M339 J: March 3rd, 2021.

HW Problem 2.8.

Density:

$$f_X(x) = \begin{cases} 0.01 & 0 < x < 80 \\ 0.03 - 0.00025x & 80 < x < 120 \end{cases}$$

$$\mathbb{E}[\underbrace{X \wedge d}_{= g(X)}] = ? \qquad d = 20.$$

$g(X)$ w/ $g(x) = x \wedge d$

In general: $$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) \underbrace{f_X(x)}_{pdf} \, dx$$

In this problem:

$$\mathbb{E}[g(X)] = \int_0^{120} (x \wedge 20) f_X(x) \, dx$$

$$= \int_0^{20} x f_X(x) \, dx + \underbrace{\int_{20}^{120} 20 f_X(x) \, dx}$$

$$20 \int_{20}^{120} f_X(x) \, dx = \mathbb{P}[X > 20] = S_X(20) = \boxed{1 - F_X(20)} = 1 - \int_0^{20} \underbrace{0.01}_{f_X(x)} \, dx = 1 - 0.2 = 0.8 \checkmark$$

$$\int_0^{20} x (0.01) \, dx = 0.01 \cdot \frac{x^2}{2} \Big|_{x=0}^{20} = 0.01 \cdot \frac{400}{2} = 0.01 \cdot 200 = 2$$

$$\Rightarrow \mathbb{E}[X \wedge 20] = 2 + 20 \cdot 0.8 = 18 \quad \blacksquare$$

"Def'n". A **parametric distribution** is a set of distribution functions each of which is fully specified via a **finite & fixed** number of parameters.

Challenge: Try to figure out an example of something non-parametric.

"Def'n". A parametric distribution is a **scale distribution** if, when one of the random variables from its set of distributions is multiplied by a POSITIVE constant, the new random variable is then the same set of distributions.

Transformation I. [Multiplying by a constant]

Say that X is a continuous r.v. w/ a pdf f_X.

Let κ be a constant. Define $\boxed{\tilde{X} := \kappa \cdot X}$ ✓

Q: What is the pdf of $\tilde{X}$? Does it exist, even?

→: $\kappa \neq 0$ If it's $\kappa = 0$, we get a degenerate $\tilde{X}$.

Let's figure out the cdf of $\tilde{X}$.

For all $x \in \mathbb{R}$:

$$F_{\tilde{X}}(x) = \mathbb{P}[\tilde{X} \leq x] = \mathbb{P}[\kappa \cdot X \leq x]$$

Case #1. $\kappa > 0$.

$$F_{\tilde{X}}(x) = \mathbb{P}\left[X \leq \frac{x}{\kappa}\right] = F_X\left(\frac{x}{\kappa}\right)$$

$$\Rightarrow \boxed{f_{\tilde{X}}(x) = \frac{1}{\kappa} \cdot f_X\left(\frac{x}{\kappa}\right)}$$

Case #2. $\kappa < 0$

$$F_{\tilde{X}}(x) = \mathbb{P}\left[X \geq \frac{x}{\kappa}\right] = 1 - F_X\left(\frac{x}{\kappa}\right)$$

$$\Rightarrow f_{\tilde{X}}(x) = -\frac{1}{\kappa} f_X\left(\frac{x}{\kappa}\right).$$

Example. $X \sim \text{Exponential}(\text{mean} = \theta)$

$\tilde{X} := \kappa \cdot X$ for some $\kappa > 0$.

Q: What's the dist'n of $\tilde{X}$?

$\rightarrow$:

$$f_{\tilde{X}}(x) = \frac{1}{\kappa} \cdot f_X\left(\frac{x}{\kappa}\right) \quad \text{for } x > 0$$

$$f_{\tilde{X}}(x) = \frac{1}{\kappa} \cdot \frac{1}{\theta} \cdot e^{-\frac{\frac{x}{\kappa}}{\theta}} = \frac{1}{\kappa\theta} \cdot e^{-\frac{x}{\kappa\theta}}$$

$$\Rightarrow \boxed{\tilde{X} \sim \text{Exponential}(\text{mean} = \kappa \cdot \theta)}$$

"Def'n". Let X be a random variable w/ a nonnegative support which has a scale distribution.

If a parameter of that scale dist'n satisfies:

1. When a member of that scale dist'n is multiplied by a positive constant, that parameter is multiplied by the same constant.
2. All the other parameters remain the same,

then that parameter is called a **scale parameter**.

Note: The Exponential dist'n has the scale parameter θ.

Example. $X \sim \text{Gamma}(\alpha, \theta)$

Then, the pdf of X is of the form

$$f_X(x) = \frac{\left(\frac{x}{\theta}\right)^{\alpha} \cdot e^{-x/\theta}}{x \cdot \Gamma(\alpha)}$$

Let $\kappa > 0$. Set $\tilde{X} := \kappa \cdot X$

$$\Rightarrow f_{\tilde{X}}(x) = \frac{1}{\kappa} \cdot f_X\left(\frac{x}{\kappa}\right) = \frac{1}{\cancel{\kappa}} \cdot \frac{\left(\frac{\frac{x}{\kappa}}{\theta}\right)^{\alpha} \cdot e^{-\frac{\frac{x}{\kappa}}{\theta}}}{\frac{x}{\cancel{\kappa}} \cdot \Gamma(\alpha)}$$

$$\Rightarrow f_{\tilde{X}}(x) = \frac{\left(\frac{x}{\kappa \cdot \theta}\right)^{\alpha} \cdot e^{-\frac{x}{\kappa \cdot \theta}}}{x \cdot \Gamma(\alpha)} \Rightarrow \tilde{X} \sim \text{Gamma}(\alpha, \kappa \cdot \theta)$$