UNIVERSITY OF TEXAS AT AUSTIN

HW Assignment 11

The Black-Scholes pricing formula.

Please, provide your **complete solution** to the following problem(s):

Problem 11.1. (2 points) Let the stock price be modeled by a lognormal distribution. Assume that the stock's volatility is strictly greater than zero. Then, the mean stock price always exceeds the median stockprice. *True or false? Why?*

Solution: **TRUE**

Problem 11.2. (2 points) Let the stock price $S(t)$ be modeled using te lognormal distribution. Define $Y(t) = S(t)^3$. Then, the random variable $Y(t)$ is lognormal itself. *True or false? Why?*

Solution: TRUE

Problem 11.3. (2 pts) Let the stochastic process $S = \{S(t), t \geq 0\}$ represent the stock price as in the Black-Scholes model. Let its volatility term be denoted by σ. Then, the volatility parameter of the process $Y(t) = 2S(t)$ is 4σ. *True or false? Why?*

Solution: FALSE
The volatility parameter of the process Y is σ.

Problem 11.4. (2 points) Under the risk-neutral probability measure, every option on a particular stock has the continuously compounded, risk-free interest rate as its mean rate of return. *True or false? Why?*

Solution: TRUE

Problem 11.5. (8 points) A non-dividend-paying stock is valued at \$75.00 per share. The time$-t$ realized return is modeled as

$$R(0,t) \sim N(\text{mean} = 0.035t, \text{variance} = 0.09t)$$

Find the probability that the time-4 stock price exceeds today's stock price.

Solution: We need to find

$$\mathbb{P}[S(4) > S(0)]$$

with

$$S(t) = S(0)e^{R(0,t)}.$$

Since $R(0,t)$ follows the normal distribution with the above parameters, we have

$$\mathbb{P}[S(4) > S(0)] = \mathbb{P}[S(0)e^{R(0,4)} > S(0)] = \mathbb{P}[R(0,4) > 0] = 1 - N\left(-\frac{0.035 \times 4}{0.3 \times 2}\right) = N(0.23) = 0.591.$$

Problem 11.6. (10 points) Consider a non-dividend-paying stock whose current price is \$40 per share. The stock's volatility equals 0.20.

The continuously compounded, risk-free interest rate equals 7%.

Using the Black-Scholes pricing formula, calculate the price of a one-year, at-the-money European call option on the above stock.

Solution: The call option's price is

$$V_C(0) = S(0)[N(d_1) - e^{-rT}N(d_2)]$$

with

$$d_1 = \frac{\sqrt{T}}{\sigma}\left(r + \frac{\sigma^2}{2}\right) = \frac{1}{0.2}\left(0.07 + \frac{0.2^2}{2}\right) = 5(0.07 + 0.02) = 0.45,$$
$$d_2 = d_1 - \sigma\sqrt{T} = 0.45 - 0.20 = 0.25.$$

Using the standard normal tables, we get

$$N(d_1) = N(0.45) = 0.6736, \quad N(d_2) = N(0.25) = 0.5987.$$

Finally, the call option's price equals

$$V_C(0) = 40(0.6736 - e^{-0.07} \times 0.5987) = 4.615033.$$

Problem 11.7. (10 points) Assume the Black-Scholes setting. Today's price of a non-dividend paying stock is \$65, and its volatility is 0.20.

The continuously compounded, risk-free interest rate is 0.055.

What is the price of a three-month, \$60-strike European put option on the above stock?

Solution: In our usual notation, the price is

$$V_P(0) = Ke^{-r\cdot T}\,N(-d_2) - S(0)N(-d_1)$$

with

$$d_1 = \frac{1}{0.2\sqrt{1/4}}\left(\ln\left(\frac{65}{60}\right) + (0.055 + \frac{1}{2}\,0.2^2)\left(\frac{1}{4}\right)\right) = 10(\ln(65/60) + (0.075)(0.25)) = 0.99,$$
$$d_2 = d_1 - 0.2\sqrt{0.25} = 0.89.$$

So,

$$V_P(0) = 60e^{-0.055\cdot\frac{1}{4}}\,(1 - 0.8133) - 65\cdot(1 - 0.8389) = 0.5922.$$

Problem 11.8. (14 points) Let $S(0) = \$100, K = \$120, \sigma = 0.3$, and $r = 0.08$.

Let $V_C(0,T)$ denote the Black-Scholes European call price for the maturity T. Does the limit of $V_C(0,T)$ as $T \to \infty$ exist? If it does, what is it?

Solution: By the Black-Scholes pricing formula, the function $V_C(0,T)$ has the form

$$V_C(0,T) = S(0)N(d_1) - Ke^{-rT}N(d_2),$$

where N denotes the distribution function of the unit normal distribution and

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)T\right],$$
$$d_2 = d_1 - \sigma\sqrt{T}.$$

As $T \to \infty$, we have that

$$d_1 \to \infty \Rightarrow N(d_1) \to 1,$$
$$e^{-rT}N(d_2) \le e^{-rT} \to 0.$$

Hence,

$$V_C(0,T) \to S(0), \text{ as } T \to \infty.$$