ii. Look @ a European put w/ strike K.

Its payoff f'tion: $v_p(s) = (K-s)_+$

The expected payoff is

$$\mathbb{E}[v_p(S(T))] = \boxed{\mathbb{E}[(K-S(T))_+]}$$

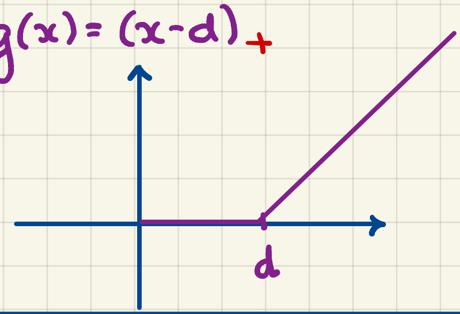
By Jensen, its lower bound is $\geq (K - \mathbb{E}[S(T)])_+$

M339D: April 10th, 2024.

iii. In classical insurance:

$$\begin{cases} X \ldots \text{(ground-up) loss, i.e., the severity r.v.} \\ d \ldots \text{deductible} \end{cases}$$

The insurer pays $(X-d)_+$, i.e., $g(x) = (x-d)_+$



By Jensen's inequality

$$\mathbb{E}[(X-d)_+] \geq (\mathbb{E}[X]-d)_+$$

The median = ?

Find $\pi_{0.5}$ such that $\boxed{\mathbb{P}[Y \leq \pi_{0.5}] = 0.5}$

$$\mathbb{P}[e^X \leq \pi_{0.5}] = 0.5$$

$$\mathbb{P}[\boxed{X} \leq \ln(\pi_{0.5})] = 0.5$$

median of $X \sim Normal(mean = m, var = \sigma^2)$

$$\Downarrow$$

$$\ln(\pi_{0.5}) = m$$

$$\Updownarrow$$

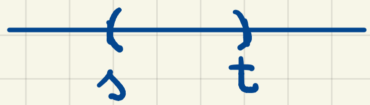
$$\boxed{\pi_{0.5} = e^m}$$

Task: How would you modify the above to any quantile of the lognormal distribution?

Log-Normal Stock Prices.

Temporarily fix a time-horizon T.

$S(t),\ t\in[0,T]$ ... time-t stock price



Define $$R(s,t) := \ln\left(\frac{S(t)}{S(s)}\right)$$

Alternatively, $$S(t) = S(s)\,e^{R(s,t)}$$

In particular, $R(0,T)$ ... realized return over $(0,T)$

We model realized returns as normal.

$R(0,T) \sim \text{Normal}(\text{mean} = m,\ \text{var} = \nu^2)$

$\Rightarrow$ $S(T)$ is lognormal

and $\mathbb{E}^*[S(T)] = S(0)\,e^{m+\frac{\nu^2}{2}}$ ★

Market model.

- Riskless asset w/ ccrfir r
- Risky asset: a non-dividend-paying stock

$\boxed{\sigma \text{... volatility}}$

Under the risk-neutral probability measure $\mathbb{E}^*[S(T)] = S(0)\,e^{rT}$ ★★

Equating ★ & ★★, we get $\boxed{m + \frac{\nu^2}{2} = rT}$ ⟵ (!)

Recall: $\text{Var}[R(0,1)] = \sigma^2$, i.e., $SD[R(0,1)] = \sigma$

$\Rightarrow$ $\text{Var}[R(0,T)] = \boxed{\sigma^2 \cdot T = \nu^2}$ ⟵

(!) $\Rightarrow$ $\boxed{m = rT - \frac{\nu^2}{2} = rT - \frac{\sigma^2 \cdot T}{2} = \left(r - \frac{\sigma^2}{2}\right)\cdot T}$

Finally,

$$\boxed{R(0,T) \sim \text{Normal}\left(\text{mean} = \boxed{\left(r-\frac{\sigma^2}{2}\right)\cdot T},\ \text{variance} = \sigma^2 \cdot T\right)}$$

Say that $\boxed{Z \sim N(0,1)}$

Then, we can express $R(0,T)$ as

$$\boxed{R(0,T) = \left(r-\frac{\sigma^2}{2}\right)\cdot T + \sigma\sqrt{T}\cdot Z}$$

Thus,

$$\boxed{S(T) = S(0)\, e^{\left(r-\frac{\sigma^2}{2}\right)\cdot T + \sigma\sqrt{T}\cdot Z}}$$

Q: What is the median of $S(T)$ under the risk-neutral probability measure $\mathbb{P}^*$?

$$\longrightarrow: \boxed{S(0)\, e^{\left(r-\frac{\sigma^2}{2}\right)\cdot T}}$$

□

Note:

$$\frac{\text{mean}}{\text{median}} = \frac{\cancel{S(0)}\cancel{e^{rT}}}{\cancel{S(0)}e^{\left(\cancel{r}-\frac{\sigma^2}{2}\right)\cdot T}} = e^{\frac{\sigma^2\cdot T}{2}}$$

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Problem Set # 13

Mean and median of the log-normal stock prices.

Problem 13.1. The current price of a non-dividend-paying stock is \$80 per share. Under the risk-neutral probability measure, its mean rate of return is 12% and its volatility is 30%.

Let $R(0,t)$ denote the realized return of this stock over the time period $[0,t]$ for any $t > 0$. Calculate $\mathbb{E}^*[R(0,2)]$.

$\rightarrow$:

$\boxed{r = 0.12}$

$\boxed{\sigma = 0.30}$

$$R(0,2) \sim Normal\left(mean = \boxed{\left(r - \frac{\sigma^2}{2}\right)\cdot 2}, var = \sigma^2 \cdot 2\right)$$

$$\mathbb{E}^*[R(0,2)] = \left(0.12 - \frac{0.09}{2}\right)\cdot 2 = (0.12 - 0.045)\cdot 2 = 0.15$$

Problem 13.2. A stock is valued at \$75.00. The continuously compounded, risk-free interest rate is 10.0% and the standard deviation of annualized returns is 25.0%. If the stock is lognormally distributed, what is the expected stock price after 2 years under the risk-neutral probability measure?

$\rightarrow$:

$$\mathbb{E}^*[S(2)] = S(0)e^{2r} = 75e^{2\cdot 0.10} = 75e^{0.20} \approx \underline{91.61}$$

Problem 13.3. A non-dividend-paying stock is valued at \$55.00 per share. Its standard deviation of annualized returns is given to be 22.0%. The continuously compounded risk-free interest rate is 12%. If the stock price is modeled using the lognormal distribution (as discussed in class), what is the median of the stock price in 3 years under the risk-neutral probability measure?

$\rightarrow$:

$$\underline{S(0)e^{\left(r - \frac{\sigma^2}{2}\right)\cdot T}} = 55e^{\left(0.12 - \frac{0.22^2}{2}\right)\cdot 3} = \underline{73.31}$$

Problem 13.4. Assume that the stock price is modeled using the lognormal distribution. The stock pays no dividends. Under $\mathbb{P}^*$, the annual mean rate of return on the stock is given to be 12% Also under $\mathbb{P}^*$, the median time$-t$ stock price is evaluated to be $S(0)e^{0.1t}$. What is the volatility parameter of this stock price?

→ :

$$\cancel{S(0)}\cancel{e}^{(r-\frac{\sigma^2}{2})t} = \cancel{S(0)}\cancel{e}^{0.1t}$$

$$r - \frac{\sigma^2}{2} = 0.1$$

$$0.12 - \frac{\sigma^2}{2} = 0.1$$

$$\frac{\sigma^2}{2} = 0.12 - 0.1 = 0.02 \Rightarrow \sigma^2 = 0.04 \Rightarrow \boxed{\sigma = 0.2}$$

Problem 13.5. The current stock price is \$100 per share. The stock price at any time $t > 0$ is modeled using the lognormal distribution. Assume that the continuously compounded risk-free interest rate equals 8%. Let its volatility equal 20%.

Find the value t^* at which the median stock price equals \$120, under the risk-neutral probability measure.

→ : median time-t stock price $= S(0)e^{(r-\frac{\sigma^2}{2})\cdot t}$

$$1.2\cancel{0} = 1\cancel{00}\, e^{(0.08 - \frac{0.04}{2})\cdot t^*}$$

$$t^* = \frac{\ln(1.2)}{0.06} = \underline{3.039}$$

Problem 13.6. The volatility of the price of a non-dividend-paying stock is 0.2. The stock price is modeled using a log-normal distribution. Under $\mathbb{P}^*$, the expected time-2 stock price is \$120. What is the median of the time-2 stock price under $\mathbb{P}^*$?