Risk-Neutral Pricing.

Start w/ $V(0) = \Delta \cdot S(0) + B$

$$V(0) = \frac{V_u - V_d}{\underbrace{S_u - S_d}_{\cancel{S(0)}(u-d)}} \cdot \cancel{S(0)} + e^{-rh} \cdot \frac{u \cdot V_d - d \cdot V_u}{u-d}$$

$$V(0) = \frac{1}{u-d}\left[V_u - V_d + e^{-rh}(u \cdot V_d - d \cdot V_u)\right]$$

$$V(0) = e^{-rh} \cdot \frac{1}{u-d}\left[e^{rh} \cdot V_u - e^{rh} \cdot V_d + u \cdot V_d - d \cdot V_u\right]$$

$$V(0) = e^{-rh} \cdot \frac{1}{u-d}\left[V_u(e^{rh} - d) + V_d(u - e^{rh})\right]$$

$$V(0) = e^{-rh} \cdot \left[V_u \cdot \boxed{\frac{e^{rh}-d}{u-d}} + V_d \cdot \boxed{\frac{u-e^{rh}}{u-d}}\right]$$

"p^*" "$1-p^*$"

Both positive (Due to the no-arbitrage condition!)

Add up to 1!

We choose to interpret the two fractions as probabilities!

We define the risk-neutral probability of the stock price going up in a single period as:

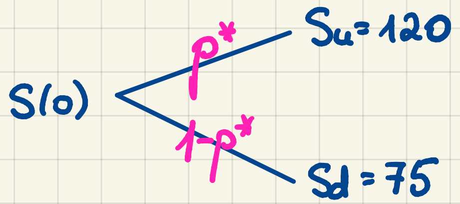
$$\boxed{p^* := \frac{e^{rh}-d}{u-d}}$$

⇒ The risk-neutral pricing formula:

$$\boxed{V(0) = e^{-rT}\left[V_u \cdot p^* + V_d(1-p^*)\right]}$$

We can generalize this principle: $\boxed{V(0) = e^{-rT} \mathbb{E}^*[V(T)]}$

Problem 9.5. [revisited]



$V_u = 20$

$V_d = 25$

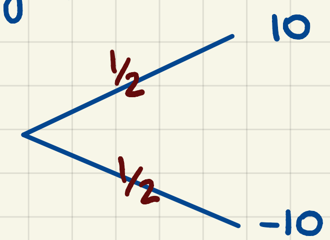
w/ $p^* = \dfrac{e^{rh} - d}{u-d} = \dfrac{S(0)e^{rh} - S_d}{S_u - S_d} = \dfrac{95e^{0.06} - 75}{120-75} = \underline{0.5749}$

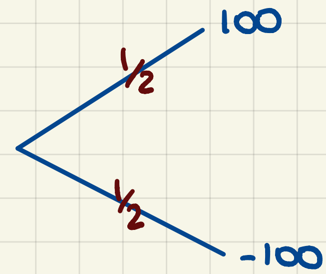
$V(0) = e^{-rT}[V_u \cdot p^* + V_d(1-p^*)]$

$V(0) = e^{-0.06}[20 \cdot p^* + 25(1-p^*)] = \underline{20.837}$ $\square$

Q: Why "risk·neutral"?

Imagine bets

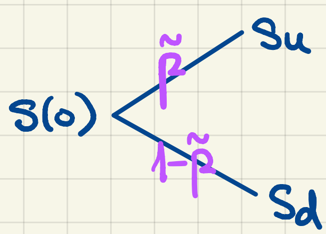




Consider a risk·neutral investor. We will find the probability $\tilde{p}$ such that they are indifferent between investing in the stock and the risk-free investment.

Say, they start w/ $S(0)$. If they invest @ the ccrfir r, then, their balance @ time h will be $S(0)e^{rh}$.

If they invest in the stock:



$\mathbb{E}[\text{Wealth}] = \mathbb{E}[S(h)] =$

$= \tilde{p} \cdot S_u + (1-\tilde{p}) \cdot S_d$ $=$

$= \tilde{p} \cdot S(0) \cdot u + (1-\tilde{p}) \cdot S(0) \cdot d$

$\tilde{p} = ?$ w/ $\cancel{S(0)}e^{rh} = \tilde{p} \cdot \cancel{S(0)} \cdot u + (1-\tilde{p}) \cdot \cancel{S(0)} \cdot d$

$$\tilde{p} \cdot u + (1-\tilde{p})d = e^{rh}$$

$$\tilde{p} \cdot u + d - \tilde{p}d = e^{rh}$$

$$\tilde{p}(u-d) = e^{rh} - d$$

$$\boxed{\tilde{p} = \frac{e^{rh}-d}{u-d} = p^*}$$

Special Case: Forward Binomial Tree

$\boxed{\sigma \ldots \text{volatility}}$

$u := e^{rh+\sigma\sqrt{h}}$

$d := e^{rh-\sigma\sqrt{h}}$

The risk-neutral probability:

$$p^* = \frac{e^{rh}-d}{u-d} = \frac{e^{rh} - e^{rh-\sigma\sqrt{h}}}{e^{rh+\sigma\sqrt{h}} - e^{rh-\sigma\sqrt{h}}} = \frac{\cancel{e^{rh}}\left(1-e^{-\sigma\sqrt{h}}\right)}{\cancel{e^{rh}}\left(e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}\right)}$$

$$p^* = \frac{1-e^{-\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}} \cdot \frac{e^{\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}}} = \frac{\cancel{e^{\sigma\sqrt{h}}-1}}{\underbrace{e^{2\sigma\sqrt{h}}-1}_{\cancel{(e^{\sigma\sqrt{h}}-1)}(e^{\sigma\sqrt{h}}+1)}}$$

$$\boxed{p^* = \frac{1}{1+e^{\sigma\sqrt{h}}}}$$

The shortcut only for the FORWARD BINOMIAL TREE.

$$\xrightarrow{h \to 0} \frac{1}{2}$$