M339J: May 4th, 2022.

60. You are given the following information about six coins:

Coin	Probability of Heads
1 – 4	0.50
5	0.25
6	0.75

A coin is selected at random and then flipped repeatedly. X_i denotes the outcome of the ith flip, where "1" indicates heads and "0" indicates tails. The following sequence is obtained:

$$S = \{X_1, X_2, X_3, X_4\} = \{1, 1, 0, 1\}$$

Calculate $E(X_5 \mid S)$ using Bayesian analysis.

(A) 0.52

(B) 0.54

(C) 0.56

(D) 0.59

(E) 0.63

61. You observe the following five ground-up claims from a data set that is truncated from below at 100:

$\boxed{d = 100}$

125 150 165 175 250

You fit a ground-up exponential distribution using maximum likelihood estimation.

Calculate the mean of the fitted distribution.

i.e., we need to find $\hat{\theta}_{MLE}$.

(A) 73

(B) 100

(C) 125

(D) 156

(E) 173

$\longrightarrow$: The log-likelihood f'tion is:

$$\ell(\theta) = \sum_{j=1}^{m} \ln(f_X(x_j;\theta)) - n\cdot\ln(S_X(d;\theta))$$

We have $X \sim$ Exponential (mean $=\theta$).

$$\ell(\theta) = \sum_{j=1}^{m} \ln\left(\frac{1}{\theta} e^{-\frac{x_j}{\theta}}\right) + n\cdot \cancel{\ln}\left(\cancel{e}^{+\frac{d}{\theta}}\right)$$

$$\ell(\theta) = \sum_{j=1}^{m}\left(-\ln(\theta) + \cancel{\ln}\left(\cancel{e}^{-\frac{x_j}{\theta}}\right)\right) + n\cdot\frac{d}{\theta}$$

$$\ell(\theta) = n\cdot(-\ln(\theta)) + \sum_{j=1}^{m}\left(-\frac{x_j}{\theta}\right) + n\cdot\frac{d}{\theta}$$

$$\ell(\theta) = -n\cdot\ln(\theta) - \frac{1}{\theta}\sum_{j=1}^{m} x_j + n\cdot\frac{d}{\theta}$$

We differentiate:

$$\ell'(\theta) = -n\cdot\frac{1}{\theta} + (+1)\cdot\frac{1}{\theta^2}\sum_{j=1}^{m} x_j + n\cdot(-1)\cdot\frac{1}{\theta^2}\cdot d = 0$$

$$-n\cdot\theta + \sum_{j=1}^{m} x_j - n\cdot d = 0$$

$$\boxed{\hat{\theta}_{MLE} = \frac{1}{n}\cdot\sum_{j=1}^{m} x_j - d = \bar{x} - d}$$

Task: Repeat for $X \sim$ Gamma $(\alpha=2, \theta)$

unknown

In this problem:

$$\hat{\theta}_{MLE} = \frac{1}{5}(125 + 150 + 165 + 175 + 250) - 100$$

$$\boxed{\hat{\theta}_{MLE} = 73}$$

$\square$

152. You are given:

(i) A sample of losses is:

600 700 900

(ii) No information is available about losses of 500 or less.

(iii) Losses are assumed to follow an exponential distribution with mean θ.

Calculate the maximum likelihood estimate of θ.

$\longrightarrow$: $\hat{\theta}_{MLE} = \frac{1}{3}(600+700+900) - 500$

$= 733\frac{1}{3} - 500 = 233\frac{1}{3}$

(A) 233

(B) 400

(C) 500

(D) 733

(E) 1233

153. DELETED

261. DELETED

262. You are given:

(i) At time 4 hours, there are 5 working light bulbs.) $\boxed{\text{truncated @ 4}}$

(ii) The 5 bulbs are observed for p more hours.

complete data

(iii) Three light bulbs burn out at times $\boxed{5, 9, \text{ and } 13}$ hours, while the remaining light bulbs are still working at time $\boxed{4 + p}$ hours.

censoring

(iv) The distribution of failure times is uniform on $(0, \omega)$.

$X \sim U(0,\omega)$

pdf: $f_X(x;\omega) = \frac{1}{\omega}$ for $x \in (0,\omega)$

(v) The maximum likelihood estimate of ω is 29.

Calculate p.

survival f'tion:

$S_X(x;\omega) = 1 - \frac{x}{\omega}$ for $x \in (0,\omega)$

(A) Less than 10

(B) At least 10, but less than 12

(C) At least 12, but less than 14

(D) At least 14, but less than 16

(E) At least 16

The likelihood function:

$$L(\omega) = \frac{1}{\left(1 - \frac{4}{\omega}\right)^5} \left(\frac{1}{\omega}\right)^3 \left(1 - \frac{4+p}{\omega}\right)^2$$

$$L(\omega) = \frac{1}{\frac{(\omega-4)^5}{\cancel{\omega^5}}} \cdot \frac{1}{\cancel{\omega^3}} \cdot \frac{(\omega-(4+p))^2}{\cancel{\omega^2}}$$

$$L(\omega) = \frac{(\omega-(4+p))^2}{(\omega-4)^5}$$

The log-likelihood:

$$\ell(\omega) = 2\ln(\omega-4-p) - 5\ln(\omega-4)$$

$$\ell'(\omega) = 2 \cdot \frac{1}{\omega-4-p} - 5 \cdot \frac{1}{\omega-4}$$

We are given that $\hat{\omega}_{MLE} = 29$

$\Rightarrow \ell'(29) = 0$

$$\frac{2}{29-4-p} = \frac{5}{29-4} = \frac{1}{5}$$

$$10 = 25-p \Rightarrow \boxed{p=15}$$

<u>**Task:**</u> Let $x_1, x_2, \dots, x_n$ be complete, unmodified data from $U(0,\theta)$. What is the MLE for θ?

40. Losses come from a mixture of an exponential distribution with mean 100 with probability p and an exponential distribution with mean 10,000 with probability $1-p$.

Losses of 100 and 2000 are observed.

Determine the likelihood function of p.

(A) $\left(\dfrac{pe^{-1}}{100}\cdot\dfrac{(1-p)e^{-0.01}}{10{,}000}\right)\cdot\left(\dfrac{pe^{-20}}{100}\cdot\dfrac{(1-p)e^{-0.2}}{10{,}000}\right)$

(B) $\left(\dfrac{pe^{-1}}{100}\cdot\dfrac{(1-p)e^{-0.01}}{10{,}000}\right)+\left(\dfrac{pe^{-20}}{100}\cdot\dfrac{(1-p)e^{-0.2}}{10{,}000}\right)$

(C) $\left(\dfrac{pe^{-1}}{100}+\dfrac{(1-p)e^{-0.01}}{10{,}000}\right)\cdot\left(\dfrac{pe^{-20}}{100}+\dfrac{(1-p)e^{-0.2}}{10{,}000}\right)$

(D) $\left(\dfrac{pe^{-1}}{100}+\dfrac{(1-p)e^{-0.01}}{10{,}000}\right)+\left(\dfrac{pe^{-20}}{100}+\dfrac{(1-p)e^{-0.2}}{10{,}000}\right)$

(E) $p\cdot\left(\dfrac{e^{-1}}{100}+\dfrac{e^{-0.01}}{10{,}000}\right)+(1-p)\cdot\left(\dfrac{e^{-20}}{100}+\dfrac{e^{-0.2}}{10{,}000}\right)$

END OF EXAMINATION

$\longrightarrow$: X is a two-point mixture of two exponentials :

$$X \sim \begin{cases} X_1 \sim \text{Exp}(\text{mean} = 100) & \text{w/ probab. } p \\ X_2 \sim \text{Exp}(\text{mean} = 10{,}000) & \text{w/ probab. } 1-p \end{cases}$$

The r.v. X has the pdf:

$$f_X(x; p) = p \cdot f_{X_1}(x) + (1-p) \cdot f_{X_2}(x)$$
$$= p\left(\frac{1}{100} e^{-\frac{x}{100}}\right) + (1-p)\left(\frac{1}{10^4} e^{-\frac{x}{10^4}}\right)$$

Our two data points are $x_1 = 100$ and $x_2 = 2000$

$$\Rightarrow L(p) = \left(p \cdot \left(\frac{1}{100} e^{-\overset{1}{\frac{100}{100}}}\right) + (1-p)\left(\frac{1}{10^4} e^{-\frac{\cancel{100}}{\cancel{10^4}\,100}}\right)\right) \cdot$$
$$\cdot \left(p \cdot \left(\frac{1}{100} e^{-\frac{20\cancel{00}}{1\cancel{00}}}\right) + (1-p)\left(\frac{1}{10^4} e^{-\frac{\cancel{2000}}{\cancel{10^4}\,5}}\right)\right)$$