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Functions of Random Vectors.

Theorem. Let $(Y_1, Y_2, \dots, Y_n)$ be a continuous random vector w/ the joint pdf $f_{Y_1,\dots,Y_n}(\cdot,\cdot,\dots,\cdot)$.

Let g be a function of n variables such that I can define

$$W = g(Y_1, Y_2, \dots, Y_n)$$

Then,

$$\mathbb{E}[W] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(y_1, y_2, \dots, y_n) f_{Y_1,\dots,Y_n}(y_1,\dots,y_n)\, dy_n \cdots dy_1$$

If the integral is well defined.

Example. Let (Y_1, Y_2) be a random pair w/ joint pdf

$$f_{Y_1,Y_2}(y_1,y_2) = 6y_1 \mathbb{1}_{\{0 \le y_1 \le y_2 \le 1\}}$$

The square of the norm of a random point from this distribution will be

$$g(Y_1, Y_2) = \underbrace{Y_1^2 + Y_2^2}_{W}$$

By our theorem:

$$\mathbb{E}[Y_1^2 + Y_2^2] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (y_1^2 + y_2^2)\, 6y_1 \mathbb{1}_{\{0 \le y_1 \le y_2 \le 1\}}\, dy_2\, dy_1$$

$$= 6\int_0^1 \int_{y_1}^1 (y_1^3 + y_1 y_2^2)\, dy_2\, dy_1$$

$$= 6\int_0^1 \left(y_1^3 y_2 + y_1 \cdot \frac{y_2^3}{3}\right)\Big|_{y_2=y_1}^{1}\, dy_1$$

$$= 6\int_0^1 \left(y_1^3(1-y_1) + \frac{y_1}{3}(1-y_1^3)\right) dy_1$$

$$= 6\int_0^1 \left(y_1^3 - y_1^4 + \frac{y_1}{3} - \frac{y_1^4}{3}\right) dy_1$$

$$= \int_0^1 (2y_1 + 6y_1^3 - 8y_1^4)\, dy_1$$

$$= 2\cdot\frac{1}{2} + 6\cdot\frac{1}{4} - 8\cdot\frac{1}{5} = \frac{9}{10}$$

$\square$

Marginal Distributions & Independence.

Theorem. Say that the r.v. $(Y_1, Y_2, \dots, Y_n)$ has the pdf $f_{Y_1,\dots,Y_n}$.

Then, for every $i = 1, \dots, n$, the random variable Y_i is continuous with this marginal density

$$f_{Y_i}(y) = \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{n-1} f_{Y_1,\dots,Y_n}(y_1, \dots, y_{i-1}, y, y_{i+1}, \dots, y_n)\, dy_n\, dy_{n-1} \cdots dy_{i+1}\, dy_{i-1} \cdots dy_1$$

Example. Continuing w/ (Y_1, Y_2) from above

Marginal of Y_1:

$$f_{Y_1}(y) = \int_{-\infty}^{\infty} f_{Y_1,Y_2}(y, y_2)\, dy_2$$

$$= \int_{-\infty}^{\infty} 6y\, \mathbb{1}_{[0 \le y \le y_2 \le 1]}\, dy_2 =$$

$$= 6y \int_y^1 dy_2 \cdot 1_{[0,1]}(y) = \underline{6y(1-y)\,1_{[0,1]}(y)}$$

Marginal of Y_2:

$$f_{Y_2}(y) = \int_{-\infty}^{\infty} f_{Y_1,Y_2}(y_1, y)\,dy_1$$

$$= \int_{-\infty}^{\infty} 6y_1 \cdot 1_{[0 \le y_1 \le y \le 1]}\,dy_1$$

$$= \int_0^y 6y_1\,dy_1 \cdot 1_{[0,1]}(y) = 6\left(\frac{y_1^2}{2}\right)\Big|_{y_1=0}^{y} 1_{[0,1]}(y)$$

$$= \underline{3y^2 \cdot 1_{[0,1]}(y)}$$

Def'n. The random variables $Y_1, \dots, Y_n$ are independent if the events

$$\{Y_i \in [a_i, b_i]\} \quad i = 1, \dots, n$$

are independent events for all (a_i, b_i) $i = 1..n$

Theorem. The Factorization Criterion.

Continuous r.v. $Y_1, \dots, Y_n$ are independent iff

$$f_{Y_1,\dots,Y_n}(y_1, \dots, y_n) = f_{Y_1}(y_1) \cdot f_{Y_2}(y_2) \cdots f_{Y_n}(y_n)$$

for $y_1, \dots, y_n \in \mathbb{R}^n$

Example. [cont'd]

$$f_{Y_1,Y_2}(y_1, y_2) = 6y_1\, 1_{[0 \le y_1 \le y_2 \le 1]}$$

$$\neq$$

$$f_{Y_1}(y_1) \cdot f_{Y_2}(y_2) = 6y_1(1-y_1)\,1_{[0,1]}(y_1) \cdot 3y_2^2 \cdot 1_{[0,1]}(y_2)$$

Theorem. Let $Y_1, Y_2, \dots, Y_n$ be independent r.v.s.
Let $g_1, g_2, \dots, g_n$ be functions such that $g_i(Y_i)$, $i=1..n$ are all well defined.

Then,

$$\mathbb{E}[g_1(Y_1)\,g_2(Y_2)\cdots g_n(Y_n)] = \mathbb{E}[g_1(Y_1)]\cdot\mathbb{E}[g_2(Y_2)]\cdots\mathbb{E}[g_n(Y_n)]$$

e.g.,
$$\mathbb{E}[\exp(Y_1+Y_2)] = \mathbb{E}[e^{Y_1}\cdot e^{Y_2}] \underset{\substack{Y_1 \text{ and } Y_2 \\ \text{independent}}}{=} \mathbb{E}[e^{Y_1}]\cdot\mathbb{E}[e^{Y_2}]$$

If Y_1 and Y_2 are identically distributed,
$$\mathbb{E}[\exp(Y_1+Y_2)] = \left(\mathbb{E}[e^{Y_1}]\right)^2$$