UNIVERSITY OF TEXAS AT AUSTIN

HW Assignment 2

Prerequisite material.

Problem 2.1. (10 points) Your goal is to price a call option on a futures contract. The movements of the futures price are modeled by a binomial tree. You are given:

(i) Each period is three months.
(ii) $u_F/d_F = 5/4$, where u_F is one plus the rate of gain on the futures price if it goes up, and d_F is one plus the rate of loss if it goes down.
(iii) The risk-neutral probability of an up move is $1/2$.
(iv) The initial futures price is 80.
(v) The continuously compounded risk-free interest rate is 5%.

Find the price of a half-year, 85-strike European call option on the futures contract.

Solution: We are given that

$$\frac{1}{2} = \frac{1-d_F}{u_F-d_F} = \frac{d_F^{-1}-1}{\frac{u_F}{d_F}-1} = \frac{d_F^{-1}-1}{\frac{5}{4}-1} \quad \Rightarrow \quad d_F^{-1} = \frac{9}{8} \quad \Rightarrow \quad d_F = \frac{8}{9} \quad \Rightarrow \quad u_F = \frac{10}{9}.$$

The prices in the futures-price tree are, thus,

$$\begin{array}{ccc} & & F_{uu} = 98.77 \\ & F_u = 88.89 & \\ F_0 = 80 & & F_{ud} = 79.01 \\ & F_d < 85 & \\ & & F_{dd} < 85 \end{array}$$

The option's price is

$$V_C(0) = e^{-0.025} \times \frac{1}{4} \times (98.77 - 85) = 3.3575.$$

Problem 2.2. (5 points) You are required to price a one-year, yen-denominated currency option on the USD. The exchange rate over the next year is modeled using a forward binomial tree with the number of periods equal to 4. Assume that the volatility of the exchange rate equals 0.1.

The continuously compounded risk-free interest rate for the yen equals 0.05, while the continuously compounded risk-free interest rate for the USD equals 0.02. What is the value of the so-called up factor u in the resulting forward binomial tree?

Solution:

$$u = e^{(r_{yen}-r_{\$})h+\sigma\sqrt{h}} = e^{(0.05-0.02)\frac{1}{4}+0.1\sqrt{\frac{1}{4}}} = 1.0592.$$

Problem 2.3. (5 points) The evolution of a market index over the following year is modeled using a four-period binomial tree. We are given that the current value of the market index equals \$144, that its volatility equals 0.25, and that it pays dividends continuously.

You are tasked with constructing a four-period forward tree for the evolution over the following year of the forward price of the above market index with delivery at time-2.

What is the **down** factor d_F in the **forward price tree** for the **futures prices** on the stock?

Solution: In our usual notation,

$$d_F = de^{-(r-\delta)h} = e^{-\sigma\sqrt{h}} = e^{-0.25\sqrt{1/4}} = 0.8825.$$

Problem 2.4. (10 points) The evolution over the following year of futures prices with delivery at time 2 on a certain commodity are modeled using a one-period forward binomial tree. The volatility is given by 0.2. The continuously compounded risk-free interest rate is given to be 0.05.

Let the current futures price equal \$50. What is the price of a one-year, \$45-strike European put on the futures contract described above?

Solution: The up and down factors in the futures tree equal

$$u_F = e^{\sigma\sqrt{h}} = e^{0.2} = 1.2214, \quad d_F = e^{-\sigma\sqrt{h}} = 0.8187.$$

The futures price at the up node is

$$50 \times 1.2214 = 61.07.$$

The futures price at the down node is

$$50/1.2214 = 40.94.$$

So, the put gets exercised in the down node with the risk neutral probability of

$$1 - p^* = \frac{u_F - 1}{u_F - d_F} = 0.55.$$

So, the futures put option price is

$$V(0) = e^{-0.05} \times 0.55 \times (45 - 40.94) = 2.12345.$$

Problem 2.5. (10 points) Consider a continuous-dividend-paying stock whose current price is \$100. The stock's dividend yield is 0.02 and its volatility is 0.25. The evolution of the price of this stock over the next half-year is modeled using a two-period forward binomial tree.

Let the continuously-compounded, risk-free interest rate be 0.04.

What is the price of a \$95-strike, half-year **American** put option on the above stock?

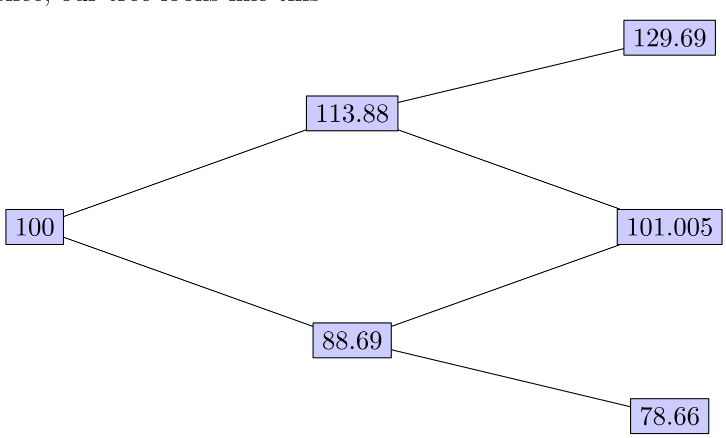
Solution: Using our "shortcut" formula for the forward binomial tree, we get that the risk-neutral probability of an up movement equals, in our usual notation,

$$p^* = \frac{1}{1 + e^{\sigma\sqrt{h}}} = \frac{1}{1 + e^{0.25\sqrt{1/4}}} = \frac{1}{1 + e^{0.125}} = 0.4688.$$

The *up* factor and the *down* factor are

$$u = e^{(r-\delta)h + \sigma\sqrt{h}} = e^{(0.04-0.02)(0.25)+0.25\sqrt{0.25}} = e^{0.13} = 1.1388,$$
$$d = e^{(r-\delta)h - \sigma\sqrt{h}} = e^{(0.04-0.02)(0.25)-0.25\sqrt{0.25}} = e^{-0.12} = 0.8869.$$

Hence, our tree looks like this



I populated the entire tree just for laughs, but I did not need to since the put is in-the-money only in the "lower" part of the tree. In the "leaves" of the tree, i.e., on the expiration date, the put option is in-the-money only in the lowest node. There, the payoff is $V_{dd} = 16.3372$.

At the *up* node the put is out-of-the-money and its continuation value is obviously equal to zero. Therefore, the American put option has the value zero, as well, i.e., $V_u^A = 0$.

At the *down* node, the continuation value of the American put is

$$CV_d = e^{-0.04(0.25)}(1-p^*)V_{dd} = 8.5921.$$

The immediate exercise value is $IE_d = 95 - S_d = 6.30796$. Thus, it's optimal to hold onto the option.

So, the price of our put is

$$V_P^A(0) = e^{-0.01}(1-p^*)V_d^A = 4.51880.$$

Problem 2.6. (10 points) Today's price of a continuous-dividend-paying stock is observed to be \$80. Its volatility is 0.2 and its dividend yield is 0.01. The evolution of this stock price over the following year is modelled using a three-period binomial tree such that the stock price can either go up by 2% or down by 1% at the end of every period. The continuously-compounded, risk-free interest rate is 0.04.

What is the price of an \$82-strike European put option on the above stock?

Solution: The up factor is given to be $u = 1.02$ while the down factor equals $d = 0.99$. The possible stock prices at the end of the year are

$$S_{uuu} = S(0)u^3 = 84.8966, \quad S_{uud} = S(0)u^2d = 82.3997,$$
$$S_{udd} = S(0)ud^2 = 79.9762, \quad \text{and} \quad S_{ddd} = S(0)d^3 = 77.6239.$$

Therefore, the possible payoffs of our European put are

$$V_{uuu} = V_{uud} = 0, \quad V_{udd} = 82 - 79.9762 = 2.0238, \quad \text{and} \quad V_{ddd} = 82 - 77.6239 = 4.37608.$$

The risk-neutral probability of the stock price going up in a single period is

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{e^{(0.04-0.01)(1/3)} - 0.99}{1.02 - 0.99} = 0.668339.$$

Hence, the put's price equals

$$V_P(0) = e^{-0.04}[2.0238(3)(0.668339)(1-0.668339)^2 + 4.37608(1-0.668339)^3] = 0.5822372$$