M339J: February 26th, 2021.

Problem. [Exam C, Spring 2007, Problem #13]

The loss severity random variable X follows the exponential dist'n w/ mean 10,000.
Determine the coefficient of variation of the excess loss random variable $Y = \max(X-30000, 0)$.

→: The probability density f'tion of X:

$$f_X(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}} \quad \text{for } x>0$$

w/ $\theta = 10000$

By def'n our coefficient of variation is $\frac{\sigma_Y}{\mu_Y}$.

Start w/ μ_Y:

$$\mu_Y = \mathbb{E}[Y] = \mathbb{E}[g(X)] \quad \text{where } g(x) = \max(x-30K, 0)$$

By def'n of expectation:

$$\boxed{\mathbb{E}[g(X)] = \int_0^{+\infty} g(x) f_X(x)\,dx}$$

$$\boxed{\mathbb{E}[X] = \int_{-\infty}^{+\infty} x f_X(x)\,dx \quad \text{by def'n}}$$

$$\mathbb{E}[g(X)] = \int_{30K}^{+\infty} (x-30K) \frac{1}{10K} e^{-\frac{x}{10K}}\,dx =$$

$$= \left[\; u = x - 30K \qquad du = dx \\ x = u + 30K \;\right]$$

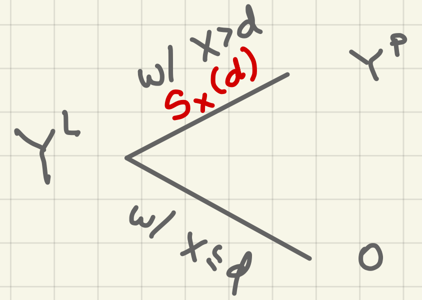
$$= \int_0^{+\infty} u \cdot \frac{1}{10K} \cdot e^{-\frac{u+30K}{10K}}\,du$$

$$= \int_0^{+\infty} u \cdot \frac{1}{10K} \cdot e^{-\frac{u}{10K}} \cdot e^{-3}\,du$$

$$= e^{-3} \underbrace{\int_0^{+\infty} u \cdot \frac{1}{10K} \cdot e^{-\frac{u}{10K}} \, du}_{\mathbb{E}[\text{Exponential}(\text{mean} = \theta = 10k)]}$$

$$= e^{-3} \cdot 10K$$

$$Y^L = (X-d)_+ = \begin{cases} X-d & \text{when } X>d \\ 0 & \text{when } X \le d \end{cases}$$



$$\mathbb{E}[Y^L] = S_X(d)\, \mathbb{E}[Y^P]$$

Q: What is the dist'n of Y^P when $X \sim Exp(\theta)$?

$\longrightarrow$:

$$Y^P = X-d \mid X>d$$

Due to the memoryless property:

$$Y^P \sim Exp(\theta)$$

$$\Rightarrow \mathbb{E}[Y^L] = e^{-\frac{d}{\theta}} \cdot \theta$$

$$Var[Y] = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2$$

$$\mathbb{E}[Y^2] = \int_{30K}^{+\infty} (x-30K)^2 \cdot \frac{1}{10K} \cdot e^{-\frac{x}{10K}} \, dx$$

$u = x - 30K \qquad du = dx$

$x = u + 30K$

$$= \int_0^{+\infty} u^2 \cdot \frac{1}{10K} \cdot e^{-\frac{u+30K}{10K}} du$$

$$= \int_0^{+\infty} u^2 \cdot \frac{1}{10K} \cdot e^{-\frac{u}{10K}} \cdot e^{-3} du$$

$$= e^{-3} \underbrace{\int_0^{+\infty} u^2 \cdot \frac{1}{10K} \cdot e^{-\frac{u}{10K}} du}_{2\cdot(10K)^2} = e^{-3} \cdot 2 \cdot (10K)^2$$

$\boxed{\text{STAM TABLES}}$

$$\Rightarrow \mathrm{Var}[Y] = 2e^{-3}(10K)^2 - (e^{-3} \cdot 10K)^2 =$$

$$= (2e^{-3} - e^{-6})(10K)^2 = 9\,709\,538.456$$

$$\Rightarrow SD[Y] = 3\,116.01$$

$\Rightarrow$ coefficient of variation: $\frac{3116.01}{(10K)e^{-3}} = 6.259$ ■

Problem. [Sample STAM Problem #162].

A loss random variable X is a two-parameter Pareto w/ $\alpha = 2$ and unspecified parameter θ.

You are given:

$$\mathbb{E}[X - 100 \mid X > 100] = \frac{5}{3} \mathbb{E}[X - 50 \mid X > 50]$$

Calculate $\mathbb{E}[X - 150 \mid X > 150] = ?$

$\rightarrow$: In general: $X \sim \text{Pareto}(\alpha, \theta)$

$$\mathbb{E}[X - d \mid X > d] = \frac{\mathbb{E}[X] - \mathbb{E}[X \wedge d]}{S_X(d)}$$

$$= \frac{\frac{\theta}{\alpha-1} - \frac{\theta}{\alpha-1}\left[1-\left(\frac{\theta}{d+\theta}\right)^{\alpha-1}\right]}{\left(\frac{\theta}{d+\theta}\right)^{\alpha}}$$

$$= \frac{\frac{\theta}{\alpha-1}\left(\cancel{1}-\cancel{1}+\left(\frac{\theta}{d+\theta}\right)^{\alpha-1}\right)}{\left(\frac{\theta}{d+\theta}\right)^{\alpha}}$$

$$= \frac{\frac{\theta}{\alpha-1}\cdot\left(\frac{\theta}{d+\theta}\right)^{\cancel{\alpha-1}}}{\left(\frac{\theta}{d+\theta}\right)^{\cancel{\alpha}\,1}} =$$

$$= \frac{\frac{\cancel{\theta}}{\alpha-1}}{\frac{\cancel{\theta}}{d+\theta}} = \frac{d+\theta}{\alpha-1} = \mathbb{E}[X-d \mid X>d]$$

In our problem, $\alpha=2$: $\underline{\mathbb{E}[X-d \mid X>d] = d+\theta}$

$$100+\theta = \frac{5}{3}(50+\theta)$$

$$300+3\theta = 250+5\theta$$

$$2\theta = 50$$

$$\theta = 25 \checkmark$$

$$\mathbb{E}[X-150 \mid X>150] = 25 + 150 = 175$$

■