Name:

M339D=M389D Introduction to Actuarial Financial Mathematics
University of Texas at Austin
In-Term Exam III
Instructor: Milica Čudina

All written work handed in by the student is considered to be
their own work, prepared without unauthorized assistance.

The University Code of Conduct

"The core values of The University of Texas at Austin are learning, discovery, freedom, leadership, individual opportunity, and responsibility. Each member of the university is expected to uphold these values through integrity, honesty, trust, fairness, and respect toward peers and community. As a student of The University of Texas at Austin, I shall abide by the core values of the University and uphold academic integrity."

"I agree that I have complied with the UT Honor Code during my completion of this exam."

Signature:

Notes: This is a closed book and closed notes exam. The maximal score on this exam is 100 points.
Time: 50 minutes

Problem 3.1. (5 points) Let the continuously compounded risk-free interest rate be 0.05. The current price of a continuous-dividend-paying stock is 80. Its dividend yield is 0.02. You model the price of this stock in half a year using a one-period binomial tree with the up factor of 1.2 and the down factor of 0.8. Consider a half-year, 90-strike European put option on the above stock. What is the investment in the shares of stock in the replicating portfolio for the put?

Solution: The two possible stock prices are $S_u = 1.2(80) = 96$ and $S_d = 0.8(80) = 64$. So, the possible put payoffs are $V_u = 0$ and $V_d = 90 - 64 = 26$. Hence,

$$\Delta = e^{-\delta h}\frac{V_u - V_d}{S_u - S_d} = e^{-0.02(1/2)}\left(\frac{0-26}{96-64}\right) = -0.8044155.$$

Problem 3.2. (5 points) Assume a positive interest rate. For any strike K, let $V_C(K)$ denote the price of a European call on stock S with expiration date T and strike price K and let $V_P(K)$ denote the price of a European put on stock S with expiration date T and strike price K.

Let $K_1 < K_2 < K_3$.

Which one of the following statements is FALSE?

(a) $V_P(K_1) \leq V_P(K_2)$

(b) $V_C(K_1) \geq V_C(K_2)$

(c) $V_C(K_2) - V_C(K_1) \leq K_2 - K_1$

(d) $V_P(K_1) - V_P(K_2) \leq K_1 - K_2$

(e) $\frac{V_P(K_2)-V_P(K_1)}{K_2-K_1} \leq \frac{V_P(K_3)-V_P(K_2)}{K_3-K_2}$

Solution: (d)

Problem 3.3. (5 points) Let the continuously compounded, risk-free interest rate be 0.04. Your co-worker Psmith considers a non-dividend-paying stock whose price in one year is modeled using a one-period binomial tree. Psmith tells you that a one-year, 93-strike European call option has the following properties:

- the payoff of the option at the up node equals 39.20;
- the number of shares of stock in the replicating portfolio for the call option equals 0.80.

What is the amount borrowed in the replicating portfolio for the call option?

Solution: From the given payoff in the up node, we know that $S_u = 39.20 + 93 = 132.20$. From the given value of $\Delta = 0.80$, we know that the call is out of the money in the down node. So,

$$0.80 = \Delta = \frac{V_u - V_d}{S_u - S_d} = \frac{39.20}{132.20 - S_d}.$$

We can now conclude that

$$S_d = 132.20 - \frac{39.20}{0.80} = 83.20.$$

Finally,

$$B = e^{-rh}\left(\frac{S_u V_d - S_d V_u}{S_u - S_d}\right) = e^{-0.04}\left(\frac{0 - 83.20(39.20)}{132.20 - 83.20}\right) = 63.95015.$$

Problem 3.4. (5 points) You are given that the price of:

- a \$50-strike, one-year European call equals \$9,
- a \$65-strike, one-year European call equals \$3.

Both options have the same underlying asset. What is the maximal price of a \$56-strike, one-year European call such that there is no arbitrage in our market model?

Solution: Using the convexity of call price with respect to the strike, we get the following answer:

$$\frac{3}{5} \times 9 + \frac{2}{5} \times 3 = \frac{27+6}{5} = 6.60.$$

Problem 3.5. (5 points) A long strangle position ...

(a) is equivalent to a short butterfly spread.
(b) can be replicated with a short call and a long put with the same strike, underlying asset and exercise date.
(c) is always strictly more expensive than the straddle on the same underlying asset and with the same exercise date.
(d) is a speculation on the stock's volatility.
(e) None of the above.

Solution: (d)

Problem 3.6. (5 points) Which one of the following positions always has an infinite upward potential in the sense that the payoff diverges to positive infinity as the argument s (standing for the final stock price) tends to positive infinity?

(a) A long call option.
(b) A short straddle.
(c) A long bull spread.
(d) A long butterfly spread.
(e) None of the above.

Solution: (a)

Problem 3.7. (10 points) Consider a non-dividend-paying stock whose current price is \$90 per share. You model the evolution of this stock price over the following year using a one-period binomial tree under the assumption that the stock price can be either \$100, or \$85 in one year.

The continuously compounded, risk-free interest rate is 0.05.

The price of a $K-$strike, one-year European straddle on the above stock, consistent with the above stock-price model, is 6.50. How much is K?

Solution: The risk-neutral probability of an up movement is

$$p^* = \frac{90e^{0.05} - 85}{100-85} = 0.6409599.$$

So, the price of our straddle satisfies one of the following three cases.

Case #1. K is between 85 and 100.

$$V(0) = e^{-0.05}[p^*(100-K) + (1-p^*)(K-85)] = 6.50.$$

We solve for K in the following equation:

$$p^*(100-K)+(1-p^*)(K-85)=6.50e^{0.05} \quad \Rightarrow \quad 100p^*-p^*K+(1-p^*)K-85(1-p^*)=6.50e^{0.05}$$
$$\Rightarrow \quad (1-2p^*)K=6.50e^{0.05}-100p^*+85(1-p^*)$$

Finally,

$$K=\frac{6.50e^{0.05}-100p^*+85(1-p^*)}{1-2p^*}=94.86499.$$

Case #2. K is greater than 100.

$$V(0)=e^{-0.05}[p^*(K-100)+(1-p^*)(K-85)]=6.50.$$

We solve for K in the following equation:

$$p^*(K-100)+(1-p^*)(K-85)=6.50e^{0.05} \quad \Rightarrow \quad p^*K-100p^*+(1-p^*)K-85(1-p^*)=6.50e^{0.05}$$
$$\Rightarrow \quad K=6.50e^{0.05}+100p^*+85(1-p^*)=101.4477$$

Case #3. K is smaller than 85. This case does not yield any acceptable solutions.

Problem 3.8. (2 points) It is never optimal to exercise an American call option on a non-dividend paying stock early. *True or false? Why?*

Solution: TRUE
See your lecture notes.

Problem 3.9. (2 points) The payoff curve of a call bear spread is never positive. *True or false? Why?*

Solution: TRUE
Draw the payoff curve.

Problem 3.10. (2 points) Consider a binomial asset-pricing model in which the length of every period equals a quarter-year. Let $i^{(4)}$ denote the nominal interest rate compounded quarterly. Then, in our usual notation, to avoid arbitrage, we must have

$$e^{\delta/4}d<1+\frac{i^{(4)}}{4}<e^{\delta/4}u. \tag{3.1}$$

True or false?

Solution: TRUE

Problem 3.11. (2 points) Arithmetic-average-price Asian put options are always worth more than the geometric-average-price Asian put options with the same strike price. *True or false? Why?*

Solution: FALSE
Since $A(T)\geq G(T)$, we have that $-A(T)\leq -G(T)$. Hence,

$$K-A(T)\leq K-G(T)$$

The positive part function is increasing itself, so

$$(K-A(T))_+\leq (K-G(T))_+.$$

Problem 3.12. (2 points) A call-on-put option is worth at least as much as an otherwise identical put-on-put option. *True or false? Why?*

Solution: FALSE
Looking at put-call parity, there is no reason for this to be true.

Problem 3.13. (5 points) You are required to price a one-year, yen-denominated currency option on the USD. The exchange rate over the next year is modeled using a forward binomial tree with the number of periods equal to 4. Assume that the volatility of the exchange rate equals 0.1.

The continuously compounded risk-free interest rate for the yen equals 0.05, while the continuously compounded risk-free interest rate for the USD equals 0.02. What is the value of the so-called up factor u in the resulting forward binomial tree?

Solution:

$$u = e^{(r_{yen}-r_{\$})h+\sigma\sqrt{h}} = e^{(0.05-0.02)\frac{1}{4}+0.1\sqrt{\frac{1}{4}}} = 1.0592.$$

Problem 3.14. (10 points) Let the continuously-compounded, risk-free interest rate be equal to 0.04. The current price of a non-dividend-paying stock is \$100. Its volatility is 0.2. The evolution of this stock over the following year is modeled using a four-period forward binomial tree. What is the price of a one-year, \$95-strike European put on the above stock consistent with the above tree?

Solution: In our usual notation, the risk-neutral probability is

$$p^* = \frac{1}{1+e^{\sigma\sqrt{h}}} = \frac{1}{1+e^{0.2\sqrt{0.25}}} = 0.4750208$$

The up and down factors are

$$u = e^{(r-\delta)h+\sigma\sqrt{h}} = e^{0.01+0.1} = e^{0.11},$$
$$d = e^{(r-\delta)h-\sigma\sqrt{h}} = e^{0.01-0.1} = e^{-0.09}.$$

Since, we are pricing a put, we move from the lowest terminal nodes upwards in the interest of efficiency.

$$S_{dddd} = S(0)d^4 = 69.76763, \quad S_{dddu} = S(0)ud^3 = 85.21438, \quad S_{dduu} = S(0)u^2d^2 = 104.0811 > K = 95.$$

So, the option is out of the money at the topmost terminal nodes as well. The time-0 European put option price is

$$V_P(0) = e^{-0.04}\left[(1-p^*)^4(95-69.76763) + 4p^*(1-p^*)^3(95-85.21438)\right] = 4.426159.$$

Problem 3.15. (10 points) Consider a one-period forward binomial model for the stock-price movement over the following year. The current stock price is $S(0) = 100$, its dividend yield is 0.05 and its volatility is 0.3 The continuously compounded risk-free interest rate is given to be 0.05.

Consider American call options on this stock with the expiration date at the end of the period/year. What is the maximal strike price K for which there is early exercise?

Solution: In our usual notation, $u = e^{0.3} = 1.35$ and $d = 0.74$. The risk-neutral probability is

$$p^* = \frac{1}{1+e^{\sigma}} = 0.425.$$

Then, the continuation value at the root node is

$$V_C(0) = e^{-0.05}[0.425(135 - K)_+ + 0.575(74 - K)_+]$$

as a function of K. The early-exercise condition is

$$100 - K > V_C(0).$$

It is evident that in order for early exercise to occur it must be that $K < 100$. So, let us focus on the possible solutions to the above inequality in the interval $(74, 100)$ first. For such K, the above inequality becomes

$$100 - K > e^{-0.05} \times 0.425(135 - K).$$

Note the absence of the "positive part"in the last expression. The K which satisfy this inequality are such that

$$100 - 54.57 > 0.596K \quad \Rightarrow \quad 76.22483 > K.$$

Problem 3.16. (10 points) Consider a non-dividend paying stock whose current price is \$100 per share. You model the evolution of the stock price over the following year using a two-period binomial tree with $u = 1.12$ and $d = 0.90$.

The continuously-compounded, risk-free interest rate is 0.04.

Consider a \$100-strike, one-year **down-and-out** call option with a barrier of \$95 on the above stock. What is the price of this option consistent with the above stock-price model?

Solution: The risk-neutral probability of a single step up in the tree equals

$$p^* = \frac{e^{0.02} - 0.90}{1.12 - 0.90} = 0.5463697.$$

The option is knocked-out if the stock price goes down in the first step to $S_d = 100(0.9) = 90$. If the stock price goes up, from the perspective of the *up* node, we have $S_{uu} = 125.44$ and $S_{ud} = 100.80$. At neither of those terminal nodes does the option get knocked out. So, the payoff of the option will be

$$\begin{aligned} V_{uu} &= 25.44, && \text{if the path } \textit{up-up} \text{ is taken,} \\ V_{ud} &= 0.80, && \text{if the path } \textit{up-down} \text{ is taken,} \\ V_{du} = V_{dd} &= 0, && \text{otherwise.} \end{aligned}$$

So, the option price is

$$V(0) = e^{-rT}[(p^*)^2 V_{uu} + p^*(1 - p^*)V_{ud}] = 7.487072.$$

Problem 3.17. (15 points) Your goal is to price a put option on a futures contract. The movements of the futures price are modeled by a binomial tree. You are given:

(i) Each period is three months.
(ii) $u_F/d_F = 5/4$, where u_F is one plus the rate of gain on the futures price if it goes up, and d_F is one plus the rate of loss if it goes down.
(iii) The risk-neutral probability of an up move is 2/3.
(iv) The initial futures price is 70.
(v) The continuously compounded risk-free interest rate is 4%.

Find the price of a half-year, 70-strike American put option on the futures contract.

Solution: We are given that

$$\frac{2}{3}=\frac{1-d_F}{u_F-d_F}=\frac{d_F^{-1}-1}{\frac{u_F}{d_F}-1}=\frac{d_F^{-1}-1}{\frac{5}{4}-1}\quad\Rightarrow\quad d_F^{-1}=\frac{7}{6}\quad\Rightarrow\quad d_F=\frac{6}{7}\quad\Rightarrow\quad u_F=\frac{15}{14}\,.$$

The prices in the futures-price tree are, thus,

$$\begin{array}{ccc} & & F_{uu}=80.37714 \\ & F_u=75 & \\ F_0=70 & & F_{ud}=64.28571 \\ & F_d=60 & \\ & & F_{dd}=51.42857 \end{array}$$

At the *up* node, we have that

$$\begin{aligned} CV_u &= e^{-0.01}\left(\frac{1}{3}\right)(5.71429)=1.885811,\\ IE_u &= 0. \end{aligned}$$

So, we conclude that it's optimal to hold onto the option at this node and $V_A^u=1.885811$. At the *down* node, we have that

$$\begin{aligned} CV_d &= e^{-0.01}\left(\left(\frac{2}{3}\right)(5.71429)+\left(\frac{1}{3}\right)(18.57143)\right)=9.900502,\\ IE_d &= 10. \end{aligned}$$

So, the option will optimally be exercised early. We get that $V_d^A=10$. The option is at-the-money at the root node. Its price is

$$V_P(0)=e^{-0.01}\left(\left(\frac{2}{3}\right)(1.885811)+\left(\frac{1}{3}\right)(10)\right)=4.544864.$$