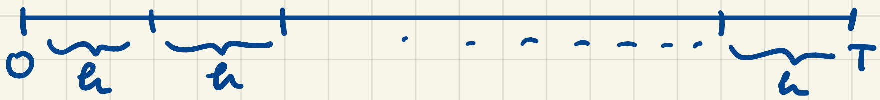
Multiple Binomial Periods.

M339D: March 27th, 2024.

T ... exercise date of a European option
n ... # of periods

the length of each period: $\boxed{h = \frac{T}{n}}$



The $(n+1)$ values are the support of $S(T)$

$\Rightarrow$ for every $k = 0, 1, \dots, n$:

k # of upsteps

$$S_{n,k} = S(0)u^k \cdot d^{n-k} = S(0)\left(\frac{u}{d}\right)^k \cdot d^n$$

Consider a European option w/ payoff function $v(\cdot)$
Then, the possible payoff values will be

$$\boxed{v_{n,k} := v(S_{n,k})}$$

Recall: Risk-neutral Pricing: $V(0) = e^{-rT}\mathbb{E}^*[V(T)]$

p^* ... the risk-neutral probability of a single upstep, i.e.,

$$\boxed{p^* = \frac{e^{rh} - d}{u - d}}$$

$\Rightarrow$ The risk-neutral probability of attaining the payoff $v_{n,k}$:

$$\boxed{\binom{n}{k}(p^*)^k(1-p^*)^{n-k}}$$

The risk-neutral option price:

$$\boxed{V(0) = e^{-rT}\sum_{k=0}^{n}\left(\binom{n}{k}(p^*)^k(1-p^*)^{n-k}\cdot v_{n,k}\right)}$$

Problem 10.2. Let the continuously compounded risk-free interest rate be 0.10. Let the initial price of a non-dividend-paying stock be $\boxed{\$100}$ per share. You use a five-period binomial tree to model the stock price over the next year. Let $\boxed{u = 1.04}$ and $\boxed{d = 0.96}$.

$n=5$

$T=1$ What is the price of a one-year, at-the-money European call option on the above stock?

$\rightarrow$: The risk-neutral probability: $h = \frac{T}{n} = \frac{1}{5}$

$$p^* = \frac{e^{rh} - d}{u-d} = \frac{e^{0.10(0.2)} - 0.96}{1.04-0.95} \approx \underline{0.7525}$$

The relevant stock prices in our tree:

$S_{5,5} = S(0)u^5 = 100(1.04)^5 = \underline{121.67}$ $\Rightarrow$ $v_{5,5} = \underline{21.67}$

$S_{5,4} = S(0)u^4 \cdot d = 100(1.04)^4 \cdot (0.96) = \underline{112.31}$ $\Rightarrow$ $v_{5,4} = \underline{12.31}$

$S_{5,3} = S(0)u^3 \cdot d^2 = 100(1.04)^3 \cdot (0.96)^2 = \underline{103.67}$ $\Rightarrow$ $v_{5,3} = \underline{3.67}$

The remaining terminal nodes are all out-o-money.

$\Rightarrow$

$$V(0) = e^{-0.10}\Big(21.67 \cdot (p^*)^5 + 12.31 \cdot 5(p^*)^4(1-p^*) + 3.67 \cdot \underset{\binom{5}{2}}{\underset{\shortparallel}{10}} \cdot (p^*)^3(1-p^*)^2\Big) = \underline{10.01821}$$

$\square$