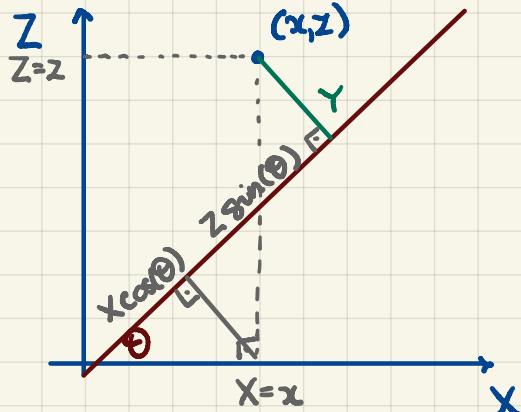


M339G: March 20th, 2024.

Bivariate Normal [cont'd].

Standard.

Start w/ a pair of independent, standard normal random variables. Say, X and Z .



$$\Rightarrow Y = X \cdot \cos \theta + Z \cdot \sin \theta$$

We know: Y is normally dist'd.

$$\begin{aligned} E[Y] &= E[X \cdot \cos \theta + Z \cdot \sin \theta] \\ (\text{linearity of } E) &= \underbrace{E[X]}_{=0} \cdot \cos \theta + \underbrace{E[Z]}_{=0} \sin \theta \\ &= 0 \end{aligned}$$

$$\begin{aligned} \text{Var}[Y] &= \text{Var}[X \cdot \cos \theta + Z \cdot \sin \theta] = (X \text{ and } Z \text{ independent}) \\ &= \cos^2(\theta) \cdot \underbrace{\text{Var}[X]}_{=1} + \sin^2(\theta) \cdot \underbrace{\text{Var}[Z]}_{=1} = 1 \end{aligned}$$

$$\Rightarrow Y \sim N(0, 1)$$

Q: What's the correlation coefficient between X and Y ?

$$\rightarrow \rho(X, Y) = \frac{\text{Cov}[X, Y]}{\underbrace{\text{SD}[X]}_{=1} \cdot \underbrace{\text{SD}[Y]}_{=1}} = \text{Cov}[X, Y]$$

$$= E[(X - \mu_X)(Y - \mu_Y)] = E[XY] - \underbrace{\mu_X \cdot \mu_Y}_{=0} \quad (\text{std normal } X \text{ and } Y)$$

$$= E[XY] =$$

$$= E[X(X \cos \theta + Z \sin \theta)] =$$

$$\begin{aligned} &= \cos \theta \cdot \underbrace{E[X^2]}_{\substack{\\ \text{Var}[X] + (\underbrace{E[X]}_1)^2}} + \sin \theta \underbrace{E[XZ]}_{\substack{\\ = 0 \text{ (independence)}}} \\ &= \underbrace{E[X] \cdot E[Z]}_{=0} = 0 \end{aligned}$$

$$\rho(X, Y) = \cos \theta$$

□

Special Cases: $\theta = 0 \Rightarrow X = Y$

$\theta = \frac{\pi}{2} \Rightarrow Y = Z$ (and X and Y are independent)

$\theta = \pi \Rightarrow Y = -X$

In general: For each correlation coefficient $-1 < \rho < 1$, there exists an angle

$$\theta = \arccos(\rho)$$

such that X and Y as above have the correlation ρ .

Alternatively,

$$Y = \rho \cdot X + \sqrt{1-\rho^2} \cdot Z$$

w/ X and Z independent and $N(0,1)$.

Joint Density: $f_{X,Y}(x,y) = \frac{1}{(1-\rho^2)\sqrt{2\pi}} \exp\left(-\frac{1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2)\right)$

Marginal Distributions. $X \sim N(0,1)$, $Y \sim N(0,1)$

Conditional Distributions: Given $X = x$, $Y \sim \text{Normal}(\rho x, 1-\rho^2)$

Given $Y = y$, $X \sim \text{Normal}(\rho y, 1-\rho^2)$

Independence: For (X, Y) w/ a standard bivariate normal w/ correlation ρ , X and Y are independent if $\rho = 0$.

Any Bivariate Normal.

Random variables U and V are said to have the bivariate normal distribution w/ parameters $\mu_u, \mu_v, \sigma_u, \sigma_v$ and ρ if

$$\left(X = \frac{U - \mu_u}{\sigma_u}, Y = \frac{V - \mu_v}{\sigma_v} \right)$$

has the standard bivariate normal dist'n w/ correlation ρ .

Note: • $\rho(U, V) = ?$

By def'n:

$$\rho(U, V) = \frac{\text{Cov}[U, V]}{\text{SD}[U] \cdot \text{SD}[V]} = \dots$$