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Rootfinding Research Prerequisites
10/7/20

Assignment 5.1
Exercises: 1, 2, 4, 8

Exercise 5.1.1

Proof. We begin by parameterizing V as follows:

$$x = t, y = t^2, z = t^3.$$

We now have that

$$v - u - u^2 = z + x^2y^2 - xy - (xy)^2 = t^3 + t^2(t^2)^2 - t \cdot t^2 - (t \cdot t^2)^2 = 0.$$

Thus W contains the image of V under ϕ and *phi* is a polynomial mapping from V to W. $\square$

Exercise 5.1.2 The image of V under ϕ consists of all three-tuples $(u, v, w) \in \mathbb{R}^3$ that satisfy

$$y - x = u - (x^2 - y) = v - (y^2) = w - (x - 3y^2) = 0$$

for some $(x, y) \in \mathbb{R}^2$. Letting the above system generate an ideal and computing a Grobner basis for said ideal with lex order $x > y > u > v > w$ yields

$$x - 3v - w, y - 3v - w, u + 2v + w, v^2 + \frac{2}{3}vw - \frac{1}{9}v + \frac{1}{9}w^2.$$

This can be further simplified using the Elimination Theorem, resulting in the image of ϕ being given by

$$u + 2v + w, v^2 + \frac{2}{3}vw - \frac{1}{9}v + \frac{1}{9}w^2.$$

Exercise 5.1.4

(5.1.4a): We begin by defining the mapping $\pi^{-1}(a, b)$ as

$$x = a, y = b, (z^2 - (a^2 + b^2 - 1)(4 - a^2 - b^2)) = 0.$$

It follows from this mapping that, given $(a, b) \in \mathbb{R}^2$, there exists at most two points in the set.

(5.1.4b): Solving the third equation in Exercise 5.1.4a for z gives

$$z = \pm\sqrt{(a^2 + b^2 - 1)(4 - a^2 - b^2)}.$$

When $a^2 + b^2 < 1$ or $a^2 + b^2 > 4$, this equation gives zero points in the reals since the result is a complex number. When $a^2+b^2 = 1$ or $a^2+b^2 = 4$, we have one point since $z = 0$. When $1 < a^2 + b^2 < 4$, we have two points.

(5.1.4c): V can be described as a torus with an inner radius of 1 and an outer radius of 4 that lies parallel to the xy-plane.

Exercise 5.1.8

(5.1.8a): Choose $q = (0, 1, 1)$ in V such that $f(q) = 1^2 + 1^3 \neq 0$. Similarly, choose $r = (2, 0, 0)$ in V such that $g(r) = 2^2 - 2 \neq 0$. Hence, neither f nor g is identically zero on V.

Note that

$$fg = (y^2 + z^3)(x^2 - x) = x^2y^2 - xy^2 + x^2z^3 - xz^3 = (xy)^2 - xy^2 + x^2z^3 - xz^3 \in \langle xy, xz\rangle.$$

Thus, fg vanishes identically on V.

(5.1.8b): Observe that

$$\begin{aligned} V_1 = V \cap \mathbf{V}(f) = \mathbf{V}(xy, xz, y^2 + z^3) = \{(0, y, -y^{\frac{2}{3}})|y \in \mathbb{R}\} \cup \{(x, 0, 0)|x \in \mathbb{R}\} \\ V_2 = V \cap \mathbf{V}(g) = \mathbf{V}(xy, xz, x^2 - x) = \{(0, y, z)|y, z \in \mathbb{R}\} \cup \{(1, 0, 0)\}. \end{aligned}$$

Using Lemma 1.1.2, we have that $\mathbf{V}(fg) = \mathbf{V}(f) \cup \mathbf{V}(g)$. Thus, using the fact that fg vanishes identically on V, we let $V = V \cap \mathbf{V}(fg)$ and have that

$$V = V \cap \mathbf{V}(fg) = V \cap (\mathbf{V}(f) \cup \mathbf{V}(g)) = (V \cap \mathbf{V}(f)) \cup (V \cap \mathbf{V}(g)) = V_1 \cup V_2.$$