Marcelo Leszynski
Rootfinding Research Prerequisites
07/13/20

Assignment 2.2
Exercises: 1, 5, 7, 8, 10, 11

Exercise 2.2.1

(2.2.1a):

$$\begin{aligned} f(x,y,z) &= x^3 + x^2 + 2x + 3y - z^2 + z && \text{(lex)} \\ &= x^3 + x^2 - z^2 + 2x + 3y + z && \text{(grlex)} \\ &= x^3 + x^2 - z^2 + 2x + 3y + z && \text{(grevlex)} \end{aligned}$$

$\text{multideg}(f) = (3,0,0)$. $\text{LM}(f) = x^3$. $\text{LT}(f) = x^3$.

(2.2.1b):

$$\begin{aligned} \text{lex} &= -3x^5yz^4 + 2x^2y^8 - xy^4 + xyz^3 && \text{LM}(f) = x^5yz^4 && \text{LT}(f) = -3x^5yz^4 && \text{multideg}(f) = (5,1,4) \\ \text{grlex} &= -3x^5yz^4 + 2x^2y^8 - xy^4 + xyz^3 && \text{LM}(f) = x^5yz^4 && \text{LT}(f) = -3x^5yz^4 && \text{multideg}(f) = (5,1,4) \\ \text{grevlex} &= 2x^2y^8 - 3x^5yz^4 + xyz^3 - xy^4 && \text{LM}(f) = x^2y^8 && \text{LT}(f) = 2x^2y^8 && \text{multideg}(f) = (2,8,0). \end{aligned}$$

Exercise 2.2.5

Proof. In order to show that grevlex is a monomial order according to Definition 1, we must prove the following conditions: first, that $>_{grevlex}$ is a linear ordering on $\mathbb{Z}^n_{\geq 0}$, second, that if $\alpha >_{grevlex} \beta$ and $\gamma \in \mathbb{Z}^n_{\geq 0}$, then $\alpha + \gamma >_{grevlex} \beta + \gamma$, third, that $>_{grevlex}$ is well-ordering on $\mathbb{Z}^n_{\geq 0}$.

(i): Let $a, b \in \mathbb{Z}^n_{\geq 0}$ with $a \neq b$. We have three cases. If $|a| > |b|$, then $a >_{grevlex} b$. Similarly, if $|b| > |a|$, then $b >_{grevlex} a$. The third case is when $|a| = |b|$. If the rightmost nonzero entry of $a - b$ is negative, then $a >_{grevlex} b$. If it is positive, this implies that the rightmost nonzero entry of $b - a$ is negative, so $b >_{grevlex} a$. Therefore, grevlex is a total order.

(ii): Let $a >_{grevlex} b$. We have two cases. If $|a| > |b|$, then $|a + c| = |a| + |c| > |b| + |c| = |b + c|$, thus $a + c >_{grevlex} b + c$. If $|a| = |b|$, then the rightmost nonzero entry of $a - b$ is negative by construction. Note that $(a + c) - (b + c) = a - b$, so it follows that the rightmost nonzero entry of $(a + c) - (b + c)$ is negative. Thus $a + c >_{grevlex} b + c$.

(iii): To show that this relation is well-ordering, we must show that an arbitrary sequence $a(1) >_{grevlex} a(2) >_{grevlex} \ldots$ is finite. Note that, for an arbitrary $|a(i)|$, there exists some m with $|a(i)| = |a(m)|$ for $i \geq m$. (This occurs because $>$ is well-ordering). This implies that the sequence must be finite since there is a finite number of $a \in \mathbb{Z}^n_{\geq 0}$ with $|a| = |a(m)|$.

Therefore, grevlex is a monomial order. $\square$

Exercise 2.2.7

(2.2.7a):

Proof. Assume, by way of contradiction, that $a < 0$. This implies that $a + a = 2a < a$. This allows us to create an infinite decreasing sequence of terms such that $0 > a > 2a > 3a > \ldots$, which contradicts part (iii) of the definition of monomial orders. $\square$

(2.2.7b):

Proof. Let x^a, x^b be arbitrary monomials such that x^a divides x^b. We wish to show that $a \le b$. x^a dividing x^b implies that there exists a monomial x^c such that $x^b = x^c x^a$. Equating exponents yields $b = c + a$, which gives $b - a = c \in \mathbb{Z}^n_{\ge 0}$. By our proof of Exercise 2.2.7a, we can conclude that $b - a \ge 0$ so $a \le b$ as desired.

A counterexample to show that the converse is not true is the monomials x^3y and x^2y^2 using lex order. Note that $x^3y >_{lex} x^2y^2$ but x^2y^2 does not divide x^3y. $\square$

(2.2.7c):

Proof. Let $a \in \mathbb{Z}^n_{\ge 0}$. We wish to show that a is the smallest element of $a + \mathbb{Z}^n_{\ge 0}$. Let $b \in \mathbb{Z}^n_{\ge 0}$ be arbitrary. Then for any $a + b \in a + \mathbb{Z}^n_{\ge 0}$, we have that x^a divides x^{a+b}, so by Exercise 2.2.7b, $a + b \ge a$. Since b is arbitrary, we can conclude that a is the smallest element of the set $a + \mathbb{Z}^n_{\ge 0}$. $\square$

Exercise 2.2.8 A matrix is in echelon form if all zero rows are below all nonzero rows, and if the first nonzero entry in a nonzero row is a 1, and is to the right of the first nonzero entries of the rows above. To incorporate the ordering given in equation (2) of the text, we define the polynomial $f_i = a_{i1}x_1 + \ldots + a_{in}x_n$ representing a row of a matrix where a_{ij} is the term on the i-th row and j-th column of a matrix with n columns.

For all $f_i \neq 0$, $\mathrm{LC}(f_i)$ corresponds to the first nonzero enntry on the i-th row, so $\mathrm{LC}(f_i) = 1$. The condition that the first nonzero entry of the i-th row is to the right of the first nonzero entries of higher rows implies that $\mathrm{LT}(f_i) > \mathrm{LT}(f_j)$ for $i < j$. Therefore we can define a matrix A to be in row echelon form when there exists an m with $1 \le m \le n$ such that $\mathrm{LT}(f_1) > \ldots > \mathrm{LT}(f_m)$, $\mathrm{LC}(f_1) = \ldots = \mathrm{LC}(f_m) = 1$, and $f_{m+1} = \ldots = f_n = 0$.

Exercise 2.2.10 This is not true for $\mathbb{Z}^n_{\ge 0}$. A counterexample is $\mathbb{Z}^3_{\ge 0}$ in which there exist an infinite number of monomials in the form $(0, x, 0)$ with $x > 0$ such that $(1, 0, 0) > (0, x, 0) > (0, 0, 1)$.

It is true for the grlex order on $\mathbb{Z}^n_{\ge 0}$ because for any $a \in \mathbb{Z}^n_{\ge 0}$, there is only a finite number of b such that $a >_{grlex} b$. This is because $a >_{grlex} b$ implies that $|a| \ge |b|$. Since for any nonnegative integer n, there is a finite number of b such that $|b| \le n$, it must follow that there are only a finite number of b such that $|b| \le |a|$.

Exercise 2.2.11

(2.2.11a):

Proof. Let $f = x_1^a + x_2^a + \ldots$ where $x_1^a > x_2^a > \ldots > x_i^a$ are ordered monomials, and let $m = x^b$. We have that $a_1 + b > a_2 + b > \ldots > a_i + b$, so x^{a_1+b} is the leading monomial of mf. Since the leading coefficient of this term is LC(f), it follows that LT(mf) $= m$LT(f). $\square$

(2.2.11b):

Proof. Let $x^{a_1} > \ldots > x^{a_n}$ be the monomials of f and let $x^{b_1} > \ldots > x^{b_n}$ be the monomials of g. We then have that $x^{a_1+b_1} \geq x^{a_i+b_i}$. This is then the leading monomial of fg, with coefficient LC(f)·LC(g). Therefore LT(fg) =LT(f)·LT(g) as desired. $\square$

(2.2.11c): No. A counterexample is given by defining $f_1 = 1, f_2 = -1, g_1 = x, g_2 = x+y$ using $>_{lex}$. Then $f_1g_1 + f_2g_2 = x - (x+y) = -y$, but

$$\mathrm{LM}(f_1) \cdot \mathrm{LM}(g_1) = \mathrm{LM}(f_1g_1) = \mathrm{LM}(f_2) \cdot \mathrm{LM}(g_2) = \mathrm{LM}(f_2g_2) = x.$$