

M378K Introduction to Mathematical Statistics

Problem Set #3

Named discrete random variables.

Problem 3.1. Source: Sample P exam, Problem #125.

An electronic system contains three cooling components that operate independently. The probability of each component's failure is 0.05. The system will overheat if and only if at least two components fail. Calculate the probability that the system will overheat.

Solution: Let us denote the number of components that fail by N . Since the components operate independently, and their failure probabilities are equal, we can model N using a binomial distribution with the number of trials equal to the number of components (i.e., $n = 3$) and the common "success" probability equal to the probability of failure (i.e., $p = 0.05$). In short,

$$N \sim b(n = 3, p = 0.05)$$

The probability that the system overheats is

$$\begin{aligned}\mathbb{P}[N \geq 2] &= \mathbb{P}[N = 2] + \mathbb{P}[N = 3] \\ &= \binom{3}{2}(0.05)^2(0.95) + \binom{3}{3}(0.05)^3 = 0.00725.\end{aligned}$$

Problem 3.2. Source: Sample P exam, Problem #462.

Each person in a large population independently has probability p of testing positive for diabetes where $0 < p < 1$. People are tested for diabetes, one person at a time, until a test is positive. Individual tests are independent. Determine the probability that m or fewer people are tested, given that n or fewer people are tested, where $1 \leq m \leq n$.

Solution: Let N' be the total number of people tested. This is a **shifted** geometric random variable with parameter p , i.e., we have that

$$N = N' - 1 = g(p)$$

We need to figure out the following probability:

$$\begin{aligned}\mathbb{P}[N' \leq m \mid N' \leq n] &= \mathbb{P}[N + 1 \leq m \mid N + 1 \leq n] \\ &= \mathbb{P}[N \leq m - 1 \mid N \leq n - 1] \\ &= \frac{\mathbb{P}[N \leq m - 1, N \leq n - 1]}{\mathbb{P}[N \leq n - 1]} \\ &= \frac{\mathbb{P}[N \leq m - 1]}{\mathbb{P}[N \leq n - 1]} \\ &= \frac{1 - \mathbb{P}[N > m - 1]}{1 - \mathbb{P}[N > n - 1]} \\ &= \frac{1 - q^m}{1 - q^n}.\end{aligned}$$

Problem 3.3. Source: Sample P exam, Problem #355.

The number of calls received by a certain emergency unit in a day is modeled by a Poisson distribution with parameter 2. Calculate the probability that on a particular day the unit receives at least two calls.

Solution: Let N denote the number of calls. We are told that $N \sim P(\lambda = 2)$. So, we have

$$\mathbb{P}[N \geq 2] = 1 - \mathbb{P}[N \leq 1] = 1 - p_0 - p_1 = 1 - e^{-\lambda} + e^{-\lambda}\lambda = 1 - 3e^{-2}.$$