UNIVERSITY OF TEXAS AT AUSTIN

HW Assignment 2

Prerequisite material.

Please, provide your justification for your response to every question in this subsection. Just the final numerical answer will receive zero credit, even if it is correct. For the graphs, it is sufficient to carefully draw the graph correctly in a clearly labeled coordinate system.

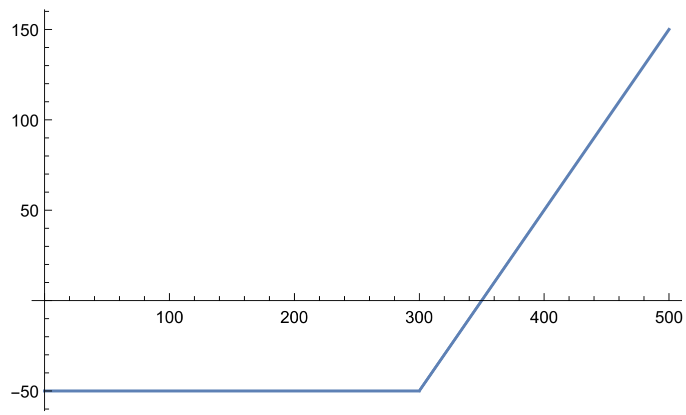
Problem 2.1. (5 points) Let the function f be given by

$$f(x) = \begin{cases} x - 300 & \text{for } x \geq 300 \\ 0 & \text{otherwise} \end{cases}$$

Draw the graph of the function g defined as

$$g(x) = f(x) - 50.$$

Solution:



Problem 2.2. (5 points) Let the function f be defined as

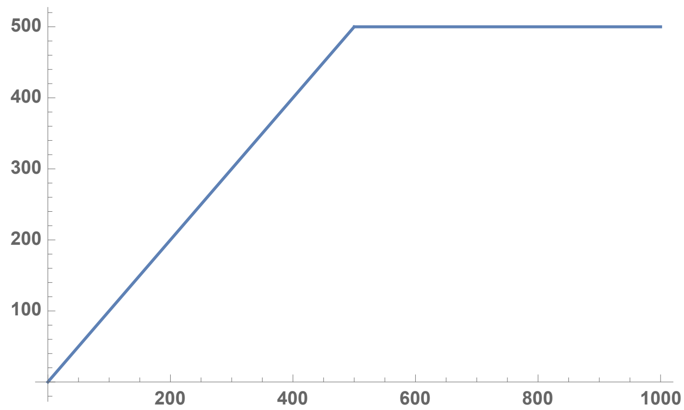
$$f(x) = x$$

Let the function g be defined as

$$g(x) = \begin{cases} 0 & \text{for } x < 500 \\ x - 500 & \text{for } x \geq 500 \end{cases}$$

Draw the graph of the function $f - g$.

Solution:



Problem 2.3. (5 points) Let $x > 0$. Then, we always have $e^x > 1 + x$. *True or false? Why?*

Solution: TRUE
By the Taylor expansion of the exponential function, we have

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

Since $x > 0$, the higher-order terms are all positive and the proposed inequality is correct.

Problem 2.4. (5 points)
We define the minimum of two values in the usual way, i.e.,

$$\min(x, y) = \begin{cases} x & \text{if } x \le y \\ y & \text{if } x \ge y \end{cases}$$

We define the maximum of two values in the usual way, i.e.,

$$\max(x, y) = \begin{cases} x & \text{if } x \ge y \\ y & \text{if } x \le y \end{cases}$$

Then, for every x and y we have that

$$x - \min(x - y, 0) = \max(x, y)$$

True or false? Why?

Solution: TRUE

$$\begin{aligned} x - \min(x - y, 0) &= \begin{cases} x - 0 = x, & \text{if } x \ge y \\ x - (x - y) = y, & \text{if } x < y \end{cases} \\ &= \max(x, y) \end{aligned}$$

Problem 2.5. (5 points)
We define the maximum of two values in the usual way, i.e.,

$$\max(x, y) = \begin{cases} x & \text{if } x \ge y \\ y & \text{if } x \le y \end{cases}$$

Then, for every x and y we have that

$$\max(x, y) = \max(x - y, 0) + y$$

True or false? Why?

Solution: TRUE
If $x \geq y$, then the left-hand side of the proposed equality equals x. On the other hand, we also have that $x - y \geq 0$. So, the right-hand side equals

$$\max(x-y,0)+y = x-y+y = x.$$

If $x < y$, then the left-hand side of the equality equals y. On the other hand, we also have that $x - y < 0$. So, the right-hand side equals

$$\max(x-y,0)+y = 0+y = y.$$

Therefore, the proposed equality is always true.

Problem 2.6. (5 points) Let $\Omega = \{a_1, a_2, a_3, a_4\}$ be an outcome space, and let $\mathbb{P}$ be a probability distribution on Ω. Assume that $\mathbb{P}[\{a_1, a_2\}] = 1/3, \mathbb{P}[\{a_2, a_3\}] = 1/4$ and $\mathbb{P}[\{a_1, a_3\}] = 1/9$. How much is $\mathbb{P}[\{a_4\}]$?

Solution: For any outcome space Ω, from the axioms of probability, we must have that $\mathbb{P}[\Omega] = 1$. In this case, $\Omega = \{a_1, a_2, a_3, a_4\}$, and so

$$\mathbb{P}[\Omega] = \mathbb{P}[\{a_1,a_2,a_3,a_4\}] = \mathbb{P}[\{a_1,a_2,a_3\}] + \mathbb{P}[\{a_4\}] = \frac{1}{2}\left(\frac{1}{3}+\frac{1}{4}+\frac{1}{9}\right) + \mathbb{P}[\{a_4\}].$$

Hence,

$$\mathbb{P}[\{a_4\}] = 1 - \frac{25}{72} = \frac{47}{72}.$$

Problem 2.7. (5 points) Let Y be a random variable such that $\mathbb{P}[Y = 2] = 1/2$, $\mathbb{P}[Y = 3] = 1/3$ and $\mathbb{P}[Y = 6] = 1/6$. What is $\mathbb{E}[\min(Y, 5)]$?

Solution:

$$\mathbb{E}[\min(Y,5)] = \frac{1}{2}(2) + \frac{1}{3}(3) + \frac{1}{6}(5) = \frac{17}{6}.$$

Problem 2.8. (5 points) A coin for which *Heads* is twice as likely as *Tails* is tossed twice, independently. If the outcomes are two consecutive *Heads*, Bertie gets a \$15 reward; if the outcomes are different, Bertie gets a \$10 reward; if the outcomes are two consecutive *Tails*, Bertie has to pay \$5. What is the expected value of Bertie's winnings?

Solution: Since *Heads* is twice as likely as *Tails*, *Heads* appears with probability $2/3$, while *Tails* appears with probability $1/3$.

Let X denote the amount Bertie wins. Then, X has the following distribution:

$$X \sim \begin{cases} 15, & \text{with probability } 4/9, \\ 10, & \text{with probability } 4/9, \\ -5, & \text{with probability } 1/9. \end{cases}$$

$$\mathbb{E}[X] = \frac{4}{9}(15) + \frac{4}{9}(10) + \frac{1}{9}(-5) = \frac{95}{9}.$$

Problem 2.9. (5 points) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be two functions given by

$$f(x) = |x - 10|$$

and

$$g(x) = \begin{cases} \min(x,4) & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Then, $g(f(7))$ equals ...

(a) 0

(b) 3

(c) 4
(d) 7
(e) None of the above

Solution: (b)

$$f(7) = |7 - 10| = |-3| = 3$$
$$g(3) = \min(3, 4) = 3$$

Problem 2.10. (5 points) Let $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$ be a probability space. We denote by p_k the probability of the elementary outcome ω_k, i.e., $p_k = \mathbb{P}[\{\omega_k\}]$ for $k = 1, \ldots, 5$. You are given that p_k/p_{k-1} is constant for $k = 2, 3, 4, 5$. You are also given that $p_1 = 16/31$. Find p_5.

(a) 1/31
(b) 2/31
(c) 4/31
(d) Not enough information is given.
(e) None of the above.

Solution: (a)
From the given recursive property, we know that, for some constant κ,

$$p_2 = \kappa p_1, \quad p_3 = \kappa^2 p_1, \quad p_4 = \kappa^3 p_1, \quad p_5 = \kappa^4 p_1.$$

We aslo know that

$$p_1 + p_2 + p_3 + p_4 + p_5 = 1.$$

So,

$$p_1(1 + \kappa + \kappa^2 + \kappa^3 + \kappa^4) = 1 \quad \Rightarrow \quad \frac{16}{31}\left(\frac{1-\kappa^5}{1-\kappa}\right) = 1 \quad \Rightarrow \quad \frac{1-\kappa^5}{1-\kappa} = \frac{31}{16} \quad \Rightarrow \quad \kappa = 1/2.$$

Finally, $p_5 = \frac{1}{2^4}\left(\frac{16}{31}\right) = \frac{1}{31}$.