M339D: April 28th, 2021.

Binomial Asset Pricing Model.

$$\begin{tikzcd} & S_u = u \cdot S(0) \\ S(0) \arrow[ur, dash] \arrow[dr, dash] & \\ & S_d = d \cdot S(0) \end{tikzcd}$$

$h = T$

The no-arbitrage condition.

$$\boxed{d < e^{(r-\delta)h} < u}$$

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Problem Set #9

Binomial option pricing.

Problem 9.1. (2 points) In the setting of the one-period binomial model, denote by i the **effective** interest rate **per period**. Let u denote the "up factor" and let d denote the "down factor" in the stock-price model. If

$$d < u \leq 1 + i$$

then there certainly is no possibility for arbitrage.

False!

Actually:

$d < e^{r \cdot h} \cdot e^{-\delta \cdot h} < u$

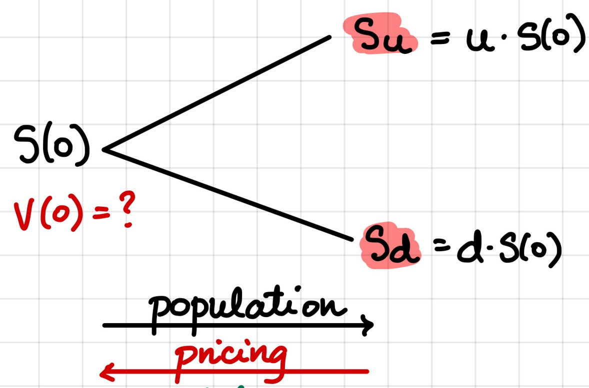
$d < (1+i) e^{-\delta \cdot h} < u$

Problem 9.2. In our usual notation, which of the parameter choices below creates a binomial model with an arbitrage opportunity?

(a) $u = 1.18, \quad d = 0.87, \quad r = 0.05, \quad \delta = 0, \quad h = 1/4$
(b) $u = 1.23, \quad d = 0.80, \quad r = 0.05, \quad \delta = 0.06, \quad h = 1/2$
(c) $u = 1.08, \quad d = 1, \quad r = 0.05, \quad \delta = 0.04, \quad h = 1$
(d) $u = 1.28, \quad d = 0.78, \quad r = \delta, \quad h = 2$
(e) None of the above.

Check if the no-arbitrage condition holds. :)

Binomial Option Pricing.



PAYOFF		REPLICATING PORTFOLIO
$V_u = v(S_u)$	$=$	$\Delta e^{\delta h} \cdot S_u + Be^{rh}$
$V_d = v(S_d)$	$=$	$\Delta e^{\delta h} \cdot S_d + Be^{rh}$

European-style derivative security w/ payoff f'tion $v(\cdot)$

In this simple model, we can replicate our derivative security w/ a portfolio w/ the following structure:

- Δ shares of stock
- B invested @ the risk-free rate

We arrive at a system of two equations w/ two unknowns.

$$\begin{aligned} \Delta e^{\delta \cdot h} \cdot S_u + \cancel{Be^{rh}} &= V_u \\ -\ \Delta e^{\delta \cdot h} \cdot S_d + \cancel{Be^{rh}} &= V_d \end{aligned} \Bigg\} -$$

$$\Delta e^{\delta \cdot h}(S_u - S_d) = V_u - V_d$$

$$\boxed{\Delta = e^{-\delta \cdot h} \cdot \frac{V_u - V_d}{S_u - S_d}}$$

unitless

$$\frac{V_u - V_d}{S_u - S_d} \cdot S_u + Be^{rh} = V_u$$

$$Be^{rh} = V_u - \frac{V_u - V_d}{\cancel{S(0)}(u-d)} \cdot u \cdot \cancel{S(0)}$$

$$B = e^{-rh} \frac{\cancel{V_u \cdot u} - V_u \cdot d - \cancel{V_u \cdot u} + V_d \cdot u}{u-d}$$

$$\boxed{B = e^{-rh} \cdot \frac{u \cdot V_d - d \cdot V_u}{u-d}}$$

By the Law of the Unique price :

$$\boxed{V(0) = \Delta \cdot S(0) + B}$$

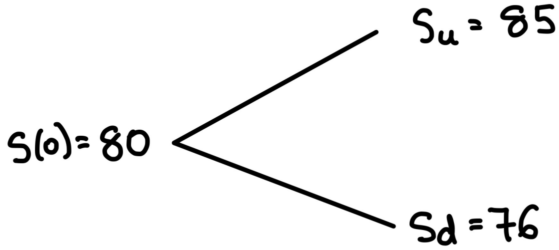
Problem 9.3. Let the continuously compounded risk-free interest rate be equal to 0.04. Consider a one-period binomial tree with every period of length one year used to model the stock price of a stock whose current price is \$80 per share. The stock pays dividends continuously with a dividend yield of 0.02. In the model, it is assumed that the stock price can either go up by \$5 or down by \$4.

You use the binomial tree to construct a replicating portfolio for a $(78, 82)-$strangle on the above stock. What is the stock investment in the replicating portfolio?

(a) Long 0.1089 shares.
(b) Long 0.33 shares.
(c) Short 0.1089 shares.
(d) Short 0.33 shares.
(e) None of the above.

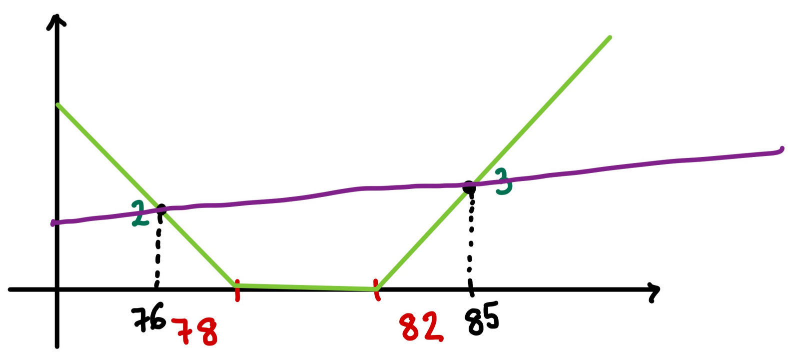
$\Delta = ?$

$$\Delta = e^{-\delta \cdot h} \cdot \frac{V_u - V_d}{S_u - S_d} = e^{-0.02} \cdot \frac{3-2}{9} = 0.1089$$



$V_u = \cancel{?}$ 85-82=3

$V_d = \cancel{?}$ 78-76=2



Problem 9.4. Let the continuously compounded risk-free interest rate be equal to 0.04. Consider a one-period binomial tree with every period of length one year used to model the stock price of a non-dividend-paying stock whose current price is \$50 per share. In the model, it is assumed that the stock price can either go up by 5% or down by 10%.

You use the binomial tree to construct a replicating portfolio for a (45, 55)−call bull spread on the above stock. What is the risk-free investment in the replicating portfolio?

(a) Borrow \$45
(b) Borrow \$43.24
(c) Lend \$45
(d) Lend \$43.24
(e) None of the above.

$B = ?$

$$B = e^{-r\cdot h}\frac{u\cdot V_d - d\cdot V_u}{u-d} = e^{-0.04}\cdot\frac{1.05\cdot 0 - 0.9(7.5)}{1.05-0.9} = -43.24$$

borrowing

$u = 1.05$
$d = 0.9$