M378K: March 31st, 2025.

Confidence Intervals for the Variance.

Consider a normal model w/ both parameters unknown, i.e., a random sample $(Y_1, \dots, Y_n)$ from $N(\mu, \sigma)$.

both unknown

A ~~good~~ unbiased point estimator for σ^2 is

$$\boxed{S^2} = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y})^2$$

Theorem. Consider a random sample $(Y_1, \dots, Y_n)$ from $N(\mu, \sigma)$. Let

$$\bar{Y} = \frac{1}{n}(Y_1 + \dots + Y_n)$$

and

$$Q^2 = \frac{1}{\sigma^2}\sum_{i=1}^{n}(Y_i - \bar{Y})^2 .$$

Then,

- $\bar{Y} \sim Normal(mean = \mu, sd = \frac{\sigma}{\sqrt{n}})$
- Q^2 is a pivotal quantity for σ^2; in fact, $Q^2 \sim \chi^2(df = n-1)$
- $\bar{Y}$ and Q^2 are INDEPENDENT.

Problem 16.3. *What is the unbiased estimator for σ^2*

$$\boxed{S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i-\bar{Y})^2}$$

Problem 16.4. *Assume a random sample $Y_1, Y_2, \ldots, Y_n$ from a normal distribution with mean μ and standard deviation σ - both unknown. What's the distribution of*

$$\boxed{Q^2 = \frac{\sum_{i=1}^{n}(Y_i-\bar{Y}_n)^2}{\sigma^2}} = \boxed{\frac{(n-1)S^2}{\sigma^2}} = \frac{\cancel{(n-1)}}{\sigma^2}\cdot\frac{1}{\cancel{n-1}}\sum_{i=1}^{n}(Y_i-\bar{Y})^2$$

$$= \frac{1}{\sigma^2}\sum_{i=1}^{n}(Y_i-\bar{Y})^2$$

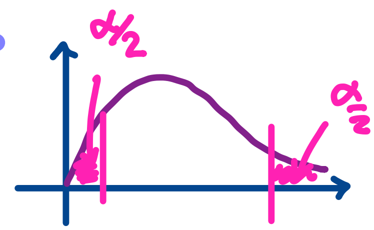
$$\boxed{\chi^2(df=n-1)}$$

A PIVOTAL QUANTITY for σ^2

Problem 16.5. *Assume that you are assigned a* confidence level $1-\alpha$. *What does it mean to find a* confidence interval *for S^2?*

$$\mathbb{P}[a \leq Q^2 \leq b] = 1-\alpha$$

$$\mathbb{P}\left[\underset{\chi^2_L}{a} \leq \frac{(n-1)S^2}{\sigma^2} \leq \underset{\chi^2_U}{b}\right] = 1-\alpha$$



Problem 16.6. *Are $\hat{\chi}^2_L$ and $\hat{\chi}^2_U$ as above uniquely defined?* No :)

We can choose a symmetric confidence interval via

$a = \chi^2_L = \text{qchisq}(\alpha/2, df=n-1)$

and

$b = \chi^2_U = \text{qchisq}(1-\alpha/2, df=n-1)$

Problem 16.7. *What's the form of the confidence interval, then?*

$$\mathbb{P}\left[\chi^2_L \leq \frac{(n-1)s^2}{\sigma^2} \leq \chi^2_U\right] = 1-\alpha$$

$$\mathbb{P}\left[\frac{1}{\chi^2_L} \geq \frac{\sigma^2}{(n-1)s^2} \geq \frac{1}{\chi^2_U}\right] = 1-\alpha$$

$$\mathbb{P}\left[\underbrace{\boxed{\frac{(n-1)s^2}{\chi^2_U}}}_{\hat{\theta}_L} \leq \sigma^2 \leq \underbrace{\boxed{\frac{(n-1)s^2}{\chi^2_L}}}_{\hat{\theta}_R}\right] = 1-\alpha$$

Problem 16.8. *Assume the above setting. Let the random sample be of size* $n = 9$. *You do the arithmetic and arrive at the estimate* $\boxed{s^2 = 7.93}$ *(based on the data set). Using the above procedure, find the* 90%*−confidence interval for* σ^2.

→: $n = 9 \implies df = 9 - 1 = 8$

$1-\alpha = 0.9 \implies \alpha = 0.10$

$\chi^2_L = \text{qchisq}(0.05, df=8) = 2.733$

$\chi^2_U = \text{qchisq}(0.95, df=8) = 15.51$

Our confidence interval:

$$\left(\frac{8\cdot 7.93}{15.51}, \frac{8\cdot 7.93}{2.733}\right) = (4.090264, 23.21259)$$

$\square$