COMPARISON OF COHOMOLOGICAL EIGENVARIETIES

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ABSTRACT. We prove that Emerton's completed cohomology and Ash–Stevens overconvergent cohomology produce isomorphic eigenvarieties. More specifically, we show that Fu's derived Emerton-Jacquet functors of completed cohomology and Ash–Stevens–Hansen's overconvergent cohomology are quasi-isomorphic as complexes of coherent sheaves. The proof proceeds by working with homology and explicitly comparing various coefficient systems involving distribution algebras, and uses the solid formalism of Clausen–Scholze. We further equate both approaches with a version of p-arithmetic homology. As a first application, the triangulation of Galois representations over Fu's eigenvarieties for $\mathrm{GL}_{n/F}$ for F a CM field extends to overconvergent cohomology, resolving many cases of Hansen's original conjecture.

1. INTRODUCTION

The theory of eigenvarieties, that is of families of p-adic automorphic forms, was initiated by the seminal papers [Col97, CM98] by Coleman and Coleman–Mazur. Since then, a plethora of different constructions have been given. Within these, one can spot two main types of constructions: the ones which, following Coleman and Coleman–Mazur, use coherent cohomology of Shimura varieties, and the ones that use singular cohomology of locally symmetric spaces in some form. This latter group of constructions again divides into two main types: the ones using cohomology of certain types of locally analytic function or distribution modules, originating in unpublished work of Stevens and further developed in [AS, Han17, Urb11, Che04, Loe11], and the construction of Emerton based on his notion of completed cohomology and the locally analytic Jacquet functor [Eme06b, Fu22]. Compared to constructions relying on Shimura varieties, the constructions using cohomology of locally symmetric spaces have the advantage that they can be defined for essentially any reductive group $\mathbf{G}/\mathbb{Q}$.

An early theme, originating with Chenevier's remarkable interpolation of the Jacquet–Langlands correspondence [Che05], is to compare different eigenvariety constructions. Chenevier's idea, which was further developed in [BC09, Han17, JN19b], is that the eigenvarieties themselves are very rigid. In particular, (iso)morphisms of eigenvarieties can essentially be constructed as long as one can match up (Zariski dense) set of points on them (in a precise technical way). This technique works really to prove that different eigenvariety constructions for the same group yield the same eigenvariety, *as long as these eigenvarieties have dense sets of points corresponding to classical automorphic representations*. This condition is essentially equivalent to the group having defect 0, and fails for large classes of groups, such as $\mathrm{Res}^F_{\mathbb{Q}}\mathrm{GL}_{n/F}$ for $n \geq 3$.

The current paper deals with a more refined question. In essence, the important output of every eigenvariety construction is a coherent sheaf $\mathscr{M}$ on the character variety $\widehat{T}$ of the maximal torus $T \subseteq G_{\mathbb{Q}_p}$, which carries an action of the Hecke operators. The eigenvariety is then, by construction, the relative spectrum of the Hecke operators over $\widehat{T}$ (viewed as a subalgebra of $\mathrm{End}_{\widehat{T}}(\mathscr{M})$). The sheaf $\mathscr{M}$ should be view as the sheaf of finite slope eigenforms for G (with regard to the particular construction that has been used). Thus, instead of asking if the eigenvarieties for two such constructions agree, one might ask if they produce the same $\mathcal{M}$. This is a stronger statement, in the sense that we do not expect it to hold even if the underlying eigenvarieties are isomorphic. For example, for $\mathbf{G} = \mathrm{GL}_{2/\mathbb{Q}}$, the coherent cohomology construction of Coleman–Mazur genuinely produces a different sheaf than the singular cohomology constructions of Ash–Stevens and Emerton, and comparing the two seems is an interesting problem involving p-adic Hodge theory.

Our main theorem here is the following, loosely stated (see Theorem 4.3.10 and discussion after):

Theorem 1.0.1. *For any reductive group $\mathbf{G}/\mathbb{Q}$ which is quasisplit at p, the eigenvariety construction of Ash–Stevens–Hansen using overconvergent cohomology of distribution modules produce the same coherent sheaf as that of Fu's 'derived' version of Emerton's construction. In particular, the corresponding eigenvarieties are isomorphic.*

As far as we know, this is the first theorem comparing the sheaves, and the first general proof that eigenvarieties are isomorphic beyond the situation when classical points are dense[1,2]. Since $\widehat{T}$ is quasi-Stein, the sheaf $\mathscr{M}$ is determined by its global sections. As a by-product of our method, we also prove the following result.

Theorem 1.0.2. *In the setting of Theorem 1.0.1, $\mathscr{M}$ can be written as the p-arithmetic homology of an explicit coefficient system.*

This relates the construction to the approach of [Tar23] using p-arithmetic homology. We regard our results as a unification of the different theories of eigenvarieties constructed from singular cohomology. As a sample application, we note that the triangulinity result for the family of Galois representations on Fu's eigenvarieties for GL_n over CM fields [McD25] immediately translate to the family of Galois representations on the corresponding overconvergent cohomology eigenvariety, constructed in [JN19a]. This implies the original [Han17, Conjecture 1.2.2] in many cases for CM fields.

Another motivation is that these constructions seemingly have different technical benefits. Emerton's completed cohomology approach has often been better for relating eigenvarieties to Galois representations, especially when coupled with patching (see [BHS17, BHS19]). On the other hand, overconvergent (co)homology has been easier to relate to classical cohomology, which can be used to bound the dimension of certain components on eigenvarieties. Having both theories can be quite useful: [McD25, §2] applies techniques from overconvergent cohomology to control Fu's boundary eigenvarieties, but stops short of a full comparison. In particular, various key calculations of *loc. cit.* are subsumed by Theorem 1.0.1.

The proof of our theorem may be described as a direct computation. In some sense, the key step is to use a result of Hill [Hil10] to rewrite completed cohomology as the cohomology of a big local system at finite level, and then start computing from there. This strategy was observed by Hansen more than a decade ago. Nevertheless, the computations from there are quite delicate, especially when it comes

[1]To be precise, it is the second theorem of its kind: In [Tar23], one of us (G.T.) introduced a new eigenvariety construction based on p-arithmetic homology of locally analytic parabolic inductions, and proved the analogue of our theorem comparing it to Hansen's construction using overconvergent homology of locally analytic function modules. Our theorem is then the first comparing two previously defined constructions.

[2]A result of this kind, but only pointwise and in the special case when $\mathbf{G}$ is compact modulo center at infinity, was proved by Loeffler [Loe11].

to extracting finite slope subspaces in both theories[3] We note that the theorem needs Fu's version of Emerton's originally construction, and thus a satisfactory theorem could not be proven before [Fu22]. We have also benefited from the approach to p-adic functional analysis using solid abelian groups ([CS19], [Bos23, Appendix A], [RJRC22]). In particular, we rephrase Fu's construction using solid functional analysis as a key conceptual step in our proof, though we have stopped short of trying to properly define a Jacquet functor in the solid framework.

The structure of the paper is as follows: Section 2 recalls preliminaries on reductive groups, locally symmetric spaces and completed cohomology. Section 3 then recalls the two eigenvariety constructions that we wish to compare, and section 4 proves the main theorem. We give a few direct applications in section 5, and finally an appendix recalls the solid functional analysis that we need for our arguments.

1.1. Notation and conventions.

We follow Weibel's book for conventions on homological algebra. In particular, we mainly use chain complexes and homological numbering conventions.

If H is a compact p-adic Lie group, let $\mathcal{O}[[H]] = \varprojlim_{H' \lhd H} \mathcal{O}[H/H']$ be the Iwasawa algebra. For a general p-adic Lie group G with an open compact subgroup $H \subset G$, we set $\mathcal{O}[[G]] := \mathcal{O}[G] \otimes_{\mathcal{O}[H]} \mathcal{O}[[H]]$. For any locally compact p-adic analytic manifold X we set $D(X) = (C^{\mathrm{la}}(X, K))^\vee$ be the distribution algebra, the strong dual of the space of locally analytic functions on G. If H is an abelian p-adic Lie group, then let $\widehat{H}$ be the rigid space of continuous characters of H, so it represents the functor $\mathrm{Sp}(A) \mapsto \mathrm{Hom}_{\mathrm{cont}}(H, A^\times)$.

For V a solid K-vector space, we set $V^\vee := \underline{\mathrm{Hom}}(V, K)$. When V is a locally convex topological vector space, $V^\vee$ will denote the strong dual.

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2. PRELIMINARIES

We fix a finite extension $K/\mathbb{Q}_p$, which will be our coefficient field throughout this paper. The ring of integers of K will be denoted by $\mathcal{O}$ (or possibly $\mathcal{O}_K$, if there is a risk of confusion), and $\varpi \in \mathcal{O}$ will denote a uniformizer.

2.1. Groups and locally symmetric spaces.

Throughout this paper, $\mathbf{G}$ will be a connected reductive group over $\mathbb{Q}$ **which is quasisplit at** p. We will need some local considerations at p and at ∞, and to define various spaces attached to $\mathbf{G}$.

[3] In fact, [Fu22, Theorem 5.2] previously related completed and overconvergent cohomology in some sense, but crucially did not address the Hecke actions at p. See also [EGH23, 9.6.7, 9.6.16, 9.6.29], which assert equivariance for the Hecke action at p, but without proof.

We start at ∞. Let $\mathbf{G}(\mathbb{R})^+$ denote the identity component of $\mathbf{G}(\mathbb{R})$ and set $\mathbf{G}(\mathbb{Q})^+ = \mathbf{G}(\mathbb{Q}) \cap \mathbf{G}(\mathbb{R})^+$. We let $K_\infty \subseteq \mathbf{G}(\mathbb{R})^+$ be a maximal compact subgroup and let $\mathbf{A}$ be the maximal $\mathbb{Q}$-split torus in the center of $\mathbf{G}$. Moreover, we set $X_\infty = \mathbf{G}(\mathbb{R})^+/(\mathbf{A}(\mathbb{R})^+K_\infty)$, and let $\overline{X}_\infty$ be the Borel–Serre bordification of X_∞ [BS73], which carries a left action by $\mathbf{G}(\mathbb{Q})^+$.

Next, we turn to p, where we need more notation. We will let $G := \mathbf{G}(\mathbb{Q}_p)$ throughout. Since $\mathbf{G}$ is quasisplit at p, we can (and will) fix a Borel subgroup $\mathbf{B} \subseteq \mathbf{G}_{\mathbb{Q}_p}$ and we let $\mathbf{B} = \mathbf{T}\mathbf{N}$ be a Levi decomposition, with opposite $\overline{\mathbf{B}} = \mathbf{T}\overline{\mathbf{N}}$. For any algebraic subgroup scheme $\mathbf{H} \subseteq \mathbf{G}_{\mathbb{Q}_p}$ written in bold, the corresponding non-bold letter H will be used to denote $\mathbf{H}(\mathbb{Q}_p)$. Let $\mathbf{S} \subseteq \mathbf{T}$ be a maximal split torus. We will let I denote a well chosen (SPECIFY WHAT YOU WANT) Iwahori subgroup of G, with Iwahori decomposition

$$I = \overline{N}_1 T_0 N_0$$

with respect to $\mathbf{B}$. MAKE SENSE OF THE SUBSCRIPTS. Moreover, put

$$T^+ = \{t \in T \mid t\overline{N}_1 t^{-1} \subseteq \overline{N}_1\}$$

and

$$T^{cpt} = \{t \in T \mid t\overline{N}_1 t^{-1} \subseteq \overline{N}_2\}.$$

We have $t^{-1}N_0 t \subseteq N_0$ for $t \in T^+$ and $t^{-1}N_0 t \subseteq N_1$ for $t \in T^{cpt}$.

Next, let X_p be the (enlarged) Bruhat–Tits building of G over $\mathbb{Q}_p$. We recall a few facts about X_p that we will need. First, X_p carries a left action of G and a G-invariant metric d. Moreover, X_p is contractible and any two points in X_p are connected by a unique geodesic [BT72, §2.5]. In particular, for $a, b \in X_p$, we may consider the renormalized geodesic $j_{a,b} : [0,1] \to X_p$ from a to b. Finally, given any compact subgroup $K_p \subseteq G$, there is a point $\alpha \in X_p$ which is fixed by all elements of K_p.

Next, we define the locally symmetric spaces that we will use. Given a compact open subgroup $K^p \subseteq \mathbf{G}(\mathbb{A}^{p,\infty})$, we define

$$\mathcal{X} := \mathbf{G}(\mathbb{Q})^+ \backslash X_\infty \times \mathbf{G}(\mathbb{A}^\infty)/K^p, \quad \overline{\mathcal{X}} := \mathbf{G}(\mathbb{Q})^+ \backslash \overline{X}_\infty \times \mathbf{G}(\mathbb{A}^\infty)/K^p$$

and

$$\mathcal{X}_p := \mathbf{G}(\mathbb{Q})^+ \backslash X_\infty \times X_p \times \mathbf{G}(\mathbb{A}^\infty)/K^p, \quad \overline{\mathcal{X}}_p := \mathbf{G}(\mathbb{Q})^+ \backslash \overline{X}_\infty \times X_p \times \mathbf{G}(\mathbb{A}^\infty)/K^p.$$

Here we equip $\mathbf{G}(\mathbb{A}^\infty)$ with the discrete topology rather than its locally profinite topology, so that the maps $X_\infty \times \mathbf{G}(\mathbb{A}^\infty) \to \mathcal{X}$, etc., are all covering maps. The action of $\mathbf{G}(\mathbb{Q})^+$ is always diagonal (from the left) and K^p acts by right translation on $\mathbf{G}(\mathbb{A}^\infty)$ and trivially on the other components. We remark that $\mathcal{X}$, $\overline{\mathcal{X}}$, $\mathcal{X}_p$ and $\overline{\mathcal{X}}_p$ all carry right actions of G, induced by right translation on $\mathbf{G}(\mathbb{A}^\infty)$.

2.2. **Homology and cohomology.** If Y is any topological space, we let $C_\bullet(Y)$ denote the complex of singular chains of Y. Since $\overline{X}_\infty \setminus X_\infty$ is the boundary of the topological manifold with boundary $\overline{X}_\infty$, the inclusion $X_\infty \to \overline{X}_\infty$ is a homotopy equivalence. It follows that $C_\bullet(\mathcal{X}) \to C_\bullet(\overline{\mathcal{X}})$ and $C_\bullet(\mathcal{X}_p) \to C_\bullet(\overline{\mathcal{X}}_p)$ are G-chain homotopy equivalences. Moreover, they are also equivariant for the action of Hecke operators away from p. Let us take a moment to recall the construction of Hecke operators in an abstract setting that will be useful for us.

Definition 2.2.1. Let Γ be a group with a subgroup $H \subseteq \Gamma$. Assume further that $\Delta \subseteq \Gamma$ is a submonoid containing H making (Δ, H) into a Hecke pair (meaning that H and $\delta H \delta^{-1}$ are commensurable for all $\delta \in \Delta$). Then, if M is a right Γ-module and N is a left Δ-module, the formula

$$U_\delta(m \otimes n) = \sum_i m\delta_i^{-1} \otimes \delta n,$$

where $H\delta H = \bigsqcup_i H\delta_i$, defines an endomorphism U_δ of $M \otimes_{\mathbb{Z}[H]} N$ by linear extension which is independent of the choice of double coset representatives δ_i and only depends on the double coset $H\delta H$.

We note that if $f_M : M \to M'$ is Γ-equivariant and $f_N : N \to N'$ is Δ-equivariant, then $f = f_M \otimes f_N : M \otimes_{\mathbb{Z}[H]} N \to M' \otimes_{\mathbb{Z}[H]} N'$ satisfies $f \circ U_\delta = U_\delta \circ f$ for all $\delta \in \Delta$. Moreover, the actions of the different U_δ are compatible in the sense that they form a left action of the Hecke algebra $\mathbf{T}(\Delta, H)$ attached to the pair (Δ, H)[4]. We will only be interested in the following two cases:

(1) $\Gamma = G$, $H = I$ and $\Delta = IT^+I$, and
(2) $\Gamma = \Delta = \mathbf{G}(\mathbb{A}^{p,\infty})$ and $H = K^p$.

Case (1) will be used mostly frequently in the paper. In that case, the Hecke algebra is well known to be commutative; we record this as a proposition.

Proposition 2.2.2. *Suppose that we are in case (1) above. Then we have $[IsI] * [ItI] = IstI$ for all $s, t \in T^+$. In particular, the Hecke algebra $\mathbf{T}(IT^+I, I)$ is isomorphic to the monoid algebra $\mathbb{Z}[T^+/T_0]$, and hence commutative.*

Proof. Give a reference/proof in this generality. □

Case (2) will only be used to construct Hecke operators away from p on the singular chain complexes above. We indicate the construction on $C_\bullet(\mathcal{X})$; the actions on $C_\bullet(\overline{\mathcal{X}})$, $C_\bullet(\mathcal{X}_p)$ and $C_\bullet(\overline{\mathcal{X}}_p)$ are constructed in the same way. The space $\mathcal{X}$ is the quotient of $\mathcal{X}' := \mathbf{G}(\mathbb{Q})^+\backslash X_\infty \times \mathbf{G}(\mathbb{A}^\infty)$ by the free action of K^p. The natural map

$$C_\bullet(\mathcal{X}') \otimes_{\mathbb{Z}[K^p]} \mathbb{Z} \to C_\bullet(\mathcal{X})$$

is then an isomorphism[5]. Now $C_\bullet(\mathcal{X}')$ carries a right action of $\mathbf{G}(\mathbb{A}^\infty)$, so by using the trivial action of $\mathbf{G}(\mathbb{A}^{p,\infty})$ on $\mathbb{Z}$ we get a (left) Hecke action of $\mathbf{T}(\mathbf{G}(\mathbb{A}^{p,\infty}), K^p)$ on $C_\bullet(\mathcal{X})$ by Definition 2.2.1.

Let us now recall the definition p-arithmetic (co)homology in the adelic setting from [Tar23]. Let $K_p \subseteq G$ be any compact open subgroup. We recall the construction of a Hecke- and K_p-equivariant chain homotopy equivalence between $C_\bullet(\overline{\mathcal{X}})$ and $C_\bullet(\overline{\mathcal{X}}_p)$ from [Tar23, §5.2]. First, we have the projection map

$$f : \overline{X}_\infty \times X_p \times \mathbf{G}(\mathbb{A}^\infty) \to \overline{X}_\infty \times \mathbf{G}(\mathbb{A}^\infty),$$

which is $\mathbf{G}(\mathbb{Q})^+ \times \mathbf{G}(\mathbb{A}^\infty)$-equivariant. Now choose $\alpha \in X_p$ which is fixed by all elements of K_p, and consider the map

$$h_\alpha : \overline{X}_\infty \times \mathbf{G}(\mathbb{A}^\infty) \to \overline{X}_\infty \times X_p \times \mathbf{G}(\mathbb{A}^\infty)$$

given by $h_\alpha(z, g) = (z, g_p\alpha, g)$, where g_p is the p-component of g. One checks directly that this is $\mathbf{G}(\mathbb{Q})^+ \times \mathbf{G}(\mathbb{A}^{p,\infty}) \times K_p$-equivariant, and that $f \circ h_\alpha$ is the identity. Moreover, the map

$$H_\alpha : \overline{X}_\infty \times X_p \times \mathbf{G}(\mathbb{A}^\infty) \times [0,1] \to \overline{X}_\infty \times X_p \times \mathbf{G}(\mathbb{A}^\infty)$$

given by $H_\alpha(z, q, g, t) = (z, j_{q,\alpha}(t), g)$ is a $\mathbf{G}(\mathbb{Q})^+ \times \mathbf{G}(\mathbb{A}^{p,\infty}) \times K_p$-equivariant homotopy from the identity to $h_\alpha \circ f$ (recall from §2.1 that $j_{q,\alpha}$ is the renormalized geodesic from q to α). It follows that

[4]As always, this is the algebra of bi-H-invariant functions on Δ which are supported on finitely many double cosets, with convolution as multiplication.

[5]Recall that, if X is a topological space with a free right action of a discrete group K, then $C_\bullet(X/K) = C_\bullet(X) \otimes_{\mathbb{Z}[K]} \mathbb{Z}$.

f induces a Hecke- and K_p-equivariant chain homotopy equivalence from $C_\bullet(\overline{\mathcal{X}}_p)$ to $C_\bullet(\overline{\mathcal{X}})$, with inverse (induced by) h_α.

We now recall the definitions of arithmetic and p-arithmetic (co)homology.

Definition 2.2.3. Let M be a complex of left K_p-modules, and let N be a complex of right K_p-modules. Set $K = K^p K_p$.

(1) We define the arithmetic homology of M to be the homology $H_*(K, M)$ of the complex $C_\bullet(K, M) := C_\bullet(\overline{\mathcal{X}}) \otimes^L_{\mathbb{Z}[K_p]} M$.
(2) We define the arithmetic cohomology of N to be the cohomology $H^*(K, N)$ of the complex $C^\bullet(K, N) := \mathrm{RHom}_{\mathbb{Z}[K_p]}(C_\bullet(\overline{\mathcal{X}}), N)$.

Definition 2.2.4. Let M be a complex of left G-modules, and let N be a complex of right G-modules.

(1) We define the p-arithmetic homology of M to be the homology $H_*(K^p, M)$ of the complex $C_\bullet(K^p, M) := C_\bullet(\overline{\mathcal{X}}_p) \otimes^L_{\mathbb{Z}[G]} M$.
(2) We define the p-arithmetic cohomology of N to be the cohomology $H^*(K^p, N)$ of the complex $C^\bullet(K^p, N) := \mathrm{RHom}_{\mathbb{Z}[G]}(C_\bullet(\overline{\mathcal{X}}_p), N)$.

Note that we make no assumption on the action of G on $\mathcal{X}_p$ or K_p on $\mathcal{X}$ being free. If G acts freely on $\mathcal{X}_p$, then $C_\bullet(\mathcal{X}_p)$ is a (bounded above) complex of free $\mathbb{Z}[G]$-modules (this will be true for K^p sufficiently small). Similarly, if K_p acts freely on $\mathcal{X}$, then $C_\bullet(\mathcal{X})$ is a (bounded above) complex of free $\mathbb{Z}[K_p]$-modules. When the actions are free, we will use $C_\bullet(K^p, M)$ to denote the actual complex $C_\bullet(\mathcal{X}_p) \otimes_{\mathbb{Z}[G]} M$, and similarly for the other notations.

We can make similar constructions for Borel–Moore and boundary homology instead. Letting

$$\partial X_\infty := \overline{X}_\infty \setminus X$$

be the boundary of the Borel–Serre bordification, we may define

$$\partial \mathcal{X} := \mathbf{G}(\mathbb{Q})^+ \backslash \partial X_\infty \times \mathbf{G}(\mathbb{A}^\infty)/K^p$$

and

$$\partial \mathcal{X}_p := \mathbf{G}(\mathbb{Q})^+ \backslash \partial X_\infty \times X_p \times \mathbf{G}(\mathbb{A}^\infty)/K^p.$$

The complexes $C_\bullet(\partial\mathcal{X})$ and $C_\bullet(\partial\mathcal{X}_p)$ then have right G-actions, and, with M, N and K_p as in Definitions 2.2.3 and 2.2.4, respectively, we define

$$C^\partial_\bullet(K, M) = C_\bullet(\partial\mathcal{X}) \otimes^L_{\mathbb{Z}[K_p]} M, \quad C^\bullet_\partial(K, N) = \mathrm{RHom}_{\mathbb{Z}[K_p]}(C_\bullet(\partial\mathcal{X}), N)$$

and

$$C^\partial_\bullet(K^p, M) = C_\bullet(\partial\mathcal{X}_p) \otimes^L_{\mathbb{Z}[K_p]} M, \quad C^\bullet_\partial(K^p, N) = \mathrm{RHom}_{\mathbb{Z}[K_p]}(C_\bullet(\partial\mathcal{X}_p), N).$$

To define Borel–Moore homology and compactly supported cohomology, we define $C^{\mathrm{BM}}_\bullet(\mathcal{X})$ to be the cone of the map $C_\bullet(\partial\mathcal{X}) \to C_\bullet(\overline{\mathcal{X}})$, and $C^{\mathrm{BM}}_\bullet(\mathcal{X}_p)$ to be the cone of $C_\bullet(\partial\mathcal{X}_p) \to C_\bullet(\overline{\mathcal{X}}_p)$. With M, N and K_p as above, we then define

$$C^{BM}_\bullet(K, M) = C^{BM}_\bullet(\mathcal{X}) \otimes^L_{\mathbb{Z}[K_p]} M, \quad C^\bullet_c(K, N) = \mathrm{RHom}_{\mathbb{Z}[K_p]}(C^{BM}_\bullet(\mathcal{X}), N)$$

and

$$C^{BM}_\bullet(K^p, M) = C^{BM}_\bullet(\mathcal{X}_p) \otimes^L_{\mathbb{Z}[K_p]} M, \quad C^\bullet_c(K^p, N) = \mathrm{RHom}_{\mathbb{Z}[K_p]}(C^{BM}_\bullet(\mathcal{X}_p), N).$$

To finish this subsection, we recall the comparison between arithmetic and p-arithmetic (co)homology. The maps f, h_α and H_α all preserve the boundary (indeed, they restrict to the identity on $\overline{X}_\infty$) and hence f induces a Hecke- and K_p-equivariant chain homotopy equivalence $C_\bullet(\partial\mathcal{X}_p) \to C_\bullet(\partial\mathcal{X})$ which sits in a commutative square

$$\begin{tikzcd} C_\bullet(\partial\mathcal{X}_p) \arrow[r] \arrow[d, "f"] & C_\bullet(\overline{\mathcal{X}}_p) \arrow[d, "f"] \\ C_\bullet(\partial\mathcal{X}) \arrow[r] & C_\bullet(\overline{\mathcal{X}}), \end{tikzcd}$$

where the horizontal maps are induced by the inclusions. It follows that f induces a Hecke- and K_p-equivariant chain homotopy equivalence $C_\bullet^{BM}(\mathcal{X}_p) \to C_\bullet^{BM}(\mathcal{X})$ as well. We then have the following comparison result.

Proposition 2.2.5. *Let $? \in \{\emptyset, BM, c, \partial\}$, let M be a complex of left K_p-modules, and let N be a complex of right K_p-modules. Then we have canonical Hecke-equivariant isomorphisms $C_\bullet^?(K, M) \cong C_\bullet^?(K^p, \mathbb{Z}[G] \otimes_{\mathbb{Z}[K_p]} M)$ and $C_?^\bullet(K, N) \cong C_?^\bullet(K^p, \mathrm{Hom}_{\mathbb{Z}[K_p]}(\mathbb{Z}[G], N))$ in the derived category of abelian groups.*

Proof. When $? = \emptyset$ this is [JNWE25, Proposition 6.3.3] (which is a special case of [Tar23, Prop. 5.2.2]), but the proof there works in general. □

2.3. **Borel–Serre complexes.** Next, we discuss the finiteness properties of arithmetic (co)homology[6]. In this paper, we will mostly work with a fixed level at p, so our discussion will reflect this choice. Namely, we fix $K_p \subseteq G$ compact open, and we fix a compact open $K^p_{fix} \subseteq \mathbf{G}(\mathbb{A}^{p,\infty})$ such that K_p acts freely on $\overline{\mathcal{X}}_{K^p_{fix}}$. The manifold with corners $\overline{\mathcal{X}}_{K^p_{fix}}/K_p$ is compact and homeomorphic to a smooth manifold with boundary by [BS73, Appendix]. In particular, it can be finitely triangulated in such a way that the boundary is a subcomplex [Mun66, Theorem 10.6]. The homotopy lifting property allows us to pull back these triangulations to get K_p-equivariant triangulations of $\overline{X}_{K^p}$ and $\partial\mathcal{X}_{K^p}$ for any $K^p \subseteq K^p_{fix}$. Taking simplicial chains, we get bounded complexes $C_\bullet^{BS}(\overline{\mathcal{X}}_{K^p})$ and $C_\bullet^{BS}(\partial\mathcal{X}_{K^p})$ whose terms are finite free right K_p-modules. They sit in a commutative diagram

$$\begin{tikzcd} C_\bullet^{BS}(\partial\mathcal{X}_{K^p}) \arrow[r] \arrow[d] & C_\bullet^{BS}(\overline{\mathcal{X}}_{K^p}) \arrow[r] \arrow[d] & C_\bullet^{BM,BS}(\mathcal{X}_{K^p}) \arrow[d] \\ C_\bullet(\partial\mathcal{X}_{K^p}) \arrow[r] & C_\bullet(\overline{\mathcal{X}}_{K^p}) \arrow[r] & C_\bullet^{BM}(\mathcal{X}_{K^p}), \end{tikzcd}$$

where, in the left square, the horizontal maps are induced by the inclusion of the boundary and the vertical maps are the inclusions of simplicial chains into singular chains (which are chain homotopy equivalences), and whole diagram is obtained by taking the cone of the left square. All maps are Hecke- and K_p-equivariant. When $K^p \subseteq K^p_{fix}$ is normal, these complexes carry natural right K^p_{fix}/K^p-actions and the maps are equivariant for these actions.

Now let M be a complex of left K_p-modules, and let N be a complex of right K_p-modules. Let K^p be arbitrary and put $K = K^pK_p$. Choose an open subgroup $K^p_0 \subseteq K^p \cap K^p_{fix}$ that is normal K^p. We set

$$C_\bullet^{BS}(K, M) := (C_\bullet^{BS}(\overline{\mathcal{X}}) \otimes^L_{\mathbb{Z}[K_p]} M) \otimes_{\mathbb{Z}[K^p/K^p_0]} \mathbb{Z}$$

[6] p-arithmetic (co)homology satisfies similar finiteness properties, but we will not need them.

and
$$C^{\bullet}_{BS}(K,N) := R\Gamma(K^p/K_0^p, \mathrm{RHom}_{\mathbb{Z}[K_p]}(C_{\bullet}^{BS}(\overline{\mathcal{X}}), N)).$$
This is independent of the choice of K_0^p, and when $K^p \subseteq K^p_{fix}$ these complexes are bounded with terms isomorphic to a finite number of copies of M (or N). If M is a $\mathbb{Q}$-vector space (which it will almost always be in this paper) and K^p is arbitrary, then the higher group (co)homology groups vanish and we again get bounded complexes, with terms now isomorphic to direct summands of a finite number of copies of M (or N). Finally, we remark that the analogous definitions and remarks apply to boundary (co)homology and Borel–Moore homology/compactly supported cohomology.

2.4. **Completed homology.** Let us now discuss completed homology. Let $E/\mathbb{Q}_p$ be a finite extension with ring of integers $\mathcal{O}$. By definition, completed homology for $\mathbf{G}$ with tame level K^p and $\mathcal{O}$-coefficients is
$$\widetilde{H}_*(K^p) := \varprojlim_{K_p'} H_*(K^pK_p', \mathcal{O}),$$
where K_p' runs over all compact open subgroups of G. It is a right $\mathcal{O}[\![G]\!]$-module, where $\mathcal{O}[\![G]\!]$ is defined as in [Sho20, Proposition 3.2] (denoted by $\mathcal{O}\langle G\rangle$ there). We can define boundary and Borel–Moore completed homology in the same way, replacing ordinary homology by boundary and Borel–Moore homology, respectively.

Proposition 2.4.1. *Let $K = K^pK_p \subseteq \mathbf{G}(\mathbb{A}^\infty)$ be a compact open subgroup.*

(1) *The complex $C_\bullet(K, \mathcal{O}[\![K_p]\!])$ with its natural right $\mathcal{O}[\![K_p]\!]$-module structure (and Hecke action) computes $\widetilde{H}_*(K^p)$ with its right $\mathcal{O}[\![K_p]\!]$-module structure (and Hecke action).*
(2) *The complex $C_\bullet(K^p, \mathcal{O}[\![G]\!])$ with its natural right $\mathcal{O}[\![G]\!]$-module structure (and Hecke action) computes $\widetilde{H}_*(K^p)$ with its right $\mathcal{O}[\![G]\!]$-module structure (and Hecke action).*

The analogous statements for boundary and Borel–Moore completed homology hold.

Proof. For ordinary completed homology, part (2) is [JNWE25, Proposition 6.3.5] (but originally proved working on this paper), and part (1) is well known (and proved en route to part (2)). The proofs for ordinary completed homology work verbatim for boundary and Borel–Moore completed homology. □

We recall from [JNWE25, Remark 6.3.6] that
$$C_\bullet(K^p, \mathcal{O}[\![G]\!]) = C_\bullet(\mathcal{X}_p) \otimes^L_{\mathbb{Z}[G]} \mathcal{O}[\![G]\!] = C_\bullet(\mathcal{X}_p) \otimes_{\mathbb{Z}[G]} \mathcal{O}[\![G]\!],$$
so we may treat $C_\bullet(K^p, \mathcal{O}[\![G]\!])$ as a genuine complex, which we will denote by $\widetilde{C}_\bullet$. Similar remarks apply to boundary and Borel–Moore homology, and we denote these complexes by $\widetilde{C}^\partial_\bullet$ and $\widetilde{C}^{BM}_\bullet$, respectively.

Let us now consider the Hecke action away from p. By the construction in [GN22, §2.1.10], the unramified Hecke action on $\widetilde{C}_\bullet$, viewed as endomorphisms in the derived category, factors through the action of a 'big' Hecke algebra $\mathbb{T} = \mathbb{T}(K^p)$. The same constructions apply to boundary and Borel–Moore homology. We record the following universality result for p-arithmetic (co)homology of $\mathcal{O}[\![G]\!]$-modules.

Proposition 2.4.2. *Let M be a complex of left $\mathcal{O}[\![G]\!]$-modules, and let N be a complex of right $\mathcal{O}[\![G]\!]$-modules.*

(1) *We have* $C_\bullet(K^p, M) \cong \widetilde{C}_\bullet \otimes^L_{\mathcal{O}[\![G]\!]} M$. *Moreover, the unramified Hecke action factors through a homomorphism* $\mathbb{T} \to \mathrm{End}_{D(\mathrm{Mod}(\mathcal{O}))}(C_\bullet(K^p, M))$.

(2) *We have* $C^\bullet(K^p, N) \cong \mathrm{RHom}_{\mathcal{O}[\![G]\!]}(\widetilde{C}_\bullet, N)$. *Moreover, the unramified Hecke action factors through a homomorphism* $\mathbb{T} \to \mathrm{End}_{D(\mathrm{Mod}(\mathcal{O}))}(C^\bullet(K^p, N))$.

The analogous statements for boundary and Borel–Moore homology hold.

Proof. For ordinary (co)homology this is [JNWE25, Proposition 6.3.7], but the proof applies in general. □

3. Eigenvariety constructions

In this section we will recall the two eigenvariety constructions that we will compare; Emerton's construction (in its derived form, due to Fu), and overconvergent (co)homology.

3.1. The Jacquet functor and the definition of the eigenvariety.

In this subsection we recall Emerton's construction of eigenvarieties from [Eme06b], using his Jacquet functor [Eme06a], and a variant of this construction. We give two definitions of Jacquet functors: one is abstract, and the other is slightly more hands on and due to Fu (and for which we can define eigenvarieties). We defer the comparison of these definitions to Section 4. **From now on, we assume that G is quasi-split at p and use the notation for such groups**. While Emerton works in considerable generality, it will be most convenient for us to only define his functors for the cases we need.

We start by recalling the dual Jacquet functor for the Borel subgroup $\mathbf{B}$, and comment on its relation to the usual definition. Let $C_\bullet$ be a complex of solid $D(G)$-modules. First we consider the homology $C_\bullet \otimes^\square_{D(N_0)} K$, where $N_0 \subset \mathbf{N}(\mathbb{Q}_p)$ is an open compact subgroup of the unipotent radical $\mathbf{N} \subset \mathbf{B}$ (this choice of N_0 does not affect the construction). This complex admits an action of the monoid T^+ via the formula

$$c \cdot z := \sum_{n \in N_0/z^{-1}N_0 z} cnz^{-1}.$$

Then the Jacquet functor is a composition of two functors

$$J_B^\vee(C_\bullet) := (C_\bullet \otimes^\square_{D(N_0)} K) \otimes^\square_{K[T^+]} \mathcal{O}(\widehat{T}),$$

where $\widehat{T}$ is the character variety of $T = \mathbf{T}(\mathbb{Q}_p)$. Plugging in the the complex $C_\bullet = C_\bullet(K^pI, D(I))$ (naturally a complex of $D(G)$-modules), we have a definition of the (dual) Jacquet functor of completed homology $J_B(C_\bullet)$, which is a complex of modules over $\mathcal{O}(\widehat{T})$ and with an action of the big Hecke algebra $\mathbb{T} = \mathbb{T}(K^p)$.

Following Fu [Fu22], we can also give a different definition of derived Jacquet functors of completed homology which is more directly related to topological vector spaces.[7] Now set $C_\bullet^{BS} := C_\bullet^{BS}(K^pI, D(I))$ for a fixed choice of triangulation. This complex does not admit an action of G on the nose, but admits an I-equivariant homotopy equivalence to a complex $C_\bullet := C(K^pI, D(I))$, which does have a G-action! Fixing homotopy inverses $C_\bullet^{BS} \underset{p}{\overset{i}{\rightleftarrows}} C_\bullet$ and $t \in T^{cpt}$, then the

[7] We work with homology, whereas [Fu22] works with cohomology. They are equivalent by [McD25, Corollary 2.10].

N_0-coinvariants $C^{BS}_\bullet(K^pI, D(I)_{N_0}) := C^{BS}_\bullet(K^pI, D(I) \otimes^{\square}_{D(N_0)} K)$ admits a right action of t via the formula

$$c \cdot U_t := \sum_{n \in N_0/t^{-1}N_0 t} cn(i \circ t^{-1} \circ p).$$

With this preparation, Fu then defines his dual Jacquet functors via

$$J^\vee_{B,\mathrm{Fu}} C^{\mathrm{BS}}_\bullet := C^{\mathrm{BS}}_\bullet(K^pI, D(I)_{N_0}) \otimes^{\square}_{K[t]} A,$$

where $A = K\{\{t^{\pm 1}\}\}$. Let $T_0 \subset T$ be a maximal compact subgroup and setting $\mathcal{W} := \widehat{T_0}$, the so-called *weight space*. Then Fu moreover showed [Fu22, Proposition 3.20] that $J^\vee_{B,\mathrm{Fu}} C^{\mathrm{BS}}_\bullet$ is a complex of coadmissible modules over $\mathcal{O}(\mathcal{W}) \widehat{\otimes}_K A \simeq \mathcal{O}(\mathcal{W} \times \mathbf{G}_m)$, and thus are (the global sections of) coherent sheaves $\mathcal{M}_\bullet$ over $\mathcal{W} \times \mathbf{G}_m$, also equipped with an action of $\mathbb{T}$. In fact the homology groups $H_i(\mathcal{M}_\bullet)$ are in fact coadmissible $\mathcal{O}(\widehat{T})$-modules by [Fu22, Lemma 3.19].

Remark 3.1.1. Typically Emerton's Jacquet functor is defined on (certain) locally analytic representations of G as a composition $V \mapsto V^{N_0} \to (V^{N_0})_{\mathrm{fs}} =: J_B(V)$, where $(U)_{\mathrm{fs}} := \mathrm{Hom}_{K[T^+]}(\mathcal{C}^{\mathrm{an}}(\widehat{T}, K), U)$. In the solid setting we could simply try to define $U_{\mathrm{fs}} := \underline{\mathrm{Hom}}_{K[t]}(A, U)$. In our cases of interest (but unclear more generally), it turns out that $W \otimes^{\square}_{K[t]} A \simeq \underline{\mathrm{Hom}}_{K[t]}(A, W^\vee)^\vee$, using [RJRC22, Theorem 3.40] along with Lemma 3.1.3. and thus the above construction is formally dual to $J_B(V)$, so our decision to work on the dual side is mainly out of convenience.

We can use this second construction to define an eigenvariety. To get an honest Hecke algebra action on a module, we need to take the homology $H_*(\mathcal{M}_\bullet)$. Consider the map $\psi : \mathbb{T} \to \mathrm{End}_{\mathcal{O}(\mathcal{W} \times \mathbb{G}_m)}(\bigoplus_i H_i(\mathcal{M}_\bullet))$, and let $\mathcal{A}$ be the $\mathcal{O}(\mathcal{W} \times \mathbb{G}_m)$-algebra by the $\mathrm{im}(\psi)$, which is a coherent sheaf of algebras over $\mathcal{W} \times \mathbb{G}_m$. We also let $\mathcal{A}^{\mathrm{red}}$ be the quotient sheaf of reduced algebras. Then we define Fu's eigenvariety via the relative spectrum

$$\mathcal{E}_{\mathrm{Fu}}(K^p) := \mathrm{Sp}_{\mathcal{W} \times \mathbb{G}_{\mathrm{m}}}(\mathcal{A}^{\mathrm{red}}).$$

If we only consider the Hecke action on $H_i(\mathcal{M}_\bullet)$, we denote the associated eigenvariety by $\mathcal{E}^i_{\mathrm{Fu}}(K^p)$.

Remark 3.1.2. The algebra $\mathcal{A}$ is the same regardless of taking the abstract Hecke algebra $\mathbb{T}^S$ or the big Hecke algebra $\mathbb{T}^S(K^p)$. See [McD25, Remark 2.33], with the main point being that $\mathbb{T}^S \to \mathbb{T}^S(K^p)$ is dense, and that $\mathcal{A}$ is locally on its support finite over $\mathcal{W}$.

We now justify that $C^{BS}_\bullet(K^pI, D(I)_{N_0}) \otimes^{\square}_{K[t]} A \cong \underline{C^{BS}_\bullet(K^pI, D(I)_{N_0}) \widehat{\otimes}_{K[t]} A}$. Recall from [RJRC22, Lemma 3.24] that the map $V \mapsto V(*)_{\mathrm{top}}$ induces an *exact* equivalence between solid and classical Frechet spaces, with $V = \underline{V(*)_{\mathrm{top}}}$. On the level of topological vector spaces, Fu's Jacquet functors take the form $C^{\mathrm{BS}}(K^pI, \overline{D(I)_{N_0}}) \widehat{\otimes}_{K[t]} A$, where $\overline{D(I)_{N_0}}$ denotes the *Hausdorff* quotient of the N_0-coinvariants. We will use the following lemma, which will also be used in §4.

Lemma 3.1.3. *Let R be a Noetherian Banach K-algebra and let M be a Banach R-module. Assume that $T : M \to M$ is a compact R-linear operator. Then $id - T$ is a Fredholm operator, i.e.* $\mathrm{Ker}(id - T)$ *and* $\mathrm{Coker}(id - T)$ *are a finitely generated R-modules and* $\mathrm{Im}(id - T)$ *is closed.*

Proof. Throughout this proof we use the canonical topology on finitely generated R-modules and its standard properties, as recalled in §A. By compactness of T, we can write $T = F + E$, where F is a finite rank operator and E has operator norm < 1. Then $id - E$ is invertible and $N := \mathrm{Ker}(F)$ is closed with the quotient M/N being finitely generated. Note that $id - E$ and $id - T$ agree on N, so if

$N' := (id - T)(N) = (id - E)(N)$, then M/N' is also finitely generated. Since $N' \subseteq \mathrm{Im}(id - T) \subseteq M$, it follows that $\mathrm{Im}(id - T)$ is closed. The statement about finite generation then follows from applying the Snake lemma to

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M/N \arrow[r] \arrow[d] & 0 , \\
0 \arrow[r] & N' \arrow[r] & M \arrow[r] & M/N' \arrow[r] & 0
\end{tikzcd}$$

where the vertical maps are (induced by) $id - T$, noting that the first vertical map is an isomorphism. □

The following related lemma will also be used in §4.

Lemma 3.1.4. *Use the setup from Lemma 3.1.3, and assume that $R \to S$ is a map of Noetherian Banach K-algebras. Write $-_S$ for base change to S, i.e. applying $- \otimes_R^{\square} S$. Then we have* $\mathrm{Coker}(id_M - T)_S \cong \mathrm{Coker}(id_{M_S} - T_S)$.

Proof. Cokernels commute with left adjoints, so $\mathrm{Coker}(id_M - T)_S \cong \mathrm{Coker}(id_{M_S} - T_S)$ follows. □

We can now prove that our solid version of Fu's construction agrees with the original one.

Proposition 3.1.5. *There is an isomorphism $C_\bullet^{BS}(K^pI, D(I)_{N_0}) \otimes_{K[t]}^{\square} A \simeq \underline{C_\bullet^{BS}(K^pI, D(I)_{N_0}) \widehat{\otimes}_{K[t]} A}$, and similarly for Borel–Moore and boundary homology.*

Proof. It suffices to show:

$$\underline{D(I)_{N_0}} \simeq D(I) \otimes_{D(N_0)}^{\square} K \tag{1}$$

$$\underline{(D(I)_{N_0})^{\oplus d} \widehat{\otimes}_{K[t]} A} \simeq \underline{(D(I)_{N_0})^{\oplus d}} \otimes_{K[t]}^{\square} A, \tag{2}$$

where the $K[t]$-module structure need not come from the conjugation action, but such that t is still a limit of compact operators.

For (1) we reduce to the Banach case. We have by definition that $D(I)_{N_0} \simeq \varprojlim D^s(I)_{N_0}$, where $D^s(I)_{N_0} \simeq (\mathcal{C}^{(s)}(I, K)^{N_0})^\vee$ is the dual of s-analytic functions. Then a key fact [Lee23, Lemma 2.3] is that $D^s(I)_{N_0}$ needs no Hausdorff closure. Interpreted differently, choosing $n_1, \ldots, n_r$ a set of topological generators for N_0, the presentation $D^s(I)^{\oplus r} \xrightarrow{R} D^s(I) \to D^s(I)_{N_0} \to 0$, where $R(\delta \otimes 1) = \sum_i (\delta n_i \otimes 1 - \delta \otimes 1)$ is in fact a strict exact sequence. The same sequence (but on the level of solid vector spaces) also presents $D^s(I)^r \to D^s(I) \to D^s(I) \otimes_{D^s(N_0)}^{\square} K$. Then since the equivalence $V \mapsto \underline{V}$ respects strict exact sequences, we get that $\underline{D^s(I)_{N_0}} \simeq D^s(I) \otimes_{D^s(N_0)}^{\square} K$. Passing to the projective limit over s gives $\underline{D(I)_{N_0}} \simeq D(I) \otimes_{D(N_0)}^{\square} K$.

For (2) we now show that $\underline{D(I)_{N_0}} \otimes_{K[t]}^{\square} A(*)_{\mathrm{top}} \simeq D(I)_{N_0} \widehat{\otimes}_{K[t]} A$. By definition, the right hand is defined to be the Hausdorff quotient of the cokernel of the map $(U \otimes 1 - 1 \otimes t) : D(I)_{N_0} \widehat{\otimes} A \to D(I)_{N_0} \widehat{\otimes} A$. In the solid setting, the analogous presentation holds for $D(I)_{N_0} \otimes_{K[t]}^{\square} A$. Moreover, writing $D(I)_{N_0} \widehat{\otimes}_{K[t]} A \simeq \varprojlim_s D^s(I)_{N_0} \widehat{\otimes}_{K[t]} A_s$ (which holds by the Mittag–Leffler property), each $D^s(I)_{N_0} \widehat{\otimes}_{K[t]} A_s$ is presented by $(U \otimes 1 - 1 \otimes t) : D^s(I)_{N_0} \widehat{\otimes}_K A_s \to D^s(I)_{N_0} \widehat{\otimes}_K A_s$, which has closed image by Lemma 3.1.3. Then the same reasoning to (1) gives $D^s(I)_{N_0} \otimes^{\square}_{K[t]} A_s \simeq D^s(I)_{N_0} \widehat{\otimes}_K A_s$, and passing to the limit yields (2). $\square$

3.2. **Eigenvarieties from overconvergent homology.** In this section, we recall the construction of eigenvarieties from overconvergent (co)homology, developed in [AS, Urb11, Han17]. This construction involves some choices, leading to many variants. The main choices are to use either cohomology or homology, and to use either locally analytic function modules or locally analytic distribution modules. In some sense these choices are dual to one another; see e.g. [Bel21, Theorem III.3.11] for a general form of Poincaré duality, and [Han17, Proposition 2.2.1] for the duality between function modules and distribution modules. Moreover, most references use smaller locally symmetric spaces than the ones from §2.1; they take the quotient of $\mathbf{G}(\mathbb{R})^+$ by K_∞ and the whole connected part $\mathbf{Z}(\mathbb{R})^+$ of the center. Here, we will use homology and distribution modules, and the locally symmetric spaces from §2.1; the main result of this paper is that this variant precisely coincides with Fu's construction. The precise construction we will use is that of [JN19a], since the distribution modules used there are orthonormalizable and commute with arbitrary base change, which turns out to be convenient (one could also use [Gul19]) later on in the comparison.

Let us now briefly recall the eigenvariety construction from [JN19a] (but only over $\mathbb{Q}_p$), adapted as mentioned above. Recall that $\mathcal{W} = \widehat{T_0}$ denotes the weight space. To each affinoid open $\Omega \subseteq \mathcal{W}$ and each $s \in (0,1)$ sufficiently close to 1 (depending on Ω), [JN19a, §3.3] constructs a Banach $\mathcal{O}(\Omega)$-module D^s_Ω such that

- D^s_Ω is orthonormalizable;
- $\Delta = IT^+I$ acts on D^s_Ω from the left, and each $t \in T^{cpt}$ acts as a compact operator;
- When $r \leq s$ we have $D^s_\Omega \subseteq D^r_\Omega$, compatibly with the actions of Δ;
- If $\Omega' \subseteq \Omega$ is another affinoid open, then there are canonical isomorphisms $D^s_\Omega \widehat{\otimes}_{\mathcal{O}(\Omega)} \mathcal{O}(\Omega') \cong D^s_{\mathcal{O}(\Omega')}$ compatible with changing s and the actions of Δ.

From this, the eigenvariety construction goes as follows (see [JN19a, §4.1]): Let $K = K^pI$, choose an element $t \in T^{cpt}$, and consider the chain homotopic complexes

$$C_\bullet(K, D^s_\Omega) \sim C^{BS}_\bullet(K, D^s_\Omega)$$

for varying Ω and s. We have a Hecke action on $C_\bullet(K, D^s_\Omega)$, which we can transfer element by element to $C^{BS}_\bullet(K, D^s_\Omega)$ via the chain homotopy equivalence (as in §3.1). In particular, U_t acts a compact operator on each of the (finitely many) terms of $C^{BS}_\bullet(K, D^s_\Omega)$, all of which satisfies Buzzard's property (Pr). Thus, we may define the Fredholm series $F^s_\Omega \in \mathcal{O}(\Omega \times \mathbb{A}^1)$ of U_t acting on the direct sum $C_*(K, D^s_\Omega)$ of the $C^{BS}_i(K, D^s_\Omega)$. One then proves that F^s_Ω is independent of s and is compatible with change of Ω, and hence comes from a Fredholm series $F \in \mathcal{O}(\mathcal{W} \times \mathbb{A}^1)$.

Having the Fredholm series $F \in \mathcal{O}(\mathcal{W} \times \mathbb{A}^1)$ we consider the corresponding spectral variety $\mathcal{Z} = \{F = 0\} \subseteq \mathcal{W} \times \mathbb{A}^1$ and its cover $\mathcal{Z} = \bigcup_{(\Omega,h)} \mathcal{Z}_{\Omega,h}$, where (Ω, h) ranges over slope data for U_t in the sense of [Han17]. By definition, if (Ω, h) is a slope datum then $C_\bullet(K, D^s_\Omega)$ has a slope $\leq h$-decomposition, with slope $\leq h$-summand $C_\bullet(K, D^s_\Omega)_{\leq h}$, which is a bounded complex of finitely generated projective $\mathcal{O}(\Omega)$-modules (and hence, a fortiori, a bounded complex of finitely generated $\mathcal{O}(\mathcal{Z}_{\Omega,h})$-modules), which turns out to be independent of s. Moreover, if (Ω, h) is a slope datum and $\Omega' \subseteq \Omega$, then (Ω', h) is a slope datum and we have $C_\bullet(K, D^s_\Omega)_{\leq h} \otimes_{\mathcal{O}(\Omega)} \mathcal{O}(\Omega') \cong C_\bullet(K, D^s_{\Omega'})_{\leq h}$. This allows us to glue the $C_\bullet(K, D^s_\Omega)_{\leq h}$ into a bounded complex $\mathscr{C}_\bullet$ of coherent sheaves on $\mathcal{Z}$. Taking its homology, we obtain a graded coherent sheaf $\mathscr{H}_*(K^p)$ on $\mathcal{W} \times \mathbb{A}^1$, which carries an action of the Hecke algebra. From this, one can build the eigenvariety as the relative spectrum of the Hecke algebra as a subalgebra of $\mathrm{End}_{\mathcal{W}\times\mathbb{A}^1}(\mathscr{H}_*(K^p))$.

3.3. **Hecke operators up to homotopy.** Both constructions of eigenvarieties feature transferring Hecke actions from one complex to another complex using a chain homotopy equivalence. Concretely (but still somewhat imprecisely), one may describe the situation as follows: we have two complexes $B = B_\bullet$ and $C = C_\bullet$ and a chain homotopy equivalence $f : B \to C$, with "chain homotopy inverse" $g : C \to B$. We can then produce a map *of abelian groups* $\mathrm{End}(B) \to \mathrm{End}(C)$ by sending $U \in \mathrm{End}(B)$ to $fUg \in \mathrm{End}(C)$. After passing to homology, this map respects composition, and this is what is used in both eigenvariety constructions. Note, however, that if we choose $U \in \mathrm{End}(B)$ and make B and C into complexes of $\mathbb{Z}[X]$-modules by letting X act by U on B and by fUg on C, then $f : B \to C$ need not be $\mathbb{Z}[X]$-linear. The goal of this section is to show that, nevertheless, B and C are quasi-isomorphic as complexes of $\mathbb{Z}[X]$-modules in this situation.

Our setup is slightly more general. We let R be a solid commutative ring and we let $B = B_\bullet$ and $C = C_\bullet$ be bounded below chain complexes of R-modules. As indicated above, we assume that we have a chain homotopy equivalence $f : B \to C$, with "chain homotopy inverse" $g : C \to B$. Assume that we have a chain map $U : B \to B$; we can then form the chain map $V : C \to C$ given by fUg. Let $R[X] := R \otimes_{\mathbb{Z}}^{\square} \mathbb{Z}[X]$ be the polynomial algebra over R in one variable X. We make B and C into complexes of $R[X]$-modules by letting X act as U and V, respectively. As mentioned above, our goal is to show that B and C are quasi-isomorphic as complexes of $R[X]$-modules.

To do this, we need to replace B and C. We make the following constructions: If M is a complex of R-modules, we let $M[X]$ be the complex $M \otimes_R^{\square} R[X]$ of $R[X]$-modules. Any R-linear map $h : M \to N$ induces an $R[X]$-linear map $h : M[X] \to N[X]$ concretely defined as

$$h\left(\sum_i m_i X^i\right) = \sum_i h(m_i) X^i.$$

We then have the following:

Proposition 3.3.1. *We have a short exact sequence*

$$0 \to B[X] \to B[X] \to B \to 0$$

of complexes of $R[X]$-modules, where the map $B[X] \to B[X]$ is "multiplication by $X - U$" (i.e. $\sum_i b_i X^i \mapsto \sum_i (b_i X^{i+1} - U(b_i) X^i)$) and the map $B[X] \to B$ is given by $\sum_i b_i X^i \mapsto \sum_i U^i(b_i)$. The analogous statement holds for C.

Proof. For injectivity of $B[X] \to B[X]$, note that $X - U$ is "monic" (the proof is the same as showing that multiplication by a monic polynomial is injective). Surjectivity of $B[X] \to B$ is clear. For exactness in middle, it is clear that the composition of the maps is 0, so it remains to prove that the kernel is contained in the image. Consider $\sum_{i=0}^n b_i X^i \in B[X]$. We wish to rewrite it as

$$\sum_{i=0}^n b_i X^i = (X - U)\left(\sum_{j=0}^{n-1} c_j X^j\right) + c_n$$

for some $c_0, \dots, c_n \in B$. If so, then the claim follows, as c_n must be $\sum_i U^i(b_i)$ (substitute $X = U$). To write it like that, we expand and see that we have to solve the usual equations

$$c_{n-1} = b_n, c_{n-2} - U(c_{n-1}) = b_{n-1}, \dots, c_0 - U(c_1) = b_1 \text{ and } c_n - U(c_0) = b_0,$$

which we can do. □

Corollary 3.3.2. *B is quasi-isomorphic to the cone $\overline{B}$ of $X - U : B[X] \to B[X]$. Explicitly, we have a $R[X]$-linear quasi-isomorphism $q_B : \overline{B} \to B$ given by*

$$q_B : B_{n-1}[X] \oplus B_n[X] \to B_n, \quad q_B\left(\sum_i a_i X^i, \sum_j b_j X^j\right) = \sum_j U^j(b_j).$$

The map $p_B : B \to \overline{B}$ given by $p_B(b) = (0, b)$ is an R-linear quasi-isomorphism which is inverse to q_B on homology. The analogous statements hold for C. Note that p_B is a chain map (an easy verification using the formula for the differential on $\overline{B}$ recalled below).

Proof. That q_B is a quasi-isomorphism is a standard consequence of the Proposition. To see that p_B is an R-linear inverse on homology, note that $q_B \circ p_B = \mathrm{id}_B$. □

At this point, let us recall that the differential of $\overline{B}$ is the map $B_{n-1}[X] \oplus B_n[X] \to B_{n-2}[X] \oplus B_{n-1}[X]$ given by the matrix

$$\begin{pmatrix} -d & 0 \\ U - X & d \end{pmatrix},$$

where d denotes the differential on B. We can now define a map $\overline{B} \to \overline{C}$ by the matrix

$$F = \begin{pmatrix} f & 0 \\ fUs & f \end{pmatrix},$$

where s is a homotopy on B satisfying $\mathrm{id}_B - gf = ds + sd$. The statement we desire is the following.

Proposition 3.3.3. *F is a quasi-isomorphism of $R[X]$-modules.*

Proof. First, we need to prove that F really is a chain map. We compute

$$\begin{pmatrix} f & 0 \\ fUs & f \end{pmatrix} \begin{pmatrix} -d & 0 \\ U - X & d \end{pmatrix} = \begin{pmatrix} -fd & 0 \\ -fUsd + f(U - X) & fd \end{pmatrix}$$

and

$$\begin{pmatrix} -d & 0 \\ V - X & d \end{pmatrix} \begin{pmatrix} f & 0 \\ fUs & f \end{pmatrix} = \begin{pmatrix} -df & 0 \\ (V - X)f + dfUs & df \end{pmatrix}.$$

We have $df = fd$, so to see that these are equal it remains to show that

$$-fUsd + f(U - X) = (V - X)f + dfUs.$$

Since $Xf = fX$, we can remove the terms involving X. This leaves us with showing that $-fUsd + fU = Vf + dfUs$. We then have

$$-fUsd + fU - Vf - dfUs = -fU(sd + ds) + fU(id - gf) = 0$$

using that f and U are chain maps, $V = fUg$ and $sd + ds = id - gf$.

To show that F is a quasi-isomorphism, consider the sequence

$$B \xrightarrow{p_B} \overline{B} \xrightarrow{F} \overline{C} \xrightarrow{q_C} C.$$

Since p_B and q_C are quasi-isomorphisms, it suffices to show that the composition is a quasi-isomorphism. We compute:

$$q_C(F(p_B(b))) = q_C\left(\begin{pmatrix} f & 0 \\ fUs & f \end{pmatrix}\begin{pmatrix} 0 \\ b \end{pmatrix}\right) = q_c\left(\begin{pmatrix} 0 \\ f(b) \end{pmatrix}\right) = f(b).$$

So $q_C \circ F \circ p_B = f$, which is a quasi-isomorphism as desired. $\square$

Together with a Tor-vanishing result carried out in Section 4.2, the above work equates our two definitions of Jacquet functors.

Corollary 3.3.4. *We have an isomorphism $J^\vee_{B,\mathrm{Fu}}(C^{\mathrm{BS}}_\bullet(K^pI, D(I))) \simeq J^\vee_B(C_\bullet(K^pI, D(I)))$.*

Proof. Applying Proposition 3.3.3 in the case of $B = C^{\mathrm{BS}}_\bullet(K^pI, D(I)_{N_0})$ and $C = C_\bullet(K^pI, D(I)_{N_0})$, we have that these complexes are $K[t]$-equivariantly isomorphic for a fixed $t \in T^{cpt}$. Thus we have $B \otimes^{\square,L}_{K[t]} A \simeq C \otimes^{\square,L}_{K[t]} A \simeq J^\vee_B(C_\bullet(K^pI, D(I)))$, where the second isomorphism holds since $t \in T^{cpt}$. Proposition 4.2.2 then implies $B \otimes^{\square,L}_{K[t]} A \simeq B \otimes^{\square}_{K[t]} A \simeq J^\vee_{B,\mathrm{Fu}}(C^{\mathrm{BS}}_\bullet(K^pI, D(I)_{N_0}))$, as required. $\square$

4. Comparing completed and overconvergent homology

This section proves the main comparison theorem. First, we prove some results about the derived functors of the finite slope functor, completing the comparison of Fu's Jacquet functors and the solid Jacquet functor. We then compare the solid Jacquet functors to overconvergent homology. In a sense, the strategy is straightforward: We will start with the object $C^{BS}_\bullet(K^pI, D(I)_{N_0}) \otimes^{\square}_{K[t]} A$ from Fu's construction and, by direct manipulations, show that it unravels to the complex $\mathscr{C}_\bullet$ in the theory of overconvergent homology.

4.1. **A formula for distribution modules.** Before we start manipulating $C^{BS}_\bullet(K^pI, D(I)_{N_0}) \otimes^{\square}_{K[t]} A$, we will prove a result relating $D(I)_{N_0}$ to the distribution modules $D_\Omega := \varprojlim_s D^s_\Omega$ in overconvergent homology. Let Ω be an open affinoid subset of weight space. Put $R = \mathcal{O}(\Omega)$, a Banach K-algebra, and let $\kappa : T_0 \to R^\times$ be the corresponding weight. The goal of this subsection is to show that $D(I)_{N_0} \otimes^{\square}_{D(T_0)} R$ and D_Ω are canonically isomorphic. As a first step, we replace $D(I)_{N_0} \otimes^{\square}_{D(T_0)} R$ by $D(I) \otimes^{\square}_{D(B_0)} R$. First we note the following: if X and Y are compact p-adic manifolds, then $D(X \times Y) \cong D(X) \otimes^{\square} D(Y)$.

Lemma 4.1.1. *Let X be a compact p-adic manifold with a faithful right action of a compact p-adic Lie group G. Assume that we have an isomorphism $X \cong Y \times G$ as right G-manifolds, where $Y \subseteq X$ is a closed submanifold, and assume that $N \subseteq G$ is a closed normal subgroup with quotient $H = G/N$. Then $D(X) \otimes^{\square}_{D(G)} D(H) \cong D(Y \times H)$.*

Proof. Upon noting that $D(X) \cong D(Y) \otimes^{\square} D(G)$ also as right $D(G)$-modules, this is clear from the remark before the lemma. $\square$

Proposition 4.1.2. *We have $D(I)_{N_0} \otimes^{\square}_{D(T_0)} R \cong D(I) \otimes^{\square}_{D(B_0)} R$. More generally, $D(I)_{N_0} \otimes^{\square}_{D(T_0)} M \cong D(I) \otimes^{\square}_{D(B_0)} M$ for any solid $D(T_0)$-module M.*

Proof. The general version reduces directly to the case $M = D(T_0)$. This case then follows from Lemma 4.1.1 (both sides are isomorphic to $D(\overline{N}_1 \times T_0)$); note that in the definition $D(I)_{N_0} = D(I) \otimes^{\square}_{D(N_0)} K$ we may think of K as the distribution algebra $D(1)$ for the trivial group 1. $\square$

Next, by definition (as recalled in §A), $D(I) \otimes^{\square}_{D(B_0)} R$ is the coequalizer of

$$D(I) \otimes^{\square} D(B_0) \otimes^{\square} R \rightrightarrows D(I) \otimes^{\square} R.$$

We are going to compare this to the diagram

$$D(I \times B_0, R) \rightrightarrows D(I, R),$$

which is R-dual to the diagram

$$C(I, R) \rightrightarrows C(I \times B_0, R),$$

where one map is $f \mapsto ((g, b) \mapsto f(gb))$ and the other is $f \mapsto ((g, b) \mapsto \kappa(b)f(g))$. By definition, the equalizer of this diagram is A_Ω. So, we want to do the following:

(1) Construct a commutative diagram

$$\begin{array}{ccc} D(I) \otimes^{\square} D(B_0) \otimes^{\square} R & \rightrightarrows & D(I) \otimes^{\square} R \\ \downarrow & & \downarrow \\ D(I \times B_0, R) & \rightrightarrows & D(I, R) \end{array}$$

where the vertical arrows are isomorphisms.

(2) Show that the coequalizer of $D(I \times B_0, R) \rightrightarrows D(I, R)$ is D_Ω (which is the R-dual of A_Ω).

Together, these two assertions would give the desired isomorphism $D(I) \otimes^{\square}_{D(B_0)} R \cong D_\Omega$. We start with the second one.

Lemma 4.1.3. *Let H be a compact p-adic analytic group. Then $C(H, R)$ and $D(H, R)$ are R-reflexive and R-duals of each other. In particular, A_Ω and D_Ω are R-reflexive, and are R-duals of each other.*

Proof. For the first part we need to equate $\underline{\mathrm{Hom}}_R(-, R)$ with the usual R-module dual. One can argue as follows: Consider $C^s(H, R)$. Letting R_0 be the unit ball for a suitable norm on R, we can write

$$C^s(H, R) \cong \widehat{\bigoplus} R := \left(\varprojlim_n \bigoplus R_0/\varpi^n \right) [1/\varpi].$$

By direct computation we then see that $\underline{\mathrm{Hom}}_R(C^s(H, R), R)$ is isomorphic to

$$\left(\prod R_0 \right) [1/\varpi]$$

with the weak topology, i.e. the topology induced from the product topology on $\prod R_0$. Taking $\underline{\mathrm{Hom}}_R(-, R)$ of this returns $\widehat{\bigoplus} R$, and this is exactly what happens on the topological side when one takes $\mathrm{Hom}_{R,cts}(-, R)$. The difference between the weak and strong topology then goes away after letting $s \to \infty$.

For the second part, use that $A_\Omega \cong C(\overline{N}_1, R)$ and $D_\Omega \cong D(\overline{N}_1, R)$. $\square$

We then have:

Proposition 4.1.4. *The coequalizer of* $D(I \times B_0, R) \rightrightarrows D(I, R)$ *is* D_Ω.

Proof. To prove this, we show that A_Ω is not only an equalizer, but in fact a split equalizer [Sta18, Tag 08WF]. The splitting is given as follows: We need to produce maps

$$C(I, R) \to A_\Omega \quad \text{and} \quad C(I \times B_0, R) \to C(I, B)$$

satisfying the properties specified in [Sta18, Tag 08WF]. If $g \in I$, we write $g = \overline{n}_g b_g$ for its Iwahori decomposition with respect to $I = \overline{N}_1 B_0$. Then the function $C(I, R) \to A_\Omega$ is given by

$$f \mapsto (g \mapsto \kappa(b_g) f(\overline{n}_g))$$

and the map $C(I \times B_0, R) \to C(I, B)$ is given by

$$F \mapsto (g \mapsto \kappa(b_g) F(g, b_g^{-1})).$$

Now contravariant functors send split equalizers to (split) coequalizers, so the Lemma 4.1.3 gives the claim. □

This finishes the proof of the second property, so we move on to the first. We want isomorphisms $D(I) \otimes^\square D(B_0) \otimes^\square R \to D(I \times B_0, R)$ and $D(I) \otimes^\square R \to D(I, R)$. These are "standard" if you replace the solid tensor product with the usual completed tensor product, and since everything in sight is a Fréchet space these two tensor products agree. To check the commutativity, the easiest thing is to compute on Dirac distributions and use continuity (note that this is computation on the underlying topological spaces). So, the two maps are given on Dirac distributions by

$$g \otimes b \otimes r \mapsto r(g, b)$$

and

$$g \otimes r \mapsto rg.$$

The maps

$$D(I) \otimes^\square D(B_0) \otimes^\square R \rightrightarrows D(I) \otimes^\square R.$$

are given by $g \otimes b \otimes r \mapsto gb \otimes r$ and $g \otimes b \otimes r \mapsto g \otimes \kappa(b) r$. The two maps

$$D(I \times B_0, R) \rightrightarrows D(I, R),$$

are given by $r(g, b) \to r.gb$ and $r(g, b) \mapsto r\kappa(b).g$. The commutativity is then a simple check.

4.2. **Derived functors of $\otimes^\square_{K[t]}$.** To get started with $C^{BS}_\bullet(K^p I, D(I)_{N_0}) \otimes^\square_{K[t]} A$, our next goal is to show that in fact

$$C^{BS}_\bullet(K^p I, D(I)_{N_0}) \otimes^\square_{K[t]} A = C^{BS}_\bullet(K^p I, D(I)_{N_0}) \otimes^{L\square}_{K[t]} A.$$

More generally, we will discuss how to compute the higher derived functors of $- \otimes^\square_{K[t]} A$. Our starting point is a solid K-vector space V which is flat for $\otimes^\square$ (over K), equipped with an endomorphism U, making it into a solid $K[t]$-module. For example, any quasiseparated solid K-vector space is flat, by [RJRC22, Lemma 3.21]. Then, just like in §3.3, one has a resolution

$$0 \to V[t] \to V[t] \to V \to 0,$$

where the map sends $\sum v_i t^i$ to $\sum (v_i t^{i+1} - U(v_i) t^i)$, and $V[t]$ is flat for $\otimes^\square_{K[t]}$. We conclude that $\mathrm{Tor}^{\square, i}_{K[t]}(V, A) = 0$ for $i \geq 2$ and that

$$\mathrm{Tor}^{\square, 1}_{K[t]}(V, A) = \mathrm{Ker}(V[t] \otimes^\square_{K[t]} A \to V[t] \otimes^\square_{K[t]} A).$$

We now want to try to make that kernel more explicit. By definition, it is given by

$$\sum_i v_i t^i \otimes a \mapsto \sum_i (v_i t^{i+1} - U(v_i)t^i) \otimes a.$$

We have an isomorphism $V[t] \otimes^{\square}_{K[t]} A \cong V \otimes^{\square} A$, and with respect to this the formula becomes

$$v \otimes a \mapsto v \otimes at - U(v) \otimes a.$$

When V is a Fréchet space, we can go one step further, writing $V \otimes^{\square} A \cong V\{\{t^{\pm 1}\}\}$. Then the formula becomes

$$\sum_i v_i t^i \mapsto \sum_i (v_i t^{i+1} - U(v_i)t^i).$$

We can summarize these calculations.

Proposition 4.2.1. *When V is a Fréchet space, the elements of $\mathrm{Tor}^{\square,1}_{K[t]}(V, A)$ can be described as the power series $\sum_i v_i t^i \in V\{\{t^{\pm 1}\}\}$ satisfying $U(v_{i+1}) = v_i$ for all i. Similarly, when V is a Banach space, the elements of $\mathrm{Tor}^{\square,1}_{K[t]}(V, A_h)$ can be described as the power series $\sum_i v_i t^i \in V\langle p^h t, p^h t^{-1}\rangle$ satisfying $U(v_{i+1}) = v_i$ for all i.*

We can then give a criterion for the vanishing of $\mathrm{Tor}^{\square,1}_{K[t]}(V, A)$. In fact, we can give a more precise criterion for the vanishing of $\mathrm{Tor}^{\square,1}_{K[t]}(V, A_h)$ when V is a Banach space.

Proposition 4.2.2. *Assume that V is a Fréchet space with an endomorphism U.*

1. *Assume that V is a Banach space and that the operator norm of U is strictly less than p^h. Then $\mathrm{Tor}^{\square,1}_{K[t]}(V, A_h) = 0$.*
2. *Assume instead that $V = \varprojlim_n V_n$ is a Fréchet space written as a projective limit of Banach spaces V_n, and that U arises as the limit of bounded operators U_n on the V_n. Then $\mathrm{Tor}^{\square,1}_{K[t]}(V, A) = 0$.*

Proof. To start, assume that V is as in (2) (which includes V being a Banach space as a special case). Then we claim that

$$\mathrm{Tor}^{\square,1}_{K[t]}(V, A) = \varprojlim_{h,n} \mathrm{Tor}^{\square,1}_{K[t]}(V_n, A_h).$$

To see this, consider the resolution $0 \to V[t] \to V[t] \to V \to 0$ as above. Then it suffices to show:

$$V \otimes^{\square} A \cong \varprojlim_{h,n} V_n \otimes^{\square} A_h \tag{3}$$

$$\ker(V \otimes^{\square} A \to V \otimes^{\square} A) \cong \varprojlim_{h,n} \ker(V_n \otimes^{\square} A_h \to V_n \otimes^{\square} A_h). \tag{4}$$

Now (3) is [RJRC22, Lemma 3.28], and (4) follows from (3) and limits commuting with other limits (the kernel in this case).

From the isomorphism $\mathrm{Tor}^{\square,1}_{K[t]}(V, A) = \varprojlim_{h,n} \mathrm{Tor}^{\square,1}_{K[t]}(V_n, A_h)$, we see that part (2) of the proposition follows from part (1), so it remains to prove part (1). Choose a norm $|-|$ on V. By Proposition 4.2.1, $\mathrm{Tor}^{\square,1}_{K[t]}(V, A_h)$ is the set of $\sum_i v_i t^i$ such that $p^{-hi} v_i$ and $v_{-i} p^{-hi}$ are bounded as $i \to +\infty$, and $U_n(v_{i+1}) = v_i$ for all i. Let λ be the operator norm of U. Then we have

$$\lambda^m \cdot |v_{m+i}| \geq |U^m(v_{m+i})| = |v_i|$$

for all i and all $m \geq 0$. But $\lambda^m \cdot |v_{m+i}| = (\lambda p^{-h})^m p^{hi} |p^{-h(i+m)} v_{i+m}| \to 0$ as $m \to +\infty$, since $\lambda < p^h$ and $|p^{-h(i+m)} v_{i+m}|$ is bounded. We conclude that $v_i = 0$ for all i, as required. $\square$

4.3. **The Main Theorem.** Our goal now is compute $C_\bullet^{BS}(D(I)_{N_0}) \otimes_{K[t]}^{\square} A$ in a way that directly compares it to overconvergent homology. The starting point is the isomorphism

$$D(I)_{N_0} \otimes_{D(T_0)}^{\square} \mathcal{O}(\Omega) \cong D(I) \otimes_{D(B_0)}^{\square} \mathcal{O}(\Omega) \cong D_\Omega$$

established in §4.1, where $\Omega \subseteq \mathcal{W}$ is an affinoid open subset of weight space. The first step is the following:

Proposition 4.3.1. *We have* $\varprojlim_\Omega D(I)_{N_0} \otimes_{D(T_0)}^{\square} \mathcal{O}(\Omega) \cong D(I)_{N_0}$, *and thus* $\varprojlim_\Omega D_\Omega \cong D(I)_{N_0}$.

Proof. We prove that $\varprojlim_\Omega D(I) \otimes_{D(B_0)}^{\square} \mathcal{O}(\Omega) \cong D(I)_{N_0}$. As noted in §4.1, $D(I) \otimes_{D(B_0)}^{\square} \mathcal{O}(\Omega)$ is the *split* coequalizer of

$$D(I) \otimes^{\square} D(B_0) \otimes^{\square} \mathcal{O}(\Omega) \rightrightarrows D(I) \otimes^{\square} \mathcal{O}(\Omega).$$

The splittings are compatible with varying Ω, so taking $\varprojlim_\Omega$ and noting that $D(T_0) = \mathcal{O}(\mathcal{W}) = \varprojlim_\Omega \mathcal{O}(\Omega)$, we see that $\varprojlim_\Omega D(I) \otimes_{D(B_0)}^{\square} \mathcal{O}(\Omega)$ is the (split) coequalizer of

$$D(I) \otimes^{\square} D(B_0) \otimes^{\square} D(T_0) \rightrightarrows D(I) \otimes^{\square} D(T_0).$$

But this coequalizer is $D(I) \otimes_{D(B_0)}^{\square} D(T_0)$ by definition, which is $D(I)_{N_0}$ by Proposition 4.1.2. $\square$

Next, we write $D_\Omega = \varprojlim_s D_\Omega^s$. Our goal now is to gradually understand $C_\bullet^{BS}(K, D_\Omega^s) \otimes_{K[t]}^{\square} A_h$. Recall the fixed compact operator U_t on the $C_\bullet^{BS}(K, D_\Omega^s)$. To simplify the notation, and to be able to use t as a coordinate on $\mathbb{G}_m$, we will write $U := U_t$ and forget the notation $t \in T^{cpt}$. With all this in mind, consider the operator

$$C_\bullet^{BS}(K, D_\Omega^s) \otimes^{\square} A_h \to C_\bullet^{BS}(K, D_\Omega^s) \otimes^{\square} A_h$$

which sends $\delta \otimes a$ to $\delta \otimes a - U(\delta) \otimes t^{-1} a$. By definition, $C_\bullet^{BS}(K, D_\Omega^s) \otimes_{K[t]}^{\square} A_h$ is the cokernel of this operator. Note that the operator is the identity minus $\delta \otimes a \mapsto U(\delta) \otimes t^{-1} a$, which is a compact operator on the Banach $\mathcal{O}(\Omega_h) := \mathcal{O}(\Omega) \otimes^{\square} A_h$-module $C_\bullet^{BS}(K, D_\Omega^s) \otimes^{\square} A_h$.

Thus, $C_\bullet^{BS}(K, D_\Omega^s) \otimes_{K[t]}^{\square} A_h$ is a finitely generated $\mathcal{O}(\Omega_h)$-module by Lemma 3.1.3. Our next goal is to establish compatibility when varying s, Ω and h. For Ω and h, compatibility follows from the following lemma.

Corollary 4.3.2. $(C_\bullet^{BS}(K, D_\Omega^s) \otimes_{K[t]}^{\square} A_h)_{\Omega,h}$ *forms a coherent sheaf on* $\mathcal{W} \times \mathbb{G}_m$.

Proof. Apply Lemma 3.1.4 to the maps $U \otimes t^{-1} : C_\bullet^{BS}(K, D_\Omega^s) \otimes^{\square} A_h \to C_\bullet^{BS}(K, D_\Omega^s) \otimes^{\square} A_h$ for varying Ω and h. $\square$

Next, we want to show independence of s for $(C(D_\Omega^s) \otimes_{K[t]}^{\square} A_h)_{\Omega,h}$. For this we revert to the standard slope decomposition techniques. One advantage of choosing the analytic distribution modules from [JN19a] is that they commute with base change even before taking slope decompositions. In particular, one has the following:

Corollary 4.3.3. *Let* $\kappa \in \Omega$ *be a closed point with residue field* K'. *Then* $(C_\bullet^{BS}(K, D_\Omega^s) \otimes_{K[t]}^{\square} A_h) \otimes_{\mathcal{O}(\Omega)} K' \cong C_\bullet^{BS}(K, D_\kappa^s) \otimes_{K[t]}^{\square} A_h$.

Proof. Since $C(D^s_\Omega) \otimes^\square_{\mathcal{O}(\Omega)} K' \cong C(D^s_\kappa)$, this is a direct consequence of Lemma 3.1.4 $\square$

To make this useful, we want to couple it with the relation to slope decomposition. We need a preliminary lemma.

Lemma 4.3.4. *We have $A_h \otimes^\square_{K[t]} A_h = A_h$. As a consequence, $M \otimes^\square_{K[t]} A_h = M$ for any solid A_h-module M.*

Proof. $A_h \otimes^\square_{K[t]} A_h$ is the cokernel of the map $\varphi : A_h \otimes^\square A_h \to A_h \otimes^\square A_h$ given by $f \otimes g \mapsto tf \otimes g - f \otimes tg$. Now note that $A_h \otimes^\square A_h$ is the ring of functions on the subset $p^{-h} \le X, Y \le p^h$ of $\mathbb{G}_m^2$ (with coordinates X and Y), and in this notation φ is the map given by multiplication by $X - Y$. Thus, the quotient is A_h as desired.

For the final statement, note that $M = M \otimes^\square_{A_h} A_h = M \otimes^\square_{A_h} (A_h \otimes^\square_{K[t]} A_h) = M \otimes^\square_{K[t]} A_h$. $\square$

Proposition 4.3.5. *Let Ω be an affinoid weight (i.e. an affinoid rigid space Ω over K with a K-map $\Omega \to \mathcal{W}$), not necessarily open in weight space. If (Ω, h) is a slope datum for $C^{BS}_\bullet(K, D^s_\Omega)$, then $C^{BS}_\bullet(K, D^s_\Omega) \otimes^\square_{K[t]} A_h \cong C^{BS}_\bullet(K, D^s_\Omega)_{\le h}$. In particular, both sides are independent of s.*

Proof. Write $M = C^{BS}_\bullet(K, D^s_\Omega)$ for simplicity. Then we have the slope decomposition $M = M_{\le h} \oplus M_{>h}$ and we need to show that $M_{>h} \otimes^\square_{K[t]} A_h = 0$ and $M_{\le h} \otimes^\square_{K[t]} A_h = M_{\le h}$.

We start by showing that $M_{>h} \otimes^\square_{K[t]} A_h = 0$. Note that $M_{>h} \otimes^\square A_h = (M \otimes^\square A_h)_{>h}$ and that the polynomial $Q(X) = X - t^{-1} \in \mathcal{O}(\Omega_h)[X]$ has slope $\le h$. We have $Q^*(X) = 1 - Xt^{-1}$ and hence, by the defining property of slope decompositions, $Q^*(U) = 1 - Ut^{-1}$ is an automorphism of $(M \otimes^\square A_h)_{>h}$. This shows that $M_{>h} \otimes^\square_{K[t]} A_h = 0$, as desired.

It remains to show that $M_{\le h} \otimes^\square_{K[t]} A_h = M_{\le h}$. By Lemma 4.3.4, it suffices to show that $M_{\le h}$ is already an A_h-module. But this is a standard property of slope decompositions arising from slope factorizations. $\square$

Corollary 4.3.6. *The coherent sheaf $(C^{BS}_\bullet(K, D^s_\Omega) \otimes^\square_{K[t]} A_h)_{\Omega,h}$ is supported on the spectral variety of U in $\mathcal{W} \times \mathbb{G}_m$ and equals the coherent sheaf constructed in the theory of overconvergent homology. In particular, it is independent of s.*

Proof. Corollary 4.3.3 and Proposition 4.3.5 show that the reduced support of $(C^{BS}_\bullet(K, D^s_\Omega) \otimes^\square_{K[t]} A_h)_{\Omega,h}$ agrees with the reduced spectral variety. The result follows by using the Coleman–Buzzard cover of the spectral variety (in the slope decomposition form) and Proposition 4.3.5 again. $\square$

At this point, it remains to study the limits over s, Ω and h. One should be a little bit careful about the order one takes the limits in. A priori, one can start with either s or h, but the limit over Ω needs to be taken after the limit over s. We start by taking the limit over h.

Proposition 4.3.7. *We have $C^{BS}_\bullet(K, D^s_\Omega) \otimes^\square_{K[t]} A = \varprojlim_h C^{BS}_\bullet(K, D^s_\Omega) \otimes^\square_{K[t]} A_h$ and $\mathrm{Tor}^{\square,1}_{K[t]}(C^{BS}_\bullet(K, D^s_\Omega), A) = \varprojlim_h \mathrm{Tor}^{\square,1}_{K[t]}(C^{BS}_\bullet(K, D^s_\Omega), A_h) = 0$. Moreover $C^{BS}_\bullet(K, D^s_\Omega) \otimes^\square_{K[t]} A$ is independent of s.*

Proof. To simplify the notation, write (in this proof only) $M_h = C^{BS}_\bullet(K, D^s_\Omega) \otimes^\square A_h$, $C_h = C^{BS}_\bullet(K, D^s_\Omega) \otimes^\square_{K[t]} A_h$ and $K_h = \mathrm{Tor}^{\square,1}_{K[t]}(C^{BS}_\bullet(K, D^s_\Omega), A_h)$, and M, C and K for the same things with A_h replaced by A. Then we have exact sequences

$$0 \to K \to M \to M \to C \to 0$$

and

$$0 \to K_h \to M_h \to M_h \to C_h \to 0.$$

Since $M = \varprojlim_h M_h$, we need to show that taking $\varprojlim_h$ over the second displayed equation (viewed as a system with respect to h) preserves exactness. Since $\varprojlim_h$ preserves kernels, we already have that $K = \varprojlim_h K_h$. By Proposition 4.2.2, $K_h = 0$ for all large enough h. Therefore, it remains to prove that $C = \varprojlim_h C_h$.

To do this, it suffices to consider h large enough so that $K_h = 0$. Then consider the short exact sequence

$$0 \to M_h \to M_h \to C_h \to 0.$$

The maps $M_{h'} \to M_h$ have dense image for all $h \leq h'$, so by topological Mittag-Leffler [RJRC22, Lemma 3.27], $R^i \varprojlim_h M_h = 0$ for all $i \geq 1$ and hence $C = \varprojlim_h C_h$ as desired. Independence of s then follows since the $C_\bullet^{BS}(K, D_\Omega^s) \otimes_{K[t]}^\square A_h$ are independent of s. $\square$

We have now established short exact sequences

$$0 \to C_\bullet^{BS}(K, D_\Omega^s) \otimes^\square A \to C_\bullet^{BS}(K, D_\Omega^s) \otimes^\square A \to C_\bullet^{BS}(K, D_\Omega^s) \otimes_{K[t]}^\square A \to 0$$

for all s and Ω. We now take the limit over s.

Proposition 4.3.8. *We have $C_\bullet^{BS}(K, D_\Omega) \otimes_{K[t]}^\square A = \varprojlim_s (C_\bullet^{BS}(K, D_\Omega^s) \otimes_{K[t]}^\square A)$. In particular, $C_\bullet^{BS}(K, D_\Omega) \otimes_{K[t]}^\square A = (C_\bullet^{BS}(K, D_\Omega^s) \otimes_{K[t]}^\square A)$ for all sufficiently large s.*

Proof. The maps $C_\bullet^{BS}(K, D_\Omega^{s'}) \otimes^\square A \to C_\bullet^{BS}(K, D_\Omega^s) \otimes^\square A$ have dense image for all $s \leq s'$, so $R^i \varprojlim_s C_\bullet^{BS}(K, D_\Omega^s) \otimes^\square A = 0$ for all $i \geq 1$. Since $C_\bullet^{BS}(K, D_\Omega) \otimes^\square A = \varprojlim_s (C_\bullet^{BS}(K, D_\Omega^s) \otimes^\square A)$, the first part follows from taking the limit over s in the displayed equation before the proposition. The second part then follows from independence of s (Proposition 4.3.7) . $\square$

Finally, we take the limit over Ω.

Proposition 4.3.9. *We have $C_\bullet^{BS}(K, D(I)_{N_0}) \otimes_{K[t]}^\square A = \varprojlim_\Omega (C_\bullet^{BS}(K, D_\Omega) \otimes_{K[t]}^\square A)$.*

Proof. There exists a cofinal system $(\Omega_n)_n$ of all Ω such that Ω_n is connected and satisfy $\Omega_n \subseteq \Omega_{n+1}$ for all n. Then the maps $C_\bullet^{BS}(K, D_{\Omega_m}) \otimes^\square A \to C_\bullet^{BS}(K, D_{\Omega_n}) \otimes^\square A$ have dense image for all $s \leq s'$, so $R^i \varprojlim_\Omega C_\bullet^{BS}(K, D_\Omega) \otimes^\square A = 0$ for all $i \geq 1$. We have $C_\bullet^{BS}(K, D(I)_{N_0}) \otimes^\square A = \varprojlim_\Omega (C_\bullet^{BS}(K, D_\Omega) \otimes^\square A)$ by Proposition 4.3.1, so the proposition follows from taking the limit over Ω in the short exact sequence

$$0 \to C_\bullet^{BS}(K, D_\Omega) \otimes^\square A \to C_\bullet^{BS}(K, D_\Omega) \otimes^\square A \to C_\bullet^{BS}(K, D_\Omega) \otimes_{K[t]}^\square A \to 0.$$

$\square$

Summing up, we have proven the following:

Theorem 4.3.10. *$C_\bullet^{BS}(K, D(I)_{N_0}) \otimes_{K[t]}^\square A$ is the coadmissible $\mathcal{O}(\mathcal{W} \times \mathbb{G}_m)$-module corresponding to the coherent sheaf $(C_\bullet^{BS}(K, D_\Omega^s) \otimes_{K[t]}^\square A_h)_{\Omega,h}$ on $\mathcal{W} \times \mathbb{G}_m$, which is independent of s and is equal to the coherent sheaf constructed via overconvergent homology.*

Proof. This is now the statement that $C_\bullet^{BS}(K, D(I)_{N_0}) \otimes_{K[t]}^{\square} A = \varprojlim_{\Omega,h} C_\bullet^{BS}(K, D_\Omega^s) \otimes_{K[t]}^{\square} A_h$, which we have proven. ☐

We now briefly summarize how all this work implies the main result. First, Proposition 3.1.5 and Corollary 3.3.4 imply $(C_\bullet^{BS}(K, D_\Omega^s) \otimes_{K[t]}^{\square} A_h)_{\Omega,h}$ computes the dual of Fu's derived Jacquet functors. Thus, Theorem 4.3.10 proves Theorem 1.0.1, although as referenced in Section 3.2 Ash–Stevens–Hansen use *cohomology* with distribution module coefficients. An application of Poincaré duality then gives the more direct comparison of relevant (co)homology groups.

Corollary 4.3.11. *We have isomorphisms*

$$\mathrm{H}_i(C(K^p, D(I)_{N_0})\widehat{\otimes}_{K[t]}A)) \simeq \mathrm{H}^{\dim X_{K^pI}-i}(C_c^\bullet(K^pI, D(I)_{N_0})\widehat{\otimes}_{K[t]}A)$$
$$\mathrm{H}_i(C^{\mathrm{BM}}(K^p, D(I)_{N_0})\widehat{\otimes}_{K[t]}A) \simeq \mathrm{H}^{\dim X_{K^pI}-i}(C^\bullet(K^pI, D(I)_{N_0})\widehat{\otimes}_{K[t]}A).$$

In particular the coherent sheaves defining the eigenvarieties from completed homology and (compactly supported) cohomology are isomorphic.

Proof. These follow from [Spa93, Theorems 10.2 and 10.4]; see also [Bel21, Theorem 4.3.11]. ☐

5. First applications

We give some initial application of our comparison theorem.

5.1. **Triangulations of eigenvarieties over CM fields.** The first is towards triangulations of Galois representations on eigenvarieties for GL_n over a CM field F. The idea is that Hansen originally formulated such a conjecture [Han17, Conjecture 1.2.2] for eigenvarieties from overconvergent cohomology, while the second-named author's recent work [McD25, Theorem 1.1] proves a result for Fu's eigenvarieties. We can thus use the comparison theorem to directly get results on Hansen's original conjecture.

We recall the context from [McD25]. Let F be a CM field, and assume that $F \supset F_0$ contains an imaginary quadratic subfield. Suppose p splits in F_0. Now let $\mathfrak{m} \subset \mathbf{T}^S(K^p)$ be a maximal ideal of the big Hecke algebra for $\mathrm{GL_n/F}$. Scholze's work [Sch15] associates to such an $\mathfrak{m}$ a Galois representation $\overline{\rho}_{\mathfrak{m}} : \mathrm{Gal}_F \to \mathrm{GL_n}(\overline{\mathbf{F}}_\mathrm{p})$. We call $\mathfrak{m}$ *non-Eisenstein* if $\overline{\rho}_{\mathfrak{m}}$ is (absolutely) irreducible, and *decomposed-generic* if $\overline{\rho}_{\mathfrak{m}}$ is decomposed-generic in the sense of Caraiani–Scholze, i.e. there is an $\ell \neq p$ such that ℓ splits completely in F, and for each $w \mid \ell$ we have $\overline{\rho}_{\mathfrak{m}}|_{\mathrm{Gal}_{F_w}}$ is unramified and $\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_w)$ has eigenvalues $\alpha_{w,1}, \dots, \alpha_{w,n}$ satisfying $\alpha_{w,i}\alpha_{w,j}^{-1} \neq q_w^{\pm 1}$ for all i, j where $q_w := \#\mathcal{O}_{F_w}/\pi_w$ denotes the size of the residue field.

Let $\mathcal{E}_{\mathrm{Fu}}^i(K^p)_{\mathfrak{m}}$ denote the reduced eigenvariety defined as the support of the coadmissible Hecke module $H_i(J_{B,\mathrm{Fu}}^\vee(C^{\mathrm{BS}}(K^pI, D(I)_{N_0})_{\mathfrak{m}}))$, and $\mathcal{E}_{\mathrm{ov}}^i(K^p)_{\mathfrak{m}}$ the eigenvariety of degree i for overconvergent homology localised at $\mathfrak{m}$. When $\mathfrak{m}$ is non-Eisenstein, there is a mapping $\mathcal{E}_{\mathrm{Fu}}^i(K^p)_{\mathfrak{m}} \hookrightarrow (\mathrm{Spf}\, R_{\overline{\rho}_{\mathfrak{m}}})^{\mathrm{rig}} \times \widehat{T_n}$, where $R_{\overline{\rho}_{\mathfrak{m}}}$ denotes the universal Galois deformation ring. In particular, for each $(x, \delta) \in \mathcal{E}_{\mathrm{Fu}}^i(K^p)_{\mathfrak{m}}(\overline{\mathbb{Q}}_p)$, there is an associated Galois representation $\rho_x : \mathrm{Gal}_F \to \mathrm{GL_n}(\overline{\mathbb{Q}}_\mathrm{p})$.

Theorem 5.1.1. *Hansen's conjecture is true for any non-Eisenstein, decomposed generic point* $(x, \delta) \in \mathcal{E}_{\mathrm{ov}}^i(K^p)_{\mathfrak{m}}$. *In other words, there exists a Galois representation* $\rho_x : \mathrm{Gal}_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ *which is trianguline at all* $\nu \mid p$ *of parameter* $\delta' \cdot \delta$, *for* δ' *an algebraic character, and for all* $\tau : F_\nu \hookrightarrow \overline{\mathbb{Q}}_p$ *we have* $\mathrm{wt}_\tau(\rho_x|_{\mathrm{Gal}_{F_\nu}}) = \mathrm{wt}_\tau(\delta)$.

Proof. Since $\mathfrak{m}$ is non-Eisenstein, we have $\mathcal{E}^i_{\mathrm{Fu},BM}(K^p)_{\mathfrak{m}} \simeq \mathcal{E}^i_{\mathrm{Fu}}(K^p)_{\mathfrak{m}}$ where the first space is the analogous eigenvariety for Borel–Moore homology. Thus by the main comparison and Corollary 4.3.11 we have $\mathcal{E}^i_{\mathrm{ov}}(K^p)_{\mathfrak{m}} \simeq \mathcal{E}^{\dim X_{K^pI}-i}_{\mathrm{Fu}}(K^p)_{\mathfrak{m}}$. The result thus follows from [McD25, Theorem 1.1]), the analogous result for $\mathcal{E}^{\dim X_{K^pI}-i}_{\mathrm{Fu}}(K^p)_{\mathfrak{m}}$. □

5.2. **p-arithmetic homology and a local-global result.** In this subsection we prove Theorem 1.0.2. In Section 4, we established that $C^{BS}_\bullet(K, D(I)_{N_0}) \otimes^{\square}_{K[t]} A$ is the bounded complex of coadmissible $\mathcal{O}(\mathcal{W} \times \mathbb{G}_m)$-modules from the theory of overconvergent homology. Moreover, $C^{BS}_\bullet(K, D(I)_{N_0})$ is a complex of acyclic modules for $- \otimes^{\square}_{K[t]} A$ by Proposition 4.2.2. Thus, we have

$$C^{BS}_\bullet(K^pI, D(I)_{N_0}) \otimes^{\square}_{K[t]} A \cong C^{BS}_\bullet(K^pI, D(I)_{N_0}) \otimes^{\square,L}_{K[t]} A \cong C_\bullet(K^pI, D(I)_{N_0}) \otimes^{\square,L}_{K[t]} A \cong$$

$$\cong C_\bullet(K^p, D(G)_{N_0}) \otimes^{\square,L}_{K[t]} A \cong C_\bullet(K^p, D(G)_{N_0} \otimes^{\square,L}_{K[t]} A)$$

in the derived category of solid $\mathcal{O}(\mathcal{W} \times \mathbb{G}_m)$-modules. In summary, we have the following result:

Theorem 5.2.1. *The (graded) coherent sheaves on $\widehat{T}$ produced by the eigenvariety constructions from Section 3 are isomorphic to $H_*(K^p, D(G)_{N_0} \otimes^{\square,L}_{K[t]} A)$.*

This result strengthens the connection of the eigenvariety to the emerging perspective on spaces of p-adic automorphic forms via a categorical version of the p-adic local Langlands correspondence (see [EGH23], especially Conjecture 9.6.18). We make tow further remarks in this direction. The first it that it is desirable to compute $D(G)_{N_0} \otimes^{\square,L}_{K[t]} A$ more explicitly. In particular, we expect that the higher Tors vanish, which amounts to following assertion:

Conjecture 5.2.2. $\mathrm{Tor}^{\square,1}_{K[t]}(D(G)_{N_0}, A) = 0.$

The second remark is that Theorem 5.2.1 allows one to prove a formula in the spirit of [EGH23, Conjecture 9.6.18] in the very restricted setting of [JNWE25, §6.4], where $\mathbf{G} = \mathrm{PGL}_{2/\mathbb{Q}}$ and we localize at a maximal ideal $\mathfrak{m}$ of the Hecke algebra satisfying conditions (1)-(4) in *loc. cit*, to which we refer for details of the setup and notation. Then Theorem 5.2.1 together with [JNWE25, Theorem 6.4.4] give with following:

Proposition 5.2.3. *Assume that we are in the situation of* [JNWE25, §6.4]. *In particular,* $\mathbf{G} = \mathrm{PGL}_{2/\mathbb{Q}}$. *Then we have*

$$H_*(K_1^p(N), D(G)_{N_0} \otimes^{\square,L}_{K[t]} A)_{\mathfrak{m}} \cong H_*(\mathfrak{X}_r, r^{univ}(1) \otimes f^!(F_{ext}(D(G)_{N_0} \otimes^{\square,L}_{K[t]} A))[-2]).$$

Appendix A. Solid functional analysis

In this appendix we collect various facts about p-adic functional analysis in the solid context.

A.1. **Foundational conventions.** In the recent literature on condensed math, two different foundational settings have emerged: the original solid framework from [CS19] or the light solid framework from Clausen–Scholze's Analytic Stacks lectures. The choice of foundations should not matter to us for the following reason for the following reason: Let Cond and LCond be the categories of condensed and light condensed sets, respectively, and let CGTop and MCGTop be the categories of compact resp. metrizably compactly generated topological spaces. Then we have a commutative diagram of functors

$$\begin{tikzcd} \mathrm{MCGTop} \arrow[r] \arrow[d] & \mathrm{LCond} \arrow[d] \\ \mathrm{CGTop} \arrow[r] & \mathrm{Cond} \end{tikzcd}$$

where the left vertical functor is the inclusion, the right vertical functor comes from pullback of sheaves (light condensed sets to condensed sets for larger cutoff cardinals) and the horizontal functors are fully faithful. It follows that MCGTop embeds fully faithfully in both condensed settings, and so it doesn't matter which setting we use. So, with that said, we just write "condensed" for now and ignore the difference.

A.2. **Some facts about solid vector spaces.** We will work in the category of solid K-vector spaces, writing $\otimes^{\square}$ for the solid tensor product and $\underline{\mathrm{Hom}}$ for the internal Hom (over K). Given solid K-algebra R, the category of solid R-modules has tensor product being the coequalizer $M \otimes_R^{\square} N$ of

$$M \otimes^{\square} R \otimes^{\square} N \rightrightarrows M \otimes^{\square} N$$

where one arrow is given by $m \otimes r \otimes n \mapsto mr \otimes n$ and the other is $m \otimes r \otimes n \mapsto m \otimes rn$, and the internal Hom $\underline{\mathrm{Hom}}_R(M, N)$ being the equalizer of

$$\underline{\mathrm{Hom}}(M, N) \rightrightarrows \underline{\mathrm{Hom}}(R \otimes^{\square} M, N)$$

where one map is $f \mapsto (r \otimes m \mapsto rf(m))$ and the other is $f \mapsto (r \otimes m \mapsto f(rm))$.

We are repeatedly use solid tensor products over R a commutative Banach K-algebra. It will be useful to more directly compare this to a completed tensor product. First recall from [RJRC22, Lemma 3.24] that $V \mapsto V(*)_{\mathrm{top}}$ induces an *exact* equivalence between solid and classical Frechet spaces, with $V = \underline{V(*)_{\mathrm{top}}}$.

Recall a Banach R-module M is called *orthonormalizable* if $M \simeq \widehat{\bigoplus}_I R$ for I some indexing set. Following [GvHH25], we then call a (solid) Fréchet R-module M *strongly countably Fréchet* if it is a limit of a countable inverse system $\{M_n\}_{n=1}^{\infty}$, where each M_n is a direct summand of an orthonormalizable R-module with countable indexing set I, and $M_{n+1} \to M_n$ has dense image for all n. By [GvHH25, Proposition 2.4.3], any strongly countably Fréchet R-module is flat for the solid tensor product over R.

Lemma A.1. *Suppose R is a noetherian K-Banach algebra, $\underline{M}$ is a strongly countably Fréchet $\underline{R}$-module, and $\underline{N}$ is either a Banach or strongly countably Fréchet R-module. Then $\underline{M} \otimes_{\underline{R}}^{\square} \underline{N} \simeq \underline{M \widehat{\otimes}_R N}$.*

Proof. When $\underline{M}, \underline{N}$ are Banach modules and $\underline{M}$ is flat for the solid tensor product over $\underline{R}$, the result follows from [GvHH25, Proposition 2.1.11] after inverting p. If both $\underline{M}, \underline{N}$ are strongly countably Fréchet, write $M = \varprojlim_i M_i$ and $N = \varprojlim_j N_j$ as limits of Banach R-modules. Then by [GvHH25, Proposition 2.4.10] gives have $\underline{M} \otimes_{\underline{R}}^{\square} \underline{N} \simeq \varprojlim_{i,j} \underline{M_i} \otimes_{\underline{R}}^{\square} \underline{N_j} \simeq \varprojlim_{i,j} \underline{M_i \widehat{\otimes}_R N_j} \simeq \underline{\varprojlim_{i,j} M_i \widehat{\otimes}_R N_j} \simeq \underline{M \widehat{\otimes}_R N}$. If instead $\underline{N}$ is Banach (but not necessarily orthonormalizable!), we need to show $\underline{M} \otimes_{\underline{R}}^{\square} \underline{N} \cong \varprojlim_n \underline{M_n} \otimes_{\underline{R}}^{\square} \underline{N}$. The proof of *loc. cit.*[8] shows it suffices to show the natural map $(\prod_n M_n) \otimes_{\underline{R}}^{\square} N \simeq$

[8]We only need flatness of M and the M_n, not N.

$\prod_n \left(M_n \otimes^{\square}_{\underline{R}} N\right)$ is an isomorphism. Since all of the M_n are orthonormalizable, we have as in the proof of *loc. cit.* $(\prod_n \widehat{\bigoplus} \underline{R}) \otimes^{\square}_{\underline{R}} N \simeq \left(\widehat{\bigoplus}_I \underline{K} \otimes^{\square}_{\underline{K}} \prod_n \underline{K} \otimes^{\square}_{\underline{K}} \underline{R}\right) \otimes^{\square}_{\underline{R}} N \simeq \widehat{\bigoplus}_I \underline{K} \otimes^{\square}_{\underline{K}} \prod_n \underline{K} \otimes^{\square}_{\underline{K}} \underline{N} \simeq \widehat{\bigoplus}_I \underline{K} \otimes^{\square}_{\underline{K}} \prod_n \underline{N} \simeq \prod_n \widehat{\bigoplus}_I \underline{K} \otimes^{\square}_{\underline{K}} N \simeq \prod_n \widehat{\bigoplus}_I \underline{R} \otimes^{\square}_{\underline{R}} N$, where we have used over $\underline{K}$ that for two countable collections of Banach spaces V_m, W_n we have $\prod_m \underline{V_m} \otimes^{\square}_{\underline{K}} \prod_n \underline{W_n} \simeq \prod_{m,n} \underline{V_m} \otimes^{\square}_{\underline{K}} \underline{W_n}$ (see [Bos23, proof of Proposition A.66]). $\square$

We now record facts about Fréchet spaces in the solid world that we will need in the main text.

Lemma A.2. *Let $V_1 \to V_2$ be a map of Fréchet spaces with dense image and let W be another Fréchet space. Then $V_1 \otimes^{\square} W \to V_2 \otimes^{\square} W$ has dense image.*

Proof. A version of this for Banach modules follows from [GvHH25, Lemma 2.4.5 and Lemma A.1.7], so we use the case when V_1, V_2 and W are Banach spaces freely. So, we write $V_1 = \varprojlim_n V_{1,n}$, $V_2 = \varprojlim_n V_{2,n}$ and $W = \varprojlim_n W_n$ all as sequential limits of Banach spaces with injective transition maps with dense image. Reindexing if necessary, we may assume $V_1 \to V_2$ induces maps $V_{1,n} \to V_{2,n}$ for all n.

First, we want to prove that the inverse systems in the presentations $V_i \otimes^{\square} W = \varprojlim_n V_{i,n} \otimes^{\square} W_n$ have injective transition maps with dense images. We factor $V_{i,m} \otimes^{\square} W_m \to V_{i,n} \otimes^{\square} W_n$ as

$$V_{i,m} \otimes^{\square} W_m \to V_{i,m} \otimes^{\square} W_n \to V_{i,n} \otimes^{\square} W_n.$$

Injectivity and density of the image then follows (the latter by applying the Banach space case of the lemma twice). It then follows that $V_i \otimes^{\square} W$ has dense image in $V_{i,n} \otimes^{\square} W_n$ for all n.

With this, we return to proving the statement of the Lemma. Using Lemma A.3, $V_1 \otimes^{\square} W \to V_2 \otimes^{\square} W$ has dense image if and only if $V_1 \otimes^{\square} W \to V_{2,n} \otimes^{\square} W_n$ has dense image for all n. Now the latter map factors through $V_{1,n} \otimes^{\square} W_n$ and $V_1 \otimes^{\square} W \to V_{1,n} \otimes W_n$ has dense image (by above), so $V_1 \otimes^{\square} W$ has dense image in $V_{2,n} \otimes^{\square} W_n$ if and only if $V_{1,n} \otimes^{\square} W_n \to V_{2,n} \otimes^{\square} W_n$ has dense image. But this is now the Banach space case, finishing the proof. $\square$

Lemma A.3. *Let $V = \varprojlim_n V_n$ be a Fréchet space, written as an inverse limit of Banach spaces V_n with injective transition maps with dense image. Then a subset $X \subseteq V$ is dense if and only if it is dense in V_n for all n.*

Proof. V is dense in V_n for all n, so if X is dense in V then it is also dense in V_n for all n. For the converse, let $|-|_n$ be a norm on V_n; without loss of generality we may assume that $|v|_n \leq |v|_{n+1}$ for all n. Then the topology on V is induced by the metric

$$d(x,y) = \sum_{n=1}^{\infty} 2^{-n} \frac{|x-y|_n}{1+|x-y|_n}.$$

From this, it is easy to see that if X is dense in V_n for all n, then X is dense in V. $\square$

Lemma A.4. *Let $(V_n)_n$ be an inverse system of Fréchet spaces such that $V_m \to V_n$ has dense image for all $m \geq n$. Then $R^i \varprojlim_n V_n = 0$ for $i \geq 1$.*

Proof. We follow Bosco's proof. By [Sch13, Lemma 3.18], it suffices to check that, for any extremally disconnected profinite set S, the system of abelian groups $(\mathrm{Map}(S, V_n))_n$ satisfies

$$R^1 \varprojlim_n \mathrm{Map}(S, V_n) = 0.$$

This, in turn, is implied by the topological Mittag-Leffler criterion, if $\mathrm{Map}(S, V_m) \to \mathrm{Map}(S, V_n)$ is a map of Fréchet spaces with dense image for all $m \geq n$. So let's prove this.

Let X be a Fréchet space with translation invariant complete metric d_X, and let S be a compact Hausdorff space. The set $\mathrm{Map}(S, X)$ inherits a vector space structure from X and has the metric of uniform convergence, given by

$$d(f, g) = \sup_{s \in S} d_X(f(s), g(s)),$$

which is complete (a uniform limit of continuous functions is continuous) and translation invariant since d_X is. So $\mathrm{Map}(S, X)$ is a Fréchet space. Moreover, if S is profinite then any $S \to X$ can be approximated to arbitrary precision by a locally constant function. Write $LC(S, X)$ for the subspace of locally constant functions.

Now assume that S is profinite and that $X \to Y$ is continuous linear map of Fréchet spaces with dense image. Then $LC(S, X) \to LC(S, Y)$ clearly has dense image, and thus $\mathrm{Map}(S, X) \to \mathrm{Map}(S, Y)$ has dense image. □

Remark A.5. The proof above is in the "old" solid setting: To prove that $R^i \varprojlim_n \mathcal{F}_n = 0$ for $i \geq 1$ for a system $\mathcal{F}_n$ of sheaves on a site, [Sch13, Lemma 3.18] asks for a basis of elements S the site satisfying $H^i(S, \mathcal{F}_n) = 0$ and $R^i \varprojlim_n \mathcal{F}_n(S) = 0$ for all n and $i \geq 1$. In the old solid setting, the extremally disconnected sets are projective and hence automatically satisfy the first condition, leaving us to check the second condition.

In the light setting, one no longer has any projective object. Instead, one can prove the following: If M is any light solid abelian group and S is any light profinite set, then $H^i(S, M) = 0$ for $i \geq 1$. To see this, we write

$$H^i(S, M) = \mathrm{Ext}^i(\mathbb{Z}[S], M) = \mathrm{Ext}^i(\mathbb{Z}[S]^{\square}, M),$$

where the first Ext is in light condensed abelian groups and the second is in light solid abelian groups, using that $-^{\square}$ is the left adjoint to the inclusion functor and that $\mathbb{Z}[S]$ is acyclic for $-^{\square}$. To finish, one uses that $\mathbb{Z}[S]^{\square}$ is a projective solid abelian group (all of these facts are in Rodriguez Camargo's notes on solid abelian groups). This allows one to prove Lemma A.4 in the light setting.

We quickly compare this to the "usual" topological Mittag-Leffler criterion: if X_n is an inverse system of complete metric spaces (and abelian groups...) with *uniformly* continuous transition maps, and $X_m \to X_n$ has dense image for all $m \geq n$, then $R^i \varprojlim_n X_n = 0$ for $i \geq 1$. Fréchet spaces satsify this, since any continuous linear map between Fréchet spaces is uniformly continuous.

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