M339D: April 14th, 2025.

Black-Scholes: Partial Expectation.

The Model.

Under the risk-neutral measure $\mathbb{P}^*$:

$$S(T) = S(0)e^{(r-\frac{\sigma^2}{2})\cdot T + \sigma\sqrt{T}\cdot Z} \quad \text{w/ } Z \sim N(0,1)$$

The Motivation.

$$V_c(0) = e^{-rT}\mathbb{E}^*[V_c(T)] = \dots$$

$$\dots = e^{-rT}\underbrace{\mathbb{E}^*[S(T)\mathbb{I}_{[S(T)\geq K]}]}_{\overset{?}{=}} - e^{-rT}\cdot K \underbrace{\boxed{\mathbb{P}^*[S(T)\geq K]}}_{= N(d_2)}$$

w/ $d_2 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + (r-\frac{\sigma^2}{2})\cdot T\right]$

$$\boxed{\mathbb{E}^*[S(T)\cdot\mathbb{I}_{[S(T)\geq K]}]}$$

Method. Use the defining formula for the expectation of a function of a r.v.
In this case, that r.v. is $Z \sim N(0,1)$

$$\{S(T)\geq K\} = \{S(0)e^{(r-\frac{\sigma^2}{2})\cdot T + \sigma\sqrt{T}\cdot Z} \geq K\}$$
$$= \{Z \geq -d_2\}$$

z... our dummy variable within the integral; it corresponds to Z

i.e., $g(z) = S(0)e^{(r-\frac{\sigma^2}{2})\cdot T + \sigma\sqrt{T}\cdot z}$ (so that $g(Z) = S(T)$)

$$\mathbb{E}^*[g(Z)\cdot\mathbb{I}_{[Z\geq -d_2]}] = \int_{-d_2}^{\infty} g(z)f_Z(z)\,dz$$

$$= \boxed{S(0)e^{rT}}\boxed{N(d_1)}$$

$/\!/$ $\mathbb{E}^*[S(T)]$ (lots of algebra/calculus)

w/ $d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + (r+\frac{\sigma^2}{2})\cdot T\right]$

The expectation under $\mathbb{P}^*$ of the call payoff:

$$\mathbb{E}^*[V_c(T)] = S(0)e^{rT} \cdot N(d_1) - K \cdot N(d_2)$$

w/ d_1 as above and $d_2 = d_1 - \sigma\sqrt{T}$

$\Rightarrow$ The Black-Scholes call price:

$$\boxed{V_c(0) = S(0) \cdot N(d_1) - Ke^{-rT} \cdot N(d_2)}$$

$\Rightarrow$ The Black-Scholes put price:

By put-call parity:

$$V_c(0) - V_p(0) = S(0) - Ke^{-rT}$$

$$\begin{aligned} V_p(0) &= V_c(0) - S(0) + Ke^{-rT} \\ &= S(0)N(d_1) - Ke^{-rT}N(d_2) \\ &\quad - S(0) \qquad + Ke^{-rT} \\ &= S(0)\underbrace{(N(d_1)-1)}_{-N(-d_1)} + Ke^{-rT}\underbrace{(1-N(d_2))}_{N(-d_2)} \end{aligned}$$

symmetry of $N(0,1)$

$$\boxed{V_p(0) = Ke^{-rT}N(-d_2) - S(0)N(-d_1)}$$

Problem 14.3. (5 pts) Assume the Black-Scholes framework. Let the current price of a non-dividend-paying stock be equal to $S(0) = 95$ and let its volatility be equal to 0.35. Consider a European call on that stock with strike 100 and exercise date in 9 months. Let the risk-free continuously compounded interest rate be 6% per annum.

Denote the price of the call by $V_C(0)$. Then,

(a) $V_C(0) < \$5.20$

(b) $\$5.20 \leq V_C(0) < \$7.69$

(c) $\$7.69 \leq V_C(0) < \$9.04$

(d) $9.04 \leq V_C(0) < \$11.25$

(e) None of the above.

$\longrightarrow$: We'll use the <u>Black-Scholes call price</u>:

$$V_c(0) = S(0) \cdot N(d_1) - Ke^{-rT} \cdot N(d_2)$$

$$\text{w/ } d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(r + \frac{\sigma^2}{2}\right) \cdot T\right]$$

$$\text{and } d_2 = d_1 - \sigma\sqrt{T}$$

✓ 1st Calculate d_1 and d_2.

✓ 2nd Use the standard normal table or 'R' (pnorm).

3rd Combine into the BS price.

$$d_1 = \frac{1}{0.35\sqrt{3/4}}\left[\ln\left(\frac{95}{100}\right) + \left(0.06 + \frac{0.35^2}{2}\right) \cdot \left(\frac{3}{4}\right)\right] = 0.1307 \approx 0.13$$

$$d_2 = d_1 - \sigma\sqrt{T} = d_1 - 0.35\sqrt{3/4} = -0.1733 \approx -0.17$$

$$N(d_1) \approx N(0.13) = 0.5517$$

$$N(d_2) \approx N(-0.17) = 0.4325$$

$$V_c(0) = 95 \cdot 0.5517 - 100e^{-0.06(3/4)} \cdot 0.4325 = 11.06$$

□

Problem 14.4. Assume the Black-Scholes setting. Let $S(0) = \$63.75$, $\sigma = 0.20$, $r = 0.055$. The stock pays no dividend and the option expires in 50 days (simplify the number of days in a year to 360).

What is the price of a \$60-strike European put?

(a) 0.66
(b) 0.55
(c) 0.44
(d) 0.37
(e) None of the above.

$\longrightarrow$:

$$d_1 = \frac{1}{0.2\sqrt{\frac{50}{360}}}\left[\ln\left(\frac{63.75}{60}\right) + \left(0.055 + \frac{0.04}{2}\right)\cdot\left(\frac{50}{360}\right)\right]$$

$$d_1 = \underline{0.9531} \approx 0.95$$

$$d_2 = d_1 - \sigma\sqrt{T} = d_1 - 0.2\sqrt{\frac{50}{360}} = \underline{0.8786} \approx 0.88$$

$$N(-d_1) = N(-0.95) = \underline{0.1711}$$

$$N(-d_2) = N(-0.88) = \underline{0.1894}$$

$$V_P(0) = Ke^{-rT}\cdot N(-d_2) - S(0)N(-d_1)$$

$$= 60e^{-0.055\left(\frac{50}{360}\right)}\cdot 0.1894 - 63.75\cdot 0.1711$$

$$= \underline{0.37}$$

$\square$