UNIVERSITY OF TEXAS AT AUSTIN

Problem set 1

Problem 1.1. Let E and F be any two events. Then, $\mathbb{P}[E \cup F] \leq \mathbb{P}[E] + \mathbb{P}[F]$. *True or false? Why?*

Solution: TRUE
By the inclusion-exclusion formula, we know that

$$\mathbb{P}[E \cup F] = \mathbb{P}[E] + \mathbb{P}[F] - \mathbb{P}[E \cap F].$$

Now, remember that $\mathbb{P}[E \cap F] \geq 0$.

Problem 1.2. Let E and F be any two events. If $\mathbb{P}[E] = \mathbb{P}[F] = \frac{2}{3}$, then E and F cannot be mutually exclusive. *True or false? Why?*

Solution: TRUE
If E and F were mutually exclusive, we would have that

$$\mathbb{P}[E \cup F] = \mathbb{P}[E] + \mathbb{P}[F] = \frac{4}{3} > 1.$$

Contradiction!

Problem 1.3. Let E and F be any two events with positive probability. If $\mathbb{P}[E|F] < \mathbb{P}[E]$, then $\mathbb{P}[F|E] < \mathbb{P}[F]$. *True or false? Why?*

Solution: TRUE
By the definition of conditiona probability, the given inequality can be rewritten as follows

$$\begin{aligned}
\mathbb{P}[E|F] < \mathbb{P}[E] \quad &\Leftrightarrow \quad \frac{\mathbb{P}[E \cap F]}{\mathbb{P}[F]} < \mathbb{P}[E] \quad \Leftrightarrow \quad \mathbb{P}[E \cap F] < \mathbb{P}[E]\mathbb{P}[F] \\
&\Leftrightarrow \quad \frac{\mathbb{P}[E \cap F]}{\mathbb{P}[E]} < \mathbb{P}[F] \quad \Leftrightarrow \quad \mathbb{P}[F|E] < \mathbb{P}[F]
\end{aligned}$$

Problem 1.4. If events E and F are independent and events F and G are independent, then E and G are independent as well. *True or false? Why?*

Solution: FALSE
What if $E = G$?

Problem 1.5. The four standard blood types are distributed in a populations as follows:

$$\begin{array}{ll} A - 42\% & O - 33\% \\ B - 18\% & AB - 7\% \end{array}$$

Assuming that people choose their mates independently of their blood type, find the probability that the people in a randomly chosen couple from this population have different blood types.

Solution: Let E denote the event that the people in a randomly chosen couple have different blood types. We have already calculated in the sample exam that $\mathbb{P}[E^c] = 0.3226$. So, $\mathbb{P}[E] = 1 - 0.3226 = 0.6774$.

Problem 1.6. Let X denote the outcome of a roll of a fair, regular icosahedron (a polyhedron with 20 faces) with numbers $1, 2, \cdots, 20$ written on its sides. Then $\mathbb{E}[X] = 15/2$. *True or false? Why?*

Solution: FALSE
Straight from the definition of expectation, we have that

$$\mathbb{E}[X] = \frac{1}{20}(1 + 2 + \cdots + 20) = \frac{1}{20} \cdot \frac{20 \cdot 21}{2} = \frac{21}{2}\,.$$

Problem 1.7. The minimum of two independent exponential random variables is also exponential. *True or false? Why?*

Solution: TRUE

Let T_1 and T_2 be two independent exponential random variables with parameters θ_1 and θ_2, respectively. Define $T = \min(T_1, T_2)$. Obviously, the support of the random variable T is $(0, \infty)$. Let us figure out the survival function of T. For every $t > 0$, we have

$$S_T(t) = \mathbb{P}[T > t] = \mathbb{P}[\min(T_1, T_2) > t] = \mathbb{P}[T_1 > t, T_2 > t].$$

Due to independence of random variables T_1 and T_2, the above equals

$$\mathbb{P}[T_1 > t]\mathbb{P}[T_2 > t] = S_{T_1}(t)S_{T_2}(t).$$

We now recall the form of the survival function of an exponential random variable (or look into our STAM tables). The above equals

$$e^{-t/\theta_1}e^{-t/\theta_2} = e^{-t(\frac{1}{\theta_1}+\frac{1}{\theta_2})}$$

Finally, we note that T must be exponential with parameter θ satisfying

$$\frac{1}{\theta} = \frac{1}{\theta_1} + \frac{1}{\theta_2}.$$