M339D : November 10th, 2023.

LogNormal Stock Prices: Tail Probabilities [cont'd].

Problem. Assume the Black-Scholes model.
Let the current stock price be \$100.

You are given:

(i) $\mathbb{P}^*[S(1/4) < 95] = 0.2358$

(ii) $\mathbb{P}^*[S(1/2) < 110] = 0.6026$.

What is the expected time-1 stock price under $\mathbb{P}^*$?

→: $\boxed{\mathbb{E}^*[S(T)] = S(0)e^{rT}}$

In this problem: $\mathbb{E}^*[S(1)] = S(0)e^{r}$

In the Black-Scholes model:

$$S(T) = S(0)\, e^{\left(r - \frac{\sigma^2}{2}\right)\cdot T + \sigma\sqrt{T}\cdot Z}$$

$\left(r - \frac{\sigma^2}{2}\right) = \mu$

$$\boxed{\mathbb{E}^*[S(1)] = 100\, e^{\mu + \frac{\sigma^2}{2}}}$$

(i) 95 is the 23.58th quantile of $S(1/4)$

The 23.58th quantile of $N(0,1)$: std normal tables: -0.72

$$95 = 100\, e^{\mu(1/4) + \sigma\sqrt{1/4}\cdot(-0.72)} \quad / :100$$

$$0.95 = e^{\mu/4 + \sigma(-0.36)}$$

$$\boxed{\ln(0.95) = 0.25\cdot\mu - 0.36\,\sigma} \qquad \text{(i)}$$

(ii) 110 is the 60.26th quantile $S(1/2)$

The 60.26th quantile of $N(0,1)$: std normal tables: 0.26

$$110 = 100\, e^{\mu(1/2) + \sigma\sqrt{1/2}\cdot(0.26)} \quad / :100$$

$$1.1 = e^{\mu/2 + \sigma\sqrt{1/2}(0.26)}$$

$$\boxed{\ln(1.1) = 0.5\mu + \sigma\sqrt{1/2}(0.26)} \qquad \text{(ii)}$$

We solve the system of two equations w/ two unknowns

$$\mu = \underline{\ ?\ } \quad \text{and} \quad \sigma = \underline{\ ?\ }$$

$$\left.\begin{aligned} &(i) \Rightarrow \cancel{0.5\mu} - 0.72\sigma = 2\ln(0.95) \\ &(ii)\ -\cancel{0.5\mu} \pm \sqrt{1/2}\,(0.26)\cdot\sigma = -\ln(1.1) \end{aligned}\right\} -$$

$$+\left(0.72 + \sqrt{1/2}\,(0.26)\right)\sigma = -2\ln(0.95) + \ln(1.1)$$

$$\sigma = \underline{0.21895} \checkmark$$

$$4(i) \Rightarrow \quad \mu = 1.44\,(0.21895) + 4\ln(0.95) = \underline{0.11011} \checkmark$$

Finally,

$$100e^{\mu + \frac{\sigma^2}{2}} = 100e^{0.11011 + \frac{(0.21895)^2}{2}} = \underline{114.3488} \ \square$$

Black-Scholes: Partial Expectations.

The Model.

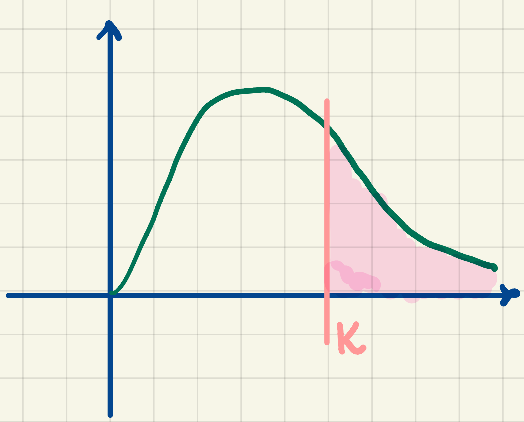
Under the risk-neutral measure $\mathbb{P}^*$:

$$\boxed{S(T) = S(0)e^{(r - \frac{\sigma^2}{2})\cdot T + \sigma\sqrt{T}\cdot Z} \quad \text{w/} \quad Z \sim N(0,1)} \leftarrow$$

Tail Probabilities.

$$\mathbb{P}^*[S(T) > \boxed{K}] = N(d_2)$$

$$\text{w/} \quad d_2 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)\cdot T\right]$$



Motivation.

Get a formula for the price of European calls and puts on a stock modeled in the Black·Scholes framework.

Idea.

$\boxed{\text{RISK·NEUTRAL PRICING}}$

$$\boxed{V(0) = e^{-rT}\,\mathbb{E}^*[V(T)]}$$

Payoff of a European option

Implementation.

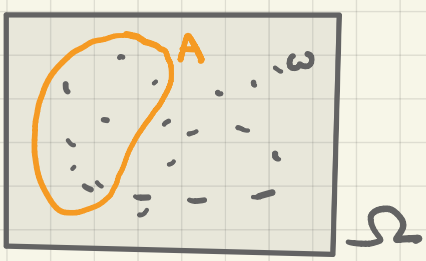
Temporarily, focus on a time·T, strike·K European call option.

The Payoff: $\boxed{V_c(T) = (S(T)-K)_+}$

Under $\mathbb{P}^*$:

$$\mathbb{E}^*[V_c(T)] = \mathbb{E}^*[(S(T)-K)_+]$$

$$= \mathbb{E}^*\left[(S(T)-K)\,\boxed{\mathbb{I}_{[S(T)\geq K]}}\right]$$



A is an event

$$\mathbb{I}_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}$$

$$\mathbb{I}_A = \begin{cases} 1 & \text{if } A \text{ happened} \\ 0 & \text{if } A \text{ did not happen} \end{cases}$$

$$\mathbb{E}[\mathbb{I}_A] = 1 \cdot \mathbb{P}[A] + 0 \cdot \mathbb{P}[A^c] = \mathbb{P}[A]$$

$$\mathbb{E}^*[V_c(T)] = \boxed{\mathbb{E}^*[S(T)\cdot\mathbb{I}_{[S(T)\geq K]}]} - K\cdot\boxed{\mathbb{P}^*[S(T)\geq K]}$$

$$\mathbb{P}^*[S(T)\geq K] = N(d_2)$$

$\mathbb{E}^*[S(T)\cdot\mathbb{I}_{[S(T)\geq K]}] = ?$

The partial expectation from the title

$$\mathbb{E}^*\left[\boxed{S(T)} \cdot \mathbb{I}_{\boxed{[S(T) \geq K]}}\right] = ?$$

$$\mathbb{E}^*\left[S(T) \cdot \mathbb{I}_{[S(T) < K]}\right] = ?$$

Method. Use the defining formula for the expectation of a function of a r.v.

In this case, the r.v. is $\boxed{Z \sim N(0,1)}$