Lecture 04
Integration

Ryan McWay[†]

[†] *Applied Economics,*
University of Minnesota

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University of Minnesota
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Last Lecture Review

- Derivatives:
  - Continutity & Differentiability
  - First & Second Derivatives
  - Derivative Rules
  - Implicit Function
  - l'Hopital's Rule
  - Taylor Series Approximation
  - Mean Value Theorem
  - Critical Points

Review Assignment

1. Problem Set 03 solutions are available on Github.
2. Any issues or problems **You** would like to discuss?

Daily Icebreaker

- Attendance via prompt:
  - Name
  - Daily Icebreaker: What 'pet peeve' do you have about something that is mundane or against a regularly accepted social norm?

Topic: Integration

Motivation

- General background
  - The total area under the curve.
  - Understanding the accumulation of the parts as a whole.
  - A core component of calculus alongside derivatives.
- Why do economists' care?
  - A tool to aggregate effects and approximate sums.
- Application in this career
  - Evaluating surplus or total welfare.

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Overview

1. Definite Integral
2. Reimann Sum
3. Fundamental Theorem of Calculus
4. Integration Rules
5. Integration by Substitution
6. Integration by Parts
7. Leibnz's Rule

1. Definite Integral

- Consider the anti-derivative $F(x)$.

$$\int_a^b f(x)dx = \lim_{x \to \Delta x} \sum_{x=a}^{b} f(x)\Delta x = F(b) - F(a) : F'(x) = f(x)$$

Demonstration: Definite Integral

Question:

$\int_1^4 (2x+3)dx$

Answer:

$$
\begin{aligned}
&= \int_1^4 (2x)dx + \int_1^4 (3)dx \\
&= \frac{2}{2}x^2 + 3x\Big|_1^4 \\
&= (4^2 + 3(4)) - (1^2 + 3(1)) \\
&= (16+12) - (1+3) \\
&= 24
\end{aligned}
$$

Demonstration: Definite Integral

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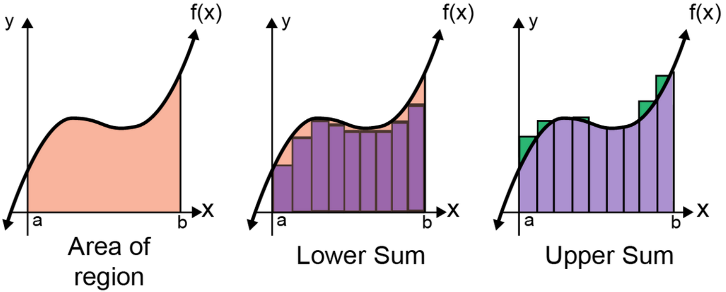
\end{aligned}
$$

2. Reimann Sum

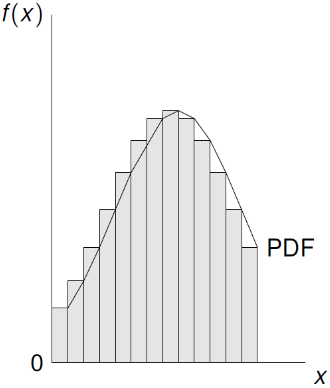
- Subdivide the interval (a, b) into N sub-intervals with endpoints x_i.

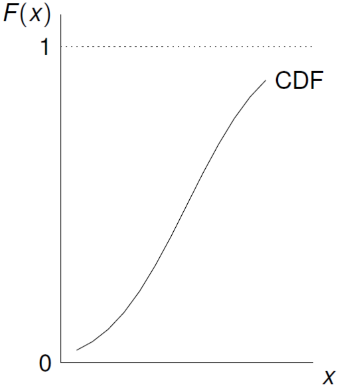
$$\lim_{\Delta \to 0} \sum_{i=1}^{N} f(x_i)\Delta = \int_a^b f(x)dx$$

2. Reimann Sum



Application: Probability and Cumulative Density Functions



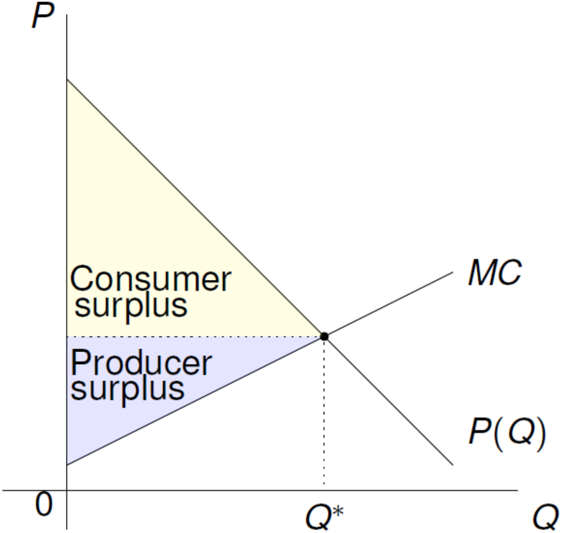


3. Fundamental Theorem of Calculus

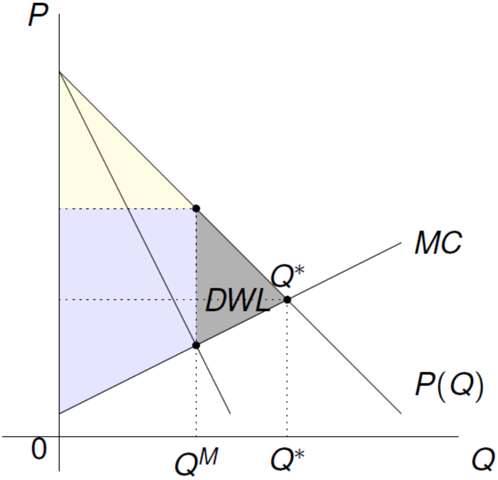
- Theorem that connects differentiation to integration.
- Let f be a continuous function on the open interval $[a, b]$. If $f(x) = F'(x)$, then:

$$\int_a^b f(x)dx = F(b) - F(a)$$

Application: Social Surplus



Application: Dead-weight Loss (DWL)



4. Integration Rules

- Constant:

$$\int a dx = ax + C$$

- Constant Multiplication:

$$\int cf(x)dx = c\int f(x)dx$$

- Reciprocal:

$$\int \frac{1}{x}dx = ln(x) + C$$

- Exponential:

$$\int e^x dx = e^x + C$$

$$\int a^x dx = \frac{a^x}{ln(a)} + C$$

Demonstration: Exponential

Question:

$\int (5^x) dx$

Answer:

$\frac{5^x}{ln(5)} + C$

Demonstration: Exponential

Question:

$\int (5^x) dx$

Answer:

$\frac{5^x}{ln(5)} + C$

4. Integration Rules

- Logarithm:
$$\int ln(x)dx = xln(x) - x + C$$
- Power Rule:
$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$
- Sum/Difference Rule:
$$\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx$$

Demonstration: Power Rule

Question:

$\int (x^5 + 3x^3 + 2x)dx$

Answer:

$\frac{1}{6}x^6 + \frac{3}{4}x^4 + x^2 + C$

Demonstration: Power Rule

Question:

$\int (x^5 + 3x^3 + 2x)dx$

Answer:

$\frac{1}{6}x^6 + \frac{3}{4}x^4 + x^2 + C$

Demonstration: Sum/Difference Rule

Question:

$\int(4x^2 + x - \frac{3}{x})dx$

Answer:

$\frac{4}{3}x^3 + \frac{1}{2}x^2 - 3ln(x) + C$

Demonstration: Sum/Difference Rule

Question:

$\int (4x^2 + x - \frac{3}{x})dx$

Answer:

$\frac{4}{3}x^3 + \frac{1}{2}x^2 - 3ln(x) + C$

Practice: Integration

1. $\int w^{-2} + 10w^{-5} - 8dw$

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Answer: Show Work

$$-w^{-1} - \frac{5}{2}w^{-4} - 8w + C$$

Practice: Integration

1. $\int w^{-2} + 10w^{-5} - 8dw$
2. $\int \frac{4}{x^2} + 2 - \frac{1}{8x^3} dx$

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1. $\int w^{-2} + 10w^{-5} - 8dw$
2. $\int \frac{4}{x^2} + 2 - \frac{1}{8x^3}dx$

Answer: Show Work

$$-4x^{-1} + 2x + \frac{1}{16}x^{-2} + C$$

Practice: Integration

1. $\int w^{-2} + 10w^{-5} - 8dw$
2. $\int \frac{4}{x^2} + 2 - \frac{1}{8x^3}dx$
3. $\int t^3 - \frac{e^{-t}-4}{e^{-t}}dt$

Practice: Integration

1. $\int w^{-2} + 10w^{-5} - 8dw$
2. $\int \frac{4}{x^2} + 2 - \frac{1}{8x^3}dx$
3. $\int t^3 - \frac{e^{-t} - 4}{e^{-t}}dt$

Answer: Show Work

$$\frac{1}{4}t^4 - t + 4e^t + C$$

5. Integration by Substitution

- Reverse chain rule from differentiation.
- Commonly referred to as u substitution.

$$\int f(g(x))g'(x)dx$$

$$\int f(u)du$$

Demonstration: Int. by Sub.

Question:

$\int x^2(3-10x^3)^4 dx$

Answer:

$$\begin{aligned} u =& 3-10x^3 \\ du =& -30x^2 dx \\ \implies dx =& \frac{-1}{30x^2} du \\ \int x^2(3-10x^3)^4 dx =& \frac{-1}{30}\int u^4 du \\ =& \frac{-1}{30}\cdot\frac{1}{5}u^5 + C \\ =& \frac{-1}{150}(3-10x^3)^5 + C \end{aligned}$$

Demonstration: Int. by Sub.

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Practice: Int. by Sub.

1. $\int \frac{1}{x^2+x} dx$

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Answer: Show Work

$$-ln(\frac{1}{x}+1)+C$$

Practice: Int. by Sub.

1. $\int \frac{1}{x^2+x} dx$
2. $\int 3(8y-1)e^{4y^2-y} dy$

Practice: Int. by Sub.

1. $\int \frac{1}{x^2+x} dx$
2. $\int 3(8y-1)e^{4y^2-y} dy$

Answer: Show Work

$$3e^{4y^2-y} + C$$

Practice: Int. by Sub.

1. $\int \frac{1}{x^2+x} dx$
2. $\int 3(8y-1)e^{4y^2-y} dy$
3. $\int \frac{2t^3+1}{(t^4+2t)^3} dt$

Practice: Int. by Sub.

1. $\int \frac{1}{x^2+x} dx$
2. $\int 3(8y-1)e^{4y^2-y} dy$
3. $\int \frac{2t^3+1}{(t^4+2t)^3} dt$

Answer: Show Work

$$\frac{-1}{4}(t^4+2t)^{-2}+C$$

6. Integration by Parts

- Reverse product rule from differentiation.
- Rarely used in economic applications, but important to know.

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

$$\int udv = u \cdot v - \int vdu$$

Demonstration: Int. by Parts

Question:

$\int ln(x)dx$

Answer:

$$
\begin{aligned}
u =& ln(x) \ , dv = 1 \\
du =& \frac{1}{x} \ , v = x \\
\int ln(x)dx =& ln(x)x - \int x \cdot \frac{1}{x} dx \\
=& x ln(x) - x + C \\
=& x(ln(x) - 1) + C
\end{aligned}
$$

Demonstration: Int. by Parts

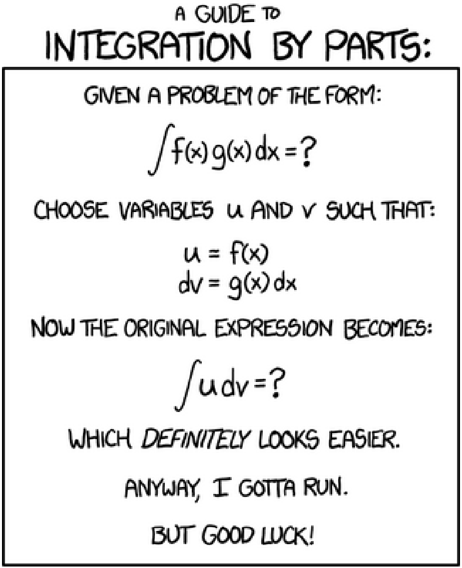
Question:

$\int ln(x)dx$

Answer:

$$
\begin{aligned}
u =& ln(x) \text{ , } dv = 1 \\
du =& \frac{1}{x} \text{ , } v = x \\
\int ln(x)dx =& ln(x)x - \int x \cdot \frac{1}{x} dx \\
=& xln(x) - x + C \\
=& x(ln(x) - 1) + C
\end{aligned}
$$

6. Integration by Parts



Practice: Int. by Parts

1. $\int (xe^{2x})dx$

Practice: Int. by Parts

1. $\int (xe^{2x})dx$

Answer: Show Work

$$\frac{(2x-1)e^{2x}}{4} + C$$

Practice: Int. by Parts

1. $\int (xe^{2x})dx$
2. $\int (2+5x)e^{\frac{1}{3}x}dx$

Practice: Int. by Parts

1. $\int (xe^{2x})dx$
2. $\int (2+5x)e^{\frac{1}{3}x}dx$

Answer: Show Work

$$(15x-39)e^{\frac{1}{3}x}+C$$

Practice: Int. by Parts

1. $\int (xe^{2x})dx$
2. $\int (2+5x)e^{\frac{1}{3}x}dx$
3. $\int x^2e^x dx$

Practice: Int. by Parts

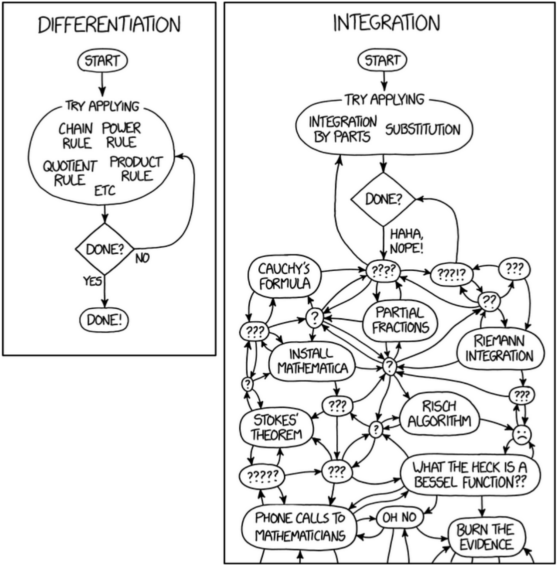
1. $\int (xe^{2x})dx$
2. $\int (2+5x)e^{\frac{1}{3}x}dx$
3. $\int x^2e^xdx$

Answer: Show Work

Int. by Parts **Twice** ...

$$e^x(x^2-2x+2)+C$$

It Can Get . . . Complicated

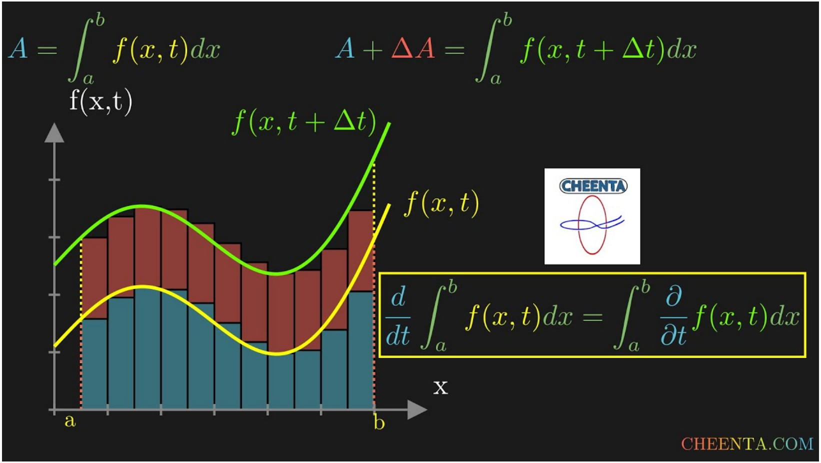


7. Leibnz's Rule

- A general rule for differentiating integrals.

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)dx = \frac{db(t)}{dt}f(b(t),t) - \frac{da(t)}{dt}f(a(t),t) = \int_{a(t)}^{b(t)} \frac{\partial f(x,t)}{\partial t}dx$$

7. Leibnz's Rule



Review

Review of Integrals

1. Definite Integral
2. Reimann Sum
3. Fundamental Theorem of Calculus
4. Integration Rules
5. Integration by Substitution
6. Integration by Parts
7. Leibnz's Rule

Assignment

- Readings on Multi-variate Calculus before Lecture 05:
  - S&B Ch. 14, 15, & 20
- Assignment:
  - Problem Set 04 (PS04)
  - Solution set will be available following end of Lecture 05
- Struggling?
  1. Read the 'Encouraged Reading'
  2. Review 'Supplementary material'
  3. Reach out directly

Integration Question 1 Answer:

◂ Question

$$-w^{-1} - \frac{5}{2}w^{-4} - 8w + C$$

Integration Question 2 Answer:

◂ Question

$$-4x^{-1} + 2x + \frac{1}{16}x^{-2} + C$$

Integration Question 3 Answer:

Question

$$\int t^3 - \frac{e^{-t} - 4}{e^{-t}} dt = \int t^3 - 1 + 4e^t dt$$
$$= \frac{1}{4}t^4 - t + 4e^t + C$$

Int. by Sub. Question 1 Answer:

◂ Question

Re-write: $$\int \frac{1}{(\frac{1}{x}+1)x^2}dx$$

$$\begin{aligned}
u =& \frac{1}{x}+1 \\
du =& \frac{-1}{x^2}dx \\
\int \frac{1}{(\frac{1}{x}+1)x^2}dx =& -\int \frac{1}{u}du \\
=& -ln(u)+C \\
=& -ln(\frac{1}{x}+1)+C
\end{aligned}$$

Int. by Sub. Question 2 Answer:

◂ Question

$$
\begin{aligned}
u &= 4y^2 - y \\
du &= (8y-1)dy \\
\int 3(8y-1)e^{4y^2-y}dy &= 3\int e^u du \\
&= 3e^u + C \\
&= 3e^{4y^2-y} + C
\end{aligned}
$$

Int. by Sub. Question 3 Answer:

◂ Question

$$\begin{aligned} u &= t^4 + 2t \\ du &= (4t^3 + 2)dt = 2(2t^3 + 1)dt \\ \int \frac{2t^3 + 1}{(t^4 + 2t)^3} &= \frac{1}{2} \int \frac{1}{u^3} du \\ &= \frac{-1}{4} (t^4 + 2t)^{-2} + C \end{aligned}$$

Int. by Parts Question 1 Answer:

◂ Question

$$
\begin{aligned}
u =& x \text{ , } dv = e^{2x} \\
du =& 1 \text{ , } v = \frac{e^{2x}}{2} \\
\int (xe^{2x}) dx =& \frac{xe^{2x}}{2} - \int \frac{e^{2x}}{2} dx \\
=& \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + C \\
=& \frac{(2x-1)e^{2x}}{4} + C
\end{aligned}
$$

Int. by Parts Question 2 Answer:

◂ Question

$$
\begin{aligned}
u =& 2+5x \text{ , } dv = e^{\frac{1}{3}x}dx \\
du =& 5dx \text{ , } v = 3e^{\frac{1}{3}x} \\
\int (2+5x)e^{\frac{1}{3}x}dx =& 3e^{\frac{1}{3}x}(2+5x) - 15\int e^{\frac{1}{3}x}dx \\
=& 3e^{\frac{1}{3}x}(2+5x) - 45e^{\frac{1}{3}x} + C \\
=& (15x-39)e^{\frac{1}{3}x} + C
\end{aligned}
$$

Int. by Parts Question 3 Answer:

◂ Question

$$
\begin{aligned}
u =& x^2 \ , \ dv = e^x dx \\
du =& 2x dx \ , \ v = e^x \\
\text{1st answer: } =& x^2 e^x - 2\int x e^x dx \\
u =& x \ , \ dv = e^x dx \\
du =& dx \ , \ v = e^x \\
\text{2nd answer: } =& x^2 e^x - 2(xe^x - e^x) + C \\
=& e^x(x^2 - 2x + 2) + C
\end{aligned}
$$