# M378K: September 22nd, 2025.

## More on the Cumulative Distribution Function.

**Example.** The Normal Distribution.

$Y \sim N(\mu, \sigma)$

The pdf is: $f_Y(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-\mu)^2}{2\sigma^2}}$ for all $y \in \mathbb{R}$

The cdf is:
$$F_Y(y) = \int_{-\infty}^{y} f_Y(u)\,du$$
$$= \int_{-\infty}^{y} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(u-\mu)^2}{2\sigma^2}}\,du$$

No ANALYTIC CDF!

**Fact.** $\frac{Y-\mu_Y}{\sigma_Y} \sim N(0,1) \sim Z$

$$Y = \mu_Y + \sigma_Y \cdot Z$$

$$F_Y(y) = \mathbb{P}[Y \leq y] = \mathbb{P}\Big[\overbrace{\frac{Y-\mu_Y}{\sigma_Y}}^{Z} \leq \frac{y-\mu_Y}{\sigma_Y}\Big]$$
$$= \mathbb{P}\Big[Z \leq \frac{y-\mu_Y}{\sigma_Y}\Big] = \Phi\left(\frac{y-\mu_Y}{\sigma_Y}\right)$$

$$\boxed{\text{CDF of } N(0,1) \ldots \Phi}$$

**Def'n.** Let $Y$ be a random variable w/ the cdf $F_Y$. For $\alpha \in (0,1)$ the $\alpha$-quantile of the dist'n of the r.v. $Y$ is defined as the number

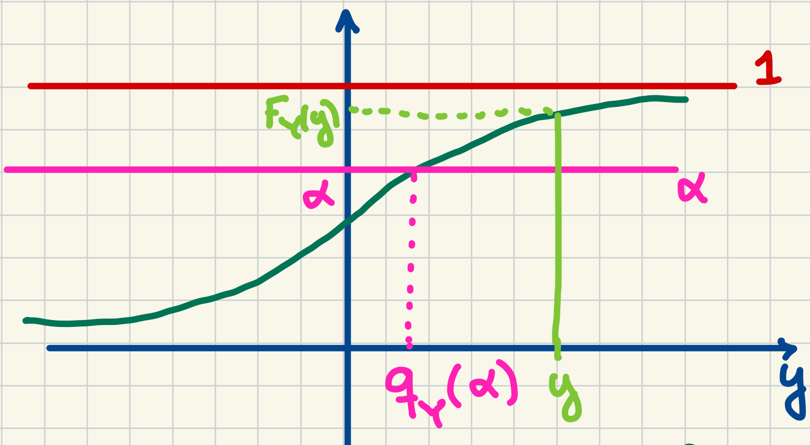
$$q_Y(\alpha) \in \mathbb{R}$$

which satisfies

$$\mathbb{P}[Y \le \boxed{q_Y(\alpha)}] = \alpha$$

$$\iff$$

$$F_Y(\boxed{q_Y(\alpha)}) = \alpha$$

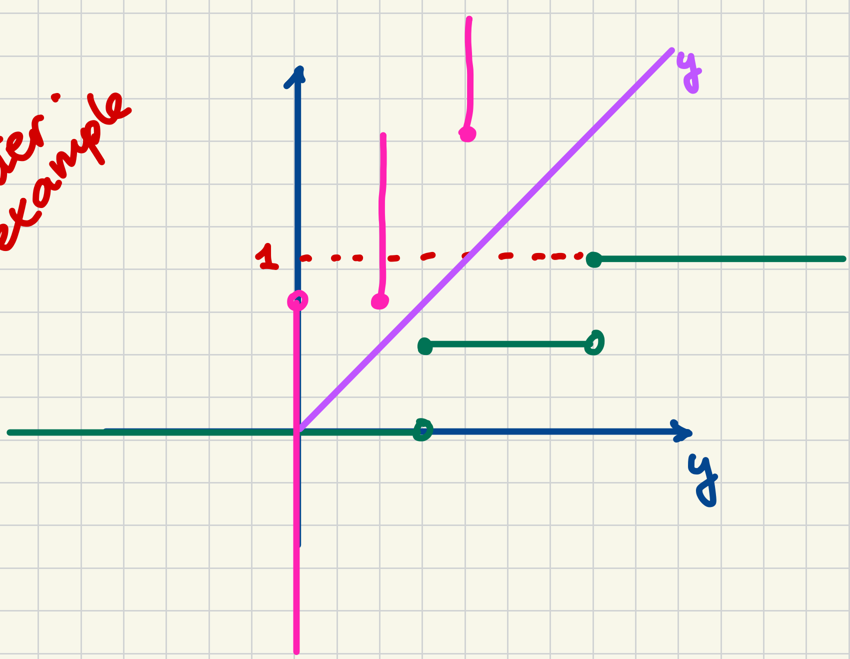



inverse function of the cdf

If $F_Y^{-1}$ exists, then $q_Y(\alpha) = F_Y^{-1}(\alpha)$.

This is, for example, the case w/ the normal.

Counter example



M378K Introduction to Mathematical Statistics

Problem Set #7

Cumulative distribution functions: Named continuous distributions.

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**Problem 7.1.** Source: Sample P exam, Problem #126.

*A company's annual profit is normally distributed with mean 100 and variance 400. Then, we can express the probability that the company's profit in a year is at most 60, given that the profit in the year is positive in terms of the standard normal cumulative distribution function denoted by Φ as*

$$\boxed{1 - \frac{\Phi(2)}{\Phi(5)}}$$

True or false? Why?

$\rightarrow$: $Y \sim N(\mu = 100, sd = \sigma = 20)$

$$\mathbb{P}[Y \leq 60 \mid Y > 0] = \frac{\mathbb{P}[0 < Y \leq 60]}{\mathbb{P}[Y > 0]}$$

$$= \frac{\mathbb{P}\left[\frac{0-100}{20} < \overbrace{\boxed{\frac{Y-100}{20}}}^{=Z} \leq \overbrace{\frac{60-100}{20}}^{-2}\right]}{\mathbb{P}\left[\frac{Y-100}{20} > \underbrace{\frac{0-100}{20}}_{-5}\right]} =$$

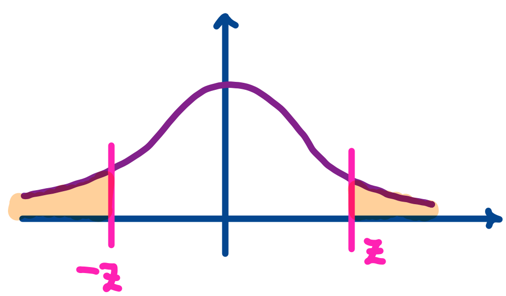
$$= \frac{\mathbb{P}[-5 < Z \leq -2]}{\mathbb{P}[-5 < Z]}$$

$$= \frac{\Phi(-2) - \Phi(-5)}{1 - \Phi(-5)}$$

$$= \frac{\cancel{1} - \Phi(2) - (\cancel{1} - \Phi(5))}{\Phi(5)}$$

$$= \frac{\Phi(5) - \Phi(2)}{\Phi(5)} = 1 - \frac{\Phi(2)}{\Phi(5)}$$

$\boxed{T}$




$\Phi(-z) = \mathbb{P}[Z > z]$

$= 1 - \mathbb{P}[Z \leq z] =$

$= 1 - \Phi(z)$

## Example. The Exponential Dist'n.

$\boxed{Y \sim E(\tau)}$

pdf... $f_Y(y) = \frac{1}{\tau} e^{-\frac{y}{\tau}} \cdot \mathbb{1}_{[0,\infty)}(y)$

cdf... $F_Y(y) = ?$

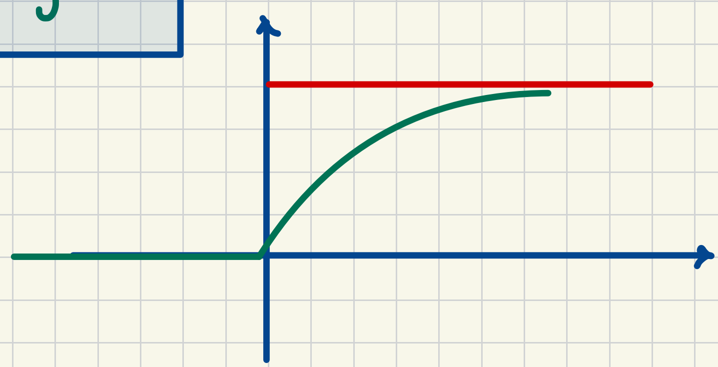
Evidently, $F_Y(y) = \underline{0}$ for $y<0$

For $y \geq 0$: $F_Y(y) = \mathbb{P}[Y \leq y] = \int_0^y \frac{1}{\tau} e^{-\frac{u}{\tau}} du$

$$= \frac{1}{\cancel{\tau}} (-\cancel{\tau}) e^{-\frac{u}{\tau}} \Big|_{u=0}^{y}$$

$$= -\left(e^{-\frac{y}{\tau}} - 1\right)$$

$$\boxed{F_Y(y) = 1 - e^{-\frac{y}{\tau}} \qquad y \geq 0}$$



**Problem 7.2.** Source: Sample P exam, Problem #267.
*The lifetime of a certain electronic device has an exponential distribution with mean* 0.50. *Calculate the probability that the lifetime of the device is greater than* 0.70, *given that it is greater than* 0.40.

→ :

$Y$ ... lifetime

$Y \sim E(\tau = 0.5)$

$$\mathbb{P}[Y > 0.7 \mid Y > 0.4] = \frac{\boxed{\mathbb{P}[Y > 0.7, Y > 0.4]}}{\mathbb{P}[Y > 0.4]}$$

$$= \frac{\mathbb{P}[Y > 0.7]}{\mathbb{P}[Y > 0.4]}$$

$$= \frac{e^{-\frac{0.7}{\tau}}}{e^{-\frac{0.4}{\tau}}} = e^{-\frac{0.3}{\tau}}$$

$$= e^{-0.6}$$

This is a special case of the memoryless property, i.e.,

$$\boxed{\mathbb{P}[Y > t+s \mid Y > t] = \mathbb{P}[Y > s]}$$