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Rootfinding Research Prerequisites
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Assignment 1.3
Exercises: 1, 2, 3, 4, 6, 7, 11, 13, 14

Exercise 1.3.1 Using row reduction, we can arrive at the system of equations

$$\begin{cases} x + 4z - 3w = 5 \\ y - 3x + 2w = -3. \end{cases}$$

From here, we let $z = t, w = u$ where t and u are arbitrary parameters, and get the parametrization

$$\begin{cases} x = 5 - 4t + 3u \\ y = -3 + 3t - 2u \\ z = t \\ w = u. \end{cases}$$

Exercise 1.3.2 Using the trigonometric identity $\cos(2t) = 2\cos^2(t) - 1$, we have that $x = cos(t)$, $y = \cos(2t) = 2\cos^2(t) - 1$. Substituting x for $\cos(t)$ into y yields the parabola $y = 2x^2 - 1$. This parametrization covers the parabola contained in the square $-1 \leq x, y \leq 1$ since $-1 \leq \cos(\theta) \leq 1$.

Exercise 1.3.3 This parametrization can be given by $x = t, y = f(t)$.

Exercise 1.3.4

(1.3.4a) We begin by solving $x = \frac{t}{1+t}$ for t, which yields $t = \frac{x}{1-x}$. Next, we substitute this value into $y = 1 - \frac{1}{t^2}$ so

$$y = 1 - \frac{1}{t^2} = 1 - \frac{(1-x)^2}{x^2} = \frac{2x-1}{x^2}.$$

Thus we have that $x^2 y = 2x - 1$, which implies that the parametrization is contained in $V(x^2 y - 2x + 1)$.

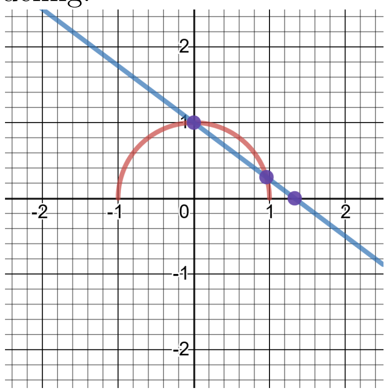
(1.3.4b) First, note that $(1, 1) \in V(x^2y - 2x + 1)$ and that $1 \neq \frac{t}{1+t}$ for all $t \in \mathbb{R}$. Thus, the parametrization excludes $(1, 1)$. To see all other points are covered by the parametrization, suppose (x, y) satisfies $x^2y - 2x + 1 = 0$ with $x \neq 1$. Observe now that $x \neq 0$, since $0 * y - 2 * 0 + 1 = 0$ is impossible. We therefore set $t = \frac{x}{1-x} \in \mathbb{R}$ and observe that $t \neq 0$ due to our constraints on x. Then $x = \frac{t}{1+t}$ by our definition of t and

$$1 - \frac{1}{t^2} = 1 - \frac{(1-x)^2}{x^2} = \frac{2x-1}{x^2} = y.$$

Note that the last equality follows from $x^2y - 2x + 1 = 0$. Therefore, (x, y) comes from the parametrization.

Exercise 1.3.6

(1.3.6a) The following is a second-dimensional representation of the concept. Note that the only point not reached by constructing lines this way is the "north pole", since that point cannot be represented by a line that intersects the sphere at two places using our current modeling.



(1.3.6b) The line from $(0, 0, 1)$ to $(u, v, 0)$ can be parametrized by $(1 - t)(0, 0, 1) + t(u, v, 0) = (tu, tv, 1 - t)$.

(1.3.6c) Substituing $x = tu, y = tv, z = 1 - t$ into $x^2 + y^2 + z^2 - 1 = 0$ yields

$$0 = (tu)^2 + (tv)^2 + (1 - t)^2 - 1 = (u^2 + v^2 + 1)t^2 - 2t = t((u^2 + v^2 + 1)t - 2).$$

Note that, since $t = 0$ corresponds to the point $(0, 0, 1)$, the other point of intersection must be given by $t = \frac{2}{u^2+v^2+1}$. Thus we have that

$$x = tu = \frac{2u}{u^2 + v^2 + 1}, y = tv = \frac{2v}{u^2 + v^2 + 1}, z = 1 - t = \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1}.$$

Exercise 1.3.7 Working in $\mathbb{R}^n$ means that the "north pole" is given by $(0, \ldots, 0, 1)$. Next, we need to define the line through the north pole and a point $(u_1, \ldots, u_{n-1}, 0)$ in the hyperplane $x_n = 0$. This can be defined by

$$(x_1, \ldots, x_n) = (1 - t)(0, \ldots, 0, 1) + t(u_1, \ldots, u_{n-1}, 0) = (tu_1, \ldots, tu_{n-1}, 1 - t).$$

We can observe that this line meets $x_1^2 + \ldots + x_n^2 - 1 = 0$ where

$$0 = (tu_1)^2 + \ldots + (tu_{n-1}^2) + (1-t)^2 - 1 = (u_1^2 + \ldots + u_{n-1}^2 + 1)t^2 - 2t = t((u_1^2 + \ldots + u_{n-1}^2 + 1)t - 2).$$

We already know that $t = 0$ is the north pole, so the other point of intersection must be given by

$$t = \frac{2}{u_1^2 + \ldots + u_{n-1}^2 + 1}$$

which gives the parametrization

$$x_i = tu_i = \frac{2u_i}{u_1^2 + \ldots + u_{n-1}^2 + 1}$$

for all $1 \leq i \leq n$.

Exercise 1.3.11

(1.3.11a) Using 1.3.8, we can change the variables $y \to x, x \to z$ to see that $x^2 = cz^2 - z^3$ is parametrized by $z = c - t^2, x = t(c - t^2)$.

(1.3.11b) Using $c = y^2$ for some fixed y, it follows that $x^2 = y^2z^2 - z^3$ is parametrized by $z = y^2 - t^2, x = t(y^2 - t^2)$. Finally, we replace y with a parameter u to get the parametrization $x = t(u^2 - t^2), y = u, z = u^2 - t^2$ of $x^2 = y^2z^2 - z^3$, which is the surface $V(x^2 - y^2z^2 + z^3)$ in $\mathbb{R}^3$.

(1.3.11c) Let $(x, y, z) \in V(x^2 - y^2z^2 + z^3)$, and define $c = y^2$. We then have that $(x, z) \in V(x^2 - cz^2 - z^3)$. Using (1.3.8c) with the change of variables $y \to x, x \to z$, there exists a t such that $z = c - t^2, x = t(c - t^2)$. Now, let $u = y$ so that $c = y^2 = u^2$. Then $x = t(u^2 - t^2), y = u, z = u^2 - t^2$ as desired.

Exercise 1.3.13 Using method 1, we have that $x = 1 + u - v, y = u + 2v, z = -1 - u + v$ satisfies $ax + by + cz = d$ if and only if

$$d = a(1 + u - v) + b(u + 2v) + c(-1 - u + v) = (a + b - c)u + (-a + 2b + c)v + (a - c).$$

We can rewrite this as the following system of equations

$$\begin{cases} a + b - c = 0 \\ -a + 2b + c = 0 \\ a - c = d. \end{cases}$$

When solved, this system yields $a = c = 1, b = d = 0$, so the equation is $x + z = 0$.

Exercise 1.3.14

(1.3.14a) Let $P, Q \in \mathbb{R}^2$. Then the line connecting P and Q is parametrized by

$$P + t(Q - P) = (1 - t)P + tQ$$

for $t \in \mathbb{R}$. Note that $t = 0$ yields P, $t = 1$ yields Q, and $0 \leq t \leq 1$ yields the line segment joining the two. Thus we have that

$$(1 - t)P + tQ \in C$$

for $0 \leq t \leq 1$.

(1.3.14b) We wish to show that, for a convex set C containing $P_1, \ldots, P_n$, we have $\sum_{i=1}^{n} t_i P_i \in C$ when $t_1, \ldots, t_n \geq 0$ and that $\sum_{i=1}^{n} t_i = 1$. We proceed by induction on n. The base case of $n = 1$ is true by observation, so we assume that the assertion is true for n. Next, we consider $P_1, \ldots, P_{n+1} \in C$ with $t_1, \ldots, t_{n+1} \geq 0$ and $\sum_{i=1}^{n+1} t_i = 1$. This implies that $t_{n+1} \leq 1$ and yields two possible cases:

Case 1: $t_{n+1} = 1$. Then $t_1 = \ldots = t_n = 0$, so $\sum_{i=1}^{n+1} t_i P_i = P_{n+1} \in C$.

Case 2: $t_{n+1} < 1$. Then $1 - t_{n+1} > 0$ and we fix $u = \frac{t_i}{1 - t_{n+1}}$ for $1 \leq i \leq n$. Then $u_i \geq 0$. Thus $\sum_{i=1}^{n} t_i = 1 - t_{n+1}$ implies $\sum_{i=1}^{nu} u_i = 1$. Using the inductive hypothesis, $P = \sum_{i=1}^{n} u_i P_i \in C$. We can then use (1.3.14a) to show that C contains the point as follows:

$$\begin{aligned}
(1 - t_{n+1})P + t_{n+1}P_{n+1} &= (1 - t_{n+1}) \sum_{i=1}^{n} u_i P_i + t_{n+1} P_{n+1} \\
&= (1 - t_{n+1}) \sum_{i=1}^{n} \frac{t_i}{1 - t_{n+1}} P_i + t_{n+1} P_{n+1} \\
&= \sum_{i=1}^{n} t_i P_i + t_{n+1} P_{n+1} \\
&= \sum_{i=1}^{n+1} t_i P_i
\end{aligned}$$

as desired.