation:

$n \cdot p = 6000(0.22) = 1320 \geq 10$

$n(1-p) = 4680 \geq 10$ ✓

$\mu_Y = \mathbb{E}[Y] = 1320$

$\sigma_Y = \sqrt{Var[Y]} = \sqrt{1320(0.78)} = 32.08738$

$$\mathbb{P}[Y \leq 1400] = \mathbb{P}\left[\underbrace{\boxed{\frac{Y-1320}{32.08738}}}_{\approx N(0,1) \sim Z} \leq \frac{1400-1320}{32.08738}\right]$$

$= N(2.49) = 0.9936$ □

pnorm(80/sigma) = 0.99367

For fun:

pbinom(1400, 6000, 0.22) = 0.9936818

Problem 12.3. You toss a fair coin 10,000 times. What is the approximate probability that the number of *Heads* exceeds the number of *Tails* by between 200 and 500 (inclusive)?

n

→ : $Y \dots$ # of Hs

$n-Y \dots$ # of Ts

$\mathbb{P}[200 \leq Y-(n-Y) \leq 500] = ?$

$= \mathbb{P}[200 \leq 2Y - 10000 \leq 500] =$

$= \mathbb{P}[10200 \leq 2Y \leq 10500] =$

$= \mathbb{P}[5100 \leq Y \leq 5250]$

$\mu_Y = \mathbb{E}[Y] = np = 5000 = n(1-p) \geq 10$

$\sigma_Y = \sqrt{5000(1/2)} = \sqrt{2500} = 50$

$\mathbb{P}\left[\frac{5100-5000}{50} \leq \boxed{\frac{Y-\mu_Y}{\sigma_Y}} \leq \frac{5250-5000}{50}\right]$ $\approx N(0,1) \sim Z$

$\approx \mathbb{P}[2 \leq Z \leq 5] = N(5) - N(2) = 1 - 0.9772 = 0.0228$ $\square$