02/10/2025.

In multiple dimensions:

Say that the random vector $(Y_1, Y_2, \dots, Y_n)$ is **jointly continuous** w/ density $f_{Y_1,\dots,Y_n}$.

Then,

$$\boxed{\mathbb{P}[Y_1 \in [a_1,b_1], Y_2 \in [a_2,b_2], \dots, Y_n \in [a_n,b_n]] = \int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_n}^{b_n} f_{Y_1,\dots,Y_n}(y_1,y_2,\dots,y_n)\,dy_n \dots dy_2\,dy_1}$$

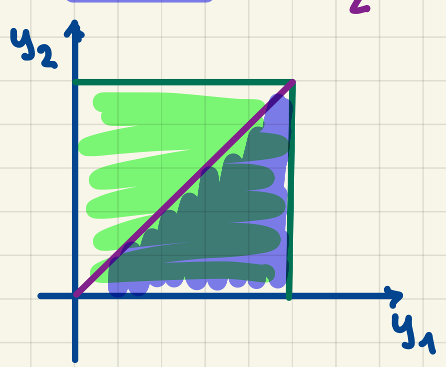
For "any $\boxed{\text{nice}}$" region $A \subseteq \mathbb{R}^n$,

$$\mathbb{P}[(Y_1, Y_2, \dots, Y_n) \in A] = \underbrace{\int\cdots\int}_{A} f_{Y_1,\dots,Y_n}(y_1,\dots,y_n)\,dy_n \dots dy_1$$

Example. (Y_1, Y_2) .... represents a point chosen @ random in a unit square $[0,1]^2$

$$\boxed{f_{Y_1,Y_2}(y_1,y_2) = 1 \cdot \mathbb{1}_{[0,1]\times[0,1]}(y_1,y_2)}$$

$\mathbb{P}[Y_1 > Y_2] = \cancel{?}\ \frac{1}{2}$

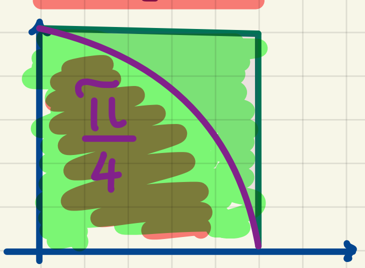


$\mathbb{P}[Y_1^2 + Y_2^2 \le 1] = ?$

$A = \{(y_1,y_2) \in [0,1]^2 : y_1^2 + y_2^2 \le 1\}$

$\mathbb{P}[(Y_1,Y_2) \in A] =$

$= \iint_A f_{Y_1,Y_2}(y_1,y_2)\,dy_2\,dy_1 = \dots = \frac{\pi}{4}$



Example. Let (Y_1, Y_2) be jointly continuous w/ pdf

$$\boxed{f_{Y_1,Y_2}(y_1,y_2)} = \begin{cases} 6y_1 & \text{for } 0 \le y_1 \le y_2 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

OR

$$\boxed{f_{Y_1,Y_2}(y_1,y_2) = 6y_1 \, \mathbb{1}_{[0 \le y_1 \le y_2 \le 1]}}$$

$$\mathbb{P}\left[Y_1 > \tfrac{1}{2}, \ Y_2 > \tfrac{1}{2}\right] =$$

$$= \int_{\frac{1}{2}}^{1} \int_{\frac{1}{2}}^{1} 6y_1 \, \mathbb{1}_{[0 \le y_1 \le y_2 \le 1]} \, dy_2 \, dy_1$$

$$= \int_{\frac{1}{2}}^{1} 6y_1 \underbrace{\boxed{\int_{y_1}^{1} dy_2}}_{=1-y_1} dy_1 = \int_{\frac{1}{2}}^{1} 6y_1(1-y_1)\, dy_1$$

$$= 6\left(\int_{\frac{1}{2}}^{1} y_1 \, dy_1 - \int_{\frac{1}{2}}^{1} y_1^2 \, dy_1\right) = 6\left(\frac{y_1^2}{2}\Bigg|_{y_1=\frac{1}{2}}^{1} - \frac{y_1^3}{3}\Bigg|_{y_1=\frac{1}{2}}^{1}\right)$$

$$= 6\left(\frac{1}{2} - \frac{1}{8} - \left(\frac{1}{3} - \frac{1}{24}\right)\right) = 6 \cdot \frac{12 - 3 - 8 + 1}{24} = \frac{1}{2} \quad \square$$

Functions of Random Vectors.

Theorem. Let $(Y_1, \dots, Y_n)$ be a continuous random vector w/ the joint pdf $f_{Y_1,\dots,Y_n}(\cdot,\dots,\cdot)$

Let g be a function of n variables such that we can define

$$W = g(Y_1, \dots, Y_n)$$

Then,

$$\mathbb{E}[W] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(y_1, \dots, y_n) \cdot f_{Y_1,\dots,Y_n}(y_1,\dots,y_n)\, dy_n \cdots dy_1$$

if the integral is well defined.

Example. (previous cont'd)

(Y_1, Y_2)

$$f_{Y_1,Y_2}(y_1,y_2) = \boxed{6y_1 \mathbb{1}_{[0 \le y_1 \le y_2 \le 1]}}$$

$$\mathbb{E}[Y_1^2 + Y_2^2] = ?$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (y_1^2+y_2^2) \cdot 6y_1 \mathbb{1}_{[0 \le y_1 \le y_2 \le 1]}\, dy_2\, dy_1$$

$$= 6 \int_0^1 \int_{y_1}^1 (y_1^2+y_2^2) \cdot y_1\, dy_2\, dy_1$$

$$= 6 \int_0^1 \int_{y_1}^1 (y_1^3 + y_1 y_2^2)\, dy_2\, dy_1$$

$$= 6 \int_0^1 \left(y_1^3 y_2 + y_1 \frac{y_2^3}{3}\right)\Big|_{y_2=y_1}^{1} dy_1$$

$$= 6 \int_0^1 \left(y_1^3(1-y_1) + y_1 \cdot \frac{1}{3} \cdot (1-y_1^3)\right) dy_1$$

$$= 6\int_0^1 \left(y_1^3 - y_1^4 + \frac{y_1}{3} - \frac{y_1^4}{3}\right) dy_1$$

$$= \int_0^1 (\underbrace{6y_1^3} - \underbrace{8y_1^4} + \underbrace{2y_1}) \, dy_1$$

$$= 6\cdot\frac{1}{4} - 8\cdot\frac{1}{5} + 2\cdot\frac{1}{2} = \frac{3}{2} - \frac{8}{5} + 1 = \frac{9}{10} \qquad \square$$

Marginal Distributions & Independence.

Theorem. Say that $(Y_1, \dots, Y_n)$ has the joint pdf $f_{Y_1,\dots,Y_n}$.

Then, for every $i = 1, \dots, n$, the random variable Y_i is also continuous with its marginal density

$$f_{Y_i}(y) := \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{n-1} f_{Y_1,\dots,Y_n}(y_1, \dots, y_{i-1}, y, y_{i+1}, \dots, y_n) \, dy_n \cdots dy_{i+1} \, dy_{i-1} \cdots dy_1$$

Example. (cont'd from above)

Marginal of Y_1:

$$f_{Y_1}(y) = \int_{-\infty}^{\infty} f_{Y_1,Y_2}(y, y_2) \, dy_2$$

$$= \int_{-\infty}^{\infty} 6y \, \mathbb{1}_{[0 \le y \le y_2 \le 1]} \, dy_2$$

$$= 6y \int_y^1 dy_2 = 6y(1-y) \, \mathbb{1}_{[0,1]}(y)$$

Marginal of Y_2:

$$f_{Y_2}(y) = \int_{-\infty}^{\infty} f_{Y_1,Y_2}(y_1, y)\, dy_1$$

$$= \int_{-\infty}^{\infty} 6y_1 \mathbb{1}_{[0 \leq y_1 \leq y \leq 1]}\, dy_1$$

$$= 6 \int_0^y y_1\, dy_1 \cdot \mathbb{1}_{[0,1]}(y) = 6 \cdot \frac{y^2}{2} \mathbb{1}_{[0,1]}(y)$$

$$= 3y^2 \mathbb{1}_{[0,1]}(y)$$

Def'n. The random variables $Y_1, \dots, Y_n$ are independent iff the events $\{Y_i \in [a_i, b_i]\}$ $i = 1..n$ are independent events for all (a_i, b_i), $i = 1..n$.

Theorem. The Factorization Criterion.

Continuous r.v. $Y_1, \dots, Y_n$ are independent

iff

$$f_{Y_1,\dots,Y_n}(y_1, \dots, y_n) = f_{Y_1}(y_1) \cdots f_{Y_n}(y_n) \quad \text{for all } y_1, \dots, y_n \in \mathbb{R}^n.$$

Corollary.

$f_{Y_1,\dots,Y_n}(y_1, \dots, y_n) = g_1(y_1) \cdots g_n(y_n)$ for all $y_1, \dots, y_n \in \mathbb{R}^n$ and some functions $g_1, \dots, g_n$ is a sufficient condition for independence.