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Lines. Planes. Hyperplanes.

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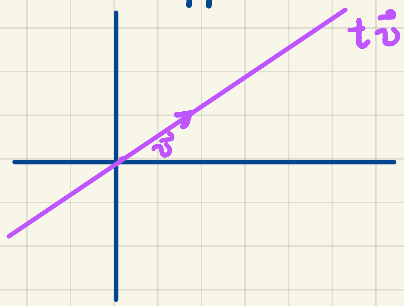
Lines in $\mathbb{R}^n$.

Start w/ $\vec{v}$, a non-zero vector in $\mathbb{R}^n$, i.e.,

$$\vec{v} = (v_1, v_2, \dots, v_n).$$

For any scalar $t \in \mathbb{R}$, the vector $t\vec{v}$ will have the same direction as $\vec{v}$ when $t>0$, the opposite direction when $t<0$, and be $\vec{0}$ when $t=0$.



If I add a vector, say $\vec{p} \neq 0$, then I get a line shifted from the origin

$$\boxed{\{ t\vec{v} + \vec{p}, \ -\infty < t < \infty \}}$$ is any line in $\mathbb{R}^n$

VECTOR EQUATION

Can be expressed as PARAMETRIC EQUATIONS:

$$\boxed{\begin{aligned} y_1 &= t \cdot v_1 + p_1 \\ y_2 &= t \cdot v_2 + p_2 \\ &\vdots \\ y_n &= t \cdot v_n \cdot p_n \end{aligned}}$$

Hyperplanes.

Consider a set of all points $(x,y) \in \mathbb{R}^2$ which satisfy the equation

$$\boxed{a \cdot x + b \cdot y + d = 0}$$

w/ a, b, and all scalars and @ least one of a and b is $\neq 0$.

Say that $b \neq 0$, then, we can rewrite the above as:

$$\boxed{y = -\frac{a}{b}x - \frac{d}{b}}$$

The equation we remember from childhood

The vector form is obtained by setting $x \leftrightarrow t$

$$(x,y) = \left(t, -\frac{a}{b} \cdot t - \frac{d}{b}\right) = t \cdot \underbrace{\left(1, -\frac{a}{b}\right)}_{\vec{v}} + \underbrace{\left(0, -\frac{d}{b}\right)}_{\vec{p}}$$

Return to: $\boxed{ax + by + d = 0}$ ✓

Define: $\vec{n} = (a, b)$

We can now write: w/ $\vec{x} = (x,y)$

$$\boxed{\vec{n} \cdot \vec{x} + d = 0}$$

Say that $\vec{p} = (p_1, p_2)$ is a point on the line.

$\Rightarrow \quad \vec{n} \cdot \vec{p} + d = 0 \quad \Rightarrow \quad \boxed{d = -\vec{n}\,\vec{p}}$

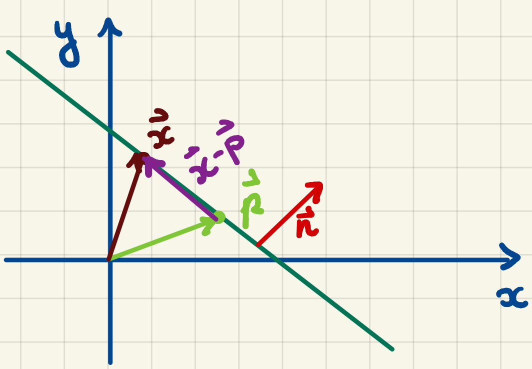
$\Rightarrow \quad \vec{n}\,\vec{x} - \vec{n}\,\vec{p} = 0$

NORMAL EQUATION

$\Rightarrow \quad \boxed{\vec{n}(\vec{x} - \vec{p}) = 0}$

$\Rightarrow$ An equivalent condition for $\vec{x}$ being on the line is that

$$\vec{n} \perp (\vec{x} - \vec{p})$$



$\vec{n}$ ... normal vector

The hyperplane is the set of all the points $\vec{x} \in \mathbb{R}^2$ which satisfy the NORMAL EQUATION.

This is the generalization of the above to $\mathbb{R}^n$.

<u>Def'n.</u> Say that $\vec{n}$ and $\vec{p}$ are vectors in $\mathbb{R}^n$ w/ $\vec{n} \neq \vec{0}$.

The set of all vectors $\vec{x}$ in $\mathbb{R}^n$ which satisfy the NORMAL EQUATION

$$\boxed{\vec{n} \cdot (\vec{x} - \vec{p}) = 0}$$

is called a hyperplane through the point $\vec{p}$ normal to the vector $\vec{n}$.