Say, we have X and Y two rnd variables.

$X_1, \dots, X_n$ and $Y_1, \dots, Y_n$ two independent random samples.

$$\beta_{X,Y} = \frac{Cov[\bar{X}, \bar{Y}]}{Var[\bar{Y}]} = \frac{\sum_{k=1}^{n} (X_k - \bar{X})(Y_k - \bar{Y})}{\sum_{k=1}^{n} (Y_k - \bar{Y})^2}$$

algebra

$$\beta_{X,Y} = \frac{\sum_{k=1}^{n} X_k Y_k - n\bar{X}\bar{Y}}{\sum_{k=1}^{n} Y_k^2 - n\bar{Y}^2}$$

Simulated stock prices:

33.29, 37.30, 40.35, 43.65, 48.90

40-strike-call payoffs:

~~X~~s: 0 0 0.35 3.65 8.90

42-strike-call payoffs:

~~X~~s: 0 0 0 1.65 6.90

For our problem: $\bar{X} = \frac{1}{5}(0.35 + 3.65 + 8.90) = 2.58$

$\bar{Y} = \frac{1}{5}(1.65 + 6.90) = 1.71$

$\beta = ?$

$$\beta = \frac{3.65 \cdot 1.65 + 8.90 \cdot 6.90 - 5 \cdot 2.58 \cdot 1.71}{0.35^2 + 3.65^2 + 8.9^2 - 5 \cdot (2.58)^2} = 0.7642$$

SAMPLE MFE

75. You are using Monte Carlo simulation to estimate the price of an option X, for which there is no pricing formula. To reduce the variance of the estimate, you use the control variate method with another option Y, which has a pricing formula.

You are given:

(i) The naive Monte Carlo estimate of the price of X has standard deviation 5.

$Var[\bar{X}] = 25$

→ (ii) The same Monte Carlo trials are used to estimate the price of Y.

(iii) The correlation coefficient between the estimated price of X and that of Y is 0.8.

$corr[\bar{X}, \bar{Y}] = 0.8$

Calculate the minimum variance of the estimated price of X, with Y being the control variate.

We need the VAR of: $\hat{X}^* = \bar{X} + \dfrac{Cov[\bar{X}, \bar{Y}]}{Var[\bar{Y}]} (\mu_Y - \bar{Y})$

The theoretical price of Y.

(A) 1.0

(B) 1.8

(C) 4.0

(D) 9.0

(E) 16.0

$$Var[\hat{X}^*] = Var[\bar{X}] + \frac{(Cov[\bar{X}, \bar{Y}])^2}{(Var[\bar{Y}])^{\cancel{2}}} \cdot \cancel{Var[\bar{Y}]}$$

$$+ 2 \cdot (-1) \cdot \frac{Cov[\bar{X}, \bar{Y}]}{Var[\bar{Y}]} \cdot Cov[\bar{X}, \bar{Y}]$$

$$Var[\hat{X}^*] = Var[\bar{X}] - \frac{(Cov[\bar{X}, \bar{Y}])^2}{Var[\bar{Y}]}$$

$$= Var[\bar{X}] - \frac{(Corr[\bar{X}, \bar{Y}])^2 \cdot Var[\bar{X}] \cdot \cancel{Var[\bar{Y}]}}{\cancel{Var[\bar{Y}]}}$$

$$Var[\hat{X}^*] = Var[\bar{X}]\left(1 - (corr[\bar{X}, \bar{Y}])^2\right)$$

In this problem:

$$Var[\hat{X}^*] = 25(1 - 0.64) = 9 \Rightarrow D.$$