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Rootfinding Research Prerequisites
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Assignment 5.3
Exercises: 2, 3, 4, 5, 7, 11

**Exercise 5.3.2**

*Proof.* We have that $\overline{f \cdot g}^G = \overline{\overline{f}^G \overline{g}^G}^G$, so the remainders of $f$ and $g$ do not need to be computed separately. Instead, for $[f] \cdot [g] \in k[x_1, \ldots, x_n]/I$, $\overline{f \cdot g}^G$ can be computed. $\square$

**Exercise 5.3.3** We can compute a Grobner basis of $I$ with lex order $x > y > z$ as

$$x^2 - z^{55}, xz^6 - z^{64}, y^3 - z^{54}, yz^6 - z^{24}, z^{67} - z^6,$$

and a basis with grlex order $x > y > z$ as

$$x^9 - x^2y^2z^4, x^2y^7 - x^2z^4, y^9 - x^7, x^7y - x^2z^3, x^4y^4 - x^2z^5, x^5z - y^6, z^6 - x^4y, x^3z^2 - y^3, y^3z - x^2.$$

**Exercise 5.3.4**

*Proof.* The case where $c = 0$ is trivial. Let $c \in k$ be nonzero, and let $r = \overline{c \cdot f}^G$ be the unique remainder such that $cf = q + r$ for some $q \in I$. It follows that $r$ is a $k$-linear combination of the monomials in the complement of $\langle \mathrm{LT}(I) \rangle$. We then divide by $c$ to get $f = \frac{q}{c} + \frac{r}{c}$. We have $\frac{q}{c} \in I$ since $c \in k$, and the monomials in $\frac{r}{c}$ are identical as the monomials in $r$. By Proposition 5.3.1, $\frac{r}{c} = \overline{f}^G$, so $\overline{c \cdot f}^G = c \cdot \overline{f}^G$. $\square$

**Exercise 5.3.5**

**(5.3.5a):** Let $I$ be defined as in the text, and let $f \in \mathbb{R}[x, y]/I$ be arbitrary. Using Proposition 5.3.4, we have that

$$[f] = c_1[1] + c_2[x] + c_3[y] + c_4[y^2]$$

is a unique representation of $f$ with $c_i \in \mathbb{R}$. Note that the mapping $\phi([f]) = (c_1, c_2, c_3, c_4)^T$ is injective between $\mathbb{R}[x, y]/I$ and $\mathbb{R}^4$.

Let $c \in \mathbb{R}$ and let

$$[g] = d_1[1] + d_2[x] + d_3[y] + d_4[y^2] \in \mathbb{R}[x, y]/I.$$

Then

$$\begin{aligned}
\phi([f] + [g]) &= \phi((c_1 + d_1)[1] + (c_2 + d_2)[x] + (c_3 + d_3)[y] + (c_4 + d_4)[y^2]) \\
&= (c_1 + d_1, c_2 + d_2, c_3 + d_3, c_4 + d_4)^T = (c_1, c_2, c_3, c_4)^T + (d_1, d_2, d_3, d_4)^T \\
&= \phi([f]) + \phi([g]),
\end{aligned}$$

and

$$\phi(c[f]) = \phi(cc_1[1] + cc_2[x] + cc_3[y] + cc_4[y^2]) = (cc_1, cc_2, cc_3, cc_4)^T = c(c_1, c_2, c_3, c_4)^T$$
$$= c\phi([f]).$$

Thus, by definition, $\phi$ is a linear map, and we conclude that $\mathbb{R}[x, y]/I \simeq \mathbb{R}^4$ by $\phi$.

**(5.3.5b):** Using the given Grobner basis, we express each product as a linear combination of $\{[1], [x], [y], [y^2]\}$ as follows:

$$\begin{aligned}
x \cdot x &= 1 \cdot (x^2 + y - 1) - y + 1 \Rightarrow [x] \cdot [x] = -[y] + [1], \\
x \cdot y &= 1 \cdot (xy - 2y^2 + 2y) + 2y^2 - 2y \Rightarrow [x] \cdot [y] = 2[y^2] - 2[y], \\
x \cdot y^2 &= y(xy - 2y^2 + 2y) + 2(y^3 - (7/4)y^2 + (3/4)y) + (3/2)y^2 - (3/2)y \\
&\Rightarrow [x] \cdot [y^2] = (3/2)[y^2] - (3/2)[y], \\
y \cdot y^2 &= 1 \cdot (y^3 - (7/4)y^2 + (3/4)y) + (7/4)y^2 - (3/4)y \Rightarrow [y] \cdot [y^2] = (7/4)[y^2] - (3/4)[y] \\
y^2 \cdot y^2 &= (y + 7/4)(y^3 - (7/4)y^2 + (3/4)y) + (37/16)y^2 - (21/16)y \\
&\Rightarrow [y^2] \cdot [y^2] = (37/16)[y^2] - (21/16)[y].
\end{aligned}$$

**(5.3.5c):** $\mathbb{R}[x, y]/I$ is not a field since

$$([x] - 2[y] + 2) \cdot [y] = [xy - 2y^2 + 2y] = 0$$

so $[y]$ is a nonzero divisor in $\mathbb{R}[x, y]$.

**(5.3.5d):** We factor and solve the last equation in the Grobner basis $G$, which results in

$$\frac{1}{4} \cdot y(4y^2 - 7y + 3) = \frac{1}{4} \cdot y(y - 1)(4y - 3) = 0$$

so $y = 0, 1, \frac{4}{3}$. Back-substituting into the first two equations of $G$ and have that

$$\mathbf{V}(I) = \{(0, 1), (0, -1), (1, 0), (\frac{3}{4}, -\frac{1}{2})\}.$$

**(5.3.5e):** Part $(ii)$ of Proposition 7 gives that $\mathbf{V}(I)$ has at most 6 points. Part $(i)$ of the proposition gives a better bound since $\mathbb{R}[x, y]/I$ is of dimension 4 so it gives a bound of 4.

**Exercise 5.3.7**

**(5.3.7a):** If $\{x^\alpha | x^\alpha \notin \langle \mathrm{LT}(I) \rangle\}$ contains $d$ elements $x^{\alpha_1}, \ldots, c^{\alpha_2}$, Proposition 4 gives that $k[x_1, \ldots, x_n]/I$ is a $k$-vector space that is isomorphic to $S =$Span$(x^{alpha_1}, \ldots, x^{alpha_2})$. Since each $\alpha_i$ is distinct, it follows that all $x^{\alpha_1}$ are linearly independent, so $S$ has dimension $d$. Because $S$ is isomorphic to $k[x_1, \ldots, x_n]/I$, it follows that $k[x_1, \ldots, x_n]/I$ is also of dimension $d$.

**(5.3.7b):** We conclude that the number of monomials in the complement of $\langle \mathrm{LT}(I) \rangle$ is independent of the choice of monomial order when that number is finite since Proposition 4 and Exercise 5.3.7a are independent of the choice of monomial order.

**Exercise 5.3.11**

**(5.3.11a):** We compute a Grobner basis for $I$ with lex order $x > y > z$ as

$$x_z + 3z^2 + z^4 - z^5, y + 3z + 6z^2 + 2z^3 + z^4 - 2z^5, -z - 3z^2 - 3z^3 - z^4 + z^6.$$

Using Theorem 5.3.6, we give the table of monomials $m_d$ of total degree $\leq d$ that are not in $\langle \mathrm{LT}(I) \rangle$ with $1 \leq d \leq 10$:

| $d$ | $m_d$ | monomial set $M_d$ |
|---|---|---|
| 1 | 2 | $\{1, z\}$ |
| 2 | 3 | $\{1, z, z^2\}$ |
| 3 | 4 | $\{1, z, z^2, z^3\}$ |
| 4 | 5 | $\{1, z, z^2, z^3, z^4\}$ |
| $\geq 5$ | 6 | $\{1, z, z^2, z^3, z^4, z^5\}$ |

**(5.3.11b):** Similarly to part (a), we compute a Grobner basis

$$x - y^2 + z^2, y^4 - 2y^2z^2 + y + z^4$$

and form a table of values:

| $d$ | $m_d$ | monomial set $M_d$ |
|---|---|---|
| 1 | 3 | $\{1, y, z\}$ |
| 2 | 6 | $\{1, y, z, yz, y^2, z^2\}$ |
| 3 | 10 | $\{1, y, z, yz, y^2, z^2, y^2z, yz^2, y^3, z^3\}$ |
| 4 | 14 | $\{1, y, z, yz, y^2, z^2, y^2z, yz^2, y^3, z^3, yz^3, y^2z^2, y^3z, z^4\}$ |
| 5 | 6 | $\{1, y, z, yz, y^2, z^2, y^2z, yz^2, y^3, z^3, yz^3, y^2z^2, y^3z, z^4, y^3z^2, y^2z^3, yz^4, z^5\}$ |
| 6 | 22 | $\{1, y, z, yz, y^2, z^2, y^2z, yz^2, y^3, z^3, yz^3, y^2z^2, y^3z, z^4, y^3z^2, y^2z^3, yz^4, z^5y^3z^3, y^2z^4, yz^5, z^6\}$ |
| 7 | 26 | $\{1, y, z, yz, y^2, z^2, y^2z, yz^2, y^3, z^3, yz^3, y^2z^2, y^3z, z^4, y^3z^2, y^2z^3, yz^4, z^5y^3z^3, y^2z^4, yz^5, z^6,$ $y^3z^4, y^2z^5, yz^6, z^7\}$ |
| $\geq 8$ | $4d - 2$ | $M_d = M_{d-1} \cup \{y^3z^{d-3}, y^2z^{d-2}, yz^{d-1}, z^d\}$ |

**(5.3.11c):** This is given ad $H(d) = 4d - 2$ by Exercise 5.3.11b, so $H(d)$ is linear.

**(6.3.11d):** We compute a Grobner basis $x^2 + y$ and form a table of values:

| $d$ | $m_d$ | monomial set $M_d$ |
|---|---|---|
| 1 | 4 | $\{1, x, y, z\}$ |
| 2 | 9 | $\{1, x, y, z, xy, xz, yz, y^2, z^2\}$ |
| 3 | 16 | $\{1, x, y, z, xy, xz, yz, y^2, z^2, xy^2, xyz, xz^2, y^3, y^2z, yz^2, z^3\}$ |
| $\geq 4$ | $(d+1)^2$ | $M_d = m_{d-1} \cup \{xy^iz^k \mid i + k = d - 1\} \cup \{y^mz^n \mid m + n = d\}$ |

**(5.3.11e):** It appears that dimension corresponds to the degree of the function $H(d)$, or in other words, each equation in the definition of a variety imposes an additional constraint that reduces the possible dimension by one.