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Rootfinding Research Prerequisites
07/30/20

Assignment 2.8
Exercises: 1-5

Exercise 2.8.1 It is true that $f \in I$.

Proof. We begin by computing a Groebner basis G in lex order

$$G = \{y^5 - z^3, -y^2 + xz, xy^3 - z^2, x^2y - z, x^3 - y\}.$$

It suffices to show that the remainder of f when divided by the terms in G is zero. Using the division algorithm, we have

$$f = 1 \cdot (y^5 - z^3) + 0 \cdot (-y^2 + xz) + 1 \cdot (xy^3 - z^2) + 0 \cdot (x^2y - z) + 0 \cdot (x^3 - y) + 0.$$

Therefore, $f \in I$. □

Exercise 2.8.2 It is not true that $f \in I$.

Proof. We begin by computing a Groebner basis G in lex order

$$G = \{2z^2 + z, y - z, xz - z\}.$$

It suffices to show that the remainder of f when divided by the terms in G is nonzero. Using the division algorithm, we have

$$f = -1 \cdot (2z^2 + z) + (-2y - 2z) \cdot (y - z) + (x^2 + x + 1) \cdot (xz - z) + 2z.$$

Therefore, $f \notin I$. □

Exercise 2.8.3 We begin by computing a Groebner basis G in lex order

$$G = \{g_1, g_2, g_3\} = \{40z^2 - 8z - 23, 3y + z - 1, 2x - 1\}.$$

Note that g_3 gives $x = \frac{1}{2}$ and that g_1 is a quadratic in terms of z, which we can solve to get $z = \frac{2 \pm 3\sqrt{26}}{20}$. Using these values to solve for y in g_2 gives the roots

$$(x, y, z) = \left(\frac{1}{2}, \frac{6 - \sqrt{26}}{20}, \frac{2 + 3\sqrt{26}}{20}\right), \left(\frac{1}{2}, \frac{6 + \sqrt{26}}{20}, \frac{2 - 3\sqrt{26}}{20}\right).$$

Exercise 2.8.4 Computing a Groebner basis G using lex order for the ideal

$$\langle x^2y - z^3, 2xy - 4z - 1, -y^2 + z, x^3 - 4yz\rangle$$

yields $G = \{1\}$. Therefore,

$$V(x^2y - z^3, 2xy - 4z - 1, -y^2 + z, x^3 - 4yz) = \emptyset.$$

Exercise 2.8.5

(2.8.5a): To find all critical points of the function $f(x, y)$, we begin by calculating the partial derivatives

$$\begin{aligned} f_x(x, y) &= 4x^3 + 4xy^2 - 8x - 3 \\ f_y(x, y) &= 4y^3 + 4x^2y - 8y - 3. \end{aligned}$$

Next, we compute a Groebner basis G for $\langle f_x(x, y), f_y(x, y)\rangle$ and have

$$G = \{g_1, g_2\} = \{x - y, 8y^3 - 8y - 3\}.$$

Note that g_2 can be factored into $(2y+1)(4y^2-2y+3)$. This, along with the fact that $x = y$ is given by $g_!$, yields three critical points

$$\left(-\frac{1}{2}, -\frac{1}{2}\right), \left(\frac{1}{4}(1-\sqrt{13}), \frac{1}{4}(1-\sqrt{13})\right), \left(\frac{1}{4}(1+\sqrt{13}), \frac{1}{4}(1+\sqrt{13})\right).$$

(2.8.5b): To determine whether the critical points are local minima/maxima or saddle points, we use the second derivative test and begin by calculating the second-order partial derivatives

$$\begin{aligned} f_{xx}(x, y) &= 12x^2 + 4y^2 - 8 \\ f_{xy}(x, y) &= 8xy \\ f_{yy}(x, y) &= 12y^2 + 4x^2 - 8. \end{aligned}$$

Next, we evaluate $D = f_{xx}f_{yy} - f_{xy}^2$ at the critical points. Remembering that $x = y$ was given to us by our Groebner basis, we have that

$$\begin{aligned} D(x, x) &= 192x^4 - 256x^2 + 64 && \text{so} \\ D\left(-\frac{1}{2}, -\frac{1}{2}\right) &= 12 > 0 && \text{is undetermined,} \\ D\left(\frac{1}{4}(1+\sqrt{13}), \frac{1}{4}(1+\sqrt{13})\right) &= 26 + 10\sqrt{13} > 0 && \text{is undetermined,} \\ D\left(\frac{1}{4}(1-\sqrt{13}), \frac{1}{4}(1-\sqrt{13})\right) &= 26 - 10\sqrt{13} < 0 && \text{is a saddle point.} \end{aligned}$$

From this, we conclude that $(\frac{1}{4}(1-\sqrt{13}), \frac{1}{4}(1-\sqrt{13}))$ is a saddle point. The status of the remaining two critical points can be ascertained as follows:

$$f_{xx}\left(-\frac{1}{2}, -\frac{1}{2}\right) = -4 < 0$$

$$f_{xx}\left(\frac{1}{4}(1+\sqrt{13}), \frac{1}{4}(1+\sqrt{13})\right) = 6 + 2\sqrt{13} > 0.$$

Thus we see that $(-\frac{1}{2}, -\frac{1}{2})$ is a local maximum and that $(\frac{1}{4}(1+\sqrt{13}), \frac{1}{4}(1+\sqrt{13}))$ is a local minimum.