Lecture 10
Optimization Day 2

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Last Lecture Review

- Unconstrained Optimization:
  - First Order Conditions
  - Second Order Conditions
- (Equality) Constrained Optimization:
  - Lagrangian Method
  - Bordered Hessian

Review Assignment

1. Problem Set 09 solutions are available on Github.
2. Any issues or problems **You** would like to discuss?

Daily Icebreaker

- Attendance via prompt:
  - Name
  - Program and track
  - Daily icebreaker subject...

(Inequality) Constrained Optimization

Motivation

- General background
  - Optimization of a function considering a constraint from other function(s).
  - Leads to a potential 'corner solution'.
- Why do economists' care?
  - This is the most typical case for optimization.
- Application in this career
  - Used throughout microeconomics.

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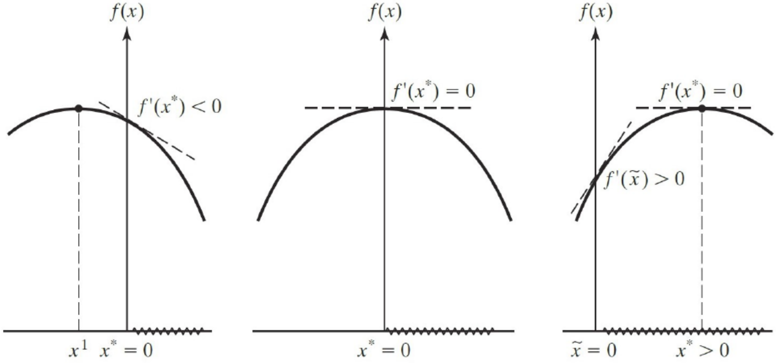
Overview

1. Minimization
2. Maximization
3. Kuhn Tucker Conditions
4. Corner Solutions
5. Quasiconcavity and Optimization

INEQUALITY



(a) Case 1 (b) Case 2 (c) Case 3

1. Minimization

- Necessary conditions for optimal in real-valued functions subject to non-negative constraints:
  - Let $f(x)$ be continuously differentiable.
  - If x^* minimizes $f(x)$ subject to $x \geq 0$, then x^* satisfies:

1. $\frac{\partial f(x)}{\partial x_i} \geq 0 \forall i = 1, \ldots, n.$
2. $x_i^* \left(\frac{\partial f(x)}{\partial x_i} \right) = 0 \forall i = 1, \ldots, n.$
3. $x_i^* \geq 0 \forall i = 1, \ldots, n.$

2. Maximization

- Necessary conditions for optimal in real-valued functions subject to non-negative constraints:
  - Let $f(x)$ be continuously differentiable.
  - If x^* maximizes $f(x)$ subject to $x \geq 0$, then x^* satisfies:

1. $\frac{\partial f(x)}{\partial x_i} \leq 0 \forall i = 1, \ldots, n$.
2. $x_i^* \left(\frac{\partial f(x)}{\partial x_i} \right) = 0 \forall i = 1, \ldots, n$.
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3. Kuhn Tucker Conditions

$$\max_{x\in\mathbb{R}^n_+} f(x) \text{ s.t. } g(x) \leq b, x \geq 0$$

$$L = f(x) - \lambda_1[g_1(x) - b_1] - \cdots - \lambda_k[g_k(x) - b_k]$$

- The constraints are 'binding' if at the optimum $g(x^*, y^*) = c$, and is said to 'slack' otherwise.
- The **three** "Kuhn-Tucker" necessary conditions (FOC) are...

1.	$\frac{\partial L}{\partial x_i^*} \leq 0$	$\frac{\partial L}{\partial \lambda_j^*} \geq 0,$	Restates constraints
2.	$x_i^* \frac{\partial L}{\partial x_i^*} = 0,$	$\lambda_j^* \frac{\partial L}{\partial \lambda_j^*} = 0,$	Complimentary slackness
3.	$x_i^* \geq 0,$	$\lambda_j^* \geq 0,$	Non-negative condition
	$\forall i = 1, \ldots, n$	$\forall j = 1, \ldots, k$	

3. Kuhn Tucker Conditions

- $\lambda_j^* \frac{\partial L}{\partial \lambda_j^*} = 0$ implies that **at least one** of the λ_j^* and $\frac{\partial L}{\partial \lambda_j^*}$ must be zero.
- If the constraint is non-binding, then $\lambda_j^* = 0$ and we give to weight to that constraint (i.e., unconstrained).

$$\frac{\partial L}{\partial \lambda_j^*} \equiv = b_j - g_j(x) > 0$$

- If $\lambda_j^* > 0$ then the constraint must be binding (i.e., constrained).

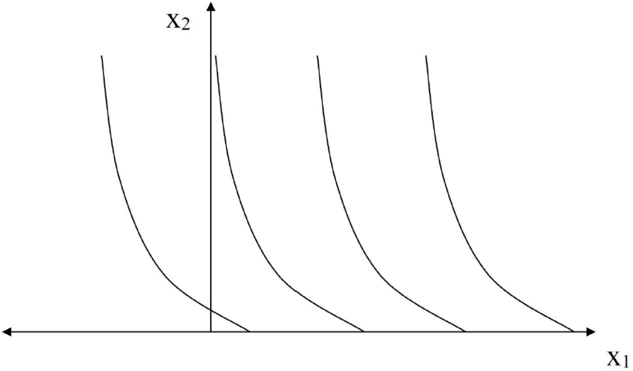
$$b_j = g_j(x)$$

- Kuhn-Tucker determine allows **any** problem to be solved as constrained or unconstrained.
- Hence, you check **both cases**.

4. Corner Solutions

4. Corner Solutions

- Quasilinear preferences (A strong preference for x_2)



5. Concavity, Convexity, and Optimization

- Convex Maximization Problem: With convex constraint sets and concave objective functions, the FOC are **both necessary and sufficient** to identify global maxima.
- Convex Minimization Problem: With convex constraint sets and convex objective functions, the FOC are **both necessary and sufficient** to identify global maxima.
- Both provide 'uniqueness' in the solution.
- So, for optimization we need:
  - Continuity on the domain.
  - Differentiablity
  - Concavity/convexity of the C^2 function is completely characterized by the second derivative.
- Quasi-concavity does **not** imply continuity...

Practice: (Inequality) Constrained Optimization

1.

Comparative Statics & Envelope Theorem

Motivation

- General background
  - Allows you determine how an optimum changes are the parameters values change.
- Why do economists' care?
  - Used primarily in macroeconomics, but in microeconomics as well.
- Application in this career
  - Used to measure policy alternatives but changing the initial conditions.

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Overview

1. The Multiplier
2. Comparative Statics
3. Unconstrained Envelope Theorem
4. Constrained Envelope Theorem

1. The Multiplier

- Consider the maximization problem:

$$\max f(x,y) \text{ s.t. } h(x,y)=a$$

- Let the solution be $(x^*(a), y^*(a))$ with the corresponding **multiplier** $\mu^*(a)$.
- Suppose x^*, y^*, and μ^* are C^1 functions of a. Then,

$$\mu^*(a) = \frac{d}{da} f(x^*(a), y^*(a))$$

- Or, for multiple variables (n) and multiple constraints (m) ...

$$\mu_j^*(a_1,\cdots,a_m) = \frac{\partial}{\partial a_j} f(x_1^*(a_1,\cdots,a_m),\ldots,x_n^*(a_1,\cdots,a_m)) \forall j=1,\ldots,m$$

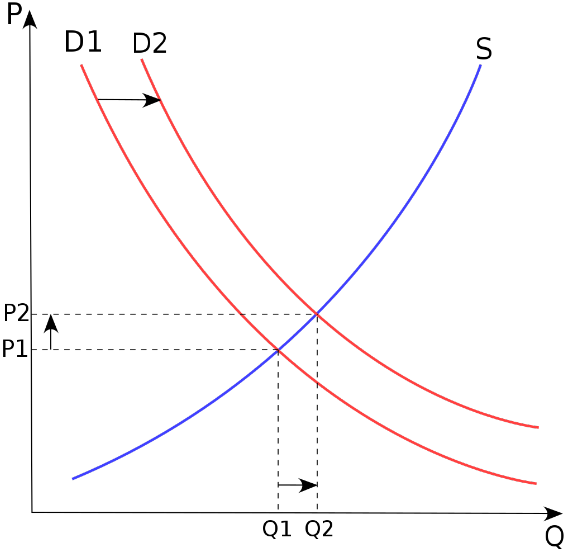
- So μ_j^* measures the sensitivity of the objective function to the constraint.

2. Comparative Statics

- Examining the change in optimization after changing an 'exogenous' parameter (a).
- In essence, the difference in two equilibrium states.
- We can use the implicit function theorem to determine a comparative static derivative.
- Adding constraints, we can apply the Envelope Theorem to generalize the following formula.

$$\begin{aligned} f(x,a) &= 0 \\ Bdx + Cda &= 0 \\ \frac{dx}{da} &= -B^{-1}C \end{aligned}$$

2. Comparative Statics



3. Unconstrained Envelope Theorem

- Let $f(x, a)$ be a C^1 function of $x \in \mathbb{R}^n$ with scalar a.
- For each possible parameter a, consider the **unconstrained** optimization problem:

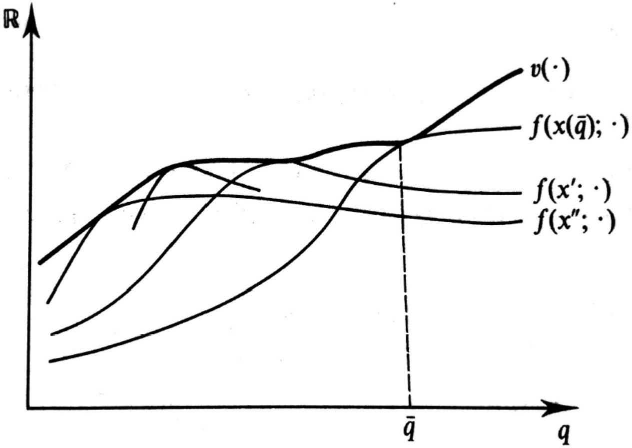
$$\max f(x, a) \text{ w.r.t. } x$$

- Let $x^*(a)$ be the solution.
- Suppose that $x^*(a)$ is a C^1 function of a.
- Then,

$$\frac{d}{da} f(x^*(a), a) = \frac{\partial}{\partial a} f(x^*(a), a)$$

3. Unconstrained Envelope Theorem

- Intuition



4. Constrained Envelope Theorem

- Let $f, h_1, \ldots, h_m : \mathbb{R}^n \times \mathbb{R}^1 \to \mathbb{R}^1$ be C^1 functions.
- Let $x^*(a) = (x_1^*(a), \ldots, x_n^*(a))$ be the solution maximizing $f(x, a)$ with respect to x given the constraint set...

$$h_1(x, a) = 0, \ldots, h_m(x, a) = 0$$

- Suppose that $x^*(a)$ and the Lagrange multipliers $\mu_1(a), \ldots, \mu_m(a)$ are C^1 functions of a.
- Then,

$$\frac{d}{da} f(x^*(a), a) = \frac{\partial L}{\partial a} f(x^*, \mu(a), a)$$

- The Lagrange multiplier is a **special case** of the envelope theorem.

Practice: Comparative Statics & Envelope Theorem

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Comparative Statics & Envelope Theorem

1. The Multiplier
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4. Constrained Envelope Theorem

Assignment

- Readings on Probability before Lecture 11:
  -
- Assignment:
  - Problem Set 10 (PS10)
  - Solution set will be available following end of Lecture 11
- Struggling?
  1. Read the 'Encouraged Reading'
  2. Review 'Supplementary material'
  3. Reach out directly