Claim. $\boxed{V_A(0) = V_B(0)}$

Proof. Assume, to the contrary, that

$$V_A(0) \neq V_B(0)$$

Without loss of generality,

$$\boxed{\underbrace{V_A(0)}_{\text{relatively cheap}} < \underbrace{V_B(0)}_{\text{relatively expensive}}}$$

Diagnosis.

M339D: 09/30/24

Propose an arbitrage portfolio:

- Long Portfolio A
- Short Portfolio B

Total Portfolio

Verify that this is, indeed, an arbitrage portfolio.

- Initial Cost(Total Portfolio) $= V_A(0) - V_B(0) < 0$

Inflow of money @ time 0

- Payoff (Total Portfolio) $= V_A(T) - V_B(T) = 0$

$$\text{Profit} = \text{Payoff} - FV_{0,T}(\text{Initial Cost})$$
$$= 0 - FV_{0,T}\big(\underbrace{V_A(0) - V_B(0)}_{<0}\big) > 0$$

Indeed, this is an ARBITRAGE PORTFOLIO!

$\Rightarrow\Leftarrow$

$\square$

Corollary. If $V_A(T) \geq V_B(T)$, then $\boxed{V_A(0) \geq V_B(0)}$

$\square$

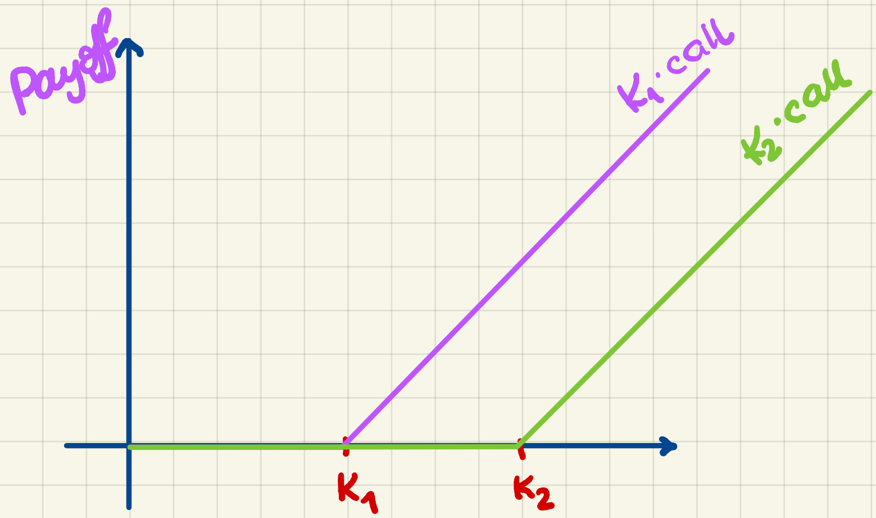
Example.

$\boxed{K_1 < K_2}$

A: one $\boxed{\text{long } K_1\text{-strike call}}$

B: one $\boxed{\text{long } K_2\text{-strike call}}$

w/ the same underlying asset and exercise date and European



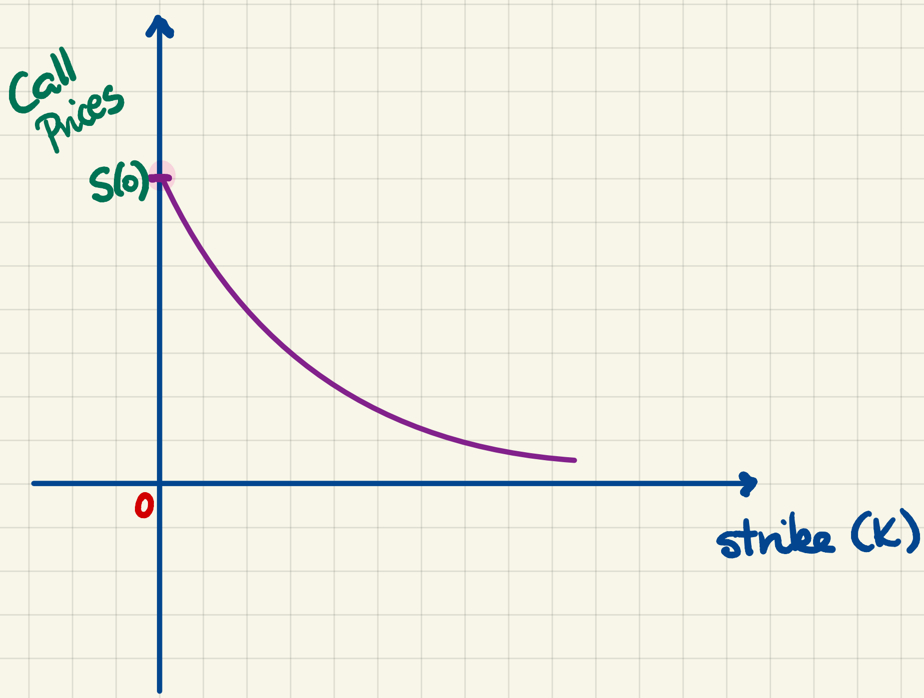
The payoff of the K_1-strike call **dominates** the payoff of the K_2-strike call

$\Downarrow$

$\boxed{\text{The } K_1\text{-call costs more than the } K_2\text{-call}}$

In Math: $\boxed{K_1 < K_2 \Rightarrow V_c(0, K_1) \geq V_c(0, K_2)}$

As a function of the strike price, call prices are <u>decreasing</u>.

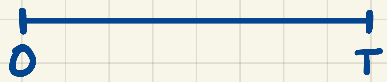


Replicating Portfolio.

Def'n. Consider a European-style derivative security. A static portfolio w/ the same payoff as that of the derivative security is called its replicating portfolio.

Note. The initial price of the derivative security is equal to the initial price of its replicating portfolio.

Example. Consider a forward contract on a non-dividend-paying stock/index.



Forward Contract: $S(T) - F$

Replicating Portfolio:
- long one share of stock
- issue a bond w/ redemption amount F and maturity date T

$$\text{Payoff (Portfolio)} = S(T) - F$$

$\Rightarrow$ The forward contract and its replicating portfolio must have the same initial cost, i.e.,

$$0 = \underbrace{S(0)}_{\text{long stock}} - \underbrace{PV_{0,T}(F)}_{\text{short bond}}$$

$$\Rightarrow PV_{0,T}(F) = S(0)$$

$$\Rightarrow \boxed{F = FV_{0,T}(S(0)) = S(0)e^{rT}}$$

$\square$