Name:

M339J/M389J: Probability Models with Actuarial Applications
The University of Texas at Austin
Practice Problems for the Final Exam
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Notes: This is a closed book and closed notes exam. The maximal score on this exam is 200 points.

Problem 4.1. (15 points) Consider a mortality study for subjects between the ages of 60 and 80. Three specific lives contributed these data to the study:

Life #1 entered the study at the age of 60 and was still alive at the end of the study.
Life #2 entered the study at the age of 60 and died between the ages of 70 and 71.
Life #3 entered the study at the age of 60 and died exactly at age 74.5.

The Gompertz law is specified via its force of mortality which has to be of the form $\mu_x = Bc^x$ where $B > 0$ and $c > 1$ are its parameters.

What are the values of the likelihood and the loglikelihood functions produced by the above data for parameter values $B = 4 \times 10^{-6}$ and $c = 1.1$?

Solution: The contribution of *Life #1* to the likelihood function is

$${}_{20}p_{60} = \exp\left(\frac{B}{\ln(c)}c^{60}(1-c^{20})\right)$$

The contribution of *Life #2* to the likelihood function is

$${}_{10}p_{60} - {}_{11}\,p_{60} = \exp\left(\frac{B}{\ln(c)}c^{60}(1-c^{10})\right) - \exp\left(\frac{B}{\ln(c)}c^{60}(1-c^{11})\right)$$

The contribution of *Life #3* to the likelihood function is

$${}_{14.5}p_{60}\mu_{74.5} = {}_{14.5}p_{60}(Bc^{14.5}) = \exp\left(\frac{B}{\ln(c)}c^{60}(1-c^{14.5})\right)(Bc^{74.5})$$

The likelihood function is the product of the above three values, i.e.,

$$L(B = 4 \times 10^{-6}, c = 1.1) = (0.929425)(0.003242234)(0.004669402) = 0.00001407084$$

The loglikelihood is

$$\ell(B = 4 \times 10^{-6}, c = 1.1) = \ln(0.00001407084) = -11.17141.$$

Problem 4.2. (5 pts) Given that the hazard rate function of a random variable X equals $h_X(x) = 1/x$ for $x > 1$, we can conclude that

(a) $0 \leq f_X(2) < 1/4$
(b) $1/4 \leq f_X(2) < 1/3$
(c) $1/3 \leq f_X(2) < 1/2$

(d) $1/2 \leq f_X(2) < 1$

(e) None of the above.

Solution: (b)

The survival function of X is

$$\begin{aligned} S_X(x) &= e^{-\int_1^x h_X(z)\,dz} \\ &= e^{-\int_1^x z^{-1}\,dz} \\ &= e^{-\ln(z)|_{z=1}^{x}} \\ &= e^{-\ln(x)} = \frac{1}{x} \qquad x \geq 1. \end{aligned}$$

So, the density of X is given by

$$f_X(x) = -S'_X(x) = \frac{1}{x^2}, \qquad x > 1.$$

In particular,

$$f_X(2) = 1/4.$$

Problem 4.3. (10 points) Denote the random variable corresponding to the age-at-death (in the absence of censoring) by X. In our usual notation, you are given the following table of observations of death-times, surrender times and times of entry into the study:

j	d_j	x_j	u_j
1	0	20	-
2	22	25	-
3	5	30	-
4	0	-	10
5	0	-	12
6	20	35	-
7	5	22	-
8	0	30	-
9	10	22	-
10	20	-	30

(a) (5 pts) Use the Kaplan-Meier estimator to estimate the probability $\mathbb{P}[X > 21]$.
(b) (5 pts) Find the Nelson-Aalen estimate for $H_X(25)$.

Solution:

(a) We need to find the Kaplan-Meier estimate of the survival function at 21. Looking at the uncensored (death) times in the table (in the x_j column) we see that $S(21) = S(20)$ as 21 falls in the interval $[20, 21)$ whose endpoints are consecutive uncensored observations.

In our usual notation, we have

$$y_1 = 20, y_2 = 22, y_3 = 25, y_4 = 30, y_5 = 35,$$
$$r_1 = 7 + 1 - 3 = 5, r_2 = 6 + 1 - 1 = 6, r_3 = 4 + 1 - 0 = 5, r_4 = 3 + 1 - 0 = 4, r_5 = 1 + 0 - 0 = 1,$$
$$s_1 = 1, s_2 = 2, s_3 = 1, s_4 = 2, s_5 = 1.$$

We will not need all of the above values in this part of the problems, but they might be useful in part (ii), so I immediately calculated all of them.

So, using the Kaplan-Meier method, our estimate of the survival function at 21 is

$$S_{10}(21) = S_{10}(y_1) = 1 - \frac{s_1}{r_1} = 1 - \frac{1}{5} = \frac{4}{5}.$$

(b) Using the method outlined in class and the numbers obtained in the solution to part (a) of this problem, we get

$$\hat{H}_X(25) = \frac{s_1}{r_1} + \frac{s_2}{r_2} + \frac{s_3}{r_3} = \frac{1}{5} + \frac{2}{6} + \frac{1}{5} = \frac{11}{15} \approx 0.7333.$$