University of Texas at Austin

HW Assignment 9

Binomial option pricing: One period.

Provide your **complete solution** to the following problems. Final answers only, without appropriate justification, will receive zero points even if correct.

Problem 9.1. (10 points) Let $S(0) = \$100$, $K = \$105$, $r = 8\%, T = 0.5$ and $\delta = 0$. Suppose that $u = 1.3$ and $d = 0.8$. Using the one-period binomial model, calculate the following:

a. (5 pts) The fair premium for a European put with the above characteristics.
b. (3 pts) The Δ in the corresponding replicating portfolio.
c. (2 pts) The amount B invested in the riskless asset in the replicating portoflio.

Solution: *Note:* See Problem 10.1.b. from the McDonald text!
The risk-neutral probability is

$$p^* = \frac{e^{rT} - d}{u - d} = 0.48.$$

a. We start by drawing the one-period tree per the described model. According to the risk-neutral pricing formula in the binomial model, we have that the time−0 price of the put is

$$\begin{aligned} V_P(0) &= e^{-rT}\left[p^* V_u + (1 - p^*)V_d\right] \\ &= e^{-0.08 \cdot 0.5}\left[0.48 \cdot 0 + 0.52 \cdot 25\right] \\ &= 12.5. \end{aligned} \tag{9.1}$$

b.

$$\Delta = \frac{0 - 25}{100(1.3 - 0.8)} = -\frac{1}{2}.$$

c.

$$B = V_P(0) - \Delta S(0) = 12.5 + \frac{100}{2} = 62.5.$$

Problem 9.2. (5 points) Consider a continuous-dividend-paying stock with the current price of \$50 and dividend yield 0.02.

The continuously-compounded, risk-free interest rateis 0.05.

You are using a one-period binomial tree to model the stock price at the end of the next quarter. You assume that the stock price can either increase by 0.04 or decrease by 0.02. What is the risk-neutral probability associated with this tree?

Solution: By definition, in our usual notation, we have

$$p^* = \frac{e^{(r-\delta)h} - d}{u - d} = \frac{e^{(0.05-0.02)(0.25)} - 0.98}{1.04 - 0.98} = 0.4588.$$

Problem 9.3. (10 points) Consider a non-dividend paying stock whose current price is \$52 per share. You model the evolution of this stock price over the following year using a one-period binomial tree under the assumption that the stock price can be either \$72 or \$42 in one year.

The continuously-compounded, risk-free interest rateis 0.05.

Consider a \$50-strike, one-year European call option on the above stock. What is the call price consistent with the above stock-price model?

Solution: The risk-neutral probability of an up movement is

$$p^* = \frac{52e^{0.05} - 42}{72 - 42} = 0.4222.$$

So, the price of our call is

$$V_C(0) = e^{-0.05}[0.4222 \times (72-50)_+ + (1-0.4222) * (42-50)_+] = 8.8355.$$

Problem 9.4. (5 points) The current price of a continuous-dividend-paying stock is $100 per share. Its volatility is given to be 0.30 and its dividend yield is 0.03.

The continuously-compounded, risk-free interest rateequals 0.06.

Consider a $95-strike European put option on the above stock with three months to expiration. Using a one-period forward binomial tree, find the price of this put option.

Solution: The up and down factors in the above model are

$$u = e^{0.03\times 0.25 + 0.3\sqrt{0.25}} = 1.17058,$$
$$d = e^{0.03\times 0.25 - 0.3\sqrt{0.25}} = 0.867188.$$

The possible stock prices at the "leaves" of the binomial tree are

$$S_u = S(0)u = 117.058 \quad \text{and} \quad S_d = dS(0) = 86.7188.$$

The risk-neutral probability of the stock price moving up in a single period is

$$p^* = \frac{1}{1 + e^{0.3\sqrt{0.25}}} = 0.46257.$$

So, the price of the European put option equals

$$V_P(0) = e^{-0.06(1/4)}\,[(95 - 86.7188)(1 - 0.46257)] = 4.38433.$$

Problem 9.5. (5 points) Consider the one-period binomial option pricing model. Let $V_C(0) > 0$ denote the price of a European call on a stock which pays continuous dividends. What is the impact on the value of European call option prices if the company decides to increase the dividend yield paid to the shareholders?

(a) The call option price will drop.

(b) The call option price will increase.

(c) The call option price will always remain constant.

(d) The impact on the price of the call cannot be determined using the binomial option pricing model.

(e) There is not enough information provided.

Solution: (a)
Let $\delta < \tilde{\delta}$ be the two dividend yields. Then, the risk-neutral price of the European call on the stock with the dividend yield δ equals

$$V_C(0) = e^{-rT}[p^*(S_u - K)_+ + (1-p^*)(S_d - K)_+]$$

with $p^* = (e^{(r-\delta)h} - d)/(u-d)$. On the other hand, the risk-neutral price of the European call on the stock with the dividend yield $\tilde{\delta}$ equals

$$\tilde{V}_C(0) = e^{-rT}[\tilde{p}^*(S_u - K)_+ + (1-\tilde{p}^*)(S_d - K)_+]$$

with $\tilde{p}^* = (e^{(r-\tilde{\delta})h} - d)/(u-d)$. We have

$$\delta < \tilde{\delta} \quad \Rightarrow \quad e^{(r-\delta)h} > e^{(r-\tilde{\delta})h} \quad \Rightarrow p^* > \tilde{p}^* \quad \Rightarrow \quad V_C(0) > \tilde{V}_C(0).$$

Problem 9.6. (15 points) Consider a continuous-dividend-paying stock whose current price is \$40 and whose dividend yield is 0.02. The price of stock in three months is modeled using a one-period binomial tree.

The continuously-compounded, risk-free interest rateis 0.06.

According to the above stock-price model, the replicating portfolio of an at-the-money, three-month European call option consists of:

- 0.6 shares of stock, and
- borrowing \$20 at the risk-free interest rate.

What is the risk-free portion of the replicating portfolio for the otherwise identical put option?

Solution: Let the stock price at the *up* node be denoted by S_u and the stock price at the *down* node by S_d. From the given value of the Δ in the replicating portfolio for the call, we conclude that the call is in-the-money at the *up* node and out-of-the-money at the *down* node. In fact,

$$0.6 = \Delta = e^{-0.02(0.25)}\frac{V_u}{S_u - S_d} = e^{-0.005}\frac{S_u - S(0)}{S(0)(u-d)} = e^{-0.005}\frac{u-1}{u-d}.$$

From the other component of the call's replicating portfolio, we get

$$20 = -B = e^{-0.06(0.25)}\frac{dV_u}{u-d} = e^{-0.015}\frac{dS(0)(u-1)}{u-d}.$$

Combining the above two equations, we get

$$20 = e^{-0.015}d(40)(0.6)e^{0.005} \quad \Rightarrow \quad d = \frac{20}{40(0.6)}e^{0.01} = 0.841708.$$

Reusing the equation for Δ, we get

$$0.6(u - 0.841708)e^{0.005} = u - 1 \quad \Rightarrow \quad u = \frac{1 - 0.6(0.841708)e^{0.005}}{1 - 0.6e^{0.005}} = 1.24044.$$

The put option is out-of-the-money at the *up* node and in-the-money at the *down* node where its payoff is $V_D^P = 40(1 - 0.841708) = 6.33168$. The risk-free portion of the put's replicating portfolio is

$$B_P =^{-0.015} \frac{1.24044(6.33168)}{1.24044 - 0.841708} = 19.4044.$$