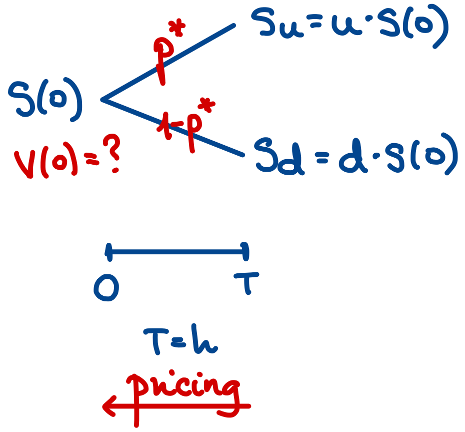
University of Texas at Austin

Binomial option pricing (review).

Problem 3.1. Let the continuously compounded risk-free interest rate be denoted by r. You are building a model for the price of a stock which pays dividends continuously with the dividend yield δ. Consider a binomial tree modeling the evolution of the stock price. Let the length of each period be h and let the up factor be denoted by u, and the down factor by d. What is the **no-arbitrage** condition for the binomial tree you are building?

$$\boxed{d < e^{(r-\delta)h} < u}$$

Problem 3.2. Set up the framework for pricing by replication in a one-period binomial tree! What is the *risk-neutral pricing* formula?



Payoff | Replicating Portfolio

$$V_u := v(S_u) = \Delta e^{\delta h}\cdot S_u + Be^{rh}$$

$$V_d := v(S_d) = \Delta e^{\delta h}\cdot S_d + Be^{rh}$$

$v(\cdot)$ ... payoff f'tion

Replicating Portfolio.

- Δ ... # of shares of stock
- B ... risk-free investment

$$\boxed{\Delta = e^{-\delta h}\frac{V_u - V_d}{S_u - S_d}}$$

$$\boxed{B = e^{-rh}\frac{u\cdot V_d - d\cdot V_u}{u-d}}$$

Pricing by Replication:

$$\boxed{V(0) = \Delta\cdot S(0) + B}$$

algebra:

$$\boxed{V(0) = e^{-rT}\left[p^*\cdot V_u + (1-p^*)\cdot V_d\right]}$$

$$\boxed{p^* = \frac{e^{(r-\delta)h} - d}{u-d}}$$ ... the risk-neutral probability

$$\boxed{V(0) = e^{-rT}\,\mathbb{E}^*[V(T)]}$$

$\delta = 0$

Problem 3.3. The current price of a certain non-dividend-paying stock is \$100 per share. You are modeling the price of this stock at the end of a quarter year using a one-period binomial tree under the assumption that the stock price can either increase by $\boxed{4\%}$, or decrease by $\boxed{2\%}$. $n=1$, $h=0.25$

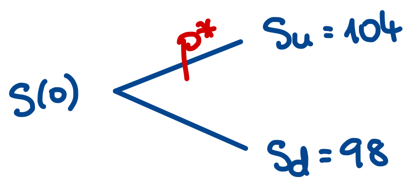
The continuously compounded risk-free interest rate is 3%. $r=0.03$

What is the price of a three-month, at-the-money European call option on the above stock consistent with the above binomial tree?

$K=S(0)$ $v_c(s)=(s-K)_+$

$\rightarrow$:

$$p^* = \frac{e^{(r-\delta)h}-d}{u-d} = \frac{e^{0.03(0.25)}-0.98}{1.04-0.98} = 0.4588$$



$\boxed{V_u = 4}$

$V_d = 0$

$$V_c(0) = e^{-0.03(0.25)}(4p^*) = 1.8215 \quad \square$$

Problem 3.4. Let the continuously compounded risk-free interest rate be equal to 0.04.

The current price of a continuous-dividend-paying stock is $80 and its dividend yield is 0.02. The stock's volatility is 0.25. You model the evolution of the stock price over the following half year using a two-period **forward** binomial tree.

$h = 0.25$

What is the price of a six-month, $82-strike European put option on the above stock consistent with the given binomial tree?

$$\boxed{\begin{aligned} u &= e^{(r-\delta)h + \sigma\sqrt{h}} \\ d &= e^{(r-\delta)h - \sigma\sqrt{h}} \end{aligned}}$$

$$\boxed{p^* = \frac{1}{1+e^{\sigma\sqrt{h}}}}$$

In this problem:

$$p^* = \frac{1}{1+e^{0.25\sqrt{0.25}}} = \frac{1}{1+e^{0.125}} = \underline{0.4688}$$

$$u = e^{(0.04-0.02)(0.25)+0.125} = e^{0.13}$$

$$d = e^{(0.04-0.02)(0.25)-0.125} = e^{-0.12}$$

Possible Final Stock Prices:

$S_{uu} = S(0)u^2 = 80 \cdot e^{0.26} > 82 = K \Rightarrow V_{uu} = 0$

$S_{ud} = S(0) u \cdot d = 80 \cdot e^{0.01} = 80.804 \Rightarrow V_{ud} = 82 - 80.804 = 1.196$

$S_{dd} = S(0) \cdot d^2 = 80 e^{-0.024} = 62.9302 \Rightarrow V_{dd} = 82 - 62.9302 = 19.0698$

$$V_P(0) = e^{-0.04(0.5)}\left[(1-p^*)^2 \cdot 19.0698 + 2p^*(1-p^*) \cdot 1.196\right] = \underline{5.85832}$$

□

$S_{ud} = S(0) \cdot u \cdot d$

$= S(0) e^{(r-\delta)(2h)}$

$= F_{0,2h}(S)$

$=$ Forward Price