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Rootfinding Research Prerequisites
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Assignment 4.3
Exercises: 2, 3, 4, 6, 8

Exercise 4.3.2

Proof. Let $f, g \in k[x_1, \dots, x_2]$ and let $f = cf_1^{a_1} \dots f_r^{a_r}$ and $g = c^l g_1^{b_1} \dots g_s^b$ be their factorizations into distinct irreducible polynomials such that $f_1, \dots, f_l$ are constant multiples of $g_1, \dots, g_l$ and for $i, j > l$, f_i is not a constant multiple of g_j. We wish to show that

$$\mathrm{lcm}(f, g) = f_1^{\max(a_1,b_1)} \dots f_l^{\max(a_l,b_l)} \cdot g_{l+1}^{b_{l+1}} \dots g_s^{b_s} \cdot f_{l+1}^{a_{l+1}} \dots f_r^{a_r}.$$

We begin by noting that $\mathrm{lcm}(f, g)$ is a common multiple of f, g by definition. We can define another arbitrary common multiple h of f and g, which implies that the irreducible factorization of h includes all irreducible factors of f, g. In other words, h contains all $f_1 = g_1, \dots, f_l = g_l, f_{l+1}, \dots, f_r, g_{l+1}, \dots, g_s$. Since the exponent of each of these irreducible factors of h must be greater than or equal to the corresponding exponents in f and g, it follows that $\mathrm{lcm}(f, g)$ must divide h. Therefore, we can conclude that $\mathrm{lcm}(f, g)$ as defined above is the least common multiple of f and g. □

Exercise 4.3.3

Proof. Let $I, J \subseteq k[x_1, \dots, x_n]$ be principal ideals. We wish to show that their intersection $I \cap J$ is also a principal ideal. To show this, we let $h \in I \cap J, I = \langle f \rangle, J = \langle g \rangle$ as defined in Proposition 13. We then have that

$$\begin{aligned} h \in I \cap J &\Leftrightarrow h \in I, h \in J \\ &\Leftrightarrow h \in \langle f \rangle, h \in \langle g \rangle \\ &\Leftrightarrow h \text{ is a multiple of } f, g \\ &\Leftrightarrow h \text{ is a multiple of } \mathrm{lcm}(f, g) && \text{by Exercise 4.3.2} \\ &\Leftrightarrow h \in \langle \mathrm{lcm}(f, g) \rangle. \end{aligned}$$

From this equivalence, it follows that $\langle f \rangle \cap \langle g \rangle = \langle \mathrm{lcm}(f, g) \rangle$. Therefore, the intersection of two principal ideals is also principal. □

Exercise 4.3.4

Proof. Let $I, J \subseteq k[x_1, \dots, x_n]$ be principal ideals and assume that $I = \langle f \rangle, J = \langle g \rangle$, and $I \cap J = \langle h \rangle$. We wish to show that $h =$lcm(f, g). Note that this equivalence is demonstrated by the equivalences given in Exercise 4.3.3, i.e., $I \cap J = \langle \mathrm{lcm}(f, g) \rangle$, together with our assumption, implies that $h = \mathrm{lcm}(f, g)$. □

Exercise 4.3.6

(4.3.6a): Let $I_l, \dots, I_r, J$ be ideals in $k[x_1, \dots, x_n]$. We wish to show that $(I_1 + I_2)J = I_1 J + I_2 J$. Defining these ideals in terms of their generators, we have $I_1 = \langle f_1, \dots, f_r \rangle, I_2 = \langle g_1, \dots, g_s \rangle$, and $J = \langle h_1, \dots, h_t \rangle$. The definition of ideal multiplication then implies that

$$\begin{aligned}
(I_1 + I_2)J &= \langle f_1, \dots, f_r, g_1, \dots, g_s \rangle \cdot \langle h_1, \dots, h_t \rangle \\
&= \langle f_i h_l, g_j h_l : 1 \leq i \leq r, 1 \leq j \leq s, 1 \leq l \leq t \rangle \\
&\text{and} \\
I_1 J + I_2 J &= \langle f_i h_l : 1 \leq i \leq r, 1 \leq l \leq t \rangle + \langle g_j h_l : 1 \leq j \leq s, 1 \leq l \leq t \rangle \\
&= \langle f_1 h_l, g_j h_l : 1 \leq i \leq r, 1 \leq j \leq s, 1 \leq l \leq t \rangle.
\end{aligned}$$

Therefore, $(I_1 + I_2)J = I_1 J + I_2 J$ as needed.

(4.3.6b): Let $I_l, \dots, I_r, J$ be ideals in $k[x_1, \dots, x_n]$. We wish to show that $(I_1 \dots I_r)^m = I_1^m \dots I_r^m$ and proceed by induction on $m \geq 1$.

Base Case: This case is easily confirmed by observation

Inductive Hypothesis: Assume that $(I_1 \dots I_r)^k = I_1^k \dots I_r^k$ for all $k \geq 1$.

Inductive Step: The inductive hypothesis implies that

$$(I_1 \dots I_r)^{k+1} = (I_1 \dots I_r)^k \cdot I_1 \dots I_r = I_1^k \dots I_r^k \cdot I_1 \dots I_r,$$

which implies that

$$(I_1 \dots I_r)^{k+1} = (I_1^k \cdot I_1) \dots (I_r^k \cdot I_r) = I_1^{k+1} \dots I_r^{k+1}$$

as needed.

Exercise 4.3.8

(4.3.8a): To compute generators for $\langle f \rangle \cap \langle g \rangle$ with f, g being defined in the text, we use Theorem 4.3.11 and compute a lex order Groebner basis for $\langle tf, (1-t)g \rangle$ with $t > x > y > z$. This turns out to be

$$\begin{aligned}
&\{x^5 + x^4 y - x^3 y^2 - x^2 y^3 + 2x^4 z^2 + 2x^3 y z^2 2x^2 y^2 z^2 - 2xy^3 z^2 + x^3 z^4 + x^2 y z^4 - xy^2 z^4 y^3 z^4, \\
&\ x^4 + tx^3 y - x^2 y^2 - txy^3 + 2x^3 z^2 - tx^3 z^2 + tx^2 y z^2 \\
&\ - 2xy^2 z^2 + txy^2 z^2 - ty^3 z^2 + x^2 z^4 tx^2 z^4 - y^2 z^4 + ty^2 z^4, \\
&\ - x^4 + tx^4 + x^2 y^2 tx^2 y^2 - 2x^3 z^2 + 2tx^3 z^2 + 2xy^2 z^2 - 2txy^2 z^2 x^2 z^4 + tx^2 z^4 + y^2 z^4 - ty^2 z^4\}.
\end{aligned}$$

Note that all polynomials in this set involve t except for the first. This means that a basis for $\langle f \rangle \cap \langle g \rangle$ can be given by

$$x^5 + x^4 y - x^3 y^2 - x^2 y^3 + 2x^4 z^2 + 2x^3 y z^2 - 2x^2 y^2 z^2 - 2xy^3 z^2 + x^3 z^4 + x^2 y z^4 - xy^2 z^4 - y^3 z^4.$$

Note that this polynomial is the same as $\mathrm{lcm}(f, g)$ by Proposition 4.3.13.

To compute generators for $\sqrt{\langle f\rangle\langle g\rangle}$, note that $\sqrt{\langle f\rangle\langle g\rangle} = \sqrt{\langle fg\rangle}$. We can use Proposition 4.2.12 to equate $\sqrt{\langle fg\rangle} = \langle (fg)_{red}\rangle$, where

$$(fg)_{red} = \frac{fg}{\gcd(fg, \frac{\partial fg}{\partial x}, \frac{\partial fg}{\partial y}, \frac{\partial fg}{\partial z})}.$$

Since we know that

$$\gcd\left(fg, \frac{\partial fg}{\partial x}, \frac{\partial fg}{\partial y}, \frac{\partial fg}{\partial z}\right) = \frac{fg \cdot \frac{\partial fg}{\partial x} \cdot \frac{\partial fg}{\partial y} \cdot \frac{\partial fg}{\partial z}}{\mathrm{lcm}(fg, \frac{\partial fg}{\partial x}, \frac{\partial fg}{\partial y}, \frac{\partial fg}{\partial z})}$$

by Proposition 4.3.14, we have that $\sqrt{\langle f\rangle\langle g\rangle} = \langle -x^3 + xy^2 - x^2z^2 + y^2z^2\rangle$.

(4.3.8b): Using Proposition 4.3.14, we have that

$$\gcd(f, g) = \frac{fg}{\mathrm{lcm}(f, g)} = x^3 - xy^2 + x^2z^2 - y^2z^2.$$

(4.3.8c): Using the definitions for p, q found in the text, we compute a lex order Groebner basis of $\langle tf, tg, (1-t)p, (1-t)q\rangle$ with t, x, y, z and get

$$\begin{aligned}
\{&x^3y - xy^3 - x^3z^2 + x^2yz^2 + xy^2z^2y^3z^2 - x^2z^4 + y^2z^4,\\
&x^4 - x^2y^2 + 2x^3z^2 - 2xy^2z^2 + x^2z^4 - y^2z^4,\\
&-y^2z + ty^2z - yz^2 + tyz^2,\\
&-xy + txy - xz + txz,\\
&-x^2 + tx^2 - yz + tyz\}.
\end{aligned}$$

Since the first two polynomials in this basis don't involve t, we have that

$$\begin{aligned}
\langle f, g\rangle \cap \langle p, q\rangle = &\langle x^3y - xy^3 - x^3z^2 + x^2yz^2 + xy^2z^2 - y^3z^2 - x^2z^4 + y^2z^4,\\
&x^4 - x^2y^2 + 2x^3z^2 - 2xy^2z^2 + x^2z^4 - y^2z^4\rangle.
\end{aligned}$$