M378K: September 3rd, 2025.

Review.

Def'n. Let E and F be two events.
Let $\mathbb{P}[E]>0$.
The conditional probability of F given E is

$$\mathbb{P}[F \mid E] = \frac{\mathbb{P}[E \cap F]}{\mathbb{P}[E]}$$

Remark. If we learn nothing about the probability of F by knowing that E happened, then we can write

$$\mathbb{P}[F \mid E] = \mathbb{P}[F]$$

$$\frac{\mathbb{P}[E \cap F]}{\mathbb{P}[E]} = \mathbb{P}[F]$$

$$\boxed{\mathbb{P}[E \cap F] = \mathbb{P}[E] \cdot \mathbb{P}[F]}$$

3. Independent events

What if knowing that an event happened in fact does **not** give any information about the probability of another event?

Definition 3.1. *We say that events E and F on Ω are independent if*

$$\mathbb{P}[E \cap F] = \mathbb{P}[E]\mathbb{P}[F].$$

In the case when E or F have a positive probability, it's possible to rewrite the above condition in a different (illustrative!) way. *How?*

Assume $\mathbb{P}[F] > 0$.

$$\mathbb{P}[E \mid F] = \mathbb{P}[E]$$

Now that we know the notion of **independence**, we can construct random variables in many creative ways.

Example 3.2. *A fair coin is tossed repeatedly and* **independently** *until the first* Heads. *Let the random variable Y represent the total number of* Tails *observed by the end of the procedure.*

What is the support of the random variable Y?

$$S_Y = \{0, 1, 2, \dots\} = \mathbb{N}_0$$

What is the **probability mass function** *of the random variable Y?*

for $y \in S_Y$:

$$p_Y(y) = \left(\frac{1}{2}\right)^y \left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^{y+1}$$

Moreover, now that we remember the definition of **conditional probability**, we can solve interesting problems such as this one:

Problem 3.1. *The number of pieces of gossip that break out in a particular high school in a week is modeled by a random variable Y with the following probability mass function:*

$$p_n := \boxed{p_Y(n) = \frac{1}{(n+1)(n+2)}} \quad \textit{for all } n \in \mathbb{N}_0.$$

(i) *Is the above a well-defined probability mass function?*

(ii) *Calculate the probability that at least one piece of gossip occurred in a week* **given** *that at most four pieces of gossip occurred.*

(i)
- $p_Y(n) > 0$ for all n ✓
- $\sum_{n=0}^{\infty} p_Y(n) \overset{?}{=} 1$

$$\sum_{n=0}^{N} p_Y(n) = \sum_{n=0}^{N} \boxed{\frac{1}{(n+1)(n+2)}}$$

$$\frac{1}{n+1} - \frac{1}{n+2} = \frac{n+2-(n+1)}{(n+1)(n+2)} = \frac{1}{(n+1)(n+2)}$$

$$= \sum_{n=0}^{N} \boxed{\left(\frac{1}{n+1} - \frac{1}{n+2}\right)}$$

$$\boxed{\frac{1}{(an+b)(cn+d)} = \frac{A}{an+b} + \frac{B}{cn+d}}$$

$$= 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cdots + \cancel{\frac{1}{N+1}} - \frac{1}{N+2}$$

$$= 1 - \frac{1}{N+2} \xrightarrow[N\to\infty]{} 1 \checkmark$$

(ii)

$$\mathbb{P}[Y \geq 1 \mid Y \leq 4] = \frac{\mathbb{P}[1 \leq Y \leq 4]}{\mathbb{P}[Y \leq 4]} =$$

$$= \frac{p_1 + p_2 + p_3 + p_4}{p_0 + p_1 + p_2 + p_3 + p_4}$$

$$= \frac{\frac{1}{2} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{4}} - \cancel{\frac{1}{5}} + \cancel{\frac{1}{5}} - \frac{1}{6}}{1 - \frac{1}{6}} = \frac{\frac{1}{3}}{\frac{5}{6}} = \frac{2}{5} \quad \square$$

Named Discrete Distributions.

Def'n. **Bernoulli trials** have two possible outcomes.

They are also known as indicators (or indicator random variables).

Usually, the outcomes are encoded as

$$\begin{cases} 1 & \text{for "success"} \\ 0 & \text{for "failure} \end{cases}$$