UNIVERSITY OF TEXAS AT AUSTIN

# Quiz #19

Please, provide your **complete solutions** to the following questions:

**Problem 19.1.** (5 points) You want to fit to the observed values

$$4, 5, 7$$

a two-parameter Pareto distribution with parameters $\alpha = 4$ and $\theta$ unknown using maximum likelihood estimation. Write down **clearly** an **explicit** expression for the loglikelihood function (of course, as a function of $\theta$).

**Solution:** Note that we are dealing with complete, individual data. We get the likelihood and loglikelihood functions:

$$L(\theta) = \prod_{j=1}^{n} f_{X_j}(x_j|\theta),$$
$$l(\theta) = \sum_{j=1}^{n} \ln[f_{X_j}(x_j|\theta)],$$

where $n = 3$ is the number of observations, and $x_1 = 4, x_2 = 5, x_3 = 7$ (as given in the problem) while $f_{X_j}(\cdot\,|\,\theta)$ denotes the density function of the Pareto distribution with parameters $\alpha = 4$ and $\theta$ (unknown) for $j = 1, 2, 3$. So, using our tables, we get

$$\begin{aligned} l(\theta) &= \sum_{j=1}^{3} \ln[\frac{\alpha\theta^{\alpha}}{(x_j+\theta)^{\alpha+1}}] \\ &= \sum_{j=1}^{3} [\ln(\alpha) + \alpha\ln(\theta) - (\alpha+1)\ln(x_j+\theta)] \\ &= 3(\ln(4) + 4\ln(\theta)) - (\alpha+1)[\ln(4+\theta) + \ln(5+\theta) + \ln(7+\theta)]\,. \end{aligned}$$

**Problem 19.2.** (10 points) Consider the following individual observed values:

$$5, 8, 10$$

of a random variable $Y$ such that $Y = X^{-1}$ with $X \sim Gamma(\alpha = 2, \theta)$.

Calculate $\hat{\theta}_{MLE}$, the Maximum Likelihood Estimate of $\theta$ based on the above observed values.

**Solution:** The above observations of $Y$ give us the corresponding observations of $X$:

$$1/5, 1/8, 1/10.$$

In our usual notation, the likelihood function is

$$\begin{aligned} L(\theta) &= \frac{(1/(5\theta))^2 e^{-1/(5\theta)}}{(1/5)\Gamma(2)} \cdot \frac{(1/(8\theta))^2 e^{-1/(8\theta)}}{(1/8)\Gamma(2)} \cdot \frac{(1/(10\theta))^2 e^{-1/(10\theta)}}{(1/10)\Gamma(2)} \\ &\propto \theta^{-6} e^{-(1/(5\theta)+1/(8\theta)+1/(10\theta))}, \end{aligned}$$

where we decided to use proportionality so that we do not have to write the part of the likelihood function which does not depend on $\theta$.

Taking the natural logarithm of the final expression above, we get that the loglikelihood can be written as

$$l(\theta) = C + (-6)\ln(\theta) - (1/(5\theta) + 1/(8\theta) + 1/(10\theta))$$

with $C$ being a constant which may depend on the observed values or the given value $\alpha = 2$, but does not depend on $\theta$. Then,

$$l'(\theta) = -\frac{6}{\theta} + \frac{1}{5\theta^2} + \frac{1}{8\theta^2} + \frac{1}{10\theta^2}.$$

Setting the above equal to zero and solving for $\theta$, we get

$$\hat{\theta} = \frac{1/5 + 1/8 + 1/10}{6} = \frac{17}{240}.$$