University of Texas at Austin

Problem Set 14

Black-Scholes pricing.

Problem 14.1. Let the stock prices be modeled using the lognormal distribution. Under the risk-neutral probability measure, the mean stock price at time-1 equals 120 and the median stock price 115. What is the risk-neutral probability that the time-1 stock price exceeds 100?

Solution: The stock price at time-1 is lognormally distributed. In fact, using our usual parameters, we can rewrite it as

$$S(1) = S(0)e^{(r-\frac{1}{2}\sigma^2)+\sigma Z}.$$

Recall that the median of $S(1)$ equals $S(0)e^{(r-\frac{1}{2}\sigma^2)}$. So, the required probability can be expressed as

$$\begin{aligned}\mathbb{P}[S(1) > 100] = \mathbb{P}[115e^{\sigma Z} > 100] &= \mathbb{P}\left[Z > \frac{1}{\sigma}\ln\left(\frac{100}{115}\right)\right]\\ &= \mathbb{P}\left[Z < \frac{1}{\sigma}\ln\left(\frac{115}{100}\right)\right] = N\left(\frac{1}{\sigma}\ln\left(\frac{115}{100}\right)\right).\end{aligned}$$

Since the mean of $S(1)$ equals $S(0)e^{(\alpha-\delta)}$, we have

$$e^{\frac{1}{2}\sigma^2} = \frac{120}{115} \quad \Rightarrow \quad \sigma = \sqrt{2\ln(1.04348)} = 0.2918.$$

So, our final answer is

$$\mathbb{P}[S(1) > 100] = N(0.48) = 0.6844.$$

Problem 14.2. (5 pts) Let the stochastic process $S = \{S(t); t \geq 0\}$ denote the stock price. The stock's rate of appreciation is 10% while its volatility is 0.30 Then,

(a) $Var[\ln(S(t))] = 0.3t$
(b) $Var[\ln(S(t))] = 0.09t^2$
(c) $Var[\ln(S(t))] = 0.09t$
(d) $Var[\ln(S(t))] = 0.09$
(e) None of the above.

Solution: (c)
The random variable $S(t)$ is lognormal so that the random variable $\ln(S(t))$ is normal with variance $0.3^2t = 0.09t$.

Problem 14.3. (5 pts) Assume the Black-Scholes framework. Let the current price of a non-dividend-paying stock be equal to $S(0) = 95$ and let its volatility be equal to 0.35. Consider a European call on that stock with strike 100 and exercise date in 9 months. Let the risk-free continuously compounded interest rate be 6% per annum.

Denote the price of the call by $V_C(0)$. Then,

(a) $V_C(0) < \$5.20$
(b) $\$5.20 \leq V_C(0) < \$7.69$
(c) $\$7.69 \leq V_C(0) < \$9.04$
(d) $9.04 \leq V_C(0) < \$11.25$
(e) None of the above.

Solution: (d)
Using the Black-Scholes formula one gets the price of about 11.06.

Problem 14.4. Assume the Black-Scholes setting. Let $S(0) = \$63.75, \sigma = 0.20, r = 0.055$. The stock pays no dividend and the option expires in 50 days (simplify the number of days in a year to 360).

What is the price of a \$60-strike European put?

(a) 0.66
(b) 0.55
(c) 0.44
(d) 0.37
(e) None of the above.

Solution: (d)
In our usual notation, the price is

$$V_P(0) = Ke^{-r\cdot T}N(-d_2) - S(0)N(-d_1)$$

with

$$\begin{aligned} d_1 &= \frac{1}{0.2\sqrt{5/36}}\left(\ln\left(\frac{63.75}{60}\right) + (0.055 + \frac{1}{2}0.2^2)\left(\frac{5}{36}\right)\right) = 0.95, \\ d_2 &= d_1 - 0.25\sqrt{0.125} = 0.88. \end{aligned}$$

So,

$$V_P(0) = 60e^{-0.055\cdot\frac{5}{36}}(1 - 0.8106) - 63.75\cdot(1 - 0.8289) = 0.37.$$