University of Texas at Austin, Department of Mathematics
M358K - Applied Statistics

The In-term One

Problem 1.1. (10 points) Write down the definition of *independence* of two *events*.

Solution: Two events A and B are said to be *independent* if

$$\mathbb{P}[A \cap B] = \mathbb{P}[A] \cdot \mathbb{P}[B]$$

Problem 1.2. (10 points) Write down the definition of the *cumulative distribution function* of a random variable.

Solution: Let X be a random variable. Its *cumulative distribution function* is a function $F_X : \mathbb{R} \to [0, 1]$ defined by

$$F_X(x) = \mathbb{P}[X \leq x], \quad \text{for every } x \in \mathbb{R}.$$

Problem 1.3. (20 points) Four balls are drawn (without replacement) from a box which contains 5 black and 6 red balls. Given that the four drawn balls were *not* all of the same color, what is the probability that there was exactly one black ball among the four.

Solution: Let A denote the event that the colors of the balls drawn are not all the same, and let B denote the event that there was exactly one black ball and three red balls. We are looking for $\mathbb{P}[B|A]$. Since $B \subseteq A$, we have

$$\mathbb{P}[B|A] = \mathbb{P}[B \cap A]/\mathbb{P}[A] = \mathbb{P}[B]/\mathbb{P}[A].$$

To compute $\mathbb{P}[A]$, we note that the event A^c consists of only two elementary outcomes: either all balls are black or all balls are red. The probability of picking all red balls is

$$\frac{\binom{6}{4}}{\binom{11}{4}} = \frac{15}{\binom{11}{4}}$$

while the probability of picking all black balls is

$$\frac{\binom{5}{4}}{\binom{11}{4}} = \frac{5}{\binom{11}{4}}.$$

Putting these two together, we obtain

$$\mathbb{P}[A] = 1 - \frac{15+5}{\binom{11}{4}} = 1 - \frac{20}{\binom{11}{4}}.$$

To compute $\mathbb{P}[B]$ we note that we can choose 3 red balls out of 6 in $\binom{6}{3} = 20$ ways and, then, for each such choice, we have 5 ways of choosing one black ball from the set of 5 black balls. Therefore,

$$\mathbb{P}[B] = \frac{100}{\binom{11}{4}}$$

Finally,

$$\mathbb{P}[B|A] = \frac{100}{\binom{11}{4} - 20} = \frac{100}{330 - 20} = \frac{100}{310} = \frac{10}{31}.$$

Problem 1.4. (25 points) *Source: "Probability" by Pitman.*
A final exam consists of multiple choice problems – each problem with 4 offered answers only one of which is correct. Before the final exam the diligent student is given a practice set of multiple choice problems. Knowing that exactly 50% of the final exam will be out of the practice set, the student works out the entire practice set and gets the correct answer to each question.

When he takes the final exam, the student proceeds to answer the known questions correctly. However, for the remaining questions, he panics and chooses the answers completely at random.

(i) (5 points) What is the probability that the student answers a randomly chosen question correctly?

(ii) (5 points) **Given** that the student answered a particular question correctly, what is the probability that he was guessing at random when he was answering that question?

Let the total number of questions in the exam be 40. Let the random variable N represent the total number of questions the student answered correctly.

(iii) (5 points) What is the expected value of N?

(iv) (5 points) What is the standard deviation of N?

(v) (5 points) What is the probability that the student scores more than 80% on the final exam? Your response can me left in sigma notation with binomial coefficients. Or, you can use 'R'.

Solution:

(i)

$$0.5(1) + 0.5(0.25) = 0.625$$

(ii)

$$\frac{0.5(0.25)}{0.625} = 0.2.$$

(iii) The random variable N can be written as

$$N = 20 + N',$$

where $N' \sim Binomial(n = 20, p = 0.25)$. So,

$$\mathbb{E}[N] = 20 + 20 * 0.25 = 15.$$

(iv)

$$Var[N] = Var[N'] = 20 * 0.25 * 0.75 = 3.75 \quad \Rightarrow \quad SD[N] = \sqrt{3.75} = 1.936492.$$

(v)

$$\mathbb{P}[N > 32] = \mathbb{P}[N' > 12] = \sum_{k=13}^{20} \binom{20}{k} (0.25)^k (0.75)^{20-k}.$$

In ‘R‘, this is $1 - pbinom(12, 20, 0.25) = 0.0001837041$.