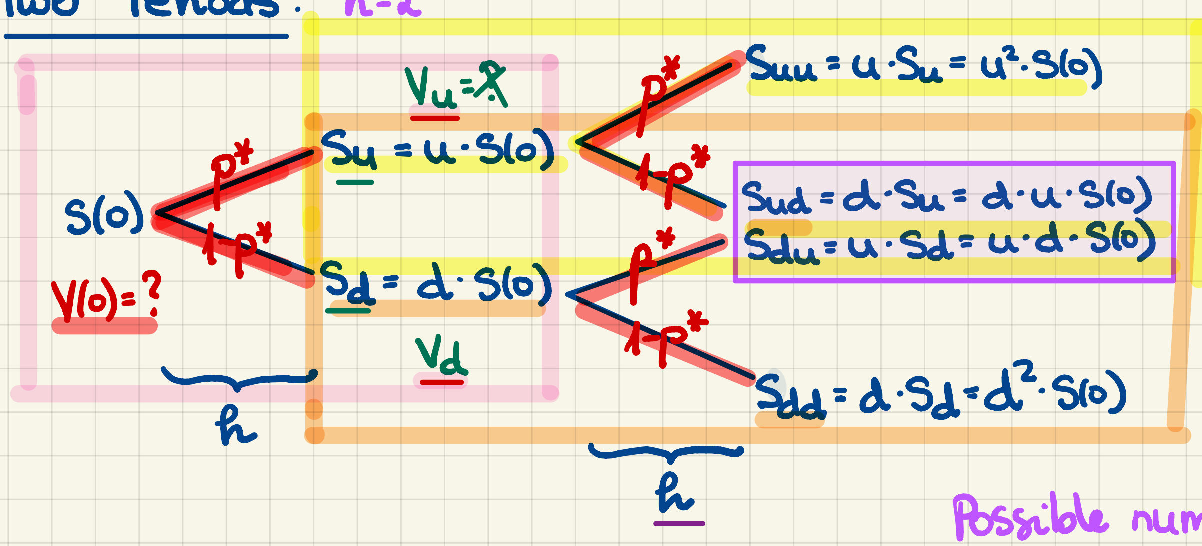
M339D: October 18th, 2023.

Binomial Option Pricing: Two Periods.

Two Periods: $n=2$



Payoff:

$V_{uu} := v(S_{uu})$

$V_{ud} := v(S_{ud})$

$V_{dd} := v(S_{dd})$

Possible number of values of $S(T)$ is $n+1=3$.

$0 \quad h = \frac{T}{2} \quad h = \frac{T}{2} \quad T$

populating the tree $\longrightarrow$

$\longleftarrow$ pricing the option

- up node:

replicating portfolio for the option:

$$\begin{cases} \Delta_u = \dfrac{V_{uu} - V_{ud}}{S_{uu} - S_{ud}} \\ B_u = e^{-rh} \cdot \dfrac{u \cdot V_{ud} - d \cdot V_{uu}}{u-d} \end{cases}$$

$\Rightarrow$ the option's value @ the up node:

$$V_u = \Delta_u \cdot S_u + B_u = e^{-rh} \cdot \left[p^* \cdot V_{uu} + (1-p^*) \cdot V_{ud}\right] \longleftarrow$$

w/ $\boxed{p^* = \dfrac{e^{rh} - d}{u-d}}$

- down node: Δ_d, B_d

$$\Rightarrow \quad V_d = \Delta_d \cdot S_d + B_d = e^{-rh}\left[p^* \cdot V_{ud} + (1-p^*) \cdot V_{dd}\right]$$

- ROOT node:

$$\Delta_0 = \frac{V_u - V_d}{S_u - S_d}$$

$$B_0 = e^{-rh} \cdot \frac{u \cdot V_d - d \cdot V_u}{u-d}$$

$$\Rightarrow \quad \boxed{V(0) = \Delta_0 \cdot S(0) + B_0}$$

From the risk-neutral "perspective":

$$V(0) = e^{-rh}\left[p^* \cdot V_u + (1-p^*) \cdot V_d\right]$$

$$V(0) = e^{-rh}\left[p^* \cdot e^{-rh}\left(p^* \cdot V_{uu} + (1-p^*) \cdot V_{ud}\right) + (1-p^*) e^{-rh}\left(p^* \cdot V_{ud} + (1-p^*) \cdot V_{dd}\right)\right]$$

$$V(0) = \underbrace{e^{-r\overset{T}{(2h)}}}_{\text{Discounting}}\underbrace{\left[(p^*)^2 \cdot V_{uu} + 2 \cdot p^*(1-p^*) V_{ud} + (1-p^*)^2 \cdot V_{dd}\right]}_{\text{Risk-Neutral Expectation of the Payoff}}$$

Generally: $\boxed{V(0) = e^{-rT}\,\mathbb{E}^*[V(T)]}$

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Problem Set #9

Binomial option pricing: Two or more periods.

Problem 9.1. For a two-period binomial model, you are given that:

(1) each period is one year; $h=1$

(2) the current price of a non-dividend-paying stock S is $S(0) = \$20$;

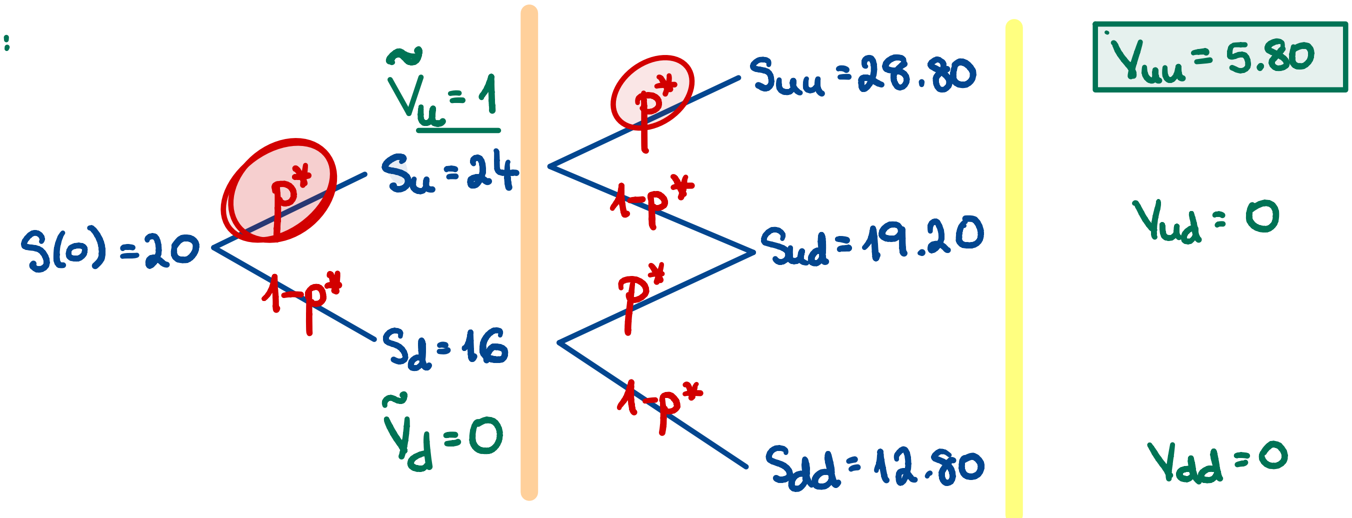
(3) $u = 1.2$, with u as in the standard notation for the binomial model;

(4) $d = 0.8$, with d as in the standard notation for the binomial model;

(5) the continuously compounded risk-free interest rate is $\boxed{r = 0.04}$.

Consider a **special** call option which pays the excess above the strike price $K = 23$ (if any!) at the end of **every** binomial period.

Find the price of this option.



Risk-Neutral Probability:

$$p^* = \frac{e^{rh} - d}{u - d} = \frac{e^{0.04} - 0.8}{1.2 - 0.8} = 0.602$$

$$\tilde{V}(0) = e^{-0.04} \cdot p^* \cdot 1 = 0.5784$$

$$V(0) = e^{-0.04(2)} (p^*)^2 \cdot 5.80 = 1.941$$

$+$

answer: the price of the special call is $\boxed{2.5194}$

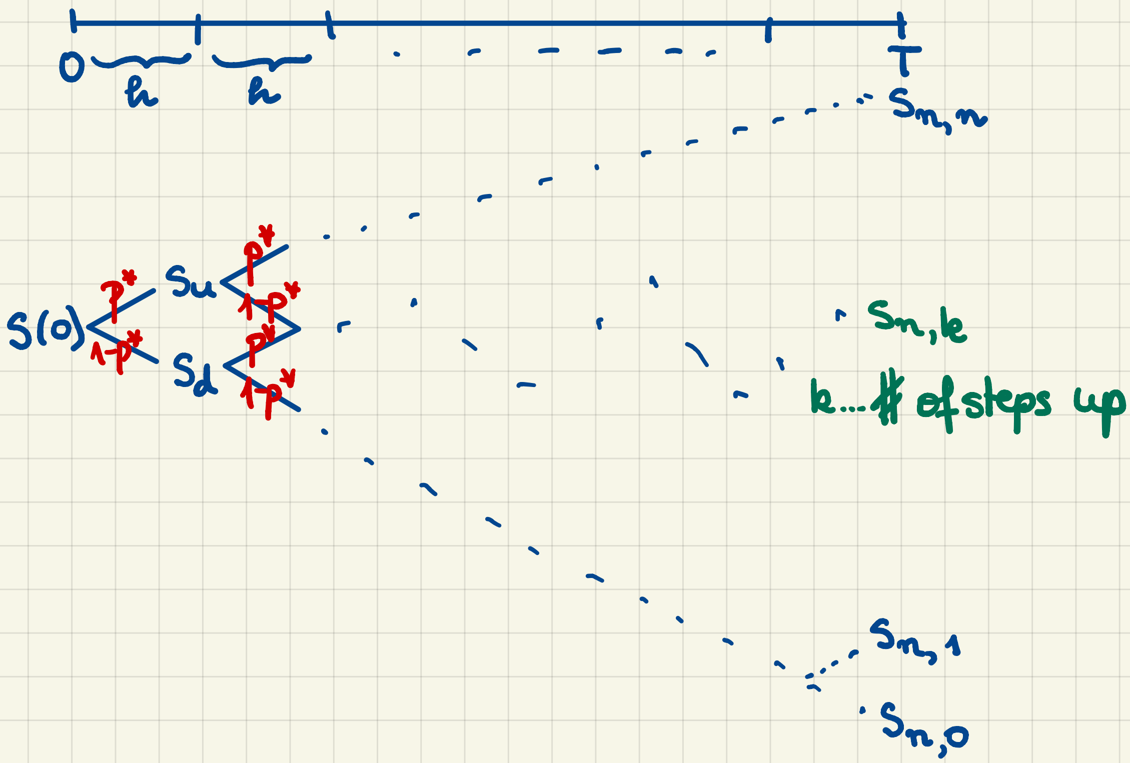
$\square$

Multiple Binomial Periods.

T … exercise date of a European option
n … # of periods

} the length of each period

$$\boxed{h = \frac{T}{n}}$$



The $(n+1)$ values are the support of $S(T)$