UNIVERSITY OF TEXAS AT AUSTIN

Problem Set #7

Binomial option pricing: Two or more periods.

Problem 7.1. For a two-period binomial model, you are given that:

(1) each period is one year;
(2) the current price of a non-dividend-paying stock S is $S(0) = \$20$;
(3) $u = 1.2$, with u as in the standard notation for the binomial model;
(4) $d = 0.8$, with d as in the standard notation for the binomial model;
(5) the continuously compounded risk-free interest rate is $r = 0.04$.

Consider a **special** call option which pays the excess above the strike price $K = 23$ (if any!) at the end of **every** binomial period.

Find the price of this option.

Solution: The risk-neutral probability is

$$p^* = \frac{e^{0.04} - 0.8}{1.2 - 0.8} = 0.6020.$$

When one constructs the two-period binomial tree, one gets

$$S_u = 24,\ S_d = 16,$$
$$S_{uu} = 28.80,\ S_{ud} = S_{dd} = 19.2,\ S_{dd} = 12.8.$$

So, the payoffs at the end of the first period are

$$V_u = 1, V_d = 0.$$

The payoffs at the end of the second period are

$$V_{uu} = 5.80, \quad V_{ud} = 0, \quad V_{dd} = 0.$$

So, taking the expected value at time 0 of the payoff with respect to the risk-neutral probability, we get that the price of this call should be

$$\begin{aligned} e^{-0.04} \times V_u \times p^* + e^{-0.04\times 2}&[V_{uu} \times (p^*)^2 + V_{ud} \times 2p^*(1-p^*)] \\ &= e^{-0.04} \times 1 \times 0.6020 + e^{-0.08}[5.8 \times 0.6020^2] \\ &= 2.51893. \end{aligned}$$

Problem 7.2. Let the continuously compounded risk-free interest rate be 0.10. Let the initial price of a non-dividend-paying stock be \$100 per share. You use a five-period binomial tree to model the stock price over the next year. Let $u = 1.04$ and $d = 0.96$.

What is the price of a one-year, at-the-money European call option on the above stock?

Solution: The risk-neutral probability is

$$p^* = \frac{e^{0.10(1/5)} - 0.96}{1.04 - 0.96} = 0.7525.$$

The relevant final possible stock prices in the binomial tree are

$$\begin{aligned} s_{5,5} &= S(0)u^5 = 100(1.04)^5 = 121.67, \\ s_{5,4} &= S(0)u^4 d = 100(1.04)^4(0.96) = 112.31, \\ s_{5,3} &= S(0)u^3 d^2 = 100(1.04)^3(0.96)^2 = 103.67. \end{aligned}$$

The remaining terminal nodes are **out of the money**.

The possible payoffs are

$$v_{5,5} = 21.67, \quad v_{5,4} = 12.31, \quad v_{5,3} = 3.67.$$

So, the price of our call is

$$V_C(0) = e^{-0.10}\left[21.67(p^*)^5 + 12.31(5)(p^*)^4(1-p^*) + 3.67(10)(p^*)^3(1-p^*)^2\right] = 10.0176.$$