M339D: December 9th, 2024.

Delta Hedging [cont'd].

Say, an agent $\boxed{\text{writes}}$ an option @ time $\cdot\, 0$. Let the value function of this option be denoted by

$$\boxed{v(s,t)}$$

At time $\cdot\,0$, they write the option $\Rightarrow$ They get $\boxed{v(S(0),0)}$

At time $\cdot\, t$, the value of the agent's position

$$\boxed{-v(s,t)} \checkmark$$

To (partially) hedge this position, they trade in the shares of the underlying stock to maintain a

delta-neutral portfolio,

i.e., a portfolio which has $\boxed{\Delta_{Port}(s,t) = 0}$

Theoretically possible

Practically NOT

At time $\cdot\, t$, let $N(s,t)$ denote the required number of shares in the portfolio to maintain Δ-neutrality.

The total value of the portfolio:

$$v_{Port}(s,t) = -v(s,t) + N(s,t)\cdot s$$

$$\frac{\partial}{\partial s}\Big| \quad \Delta_{Port}(s,t) = -\Delta(s,t) + N(s,t) = 0$$

$$\boxed{N(s,t) = \Delta(s,t)} \checkmark\checkmark$$

Example. An agent writes a call option @ time $\cdot\, 0$.

At time $\cdot\, t$, the agent's unhedged position is

$$-v_C(s,t)$$

$\Rightarrow$

$N(s,t) = \Delta_C(s,t)$ in the Δ-hedge

$\Rightarrow$ In particular, @ time $\cdot\, 0$:

The agent must long this much of a share.

$$N(S(0),0) = \Delta_C(S(0),0) > 0$$

$\Rightarrow$ <u>The total position:</u>

$$v_{Port}(S(0),0) = -v_c(S(0),0) + \Delta_c(S(0),0)\,S(0)$$

<u>**Example.**</u> An agent writes a put @ time $\cdot$ 0.

At any time $\cdot t$, their unhedged position is

$$-v_p(s,t)$$

$\Rightarrow$ They must maintain $N(s,t) = \Delta_p(s,t) = -N(-d_1(s,t)) < 0$ in the Δ-hedge

$\Rightarrow$ They must short a fraction of a share.

$\Rightarrow$ <u>Their total initial position:</u>

$$v_{Port}(S(0),0) = -\underline{v_p(S(0),0)} + \underline{\Delta_p(S(0),0)}\cdot S(0)$$

In the <u>Black $\cdot$ Scholes model</u>:

$$v_{Port}(S(0),0) = -\left(Ke^{-rT}N(-d_2(S(0),0)) - \cancel{S(0)\,N(-d_1(S(0),0))}\right) + \cancel{\left(-N(-d_1(S(0),0))\right)}\cdot S(0)$$

$$= -Ke^{-rT}N(-d_2(S(0),0)) \quad \square$$