

## M378K Introduction to Mathematical Statistics

### Problem Set #19

### Hypothesis testing.

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**Problem 19.1.** An instructor of a massive online course claims that students solve at most 20 problems per week (on average). To verify this conviction, the instructor intends to conduct a hypothesis test.

What are the null and alternative hypotheses in this case?

**Solution:** Let  $\mu$  denote the population mean, i.e., the mean number of solved problems (in a week).

$$H_0 : \mu = 20 \quad \text{vs.} \quad H_a : \mu < 20.$$

With a sample size of 256, what is the test statistics appropriate the test the above claim? What is its (approximate) distribution under the null?

**Solution:** With such a large sample size ( $n = 256$ ), we can use the CLT and the appropriate test statistic is

$$Z = \frac{\bar{Y} - \mu_0}{\frac{S}{\sqrt{n}}} \approx N(0, 1)$$

Note that with  $n = 256$ , I am comfortable using the standard normal above (instead of the - more conservative -  $t$ -distribution).

Say that the sample average equals 19.7 and that the sample variance equals 9. What is the  $p$ -value associated with these data?

**Solution:**

$$\mathbb{P} \left[ \frac{\bar{Y} - \mu_0}{\frac{S}{\sqrt{n}}} < \frac{19.7 - 20}{\frac{3}{\sqrt{256}}} \right] \approx \mathbb{P}[Z < -1.6] = 0.0548.$$

Assume that the given significance level is 5%. What would the decision be?

**Solution:** Fail to reject the null hypothesis.

**Problem 19.2.** A candy-cane twisting machine is considered defective if at least 10% of the candy canes crack or break in the twisting process. A random sample of 100 candy canes was collected and it was found that it contained 12 cracked candy canes. You believe that the machine is defective. Formulate and conduct the relevant hypothesis test with a 2% significance level.

**Solution:** Our hypotheses are

$$H_0 : p = 0.10 \quad \text{vs.} \quad H_a : p > 0.10$$

The appropriate test statistic is

$$\frac{\hat{P} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$$

which is approximately standard normal (under the null hypothesis). So, to calculate the  $p$ -value, we do the following

$$\mathbb{P} \left[ Z > \frac{0.12 - 0.1}{\sqrt{\frac{0.1(1-0.1)}{100}}} = 0.6667 \right] = 0.2514.$$

Obviously, the obtained  $p$ -value is larger than 0.02, and so we fail to reject the null hypothesis.

**Problem 19.3.** Consider a poll ahead of an election with two candidates: A and B. Let  $p$  denote the population proportion of voters who will vote for A. We want to conduct a hypothesis test on whether candidate A will win, i.e., our hypotheses are

$$H_0 : p = 0.5 \quad \text{vs.} \quad H_a : p < 0.5$$

Let our significance level be 5%. What is the rejection region (RR) for a sample size of 10?

**Solution:** Evidently, in this case, we cannot use the large-sample approach with the normal approximation to the binomial. Luckily, we have 'R' at our disposal. Under the null hypothesis, our sample count  $T$  is binomial with the number of trials equal to  $n = 10$  and the success probability equal to 0.5.

From the structure of the alternative hypothesis, we know that the rejection region (RR) will be of the form  $[0, t^*]$  for some critical value  $t^*$  such that

$$\mathbb{P}[T \leq t^*] \leq \alpha < \mathbb{P}[T \leq t^* + 1]$$

Let's examine the probabilities that  $T \leq t$  produced by 'R' (as we vary  $t$  from 0 to, say,  $n = 4$ ). The "code" is `pbinom(0:4, size=10, p=0.5)`. We output is

0.0009765625, 0.0107421875, 0.0546875000, 0.1718750000, 0.3769531250

We conclude that our rejection region is  $RR = [0, 1]$ .