Day 12 Notes - optimization 3

Tuesday, August 24, 2021 9:03 AM

Envelope Theorem

objective function $F(x, y, \alpha)$

maximized s.t. $G(x, y, \alpha) = 0$

$$V(\alpha) = F(x^*(\alpha), y^*(\alpha), \alpha)$$

$$\frac{dV(\alpha)}{d\alpha} = \underbrace{\frac{\partial F}{\partial \alpha}}_{\text{direct effect of } \alpha \text{ on } F} - \underbrace{\lambda}_{\text{"value" of changing the constraint}} \underbrace{\frac{\partial G}{\partial \alpha}}_{\text{direct effect on the constraint}}$$

Proof (Varian p.502)

$$\frac{dV}{d\alpha} = \frac{\partial F}{\partial x} \cdot \frac{\partial x}{\partial \alpha} + \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial \alpha} + \frac{\partial F}{\partial \alpha}$$

FOC's: $\frac{\partial F}{\partial x} = \lambda \frac{\partial G}{\partial x}$

$\frac{\partial F}{\partial y} = \lambda \frac{\partial G}{\partial y}$

$$\Rightarrow \frac{dV}{d\alpha} = \lambda \left[\frac{\partial G}{\partial x} \cdot \frac{\partial x}{\partial \alpha} + \frac{\partial G}{\partial y} \cdot \frac{\partial y}{\partial \alpha} \right] + \frac{\partial F}{\partial \alpha}$$

Now see the constraint

$G(x^*, y^*, \alpha) = 0$, Differentiate both sides:

$$\frac{\partial G}{\partial x} \cdot \frac{\partial x^*}{\partial \alpha} + \frac{\partial G}{\partial y} \cdot \frac{\partial y^*}{\partial \alpha} + \frac{\partial G}{\partial \alpha} = 0 \quad *$$

$$\Rightarrow \frac{dV}{d\alpha} = \lambda \left[-\frac{\partial G}{\partial \alpha} \right] + \frac{\partial F}{\partial \alpha}$$

$$= \frac{\partial F}{\partial \alpha} - \lambda \frac{\partial G}{\partial \alpha} \qquad \text{QED.}$$

Note: this also works w/ inequality constraints as long as they're binding at the optimum.

Micro Prelim - summer 2017

Question I.1

Consider a consumer with the following utility function for three goods, x_1, x_2 and x_3:

$$u = \beta_1 \ln(x_1) + \beta_2 \ln(x_2) + \beta_3 x_3, \quad \text{with } \beta_1 > 0, \beta_2 > 0 \text{ and } \beta_3 > 0,$$

where ln( ) is the natural logarithm function.

(a) Derive the Walrasian demands functions for x_1, x_2 and x_3. Assume that both wealth (w) and all prices (p_1, p_2 and p_3) are strictly greater than 0. For this part, you can assume an interior solution (strictly positive demand for all three goods).

(b) Are the Walrasian demands that you derived for part (a) necessarily positive when all prices and wealth are strictly greater than 0? The answer to this question should be very brief. Also, the Walrasian demand functions you just derived correspond to a particular type of preferences. What is this type of preferences? Just state the answer, there is no need to demonstrate it.

$$\max_{x_1, x_2, x_3} \beta_1 \ln x_1 + \beta_2 \ln x_2 + \underbrace{\beta_3 x_3}_{\text{numeraire}}$$

s.t. $p_1 x_1 + p_2 x_2 + p_3 x_3 \le w$ (= b.c. of Walras' Law)

$x_1, x_2, x_3 > 0$ ← assum

①: objective fn is strictly increasing in all arguments. By Walras' Law, the Budget constraint will hold w/ Equality

②: Write down Kuhn-Tucker conditions

$\frac{\beta_1}{x_1} \le \lambda p_1$, $x_1 \ge 0$, C.S.

$\frac{\beta_2}{x_2} \le \lambda p_2$, $x_2 \ge 0$, C.S.

$$\frac{\beta_1}{x_1} \le \lambda p_1, \quad \underbrace{x_1 \ge 0}, \quad C.S.$$

$$\frac{\beta_2}{x_2} \le \lambda p_2, \quad \underbrace{x_2 \ge 0}, \quad C.S.$$

$$\beta \le \lambda p_3, \quad x_3 \ge 0, C.S.$$

these are > so C.S. ⟹ = for the ~~first~~ cond'ns

$$p_1x_1 + p_2x_2 + p_3x_3 = W$$

③ C.S. and interior soln ⟹

$$\frac{\beta_1}{x_1} = \lambda p_1$$

$$\frac{\beta_2}{x_2} = \lambda p_2$$

$$\beta_3 = \lambda p_3$$

and. B.C.

④ $\lambda = \frac{\beta_3}{p_3}$

⑤ $\beta_1 = \left(\frac{\beta_3}{p_3}\right) \cdot p_1 \cdot x_1$

$$\beta_2 = \left(\frac{\beta_3}{p_3}\right) \cdot p_2 \cdot x_2$$

$$\Rightarrow \boxed{x_1^* = \frac{\beta_1}{p_1} \cdot \frac{p_3}{\beta_3}} \leftarrow >0$$

$$\boxed{x_2^* = \frac{\beta_2}{p_2} \cdot \frac{p_3}{\beta_3}} \leftarrow >0$$

⑥ Plug x_1^*, x_2^* into B.C. and solve for x_3

$$p_1\left(\frac{\beta_1}{p_1} \cdot \frac{p_3}{\beta_3}\right) + p_2\left(\frac{\beta_2}{p_2} \cdot \frac{p_3}{\beta_3}\right) + p_3x_3 = W$$

$$\Rightarrow x_3 = \frac{W - \beta_1\left(\frac{p_3}{\beta_3}\right) - \beta_2\left(\frac{p_3}{\beta_3}\right)}{p_3}$$

$$\boxed{x_3^* = \frac{W}{p_3} - \frac{\beta_1}{\beta_3} - \frac{\beta_2}{\beta_3}}$$

not necessarily >0

If you want: solve for conditions on the β's and p's that give an interior sol'n.

Quasilinear preferences

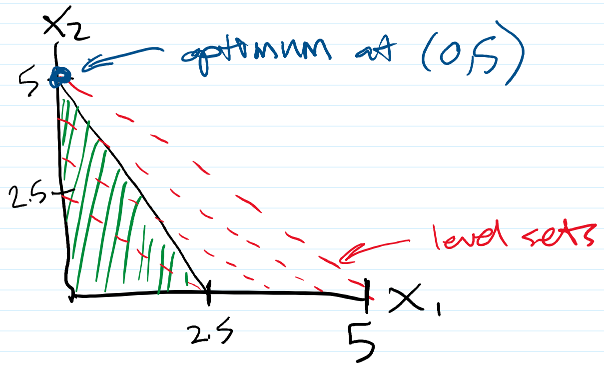
Demand for numeraire depends only on its <u>own</u> price and wealth.

Demand for <u>other</u> goods are independent of wealth.

Example: corner solution

$$\max_{x_1, x_2} \; x_1 + x_2 \quad \text{s.t. } 2x_1 + x_2 \le 5$$

linear objective fn and constraint (linear programming)



① Kuhn-Tucker Conditions (F.O.C.'s)

(1) $1 \le \lambda \cdot 2$, $x_1 \ge 0$, c.s.

(2) $1 \le \lambda$, $x_2 \ge 0$, c.s.

(3) $2x_1 + x_2 \le 5$, $\lambda \ge 0$, c.s.

② (1) and (2) imply:

$\lambda \ge \frac{1}{2}$ and $\lambda \ge 1$

$\Rightarrow \lambda > 0$

③ $\lambda > 0$ and c.s. imply:

$2x_1 + x_2 = 5$ (w/ equality)

④ Consider (1) and (2)

suppose (1) holds w/ equality.
Then $\lambda = \frac{1}{2}$. But that contradicts (2) which says $\lambda \ge 1$.

So c.s. requires $x_1 = 0$

⑤ Plug $x_1 = 0$ into constraint

$2 \cdot 0 + x_2 = 5$

$\Rightarrow x_2 = 5$

⑥ Solution: $\boxed{x_1 = 0,\ x_2 = 5}$

What about λ?

We know $\lambda \geq 1$. We also know $x_2 = 5 > 0$.
So C.S. requires $\boxed{\lambda = 1.}$

Example: Cost minimization

$\min_{z \geq 0} \; c = r_1 z_1 + r_2 z_2$ r_1, r_2 input prices] $r_1, r_2 > 0$

s.t. $\underbrace{\dfrac{\sqrt{z_1 z_2}}{q_1^2 + q_2^2}}_{\text{output distance function}} \geq 1$ z_1, z_2 inputs

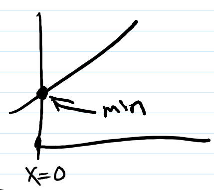
q_1, q_2 outputs → fixed

Kuhn-Tucker conditions:

$$\begin{cases} r_1 \geq \lambda \cdot \underbrace{\left(\dfrac{\frac{1}{2}(z_1 z_2)^{-1/2}}{q_1^2 + q_2^2} \cdot z_2\right)}_{= \frac{\partial}{\partial z_1}\left(\frac{\sqrt{z_1 z_2}}{q_1^2 + q_2^2}\right)}, \quad z_1 \geq 0, \quad \text{complementary slackness} \\ r_2 \geq \lambda \cdot \left(\dfrac{\frac{1}{2}(z_1 z_2)^{-1/2} \cdot z_1}{q_1^2 + q_2^2}\right), \quad z_2 \geq 0, \quad \text{C.S.} \\ \dfrac{\sqrt{z_1 z_2}}{q_1^2 + q_2^2} \geq 1, \quad \lambda \geq 0, \quad \text{C.S.} \end{cases}$$

$\geq$ ← minimization

$[G(z, q) \geq c$ so $\lambda \geq 0]$



Aside: Sign of λ (Chiang)

If you have maximization problem, set it up like $\underline{G(x,y) \leq c}$ → increasing c relaxes the constraint, which allows for a higher max, so $\lambda \geq 0$.

If you have a minimization problem, set it up like: $\underline{G(x,y) \geq c}$ → increasing c tightens the constraint, so you get a worse minimum (higher), so $\lambda \geq 0$.

If you did: Min: $G(x,y) \leq c$

→ then increasing c gives a better min, so $\lambda \leq 0$. → you can choose bec. $-G(x,y) \leq -c$ is an equivalent constraint

$$
\begin{cases}
r_1 \geq \lambda \cdot \left(\frac{\frac{1}{2}(z_1 z_2)^{-1/2}}{q_1^2 + q_2^2} \cdot z_2 \right) & , z_1 \geq 0, \text{ C.S} \quad (1) \\
r_2 \geq \lambda \cdot \left(\frac{\frac{1}{2}(z_1 z_2)^{-1/2} \cdot z_1}{q_1^2 + q_2^2} \right) & , z_2 \geq 0, \text{ C.S.} \quad (2) \\
\frac{\sqrt{z_1 z_2}}{q_1^2 + q_2^2} \geq 1 & , \lambda \geq 0, \text{ C.S.} \quad (3)
\end{cases}
$$

Find an interior solution. So $z_1, z_2 > 0$ and (1) and (2) will hold with equality.

Take the ratio of (1) and (2).

$$\Rightarrow \frac{r_1}{r_2} = \frac{z_2}{z_1} \Rightarrow z_2 = \frac{r_1}{r_2} z_1$$

Now plug into B.C.

$$\frac{\sqrt{z_1 \cdot \frac{r_1}{r_2} z_1}}{q_1^2 + q_2^2} \geq 1$$

$$\Rightarrow \frac{z_1 \cdot \sqrt{\frac{r_1}{r_2}}}{q_1^2 + q_2^2} \geq 1 \qquad (4)$$

Let's get rid of the inequality. It will hold with equality if $\lambda > 0$ (by C.S.).

(1) says $r_1 = \lambda \cdot (\quad)$ but we know $r_1 > 0$.
So $\lambda \neq 0$. $\therefore \lambda > 0$
and (4) holds w/ equality.

$$(4) \Rightarrow z_1 = \frac{q_1^2 + q_2^2}{\sqrt{\frac{r_1}{r_2}}} = \frac{\sqrt{r_2}}{\sqrt{r_1}} (q_1^2 + q_2^2)$$

$$z_2 = \frac{q_1^2 + q_2^2}{\sqrt{\frac{r_2}{r_1}}}$$

Cost function: $C(q_1, q_2, r_1, r_2) = r_1 z_1^* + r_2 z_2^*$

$$= \sqrt{r_1 r_2}\,(q_1^2 + q_2^2) + \sqrt{r_1 r_2}\,(q_1^2 + q_2^2)$$

$$\boxed{C(q, r) = 2\sqrt{r_1 r_2}\,(q_1^2 + q_2^2)}$$

Comparative Statics

envelope thm: how does value fn vary with the parameters?

can also look at: consumer demand
factor demand
output
→ how do the optimal choices vary with parameters?

Cost minimization (Varian, ch. 4)

$\min_z r \cdot z$ s.t. $f(z) = q$

f: production function
q: output (fixed)

FOC's

$$r_1 = \lambda \frac{\partial f}{\partial z_1}$$

$$r_2 = \lambda \frac{\partial f}{\partial z_2}$$

$$f(z) = q$$

Look at $\frac{r_1}{r_2}$: $\frac{r_1}{r_2} = \frac{\frac{\partial f}{\partial z_1}}{\frac{\partial f}{\partial z_2}}$ ← marginal product of z_1

Ex: $\frac{w}{r} = \frac{MP_L}{MP_K} = MRS_{LK}$

Now let's vary some parameters.

Note: we haven't solved for z_1^* or z_2^* but the implicit function theorem allows us to know the sensitivity of z_1^* and z_2^* to the parameters.

First let's look at r_1.

FOC: $r_1 = \lambda f_1$
$r_2 = \lambda f_2$
$f(z) = q$

Re-write:

$$r_1 = \lambda \frac{\partial f(z_1^*(r_1, r_2, q), z_2^*(r_1, r_2, q))}{\partial z_1} = \lambda f_1(z_1^*, z_2^*)$$

$$r_2 = \lambda \frac{\partial f(z_1^*(r_1,r_2,q), z_2^*(r_1,r_2,q))}{\partial z_2}$$

$$f(z_1^*(r_1,r_2,q), z_2^*(r_1,r_2,q)) = q$$

Note: λ is also a function of r_1, r_2, q

→ so need to apply product rule

Differentiate FOCs wrt. r_1:

$$\begin{cases} 1 = \lambda\left(f_{11}\cdot\frac{\partial z_1}{\partial r_1} + f_{12}\cdot\frac{\partial z_2}{\partial r_1}\right) + \frac{\partial \lambda}{\partial r_1}\cdot f_1 \\ 0 = \lambda\left(f_{21}\cdot\frac{\partial z_1}{\partial r_1} + f_{22}\frac{\partial z_2}{\partial r_1}\right) + \frac{\partial \lambda}{\partial r_1}\cdot f_2 \\ f_1\cdot\frac{\partial z_1}{\partial r_1} + f_2\frac{\partial z_2}{\partial r_1} = 0 \end{cases}$$

This is a linear system in $\frac{\partial z_1}{\partial r_1}, \frac{\partial z_2}{\partial r_1}, \frac{\partial \lambda}{\partial r_1}$

$$\begin{cases} f_1\frac{\partial z_1}{\partial r_1} + f_2\frac{\partial z_2}{\partial r_1} = 0 \\ f_1\cdot\frac{\partial \lambda}{\partial r_1} + \lambda f_{11}\frac{\partial z_1}{\partial r_1} + \lambda f_{12}\frac{\partial z_2}{\partial r_1} = 1 \\ f_2\cdot\frac{\partial \lambda}{\partial r_2} + \lambda f_{21}\frac{\partial z_1}{\partial r_1} + \lambda f_{22}\frac{\partial z_2}{\partial r_1} = 0 \end{cases}$$

Matrix form:

$$\underbrace{\begin{bmatrix} 0 & f_1 & f_2 \\ f_1 & \lambda f_{11} & \lambda f_{12} \\ f_2 & \lambda f_{21} & \lambda f_{22} \end{bmatrix}}_{\text{bordered Hessian}} \begin{bmatrix} \frac{\partial \lambda}{\partial r_1} \\ \frac{\partial z_1}{\partial r_1} \\ \frac{\partial z_2}{\partial r_1} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

What is a solution to the system?
Let's use Cramer's Rule.

$$\frac{\partial z_1}{\partial r_1} = \frac{\begin{vmatrix} 0 & 0 & f_2 \\ f_1 & 1 & \lambda f_{12} \\ f_2 & 0 & \lambda f_{22} \end{vmatrix}}{\begin{vmatrix} 0 & f_1 & f_2 \\ f_1 & \lambda f_{11} & \lambda f_{12} \\ f_2 & \lambda f_{21} & \lambda f_{22} \end{vmatrix}}$$

This is a formula for $\frac{\partial z_1}{\partial r_1}$.

How do we interpret this?
This is where second order conditions come in.
(Varian p. 499)

If there are n choice variables, and one constraint, the bordered Hessian will be a $n+1$ by $n+1$ matrix. In this case, we have to look at the determinants of various submatrices of the bordered Hessian. We illustrate this calculation in the case of a 4 by 4 bordered Hessian. Denote the border terms by b_i and the Hessian terms by h_{ij}, as above, so that the bordered Hessian is

$$\begin{pmatrix} 0 & b_1 & b_2 & b_3 & b_4 \\ b_1 & h_{11} & h_{12} & h_{13} & h_{14} \\ b_2 & h_{21} & h_{22} & h_{23} & h_{24} \\ b_3 & h_{31} & h_{32} & h_{33} & h_{34} \\ b_4 & h_{41} & h_{42} & h_{43} & h_{44} \end{pmatrix}.$$

Consider the case of a regular maximum, where the Hessian is negative definite subject to constraint, and assume $b_1 \neq 0$. Then an equivalent set of second-order conditions is that the following determinant conditions must hold:

$$\det \begin{pmatrix} 0 & b_1 & b_2 \\ b_1 & h_{11} & h_{12} \\ b_2 & h_{21} & h_{22} \end{pmatrix} > 0$$

← bordered Hessian

$$\det \begin{pmatrix} 0 & b_1 & b_2 & b_3 \\ b_1 & h_{11} & h_{12} & h_{13} \\ b_2 & h_{21} & h_{22} & h_{23} \\ b_3 & h_{31} & h_{32} & h_{33} \end{pmatrix} < 0$$

$$\det \begin{pmatrix} 0 & b_1 & b_2 & b_3 & b_4 \\ b_1 & h_{11} & h_{12} & h_{13} & h_{14} \\ b_2 & h_{21} & h_{22} & h_{23} & h_{24} \\ b_3 & h_{31} & h_{32} & h_{33} & h_{34} \\ b_4 & h_{41} & h_{42} & h_{43} & h_{44} \end{pmatrix} > 0.$$

The same pattern holds for an arbitrary number of factors. We express this condition by saying that *naturally ordered principal minors of the bordered Hessian must alternate in sign.*

The analogous second-order condition for a regular local minimum is that the same set of determinants must all be *negative.*

For a constrained minimum, we need the determinant of the bordered Hessian to be negative.

$$\frac{\partial z_1}{\partial r_1} = \frac{\overbrace{\begin{vmatrix} 0 & 0 & f_2 \\ f_1 & 1 & \lambda f_{12} \\ f_2 & 0 & \lambda f_{22} \end{vmatrix}}^{C}}{\underbrace{\begin{vmatrix} 0 & f_1 & f_2 \\ f_1 & \lambda f_{11} & \lambda f_{12} \\ f_2 & \lambda f_{21} & \lambda f_{22} \end{vmatrix}}_{B}}$$

← we know this is negative by the S.O.C.

Just look at the numerator.

$|C| = f_2^2$

So $\boxed{\frac{\partial z_1}{\partial r_1}} = \frac{f_2^2}{|B|} = \frac{+}{-} < 0$

So factor demand goes down whenever that factor's price goes up.