M378K Introduction to Mathematical Statistics
Problem Set #13
Bias. MSE.

Problem 13.1. Source: "Mathematical Statistics with Applications" by Wackerly, Mendenhall, Scheaffer.
Let $Y_1, , Y_2, Y_3$ be a random sample from $E(\tau)$. Consider the following five estimators of τ:

$$\hat{\theta}_1 = Y_1, \quad \hat{\theta}_2 = \frac{Y_1 + Y_2}{2}, \quad \hat{\theta}_3 = \frac{Y_1 + 2Y_2}{3}, \quad \hat{\theta}_4 = 3Y_{(1)}, \quad \hat{\theta}_5 = \bar{Y}.$$

Which ones of these estimators are unbiased? Among the unbiased ones which one has the smallest variance?

Problem 13.2. *Suppose that the two estimators* $\hat{\theta}_1$ *and* $\hat{\theta}_2$ *are unbiased. We know that* $\mathrm{Var}[\hat{\theta}_1] = \sigma_1^2$ *and* $\mathrm{Var}[\hat{\theta}_2] = \sigma_2^2$.

Consider the estimator all the estimators that can be obtained as convex combinations of $\hat{\theta}_1$ *and* $\hat{\theta}_2$*, i.e., all the estimators of the form*

$$\hat{\theta} = \alpha\hat{\theta}_1 + (1-\alpha)\hat{\theta}_2.$$

What can you say about the bias *of estimators* $\hat{\theta}$ *of the form above? Assuming that* $\hat{\theta}_1$ *and* $\hat{\theta}_2$ *are* **independent***, for which weight* α *is the variance minimal?*

Problem 13.3. *Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from a continuous distribution with probability density function*

$$f_Y(y) = \frac{\alpha y^{\alpha-1}}{\theta^\alpha} \mathbf{1}_{[0,\theta]}(y)$$

with a known parameter $\alpha > 0$ and an unknown parameter $\theta > 0$. We propose the estimator $\hat{\theta} = \max(Y_1, \ldots, Y_n)$. Is this estimator unbiased*? If not, how would you modify it to create an* unbiased *estimator? What is the* **mean-squared error** *of the* unbiased *estimator you obtained?*