M378K Introduction to Mathematical Statistics
Problem Set #10
The Normal Distribution.

Definition 10.1. *The* moment-generating function (mgf) m_Y *for a random variable* Y *is defined as*

$$m_Y(t) = \mathbb{E}[e^{tY}]$$

for all t *for which the above expectation exists. In fact, we say that the moment-generating function* **exists** *there exists a positive number* b *such that* $m_Y(t)$ *is finite for all* t *such that* $|t| \leq b$*.*

Proposition 10.2.

1. *If* m_Y *exists for a certain probability distribution, then it is unique.*
2. *If* m_{Y_1} *and* m_{Y_2} *are equal on an interval, then* $Y_1 \stackrel{(d)}{=} Y_2$*.*

Corollary 10.3. *Let* Y_1 *and* Y_2 *be independent and normally distributed. Define* $Y = Y_1 + Y_2$*. Then, the distribution of* X *is ...*

Proof. Note that $Y_i \sim N(\mu = mu_i, \sigma_i)$ for $i = 1, 2$. Now, let's look at the mgf of Y. Then, since Y_1 and Y_2 are independent, we have

$$m_Y(t) = m_{Y_1}(t)m_{Y_2}(t).$$

We can now use the fact that for any $X \sim N(\mu, \sigma)$,

$$m_X(t) = e^{\mu t}m_Z(\sigma t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

Hence,

$$m_Y(t) = e^{\mu_1 t + \frac{\sigma_1^2 t^2}{2}} \times e^{\mu_2 t + \frac{\sigma_2^2 t^2}{2}} = e^{(\mu_1 + \mu_2)t + \frac{(\sigma_1^2 + \sigma_2^2)t^2}{2}}$$

We can conclude that $Y \sim N(\mu_1 + \mu_2, \sqrt{\sigma_1^2 + \sigma_2^2})$. $\square$

Problem 10.1. *Two scales are used to measure the mass m of a precious stone. The first scale makes an error in measurement which we model by a normally distributed random variable X_1 with mean $\mu_1 = 0$ and standard deviation $\sigma_1 = 0.04m$. The second scale is more accurate. We model its error by a normal random variable X_2 with mean $\mu_2 = 0$ and standard deviation $\sigma_2 = 0.03m$.*

We assume that the measurements made using the two different scales are independent, i.e., that the random variables X_1 and X_2 are independent.

To get our final estimate of the mass of the stone, we take the average of the two results from the two different scales, i.e., we define $Y = \frac{X_1+X_2}{2}$.

(i) What is the distribution of the random variable Y? State the **name** *of its distribution and the* **values** *of the parameters.*

(ii) What is the probability that the error Y we get is within $0.005m$ of the actual mass of the stone? Namely, calculate

$$\mathbb{P}[|Y| < 0.005m].$$

Corollary 10.4. *Let $Y_1, \dots, Y_n$ be independent and identically distributed. Assume that $Y_1 \sim N(\mu, \sigma)$. Define*

$$S = Y_1 + Y_2 + \cdots + Y_n$$

Then, the distribution of S is ...

Proof. $\square$