UNIVERSITY OF TEXAS AT AUSTIN

HW Assignment 3

Prerequisite material. Long/short positions. Short sales.

Provide your **complete solution** to the following problems. Final answers only, without appropriate justification, will receive zero points even if correct.

Problem 3.1. (5 points) Let the accumulation function be given by

$$a(t) = (1 + 0.05)^{2t}(1 + 0.02)^{t/3}$$

Then, we can say the following about the continuously compounded, risk-free interest rate r associated with the above accumulation function:

(a) $r = 0.11$
(b) $r = (\ln(1.05))^2 + (\ln(1.02))^{1/3}$
(c) $r = 2\ln(1.05) + \frac{1}{3}\ln(1.02)$
(d) The continuously compounded, risk-free interest rate is not constant.
(e) None of the above

Solution: (c)

$$r = \frac{d}{dt}\ln(a(t)) = \frac{d}{dt}\ln[(1 + 0.05)^{2t}(1 + 0.2)^{t/3}] = \frac{d}{dt}[2t\ln(1.05) + \frac{t}{3}\ln(1.02)] = 2\ln(1.05) + \frac{1}{3}\ln(1.02).$$

Problem 3.2. (5 points) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be two functions given by

$$f(x) = |x - 10|$$

and

$$g(x) = \begin{cases} \min(x, 4) & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Then, $f(g(7) + 1)$ equals ...

(a) 0
(b) 3
(c) 4
(d) 5
(e) None of the above

Solution: (d)

$$f(g(7) + 1) = f(4 + 1) = f(5) = 5.$$

Problem 3.3. (5 points) *Source: Sample P Exam, Problem #198.*
In a certain group of cancer patients, each patient's cancer is classified in exactly one of the following five stages: stage 0, stage 1, stage 2, stage 3, or stage 4. You know the following:

(i) 75% of the patients in the group have stage 2 or lower;
(ii) 80% of the patients in the group have stage 1 or higher;
(iii) 80% of the patients in the group have stage 0, 1, 3, or 4.

One patient from the group is randomly selected. Calculate the probability that the selected patient's cancer is stage 1.

(a) 1/4

(b) 1/5

(c) 7/20

(d) Not enough information is given.

(e) None of the above.

Solution: (c)
Define $p_k = \mathbb{P}[$a randomly chosen patien has cancer stage $i]$, for $i = 0, 1, 2, 3, 4$. By the definition of probability, we have that

$$p_0 + p_1 + p_2 + p_3 + p_4 = 1.$$

From condition (i), we conclude that

$$p_0 + p_1 + p_2 = 0.75.$$

From condition (ii), we conclude that

$$p_1 + p_2 + p_3 + p_4 = 1 - p_0 = 0.80 \quad \Rightarrow \quad p_0 = 0.2.$$

From condition (iii), we conclude that

$$p_0 + p_1 + p_3 + p_4 = 1 - p_2 = 0.80 \quad \Rightarrow \quad p_2 = 0.2.$$

So, $p_1 = 0.75 - 0.2 - 0.2 = 0.35 = 7/20$.

Problem 3.4. (5 points) *Source: Sample P exam, Problem #201.*
A theme park conducts a study of families that visit the park during a year. The fraction of such families of size m is $\frac{8-m}{28}$, for $m = 1, 2, 3, 4, 5, 6$, and 7.

For a family of size m that visits the park, the number of members of the family that ride the roller coaster follows a discrete uniform distribution on the set $\{1, \dots, m\}$.

Calculate the probability that a family visiting the park has exactly six members, **given** that exactly five members of the family ride the roller coaster.

(a) 0.17

(b) 0.21

(c) 0.24

(d) 0.31

(e) None of the above.

Solution: (d)
Let M be the size of a randomly chosen family and let N denote the number of family members who ride the roller coaster. We seek $\mathbb{P}[M = 6|N = 5]$. By the definition of conditional probability,

$$\mathbb{P}[M = 6|N = 5] = \frac{\mathbb{P}[M = 6, N = 5]}{\mathbb{P}[N = 5]}.$$

Again, by the definition of conditional probability

$$\mathbb{P}[M = 6, N = 5] = \mathbb{P}[M = 6]\mathbb{P}[N = 5|M = 6] = \frac{8-6}{28}\left(\frac{1}{6}\right) = \frac{2}{28}\left(\frac{1}{6}\right).$$

By the *Law of Total Probability*, we have

$$\begin{aligned}
\mathbb{P}[N = 5] &= \mathbb{P}[M = 1]\mathbb{P}[N = 5|M = 1] + \mathbb{P}[M = 2]\mathbb{P}[N = 5|M = 2] + \mathbb{P}[M = 3]\mathbb{P}[N = 5|M = 3] \\
&\quad + \mathbb{P}[M = 4]\mathbb{P}[N = 5|M = 4] + \mathbb{P}[M = 5]\mathbb{P}[N = 5|M = 5] + \mathbb{P}[M = 6]\mathbb{P}[N = 5|M = 6] \\
&\quad + \mathbb{P}[M = 7]\mathbb{P}[N = 5|M = 7] \\
&= 0 + 0 + 0 + 0 + \frac{8-5}{28}\left(\frac{1}{5}\right) + \frac{8-6}{28}\left(\frac{1}{6}\right) + \frac{8-7}{28}\left(\frac{1}{7}\right) \\
&= \frac{3}{28}\left(\frac{1}{5}\right) + \frac{2}{28}\left(\frac{1}{6}\right) + \frac{1}{28}\left(\frac{1}{7}\right).
\end{aligned}$$

Finally,

$$\mathbb{P}[M=6|N=5] = \frac{\mathbb{P}[M=6,N=5]}{\mathbb{P}[N=5]} = \frac{\frac{2}{28}\left(\frac{1}{6}\right)}{\frac{3}{28}\left(\frac{1}{5}\right)+\frac{2}{28}\left(\frac{1}{6}\right)\frac{1}{28}\left(\frac{1}{7}\right)} = \frac{\frac{2}{6}}{\frac{3}{5}+\frac{2}{6}+\frac{1}{7}} \approx 0.3097345.$$

Problem 3.5. (4 points) Consider the functions $f:[0,\infty)\to\mathbb{R}$ and $g:[0,\infty)\to\mathbb{R}$. Let f be given by

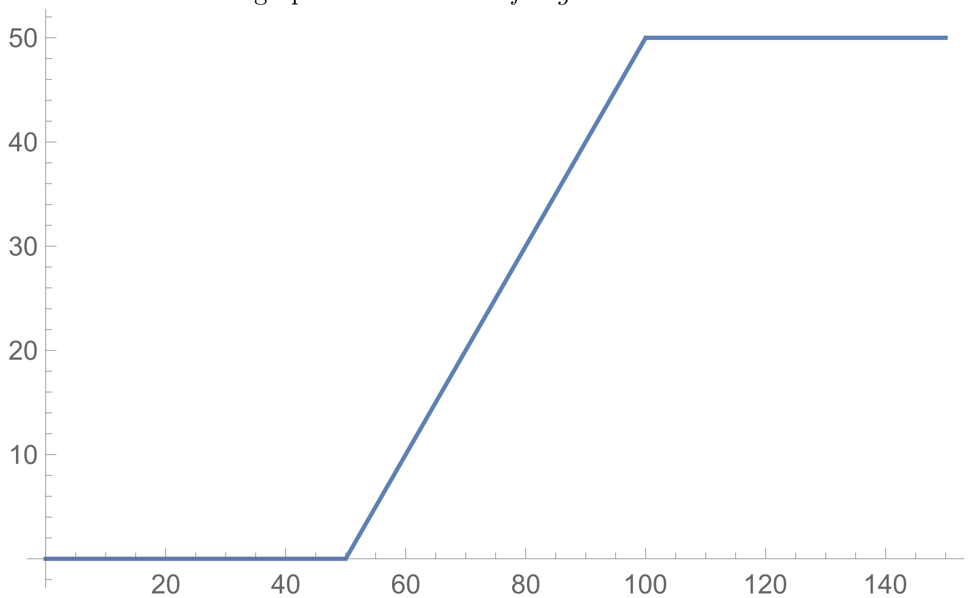
$$f(x)=\max(x-50,0).$$

Let g be given by

$$g(x)=\max(x-100,0).$$

What can you say about the monotonicity of the function $f-g$? Remember to justify your answer!

Solution: Here is the graph of the function $f-g$:



For any $x_1 < x_2$, we see that $f(x_1)-g(x_1) \leq f(x_2)-g(x_2)$.

Problem 3.6. (5 points) Complete the following definition:

A financial portfolio is said to be *long with respect to an underlying asset* if
Solution:
its payoff/profit function is increasing as a function of the final asset price.

Problem 3.7. (5 points) Complete the following definition:

A financial portfolio is said to be *short with respect to an underlying asset* if
Solution:
its payoff/profit function is decreasing as a function of the final asset price.

Problem 3.8. (3 points) Consider an outright purchase of a share of continuous-dividend-paying stock whose current price is \$80 per share and whose dividend yield is 0.02. Let the continuously compounded, risk-free interest rate be equal to 0.04. What is the time-2 break-even stock price for this investment?

Solution: In our usual notation, the break-even price is

$$S(0)e^{(r-\delta)T} = 80e^{(0.04-0.02)(2)} = 80e^{0.04} = 83.2649$$

Problem 3.9. (3 points) Bertram sells short 10 shares of a continuous-dividend-paying stock. The time-0 price of this stock is \$100 and its dividend yield is 0.03. Assume that the continuously compounded, risk-free interest rate equals 0.06. If Bertram closes the short sale in six months, what is his break-even final stock price?

Solution: In our usual notation, the break-even price for Bertram's short sale equals

$$S(0)e^{(r-\delta)T} = 100e^{(0.06-0.03)(0.5)} = 100e^{0.015} = 101.511.$$

Problem 3.10. (5 points) Let the current price of a non-dividend-paying stock be \$40.

You model the distribution of the time−1 price of the above stock as follows:

$$S(1) \sim \begin{cases} 42, & \text{with probability } 1/4, \\ 38, & \text{with probability } 1/2, \\ 36, & \text{with probability } 1/4. \end{cases}$$

The continuously compounded, risk-free interest rate is 0.04.

What is your expected profit under the above model, if you short sell one share of stock at time−0 and intend to close the short sale at time−1?

(a) 1.50

(b) 2.17

(c) 2.54

(d) 3.13

(e) None of the above.

Solution: (d)
The initial cost is $-S(0)$ and the payoff is $-S(1)$. So, with $T = 1$, the profit equals

$$-S(T) + S(0)e^{rT}.$$

Thus, the expected profit equals

$$-\mathbb{E}[S(T)] + S(0)e^{rT}.$$

According to the given model for the stock price, we have

$$\mathbb{E}[S(T)] = 42\left(\frac{1}{4}\right) + 38\left(\frac{1}{2}\right) + 36\left(\frac{1}{4}\right) = 38.5.$$

Finally, the expected profit is

$$-38.5 + 40e^{0.04} = 3.132431.$$

Problem 3.11. (5 points) The current price of a continuous-dividend paying stock is \$100 per share. Its dividend yield is 0.02. You purchase one share of this stock. You do not intend to make any further trades over the next year. You intend to liquidate your investment at the end of the year.

You model the stock price at the end of the year to be distributed as follows:

$$S(T) \sim \begin{cases} 90 & \text{with probability } 1/6 \\ 100 & \text{with probability } 1/2 \\ 105 & \text{with probability } 1/3 \end{cases}$$

The continuously compounded, risk-free interest rate is 0.01.

What is the expected profit of your investment?

(a) 0

(b) 1.015

(c) 101.005

(d) 102.02

(e) None of the above.

Solution: (b)
The expected stock price is

$$\mathbb{E}[S(T)] = 90\left(\frac{1}{6}\right) + 100\left(\frac{1}{2}\right) + 105\left(\frac{1}{3}\right) = 100.$$

Due to continuous reinvestment of dividends in the same stock, the number of shares of stock owned at the end of the year is $e^{0.02}$. So, the expected wealth/payoff is $100e^{0.02}$. Finally, the expected profit is

$$100e^{0.02} - 100e^{0.01} = 1.015117.$$