Name:

M339D=M389D Introduction to Actuarial Financial Mathematics
University of Texas at Austin
In-Term Exam II
Instructor: Milica Čudina

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Signature:

The maximum number of points on this exam is 80. Yes, you can get up to 10 "extra" points on the exam.

Provide your **complete solution** to the following problems. Final answers only, without appropriate justification, will receive zero points even if correct.

2.1. **DEFINITIONS.**

Problem 2.1. (10 points) Write the definition of an **arbitrage portfolio**.

Problem 2.2. (5 points) Write the definition of a **replicating portfolio** of a European option.

2.2. **TRUE/FALSE QUESTIONS.**

Problem 2.3. (2 points) Assume that the continuously compounded, risk-free interest rate is strictly positive. The forward price of a non-dividend-paying stock is always strictly increasing with respect to the delivery date. *True or false? Why?*

Solution: TRUE
The forward price is $F_{0,T} = S(0)e^{rT}$ as established in class.

Problem 2.4. (2 points) You are using a binomial asset-pricing model to model the evolution of the price of a particular stock. Then, the Δ in the replicating portfolio of a single call option on that stock never exceeds 1. *True or false? Why?*

Solution: TRUE
The call's Δ will always be between 0 and 1.

Problem 2.5. (2 points) You are using a binomial asset-pricing model to model the evolution of the price of a particular stock. Then, the Δ in the replicating portfolio of a single put option on that stock is between -1 and 0. *True or false? Why?*

Solution: TRUE
The puts's Δ will always be between -1 and 0.

Problem 2.6. (2 points) You are using a binomial asset-pricing model to model the evolution of the price of a particular stock. Then, the risk-free component in the replicating portfolio of a single put option on that stock should be interpreted as lending. *True or false? Why?*

Solution: TRUE
The put's B will always be positive and should be intrpreted as lending.

Problem 2.7. (2 points) You are using a binomial asset-pricing model to model the evolution of the price of a particular stock. Then, the risk-free component in the replicating portfolio of a single call option on that stock should be interpreted as lending. *True or false? Why?*

Solution: FALSE
The call's B will always be negative and should be intrpreted as borrowing.

2.3. **FREE-RESPONSE PROBLEMS.**

Problem 2.8. (5 points) A portfolio consists of the following:

- one **short** one-year, $50-$strike call option with price equal to \$8.50,
- one **long** one-year, $60-$strike put option with price equal to \$6.75

All of the options are European and with the same underlying asset.

Assume that the continuously compounded, risk-free interest rate equals 0.04.

What is the portfolio's profit if the final price of the underlying asset equals \$55?

Solution:

$$-(55-50)_+ + (60-55)_+ + (8.50-6.75)e^{0.04} = 1.82$$

Problem 2.9. (10 points) A derivative security has the payoff function given by

$$v(s) = \begin{cases} 10 & \text{if } s < 90 \\ 0 & \text{if } 90 \leq s < 100 \\ 20 & \text{if } 100 \leq s \end{cases}$$

Its exercise date is in one year. You model the time-1 price of the underlying asset as

$$S(1) \sim \begin{cases} 85 & \text{with probability } 1/4 \\ 95 & \text{with probability } 1/2 \\ 105 & \text{with probability } 1/4 \end{cases}$$

What is the expected payoff of the above derivative security?

Solution:

$$10\left(\frac{1}{4}\right) + 20\left(\frac{1}{4}\right) = \frac{30}{4} = 7.5$$

Problem 2.10. (5 points) Consider a non-dividend-paying stock whose current price equals \$50 per share. A pair of one-year European calls on this stock with strikes of \$40 and \$50 is available in the market for the observed prices of \$2 and \$4, respectively.

The continuously-compounded, risk-free interest rate is given to be 10%.

George suspects that there exists an arbitrage portfolio in the above market consisting of the following components:

- the **long** \$40-strike call,
- the **written** \$50-strike call.

What is the minimum **gain** from this suspected arbitrage portfolio?

Solution: The initial cost of this portfolio is $2-4=-2$. The lower bound on the payoff is zero. The lower bound on the gain is, hence,

$$2e^{0.1} = 2.21$$

Problem 2.11. (10 points) Let the current price of a market index be \$80. Consider a European six-month, at-the-money call option on this market index.

We model the price of the market index in half a year as follows:

$$S(1/2) \sim \begin{cases} 78 & \text{with probability } 1/6 \\ 82 & \text{with probability } 1/2 \\ 84 & \text{with probability } 1/3 \end{cases}$$

What is the expected payoff of this call option?

Solution: Since the option is at-the-money, the strike price is \$80. We have

$$V_C(1/2) = (S(1/2) - 80)_+ \sim \begin{cases} 0 & \text{with probability } 1/6 \\ 2 & \text{with probability } 1/2 \\ 4 & \text{with probability } 1/3 \end{cases}$$

So,

$$\mathbb{E}[V_C(T)] = 2\left(\frac{1}{2}\right) + 4\left(\frac{1}{3}\right) = \frac{7}{3}.$$

Problem 2.12. (10 points) Let the current price of a non-dividend-paying stock be \$40. A market maker writes a \$38-strike, three-month call option on this stock. The option's price is \$2.72. The market-maker simultaneously buys one share of the underlying stock.

The continuously compounded, risk-free interest rate is 0.04.

For which final value of the stock price will the market maker break even?

Solution: The initial cost of the portfolio is $40 - 2.72 = 37.28$. This is a covered call, so the expression for the payoff is, in our usual notation,

$$-(S(T) - K)_+ + S(T) = \min(S(T), K).$$

In this problem, the payoff function for the portfolio is, therefore, $v(s) = min(s, 38)$. We need to solve for s in

$$\min(s, 38) - 37.28e^{0.04/4} = 0 \quad \Rightarrow \quad \min(s, 38) = 37.65467 \quad \Rightarrow \quad s = 37.65467.$$

Problem 2.13. (5 points) A non-dividend-paying stock is currently priced at \$100 per share. One-year, \$102-strike European call and put options on this stock have equal prices. What is the continuously-compounded annual risk-free rate of interest?

Solution: By put-call parity, we have

$$K = F_{0,T}(S) = S(0)e^{rT} \quad \Rightarrow \quad r = \frac{1}{T}\ln\left(\frac{K}{S(0)}\right).$$

So,

$$r = \frac{1}{T}\ln\left(\frac{K}{S(0)}\right) = \ln(102/100) = 0.01980263.$$

Problem 2.14. (15 points) Consider a non-dividend-paying stock whose current price is \$90 per share. You model the evolution of this stock price over the following year using a one-period binomial tree under the assumption that the stock price can be either \$100, or \$85 in one year.

The continuously compounded, risk-free interest rate is 0.05.

The price of a $K-$strike, one-year European straddle on the above stock, consistent with the above stock-price model, is 6.50. How much is K?

Solution: The risk-neutral probability of an up movement is

$$p^* = \frac{90e^{0.05} - 85}{100 - 85} = 0.6409599.$$

So, the price of our straddle satisfies one of the following three cases.

Case #1. K is between 85 and 100.

$$V(0) = e^{-0.05}[p^*(100 - K) + (1 - p^*)(K - 85)] = 6.50.$$

We solve for K in the following equation:

$$\begin{aligned} p^*(100 - K) + (1 - p^*)(K - 85) = 6.50e^{0.05} \quad &\Rightarrow \quad 100p^* - p^*K + (1 - p^*)K - 85(1 - p^*) = 6.50e^{0.05} \\ &\Rightarrow \quad (1 - 2p^*)K = 6.50e^{0.05} - 100p^* + 85(1 - p^*) \end{aligned}$$

Finally,

$$K = \frac{6.50e^{0.05} - 100p^* + 85(1 - p^*)}{1 - 2p^*} = 94.86499.$$

Case #2. K is greater than 100.

$$V(0) = e^{-0.05}[p^*(K - 100) + (1 - p^*)(K - 85)] = 6.50.$$

We solve for K in the following equation:

$$\begin{aligned} p^*(K - 100) + (1 - p^*)(K - 85) = 6.50e^{0.05} \quad &\Rightarrow \quad p^*K - 100p^* + (1 - p^*)K - 85(1 - p^*) = 6.50e^{0.05} \\ &\Rightarrow \quad K = 6.50e^{0.05} + 100p^* + 85(1 - p^*) = 101.4477 \end{aligned}$$

Case #3. K is smaller than 85. This case does not yield any acceptable solutions.

2.4. MULTIPLE-CHOICE QUESTIONS.

Problem 2.15. (5 points) Consider a one-year, \$45-strike European call option and a one-year, \$55-strike European put option on the same underlying asset. You observe that the time-0 stock price equals \$40 while the time-1 stock price equals \$50. Then,

(a) both of the options are out-of-the-money at expiration.
(b) both of the options are in-the-money at expiration.
(c) the call is out-of-the-money and the put is in-the-money at expiration.
(d) the put is out-of-the-money and the call is in-the-money at expiration.
(e) both options are at-the-money at expiration.

Solution: (b)