M339W: September 13th, 2021: Part II

Review:

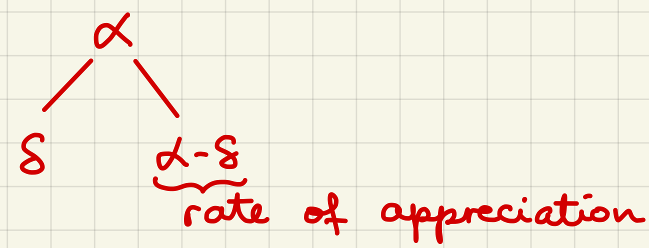
Parameters.

- α ... mean rate of return (per annum)

  satisfies $\mathbb{E}[S(T)] = S(0)e^{(\alpha-\delta)\cdot T}$

  w/ δ ... dividend yield

  Note:



- σ ... volatility,

  i.e., the standard deviation of realized returns on an annual basis

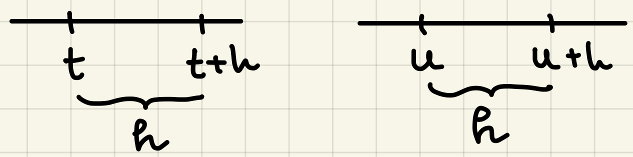
Realized returns. For any $t, h > 0$, we set

$$R(t,t+h) = \ln\left(\frac{S(t+h)}{S(t)}\right)$$

Modeling Assumptions. In continuous time:

We want them to be inherited from the binomial tree.

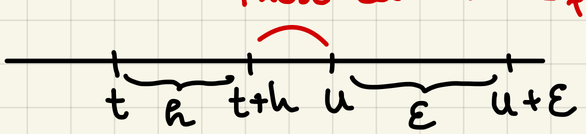
i.



We require that $R(t,t+h)$ and $R(u,u+h)$ be identically distributed.

ii.

These can be equal!

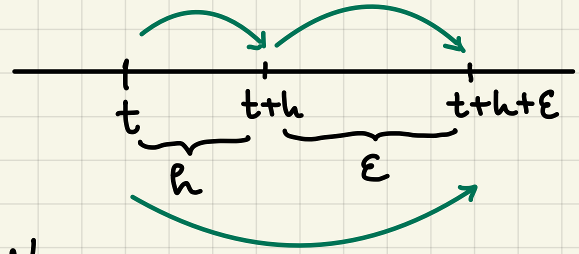


$t, h, u, \varepsilon > 0$

We require that $R(t,t+h)$ and $R(u,u+\varepsilon)$ be independent.

Property. iii.

$t, h, \varepsilon > 0$



By def'n:

$$R(t, t+h+\varepsilon) = \ln\left(\frac{S(t+h+\varepsilon)}{S(t)}\right)$$

$$= \ln\left(\frac{S(t+h+\varepsilon)}{S(t+h)} \cdot \frac{S(t+h)}{S(t)}\right)$$

$$= \underbrace{\ln\left(\frac{S(t+h+\varepsilon)}{S(t+h)}\right)}_{} + \ln\left(\frac{S(t+h)}{S(t)}\right)$$

$$= R(t+h, t+h+\varepsilon) + R(t, t+h)$$

$$\boxed{R(t, t+h+\varepsilon) = R(t, t+h) + R(t+h, t+h+\varepsilon)}$$

Realized Returns are ADDITIVE.

Q: Which probabilistic model would you propose for our realized returns?

→: Think back to our discussion of what happens in the limit w/ a binomial tree :)

⇒ We decide to model our realized returns $R(t, t+h)$ as normally distributed,

i.e.,

$$R(t, t+h) \sim \text{Normal}(\text{mean} = m, \text{variance} = v^2)$$

Note: By def'n:

$$S(t+h) = S(t)e^{R(t, t+h)}$$

Moment Generating Functions.

For any random variable Y,
and for independent arguments denoted by t,
we define the moment generating f'tion of Y as the following function of t:

$$M_Y(t) := \mathbb{E}\left[e^{Y \cdot t}\right]$$

for all t such that the expectation exists, i.e., when it is finite

Note: • $M_Y(0) = 1$

$\Rightarrow$ @ least $t=0$ is in the domain of M_Y

Goal: Understanding e^X where $X \sim \text{Normal}(\text{mean}=m, \text{var}=\nu^2)$.

$\rightarrow$: Recall the standard normal : $Z \sim N(0,1)$
Then, our X can be expressed as:

$$X = m + \nu \cdot Z$$

In general: Take constants a and b;
define $\tilde{Y} = a \cdot Y + b$ w/ Y any r.v.

By def'n:
$$\begin{aligned} M_{\tilde{Y}}(t) &= \mathbb{E}\left[e^{t \cdot \tilde{Y}}\right] \\ &= \mathbb{E}\left[e^{t(a \cdot Y + b)}\right] \\ &= \mathbb{E}\left[e^{t \cdot a \cdot Y} \cdot e^{t \cdot b}\right] \\ &= e^{t \cdot b} \cdot \mathbb{E}\left[e^{t \cdot a \cdot Y}\right] \\ &= e^{t \cdot b} \cdot M_Y(a \cdot t) \end{aligned}$$

In particular: Let $X \sim \text{Normal}(\text{mean}=m, \text{var}=\nu^2)$.

$$\Rightarrow M_X(t) = e^{m \cdot t} \cdot M_Z(\nu \cdot t) = e^{m \cdot t} \cdot e^{\frac{\nu^2 \cdot t^2}{2}}$$

Recall: $M_Z(s) = e^{\frac{s^2}{2}}$

$$\Rightarrow M_X(t) = e^{m \cdot t + \frac{\nu^2 \cdot t^2}{2}}$$