M339W: April 19th, 2021.

Exchange Options.

T... exercise date

two risky assets $\begin{cases} S \text{ ... underlying asset} \\ Q \text{ ... strike asset} \end{cases}$

For an exchange call:

payoff: $V_{EC}(T, S, Q) = (S(T) - Q(T))_+$

For an exchange put:

payoff: $V_{EP}(T, S, Q) = (Q(T) - S(T))_+$

$\Rightarrow$ We have a special symmetry:

$$\boxed{V_{EC}(T, S, Q) = V_{EP}(T, Q, S)}$$

$\Rightarrow$ The time-0 prices of the two options must also be equal:

$$\boxed{V_{EC}(0, S, Q) = V_{EP}(0, Q, S)}$$

$\Rightarrow$ It suffices to develop the Black-Scholes pricing formula for exchange calls.

- S... underlying: δ_S... dividend yield
  σ_S... volatility

Because we're pricing, we have to consider S under the risk-neutral measure:

$$S(T) = S(0)\, e^{\left(r - \delta_S - \frac{\sigma_S^2}{2}\right)\cdot T + \sigma_S\sqrt{T}\cdot Z_S} \quad \text{w/ } \boxed{Z_S \sim N(0,1)}$$

- Q... strike: δ_Q... dividend yield
  σ_Q... volatility

Under the risk-neutral measure:

$$Q(T) = Q(0)\, e^{\left(r - \delta_Q - \frac{\sigma_Q^2}{2}\right)\cdot T + \sigma_Q\sqrt{T}\cdot Z_Q} \quad \text{w/ } \boxed{Z_Q \sim N(0,1)}$$

w/ ρ... the correlation coefficient between Z_S and Z_Q ⟵

Black-Scholes Pricing Formula.

$$V_{EC}(0,S,Q) = F^P_{0,T}(S)\cdot N(d_1) - F^P_{0,T}(Q)\cdot N(d_2)$$

w/ $$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{F^P_{0,T}(S)}{F^P_{0,T}(Q)}\right) + \frac{1}{2}\sigma^2\cdot T\right]$$

and $$d_2 = d_1 - \sigma\sqrt{T}$$

where $$\sigma^2 = \sigma_S^2 + \sigma_Q^2 - 2\rho\,\sigma_S\,\sigma_Q$$

<u>Note</u>: • $\begin{cases} S(t), & t\geq 0 \\ Q(t), & t\geq 0 \end{cases}$

For every t:

$$\mathrm{Var}\left[\ln\left(\frac{S(t)}{Q(t)}\right)\right] = \mathrm{Var}\left[\ln(S(t)) - \ln(Q(t))\right] = (\text{under } \mathbb{P}^*)$$

$$= \mathrm{Var}\Big[\underbrace{\boxed{\ln(S(0)) + (r-\delta_S-\tfrac{\sigma_S^2}{2})\cdot t - \ln(Q(0)) - (r-\delta_Q-\tfrac{\sigma_Q^2}{2})\cdot t}}_{\text{deterministic}} + \sigma_S\sqrt{t}\cdot Z_S - \sigma_Q\sqrt{t}\cdot Z_Q\Big]$$

$$= \mathrm{Var}\left[\sigma_S\cdot\sqrt{t}\cdot Z_S - \sigma_Q\cdot\sqrt{t}\cdot Z_Q\right]$$

$$= t\cdot\mathrm{Var}\left[\sigma_S\cdot Z_S - \sigma_Q\cdot Z_Q\right]$$

$$= t\Big(\sigma_S^2\cdot\underbrace{\mathrm{Var}[Z_S]}_{1} + \sigma_Q^2\cdot\overbrace{\mathrm{Var}[Z_Q]}^{1} - 2\cdot\sigma_S\cdot\sigma_Q\cdot\underbrace{\mathrm{Cov}[Z_S,Z_Q]}_{=\rho}\Big)$$

$$= t\underbrace{(\sigma_S^2+\sigma_Q^2-2\sigma_S\cdot\sigma_Q\cdot\rho)}_{\sigma^2}$$

• <u>Favorite special case:</u> both S and Q pay dividends continuously

$$V_{EC}(0,S,Q) = S(0)e^{-\delta_S\cdot T}\cdot N(d_1) - Q(0)e^{-\delta_Q\cdot T}\cdot N(d_2)$$

w/ $$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)e^{-\delta_S\cdot T}}{Q(0)e^{-\delta_Q\cdot T}}\right) + \frac{1}{2}\sigma^2\cdot T\right]$$

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{Q(0)}\right) + (\overset{\nearrow r}{\delta_Q} - \overset{\nearrow \delta}{\delta_S} + \frac{\sigma^2}{2})\cdot T\right]$$

and $\quad d_2 = d_1 - \sigma\sqrt{T}$

w/ $\quad \sigma^2 = \sigma_s^2 + \sigma_a^2 - 2\rho\sigma_s \cdot \sigma_a$

50. Assume the Black-Scholes framework.

You are given the following information for a stock that pays dividends continuously at a rate proportional to its price.

(i) The current stock price is 0.25.

(ii) The stock's volatility is 0.35.

(iii) The continuously compounded expected rate of stock-price appreciation is 15%.

Calculate the upper limit of the 90% lognormal confidence interval for the price of the stock in 6 months.

(A) 0.393

(B) 0.425

(C) 0.451

(D) 0.486

(E) 0.529

51-53. DELETED

54. Assume the Black-Scholes framework. Consider two nondividend-paying stocks whose time-t prices are denoted by $S_1(t)$ and $S_2(t)$, respectively.

You are given:

(i) $S_1(0) = 10$ and $S_2(0) = 20$.

(ii) Stock 1's volatility is 0.18. $\sigma_1 = 0.18$

(iii) Stock 2's volatility is 0.25. $\sigma_2 = 0.25$

(iv) The correlation between the continuously compounded returns of the two stocks is –0.40. $\rho = -0.4$

(v) The continuously compounded risk-free interest rate is 5%. $\boxed{r = 0.05}$

(vi) A one-year European option with payoff $\max\{\underbrace{\boxed{\min[2S_1(1), S_2(1)]}}_{= Y(1)} - 17, 0\}$ has a current (time-0) price of $\boxed{1.632}$. SO

"SPECIAL PUT"

Consider a European option that gives its holder the right to sell either two shares of Stock 1 or one share of Stock 2 at a price of 17 one year from now.

"STRIKE" T=1

Calculate the current (time-0) price of this option.

The payoff of the "special put":

$$\Big(17 - \underbrace{\min(2\cdot S_1(1), S_2(1))}_{=:Y(1)}\Big)_+ \qquad \text{(SP)}$$

This payoff, indeed, looks like a payoff of a put option on Y w/ strike 17.

(A) 0.67

(B) 1.12

(C) 1.49

(D) 5.18

(E) 7.86

55. Assume the Black-Scholes framework. Consider a 9-month at-the-money European put option on a futures contract. You are given:

(i) The continuously compounded risk-free interest rate is 10%.

(ii) The strike price of the option is 20.

(iii) The price of the put option is 1.625.

If three months later the futures price is 17.7, what is the price of the put option at that time?

(A) 2.09

(B) 2.25

(C) 2.45

(D) 2.66

(E) 2.83

56-76. DELETED

(vi) gives us the price of the corresponding call option on Y.

Using put-call parity:

$$\underbrace{V_{SO}(0)}_{\overset{\text{(vi)}}{=}\ 1.632} - \underbrace{V_{SP}(0)}_{=\ ?} = \boxed{F^P_{0,T}(Y)} - \underbrace{PV_{0,T}(K)}_{=\ 17\cdot e^{-0.05}}$$

The price to be paid @ time 0 in order to get $Y(1) = \min(2S_1(1), S_2(1))$ @ time 1.

$$Y(1) = \min(2S_1(1), S_2(1))$$

$$= S_2(1) + \min(2S_1(1) - S_2(1), 0)$$

$$= \underbrace{S_2(1)}_{\text{prepaid forward}} - \max(S_2(1) - 2S_1(1), 0)$$

exchange call w/ underlying S_2 and strike asset $2\cdot S_1$

$$2\cdot S_1(T) = 2\cdot S_1(0)\, e^{(r-\delta_1-\frac{\sigma_1^2}{2})\cdot T + \sigma_1\sqrt{T}\cdot Z_1}$$

$\Rightarrow$ $(2S_1)$ has the same δ_1 and σ_1 as the original stock S_1.

Time 0:

$$F^P_{0,1}(S_2) = S_2(0)$$

$$\boxed{V_{EC}(0, S_2, 2\cdot S_1) = ?}$$

$$\sigma^2 = (0.18)^2 + (0.25)^2 - 2(0.18)(0.25)(-0.4)$$
$$\sigma = 0.3618$$

$$V_{EC}(0) = 2.856$$

$V_{SP}(0) = 0.65$ (using the std normal tables).