M339J *Probability models*
University of Texas at Austin
HW #6
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Note: You **must** show all your work. Numerical answers without a proper explanation or a clearly written down path to the solution will be assigned zero points.

Problem 6.1. (5 points)Let X be the ground-up loss random variable. Assume that X has the exponential distribution with mean $5,000$. Let B denote the expected payment per loss on behalf of an insurer which wrote a policy with a deductible of $1,500$ and with no maximum policy payment. What is the value of B?

Solution: Using our tables,

$$B = \mathbb{E}[(X-1500)_+] = \mathbb{E}[X] - \mathbb{E}[X \wedge 1500] = \theta - \theta(1 - e^{-1500/\theta}) = \theta\, e^{-1500/\theta} = 5000e^{-3/10} \approx 3704.$$

Problem 6.2. (5 points) The ground-up loss random variable is denoted by X. An insurance policy on this loss has a **franchise** deductible of d and no policy limit. Then, the expected **policyholder** payment per loss equals ...

(a) $\mathbb{E}[X\mathbb{I}_{[X\leq d]}]$
(b) $\mathbb{E}[X \wedge d]$
(c) $\mathbb{E}[(X-d)_+]$
(d) $\mathbb{E}[X \wedge d] - d$
(e) None of the above.

Solution: (a)

Problem 6.3. (5 points) Let $X \sim Pareto(\alpha = 3, \theta = 3000)$. Assume that there is a deductible of $d = 5000$. Find the loss elimination ratio.

Solution: Using the tables, we get

$$\mathbb{E}[X] = \frac{\theta}{\alpha - 1} = \frac{3000}{3-1} = 1500,$$
$$\mathbb{E}[X \wedge d] = \frac{\theta}{\alpha - 1}\left[1 - (\tfrac{\theta}{d+\theta})^{\alpha-1}\right] = \frac{3000}{3-1}\left[1 - (\tfrac{3000}{5000+3000})^{3-1}\right] = 1500[1 - (\tfrac{3}{8})^2].$$

Finally, the loss elimination ratio is

$$[1 - (\tfrac{3}{8})^2] = \tfrac{55}{64}\,.$$

Problem 6.4. (15 points) Assume that the severity random variable X is exponentially distributed with mean 1400. There is an insurance policy to cover this loss. This insurance policy has

(1) a deductible of 200 per loss,
(2) the coinsurance factor of $\alpha = 0.25$, and
(3) the maximum amount payable by the insurer per loss of 700.

Calculate the expected value of the per payment random variable Y^P under this policy.

Solution: Due to the memoryless property of the exponential distribution, we have

$$X - 200 \,|\, X > 200 \sim \mathit{Exponential}(\theta = 1400).$$

Due to the fact that the exponential distribution is a scale distribution, when we introduce the coinsurance factor, we get

$$0.25(X - 200) \,|\, X > 200 \sim \mathit{Exponential}(\theta^* = 0.25 * 1400 = 350).$$

Hence, using our tables with $Y \sim \mathit{Exponential}(\theta = 350)$,

$$\mathbb{E}[Y^P] = \mathbb{E}[Y \wedge 700] = 350(1 - e^{-700/350}) \approx 302.63.$$

Problem 6.5. (10 points) Let Y be lognormal with parameters $\mu = 1$ and $\sigma = 2$.

Define $\tilde{Y} = 3Y$.

Find the median of $\tilde{Y}$, i.e., find the value m such that $\mathbb{P}[\tilde{Y} \leq m_Y] = 1/2$.

Solution: In class, we showed that Y is lognormal with parameters $\mu^* = \mu + \ln(3)$ and $\sigma^* = \sigma$. So, Y can be written as $Y = e^Z$ where $Z \sim N(\mu^*, (\sigma^*)^2)$. Hence, with m_Y denoting the median of Y, we have

$$\begin{aligned} 1/2 &= \mathbb{P}[Y \leq m_Y] \\ &= \mathbb{P}[e^X \leq m_Y] \\ &= \mathbb{P}[X \leq \ln(m_Y)]. \end{aligned}$$

Since X is normal with mean μ^* (and the mean and the median of a normal r.v. are one and the same), we conclude that

$$\ln(m_Y) = 1 + \ln(3) \quad \Rightarrow \quad m_y = 3e \approx 8.15.$$

Problem 6.6. (10 points) In the notation of our tables, let X be a Weibull random variable with parameters $\theta = 20$ and $\tau = 2$.

Define $Y = 5X$ and denote the coefficient of variation of Y by CV_Y. Find CV_Y.

Hint: The following facts you may have forgotten from probability could be useful:

$$\begin{aligned} \Gamma(1/2) &= \sqrt{\pi}, \\ \Gamma(1) &= 1, \\ \Gamma(\alpha + 1) &= \alpha\Gamma(\alpha), \quad \text{for all } \alpha. \end{aligned}$$

Solution: The Weibull distribution has the scale parameter θ. So,

$$Y \sim Weibull(\theta = 100, \tau = 2).$$

Using our tables, we get

$$\begin{aligned}
\mathbb{E}[Y] &= \theta\,\Gamma(1+\frac{1}{\tau})\\
&= \theta\,\Gamma(1+\frac{1}{2})\\
&= \theta\cdot\frac{1}{2}\Gamma(1/2)\\
&= \theta\cdot\frac{1}{2}\sqrt{\pi},
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}[Y^2] &= \theta^2\,\Gamma(1+\frac{2}{\tau})\\
&= \theta^2\,\Gamma(1+\frac{2}{2})\\
&= \theta^2\cdot 1\cdot\Gamma(1)\\
&= \theta^2.
\end{aligned}$$

So,

$$\begin{aligned}
Var[Y] &= \mathbb{E}[Y^2] - \mathbb{E}[Y]^2\\
&= \theta^2 - \theta^2\cdot\frac{\pi}{4}\\
&= \theta^2(1-\frac{\pi}{4})\\
&= \frac{\theta^2}{4}(4-\pi).
\end{aligned}$$

Finally,

$$CV_Y = \frac{\frac{\theta}{2}\sqrt{4-\pi}}{\theta\cdot\frac{\sqrt{\pi}}{2}} = \sqrt{\frac{4-\pi}{\pi}} \approx 0.5227.$$

Note that we never used the exact value of θ to get the final answer.

Also, note that one can immediately realize that

$$CV_Y = \frac{\sqrt{Var[Y]}}{\mathbb{E}[Y]} = \frac{\sqrt{Var[5X]}}{\mathbb{E}[5X]} = \frac{5\sqrt{Var[X]}}{5\mathbb{E}[X]} = CV_X$$

and then just use the definition of X to get the desired coefficient of variation; there is no need to know anything about the distribution of Y.