M339W: March 24th, 2021.

Black-Scholes Pricing.

In general, under a probability measure $\mathbb{P}$:

$$S(T) = S(0)e^{(\alpha - \delta - \frac{\sigma^2}{2})\cdot T + \sigma\sqrt{T}\cdot Z} \quad w/ \quad Z \sim N(0,1)$$

where α ... mean rate of return

In particular, under the risk-neutral measure $\mathbb{P}^*$:

$$S(T) = S(0)e^{(r - \delta - \frac{\sigma^2}{2})\cdot T + \sigma\sqrt{T}\cdot Z} \quad w/ \quad Z \sim N(0,1)$$

Under a probab. measure $\mathbb{P}$:

$$\mathbb{E}[V_C(T)] = S(0)e^{(\alpha-\delta)\cdot T}\cdot N(\hat{d}_1) - K\cdot N(\hat{d}_2)$$

w/ $\hat{d}_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + (\alpha - \delta + \frac{\sigma^2}{2})\cdot T\right]$

and $\hat{d}_2 = \hat{d}_1 - \sigma\sqrt{T}$.

Remember the risk-neutral pricing principle:

$$\boxed{V(0) := e^{-rT}\mathbb{E}^*[\underbrace{V(T)}_{\text{payoff of a European option}}]}$$

$\Rightarrow$ We get the Black-Scholes price of the European call as:

$$V_C(0) = e^{-rT}\cdot\mathbb{E}^*[V_C(T)] =$$
$$= \cancel{e^{-rT}}\left(S(0)e^{(\cancel{r}-\delta)\cdot T}\cdot N(d_1) - KN(d_2)\right)$$

$$\Rightarrow \boxed{\begin{aligned} V_c(0) &= S(0)e^{-\delta\cdot T}\cdot N(d_1) - Ke^{-rT}\cdot N(d_2) \\ \text{w/ } d_1 &= \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(r-\delta+\frac{\sigma^2}{2}\right)\cdot T\right] \\ \text{and } d_2 &= d_1 - \sigma\sqrt{T} \end{aligned}}$$

Q: What do we do to price the put?

$\rightarrow$: Put · Call Parity:

$$V_c(0) - V_p(0) = F^P_{0,T}(S) - PV_{0,T}(K)$$

$$\begin{aligned} V_p(0) &= -S(0)e^{-\delta T} + Ke^{-rT} \\ &\quad + S(0)e^{-\delta T}N(d_1) - Ke^{-rT}\cdot N(d_2) \\ &= S(0)e^{-\delta T}\big(\underbrace{-1+N(d_1)}_{-N(-d_1)}\big) + Ke^{-rT}\big(\underbrace{1-N(d_2)}_{N(-d_2)}\big) \end{aligned}$$

$$\Rightarrow \boxed{V_p(0) = Ke^{-rT}\cdot N(-d_2) - S(0)e^{-\delta T}N(-d_1)}$$

Sample IFM: Part I: Advanced

6. You are considering the purchase of 100 units of a 3-month 25-strike European call option on a stock. $\boxed{T = 1/4}$ $\boxed{K = 25}$

You are given:

(i) The Black-Scholes framework holds.
(ii) The stock is currently selling for 20. $\boxed{S(0) = 20}$
(iii) The stock's volatility is 24%. $\boxed{\sigma = 0.24}$
(iv) The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 3%. $\boxed{\delta = 0.03}$
(v) The continuously compounded risk-free interest rate is 5%. $\boxed{r = 0.05}$

Calculate the price of the block of 100 options.

The Usual Steps:

(A) 0.04

(B) 1.93

(C) 3.63

(D) 4.22

(E) 5.09

1st Get d_1 and d_2

2nd Use the standard normal tables, or the Prometric calculator, or R to get $N(d_1)$ and $N(d_2)$

3rd Calculate $V_c(0)$ = B·S Price

7. Company A is a U.S. international company, and Company B is a Japanese local company. Company A is negotiating with Company B to sell its operation in Tokyo to Company B. The deal will be settled in Japanese yen. To avoid a loss at the time when the deal is closed due to a sudden devaluation of yen relative to dollar, Company A has decided to buy at-the-money dollar-denominated yen put of the European type to hedge this risk.

You are given the following information:

(i) The deal will be closed 3 months from now.

(ii) The sale price of the Tokyo operation has been settled at 120 billion Japanese yen.

(iii) The continuously compounded risk-free interest rate in the U.S. is 3.5%.

(iv) The continuously compounded risk-free interest rate in Japan is 1.5%.

(v) The current exchange rate is 1 U.S. dollar = 120 Japanese yen.

(vi) The daily volatility of the yen per dollar exchange rate is 0.261712%.

(vii) 1 year = 365 days; 3 months = ¼ year.

Calculate Company A's option cost.

$\longrightarrow$: $1^{\underline{st}}$

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right)\cdot T\right]$$

$$d_1 = \frac{1}{0.24\sqrt{1/4}}\left[\ln\left(\frac{20}{25}\right) + \left(0.05 - 0.03 + \frac{(0.24)^2}{2}\right)\cdot \frac{1}{4}\right]$$

$$\Rightarrow \quad d_1 = -1.7579$$

$$\Rightarrow \quad d_2 = d_1 - \sigma\sqrt{T} = -1.7579 - 0.24\left(\tfrac{1}{2}\right) = -1.8779$$

$2^{\underline{nd}}$

$$N(d_1) = 0.03938$$

$$N(d_2) = 0.030197$$

$3^{\underline{rd}}$

$$\begin{aligned} V_c(0) &= S(0)e^{-\delta T}\cdot N(d_1) - Ke^{-rT}\cdot N(d_2) \\ &= 20e^{-0.03(1/4)}\cdot 0.03938 - 25e^{-0.05(1/4)}\cdot 0.030197 \\ &= 0.03617 \end{aligned}$$

$\Rightarrow$ Our answer: $100\cdot V_c(0) = 3.617$

At home: use the std normal tables: answer $= 3.499$?

MFE: Spring 2007.

$\delta = 0$

8. Let $S(t)$ denote the price at time t of a stock that pays no dividends. The Black-Scholes framework holds. Consider a European call option with exercise date T, $T > 0$, and exercise price $\boxed{S(0)e^{rT}}$, where r is the continuously compounded risk-free interest rate.

$\overset{\shortparallel}{K}$ ✓

You are given:

(i) $S(0) = \$100$

(ii) $T = 10$

(iii) $\mathrm{Var}[\ln S(t)] = 0.4t, \quad t > 0.$

Determine the price of the call option.

(A) \$7.96

(B) \$24.82

(C) \$68.26

(D) \$95.44

(E) There is not enough information to solve the problem.

1st

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{\cancel{S(0)}}{\cancel{S(0)}e^{rT}}\right) + \left(r + \frac{\sigma^2}{2}\right)\cdot T\right]$$

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\cancel{-rT} + \cancel{rT} + \frac{\sigma^2 \cdot T}{2}\right]$$

$$\boxed{d_1 = \frac{\sigma\sqrt{T}}{2}} \Rightarrow d_2 = d_1 - \sigma\sqrt{T} = \frac{\sigma\sqrt{T}}{2} - \sigma\sqrt{T}$$

$$\Rightarrow \boxed{d_2 = -\frac{\sigma\sqrt{T}}{2}}$$

2nd

$$N(d_1) = N\left(\frac{\sigma\sqrt{T}}{2}\right) \quad \text{and} \quad N(d_2) = N(-d_1) = 1 - N(d_1)$$

3rd

$$V_c(0) = S(0)N(d_1) - S(0)\cancel{e^{r\cdot T}}\cdot\cancel{e^{-r\cdot T}}\cdot N(d_2)$$

$$V_c(0) = S(0)N(d_1) - S(0)(1-N(d_1))$$

$$\boxed{V_c(0) = S(0)\left(2\cdot N(d_1) - 1\right)}$$

Given : $\mathrm{Var}[\ln(S(t))] = 0.4\cdot t$

$$\ln(S(t)) = \ln(S(0)) + \underbrace{\left(r - \frac{\sigma^2}{2}\right)\cdot T + \sigma\sqrt{T}\cdot Z}_{R(0,t)} \quad Z\sim N(0,1)$$

In general: $\mathrm{Var}[\ln(S(t))] = \mathrm{Var}[\sigma\sqrt{T}\cdot Z] = \sigma^2\cdot T$

$\Rightarrow$ In our problem : $\sigma^2 = 0.4$

$$\Rightarrow \quad d_1 = \frac{\sigma\sqrt{T}}{2} = \frac{\sqrt{0.4}\sqrt{10}}{2} = \frac{\sqrt{4}}{2} = 1$$

$$\Rightarrow \quad V_c(0) = 100\left(2\cdot N(1) - 1\right) = \text{consult tables @ home}$$

$= 68.26$ ■

3. You are asked to determine the price of a European put option on a stock. Assuming the Black-Scholes framework holds, you are given:

(i) The stock price is \$100.

(ii) The put option will expire in 6 months.

(iii) The strike price is \$98.

(iv) The continuously compounded risk-free interest rate is $r = 0.055$.

(v) $\delta = 0.01$

(vi) $\sigma = 0.50$

Calculate the price of this put option.

(A) \$3.50

(B) \$8.60

(C) \$11.90

(D) \$16.00

(E) \$20.40