M378K: January 16^{th}, 2026.

M378K Introduction to Mathematical Statistics

Problem Set #2

Discrete random variables.

2.1. Probability mass function.

Recall the following definition from the last class:

Definition 2.1. *Given a set B, we say that a random variable Y is $B-$valued if*

$$\mathbb{P}[Y \in B] = 1.$$

We reserve special terminology for random variables Y depending on the cardinality of the set B from the above definition. In particular, we have the following definition:

Definition 2.2. *A random variable Y is said to be discrete if there exists a set S such that :*

- *Y is $S-$valued, and*
- *S is either* **finite** *or* **countable**.

Problem 2.1. *Provide an example of a* **discrete** *random variable.*

- coin toss: $S = \{H, T\}$
- rolling a die: $S = \{1,2,3,4,5,6\}$
- # of trials until first H: $S = \mathbb{N} = \{1,2,3,\dots\}$

Our next task is to try to keep track of the probabilities that Y takes specific values from S. In order to be more "economical", we introduce the following concept:

"**Definition 2.3.**" *The support S_Y of a random variable Y is the* **smallest** *set S such that Y is $S-$valued.*

Problem 2.2. *What is the* **support** *of the random variable you provided as an example in the above problem?*

Problem 2.3. *Let $y \in S_Y$ where Y is a discrete random variable. Is it possible to have $\mathbb{P}[Y = y] = 0$?*

$\boxed{\text{No!}}$ Assume, to the contrary, that such a $\tilde{y}$ exists, i.e., $\tilde{y} \in S_Y$ and $\mathbb{P}[Y=\tilde{y}]=0$.

Set $\tilde{S}_Y = S_Y \setminus \{\tilde{y}\}$.

Then, $\mathbb{P}[Y \in \tilde{S}_Y] = \mathbb{P}[Y \in S_Y] - \mathbb{P}[Y=\tilde{y}] = \mathbb{P}[Y \in S_Y] = 1$

BUT: $\tilde{S}_Y \subset S_Y$ $\square$

Usually, we are interested in calculating and modeling probabilities that look like this

$$\mathbb{P}[Y \in A] \quad \text{for some } A \subseteq S_Y.$$

Note that, if we know the probabilities of the form

$$\mathbb{P}[Y = y] \quad \text{for all } y \in S_Y,$$

then we can calculate any probability of the above form. *How?*

$$\mathbb{P}[Y \in A] = \sum_{y \in A} \mathbb{P}[Y = y]$$

So, if we "tabulate" the probabilities of the form $\mathbb{P}[Y = y]$ for all $y \in S_Y$, we have sufficient information to calculate any probability of interest to do with the random variable Y. This observation motivates the following definition:

Definition 2.4. *The probability mass function (pmf) of a* **discrete** *random variable Y is the function $p_Y : S_Y \to \mathbb{R}$ defined as*

$$p_Y(y) = \mathbb{P}[Y = y] \quad \textit{for all } y \in S_Y.$$

Can you think of different ways in which to display the pmf?

- a distribution table.

y	y_1	y_2	.....	y_k	....
$p_Y(y)$	p_1	p_2	....	p_k	.....

- roll of a die: $p_Y(k) = \frac{1}{6}$, $k = 1, \dots, 6$

- formula

What is the pmf of the random variable which you provided as an example above?

What are the immediate properties of every pmf?

- $p_Y(y) > 0$ for all $y \in S_Y$
- $\sum_{y \in S_Y} p_Y(y) = 1$

Does the "reverse" hold, i.e., if a function p_Y satisfies you stated, is it always a pmf of **some** random variable?

✓

Problem 2.4. *The number of pieces of gossip that break out in a particular high school in a week is modeled by a random variable Y with the following probability mass function:*

$$p_Y(n) = \frac{1}{(n+1)(n+2)} \quad \textit{for all } n \in \mathbb{N}_0. = \{0, 1, 2, \dots\}$$

Is the above a well-defined probability mass function?

$\rightarrow$:

- $p_Y(n) > 0$ for all $n \in \mathbb{N}_0$ ✓
- $\sum_{n=0}^{\infty} p_Y(n) \overset{?}{=} 1$

$$\underbrace{\sum_{n=0}^{N} p_Y(n)}_{} \xrightarrow[N\to\infty]{?} 1$$

$$= \sum_{n=0}^{N} \boxed{\frac{1}{(n+1)(n+2)}}$$

$$\boxed{\frac{1}{n+1} - \frac{1}{n+2}} = \frac{n+2-(n+1)}{(n+1)(n+2)} = \boxed{\frac{1}{(n+1)(n+2)}}$$

$$= \sum_{n=0}^{N} \left(\frac{1}{n+1} - \frac{1}{n+2} \right)$$

$$= \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) \cancel{+} \dots \cancel{+} \left(\cancel{\frac{1}{N+1}} - \frac{1}{N+2}\right)$$

$$= 1 - \frac{1}{N+2} \xrightarrow[N\to\infty]{} 1$$

□