M339W: March 1st, 2021.

LogNormal Distribution.

Def'n. Let $X \sim \text{Normal}(\text{mean} = m, \text{variance} = \nu^2)$.

Define $\boxed{Y = e^X}$.

We say that Y is lognormally distributed.

$$\mathbb{E}[Y] = \mathbb{E}[e^X] = \mathbb{E}[e^{m+\nu \cdot Z}] = e^m \cdot \mathbb{E}[e^{\nu \cdot Z}] = e^{m + \frac{\nu^2}{2}}$$

Let $Z \sim N(0,1)$: $X \stackrel{(d)}{=} m + \nu \cdot Z$ $\qquad M_Z(t) = e^{t^2/2}$

$\mathbb{E}[X] = m$

Caveat: $\mathbb{E}[e^X] \geq e^{\mathbb{E}[X]}$

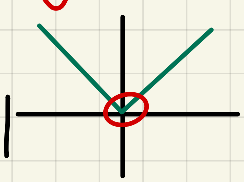
This is a special case of Jensen's Inequality.

Thm. If X is a random variable and g is a convex function such that $g(X)$ is well defined and $\mathbb{E}[g(X)]$ exists, then

$$\mathbb{E}[g(X)] \geq g(\mathbb{E}[X])$$

Examples. (i) $g(x) = |x|$

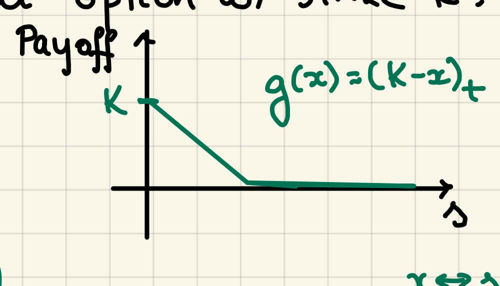
$\Rightarrow \mathbb{E}[|X|] \geq |\mathbb{E}[X]|$



(ii) Consider a European put option w/ strike K.

The expected payoff is:

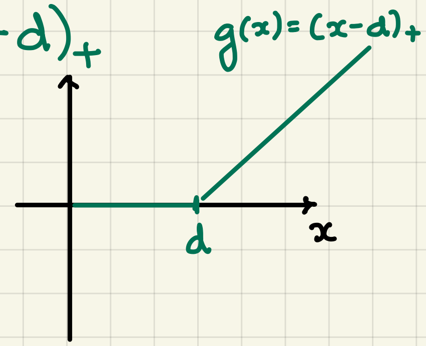
$$\mathbb{E}[\underbrace{(K - S(T))_+}_{= V_P(T)}] \geq (K - \mathbb{E}[S(T)])_+$$



(iii) In classical insurance:

$\begin{cases} X \ldots \text{(ground-up) loss, i.e., severity r.v.} \\ d \ldots \text{deductibe} \end{cases}$

The insurer pays: $(X-d)_+$



Jensen's inequality:

$$\mathbb{E}\left[(X-d)_+\right] \geq \left(\mathbb{E}[X]-d\right)_+$$

Q: What's the median of Y?

→: Looking for a^* such that

$$\boxed{F_Y(a^*) = \frac{1}{2}}$$

$$\Rightarrow \quad \frac{1}{2} = F_Y(a^*) \underset{\substack{\uparrow \\ \text{by the} \\ \text{def'n of the cdf}}}{=} \mathbb{P}[Y \leq a^*] \underset{\substack{\uparrow \\ \text{by the def'n of } Y}}{=} \mathbb{P}[e^X \leq a^*]$$

$$\underset{\substack{\uparrow \\ \ln \text{ is increasing}}}{=} \mathbb{P}[X \leq \ln(a^*)]$$

Note:

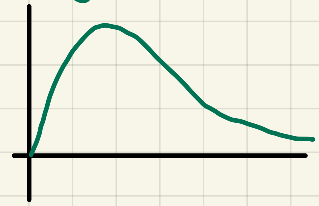
$\boxed{\ln(a^*) \text{ is the median of } X}$

$X \sim N(\text{mean} = m, \text{var} = \nu^2)$

$$\Rightarrow \ln(a^*) = m$$

$$\Rightarrow \text{Median of } Y \text{ is } e^m.$$

Mean of the lognormal $\geq$ median of the lognormal



Log-Normal Stock Prices.

$S(t)$, $t \geq 0$ ... the time-t stock price

$R(0,t)$, $t \geq 0$ ... realized returns.

In particular, @ time T:

$$S(T) = S(0)e^{R(0,T)} \iff R(0,T) = \ln\left(\frac{S(T)}{S(0)}\right)$$

We settled on the <u>normal</u> distribution to model realized returns, i.e.,

$R(0,T) \sim \text{Normal}(\text{mean} = m, \text{variance} = v^2)$.

$\Rightarrow$ $S(T)$ is <u>lognormally</u> distributed.