M339W: April 7th, 2021.

Implied Volatility.

- We can observe call/put prices in our market.
- Assume the Black-Scholes model

$\Rightarrow$ we do have nice formulae for call/put prices with the arguments $(s, t, r, \delta, \sigma)$

Say that the value of the option at a particular time is

$$\boxed{v(s, t, r, \delta, \sigma)}$$

✓ ✓ ✓ ✓

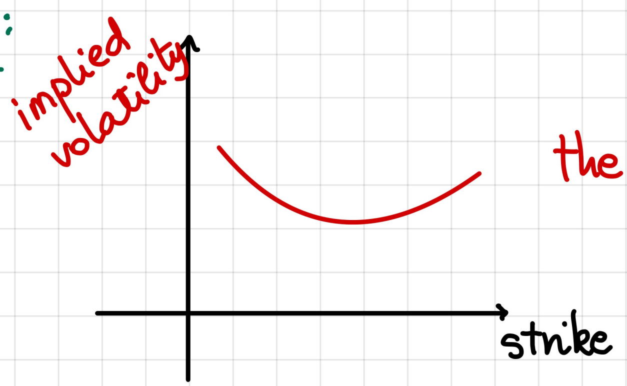
Assume they are given/observed.

Consider our pricing formula as a function of the volatility σ.

We invert the Black-Scholes pricing formula and get the σ which is called the IMPLIED VOLATILITY.

Theoretically: If all of the above assumptions are true, the observed call prices for varying strikes K would give us the same value of the implied volatility.

Practically:



the effect is called the "volatility smile".

$T=1$

17. Assume the Black-Scholes framework. Consider a one-year at-the-money European put option on a nondividend-paying stock.

$\delta = 0$

You are given:

(i) The ratio of the put option price to the stock price is less than 5%.

(ii) Delta of the put option is –0.4364.

(iii) The continuously compounded risk-free interest rate is 1.2%.

Determine the stock's volatility.

(A) 12%

(B) 14%

(C) 16%

(D) 18%

(E) 20%

(ii) $\Delta_p(S(0),0) = -0.4364$

$= -\cancel{e^{-\delta(T)}} \cdot N(-d_1(S(0),0))$

$= -N(-d_1(S(0),0))$

$\Rightarrow \boxed{N(-d_1(S(0),0)) = 0.4364}$

$\Rightarrow N(d_1(S(0),0)) = 1 - 0.4364 = 0.5636$

$\Rightarrow \boxed{d_1(S(0),0)} = N^{-1}(0.5636) = 0.16$

$\| $

$\frac{1}{\sigma\sqrt{T}}\left[\cancel{\ln\left(\frac{S(0)}{K}\right)} + \left(r - \cancel{\delta} + \frac{\sigma^2}{2}\right)\cancel{T}\right]$

$\delta = 0$

at the money

$\frac{1}{\sigma}\left(r + \frac{\sigma^2}{2}\right) = 0.16 \quad / \cdot \sigma$

$0.012 + \frac{\sigma^2}{2} = 0.16 \cdot \sigma \quad / \cdot 2$

$\sigma^2 - 0.32\sigma + 0.024 = 0$

$\Rightarrow \quad \sigma_1 = 0.12 \quad$ and $\quad \sigma_2 = 0.2$

(i) $\frac{v_p(S(0),0)}{S(0)} < 0.05$

no dividends : $\delta = 0$

$$\frac{\cancel{K}e^{-r\cdot T}\cdot N(-d_2(S(0),0)) - \cancel{S(0)}\cdot N(-d_1(S(0),0))}{\cancel{S(0)}} < 0.05$$

at-the-money
$S(0)=K$

$$e^{-r}\cdot N(-d_2(S(0),0)) - \underbrace{N(-d_1(S(0),0))}_{0.4364} < 0.05$$

$r=0.012$

$$e^{-r} N(-d_2(S(0),0)) < 0.4864$$

$$N(-d_2(S(0),0)) < e^{0.012}\cdot 0.4864$$

Remember: $d_2 = d_1 - \sigma\sqrt{T}$ (in general)

$\Rightarrow$ In this problem : $d_2(S(0),0) = 0.16 - \sigma$

Test: $N(\sigma - 0.16) < e^{0.012}\cdot 0.4864$

w/ $\sigma_1 = 0.12$ and $\sigma_2 = 0.2$

Finally : choose $\sigma_1 = 0.12$

Example. Consider:

- ✓ an at-the-money call/put with $r=\delta$; or
- ✓ a call/put on a non-dividend-paying stock ; or
  w/ strike $K = S(0)e^{r\cdot T}$
- any choice of given values such that
  $F^P_{0,T}(S) = PV_{0,T}(K)$ (*)

If this is the case:

$$d_1(S(0),0) = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right)\cdot T\right]$$

$$= \frac{1}{\sigma\sqrt{T}}\left[\underbrace{\cancel{\ln\left(\frac{S(0)e^{-\delta\cdot T}}{Ke^{-r\cdot T}}\right)}}_{=0} + \frac{\sigma^2}{2}\cdot T\right] = \frac{\sigma\sqrt{T}}{2}$$

$$\Rightarrow \quad d_2(S(0),0) = d_1(S(0),0) - \sigma\sqrt{T} = \frac{\sigma\sqrt{T}}{2} - \sigma\sqrt{T} = -\frac{\sigma\sqrt{T}}{2}$$

$\Rightarrow$ The call price is

$$v_c(S(0),0) = \underline{S(0)e^{-\delta\cdot T}} \cdot N(d_1(S(0),0)) - \underline{Ke^{-rT}} \cdot N(d_2(S(0),0))$$

$$= S(0)e^{-\delta\cdot T}\left(N\left(\frac{\sigma\sqrt{T}}{2}\right) - N\left(-\frac{\sigma\sqrt{T}}{2}\right)\right)$$

$$= S(0)e^{-\delta\cdot T}\left(2N\left(\frac{\sigma\sqrt{T}}{2}\right) - 1\right)$$

Given the price of the call, we can invert the price function to get the implied volatility. $\blacksquare$