Day 8 Notes

Wednesday, August 18, 2021 9:04 AM

Math tricks

sum of geometric series

$$\sum_{i=0}^{\infty} r^i \quad \text{where } |r|<1$$

$$1+r+r^2+r^3+\dots$$

$$\text{Sum} = \frac{1}{1-r}$$

$$S_0 = \sum_{i=0}^{\infty} r^i = 1+r+r^2+\dots$$

$$S_1 = \sum_{i=1}^{\infty} r^i = r+r^2+r^3+\dots$$

$\boxed{S_0}$

Two facts 1) $S_0 = S_1 + 1$

2) $S_1 = rS_0$

$$S_0 - \underbrace{S_1}_{1} = 1$$

$$\downarrow$$
$$rS_0$$

$$\Rightarrow S_0(1-r) = 1$$

$$\Rightarrow S_0 = \frac{1}{1-r}$$

Systems

$$Ax = b$$

solve for x.

A: coefficient matrix

or a linear transformation

→ a function mapping x to b. It must satisfy:

$$f : X \to Y$$

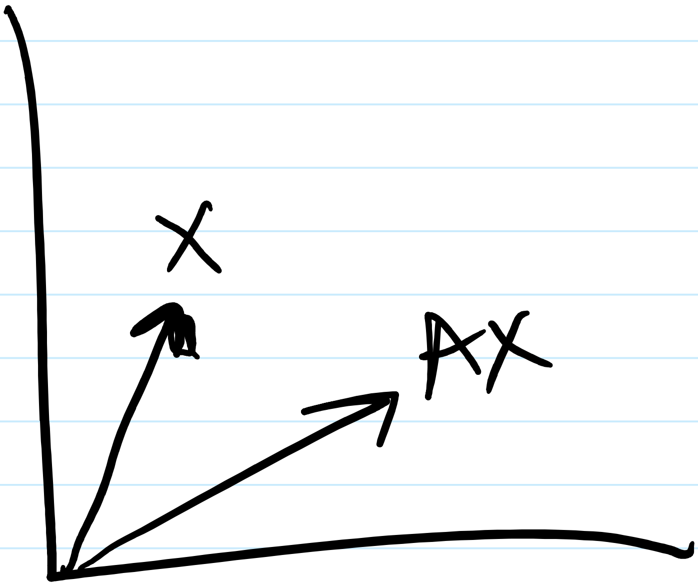
where $f(\alpha x + \beta z)$

$$= \alpha f(x) + \beta f(z)$$

If A is square then it is called a <u>linear operator</u> and it maps a space into itself, e.g. $A: \mathbb{R}^3 \to \mathbb{R}^3$

$$\underset{3\times 3}{A}\ \underset{3\times 1}{x} = \underset{3\times 1}{b}$$

$$x \in \mathbb{R}^3 \to b \in \mathbb{R}^3$$

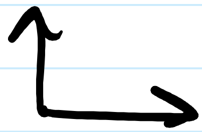


Correlation matrix

$$R = \begin{bmatrix} 1 & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{12} & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

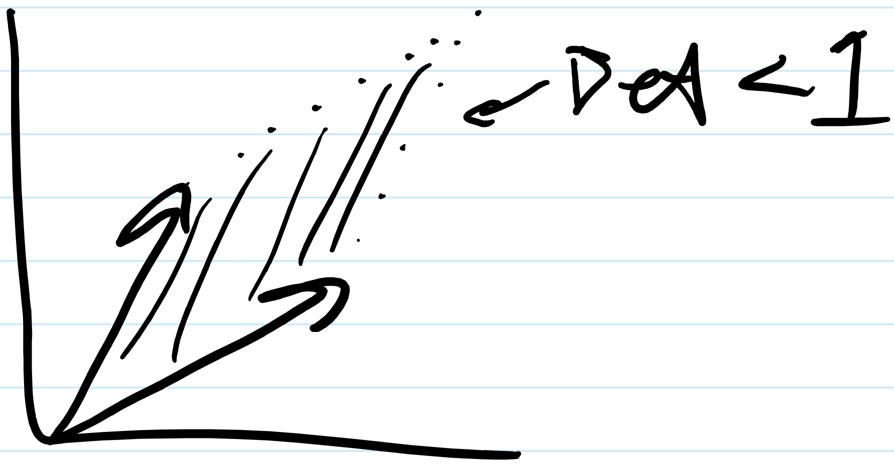
If all vars are uncorrelated then $\mathrm{Det}(R) = 1$. In that case $R = I$.

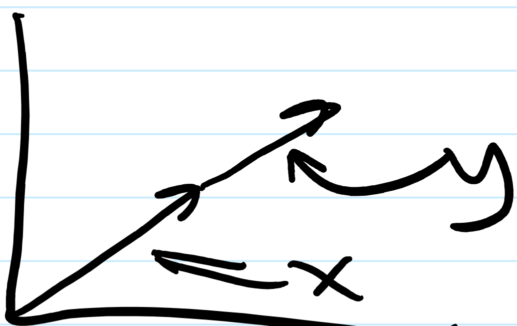
otherwise $|R| < 1$.



uncorrelated = orthogonal

Det is a unit cube





$y = \alpha X$

Then $|R| = 0$.

The correl matrix is singular, so R^{-1} DNE.

$(X'X)$ — $X'X$ is a rescaling of correlation matrix

$\frac{X'X}{n}$ is the sample covariance (if X is in deviations from means)

Determinant of an $n \times n$ matrix is HD(n).

So $Det(\alpha A) = \alpha^n Det(A)$

which comes from:

$Det\, B = \alpha\, Det\, A$ if B is just A with one column or row multiplied by α.

by α.

Systems

$$Ax = b$$

Gaussian elimination

$$\left[\begin{array}{c:c} A & b \end{array}\right]$$

Perform elementary row operations until A is in row echelon form.

Row-echelon form

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 6 & 6 & 8 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

pivots

Reduced R-E form

1) Pivots are 1
2) all other entries in a pivot column are 0

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 5 \\ 0 & 1 & 6 & 0 & 8 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

zeros

pivots = 1

A

B

C

D

E

A — Red. R.E.F.

B — REF

C — REF

D — Red. REF

E — REF

Gauss-Jordan

$[A \vdots b]$

step 3: STOP

$$\left[\begin{array}{ccc:c} 1 & 0 & 1 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

STOP No solution

$$\left[\begin{array}{ccc:c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

1) $a_1x + b_1y + c_1z = d_1$

2) $a_2x + b_2y + c_2z = d_2$

3) $a_3x + b_3y + c_3z = d_3$

$1 \cdot z = 5$

so $z = 5$

$y = 4$

$x = 3$

Mult (1) by 5

$$5a_1x + 5b_1y + 5c_1z = 5d_1$$

Add (1) to (2)

$$a_1x + a_2x + b_1y + b_2y + c_1z + c_2z$$
$$= d_1 + d_2$$

Ex | Solve:

$$4y + z = 2$$
$$2x + 6y - 2z = 3$$
$$4x + 8y - 5z = 4$$

$$\begin{array}{c} \quad A \qquad\qquad\qquad b \\ \left[\begin{array}{ccc:c} 0 & 4 & 1 & 2 \\ 2 & 6 & -2 & 3 \\ 4 & 8 & -5 & 4 \end{array}\right] \end{array}$$

$R_1 \leftrightarrow R_2$

$$\left[\begin{array}{ccc|c} 2 & 6 & -2 & 3 \\ 0 & 4 & 1 & 2 \\ 4 & 8 & -5 & 4 \end{array}\right]$$

$-2 \cdot R_1 + R_3$

$$\left[\begin{array}{ccc|c} 2 & 6 & -2 & 3 \\ 0 & 4 & 1 & 2 \\ 0 & -4 & -1 & -2 \end{array}\right]$$

$1 \cdot R_2 + R_3$

$$\left[\begin{array}{ccc|c} 2 & 6 & -2 & 3 \\ 0 & 4 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

<u>Row echelon form</u>

$-\frac{3}{2} R_2 + R_1$

$$\left[\begin{array}{ccc|c} 2 & 0 & -3.5 & 0 \\ 0 & 4 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc:c} 0 & 0 & 0 & 0 \end{array}\right]$$

$$\frac{1}{2}R_1 \sim \left[\begin{array}{ccc:c} 1 & 0 & -7/4 & 0 \\ 0 & 4 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\frac{1}{4}R_2 \sim \left[\begin{array}{ccc:c} 1 & 0 & -7/4 & 0 \\ 0 & 1 & 1/4 & 1/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Reduced REF.

$$x \quad -\frac{7}{4}z = 0$$

$$y + \frac{1}{4}z = 1/2$$

$$x = \frac{7}{4}z$$

$y = \frac{1}{2} - \frac{1}{4} z$

choose z.

→ infinitely many solutions.

Cramer's Rule

Let $Ax = b$ be a system of n equations in n unknowns. Then

$$x_i = \frac{\det A^i}{\det A}$$

A^i has the ith column of A replaced by b.

A replaced by b.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 3 & 3 \end{pmatrix} \quad b = \begin{pmatrix} 9 \\ 9 \\ 9 \end{pmatrix}$$

$$A^1 = \begin{pmatrix} 9 & 2 & 3 \\ 9 & 1 & 0 \\ 9 & 3 & 3 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 1 & 9 & 3 \\ 0 & 9 & 0 \\ 0 & 9 & 3 \end{pmatrix}$$

$$x^1 = \frac{\det\begin{pmatrix} 9 & 2 & 3 \\ 9 & 1 & 0 \\ 9 & 3 & 3 \end{pmatrix}}{\det\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 3 & 3 \end{pmatrix}}$$

$(0\ 3\ 3)$

Cramer's Rule

Ex

$$A = \begin{pmatrix} 7 & -2 \\ 3 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

$$A^1 = \begin{pmatrix} 3 & -2 \\ 5 & 1 \end{pmatrix} \qquad A^2 = \begin{pmatrix} 7 & 3 \\ 3 & 5 \end{pmatrix}$$

$$\det A = 7 \cdot 1 - (-2) \cdot 3$$
$$= 7 + 6 = 13$$

$$\det A^1 = 3 \cdot 1 - (-2)5$$
$$= 13$$

$$\det A^2 = 7 \cdot 5 - 3 \cdot 3$$
$$= 26$$

$$= 26$$

$$x_1 = \frac{\det A^1}{\det A} = \frac{13}{13} = 1$$

$$x_2 = \frac{\det A^2}{\det A} = \frac{26}{13} = 2$$

Check:

$7 \cdot 1 - 2 \cdot 2 = 3$ ✓

$3 \cdot 1 + 2 = 5$ ✓

Homogeneous systems

$Ax = 0$ the set of all solutions is called the nullspace

Trivial solution: $x = 0$.

If $Ax=0$ has only the trivial solution, the columns of A are called linearly independent. Otherwise if $\text{nullsp}(A) \neq \{0\}$, the columns are linearly dependent. Then at least column is a linear combo. of the others.

$$c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$$

Suppose not all $v_1, \cdots, v_n$ are zero.

Then $c_1v_1 + c_2v_2 + \cdots + c_{n-1}v_{n-1}$

$$= -c_n v_n$$

$$\Rightarrow \underbrace{v_n}_{} = -\frac{c_1}{c_n} v_1 - \frac{c_2}{c_n} v_2 - \cdots - \frac{c_{n-1}}{c_n} v_{n-1}$$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ lin ind.

$$0x_1 + 1x_2 = b_1$$
$$1x_1 + 0x_2 = b_2$$

homogeneous:

$$\left.\begin{aligned} x_2 &= 0 \\ x_1 &= 0 \end{aligned}\right\} \text{only trivial solution}$$

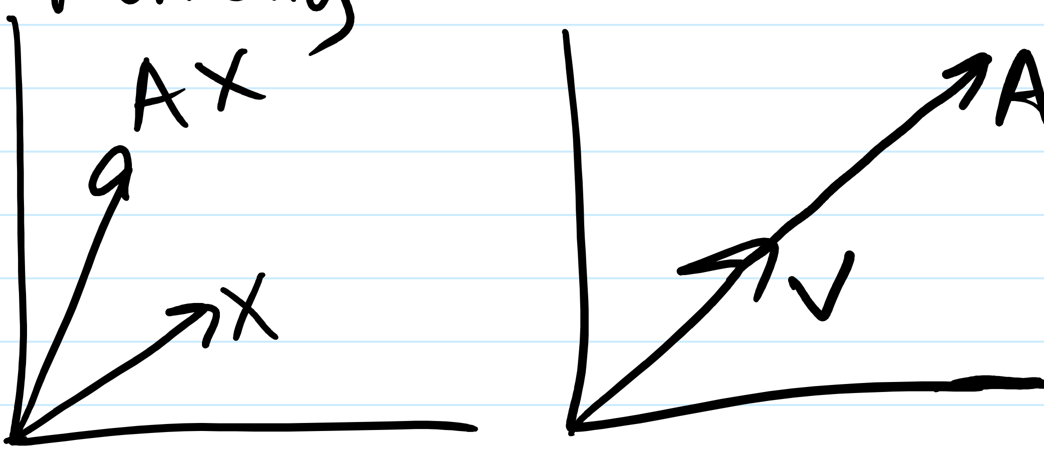
Eigenvalues and Eigenvectors

Let A be $n \times n$.

Then $\lambda \in \mathbb{C}$ is an <u>eigenvalue</u> of A iff $\exists v \neq 0$ such that $Av = \lambda v$

And v is called an <u>eigenvector</u> corresponding to λ.

Normally



$$Ax = \lambda x$$

$$\iff Ax = \lambda I x$$

$\iff Ax - \lambda I x = 0$

$\iff (A - \lambda I)x = 0$

This has nontrivial solutions iff $(A - \lambda I)$ is singular.

So $\text{Det}(A - \lambda I) = 0$

That leads to the characteristic equation

$\text{Det}(A - \lambda I) = 0$. We want to solve ~~it~~ for λ.

Ex $2x_1 + 3x_2 = \lambda x_1$

$4x_1 + 3x_2 = \lambda x_2$

Or $\begin{pmatrix} 2 & 3 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

char equation:

$$\det \begin{pmatrix} 2-\lambda & 3 \\ 4 & 3-\lambda \end{pmatrix} = 0$$

$$(2-\lambda)(3-\lambda) - 3 \cdot 4 = 0$$

$$6 - 2\lambda - 3\lambda + \lambda^2 - 12 = 0$$

$$\boxed{\lambda^2 - 5\lambda - 6 = 0}$$

quadratic equation

The 2 solutions are the eigenvalues of A.

In general the char. eq'n

In general the char. eq'n will be an n'th order polynomial in λ, and will have n roots.

The roots can be repeated, or 0, or complex.

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a = 1$
$b = -5$
$c = -6$

$$\lambda_1 = \frac{5 + \sqrt{25 + 24}}{2}$$

$$= \frac{5 + \sqrt{49}}{2}$$

$$= \frac{5 + 7}{2} = 6$$

$$\lambda_2 = \frac{5 - \sqrt{49}}{2}$$

$$= \frac{5-7}{2} = -1$$

Eigenvectors

$\lambda = 6$

$$\begin{pmatrix} 2-6 & 3 \\ 4 & 3-6 \end{pmatrix} x = 0$$

$$-4x_1 + x_2 = 0$$

$$4x_1 \quad 3x_2 = 0$$

$$\left(\begin{array}{cc:c} -4 & 3 & 0 \\ 4 & -3 & 0 \end{array}\right)$$

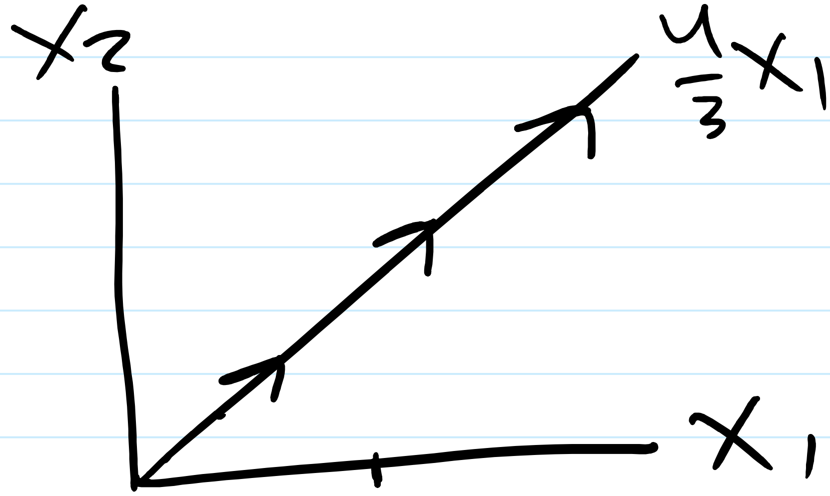
$$\underset{R_1+R_2}{\sim} \left(\begin{array}{cc:c} -4 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$-4x_1 + 3x_2 = 0$$

$$3x_2 = 4x_1$$

$$x_2 = \frac{4}{3}x_1$$

any $\begin{pmatrix} x_1 \\ 4/3\, x_1 \end{pmatrix}$ will be a solution for $\lambda = 6$.



Eigenvalue decomposition
or "spectral decomposition"

A square matrix is called
"diagonalizable" if $\exists$ a

"diagonalizable" if $\exists$ a invertible matrix C and a diagonal matrix Λ such that $\Lambda = C^{-1}AC$.

$$\Rightarrow C\Lambda C^{-1} = A$$

The columns of C are the eigenvectors (scaled to have length 1) and the diagonal elements of Λ are corresponding eigenvalues.

$$AC = C\Lambda$$

$$\begin{bmatrix} Ac_1 & Ac_2 & \cdots & Ac_n \end{bmatrix} = \begin{bmatrix} \lambda_1 c_1 & \lambda_2 c_2 & \cdots & \lambda c_n \end{bmatrix}$$

This says:

$Ac_1 = \lambda c_1$ ← def'n of eigenvalues/vectors

$Ac_2 = \lambda c_2$

$Ac_3 = \lambda c_3$

$\vdots$

A matrix is diagonalizable iff its <u>eigenvectors</u> are linearly independent.

<u>Positive and negative definite matrices</u>

A symmetric matrix is positive definite

A, matrix is positive definite if $\forall x \neq 0$, $x'Ax > 0$.

If A is positive definite:

- all eigenvalues are positive.
- $\exists$ a matrix W such that $A = W'W$. W is $m \times n$. (necessary <u>and</u> sufficient)

If A pos def then $A = W'W$ for some W.

If we have a matrix $X'X$, then it is <u>positive</u> <u>semidef.</u> and positive definite if the <u>columns</u> of X are the columns of X are linearly independent.

Let A be pos. def. Then

$A = W'W$ so

$$x'Ax = x'W'Wx$$
$$= (Wx)'Wx$$
$$= \text{a sum of squares}$$
$$= (Wx)_1^2 + (Wx)_2^2 + (Wx)_3^2 + \cdots$$

$$> 0$$

Vector Spaces

scalar mult: $\odot$

Vector addition: $\oplus$

$w, u, v \in S$, $a, b \in F$ (e.g. $\mathbb{R}$)

1) $a \odot u \oplus b \odot v \in S$

2) $u \oplus v = v \oplus u$

3) $u \oplus (v \oplus w) = (u \oplus v) \oplus w$

4) $\exists\, 0 \in S$ s.t. $u \oplus 0 = u$
$\forall u \in S$

5) $\forall u \in S$, $\exists -u$ s.t. $u \oplus -u = 0$

6) $(ab) \odot u = a \odot (b \odot u)$

7) $(a+b) \odot u = (a \odot u) \oplus (b \odot u)$

8) $a \odot (u \oplus v) = (a \odot u) \oplus (a \odot v)$

9) $1 \odot u = u \quad \forall u \in S$

Examples: $\mathbb{R}^n$ is a vector space where $\odot$ is just multiplying a vector by a constant and $\oplus$ is just adding vectors.

The set of $m \times n$ matrices is a vector space.

The set of <u>random variables</u> is a vector space.

The set of continuous fns on a closed interval.

$f(x) = 0$ is the zero vector.

$-f = -1 \cdot f$