M339J *Probability models*
University of Texas at Austin
HW V
Instructor: Milica Čudina

Note: You **must** show all your work. Numerical answers without a proper explanation or a clearly written down path to the solution will be assigned zero points.

Problem 5.1. (5 points) Let the ground-up loss X be exponentially distributed with mean \$800. An insurance policy has an ordinary deductible of \$100 and the maximum amount payable per loss of \$2500.
Find the expected value of the amount paid (by the insurance company) **per positive payment.**

Solution: We are given $X \sim Exponential(\theta = 800)$, the deductible $d = 100$ and the policy limit $u - d = 2500$. We need to calculate $\mathbb{E}[Y^P]$ where $Y^P = Y^L \,|\, Y^L > 0$ and

$$\begin{aligned} Y^L &= \begin{cases} (X-d)_+, & X < u, \\ u - d, & X \geq u \end{cases} \\ &= (X \wedge u - d)_+. \end{aligned}$$

By the memoryless property of the exponential distribution, we have that

$$Y = X - d \,|\, X > d$$

is also exponential with mean 800. So, using our tables, we get

$$\mathbb{E}[Y^P] = \mathbb{E}[Y \wedge (u-d)] = \mathbb{E}[Y \wedge 2500] = 800(1 - e^{-2500/800}) \approx 764.85.$$

Problem 5.2. (5 pts)Let X be the ground-up loss random variable. Assume that X has the exponential distribution with mean $5,000$. Let B denote the expected payment per loss on behalf of an insurer which wrote a policy with a deductible of $1,500$ and with no maximum policy payment. What is the value of B?

Solution: Using our tables,

$$B = \mathbb{E}[(X-1500)_+] = \mathbb{E}[X] - \mathbb{E}[X \wedge 1500] = \theta - \theta(1 - e^{-1500/\theta}) = \theta\, e^{-1500/\theta} = 5000e^{-3/10} \approx 3704.$$

Problem 5.3. (10 points) Let X have a two-point mixture distribution. More precisely, with probability $1/3$, X has the Pareto distribution with parameters $\alpha = 3$ and $\theta = 10$ and with probability $2/3$, X has the Gamma distribution with parameters $\alpha = 2$ and $\theta = 8$.

Find $Var[X]$.

Solution: Let $X_1 \sim Pareto(\alpha = 3, \theta = 10)$, and $X_2 \sim Gamma(\alpha = 2, \theta = 8)$.
Then,

$$\begin{aligned}
\mathbb{E}[X_1] &= \frac{10}{3-1} = 5, \\
\mathbb{E}[X_1^2] &= \frac{10^2 \cdot 2}{(3-1)(3-2)} = 100, \\
\mathbb{E}[X_2] &= 2 \cdot 8 = 16, \\
\mathbb{E}[X_2^2] &= 8^2(2+1) \cdot 2 = 384.
\end{aligned}$$

So,

$$\begin{aligned}
\mathbb{E}[X] &= \frac{1}{3} \cdot \mathbb{E}[X_1] + \frac{2}{3} \cdot \mathbb{E}[X_2] \\
&= \frac{1}{3} \cdot 5 + \frac{2}{3} \cdot 16 \\
&= \frac{37}{3}, \\
\mathbb{E}[X^2] &= \frac{1}{3} \cdot \mathbb{E}[X_1^2] + \frac{2}{3} \cdot \mathbb{E}[X_2^2] \\
&= \frac{1}{3} \cdot 100 + \frac{2}{3} \cdot 384 \\
&= \frac{868}{3}.
\end{aligned}$$

Finally,

$$Var[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{868}{3} - \frac{37^2}{3^2} = \frac{1235}{9}.$$

Problem 5.4. (10 points) *Source: Sample C Exam Problem #100.*
Let X have the following cumulative distribution function

$$F_X(x) = 1 - 0.8e^{-0.02x} - 0.2e^{-0.001x}, \qquad x \geq 0.$$

Let $u = 1000$.
Find $\mathbb{E}[X \wedge u]$.

Solution: If we wanted to use "brute force", we would need to calculate

$$\begin{aligned}
\mathbb{E}[X \wedge u] &= \int_0^\infty (x \wedge u) f_X(x)\, dx \\
&= \int_0^u x f_X(x)\, dx + \int_u^\infty u f_X(x)\, dx
\end{aligned}$$

with

$$f_X(x) = F'_X(x) = 0.016e^{-0.02x} + 0.0002e^{-0.001x}.$$

This would involve looking at (at least) 4 integrals and possibly some integration-by-parts.

So, let us take a step back and look at the given distribution function once again.
We can rewrite is as:

$$F_X(x) = 0.8(1 - e^{-0.02x}) + 0.2(1 - e^{-0.001x}), \qquad x \geq 0,$$

and recognize that it has the form of a 2-point mixture of two random variables X_1 and X_2 with

$$\begin{aligned} X_1 &\sim \mathit{Exponential}(\theta = 50) \quad \text{with probability} \quad a_1 = 0.8, \\ X_2 &\sim \mathit{Exponential}(\theta = 1000) \quad \text{with probability} \quad a_2 = 0.2. \end{aligned}$$

Now, we realize that we can use

$$\mathbb{E}[X \wedge u] = 0.8\mathbb{E}[X_1 \wedge u] + 0.2\mathbb{E}[X_2 \wedge u].$$

From the tables, and with $u = 1000$, we have

$$\begin{aligned} \mathbb{E}[X_1 \wedge u] &= 50(1 - e^{-1000/50}) = 50(1 - e^{-20}), \\ \mathbb{E}[X_2 \wedge u] &= 1000(1 - e^{-1000/1000}) = 1000(1 - e^{-1}). \end{aligned}$$

Finally,

$$\mathbb{E}[X \wedge 1000] = 0.8 \cdot 50(1 - e^{-20}) + 0.2 \cdot 1000(1 - e^{-1}) \approx 166.4241.$$

Alternatively, if you remembered to use equation (3.9) from *"Loss Models"(3rd Ed)*, you could also get

$$\mathbb{E}[X \wedge u] = \int_0^u S_X(x)\, dx = \int_0^u (1 - F_X(x))\, dx.$$

In this scenario, you get two rather simple integrals and (of course) the same final answer.

Problem 5.5. (10 points) Let Y be lognormal with parameters $\mu = 1$ and $\sigma = 2$.
Define $\tilde{Y} = 3Y$.
Find the median of $\tilde{Y}$, i.e., find the value m such that $\mathbb{P}[\tilde{Y} \leq m_Y] = 1/2$.

Solution: In class, we showed that Y is lognormal with parameters $\mu^* = \mu + \ln(3)$ and $\sigma^* = \sigma$. So, Y can be written as $Y = e^Z$ where $Z \sim N(\mu^*, (\sigma^*)^2)$. Hence, with m_Y denoting the median of Y, we have

$$\begin{aligned} 1/2 &= \mathbb{P}[Y \leq m_Y] \\ &= \mathbb{P}[e^X \leq m_Y] \\ &= \mathbb{P}[X \leq \ln(m_Y)]. \end{aligned}$$

Since X is normal with mean μ^* (and the mean and the median of a normal r.v. are one and the same), we conclude that

$$\ln(m_Y) = 1 + \ln(3) \quad \Rightarrow \quad m_y = 3e \approx 8.15.$$

Problem 5.6. (10 points) In the notation of our tables, let X be a Weibull random variable with parameters $\theta = 20$ and $\tau = 2$.
Define $Y = 5X$ and denote the coefficient of variation of Y by CV_Y. Find CV_Y.

Hint: The following facts you may have forgotten from probability could be useful:

$$\begin{aligned}
\Gamma(1/2) &= \sqrt{\pi}, \\
\Gamma(1) &= 1, \\
\Gamma(\alpha+1) &= \alpha\Gamma(\alpha), \quad \text{for all } \alpha.
\end{aligned}$$

Solution: The Weibull distribution has the scale parameter θ. So,

$$Y \sim Weibull(\theta = 100, \tau = 2).$$

Using our tables, we get

$$\begin{aligned}
\mathbb{E}[Y] &= \theta\,\Gamma(1 + \frac{1}{\tau}) \\
&= \theta\,\Gamma(1 + \frac{1}{2}) \\
&= \theta \cdot \frac{1}{2}\Gamma(1/2) \\
&= \theta \cdot \frac{1}{2}\sqrt{\pi},
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}[Y^2] &= \theta^2\,\Gamma(1 + \frac{2}{\tau}) \\
&= \theta^2\,\Gamma(1 + \frac{2}{2}) \\
&= \theta^2 \cdot 1 \cdot \Gamma(1) \\
&= \theta^2.
\end{aligned}$$

So,

$$\begin{aligned}
Var[Y] &= \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 \\
&= \theta^2 - \theta^2 \cdot \frac{\pi}{4} \\
&= \theta^2(1 - \frac{\pi}{4}) \\
&= \frac{\theta^2}{4}(4 - \pi).
\end{aligned}$$

Finally,

$$CV_Y = \frac{\frac{\theta}{2}\sqrt{4-\pi}}{\theta \cdot \frac{\sqrt{\pi}}{2}} = \sqrt{\frac{4-\pi}{\pi}} \approx 0.5227.$$

Note that we never used the exact value of θ to get the final answer.

Also, note that one can immediately realize that

$$CV_Y = \frac{\sqrt{Var[Y]}}{\mathbb{E}[Y]} = \frac{\sqrt{Var[5X]}}{\mathbb{E}[5X]} = \frac{5\sqrt{Var[X]}}{5\mathbb{E}[X]} = CV_X$$

and then just use the definition of X to get the desired coefficient of variation; there is no need to know anything about the distribution of Y.