

Lecture 04

Integration

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LAST LECTURE REVIEW

- ▶ Derivatives:
 - ▶ Continuity & Differentiability
 - ▶ First & Second Derivatives
 - ▶ Derivative Rules
 - ▶ Implicit Function
 - ▶ l'Hopital's Rule
 - ▶ Taylor Series Approximation
 - ▶ Mean Value Theorem
 - ▶ Critical Points

REVIEW ASSIGNMENT

1. Problem Set 03 solutions are available on Github.
2. Any issues or problems **You** would like to discuss?

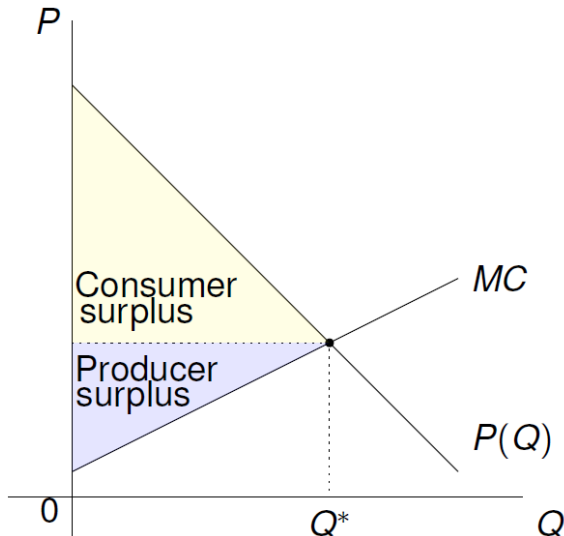
Topic: Integration

3. FUNDAMENTAL THEOREM OF CALCULUS

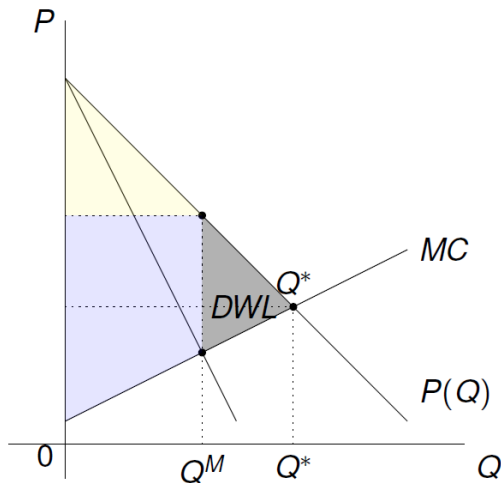
- ▶ Theorem that connects differentiation to integration.
- ▶ Let f be a continuous function on the open interval $[a, b]$. If $f(x) = F'(x)$, then:

$$\int_a^b f(x)dx = F(b) - F(a)$$

APPLICATION: SOCIAL SURPLUS



APPLICATION: DEAD-WEIGHT LOSS (DWL)



4. INTEGRATION RULES

- Constant:

$$\int a dx = ax + C$$

- Constant Multiplication:

$$\int cf(x)dx = c \int f(x)dx$$

- Reciprocal:

$$\int \frac{1}{x} dx = \ln(x) + C$$

- Exponential:

$$\int e^x dx = e^x + C$$

$$\int a^x dx = \frac{a^x}{\ln(a)} + C$$

DEMONSTRATION: EXPONENTIAL

Question:

$$\int (5^x) dx$$

Answer:

$$\frac{5^x}{\ln(5)} + C$$

DEMONSTRATION: EXPONENTIAL

Question:

$$\int (5^x) dx$$

Answer:

$$\frac{5^x}{\ln(5)} + C$$

4. INTEGRATION RULES

- Logarithm:

$$\int \ln(x) dx = x \ln(x) - x + C$$

- Power Rule:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

- Sum/Difference Rule:

$$\int (f(x) \pm g(x)) dx = \int f(x) dx \pm \int g(x) dx$$

DEMONSTRATION: POWER RULE

Question:

$$\int (x^5 + 3x^3 + 2x) dx$$

Answer:

$$\frac{1}{6}x^6 + \frac{3}{4}x^4 + x^2 + C$$

DEMONSTRATION: POWER RULE

Question:

$$\int (x^5 + 3x^3 + 2x) dx$$

Answer:

$$\frac{1}{6}x^6 + \frac{3}{4}x^4 + x^2 + C$$

DEMONSTRATION: SUM/DIFFERENCE RULE

Question:

$$\int (4x^2 + x - \frac{3}{x}) dx$$

Answer:

$$\frac{4}{3}x^3 + \frac{1}{2}x^2 - 3\ln(x) + C$$

DEMONSTRATION: SUM/DIFFERENCE RULE

Question:

$$\int (4x^2 + x - \frac{3}{x}) dx$$

Answer:

$$\frac{4}{3}x^3 + \frac{1}{2}x^2 - 3\ln(x) + C$$

PRACTICE: INTEGRATION

1. $\int w^{-2} + 10w^{-5} - 8dw$

PRACTICE: INTEGRATION

1. $\int w^{-2} + 10w^{-5} - 8dw$

Answer: [◀ Show Work](#)

$$-w^{-1} - \frac{5}{2}w^{-4} - 8w + C$$

PRACTICE: INTEGRATION

1. $\int w^{-2} + 10w^{-5} - 8dw$

2. $\int \frac{4}{x^2} + 2 - \frac{1}{8x^3} dx$

PRACTICE: INTEGRATION

1. $\int w^{-2} + 10w^{-5} - 8dw$

2. $\int \frac{4}{x^2} + 2 - \frac{1}{8x^3} dx$

Answer: [◀ Show Work](#)

$$-4x^{-1} + 2x + \frac{1}{16}x^{-2} + C$$

PRACTICE: INTEGRATION

1. $\int w^{-2} + 10w^{-5} - 8dw$

2. $\int \frac{4}{x^2} + 2 - \frac{1}{8x^3} dx$

3. $\int t^3 - \frac{e^{-t}-4}{e^{-t}} dt$

5. INTEGRATION BY SUBSTITUTION

- ▶ Reverse chain rule from differentiation.
- ▶ Commonly referred to as u substitution.

$$\int f(g(x))g'(x)dx$$
$$\int f(u)du$$

DEMONSTRATION: INT. BY SUB.

Question:

$$\int x^2(3 - 10x^3)^4 dx$$

Answer:

$$u = 3 - 10x^3$$

$$du = -30x^2 dx$$

$$\implies dx = \frac{-1}{30x^2} du$$

$$\int x^2(3 - 10x^3)^4 dx = \int u^4 \left(\frac{-1}{30}\right) du$$

$$= \frac{-1}{30} \cdot \frac{1}{5} u^5 + C$$

$$= \frac{-1}{150} (3 - 10x^3)^5 + C$$

DEMONSTRATION: INT. BY SUB.

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$$= \frac{-1}{150} (3 - 10x^3)^5 + C$$

PRACTICE: INT. BY SUB.

1. $\int \frac{1}{x^2+x} dx$

PRACTICE: INT. BY SUB.

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Answer: [◀ Show Work](#)

$$-\ln\left(\frac{1}{x} + 1\right) + C$$

PRACTICE: INT. BY SUB.

1. $\int \frac{1}{x^2+x} dx$

2. $\int 3(8y-1)e^{4y^2-y} dy$

PRACTICE: INT. BY SUB.

1. $\int \frac{1}{x^2+x} dx$

2. $\int 3(8y-1)e^{4y^2-y} dy$

Answer: [◀ Show Work](#)

$$3e^{4y^2-y} + C$$

6. INTEGRATION BY PARTS

- ▶ Reverse product rule from differentiation.
- ▶ Rarely used in economic applications, but important to know.

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$
$$\int u dv = u \cdot v - \int v du$$

DEMONSTRATION: INT. BY PARTS

Question:

$$\int \ln(x) dx$$

Answer:

$$u = \ln(x) , dv = 1$$

$$du = \frac{1}{x} , v = x$$

$$\begin{aligned}\int \ln(x) dx &= \ln(x)x - \int x \cdot \frac{1}{x} dx \\ &= x \ln(x) - x + C \\ &= x(\ln(x) - 1) + C\end{aligned}$$

DEMONSTRATION: INT. BY PARTS

Question:

$$\int \ln(x) dx$$

Answer:

$$u = \ln(x) , dv = 1$$

$$du = \frac{1}{x} , v = x$$

$$\begin{aligned} \int \ln(x) dx &= \ln(x)x - \int x \cdot \frac{1}{x} dx \\ &= x \ln(x) - x + C \\ &= x(\ln(x) - 1) + C \end{aligned}$$

PRACTICE: INT. BY PARTS

1. $\int (xe^{2x})dx$

PRACTICE: INT. BY PARTS

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Answer: [◀ Show Work](#)

$$\frac{(2x - 1)e^{2x}}{4} + C$$

PRACTICE: INT. BY PARTS

1. $\int (xe^{2x})dx$

2. $\int (2 + 5x)e^{\frac{1}{3}x}dx$

PRACTICE: INT. BY PARTS

1. $\int (xe^{2x})dx$

2. $\int (2 + 5x)e^{\frac{1}{3}x}dx$

Answer: ◀ Show Work

$$(15x - 39)e^{\frac{1}{3}x} + C$$

DIFFERENTIATION

```

graph TD
    START([START]) --> TRY([TRY APPLYING  
CHAIN RULE  
POWER RULE  
QUOTIENT RULE  
PRODUCT RULE  
ETC])
    TRY --> DONE1{DONE?}
    DONE1 -- YES --> DONE2([DONE!])
    DONE1 -- NO --> TRY
  
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INTEGRATION

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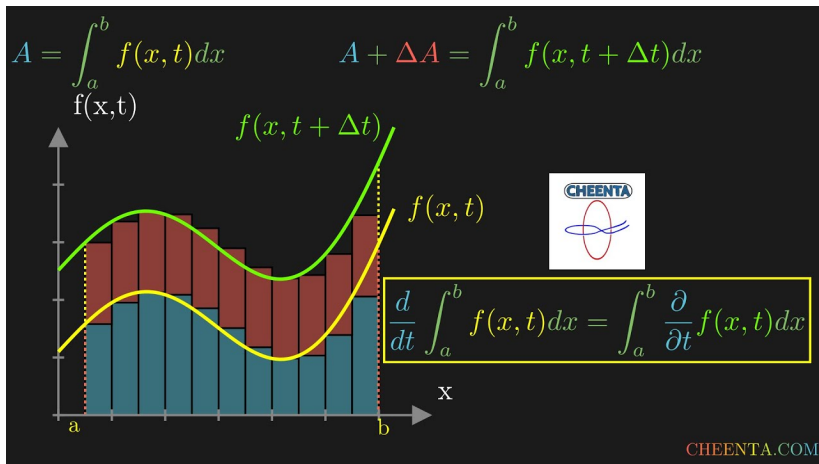
graph TD
    START([START]) --> TRY([TRY APPLYING  
INTEGRATION BY PARTS  
SUBSTITUTION])
    TRY --> DONE1{DONE?}
    DONE1 -- "HAHA, NOPE!" --> CAUCHY([CAUCHY'S FORMULA])
    DONE1 -- "HAHA, NOPE!" --> PARTIAL([PARTIAL FRACTIONS])
    DONE1 -- "HAHA, NOPE!" --> RIEMANN([RIEMANN INTEGRATION])
    DONE1 -- "HAHA, NOPE!" --> RISCH([RISCH ALGORITHM])
    DONE1 -- "HAHA, NOPE!" --> STOKES([STOKES' THEOREM])
    DONE1 -- "HAHA, NOPE!" --> BESSEL([WHAT THE HECK IS A BESSEL FUNCTION??])
    CAUCHY --> PARTIAL
    PARTIAL --> RIEMANN
    RIEMANN --> RISCH
    RISCH --> STOKES
    STOKES --> BESSEL
    BESSEL --> OHNO([OH NO])
    OHNO --> PHONE([PHONE CALLS TO MATHEMATICIANS])
    PHONE --> BURN([BURN THE EVIDENCE])
    BURN --> START
  
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7. LEIBNZ'S RULE

- A general rule for differentiating integrals.

$$\frac{d}{dt} \int_{a(t)}^{b(t)} f(x, t) dx = \frac{db(t)}{dt} f(b(t), t) - \frac{da(t)}{dt} f(a(t), t) = \int_{a(t)}^{b(t)} \frac{\partial f(x, t)}{\partial t} dx$$

7. LEIBNZ'S RULE



Review

REVIEW OF INTEGRALS

1. Definite Integral
2. Reimann Sum
3. Fundamental Theorem of Calculus
4. Integration Rules
5. Integration by Substitution
6. Integration by Parts
7. Leibnz's Rule

INTEGRATION QUESTION 1 ANSWER:

[◀ QUESTION](#)

$$-w^{-1} - \frac{5}{2}w^{-4} - 8w + C$$

INTEGRATION QUESTION 2 ANSWER:

[◀ QUESTION](#)

$$-4x^{-1} + 2x + \frac{1}{16}x^{-2} + C$$

INTEGRATION QUESTION 3 ANSWER:

◀ QUESTION

$$\begin{aligned}\int t^3 - \frac{e^{-t} - 4}{e^{-t}} dt &= \int t^3 - 1 + 4e^t dt \\ &= \frac{1}{4}t^4 - t + 4e^t + C\end{aligned}$$

INT. BY SUB. QUESTION 1 ANSWER:

◀ QUESTION

Re-write: $\int \frac{1}{(\frac{1}{x} + 1)x^2} dx$

$$u = \frac{1}{x} + 1$$

$$du = -\frac{1}{x^2} dx$$

$$\begin{aligned} \int \frac{1}{(\frac{1}{x} + 1)x^2} dx &= - \int \frac{1}{u} du \\ &= -\ln(u) + C \\ &= -\ln\left(\frac{1}{x} + 1\right) + C \end{aligned}$$

INT. BY SUB. QUESTION 2 ANSWER:

◀ QUESTION

$$\begin{aligned}u &= 4y^2 - y \\ du &= (8y - 1)dy \\ \int 3(8y - 1)e^{4y^2 - y} dy &= 3 \int e^u du \\ &= 3e^u + C \\ &= 3e^{4y^2 - y} + C\end{aligned}$$

INT. BY SUB. QUESTION 3 ANSWER:

◀ QUESTION

$$\begin{aligned}u &= t^4 + 2t \\du &= (4t^3 + 2)dt = 2(2t^3 + 1)dt \\ \int \frac{2t^3 + 1}{(t^4 + 2t)^3} &= \frac{1}{2} \int \frac{1}{u^3} du \\ &= \frac{-1}{4} (t^4 + 2t)^{-2} + C\end{aligned}$$

INT. BY PARTS QUESTION 1 ANSWER:

◀ QUESTION

$$u = x, dv = e^{2x}$$

$$du = 1, v = \frac{e^{2x}}{2}$$

$$\begin{aligned}\int (xe^{2x})dx &= \frac{xe^{2x}}{2} - \int \frac{e^{2x}}{2}dx \\ &= \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + C \\ &= \frac{(2x - 1)e^{2x}}{4} + C\end{aligned}$$

INT. BY PARTS QUESTION 2 ANSWER:

◀ QUESTION

$$u = 2 + 5x, dv = e^{\frac{1}{3}x} dx$$

$$du = 5dx, v = 3e^{\frac{1}{3}x}$$

$$\begin{aligned}\int (2 + 5x)e^{\frac{1}{3}x} dx &= 3e^{\frac{1}{3}x}(2 + 5x) - 15 \int e^{\frac{1}{3}x} dx \\ &= 3e^{\frac{1}{3}x}(2 + 5x) - 45e^{\frac{1}{3}x} + C \\ &= (15x - 39)e^{\frac{1}{3}x} + C\end{aligned}$$

INT. BY PARTS QUESTION 3 ANSWER:

◀ QUESTION

$$u = x^2, dv = e^x dx$$

$$du = 2x dx, v = e^x$$

$$\text{1st answer: } = x^2 e^x - 2 \int x e^x dx$$

$$u = x, dv = e^x dx$$

$$du = dx, v = e^x$$

$$\begin{aligned} \text{2nd answer: } &= x^2 e^x - 2(xe^x - e^x) + C \\ &= e^x(x^2 - 2x + 2) + C \end{aligned}$$