UNIVERSITY OF TEXAS AT AUSTIN, DEPARTMENT OF MATHEMATICS
M358K - Applied Statistics

HOMEWORK #6

Problem 6.1. You are rolling a fair dodecahedral (20−sided) die with numbers $1 - 20$ noted its sides. You win if the number rolled is prime and you lose if it is not. What is the probability that you win at most 200 times in 500 rolls? Calculate your answer **both**:

(i) (4 points) using **R**, and
(ii) (6 points) using the normal approximation.

Solution: The prime numbers smaller than or equal to 20 are: $2, 3, 5, 7, 11, 13, 17, 19$. So, the probability of the number rolled being prime in a single roll is $8/20 = 2/5 = 0.4$. Hence, the number X of wins in 500 rolls has the following distribution:

$$X \sim Binomial(n = 500, p = 0.4)$$

In **R**, we use the command

```
pbinom(200, size=500, prob = 0.4)
```

to get 0.5194108.

The mean and the variance of X are

$$\mathbb{E}[X] = np = 500(0.4) = 200 \quad \text{and} \quad Var[X] = np(1-p) = 120.$$

So, the standard deviation of X is $SD[X] = 2\sqrt{30}$. Using the normal approximation to the binomial (with the continuity correction), we get

$$\mathbb{P}[X \leq 200] = \mathbb{P}[X < 200.5] = \mathbb{P}\left[\frac{X - 200}{2\sqrt{30}} < \frac{200.5 - 200}{2\sqrt{30}}\right] = \mathbb{P}[Z < 0.0456] = 0.5182$$

where $Z \sim N(0, 1)$, and I used **R** to obtain the final probability. The exact command is

```
pnorm(0.0456)
```

Problem 6.2. A biased coin (probability of *heads* is 0.7) is tossed 1000 times. Write down the exact expression for the probability that more than 750 *heads* have been observed.

(i) (4 points) Use **R** to calculate this probability.
(ii) (6 points) Use the normal approximation to approximate this probability.

Solution: The random variable X which equals to the number of heads is binomial with probability $p = 0.7$ and $n = 1000$. We are interested in the probability $\mathbb{P}[X > 750]$. If we split this probability among the elementary outcomes which are > 750, we get

$$\mathbb{P}[X > 750] = \sum_{i=751}^{1000} \mathbb{P}[X = i] = \sum_{i=751}^{1000} \binom{1000}{i} (0.7)^i (0.3)^{1000-i}.$$

In **R**, we use the command

```
1-pbinom(750, size=1000, prob = 0.7)
```

to get 0.0001985473.

According to the normal approximation to the binomial distribution, the random variable

$$X' = \frac{X - \mathbb{E}[X]}{\sqrt{\text{Var}[X]}} = \frac{X - 700}{\sqrt{1000 \cdot 0.7 \cdot 0.3}},$$

is approximately normally distributed with mean 0 and standard variation 1. Therefore (note the continuity correction),

$$\mathbb{P}[X > 750] = \mathbb{P}[X' \geq \frac{750.5 - 700}{\sqrt{210}}] \approx \mathbb{P}[Z \geq 3.48483],$$

where $Z \sim N(0, 1)$ is normally distributed with mean 0 and standard variation 1. Table look-up or your computer will give you that this probability is approximately equal to 0.0002. In fact, in **R**, the command

```
1-pnorm(3.48483)
```

gives us 0.0002462249.

Problem 6.3. $(2 + 2 + 4 + 1 + 1 = 10$ points) Solve **Exercise 6.8** from the textbook.

Problem 6.4. $(4 + 4 = 8$ points) Solve **Exercise 6.14** from the textbook.

Problem 6.5. (4 points) Solve **Exercise 6.16** from the textbook.

Problem 6.6. (4 points) In a random sample of 1000 small children, it was found that 880 of them observe Halloween. Find the 80%-confidence interval for the population proportion of childred who observe Halloween.

Solution: Let p denote the population parameter denoting the probability that a randomly chosen child from the population observes Halloween. The point estimate for p based on our data is $\hat{p} = \frac{880}{1000} = 0.88$.

The critical value z^* associated with the 80% confidence level is

$$z^* = \Phi^{-1}(0.90) = 1.28.$$

In our usual notation, the standard error is

$$\sqrt{\frac{\hat{p}(1 - \hat{p})}{n}} = \sqrt{\frac{0.88(0.12)}{1000}} = 0.01028.$$

Hence, the margin of error is $1.28(0.01028) = 0.01315$. The 80%-confidence interval is

$$p = 0.88 \pm 0.01315.$$

Problem 6.7. (4 points) Let p denote the population proportion. How large should the sample size be so that one is at least 95% confident that the true parameter p is within a 0.02 margin of error from the point estimate?

Solution: Let m denote the upper bound on the margin of error. Then,

$$n \geq \left(\frac{z^*}{2m}\right)^2 = \left(\frac{1.96}{2(0.02)}\right)^2 = (49)^2 = 2401.$$