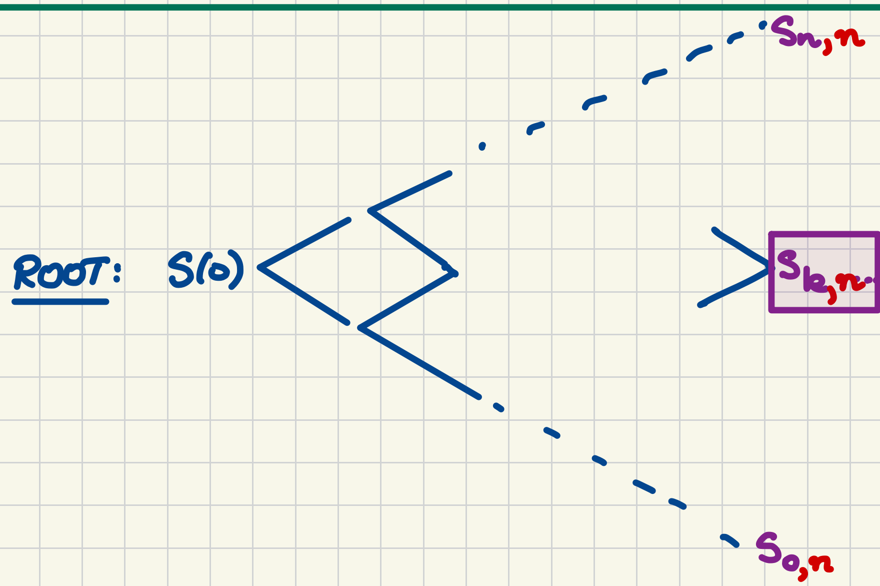
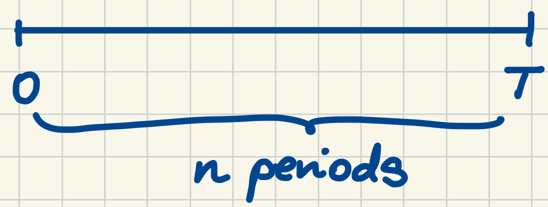
M339D: November 7th, 2025.

The Pre-Limit n-Period Binomial Tree.



$s_{k,n}$ … the stock price resulting from k steps up



$h_n = \frac{T}{n}$

u_n … up factor

d_n … down factor

e.g., forward tree

$$u_n = exp\left(r \cdot \frac{T}{n} + \sigma\sqrt{\frac{T}{n}}\right)$$

$$d_n = exp\left(r \cdot \frac{T}{n} - \sigma\sqrt{\frac{T}{n}}\right)$$

$$s_{k,n} = S(0)\, u_n^k \cdot d_n^{n-k} = S(0) \cdot \left(\frac{u_n}{d_n}\right)^{k} \cdot d_n^{n}$$

k … corresponds to a realization of the binomial random variable w/ n trials and success probability

$$p_n^* = \frac{e^{r\left(\frac{T}{n}\right)} - d_n}{u_n - d_n}$$

e.g., in the forward tree

$$\boxed{p_n^* = \frac{1}{1+e^{\sigma\sqrt{\frac{T}{n}}}}} \xrightarrow[n\to\infty]{} \frac{1}{2}$$

Say, X_n ... # of steps up in n periods

$X_n \sim$ Binomial (# of trials $= n$, success probability $= p_n^*$)

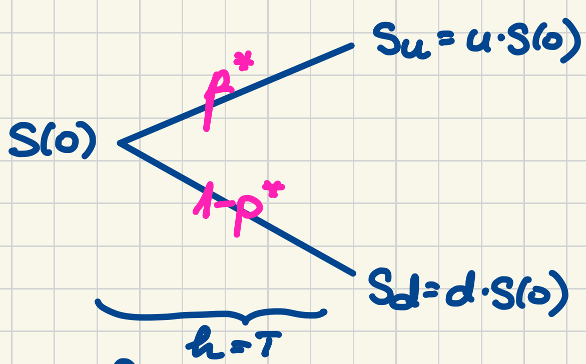
Q: Can we simply use the normal approximation to the binomial?

$$\boxed{\text{☹} \quad p_n^* \text{ depends on } n}$$

Subjective Probability.

Recall: When pricing, we use $\boxed{\mathbb{P}^* \text{ ... the risk-neutral}}$ probability measure

associated w/ $\boxed{p^* = \frac{e^{rh} - d}{u - d}}$



Q: If we invest in one share of this non-dividend-paying stock @ time-0, what is the expected wealth @ time-T under $\mathbb{P}^*$?

$\longrightarrow$: $\mathbb{E}^*[S(T)] = S(0)e^{rT}$ $\square$

In Contrast:

There can be a subjective probability measure $\mathbb{P}$.
We can think about the quality of our investment under that probability measure, i.e.,

$$\mathbb{E}[S(T)] = S(0)e^{\alpha \cdot T}.$$

We call this α the mean rate of return.
In a binomial tree, we can express the "true probability" of a step up as

$$\boxed{p = \frac{e^{\alpha h} - d}{u - d}}$$

The mean rate of return of the stock under the risk-neutral measure $\mathbb{P}^*$ is $\underline{r}$.

Moment Generating Functions.

For a random variable Y, and for an independent argument denoted by t, we define the moment generating function (mgf) of Y as this function of t:

$$M_Y(t) := \mathbb{E}\left[e^{Y \cdot t}\right]$$ for all t such that the expectation existing

Q: $M_Y(0) = \underline{1}$ $\Rightarrow$ @ least $t=0$ is in the domain.

We say that the mgf exists if it is finite for t such that $|t| \leq b$ for a $b>0$.

Goal: To understand e^X w/ $X \sim \text{Normal}(\text{mean}=m, \text{var}=\nu^2)$

Recall: In terms of $Z \sim N(0,1)$,

$$\boxed{X = m + \nu Z}$$ ✓

Fact: ✓ $\boxed{M_Z(t) = e^{\frac{t^2}{2}}}$ for all $t \in \mathbb{R}$

$\Rightarrow$ For any normal X:

$$M_X(t) = \mathbb{E}\left[e^{Xt}\right] = \mathbb{E}\left[e^{(m+\nu Z)t}\right]$$

$$= \mathbb{E}\left[e^{mt} e^{\nu t \cdot Z}\right]$$

$$= e^{mt} \cdot \mathbb{E}\left[e^{\nu t \cdot Z}\right]$$

$$= e^{mt} \cdot M_Z(\nu t)$$

$$= e^{mt} \cdot e^{\frac{\nu^2 t^2}{2}} = e^{mt + \frac{\nu^2 t^2}{2}}$$