M378K: April 9th, 2025.

More about Maximum Likelihood Estimation.

Def'n. An estimator $\hat{\Theta} = \hat{\Theta}(y_1, \dots, y_n)$ is called the **maximum likelihood estimator (MLE)** if it satisfies that for any $\hat{\Theta}' = \hat{\Theta}'(y_1, \dots, y_n)$ we have:

$$\boxed{L(\hat{\Theta}; y_1, \dots, y_n) \geq L(\hat{\Theta}'; y_1, \dots, y_n) \quad \text{for all } y_1, y_2, \dots, y_n}$$

Example. $Y_1, Y_2, \dots, Y_n \sim U(0, \theta)$ $\quad \theta > 0$ unknown

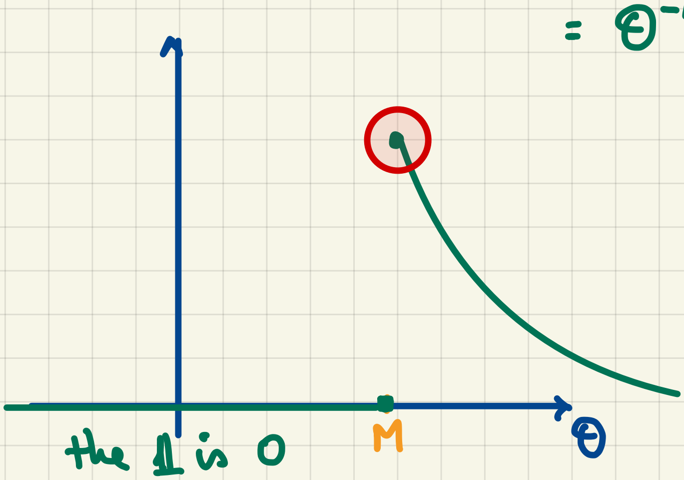
The pdf of $U(0,\theta)$: $f^{\theta}(y) = \frac{1}{\theta} \mathbb{1}_{[0,\theta]}(y) = \frac{1}{\theta} \mathbb{1}_{\{0 \leq y \leq \theta\}}$

The likelihood:

$$\begin{aligned}
L(\theta; y_1, \dots, y_n) &= f^{\theta}(y_1) \cdots f^{\theta}(y_n) \\
&= \left(\frac{1}{\theta} \mathbb{1}_{\{0 \leq y_1 \leq \theta\}}\right) \cdots \left(\frac{1}{\theta} \mathbb{1}_{\{0 \leq y_n \leq \theta\}}\right) \\
&= \left(\frac{1}{\theta}\right)^n \mathbb{1}_{\{0 \leq y_1, \dots, y_n \leq \theta\}} \\
&= \left(\frac{1}{\theta}\right)^n \mathbb{1}_{\{0 \leq \min(y_1, \dots, y_n)\}} \mathbb{1}_{\{\max(y_1, \dots, y_n) \leq \theta\}}
\end{aligned}$$

"assume $y_1, \dots, y_n \geq 0$"

$$= \theta^{-n} \cdot \mathbb{1}_{\{\underbrace{\max(y_1, \dots, y_n)}_{y_{(n)} \,=:\, M} \leq \theta\}}$$



$$\boxed{\hat{\Theta}_{MLE} = \max(y_1, \dots, y_n) = y_{(n)}}$$

Q: Is this the same as the moment matching estimator?

→: Matching the theoretical to the sample mean

$$\frac{\theta}{2} = \bar{Y}$$

$$\Rightarrow \boxed{\hat{\theta}_{MM} = 2\bar{Y}}$$

Example. Let $Y_1, \dots, Y_n$ be a random sample from a distribution w/ density

$$f^{\theta}(y) = \theta y^{\theta-1} \cdot \underbrace{\mathbb{1}_{[0,1]}(y)}$$

for some unknown positive parameter θ.

Find the MLE for θ.

→:

$$L(\theta; y_1, \dots, y_n) = \prod_{i=1}^{n} f^{\theta}(y_i)$$

$$= \prod_{i=1}^{n} \left(\theta \cdot y_i^{\theta-1}\right) = \theta^n \cdot \left(\prod_{i=1}^{n} y_i\right)^{\theta-1}$$

$$\ell(\theta; y_1, \dots, y_n) = n \cdot \ln(\theta) + (\theta-1) \cdot \ln\left(\prod_{i=1}^{n} y_i\right)$$

$$= n \cdot \ln(\theta) + (\theta-1)\left(\sum_i \ln(y_i)\right)$$

$$\ell'(\theta; y_1, \dots, y_n) = n \cdot \frac{1}{\theta} + \sum_{i=1}^{n} \ln(y_i) = 0$$

$$\frac{n}{\theta} = -\sum_{i=1}^{n} \ln(y_i)$$

$$\boxed{\hat{\theta}_{MLE} = -\frac{n}{\sum_{i=1}^{n} \ln(y_i)}}$$

□

Example. $\boxed{\text{CAS Exam 3, Spring 2007}}$

Consider a random sample $Y_1, \dots, Y_n$ from a distribution w/ the pdf

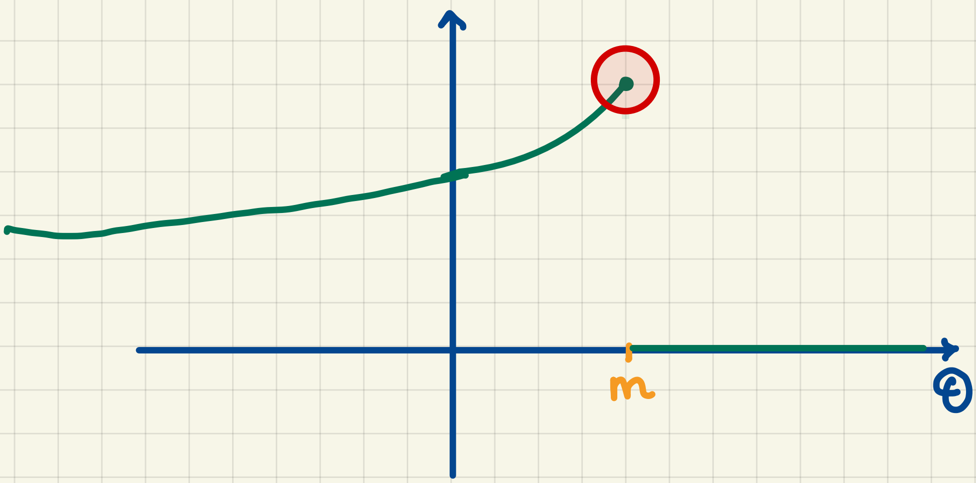
$$f^{\theta}(y) = e^{-y+\theta} \cdot \mathbb{1}_{(\theta,\infty)}(y)$$

Find the MLE for θ.

$\rightarrow$:

$$L(\theta; y_1, \dots, y_n) = \prod_{i=1}^{n} \left(e^{-y_i+\theta} \mathbb{1}_{(\theta,\infty)}(y_i) \right)$$

$$= e^{-\sum_{i=1}^{n} y_i + n\theta} \cdot \mathbb{1}_{\{\underbrace{\min(y_1, \dots, y_n)}_{\underbrace{y_{(1)}}_{m}} > \theta\}}$$



$$\boxed{\hat{\theta}_{MLE} = y_{(1)}}$$