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Rootfinding Research Prerequisites
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Assignment 4.2
Exercises: 1-5

Exercise 4.2.1

Proof. Let k be a field. We wish to show that $\sqrt{\langle x^n, y^m\rangle} = \langle x, y\rangle$ for any positive integers n and m.

($\subseteq$): Let $f \in \sqrt{\langle x^n, y^m\rangle}$. We then have that $f^p = Ax^n + By^m$ for some $p \geq 1$. This implies that $f(0,0) = 0$. We can then write $f = xf_1 + yf_2 + r$ with $r \in k$, and our previous statement implies that $r = 0$. Therefore, $f \in \langle x, y\rangle$.

($\supseteq$): Note that $x^n, y^m \in \langle x^n, y^m\rangle$ which implies $x, y \in \sqrt{\langle x^n, y^m\rangle}$ by definition. It follows that $\langle x, y\rangle$ by Lemma 4.2.5. $\square$

Exercise 4.2.2 The given proposition is not necessarily true. We show this by fixing $f = x^2$ and $g = x^3$. We then have that $\langle f^2, g^3\rangle = \langle x^4, x^9\rangle = \langle x^4\rangle$, but $\langle f, g\rangle = \langle x^2, x^3\rangle = \langle x^2\rangle$. This demonstrates that $x \in \sqrt{\langle x^4\rangle} = \sqrt{I}$, but $x \notin \langle x^2\rangle = \langle f, g\rangle$. Therefore $\sqrt{I} \not\subseteq \langle f, g\rangle$ in all cases.

Exercise 4.2.3

Proof. We begin by showing that $V(x^2+1)$ is the empty variety, which can be quickly verified by noting that $x^2 + 1 \in \mathbb{R}[x]$ has no roots in $\mathbb{R}$. Thus, $V(x^2 + 1) = \emptyset$.

Next, we show that $\langle x^2+1\rangle \subseteq \mathbb{R}[x]$ is a radical ideal. This is done by recognizing that this polynomial is irreducible as a result of any nontrivial factorization in $\mathbb{R}[x]$ involving linear factors. These linear factors would result in roots in $\mathbb{R}$, which is a contradiction. Next, we suppose that $f \in \mathbb{R}[x]$ satisfies $f^m \in \langle x^2 + 1\rangle$. This implies that $x^2 + 1$ is an irreducible factor of f^m. It follows that $x^2 + 1$ must be an irreducible factor of f, thus we have that $f \in \langle x^2 + 1\rangle$ and that $\langle x^2 + 1\rangle$ is radical. $\square$

Exercise 4.2.4 Let I be an ideal in $k[x_1, \ldots, x_n]$ where k is an arbitrary field.

(4.2.4a):

Proof. We wish to show that $\sqrt{I}$ is a radical ideal and proceed directly. Suppose that $f^m \in \sqrt{I}$. By the definition of a radical ideal, it follows that there exists some $n \geq 1$ such that $(f^m)^n = f^{mn} \in I$, implying $f \in \sqrt{I}$. Therefore, $\sqrt{I}$ is radical. $\square$

(4.2.4b):

Proof. We wish to show that I is radical if and only if $I = \sqrt{I}$ and proceed directly.

($\Rightarrow$): Assuming that I is radical, we have that $I \subseteq \sqrt{I}$ by Lemma 4.2.5. To show the other inclusion, let $f \in \sqrt{I}$. This implies that there exists some $m \geq 1$ such that $f^m \in I$. Since I is radical by assumption, we conclude that $f \in I$. Therefore $I = \sqrt{I}$.

($\Leftarrow$): This follows from Exercise 4.2.4a $\square$

(4.2.4c):

Proof. Exercise 4.2.4a implies that $\sqrt{I}$ is radical. We can then make the substitution $I = \sqrt{I}$ in Exercise 4.2.4b to show that $\sqrt{I} = \sqrt{\sqrt{I}}$ as desired. $\square$

Exercise 4.2.5

Proof. We begin by proving that the mappings I and V are inclusion-reversing and proceed directly. Working on the mapping I, we let A and B be affine varieties in k^n. Assuming that $A \subseteq B$, any polynomial vanishing on B must vanish on A, so $I(B) \subseteq I(A)$. Conversely, we assume that $I(B) \subseteq I(A)$. Since we can define B in terms of polynomials $f_1, \ldots, f_m \in k[x_1, \ldots, x_n]$, we have that $f_1, \ldots, f_m \in I(B) \subseteq I(A)$ so all polynomials f_i vanish on A. Since B consists of all common zeros of the polynomials f_i, it follows that $A \subseteq B$.

Next, we work on the mapping V. Let I, J be ideals and assume $I \subseteq J$. Let $a \in V(J)$, so $f(a) = 0$ for all $f \in J$. Since $I \subseteq J$, we can conclude that $f(a) = 0$ for all $f \in I$, so $a \in V(I)$. Since a was arbitrary, we conclude that $V(J) \subseteq V(I)$.

Finally, to show that $V(\sqrt{I}) = V(I)$, we have that $I \subseteq \sqrt{I}$ by Lemma 4.2.5, so $V(\sqrt{I}) \subseteq V(I)$. To show the other inclusion, we let $a \in V(I)$ and $f \in \sqrt{I}$. We then have that $f^m \in I$ for some $m \geq 1$, so it follows that $f^m(a) = (f(a))^m = 0$. This implies that $f(a) = 0$, which gives that $a \in V(\sqrt{I})$. Therefore $V(I) \subseteq V(\sqrt{I})$ as needed. $\square$