Required Returns [cont'd]

W: April 24th, 2019.

- Start w/ an arbitrary portfolio P.

  Its Sharpe ratio is denoted by $\eta_P = \frac{\mathbb{E}[R_P] - r_f}{\sigma_P}$

- Consider an investment I.
- Construct a new portfolio:

P'
- **keep** the "old" portfolio P,
- **borrow** $x \cdot$(value of P) @ the risk-free interest rate r_f

and

- **invest** the proceeds of the loan in the investment I

Assume that the weight x is small!

$\Rightarrow$ The new return: $\boxed{R_{P'} = R_P - x \cdot r_f + x \cdot R_I}$

- The new excess return:

$$\mathbb{E}[R_{P'}] - r_f = \underbrace{\mathbb{E}[R_P]} - x \cdot r_f + x \cdot \mathbb{E}[R_I] \underbrace{- r_f}$$

$$= \underbrace{\mathbb{E}[R_P] - r_f}_{\text{the excess return of the "old" portfolio P}} + x \underbrace{\left(\mathbb{E}[R_I] - r_f\right)}_{\text{the excess return of investment I}} \qquad (\mathbb{E})$$

· The SD of the return of portfolio P':

$$Var[R_{P'}] = Var[R_P \underbrace{- x \cdot r_f}_{\text{constant}} + x \cdot R_I]$$

$$= Var[R_P + x \cdot R_I]$$

$$Var[R_{P'}] = Var[R_P] + 2 \cdot x \cdot Cov[R_P, R_I] + \underbrace{x^2 \cdot Var[R_I]}_{\text{can be diversified away by } x \text{ small!}}$$

$$f(y) = \sqrt{y} = y^{1/2}$$

$$f'(y) = \frac{1}{2} \frac{1}{\sqrt{y}}$$

By the Taylor approximation:

$$f(y_0 + dy) = f(y_0) + \frac{1}{2} \cdot \frac{1}{\sqrt{y_0}} dy + \text{lower order terms}$$

$y_0 \leftrightarrow Var[R_P]$

$dy \leftrightarrow Var[R_{P'}] - Var[R_P] = 2 \cdot x \cdot Cov[R_P, R_I]$

$\Rightarrow$

$$SD[R_{P'}] = \sqrt{Var[R_{P'}]}$$

$$= \sqrt{Var[R_P]} + \frac{1}{\cancel{2}} \cdot \frac{1}{\sqrt{Var[R_P]}} (\cancel{2} x \cdot Cov[R_P, R_I])$$

$\Rightarrow$

$$SD[R_{P'}] = SD[R_P] + x \cdot \frac{1}{\cancel{SD[R_P]}} \cdot \cancel{SD[R_P]} \cdot SD[R_I] \cdot corr[R_P, R_I]$$

2.

$\Rightarrow$

$$SD[R_{P'}] = \underbrace{SD[R_P]}_{\text{the volatility of the "old" portfolio } P} + \underbrace{x \cdot SD[R_I] \cdot \mathrm{corr}[R_P, R_I]}_{\text{the "incremental" risk due to the addition of investment } I} \qquad (\sigma)$$

Putting ($\mathbb{E}$) & (σ) together, in order for portfolio P' to be an improvement, we must have: $\boxed{x>0}$

$$\cancel{x}(\mathbb{E}[R_I] - r_f) > \cancel{x} \cdot SD[R_I] \cdot \mathrm{corr}[R_P, R_I] \cdot \eta_P$$

The Sharpe ratio of portfolio P

the effect of staying on the line through P w/ the same increase in risk

$\Rightarrow$

$$\mathbb{E}[R_I] - r_f > SD[R_I] \cdot \mathrm{Corr}[R_P, R_I] \cdot \frac{\mathbb{E}[R_P] - r_f}{SD[R_P]}$$

$$\mathbb{E}[R_I] > r_f + \frac{SD[R_I]}{SD[R_P]} \cdot \mathrm{Corr}[R_P, R_I] \cdot (\mathbb{E}[R_P] - r_f)$$

$=: \beta_I^P$ ... the beta of the investment I w/ portfolio P

<u>Def'n.</u> $r_I := r_f + \beta_I^P(\mathbb{E}[R_P] - r_f)$ ... the required return of investment I given portfolio P.

<u>Note</u>: A portfolio P^* is <u>EFFICIENT</u> if no other portfolio outperforms it w/ respect to the interplay between volatility & expected return.

Consider an investment I such that:

$$\mathbb{E}[R_I] > r_I = r_f + \beta_I^{P^*}(\mathbb{E}[R_P] - r_f)$$

$\Rightarrow$ you want to invest additionally to portfolio P^* in investment I

This is a contradiction w/ the fact that portfolio P^* is efficient.

$\Rightarrow$ For any security I:

$$\mathbb{E}[R_I] = r_I = r_f + \beta_I^{eff}(\mathbb{E}[R_{eff}] - r_f)$$

β of investment I w/ an efficient portfolio

the return of the efficient portfolio

The Capital Asset Pricing Model (CAPM)

Assumptions:

1. The investors can buy/sell all securities at competitive market prices w/ no transaction costs (no commissions, no bid-ask spread!). Both borrowing and lending are done @ the same risk-free interest rate. (no friction!)

2. Investors hold only **efficient portfolios** of traded securities, i.e., portfolios that yield the maximum expected return for a given level of volatility.

3. HOMOGENEOUS EXPECTATIONS.

   All investors have homogeneous beliefs regarding:
   - expected returns of securities,
   - volatilities,
   - correlations.