

M378K: November 1<sup>st</sup>, 2024.

## More on Pivotal Quantities & Confidence Intervals.

Example.

$N(\mu, 1)$

Random Sample  $Y_1, Y_2, \dots, Y_n$

1<sup>st</sup> Propose the pivotal quantity :

$$U = \frac{\bar{Y} - \mu}{\frac{1}{\sqrt{n}}} = \sqrt{n}(\bar{Y} - \mu)$$

2<sup>nd</sup> Confidence level :  $C = 0.95$

Significance level :  $\alpha = 0.05$

$$U \sim N(0, 1)$$

$$a = q_u(\alpha/2) = -1.96 =: -z^*$$

$$b = q_u(1 - \alpha/2) = 1.96 =: z^*$$

$$\mathbb{P}[a \leq U \leq b] = 1 - 0.05 = 0.95$$

$$\mathbb{P}[-z^* \leq \sqrt{n}(\bar{Y} - \mu) \leq z^*] = 0.95$$

3<sup>rd</sup>  $\mathbb{P}\left[-z^* \cdot \left(\frac{1}{\sqrt{n}}\right) \leq \bar{Y} - \mu \leq z^* \cdot \left(\frac{1}{\sqrt{n}}\right)\right] = 0.95$

$$\mathbb{P}\left[\underbrace{\bar{Y} - z^* \cdot \left(\frac{1}{\sqrt{n}}\right)}_{\hat{\theta}_L} \leq \mu \leq \underbrace{\bar{Y} + z^* \cdot \left(\frac{1}{\sqrt{n}}\right)}_{\hat{\theta}_R}\right] = 0.95$$

Example.  $Y_1, \dots, Y_n$  a random sample from  $N(\mu, \sigma_0)$   
known

$$CI = \left( \bar{Y} - z^* \cdot \frac{\sigma_0}{\sqrt{n}}, \bar{Y} + z^* \cdot \frac{\sigma_0}{\sqrt{n}} \right)$$

## M378K Introduction to Mathematical Statistics

### Problem Set #14

#### Confidence intervals.

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**Problem 14.1.** Suppose that the thumb sizes of the US males are following a normal distribution with an unknown mean  $\mu$  and standard deviation  $\sigma = 20$  on the LDI - scale (Lauretski's Digital Index - LDI - from 50 to 280). The US Department of Thumbs and Toes (DTT) reports that the mean thumb size in the country is  $\mu = 150$ . Being the chairman of the Faculty of Thumbs of the local university you see an excellent opportunity here and decide to conduct your own study of the size of the average American thumb.

- (i) After carefully collecting a random sample of 100 American thumbs you obtain the following sample mean:  $\bar{x} = 153$ . This result doesn't seem to be compatible with the DTT report so you decide to construct a 95%-confidence interval for the unknown parameter  $\mu$  based on your study. What is your confidence interval?

→ :  $z^* = 1.96$      $n = 100$

$$\mu = \bar{x} \pm z^* \cdot \left( \frac{\sigma}{\sqrt{n}} \right) = 153 \pm 1.96 \cdot \left( \frac{20}{10} \right)$$
$$= 153 \pm 3.92$$



- (ii) Now, you dream about achieving fame and fortune by being the first person ever to estimate the mean thumb size up to  $\pm 0.1$ . How large a sample size do you need for that?

Example.  $Y_1, \dots, Y_n \sim E(\tau)$

$\bar{Y} = \frac{1}{n}(\underbrace{Y_1 + \dots + Y_n})$  is not a pivotal quantity