Lecture 03
Derivatives

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Mathematics Review Course, Summer 2023
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Last Lecture Review

- Set Theory:
  - Set Operators
  - de Margan's Law
  - Cartesian Product
  - Convex Sets
  - Bounded Sets
  - Compact Sets
- Topology:
  - Supremum and Infimum
  - Separating Hyperplane Theorem

Review Assignment

1. Problem Set 02 solutions are available on Github.
2. Any issues or problems **You** would like to discuss?

Daily Icebreaker

- Attendance via prompt:
  - Name
  - Program and track
  - Daily icebreaker subject...

Topic: Derivatives

Motivation

- General background
  - Understanding a rate of change.
  - A core component of calculus alongside integration.
- Why do economists' care?
  - How we determine a marginal effect (e.g., coefficient of interest).
  - Heavily used throughout theory.
- Application in this career
  - The main math tool you will use throughout microeconomic theory.

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Overview

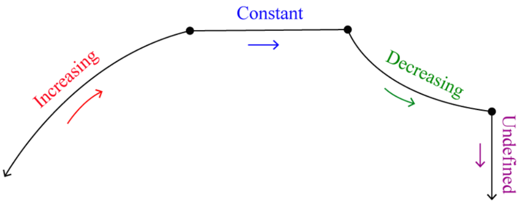
1. Continutity & Differentiability
2. First Derivative
3. Second Derivative
4. Derivative Rules
5. Implicit Function
6. l'Hopital's Rule
7. Taylor Series Approximation
8. Mean Value Theorem
9. Convexity

1. Continuity and Differentiability

- Continuous: A function $f : \mathbb{R} \to \mathbb{R}$ is continuous at point $p \in \mathbb{R} \iff \forall \varepsilon > 0 \exists \delta > 0 : |x - p| < \delta \implies |f(x) - f(p)| < \varepsilon$.
  - E.g., All x uniquely maps to $f(x)$ at all x.
- Differentiable: A function f is differentiable at x if and only if a limit exists. The entire function is differentiable if it is differentiable for all points of $x \in R$.
- Differentiable $\implies$ continuous.
- Continuous $\not\Longrightarrow$ differentiable.
- $C^1 = f'$ is continuously differentiable.
- $C^2 = f''$ is twice continuously differentiable.

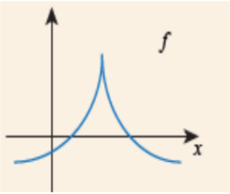
Slope Change

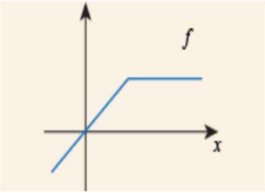
- Increasing: $f'(x) > 0 \forall x \in [a, b]$.
- Decreasing: $f'(x) < 0 \forall x \in [a, b]$.
- Monotonically Increasing: $f'(x) \geq 0 \forall x \in \mathbb{R}$.
- Strictly Increasing: $f'(x) > 0 \forall x \in \mathbb{R}$.

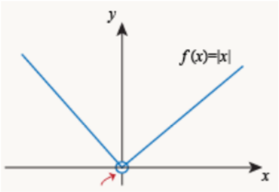


Continuous but **Not** Differentiable

- Sharp points.
- Edges.
- Jumps/holes.

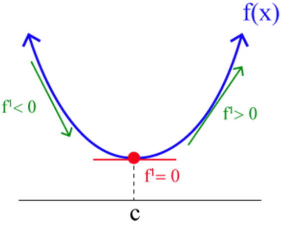




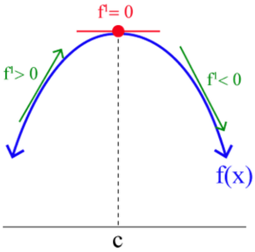


2. First Derivative

$$f'(x_0) \equiv \frac{df}{dx}(x_0) \equiv \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}$$



Relative Minimum



Relative Maximum

Common First Derivatives

- Constant: $a' = 0$
- Base Variable: $(x^a)' = ax^{a-1}$
- Base Constant: $(a^x)' = a^x ln(a)$
- Exponent Variable: $(e^x)' = e^x$
- Logarithmic: $ln(x)' = \frac{1}{x}$

3. Second Derivative

$$f''(x_0) \equiv \frac{d}{dx}\left(\frac{df}{dx}\right)(x_0) \equiv \frac{d^2f}{dx^2}(x_0)$$

- Can be taken at higher orders, but an unlikely application in economics.

4. Derivative Rules

- Sum Rule

$$[f(x) \pm g(x)]' \equiv f'(x) \pm g'(x)$$

- Power Rule

$$[\alpha x^n]' \equiv n\alpha x^{n-1}$$

- Product Rule

$$[f(x)g(x)]' \equiv f'(x)g(x) + f(x)g'(x)$$

- Quotient Rule

$$\left[\frac{f(x)}{g(x)}\right]' \equiv \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

4. Derivative Rules

- Inverse Rule
  - When $f(x)$ is monotone, differentiable, $f'(x) \neq 0$, and $f^{-1\prime}(x)$ is differentiable.

$$[f^{-1}(x)]' \equiv \frac{1}{f'(x)}$$

- Chain Rule

$$\frac{d}{dx}h(g(x)) \equiv h'(g(x))g'(x)$$

5. Implicit Function

- Implicit Function Theorem requires invoking the Jacobian matrix for partial derivatives. This involves knowledge of matrices and multivariate calculus covered later in the course. ▸ Implicit Theorem
- Sometimes y cannot be expressed as an explicit function of x.
- But we still can calculate $\frac{dy}{dx}$ ... implicitly.
- Ex. $(y = 5x^2 - 9e^y)dx$.

Answer:

$$\begin{aligned}
\frac{d}{dx}y + \frac{d}{dx}(9e^y) &= \frac{d}{dx}5x^2 \\
\frac{dy}{dx} + \frac{dy}{dx}(9e^y) &= 10x \\
\frac{dy}{dx} &= \frac{10x}{1+9e^y}
\end{aligned}$$

6. l'Hopital's Rule

- Applies when:
  - $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$
  - $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$
- Both $f(x)$ and $g(x)$ need to be differentiable over the interval $I : a \in I$.
- In both scenarios, we assume that the denominator does not equal 0 or ∞.

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(x)}{g'(x)}$$

7. Taylor Series Approximation

- Taylor Series:

$$
\begin{aligned}
f(x) =& f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \cdots \\
=& \sum_{k=0}^{n} \frac{f^k(a)}{k!}(x-a)^k
\end{aligned}
$$

- Use Taylor Series to approximate with a remainder:

$$
\begin{aligned}
f(x_0 + \Delta x) \approx& f(x_0) + f'(x_0)\Delta x \\
R(\Delta x, x_0) =& f(x_0 + \Delta x) - f(x_0) + f'(x_0)\Delta x
\end{aligned}
$$

7. Taylor Series Approximation

- We can approximate to the $(k+1)$ order of derivatives.

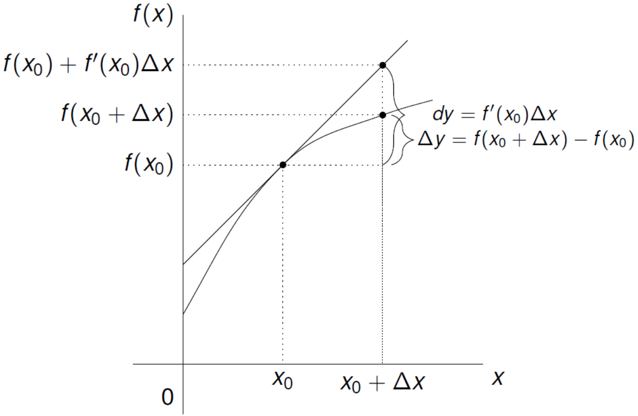
$$f(x_0+\Delta x) = f(x_0) + f'(x_0)\Delta x + \frac{1}{2!}f''(x_0)(\Delta x)^2 + \dots$$

$$+ \frac{1}{k!}f^k(x_0)(\Delta x)^k + R_k(\Delta x, x_0)$$

$$R_k(\Delta x, x_0) = \frac{f^{(k+1)}(c^*)}{(k+1)!}(\Delta x)^{k+1} \text{ , } c^* \in (x_0, x_0+\Delta x)$$

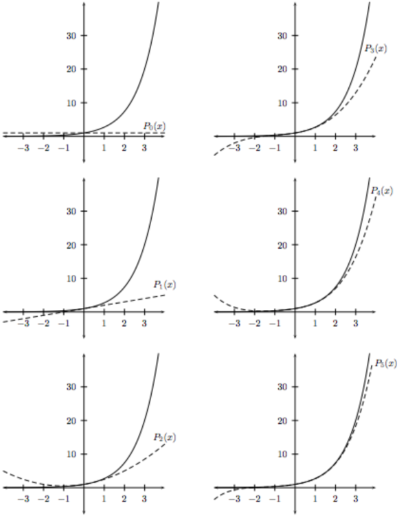
$$\lim_{\Delta x \to 0} \frac{R_k(\Delta x, x_0)}{(\Delta x)^k} \to 0$$

7. Taylor Series Approximation



$$\Delta y \approx dy = f'(x_0)\Delta x$$

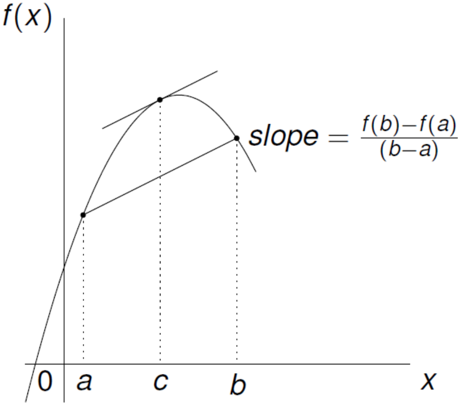
7. Taylor Series Approximation



8. Mean Value Theorem

- Let $f : U \to \mathbb{R}$ be a C^1 function over the interval $U \subset \mathbb{R}$.

$$\forall a, b \in U \exists c : a \leq c \leq b : f'(c) = \frac{f(b) - f(a)}{b - a}$$



9. Convexity & Critical Points

Weierstrass Theorem:
A continuous function $f(\cdot)$ over a closed and bounded interval $[a, b]$ attains both a local maximum and minimum.

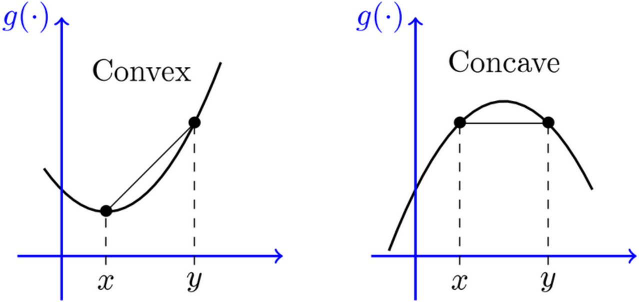
- Concave function:

$$\forall x, y \in I : f(y) - f(x) \leq f'(x)(y-1) \vee f''(x) \leq 0$$

- Convex function:

$$\forall x, y \in I : f(y) - f(x) \geq f'(x)(y-1) \vee f''(x) \geq 0$$

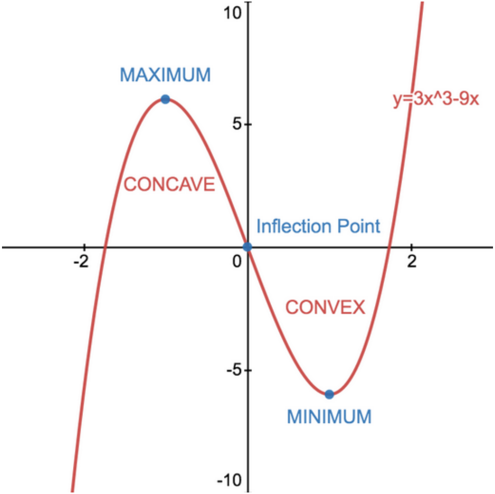
Concave Up and Concave Down



9. Convexity & Critical Points

- Critical Points: Values of x where $f'(x) = 0$ or is undefined.
- Local Max/Min (over interval I):
  $x_0, x \in I : f(x_0) \geq (\leq) f(x) \forall x$.
- Global Max/Min (over domain f):
  $x_0, x \in f : f(x_0) \geq (\leq) f(x) \forall x$.

Maximums and Minimums



Practice: Derivatives

1.

Review of Derivatives

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Assignment

- Readings on Integration before Lecture 04:
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- Assignment:
  - Problem Set 03 (PS03)
  - Solution set will be available following end of Lecture 04
- Struggling?
  1. Read the 'Encouraged Reading'
  2. Review 'Supplementary material'
  3. Reach out directly