Risk-neutral Pricing [cont'd]

$$V(0) = \underbrace{\Delta \cdot S(0) + B}_{\text{pricing by replication}}$$

$$= e^{-rT}\left[p^* \cdot V_u + (1-p^*) \cdot V_d\right]$$

where p^*... the risk-neutral probability of the stock price going up in a single period, i.e.,

$$\boxed{p^* = \frac{e^{rh}-d}{u-d}}$$

$\Rightarrow$ The risk-neutral pricing formula:

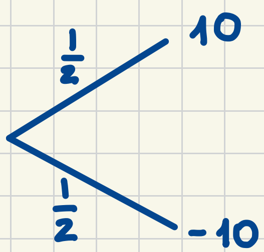
$$V(0) = \underbrace{e^{-rT}}_{\text{discounting}} \underbrace{\boxed{\left[p^* \cdot V_u + (1-p^*) V_d\right]}}_{\text{expected payoff under the risk-neutral probability}}$$

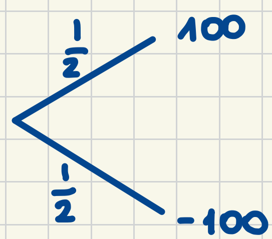
We can (and do) generalize this principle:

$$\boxed{V(0) = e^{-rT}\, \mathbb{E}^*[V(T)]}$$

Q: Why "risk-neutral"?

Imagine bets:





Consider a risk-neutral investor, i.e., one who is indifferent to risk and cares only about expectation.

Q: What is the probability $\tilde{p}$ such that, for a specific stock-price tree, this investor is indifferent between investing in the stock and the risk-free investment?

→ : Say, they start w/ $S(0)$.

If the invest @ the ccrfir r, then their balance @ time h is $S(0)e^{rh}$

If they invest in the stock:

$$\begin{tikzcd} & S_u \\ S(0) \arrow[ur, "\tilde{p}"] \arrow[dr, "1-\tilde{p}"'] & \\ & S_d \end{tikzcd}$$

$$\mathbb{E}[\text{Wealth}] = \mathbb{E}[S(h)]$$
$$= \tilde{p} \cdot S_u + (1-\tilde{p}) \cdot S_d$$
$$= \tilde{p} \cdot u \cdot S(0) + (1-\tilde{p}) \cdot d \cdot S(0)$$
$$= \left(\tilde{p} \cdot u + (1-\tilde{p}) \cdot d\right) S(0) = S(0)e^{rh}$$

$$\tilde{p} \cdot u + (1-\tilde{p}) \cdot d = e^{rh}$$
$$\tilde{p}(u-d) = e^{rh} - d$$
$$\Rightarrow \boxed{\tilde{p} = \frac{e^{rh}-d}{u-d} = p^*}$$

□

Problem 9.5 [revisited]

$$\begin{tikzcd} & S_u = 120 \\ S(0) = 95 \arrow[ur, "p^*"] \arrow[dr, "1-p^*"'] & \\ & S_d = 75 \end{tikzcd}$$

Payoff

$V_u = 20$

$V_d = 25$

$r = 0.06$

$\boxed{v(s) = |s-100|}$

$$p^* = \frac{e^{rh}-d}{u-d} \cdot \frac{S(0)}{S(0)} = \frac{S(0)e^{rh} - S_d}{S_u - S_d} = \frac{95e^{0.06} - 75}{120-75} = \underline{0.5749}$$

$$V(0) = e^{-rT}\left[V_u \cdot p^* + V_d (1-p^*)\right]$$
$$= e^{-0.06}\left[20 \cdot p^* + 25(1-p^*)\right] = \underline{20.84}$$

□

Special Case: Forward Binomial Tree.

$$\boxed{\sigma \ldots \text{volatility}}$$

$$\boxed{\begin{aligned} u &:= e^{rh+\sigma\sqrt{h}} \\ d &:= e^{rh-\sigma\sqrt{h}} \end{aligned}}$$

The risk-neutral probability:

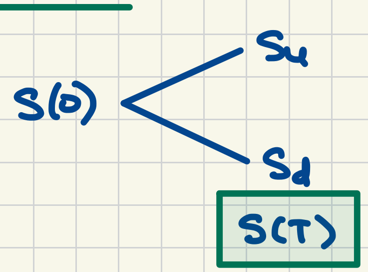
$$p^* = \frac{e^{rh}-d}{u-d} = \frac{\overset{1}{\cancel{e^{rh}}} - e^{\cancel{rh}-\sigma\sqrt{h}}}{e^{\cancel{rh}+\sigma\sqrt{h}} - e^{\cancel{rh}-\sigma\sqrt{h}}}$$

$$p^* = \frac{1-e^{-\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}}-e^{-\sigma\sqrt{h}}} \cdot \frac{e^{\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}}} = \frac{\overset{1}{\cancel{e^{\sigma\sqrt{h}}-1}}}{\underbrace{e^{2\sigma\sqrt{h}}-1}_{(e^{\sigma\sqrt{h}}+1)\cancel{(e^{\sigma\sqrt{h}}-1)}}}$$

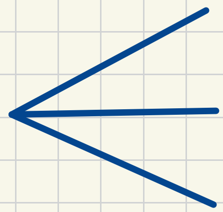
$$\boxed{p^* = \frac{1}{1+e^{\sigma\sqrt{h}}}} \xrightarrow[h\to 0]{} \frac{1}{2}$$

The shortcut $\boxed{\text{ONLY}}$ for the FORWARD binomial tree.

Motivation.



Q: How can we make the model for S(T) richer, but still "interpretable"?



k-nary tree