M378K Introduction to Mathematical Statistics
Problem Set #6
Cumulative distribution functions.

Problem 6.1. Source: Sample P exam, Problem #342.
Consider a Poisson distributed random variable X. As usual, let's denote its cumulative distribution function by F_X. You are given that

$$\frac{F_X(2)}{F_X(1)} = 2.6$$

Calculate the expected value of the random variable X.

Solution: Let us denote, as we usually do, the parameter of the random variable X by λ. We know that the expectation of X is equal to it parameter value. So, if we figure out λ, we are done.

The probability mass function of X can be expressed as

$$p_X(k) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad \text{for } k = 0, 1, 2, \ldots$$

So, the given equality can be rewritten as

$$\begin{aligned}
\frac{p_X(0) + p_X(1) + p_X(2)}{p_X(0) + p_X(1)} = 2.6 \quad &\Rightarrow \quad \frac{e^{-\lambda} + \lambda e^{-\lambda} + \frac{\lambda^2}{2}e^{-\lambda}}{e^{-\lambda} + \lambda e^{-\lambda}} = 2.6 \\
&\Rightarrow \quad 1 + \lambda + \tfrac{\lambda^2}{2} = 2.6(1+\lambda) \\
&\Rightarrow \quad 5\lambda^2 - 16\lambda - 16 = 0
\end{aligned}$$

Solving the quadratic, we obtain

$$\lambda_{1,2} = \frac{16 \pm \sqrt{256 + 320}}{10} = \frac{16 \pm \sqrt{576}}{10} = \frac{16 \pm 24}{10}$$

Of course, the Poisson parameter must be positive, so we only keep

$$\lambda = \frac{16 + 24}{10} = 4.$$

Problem 6.2. *Consider a random variable Y whose cumulative distribution function is given by*

$$F_Y(y) = \begin{cases} 0, & \text{for } y < 0 \\ y^4, & \text{for } 0 \leq y < 1 \\ 1, & \text{for } 1 \leq y \end{cases}$$

Calculate the expectation of the random variable Y.

Solution: Evidently, the random variable Y is continuous. (*Why?*) By definition,

$$\mathbb{E}[Y] = \int_{infty}^{\infty} y f_Y(y)\, dy$$

where f_Y stands for the probability density function of Y. We can calculate the pdf as

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

wherever the derivative exists. In this problem, we get

$$f_Y(y) = 4y^3 \mathbf{1}_{[0,1]}(y)$$

Finally, the expectation equals

$$\mathbb{E}[Y] = 4\int_0^1 y \cdot y^3\, dy = 4\int_0^1 y^4\, dy = 4\left(\frac{y^5}{5}\right)|_{y=0}^1 = \frac{4}{5}\,.$$