Day 13 Notes

Wednesday, August 25, 2021 9:08 AM

Stat 5101

Probability

Reference: Degroot and Schervish "probability and statistics"

"outcomes": the "basic" set of possibilities of the experiment

Event: a set of outcomes, and a subset of the sample space.

$\binom{n}{k}$ "n choose k"

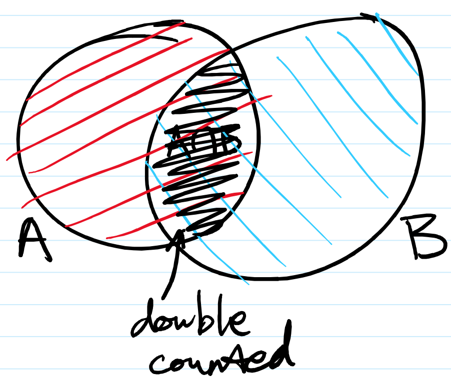
${}_nC_k$

→ the number of subsets of size k within a set of size n.

→ the number of "combinations" of k elements.

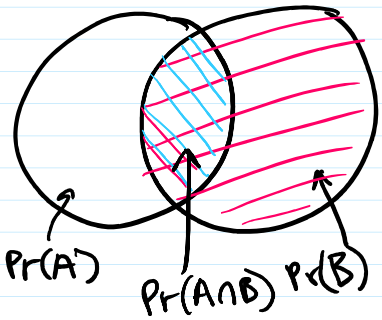
Property #2 should be:

$0 \leq Pr(A) \leq 1$



Conditional Probability

$Pr(A|B)$ probability of A given B



$$Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)}$$

Independent Events

Prob. of one occurring is not affected by whether the other occured.

A and B are independent iff $Pr(A|B) = Pr(A)$

or $Pr(A \cap B) = Pr(A) Pr(B)$

Bayes' Rule

$$Pr(A|B) = \frac{Pr(B|A) \cdot Pr(A)}{Pr(B)}$$

$Pr(A), Pr(B)$ prior probabilities

$Pr(A|B)$ posterior prob. of A

Random Variables

$X: S \to \mathbb{R}$

An R.V. assigns a real number to each outcome $x \in S$.

X the random variable

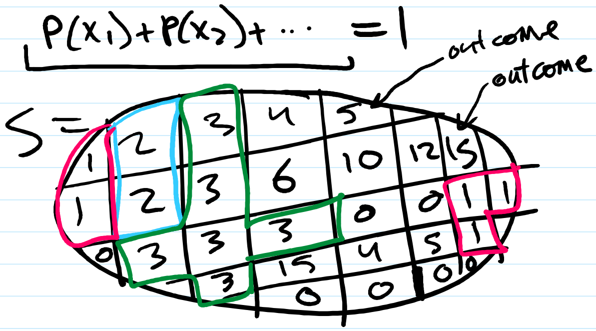
x the realized value

$Pr(x) \equiv Pr(X = x)$

Discrete : a finite or countably infinite set of possible values.

$P(X=x_i) \in [0,1] \ \forall i$

$\underbrace{P(x_1)+P(x_2)+\cdots}$ $=1$



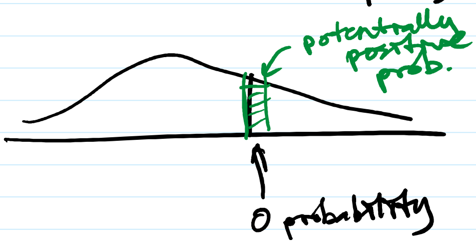
Continuous R.V.

For any $y \in \mathbb{R}$

$Pr(X=y)=0$ (bec. y is so "tiny")

We can only assign positive probability to intervals.

So $Pr(y-\varepsilon < X < y+\varepsilon) = 0.001$ (example)



PMF - discrete

Ex coins

x	$f(x)$
0	1/8
1	3/8
2	3/8
3	1/8

Continuous: PDF

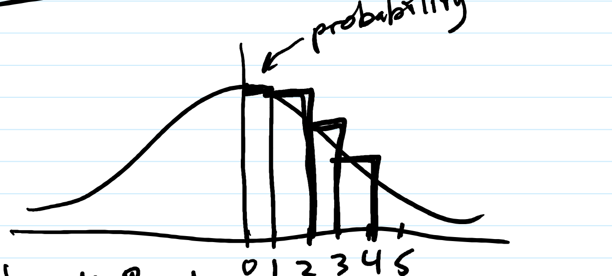
$f(x)$ is the pdf.

$$\int_a^b f(x)\,dx = \Pr(a \le X \le b)$$

and $$\underbrace{\int_{-\infty}^{\infty} f(x)\,dx = 1}$$

equiv to $P(x_1)+P(x_2)+\cdots=1$ in the discrete case

PDF value

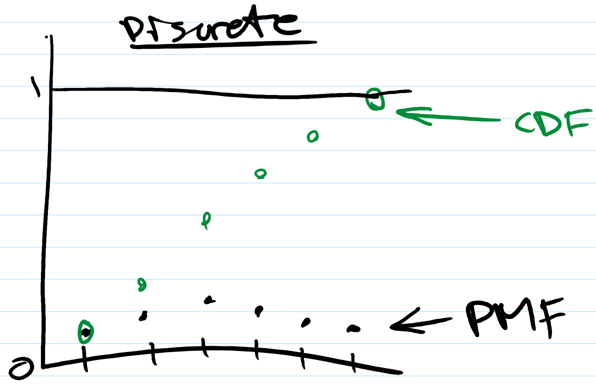


break $\mathbb{R}$ into intervals, each w/ constant value of the PDF within it.

$f(x) = 0.10$ for $x \in (1,2)$

then $\Pr(X \in (1,2)) = f(x)$

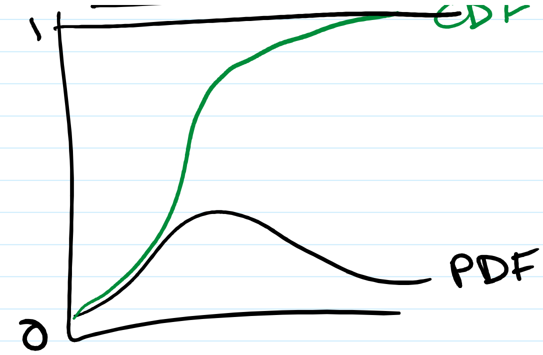
Discrete



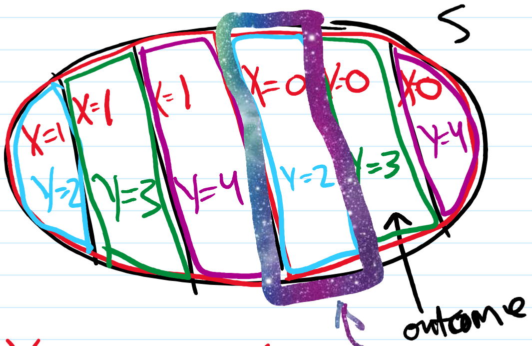
$CDF(-\infty) = 0$

$CDF(\infty) = 1$

Continuous



$$\frac{d}{dx}\text{CDF} = \text{PDF}$$



X partitions S into two events, X = 0 and X = 1

$Pr(\{X=0\} \cap \{Y=2\})$

Y partitions S into 3 events
Y = 2
Y = 3
Y = 4

What is the probability of $\{X=0\} \cap \{Y=2\}$?

If two RV's are independent

$$f(x,y) = f_X(x) f_Y(y)$$

continuous case:

$$f(x,y) = f_X(x) f_Y(y)$$

pdf $\qquad$ pdf $\qquad$ pdf

$$CDF = F(x,y) = Pr(X \le x \cap Y \le y)$$

Example: recode (H, T) as $(1, 0)$

Contingency table

coin Y \ die X	1	2	3	4	5	6	Total
H(1)	a	b	c	d	e	f	α
T(0)	g	h	i	j	k	l	β
Total	γ	δ	ε	ζ	θ	φ	ω

$f_Y(1), f_Y(0)$

Marginal dist. of Y.

$f_X(1), f_X(2), \ldots$

Marginal distribution of X.

"The distribution ..."

$f(1,1) = a$

$f(0,3) = i$

$f(0,6) = l$

$f(0,6) = 1$

"The distribution of X, ignoring Y"

$w = ?$

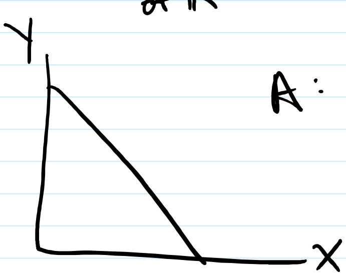
$a = ?$ $\frac{1}{12} = \frac{1}{2} \cdot \frac{1}{6}$

$a = b = \cdots = l = \frac{1}{12}$

Joint PDF

$$Pr((x,y) \in A) = \iint_A f(x,y)\,dA$$

region of $\mathbb{R}^2$



$A : \{(x,y) : x+y \leq 1\}$

Example

$$f(x,y) = \begin{cases} \frac{2}{3}(x+2y), & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find $P(X \leq \frac{1}{2} \text{ and } Y \leq \frac{1}{2})$

$$= \int_0^{0.5}\int_0^{0.5} \frac{2}{3}(x+2y)\,dx\,dy$$

$$= \frac{2}{3}\int_0^{0.5}\left[\int_0^{0.5}(x+2y)\,dx\right]dy$$

$$= \frac{2}{3}\int_0^{0.5}\left[\frac{x^2}{2}+2yx\right]_0^{0.5} dy$$

$$= \frac{2}{3}\int_0^{0.5}\left(\frac{0.25}{2}+y-0\right)dy$$

$$= \frac{2}{3}\left[0.125y+\frac{y^2}{2}\right]_0^{0.5}$$

$$= \frac{2}{3}\cdot\left[0.0625+0.125\right]$$

$$= \frac{2}{3}\cdot\left[\frac{1}{16}+\frac{1}{8}\right]$$

$$= \frac{2}{3}\cdot\frac{3}{16} = \frac{1}{8} \checkmark$$

Marginal distributions

$$f_X(x) = \int^{\infty} f(x,y)\,dy$$

$$f_X(x) = \underbrace{\int_{-\infty} f(x,y)\,dy}_{\text{summing } f(x,y) \text{ over all } y}$$

Conditional Distributions

$$f(y|x) = \frac{f(x,y)}{f_x(x)} \leftarrow \frac{Pr(X=x \cap Y=y)}{Pr(X=x)}$$

$$= Pr(Y=y \,|\, X=x)$$

X, Y are independent

iff $f(y|x) = f_y(y)$

and $f(x|y) = f_X(x)$

Ex] Ans: No, not independent.

If they were, then we'd need

$$f(1|1) = f(1|0)$$

$$f(0|1) = f(0|0)$$

Expectation

$E(X) = \sum_{x_i} x_i \underbrace{f(x_i)}_{\text{pmf}}$ (discrete)

$E(X) = \int_{-\infty}^{\infty} x \underbrace{f(x)}_{\text{pdf}} \, dx$ (contin.)

Properties

If $g(X)$ is a function of X then $E(g(X)) = \sum_i g(x_i) f(x_i)$

or $E[g(x)] = \int_{-\infty}^{\infty} g(x) f(x) \, dx$

$G = g(x)$ is its own r.v. - but this says we don't need to find the distribution of G in order to find its expectation.

- Linearity: $E(AX+b) = A\,E(X) + b$ for any constants A, b.

Moments

$E(X^k)$ is the k^{th} raw moment of X.

1st moment is the expected value/mean.

$E[(X-\mu)^k]$ is the k^{th}

$E[(X-\mu)^k]$ is the kth central moment of X.

Eg. $E[(X-\mu)^2]$ is the 2nd central moment and is known as the variance.

$$E[(X-\mu)^2] = E[X^2 - 2\mu X + \mu^2]$$

$$= E(X^2) - 2\mu E(X) + \mu^2$$

(but $\mu = E(X)$)

$$\boxed{Var(X) = \underbrace{E(X^2)}_{\text{2nd raw moment}} - \underbrace{E(X)^2}_{\text{1st raw moment}}}$$

Properties

HOMEWORK

$$\boxed{Var(aX+bY) = a^2 Var(X) + b^2 Var(Y) + 2ab\,Cov(X,Y)}$$

$$Cov(X,Y) = E[(X-EX)(Y-EY)]$$

$$\Rightarrow Var(X) = Cov(X,X)$$

Standard deviation: $sd(X)$ or σ_X

$$\sigma_X = \sqrt{Var(X)}$$

$$sd(aX) = |a|\,sd(X)$$

Example:

$\text{Var}(X) = ?$

Mean: $0.3 \cdot 3 + 0.4 \cdot 4 + 0.3 \cdot 5$

$= 0.09 + 1.6 + 1.5 = 4$

$\text{Var}(X) = (3-4)^2 \cdot 0.3 + (4-4)^2 \cdot 0.4$

$+ (5-4)^2 \cdot 0.3$

$= 0.6$

$\sigma_X = \sqrt{0.6} = 0.77$

Covariance

X, Y independent $\Longrightarrow \text{cov}(X,Y) = 0$

$\not\Longleftarrow$

covariance is bilinear:

linear in each argument

$\text{cov}(aX+bY, Z) =$

$a\,\text{cov}(X,Z) + b\,\text{cov}(Y,Z)$

$|\mathrm{cov}(X,Y)| \le \sigma_X \sigma_Y$ (a version of the Cauchy-Schwarz inequality)

Recall: $\mathrm{cov}(X,Y)$ is an <u>inner product</u> on the vector space of random variables.

<u>Correlation</u>

Normalized covariance

$$\mathrm{corr}(X,Y) = \frac{\mathrm{cov}(X,Y)}{\sigma_X \sigma_Y}$$

$$= \mathrm{cov}\left(\frac{X}{\sigma_X}, \frac{Y}{\sigma_Y}\right)$$

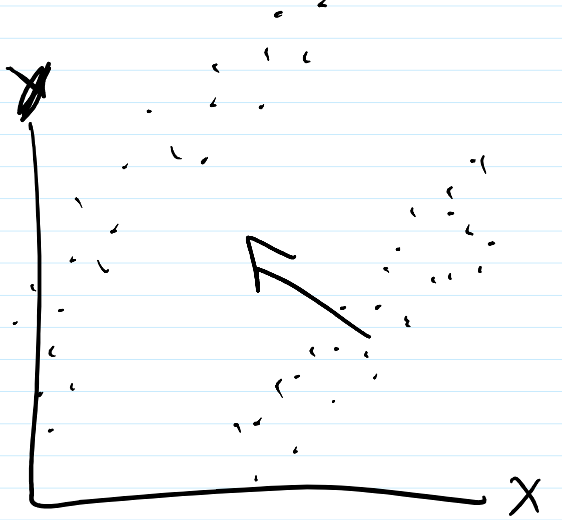
deviation scores

$\mathrm{corr} \in [-1,1]$

$\mathrm{corr}(X,Y) = 0 \equiv$ uncorrelated

$\not\Rightarrow$ independence

$$\mathrm{corr}(aX+b, Y) = \mathrm{corr}(X, Y)$$



these will have the same correlation

Consider $\mathrm{Corr}(aX, bY)$

$$= \frac{\mathrm{cov}(aX, bY)}{\mathrm{sd}(aX)\,\mathrm{sd}(bX)}$$

$$= \frac{ab\,\mathrm{cov}(X,Y)}{|a|\sigma_X |b|\sigma_Y}$$

$$= \pm \frac{\mathrm{cov}(X,Y)}{\sigma_X \sigma_Y} = \pm\mathrm{corr}(X,Y)$$

Therefore $\mathrm{corr}(aX, bY) = \mathrm{corr}(X,Y)$ if a,b are both positive

if a,b are
both positive
or negative

$$= -\operatorname{corr}(x,y)$$

if a and b
have opposite signs

So in particular:

$$\operatorname{Corr}(X,-Y) = -\operatorname{corr}(X,Y)$$

Conditional Expectation

$$E(Y|X) = \int_{-\infty}^{\infty} y \underbrace{f(y|x)}_{\text{replace pdf w/ the conditional pdf}} dy$$

In econometrics we care alot about the CEF (cond. exp. fn) which is $E(Y|X)$ viewed as a fn of X.

conditional variance

$$Var(Y|X) = E\left[(Y-\mu)^2 | X\right]$$

Law of Iterated Expectations (LIE)

$$E_X\left(E(Y|X)\right) = E(Y)$$