Problem Set 1

APEC Math Review

August 2020

1. (Simon & Blume Exercise A1.3) Write out careful proofs of the following properties of operations.

   (a) $(A \cap B)^c = A^c \cup B^c$

   (b) $(A \cup B)^c = A^c \cap B^c$

   (c) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

A1.3 *a)* $x \in (A \cap B)^c \iff x \notin A \cap B \iff x \notin A$ or $x \notin B \iff x \in A^c$ or $x \in B^c$ $x \in A^c \cup B^c$.

b) $x \in (A \cup B)^c \iff x \notin A \cup B \iff x \notin A$ and $x \notin B \iff x \in A^c$ and $x \in B^c \iff x \in A^c \cap B^c$.

c) $x \in A \cap (B \cup C)$

$\iff x \in A$ and $x \in B \cup C$

$\iff (x \in A$ and $x \in B)$ or $(x \in A$ and $x \in C)$

$\iff x \in A \cap B$ or $x \in A \cap C$

$\iff x \in (A \cap B) \cup (A \cap C)$.

2. (Jehle Reny, Exercise 1.4) The strict preference relation $\succ$ is defined by

$$x \succ y \Leftrightarrow x \succsim y \, but \, not \, y \succsim x$$

The indifference relation $\sim$ is defined by

$$x \sim y \Leftrightarrow x \succsim y \, and \, y \succsim x$$

Prove that if $\succsim$ is a transitive, then the relation $\succ$ is transitive and the relation $\sim$ is transitive.

1.4 To get you started, take the indifference relation. Consider any three points $\mathbf{x}^i \in X$, $i = 1, 2, 3$, where $\mathbf{x}^1 \sim \mathbf{x}^2$ and $\mathbf{x}^2 \sim \mathbf{x}^3$. We want to show that $\mathbf{x}^1 \sim \mathbf{x}^2$ and $\mathbf{x}^2 \sim \mathbf{x}^3 \Rightarrow \mathbf{x}^1 \sim \mathbf{x}^3$. By definition of $\sim$, $\mathbf{x}^1 \sim \mathbf{x}^2 \Rightarrow \mathbf{x}^1 \succsim \mathbf{x}^2$ and $\mathbf{x}^2 \succsim \mathbf{x}^1$. Similarly, $\mathbf{x}^2 \sim \mathbf{x}^3 \Rightarrow \mathbf{x}^2 \succsim \mathbf{x}^3$ and $\mathbf{x}^3 \succsim \mathbf{x}^2$. By transitivity of $\succsim$, $\mathbf{x}^1 \succsim \mathbf{x}^2$ and $\mathbf{x}^2 \succsim \mathbf{x}^3 \Rightarrow \mathbf{x}^1 \succsim \mathbf{x}^3$. Keep going.

3. (Jehle Reny, Exercise A1.16) Let S and T to be convex sets. Prove that each of the following is also a convex set:

   (a) $-S \equiv \{\mathbf{x} \,|\, \mathbf{x} = -\mathbf{s}, \mathbf{s} \in S\}$.

(b) $S - T \equiv \{\mathbf{x} \,|\, \mathbf{x} = \mathbf{s} - \mathbf{t}, \mathbf{s} \in S, \mathbf{t} \in T\}$.

4. Let $A \equiv \{(\mathbf{x}, y) \,|\, \mathbf{x} \in D, f(\mathbf{x}) \geq y\}$ be the set of points on and below the graph of $f : D \to R$ where $D \subset \mathbb{R}^n$ is a convex set and $R \subset \mathbb{R}$, then f is a concave function if and only if A is a convex set.

Assume f is a concave function. Then for $\mathbf{x}^t \equiv t\mathbf{x}^1 + (1-t)\mathbf{x}^2$ and by the definition of concave functions,

$$f(\mathbf{x}^t) \geq tf(\mathbf{x}^1) + (1-t)f(\mathbf{x}^2) \quad \text{for all } \mathbf{x}^1, \mathbf{x}^2 \in D, \text{ and } t \in [0,1]. \tag{P.1}$$

Take any two points $(\mathbf{x}^1, y^1) \in A$ and $(\mathbf{x}^2, y^2) \in A$. By definition of A,

$$f(\mathbf{x}^1) \geq y^1 \quad \text{and} \quad f(\mathbf{x}^2) \geq y^2. \tag{P.2}$$

To prove that A is a convex set, we must show that the convex combination $(\mathbf{x}^t, y^t) \equiv (t\mathbf{x}^1 + (1-t)\mathbf{x}^2, ty^1 + (1-t)y^2)$ is also in A for all $t \in [0,1]$. Because D is a convex set by assumption, we know $\mathbf{x}^t \in D$ for all $t \in [0,1]$. Thus, we need only show that $f(\mathbf{x}^t) \geq y^t$ for all $t \in [0,1]$ to establish $(\mathbf{x}^t, y^t) \in A$. But that is easy. From (P.2), we know that $f(\mathbf{x}^1) \geq y^1$ and $f(\mathbf{x}^2) \geq y^2$. Multiplying the first of these by $t \geq 0$ and the second by $(1-t) \geq 0$ gives us $tf(\mathbf{x}^1) \geq ty^1$ and $(1-t)f(\mathbf{x}^2) \geq (1-t)y^2 \ \forall\ t \in [0,1]$. Adding these last two inequalities together gives us

$$tf(\mathbf{x}^1) + (1-t)f(\mathbf{x}^2) \geq ty^1 + (1-t)y^2.$$

Using (P.1), and remembering that $y^t \equiv ty^1 + (1-t)y^2$, gives us

$$f(\mathbf{x}^t) \geq y^t.$$

Thus, $(\mathbf{x}^t, y^t) \in A$, so A is a convex set.

That completes the first part of the proof and establishes that f concave $\Rightarrow A$ is a convex set. We need to prove the second part next.

Second part: A convex $\Rightarrow f$ concave.

Here we assume that A is a convex set and must show that f is therefore a concave function. The strategy for this part of the proof is to pick *any* two points in the domain D of f but two *particular* points in the set A, namely, the two points in A that are on, rather than beneath, the graph of f corresponding to those two points in its domain. If we can use the convexity of the set A to establish that f must satisfy the definition of a concave function at these two points in its domain, we will have established the assertion in general because those two points in the domain are chosen arbitrarily.

Choose $\mathbf{x}^1 \in D$ and $\mathbf{x}^2 \in D$, and let y^1 and y^2 satisfy

$$y^1 = f(\mathbf{x}^1) \text{ and } y^2 = f(\mathbf{x}^2). \tag{P.3}$$

The points $(\mathbf{x}^1, y^1)$ and $(\mathbf{x}^2, y^2)$ are thus in A because they satisfy $\mathbf{x}^i \in D$ and $f(\mathbf{x}^i) \geq y^i$ for each i. Now form the convex combination of these two points, $(\mathbf{x}^t, y^t)$. Because A is a convex set, $(\mathbf{x}^t, y^t)$ is also in A for all $t \in [0,1]$. Thus,

$$f(\mathbf{x}^t) \geq y^t. \tag{P.4}$$

Now $y^t \equiv ty^1 + (1-t)y^2$, so we can substitute for y^i from (P.3) and write

$$y^t = tf(\mathbf{x}^1) + (1-t)f(\mathbf{x}^2). \tag{P.5}$$

Combining (P.4) and (P.5), we have $f(\mathbf{x}^t) \geq tf(\mathbf{x}^1) + (1-t)f(\mathbf{x}^2) \ \forall\ t \in [0,1]$, so f is a concave function.