Math Review
Summer 2017

Topic 6

6. Matrices and linear algebra

6.1. Introduction to matrices and matrix operations

Most of you have had some basic linear algebra at this point. This should be smooth sailing. The goal is to give you the fundamental notions and instruments in linear algebra that may be helpful as you progress through the program. Matrices are very valuable for economists. Just think of how much we've used matrices already so far… when totally differentiating, for the Hessian, for the gradient, etc. The usefulness of matrices pans higher and further.

A matrix is a rectangular array of real numbers. It is usually written as:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

A m x n matrix will have m rows and n columns. Recall that a m x 1 matrix is called a column vector. A 1 x n matrix is called a row vector. A matrix is generally denoted with a bold letter (*forgive me if some are not bolded by mistake)*

You may sometime need to refer to an element in a matrix. For example, if you have the matrix given by:

$$\boldsymbol{A} = \begin{bmatrix} 4 & 9 \\ 5 & 8 \\ 8 & 13 \end{bmatrix}$$

Then, $a_{11} = 4$ and $a_{32} = 13$

Q: What is a_{21}, $a_{22,}$and a_{12}?

$\mathcal{A}$:

A n x n matrix is called a square matrix of order n. There are some special types of square matrices.

A matrix where all the entries above the main diagonal are zero is called *lower triangular*:

A matrix where all the entries above the main diagonal are zero is called *upper triangular*:

A *diagonal matrix* is one in which all the entries off the main diagonal are zero:

We also sometimes need to use the following 'special' types of matrices. They are the:

(1) **Zero matrix**, we denote by **0** the null matrix which contains zeros only.

$\mathcal{Q}$: What are these dimensions?

$\mathcal{A}$:

(2) The other one is the **identity matrix.** The identity matrix is a matrix $I = I_n$ of size n x n whose elements are all units on the diagonal and zeroes on the other places.

One more matrix to recall:

(3) The matrix $\boldsymbol{A}$ is an idempotent matrix if it is equal to its square:

6.1.1. Basic operations.

If A and B are both m x n matrices then $\boldsymbol{A} + \boldsymbol{B}$ is obtained by adding the corresponding elements of the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$. The matrix $\boldsymbol{A} - \boldsymbol{B}$ is obtained by subtracting the corresponding elements of the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$. In matrix notation we have:

$$\boldsymbol{A}+\boldsymbol{B} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{bmatrix}$$

$$\boldsymbol{A}-\boldsymbol{B} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} - \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} - b_{11} & a_{12} - b_{12} & \cdots & a_{1n} - b_{1n} \\ a_{21} - b_{21} & a_{22} - b_{22} & \cdots & a_{2n} - b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} - b_{m1} & a_{m2} - b_{m2} & \cdots & a_{mn} - b_{mn} \end{bmatrix}$$

Given the following matrices:

$$\boldsymbol{A} = \begin{bmatrix} 2 & 8 \\ 0 & 7 \\ 8 & 5 \end{bmatrix} ; \boldsymbol{B} = \begin{bmatrix} 11 & -2 \\ 4 & 9 \\ 0 & -1 \end{bmatrix}; \quad \boldsymbol{C} = \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix} ; \boldsymbol{D} = \begin{bmatrix} 3 & 9 & 9 \\ 6 & 7 & 1 \end{bmatrix}; \boldsymbol{E} = \begin{bmatrix} 0 & 2 & 0 \\ 10 & 9 & 3 \end{bmatrix}$$

if defined, find:

(1) $\boldsymbol{A} + \boldsymbol{B}$

(2) $\boldsymbol{B} + \boldsymbol{C}$

(3) $\boldsymbol{B} - \boldsymbol{A}$

(4) $\boldsymbol{C} - \boldsymbol{E}$

(5) $\boldsymbol{E} + \boldsymbol{D}$

Other key properties that may be worth remembering:

(i)

(ii)

(iii)

(iv)

(v)

(vi)

(vii)

(viii)

Multiplication by a scalar:

If $\boldsymbol{A}$ is a m x n matrix and c is any scalar (i.e. a number), $c\boldsymbol{A}$ is obtained by multiplying each element of **A** by c. In matrix notation:

$$c\boldsymbol{A} = c.\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} c \text{ x } a_{11} & c \text{ x } a_{12} & \cdots & c \text{ x } a_{1n} \\ c \text{ x } a_{21} & c \text{ x } a_{22} & \cdots & c \text{ x } a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c \text{ x } a_{m1} & c \text{ x } a_{m2} & \cdots & c \text{ x } a_{mn} \end{bmatrix}$$

Matrix Multiplication:

Recall that we need that the 'inside dimension' of the matrices being multiplied to *equate* or *match*. That is we can multiply $\boldsymbol{AB}$ only if we have the form $\boldsymbol{A_{m\,x\,p}}$ **and** $\boldsymbol{B_{p\,x\,n}}$. Denote the subscript to be the dimension of the matrices here.

$\mathcal{Q}$: Find $\boldsymbol{AB}$, where $\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 5 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} 3 & 6 & 4 \\ 2 & 5 & 8 \\ 7 & 1 & 9 \end{bmatrix}$

$\mathcal{A}$:

Quick $\mathcal{Q}$: Can you also find $\boldsymbol{BA}$?

$\mathcal{Q}$: Given the dimension below, which matrices can you multiply? Write out the matrices with their dimensions.

$$\boldsymbol{A}_{3x4},\ \ \boldsymbol{B}_{4x6},\ \ \boldsymbol{C}_{6x3}$$

$\mathcal{A}$:

Try at home:

Now that you remember your basic operations, which of the following matrices are idempotent?

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \boldsymbol{C} = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

Ans: $\boldsymbol{A}$ and $\boldsymbol{C}$

Let's remember two more properties of matrices known as the:

(1) Distributive property:

(2) Associative Addition:

6.2. Transpose and trace

Definition.

Q: Consider the following matrices, what are the respective transpose?

$$\boldsymbol{A} = \begin{bmatrix} 2 & 7 \\ 3 & 0 \end{bmatrix}$$

$$\boldsymbol{B} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

$$\boldsymbol{C} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Properties of transposed matrices:

(i)

(ii)

(iii)

(iv)

Symmetric matrix:
Definition.

Trace of a matrix:
Definition.

$\mathcal{Q}$:For example, looking at the definition, what is the trace of this matrix?

$$Tr\begin{bmatrix} 1 & 2 & 3 \\ 10 & 20 & 30 \\ 100 & 200 & 300 \end{bmatrix}$$

$\mathcal{A}$:

6.3. Determinants, inverses and related properties

6.3.1 Determinants

When we covered concavity in n-space, we touched on determinants. We will review quickly if you think it is helpful and move forward.

For a 2 x 2 matrix: $det\ \boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21}$

Q: Use the definition to find the determinant of the following:

$$\boldsymbol{C} = \begin{bmatrix} 4 & 6 \\ 3 & 8 \end{bmatrix}$$

$\mathcal{A}$:

For a 3 x 3 matrix: $\det \boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}2_{32} + a_{12}a_{21}a_{33} - a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

Q: Let $\boldsymbol{A} = \begin{bmatrix} 6 & 1 & 1 \\ 4 & -2 & 5 \\ 2 & 8 & 7 \end{bmatrix}$. Find the determinant of this matrix $\boldsymbol{A}$.

$\mathcal{A}$:

Try at home:

Find the determinant of the following matrices:

$$\boldsymbol{A} = \begin{bmatrix} 3 & 5 \\ 1 & 14 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} -7 & 5 \\ -14 & 4 \end{bmatrix}, \boldsymbol{C} = \begin{bmatrix} 3 & -7 \\ 1 & -14 \end{bmatrix}$$

$$det\ \boldsymbol{A} = 7, det\ \boldsymbol{B} = 42, det\ \boldsymbol{C} = \mathbf{-35}$$

$$\boldsymbol{D} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 1 \\ 0 & 3 & -1 \end{bmatrix}, \ \boldsymbol{E} = \begin{bmatrix} 5 & -2 & 1 \\ 0 & 3 & -1 \\ 2 & 0 & 7 \end{bmatrix}$$

$$det\ \boldsymbol{D} = 1, \det \boldsymbol{E} = 103$$

Like I mentioned before, I think is it quasi impossible that you will be asked to compute the determinants for matrices bigger than 3 x 3. If you are, probably a TA is feeling like getting you thinking about it a little more. I doubt you'll see it on an exam, ever. Now if it comes up, … uhm, good luck?

Properties of determinants:

Note that we will not be proving any of these. Proofs are easily available is so desired.

Let A and B be square n x n matrices. Then:

6.3.2. Inverses

Intuition:

Definition. If $\boldsymbol{A}$ is a square n x n matrix, and if there exists an n x n matrix $\boldsymbol{B}$ such that:

$$\boldsymbol{AB} = \boldsymbol{BA} = \boldsymbol{I},$$

then $\boldsymbol{A}$ is invertible and $\boldsymbol{B}$ is the inverse of $\boldsymbol{A}$. Matrix $\boldsymbol{B}$ can also be denoted $\boldsymbol{A^{-1}}$.
If matrix B does not exist then A is singular.

(Uniqueness of Inverse) Theorem:

Properties of Inverses

Finding inverses

If the 2 x 2 square matrix given by $\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible, then the inverse given by:

$\boldsymbol{A}^{-\mathbf{1}} =$

Q:Find the inverse of matrix $\boldsymbol{D} = \begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}$

$\mathcal{A}$:

Recall one of the properties of inverses, that is $\boldsymbol{A}$ x $\boldsymbol{A}^{-\mathbf{1}} = I$.

Q: Is that true for the case above? Can you check?

$\mathcal{A}$:

Inverses may not always exist! This happens with any instance where the determinant is equal to 0 (and we cannot divide by 0). Think of a matrix given by: $\begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$. What do we call this matrix?

Elementary matrix:
<u>Definition</u>.

Solve for $\boldsymbol{A}^{-1}$, by following the following steps:

1) Find a sequence of elementary row operations that reduce $\boldsymbol{A}$ to the identity matrix.
2) Perform the same sequence of operations on $\boldsymbol{I}_{\boldsymbol{n}}$ to obtain $\boldsymbol{A}$.

I'll work through an example and you can do the next.

Example. Let $\boldsymbol{B} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$, find $\boldsymbol{B}^{-1}$.

To solve for $\boldsymbol{B}^{-1}$, remember that we want to reduce the given $\boldsymbol{B}$, to the 3 x 3 identity matrix using row operations while doing these operations simultaneously to I, to produce $\boldsymbol{B}^{-1}$.

Let $\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$, find $\boldsymbol{A}^{-1}$.

To try at home:

Find the inverse of the following matrices:

$$\boldsymbol{E} = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}$$

$$Ans: \boldsymbol{E}^{-1} = \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix}$$

$$\boldsymbol{G} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$Ans: \boldsymbol{G}^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

$$\boldsymbol{K} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}$$

$$\boldsymbol{K}^{-1} = \frac{1}{11}\begin{bmatrix} 12 & -6 & -1 \\ \frac{5}{2} & \frac{3}{2} & -\frac{5}{2} \\ -2 & 1 & 2 \end{bmatrix}$$

$$\boldsymbol{L} = \begin{bmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{bmatrix}$$

$$\boldsymbol{L}^{-1} = \begin{bmatrix} -2 & 8 & -5 \\ 3 & -11 & 7 \\ 9 & -34 & 21 \end{bmatrix}$$

6.4. Systems of linear equations, independence and rank

6.4.1. System of linear equations

You will find system of linear equations widely used in economics. To begin, consider a simple arbitrary system of two linear equations given by:

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

From the above, we can let $\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. This can be denoted as the matrix of *known* coefficients.

The vector of unknown variables is given as $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
and the vector of constants is $\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. In matrix form, this is the familiar $\boldsymbol{Ax} = \boldsymbol{b}$
Suppose you have a system of m linear equations and n unknowns, it can be written as:

Our goal is often to solve this system of linear equations and the most basic method is the **Guassian elimination**.

Gauss-Jordan Elimination

To solve a system of linear equations we can reduce the systems associated augmented matrix to reduced row-echelon form:

Reduced row-echelon form:
Definition.

$\mathcal{Q}$: Which of these are in reduced row echelon form?

$$\boldsymbol{A}=\begin{bmatrix}1&0&0&1\\0&1&0&2\\0&0&1&3\end{bmatrix}\quad \boldsymbol{B}=\begin{bmatrix}1&0&-3&0\\0&1&4&2\\0&0&0&1\end{bmatrix}\quad \boldsymbol{C}=\begin{bmatrix}1&0&0\\0&2&0\\0&0&1\end{bmatrix}$$

$$\boldsymbol{D}=\begin{bmatrix}1&-2&0\\0&0&1\\0&0&0\end{bmatrix}\quad \boldsymbol{E}=\begin{bmatrix}1&0&2\\0&1&0\\0&0&1\end{bmatrix}$$

$\mathcal{A}$:

To solve a system of linear equations using Gauss-Jordan elimination we apply the following algorithm to the augmented matrix, $\boldsymbol{A}$:

1. Write the augmented matrix of the system.

2. Use row operations to transform the augmented matrix in the form described below, which is called the *reduced row echelon form (RREF).*
   (a) The rows (if any) consisting entirely of zeros are grouped together at the bottom of the matrix.
   (b) In each row that does not consist entirely of zeros, the leftmost nonzero element is a 1 (called a leading 1 or a pivot).
   (c) Each column that contains a leading 1 has zeros in all other entries.
   (d) The leading 1 in any row is to the left of any leading 1's in the rows below it.

3. Stop process in step 2 if you obtain a row whose elements are all zeros except the last one on the right. In that case, the system is inconsistent and has no solutions. Otherwise, finish step 2 and read the solutions of the system from the final matrix.

Note: When doing step 2, row operations can be performed in any order. Try to choose row operations so that as few fractions as possible are carried through the computation. This makes calculation easier when working by hand.

Example. Solve the following system by Gauss –Jordan elimination.

$$\begin{gathered} x_1 - 2x_2 + 3x_3 + 2x_4 + x_5 = 10 \\ 2x_1 - 4x_2 + 8x_3 + 3x_4 + 10x_5 = 7 \\ 3x_1 - 6x_2 + 10x_3 + 6x_4 + 5x_5 = 27 \end{gathered}$$

Q: Try this example (*slightly simpler actually – maybe, I don't know, you tell me ☺!*)

Solve the following system by using the Gauss-Jordan elimination method.

$$\begin{aligned} 4y + z &= 2 \\ 2x + 6y - 2z &= 3 \\ 4x + 8y - 5z &= 4 \end{aligned}$$

To try at home:

Solve the following systems by using the Gauss-Jordan elimination method.

$$\begin{cases} x + y + z = 5 \\ 2x + 3y + 5z = 8 \\ 4x + 5z = 2 \end{cases}$$

$$Ans: x = 3, y = 4, z = -2$$

$$\begin{cases} x + 2y - 3z = 2 \\ 6x + 3y - 9z = 6 \\ 7x + 14y - 21z = 13 \end{cases}$$

$$Ans: No\ solutions, inconsistent\ system$$

$$\begin{cases} A + B + 2C = 1 \\ 2A - B + D = -2 \\ A - B - C - 2D = 4 \\ 2A - B + 2C - D = 0 \end{cases}$$

$$Ans: A = 1, B = 2, C = -1, D = -2$$

6.4.2. Cramer's Rule

Now, there is a second way that we can solve a system of linear equations where $\boldsymbol{A}$ is a square matrix is via Cramer's Rule:

(Cramer's Rule) Theorem.

Example. Solve this system using Cramer's rule:

$$7x_1 - 2x_2 = 3$$
$$3x_1 + x_2 = 5$$

$\mathcal{Q}$: Use Cramer's Rule to solve this system:

$$3x - y = 7$$
$$-5x + 4y = -2$$

$\mathcal{A}$:

Let's walk through an example of Cramer's rule with a 3 x 3 matrix together. The steps are very similar, so if you want, you can try a couple at home as well. I think it might be a little repetitive to have you guys do a 3 x 3 matrix exercise in class today though.

Example. Consider the following system of equations. Use Cramer's rule to solve for x, y and z.

$$2x + y + z = 3$$
$$x - y - z = 0$$
$$x + 2y + z = 0$$

Try at home:

Use Cramer's Rule to solve these system:

$$\begin{cases} 4x - 3y = 11 \\ 6x + 5y = 7 \end{cases}$$

$$Ans: x = 2, y = -1$$

$$\begin{cases} 3x + 5y = -7 \\ x + 4y = -14 \end{cases}$$

$$Ans: x = 6, y = -5$$

Solve for z using Cramer's Rule:

$$\begin{cases} 2x + y + z = 1 \\ x - y + 4z = 0 \\ x + 2y - 2z = 3 \end{cases}$$

$$Ans: z = 2$$

Every system of linear equations has either

(a) a unique solution,
(b) no solution, or
(c) infinitely many solutions.

If a system of equations has *no solutions it is said to be inconsistent*, and if a system of equations has *at least one solution it is said to be consistent.* A very important type of linear systems is that of a *homogenous linear system*:

Homogeneous system of linear equations:
Definition.

As you can deduce from looking at the matrix, every homogenous system has the trivial solution $x_1 = x_2 = \cdots = x_n = 0$. A nontrivial solution is any other solution.

Theorem.

6.4.3 Linear Dependence/Independence

Linear dependence:
<u>Definition</u>:

We have the following vectors:

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, x_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, x_3 = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$$

$\mathcal{Q}$: Consider the definition, are these vectors linearly dependent or linearly independent?

$\mathcal{A}$:

6.4.5. Rank of a matrix

Definition. The rank of matrix $\boldsymbol{A}$ is the equivalent to the number of non-zero rows when $\boldsymbol{A}$ is in its reduced row-echelon form. If the reduced row-echelon form has no zeros, then we say that **matrix $\boldsymbol{A}$ has full rank**.

Theorem:

6.6. Eigenvalues, eigenvectors and Markov processes

Definition:

An example can clear things up as eigenvalues and eigenvectors are usually confusing for many.[1]

Example. What is the characteristic equation, eigenvectors, and eigenvalues for the following linear system:

$$2x_1 + 3x_2 = \lambda x_1$$
$$4x_1 + 3x_2 = \lambda x_2.$$

[1] *I would refer you here for a deeper explanation (with visuals) of the mechanics of eigenvalues*: http://setosa.io/ev/eigenvectors-and-eigenvalues/

$\mathcal{Q}$: What is the characteristic equation, eigenvectors, and eigenvalues for the following linear system:

$$\begin{aligned} 2x_1 + 2x_2 &= \lambda x_1 \\ 5x_1 - x_2 &= \lambda x_2. \end{aligned}$$

$\mathcal{A}$:

6.5. (*Some more*) Matrix Calculus

So far, we've already done plenty of calculus with matrices when we covered real valued functions. In this section, the goal is to summarize main results of what is happening when we are undertaking derivatives of matrices. We won't spend too much time on this unless the class wants to. Let's go through it and work through an example to see what the **main results** are saying. Some stuff here is more of **nice to know** but still intuitive.

Example. Let $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Given:

$$y_1 = x_1^2 - x_2$$
$$y_2 = x_3^2 + 3x_2$$

Find $\frac{dy}{dx}$.

Generally, if y is an m-element vector, and x is an n-element vector, then the first-order partial derivatives is given by a m x n matrix.

Please note that <u>many</u> resources (especially in the discipline of engineering) report this matrix of first order derivatives as the transpose of what we have… but in economics and statistics, we most usually use what I am presenting. Just something to keep in mind.

Main results:

$\boldsymbol{y}$	$\frac{d\boldsymbol{y}}{d\boldsymbol{x}}$
$\boldsymbol{Ax}$	$\boldsymbol{A}$
$\boldsymbol{x^T A}$	$\boldsymbol{A^T}$
$\boldsymbol{x^T x}$	$2\boldsymbol{x^T}$

Let's apply these results to a simple example to see what is happening:

$$Let\ \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, x^T = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$$

<u>The Chain Rule for Vector Functions</u>:

Let $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_r \end{bmatrix}, z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_m \end{bmatrix}$, where z is a function of y which in turn is a function of x.

Find $\frac{\partial z}{\partial x}$.