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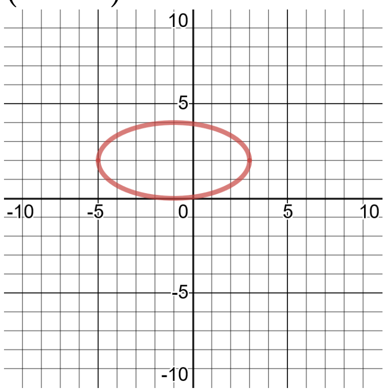
Rootfinding Research Prerequisites
05/17/20

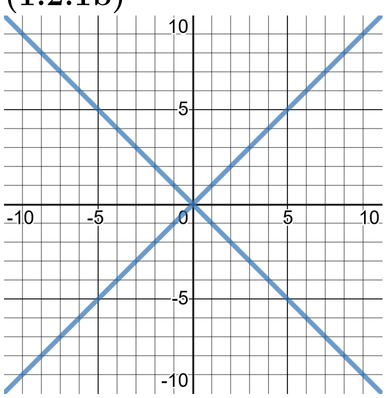
Assignment 1.2
Exercises: 1, 2, 3, 4, 5, 6, 8, 9, 15

Exercise 1.2.1

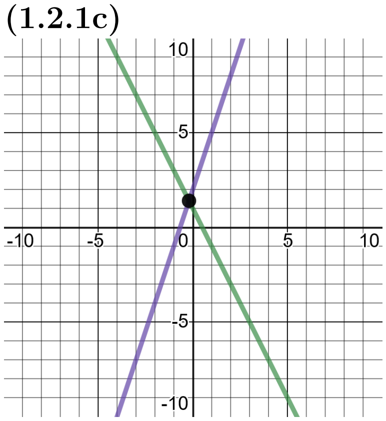
(1.2.1a)



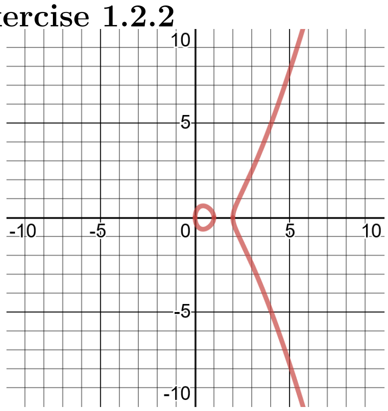
(1.2.1b)



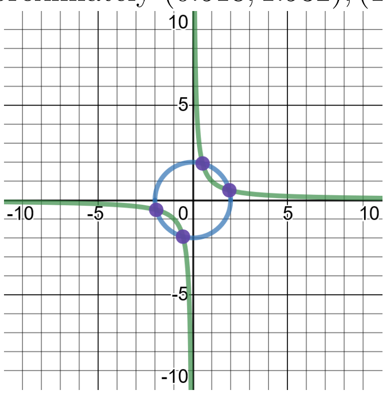
(1.2.1c)



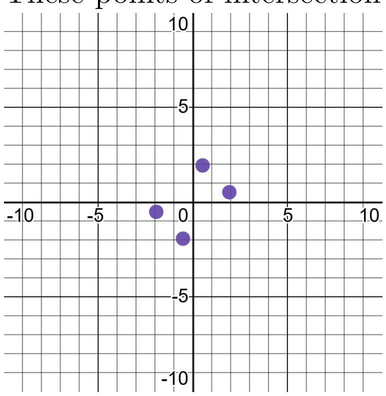
Exercise 1.2.2



Exercise 1.2.3 The following is $V(x^2+y^2-4) \cap V(xy-2)$, which is the intersection of the two affine varieties represented by two different colors. The points of intersection are approximately $(0.518, 1.932), (1.932, 0.518), (-0.518, -1.932), (-1.932, -0.518)$.



These points of intersection are equal to the affine variety $V(x^2+y^2-4, xy-1)$.



Exercise 1.2.4

(1.2.4a) A unit sphere centered at the origin.

(1.2.4b) A cylinder whose cross-sections parallel to the xy-plane are unit circles centered at (0,0) in the xy-plane.

(1.2.4c) A point $(-2, 1.5, 0)$.

(1.2.4d) The union of the yz-plane and a parabolic cylinder perpendicular to the yz-plane above $y = z^2$.

(1.2.4e) The union of the yz-plane and a twisted cubic given by $V(x^3 - z, x^2 - y)$.

(1.2.4f) A circle with radius $\frac{\sqrt{3}}{2}$ centered at $(0, 0, \frac{1}{2})$ that lies on the plane $z = \frac{1}{2}$.

Exercise 1.2.5 The affine variety in $\mathbb{R}^3$ given by $V((x-2)(x^2-y), y(x^2-y), (z+1)(x^2-y))$ can be described as the union $V(x-2, y, z+1) \cup V(x^2 - y)$, which is the union of the point $(2, 0, -1)$ and a parabolic cylinder which is perpendicular to the xy-plane above the parabola $y = x^2$.

Exercise 1.2.6

(1.2.6a)

Proof. Define V to be the affine variety $V(f_1, \ldots, f_n)$ with each $f_i = x_i - a_i$ for all $1 \leq i \leq n$. Then $V = \{(b_1, \ldots, b_n) | f_i(b_1, \ldots, b_n) = 0, 1 \leq i \leq n\}$. Thus we have that $(b_1, \ldots, b_n) \in V$ implies $b_i - a_i = 0$ for $1 \leq i \leq n$. This results in $(b_1, \ldots, b_n) = (a_1, \ldots, a_n)$ so $V = \{(a_1, \ldots, a_n)\}$.

Since this works for an arbitrary point $(a_1, \ldots, a_n)$, this argument will hold for any point. Therefore, a single point $(a_1, \ldots, a_n) \in k^n$ is an affine variety. □

(1.2.6b)

Proof. Let S be a finite subset of k^n with r points such that $S = \{p_1, \ldots, p_r\}$, $p_i \in k^n$ for $1 \leq i \leq r$.

By (1.2.6a), we have that each set of one point $\{p_i\}$ is an affine variety V_i. Lemma 2 indicates that finite unions of affine varieties are also affine varieties, so taking the union of finitely many p_i yields

$$V = \bigcup_{i=1}^{r} V_i = \bigcup_{i=1}^{r} \{p_i\} = S.$$

Since S is arbitrary, we conclude that every finite subset S of a field k^n is an affine variety. □

Exercise 1.2.8

Proof. Let $f(x, y) \in \mathbb{R}[x, y]$ vanish on $X = \{(x, x) \in \mathbb{R}^2 | x \neq 1\}$. Then $g(t) = f(t, t) \in \mathbb{R}[t]$ vanishes at all points such that $t = x \neq 1$. This implies that $g(t)$ has infinitely many roots, so $g(t)$ must therefore be the zero polynomial. Note, however, that if $g(t) = 0$ is the zero polynomial, then $g(1) = f(1, 1) = 0$. Thus, we conclude that X is not an affine variety. □

Exercise 1.2.9

Proof. Assume, by way of contradiction, that $R = \{(x, y) \in \mathbb{R}^2 | y > 0\}$ is a variety. Then $R = V(f_1, \ldots f_n)$ where each f_i is a polynomial in $\mathbb{R}[x, y]$. We consider $f_1(x, y)$ and let $y = y_0 > 0$. Note that

$$f_1(x, y_0) = \sum_{i=0}^{N} g_i(y_0) x^i$$

vanishes at all $x \in \mathbb{R}$. This means that $g_i(y_0) = 0$ is the zero polynomial for all i. This holds for all $y_0 > 0$. Thus, f_1 is the zero polynomial. This can be applied to all f_i so $f_i = 0$ is the zero polynomial for all i. Now $R = V(0, \ldots 0) = \mathbb{R}^2$, and we arrive at a contradiction. Therefore R is not an affine variety. □

Exercise 1.2.15

(1.2.15a)

Proof. The base cases have been shown for both unions and intersections of two affine varieties. Thus, we can rewrite finite intersections and unions as follows: $V_1 \cup \ldots \cup V_m = (V_1 \cup \ldots \cup V_{m-1}) \cup V_m$ and $V_1 \cap \ldots \cap V_m = (V_1 \cap \ldots \cap V_{m-1}) \cap V_m$, so our result follows from induction on m. □

(1.2.15b)

Proof. Consider the affine variety $V_i = \{i\} \subseteq \mathbb{R}$. Note that $\bigcup_{i=1}^{\infty} = \mathbb{N}$. Note that this set is infinite, but not equal to $\mathbb{R}$, so it is not an affine variety. □

(1.2.15c)

Proof. Define $V = \mathbb{R}, W = \{0\}$. Then $V \backslash W = \mathbb{R} - \{0\}$ which is an infinite set not equal to $\mathbb{R}$. Thus, similarly to (1.2.15b), $V \backslash W$ is not an affine variety. □

(1.2.15d)

Proof. Let $V = V(f_1, \ldots, f_s)$ with $f_i \in k[x_1, \ldots, x_n]$ and let $W = V(g_1, \ldots, g_t)$ with $g_j \in k[x_1, \ldots, x_m]$. Furthermore, define $y_1, \ldots, y_m$ to be new variables and $h_j = g_j(y_1, \ldots, y_m)$.

We wish to show that

$$V \times W = V(f_1, \ldots, f_s, h_1, \ldots, h_t) \subseteq k^n \times k^m$$

Note that, if $f(a, b) \in V \times W$, then $f_i(a, b) = f_i(a) = 0$ and $h_j(a, b) = h_j(b) = 0$ since $a \in V, b \in W$. Thus $(a, b) \in V(f_1, \ldots, f_s, h_1, \ldots, h_t)$.

Conversely, if $(a, b) \in V(f_1, \ldots, f_s, h_1, \ldots, h_t)$, then $f_i(a) = 0$ for $i = 1, \ldots, s$, and $g_j(b) = 0$ for $j = 1, \ldots, t$. Therefore $a \in V$ and $b \in W$.

Thus we conclude that the cartesian product of two affine varieties is an affine variety. $\square$