M339D: April 8th, 2022.

The Binomial Asset Pricing Model.

$S(0)$ ... the observable initial asset price

0 ———————— T

T ... time horizon (i.e., **exercise** or **expiration date** of an option)

By convention: $u > d$

$$S(0) \;\to\; \boxed{S_u = u \cdot S(0)} \text{ up node}$$

$$S(0) \;\to\; \boxed{S_d = d \cdot S(0)} \text{ down node}$$

ROOT NODE: $S(0)$

h ... length of a single period

one-period: $S(T) = S(h)$ ... a r.v. denoting the time-T stock price w/ two possible values: S_u and S_d

Market Model.

- **riskless asset**: @ the ccrfir $\boxed{r}$
- **risky asset**: continuous-dividend paying asset w/ dividend yield δ

Imagine investing in one share of stock @ time-0:

$\boxed{S(0)}$, over one period of length h:

	Wealth
$S_u = u \cdot S(0)$	$e^{\delta \cdot h} \cdot S_u = e^{\delta \cdot h} \cdot u \cdot S(0)$
- - - - - -	$S(0)e^{rh}$
$S_d = d \cdot S(0)$	$e^{\delta \cdot h} \cdot S_d = e^{\delta \cdot h} \cdot d \cdot S(0)$

The no-arbitrage condition:

$$e^{\delta h} \cdot d \cdot \cancel{S(0)} < \cancel{S(0)}\, e^{rh} < e^{\delta \cdot h} \cdot u \cdot \cancel{S(0)} \quad \Big/ : e^{\delta h}$$

$$\boxed{d < e^{(r-\delta)h} < u}$$

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Problem Set #9

Binomial option pricing.

Problem 9.1. (2 points) In the setting of the one-period binomial model, denote by i the **effective** interest rate **per period**. Let u denote the "up factor" and let d denote the "down factor" in the stock-price model. If

$$\boxed{d < u \leq 1 + i}$$

Actually: $d < e^{rh} \cdot e^{-\delta h} < u$

$$\boxed{d < (1+i)e^{-\delta h} < u}$$

then there certainly is no possibility for arbitrage.

False!

Problem 9.2. In our usual notation, which of the parameter choices below creates a binomial model with an arbitrage opportunity?

(a) $u = 1.18,\quad d = 0.87,\quad r = 0.05,\quad \delta = 0,\quad h = 1/4$
(b) $u = 1.23,\quad d = 0.80,\quad r = 0.05,\quad \delta = 0.06,\quad h = 1/2$
(c) $u = 1.08,\quad d = 1,\quad r = 0.05,\quad \delta = 0.04,\quad h = 1$
(d) $u = 1.28,\quad d = 0.78,\quad r = \delta,\quad h = 2$
(e) None of the above.

Binomial Option Pricing.

Stock-price tree.

$$S(0) \quad \begin{cases} S_u = \boxed{u} \cdot S(0) \\ S_d = \boxed{d} \cdot S(0) \end{cases}$$

populating
the tree

We want to price a European-style derivative security w/ the exercise date @ the end of the period.

It is completely determined by its payoff f'tion : $v(\cdot)$

e.g., for the call: $v_c(s) = (s-K)_+$

for the put: $v_p(s) = (K-s)_+$

$\Rightarrow$ The payoff the derivative security is a random variable :

$$\boxed{V(T) := v(S(T))}$$

payoff

$S(0)$ — S_u : $V_u = v(S_u)$

$V(0) = ?$ — S_d : $V_d = v(S_d)$

e.g., for a call option

$V_u = (S_u - K)_+$

$V_d = (S_d - K)_+$

populating $\rightarrow$

$\leftarrow$ pricing

In the binomial model, any derivative security can be replicated w/ a portfolio of this form:

- $\underline{\quad \Delta \quad}$ shares of stock $\begin{cases} \Delta > 0 & \text{buying} \\ \Delta = 0 & \text{"nothing"} \\ \Delta < 0 & \text{shorting} \end{cases}$
- $\underline{\quad B \quad}$ @ the ccrfir r $\begin{cases} B > 0 & \text{lending (buying a bond)} \\ B = 0 & \text{"nothing"} \\ B < 0 & \text{borrowing (issuing a bond)} \end{cases}$

$\uparrow$

$\boxed{\text{time} \cdot 0 \text{ holdings}}$