M339 W: January 24^{th}, 2022.

The Strong Law of Large Numbers (SLLN).

Let $\{X_k, k=1,2,\dots\}$ be a sequence of independent, identically distributed random variables.

Assume: $\mu_X := \mathbb{E}[X_1] < \infty$.

Then,

$$\boxed{\frac{X_1 + X_2 + \dots + X_n}{n} \xrightarrow[n\to\infty]{} \mu_X}$$

If a function g is such that

$g(X_1)$ is well defined

and $\mathbb{E}[g(X_1)] < \infty$,

then,

$$\boxed{\frac{g(X_1) + g(X_2) + \dots + g(X_n)}{n} \xrightarrow[n\to\infty]{} \mathbb{E}[g(X_1)]}$$

Monte Carlo.

Recipe:
- Draw simulated values of a random variable from a particular distribution.
- Apply a function to the simulated values.
- Calculate the arithmetic average of the obtained quantities.

We get a value which is "close to" the theoretical expected value.

About precision:

$$Var\left[\frac{X_1 + X_2 + \dots + X_n}{n}\right] = \frac{1}{n^2} Var[X_1 + X_2 + \dots + X_n] \quad \text{independence}$$

$$= \frac{1}{n^2}\sum_{k=1}^{n} Var[X_k] = \frac{1}{n^{\cancel{2}}} \cdot \cancel{n} \cdot Var[X_1]$$

$$SD\left[\frac{X_1 + \cdots + X_n}{n}\right] = \frac{SD[X_1]}{\sqrt{n}}$$

To increase precision by a factor of η, we must increase the number of variates by a factor of η^2.

Risk-Neutral Pricing.

$v(\cdot)$... the value function of a (for simplicity) European option

$S(T)$... the time-T stock price

$V(T)$... payoff of the European option, i.e.,

$$\boxed{V(T) = v(S(T))}$$

$\mathbb{P}^*$... the risk-neutral probability measure

$$\boxed{\underbrace{V(0)}_{\text{time-0 price of option}} = e^{-rT}\mathbb{E}^*[V(T)]}$$

Monte Carlo Pricing.

Recipe:

- From the risk-neutral probability dist'n, simulate the stock-price paths.
- Apply the payoff function to the simulated stock-price paths.

  Get: The simulated values of the payoff. Call them:

  $$v_1, v_2, \dots, v_n$$

- Calculate the arithmetic average:

  $$\bar{v} = \frac{v_1 + v_2 + \cdots + v_n}{n}$$

  Note: "Close to" the expected risk-neutral payoff.

- Finally, $\boxed{e^{-rT} \cdot \bar{v}}$ is the Monte Carlo price.