Optimization - Day 2

Math Review : Aug- 17 – 2022

Mwaso Mnensa

Solution for yesterday’s exercise

$$- ax_1^2 - bx_2^2$$

subject to:

$$x_1 + x_2 = 1$$

Form the Lagrangean:

$$L\left(x_1, x_2, \lambda\right) = - ax_1^2 - bx_2^2 - \lambda(x_1 + x_2 - 1)$$

This can be equivalently expressed as:

$$L\left(x_1, x_2, \lambda\right) = - ax_1^2 - bx_2^2 + \lambda(1 - x_1 - x_2)$$

Can we check for the constraint qualification?

Denote the constraint as $h\left(x_1, x_2\right) = x_1 + x_2$. The first order conditions for the critical points are:

$h_{x_1} = 1$; $h_{x_2} = 1$, which implies that the point $(1, 1)$ is a critical point of $h(x_1, x_2)$. But note that $h(1, 1) \neq 1$ so the only critical point $(1, 1)$ is not in the constraint set and $h(x_1, x_2)$ satisfies the constraint qualification.

The first order conditions (FOC) are given as:

$$\frac{\partial L\left(x_1, x_2\right)}{\partial x_1} = -2ax_1 - \lambda = 0 \tag{1}$$

$$\frac{\partial L\left(x_1, x_2\right)}{\partial x_2} = -2bx_2 - \lambda = 0 \tag{2}$$

$$\frac{\partial L\left(x_1, x_2\right)}{\partial \lambda} = x_1 + x_2 - 1 = 0 \tag{3}$$

We are now ready to solve for $x_1,\ x_2,\ \lambda$

From (1) and (2) we have

$$\frac{-2ax_1}{-2bx_2} = \frac{\lambda}{\lambda}$$

$$ax_1 = bx_2$$

$$x_1 = \frac{bx_2}{a}$$

We can plug in x_1 in our equation (3)

$$x_1 + x_2 - 1 = 0$$

$$\frac{bx_2}{a} + x_2 = 1$$

$$bx_2 + ax_2 = a$$

$$x_2(a + b) = a$$

$$x_2 = \frac{a}{a+b}$$

$$x_1 = \frac{bx_2}{a} = \frac{b}{a} * \frac{a}{a+b} = \frac{b}{a+b}$$

Problem 2

$$x_1^2 x_2$$

subject to

$$2x_1^2 + x_2^2 = 3$$

$$L = x_1^2 x_2 - \lambda(2x_1^2 + x_2^2 - 3)$$

Note, this is equivalent to:

$$L = x_1^2 x_2 + \lambda(3 - 2x_1^2 - x_2^2)$$

The first order conditions are given by:

$$\frac{\partial L(x_1,x_2)}{\partial x_1} = 2x_1x_2 - \lambda 4x_1 = 0 \tag{1}$$

$$\frac{\partial L(x_1,x_2)}{\partial x_2} = x_1^2 - \lambda 2x_2 = 0 \tag{2}$$

$$\frac{\partial L(x_1,x_2)}{\partial \lambda} = 2x_1^2 + x_2^2 - 3 = 0 \tag{3}$$

Using (1) and (2) we have:

$$\frac{2x_1x_2}{x_1^2} = \frac{\lambda 4x_1}{\lambda 2x_2}$$

$$\frac{2x_2}{x_1} = \frac{2x_1}{x_2}$$

$$x_1^2 = x_2^2$$

Plugging this last result in (3), we have:

$$2x_1^2 + x_1^2 - 3 = 0$$
$$x_1^2 = 1$$
$$x_1 = \pm 1$$

For these values of x_1, x_2, we have $2x_1x_2 = \lambda 4x_1$ from (2). This implies $\lambda = \pm\frac{1}{2}$

It may feel nice to stop here, but there may be other solutions. We have:

$$2x_1x_2 - \lambda 4x_1 = 0$$

$x_1\left(x_2 - 2\lambda\right) = 0$, we have $x_1 = 0,\ x_2 = 2\lambda$

If $x_1 = 0, \rightarrow 2x_1^2 + x_2^2 - 3 = 0$, means that $x_2^2 = 3$.

Then, $x_2 = \pm\sqrt{3}$

We have found that $x_1 = 0$ and we also have, $x_1^2 - \lambda 2x_2 = 0$

For this to hold, $\lambda = 0$.

Let's recap all the solutions:

$\left(1, 1, \frac{1}{2}\right), \left(-1, -1, -\frac{1}{2}\right), \left(1, -1, -\frac{1}{2}\right), \left(-1, 1, \frac{1}{2}\right), \left(0, \sqrt{3}, 0\right) and\ (0, -\sqrt{3}, 0)$

Note that, we have to carry out a second derivative test to check which of these yield the maximum solution. If that does not work, we can plug in the values of x_1 and x_2 and check which yields the highest 'output.'

Exercise

Solve for x_1 and x_2

$x_1^2 + x_2^2$, subject to $x_1 x_2 = 1$

$x_1 x_2$, subject to $x_1^2 + x_2^2 = 1$

Kuhn Tucker Conditions

In economics it is much more common to start with inequality constraints of the form $g(x, y) \leq c$, for instance in our consumer problem, we would have, $p_1x_1 + p_2x_2 \leq w$.

The constraint is said to be **binding** if at the optimum $g(x^*, y^*) = c$, and it is said to be slack otherwise. Luckily for us, with a small tweak, the Lagrangean can still be used with the same FOC's except now we have **three "Kuhn-Tucker" necessary conditions** for each inequality constraint.

$$\frac{\partial L}{\partial \lambda} = c - g(x^*, y^*) \geq 0$$

$$\lambda^* \geq 0$$

$$\frac{\partial L}{\partial \lambda}\lambda^* = \lambda^*\left[c - g(x^*, y^*)\right] = 0$$

Proof of Kuhn-Tucker can be found on pp. 959-960 Mas-Colell.

Note:

- The first condition is just a restatement of the constraint.
- The second condition says that λ^* is always non-negative.
- The third condition says that either λ^* or $c - g(x^*, y^*)$ must be zero.
- If $\lambda^* = 0$, intuitively, we are not giving any weight to the constraint. The problem can really turn into an unconstrained optimization problem.
- If $\lambda^* > 0$, then the constraint must be binding then the problem turns into the standard Lagrangean considered above.

Therefore, to be thorough, check if:

- There is a corner solution
- Check if and solve for when the Lagrange multiplier is zero
- Solve for $\lambda > 0$

- Luckily for us, this tends to not always be necessary

More exercises

Consider this profit maximization problem with two outputs and two inputs constrained by an input distance function given by $D_1(q, z)$.

$$\pi(p, r) = p_1 q_1 + p_2 q_2 - r_1 z_1 - r_2 z_2$$

subject to:

$$D_I(q, z) = \frac{\sqrt{z_1 z_2}}{q_1^2 + q_2^2} \geq 1$$

Exercise

Consider a simple utility function for two goods, x_1 and x_2:

$$u = (x_1 - \gamma_1)^{\beta_1}(x_2 - \gamma_2)^{\beta_2}, \qquad \text{with } \beta_1 > 0, \beta_2 > 0.$$

The parameters γ_1 and γ_2 are constants, but they could be positive or negative.

(a) Someone suggests that "it would be okay" to assume that $\beta_1 + \beta_2 = 1$. Yet someone else claims that this restriction may be unrealistic. Which person is correct? Briefly explain your answer.

(b) Suppose that the person with the above utility function is struck by lightning. He survives, except now his utility function is $u = \beta_1 \log(x_1 - \gamma_1) + \beta_2 \log(x_2 - \gamma_2)$. Will his consumption decisions change as a result of being struck by lightning? Briefly explain your answer.

(c) Returning to the original utility function, assume that the consumer faces the budget constraint $p_1 x_1 + p_2 x_2 = w$, where $p_1 > 0$, $p_2 > 0$ are prices and $w > 0$ is wealth. Set up the Lagrangean and solve for the two first-order conditions. Then use the budget constraint to solve for the Walrasian demands of both goods. These should be functions of p_1, p_2, w, β_1, β_2, γ_1, and γ_2. Finally, use your answer to part (a) to simplify your answer.