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Rootfinding Research Prerequisites
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Assignment 3.1
Exercises: 1, 3, 4, 7

Exercise 3.1.1

(3.1.1a):

Proof. By construction, we observe that $0 \in I_l$. Furthermore, I_l is closed under addition since both I and $k[x_{l+1}, \ldots, x_n]$ are. To show I_l is closed under multiplication in $k[x_{l+1}, \ldots, x_n]$, we let $f \in I_l$ and $h \in k[x_{l+1}, \ldots, x_n]$. Since $f \in I$ and $k[x_{l+1}, \ldots, x_n] \subseteq k[x_1, \ldots, x_n]$, it follows that $hf \in I$. It also follows that $hf \in k[x_{l+1}, \ldots, x_n]$ since $h, f \in k[x_{l+1}, \ldots, x_n]$ and $k[x_{l+1}, \ldots, x_n]$ is a ring. Therefore $hf \in I \cap k[x_{l+1}, \ldots, x_n] = I_l$. □

(3.1.1b):

Proof. We have that

$$\begin{aligned}
(I_l)_1 &= I_l \cap k[x_{l+2}, \ldots, x_n] \\
&= (I \cap k[x_{l+1}, \ldots, x_n]) \cap k[x_{l+2}, \ldots, x_n] \\
&= I \cap k[x_{l+2}, \ldots, x_n] && \text{since } k[x_{l+2}, \ldots, x_n] \subseteq k[x_{l+1}, \ldots, x_n] \\
&= I_{l+1}
\end{aligned}$$

□

Exercise 3.1.3 We begin by computing a Groebner basis G for the ideal

$$\langle x^2 + 2y^2 - 2, x^2 + xy + y^2 - 2 \rangle$$

using lex order with $x > y$ and find that

$$G = \{g_1, g_2, g_3\} = \{x^2 + 2y^2 - 2, xy - y^2, 3y^3 - 2y\}.$$

Solving g_3 for y yields $y = 0, \pm\sqrt{2/3}$. Substituting $y = 0$ into g_1 and g_2 yields $x = \pm\sqrt{2}$, while substituting $y = \pm\sqrt{2/3}$ into both g_1 and g_2 yields $x = \pm\sqrt{2/3}$. Therefore the solutions to the system are

$$(x, y) = \pm(\sqrt{2}, 0), \pm(\sqrt{2/3}, \sqrt{2/3}).$$

Note that all of these solutions lie in $\mathbb{C}^2$, and none exist in $\mathbb{Q}^2$.

Exercise 3.1.4 We begin by computing a Groebner basis G for the ideal

$$\langle x^2 + y^2 + z^2 - 4, x^2 + 2y^2 - 5, xz - 1 \rangle$$

using lex order with $x > y > z$ and find that

$$G = \{g_1, g_2, g_3\} = \{x + 2z^3 - 3z, y^2 - z^2 - 1, 2z^4 - 3z^2 + 1\}.$$

The Elimination Theorem then gives that

$$\begin{aligned} I_1 &= \langle y^2 - z^2 - 1, 2z^4 - 3z^2 + 1\rangle \subseteq \mathbb{Q}[y, z] \\ I_2 &= \langle 2z^4 - 3z^2 + 1\rangle \subseteq \mathbb{Q}[z]. \end{aligned}$$

To find solutions over $\mathbb{Q}$, we use g_3 to find values $z = \pm 1, \pm\frac{1}{\sqrt{2}}$. Over Q, we ignore all irrational values for z and thus we have $z = \pm 1$. Substituting these values into g_2 yields $y = \pm\sqrt{2}$. Since these values are irrational, we see that there are no solutions to the system over $\mathbb{Q}$.

Exercise 3.1.7

(3.1.7a): Computing a Groebner basis for the ideal

$$I = \langle t^2 + x^2 + y^2 + z^2, t^2 + 2x^2 - xy - z^2, t + y^3 - z^3\rangle$$

using lex order with $t > x > y > z$ yields a Groebner basis $G = \{g_1, g_2, g_3, g_4, g_5\}$ defined by

$$\begin{aligned} g_1 &= t + y^3 - z^3 \\ g_2 &= x^2 + y^6 - 2y^3z^3 + y^2 + z^6 + z^2 \\ g_3 &= xy + y^6 - 2y^3z^3 + 2y^2 + z^6 + 3z^2 \\ g_4 &= 3xz^2 + xz^6 - y^{11} + 4y^8z^3 - 5y^7 - 5y^5z^6 - 3y^5z^2 \\ &\quad + 10y^4z^3 - 5y^3 + 2y^2z^9 + 6y^2z^5 - 3yz^6 - 7yz^2 \\ g_5 &= y^{12} - 4y^9z^3 + 5y^8 + 6y^6z^6 + 6y^6z^2 - 10y^5z^3 + 5y^4 \\ &\quad - 4y^3z^9 - 12y^3z^5 + 5y^2z^6 + 13y^2z^2 + z^{12} + 6z^8 + 9z^4. \end{aligned}$$

Since only g_1 involves t, the remaining $\{g_2, g_3, g_4, g_5\}$ form a Groebner basis for $I \cap k[x, y, z]$ in lex order with $x > y > z$ by the Elimination Theorem. g_4 has a total degree of 12.

(3.1.7b): Computing a Groebner basis for $I \cap \mathbb{Q}[x, y, z]$ in grevlex order with $x > y > z$ yields

$$\begin{aligned} h_1 &= x^2 - xy - y^2 - 2z^2 \\ h_2 &= y^6 - 2y^3z^3 + z^6 + xy + 2y^2 + 3z^2. \end{aligned}$$

(3.1.7c): We wish to show that $\{g_1, h_1, h_2\}$ is a Groebner basis for the elimination order $>_1$. Let $f \in I$ be arbitrary. We then have two cases. In the first case, suppose the $\mathrm{LT}_{>_1}(f)$ involves t. Then $\mathrm{LT}_{>_1}(f)$ is divisible by $t = \mathrm{LT}_{>_1}(g_1)$.

In the second case, $\mathrm{LT}_{>_1}(f)$ does not involve t. That means that no other monomial term of f involves t because of the ordering we have chosen. It follows that $f \in I \cap k[x, y, z]$. Using Exercise 3.1.7b, we conclude that $\mathrm{LT}_{>_{grevlex}}(f)$ is divisible by either $\mathrm{LT}_{>_{grevlex}}(h_1)$ or $\mathrm{LT}_{>_{grevlex}}(h_2)$. Note that both $>_1$ and $>_{grevlex}$ agree on $k[x, y, z]$, so it must be that $\mathrm{LT}_{>_1}(f)$ is divisible by either $\mathrm{LT}_{>_1}(h_1)$ or $\mathrm{LT}_{>_1}(h_2)$.

Therefore, in all cases, an arbitrary $f \in I$ has a leading term $\mathrm{LT}_{>_1}(f)$ that is divisible by at least one of $\mathrm{LT}_{>_1}(g_1)$, $\mathrm{LT}_{>_1}(h_1)$, or $\mathrm{LT}_{>_1}(h_2)$. Since $g_1, h_1, h_2 \in I$, we can conclude that $\{g_1, h_1, h_2\}$ is a Groebner basis for I on $>_1$.