Extra example ☺

To clarify some of the confusions from yesterday, I thought to present one more example which also helped me personally when I was putting it together.

With a single output like yesterday, the mapping was to $\mathbb{R}$, that is why we found a single equation for total derivative. The Jacobian is used to find the total derivative when we have a mapping that is is $\mathbb{R}^n \to \mathbb{R}^m$.

For instance, define $f: \mathbb{R}^3 \to \mathbb{R}^4$ to be given by:

$$f(x, y, z) = (x + 2y + 3z, xyz^3, \ln(x^2y), e^{2xy^2}y^2)$$

1. What is the dimension of this Jacobian?

2. Find the Jacobian Df.

3. Use the Jacobian to write out matrices that give the total derivative of f.

4. Write out the total derivative equation for each function f_i in f.

1. This is a 4 x 3 Jacobian matrix

2. The Jacobian is given by:

$$Df = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \\ \frac{\partial f_4}{\partial x} & \frac{\partial f_4}{\partial y} & \frac{\partial f_4}{\partial z} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ yz^3 & xz^3 & 3xyz^2 \\ \frac{2x}{x^2y} & \frac{1}{x^2y} & 0 \\ 2y^4e^{2xy^2} & 0 & e^{2xy^2}(2y + 4xy^3) \end{bmatrix}$$

$$\frac{\partial e^{2xy^2}y^2}{\partial y} = e^{2xy^2}(2y) + y^2\left(4xye^{2xy^2}\right) = e^{2xy^2}(2y + 4xy^3)$$

3. The total derivative can be expressed as:

$$\begin{bmatrix} df_1 \\ df_2 \\ df_3 \\ df_4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ yz^3 & xz^3 & 3xyz^2 \\ \frac{2x}{x^2y} & \frac{1}{x^2y} & 0 \\ 2y^4e^{2xy^2} & 0 & e^{2xy^2}(2y + 4xy^3) \end{bmatrix} * \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}$$

4. The total derivative equation for each function f_i in f.

$$df_1 = dx + 2dy + 3dz$$

$$df_2 = yz^3dx + xz^3dy + 3xyz^2dz$$

$$df_3 = \frac{2x}{x^2y}dx + \frac{1}{x^2y}dy$$

$$df_4 = 2y^4e^{2xy^2}dx + e^{2xy^2}(2y + 4xy^3)\ dz$$