M339D: October 17^{th}, 2025.

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Problem Set #9

Binomial option pricing.

Problem 9.1. In the setting of the one-period binomial model, denote by i the **effective** interest rate **per period**. Let u denote the "up factor" and let d denote the "down factor" in the stock-price model. If

$$d < u \leq 1 + i$$

then there certainly is no possibility for arbitrage.

T/F?

No-arbitrage Condition

$$d < e^{rh} < u$$

$$\boxed{d < 1+i < u}$$

This is the fixed statement!

Problem 9.2. In our usual notation, does this parameter choice create a binomial model with an arbitrage opportunity?

$u = 1.18, \quad d = 0.87, \quad r = 0.05, \quad \delta \cancel{=} 0, \quad h = 1/4$

No, it doesn't

→:

Check:

$$d = 0.87 < e^{rh} = \underbrace{e^{0.05(1/4)}}_{e^{0.0125}} < u = 1.18$$

?? 1.0125

Taylor Expansion of e^x:

$$\boxed{e^x = 1 + x} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

Q: What if $\tilde{d} = 1.01$?

This is still OK!

Binomial Option Pricing [cont'd].

		PAYOFF		REPLICATING PORTFOLIO
$S(0)$, $V(0)=?$	$S_u = u\cdot S(0)$	$V_u := v(S_u)$	$=$	$\Delta\cdot S_u + Be^{rh}$
	$S_d = d\cdot S(0)$	$V_d := v(S_d)$	$=$	$\Delta\cdot S_d + Be^{rh}$

population $\rightarrow$

$\leftarrow$ pricing

Goal: Pricing a European option w/ the exercise date @ the end of the tree and w/ payoff function $v(\cdot)$

$\Rightarrow$ Payoff of option: $\boxed{V(T) = v(S(T))}$

In the binomial model, any derivative security can be REPLICATED w/ a portfolio of this form:

@ time 0

- Δ shares of stock
  - $\Delta > 0$ buying
  - $\Delta = 0$ "nothing"
  - $\Delta < 0$ short-selling

and

- B @ ccrfir
  - $B > 0$ lending (buying a bond)
  - $B = 0$ "nothing"
  - $B < 0$ borrowing (issuing a bond)

If we can calculate Δ and B, then $\boxed{V(0) = \Delta\cdot S(0) + B}$

We get a system of two eq'ns w/ two unknowns:

$$\left.\begin{aligned} \Delta\cdot S_u + \cancel{Be^{rh}} &= V_u \\ -\ \Delta\cdot S_d + \cancel{Be^{rh}} &= -\ V_d \end{aligned}\right\} -$$

$$\Delta\,(S_u - S_d) = V_u - V_d$$

$$\boxed{\Delta = \frac{V_u - V_d}{S_u - S_d}}$$

unitless/shares of stock

$$\frac{V_u - V_d}{S_u - S_d} \cdot \underbrace{S_u} + Be^{rh} = V_u$$

$$Be^{rh} = V_u - \frac{V_u - V_d}{\cancel{S(0)}(u-d)} \cdot u \cdot \cancel{S(0)} = \frac{\cancel{u \cdot V_u} - d \cdot V_u - \cancel{u \cdot V_u} + u \cdot V_d}{u-d}$$

$$\boxed{B = e^{-rh} \cdot \frac{u \cdot V_d - d \cdot V_u}{u-d}}$$

cash($)

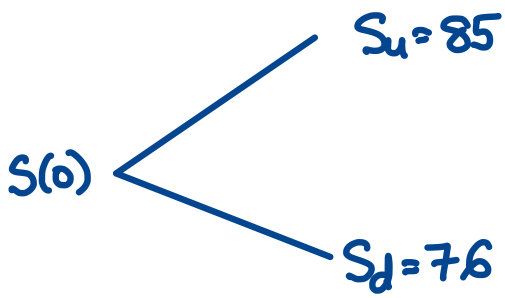
Problem 9.3. Let the continuously compounded risk-free interest rate be equal to 0.04. Consider a one-period binomial tree with every period of length one year used to model the stock price of a stock whose current price is \$80 per share. In the model, it is assumed that the stock price can either go up by \$5 or down by \$4.

You use the binomial tree to construct a replicating portfolio for a 78-strike call option on the above stock. What is the stock investment in the replicating portfolio?

→ :

$$\Delta = \frac{V_u - V_d}{S_u - S_d} = \frac{7}{85-76} = \frac{7}{9}$$

$\square$



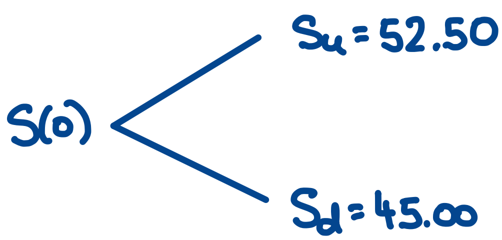
Payoff

$V_u = (85-78)_+ = 7$

$V_d = (76-78)_+ = 0$

Problem 9.4. Let the continuously compounded risk-free interest rate be equal to 0.04. Consider a one-period binomial tree with every period of length one year used to model the stock price of a non-dividend-paying stock whose current price is \$50 per share. In the model, it is assumed that the stock price can either go up by 5% or down by 10%.

You use the binomial tree to construct a replicating portfolio for a 45−strike call on the above stock. What is the risk-free investment in the replicating portfolio?



PAYOFF:

$V_u = (52.5 - 45)_+ = 7.5$

$V_d = 0$

$$B = e^{-rh} \cdot \frac{u \cdot V_d - d \cdot V_u}{u - d}$$

$$B = e^{-0.04} \cdot \frac{1.05 \cdot (0) - 0.9 \cdot 7.5}{1.05 - 0.90}$$

$$B = - e^{-0.04} \cdot \frac{6.75}{0.15} = \dots = \underline{-43.24}$$

$\square$