M339W: February 8th, 2021.

Realized Returns [cont'd].

With an agent's subjective probabilistic model for the return of a stock (or the stock price), we can do the following.

Temporarily fix a time $\cdot T$ (of some importance; you want to assess your wealth @ that time).

Say, you invest in one share of a continuous paying stock @ time $\cdot 0$ w/ δ dividend yield.

Let $S(t), t \geq 0$ denote the time $\cdot t$ stock price.

In particular, $S(T)$ denotes the stock price @ the end of your time horizon. The probabilistic model will be a distribution of this random variable. In particular, the mean time $\cdot T$ stock price is

$$\mathbb{E}[S(T)].$$

Q: What is your wealth @ time $\cdot T$?

$\longrightarrow$: $e^{\delta \cdot T} \cdot S(T)$

$\blacksquare$

$\Rightarrow$ Your expected wealth will be

$$\boxed{e^{\delta \cdot T} \cdot \mathbb{E}[S(T)]}$$

Def'n. We define the **mean rate of return**, usually denoted by α, as the constant satisfying:

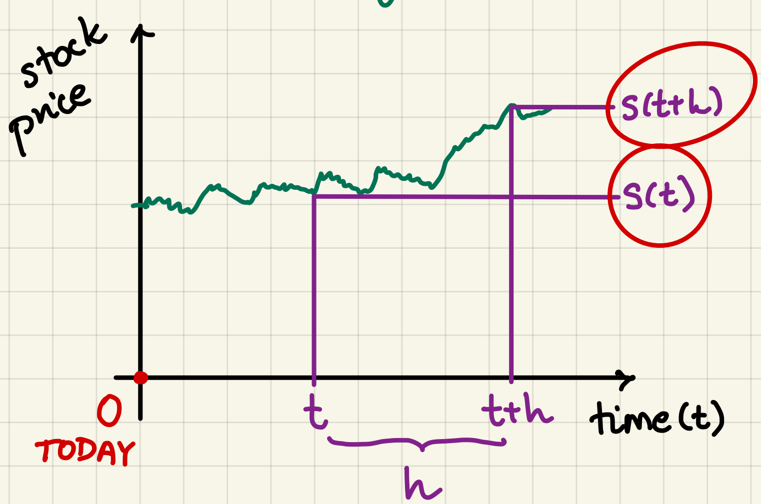
$$\boxed{S(0)e^{\alpha \cdot T} = e^{\delta \cdot T} \cdot \mathbb{E}[S(T)]}$$

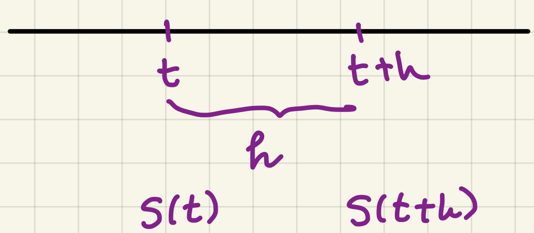
Note:
- We assume a constant α independent of the time horizon T.
- $$\mathbb{E}[S(T)] = S(0)e^{(\alpha-\delta)\cdot T}$$

  $(\alpha-\delta)$: mean rate appreciation.

α — mean rate of return

$$\alpha = \underbrace{\delta}_{\text{dividend yied}} + \underbrace{(\alpha-\delta)}_{\text{mean rate of appreciation}}$$





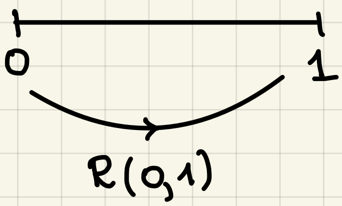
Def'n. For every $t, h > 0$, we define the realized return as $R(t, t+h)$ which satisfies

$$S(t+h) = S(t)\, e^{R(t,t+h)}$$

$$\iff \quad R(t, t+h) = \ln\left(\frac{S(t+h)}{S(t)}\right)$$

Note: **Recall:** the standard deviation of realized returns over any time period of leghth one year was called the $\boxed{\text{volatility}}$; it's usually denoted by σ

$\implies$ We should have:



$$\mathrm{Var}[R(0,1)] = \sigma^2 \qquad SD[R(0,1)] = \sigma$$

Our model requirements:

- We want our model for $R(t, t+h)$, $t, h > 0$, to inherit the nice properties we had for the returns in the binomial tree.
- We want to be able to interpret it as a limiting model for the binomial tree.
- We want its parametrization to be interpretable in terms of α and σ.

Think about: Which probabilistic model would you suggest for $R(t, t+h)$, $t, h > 0$?