Marcelo Leszynski
Rootfinding Research Prerequisites
08/10/20

Assignment 4.1
Exercises: 2, 4, 7, 8

Exercise 4.1.2 We first begin by calculating $V(J)$ and note that, for any $(a,b) \in V(J)$,

$$a^2 + b^2 - 1 = b - 1 = 0 \Rightarrow b = 1 \Rightarrow a = 0$$

so $V(J) = \{(0,1)\}$. Using this variety, we see that the polynomial $f = x$ vanishes at the point $(0,1)$, so $f \in I(V(J))$. To show that $f \notin J$, we compute the Groebner basis G for J with lex order and $x > y$ and have $G = \{x^2, y-1\}$. None of the terms in G divide $f = x$, so dividing f by G yields a nonzero remainder and $f = x \notin \langle G \rangle = J$.

Exercise 4.1.4

Proof. Let k be an algebraically closed field. We wish to show that k must be infinite and proceed by contradiction. Assume, by way of contradiction, that k is finite such that $k = \{a_1, \ldots, a_n\}$. We then have that the polynomial $f = (x-a_1)\ldots(x-a_n)+1$ is nonconstant and satisfies $f(a_1) = \ldots = f(a_n) = 1$. Since k is algebraically closed, we know that there exists some $a \in k$ with $f(a) = 0$. We then have that $a = a_i$ for some $1 \leq i \leq n$, which gives $0 = f(a) = f(a_i) = 1$, which is a contradiction. Therefore, k must be infinite. □

Exercise 4.1.7 Credit to:

https://web.ma.utexas.edu/users/allcock/expos/nullstellensatz3.pdf

https://pdfs.semanticscholar.org/34b1/0b8c7deb313c12e729e1663a0b407dd3e80c.pdf

(2) $\Rightarrow$ (3): Let $k[t_1, \ldots, t_n]$ be a polynomial ring and let L be a field such that $k \subseteq L$. This implies that the terms $a_1, \ldots, a_n \in L$ give a map

$$g(t_1, \ldots, t_n) \in k[t_1, \ldots, t_n] \to g(a_1, \ldots, a_n) \in L$$

which preserves addition and multiplication. We apply this idea to the polynomial ring $k[x_1, \ldots, x_n, y]$ over the field of rational functions $k(x_1, \ldots, x_n)$. We can evaluate this ring at terms $x_1, \ldots, x_n, 1/f$ with $1/f \in k(x_1, \ldots, x_n)$ to yield the map

$$g(x_1, \ldots, x_n, y) \in k[x_1, \ldots, x_n, y] \to g(x_1, \ldots, x_n, 1/f) \in k(x_1, \ldots, x_n).$$

Note that this mapping preserves addition and multiplication and is also the map of the identity on $k[x_1, \ldots, x_n]$. Proceeding from equation (2) in the text, we can evaluate the given polynomial as follows:

$$\begin{aligned}
1 &= \sum_{i=1}^{s} p_1(x_1,\ldots,x_n,y)f_i + q(x_1,\ldots,x_n,y)(1-yf) \\
&= \sum_{i=1}^{s} p_1(x_1,\ldots,x_n,1/f)f_i + q(x_1,\ldots,x_n,1/f)(1-(1/f)f) \\
&= \sum_{i=1}^{s} p_1(x_1,\ldots,x_n,1/f)f_i + q(x_1,\ldots,x_n,1/f)\cdot 0 \\
&= \sum_{i=1}^{s} p_1(x_1,\ldots,x_n,1/f)f_i.
\end{aligned}$$

The resulting equation given above can be used to justify equation (3) in the text.

(3) $\Rightarrow$ (4): To show this justification, we further analyze $p_i(x_1,\ldots,x_n,1/f)$. Note that

$$p_i(x_1,\ldots,x_n,y) = \sum_{l=1}^{m_i} h_l(x_1,\ldots,x_n)y^l.$$

Using our previous mapping, this evaluates to

$$p_i(x_1,\ldots,x_n,y) = p_i(x_1,\ldots,x_n,1/f) = \sum_{l=1}^{m_i} h_l(x_1,\ldots,x_n)1/f^l \in k(x_1,\ldots,x_n)$$

It follows that

$$f^{m_i}p_i(x_1,\ldots,x_n,1/f) = \sum_{l=1}^{m_i} h_l(x_1,\ldots,x_n)f^{m_i-l}$$

is a polynomial in $x_1,\ldots,x_n$. We can arrive at equation (4) by fixing $m = \max(m_1,\ldots,m_s)$ and multiplying each side of (3) by f^m.

Exercise 4.1.8

(4.1.8a):

Proof. Let $g = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$ be a polynomial of degree n in x, and define the homogenization g^h of g with respect to some variable y as the polynomial $g^h = a_0x^n + a_1x^{n-1}y + \ldots a_{n-1}xy^{n-1} + a_ny^n$. We wish to show that g has a root in k if and only if there exists some $(a,b) \in k^2$ such that $(a,b) \neq (0,0)$ and $g^h(a,b) = 0$ Before we proceed, we note that

$$g^h(x,y) = \sum_{i=0}^{n} a_ix^{n-i}y^i = \sum_{i=0}^{n} a_i\left(\frac{x}{y}\right)^{n-i}y^n = y^ng^h\left(\frac{x}{y},1\right) = y^ng\left(\frac{x}{y}\right).$$

($\Rightarrow$): Assume that g has a root $r \in k$. Using the above notation, we have that

$$0 = g(r) = 1^n g(r/1) = g^h(r, 1),$$

so $(r, 1)$ is a root of g^h.

($\Leftarrow$): Assume that there is some nontrivial root $(r_1, r_2) \neq (0, 0)$ of g^h. We now divide the proof into two cases: in the first case, $r_2 = 0$. Then $0 = g^h(r_1, 0) = a_0 r_1^n$, implying that $r_1 = 0$. This is a contradiction since $(r_1, r_2) \neq (0, 0)$, so this case can never hold. In the second case, $r_2 \neq 0$, so we have

$$0 = g^h(r_1, r_2) = r_2^n g(r_1/r_2),$$

implying that $g(r_1, r_2) = 0$. □

(4.1.8b):

Proof. Let g be a nonconstant polynomial with no root in k of degree n. Let $f(x, y) = g^h$ be the homogenization of g. Then f vanishes at $(0, 0)$ since every term of f is of degree $n > 0$. We know by Exercise 4.1.8a that any other root (a, b) of f with $f(a, b) = 0, (a, b) \neq 0$ implies that g has a root in k. By construction of g, we know that this cannot occur. Therefore $(0, 0)$ is the only solution of $f = 0$. □

(4.1.8c):

Proof. Base Case: Let $f_2 = f$ be given by f in Exercise 4.1.8b. This proves the base case for our inductive proof of $n = 2$.

Inductive Hypothesis: We proceed by induction and assume there exists some $f_m \in k[x_1, \ldots, x_m]$ such that $f_m(a_1, \ldots, a_m) = 0$ if and only if $a_1 = \ldots = a_m = 0$ for $(a_1, \ldots, a_m) \in k^m$.

Inductive Step: Define $f_{m+1} = f_2(f_m(x_1, \ldots, x_m), x_{m+1})$ and let $(a_1, \ldots, a_m, b) \in k^{m+1}$. We then have that

$$\begin{aligned} 0 = f_{m+1}(a_1, \ldots, a_m, b) \Leftrightarrow 0 = f_2(f_m(a_1, \ldots, a_m) b) &\Leftrightarrow f_m(a_1, \ldots, a_m) = 0 \text{ and } b = 0 \\ &\Leftrightarrow a_1 = \ldots = a_m = 0 \text{ and } b = 0. \end{aligned}$$

□

(4.1.8d):

Proof. Let $W = V(g_1, \ldots, g_s)$ be a variety in k^n with k not being algebraically closed, and let $f_s \in k[y_1, \ldots, y_s]$ be given by Exercise 4.1.8c. Note the existence of f_s is due to k not being algebraically closed. We can set

$$\begin{aligned} h &= f_s(g_1, \ldots, g_s) \in k[x_1, \ldots, x_n], \\ a &= (a_1, \ldots, a_n) \in k^n \end{aligned}$$

which gives

$$h(a) = 0 \Leftrightarrow f_s(g_1(a), \ldots, g_s(a)) = 0 \Leftrightarrow g_1(a) = \ldots = g_s(a) = 0.$$

This proves that $W = V(h)$ as desired. $\square$