

M3392: November 6<sup>th</sup>, 2023.

## Log-Normal stock Prices.

Temporarily fix a time-horizon  $T$ .

$S(t)$ ,  $t \in [0, T]$  ... time  $t$  stock price

$$\xrightarrow{\quad} t$$

Define:

$$R(s, t) := \ln\left(\frac{S(t)}{S(s)}\right)$$

Equivalently:

$$S(t) = S(s) e^{R(s, t)}$$

In particular:  $R(0, T)$  ... realized return over  $(0, T)$

We model realized returns as normal.

$$R(0, T) \sim \text{Normal}(\text{mean} = m, \text{variance} = \sigma^2)$$



$\Rightarrow S(T)$  is lognormal

and

$$\mathbb{E}^*[S(T)] = S(0) e^{m + \frac{\sigma^2}{2}}$$



## Market model.

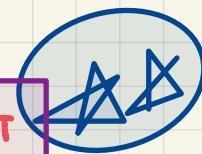
- Riskless Asset w/ ccfir  $r$
- Risky Asset : a non-dividend-paying stock

$\sigma$ ... volatility



Under the risk-neutral measure

$$\mathbb{E}^*[S(T)] = S(0) e^{rT}$$



Equating



to



$$m + \frac{\sigma^2}{2} = rT$$



Consider:  $\text{Var}[R(0,T)] = \nu^2 = ?$   $\nu^2 = \sigma^2 \cdot T$

Recall:  $\text{Var}[R(0,1)] = \sigma^2$ , i.e.,  $\text{SD}[R(0,1)] = \sigma$

$$\text{Var}[R(0,2)] = \text{Var}[R(0,1)] + \text{Var}[R(1,2)] = 2\text{Var}[R(0,1)] = 2\sigma^2$$

$$m = rT - \frac{\sigma^2 \cdot T}{2} = (r - \frac{\sigma^2}{2}) \cdot T$$

Finally:

$R(0,T) \sim \text{Normal}(\text{mean} = (r - \frac{\sigma^2}{2}) \cdot T, \text{variance} = \sigma^2 \cdot T)$

Say that  $Z \sim N(0,1)$

Then, we can express  $R(0,T)$  as

$$R(0,T) = (r - \frac{\sigma^2}{2}) \cdot T + \sigma\sqrt{T} \cdot Z$$

Thus,

$$S(T) = S(0)e^{(r - \frac{\sigma^2}{2}) \cdot T + \sigma\sqrt{T} \cdot Z}$$

Q: What is the median of  $S(T)$  under  $P^*$ ?

→:  $S(0)e^{(r - \frac{\sigma^2}{2}) \cdot T}$  ✓

Note: 
$$\frac{\text{mean}}{\text{median}} = \frac{S(0)e^{rT}}{S(0)e^{(r - \frac{\sigma^2}{2}) \cdot T}} = e^{\frac{\sigma^2 \cdot T}{2}}$$

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## Problem Set # 100

Mean and median of the log-normal stock prices.

**Problem 100.1.** The current price of a non-dividend-paying stock is \$80 per share. Under the risk-neutral probability measure, its mean rate of return is 12% and its volatility is 30%.

Let  $R(0, t)$  denote the realized return of this stock over the time period  $[0, t]$  for any  $t > 0$ . Calculate  $E^*[R(0, 2)]$ .

→:

$$r = 0.12$$

$$\sigma = 0.30$$

$$R(0, 2) \sim \text{Normal}(\text{mean} = (r - \frac{\sigma^2}{2}) \cdot 2, \text{var} = \sigma^2 \cdot 2)$$

$$E^*[R(0, 2)] = (0.12 - \frac{0.09}{2}) \cdot 2 = 0.15$$

□

**Problem 100.2.** A stock is valued at \$75.00. The continuously compounded, risk-free interest rate is 10.0% and the standard deviation of annualized returns is 25.0%. If the stock is lognormally distributed, what is the expected stock price after 2 years under the risk-neutral probability measure?

→:

$$E^*[S(2)] = S(0)e^{2r} = 75e^{0.20} \approx 91.605$$

□

**Problem 100.3.** A non-dividend-paying stock is valued at \$55.00 per share. Its standard deviation of annualized returns is given to be 22.0%. The continuously compounded risk-free interest rate is 12%. If the stock price is modeled using the lognormal distribution (as discussed in class), what is the median of the stock price in 3 years under the risk-neutral probability measure?

$$S(0)e^{(r - \frac{\sigma^2}{2}) \cdot T} = 55e^{(0.12 - \frac{0.22^2}{2}) \cdot 3} = 73.313$$

□

**Problem 100.4.** Assume that the stock price is modeled using the lognormal distribution. The stock pays no dividends. Under  $\mathbb{P}^*$ , the annual mean rate of return on the stock is given to be 12%. Also under  $\mathbb{P}^*$ , the median time- $t$  stock price is evaluated to be  $S(0)e^{0.1t}$ . What is the volatility parameter of this stock price?

$$\rightarrow: S(0)e^{(r-\frac{\sigma^2}{2}) \cdot t} = S(0)e^{0.1t}$$

$$r - \frac{\sigma^2}{2} = 0.1$$

$$0.12 - \frac{\sigma^2}{2} = 0.1$$

$$\frac{\sigma^2}{2} = 0.02 \Rightarrow \sigma^2 = 0.04 \Rightarrow \sigma = 0.20$$

**Problem 100.5.** The current stock price is \$100 per share. The stock price at any time  $t > 0$  is modeled using the lognormal distribution. Assume that the continuously compounded risk-free interest rate equals 8%. Let its volatility equal 20%.

Find the value  $t^*$  at which the median stock price equals \$120, under the risk-neutral probability measure.

$$\rightarrow: \text{median time-}t \text{ stock price} = S(0)e^{(r-\frac{\sigma^2}{2}) \cdot t}$$

$$120 = 100e^{(0.08 - \frac{0.04}{2}) \cdot t^*}$$

$$1.2 = \frac{\ln(1.2)}{0.06} = 3.039 \quad \square$$

**Problem 100.6.** The volatility of the price of a non-dividend-paying stock is 0.2. The stock price is modeled using a log-normal distribution. Under  $\mathbb{P}^*$ , the expected time-2 stock price is \$120. What is the median of the time-2 stock price under  $\mathbb{P}^*$ ?

$$\rightarrow: \mathbb{E}^*[S(2)] e^{-\frac{2\sigma^2}{2}} = 120e^{-0.04} = 115.295 \quad \square$$