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Binomial option pricing (review).

Problem 3.1. Let the continuously compounded risk-free interest rate be denoted by r. You are building a model for the price of a stock which pays dividends continuously with the dividend yield δ. Consider a binomial tree modeling the evolution of the stock price. Let the length of each period be h and let the up factor be denoted by u, and the down factor by d. What is the **no-arbitrage** condition for the binomial tree you are building?

Solution:

$$d < e^{(r-\delta)h} < u \tag{3.1}$$

Problem 3.2. Set up the framework for pricing by replication in a one-period binomial tree! What is the *risk-neutral pricing* formula?

Problem 3.3. The current price of a certain non-dividend-paying stock is \$100 per share. You are modeling the price of this stock at the end of a quarter year using a one-period binomial tree under the assumption that the stock price can either increase by 4%, or decrease by 2%.

The continuously compounded risk-free interest rate is 3%.

What is the price of a three-month, at-the-money European call option on the above stock consistent with the above binomial tree?

Solution: The risk-neutral probability is

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{e^{0.0075} - 0.98}{1.04 - 0.98} = 0.4588. \tag{3.2}$$

At expiration, the option is only in-the-money at the *up node* and its payoff at that node equals \$4. So, by the risk-neutral pricing formula, we have

$$V_C(0) = e^{-0.0075}(4)(0.4588) = 1.82149. \tag{3.3}$$

Problem 3.4. Let the continuously compounded risk-free interest rate be equal to 0.04.

The current price of a continuous-dividend-paying stock is \$80 and its dividend yield is 0.02. The stock's volatility is 0.25. You model the evolution of the stock price over the following half year using a two-period **forward** binomial tree.

What is the price of a six-month, \$82-strike European put option on the above stock consistent with the given binomial tree?

Solution: This is a forward binomial tree, so we can use a "shortcut" to calculate the risk-neutral probability

$$p^* = \frac{1}{1+e^{\sigma\sqrt{h}}} = \frac{1}{1+e^{0.125}} = 0.4688. \tag{3.4}$$

The up and down factors in the above forward binomial tree are

$$\begin{aligned} u &= e^{(r-\delta)h+\sigma\sqrt{h}} = e^{0.005+0.125} = e^{0.13}, \\ d &= e^{(r-\delta)h-\sigma\sqrt{h}} = e^{0.005-0.125} = e^{-0.12}. \end{aligned} \tag{3.5}$$

So, the possible stock prices at the end of the second period are

$$S_{uu} = S(0)u^2 = 80e^{0.26} > 82, \quad S_{ud} = 80e^{0.01} = 80.804, \quad S_{dd} = S(0)d^2 = 80e^{-0.24} = 62.9302. \tag{3.6}$$

Finally, the option price equals

$$V_P(0) = e^{-0.04(0.5)}\left[2(0.4688)(1-0.4688)(82-80.804) + (1-0.4688)^2(81-62.9302)\right] = 5.85832. \tag{3.7}$$

Problem 3.5. The current price of a continuous-dividend-paying stock is \$100 per share. Its volatility is given to be 0.2 and its dividend yield is 0.03.

The continuously compounded risk-free interest rate equals 0.06.

Consider a \$95-strike European put option on the above stock with nine months to expiration. Using a three-period forward binomial tree, find the price of this put option.

(a) \$2.97
(b) \$3.06
(c) \$3.59
(d) \$3.70
(e) None of the above.

Solution: (c)
The up and down factors in the above model are

$$u = e^{0.03\times 0.25+0.2\sqrt{0.25}} = 1.1135,$$
$$d = e^{0.03\times 0.25-0.2\sqrt{0.25}} = 0.9116.$$

The relevant possible stock prices at the "leaves" of the binomial tree are

$$S_{ddd} = d^3S(0) = 100(0.9116)^3 = 75.7553,$$
$$S_{ddu} = d^2uS(0) = 92.5335.$$

The remaining two final states of the world result in the put option being out-of-the-money at expiration.

The risk-neutral probability of the stock price moving up in a single period is

$$p^* = \frac{1}{1+e^{0.2\sqrt{0.25}}} = 0.475.$$

So, the price of the European put option equals

$$V_P(0) = e^{-0.06(3/4)}\left[(95-75.7553)(1-0.475)^3 + (95-92.5335)(3)(1-0.475)^2(0.475)\right] = 3.5884.$$

Problem 3.6. Consider a non-dividend-paying stock whose current price is \$100 per share. Its volatility is given to be 0.25. You model the evolution of the stock price over the following year using a two-period forward binomial tree.

The continuously compounded risk-free interest rate is 0.04.

Consider a \$110-strike, one-year **down-and-in** put option with a barrier of \$90 on the above stock. What is the price of this option consistent with the above stock-price model?

(a) About \$10.23
(b) About \$11.55
(c) About \$11.78
(d) About \$11.90
(e) None of the above.

Solution: (d)
The up and down factors in the above forward binomial tree are

$$u = e^{0.02+0.25/\sqrt{2}} = 1.2175, \quad d = 0.8549.$$

Of course, in the exam, you would not necessarily populate the entire stock-price tree, since you would want to work efficiently and only consider the nodes you need for pricing.

The option is knocked-in only if the stock price goes down in the first step. So, the payoff of the option will be

$$\begin{aligned} V_{du} &\approx 5.9159, &&\text{if the path } \textit{down-up} \text{ is taken,}\\ V_{dd} &\approx 36.0146, &&\text{if the path } \textit{down-down} \text{ is taken,}\\ V_{uu} = V_{ud} &= 0, &&\text{otherwise.} \end{aligned}$$

The risk-neutral probability of a single step up in the tree equals

$$p^* = \frac{1}{1+e^{0.25/\sqrt{2}}} \approx 0.4559.$$

So, the option price is

$$V(0) = e^{-rT}[(1-p^*)^2 V_{dd} + p^*(1-p^*)V_{du}] = 11.91.$$

Problem 3.7. The current stock price is observed to be \$100 per share. The stock is projected to pay dividends continuously at the rate proportional to its price with the dividend yield of 0.03. The stock's volatility is given to be 0.23. You model the evolution of the stock price using a two-period forward binomial tree with each period of length one year.

The continuously compounded risk-free interest rate is given to be 0.04.

What is the price of a two-year, \$101-strike **American** put option on the above stock consistent with the above stock-price tree?

(a) About \$6.62
(b) About \$8.34
(c) About \$8.83
(d) About \$11.11
(e) None of the above.

Solution: (d)
The up and down factors in the forward tree are

$$u = e^{0.01+0.23} = 1.2712, \quad \text{andd} = e^{-0.22} = 0.8025.$$

Of course, in the exam, you would not necessarily populate the entire stock-price tree, since you would want to work efficiently and only consider the nodes you need for pricing.

The risk-neutral probability equals

$$p^* = \frac{1}{1+e^{0.23}} = 0.4428.$$

Evidently, the option produces a positive payoff only in the down-down node; the value of this payoff is $V_{dd} = 101 - 64.40 = 36.60$.

The continuation value at the down node is, hence,

$$CV_d = e^{-0.04} \times (1 - 0.4428) \times 36.60 = 15.57.$$

On the other hand, the value of immediate exercise at the down node equals $IE_d = 101 - 80.25 = 20.75$. So, it is optimal to exercise the American put in the down node and the value of the American put equals $V_d^P = 20.75$.

In the up node, both the continuation value and the immediate exercise value are zero. So, the initial price of the American put is

$$V_P(0) = e^{-0.04} \times (1 - 0.4428) \times 20.75 = 11.11$$