Lecture 06
Matrices

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Mathematics Review Course, Summer 2023
University of Minnesota
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Last Lecture Review

- Multi-variate Calculus:
  - Partial Derivatives
  - Total Differentiation
  - Gradients
  - Implicit Partial Derivatives

Review Assignment

1. Problem Set 05 solutions are available on Github.
2. Any issues or problems **You** would like to discuss?

Daily Icebreaker

- Attendance via prompt:
  - Name
  - Daily Icebreaker: Magic is real and you can crossover into the world of TV. What TV show would you want to live in?

Topic: Matrices

Motivation

- General background
  - A structured way to capture a series of related numbers.
  - Condense a system of equations.
  - This is the original conception of a 'spreadsheet' like you might find in Excel.
- Why do economists' care?
  - We primarily work with tabular datsets.
  - This is the main way with conceptualize information and manipulate it in practice.
- Application in this career
  - Matrices are throughout applied work as they are the foundation of both data storage, as well as how statistical software performs operations.

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Overview

1. Foundations of Matrices
2. Matrix Operators
3. Rank
4. Special Matrices
5. The Determinant
6. Trace
7. Matrix Decomposition
8. Positive and Negative Definite Matrices
9. Linear Independence
10. Chain Rule for Vectors

1. Foundations of Matrices

- Scalar: Single number. e.g., [5]
- Vector: Either a $k \times 1$ column or $1 \times k$ row.

$$a = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{pmatrix}$$

- Matrix: A $k \times r$ rectangular array.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kr} \end{bmatrix}$$

Vectors are both Direction and Magnitude

1. Foundations of Matrices

- In a matrix, a row is denoted i and the column j and is written as such in a cell: c_{ij}.
- To use linear operators on two or more matrices, you must be sure that the result will be a **conformable** matrix.
- For multiplication, it must be that $n = s$ (e.g., the columns of A must match the rows of B):

$$\text{If } A_{m\times n} \times B_{s\times p} = AB_{m\times p}$$

$$\begin{array}{cc} \text{A} & \text{B} \\ (2 \times 2) & (2 \times 3) \\ \begin{bmatrix} 7 & 1 \\ 3 & 4 \end{bmatrix} & \begin{bmatrix} 7 & 1 & 0 \\ 3 & 4 & -1 \end{bmatrix} \end{array}$$

2. Matrix Operators

- Transpose: Flip a matrix about the diagonal.
- Ex., given the a and A previously defined.

$$a^T = \begin{pmatrix} a_1 & a_2 & \cdots & a_k \end{pmatrix}$$

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{k1} \\ a_{12} & a_{22} & \cdots & a_{k2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1r} & a_{2r} & \cdots & a_{kr} \end{bmatrix}$$

2. Matrix Operators

- Partition: Divide a matrix into column or row vectors or into smaller matrices.

$$A = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & \cdots & a_r \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{k1} \end{bmatrix} \cdots \begin{bmatrix} a_{1r} \\ a_{2r} \\ \vdots \\ a_{kr} \end{bmatrix}$$

2. Matrix Operators

- Addition:
  - Commutes: $A + B = B + A$
  - Associate: $A + (B + C) = (A + B) + C$
  - Distributive: $(A + B)^T = A^T + B^T$

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1r} + b_{1r} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2r} + b_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} + b_{k1} & a_{k2} + b_{k2} & \cdots & a_{kr} + b_{kr} \end{bmatrix}$$

2. Matrix Operators

- Multiplication by constant

$$A \cdot c = \begin{bmatrix} a_{11}c & a_{12}c & \cdots & a_{1r}c \\ a_{21}c & a_{22}c & \cdots & a_{2r}c \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1}c & a_{k2}c & \cdots & a_{kr}c \end{bmatrix}$$

- Inner product of two $k \times 1$ vectors

$$a^T \cdot b = a_1 \cdot b_1 + a_2 \cdot b_2 + \cdots + a_k \cdot b_k = \sum_{j=1}^{k} a_j b_j$$

2. Matrix Operators

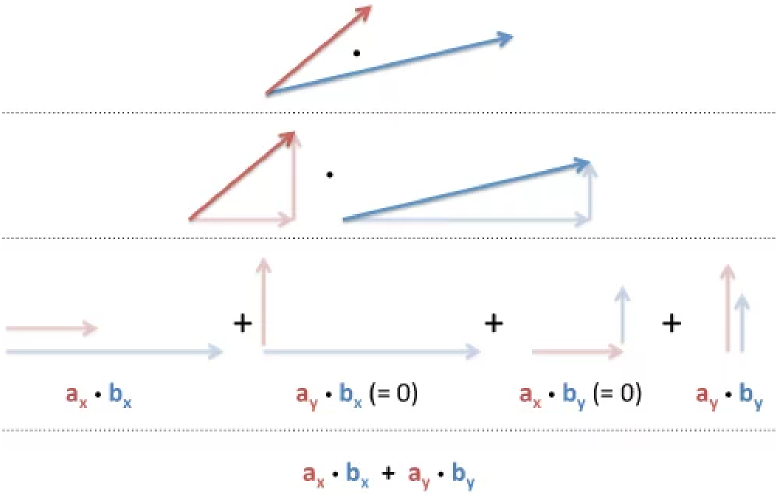
- Dot Product: For two vectors a and b the **inner product** (or scalar product) is defined as:

$$a \cdot b = \sum_{i=1}^{n} a_i b_i = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

- A useful identity is to consider the dot product of Euclidean vectors when they are orthogonal (i.e., right angles, or 90° or $\theta = \frac{\pi}{2}$.

$$\begin{aligned}
a \cdot b =& ||a|| \cdot ||b|| cos(\theta) \\
a \cdot b =& ||a|| \cdot ||b|| cos(\frac{\pi}{2}) \\
a \cdot b =& ||a|| \cdot ||b|| 0 \\
a \cdot b =& 0
\end{aligned}$$

2. Matrix Operators



Demonstration: Dot Product

Question:

Find dot product of $a = \begin{bmatrix} 0 & 4 & -2 \end{bmatrix} \cdot b = \begin{bmatrix} 2 \\ -1 \\ 7 \end{bmatrix}$

Answer:

$$\begin{aligned} a \cdot b =& (0)(2) + 4(-1) + (-2)(7) \\ =& -18 \end{aligned}$$

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PRACTICE: DOT PRODUCT

1. Find dot product $a \cdot b$ for $a = \begin{bmatrix} 9 & 5 & -4 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} -3 & -2 & 7 & -1 \end{bmatrix}$.

Practice: Dot Product

1. Find dot product $a \cdot b$ for $a = \begin{bmatrix} 9 & 5 & -4 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} -3 & -2 & 7 & -1 \end{bmatrix}$.

Answer: Show Work

$$a \cdot b = -67$$

Practice: Dot Product

1. Find dot product $a \cdot b$ for $a = \begin{bmatrix} 9 & 5 & -4 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} -3 & -2 & 7 & -1 \end{bmatrix}$.
2. Find dot product $a \cdot b$ for $a = \begin{bmatrix} 2 & -3 & 4 & 15 & -3 \end{bmatrix}$ and $b = \begin{bmatrix} 8 & 8 & 2 & -10 & 0 \end{bmatrix}$.

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Answer: Show Work

$$a \cdot b = -150$$

Practice: Dot Product

1. Find dot product $a \cdot b$ for $a = \begin{bmatrix} 9 & 5 & -4 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} -3 & -2 & 7 & -1 \end{bmatrix}$
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3. Find dot product $a \cdot b$ for $a = \begin{bmatrix} 2x & 2xy & 4z \end{bmatrix}$ and $b = \begin{bmatrix} x & 3y & xz \end{bmatrix}$.

Practice: Dot Product

1. Find dot product $a \cdot b$ for $a = \begin{bmatrix} 9 & 5 & -4 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} -3 & -2 & 7 & -1 \end{bmatrix}$
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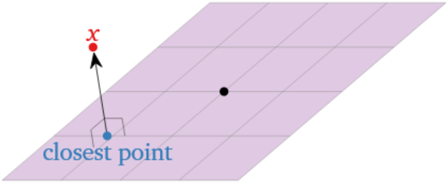
Answer: Show Work

$$a \cdot b = 2x^2 + 6xy^2 + 4xz^2$$

2. Matrix Operators

- Orthogonality: A perpendicular (i.e., right angle) in n dimensions.
- Orthogonal vectors are $a^T \cdot b = 0$

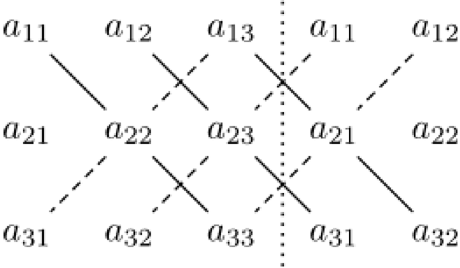


2. Matrix Operators

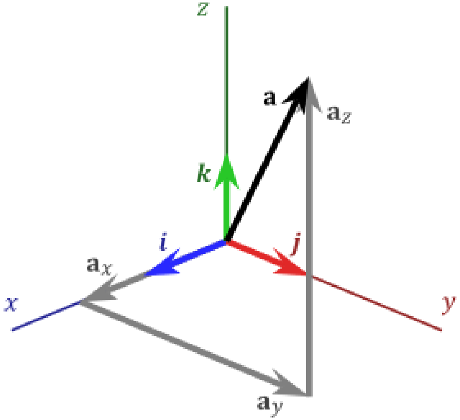
- Cross Product: Vector product (or outer product) is a new vector perpendicular to both input vectors $a = \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix}$ and $b = \begin{pmatrix} b_1 & b_2 & b_3 \end{pmatrix}$.

$$\begin{aligned} a \times b =& (a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) \times (b_1\vec{i} + b_2\vec{j} + b_3\vec{k}) \\ =& (a_2b_3 - a_3b_2)\vec{i} + (a_3b_1 - a_1b_3)\vec{j} + (a_1b_2 - a_2b_1)\vec{k} \end{aligned}$$

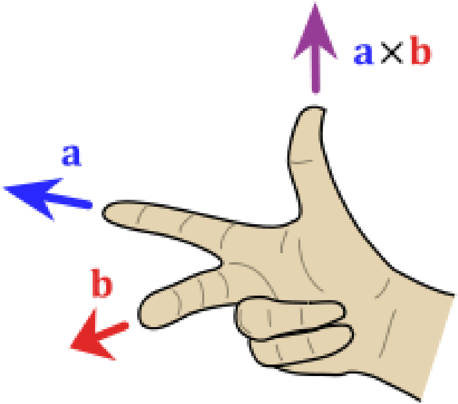
- Sarrus's Rule



2. Matrix Operators



Right Hand Rule



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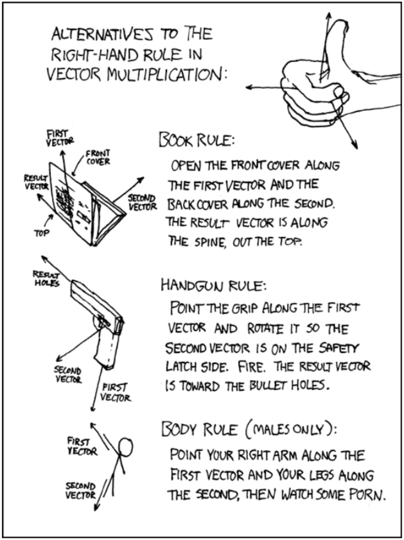

Demonstration: Cross Product

Question:

Find the cross product of $v = \begin{bmatrix} 0 & 4 & -2 \end{bmatrix} \times w = \begin{bmatrix} 3 & -1 & 5 \end{bmatrix}$

Answer:

$$v \times w = \left|\begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 4 & -2 \\ 3 & -1 & 5 \end{matrix}\right| \begin{matrix} \vec{i} & \vec{j} \\ 0 & 4 \\ 3 & -1 \end{matrix}$$

$$= 20\vec{i} - 6\vec{j} + 0\vec{k} - 0\vec{j} - 2\vec{i} - 12\vec{k}$$

$$= 18\vec{i} - 6\vec{j} - 12\vec{k}$$

$$= \begin{pmatrix} 18 & -6 & -12 \end{pmatrix}$$

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$$= 18\vec{i} - 6\vec{j} - 12\vec{k}$$

$$= \begin{pmatrix} 18 & -6 & -12 \end{pmatrix}$$

PRACTICE: CROSS PRODUCT

1. Find cross product $w \times v$ for $w = \begin{bmatrix} 1 & 6 & -8 \end{bmatrix}$ and $v = \begin{bmatrix} 4 & -2 & -1 \end{bmatrix}$.

Practice: Cross Product

1. Find cross product $w \times v$ for $w = \begin{bmatrix} 1 & 6 & -8 \end{bmatrix}$ and $v = \begin{bmatrix} 4 & -2 & -1 \end{bmatrix}$.

Answer: Show Work

$$w \times v = -22\vec{i} - 31\vec{j} - 26\vec{k}$$

Practice: Cross Product

1. Find cross product $w \times v$ for $w = \begin{bmatrix} 1 & 6 & -8 \end{bmatrix}$ and $v = \begin{bmatrix} 4 & -2 & -1 \end{bmatrix}$.
2. Find cross product $a \times b$ for $a = \begin{bmatrix} 2 & 1 & -1 \end{bmatrix}$ and $b = \begin{bmatrix} -3 & 4 & 1 \end{bmatrix}$.

Practice: Cross Product

1. Find cross product $w \times v$ for $w = \begin{bmatrix} 1 & 6 & -8 \end{bmatrix}$ and $v = \begin{bmatrix} 4 & -2 & -1 \end{bmatrix}$.
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Answer: Show Work

$$a \times b = 5\vec{i} + \vec{j} + 11\vec{k}$$

Practice: Cross Product

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2. Find cross product $a \times b$ for $a = \begin{bmatrix} 2 & 1 & -1 \end{bmatrix}$ and $b = \begin{bmatrix} -3 & 4 & 1 \end{bmatrix}$.
3. Find cross product $a \times b$ for $a = \begin{bmatrix} 3 & -2 & -1 & 4 \end{bmatrix}$ and $b = \begin{bmatrix} 5 & 1 & -2 & 3 \end{bmatrix}$.

Practice: Cross Product

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3. Find cross product $a \times b$ for $a = \begin{bmatrix} 3 & -2 & -1 & 4 \end{bmatrix}$ and $b = \begin{bmatrix} 5 & 1 & -2 & 3 \end{bmatrix}$.

Answer: Show Work

$$a \times b = 12\vec{i} - 12\vec{j} + 30\vec{k} + 4\vec{l}$$

2. Matrix Operators

- Matrix Multiplication is summing the multiplication of the columns and the rows into a new matrix.
- Conformable: $[k \times r] \times [r \times s] = [r \times s]$
- Multiplication is not commutative $A \times B \neq B \times A$

$$A \times B = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_k^T \end{bmatrix} \times \begin{bmatrix} b_1 & b_2 & \cdots & b_s \end{bmatrix}$$

$$= \begin{bmatrix} a_1^T b_1 & a_1^T b_2 & \cdots & a_1^T b_s \\ a_2^T b_1 & a_2^T b_2 & \cdots & a_2^T b_s \\ \vdots & \vdots & \ddots & \vdots \\ a_k^T b_1 & a_k^T b_2 & \cdots & a_k^T b_s \end{bmatrix}$$

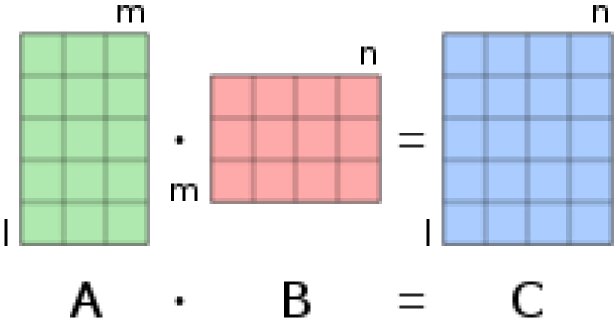
2. Matrix Operators

- Matrix multiplication: $A_{m\times n}B_{n\times p} = C_{m\times p}$

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}, B = \begin{bmatrix} b_{11} & \cdots & b_{1p} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{np} \end{bmatrix}$$

$$C = \begin{bmatrix} a_{11}b_{11} + \cdot + a_{1n}b_{n1} & \cdots & a_{11}b_{1p} + \cdot + a_{1n}b_{np} \\ \vdots & \ddots & \vdots \\ a_{m1}b_{11} + \cdot + a_{mn}b_{n1} & \cdots & a_{m1}b_{1p} + \cdot + a_{mn}b_{np} \end{bmatrix}$$

2. Matrix Operators



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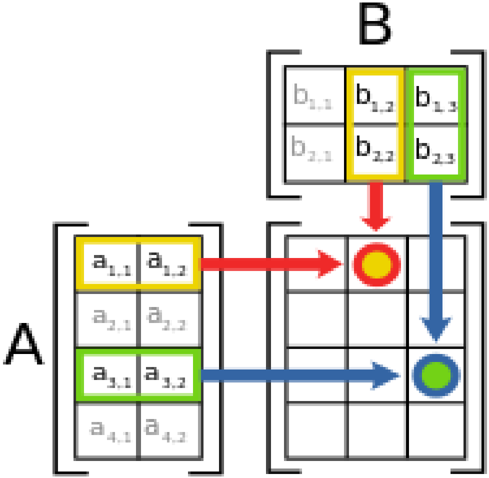

Demonstration: Matrix Multiplication

Question:

Determine CD given $C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$

Answer:

$$
\begin{aligned}
CD &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \\
&= \begin{bmatrix} 1(5) + (2)(7) & 1(6) + (2)(8) \\ (3)(5) + (4)(7) & (3)(6) + (4)(8) \end{bmatrix} \\
&= \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}
\end{aligned}
$$

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&= \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}
\end{aligned}
$$

Practice: Matrix Multiplication

1. Find AB given $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 \\ -2 & 1 \end{bmatrix}$.

Practice: Matrix Multiplication

1. Find AB given $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 \\ -2 & 1 \end{bmatrix}$.

Answer: Show Work

$$AB = \begin{bmatrix} -1 & 6 \\ 4 & 9 \end{bmatrix}$$

Practice: Matrix Multiplication

1. Find AB given $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 \\ -2 & 1 \end{bmatrix}$.
2. Find AB given $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 8 \\ 2 & 9 \\ 1 & 7 \end{bmatrix}$.

Practice: Matrix Multiplication

1. Find AB given $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 \\ -2 & 1 \end{bmatrix}$.
2. Find AB given $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 8 \\ 2 & 9 \\ 1 & 7 \end{bmatrix}$.

Answer: Show Work

$$|2 \times 2| \times |3 \times 2| = \text{Does not conform}$$

Practice: Matrix Multiplication

1. Find AB given $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 \\ -2 & 1 \end{bmatrix}$.
2. Find AB given $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 8 \\ 2 & 9 \\ 1 & 7 \end{bmatrix}$.
3. Find AB given $A = \begin{bmatrix} 3 & -6 & 9 \\ 10 & 1 & 0 \\ -8 & 2 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 9 & 1 & 7 & -5 \\ 3 & 5 & 0 & 6 \\ 4 & 1 & 2 & 2 \end{bmatrix}$.

Practice: Matrix Multiplication

1. Find AB given $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 \\ -2 & 1 \end{bmatrix}$.
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3. Find AB given $A = \begin{bmatrix} 3 & -6 & 9 \\ 10 & 1 & 0 \\ -8 & 2 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 9 & 1 & 7 & -5 \\ 3 & 5 & 0 & 6 \\ 4 & 1 & 2 & 2 \end{bmatrix}$.

Answer: Show Work

$$AB = \begin{bmatrix} 45 & -18 & 39 & -33 \\ 93 & 15 & 70 & -44 \\ -74 & 0 & -60 & 48 \end{bmatrix}$$

2. Matrix Operators

- Inversion: Solve when when matrix is square and non-singular ($\#rows(A) = \#cols(A)$).
- Let $B = A^{-1}$ be the inverse of full-rank $k \times k$ matrix A.
- This satisfies $AB = I_k$.
- Suppose $n \times n$ matrix A is invertible. Then it is non-singular and has a unique solution:

$$\begin{aligned} Ax &= b \\ x &= A^{-1}b \end{aligned}$$

- This is important for determining coefficients in OLS.

3. Rank

- Rank: Number of non-zero rows in the row echelon form.
- $rank = \min(m, n)$
  - Full Rank: # of rows = # of columns
  - $rankA \leq \#rows(A)$
  - $rankA \leq \#cols(A)$

4. Trace

- Trace: Sum of the diagonal elements of $k \times k$ matrix A
  - $tr(cA) = ctr(A)$
  - $tr(A^T) = tr(A)$
  - $tr(A + B) = tr(A) + tr(B)$
  - $tr(I_k) = k$
  - If conformable, $tr(AB) = tr(BA)$

$$tr(A) = \sum_{i=1}^{k} a_{ii}$$

5. Special Matrices

- Square Matrix: $k = r$
  - Square matrices are symmetric $A = A^T$
  - Called a **diagonal** if all off diagonal elements are zero.
  - Called an **upper diagonal** (or lower) if all elements below (above) the diagonal are zero.
  - Idempotent: $B^2 = BB = B$
- Identity Matrix: Diagonal matrix with only 1's as values in diagonal.

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

- Zero Matrix: A null matrix with only zeros.
  - E.g., $[0]$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

6. The Determinant

- A matrix A is non-singular iff its determinant is non-zero.

$$det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

$$- a_{13}a_{22}a_{31} - a_{21}a_{12}a_{33} - a_{11}a_{23}a_{32}$$

- $detA^T = detA$
- $det(A \cdot B) = detA \cdot detB$
- $det(A + B) \neq detA + detB$

6. The Determinant

- Minor of Matrix: A determinant of a smaller square matrix cut from A by removing one or more rows and columns.

$$
\begin{aligned}
|A| &= a_{11}|M_{11}| - a_{12}|M_{12}| + a_{13}|M_{13}| \\
&= a_{11}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} - a_{12}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + a_{13}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \\
&= a_{11}\cdot\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\cdot\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\cdot\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}
\end{aligned}
$$

- Adjacent of Matrix: $adjA = (-1)^{i+j} \times det(\text{minor of } i, j)$.
- A non-singular matrix has the inversion:

$$A^{-1} = \frac{1}{detA} \cdot adjA$$

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Demonstration: Determinant

Question:

$$det\begin{pmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{pmatrix}$$

Answer:

$$\begin{aligned} det(\cdot) =& 7(-6-(-4)) - (0-(-3)) + (0-(-9)) \\ =& 7(-2) - 2(3) + 1(9) \end{aligned}$$

$$= -11$$

Demonstration: Determinant

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Practice: Determinants

1. $det \begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix}$

Practice: Determinants

1. $det \begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix}$

Answer: Show Work

$$det(\cdot) = -1$$

Practice: Determinants

1. $det \begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix}$.
2. $det \begin{pmatrix} 1 & 2 & 4 \\ 2 & 7 & 3 \\ 3 & 1 & -5 \end{pmatrix}$.

Practice: Determinants

1. $det \begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix}$.
2. $det \begin{pmatrix} 1 & 2 & 4 \\ 2 & 7 & 3 \\ 3 & 1 & -5 \end{pmatrix}$.

Answer: Show Work

$$det(\cdot) = -76$$

Practice: Determinants

1. $det \begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix}$.
2. $det \begin{pmatrix} 1 & 2 & 4 \\ 2 & 7 & 3 \\ 3 & 1 & -5 \end{pmatrix}$.
3. $det \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 1 \\ 3 & 4 & -5 \end{pmatrix}$.

Practice: Determinants

1. $det\begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix}$.
2. $det\begin{pmatrix} 1 & 2 & 4 \\ 2 & 7 & 3 \\ 3 & 1 & -5 \end{pmatrix}$.
3. $det\begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 1 \\ 3 & 4 & -5 \end{pmatrix}$.

Answer: Show Work

$$det(\cdot) = -8$$

7. Matrix Decomposition

- A matrix can be decomposed into its eigenvector.
- Eignevector is the vector which transforms (e.g., rotates, stretches) another vector by a constant factor.
- Eigenvectors = c_i.
- Eigenvalues = λ_i

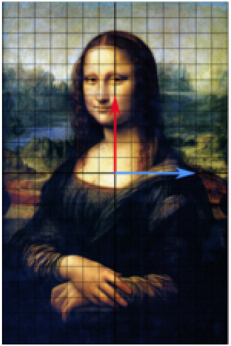
$$C = \begin{pmatrix} c_1 & c_2 & \cdots & c_k \end{pmatrix}$$

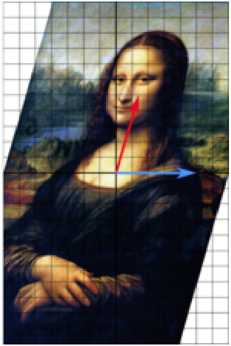
$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k \end{pmatrix}$$

$$Ac_k = \lambda_k c_k$$

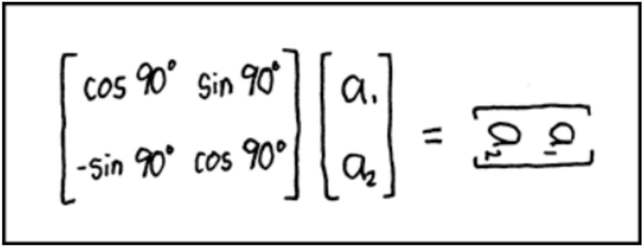
$$AC = C\Lambda$$

Linear Transformation





Linear Transformation



7. Matrix Decomposition

- The vectors are orthogonal.
- $c_i^T c_i = 1 \implies C^T C = I \implies CC^T = CC^{-1} = I.$
- The diagonalization of A is
$$C^T AC = C^T C\Lambda = I\Lambda = \Lambda$$

- The spectral decomposition
$$A = C\Lambda C^T = \sum_{i=1}^{k} \lambda_i c_i c_i^T$$

8. Positive and Negative Definite Matrices

- Positive Definite: Iff for $k \times k$ real symmetric matrix A, $\forall c \neq 0 \in \mathbb{R}^k$, $c^T Ac > 0$.
- Negative Definite: Iff for $k \times k$ real symmetric matrix A, $\forall c \neq 0 \in \mathbb{R}^k$, $c^T Ac < 0$.
- Semi-definite: A weak inequality $\geq, \leq$ in either case.
- Negative Semi-definite: All diagonal elements must be ≤ 0.

8. Positive and Negative Definite Matrices

How to determine if you have a **positive** definite matrix?

1. All eigenvalues > 0.
2. Sylvester's Criterion: All upper left determinants (e.g., sub-matrices) must be > 0.
3. Every pivot must be > 0.
4. A has independent columns (e.g., $S = A^T A$)
5. $x^T S x > 0 \forall x : x \neq 0$

9. Linear Independence

- Homogeneous System: Guaranteed to have at least one solution $x_i = 0 \forall i$ when $b_i = 0 \forall i$

$$A \begin{pmatrix} x_1 \\ \cdots \\ x_n \end{pmatrix} = 0$$

- Linearly Dependent: Iff there is a non-zero solution.
  - Means one of the column vectors a_n can be written as a linear combination of the other vectors.
  - Implies infinite solutions.
  - Short Rank ($\#rows(A) < \#cols(A)$ must be linearly dependent.
- Linearly Independent: Iff the only solution is the zero solution.
- Singular: When a square matrix has a non-zero solution.

10. Chain Rule for Vectors

- Let x, y, z be vectors such that z is a function of y, and y is a function of x.
- We can apply the chain rule noting that with vectors we must chain the results **from the left**:

$$\frac{\partial z}{\partial x} = \frac{\partial y}{\partial x}\frac{\partial z}{\partial y} = \frac{\partial z}{\partial y}\frac{\partial y}{\partial x}$$

10. Chain Rule for Vectors

$$\frac{\partial z}{\partial x} = \begin{bmatrix} \sum_{q=1}^{r} \frac{\partial z_1}{\partial y_q}\frac{\partial y_q}{\partial x_1} & \cdots & \sum_{q=1}^{r} \frac{\partial z_1}{\partial y_q}\frac{\partial y_q}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \sum_{q=1}^{r} \frac{\partial z_m}{\partial y_q}\frac{\partial y_q}{\partial x_1} & \cdots & \sum_{q=1}^{r} \frac{\partial z_m}{\partial y_q}\frac{\partial y_q}{\partial x_n} \end{bmatrix}$$

Review

Review of Matrices

1. Foundations of Matrices
2. Matrix Operators
3. Rank
4. Special Matrices
5. The Determinant
6. Trace
7. Matrix Decomposition
8. Positive and Negative Definite Matrices
9. Linear Independence
10. Chain Rule for Vectors

Assignment

- Readings on Linear Algebra before Lecture 07:
  - S&B Ch. 7, 8, & 9
- Assignment:
  - Problem Set 06 (PS06)
  - Solution set will be available following end of Lecture 07
- Struggling?
  1. Read the 'Encouraged Reading'
  2. Review 'Supplementary material'
  3. Reach out directly

Dot Product Question 1 Answer:

◂ Question

$$
\begin{aligned}
a \cdot b =& 9(-3) + 5(-2) + (-4)7 + 2(-1) \\
=& -67
\end{aligned}
$$

Dot Product Question 2 Answer:

◂ Question

$$\begin{aligned} a \cdot b =& 2(8) + (-3)(8) + 4(2) + 15(-10) + (-3)0 \\ =& -150 \end{aligned}$$

Dot Product Question 3 Answer:

◂ Question

$$a \cdot b = 2x^2 + 6xy^2 + 4xz^2$$

Cross Product Question 1 Answer:

◂ Question

$$\begin{aligned} w \times v &= -6\vec{i} - 32\vec{j} - 2\vec{k} + 1\vec{j} - 16\vec{i} - 24\vec{k} \\ &= -22\vec{i} - 31\vec{j} - 26\vec{k} \end{aligned}$$

Cross Product Question 2 Answer:

◂ Question

$$
\begin{aligned}
a \times b =& (1)(1)\vec{i} + (-1)(-3)\vec{j} + 2(1)\vec{k} \\
& -2(1)\vec{j} - (-1)4\vec{i} - (1)(-3)\vec{k} \\
=& 5\vec{i} + \vec{j} + 11\vec{k}
\end{aligned}
$$

Cross Product Question 3 Answer:

◂ Question

$$
\begin{aligned}
a \times b =& 4\vec{i} - 3\vec{j} + 20\vec{k} + 3\vec{l} \\
&+10\vec{k} - 9\vec{j} + 8\vec{i} + 1\vec{l} \\
=& 12\vec{i} - 12\vec{j} + 30\vec{k} + 4\vec{l}
\end{aligned}
$$

Matrix Mult. Question 1 Answer:

◂ Question

$$AB = \begin{bmatrix} 3-4 & 4+2 \\ 6-2 & 8+1 \end{bmatrix}$$
$$= \begin{bmatrix} -1 & 6 \\ 4 & 9 \end{bmatrix}$$

Matrix Mult. Question 2 Answer:

◂ Question

$$|2 \times 2| \times |3 \times 2| = \text{Does not conform}$$

Matrix Mult. Question 3 Answer:

Question

$$AB = \begin{bmatrix} 27 - 18 + 36 & 3 - 30 + 9 & 21 + 0 + 18 & -15 - 36 + 18 \\ 90 + 3 + 0 & 10 + 5 + 0 & 70 + 0 + 0 & -50 + 6 + 0 \\ -72 + 6 - 8 & -8 + 10 - 2 & -56 + 0 - 4 & 40 + 12 - 4 \end{bmatrix}$$

$$= \begin{bmatrix} 45 & -18 & 39 & -33 \\ 93 & 15 & 70 & -44 \\ -74 & 0 & -60 & 48 \end{bmatrix}$$

Determinants Question 1 Answer:

◂ Question

$$
\begin{aligned}
det(\cdot) =& 1 - 2 \\
=& -1
\end{aligned}
$$

Determinants Question 2 Answer:

◂ Question

$$
\begin{aligned}
det(\cdot) =& 1(-35 - 3) - 2(-10 - 9) + 4(2 - 21) \\
=& -76
\end{aligned}
$$

Determinants Question 3 Answer:

Question

$$\begin{aligned} det(\cdot) =& 1(-15-4) - 1(-10-3) + 2(8-9) \\ =& -8 \end{aligned}$$