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Rootfinding Research Prerequisites
07/13/20

Assignment 1.4
Exercises: 2, 3, 6, 7, 9

Exercise 1.4.2

Proof. Let $I \subseteq k[x_1, \ldots, x_n]$ be an ideal, and let $f_1, \ldots, f_s \in k[x_1, \ldots, x_n]$. We wish to show that

$$f_1, \ldots, f_s \in I \Leftrightarrow \langle f_1, \ldots, f_s \rangle \subseteq I.$$

($\Rightarrow$): Assume that $f_1, \ldots, f_s \in I$ and let $f \in \langle f_1, \ldots, f_s \rangle$ be arbitrary. Note that $f = h_1 f_1 + \ldots + h_s f_s$ for some $h_1, \ldots, h_s \in k[x_1, \ldots, x_n]$. Since I is an ideal, we know by definition that each $h_i f_i \in I$ and that $h_1 f_1 + \ldots + h_s f_s \in I$. Therefore, $\langle f_1, \ldots f_s \rangle \subseteq I$.

($\Leftarrow$): Assume that $\langle f_1, \ldots, f_s \rangle \subseteq I$. Note that $f_1 \in \langle f_1, \ldots, f_s \rangle$ since

$$f_1 = 1 \cdot f_1 + 0 \cdot f_2 + \ldots + 0 \cdot f_s \in \langle f_1, \ldots, f_s \rangle.$$

Thus $f_1 \in \langle f_1, \cdots, f_s \rangle \subseteq I$. This conclusion can be generalized for any polynomial f_i in the basis $\langle f_1, \ldots, f_s \rangle$. Therefore, $f_1, \ldots, f_s \in I$. □

Exercise 1.4.3

(1.4.3a): ($\subseteq$): Note that $x \in \langle x+y, x-y \rangle$ since $x = \frac{1}{2} \cdot (x+y) + \frac{1}{2} \cdot (x-y)$. Similarly, $y \in \langle x+y, x-y \rangle$ since $y = \frac{1}{2} \cdot (x+y) + \frac{-1}{2} \cdot (x-y)$. Thus, by Exercise 1.4.2,

$$x, y \in \langle x+y, x-y \rangle \Rightarrow \langle x, y \rangle \subseteq \langle x+y, x-y \rangle.$$

($\supseteq$): Note that $x+y \in \langle x, y \rangle$ since $x+y = 1 \cdot x + 1 \cdot y$. Similarly, $x-y \in \langle x, y \rangle$ since $x - y = 1 \cdot x - 1 \cdot y$. Thus, by Exercise 1.4.2,

$$x+y, x-y \in \langle x, y \rangle \Rightarrow \langle x+y, x-y \rangle \subseteq \langle x, y \rangle.$$

Therefore $\langle x+y, x-y \rangle = \langle x, y \rangle$ in $\mathbb{Q}[x, y]$.

(1.4.3b): ($\subseteq$): Note that

$$\begin{aligned} x + xy &= 1 \cdot x + x \cdot y \in \langle x, y \rangle \\ y + xy &= y \cdot x + 1 \cdot y \in \langle x, y \rangle \\ x^2 &= x \cdot x + 0 \cdot y \in \langle x, y \rangle \\ y^2 &= 0 \cdot x + y \cdot y \in \langle x, y \rangle. \end{aligned}$$

Thus, by Exercise 1.4.2, $\langle x + xy, y + xy, x^2, y^2 \rangle \subseteq \langle x, y \rangle$.

($\supseteq$): Note that

$$\begin{aligned} x &= (1-y)\cdot(x+xy) + y\cdot(y+xy) + 0\cdot x^2 + (-1)\cdot y^2 \in \langle x+xy, y+xy, x^2, y^2\rangle \\ y &= x\cdot(x+xy) + (1-x)\cdot(y+xy) + (-1)\cdot x^2 + 0\cdot y^2 \in \langle x+xy, y+xy, x^2, y^2\rangle. \end{aligned}$$

Thus, by Exercise 1.4.2, $\langle x, y\rangle \subseteq \langle x+xy, y+xy, x^2, y^2\rangle$. Therefore, $\langle x+xy, y+xy, x^2, y^2\rangle = \langle x, y\rangle$ in $\mathbb{Q}[x,y]$.

(1.4.3c): ($\subseteq$): Note that

$$\begin{aligned} 2x^2+3y^2-11 &= 2\cdot(x^2-4) + 3\cdot(y^2-1) \in \langle x^2-4, y^2-1\rangle \\ x^2-y^2-3 &= 1\cdot(x^2-4) + (-1)\cdot(y^2-1) \in \langle x^2-4, y^2-1\rangle. \end{aligned}$$

Thus, by Exercise 1.4.2, $\langle 2x^2+3y^2-11, x^2-y^2-3\rangle \subseteq \langle x^2-4, y^2-1\rangle$.

($\supseteq$): Note that

$$\begin{aligned} x^2-4 &= \frac{1}{5}\cdot 2x^2+3y^2-11 + \frac{3}{5}\cdot x^2-y^2-3 \in \langle 2x^2+3y^2-11, x^2-y^2-3\rangle \\ y^2-1 &= \frac{1}{5}\cdot 2x^2+3y^2-11 + \frac{-2}{5}\cdot x^2-y^2-3 \in \langle 2x^2+3y^2-11, x^2-y^2-3\rangle. \end{aligned}$$

Thus, by Exercise 1.4.2, $\langle x^2-4, y^2-1\rangle \subseteq \langle 2x^2+3y^2-11, x^2-y^2-3\rangle$. Therefore $\langle 2x^2+3y^2-11, x^2-y^2-3\rangle = \langle x^2-4, y^2-1\rangle$ in $\mathbb{Q}[x,y]$.

Exercise 1.4.6

(1.4.6a): Note that $x^n \in \langle x\rangle$ for all $n \in \mathbb{N}$. Each term in the set of x^ns described this way is linearly independent from the others as long as we are multiplying by elements of k, so $\langle x\rangle$ has infinite dimension. Thus, any vector space basis of I over k is infinite.

(1.4.6b): $0 = y\cdot x + (-x)\cdot y$.

(1.4.6c): $0 = f_j\cdot f_i + (-f_i)\cdot f_j$.

(1.4.6d): $f = x^2+xy+y^2 = (x+y)\cdot x + y\cdot y = x\cdot x + (x+y)\cdot y$.

(1.4.6e): We wish to show that x and $x+x^2, x^2$ are minimal bases for the same ideal of $k[x]$.

Proof. We begin by showing that x and $x+x^2, x^2$ are bases for the same ideal. Note that

$$\begin{aligned} x &= 1\cdot(x+x^2) + (-1)\cdot x^2 \in \langle x+x^2, x^2\rangle \\ x^2 &= x\cdot x \in \langle x\rangle \\ x+x^2 &= x + x\cdot x \in \langle x\rangle \end{aligned}$$

Thus, by Exercise 1.4.2 we can conclude that $\langle x\rangle = \langle x+x^2, x^2\rangle$.

Next, we show that x and $x+x^2, x^2$ are both minimal bases. Note that x is a minimal basis because the only proper subset is the empty set, which cannot describe the same ideal of $k[x]$. We can see that $x+x^2, x^2$ is a minimal basis by noting that $x \notin \langle x^2 \rangle$ since x^2 does not divide x. Similarly, we can conclude that $x \notin \langle x+x^2 \rangle$. Hence $x+x^2, x^2$ is a minimal basis.

We therefore conclude that the bases x and $x+x^2, x^2$ both describe the same ideal and are both minimal. $\square$

Exercise 1.4.7

Proof. Note that, since m, n are both positive, $V(x^n, y^m) = \{(0,0)\}$. All that remains is to show that $I(\{(0,0)\}) = \langle x, y \rangle$. To prove one direction, we see that any polynomial in the form $A(x,y)x + B(x,y)y$ vanishes at the origin. For the other direction, suppose that $f = \sum_{ij} a_{ij}x^i y^j \in I(\{(0,0)\})$. Then $a_{00} = f(0,0) = 0$ and we have that

$$\begin{aligned} f &= a_{00} + \sum_{i,j \neq 0,0} a_{ij}x^i y^j \\ &= 0 + \left(\sum_{i,j,i>0} a_{ij}x^{i-1}y^j \right) x + \left(\sum_{j>0} a_{0j}y^{j-1} \right) y \in \langle x, y \rangle. \end{aligned}$$

Thus $I(V(x^n, y^m)) = I(\{(0,0)\}) = \langle x, y \rangle$. $\square$

Exercise 1.4.9

(1.4.9a): By definition, we have that $f(x,y,z) = y^2 - xz \in I(V)$ if and only if $f = 0$ on V. Let (a,b,c) be an arbitrary point contained in V. By the book's given parameterization, we have that $(a,b,c) = (t, t^2, t^3)$ for some $t \in \mathbb{R}$. Hence $b^2 - ac = (t^2)^2 - t(t^3) = t^4 - t^4 = 0$. Since (a,b,c) is arbitrary, we now have that $y^2 - xz = 0$ on V so $y^2 - xz \in I(V)$.

(1.4.9b): We wish to show that $y^2 - xz \in \langle y - x^2, z - x^3 \rangle$, or in other words, $y^2 - xz = h_1(y - x^2) + h_2(z - x^3)$ where $h_1, h_2 \in \mathbb{R}[x,y,z]$. We begin by expanding the terms of $y^2 - xz$ as follows:

$$\begin{aligned} y^2 = x^0 y^2 z^0 &= x^0(x^2 + (y - x^2))^2 (x^3 + (z - x^3))^0 \\ &= (x^4 + 2x^2(y - x^2) + y^2 - 2x^2 y + x^4) \\ &= (2x^2 + (y - x^2))(y - x^2) + x^4 \\ &= (y + x^2)(y - x^2) + x^4 \end{aligned}$$

and

$$\begin{aligned} xz = x^1 y^0 z^1 &= x(x^2 + (y - x^2))^0 (x^3 + (z - x^3))^1 \\ &= x(z - x^3) + x^4. \end{aligned}$$

Thus

$$\begin{aligned} y^2 - xz &= ((y + x^2)(y - x^2) + x^4) - (x(z - x^3) + x^4) \\ &= (y + x^2)(y - x^2) + (-x)(z - x^3) + x^4 - x^4 \\ &= h_1 \cdot (y - x^2) + h_2 \cdot (z - x^3) \end{aligned}$$

where $h_1 = y + x^2$ and $h_2 = -x$. Therefore $y^2 - xz \in \langle y - x^2, z - x^3 \rangle$ as desired.