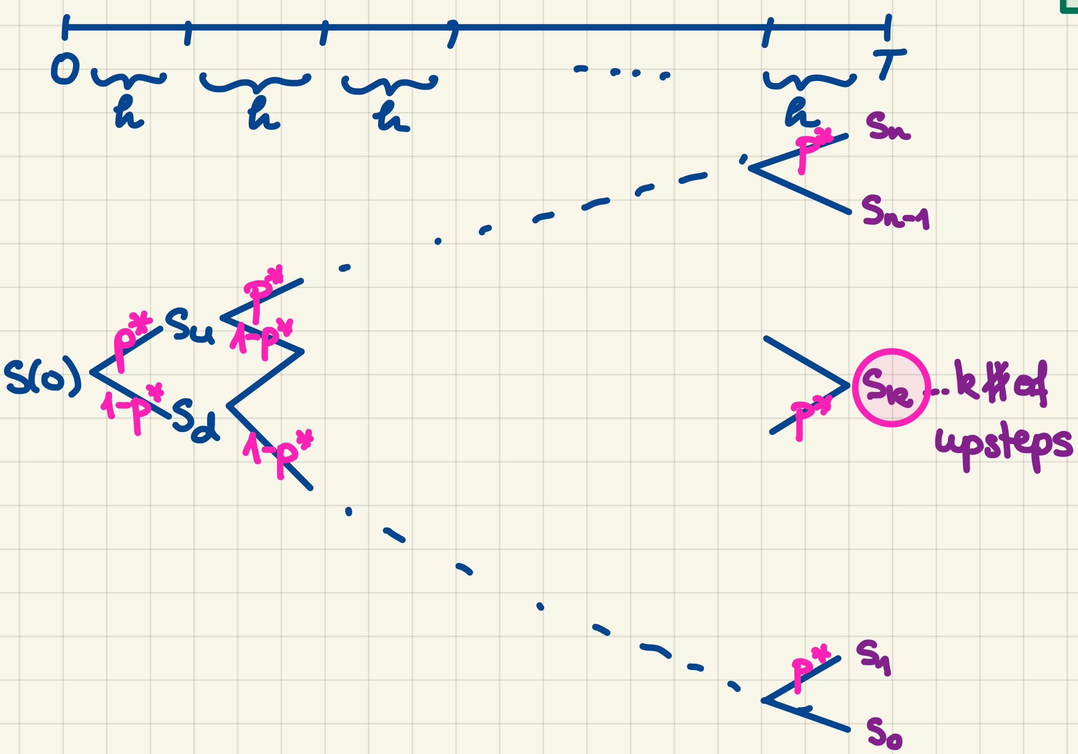
M339D: March 24th, 2025.

Multiple Binomial Periods.

T... exercise date of a European option
n... # of periods

the length of each period: $\boxed{h = \frac{T}{n}}$



$v_n = v(S_n)$

$\boxed{v_k = v(S_k)}$

$v_1 = v(S_1)$
$v_0 = v(S_0)$

The n+1 values are the support of S(T)

$\Rightarrow$ for every $k = 0, 1, \dots, n$:

$$s_k = S(0) \cdot u^k \cdot d^{n-k} = S(0) \left(\frac{u}{d}\right)^k \cdot d^n$$

k # of upsteps

Consider a European option w/ payoff f'tion $v(\cdot)$.
Then, the possible values of the payoff will be

$\boxed{v_k = v(s_k)}$ for all $k = 0, \dots, n$

Risk-neutral Pricing:

$$V(0) = e^{-rT} \mathbb{E}^*[V(T)] \quad \text{w/} \quad \boxed{p^* = \frac{e^{rh} - d}{u - d}}$$

$\Rightarrow$ The risk-neutral probability of reaching the payoff v_k is

$$\binom{n}{k}(p^*)^k(1-p^*)^{n-k}$$

$\boxed{k = 0, 1, \dots, n}$

$\Rightarrow$ The risk-neutral option price:

$$\boxed{V(0) = e^{-rT}\sum_{k=0}^{n}\left(\binom{n}{k}(p^*)^k(1-p^*)^{n-k}v_k\right)}$$

Problem 10.2. Let the continuously compounded risk-free interest rate be 0.10. Let the initial price of a non-dividend-paying stock be \$100 per share. You use a five-period binomial tree to model the stock price over the next year. Let $u = 1.04$ and $d = 0.96$.

$T=1$ $n=5$ $h=1/5=0.2$

What is the price of a one-year at-the-money European call option on the above stock?

→: The Risk-Neutral Probability:

$$p^* = \frac{e^{rh}-d}{u-d} = \frac{e^{0.1(0.2)}-0.96}{1.04-0.96} = 0.7525$$

The relevant final stock prices in our tree:

$$S_5 = S(0)u^5 = 100(1.04)^5 = 121.67 \qquad v_5 = 21.67$$

$$S_4 = S(0)u^4 d = 100(1.04)^4 \cdot (0.96) = 112.31 \qquad v_4 = 12.31$$

$$S_3 = S(0)u^3 d^2 = 100(1.04)^3(0.96)^2 = 103.67 \qquad v_3 = 3.67$$

The remaining terminal nodes are all out-o-money.

$\Rightarrow$

$$V(0) = e^{-0.10}\left(21.67 \cdot (p^*)^5 + 12.31 \cdot 5(p^*)^4(1-p^*) + 3.67 \underbrace{\binom{5}{2}}_{10}(p^*)^3(1-p^*)^2\right) = 10.01821$$

$\square$