M339J: Probability models with actuarial applications
University of Texas at Austin
HW #8
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Note: You **must** show all your work. Numerical answers without a proper explanation or a clearly written down path to the solution will be assigned zero points.

Problem 8.1. (10 points) The aggregate loss random variable S has a compound Poisson claims distribution, i.e., let the frequency random variable N have the Poisson distribution. You are given that

i. Individual claim amounts may only be equal to 1, 2, or 3.
ii. $\mathbb{E}[S] = 56$
iii. $Var[S] = 126$
iv. The rate of the Poisson claim count random variable is $\lambda = 29$.

Determine the probability mass function of the claim amounts.

Solution: We are given that, in our usual notation,

$$N \sim Poisson(\lambda = 29),$$

and that the support of the severity r.v. X is $\{1, 2, 3\}$.

Moreover,

$$56 = \mathbb{E}[S] = \mathbb{E}[N]\mathbb{E}[X] = 29(p_X(1) + 2p_X(2) + 3p_X(3)),$$

and

$$126 = Var[S] = 29\mathbb{E}[X^2] = 29(p_X(1) + 4p_X(2) + 9p_X(3)).$$

Together with the law of total one, i.e.,

$$p_X(1) + p_X(2) + p_X(3) = 1,$$

we get three equations with three unknowns. The solution is

$$p_X(1) = 10/29, p_X(2) = 11/29, p_X(3) = 8/29.$$

Problem 8.2. (15 points) In the compound model for aggregate claims, let the frequency random variable N be negative binomial with parameters $r = 2$ and $\beta = 4$, and let the common distribution of the i.i.d. severity random variables $\{X_j; j = 1, 2, \dots\}$ be given by the probability (mass) function $p_X(1) = 0.3$ and $p_X(2) = 0.7$.
Let our usual assumptions hold, i.e., let N be independent of $\{X_j; j = 1, 2, \dots\}$.
Define the aggregate loss as $S = \sum_{j=1}^{N} X_j$.
Calculate $\mathbb{P}[S \leq 3]$.

Solution: Evidently,

$$\mathbb{P}[S \leq 3] = \mathbb{P}[S=0] + \mathbb{P}[S=1] + \mathbb{P}[S=2] + \mathbb{P}[S=3]$$

We simplify one probability at a time, using independence of N and $\{X_j; j \geq 1\}$:

$$\begin{aligned}
\mathbb{P}[S=0] &= \mathbb{P}[N=0] = (1+\beta)^{-r} = 5^{-2} = 1/25 = 0.04,\\
\mathbb{P}[S=1] &= \mathbb{P}[N=1, X_1=1] = \mathbb{P}[N=1]\mathbb{P}[X_1=1] = p_N(1)\, p_X(1),\\
\mathbb{P}[S=2] &= \mathbb{P}[N=1, X_1=2] + \mathbb{P}[N=2, X_1=1, X_2=1]\\
&= p_N(1)\, p_X(2) + p_N(2)\, p_X(1)^2,\\
\mathbb{P}[S=3] &= \mathbb{P}[N=2, X_1=2, X_2=1] + \mathbb{P}[N=2, X_1=1, X_2=2] + \mathbb{P}[N=3, X_1=X_2=X_3=1]\\
&= 2p_N(2)\, p_X(2)\, p_X(1) + p_N(3)\, p_X(1)^3.
\end{aligned}$$

So,

$$\begin{aligned}
\mathbb{P}[S \leq 3] &= 0.04 + p_N(1)\, p_X(1) + p_N(1)\, p_X(2) + p_N(2)\, p_X(1)^2 + 2p_N(2)\, p_X(2)\, p_X(1) + p_N(3)\, p_X(1)^3\\
&= 0.04 + p_N(1) + p_N(2)[p_X(1)^2 + 2p_X(2)\, p_X(1)] + p_N(3)\, p_X(1)^3\\
&= 0.04 + p_N(1) + p_N(2)[(p_x(1) + p_X(2))^2 - p_X(2)^2] + p_N(3)\, p_X(1)^3\\
&= 0.04 + p_N(1) + p_N(2)[1 - p_X(2)^2] + p_N(3)\, p_X(1)^3.
\end{aligned}$$

Since

$$\begin{aligned}
p_N(1) &= r\beta^1(1+\beta)^{-(r+1)} = 2 \cdot 4 \cdot 5^{-3} = 0.064,\\
p_N(2) &= \frac{1}{2}\, r(r+1)\beta^2(1+\beta)^{-(r+2)} = \frac{1}{2} \cdot 2 \cdot 3 \cdot 4^2 \cdot 5^{-4} = 0.0768\\
p_N(3) &= \frac{1}{2 \cdot 3}\, r(r+1)(r+2)\beta^3(1+\beta)^{-(r+3)} = \frac{1}{2 \cdot 3} \cdot 2 \cdot 3 \cdot 4 \cdot 4^3 \cdot 5^{-5} = 4^4/5^5 = 0.08192,
\end{aligned}$$

we get

$$\mathbb{P}[S \leq 3] = 0.04 + 0.064 + 0.0768(1 - 0.7^2) + 0.08192 \cdot 0.3^3 = 0.14537984.$$

Problem 8.3. (5 pts) In the compound model for aggregate claims, let the frequency random variable N have the Poisson distribution with mean 5. Moreover, let the common distribution of the i.i.d. severity random variables $\{X_j; j = 1, 2, \dots\}$ be the two-parameter Pareto with parameters $\alpha = 3$ and $\theta = 10$. Let our usual assumptions hold, i.e., let N be independent of $\{X_j; j = 1, 2, \dots\}$.
Define the aggregate loss as $S = \sum_{j=1}^{N} X_j$.
What is the variance of S?

Solution: We will use the formula

$$Var[S] = \mathbb{E}[N]Var[X] + Var[N]\mathbb{E}[X]^2.$$

We are given that

$$\mathbb{E}[N] = Var[N] = 5.$$

So,

$$Var[S] = 5(Var[X] + \mathbb{E}[X]^2) = 5(\mathbb{E}[X^2] - \mathbb{E}[X]^2 + \mathbb{E}[X]^2) = 5\mathbb{E}[X^2].$$

Using our tables, we get

$$\mathbb{E}[X^2] = \frac{\theta^2 \cdot 2!}{(\alpha-1)(\alpha-2)} = \frac{10^2 \cdot 2}{(3-1)(3-2)} = 100.$$

Finally, $Var[S] = 5 \cdot 100 = 500$.

Problem 8.4. (10 points) We are using the aggregate loss model and our usual notation. The frequency random variable N is assumed to be Poisson distributed with mean equal to 1. The severity random variable is assumed to have the following probability mass function:

$$p_X(100) = 3/5, \quad p_X(200) = 3/10, \quad p_X(300) = 1/10.$$

Find the probability that the total aggregate loss **exactly** equals 300.

Solution: If we focus on the event that $\{S = 300\}$, we know that the number of losses must be $1, 2$ or 3.

$$\begin{aligned}
\mathbb{P}[S = 300] &= \mathbb{P}[S = 300 \,|\, N = 1]\mathbb{P}[N = 1] + \mathbb{P}[S = 300 \,|\, N = 2]\mathbb{P}[N = 2] + \mathbb{P}[S = 300 \,|\, N = 3]\mathbb{P}[N = 3] \\
&= \mathbb{P}[X_1 = 300 \,|\, N = 1]\mathbb{P}[N = 1] + \mathbb{P}[X_1 + X_2 = 300 \,|\, N = 2]\mathbb{P}[N = 2] \\
&\quad + \mathbb{P}[X_1 + X_2 + X_3 = 300 \,|\, N = 3]\mathbb{P}[N = 3] \\
&= p_X(300)p_N(1) + 2p_X(100)p_X(200)p_N(2) + (p_X(100))^3 p_N(3) \\
&= \frac{1}{10}e^{-1} + 2\left(\frac{3}{5}\right)\left(\frac{3}{10}\right)e^{-1}\left(\frac{1}{2}\right) + \left(\frac{3}{5}\right)^3 e^{-1}\left(\frac{1}{6}\right) = 0.316e^{-1} = 0.11625.
\end{aligned}$$

Problem 8.5. (10 points) In the compound model for aggregate claims, let the frequency random variable N be negative binomial with parameters $r = 15$ and $\beta = 5$.

Moreover, let the common distribution of the i.i.d. severity random variables $\{X_j; j = 1, 2, \dots\}$ be the two-parameter Pareto with $\alpha = 3$ and $\theta = 10$.

Let our usual assumptions hold, i.e., let N be independent of $\{X_j; j = 1, 2, \dots\}$. The insurer is interested in finding the total premium π such that the aggregate losses exceed it with the probability less than or equal to 5%. Using the normal approximation, find π such that

$$\mathbb{P}[S > \pi] = 0.05.$$

Solution: Let $\mu_S = \mathbb{E}[S]$ and $\sigma_S = \sqrt{Var[S]}$. Then, using the normal approximation, we have

$$0.05 = \mathbb{P}[S > \pi] = \mathbb{P}\left[\frac{S - \mu_S}{\sigma_S} > \frac{\pi - \mu_S}{\sigma_S}\right] \approx 1 - \Phi\left(\frac{\pi - \mu_S}{\sigma_S}\right)$$

where Φ denotes the c.d.f. of the standard normal distribution. From the tables for Φ, we get

$$\pi = \mu_S + 1.645\sigma_S.$$

From the given information on the severity r.v.s, we obtain

$$\begin{aligned}
\mathbb{E}[X] &= \frac{\theta}{\alpha-1} = \frac{10}{3-1} = 5,\\
Var[X] &= \frac{\theta^2 \cdot 2}{(\alpha-1)(\alpha-2)} - (\frac{\theta}{\alpha-1})^2 = \frac{\theta^2 \cdot \alpha}{(\alpha-1)^2(\alpha-2)} = \frac{10^2 \cdot 3}{(3-1)^2(3-2)} = 75,\\
\mathbb{E}[N] &= r\beta = 75,\\
Var[N] &= r\beta(1+\beta) = 450.
\end{aligned}$$

So,

$$\begin{aligned}
\mu_S &= \mathbb{E}[S] = \mathbb{E}[X]\mathbb{E}[N] = 5 \cdot 75 = 375,\\
\sigma_S^2 &= Var[S] = Var[X]\mathbb{E}[N] + Var[N]\mathbb{E}[X]^2 = 75 \cdot 75 + 450 \cdot 5^2 = 16,875.
\end{aligned}$$

Hence,

$$\pi = 375 + 1.645 \cdot \sqrt{16875} \approx 588.692.$$