UNIVERSITY OF TEXAS AT AUSTIN

Problem set 4

Expectation. Moments. Coefficient of Variation.

Problem 4.1. Let X be a random variable with a finite expectation. Consider the function

$$g(a) = \mathbb{E}[(X-a)^2]$$

defined for all a such that the expectation exists. For which value a does the function g attain its minimum?

Solution: We can expand the expectation in the definition of g as follows:

$$g(a) = \mathbb{E}[X^2] - 2a\mathbb{E}[X] + a^2.$$

This is a quadratic in a with the positive leading term, and we can find its minimum through differentiation. We get

$$g'(a) = -2\mathbb{E}[X] + 2a.$$

Equating the above to zero, we obtain that the minimizer of g is $\mathbb{E}[X]$.

Problem 4.2. *Source: "Probability and Statistical Inference" by Hogg, Tanis, and Zimmerman.* An insurance agent receives a bonus if the loss ratio L on the business is less than 0.5 where L is the total losses X divided by the total premiums where the total premiums are equal exactly to 3. The bonus equals

$$\frac{0.5-L}{10} \tag{4.1}$$

if it occurs (and it is, obviously, zero otherwise). Let X (in 100K) have the probability density function

$$f_X(x) = 3x^{-4}, \quad x > 1.$$

What is the expected value of the bonus?

Solution: By definition, the expected value of the bonus, in 100K, is

$$\mathbb{E}\left[\left(\frac{0.5-L}{10}\right)\mathbb{I}_{[L<0.5]}\right] = \frac{1}{10}\mathbb{E}\left[(0.5-L)\,\mathbb{I}_{[L<0.5]}\right] = \frac{1}{10}\mathbb{E}\left[\left(0.5-\tfrac{X}{3}\right)\mathbb{I}_{[X/3<0.5]}\right] = \frac{1}{30}\mathbb{E}\left[(1.5-X)\,\mathbb{I}_{[X<1.5]}\right]$$

Now, we integrate

$$\begin{aligned}\mathbb{E}\left[(1.5-X)\,\mathbb{I}_{[X<1.5]}\right] &= 3\int_1^{1.5}(1.5-x)x^{-4}\,dx = 4.5\int_1^{1.5}x^{-4}\,dx - 3\int_1^{1.5}x^{-3}\,dx \\ &= -\frac{4.5}{3}\left[x^{-3}\right]_{x=1}^{1.5} + \frac{3}{2}\left[x^{-2}\right]_{x=1}^{1.5} = -1.5(1.5^{-3}-1) + 1.5(1.5^{-2}-1) = 0.2222222.\end{aligned}$$

Hence, the expected total value of the bonus is

$$\frac{0.2222222}{30}(100000) \approx 740.74.$$

Problem 4.3. The manufacturer claims that the lifetime of an espresso machine is uniform between 0 and 4. The coffee shop replaces the machine either at the time of failure or at time 3, whichever occurs first. What is the variance of the replacement time?

Solution: Let the lifetime of the machine be $T \sim U(0,4)$. We need to calculate $\mathrm{Var}[T \wedge 3]$. We have

$$\begin{aligned}\mathbb{E}[T\wedge 3] &= \int_0^3 t\left(\frac{1}{4}\right)dt + \int_3^4 3\left(\frac{1}{4}\right)dt = \frac{1}{4}\left[\frac{t^2}{2}\right]_{t=0}^3 + \frac{3}{4} = \frac{1}{4}\left(\frac{9}{2}\right) + \frac{3}{4} = \frac{15}{8}\,, \\ \mathbb{E}[(T\wedge 3)^2] &= \int_0^3 t^2\left(\frac{1}{4}\right)dt + \int_3^4 3^2\left(\frac{1}{4}\right)dt = \frac{1}{4}\left[\frac{t^3}{3}\right]_{t=0}^3 + \frac{9}{4} = \frac{9}{4} + \frac{9}{4} = \frac{9}{2}.\end{aligned}$$

So, the variance is

$$\mathrm{Var}[T \wedge 3] = \frac{9}{2} - \left(\frac{15}{8}\right)^2 = 0.984375.$$

Problem 4.4. Consider two independent random variables X and Y which have the same mean. You are given that coefficient of variation of X equals 5 and the coefficient of variation of Y equals 12. What is the coefficient of variation of the sum of X and Y?

Solution: Let $\mu = \mathbb{E}[X] = \mathbb{E}[Y]$. Then, $\sigma_X = SD[X] = 5\mu$ and $\sigma_Y = SD[Y] = 12\mu$. TDue to their independence, the variance of the sum of the two random variables is

$$Var\,[X+Y] = Var[X] + Var[Y].$$

In terms of μ, the variance of the sum can be rewritten as

$$Var\,[X+Y] = 25\mu^2 + 144\mu^2 = 169\mu^2.$$

So, the standard deviation of the sum can be expressed as 13μ. Hence, the coefficient of variation of the sum equals $13/2$.