M339J: February 9^{th}, 2022.

Tail Formula for the Expectation.

Let Y be a nonnegative continuous random variable.
Then, we have that

$$\mathbb{E}[Y] = \int_0^{+\infty} S_Y(y)\,dy \qquad (\star)$$

$\longrightarrow$: We know that, by definition,

$$\mathbb{E}[Y] = \int_0^{+\infty} y \cdot f_Y(y)\,dy.$$

If we can show that the right-hand side in $(\star)$ equals the integral above, we're done.

$$\int_0^{+\infty} S_Y(y)\,dy \underset{\text{by defn.}}{=} \int_0^{+\infty} \mathbb{P}[Y>y]\,dy = \int_0^{+\infty} \int_y^{+\infty} f_Y(u)\,du\,dy$$

Now, we switch the integrals!

$$\int_0^{+\infty} \int_0^{u} f_Y(u)\,dy\,du = \int_0^{+\infty} f_Y(u) \overbrace{\left(\int_0^{u} dy\right)}^{=u} du = \int_0^{+\infty} f_Y(u) \cdot u\,du = \mathbb{E}[Y]$$

$\blacksquare$

For discrete random variables, we focus on $\mathbb{N}_0$-valued r.v.s.

$$\boxed{\mathbb{E}[Y] = \sum_{k=0}^{+\infty} S_X(k)}$$

Example. [The Weibull Distribution]

The random variable X has the Weibull distribution if its cdf is of the form

$$F_X(x) = 1 - e^{-\left(\frac{x}{\theta}\right)^{\tau}}.$$

Special case: Say that $\tau = 1$. Then, we get the exponential dist'n.

Choose $\theta = 1$ and $\tau = 2$. What is $\mathbb{E}[X]$?

$\longrightarrow$:

$$\mathbb{E}[X] = \int_0^{+\infty} S_X(x)\,dx$$

$$= \int_0^{+\infty} e^{-x^2}\,dx$$

Resembles the density of the standard normal:

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad \text{for } x \in \mathbb{R}$$

$$u = x\sqrt{2} \rightarrow \begin{cases} du = \sqrt{2}\,dx \\ x = \frac{u}{\sqrt{2}} \end{cases}$$

$$\int_0^{+\infty} e^{-\frac{u^2}{2}} \frac{1}{\sqrt{2}}\,du = \frac{1}{\sqrt{2}} \boxed{\int_0^{+\infty} e^{-\frac{u^2}{2}}\,du}$$

$$= \frac{1}{\sqrt{2}} \frac{\sqrt{\pi}}{\sqrt{\pi}} \int_0^{+\infty} e^{-\frac{u^2}{2}}\,du =$$

$$= \sqrt{\pi} \underbrace{\frac{1}{\sqrt{2\pi}} \int_0^{+\infty} e^{-\frac{u^2}{2}}\,du}_{=\frac{1}{2}} = \frac{\sqrt{\pi}}{2} \quad \blacksquare$$

Strong Law of Large Numbers.

Let $\{X_k,\ k=1,2,\dots\}$ be a sequence of independent, identically distributed r.v.

Assume: the expected value exists and it is finite.
Set $\boxed{\mu_X := \mathbb{E}[X_1]} < \infty$

Then,

$$\boxed{\frac{X_1 + X_2 + \cdots + X_n}{n} \xrightarrow[n\to\infty]{} \mu_X}$$

If a function g is such that $g(X_1)$ is well-defined, we also have

$$\boxed{\frac{g(X_1) + g(X_2) + \cdots + g(X_n)}{n} \xrightarrow[n\to\infty]{} \mathbb{E}[g(X_1)]}$$

Monte Carlo Simulations

... take advantage of the Law of Large Numbers.

If $x_1, x_2, \dots, x_n$ are simulated values of some r.v. X then, for large n, $\dfrac{x_1 + x_2 + \cdots + x_n}{n}$ is "close to" μ_X.