# Notes on Competitiveness and Carbon Taxes

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**Abstract**

Blah blah blah

## 1 Introduction

Our goal is to develop analytic models to study carbon carbon taxes in an international setting. We begin in section 2 by extending results from our earlier work, Elliott et. al. (2010). This model allows us to evaluate a consumption and extraction tax. We then turn to a more sophisticated model in section 3, which includes a manufacturing sector. In this model we can study a production tax together with a consumption and extraction tax. Our main question is what combination of taxes is unilaterally optimal for a country, given a goal for reducing global carbon emissions.

## 2 Simple Model

There are two countries, home ($\mathcal{H}$) and foreign ($\mathcal{F}$), endowed with fossil fuel deposits ($E$ and $E^*$) and labor ($L$ and $L^*$, measured in efficiency units). Note that variables related to $\mathcal{F}$ are denoted with a $^*$. Fuel deposits are extracted using labor to produce energy (the $e$-good) and carbon emissions. Each country can also produce another good (the $l$-good) using only labor. Consumers have preferences over the $e$-good and the $l$-good. Both are costlessly traded. Nonetheless, after-tax prices of the $e$-good may differ across countries due to carbon taxes. The law of one price holds for the $l$-good.

## 2.1 Basic Setup

We parameterize the model with a Cobb-Douglas production functions for extraction of the $e$-good and constant elasticity of substitution preferences over the two goods. We allow the parameters to differ by country.

### 2.1.1 Production

Labor's share in the extraction of energy is $\beta$:

$$Q_e = (L_e/\beta)^{\beta} E^{1-\beta}. \tag{1}$$

(We derive this production function from more primitive assumptions about extraction in the Appendix.) Production of the $l$-good in a given country is given by:

$$Q_l = L_l.$$

Since we measure labor in efficiency units, this formulation can capture differences in $l$-sector productivity across countries.

Consider the problem of a representative $e$-good extraction firm in $\mathcal{H}$ that owns energy deposits $E$. Facing a price for output of $p_e$ and a cost of labor $w$, it solves:

$$\max_{l_e} \{p_e q_e - w l_e\},$$

subject to the production function:

$$q_e = (l_e/\beta)^{\beta} E^{1-\beta}.$$

The solution is to hire labor:

$$L_e = \beta \left(\frac{p_e}{w}\right)^{1/(1-\beta)} E$$

and to extract a quantity:

$$Q_e = \left(\frac{p_e}{w}\right)^{\beta/(1-\beta)} E.$$

The elasticity of energy supply is $\varepsilon_S = \beta/(1-\beta)$.

The problem is the same in $\mathcal{F}$, replacing $\beta$ with $\beta^*$, $w$ with $w^*$, $E$ with $E^*$, and taking into account any differences in the price faced by producers there. The solution in $\mathcal{F}$ is $L_e^*$ and $Q_e^*$.

### 2.1.2 Consumption

Preferences are parameterized as:

$$U(C_e, C_l) = \left(\alpha^{1/\sigma} C_e^{(\sigma-1)/\sigma} + (1-\alpha)^{1/\sigma} C_l^{(\sigma-1)/\sigma}\right)^{\sigma/(\sigma-1)}. \tag{2}$$

The parameter $\sigma$ is the elasticity of substitution in consumption between the $e$-good and the $l$-good while $\alpha$ governs the share of spending on the $e$-good. In the special case of $\sigma = 1$ preferences simplify to the Cobb-Douglas case considered in Elliott et. al. (2010):

$$U = (C_e)^{\alpha} (C_l)^{1-\alpha}.$$

For $\sigma = 0$ we get Leontief preferences:

$$U = \min\left\{\frac{C_e}{\alpha}, \frac{C_l}{1-\alpha}\right\}.$$

By analogy to the parameter $\beta \in [0,1]$ on the supply side, we can define $\rho = \sigma/(\sigma+1)$ on the demand side, a parameter which also stays in the unit interval.

Consumer income $Y$ comes from labor $wL$, from rents on energy deposits $rE$, and from any tax revenue $T$, which is rebated lump-sum. Consider the problem of a representative consumer in $\mathcal{H}$. Taking income $Y$ and prices $p_e$ and $p_l$ as given, the consumer solves:

$$\max_{c_e, c_l} \{U(c_e, c_l)\},$$

subject to the budget constraint:

$$Y = p_e c_e + p_l c_l.$$

The solution is:

$$C_e = \frac{\alpha}{p_e}\left(\frac{p_e}{p}\right)^{-(\sigma-1)} Y,$$

and

$$C_l = \frac{1-\alpha}{p_l}\left(\frac{p_l}{p}\right)^{-(\sigma-1)} Y,$$

with the price index given by:

$$p = \left(\alpha p_e^{-(\sigma-1)} + (1-\alpha)p_l^{-(\sigma-1)}\right)^{-1/(\sigma-1)}. \qquad (3)$$

The problem is the same in $\mathcal{F}$, replacing $\sigma$ with $\sigma^*$, $\alpha$ with $\alpha^*$, $Y$ with $Y^*$, and taking into account any differences in the price $p_e$ faced by consumers there. The solutions in $\mathcal{F}$ are $C_e^*$ and $C_l^*$, along with $p^*$.

#### 2.1.3 Identities and Constraints

Labor is perfectly mobile between producing the $e$-good $L_e$ and the $l$-good, $L_l$:

$$L = L_e + L_l.$$

Labor and deposits are immobile across countries. The resource constraints are:

$$Q_e^W = Q_e + Q_e^* = C_e + C_e^*,$$

and

$$Q_l^W = Q_l + Q_l^* = C_l + C_l^*.$$

Emissions arise from extracting and consuming energy so global carbon emissions are equal to $Q_e^W$.

#### 2.1.4 Equilibrium

We consider a competitive equilibrium, consisting of wages and prices such that consumers maximize utility, producers maximize profit, and markets clear. It is convenient to take $\mathcal{F}$'s wage $w^* = 1$ as the numeraire. Furthermore, we will restrict parameters so that the equilibrium is one in which both countries produce some of the $l$-good. In that situation wages are the same in both countries so that $p_l = w = w^* = 1$. (We can later derive the conditions under which both countries do, in fact, produce the $l$-good.)

### 2.2 Carbon Taxes

We introduce both extraction taxes and consumption taxes on the $e$-good. We let $t_e$ denote the level of an ad-valorem extraction tax in $\mathcal{H}$ (and $t_e^*$ in $\mathcal{F}$).[1] We let $t_c$ denote the level of an ad-valorem consumption tax in $\mathcal{H}$ (and $t_c^*$ in $\mathcal{F}$).

[1] In Elliott et. al. (2010), we referred to this extraction tax as a production tax.

### 2.2.1 Taxes on Extraction

With an extraction tax $t_e$, the producer of the $e$-good in $\mathcal{H}$ faces a net price of $p_e/(1+t_e)$, so that:

$$Q_e = \left(\frac{p_e}{1+t_e}\right)^{\beta/(1-\beta)} E.$$

(Note that we have imposed the condition that $w = 1$.)

After-tax revenue in the sector is:

$$R_e = \frac{p_e}{1+t_e} Q_e = \left(\frac{p_e}{1+t_e}\right)^{1/(1-\beta)} E,$$

which is shared between wages payed to labor:

$$wL_e = L_e = \beta R_e$$

and rents payed to owners of energy deposits:

$$rE = (1-\beta)\, R_e.$$

Total pre-tax revenue is:

$$p_e Q_e = (1+t_e)\, R_e = (1+t_e) \left(\frac{p_e}{1+t_e}\right)^{1/(1-\beta)} E,$$

split between after-tax revenue $R_e$ and extraction tax revenue:

$$T_e = \tau R_e = t_e \left(\frac{p_e}{1+t_e}\right)^{1/(1-\beta)} E.$$

### 2.2.2 Taxes on Consumption

With a consumption tax $t_c$, consumers face an after-tax price for the $e$-good of $(1+t_c)p_e$, hence after-tax spending is:

$$(1+t_c)p_e C_e = \alpha \left(\frac{(1+t_c)p_e}{p}\right)^{-(\sigma-1)} Y,$$

with the price of a consumption bundle given by:

$$p = \left(\alpha \left((1+t_c)p_e\right)^{-(\sigma-1)} + (1-\alpha)\right)^{-1/(\sigma-1)}. \tag{4}$$

(Note that we have imposed the condition that $p_l = 1$.)

Total spending on the $e$-good can be expressed as:

$$(1+t_c)p_eC_e = \frac{\alpha\left((1+t_c)p_e\right)^{-(\sigma-1)}}{\alpha\left((1+t_c)p_e\right)^{-(\sigma-1)}+1-\alpha}Y, \tag{5}$$

split between pre-tax consumption spending $p_eC_e$ and tax revenue:

$$T_c = t_cp_eC_e = \frac{t_c}{1+t_c}\frac{\alpha\left((1+t_c)p_e\right)^{-(\sigma-1)}}{\alpha\left((1+t_c)p_e\right)^{-(\sigma-1)}+1-\alpha}Y.$$

### 2.2.3 Income

So far we have simply taken a country's GDP $Y$ (and $Y^*$ in $\mathcal{F}$) as given. Adding together labor income, rents on the $E$ factor, and tax revenue from the two carbon taxes, income in $\mathcal{H}$ is:

$$Y = wL + rE + T_e + T_c = L + (1-\beta+t_e)\,R_e + t_cp_eC_e.$$

Certain shares of GDP prove useful in simplifying and calibrating the model. We denote after-tax spending on energy consumption as a share of GDP by:

$$\pi_c = \frac{(1+t_c)\,p_eC_e}{Y}.$$

Similarly, we denote pre-tax revenue of the energy extraction sector as a share of GDP is:

$$\pi_e = \frac{p_eQ_e}{Y}.$$

Returns to the $L$ factor as a share of GDP are:

$$\pi_L = \frac{wL}{Y} = \frac{L}{Y}.$$

These shares are connected in several ways. First, to satisfy our assumption that both countries produce the $l$-good we need $L_e < L$, hence:

$$\frac{\beta}{1+t_e}\pi_e < \pi_L.$$

Second, for the income identity we need:

$$1-\pi_L-\frac{1-\beta}{1+t_e}\pi_e = \frac{t_e}{1+t_e}\pi_e + \frac{t_c}{1+t_c}\pi_c,$$

hence:

$$1-\pi_L = \frac{1-\beta+t_e}{1+t_e}\pi_e + \frac{t_c}{1+t_c}\pi_c$$

The analogs of these two expressions must also hold in $\mathcal{F}$.

### 2.2.4 Equilibrium with Taxes

We can calculate the competitive equilibrium with taxes by finding the world energy price $p_e$ that clears the energy market. By Walras law the $l$-good market will also clear. Equivalently, we can find the price that equate pre-tax consumption spending with pre-tax revenue of the energy sector:

$$p_e\left(Q_e + Q_e^*\right) = p_e\left(C_e + C_e^*\right). \tag{6}$$

Using our share expressions, market clearing becomes:

$$\pi_e Y + \pi_e^* Y^* = \frac{\pi_c Y}{1+t_c} + \frac{\pi_c^* Y^*}{1+t_c^*}.$$

Thus, at the equilibrium price we have an additional restriction on these shares:

$$\pi_c^* = \left(\frac{1+t_c^*}{Y^*}\right)\left(\pi_e Y + \pi_e^* Y^* - \frac{\pi_c Y}{1+t_c}\right).$$

In general a solution needs to be computed numerically. But before setting aside the analytics, we can show some results about taxation and can solve the model in special cases.

### 2.2.5 Tax Equivalence Results

We consider two results.

**Uniform Taxes with Identical Preferences** First we show that if demand parameters are identical across countries the effect on global emissions of a uniform carbon tax is the same whether it is applied to extraction or consumption. There is an income effect, however, which is why we need to impose identical demand parameters. If the tax is applied to consumption it leaves the country that is a net importer of the $e$-good with higher income, while if applied to extraction it leaves that country with lower income.

To demonstrate this result, start with a uniform consumption tax $t_c$ on the $e$-good and no extraction tax. Imposing $\alpha = \alpha^*$ and $\sigma = \sigma^*$, the global equilibrium conditions can be written as:

$$\frac{\alpha}{1+t_c}\frac{((1+t_c)p_e)^{-(\sigma-1)}}{\alpha\left((1+t_c)p_e\right)^{-(\sigma-1)}+(1-\alpha)}Y^W = p_e^{1/(1-\beta)}E + p_e^{1/(1-\beta^*)}E^*,$$

where world income can be expressed as:

$$\begin{aligned} Y^W &= L^W + (1-\beta)\,p_e^{1/(1-\beta)}E + (1-\beta^*)\,p_e^{1/(1-\beta^*)}E^* + tp_eC_e^W \\ &= L^W + (1-\beta+t_c)\,p_e^{1/(1-\beta)}E + (1-\beta^*+t_c)\,p_e^{1/(1-\beta^*)}E^* \end{aligned}$$

Let's conjecture that the equilibrium price with an extraction tax is $\bar{p}_e = (1+t_c)p_e$. Substituting this new price into the conditions above, we get:

$$\frac{\alpha}{1+t_c}\frac{\bar{p}_e^{-(\sigma-1)}}{\alpha\bar{p}_e^{-(\sigma-1)}+(1-\alpha)}Y^W = \left(\frac{\bar{p}_e}{1+t_c}\right)^{1/(1-\beta)}E + \left(\frac{\bar{p}_e}{1+t_c}\right)^{1/(1-\beta^*)}E^*,$$

with

$$Y^W = L^W + (1-\beta+t_c)\left(\frac{\bar{p}_e}{1+t_c}\right)^{1/(1-\beta)}E + (1-\beta^*+t_c)\left(\frac{\bar{p}_e}{1+t_c}\right)^{1/(1-\beta^*)}E^*,$$

exactly the global equilibrium conditions for a uniform extraction tax $t_e = t_c$ on extraction of the $e$-good and no consumption tax.

While the equilibrium price is higher with the extraction tax, global energy supply is the same in either case:

$$\begin{aligned} Q_e^W &= p_e^{\beta/(1-\beta)}E + (1-\beta^*)\,p_e^{\beta^*/(1-\beta^*)}E^* \\ &= \left(\frac{\bar{p}_e}{1+t_c}\right)^{\beta/(1-\beta)}E + \left(\frac{\bar{p}_e}{1+t_c}\right)^{\beta^*/(1-\beta^*)}E^*. \end{aligned}$$

The two policies achieve the same goal in reducing global emissions, but they do not lead to the same global distribution of income. In the case of a uniform consumption tax, income in $\mathcal{H}$ is:

$$\begin{aligned} Y &= L + (1-\beta)\,p_e^{1/(1-\beta)}E + tp_eC_e \\ &= L + (1-\beta+t_c)\,p_e^{1/(1-\beta)}E + t_c\left(p_eC_e - p_eQ_e\right). \end{aligned}$$

In the case of a uniform extraction tax, it is:

$$Y = L + (1 - \beta + t_c) \left( \frac{\bar{p}_e}{1 + t_c} \right)^{1/(1-\beta)} E.$$

Thus, if $\mathcal{H}$ is a net importer of the $e$-good, it will have higher income under a uniform consumption tax. Since a consumer's after-tax price of energy is the same in either case, welfare is higher in $\mathcal{H}$ under a consumption tax if it is a net importer of the $e$-good.

**Home Taxes and Border Adjustments** Now consider the same comparison, but with a tax only in $\mathcal{H}$. We can drop the restriction of identical preferences, since all the action is now in $\mathcal{H}$. First, consider a tax $t_c$ on consumption of the $e$-good. At an equilibrium price $p_e$, the quantity of energy extracted in $\mathcal{H}$ is:

$$Q_e = p_e^{\beta/(1-\beta)} E$$

and consumption spending is:

$$p_e C_e = \frac{1}{1 + t_c} \frac{((1 + t_c)p_e)^{-(\sigma-1)}}{((1 + t_c)p_e)^{-(\sigma-1)} + (1 - \alpha)/\alpha} Y,$$

where

$$Y = L + (1 - \beta)\, p_e^{1/(1-\beta)} E + \frac{t_c}{1 + t_c} \frac{((1 + t_c)p_e)^{-(\sigma-1)}}{((1 + t_c)p_e)^{-(\sigma-1)} + (1 - \alpha)/\alpha} Y.$$

Extraction in $\mathcal{F}$ is:

$$Q_e^* = p_e^{\beta^*/(1-\beta^*)} E^*$$

and consumption spending in $\mathcal{F}$ is:

$$p_e C_e^* = \frac{p_e^{-(\sigma^*-1)}}{p_e^{-(\sigma^*-1)} + (1 - \alpha^*)/\alpha^*} Y^*,$$

where

$$Y^* = L^* + (1 - \beta^*)\, p_e^{1/(1-\beta^*)}.$$

Let's try to reinterpret these conditions as an extraction tax $t_e = t_c$ with border tax adjustments (and no consumption tax). Letting $\bar{p}_e = (1 + t_c)p_e$

be the equilibrium price in $\mathcal{H}$ under the extraction tax interpretation, we have:

$$Q_e = \left(\frac{\bar{p}_e}{1+t_c}\right)^{\beta/(1-\beta)} E$$

and

$$\bar{p}_e C_e = \alpha \frac{\bar{p}_e^{-(\sigma-1)}}{\alpha \bar{p}_e^{-(\sigma-1)} + (1-\alpha)} Y,$$

where

$$\begin{aligned} Y &= L + (1-\beta)\left(\frac{\bar{p}_e}{1+t_c}\right)^{1/(1-\beta)} E + \frac{t_c}{1+t_c} \alpha \frac{\bar{p}_e^{-(\sigma-1)}}{\alpha \bar{p}_e^{-(\sigma-1)} + (1-\alpha)} Y \\ &= L + (1-\beta+t_c)\left(\frac{\bar{p}_e}{1+t_c}\right)^{1/(1-\beta)} E + \frac{t_c}{1+t_c}\left(\bar{p}_e C_e - \bar{p}_e Q_e\right) \\ &= L + (1-\beta+t_c)\left(\frac{\bar{p}_e}{1+t_c}\right)^{1/(1-\beta)} E + t_c\left(p_e C_e - p_e Q_e\right). \end{aligned}$$

The last term of the last expression shows the revenue (or cost) of the border tax adjustment, at rate $t$. Thus, the border tax adjustment is either a tariff on $\mathcal{H}$'s net imports of the $e$-good or a tax rebate on $\mathcal{H}$'s net exports of the $e$-good. The border taxes are collected on net imports or paid on net exports of the $e$-good valued at the original equilibrium price $p_e$. The price of the $e$-good in $\mathcal{F}$ is $p_e = \bar{p}_e/(1+t)$, which is lower than the price in $\mathcal{H}$ due to the border tax adjustment. It is the price $\mathcal{F}$ would face if $\mathcal{H}$ had imposed a consumption tax.

### 2.2.6 Special Cases

To get analytical tractability, we consider the case of common parameters and unit elastic demand, as in Elliott et. al. (2010). Suppose $\beta = \beta^*$, $\sigma = \sigma^* = 1$, $\alpha = \alpha^*$, $t_e = t_e^* \geq 0$, and $t_c = t_c^* = 0$. The right-hand side of (6) becomes:

$$p_e Q_e^W = (1+t_e)^{-\beta/(1-\beta)} p_e^{1/(1-\beta)} E^W.$$

The left-hand side is:

$$p_e C_e^W = \frac{\alpha}{1+t_c-\alpha t_c}\left(L^W + (1-\beta+t_e)\left(\frac{p_e}{1+t_e}\right)^{1/(1-\beta)} E^W\right).$$

Equating the two, we obtain the equilibrium price:

$$\frac{p_e}{1+t_e} = \left(\frac{\alpha}{(1+t_e)(1+t_c)(1-\alpha)+\alpha\beta}\frac{L^W}{E^W}\right)^{1-\beta}.$$

Thus, world energy supply is:

$$Q_e^W = \left(\frac{\alpha}{(1+t_e)(1+t_c)(1-\alpha)+\alpha\beta}\right)^{\beta} \left(L^W\right)^{\beta} \left(E^W\right)^{1-\beta}.$$

In the case of $t_c = 0$, we match the result in Elliott et. al. (2010). We can also verify in this setting that a consumption tax can acheive the same outcome as a extraction tax at the same ad valorem rate. In general the two taxes augment each other to lower world energy supply.

## 2.3 Solving the Model

To solve the model in a realistic setting, we calibrate it to aggregate income $Y$ (and $Y^*$ in $\mathcal{F}$) together with the shares (in GDP) of energy extraction and consumption, $\pi_e$ and $\pi_c$ (and $\pi_e^*$ and $\pi_c^*$ in $\mathcal{F}$). By calibrating to these values, we will not need to know $\alpha$, $L$, or $E$ (nor their analogs in $\mathcal{F}$).

We then consider how the equilibrium values of endogenous variables change with respect to a change in carbon taxes. All such changes are denoted with a "hat", indicating the proportional change from the baseline (e.g. for a variable whose baseline value is $x$, if its value becomes $x'$ when carbon taxes are changed, then we denote $\hat{x} = x'/x$).

### 2.3.1 Equilibrium Energy Price Change

First consider total pre-tax revenue in $\mathcal{H}$'s energy sector after a change in taxes:

$$\begin{aligned} \pi_e' Y' &= p_e' Q_e' = (1+t_e')^{-\beta/(1-\beta)} p_e'^{1/(1-\beta)} E \\ &= \pi_e Y \left(\frac{1+t_e'}{1+t_e}\right)^{-\beta/(1-\beta)} \hat{p}_e^{1/(1-\beta)}. \end{aligned}$$

It follows that the change in pre-tax energy revenue is:

$$\hat{\pi}_e \hat{Y} = \left(\frac{1+t_e'}{1+t_e}\right)^{-\beta/(1-\beta)} \hat{p}_e^{1/(1-\beta)}. \tag{7}$$

(The same equation works for $\mathcal{F}$, with *'s in the appropriate places.)

The change in the share of after-tax consumption spending on the $e$-good is:

$$\hat{\pi}_c = \frac{\left(\frac{1+t_c'}{1+t_c}\right)^{-(\sigma-1)} \hat{p}_e^{-(\sigma-1)}}{\pi_c \left(\frac{1+t_c'}{1+t_c}\right)^{-(\sigma-1)} \hat{p}_e^{-(\sigma-1)} + 1 - \pi_c}. \tag{8}$$

(Again, there is an analogous equation for $\mathcal{F}$.)

After a change in taxes, income is:

$$\begin{aligned} Y' &= L + (1-\beta) R_e' + t_e' R_e' + t_c' p_e' C_e' \\ &= L + \frac{1-\beta}{1+t_e'} \pi_e' Y' + \frac{t_e'}{1+t_e'} \pi_e' Y' + \frac{t_c'}{1+t_c'} \pi_c' Y'. \end{aligned}$$

Thus:

$$\hat{Y} = \pi_L + \frac{1-\beta+t_e'}{1+t_e'} \pi_e \hat{\pi}_e \hat{Y} + \frac{t_c'}{1+t_c'} \pi_c \hat{\pi}_c \hat{Y}, \tag{9}$$

(Again, there is an analogous equation for the change in income in $\mathcal{F}$.)

Finally, the goods market clearing condition is:

$$p_e' Q_e' + p_e' Q_e^{*\prime} = p_e' C_e' + p_e' C_e^{*\prime},$$

or

$$\pi_e Y \hat{\pi}_e \hat{Y} + \pi_e^* Y^* \hat{\pi}_e^* \hat{Y}^* = \frac{\pi_c Y \hat{\pi}_c \hat{Y}}{1+t_c'} + \frac{\pi_c^* Y^* \hat{\pi}_c^* \hat{Y}^*}{1+t_c^{*\prime}} \tag{10}$$

(This equation applies for the world, so there is no analog in $\mathcal{F}$.)

Given initial and final levels of carbon taxes in $\mathcal{H}$ ($t_c$, $t_c'$, $t_e$, $t_e'$) and $\mathcal{F}$, we can solve the system (including the analogs in $\mathcal{F}$) consisting of (7), (8), (9), and (10) for $\hat{\pi}_e$, $\hat{\pi}_e^*$, $\hat{\pi}_c$, $\hat{\pi}_c^*$, $\hat{Y}$, $\hat{Y}^*$, and the change in the equilibrium world price of energy $\hat{p}_e$. We need only the baseline data on $Y$, $\pi_c$, $\pi_L$, and $\pi_e$ (along with $Y^*$, $\pi_c^*$, $\pi_L^*$, and $\pi_e^*$ in $\mathcal{F}$) together with parameters $\beta$ and $\sigma$ ($\beta^*$ and $\sigma^*$ in $\mathcal{F}$).

### 2.3.2 Carbon Leakage

Having determined the equilibrium change in the world energy price, we can calculate the change in world energy production. For cases in which only $\mathcal{H}$ imposes a tax, we can compute several forms of carbon leakage. For an increase in $\mathcal{H}$'s extraction tax, extraction leakage is the increase in $\mathcal{F}$'s extraction of energy as a share of $\mathcal{H}$'s reduction in extraction of energy:

$$l_Q = \frac{(Q_e^{*\prime} - Q_e^*)}{(Q_e - Q_e^\prime)} = \left(\frac{\hat{Q}_e^* - 1}{1 - \hat{Q}_e}\right) \frac{\pi_e^* Y^*}{\pi_e Y},$$

where

$$\hat{Q}_e = \frac{\hat{\pi}_e \hat{Y}}{\hat{p}_e},$$

and

$$\hat{Q}_e^* = \frac{\hat{\pi}_e^* \hat{Y}^*}{\hat{p}_e}.$$

For an increase in $\mathcal{H}$'s consumption tax, consumption leakage is the increase in $\mathcal{F}$'s energy consumption as a share of $\mathcal{H}$'s reduction in energy consumption:

$$l_C = \frac{(C_e^{*\prime} - C_e^*)}{(C_e - C_e^\prime)} = \left(\frac{\hat{C}_e^* - 1}{1 - \hat{C}_e}\right) \frac{\pi_c^* Y^*/(1 + t_c^*)}{\pi_c Y/(1 + t_c)},$$

where

$$\hat{C}_e = \frac{\hat{\pi}_c \hat{Y}}{\hat{p}_e \frac{1+t_c^\prime}{1+t_c}},$$

and

$$\hat{C}_e^* = \frac{\hat{\pi}_c^* \hat{Y}^*}{\hat{p}_e \frac{1+t_c^{*\prime}}{1+t_c^*}}.$$

## 2.4 Tax Policy to Achieve a Global Emission Goal

We want to compare tax policies in $\mathcal{H}$ that achieve a set goal of emission reduction for the world. To keep this exercise as simple as possible, we will assume that $\sigma = \sigma^*$, $\beta = \beta^*$, and $t_e = t_c = 0$ (i.e. we'll start from a world without taxes).

### 2.4.1 Supply Side

Let $\hat{Q}_e^W = G < 1$ be the goal set for reducing world emissions. Looking at energy extraction, for any $t_e^\prime \geq 0$, the change in energy price that would achieve that goal satisfies:

$$\begin{aligned}
G &= \frac{Q_e \hat{Q}_e + Q_e^* \hat{Q}_e^*}{Q_e^W} \\
&= \frac{Q_e \left(1 + t_e'\right)^{-\beta/(1-\beta)} \hat{p}_e^{\beta/(1-\beta)} + Q_e^* \hat{p}_e^{\beta/(1-\beta)}}{Q_e^W} \\
&= \omega_e \left(1 + t_e'\right)^{-\beta/(1-\beta)} \hat{p}_e^{\beta/(1-\beta)} + \left(1 - \omega_e\right) \hat{p}_e^{\beta/(1-\beta)},
\end{aligned}$$

where

$$\omega_e = \frac{\pi_e Y}{\pi_e Y + \pi_e^* Y^*} = \frac{p_e Q_e}{p_e Q_e^W}$$

is $\mathcal{H}$'s share of world energy extraction. Given $G$ (which will be held fixed in this problem), we can express the required change in the energy price, as dictated by the supply side $\hat{p}_e^S$, as a function of the extraction tax:

$$\hat{p}_e^S(t_e') = \left( \frac{G}{\omega_e \left(1 + t_e'\right)^{-\beta/(1-\beta)} + \left(1 - \omega_e\right)} \right)^{(1-\beta)/\beta}. \tag{11}$$

Note that the required change in energy prices moves in the same direction as $G$. Given the tax rate on extraction, a more ambitious goal for reducing emissions (a smaller $G$) requires that producers face a steeper decline in the energy price. Our focus is on how the extraction tax alters incentives for producing energy. We see from (11) that, given $G$, a higher extraction tax in $\mathcal{H}$ means energy prices fall by less, or even rise. As the tax rate rises from 0 to $\infty$, the energy price change increases from $\hat{p}_e^S(0) = G^{(1-\beta)/\beta}$ to $\hat{p}_e^S(\infty) = \left(1 - \omega_e\right)^{-(1-\beta)/\beta} \hat{p}_e^S(0)$. The extent of this rise is governed by the fraction of world energy extraction $\omega_e$ initially taking place in $\mathcal{H}$, as that determines the size of the tax base.

### 2.4.2 Demand Side

Turning to the consumption side, we want to calculate the change in the energy price that, along with a tax on energy consumption $t_c'$ (in $\mathcal{H}$), reduces world consumption by the factor $G$. The decline in global consumption can be expressed as:

$$G = \hat{C}_e^W = \frac{C_e \hat{C}_e + C_e^* \hat{C}_e^*}{C_e^W} = \omega_c \hat{C}_e + \left(1 - \omega_c\right) \hat{C}_e^*,$$

where

$$\omega_c = \frac{\pi_c Y}{\pi_c Y + \pi_c^* Y^*} = \frac{p_e C_e}{p_e C_e^W}.$$

We can write:

$$\hat{C}_e^W(t_c', t_e', \hat{p}_e) = \omega_c \frac{\hat{\pi}_c(t_c', \hat{p}_e)}{(1+t_c')\,\hat{p}_e} \hat{Y}(t_c', t_e', \hat{p}_e) + (1-\omega_c)\,\frac{\hat{\pi}_c^*(\hat{p}_e)}{\hat{p}_e} \hat{Y}^*(\hat{p}_e), \qquad (12)$$

where $\hat{\pi}_c$ is given by (8) with $t_c = 0$:

$$\hat{\pi}_c(t_c', \hat{p}_e) = \frac{(1+t_c')^{-(\sigma-1)}\,\hat{p}_e^{-(\sigma-1)}}{\pi_c\,(1+t_c')^{-(\sigma-1)}\,\hat{p}_e^{-(\sigma-1)} + 1 - \pi_c}$$

and

$$\hat{\pi}_c^*(\hat{p}_e) = \frac{\hat{p}_e^{-(\sigma-1)}}{\pi_c^* \hat{p}_e^{-(\sigma-1)} + 1 - \pi_c^*}.$$

We now turn to the terms for the change in income.

Adjusting to lower global emissions generates income changes in both countries. Changes in the energy price affect revenue in the energy sector. Furthermore, in $\mathcal{H}$ there is income generated by tax revenue. From (9) and (7), the change in $\mathcal{H}$'s income is:

$$\hat{Y} = \pi_L + \frac{1-\beta+t_e'}{1+t_e'} \pi_e\,(1+t_e')^{-\beta/(1-\beta)}\,\hat{p}_e^{1/(1-\beta)} + \frac{t_c'}{1+t_c'} \pi_c \hat{\pi}_c \hat{Y},$$

or

$$\hat{Y}(t_c', t_e', \hat{p}_e) = \frac{1+t_c'}{1+t_c' - t_c' \pi_c \hat{\pi}_c(t_c', \hat{p}_e)} \left( \pi_L + \frac{1-\beta+t_e'}{1+t_e'} \pi_e\,(1+t_e')^{-\beta/(1-\beta)}\,\hat{p}_e^{1/(1-\beta)} \right). \qquad (13)$$

Since carbon taxes do not generate any revenue abroad, the change in $\mathcal{F}$'s income is simply:

$$\hat{Y}^*(\hat{p}_e) = \pi_L^* + (1-\beta)\,\pi_e^* \hat{p}_e^{1/(1-\beta)}. \qquad (14)$$

#### 2.4.3 Welfare

Welfare in $\mathcal{H}$ is given by:

$$W = \frac{Y}{p},$$

where we use equation (4) for the aggregate price level, $p$. The change in welfare, when taxes are introduced, is:

$$\hat{W} = \frac{W'}{W} = \frac{\hat{Y}}{\hat{p}}.$$

We already have an expression (13) for $\hat{Y}$, so we just need an expression for the change in the aggregate price level to evaluate welfare. From (4), the change in the price level will be:

$$\hat{p} = \left( \frac{\alpha \left((1+t_c')p_e'\right)^{-(\sigma-1)} + (1-\alpha)}{\alpha p_e^{-(\sigma-1)} + 1 - \alpha} \right)^{-1/(\sigma-1)},$$

which simplifies to:

$$\hat{p} = \left( \pi_c \left((1+t_c')\, \hat{p}_e\right)^{-(\sigma-1)} + (1-\pi_c) \right)^{-1/(\sigma-1)}.$$

Thus, the change in welfare is:

$$\hat{W} = \frac{\left( \frac{\pi_c (1+t_c')^{-(\sigma-1)} (\hat{p}_e)^{-(\sigma-1)} + 1 - \pi_c}{\pi_c (1+t_c')^{-\sigma} (\hat{p}_e)^{-(\sigma-1)} + 1 - \pi_c} \right) \left( \pi_L + (1 + t_e' - \beta)\, \pi_e\, (1+t_e')^{-1/(1-\beta)}\, \hat{p}_e^{1/(1-\beta)} \right)}{\left( \pi_c \left((1+t_c')\, \hat{p}_e\right)^{-(\sigma-1)} + (1-\pi_c) \right)^{-1/(\sigma-1)}},$$

or:

$$\hat{W}(t_c', t_e', \hat{p}_e) = \frac{\left( \pi_c (1+t_c')^{-(\sigma-1)}\, \hat{p}_e^{-(\sigma-1)} + (1-\pi_c) \right)^{\sigma/(\sigma-1)}}{\pi_c (1+t_c')^{-\sigma}\, \hat{p}_e^{-(\sigma-1)} + 1 - \pi_c} \times \left( \pi_L + (1 + t_e' - \beta)\, \pi_e\, (1+t_e')^{-1/(1-\beta)}\, \hat{p}_e^{1/(1-\beta)} \right). \quad (15)$$

Given $\hat{p}_e$, it is easy to show that $\hat{W}$ is decreasing in both $t_c'$ and $t_e'$.

#### 2.4.4 Optimization

Since the goal $G$ for reduction in global emissions is fixed, we can substitute $\hat{p}_e = \hat{p}_e^S(t_e')$ from (11) into (15) to obtain an expression for the change in wefare that depends only on tax rates:

$$\hat{W}\left(t_c', t_e', \hat{p}_e^S(t_e')\right) = f(t_c', t_e').$$

We want to choose tax rates that maximize this expression, subject to the constraint that world demand for energy also falls by the factor $G$. We can write this constraint as:

$$G = \hat{C}_e^W(t'_c, t'_e, \hat{p}_e^S(t'_e)) = g(t'_c, t'_e),$$

where $\hat{C}_e^W(t'_c, t'_e, \hat{p}_e)$ is given in (12). The optimization problem reduces to maximizing the lagrangian:

$$\mathcal{L} = f(t'_c, t'_e) - \lambda \left[ G - g(t'_c, t'_e) \right],$$

where $\lambda$ is the lagrangian multiplier.

# 3 Model with a Manufacturing Sector

Up to this point we have modeled the extraction and consumption of carbon-based energy, but not its use in industry and its embodiment in tradable manufactured goods. With our Simple Model we could discuss extraction and consumption taxes, but not production taxes. To address this shortcoming, we now add a manufacturing sector to Simple Model, with two-way trade in differentiated manufactures.

To keep the analysis as compact as possible, we assume the two countries $\mathcal{H}$ and $\mathcal{F}$ differ only in their endowment of labor $L$ vs. $L^*$, fossil fuel deposits $E$ vs. $E^*$, preference parameters $\alpha$ vs. $\alpha^*$, and labor share in energy extraction $\beta$ vs. $\beta^*$. The reason for maintaining heterogeneity in $\beta$ is to capture differences between a country like Saudi Arabia with energy that is cheap to extract (small $\beta$) and a country like Canada with energy that is costly to extract (large $\beta$).

## 3.1 Basic Setup

Manufactures come in a continuum of varieties, $j \in [0, 1]$. These varieties enter preferences through a symmetric aggregator, with constant elasticity of substitution $\rho$, to form the manufactured good, the $m$-good. The $m$-good, in turn, enters preferences in Cobb-Douglas combination with the $e$-good (to form the composit $c$-good), with share $\eta$ on the $m$-good. (Simple Model reemerges when $\eta = 0$.) Preferences can thus be represented by the following utility funtction:

$$U(C_e, C_l, C_m) = \left( \alpha^{1/\sigma} C_c^{(\sigma-1)/\sigma} + (1-\alpha)^{1/\sigma} C_l^{(\sigma-1)/\sigma} \right)^{\sigma/(\sigma-1)}, \qquad (16)$$

where consumption of the $c$-good is:

$$C_c = \left(\frac{C_m}{\eta}\right)^{\eta}\left(\frac{C_e}{1-\eta}\right)^{1-\eta}$$

and consumption of the $m$-good is:

$$C_m = \left(\int_0^1 C_m(j)^{(\rho-1)/\rho}dj\right)^{\rho/(\rho-1)}.$$

Other aspects of the model, such as the extraction of energy and the production of the $l$-good, are identical to Simple Model. For example, the aggregate price index can be taken from (3) with $p_c$ in place of $p_e$:

$$p = \left(\alpha p_c^{-(\sigma-1)} + (1-\alpha)p_l^{-(\sigma-1)}\right)^{-1/(\sigma-1)}. \tag{17}$$

The new features are in the modeling of production and trade in manufactures.

#### 3.1.1 Trade in Manufactures

Our formulation of production and trade in manufactures is a version of the Ricardian model introduced by Dornbusch, Fisher, and Samuelson (1977). Individual varieties of the $m$-good are costlessly traded. The production function for manufacturing variety $j$ in $\mathcal{H}$ is:

$$Q_m(j) = A(j)\left(\frac{L_m(j)}{\gamma}\right)^{\gamma}\left(\frac{M_e(j)}{1-\gamma}\right)^{1-\gamma},$$

where $A(j)$ is $\mathcal{H}$'s productivity in variety $j$, $M_e$ is energy input, and $\gamma$ is labor's share. We imagine that many price-taking producers have access to the technology to produce each variety $j$. In $\mathcal{F}$ the production function has the same form, with productivity $A^*(j)$.

Labor is perfectly mobile, within each country, across all sectors and varieties. The labor constraint in $\mathcal{H}$ is:

$$L_e + L_l + \int_0^1 L_m(j)dj = L.$$

A similar condition holds in $\mathcal{F}$.

We parameterize productivity for each variety as:

$$A(j) = \frac{A}{j^{1/\theta}}$$

in $\mathcal{H}$ and:

$$A^*(j) = \frac{A^*}{(1-j)^{1/\theta}}$$

in $\mathcal{F}$. Here, the parameters $A$ and $A^*$ capture absolute advantage in $\mathcal{H}$ and $\mathcal{F}$. Looking at the relative productivity of the two countries in producing variety $j$:

$$R(j) = \frac{A(j)}{A^*(j)} = \frac{A}{A^*}\left(\frac{j}{1-j}\right)^{-1/\theta}. \tag{18}$$

For $j < j'$, $\mathcal{H}$ has a comparative advantage in variety $j$ and $\mathcal{F}$ in $j'$. The parameter $\theta$ captures (inversely) the strength of comparative advantage. As $\theta \to \infty$ relative productivity does not vary across varieties. As will become apparent below, we need to restrict $\theta > \rho - 1$.

**Specialization in Manufactures** Since there are no trade costs, specialization in production of manufactures is detached from country-level demand. We can therefore determine which country produces which variety without solving the entire model.

Due to trade in energy, prices $p_e$ are the same in each country. A bundle of inputs costs $w^\gamma p_e^{1-\gamma}$ in $\mathcal{H}$ and $w^{*\gamma} p_e^{1-\gamma}$ in $\mathcal{F}$. Hence $\mathcal{H}$ will produce varieties $j$ for which $R(j) \geq (w/w^*)^\gamma$ and $\mathcal{F}$ will produce the rest. In other words, $\mathcal{H}$ produces all varieties in the interval $[0, \bar{j}]$ and $\mathcal{F}$ in the interval $(\bar{j}, 1]$ where:

$$\bar{j} = \frac{A^\theta w^{-\gamma\theta}}{A^\theta w^{-\gamma\theta} + A^{*\theta} w^{*-\gamma\theta}}. \tag{19}$$

(Whether $\mathcal{H}$ or $\mathcal{F}$ produces variety $\bar{j}$ doesn't matter since it is only one point on a continuum.)

Given specialization, we can solve for the competitive price of each manufactured variety. For $j \leq \bar{j}$, the variety is produced in $\mathcal{H}$, so that:

$$p_m(j) = \frac{w^\gamma p_e^{1-\gamma}}{A(j)}, \tag{20}$$

while for $j > \bar{j}$, the variety is produced in $\mathcal{F}$, with:

$$p_m(j) = \frac{w^{*\gamma} p_e^{1-\gamma}}{A^*(j)}.$$

With no trade cost we have the law of one price; these variety-level prices apply for consumers in either country.

**Price Index for Manufactues** The price index for the $m$-good is:

$$\begin{aligned} p_m &= \left( \int_0^1 p_m(j)^{-(\rho-1)} dj \right)^{-1/(\rho-1)} \\ &= \left( \int_0^{\bar{j}} \left( w^{\gamma} p_e^{1-\gamma} / A(j) \right)^{-(\rho-1)} dj + \int_{\bar{j}}^1 \left( w^{*\gamma} p_e^{1-\gamma} / A^*(j) \right)^{-(\rho-1)} dj \right)^{-1/(\rho-1)} \end{aligned} \quad (21)$$

To solve this integral and to understand the properties of this price index, consider varieties produced in $\mathcal{H}$. The lowest price of any variety produced in $\mathcal{H}$ approaches 0 and the highest price is given by (20) evaluated at $j = \bar{j}$:

$$\bar{p} = p_m(\bar{j}) = \frac{\bar{j}^{1/\theta} w^{\gamma} p_e^{1-\gamma}}{A}, \quad (22)$$

which can be inverted to get:

$$\bar{j} = \left( \frac{\bar{p} A}{w^{\gamma} p_e^{1-\gamma}} \right)^{\theta}.$$

If we change the variable of integration from $j$ to $p_m(j)$ in the expression for the average of $p_m(j)^{-(\rho-1)}$ across varieties produced in $\mathcal{H}$, we get a simple result:

$$\begin{aligned} \frac{1}{\bar{j}} \int_0^{\bar{j}} p_m(j)^{-(\rho-1)} dj &= \left( \frac{\bar{p} A}{w^{\gamma} p_e^{1-\gamma}} \right)^{-\theta} \int_0^{\bar{p}} p^{-(\rho-1)} \theta \left( \frac{p A}{w^{\gamma} p_e^{1-\gamma}} \right)^{\theta-1} \frac{A}{w^{\gamma} p_e^{1-\gamma}} dp \\ &= \bar{p}^{-\theta} \int_0^{\bar{p}} \theta p^{\theta-\rho} dp = \frac{\theta}{\theta - (\rho-1)} \bar{p}^{-(\rho-1)}. \end{aligned}$$

Now consider varieties produced by $\mathcal{F}$. The prices of these goods also vary from 0 to $\bar{p}$, where for $\mathcal{F}$ we can write:

$$1 - \bar{j} = \left( \frac{\bar{p} A^*}{w^{*\gamma} p_e^{1-\gamma}} \right)^{\theta}.$$

Thus, the average of $p_m(j)^{-(\rho-1)}$ across varieties produced in $\mathcal{F}$ is:

$$\begin{aligned}\frac{1}{1-\bar{j}}\int_{\bar{j}}^{1} p_m(j)^{-(\rho-1)}dj &= \left(\frac{\bar{p}A^*}{w^{*\gamma}p_e^{1-\gamma}}\right)^{-\theta}\int_0^{\bar{p}} p^{-(\rho-1)}\theta\left(\frac{pA^*}{w^{*\gamma}p_e^{1-\gamma}}\right)^{\theta-1}\frac{A^*}{w^{*\gamma}p_e^{1-\gamma}}dp \\ &= \frac{\theta}{\theta-(\rho-1)}\bar{p}^{-(\rho-1)},\end{aligned}$$

exactly the same as for $\mathcal{H}$. Since these averages are the same, independent of where the varieties are produced, the overall price index (21) can be expressed as:

$$p_m = \phi\bar{p},$$

where

$$\phi = \left(\frac{\theta}{\theta-(\rho-1)}\right)^{-1/(\rho-1)},$$

is finite if $\theta > \rho - 1$ as we have assumed.[2] Substituting in for $\bar{p}$ from (22) and applying (19), we get:

$$\begin{aligned}p_m &= \phi w^{\gamma}p_e^{1-\gamma}\frac{1}{A}\left(\frac{A^{\theta}w^{-\gamma\theta}}{A^{\theta}w^{-\gamma\theta}+A^{*\theta}w^{*-\gamma\theta}}\right)^{1/\theta} \\ &= \phi p_e^{1-\gamma}\left(A^{\theta}w^{-\gamma\theta}+A^{*\theta}w^{*-\gamma\theta}\right)^{-1/\theta}.\end{aligned}$$

Because the relevant average price is independent of where a variety is produced, $\bar{j}$ given in (19) is also the share of world spending on the $m$-good devoted to producers in $\mathcal{H}$ and $1-\bar{j}$ the share devoted to producers in $\mathcal{F}$. Note that this share does not depend on on $\rho$.[3] The so-called trade elasticity, giving the response of trade shares to factor costs, is $\theta$.

### 3.1.2 Equilibrium

As in Simple Model, we take $\mathcal{F}$'s wage $w^* = 1$ as the numeraire and consider only equilibria in which each country produces positive quantities of the $l$-good. In such equilibria we have $p_l = w = w^* = 1$. (Even though they

[2] If preferences over varieties are Cobb Douglas ($\rho = 1$) then $\phi = e^{-1/\theta}$. Our result, about a particular average of prices being independent of which country produces the good in equilibrium, is a special case of a more general result. It is easy to show that, whether produced in $H$ or $F$, the fraction of varieties sold at a price $p_m(j) \leq p$, which we can treat as the price distribution, is $(p/\bar{p})^{\theta}$.

[3] Other than determining the value of $\phi$, the value of the parameter $\rho$ is irrelevant to all that follows.

equal 1, we sometimes leave wages in the formulas that follow for clarity.) Returning to earlier formulas, we now have:

$$\bar{j} = \frac{A^{\theta}}{A^{\theta} + A^{*\theta}} \tag{23}$$

and

$$p_m = \phi p_e^{1-\gamma} \left(A^{\theta} + A^{*\theta}\right)^{-1/\theta}. \tag{24}$$

Thus, solving for the competitive equilibrium in this model reduces to finding the price of energy $p_e$ together with the allocation of labor across the three sectors in each country. We assume trade balance.

**Income** Turning to the determination of income, we have labor income $wL$ and income from rents on energy resources:

$$rE = (1-\beta)\, p_e Q_e = (1-\beta)\, p_e^{1/(1-\beta)} E.$$

Thus, aggregate income in $\mathcal{H}$ is:

$$Y = wL + (1-\beta)\, p_e^{1/(1-\beta)} E, \tag{25}$$

exactly as in Simple Model. The analog holds for $\mathcal{F}$:

$$Y^* = wL^* + (1-\beta^*)\, p_e^{1/(1-\beta^*)} E^*. \tag{26}$$

World income is the total payment to all primary factors $L$, $L^*$, $E$, and $E^*$:

$$Y^W = L^W + (1-\beta)\, p_e^{1/(1-\beta)} E + (1-\beta^*)\, p_e^{1/(1-\beta^*)} E^*. \tag{27}$$

**Demand** With trade balance each country's income is also its total spending (measured in terms of the numeraire, the $l$-good, whose price is $p_l = 1$). The $e$-good and $m$-good combine with a Cobb-Douglas aggregator into what we have called the composite or $c$-good. The price of the $c$-good is:

$$p_c = p_c^* = p_m^{\eta} p_e^{1-\eta} = \phi^{\eta} p_e^{1-\eta\gamma} \left(A^{\theta} + A^{*\theta}\right)^{-\eta/\theta}. \tag{28}$$

We are back in the demand system of Simple Model but with $p_c$ replacing $p_e$. In particular, defining:

$$D(p) = \frac{\alpha p^{-(\sigma-1)}}{\alpha p^{-(\sigma-1)} + (1-\alpha)}, \tag{29}$$

spending by $\mathcal{H}$ on the $c$-good is:

$$p_c C_c = D(p_c)Y.$$

We thus know how consumers allocate spending across the $c$-good and the $l$-good given prices. We also know that a fraction $\eta$ of spending on the $c$-good is allocated to the $m$-good, with the rest spent on the $e$-good. In $\mathcal{F}$ we have:

$$D^*(p) = \frac{\alpha^* p^{-(\sigma-1)}}{\alpha^* p^{-(\sigma-1)} + (1-\alpha^*)}$$

and:

$$p_c C_c^* = D^*(p_c)Y^*.$$

**Equilibrium Energy Price** We now have all the ingredients to find the price of energy that equates world supply and demand. Rather than equating quantities of energy, we will equate the value of those quantities. World spending on energy consists of world spending on consumption of the $e$-good plus world spending on energy intermediates, which represent a fraction $1-\gamma$ of world spending on the $m$-good. We can therefore express total spending on energy in terms of spending on the composite good:

$$\begin{aligned} p_e\left(C_e + C_e^*\right) + p_e\left(M_e + M_e^*\right) &= (1-\eta\gamma)\, p_c C_c + (1-\eta\gamma)\, p_c C_c^* \\ &= (1-\eta\gamma)\left(D(p_c)Y + D^*(p_c)Y^*\right). \end{aligned}$$

Equating the value of world demand to the value of world energy supply (which is unchanged from Simple Model), the equilibrium price must satisfy:

$$p_e^{1/(1-\beta)} E + p_e^{1/(1-\beta^*)} E^* = (1-\eta\gamma)\left(D(p_c)Y + D^*(p_c)Y^*\right),$$

with $p_c$ given by (28), $Y$ given by (25), and $Y^*$ by (26).

**Employment** What about the allocation of labor? We'll start with employment in extraction. Having determined the equilibrium energy price, we get:

$$L_e = \beta\left(\frac{p_e}{w}\right)^{1/(1-\beta)} E = \beta p_e^{1/(1-\beta)} E,$$

and likewise for $L_e^*$.

In solving for the equilibrium energy price, we've also solved for world income $Y^W$ and the price of the composite good, $p_c$. Thus, from the demand system (with wages equal to 1) we know that world demand for labor in the $m$-sector is:

$$L_m^W = \gamma\eta\left(D(p_c)Y + D^*(p_c)Y^*\right),$$

with a fraction $\bar{j}$ employed in $\mathcal{H}$, so that:

$$L_m = \frac{A^\theta}{A^\theta + A^{*\theta}} L_m^W.$$

To support this equilibrium, we need positive employment in the $l$-sector in both $\mathcal{H}$ and $\mathcal{F}$. Thus we require:

$$L > L_e + L_m$$

and

$$L^* > L_e^* + L_m^*.$$

As a check, let's calculate world employment in the $l$-sector:

$$\begin{aligned}
L_l^W &= L^W - L_e^W - L_m^W \\
&= L^W - \beta p_e^{1/(1-\beta)} E - \beta^* p_e^{1/(1-\beta^*)} E^* - \gamma\eta\left(D(p_c)Y + D^*(p_c)Y^*\right) \\
&= Y^W - p_e^{1/(1-\beta)} E - p_e^{1/(1-\beta^*)} E^* - \gamma\eta\left(D(p_c)Y + D^*(p_c)Y^*\right) \\
&= Y^W - p_e M_e^W - p_e C_e^W - \gamma\eta\left(D(p_c)Y + D^*(p_c)Y^*\right) \\
&= Y^W - \left[(1-\gamma)\eta + (1-\eta) + \gamma\eta\right]\left(D(p_c)Y + D^*(p_c)Y^*\right) \\
&= \left(1 - D(p_c)\right)Y + \left(1 - D^*(p_c)\right)Y^*.
\end{aligned}$$

Note that the last term is simply world spending on the $l$-good, all of which is paid to labor $L_l^W$, thus confirming the logic of the model.

## 3.2 Carbon Taxes

We now have three levels at which to tax carbon: extraction of energy, production (of manufactures), and consumption of energy (directly or as embodied in manufactures). As a notational convention, we will treate $p_e$ and $p_m$ as world prices, while we treat the price of the $c$-good as being potentially country specific, due to taxes.

The ad valorem extraction tax, $t_e$, is unchanged from our formulation in the simple model. It has the effect of reducing the extraction of energy where it is applied, reducing world energy supply, and increasing the energy price.

The ad valorem production tax $t_p$ raises the cost of the energy intermediate for manufacturers. Since direct consumption of the $e$-good can be thought of as household production (driving a car to produce transportation services or running a furnace to produce heating services), we want the production tax to apply there as well. The production tax is straightforward since it simply raises the price of energy (for all users of energy) from $p_e$ to $(1+t_p)p_e$.

In Simple Model we had an ad valorem tax $t_c$ on direct consumption of energy, raising the price faced by consumers from $p_e$ to $(1+t_c)p_e$. We now need to expand the tax base to also cover the energy embodied in manufactures. The tax on consumption of the $m$-good is more complicated since we want it to mimic the incentive effects, on both producers and consumers, of the production tax. We discuss this issue below. The key distinction between a production and a consumption tax arises when these taxes are applied in only one country, say $\mathcal{H}$. In this case, the consumption tax, unlike the production tax, applies to $\mathcal{H}$'s imports of manufactures and does not apply to its manufacturing exports.

### 3.2.1 Implementing the Consumption Tax

Consider a firm producing variety $j$ of the $m$-good in $\mathcal{H}$. It takes as given the competitive prices for this good, $p(j)$ in $\mathcal{H}$ and $p^*(j)$ in $\mathcal{F}$, inclusive of carbon taxes on consumption. The consumption tax is imposed at an ad valorem rate of $t_c$ and $t_c^*$ on the value of energy embodied in the goods sold in $\mathcal{H}$ and $\mathcal{F}$, respectively. Consumers must purchase the good at a discount, so that after paying the consumption tax the total cost to a consumer in $\mathcal{H}$ is $p_m(j)$ and to a consumer in $\mathcal{F}$ is $p_m^*(j)$. (It simplifies notation to define the price of manufactured goods inclusive of taxes.)

First consider the firm's sales to consumers in $\mathcal{H}$. The firm's problem is to choose inputs of labor $l$ and energy $e$ so as to maximize revenue per unit, taking into account that the consumption tax is not revenue:

$$\max_{l,e}\left\{(p_m(j) - t_c p_e e) - wl - p_e e\right\}$$

subject to:

$$A(j)\left(\frac{l}{\gamma}\right)^{\gamma}\left(\frac{e}{1-\gamma}\right)^{1-\gamma} = 1.$$

Since the firm takes $p_m(j)$ as given, we can reformulate this problem as cost minimization:

$$\min_{l,e} \{wl + (1 + t_c)\, p_e e\}$$

subject to

$$A(j) \left(\frac{l}{\gamma}\right)^{\gamma} \left(\frac{e}{1-\gamma}\right)^{1-\gamma} = 1.$$

The solution to this problem is a unit cost function:

$$c(j) = \frac{w^{\gamma} \left((1 + t_c)\, p_e\right)^{1-\gamma}}{A(j)}$$

and associated input shares:

$$wl(j) = \gamma c(j)$$

$$(1 + t_c)\, p_e e(j) = (1 - \gamma)\, c(j).$$

The competitive zero-profit condition is simply:

$$p_m(j) = c(j) = (1 + t_c)^{1-\gamma} \frac{w^{\gamma} p_e^{1-\gamma}}{A(j)}.$$

For the firm's sales to consumers in $\mathcal{F}$ the same derivation applies, so that:

$$p_m^*(j) = (1 + t_c^*)^{1-\gamma} \frac{w^{\gamma} p_e^{1-\gamma}}{A(j)}.$$

For a firm in $\mathcal{F}$ producing good $j'$, the price of its good (again, inclusive of the consumption tax) to consumers in $\mathcal{F}$ is:

$$p_m^*(j') = (1 + t_c^*)^{1-\gamma} \frac{w^{*\gamma} p_e^{1-\gamma}}{A^*(j)},$$

and to consumers in $\mathcal{H}$ is:

$$p_m(j') = (1 + t_c)^{1-\gamma} \frac{w^{*\gamma} p_e^{1-\gamma}}{A^*(j)}.$$

Note that the consumption tax will give producers the same incentives to reduce energy inputs as would a production tax. Thus, it is not a simple ad valorem tax on consumption. Rather, it requires information on the actual energy used in production. Nonetheless, due to the Cobb-Douglas technology, in equilibrium it will appear as an ad valorem tax. For sales in $\mathcal{H}$ the ad valorem rate is $t_m$ on consumption of the $m$-good, with $1 + t_m = (1 + t_c)^{1-\gamma}$. The calculation of tax revenue differs from what would be obtained from an ad valorem tax at rate $t_m$, however. For example, the tax revenue generated in $\mathcal{H}$ on consumption of variety $j$ produced in $\mathcal{H}$ is:

$$T_c(j) = t_c p_e M_e^{HH}(j) = \frac{t_c}{1 + t_c}(1 - \gamma) p_m(j) C_m(j),$$

where $M_e^{HH}(j)$ represents the quantity of the energy intermediate used by producers of $j$ in $\mathcal{H}$ to supply consumers in $\mathcal{H}$.

A firm in $\mathcal{F}$ producing some variety $j'$ to sell in $\mathcal{H}$ will face similar considerations due to the consumption tax. The tax revenue generated in $\mathcal{H}$ on consumption of variety $j'$ produced in $\mathcal{F}$ is

$$T_c(j') = t_c p_e M_e^{HF}(j') = \frac{t_c}{1 + t_c}(1 - \gamma) p_m(j') C_m(j'),$$

where $M_e^{HF}(j)$ represents the quantity of the energy intermediate used by producers of $j'$ in $\mathcal{F}$ to supply consumers in $\mathcal{H}$.

### 3.2.2 Uniform Taxes

Suppose both countries impose carbon taxes at the same rate. We will continue to assume that the world equilibrium remains one in which both countries continue to produce positive quantities of the $l$-good. We will consider taxes one by one, but accomodate the possibility that an extraction tax is imposed together with either a production or consumption tax.

**Extraction Tax** With an extraction tax, the extraction sector behaves as in Simple Model. World extraction is:

$$Q_e^W = \left(\frac{p_e}{1 + t_e}\right)^{\beta/(1-\beta)} E + \left(\frac{p_e}{1 + t_e}\right)^{\beta^*/(1-\beta^*)} E^*.$$

After-tax revenue of $\mathcal{H}$ in the $e$-sector is:

$$R_e = \frac{p_e}{1 + t_e} Q_e = \left(\frac{p_e}{1 + t_e}\right)^{1/(1-\beta)} E,$$

generating tax revenue:

$$T_e = t_e R_e = t_e \left(\frac{p_e}{1+t_e}\right)^{1/(1-\beta)} E.$$

Parallel equations hold for $\mathcal{F}$. Given its effect on the equilibrium price of energy, the extraction tax has no other effects that interact with the production or consumption tax.

**Production Tax** As explained above, a production tax is just a tax on using energy for either production or consumption. Energy goods now cost $p_e(1+t_p)$ in either country, whether used as intermediates by manufacturers or as consumption goods by households.

With a tax on energy inputs, producers substitute away from them toward the labor input. The cost of producing manufactures rises by the factor $(1+t_p)^{1-\gamma}$ in both countries. Specialization in the production of manufactures remains unchanged, with $\bar{j}$ still given by (23). The price of manufactures rises to:

$$p_m = \phi(p_e\,(1+t_p))^{1-\gamma}\left(A^\theta + A^{*\theta}\right)^{-1/\theta}.$$

The statements above treat $p_e$ as fixed. Of course the reduced demand leads to a fall in the price of energy, dampening the effects.

The production tax also raises the price of energy consumed directly by households. The price of the composite good becomes:

$$p_c = p_m^\eta((1+t_p)\,p_e)^{1-\eta} = \phi^\eta((1+t_p)\,p_e)^{1-\eta\gamma}\left(A^\theta + A^{*\theta}\right)^{-\eta/\theta}, \tag{30}$$

Given $p_c$, spending on the $c$-good is as above. Hence consumption spending on the $e$-good in $\mathcal{H}$ is:

$$(1+t_p)p_e C_e = (1-\eta)\,D(p_c)Y,$$

with the spending share $D(p_c)$ given by (29). The analog holds in $\mathcal{F}$.

Production tax revenue consists of the taxes collected from firms and those collected from households. The total in $\mathcal{H}$ is:

$$\begin{aligned} T_p &= t_p p_e M_e^{WH} + t_p p_e C_e \\ &= \frac{t_p}{1+t_p}\left[(1-\gamma)\frac{A^\theta}{A^\theta + A^{*\theta}}\eta\,(D(p_c)Y + D^*(p_c)Y^*) + (1-\eta)\,D(p_c)Y\right], \end{aligned}$$

Where $M_e^{WH}$ is the quantity of the energy input used by producers in $\mathcal{H}$ to supply the world. In $\mathcal{F}$:

$$T_p^* = \frac{t_p}{1+t_p}\left[(1-\gamma)\frac{A^{*\theta}}{A^\theta + A^{*\theta}}\eta\left(D(p_c)Y + D^*(p_c)Y^*\right) + (1-\eta)\,D(p_c)Y^*\right].$$

Hence, tax revenue for the world is:

$$T_p^W = \frac{t_p}{1+t_p}\left(1-\gamma\eta\right)\left(D(p_c)Y + D^*(p_c)Y^*\right).$$

Income in $\mathcal{H}$ is given by:

$$Y = wL + rE + T_e + T_p = L + (1-\beta+t_e)\,R_e + T_p,$$

while in $\mathcal{F}$:

$$Y^* = L^* + (1-\beta^*+t_e)\,R_e^* + T_p^*.$$

**Consumption Tax** As explained above, a consumption tax raises the price faced by households for both the $m$-good and the $e$-good. In considering a consumption tax at rate $t_c$ we will set $t_p = 0$ (below we consider a combination of the two taxes). We put no restriction on $t_e$. Since $t_p = 0$, we are back to:

$$p_m = \phi(1+t_c)^{1-\gamma}p_e^{1-\gamma}\left(A^\theta + A^{*\theta}\right)^{-1/\theta}.$$

A household faces prices $(1+t_c)p_e$ and $p_m$ (recall that the price for manufactures is inclusive of the consumption tax). The after-tax price of the $c$-good is thus:

$$p_c = (1+t_c)^{1-\eta}p_m^\eta p_e^{1-\eta} = \phi^\eta\left((1+t_c)p_e\right)^{1-\eta\gamma}\left(A^\theta + A^{*\theta}\right)^{-\eta/\theta},$$

which is the same as (30) if $t_c = t_p$. Given $p_c$, spending by households on the $e$-good in $\mathcal{H}$ is:

$$(1+t_c)p_eC_e = (1-\eta)\,D(p_c)Y.$$

On the $m$-good households spend:

$$p_mC_m = \eta D(p_c)Y,$$

so that:

$$(1+t_c)p_eM_e^{HH} = (1-\gamma)\,p_mC_m = (1-\gamma)\,\eta D(p_c)Y.$$

Tax revenue in $\mathcal{H}$ is:

$$\begin{aligned} T_c &= \frac{t_c}{1+t_c}(1-\gamma)\,\eta D(p_c)Y + \frac{t_c}{1+t_c}(1-\eta)\,D(p_c)Y \\ &= \frac{t_c}{1+t_c}(1-\gamma\eta)\,D(p_c)Y, \end{aligned}$$

while for the world:

$$T_c^W = \frac{t_c}{1+t_c}(1-\gamma\eta)\left(D(p_c)Y + D^*(p_c)Y^*\right).$$

Income in $\mathcal{H}$ is given by:

$$Y = L + (1-\beta+t_e)\,R_e + T_c.$$

**Tax Equivalence** Equilibrium amounts to a price $p_e$ that equates world supply and demand for energy. The value of world supply, given a uniform extraction tax, is:

$$p_e Q_e^W = (1+t_e)^{-\beta/(1-\beta)}\,p_e^{1/(1-\beta)}E + (1+t_e)^{-\beta^*/(1-\beta^*)}\,p_e^{1/(1-\beta^*)}E^*,$$

so at the equilbrium price:

$$p_e Q_e^W = p_e C_e^W + p_e M_e^W,$$

where:

$$M_e^W = M_e^{WH} + M_e^{WF},$$

is total energy intermediates used in the world. For the demand side of this equation, we want to show that a uniform production tax is equivalent to a uniform consumption tax at rate $t_c = t_p = t$.

Under either a production tax or a consumption tax, the value of world demand for energy by households is:

$$p_e C_e^W = \frac{1}{1+t}(1-\eta)\left(D(p_c)Y + D^*(p_c)Y^*\right),$$

and the value of world demand for energy by firms is:

$$p_e M_e^W = \frac{1}{1+t}(1-\gamma)\,\eta\left(D(p_c)Y + D^*(p_c)Y^*\right),$$

so that total demand is:

$$p_e Q_e^W = \frac{1}{1+t}\left(1-\eta\gamma\right)\left(D(p_c)Y + D^*(p_c)Y^*\right),$$

with $p_c$, in either case, given by (30). World income $Y^W$ under either tax is:

$$Y^W = L^W + \left(1-\beta+t_e\right)R_e + \left(1-\beta^*+t_e\right)R_e^* + T^W,$$

where total tax revenue from the production or consumption tax is:

$$T^W = \frac{t}{1+t}\left(1-\gamma\eta\right)\left(D(p_c)Y + D^*(p_c)Y^*\right)$$

Thus, the equilibrium price of energy is the same under a production tax at rate $t_p = t$ or a consumption tax at rate $t_c = t$. There is a difference in the distribution of tax revenue, however, since

$$T_p = \frac{t}{1+t}\left[\left(1-\gamma\right)\frac{A^\theta}{A^\theta + A^{*\theta}}\eta\left(D(p_c)Y + D^*(p_c)Y^*\right) + \left(1-\eta\right)D(p_c)Y\right]$$

need not equal

$$T_c = \frac{t}{1+t}\left(1-\gamma\eta\right)D(p_c)Y.$$

Home will do better under a uniform production tax iff:

$$\frac{A^\theta}{A^\theta + A^{*\theta}}Y^W > Y,$$

i.e. if it has a comparative advantage in manufactures.

### 3.2.3 Taxes in Home

Suppose only $\mathcal{H}$ taxes carbon. We will continue to assume that the world equilibrium remains one in which both countries continue to produce positive quantities of the $l$-good. We will consider taxes one by one, but accomodate the possibility that an extraction tax is imposed together with either a production or consumption tax. As above, we do not consider the simultaneous imposition of both a production and consumption tax.

**Extraction Tax** With an extraction tax, the extraction sector behaves as in Simple Model. Extraction in $\mathcal{H}$ is:

$$Q_e = \left(\frac{p_e}{1+t_e}\right)^{\beta/(1-\beta)} E,$$

generating after-tax revenue in the $e$-sector of:

$$R_e = \frac{p_e}{1+t_e} Q_e = \left(\frac{p_e}{1+t_e}\right)^{1/(1-\beta)} E.$$

Tax revenue is:

$$T_e = t_e R_e = t_e \left(\frac{p_e}{1+t_e}\right)^{1/(1-\beta)} E.$$

In $\mathcal{F}$ we have:

$$Q_e^* = p_e^{\beta^*/(1-\beta^*)} E^*.$$

Since there are no taxes in $\mathcal{F}$, we have:

$$Y^* = L^* + (1-\beta^*)\, p_e^{1/(1-\beta^*)} E^*. \tag{31}$$

Given its effect on the equilibrium price of energy, the extraction tax has no other effects that interact with the production or consumption tax.

**Production Tax** As explained above, a production tax is just a tax on using energy for either production or consumption. Energy goods now cost $p_e(1+t_p)$ whether used as intermediates by manufacturers in $\mathcal{H}$ or as consumption goods by households in $\mathcal{H}$.

The cost of producing manufactures in $\mathcal{H}$ rises by the factor $(1+t_p)^{1-\gamma}$. Now $\mathcal{H}$ produces all varieties $j$ for which:

$$R(j) \geq (1+t_p)^{1-\gamma},$$

where relative productivity $R(j)$ is given by (18). Thus $\mathcal{H}$ produces all varieties in the interval $[0, \bar{j}]$, where now:

$$\bar{j} = \frac{A^\theta (1+t_p)^{-\theta(1-\gamma)}}{A^\theta (1+t_p)^{-\theta(1-\gamma)} + A^{*\theta}}. \tag{32}$$

The production tax alters how countries specialize across varieties of manufactures.

With a production tax, the maximum price of a manufacturing variety becomes:

$$\begin{aligned}\bar{p} &= \frac{\bar{j}^{1/\theta} p_e^{1-\gamma} \left(1+t_p\right)^{1-\gamma}}{A} \\ &= p_e^{1-\gamma} \left(A^{\theta} \left(1+t_p\right)^{-\theta(1-\gamma)} + A^{*\theta}\right)^{-1/\theta}.\end{aligned}$$

We still have that the manufacturing price index is $p_m = \phi\bar{p}$, so that when we substitute out $\bar{p}$:

$$p_m = \phi p_e^{1-\gamma} \left(A^{\theta} \left(1+t_p\right)^{-\theta(1-\gamma)} + A^{*\theta}\right)^{-1/\theta}.$$

Holding $p_e$ fixed, the production tax in $\mathcal{H}$ makes manufactures more expensive to consumers in both countries.

The production tax also raises the price of energy consumed directly by households. The after-tax price of the composite good in $\mathcal{H}$ becomes:

$$p_c = p_m^{\eta} p_e^{1-\eta} (1+t_p)^{1-\eta} = \phi^{\eta} p_e^{1-\eta\gamma} (1+t_p)^{1-\eta} \left(A^{\theta} \left(1+t_p\right)^{-\theta(1-\gamma)} + A^{*\theta}\right)^{-\eta/\theta}, \tag{33}$$

while in $\mathcal{F}$ it is:

$$p_c^* = p_m^{\eta} p_e^{1-\eta} = \phi^{\eta} p_e^{1-\eta\gamma} \left(A^{\theta} \left(1+t_p\right)^{-\theta(1-\gamma)} + A^{*\theta}\right)^{-\eta/\theta}. \tag{34}$$

Given $p_c$, spending on the $c$-good is as above. Hence consumption spending on the $e$-good is:

$$(1+t_p) p_e C_e = (1-\eta)\, D(p_c) Y,$$

with the spending share $D(p_c)$ given by (29). Production tax revenue consists of the taxes collected from firms and those collected from households. The total is:

$$\begin{aligned}T_p &= t_p p_e M_e^{WH} + t_p p_e C_e \\ &= \frac{t_p}{1+t_p} \left[(1-\gamma)\, \bar{j}\eta \left[D(p_c) Y + D^*(p_c^*) Y^*\right] + (1-\eta)\, D(p_c) Y\right],\end{aligned}$$

with $\bar{j}$ given by (32), and income in $\mathcal{H}$ given by:

$$Y = wL + rE + T_e + T_p = L + (1-\beta+t_e)\, R_e + T_p. \tag{35}$$

**Consumption Tax** As explained above, a consumption tax raises the price of both the $m$-good and the $e$-good to households in $\mathcal{H}$. In considering a consumption tax at rate $t_c$ we continue to assume $t_p = 0$, turning only later to the case of both taxes. We put no restriction on $t_e$. Since $t_p = 0$, we are back to:

$$p_m = \phi(1+t_c)^{1-\gamma} p_e^{1-\gamma} \left(A^\theta + A^{*\theta}\right)^{-1/\theta}.$$

Since the household in $\mathcal{H}$ faces prices $(1+t_c)p_e$ and $p_m$, the after-tax price of the $c$-good is:

$$\begin{aligned} p_c &= p_m^\eta \left(p_e(1+t_c)\right)^{1-\eta} \\ &= \phi^\eta p_e^{1-\eta\gamma}(1+t_c)^{1-\eta\gamma} \left(A^\theta + A^{*\theta}\right)^{-\eta/\theta}, \end{aligned} \tag{36}$$

while the price in $\mathcal{F}$ is simply:

$$p_c^* = \phi^\eta p_e^{1-\eta\gamma} \left(A^\theta + A^{*\theta}\right)^{-\eta/\theta}. \tag{37}$$

Given $p_c$, spending by households on the $e$-good is:

$$(1+t_c)p_e C_e = (1-\eta)\, D(p_c)Y,$$

while on the $m$-good:

$$p_m C_m = \eta D(p_c)Y.$$

Spending on the energy input in $\mathcal{H}$ for sales of manufactures in $\mathcal{H}$ is:

$$(1+t_c)p_e M_e^{HH} = (1-\gamma)\, \bar{j}\eta D(p_c)Y.$$

Since there is no consumption tax in $\mathcal{F}$, total spending on energy inputs in $\mathcal{H}$ is:

$$p_e M_e^{WH} = p_e M_e^{HH} + p_e M_e^{FH} = (1-\gamma)\, \bar{j}\eta \left[\frac{1}{1+t_c} D(p_c)Y + D^*(p_c^*)Y^*\right].$$

Total spending on energy inputs in $\mathcal{F}$ is:

$$p_e M_e^{WF} = p_e M_e^{HF} + p_e M_e^{FF} = (1-\gamma)\,(1-\bar{j})\,\eta \left[\frac{1}{1+t_c} D(p_c)Y + D^*(p_c^*)Y^*\right],$$

so that world spending on energy inputs is:

$$p_e M_e^W = (1-\gamma)\,\eta \left[\frac{1}{1+t_c} D(p_c)Y + D^*(p_c^*)Y^*\right].$$

Tax revenue in $\mathcal{H}$ is:

$$\begin{aligned} T_c &= t_c p_e M_e^{HH} + t_c p_e M_e^{HF} + t_c p_e C_e \\ &= \frac{t_c}{1+t_c} (1-\gamma)\, \eta D(p_c) Y + \frac{t_c}{1+t_c} (1-\eta)\, D(p_c) Y \\ &= \frac{t_c}{1+t_c} (1-\eta\gamma)\, D(p_c) Y. \end{aligned}$$

Income in $\mathcal{H}$ is given by:

$$Y = L + (1-\beta+t_e)\, R_e + T_c. \tag{38}$$

**Equilibrium with Taxes** Equilibrium amounts to a price $p_e$ that equates world supply and demand for energy, both valued at world prices. The value of world supply is:

$$p_e Q_e^W = (1+t_e)^{-\beta/(1-\beta)}\, p_e^{1/(1-\beta)} E + p_e^{1/(1-\beta^*)} E^*.$$

At the equilbrium price we have:

$$p_e Q_e^W = p_e C_e^W + p_e M_e^W.$$

For the demand side of this equation, we separately consider the case of the production tax and the consumption tax.

Under a production tax in home, the value of world demand for energy by households is:

$$p_e C_e^W = (1-\eta) \left[ \frac{1}{1+t_p} D(p_c) Y + D^*(p_c^*) Y^* \right],$$

and the value of world demand for energy by firms is:

$$p_e M_e^W = (1-\gamma)\, \eta \left[ D(p_c) Y + D^*(p_c^*) Y^* \right] \left( \frac{1}{1+t_p} \bar{\jmath} + (1-\bar{\jmath}) \right),$$

with $p_c$ given by (33), $p_c^*$ given by (34), $\bar{\jmath}$ given by (32), $Y$ given by (35), and $Y^*$ given by (31).

Under a consumption tax in home:

$$p_e C_e^W = (1-\eta) \left[ \frac{1}{1+t_c} D(p_c) Y + D^*(p_c^*) Y^* \right]$$

and

$$p_e M_e^W = (1-\gamma)\,\eta\left[\frac{1}{1+t_c}D(p_c)Y + D^*(p_c^*)Y^*\right].$$

Thus:

$$p_e C_e^W + p_e M_e^W = (1-\eta\gamma)\left[\frac{1}{1+t_c}D(p_c)Y + D^*(p_c^*)Y^*\right],$$

with $p_c$ given by (36), $p_c^*$ given by (37), $Y$ given by (38), and $Y^*$ given by (31).

**Border Adjustments** We now show that a production tax at rate $t_p$, with border tax adjustments (BTA's), is equivalent to a consumption tax at rate $t_c = t_p$. Starting with the production tax, let's now add a tax on the value of energy embodied in $\mathcal{H}$'s imports of the $m$-good, at rate $t_p$, and a tax rebate on the value of energy embodied in $\mathcal{H}$'s exports of the $m$-good. Taking into account the tax on imports, to supply variety $j$ to consumers in $\mathcal{H}$, a firm in $\mathcal{F}$ must charge

$$\frac{w^{*\gamma}p_e^{1-\gamma}(1+t_p)^{1-\gamma}}{A^*(j)} = \frac{(p_e\,(1+t_p))^{1-\gamma}}{A^*(j)},$$

while, taking into account the production tax, a firm in $\mathcal{H}$ charges

$$\frac{w^{\gamma}(p_e\,(1+t_p))^{1-\gamma}}{A(j)} = \frac{(p_e\,(1+t_p))^{1-\gamma}}{A(j)}.$$

Thus, the firm in $\mathcal{H}$ supplies variety $j$ if and only if it can produce that variety more efficiently:

$$R(j) = \frac{A(j)}{A^*(j)} \geq 1,$$

i.e. if and only if:

$$j \leq \bar{j} = \frac{A^\theta}{A^\theta + A^{*\theta}}.$$

Similarly, taking into account the tax rebate on exports, firms from $\mathcal{H}$ and $\mathcal{F}$ both act as if they are untaxed when supplying consumers in $\mathcal{F}$. The same $\bar{j}$ applies, which is also the $\bar{j}$ for the case of a consumption tax.

Now, consider the price of the $m$-good (recall that it includes taxes) faced by consumers in each country. Taking into account how $\bar{j}$ responds to the BTA's, the price of the $m$-good in $\mathcal{F}$ is simply:

$$p_m = \phi p_e^{1-\gamma} \left(A^\theta + A^{*\theta}\right)^{-1/\theta}$$

while in $\mathcal{H}$, since every supplier must cover taxes on the energy input, the price is:

$$p_m = \phi(p_e (1 + t_p))^{1-\gamma} \left(A^\theta + A^{*\theta}\right)^{-1/\theta}.$$

The price of the composite good in $\mathcal{H}$ is:

$$p_c = p_m^\eta \left((1 + t_c) p_e\right)^{1-\eta} = \phi^\eta (p_e (1 + t_c))^{1-\eta\gamma} \left(A^\theta + A^{*\theta}\right)^{-\eta/\theta},$$

while in $\mathcal{F}$ it is:

$$p_c^* = p_m^\eta p_e^{1-\eta}.$$

Again, the production tax with BTA's gives the same outcomes as the consumption tax.

Let's consider the determination of the equilibrium energy price under a production tax with BTA's. With BTA's, the global value of demand for energy by households is:

$$p_e C_e^W = (1 - \eta) \left[\frac{1}{1 + t_p} D(p_c) Y + D^*(p_c^*) Y^*\right],$$

while the global value of demand for energy inputs is

$$p_e M_e^W = (1 - \gamma) \eta \left[\frac{1}{1 + t_p} D(p_c) Y + D^*(p_c^*) Y^*\right].$$

Thus the total value of global demand for energy is:

$$p_e C_e^W + p_e M_e^W = (1 - \eta\gamma) \left[\frac{1}{1 + t_p} D(p_c) Y + D^*(p_c^*) Y^*\right],$$

exactly the same as with a consumption tax at rate $t_c = t_p$. It follows that the equilibrium energy price is the same.

Note that $\mathcal{H}$ gets tax revenue only on energy consumed directly by households in $\mathcal{H}$ and energy embodied in the $m$-good consumed in $\mathcal{H}$, just as with a consumption tax. Thus tax revenue and hence income will be the same under a production tax with BTA's as under a consumption tax.

**Welfare Results** We now ask whether a consumption tax (which we have shown to be a production tax with BTA's) has any welfare advantages for home over a production tax without BTA's. Note that in making this comparison, we should not set $t_c = t_p$. Rather, we should hold $p_e$ fixed in this comparison, since by doing so we fix global emissions of carbon. Thus, we want to ask, fixing global emissions, is it advantageous for $\mathcal{H}$ to impose a consumption tax at rate $t_c$ rather than a production tax at rate $t_p$. Equivalently, is it advantageous for $\mathcal{H}$ to add BTA's to a production tax, changing the tax rate from $t_p$ to $t_c$ so as to acheive the same level of global emissions?

By fixing $p_e$ we obtain some nice simplifications. First, income in $\mathcal{F}$ is fixed while in $\mathcal{H}$ it will change only to the extent of a change in tax revenue. Second, of the three goods prices, only the price of the $m$-good will differ across these two tax regimes, since the price of the $e$-good and the $l$-good are fixed.

**Home Dominates Manufacturing** To get sharp results, let's consider an extreme case. Suppose $A^* = 0$, so that $\mathcal{H}$ produces all of the $m$-good for the world. We want to compare the equilibrium under a production tax with the equilibrium under a consumption tax.

Under a production tax:

$$p_m = \frac{\phi}{A} \left( (1 + t_p) \, p_e \right)^{1-\gamma} .$$

Hence the price of the $c$-good in $\mathcal{H}$ is:

$$p_c = \left( \frac{\phi}{A} \right)^{\eta} \left( (1 + t_p) \, p_e \right)^{1-\eta\gamma} .$$

Since the price of the $m$-good is the same everywhere, while the $e$-good is untaxed in $F$, consumers there pay:

$$p_c^* = \left( \frac{\phi}{A} \right)^{\eta} (1 + t_p)^{\eta} \, p_e^{1-\eta\gamma}.$$

The total value of demand for energy intermediates is:

$$p_e M_e^W = (1 - \gamma) \, \eta \frac{1}{1 + t_p} \left[ D(p_c) Y + D^*(p_c^*) Y^* \right],$$

while the total value of demand for energy by households is:

$$p_e C_e^W = (1-\eta)\left[\frac{1}{1+t_p} D(p_c)Y + D^*(p_c^*)Y^*\right].$$

Thus, the total value of demand for energy is:

$$p_e M_e^W + p_e C_e^W = (1-\eta\gamma)\left[\frac{1}{1+t_p} D(p_c)Y + D^*(p_c^*)Y^*\right] - \frac{t_p}{1+t_p}(1-\gamma)\,\eta D^*(p_c^*)Y^*$$

Finally, tax revenue in $\mathcal{H}$ is:

$$T_p = \frac{t_p}{1+t_p}(1-\eta\gamma)\,D(p_c)Y + \frac{t_p}{1+t_p}(1-\gamma)\,\eta D^*(p_c^*)Y^*$$

Under a consumption tax, consumers in $\mathcal{H}$ pay:

$$p_m = \frac{\phi}{A}\left((1+t_c)\,p_e\right)^{1-\gamma},$$

so that the price of the $c$-good in $\mathcal{H}$ is:

$$p_c = \left(\frac{\phi}{A}\right)^{\eta}\left((1+t_c)\,p_e\right)^{1-\eta\gamma}.$$

Consumers in $\mathcal{F}$ pay only:

$$p_c^* = \left(\frac{\phi}{A}\right)^{\eta} p_e^{1-\eta\gamma}.$$

The total value of demand for energy is:

$$p_e M_e^W + p_e C_e^W = (1-\eta\gamma)\left[\frac{1}{1+t_c} D(p_c)Y + D^*(p_c^*)Y^*\right].$$

Tax revenue is:

$$T_c = \frac{t_c}{1+t_c}(1-\eta\gamma)\,D(p_c)Y.$$

Let's compare the two tax regimes. If $t_p = t_c$, which tax regime would lead to more spending on energy? If the value of spending on energy would be higher under the consumption tax, then we can conclude that $t_p < t_c$ in order for $p_e$ to be the same in both regimes. Under the hypothetical in which $t_p = t_c$, there are three differences in the value of world energy demand. First, unlike the consumption tax, the production tax raises the price of manufactures everywhere. Hence spending under the production tax is lower by the amount $\frac{t_p}{1+t_p}(1-\gamma)\,\eta D^*(p_c^*)Y^*$. Second, for the same reason, the price of the $c$-good is lower in $\mathcal{F}$ under the consumption tax, which leads to more spending on energy. Third, tax revenue is higher under the production tax by the amount $\frac{t_p}{1+t_p}(1-\gamma)\,\eta D^*(p_c^*)Y^*$, which raises income by that amount in $\mathcal{H}$. Note that this third effect, which would lead to greater spending on energy under the production tax, is more than offset by the first effect. Thus, overall, spending on energy is lower under the production tax regime if $t_p = t_c$. To equate supply and demand in both regimes at the same energy price $p_e$, it must be that $t_c > t_p$. But, from the price equations, we see that if $t_c > t_p$ then the price of the $c$-good is higher under the consumption tax. Since income is higher under the production tax, and prices are equal or lower under the production tax, welfare must be higher under the prodution tax.

What we've shown is that in an extreme case in which $\mathcal{H}$ is the dominant manufacturing country, the home country prefers a production tax. It would strictly prefer not to impose BTA's. The weakness of this extreme case is that, by setting $A^* = 1$, we've ruled out an argument for BTA's, which is that they prevent a loss in $\mathcal{H}$'s market share in manufacturing (in this extreme case $\mathcal{H}$'s market share is fixed at 1). But, since in this extreme case $\mathcal{H}$ stictly prefers a production tax, by continuity it will also likely prefer a production tax if $\mathcal{F}$'s market share in manufacturing is not too large.

Finally, note that $\mathcal{F}$ will prefer the consumption tax since its income is invariant to taxes and the price of the $c$-good is lower with a consumption tax. The consumption tax removes any tax burden to $\mathcal{F}$.

### 3.3 Partial Border Tax Adjustments

We now consider a scenario in which $\mathcal{H}$ imposes a production tax together with some level of border tax adjustments (a tax on imports of the $m$-good and a rebate on exports of the $m$-good). The two policy levers are thus the production tax rate $t_p \geq 0$ and the border tax adjustment rate $t_b \in [0, t_p]$. The production tax raises the effective cost of energy to consumers in $\mathcal{H}$ and of energy intermediates to producers in $\mathcal{H}$ by the factor $1 + t_p$. The border tax adjustment has two parts: (i) a tariff of $t_b$ on the carbon content of $\mathcal{H}$'s imports of the $m$-good, effectively raising the cost of energy intermediates to producers in $\mathcal{F}$ selling in $\mathcal{H}$ by the factor $1+t_b$ and (ii) a rebate to producers in $\mathcal{H}$ on a portion of the production tax paid on the carbon content of their exports of the $m$-good, effectively lowering their costs of energy intermediates by a factor $1/(1+t_b)$ from what that cost would be with the production tax on its own.

If the border tax adjustment is set to $t_b = 0$ then the policy is one of a pure production tax at rate $t_p$. If the border tax adjustment is set to $t_b = t_p$ then the policy is one of a pure consumption tax at rate $t_b$. We can also consider cases between these two extremes. Setting $t_p > 0$ in combination with $t_b \in (0, t_p)$ is equivalent to a consumption tax at rate:

$$\tilde{t}_c = t_b$$

combined with a production tax at rate[4]:

$$\tilde{t}_p = \frac{1+t_p}{1+t_b} - 1 = \frac{t_p - t_b}{1+t_b}.$$

This production tax, which is applied on top of the consumption tax, yields an effective tax on production in $\mathcal{H}$ for domestic consumption of:

$$\left(1+\tilde{t}_p\right)\left(1+\tilde{t}_c\right) - 1 = \frac{1+t_p}{1+t_b}\left(1+t_b\right) - 1 = t_p.$$

### 3.3.1 Specialization and Prices

As explained above, a production tax is a tax on using energy for either production or consumption. Energy goods now cost $p_e(1+t_p)$ in $\mathcal{H}$ whether used there as an intermediate by manufacturers (to supply consumers in $\mathcal{H}$) or directly as a consumption good by households in $\mathcal{H}$.

Consider producers in either country supplying manufactures to consumers in $\mathcal{H}$. For producers in $\mathcal{H}$ the production tax raises costs by the factor $(1+t_p)^{1-\gamma}$. For producers in $\mathcal{F}$ the border tax raises costs by the factor $(1+t_b)^{1-\gamma}$. Thus producers in $\mathcal{H}$ supply all varieties $j$ for which:

$$R(j) \geq \left(\frac{1+t_p}{1+t_b}\right)^{1-\gamma},$$

[4] We thank David Weisbach's colleague ??? for this interpretation of partial BTA's.

i.e. all varieties in the interval $[0, \bar{j}]$, where, from (18):

$$\bar{j} = \frac{A^{\theta}\left(\frac{1+t_p}{1+t_b}\right)^{-\theta(1-\gamma)}}{A^{\theta}\left(\frac{1+t_p}{1+t_b}\right)^{-\theta(1-\gamma)} + A^{*\theta}} = \frac{A^{\theta}\left(1+\tilde{t}_p\right)^{-\theta(1-\gamma)}}{A^{\theta}\left(1+\tilde{t}_p\right)^{-\theta(1-\gamma)} + A^{*\theta}}. \tag{39}$$

These taxes alter how countries specialize across varieties of manufactures, reducing $\mathcal{H}$'s share if $t_p > t_b$ (i.e. if $\tilde{t}_p > 0$). To illustrate the symmetry, we can also write:

$$\bar{j} = \frac{A^{\theta}\left(1+t_p\right)^{-\theta(1-\gamma)}}{A^{\theta}\left(1+t_p\right)^{-\theta(1-\gamma)} + A^{*\theta}\left(1+t_b\right)^{-\theta(1-\gamma)}}.$$

Just as the production tax hits $\mathcal{H}$, so the border tax adjustment hits $\mathcal{F}$.

Now consider producers in either country supplying manufactures to consumers in $\mathcal{F}$. For producers in $\mathcal{H}$ the production tax, together with the border rebate of taxes on exports, raises costs by the factor:

$$\left((1+t_p)/(1+t_b)\right)^{1-\gamma} = \left(1+\tilde{t}_p\right)^{1-\gamma}.$$

Producers in $\mathcal{F}$ face no taxes. Thus (39) is the cutoff determining which country supplies which varieties to consumers anywhere.

The maximum price of a manufacturing variety supplied to consumers in $\mathcal{H}$ is:

$$\begin{aligned}\bar{p} &= \frac{\bar{j}^{1/\theta} p_e^{1-\gamma}\left(1+t_p\right)^{1-\gamma}}{A} \\ &= p_e^{1-\gamma}\left(A^{\theta}\left(1+t_p\right)^{-\theta(1-\gamma)} + A^{*\theta}\left(1+t_b\right)^{-\theta(1-\gamma)}\right)^{-1/\theta}.\end{aligned}$$

We still have that the manufacturing price index is $p_m = \phi\bar{p}$, so that when we substitute out $\bar{p}$:

$$p_m = \phi p_e^{1-\gamma}\left(A^{\theta}\left(1+t_p\right)^{-\theta(1-\gamma)} + A^{*\theta}\left(1+t_b\right)^{-\theta(1-\gamma)}\right)^{-1/\theta}.$$

Holding $p_e$ fixed, the production tax and border tax adjustment in $\mathcal{H}$ makes manufactures more expensive to consumers there. What about for consumers in $\mathcal{F}$? The maximum price there is:

$$\bar{p}^* = \frac{\bar{j}^{1/\theta} p_e^{1-\gamma}\left(\frac{1+t_p}{1+t_b}\right)^{1-\gamma}}{A},$$

so that

$$p_m^* = \phi p_e^{1-\gamma} \left( A^\theta \left( \frac{1+t_p}{1+t_b} \right)^{-\theta(1-\gamma)} + A^{*\theta} \right)^{-1/\theta} = (1+t_b)^{-(1-\gamma)} p_m.$$

The production tax also raises the price of energy consumed directly by households in $\mathcal{H}$. The after-tax price of the composite good in $\mathcal{H}$ becomes:

$$\begin{aligned} p_c &= p_m^\eta p_e^{1-\eta} (1+t_p)^{1-\eta} \\ &= \phi^\eta p_e^{1-\eta\gamma} (1+t_p)^{1-\eta} \left( A^\theta (1+t_p)^{-\theta(1-\gamma)} + A^{*\theta} (1+t_b)^{-\theta(1-\gamma)} \right)^{-\eta/\theta} \end{aligned} \tag{40}$$

Since there is no production tax in $\mathcal{F}$, the after-tax price of the composite good there is:

$$p_c^* = \phi^\eta p_e^{1-\eta\gamma} \left( A^\theta \left( \frac{1+t_p}{1+t_b} \right)^{-\theta(1-\gamma)} + A^{*\theta} \right)^{-\eta/\theta}. \tag{41}$$

Given $p_c$, spending on the $c$-good is $D(p_c)Y$, with spending share $D(p_c)$ given by (29). Hence consumption spending on the $e$-good is:

$$(1+t_p) p_e C_e = (1-\eta)\, D(p_c) Y.$$

In $F$ we have:

$$p_e C_e^* = (1-\eta)\, D^*(p_c^*) Y^*.$$

### 3.3.2 Income

If $\mathcal{H}$ also imposes an extraction tax, world energy production is:

$$Q_e^W = \left( \frac{p_e}{1+t_e} \right)^{\beta/(1-\beta)} E + p_e^{\beta^*/(1-\beta^*)} E^*.$$

Since there are no taxes in $\mathcal{F}$, total income there is:

$$Y^* = L^* + (1-\beta^*)\, p_e^{1/(1-\beta^*)} E^*. \tag{42}$$

In $\mathcal{H}$, income is:

$$Y = L + (1-\beta+t_e) \left( \frac{p_e}{1+t_e} \right)^{1/(1-\beta)} E + T_p + T_b,$$

where $T_p$ is revenue from the production tax and $T_b$ is revenue (positive or negative) from border tax adjustments.

To calculate tax revenue, we need expressions for spending on energy intermediates. Spending on the energy input in $\mathcal{H}$ for sales of manufactures in $\mathcal{H}$ is:

$$(1+t_p)p_e M_e^{HH} = (1-\gamma)\,\bar{j}\eta D(p_c)Y,$$

while for $\mathcal{H}$'s sales of manufactures in $\mathcal{F}$ the border adjustment applies:

$$\left(1+\tilde{t}_p\right)p_e M_e^{FH} = \frac{1+t_p}{1+t_b}p_e M_e^{FH} = (1-\gamma)\,\bar{j}\eta D^*(p_c^*)Y^*.$$

Spending on the energy input in $\mathcal{F}$ for sales of manufactures in $\mathcal{H}$ is:

$$(1+t_b)\,p_e M_e^{HF} = (1-\gamma)\,(1-\bar{j})\,\eta D(p_c)Y,$$

while for $\mathcal{F}$'s sales of manufactures in $\mathcal{F}$ no tax applies:

$$p_e M_e^{FF} = (1-\gamma)\,(1-\bar{j})\,\eta D^*(p_c^*)Y^*.$$

Production tax revenue consists of the taxes collected from firms and those collected from households. The total is:

$$\begin{aligned}
T_p &= t_p p_e M_e^{HH} + t_p p_e M_e^{FH} + t_p p_e C_e \\
&= \frac{t_p}{1+t_p}(1-\gamma)\,\bar{j}\eta D(p_c)Y + \frac{t_p}{1+\tilde{t}_p}(1-\gamma)\,\bar{j}\eta D^*(p_c^*)Y^* + \frac{t_p}{1+t_p}(1-\eta)\,D(p_c)Y \\
&= \frac{t_p}{1+t_p}\left[(1-\gamma)\,\bar{j}\eta D(p_c)Y + (1-\eta)\,D(p_c)Y + (1+t_b)\,(1-\gamma)\,\bar{j}\eta D^*(p_c^*)Y^*\right],
\end{aligned}$$

with $\bar{j}$ given by (39). The revenue (positive or negative) from border tax adjustments is:

$$\begin{aligned}
T_b &= t_b p_e M_e^{HF} - \left(t_p - \tilde{t}_p\right)p_e M_e^{FH} \\
&= \frac{t_b}{1+t_b}(1-\gamma)\,(1-\bar{j})\,\eta D(p_c)Y - \frac{t_p-\tilde{t}_p}{1+\tilde{t}_p}(1-\gamma)\,\bar{j}\eta D^*(p_c^*)Y^* \\
&= \frac{t_b}{1+t_b}\left[(1-\gamma)\,(1-\bar{j})\,\eta D(p_c)Y - (1+t_b)\,(1-\gamma)\,\bar{j}\eta D^*(p_c^*)Y^*\right]
\end{aligned}$$

Adding in tax revenue, income in $\mathcal{H}$ is given by:

$$\begin{aligned}
Y &= L + (1-\beta+t_e)\,R_e \\
&\quad + \left(\frac{t_p}{1+t_p}\left[(1-\gamma)\,\bar{j}\eta + (1-\eta)\right] + \frac{t_b}{1+t_b}(1-\gamma)\,(1-\bar{j})\,\eta\right)D(p_c)Y \\
&\quad + \left(\frac{t_p-t_b}{1+t_p}\right)(1-\gamma)\,\bar{j}\eta D^*(p_c^*)Y^*. \qquad (43)
\end{aligned}$$

### 3.3.3 Equilibrium with Partial BTA's

Equilibrium consists of a price $p_e$ that equates world supply and demand for energy. Valuing both at world prices, we have:

$$(1+t_e)^{-\beta/(1-\beta)}\, p_e^{1/(1-\beta)} E + p_e^{1/(1-\beta^*)} E^* = p_e C_e^W + p_e M_e^W.$$

The value of world demand for energy by households is:

$$p_e C_e^W = (1-\eta)\left(\frac{1}{1+t_p} D(p_c) Y + D^*(p_c^*) Y^*\right), \tag{44}$$

and the value of world demand for energy by firms is:

$$\begin{aligned}
p_e M_e^W &= p_e M_e^{HW} + p_e M_e^{FW} \\
&= (1-\gamma)\,\eta \left[\left(\frac{\bar{j}}{1+t_p} + \frac{1-\bar{j}}{1+t_b}\right) D(p_c) Y + \left(\frac{\bar{j}}{1+\tilde{t}_p} + 1 - \bar{j}\right) D^*(p_c^*) Y^*\right] \\
&= (1-\gamma)\,\eta \left(\frac{1+t_b}{1+t_p}\bar{j} + 1 - \bar{j}\right)\left(\frac{1}{1+t_b} D(p_c) Y + D^*(p_c^*) Y^*\right). \qquad (45)
\end{aligned}$$

with $p_c$ given by (40), $p_c^*$ given by (41), $\bar{j}$ given by (39), $Y$ given by (43), and $Y^*$ given by (42).

## 3.4 Optimal Partial BTA's

We now consider what level of border tax adjustment is optimal for $\mathcal{H}$ given some goal $\hat{Q}_e^W = G < 1$ for reduction of global emissions. The two policy levers are the production tax $t_p' \geq 0$ and the border tax adjustment $t_b' \in [0, t_p']$. For now we ignore extraction taxes, setting $t_e' = 0$. We will work out the answer in the form of changes from a baseline that can be calibrated to data. For simplicity, we assume no taxes in this baseline. We also set $\beta = \beta^*$.

### 3.4.1 Properties of the Baseline

In the baseline, income in $\mathcal{H}$ is:

$$Y = L + rE = L + (1-\beta) p_e Q_e$$

so that, dividing both sides by $Y$:

$$1 - \pi_L = (1-\beta)\pi_e,$$

where

$$\pi_e = \frac{p_e Q_e}{Y}$$

is the share of the energy or extraction sector in GDP. The same holds in $\mathcal{F}$:

$$1 - \pi_L^* = (1-\beta)\pi_e^*.$$

The cutoff manufacturing variety in the baseline is:

$$\bar{j} = \frac{A^\theta}{A^\theta + A^{*\theta}}.$$

In data, the analog of $\bar{j}$ is the fraction of spending on manufactures devoted to those produced in $\mathcal{H}$. We also need an expression for how $\bar{j}$ evolves from its baseline when we introduce carbon taxes. From (39) we get the convenient expression:

$$\bar{j}'(t_b', t_p') = \frac{\bar{j}\left(\frac{1+t_p'}{1+t_b'}\right)^{-\theta(1-\gamma)}}{\bar{j}\left(\frac{1+t_p'}{1+t_b'}\right)^{-\theta(1-\gamma)} + 1 - \bar{j}}. \tag{46}$$

Our definition of the consumption share is now:

$$\pi_c = \frac{p_c C_c}{Y} = D(p_c)$$

in $\mathcal{H}$ and

$$\pi_c^* = D^*(p_c^*)$$

in $\mathcal{F}$, i.e. 1 minus the share of income spent on the $l$-good (reducing to Simple Model for the case of $\eta = 0$). Applying (29) we have:

$$\pi_c = \frac{\alpha p_c^{-(\sigma-1)}}{\alpha p_c^{-(\sigma-1)} + 1 - \alpha}.$$

When we introduced carbon taxes, this share in $H$ evolves to:

$$\pi_c' = \frac{\pi_c \hat{p}_c^{-(\sigma-1)}}{\pi_c \hat{p}_c^{-(\sigma-1)} + 1 - \pi_c}. \tag{47}$$

In $F$ it evolves to:

$$\pi_c^{*\prime} = \frac{\pi_c^*\left(\hat{p}_c^*\right)^{-(\sigma-1)}}{\pi_c^*\left(\hat{p}_c^*\right)^{-(\sigma-1)} + 1 - \pi_c^*}. \tag{48}$$

Carbon taxes do not show up directly in these expressions because $\hat{p}_c$ and $\hat{p}_c^*$ already embody the effect of the tax on the spending share.

### 3.4.2 Energy Supply

Recall that $G$ represents the goal for a reduction in global emissions. Because there is no extraction tax, the supply side is simply:

$$G = \omega_e \hat{p}_e^{\beta/(1-\beta)} + (1-\omega_e)\,\hat{p}_e^{\beta/(1-\beta)}, \tag{49}$$

where

$$\omega_e = \frac{\pi_e Y}{\pi_e Y + \pi_e^* Y^*} = \frac{p_e Q_e}{p_e Q_e^W}.$$

The global emission goal $G$ nails down the change in the energy price:

$$\hat{p}_e = G^{(1-\beta)/\beta}, \tag{50}$$

which we can henceforth take as given.

### 3.4.3 Prices

With the change in the energy price nailed down, we can solve for the other prices. From (40) we have in $\mathcal{H}$:

$$\hat{p}_c = \hat{p}_m^{\eta} \hat{p}_e^{1-\eta} (1+t_p')^{1-\eta}$$

with $\hat{p}_m$ given by

$$\hat{p}_m = \hat{p}_e^{1-\gamma} \left( \bar{j}\left(1+t_p'\right)^{-\theta(1-\gamma)} + (1-\bar{j})\left(1+t_b'\right)^{-\theta(1-\gamma)} \right)^{-1/\theta}.$$

Combining these pieces, the change in the price of the $c$-good in $\mathcal{H}$ is:

$$\hat{p}_c(t_b', t_p') = \hat{p}_e^{1-\gamma\eta} (1+t_p')^{1-\eta} \left( \bar{j}\left(1+t_p'\right)^{-\theta(1-\gamma)} + (1-\bar{j})\left(1+t_b'\right)^{-\theta(1-\gamma)} \right)^{-\eta/\theta}. \tag{51}$$

The change in the price of the $c$-good in $\mathcal{F}$ reflects the fact that neither energy used in final consumption nor manufactures produced in $\mathcal{F}$ for domestic consumption are taxed:

$$\hat{p}_c^*(t_b', t_p') = \hat{p}_e^{1-\gamma\eta} \left( \bar{j}\left(\frac{1+t_p'}{1+t_b'}\right)^{-\theta(1-\gamma)} + (1-\bar{j}) \right)^{-\eta/\theta}. \tag{52}$$

We can substitute (51) into (47) to get:

$$\pi_c'(t_b', t_p') = \frac{\pi_c \hat{p}_c(t_b', t_p')^{-(\sigma-1)}}{\pi_c \hat{p}_c(t_b', t_p')^{-(\sigma-1)} + 1 - \pi_c}, \tag{53}$$

and (52) into (48) to get:

$$\pi_c^{*\prime}(t_b', t_p') = \frac{\pi_c^* \hat{p}_c^*(t_b', t_p')^{-(\sigma-1)}}{\pi_c^* \hat{p}_c^*(t_b', t_p')^{-(\sigma-1)} + 1 - \pi_c^*}. \tag{54}$$

#### 3.4.4 Income

For the change in $\mathcal{H}$'s income, we can rewrite (43) as:

$$\begin{aligned}\hat{Y} &= \pi_L + (1-\beta)\,\pi_e \hat{p}_e^{1/(1-\beta)} \\ &\quad + \left(\frac{t_p'}{1+t_p'}\left((1-\gamma)\,\bar{j}'\eta + (1-\eta)\right) + \frac{t_b'}{1+t_b'}\,(1-\gamma)\,(1-\bar{j}')\,\eta\right)\pi_c'\hat{Y} \\ &\quad + \left(\frac{t_p'-t_b'}{1+t_p'}\right)(1-\gamma)\,\bar{j}'\eta\pi_c^{*\prime}\frac{Y^*}{Y}\hat{Y}^*.\end{aligned}$$

Hence:

$$\hat{Y}(t_b', t_p') = \frac{\pi_L + (1-\beta)\,\pi_e \hat{p}_e^{1/(1-\beta)} + \left(\frac{t_p'-t_b'}{1+t_p'}\right)(1-\gamma)\,\bar{j}'\eta\pi_c^{*\prime}\frac{Y^*}{Y}\hat{Y}^*}{1 - \left(\frac{t_p'}{1+t_p'}\left((1-\gamma)\,\bar{j}'\eta + (1-\eta)\right) + \frac{t_b'}{1+t_b'}\,(1-\gamma)\,(1-\bar{j}')\,\eta\right)\pi_c'}, \tag{55}$$

where $\bar{j}' = \bar{j}'(t_b', t_p')$ given by (46), $\pi_c' = \pi_c'(t_b', t_p')$ by (53), $\pi_c^{*\prime} = \pi_c^{*\prime}(t_b', t_p')$ by (54), and the change in $\mathcal{F}$'s income:

$$\hat{Y}^* = \pi_L^* + (1-\beta)\,\pi_e^* \hat{p}_e^{1/(1-\beta)}. \tag{56}$$

Everywhere we evaluate $\hat{p}_e$ using (51).

#### 3.4.5 Demand

Combining (44) and (45), with taxes world spending on energy becomes:

$$\begin{aligned}X_e^{W\prime} &= p_e' C_e^{W\prime} + p_e' M_e^{W\prime} = (1-\eta)\left(\frac{1}{1+t_p'}D(p_c')Y' + D^*(p_c^{*\prime})Y^{*\prime}\right) \\ &\quad + (1-\gamma)\,\eta\left(\frac{1+t_b'}{1+t_p'}\bar{j}' + 1 - \bar{j}'\right)\left(\frac{1}{1+t_b'}D(p_c')Y' + D^*(p_c^{*\prime})Y^{*\prime}\right) \\ &= \left(\frac{1-\eta}{1+t_p'} + \left(\frac{1-\gamma}{1+t_b'}\right)\eta\left(\frac{1+t_b'}{1+t_p'}\bar{j}' + 1 - \bar{j}'\right)\right)D(p_c')Y' \\ &\quad + \left((1-\eta) + (1-\gamma)\,\eta\left(\frac{1+t_b'}{1+t_p'}\bar{j}' + 1 - \bar{j}'\right)\right)D^*(p_c^{*\prime})Y^{*\prime}.\end{aligned}$$

Thus, the change in world energy spending can be written as:

$$\begin{aligned}\hat{X}_e^W(t_b', t_p') &= \left(\frac{1-\eta}{1+t_p'} + \left(\frac{1-\gamma}{1+t_b'}\right)\eta\left(\frac{1+t_b'}{1+t_p'}\bar{j}' + 1 - \bar{j}'\right)\right)\frac{\omega_e}{\pi_e}\pi_c'\hat{Y} \\ &\quad + \left((1-\eta) + (1-\gamma)\,\eta\left(\frac{1+t_b'}{1+t_p'}\bar{j}' + 1 - \bar{j}'\right)\right)\frac{\omega_e^*}{\pi_e^*}\pi_c^{*\prime}\hat{Y}^* \quad (57)\end{aligned}$$

where $\bar{j}' = \bar{j}'(t_b', t_p')$ is given by (46), $\pi_c' = \pi_c'(t_b', t_p')$ by (53), $\pi_c^{*\prime} = \pi_c^{*\prime}(t_b', t_p')$ by (54), $\hat{Y} = \hat{Y}(t_b', t_p')$ by (55), $\hat{Y}^*$ by (56), and all evaluated at (52).

### 3.4.6 Welfare

Welfare in $\mathcal{H}$ is:

$$W = \frac{Y}{p},$$

where the aggregate price index is from (17). We want to select the tax pair that maximizes:

$$\hat{W} = \frac{W'}{W} = \frac{\hat{Y}}{\hat{p}}.$$

We already have an expression (55) for $\hat{Y}$. From (17), the change in the aggregate price level can be written as:

$$\hat{p}(t_b', t_p') = \left(\pi_c \hat{p}_c(t_b', t_p')^{-(\sigma-1)} + (1-\pi_c)\right)^{-1/(\sigma-1)}, \quad (58)$$

with $\hat{p}_c(t_b', t_p')$ given by (51).

As a result of these substitutions, we can express the change in welfare as a function of just the two policy instruments:

$$\hat{W}(t_b', t_p') = \frac{\hat{Y}(t_b', t_p')}{\hat{p}(t_b', t_p')}.$$

The constraint is that global consumption of energy decline by the same amount as supply:

$$g(t_b', t_p') = \frac{\hat{X}_e^W(t_b', t_p')}{\hat{p}_e} = G,$$

where $\hat{p}_e$ is given by (52). Our problem reduces to maximizing the Lagrangian:

$$\mathcal{L} = \hat{W}(t_b', t_p') + \lambda\left[G - g(t_b', t_p')\right],$$

where $\lambda$ is the lagrange multiplier.

### 3.4.7 Special Cases

To get some insight into this problem we consider several extreme cases. We focus on the determinants of the change in world demand for energy, as given by (57).

**Only Direct Consumption of Energy** The simplest case is $\eta = 0$ so that manufactures are not consumed. Hence, energy is consumed only directly by households, as in Simple Model. In this case (57) simplifies to:

$$\hat{X}_e^W = \frac{1}{1+t_p'}\frac{\omega_e}{\pi_e}\pi_c'\hat{Y} + \frac{\omega_e^*}{\pi_e^*}\pi_c^{*\prime}\hat{Y}^*.$$

How is this expression altered by a shift in demand? Inspection of (47) shows that $\pi_c'$ is increasing in $\pi_c$ and of (48) that $\pi_c^{*\prime}$ is increasing in $\pi_c^*$. Thus, a shift in preference for energy from $\mathcal{F}$ to $\mathcal{H}$ (an increase in $\pi_c$ together with a decrease in $\pi_c^*$) will increase the tax base in $\mathcal{H}$ so that a *lower* production tax rate in $\mathcal{H}$ will acheive the same reduction in world energy consumption, given $\hat{Y}$.

Taking account of the change in income in $\mathcal{H}$ strengthens this effect. Note that (55) reduces to:

$$\hat{Y} = \frac{\pi_L + (1-\beta)\,\pi_e\hat{p}_e^{1/(1-\beta)}}{1 - \frac{t_p'}{1+t_p'}\pi_c'},$$

which is increasing in $\pi_c'$.

Another case delivering essentially the same result is if $\eta > 0$ but $\gamma = 1$. In this case the manufacturing sector is active, but only labor is used to produce manufactures. Then (57) simplifies to:

$$\hat{X}_e^W = (1-\eta)\left(\frac{1}{1+t_p'}\frac{\omega_e}{\pi_e}\pi_c'\hat{Y} + \frac{\omega_e^*}{\pi_e^*}\pi_c^{*\prime}\hat{Y}^*\right).$$

The only difference is the multiplicative term, which represents direct spending on energy as a share of spending on the $c$-good. As above, income in $\mathcal{H}$ rises with an increase in $\pi_c'$, now according to:

$$\hat{Y} = \frac{\pi_L + (1-\beta)\,\pi_e\hat{p}_e^{1/(1-\beta)}}{1 - \frac{t_p'}{1+t_p'}(1-\eta)\,\pi_c'}.$$

**Only Indirect Consumption of Energy** Now consider $\eta = 1$, so that there is no direct spending on energy. Energy is demanded only indirectly as it is embodied in manufactures. In this case (57) reduces to:

$$\hat{X}_e^W = (1-\gamma)\left(\frac{1+t_b'}{1+t_p'}\bar{j}' + 1 - \bar{j}'\right)\left(\frac{1}{1+t_b'}\frac{\omega_e}{\pi_e}\pi_c'\hat{Y} + \frac{\omega_e^*}{\pi_e^*}\pi_c^{*\prime}\hat{Y}^*\right).$$

In this case, consider an improvement in $\mathcal{H}$'s comparative advantage in manufactures. In particular, raising $\bar{j}$ will, via (46), lead to a rise in $\bar{j}'$. The resulting increase in the tax base will, given $t_b'$, allow for a *lower* production tax rate to acheive the same reduction in world demand for energy. In this case we can ignore $\hat{Y}$ and $\hat{Y}^*$ since they don't vary with $\bar{j}$.

We can now return to the analysis of a shift in demand, imposing initial symmetry across countries to keep things simple ($\pi_e = \pi_e^*$ and $\omega_e = \omega_e^*$, so that $Y = Y^*$):

$$\hat{X}_e^W = (1-\gamma)\left(\frac{1}{1+t_p'}\bar{j}' + 1 - \bar{j}'\right)\left(\pi_c'\hat{Y} + \pi_c^{*\prime}\hat{Y}^*\right),$$

where we have assumed no border adjustments so that $t_b' = 0$. In this case, it appears that a shift in preference for energy from $\mathcal{F}$ to $\mathcal{H}$ will raise overall energy demand, requiring a *higher* production tax rate to reduce energy demand by the same amount. To confirm this intuition, consider the change in income (imposing $Y = Y^*$):

$$\hat{Y} = \frac{\pi_L + (1-\beta)\,\pi_e\hat{p}_e^{1/(1-\beta)} + (1-\gamma)\,\frac{t_p'}{1+t_p'}\bar{j}'\pi_c^{*\prime}\hat{Y}^*}{1-(1-\gamma)\,\frac{t_p'}{1+t_p'}\bar{j}'\pi_c'}.$$

Consider how the income change is affected by changes in $\pi_c'$ and $\pi_c^{*\prime}$:

$$\frac{\partial\hat{Y}}{\partial\pi_c'} = \frac{(1-\gamma)\frac{t_p'}{1+t_p'}\bar{j}'}{\left(1-(1-\gamma)\frac{t_p'}{1+t_p'}\bar{j}'\pi_c'\right)^2}$$

$$\frac{\partial\hat{Y}}{\partial\pi_c^{*\prime}} = \frac{(1-\gamma)\frac{t_p'}{1+t_p'}\bar{j}'\hat{Y}^*}{1-(1-\gamma)\frac{t_p'}{1+t_p'}\bar{j}'\pi_c'}$$

Noting that $1-(1-\gamma)\frac{t_p'}{1+t_p'}\bar{j}'\pi_c' < 1$ and $\hat{Y}^* < 1$, it follows that $\frac{\partial\hat{Y}}{\partial\pi_c'} > \frac{\partial\hat{Y}}{\partial\pi_c^{*\prime}}$. Thus, a shift in energy preference from $\mathcal{F}$ to $\mathcal{H}$ tends to increase $\hat{Y}$.

Taking into account that $\hat{Y}^*$ is invariable to energy preference, it must then be that this shift increases $\left(\pi_c' \hat{Y} + \pi_c^{*\prime} \hat{Y}^*\right)$. To maintain the same goal for emissions reduction, $\left(\frac{1}{1+t_p'} \bar{j}' + 1 - \bar{j}'\right)$ must fall to compensate. How should $t_p'$ change to achieve this decrease? To determine this, substitute (46) into the expression. This yields

$$\begin{aligned}\left(\frac{1}{1+t_p'} \bar{j}' + 1 - \bar{j}'\right) &= 1 - \frac{t_p'}{1+t_p'} \left(\frac{\bar{j}(1+t_p')^{-\theta(1-\gamma)}}{\bar{j}(1+t_p')^{-\theta(1-\gamma)} + 1 - \bar{j}}\right) \\ &= 1 - \frac{t_p'}{(1+t_p')\left(1 + \bar{j}(1-\bar{j})(1+t_p')^{\theta(1-\gamma)}\right)}\end{aligned}$$

Now take a derivative with respect to $t_p'$ to get (after rearranging):

$$\bar{j}(1-\bar{j})(1+t_p')^{\theta(1-\gamma)} \left(\theta(1-\gamma)t_p' - 1\right) - 1$$

When this derivative is positive, a *decrease* in $t_p'$ is needed to balance the increase in the consumption term. When it is negative, an *increase* in $t_p'$ is needed. An exact solution must be determined numerically after asserting values for parameters, but immediately one can see that if $t_p' \leq \frac{1}{\theta(1-\gamma)}$, which for reasonable values of $\theta$ and $\gamma$ is true for tax rates of up to and greater than 100%, an *increase* in $t_p'$ is unequivocally needed to maintain the same level of emissions reduction as energy preference shifts from $\mathcal{F}$ to $\mathcal{H}$.

## 3.5 Leakage Under Partial BTA's

Imposition of carbon taxes typically results in production and consumption leakage; $\mathcal{F}$ increases production and consumption of energy, somewhat offsetting the reductions in $\mathcal{H}$. Here we derive formulas for leakage resulting from introducing partial BTA's, continuing to assume baseline taxes are zero.

We start by summarizing results from above that are the key ingredients for leakage calculations. There are six ways in which energy is used throughout the world: for production of manufactures in either country for delivery to either country and directly for consumption in either country.

After imposing partial BTA's, the value of the energy used in $\mathcal{H}$ to supply the domestic market is:

$$p_e' M_e^{HH\prime} = (1-\gamma)\, \bar{j}' \eta \frac{\pi_c' Y'}{1+t_p'}$$

while for the foreign market:

$$p_e' M_e^{FH\prime} = (1-\gamma)\,\bar{j}'\eta \frac{\pi_c^{*\prime} Y^{*\prime}}{1+\tilde{t}_p'}.$$

The value of the energy used in $\mathcal{F}$ to supply manufactures to $\mathcal{H}$ is:

$$p_e' M_e^{HF\prime} = (1-\gamma)\,(1-\bar{j}')\,\eta \frac{\pi_c' Y'}{1+t_b'},$$

while in producing for $\mathcal{F}$:

$$p_e' M_e^{FF\prime} = (1-\gamma)\,(1-\bar{j}')\,\eta \pi_c^{*\prime} Y^{*\prime}.$$

Direct demand for energy by households in $\mathcal{H}$ is:

$$p_e' C_e' = (1-\eta)\,\frac{\pi_c' Y'}{1+t_p'},$$

while in $\mathcal{F}$:

$$p_e' C_e^{*\prime} = (1-\eta)\,\pi_c^{*\prime} Y^{*\prime}.$$

We can exploit the equations above, together with (46), to derive expressions for the changes in the quantity of energy used for each of these six purposes:

$$\hat{M}_e^{HH} = \left(\frac{\left(1+\tilde{t}_p'\right)^{-\theta(1-\gamma)}}{\bar{j}\left(1+\tilde{t}_p'\right)^{-\theta(1-\gamma)} + 1 - \bar{j}}\right)\frac{\hat{\pi}_c \hat{Y}}{\left(1+t_p'\right)\hat{p}_e}, \tag{59}$$

$$\hat{M}_e^{FH} = \left(\frac{\left(1+\tilde{t}_p'\right)^{-\theta(1-\gamma)}}{\bar{j}\left(1+\tilde{t}_p'\right)^{-\theta(1-\gamma)} + 1 - \bar{j}}\right)\frac{\hat{\pi}_c^* \hat{Y}^*}{\left(1+\tilde{t}_p'\right)\hat{p}_e}, \tag{60}$$

$$\hat{C}_e = \frac{\hat{\pi}_c \hat{Y}}{\left(1+t_p'\right)\hat{p}_e}, \tag{61}$$

$$\hat{M}_e^{HF} = \left(\frac{1}{\bar{j}\left(1+\tilde{t}_p'\right)^{-\theta(1-\gamma)} + 1 - \bar{j}}\right)\frac{\hat{\pi}_c \hat{Y}}{\left(1+t_b'\right)\hat{p}_e}, \tag{62}$$

$$\hat{M}_e^{FF} = \left(\frac{1}{\bar{j}\left(1+\tilde{t}_p'\right)^{-\theta(1-\gamma)} + 1 - \bar{j}}\right)\frac{\hat{\pi}_c^* \hat{Y}^*}{\hat{p}_e}, \tag{63}$$

and

$$\hat{C}_e^* = \frac{\hat{\pi}_c^* \hat{Y}^*}{\hat{p}_e}. \tag{64}$$

We now turn to formulas for production leakage and consumption leakage, in turn.

### 3.5.1 Production Leakage

Consider production leakage first, where production includes the energy used in manufacturing together with direct consumption of energy (which we interpret as household production). Production leakage represents the increase in energy used in production in the $\mathcal{F}$ divided by the decline in energy used in production in $\mathcal{H}$:

$$\begin{aligned} l_P &= \frac{\left(M_e^{HF\prime} + M_e^{FF\prime} + C_e^{*\prime}\right) - \left(M_e^{HF} + M_e^{FF} + C_e^*\right)}{\left(M_e^{HH} + M_e^{FH} + C_e\right) - \left(M_e^{HH\prime} + M_e^{FH\prime} + C_e^\prime\right)} \\ &= \frac{p_e M_e^{HF}\left(\hat{M}_e^{HF} - 1\right) + p_e M_e^{FF}\left(\hat{M}_e^{FF} - 1\right) + p_e C_e^*\left(\hat{C}_e^* - 1\right)}{p_e M_e^{HH}\left(1 - \hat{M}_e^{HH}\right) + p_e M_e^{FH}\left(1 - \hat{M}_e^{FH}\right) + p_e C_e\left(1 - \hat{C}_e\right)}. \end{aligned}$$

Plugging in the expressions for the six forms of spending on energy above:

$$l_P = \frac{(1-\gamma)\,\eta\,(1-\bar{j})\left[\pi_c Y\left(\hat{M}_e^{HF} - 1\right) + \pi_c^* Y^*\left(\hat{M}_e^{FF} - 1\right)\right] + (1-\eta)\,\pi_c^* Y^*\left(\hat{C}_e^* - 1\right)}{(1-\gamma)\,\eta\bar{j}\left[\pi_c Y\left(1 - \hat{M}_e^{HH}\right) + \pi_c^* Y^*\left(1 - \hat{M}_e^{FH}\right)\right] + (1-\eta)\,\pi_c Y\left(1 - \hat{C}_e\right)}.$$

Defining the $\mathcal{H}$'s share of world spending on the $c$-good by:

$$\omega_c = \frac{\pi_c Y}{\pi_c Y + \pi_c^* Y^*},$$

we can write:

$$l_P = \frac{(1-\gamma)\,\eta\,(1-\bar{j})\left[\left(\omega_c \hat{M}_e^{HF} + (1-\omega_c)\,\hat{M}_e^{FF}\right) - 1\right] + (1-\eta)\,(1-\omega_c)\,\pi_c^* Y^*\left(\hat{C}_e^* - 1\right)}{(1-\gamma)\,\eta\bar{j}\left[1 - \left(\omega_c \hat{M}_e^{HH} + (1-\omega_c)\,\hat{M}_e^{FH}\right)\right] + (1-\eta)\,\omega_c\left(1 - \hat{C}_e\right)}$$

Plugging in equation (59) to (64) yields our formula for production leakage.

### 3.5.2 Consumption Leakage

Now consider consumption leakage, where consumption includes both the energy consumed directly by households as well as the energy embodied in consumption of manufactures. This form of leakage is simply a rearrangement of the terms in the formula for production leakage:

$$l_C = \frac{\left(M_e^{FH\prime} + M_e^{FF\prime} + C_e^{*\prime}\right) - \left(M_e^{FH} + M_e^{FF} + C_e^{*}\right)}{\left(M_e^{HH} + M_e^{HF} + C_e\right) - \left(M_e^{HH\prime} + M_e^{HF\prime} + C_e^{\prime}\right)}.$$

Plugging in the expressions for the six forms of spending on energy above:

$$l_C = \frac{(1-\gamma)\,\eta\pi_c^* Y^* \left[\bar{j}\left(\hat{M}_e^{FH} - 1\right) + (1-\bar{j})\left(\hat{M}_e^{FF} - 1\right)\right] + (1-\eta)\,\pi_c^* Y^* \left(\hat{C}_e^* - 1\right)}{(1-\gamma)\,\eta\pi_c Y \left[\bar{j}\left(1 - \hat{M}_e^{HH}\right) + (1-\bar{j})\left(1 - \hat{M}_e^{HF}\right)\right] + (1-\eta)\,\pi_c Y \left(1 - \hat{C}_e\right)}$$

or

$$l_C = \frac{(1-\gamma)\,\eta\,(1-\omega_c)\left[\left(\bar{j}\hat{M}_e^{FH} + (1-\bar{j})\,\hat{M}_e^{FF}\right) - 1\right] + (1-\eta)\,(1-\omega_c)\left(\hat{C}_e^* - 1\right)}{(1-\gamma)\,\eta\omega_c\left[1 - \left(\bar{j}\hat{M}_e^{HH} + (1-\bar{j})\,\hat{M}_e^{HF}\right)\right] + (1-\eta)\,\omega_c\left(1 - \hat{C}_e\right)}$$

Plugging in equation (59) to (64) yields our formula for consumption leakage.

### 3.5.3 A Modified Leakage Formula

Suppose we redefine leakage to be the increase in emissions abroad relative to the decline in *global* emissions. This definition has the convenient property that the denominator is the same for production leakage, consumption leakage, or even extraction leakage. Another advantage is that the denominator is always positive given a set of taxes that reduce global emissions.

Consider this formulation, which we denote by $\tilde{l}$, as it relates to production leakage:

$$\begin{aligned}
\tilde{l}_P &= \frac{p_e M_e^{HF}\left(\hat{M}_e^{HF} - 1\right) + p_e M_e^{FF}\left(\hat{M}_e^{FF} - 1\right) + p_e C_e^*\left(\hat{C}_e^* - 1\right)}{p_e Q_e^W (1-G)} \\
&= \frac{(1-\gamma)\,\eta\,(1-\bar{j})\left[\pi_c Y\left(\hat{M}_e^{HF} - 1\right) + \pi_c^* Y^*\left(\hat{M}_e^{FF} - 1\right)\right] + (1-\eta)\,\pi_c^* Y^*\left(\hat{C}_e^* - 1\right)}{p_e C_e^W (1-G)} \\
&= \frac{(1-\gamma)\,\eta\,(1-\bar{j})\left[\left(\omega_c\hat{M}_e^{HF} + (1-\omega_c)\,\hat{M}_e^{FF}\right) - 1\right] + (1-\eta)\,(1-\omega_c)\,\pi_c^* Y^*\left(\hat{C}_e^* - 1\right)}{(1-\eta\gamma)\,(1-G)}.
\end{aligned}$$

For consumption leakage:

$$\tilde{l}_C = \frac{(1-\gamma)\,\eta\,(1-\omega_c)\left[\left(\bar{j}\hat{M}_e^{FH} + (1-\bar{j})\,\hat{M}_e^{FF}\right) - 1\right] + (1-\eta)\,(1-\omega_c)\left(\hat{C}_e^{*} - 1\right)}{(1-\eta\gamma)\,(1-G)}$$

Modified leakage is related to the standard leakage expression $l$ via:

$$\tilde{l} = \frac{l}{1-l}, \tag{65}$$

or looked at the other way around:

$$l = \frac{\tilde{l}}{1+\tilde{l}}.$$

We now turn to the value of these various leakage measures in special cases of the model.

### 3.5.4 Analysis of Special Cases

Consider the special case of $\eta = 1$ so that energy is consumed only indirectly. Production leakage simplifies to:

$$l_P = \left(\frac{1-\bar{j}}{\bar{j}}\right)\frac{\left(\omega_c\hat{M}_e^{HF} + (1-\omega_c)\,\hat{M}_e^{FF}\right) - 1}{1 - \left(\omega_c\hat{M}_e^{HH} + (1-\omega_c)\,\hat{M}_e^{FH}\right)},$$

and the modified formula is:

$$\tilde{l}_P = (1-\bar{j})\,\frac{\left(\omega_c\hat{M}_e^{HF} + (1-\omega_c)\,\hat{M}_e^{FF}\right) - 1}{1-G}.$$

Consumption leakage simplifies to:

$$l_C = \left(\frac{1-\omega_c}{\omega_c}\right)\frac{\left(\bar{j}\hat{M}_e^{FH} + (1-\bar{j})\,\hat{M}_e^{FF}\right) - 1}{1 - \left(\bar{j}\hat{M}_e^{HH} + (1-\bar{j})\,\hat{M}_e^{HF}\right)}$$

and

$$\tilde{l}_C = (1-\omega_c)\,\frac{\left(\bar{j}\hat{M}_e^{FH} + (1-\bar{j})\,\hat{M}_e^{FF}\right) - 1}{1-G}$$

**Pure Consumption Tax** Suppose we have a pure consumption tax, $t_b' = t_p' = \tilde{t}_c'$ and $\tilde{t}_p' = 0$. In this case (59) simplifies to:

$$\hat{M}_e^{HH} = \frac{\hat{X}_e^{HH}}{\left(1+\tilde{t}_c'\right)\hat{p}_e} = \frac{\hat{\pi}_c \hat{Y}}{\left(1+\tilde{t}_c'\right)\hat{p}_e},$$

(60) to:

$$\hat{M}_e^{FH} = \frac{\hat{X}_e^{FH}}{\hat{p}_e} = \frac{\hat{\pi}_c^* \hat{Y}^*}{\hat{p}_e},$$

(62) to:

$$\hat{M}_e^{HF} = \frac{\hat{X}_e^{HF}}{\left(1+\tilde{t}_c'\right)\hat{p}_e} = \frac{\hat{\pi}_c \hat{Y}}{\left(1+\tilde{t}_c'\right)\hat{p}_e},$$

and (63) to:

$$\hat{M}_e^{FF} = \frac{\hat{X}_e^{FF}}{\hat{p}_e} = \frac{\hat{\pi}_c^* \hat{Y}^*}{\hat{p}_e}.$$

Price changes are given by:

$$\hat{p}_c = \hat{p}_e^{1-\gamma}\left(\bar{j}\left(1+\tilde{t}_c'\right)^{-\theta(1-\gamma)} + (1-\bar{j})\left(1+\tilde{t}_c'\right)^{-\theta(1-\gamma)}\right)^{-1/\theta} = \hat{p}_e^{1-\gamma}\left(1+\tilde{t}_c'\right)^{(1-\gamma)}$$

and

$$\hat{p}_c^* = \hat{p}_e^{1-\gamma}.$$

The resulting changes in consumption shares are:

$$\hat{\pi}_c = \frac{\hat{p}_c^{-(\sigma-1)}}{\pi_c \hat{p}_c^{-(\sigma-1)} + 1 - \pi_c}$$

and

$$\hat{\pi}_c^* = \frac{\hat{p}_e^{-(\sigma-1)(1-\gamma)}}{\pi_c^* \hat{p}_e^{-(\sigma-1)(1-\gamma)} + 1 - \pi_c^*}.$$

Production leakage reduces to:

$$l_P = \left(\frac{1-\bar{j}}{\bar{j}}\right)\frac{\left(\omega_c \frac{\hat{\pi}_c \hat{Y}}{\left(1+\tilde{t}_c'\right)\hat{p}_e} + (1-\omega_c)\frac{\hat{\pi}_c^* \hat{Y}^*}{\hat{p}_e}\right) - 1}{1 - \left(\omega_c \frac{\hat{\pi}_c \hat{Y}}{\left(1+\tilde{t}_c'\right)\hat{p}_e} + (1-\omega_c)\frac{\hat{\pi}_c^* \hat{Y}^*}{\hat{p}_e}\right)} = -\left(\frac{1-\bar{j}}{\bar{j}}\right).$$

Noting that

$$\begin{aligned}
\omega_c \frac{\hat{\pi}_c \hat{Y}}{\left(1+\tilde{t}_c'\right)\hat{p}_e} + (1-\omega_c)\frac{\hat{\pi}_c^* \hat{Y}^*}{\hat{p}_e} &= \omega_c \hat{M}_e^{HW} + (1-\omega_c)\hat{M}_e^{FW} = \\
&= \frac{p_e M_e^{HW}}{p_e M_e^W}\hat{M}_e^{HW} + \frac{p_e M_e^{FW}}{p_e M_e^W}\hat{M}_e^{FW} \\
&= \frac{M_e^{HW'}}{M_e^W} + \frac{M_e^{FW'}}{M_e^W} \\
&= \frac{M_e^{W'}}{M_e^W} = G
\end{aligned}$$

modified production leakage is:

$$\tilde{l}_P = -\left(1-\bar{j}\right).$$

Of course this result also follows from (65). With a pure consumption tax, production leakage is always negative as $\mathcal{F}$ reduces its production of emissions along with $\mathcal{H}$. The magnitude (of the reduction in $\mathcal{F}$'s emissions) depends (negatively) on $\mathcal{H}$'s initial share in the global production of manufactures. We learn nothing new from the production leakage measure in this setting.

Consumption leakage is more relevant in this setting. Evaluating consumption leakage:

$$\begin{aligned}
l_C &= \left(\frac{1-\omega_c}{\omega_c}\right)\frac{\left(\bar{j}\frac{\hat{\pi}_c^*\hat{Y}^*}{\hat{p}_e} + \left(1-\bar{j}\right)\frac{\hat{\pi}_c^*\hat{Y}^*}{\hat{p}_e}\right)-1}{1-\left(\bar{j}\frac{\hat{\pi}_c\hat{Y}}{\left(1+\tilde{t}_c'\right)\hat{p}_e} + \left(1-\bar{j}\right)\frac{\hat{\pi}_c\hat{Y}}{\left(1+\tilde{t}_c'\right)\hat{p}_e}\right)} \\
&= \left(\frac{1-\omega_c}{\omega_c}\right)\frac{\frac{\hat{\pi}_c^*\hat{Y}^*}{\hat{p}_e}-1}{1-\frac{\hat{\pi}_c\hat{Y}}{\left(1+\tilde{t}_c'\right)\hat{p}_e}} \\
&= \left(\frac{1-\omega_c}{\omega_c}\right)\frac{\frac{\hat{p}_e^{-(\sigma-1)(1-\gamma)-1}}{\pi_c^*\hat{p}_e^{-(\sigma-1)(1-\gamma)}+1-\pi_c^*}\hat{Y}^*-1}{1-\frac{\left(\hat{p}_e\left(1+\tilde{t}_c'\right)\right)^{-(\sigma-1)(1-\gamma)-1}}{\pi_c\left(\hat{p}_e\left(1+\tilde{t}_c'\right)\right)^{-(\sigma-1)(1-\gamma)}+1-\pi_c}\hat{Y}},
\end{aligned}$$

In this setting, $\hat{p}_e < 1$ (as dictated by the global emissions reduction goal) and $\hat{p}_e\left(1+\tilde{t}_c'\right) > 1$ (as dictated by $\tilde{t}_c'$ being sufficient to acheive the global emissions goal through a reduction in demand for manufactures in $\mathcal{H}$). Its easier to see using the modified leakage formula:

$$\tilde{l}_C = \left(\frac{1-\omega_c}{1-G}\right)\left(\frac{\hat{p}_e^{-(\sigma-1)(1-\gamma)-1}}{\pi_c^* \hat{p}_e^{-(\sigma-1)(1-\gamma)} + 1 - \pi_c^*}\hat{Y}^* - 1\right).$$

Consumption leakage is determined by the energy price decline and the extent to which that decline leads $\mathcal{F}$ to substitute away from the $l$-good into manufactures. Consumption leakage is also proportional to $\mathcal{F}$'s initial share $1-\omega_c$ of global spending on manufactures.

Substituting in the energy price decline dictated by the global emission goal:

$$\tilde{l}_C = \left(\frac{1-\omega_c}{1-G}\right)\left(\frac{G^{-(\sigma-\sigma\gamma+\gamma)(1-\beta)/\beta}}{\pi_c^* G^{-(\sigma-1)(1-\gamma)(1-\beta)/\beta} + 1 - \pi_c^*}\left(\pi_L^* + (1-\beta)\,\pi_e^* G^{1/\beta}\right) - 1\right). \tag{66}$$

A nice feature of this leakage formula is that the tax rate itself doesn't enter, as it is subsumed in $G$. For a given $G$, the larger is $(\sigma-\sigma\gamma+\gamma)(1-\beta)/\beta$, the greater the consumption leakage resulting from a pure consumption tax. Consumption leakage is increasing in the demand elasticity $\sigma$ and decreasing in the extraction elasticity $\beta/(1-\beta)$. A smaller extraction elasticity implies a larger price decline to acheive the global goal, and a higher demand elasticity implies a greater increase in $\mathcal{F}$'s consumption of embodied energy for a given price decline.

**Pure Production Tax** Suppose we have a pure production tax, $t_b' = 0$ and $\tilde{t}_p' = t_p'$. In this case (59) simplifies to:

$$\hat{M}_e^{HH} = \left(\frac{\left(1+t_p'\right)^{-\theta(1-\gamma)-1}}{\bar{j}\left(1+t_p'\right)^{-\theta(1-\gamma)} + 1 - \bar{j}}\right)\frac{\hat{\pi}_c \hat{Y}}{\hat{p}_e},$$

(60) to:

$$\hat{M}_e^{FH} = \left(\frac{\left(1+t_p'\right)^{-\theta(1-\gamma)-1}}{\bar{j}\left(1+t_p'\right)^{-\theta(1-\gamma)} + 1 - \bar{j}}\right)\frac{\hat{\pi}_c^* \hat{Y}^*}{\hat{p}_e},$$

(62) to:

$$\hat{M}_e^{HF} = \left(\frac{1}{\bar{j}\left(1+t_p'\right)^{-\theta(1-\gamma)} + 1 - \bar{j}}\right)\frac{\hat{\pi}_c \hat{Y}}{\hat{p}_e},$$

and (63) to:

$$\hat{M}_e^{FF} = \left( \frac{1}{\bar{j}\left(1+t_p'\right)^{-\theta(1-\gamma)} + 1 - \bar{j}} \right) \frac{\hat{\pi}_c^* \hat{Y}^*}{\hat{p}_e}.$$

Price changes are given by:

$$\hat{p}_c = \hat{p}_c^* = \hat{p}_e^{1-\gamma} \left( \bar{j}\left(1+t_p'\right)^{-\theta(1-\gamma)} + (1-\bar{j}) \right)^{-1/\theta}.$$

$$\hat{p}_c^{-\theta} = \hat{p}_e^{-\theta(1-\gamma)} \left( \bar{j}\left(1+t_p'\right)^{-\theta(1-\gamma)} + (1-\bar{j}) \right)$$

In this setting (unlike the setting of the pure consumption tax) production leakage should be a useful measure. Modified production leakage is

$$\begin{aligned}
\tilde{l}_P &= \left( \frac{1-\bar{j}}{1-G} \right) \left( \left( \frac{1}{\bar{j}\left(1+t_p'\right)^{-\theta(1-\gamma)} + 1 - \bar{j}} \right) \left( \omega_c \frac{\hat{\pi}_c \hat{Y}}{\hat{p}_e} + (1-\omega_c) \frac{\hat{\pi}_c^* \hat{Y}^*}{\hat{p}_e} \right) - 1 \right) \\
&= \left( \frac{1-\bar{j}}{1-G} \right) \left( \frac{G^{-(1-\beta)/\beta}}{\bar{j}\left(1+t_p'\right)^{-\theta(1-\gamma)} + 1 - \bar{j}} \hat{X}_c^W - 1 \right).
\end{aligned}$$