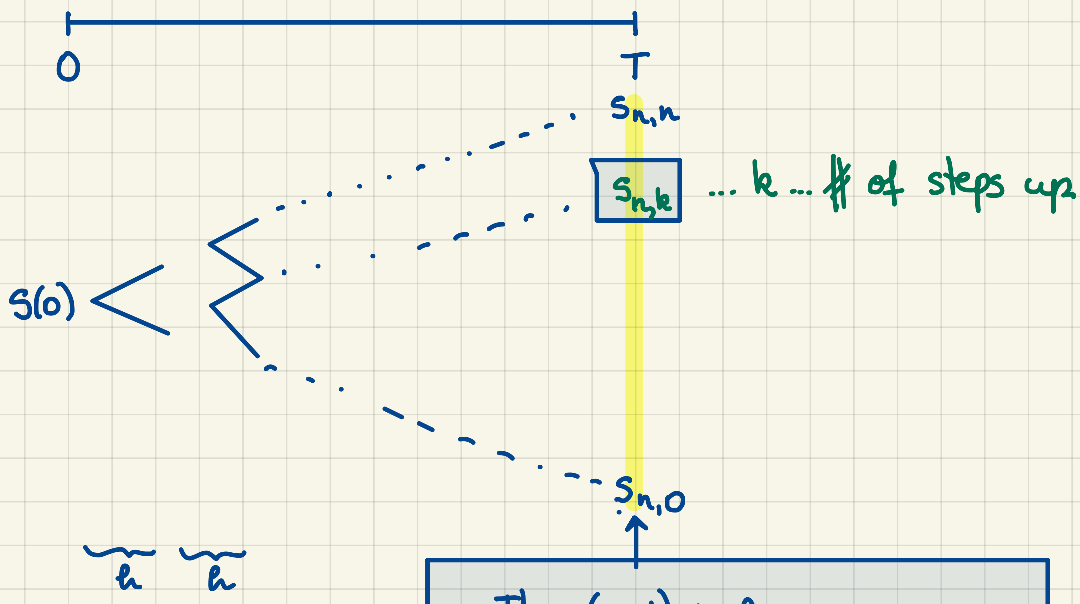
M339D : November 2^{nd}, 2022.

Multiple Binomial Periods.

T... exercise date of a European option
n... # of periods

the length of each period $\boxed{h = \frac{T}{n}}$



The $(n+1)$ values are the support of the r.v. $S(T)$.

$\Rightarrow$ for every $k = 0, 1, \dots, n$,

# of upsteps

$$S_{n,k} = S(0)\, u^k \cdot d^{n-k} = S(0)\left(\frac{u}{d}\right)^k \cdot d^n$$

Consider a European option w/ payoff f'tion $v(\cdot)$.
Then, the possible payoff values will be

$$\boxed{v_{n,k} = v(S_{n,k})}$$

p^*... the risk-neutral probability of an upstep,
i.e.,

$$\boxed{p^* = \frac{e^{rh} - d}{u - d}}$$

⇒ The risk-neutral probability of attaining the payoff $v_{n,k}$ is:

$$\binom{n}{k}(p^*)^k(1-p^*)^{n-k}$$

The risk-neutral option price:

$$\boxed{\begin{aligned} V(0) &= e^{-rT}\,\mathbb{E}^*[V(T)] \\ &= e^{-rT}\sum_{k=0}^{n}\left(\left(\binom{n}{k}(p^*)^k(1-p^*)^{n-k}\right)\cdot v_{n,k}\right) \end{aligned}}$$

Problem 7.2. Let the continuously compounded risk-free interest rate be 0.10. Let the initial price of a non-dividend-paying stock be \$100 per share. You use a five-period binomial tree to model the stock price over the next year. Let $u = 1.04$ and $d = 0.96$.

What is the price of a one-year, at-the-money European call option on the above stock?

$n=5$

$K=100$

$T=1$ $\longrightarrow$:

$h=\frac{1}{5}$

Risk-neutral Probability:

$$p^* = \frac{e^{rh}-d}{u-d} = \frac{e^{0.10(1/5)}-0.96}{1.04-0.96} = \underline{0.7527}$$

The relevant final stock prices in our tree are:

$$S_{5,5} = S(0)\cdot u^5 = 100(1.04)^5 = \underline{121.66} \Rightarrow \underline{v_{5,5} = 21.66}$$

$$S_{5,4} = S(0)\cdot u^4 \cdot d = 100(1.04)^4(0.96) = 112.31 \Rightarrow \underline{v_{5,4} = 12.31}$$

$$S_{5,3} = S(0)\cdot u^3 \cdot d^2 = 100(1.04)^3(0.96)^2 = 103.66 \Rightarrow \underline{v_{5,3} = 3.66}$$

the remaining terminal nodes are all out-o-money.

$\Rightarrow$

$$V_c(0) = e^{-0.10}\Big(21.66\cdot(p^*)^5 + 12.31\cdot 5\cdot (p^*)^4(1-p^*) + 3.66\cdot 10\,(p^*)^3(1-p^*)^2\Big) = \underline{10.01}$$

$\square$