University of Texas at Austin

Homework assignment 2

The covariance formula.

Please, provide your **complete solutions** to the following problems:

Problem 2.1. (5 points) The random vector (X_1, X_2) is jointly normal. Its marginal distributions are:

$$X_1 \sim N(\text{mean} = 0, \text{variance} = 4), \quad X_2 \sim N(\text{mean} = 1, \text{variance} = 1).$$

The correlation coefficient is given to be

$$corr[X_1, X_2] = -0.2.$$

What is the variance of the random variable $X = 3X_1 - 2X_2$?

Solution:
The variance of X is

$$\begin{aligned} Var[X] &= 9Var[X_1] + 4Var[X_2] - 2(3)(2)Cov[X_1, X_2] \\ &= 9(4) + 4(1) + 12(2)(1)(0.2) = 44.8. \end{aligned}$$

Problem 2.2. (5 points) The random vector (X_1, X_2) is jointly normal. Its marginal distributions are:

$$X_1 \sim N(\text{mean} = 1, \text{sd} = 1), \quad X_2 \sim N(\text{mean} = 2, \text{sd} = 2).$$

The correlation coefficient is given to be

$$corr[X_1, X_2] = 0.2.$$

What is the standard deviation of the random variable $X = X_1 - 2X_2$?

Solution: The variance of X is

$$\begin{aligned} Var[X] &= Var[X_1] + 4Var[X_2] - 2(2)Cov[X_1, X_2] \\ &= 1 + 4(2)^2 - 2(2)(1)(2)(0.2) = 15.4. \end{aligned}$$

So, the standard deviation of X is $\sigma_X = \sqrt{15.4} = 3.924283$.

Problem 2.3. (10 points) An actuary is analyzing total claims for the company. The number of claims of Type #1 has expectation 100 and variance 100. Each such claim has the exact amount of 40. The number of claims of Type #2 has expectation 100 and variance 144. Each such claim has the exact amount of 10. The correlation between the number of claims of Type #1 and those of Type #2 is 0.25. What is the variance of the total **amount** of all the claims?

Solution: Let $N_i, i = 1, 2$ denote the number of claims of Type $\#i$. Then, the total amount of all the claims is $X = 40N_1 + 10N_2$. Hence,

$$\begin{aligned} Var[X] = Var[40N_1 + 10N_2] &= 40^2Var[N_1] + 10^2Var[N_2] + 2(40)(10)Cov[N_1, N_2] \\ &= 40^2Var[N_1] + 10^2Var[N_2] + 2(40)(10)SD[N_1]SD[N_2]corr[N_1, N_2] \\ &= 40^2(100) + 10^2(144) + 2(40)(10)(10)(12)(0.25) = 198400. \end{aligned}$$

Problem 2.4. (15 points) The annual profits that Company A and Company B earn follow a bivariate normal distribution. Company A's annual profit has mean 2000 and standard deviation 1000. Company B's annual profit has mean 1000 and standard deviation 500. The correlation coefficient between these annual profits is 0.80. Calculate the probability that Company B's annual profit is less than Company A's annual profit.

Solution: Let X_A be the annual profit of Company A and let X_B be the annual profit of Company B. We are looking for the probability

$$\mathbb{P}[X_A > X_B] = \mathbb{P}[X_A - X_B > 0].$$

The random variable $X_A - X_B$ is normally distributed with the following parameters

$$\mathbb{E}[X_A - X_B] = \mathbb{E}[X_A] - \mathbb{E}[X_B] = 2000 - 1000 = 1000,$$

$$Var[X_A - X_B] = Var[X_A] + Var[X_B] - 2Cov[X_A, X_B] = 1000^2 + 500^2 - 2(1000)(500)(0.8) = 450000.$$

So,

$$\mathbb{P}[X_A - X_B > 0] = \mathbb{P}\left[\frac{X_A - X_B - 1000}{\sqrt{450000}} > \frac{0 - 1000}{\sqrt{450000}}\right] = \mathbb{P}[Z > -1.490712] \approx N(1.49) = 0.9319.$$

where $Z \sim N(0,1)$.

Problem 2.5. (15 points) The random vector (X_1, X_2, X_3) is jointly normal. Its marginal distributions are:

$$X_1 \sim N(\text{mean} = 0, \text{variance} = 4), X_2 \sim N(\text{mean} = 1, \text{variance} = 1), X_3 \sim N(\text{mean} = -1, \text{variance} = 9).$$

The correlation coefficients are given to be

$$corr[X_1, X_2] = 0.3, corr[X_2, X_3] = 0.4, corr[X_1, X_3] = -0.3.$$

What is the distribution of the random variable $X = X_1 - X_2 + 2X_3$? Please, provide the **name** of the distribution, as well as the **values** of its parameters.

Solution:

(2 points) The linear combination of jointy normal random variables in normally distributed itself. Now, we need to identify the mean and the variance of this normal distribution.

(3 points) The mean of X is

$$\mathbb{E}[X] = \mathbb{E}[X_1] - \mathbb{E}[X_2] + 2\mathbb{E}[X_3] = 0 - 1 + 2(-1) = -3.$$

(10 points) The variance of X is

$$\begin{aligned}Var[X] &= Var[X_1] + Var[X_2] + 4Var[X_3] - 2Cov[X_1, X_2] + 4Cov[X_1, X_3] - 4Cov[X_2, X_3]\\ &= 4 + 1 + 36 - 2(2)(1)(0.3) + 4(2)(3)(-0.3) - 4(1)(3)(0.4) = 27.8.\end{aligned}$$