UNIVERSITY OF TEXAS AT AUSTIN

HW Assignment 7

Binomial option pricing.

7.1. **The forward binomial tree.** Please, provide your *final answer only* to the following problem.

Problem 7.1. (5 points) Assume that the a stock price is modeled using a one-period forward binomial tree with the length of a single period equal to three months. According to this model, the stock price can take either the value of \$50, or the value of \$40 in exactly three months. Calculate the volatility of the stock price.

(a) About 0.22
(b) About 0.28
(c) About 0.30
(d) About 0.32
(e) None of the above.

Solution: (a)

$$e^{2\sigma\sqrt{h}} = S_u/S_d \quad \Rightarrow \quad \sigma = \frac{1}{2\sqrt{h}}\ln(S_u/S_d) = \frac{1}{2\sqrt{1/4}}\ln(50/40) = \ln(50/40) = 0.2231$$

7.2. **Alternative binomial trees.** Please, provide your **complete** solutions to the following problem(s):

Problem 7.2. Cox-Ross-Rubinstein (CRR)
The Cox-Ross-Rubinstein model is a binomial tree in which the up and down factors are given as

$$u = e^{\sigma\sqrt{h}}, \quad d = e^{-\sigma\sqrt{h}},$$

where σ denotes the volatility parameter and h stands for the length of a single period in a tree.

a. (2 points) What is the ratio S_u/S_d?
Solution: $S_u/S_d = e^{2\sigma\sqrt{h}}$.

b. (2 points) What is the (as simplified as possible) expression for the risk-neutral probability of the stock price going up in a single step?
Solution:

$$p^* = \frac{e^{rh} - e^{-\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}} = \frac{e^{rh+\sigma\sqrt{h}} - 1}{e^{2\sigma\sqrt{h}} - 1}$$

Substantial further simplification is impossible.

c. (2 points) Express S_{ud} in terms of $S(0), \sigma$ and h in a CRR tree.
Solution: $S_{ud} = S(0)$

d. (5 points) As was the case with the forward tree, the *no-arbitrage* condition for the binomial asset-pricing model is satisfied for the CRR tree regardless of the specific values of σ, r and h. *True or false?*
Solution: FALSE
Counterexamples will vary.

Problem 7.3. The Jarrow-Rudd model.
The **Jarrow-Rudd** model (aka, the lognormal binomial tree) is a binomial tree in which the up and down factors are defined as follows

$$u = e^{\left(r-\frac{\sigma^2}{2}\right)h+\sigma\sqrt{h}}, \quad d = e^{\left(r-\frac{\sigma^2}{2}\right)h-\sigma\sqrt{h}},$$

where

- r stands for the continuously-compounded, risk-free interest rate,

- δ is the stock's dividend yield,
- σ denotes the volatility parameter, and
- h stands for the length of a single period in a tree.

Answer the following questions:

a. (2 points) What is the ratio S_u/S_d?
Solution: $S_u/S_d = e^{2\sigma\sqrt{h}}$.

b. (2 points) What is the (as simplified as possible) expression for the risk-neutral probability of the stock price going up in a single step?
Solution:

$$p^* = \frac{e^{rh} - e^{\left(r-\frac{\sigma^2}{2}\right)h-\sigma\sqrt{h}}}{e^{\left(r-\frac{\sigma^2}{2}\right)h+\sigma\sqrt{h}} - e^{\left(r-\frac{\sigma^2}{2}\right)h-\sigma\sqrt{h}}} = \frac{1 - e^{-\frac{\sigma^2 h}{2}-\sigma\sqrt{h}}}{e^{-\frac{\sigma^2 h}{2}+\sigma\sqrt{h}} - e^{-\frac{\sigma^2 h}{2}-\sigma\sqrt{h}}}.$$

Substantial further simplification is impossible.

c. (5 points) As was the case with the forward tree, the *no-arbitrage* condition for the binomial asset-pricing model is satisfied for the Jarrow-Rudd tree regardless of the specific values of σ, δ, r and h. *True or false?*
Solution: FALSE
Counterexamples will vary.

7.3. **Multi-period binomial option pricing: European options.** Please, provide your **complete** solutions to the following problem:

Problem 7.4. (10 points) The current price of a non-dividend-paying stock is \$100 per share. Its volatility is given to be 0.2.

The continuously compounded risk-free interest rate equals 0.04.

Consider a \$95-strike European put option on the above stock with nine months to expiration. Using a three-period forward binomial tree, find the price of this put option.

Solution: The up and down factors in the above model are

$$u = e^{0.04\times 0.25+0.2\sqrt{0.25}} = 1.116278,$$
$$d = e^{0.04\times 0.25-0.2\sqrt{0.25}} = 0.9139312.$$

The relevant possible stock prices at the "leaves" of the binomial tree are

$$S_{ddd} = d^3 S(0) = 100(0.9139312)^3 = 76.33795,$$
$$S_{ddu} = d^2 u S(0) = 93.23938.$$

The remaining two final states of the world result in the put option being out-of-the-money at expiration.

The risk-neutral probability of the stock price moving up in a single period is

$$p^* = \frac{1}{1+e^{0.2\sqrt{0.25}}} = 0.475.$$

So, the price of the European put option equals

$$V_P(0) = e^{-0.04(3/4)}\left[(95-76.33795)(1-0.475)^3 + (95-93.23938)(3)(1-0.475)^2(0.475)\right] = 3.29172.$$

7.4. **Strong Law of Large Numbers. Monte Carlo.**

Problem 7.5. (10 points) Let $\{Y_n, n \in \mathbb{N}\}$ be a sequence of independent, identically distributed random variables. Assume that $Y_1 = e^X$ where X is a standard normal random variable. Use the Strong Law of Large Numbers to find the following limit

$$\lim_{n\to\infty}\left(\prod_{i=1}^{n} Y_i\right)^{1/n} = \lim_{n\to\infty}\left(Y_1 \cdot Y_2 \cdots Y_n\right)^{1/n}.$$

Hint: Note that for every n, $Y_n = e^{X_n}$ where $\{X_n, n \in \mathbb{N}\}$ is a sequence of independent identically distributed standard normal random variables. Then, it helps to modify the product in the limit above and use the continuity of the exponential function.

Solution: For every $n \in \mathbb{N}$,

$$(\prod_{i=1}^{n} Y_i)^{1/n} = (\prod_{i=1}^{n} e^{X_i})^{1/n} = \exp\{\frac{1}{n}\sum_{i=1}^{n} X_i\}.$$

By the SLLN, with probability 1,

$$\frac{1}{n}\sum_{i=1}^{n} X_i \to \mathbb{E}[X_1] = 0, \text{ as } n \to \infty.$$

So, thanks to the continuity of the exponential function

$$(\prod_{i=1}^{n} Y_i)^{1/n} \to e^0 = 1, \text{ as } n \to \infty$$

with probability 1.

Problem 7.6. (5 points) You use *Monte Carlo* to simulate values from a normal distribution with mean 0 and variance 4. Your plan is to use 10000 simulations. What is the variance of the *Monte Carlo* simulations?

Solution: Let $n = 10000$. Then, every *Monte Carlo* simulation will be of the form

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

where $X_i \sim Normal(mean = 0, var = 4)$ for all $i = 1, \ldots, n$. We have

$$Var[\bar{X}] = \frac{1}{n^2}\sum_{i=1}^{n} Var[X_i] = \frac{Var[X_1]}{n} = \frac{4}{10000} = 0.0004.$$