Name:

M339D=M389D Introduction to Actuarial Financial Mathematics
University of Texas at Austin
In-Term Exam II
Instructor: Milica Čudina

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Signature:

The maximum number of points on this exam is 100.

Provide your **complete solution** to the following problems. Final answers only, without appropriate justification, will receive zero points even if correct.

2.1. **DEFINITIONS.**

Problem 2.1. (10 points) Write the definition of an **arbitrage portfolio**.

Problem 2.2. (5 points) Write the definition of a **replicating portfolio** of a European option.

2.2. **TRUE/FALSE QUESTIONS.**

Problem 2.3. (2 points) Assume that the continuously compounded, risk-free interest rate is strictly positive. The forward price of a non-dividend-paying stock is always strictly increasing with respect to the delivery date. *True or false? Why?*

Solution: TRUE
The forward price is $F_{0,T} = S(0)e^{rT}$ as established in class.

Problem 2.4. (2 points) You are using a binomial asset-pricing model to model the evolution of the price of a particular stock. Then, the Δ in the replicating portfolio of a single call option on that stock never exceeds 1. *True or false? Why?*

Solution: TRUE
The call's Δ will always be between 0 and 1. More precisely, consider

$$\Delta_C = \frac{V_u - V_d}{S_u - S_d} = \frac{(S_u - K)_+ - (S_d - K)_+}{S_u - S_d}.$$

The numerator is between 0 and $S_u - S_d$ which completes the proof.

Problem 2.5. (2 points) You are using a binomial asset-pricing model to model the evolution of the price of a particular stock. Then, the Δ in the replicating portfolio of a single put option on that stock is between -1 and 0. *True or false? Why?*

Solution: TRUE
The put's Δ will always be between -1 and 0. By definition,

$$\Delta_P = \frac{V_u - V_d}{S_u - S_d} = \frac{(K - S_u)_+ - (K - S_d)_+}{S_u - S_d}.$$

The numerator is non-positive and at least $S_d - S_u$.

Problem 2.6. (2 points) You are using a one-period binomial asset-pricing model to model the evolution of the price of a particular stock. Assume that, in our usual notation, $S_d < K < S_u$ for a European put option. Then, the risk-free component in the replicating portfolio of a single put option on that stock should be interpreted as lending. *True or false? Why?*

Solution: TRUE
The put's B will always be positive and should be interpreted as lending. Indeed, by definition,

$$B_P = e^{-rh}\frac{uV_d - dV_u}{u-d}\,.$$

Since the option is in-the-money in the *down* node and out-of-the-money in the *up* node, we have

$$B_P = e^{-rh}\frac{u(K-S_d)}{u-d} > 0.$$

Problem 2.7. (2 points) In the setting of the one-period binomial model, denote by i the **effective** interest rate **per period**. Let u denote the "up factor" and let d denote the "down factor" in the stock-price model.
If

$$d < u \leq 1+i$$

then there certainly is no possibility for arbitrage. *True or false? Why?*

Solution: FALSE
The no-arbitrage condition is

$$d < 1+i < u$$

2.3. **FREE-RESPONSE PROBLEMS.**

Problem 2.8. (5 points) A portfolio consists of the following:

- two **short** one-year, $50-$strike call options with price equal to \$8.50,
- three **long** one-year, $60-$strike put option with price equal to \$6.75

All of the options are European and with the same underlying asset.

Assume that the continuously compounded, risk-free interest rate equals 0.02.

What is the portfolio's profit if the final price of the underlying asset equals \$55?

Solution:

$$-2(55-50)_+ + 3(60-55)_+ + (2(8.50) - 3(6.75))e^{0.02} = 1.684346$$

Problem 2.9. (10 points) A derivative security has the payoff function given by

$$v(s) = \begin{cases} 10 & \text{if } s < 90 \\ 0 & \text{if } 90 \leq s < 100 \\ 20 & \text{if } 100 \leq s \end{cases}$$

Its exercise date is in one year. You model the time-1 price of the underlying asset as

$$S(1) \sim \begin{cases} 85 & \text{with probability } 1/4 \\ 95 & \text{with probability } 1/2 \\ 105 & \text{with probability } 1/4 \end{cases}$$

What is the expected payoff of the above derivative security?

Solution:

$$10\left(\frac{1}{4}\right) + 20\left(\frac{1}{4}\right) = \frac{30}{4} = 7.5$$

Problem 2.10. (10 points) A non-dividend-paying stock is currently priced at \$100 per share. One-year, \$102-strike European call and put options on this stock have equal prices. What is the continuously-compounded annual risk-free rate of interest?

Solution: By put-call parity, we have

$$K = F_{0,T}(S) = S(0)e^{rT} \quad \Rightarrow \quad r = \frac{1}{T}\ln\left(\frac{K}{S(0)}\right).$$

So,

$$r = \frac{1}{T}\ln\left(\frac{K}{S(0)}\right) = \ln(102/100) = 0.01980263.$$

Problem 2.11. (15 points) Consider a non-dividend-paying stock whose current price is \$80 per share. The stock has the volatility equal to 0.20.

Let the continuously-compounded, risk-free interest rate be equal to 0.04.

You model the evolution of this stock over the next quarter with a **forward** binomial tree.

What is the price of a \$75-strike, three-month call on the above stock consistent with this model?

Solution: In our usual notation, in the forward binomial tree, the risk-neutral probability is

$$p^* = \frac{1}{1+e^{\sigma\sqrt{h}}} = \frac{1}{1+e^{0.2\sqrt{1/4}}} = \frac{1}{1+e^{0.1}} = 0.4750208$$

The *up* and *down* factors are

$$u = e^{rh+\sigma\sqrt{h}} = e^{0.04(1/4)+0.1} = e^{0.11},$$
$$d = e^{rh-\sigma\sqrt{h}} = e^{0.04(1/4)-0.1} = e^{-0.09}.$$

Hence, the two possible stock prices at the end of the period are $S_u = 80e^{0.11} = 89.30225$ and $S_d = 80e^{-0.09} = 73.11449$. So, the option is in the money only in the *up* node where the payoff equals

$$V_u = (S_u - K)_+ = 14.30225.$$

By the risk neutral pricing formula, we have that

$$V_C(0) = e^{-0.04(1/4)}(0.4750208)(14.30225) = 6.726264.$$

Alternatively, the replicating portfolio has the following components

$$\Delta = \frac{V_u - V_d}{S_u - S_d} = \frac{14.30225}{89.30225 - 73.1149} = 0.8835227,$$
$$B = e^{-rh}\frac{uV_d - dV_u}{u-d} = -e^{0.01}\frac{e^{-0.09}(14.30225)}{e^{0.11}-e^{-0.09}} = -63.95555.$$

So,

$$V_C(0) = \Delta S(0) + B = 0.8835227(80) + 63.95555 = 6.726264.$$