Recall that Y_1 and Y_2 are assumed to be independent random variables. With this in mind, we get:

$$m_Y(t) = \boxed{\mathbb{E}[e^{tY_1}]} \cdot \boxed{\mathbb{E}[e^{tY_2}]}$$

Finally, using the definition of a m.g.f., we have

$$\boxed{m_Y(t) = m_{Y_1}(t) \cdot m_{Y_2}(t)}$$

M378K: October 2nd, 2024.

□

Example 7.11. *Let $Y \sim b(n,p)$. What is the moment generating function of Y?*

$\rightarrow$: $m_Y(t) = ?$

$Y = I_1 + I_2 + \cdots + I_n$ w/ I_j, $j = 1..n$ independent and Bernoulli(p)

$$m_Y(t) = m_{I_1}(t) \cdot m_{I_2}(t) \cdots m_{I_n}(t) \underset{\substack{\uparrow \\ \text{identically} \\ \text{dist'd}}}{=} (m_{I_1}(t))^n = ((1-p) + pe^t)^n$$

□

Example 7.12. *Let $N \sim Poisson(\lambda)$. What is the moment generating function m_N of N?*

$\rightarrow$: $$m_N(t) = \mathbb{E}[e^{t \cdot N}] = \sum_{n=0}^{\infty} e^{t \cdot n} p_N(n) = \sum_{n=0}^{\infty} e^{t \cdot n} e^{-\lambda} \frac{\lambda^n}{n!}$$

$$= e^{-\lambda} \cdot \sum_{n=0}^{\infty} \frac{e^{tn} \cdot \lambda^n}{n!} = e^{-\lambda} \cdot \sum_{n=0}^{\infty} \frac{(e^t \cdot \lambda)^n}{n!} = e^{-\lambda} e^{e^t \cdot \lambda} = e^{\lambda(e^t - 1)} \checkmark$$

□

Example 7.13. *Let $Z \sim N(0,1)$. What is the moment generating function m_Z of Z?*

$\rightarrow$: $$m_Z(t) = \mathbb{E}[e^{t \cdot Z}] = \int_{-\infty}^{\infty} e^{t \cdot z} \varphi(z)\, dz = \int_{-\infty}^{\infty} e^{t \cdot z} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\, dz$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2} + t \cdot z - \frac{t^2}{2}} e^{\frac{t^2}{2}}\, dz$$

density of $N(t, \sigma = 1)$

$$= e^{t^2/2} \underbrace{\int_{-\infty}^{\infty} \boxed{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-t)^2}}\, dz}_{=1}$$

□

$$\boxed{m_Z(t) = e^{t^2/2}} \checkmark$$

4

Example 7.14. *Let the random variable Y have the mgf m_Y. Define $X = aY + b$ for some constants a and b. Express the mgf m_X of X in terms of m_Y, a and b.*

→: $m_X(t) = \mathbb{E}[e^{t\cdot X}] = \mathbb{E}[e^{t(a\cdot Y+b)}] =$

$= \mathbb{E}[e^{taY}\cdot e^{tb}] = e^{tb}\,\mathbb{E}[e^{ta\,Y}] = e^{tb}\cdot m_Y(ta)$ □

Example 7.15. *Let $X \sim N(\mu, \sigma^2)$. What is the moment generating function m_X of X?*

→: $X = \underbrace{\mu}_{b} + \underbrace{\sigma}_{a}\cdot Z$ w/ $Z \sim N(0,1)$

$m_X(t) = e^{\mu\cdot t}\cdot \boxed{m_Z(\sigma\cdot t)} = e^{\mu t}\cdot e^{\frac{\sigma^2 t^2}{2}} = \boxed{e^{\mu t + \frac{\sigma^2 t^2}{2}}}$ □

Problem 7.2. *A random variable Y is said to be lognormal if there exists a normally distributed random variable $X \sim N(\mu, \sigma^2)$ such that $Y \overset{(d)}{=} e^X$. Express the mean and the variance of the lognormal r.v. Y in terms of the parameters μ and σ.*

→: $\mathbb{E}[Y] = \mathbb{E}[e^X] = \mathbb{E}[e^{1\cdot X}] = m_X(1) = e^{\mu + \frac{\sigma^2}{2}}$

$\mathbb{E}[Y^2] = \mathbb{E}[(e^X)^2] = \mathbb{E}[e^{2\cdot X}] = m_X(2) = e^{2\mu + \frac{4\sigma^2}{2}} = e^{2(\mu+\sigma^2)}$

$\mathrm{Var}[Y] = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2 = e^{2(\mu+\sigma^2)} - e^{2\mu+\sigma^2}$

$= e^{2\mu+\sigma^2}(e^{\sigma^2} - 1)$ □

Proposition 7.16.

1. *If m_Y exists for a certain probability distribution, then it is unique.*
2. *If m_{Y_1} and m_{Y_2} are equal on an interval, then $\boxed{Y_1 \overset{(d)}{=} Y_2}$.* equally distributed

Corollary 7.17. *Let X_1 and X_2 be independent and normally distributed. Define $X = X_1 + X_2$. Then, the distribution of X is ...*

Proof. $X_i \sim N(\mu = \mu_i, \sigma_i^2)$ for $i = 1, 2$

→: $m_X(t) = \mathbb{E}[e^{tX}] = \mathbb{E}[e^{t(X_1+X_2)}]$

$= \mathbb{E}[e^{tX_1}\cdot e^{tX_2}] = m_{X_1}(t)\cdot m_{X_2}(t)$

independent

$= e^{\mu_1 t + \frac{\sigma_1^2 t^2}{2}}\cdot e^{\mu_2 t + \frac{\sigma_2^2 t^2}{2}}$

$= e^{\overbrace{(\mu_1+\mu_2)}^{\mu}\cdot t + \frac{1}{2}\overbrace{(\sigma_1^2+\sigma_2^2)}^{\sigma^2} t^2}$

$\boxed{X \sim N(\mu_1+\mu_2,\ \sigma = \sqrt{\sigma_1^2+\sigma_2^2})}$ □

□

Corollary 7.18. *Let N_1 and N_2 be independent and Poisson distributed. Define $N = N_1 + N_2$. Then, the distribution of N is ...*

Proof. $N_i \sim Poisson(\lambda_i)$ for $i = 1, 2$

$\rightarrow$:

$$m_N(t) = m_{N_1}(t) \cdot m_{N_2}(t) = e^{\lambda_1(e^t-1)} \cdot e^{\lambda_2(e^t-1)}$$

$$= e^{\overset{\lambda}{(\lambda_1+\lambda_2)}(e^t-1)}$$

$$\boxed{N \sim \text{Poisson}(\lambda = \lambda_1 + \lambda_2)}$$

□

□