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Rootfinding Research Prerequisites
07/30/20

Assignment 2.5
Exercises: 1, 7, 8, 10

Exercise 2.5.1 Consider the polynomial

$$g = y^2z^4 - z^2 = 0 \cdot (xy^2 - xz + y) + 1 \cdot (xy - z^2) - y \cdot (x - yz^4) \in I.$$

Then $\mathrm{LT}(g) = y^2z^4$ which is not divisible by $\mathrm{LT}(g_1)$, $\mathrm{LT}(g_2)$, or $\mathrm{LT}(g_3)$. Therefore $\mathrm{LT}(g) \notin \langle \mathrm{LT}(g_1), \mathrm{LT}(g_2), \mathrm{LT}(g_3) \rangle$.

Exercise 2.5.7 The given set is not a Groebner basis. A counterexample can be seen by considering the ideal $I = \langle x^4y^2 - z^5, x^3y^3 - 1, x^2y^4 - 2z \rangle$ and the polynomial

$$g = -2y^2z^5 + 2x^2z + xy = 2y^2 \cdot (x^4y^2 - z^5) - xy \cdot (x^3y^3 - 1) - x^2 \cdot (x^2y^4 - 2z) \in I.$$

Using grlex order, we have that $\mathrm{LT}(g) = -2y^2z^5$, which is not divisible by any of the leading terms in the given set of generators for I. Therefore $\mathrm{LT}(g)\ /in\langle x^4y^2, x^3y^3, x^2y^4 \rangle$ and by definition the given set is not a Groebner basis for the generated ideal.

Exercise 2.5.8 The basis $\langle x - z^2, y - z^3 \rangle$ is a Groebner basis for lex order.

Proof. Specifically, we claim that $\langle \mathrm{LT}(I) \rangle = \langle \mathrm{LT}(x - z^2), \mathrm{LT}(y - z^3) \rangle = \langle x, y \rangle$. The $(\supseteq)$ inclusion is obvious by the closure of ideals. To show the other inclusion, it suffices to show that, for any polynomial $f \in I - \{0\}$, $\mathrm{LT}(f)$ is divisible by x or y.

Suppose, by way of contradiction, that this was not the case. Then $\mathrm{LT}(f)$ is a power of z. Since we are using lex ordering with $x > y > z$, the multidegree of f is of the form $(0, 0, n)$ with $n \in \mathbb{Z}_{>0}$. We can use a mapping to convert from (x, y, z) to (z^2, z^3, z), which lets f map to itself, but $f \in I = \langle x - z^2, y - z^3 \rangle = \langle z^2 - z^2, z^3 - z^3 \rangle$ suggests that f maps to zero. This is a contradiction, so it must be that all polynomials in I have a leading term that is divisible by either x or y. □

Exercise 2.5.10

Proof. Let $G = \{g_1, \ldots, g_n\}$ be a finite set in I where g_1 is a generator for I. By the secondary (informal) definition of Groebner bases, it suffices to show that $f \in I - \{0\}$ implies that $\mathrm{LT}(f)$ is divisible by some $\mathrm{LT}(g_i)$ with $1 \leq i \leq n$.

g_1 being a generator for I and $f \in I$ imply that $f = hg_1$ for some polynomial h. We then have that $\mathrm{LT}(f) = \mathrm{LT}(hg_1) = \mathrm{LT}(h)\mathrm{LT}(g_1)$ by Lemma 2.4.3. Thus, $\mathrm{LT}(f)$ is divisible by $\mathrm{LT}(g_1)$. □