Lecture 07
Linear Algebra

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Last Lecture Review

- Matrices:
  - Matrix Operators
  - Rank Trace
  - The Determinant
  - Positive and Negative Definite Matrices
  - Linear Independence

Review Assignment

1. Problem Set 06 solutions are available on Github.
2. Any issues or problems **You** would like to discuss?

Daily Icebreaker

- Attendance via prompt:
  - Name
  - Program and track
  - Daily icebreaker subject...

Topic: Linear Algebra

Motivation

- General background
  - The algebra of manipulating matrices.
  - A compact, efficient method for working with systems of equations (e.g., a spreadsheet).
- Why do economists' care?
  - These is the foundation of how economists manipulate data.
- Application in this career
  - Important applications in econometrics.
  - Useful whenever you are creating or modifying an estimator.

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Overview

1. Systems of Linear Equations
2. Gaussian Elimination
3. Linear Operators
4. Existence of a Solution
5. Cramer's Rule
6. Eigenvalues and the Characteristic Equation
7. Leading Principle Minors
8. Regression as a Matrix
9. Centering Matrix
10. Residual Maker

1. Systems of Linear Equations

- Linear Functions: $y = Ax$ with elements y_i of y such that $y_i = a_i^T x$.

$$a_i = \frac{\partial y_i}{\partial x} = \frac{\partial}{\partial x}(a_i^T x)$$

- Linear System: m equations for n unknown variables.

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

2. Gaussian Elimination

- Augmented Matrix

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n}|b_1 \\ a_{21} & a_{22} & \cdots & a_{2n}|b_2 \\ \vdots & \vdots & \ddots & \vdots|\vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn}|b_m \end{pmatrix}$$

2. Gaussian Elimination

- Row Echelon Form
  - Interchange any two rows.
  - Change row by adding a multiple of another row.
  - Multiply each element in a row by a non-zero scalar.

$$\begin{pmatrix} 1 & 0 & \cdots & 0|b_1 \\ 0 & 1 & \cdots & 0|b_2 \\ \vdots & \vdots & \ddots & \vdots|\vdots \\ 0 & 0 & \cdots & 1|b_m \end{pmatrix}$$

3. Linear Operators

- Product Rule:

$$\frac{\partial a^T b}{\partial x} = \frac{\partial a^T}{\partial x} b + \frac{\partial b^T}{\partial x} a$$

- Quadratic Form:

$$x^T A x = \sum_{i=1}^{n} \sum_{i=1}^{n} x_i x_j a_{ij}$$

4. Existence of Solution

- Important to know that a solution for a system exists.
- A system of linear equations with coefficient matrix A and an augmented matrix $\hat{A}$ has a solution iff:

$$rank\hat{A} = rankA$$

- There are infinite solutions if $\#rows(A) < \#cols(A)$.
  - More unknown variables than observations.
- Non-singular Square Matrix: Ensure only one solutions exists iff

$$\#rows(A) = \#cols(A) = rank(A)$$

5. Cramer's Rule

- A unique solution $x = (x_1, \cdots, x_n)$ for $n \times n$ system $Ax = b$ is

$$x_i = \frac{detB_i}{detA} \forall i = 1, \ldots, n$$

- Where B_i replaces the i'th column of A with the b vector.

6. Eigenvalues and the Characteristic Equation

- Characteristic Equation

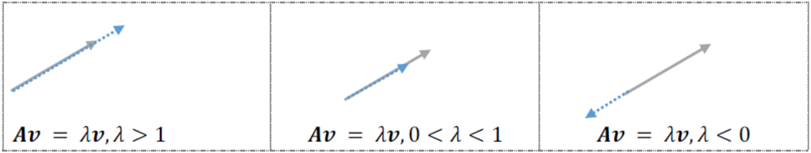
$$Ac = \lambda c$$

- Characteristic vectors: (c, λ)
- Eigenvectors: c
- Eigenvalues: λ

$$Ac = \lambda I c \iff (A - \lambda I)c = 0$$

- Homogeneous system has non-zero solution if it is singular and has zero determinant: $det(A - \lambda I) = 0$

Eigenvectors Scaled by Eigenvalues



7. Leading Principle Minors

- A way to test for matrix definiteness.
- Leading Principal Sub-matrix: Let A be a $N \times N$ matrix. The K'th order principal sub-matrix of A obtained by deleting the last $N - K$ rows and the last $N - K$ columns.
- Leading Principal Minor: The determinant of the K'th order leading principal sub-matrix.
- Ex., for a 3×3 matrix, the leading principal minors are:

$$|a_{11}|, \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

7. Leading Principle Minors

- Positive Definite: Iff all N leading principal minors are ≥ 0.
- Negative Definite: Iff the N leading principal minors alternate signs

$$det(A_1) < 0, det(A_2) > 0, det(A_3) < 0 \text{ , etc.}$$

- Indefinite: Leading principal minors follow any other order.

8. Regression as a Matrix

- A linear OLS model:

$$Y_{N\times 1} = X_{N\times K}\beta_{K\times 1} + e_{N\times 1}$$

- With the goal of selecting $\hat{\beta}$ that minimizes squared predicted errors.

$$\hat{e}^T\hat{e} = (Y - X\hat{\beta})^T(Y - X\hat{\beta})$$

8. Regression as a Matrix

- So, taking the first derivative we can get:

$$X^T X\hat{\beta} - X^T Y = 0$$
$$\Longrightarrow$$
$$-X^T(Y - X\hat{\beta}) = -X^T\hat{e} = 0$$

- And the solution to OLS is:
$$\hat{\beta} = (X^T X)^{-1} X^T Y$$

9. Centering Matrix

- It is common in statistics to transform data to be 'deviations from the mean'.
- This can be done by creating the "centering matrix".
- First, create a multiplier $\frac{1}{N}ii^T$.

$$\frac{1}{N}ii^T = \frac{1}{N}\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}\begin{pmatrix}1 & 1 & \cdots & 1\end{pmatrix} = \begin{pmatrix}\frac{1}{N} & \frac{1}{N} & \cdots & \frac{1}{N}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{1}{N} & \frac{1}{N} & \cdots & \frac{1}{N}\end{pmatrix}$$

- Then define vectors of means.

$$i\bar{x} = \frac{1}{N}ii^T x = \begin{pmatrix}\frac{1}{N} & \frac{1}{N} & \cdots & \frac{1}{N}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{1}{N} & \frac{1}{N} & \cdots & \frac{1}{N}\end{pmatrix}\begin{pmatrix}x_1\\x_2\\\vdots\\x_N\end{pmatrix} = \begin{pmatrix}\bar{x}\\\bar{x}\\\vdots\\\bar{x}\end{pmatrix}$$

9. Centering Matrix

- The vector of derivatives can be expressed as:

$$x - i\bar{x} = \begin{pmatrix} x_1 - \bar{x} \\ x_2 - \bar{x} \\ \vdots \\ x_N - \bar{x} \end{pmatrix} = x - \frac{1}{N} i i^T x = (I - \frac{1}{N} i i^T) x = M^0 x$$

- Summing these derivatives can be written as follows:

$$\sum_{i=1}^{N} (x_i - x)^2 = (x - i\bar{x})^T (x - i\bar{x}) = (M^0 x)^T (M^0 x) = x^T M^{0T} M^0 x$$

10. Residual Maker

- Define
$$M = I - X(X^T X)^{-1} X^T$$
- Then
$$\hat{e} = Y - X\hat{\beta} = (I - X(X^T X)^{-1} X^T)Y = MY$$
- Residual Maker: M

Practice: Linear Algebra

1.

Review of Linear Algebra

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Assignment

- Readings on Functions before Lecture 08:
  -
- Assignment:
  - Problem Set 07 (PS07)
  - Solution set will be available following end of Lecture 08
- Struggling?
  1. Read the 'Encouraged Reading'
  2. Review 'Supplementary material'
  3. Reach out directly