UNIVERSITY OF TEXAS AT AUSTIN

Lecture 9
Bounds on option prices. Bull spreads

9.1. **Introduction.** As was the case with put-call parity, we can discover further properties of option prices stemming from the no-arbitrage assumption only. These properties are not going to be equalities (as was the case with put-call parity), but rather they will be inequalities which we can classify into three different categories. The first category is a set of upper and lower bounds one can obtain on European call and put prices by simply using the put-call parity and the payoff structure. The second and third categories rely on the analysis of vanilla option prices as functions of the strike price with all the other parameters/arguments fixed. Here, we will incounter monotonicty and convexity inequalities.

9.2. **Bounds on option prices.** To ensure we get the most general statements we can, let us denote by $V_C(t)$ and $V_P(t)$ the values of European calls and puts at time $t \in [0, T]$, respectively. We temporarily fix the underlying asset S, their common exercise date T and strike price K.

Due to put-call parity, which is valid at any time t, and not merely at time 0 (at which we usually use it!), we have

$$V_C(t) - V_P(t) = F_{t,T}^P(S) - Ke^{-r(T-t)}.$$

Since both the put- and the call-option prices must be nonnegative at all times, we conclude that
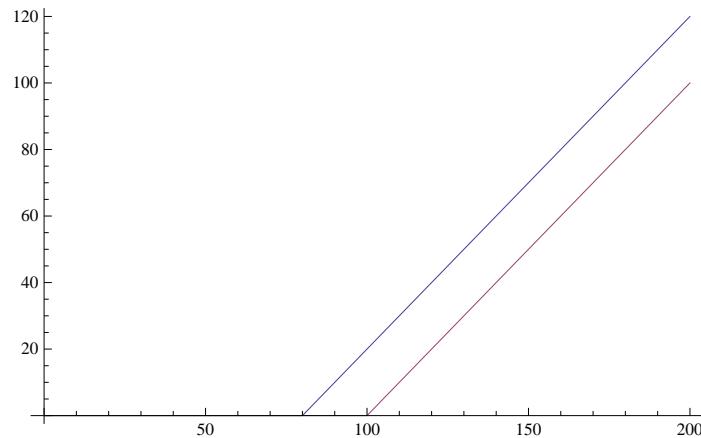
$$V_C(t) \geq (F_{t,T}^P(S) - Ke^{-r(T-t)}))_+.$$

On the other hand, trivially,

$$S(T) \geq (S(T) - K)_+ = V_C(T).$$

So, $F_{t,T}^P(S) \geq V_C(t)$, and

$$F_{t,T}^P(S) \geq V_C(t) \geq (F_{t,T}^P(S) - Ke^{-r(T-t)}))_+.$$

9.3. **Monotonicity inequalities.** Let us temporarily fix the underlying asset and the exercise date T. Consider two strike prices $K_1 < K_2$. Looking at the payoffs of the two calls with these strikes, we see that the payoff of the K_1-call dominates the payoff of the K_2-call.
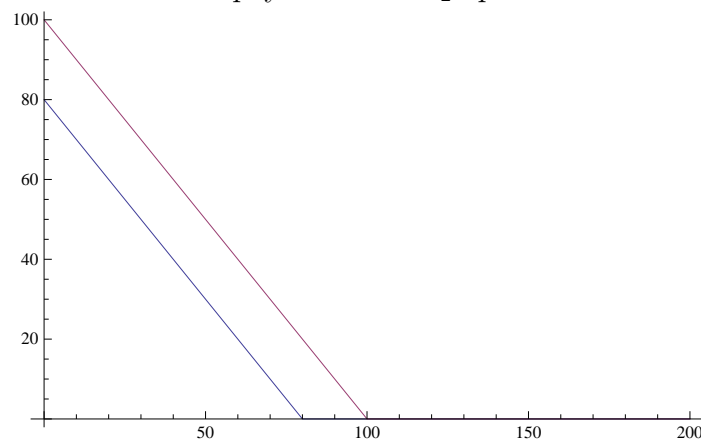
Algebraically,

$$V_C(T, K_1) = (S(T) - K_1)_+ \geq (S(T) - K_2)_+ = V_C(T, K_2).$$

Due to the no-arbitrage principle, we conclude that

$$K_1 \leq K_2 \quad \Rightarrow \quad V_C(0, K_1) \geq V_C(0, K_2).$$

So, European call prices are **decreasing** with respect to the strike price.

Similarly, considering the payoffs of the two puts with strikes K_1 and K_2, we see that the payoff of the K_1−put lies below the payoff of the K_2−put.

What is evident from the figure above can also be shown as

$$V_P(T, K_1) = (K_1 - S(T))_+ \leq (K_2 - S(T))_+ = V_P(T, K_2).$$

Again, because of the no-arbitrage principle, we get

$$K_1 \leq K_2 \quad \Rightarrow \quad V_P(0, K_1) \leq V_P(0, K_2).$$

Thus, European put prices are **increasing** with respect to the strike price.

9.4. **Spreads.** Broadly speaking, an option *spread* is a position consisting of either only calls, or only puts, in which some options are purchased and some written. There are exceptions to this "rule", but most spreads can be constructed this way.

9.4.1. *Bull spreads.* A bull spread can be constructed in multiple ways using vanilla options. One possible construction is the *call* bull spread in which one buys a call and sells another call with the same underlying asset and the expiration date T, but with a **higher** strike price. Indeed, if we denote the strike prices of the two calls by K_1 and K_2 (with $K_1 < K_2$), the payoff of the portfolio described above is

$$V_{CB}(T) = (S(T) - K_1)_+ - (S(T) - K_2)_+$$

$$= \begin{cases} 0, & \text{if } S(T) < K_1 \\ S(T) - K_1, & \text{if } K_1 \le S(T) < K_2 \\ (S(T) - K_1) - (S(T) - K_2), & \text{if } K_2 \le S(T). \end{cases}$$

$$= \begin{cases} 0, & \text{if } S(T) < K_1 \\ S(T) - K_1, & \text{if } K_1 \le S(T) < K_2 \\ K_2 - K_1, & \text{if } K_2 \le S(T). \end{cases}$$

The payoff diagram for the call bull spread looks like this:
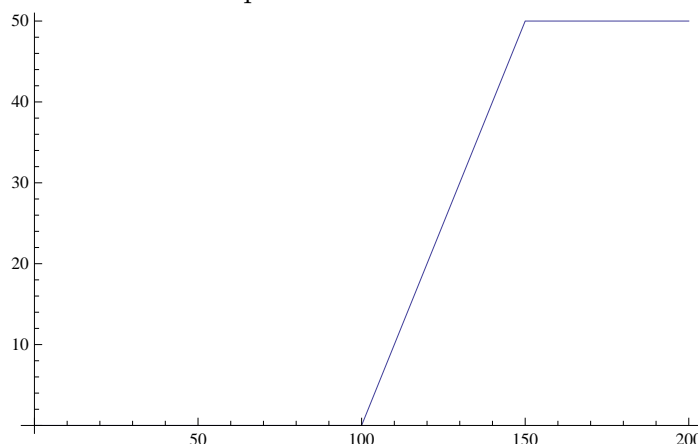
Note that the call bull spread is a **long** position with respect to the underlying asset's price – one would expect this from a position with "bull" in its name – and that it is always nonnegative.

Example 9.1. Assume that the decrease of call prices with respect to the strike price does not hold. More precisely, assume that there exists a pair of strikes such that

$$K_1 \le K_2 \quad \text{and} \quad V_C(0, K_1) < V_C(0, K_2).$$

We suspect the above situation yields an arbitrage opportunity. Indeed, if we acquire a call bull spread consisting of calls with strike K_1 and K_2, then the initial cost of this position equals $V_C(0, K_1) - V_C(0, K_2) < 0$. So, there is an initial **inflow** of money. On the other hand, the payoff at time$-T$ is always nonnegative as we have just seen. Therefore, our call bull spread is really an arbitrage portfolio.

Using put-call parity, we can show that the same arbitrage portfolio would also work if the increase in the put prices with respect to the strike price were not true.

Similarly to the above construction, one obtains a *put* bull spread by buying a put with strike K_1 and selling an otherwise identical put with strike K_2. The payoff at time$-T$ is,
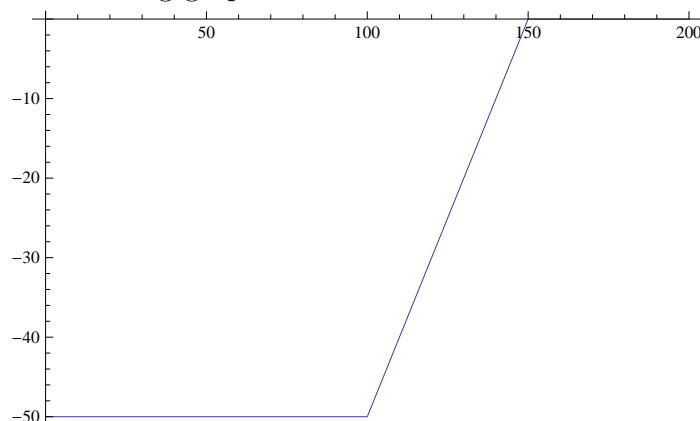
then,

$$V_{PB}(T) = (K_1 - S(T))_+ - (K_2 - S(T))_+$$

$$= \begin{cases} -(K_2 - K_1), & \text{if } S(T) < K_1 \\ -(S(T) - K_2), & \text{if } K_1 \le S(T) < K_2 \\ 0, & \text{if } K_2 \le S(T). \end{cases}$$

$$= \begin{cases} K_1 - K_2, & \text{if } S(T) < K_1 \\ S(T) - K_2, & \text{if } K_1 \le S(T) < K_2 \\ 0, & \text{if } K_2 \le S(T). \end{cases}$$

The payoff curve is similar to the one for the call bull spread but translated down by $K_2 - K_1$, i.e.,

$$V_{CB}(T) - V_{PB}(T) = K_2 - K_1,$$

as we can see from the following graph.

As was the case with the call bull spread, the payoff of the put bull spread is increasing with respect to the underlying asset's price, but this time the payoff is never strictly positive.

We began the discussion of bull spreads by stating that they can be constructed in two equivalent ways: using either only calls or only puts. However, the payoff curves we obtained are distinct. So, let us consider the profits associated with the two positions. They would be a more natural criterion for the equivalence of the two positions.

The profit of the call bull spread is obtained by taking into account the initial option prices, and it equals

$$V_{CB}(T) - FV_{0,T}(V_C(0, K_1)) + FV_{0,T}(V_C(0, K_2)).$$

Analogously, the profit of the put bull spread is initial option prices, and it equals

$$V_{PB}(T) - FV_{0,T}(V_P(0, K_1)) + FV_{0,T}(V_P(0, K_2)).$$

The difference between the two profits is

$$V_{CB}(T) - V_{PB}(T) - FV_{0,T}(V_C(0, K_1) - V_P(0, K_1)) + FV_{0,T}(V_C(0, K_2) - V_P(0, K_2)).$$

Recalling the put-call parity as well as the payoff curves above and the fact that they differ by exactly $K_2 - K_1$ always we get that the difference between the two profits equals

$$K_2 - K_1 - FV_{0,T}(F_{0,T}^P(S) - PV_{0,T}(K_1)) + FV_{0,T}(F_{0,T}^P(S) - PV_{0,T}(K_2)) = 0.$$