M378K: September 27th, 2024.

Functions of Random Vectors.

The cdf-Method.

Example. Let $Y \sim U(0,1)$

Let $\tilde{Y} = Y^{\varkappa}$ w/ $\varkappa > 1$

$y \in (0,1)$: $F_{\tilde{Y}}(y) = \mathbb{P}[Y^{\varkappa} \leq y]$

$= \mathbb{P}[Y \leq y^{1/\varkappa}] = F_Y(y^{1/\varkappa})$

$\Rightarrow$ for $0 < y < 1$

$$f_{\tilde{Y}}(y) = F'_{\tilde{Y}}(y) = \frac{d}{dy} F_Y(y^{1/\varkappa}) = \frac{d}{dy}(y^{1/\varkappa})$$

$$= \frac{1}{\varkappa} \cdot y^{\frac{1}{\varkappa} - 1}$$

$\square$

CDF-Method in 2D.

Goal: We want to find the density f_W of a r.v. $\boxed{g(Y_1, Y_2)}$ where (Y_1, Y_2) are jointly continuous w/ pdf $\boxed{f_{Y_1,Y_2}}$

$$F_W(w) = \mathbb{P}[W \leq w] = \mathbb{P}[g(Y_1,Y_2) \leq w] = \mathbb{P}[(Y_1,Y_2) \in A]$$

$$\boxed{A = \{(y_1,y_2) \in \mathbb{R}^2 : g(y_1,y_2) \leq w\}}$$

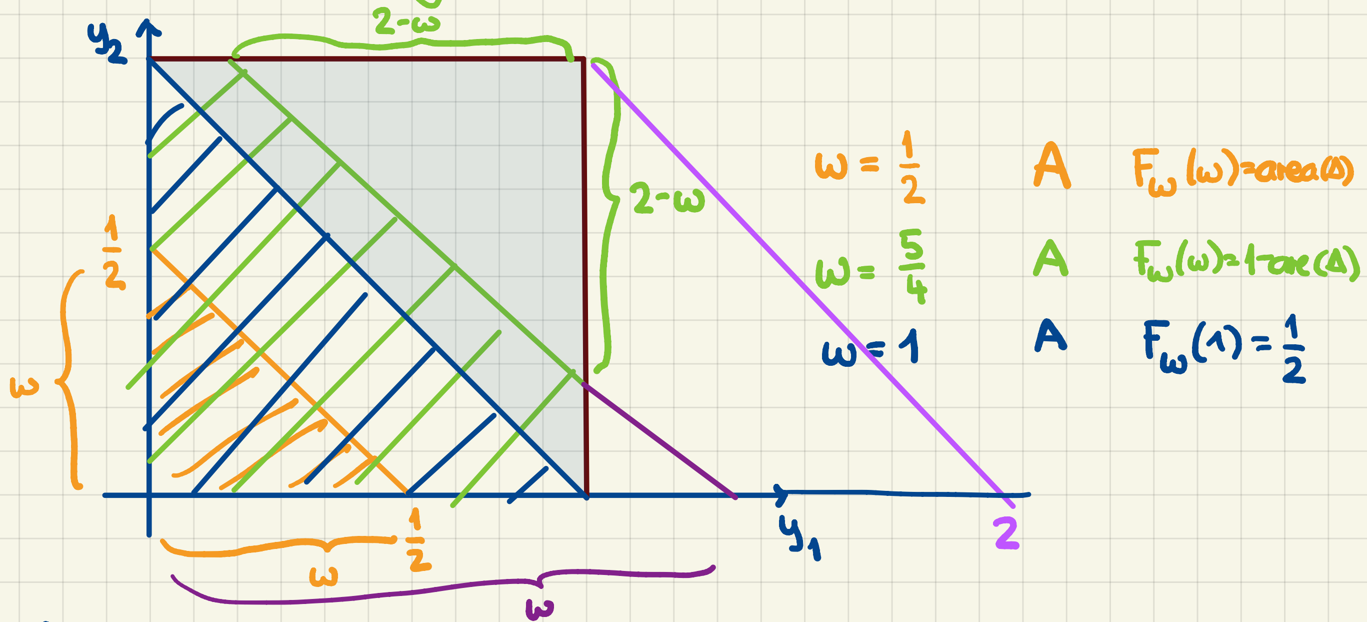
$$\boxed{F_W(w) = \iint_A f_{Y_1,Y_2}(y_1,y_2)\,dy_2\,dy_1}$$

Example. Say (Y_1, Y_2) represent points chosen @ random in a unit square $[0,1]\times[0,1] = [0,1]^2$

$$\boxed{f_{Y_1,Y_2}(y_1,y_2) = \mathbb{1}_{[0,1]^2}(y_1,y_2)}$$

Define $\boxed{W = Y_1 + Y_2}$

i.e., $g(y_1,y_2) = y_1 + y_2$



for $w < 0$: $F_W(w) = 0$ ←

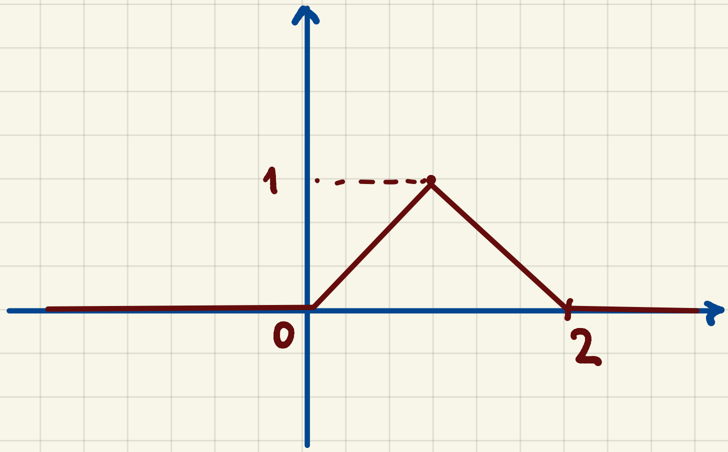
for $0 \le w < 1$: $F_W(w) = \frac{1}{2}w^2$

for $w = 1$: $F_W(1) = \frac{1}{2}$

for $1 < w \le 2$: $F_W(1) = 1 - \frac{(2-w)^2}{2} = -1 + 2w - \frac{1}{2}w^2$

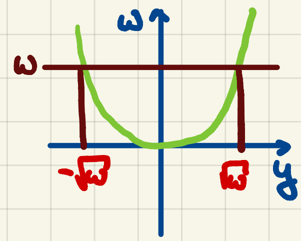
for $w > 2$: $F_W(w) = 1$

$$f_W(w) = \begin{cases} 0 & w < 0 \\ w & w \in [0,1) \\ 2-w & w \in [1,2] \\ 0 & w > 2 \end{cases}$$



Example. Let $\boxed{Y \sim N(0,1)}$

Set $W = Y^2$, i.e., $W = g(Y)$ w/ $g(y) = y^2$



For all $w \leq 0$: $F_W(w) = 0$

For all $w > 0$:
$$F_W(w) = \mathbb{P}[W \leq w] = \mathbb{P}[Y^2 \leq w]$$
$$= \mathbb{P}[-\sqrt{w} \leq Y \leq \sqrt{w}]$$
$$= F_Y(\sqrt{w}) - F_Y(-\sqrt{w})$$

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for $w > 0$:
$$f_W(w) = \frac{d}{dw}\left(F_Y(\sqrt{w}) - F_Y(-\sqrt{w})\right)$$
$$= \frac{1}{2\sqrt{w}} f_Y(\sqrt{w}) + \left(+\frac{1}{2\sqrt{w}}\right) \cdot f_Y(-\sqrt{w})$$

$$f_Y(z) = \varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \quad \text{for all } z \in \mathbb{R}$$

$$f_W(w) = \frac{1}{\cancel{2}\sqrt{w}} \cdot \frac{1}{\sqrt{2\pi}} \cdot \cancel{2} \cdot e^{-\frac{w}{2}} \qquad \text{for } w > 0$$

$$\boxed{f_W(w) = \frac{1}{\sqrt{2\pi w}} e^{-\frac{w}{2}} \cdot 1_{(0,\infty)}(w)}$$

W is said to have the χ^2-dist'n w/ 1 degree of freedom

$$\boxed{W \sim \chi^2(df = 1)}$$

More generally, for $Y_1, \dots, Y_k$ independent and $N(0,1)$ all

$$\boxed{X = Y_1^2 + Y_2^2 + \dots + Y_k^2 \sim \chi^2(df = k)}$$