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Rootfinding Research Prerequisites
10/07/20

Assignment 5.4
Exercises: 2, 4, 5, 8, 11

Exercise 5.4.2

(5.4.2a): Let V be defined as in the text, and let $\phi = [f], f \in \mathbb{C}[x_1, \dots, x_n]$. It follows that

$$\mathbf{V}_V(\phi) = \{(x_1, \dots, x_n) \in V | \phi(x_1, \dots, x_n) = 0\} = \mathbf{V}(f_1, \dots, f_s, f)$$

for $f_1, \dots, f_s \in V$. We then have

$$\begin{aligned} \mathbf{V}_V(\phi) = 0 &\Leftrightarrow \langle f_1, \dots, f_s, f \rangle = \mathbb{C}[x_1, \dots, x_n] && \text{Weak Nullstellensatz} \\ &\Leftrightarrow h_1 f_1 + \dots + h_s f_s + hf = 1 && \text{for some } h_1, \dots, h_s, h \in \mathbb{C}[x_1, \dots, x_n] \\ &\Leftrightarrow [h]\phi = [1] \in \mathbb{C}[V] && \text{since } [h_1 f_1 + \dots + h_s f_s] = [0] \in \mathbb{C}[V] \\ &\Leftrightarrow \phi \text{ is invertible in } \mathbb{C}[V]. \end{aligned}$$

(5.4.2b): The statement above is not true over $\mathbb{R}$ as can be seen by the counterexample $V = \mathbf{V}(y) \subseteq \mathbb{R}^2, \phi = x^2 + 1$. It follows that $\mathbf{V}_V(\phi) = \emptyset$ since ϕ has no root in $\mathbb{R}^2$.

It is also true that ϕ is not invertible in $\mathbb{R}[V]$. Supposing that this is not the case, there exists some $\psi = [g] \in \mathbb{R}[V]$ such that $\psi\phi = [1]$. By proposition 5.1.2, this implies that $(x^2 + 1) \cdot g(x, y) - 1 = h(x, y) \cdot y$ for some $h(x, y) \in \mathbb{R}[x, y]$. This is equivalent to saying that $(x^2 + 1) \cdot g(x, 0) = 1$, which is impossible since $g(x, 0)$ is a polynomial in $\mathbb{R}[x]$. Therefore, ϕ is not invertible in $\mathbb{R}[V]$.

Exercise 5.4.4 Let V be as defined in the text, and define the mappings $\alpha : k \to V, \beta : V \to k$ as $\alpha(x) = (x, x^n, x^m)$ and $\beta(x, y, z) = x$, respectively. We take their compositions and have that

$$\begin{aligned} (\alpha \circ \beta)(x, y, z) &= \alpha(x) = (x, x^n, x^m) = (x, y, z), (x, y, z) \in V, \\ (\beta \circ \alpha)(x) &= \beta(x, x^n, x^m) = x. \end{aligned}$$

Therefore $\alpha \circ \beta = id_V$ and $\beta \circ \alpha = id_k$, so V is isomorphic as a variety to k.

Exercise 5.4.5 The processes are similar enough that we demonstrate only that the surface $V \in k^3$ defined by $x - f(y, z) = 0$ is isomorphic as a variety to k^2. We do so by defining the polynomial mappings $\alpha : k^2 \to V$ and $\beta : V \to k^2$ as $\alpha(y, z) = (f(y, z)y, z)$ and $\beta(x, y, z) = (y, z)$, respectively. Taking the compositions yields

$$\begin{aligned} (\alpha \circ \beta)(x, y, z) &= \alpha(y, z) = (f(y, z), y, z) = (x, y, z), (x, y, z) \in V \\ (\beta \circ \alpha)(y, z) &= \beta(f(y, z), y, z) = (y, z). \end{aligned}$$

Therefore $\alpha \circ \beta = id_V$ and $\beta \circ \alpha = id_{k^2}$, so it follows that V is isomorphic as a variety to k^2.

Exercise 5.4.8

(5.4.8a): Let Q_1 and Q_2 be as in the text. The pencil of surfaces determined by Q_1 and Q_2 is given by

$$\{Q_2\} \cup \{F_c = \mathbf{V}(f_1 + cf_2) | c \in R\}.$$

Fixing $c = -1$ constrains F_{-1} to be

$$0 = (x^2 + y^2 + z^2 - 1) - (x^2 - x + \frac{1}{4} - 3y^2 - 2z^2) = x + 4y^2 + 3z^3 - \frac{5}{4}$$

Using Exercise 5.4.5, the surface $Q = F_{-1} = \mathbf{V}(x + 4y^2 + 3z^3 - \frac{5}{4})$ is isomorphic as a variety to $\mathbb{R}^2$.

(5.4.8b): Q_1 is the unit sphere and Q_2 is a cone with its vertex at $(\frac{1}{2}, 0, 0)$. Thus, their intersections are two ellipses in planes parallel to the (y, z)-plane.

Exercise 5.4.11

(5.4.11a): To begin, we show that $\mathbf{I}(\mathbf{V}(z - x^2 - y^2)) = \langle z - x^2 - y^2 \rangle$. Let $f \in \mathbf{I}(\mathbf{V}(z - x^2 - y^2))$ and divide it by $z - x^2 - y^2$ using lex order $z > x > y$. The result is that $f = q(x, y, z)(z - x^2 + y^2) + r(x, y)$. We can now substitute $x^2 + y^2$ for z and have $r(x, y) = 0$. Since $\mathbb{R}$ is an infinite field, $r(x, y)$ must be the zero polynomial, so $f \in \langle z - x^2 - y^2 \rangle$. The other inclusion is obvious by observation, so we have that $\mathbf{I}(\mathbf{V}(z - x^2 - y^2)) = \langle z - x^2 - y^2 \rangle$.

Next, we let $V = \mathbf{V}(z - x^2 - y^2)$. Then $\mathbf{V}_V([x-1], [y-1])$ consists of the points in V such that $x - 1 \in \mathbf{I}(V)$ and $y - 1 \in \mathbf{I}(V)$. Note that $x-1, y-1$ are not divisible by $z - x^2 - y^2$, so it must be that $x = 1, y = 1$ and $z = 1^2 + 1^2 = 2$. Therefore $W = \{(1, 1, 2)\} = \mathbf{V}_V([x - 1], [y - 1])$. Using Proposition 3.3, we have $\langle [x - 1], [y - 1] \rangle \subseteq \mathbf{I}_V(\mathbf{V}_V([x - 1], [y - 1])) = \mathbf{I}_V(W)$.

(5.4.11b): Define the mappings $\alpha : V \to \mathbb{R}^2$ and $\beta : \mathbb{R}^2 \to V$ by $\alpha(x, y, z) = (x, y)$ and $\beta(x, y) = (x, y, x^2 + y^2)$, respectively. Then the compositions

$$\begin{aligned} (\alpha \circ \beta)(x, y) &= \alpha(x, y, x^2 + y^2) = (x, y), \\ (\beta \circ \alpha)(x, y, z) &= \beta(x, y) = (x, y, x^2 + y^2) = (x, y, z), (x, y, z) \in V \end{aligned}$$

imply that V is isomorphic as a variety to $\mathbb{R}^2$. Furthermore, let $I = \mathbf{I}_V(W) \subseteq \mathbb{R}[V]$ and note that $W = \mathbf{V}_V(I)$ by Proposition 5.5.3. Using Exercise 5.5.9 as per the given hint, we have $\alpha(W) = \mathbf{V}(\beta^*(I))$, such that $\beta^*(I) \subseteq \mathbf{I}(\alpha(W))$. Note that $W = \{(1, 1, 2)\}$, so $\alpha(W) = \{(1, 1)\}$, which yields $\beta^*(I) \subseteq \mathbf{I}(\alpha(W)) = \mathbf{I}(\{1, 1\}) = \langle x - 1, y - 1 \rangle$.

Finally, since $\alpha^* : \mathbb{R}[x, y] \to \mathbb{R}[V]$ is a ring isomorphism with an inverse β^*, there exists an injective, inclusion-preserving correspondence between ideals of $\mathbb{R}[x, y]$ and $\mathbb{R}[V]$. Observe that $\langle x - 1, y - 1 \rangle$ is the smallest ideal of $\mathbb{R}[x, y]$ containing $x - 1, y - 1$, it follows that $\alpha^*(\langle x - 1, y - 1 \rangle)$ is the smallest ideal of $\mathbb{R}[V]$ which contains $\alpha^*(x - 1) = [x - 1]$ and $\alpha^*(y - 1) = [y - 1]$. Thus $\alpha^*(\langle x - 1, y - 1 \rangle) = \langle [x - 1], [y - 1] \rangle$, which implies

$$\mathbf{I}_V(W) = I = \alpha^*(\beta^*(I)) \subseteq \alpha^*(\langle x - 1, y - 1 \rangle) = \langle [x - 1], [y - 1] \rangle.$$

Therefore, we can conclude that $\langle [x - 1], [y - 1] \rangle = \mathbf{I}_V(W)$.