University of Texas at Austin

Problem Set # 2

Sequences of events.

Problem 2.1. Roger is playing darts. His throws are all mutually independent, and he has a probability 0.3 of hitting the bull's eye in any single throw. How many darts should Roger throw so that there is at least an 80% probability of hitting bull's eye at least once?

Solution: Let us label Roger's throws so that the event A_i stands for him hitting bull's eye in the i^{th} throw. Then,

$$\mathbb{P}[A_i] = 0.3, \qquad \text{for every } i = 1, 2, 3, \dots$$

On the other hand, let B_n denote the event that Roger did not ever hit bull's eye **in the first n throws**, i.e.,

$$B_n = A_1^c \cap A_2^c \cap \cdots \cap A_n^c.$$

Since the throws are independent, we have that

$$\mathbb{P}[B_n] = 0.7^n, \text{ for every } n \in \mathbb{N}.$$

The probability that Roger hits bull's eye at least once in the first n throws is

$$\mathbb{P}[B_n^c] = 1 - \mathbb{P}[B_n] = 1 - 0.7^n.$$

Note that the quantity in the last display is *decreasing* in the variable n. So, in order to have the probability of hitting bull's eye at least once in the first n throw greater than or equal to 80%, we should have

$$1 - 0.7^n \geq 0.8,$$

i.e.,

$$0.2 \geq 0.7^n \;\Rightarrow\; \ln(0.2) \geq n\ln(0.7) \Rightarrow \; n \geq 4.5.$$

Since $n \in \mathbb{N}$, we conclude that $n \geq 5$.

Problem 2.2. Audrey and Evie take turns tossing a fair coin with Audrey having the first turn. Whoever gets *Heads* first wins the game.

(i) What's the probability that Evie wins the game?
(ii) Is it possible to weight the coin so that the game is fair, i.e., with what probability p should *Heads* appear so that Audrey and Evie are equally likely to win the game?

Solution: This situation is modeled well by a geometric distribution. Instead of considering a fair coin, let's take a look at a coin whose probability of *Heads* equals p. The outcome space whose elementary outcomes correspond to the first time that *Heads* was observed is

$$\Omega = \{1, 2, 3, \dots\}.$$

Let $p_i, i = 1, 2, 3, \dots$ denote the probability that the elementary outcome i happens, i.e., the probability that *Heads* comes up for the first time on the i^{th} toss. Then,

$$p_i = (1-p)^{i-1}p \quad \text{for } i = 1, 2, 3, \dots$$

Since Evie wins if the *Heads* appears on an **even** turn, the probability of her winning the game is

$$\begin{aligned} p_2 + p_4 + p_6 + \ldots &= (1-p)^{2-1}p + (1-p)^{4-1}p + (1-p)^{6-1}p \ldots \\ &= p(1-p)(1 + (1-p)^2 + (1-p)^4 + \ldots) \\ &= p(1-p)\sum_{k=0}^{\infty}\left((1-p)^2\right)^k \end{aligned}$$

$$= \frac{p(1-p)}{1-(1-p)^2} = \frac{p(1-p)}{p(2-p)} = \frac{1-p}{2-p}\,.$$

In the first case, the coin is fair, i.e., $p = 1/2$. So, the answer is

$$\frac{1-\frac{1}{2}}{2-\frac{1}{2}} = \frac{1}{3}\,.$$

Finally, let's try to find the p for which the game is "fair" in the sense that both Evie and Audrey are equally likely to win. We have to solve for p in

$$\frac{1-p}{2-p} = \frac{1}{2} \quad \Rightarrow \quad 2(1-p) = 2-p \quad \Rightarrow \quad 2p = p.$$

We see that such a p does not exist!

Problem 2.3. *Source: Problem #1.5.6 from Pitman.*
Suppose you roll a fair six-sided die repeatedly until the first time you roll a number that you have rolled before. Let p_r denote the probability that you roll the die exactly r times.

(i) Without calculation, write down the value of $p_1 + p_2 + ... + p_{10}$. **Explain.**
(ii) For each $r = 1, 2, \ldots$ calculate the probability p_r.
(iii) Verify arithmetically that your response to (i) was correct.

Solution:

(i) Since there are six sides on the die, even in the "worst" possible case, the six possibilities are exhausted in the first six rolls of the die. Hence, an already obtained value must have happened at the latest in the seventh roll. Thus,

$$p_1 + p_2 + ... + p_{10} = 1.$$

(ii) Using he same reasoning as above, we have that $p_8 = p_9 = \cdots = 0$. Now, for the more interesting cases:

$p_1 = 0$ (because there is not yet a roll to repeat),

$p_2 = \frac{1}{6}$ (because exactly the result of the first roll must be repeated),

$p_3 = \frac{5}{6} \times \frac{2}{6}$ (because the 2^{nd} isn't allowed to match the first, and the 3^{rd} must match one of the already rolled)

$p_4 = \frac{5}{6} \times \frac{4}{6} \times \frac{3}{6}$

$p_5 = \frac{5}{6} \times \frac{4}{6} \times \frac{3}{6} \times \frac{4}{6}$

$p_6 = \frac{5}{6} \times \frac{4}{6} \times \frac{3}{6} \times \frac{2}{6} \times \frac{5}{6}$

$p_7 = \frac{5}{6} \times \frac{4}{6} \times \frac{3}{6} \times \frac{2}{6} \times \frac{1}{6} \times 1$

(iii) The easiest way to do this appears to be to work backwards and add up the final fractions in the products.

An alternative - very very formal!!! - solution:
The probability p_1 that one roll is necessary is 0, since you need at least two tosses to *repeat* a number. For $r = 2$, we are looking for a probability that the first two tosses are the same:

$$p_2 = \tfrac{1}{6} \times \tfrac{1}{6} + \tfrac{1}{6} \times \tfrac{1}{6} + \tfrac{1}{6} \times \tfrac{1}{6} + \tfrac{1}{6} \times \tfrac{1}{6} + \tfrac{1}{6} \times \tfrac{1}{6} + \tfrac{1}{6} \times \tfrac{1}{6} = \tfrac{1}{6}.$$

To treat the general case, let us consider the events

$$B_r = \{ \text{ first repetition happens at time } r \ \}, \text{ so that } p_r = \mathbb{P}[B_r],$$

as well as

$$A_r = \{ \text{ no repetitions in the first } r-1 \text{ tosses}\},$$

for $r = 1, 2, 3, \dots$.

It is clear that $p_r = 0$, for $r \geq 8$. Indeed, in any 7 tosses, at least two will be the same, so the game will have stopped by the 8^{th} cointoss. Also, note that

$$B_r = A_{r-1} \setminus A_r$$

so that $p_r = \mathbb{P}[A_{r-1}] - \mathbb{P}[A_r]$, since $A_r \subseteq A_{r-1}$. It is clear from here that $p_1 + p_2 + \dots + p_{10} = \mathbb{P}[A_1] - \mathbb{P}[A_{10}] = 1$, because $\mathbb{P}[A_1] = 1$ (no repetitions if die rolled only once) and $\mathbb{P}[A_{10}] = 0$ (certainly at least one repetition in 10 rolls).

To compute the individual probabilities p_r, we need to compute $\mathbb{P}[A_r]$, for $r = 1, 2, \dots, 6$. For that, need to count the number of possible outcomes of r tosses which to not repeat. The total number $\#A_r$ of sequences of r tosses without repetition is exactly the number of ways to choose r distinct numbers numbers from $\{1, 2, \dots, 6\}$ and put them in order. Clearly, we can do this in $6 \times 5 \times \dots \times (6 - r + 1)$ ways. On the other hand, the number of ways $\#\Omega$ we can choose r numbers from $\{1, 2, \dots, 6\}$ is 6^r. Therefore,

$$\mathbb{P}[A_r] = \tfrac{6 \times 5 \times \dots \times (6-r+1)}{6^r}.$$

It follows that

$$\mathbb{P}[B_r] = \tfrac{6}{6} \times \tfrac{5}{6} \times \dots \times \tfrac{6-r+2}{6}(1 - \tfrac{6-r+1}{6}) = \tfrac{6}{6} \times \dots \times \tfrac{6-r+2}{6} \times \tfrac{r-1}{6}.$$