Name:

M339J/M389J: Probability Models with Actuarial Applications
The University of Texas at Austin
Sample Problems for In-Term Exam III
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Notes: This is a closed book and closed notes exam. The maximal score on this exam is 100 points.

3.1. **Free-response problems.** *Please, explain carefully all your statements and assumptions. Numerical results or single-word answers without an explanation (even if they're correct) are worth 0 points.*

Problem 3.1. A Poisson distribution is used to fit frequency data. You are given that:

- there was a total of 20 observations less than or equal to 1;
- there were 6 observations equal to 2;
- there were 4 observations equal to 3;
- there were no observations greater than or equal to 4.

What is the maximum likelihood estimate for the mean of this Poisson distribution?

Problem 3.2. (10 points) In the compound model for aggregate claims, let the frequency random variable N have the Poisson distribution with mean 1.

Let the common distribution of the i.i.d. severity random variables $\{X_j; j = 1, 2, \dots\}$ be given by the following p.m.f.

$$p_X(100) = 1/2, p_X(200) = 3/10, p_X(300) = 1/5.$$

Let our usual assumptions hold, i.e., let N be independent of $\{X_j; j = 1, 2, \dots\}$.
Define the aggregate loss as $S = \sum_{j=1}^{N} X_j$.
Find the expected value of the **policyholder's** payment for a stop-loss insurance policy with an ordinary deductible of 200, i.e., calculate $\mathbb{E}[S \wedge 200]$.

Problem 3.3. (10 points) The frequency random variable N is assumed to have a Poisson distribution with a mean of 2. Individual claim severity random variable X has the following probability mass function

$$p_X(100) = 0.6, \quad p_X(200) = 0.3, \quad p_X(300) = 0.1.$$

Let the above be the common distribution of the i.i.d. severity random variables $\{X_j; j = 1, 2, \dots\}$, and Let our usual assumptions hold, i.e., let N be independent of $\{X_j; j = 1, 2, \dots\}$. Define the aggregate loss as $S = \sum_{j=1}^{N} X_j$. Calculate the probability that S is exactly equal to 3.

Problem 3.4. (6 points) In the compound model for aggregate claims, let the frequency random variable N have the probability (mass) function

$$p_N(0) = 0.4, p_N(1) = 0.3, p_N(2) = 0.2, p_N(4) = 0.1.$$

Moreover, let the common distribution of the i.i.d. severity random variables $\{X_j; j = 1, 2, \dots\}$ be given by the probability (mass) function $p_X(1) = 0.3$ and $p_X(2) = 0.7$.
Let our usual assumptions hold, i.e., let N be independent of $\{X_j; j = 1, 2, \dots\}$.
Define the aggregate loss as $S = \sum_{j=1}^{N} X_j$.
Calculate $\mathbb{E}[(S-2)_+]$.

Problem 3.5. (15 points) A sizeable village of log cabins is insured. Aggregate losses have a compound Poisson distribution. The expected number of losses is 25 per year. Loss amounts, regardless of the type of wood used to construct the cabin, follow an exponential distribution with mean 2000.

The insurer seeks to reduce the cost of insurance so they decide to do two things:

- They stop insuring pine cabins which make up one fifth of the total number of cabins.
- They impose an ordinary deductible of 500 per loss.

What are the expected value and the variance of the total aggregate amount paid by the insurer after the above modifications are implemented?

3.2. **MULTIPLE CHOICE QUESTIONS.**

Problem 3.6. (5 points) Let S be the aggregate claims random variable. You are given the following:
(i) $\mathbb{E}[(S-100)_+] = 15$,
(ii) $\mathbb{E}[(S-120)_+] = 10$,
(iii) $\mathbb{P}[80 < S \leq 120] = 0$.
Find $\mathbb{E}[(S-105)_+]$.

(a) $\mathbb{E}[(S-105)_+] \approx 10.75$
(b) $\mathbb{E}[(S-105)_+] \approx 12.75$
(c) $\mathbb{E}[(S-105)_+] \approx 13.75$
(d) $\mathbb{E}[(S-105)_+] \approx 15.75$
(e) None of the above

Problem 3.7. (5 points) Consider the following individual observed values:

$$5, 8, 10$$

of a random variable X whose distribution function is given by $F_X(x) = 1 - (1/x)^p$ for $x > 1$ and an unknown parameter p. Let $\hat{p}$ denote the Maximum Likelihood Estimate of p based on the above observed values. Then,

(a) $\hat{p} \approx \frac{3}{2}$
(b) $\hat{p} \approx \frac{3\ln(20)}{2}$
(c) $\hat{p} \approx \frac{3\ln(20)}{4}$
(d) $\hat{p} \approx \frac{3}{2\ln(20)}$
(e) None of the above

Problem 3.8. (5 points) Consider the following individual observed values

$$5, 10$$

and the one right censored value 8 of a random variable X whose distribution function is given by $F_X(x) = 1 - (1/x)^p$ for $x > 1$ and an unknown parameter p. Find $\hat{p}$, i.e., the maximum likelihood estimate of p, based on the above observed values.

(a) $\hat{p} \approx \frac{1}{\ln(20)}$
(b) $\hat{p} \approx \frac{3\ln(20)}{2}$
(c) $\hat{p} \approx \frac{3}{4\ln(20)}$
(d) $\hat{p} \approx \frac{3}{2\ln(20)}$
(e) None of the above

Problem 3.9. You fit a distribution with the following density function:

$$f_X(x) = (p+1)x^p, \quad 0 < x < 1, p > -1.$$

As usual, your observations are denoted by

$$x_1, x_2, \ldots, x_n.$$

What is the expression for the maximum likelihood estimate of the parameter p?

(a) $-\frac{n}{\sum_{i=1}^{n}\ln(x_i)} - 1$
(b) $-\frac{1}{n}\sum_{i=1}^{n}\ln(x_i)$
(c) $\frac{1}{n}\sum_{i=1}^{n} x_i$
(d) $\frac{\sum_{i=1}^{n}\ln(x_i)}{n} - 1$
(e) None of the above.

Problem 3.10. You have observed the following three loss amounts:

$$190, \quad 90, \quad 60$$

Four other loss amounts are known to be less than or equal to 60. Losses follow an inverse exponential distribution. Calculate the maximum likelihood estimate of the population mode based on the above data.

(a) 10.125
(b) 12.378
(c) 15.044
(d) 20.232
(e) None of the above.

Problem 3.11. *Source: Sample STAM Problem#196.*
You are given the following 10 bodily injury losses (before the deductible is applied):

Loss amount	Number of losses	Policy limit
200	3	500
400	4	800
> 800	3	800

Past experience indicates that these losses follow a two-parameter Pareto distribution with parameters α unknown and $\theta = 1,000$. Calculate the maximum likelihood estimate of α.

(a) 1.9145
(b) 2.307
(c) 2.853
(d) 3.089
(e) None of the above.

Problem 3.12. *Source: Sample STAM Problem #152.*
You are given the following information:

(i) A sample of losses is: $400, 600, 700, 900, 1000$
(ii) No information is available about losses of 300 or less.
(iii) Losses are assumed to follow an exponential distribution with mean θ.

Calculate the maximum likelihood estimate of θ based on the above data.

(a) 420
(b) 520
(c) 720
(d) 920
(e) None of the above.