M339J: February 28^{th}, 2022.

Continuous Mixing.

- Start w/ a r.v.

  Λ ... plays the role of a mixing parameter

  Let f_Λ be its probability density function.

- Suppose that, conditional on $\Lambda = \lambda$, the r.v. X

  has the pdf $f_{X|\Lambda}(x|\lambda)$

  and the cdf $F_{X|\Lambda}(x|\lambda)$.

Then, the unconditional pdf of X is

$$\boxed{f_X(x) = \int f_{X|\Lambda}(x|\lambda) f_\Lambda(\lambda)\, d\lambda}\,,$$

and the unconditional cdf of X is

$$\boxed{F_X(x) = \int F_{X|\Lambda}(x|\lambda) f_\Lambda(\lambda)\, d\lambda}$$

Note:

- $\mathbb{E}[X^k] = \mathbb{E}\left[\mathbb{E}[X^k | \Lambda]\right]$
- $Var[X] = \mathbb{E}[Var[X|\Lambda]] + Var[\mathbb{E}[X|\Lambda]]$

Problem. Assume that

$$\begin{cases} X | \Lambda = \lambda \sim \text{Exponential}(\text{mean} = \lambda) \\ \text{w/ } \Lambda \sim \text{Uniform}(50, 100). \end{cases}$$

Find the unconditional coefficient of variation of X.

$\longrightarrow$: coeff. of variation $= \dfrac{\sigma_X}{\mu_X}$

Focus on μ_X.

$$\mathbb{E}[X] = \mathbb{E}\big[\underbrace{\mathbb{E}[X|\Lambda]}\big]$$

Always a function of Λ.

In this problem: $\mathbb{E}[X|\Lambda] = \Lambda$

$X|\Lambda \sim Exp(mean = \Lambda)$

$$\mathbb{E}[X] = \mathbb{E}[\Lambda] = 75$$

Focus on σ_X^2.

$$\mathrm{Var}[X] = \mathbb{E}\big[\underbrace{\mathrm{Var}[X|\Lambda]}\big] + \mathrm{Var}\big[\mathbb{E}[X|\Lambda]\big]$$

$$= \mathbb{E}[\quad \Lambda^2 \quad] + \mathrm{Var}[\quad \Lambda \quad]$$

$$= \mathrm{Var}[\Lambda] + (\mathbb{E}[\Lambda])^2 + \mathrm{Var}[\Lambda]$$

$$= 2\cdot \mathrm{Var}[\Lambda] + (\mathbb{E}[\Lambda])^2$$

$$= 2\cdot \frac{50^2}{12} + 75^2 = 6041.67$$

$$\sigma_X = \sqrt{6041.67} = 77.73$$

$$\text{coeff. of var} = \frac{77.73}{75} = 1.0364$$

Problem. Let X have a mixture dist'n w/ the mixing variable Λ.

$$X \mid \Lambda = \lambda \sim \text{Exponential}\left(mean = \frac{1}{\lambda}\right)$$

w/ $\Lambda \sim \text{Exponential}(mean = \theta = 5)$.

Find the unconditional probability $\mathbb{P}[X \le 3]$.

$\longrightarrow$: $\mathbb{P}[X \le 3] = F_X(3)$.

By def'n: for $x > 0$:

$$F_X(x) = \int_0^{+\infty} F_{X|\Lambda}(x|\lambda)\cdot f_\Lambda(\lambda)\, d\lambda$$

w/ • $f_\Lambda(\lambda) = \frac{1}{\theta} e^{-\frac{\lambda}{\theta}}$

• $F_{X|\Lambda}(x|\lambda) = 1 - e^{-\frac{x}{\frac{1}{\lambda}}} = 1 - e^{-\lambda \cdot x}$

$$F_X(x) = \int_0^{+\infty} (1-e^{-\lambda\cdot x}) \cdot \frac{1}{\theta} e^{-\frac{\lambda}{\theta}} \, d\lambda$$

$$= \frac{1}{\theta}\int_0^{+\infty} (1-e^{-\lambda\cdot x}) e^{-\frac{\lambda}{\theta}} \, d\lambda$$

$$= \boxed{\frac{1}{\theta}\int_0^{+\infty} e^{-\frac{\lambda}{\theta}} \, d\lambda} - \frac{1}{\theta}\boxed{\int_0^{+\infty} e^{-\lambda(x+\frac{1}{\theta})} \, d\lambda}$$

$$\frac{1}{\theta}\int_0^{+\infty} e^{-\frac{\lambda}{\theta}} \, d\lambda = 1$$

$$\int_0^{+\infty} e^{-\lambda(x+\frac{1}{\theta})} \, d\lambda = \underbrace{-\frac{1}{x+\frac{1}{\theta}} e^{-\lambda(x+\frac{1}{\theta})} \Big|_{\lambda=0}^{+\infty}}_{= +\frac{1}{x+\frac{1}{\theta}}(0+1)}$$

$$F_X(x) = 1 - \frac{1}{\theta} \cdot \frac{1}{x+\frac{1}{\theta}}$$

$$\boxed{\boxed{F_X(x) = 1 - \frac{1}{x\cdot\theta+1}}}$$

In this problem: $F_X(3) = 1 - \frac{1}{3(5)+1} = \frac{15}{16}$ ■

$$\boxed{X \sim \text{Pareto}\left(\alpha^* = 1, \theta^* = \frac{1}{\theta}\right)}$$