Lecture 03
Derivatives

Ryan McWay[†]

[†] *Applied Economics,*
University of Minnesota

Mathematics Review Course, Summer 2023
University of Minnesota
August 9th, 2023

Last Lecture Review

- Set Theory:
  - Set Operators
  - De Morgan's Law
  - Cartesian Product
  - Convex Sets
  - Bounded & Compact Sets
- Topology:
  - Supremum and Infimum and Limits
  - Separating Hyperplane Theorem

Review Assignment

1. Problem Set 02 solutions are available on Github.
2. Any issues or problems **You** would like to discuss?

Daily Icebreaker

- Attendance via prompt:
  - Name
  - Daily Icebreaker: You are a late night show host. Who is the first celebrity you would invite to interview?

Topic: Derivatives

Motivation

- General background
  - Understanding a rate of change.
  - A core component of calculus alongside integration.
- Why do economists' care?
  - How we determine a marginal effect (e.g., coefficient of interest).
  - Heavily used throughout theory.
- Application in this career
  - The main math tool you will use throughout the microeconomic theory (alongside optimization).

Motivation

- General background
  - Understanding a rate of change.
  - A core component of calculus alongside integration.
- Why do economists' care?
  - How we determine a marginal effect (e.g., coefficient of interest).
  - Heavily used throughout theory.
- Application in this career
  - The main math tool you will use throughout the microeconomic theory (alongside optimization).

Motivation

- General background
  - Understanding a rate of change.
  - A core component of calculus alongside integration.
- Why do economists' care?
  - How we determine a marginal effect (e.g., coefficient of interest).
  - Heavily used throughout theory.
- Application in this career
  - The main math tool you will use throughout the microeconomic theory (alongside optimization).

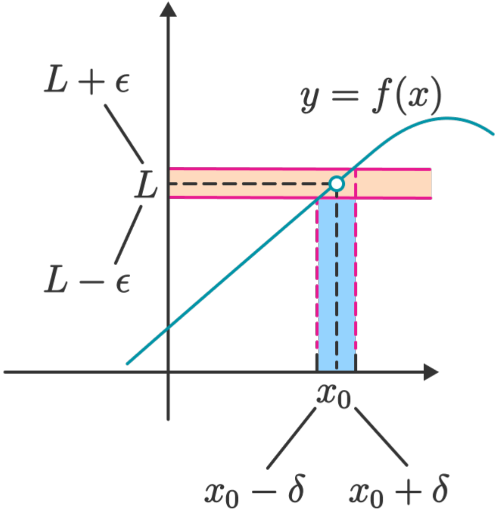
Overview

1. Continutity & Differentiability
2. First Derivative
3. Second Derivative
4. Derivative Rules
5. Implicit Function
6. l'Hopital's Rule
7. Taylor Series Approximation
8. Mean Value Theorem
9. Critical Points

1. Continuity and Differentiability

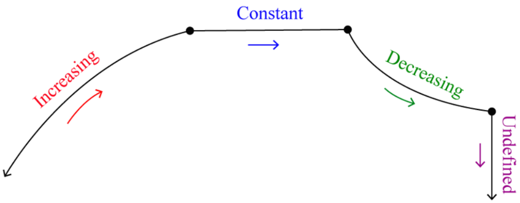
- Continuous: A function $f : \mathbb{R} \to \mathbb{R}$ is continuous at point $p \in \mathbb{R} \iff \forall \varepsilon > 0 \exists \delta > 0 : |x - p| < \delta \implies |f(x) - f(p)| < \varepsilon$.
  - E.g., All x uniquely maps to $f(x)$ at all x.
- Differentiable: A function f is differentiable at x if and only if (iff) a limit exists. The entire function is differentiable if it is differentiable for all points of $x \in \mathbb{R}$.
- Differentiable $\implies$ continuous.
- Continuous $\not\Longrightarrow$ differentiable.
- $C^1 = f'$ is continuously differentiable.
- $C^2 = f''$ is twice continuously differentiable.

Continuity



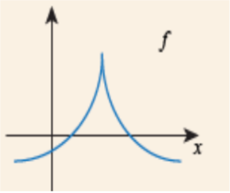
Slope Change

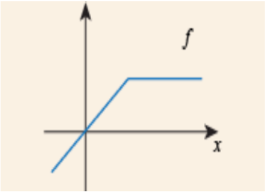
- Increasing: $f'(x) > 0 \forall x \in [a, b]$.
- Decreasing: $f'(x) < 0 \forall x \in [a, b]$.
- Monotonically Increasing: $f'(x) \geq 0 \forall x \in \mathbb{R}$.
- Strictly Increasing: $f'(x) > 0 \forall x \in \mathbb{R}$.

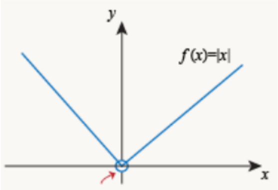


CONTINUOUS BUT **NOT** DIFFERENTIABLE

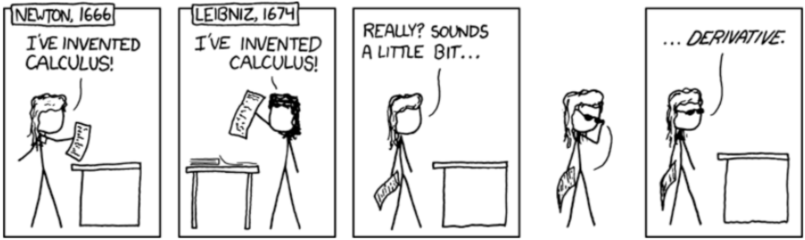
- Sharp points.
- Edges.
- Jumps/holes.





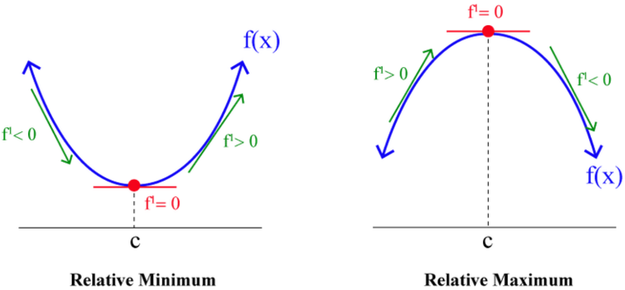


2. Newton or Leibniz



2. First Derivative

$$f'(x_0) \equiv \frac{df}{dx}(x_0) \equiv \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$



Common First Derivatives

- Constant: $a' = 0$
- Base Variable: $(x^a)' = ax^{a-1}$
- Base Constant: $(a^x)' = a^x ln(a)$
- Exponent Variable: $(e^x)' = e^x$
- Logarithmic: $ln(x)' = \frac{1}{x}$

3. Second Derivative

$$f''(x_0) \equiv \frac{d}{dx}\left(\frac{df}{dx}\right)(x_0) \equiv \frac{d^2f}{dx^2}(x_0)$$

- Can be taken at higher orders, but rarely applied within first year coursework.

4. Derivative Rules

- Sum Rule

$$[f(x) \pm g(x)]' \equiv f'(x) \pm g'(x)$$

- Power Rule

$$[\alpha x^n]' \equiv n\alpha x^{n-1}$$

- Product Rule

$$[f(x)g(x)]' \equiv f'(x)g(x) + f(x)g'(x)$$

- Quotient Rule

$$\left[\frac{f(x)}{g(x)}\right]' \equiv \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

Demonstration: Product Rule

Question:
Find $\frac{df(x)}{dx}$ for $f(x) = (3x - 2x^2)(5 + 4x)$.

Answer:
Note the two functions. $f(x) = (3x - 2x^2)$ and $g(x) = (5 + 4x)$.
By the product rule:

$$\begin{aligned} f'(x) =& (3x - 2x^2)(4) + (3 - 4x)(5 + 4x) \\ =& -24x^2 + 4x + 15 \end{aligned}$$

Demonstration: Product Rule

Question:
Find $\frac{df(x)}{dx}$ for $f(x) = (3x - 2x^2)(5 + 4x)$.

Answer:
Note the two functions. $f(x) = (3x - 2x^2)$ and $g(x) = (5 + 4x)$.
By the product rule:

$$\begin{aligned} f'(x) =& (3x - 2x^2)(4) + (3 - 4x)(5 + 4x) \\ =& -24x^2 + 4x + 15 \end{aligned}$$

Demonstration: Quotient Rule

Question:
Find $\frac{dh(z)}{dz}$ for $h(z) = \frac{4\sqrt{z}}{z^2-2}$.

Answer:
Again, identify the two functions $f(z) = (4\sqrt{z})$ and $g(z) = (z^2 - 2)$. Then by the rule we have:

$$h'(z) = \frac{4(1/2)x^{\frac{-1}{2}}(x^2-2) - 4x^{\frac{1}{2}}(2x)}{(x^2-2)^2}$$

$$= \frac{-6x^{\frac{3}{2}} - 4x^{\frac{-1}{2}}}{(x^2-2)^2}$$

Note: Quotients can be done as products.

Demonstration: Quotient Rule

Question:

Find $\frac{dh(z)}{dz}$ for $h(z) = \frac{4\sqrt{z}}{z^2-2}$.

Answer:

Again, identify the two functions $f(z) = (4\sqrt{z})$ and $g(z) = (z^2 - 2)$. Then by the rule we have:

$$h'(z) = \frac{4(1/2)x^{\frac{-1}{2}}(x^2-2) - 4x^{\frac{1}{2}}(2x)}{(x^2-2)^2}$$
$$= \frac{-6x^{\frac{3}{2}} - 4x^{\frac{-1}{2}}}{(x^2-2)^2}$$

Note: Quotients can be done as products.

4. Derivative Rules

- Inverse Rule
  - When $f(x)$ is monotone, differentiable, $f'(x) \neq 0$, and $f^{-1\prime}(x)$ is differentiable.

$$[f^{-1}(x)]' \equiv \frac{1}{f'(x)}$$

- Chain Rule

$$\frac{d}{dx}h(g(x)) \equiv h'(g(x))g'(x)$$

Demonstration: Chain Rule

Question:

Find $\frac{df(x)}{dx}$ for $g(x) = ln(x^{-4} + x^4)$.

Answer:

$$g'(x) = \frac{1}{x^{-4} + x^4}(-4x^{-5} + 4x^3) = \frac{-4x^{-5} + 4x^3}{x^{-4} + x^4}$$

DEMONSTRATION: CHAIN RULE

Question:
Find $\frac{df(x)}{dx}$ for $g(x) = ln(x^{-4} + x^4)$.

Answer:

$$g'(x) = \frac{1}{x^{-4} + x^4}(-4x^{-5} + 4x^3) = \frac{-4x^{-5} + 4x^3}{x^{-4} + x^4}$$

Practice: First Derivatives

1. $f(x) = xe^{3x}$

Practice: First Derivatives

1. $f(x) = xe^{3x}$

Answer: Show Work

$$f'(x) = (1 + 3x)x^{3x}$$

Practice: First Derivatives

1. $f(x) = xe^{3x}$
2. $f(x) = ln(x^4 + 2)^2$

Practice: First Derivatives

1. $f(x) = xe^{3x}$
2. $f(x) = ln(x^4 + 2)^2$

Answer: Show Work

$$f'(x) = \frac{8x^3 ln(x^4 + 2)}{x^4 + 2}$$

Practice: First Derivatives

1. $f(x) = xe^{3x}$
2. $f(x) = ln(x^4 + 2)^2$
3. $f(x) = \left(\frac{x+4}{x-3}\right)^{2/3}$

Practice: First Derivatives

1. $f(x) = xe^{3x}$
2. $f(x) = ln(x^4 + 2)^2$
3. $f(x) = \left(\frac{x+4}{x-3}\right)^{2/3}$

Answer: Show Work

$$f'(x) = \frac{-14}{3(x+4)^{1/3}(x-3)^{5/3}}$$

5. Implicit Function

- Implicit Function Theorem requires invoking the Jacobian matrix for partial derivatives. This involves knowledge of matrices and multivariate calculus covered later in the course.
- Sometimes y cannot be expressed as an explicit function of x.
- But we still can calculate $\frac{dy}{dx}$ … implicitly.

Demonstration: Implicit Functions

Question:

Find $\frac{dy}{dx}$ for $y = 5x^2 - 9e^y$.

Answer:

$$\begin{aligned}\frac{dy}{dx}(y) &= \frac{dy}{dx}5x^2 - \frac{dy}{dx}(9e^y)\\ \frac{dy}{dx} &= 10x - (9e^y)\frac{dy}{dx}\\ \frac{dy}{dx}(1+9e^y) &= 10x\\ \frac{dy}{dx} &= \frac{10x}{1+9e^y}\end{aligned}$$

Demonstration: Implicit Functions

Question:

Find $\frac{dy}{dx}$ for $y = 5x^2 - 9e^y$.

Answer:

$$\begin{aligned}
\frac{dy}{dx}(y) =& \frac{dy}{dx}5x^2 - \frac{dy}{dx}(9e^y) \\
\frac{dy}{dx} =& 10x - (9e^y)\frac{dy}{dx} \\
\frac{dy}{dx}(1 + 9e^y) =& 10x \\
\frac{dy}{dx} =& \frac{10x}{1 + 9e^y}
\end{aligned}$$

Practice: Implicit Functions

1. Find $\frac{dy}{dx}$ for $x^2y^3 - xy = 10$.

Practice: Implicit Functions

1. Find $\frac{dy}{dx}$ for $x^2y^3 - xy = 10$.

Answer: Show Work

$$\frac{dy}{dx} = \frac{-2xy^3 + y}{3x^2y^2 - x}$$

Practice: Implicit Functions

1. Find $\frac{dy}{dx}$ for $x^2y^3 - xy = 10$.
2. Find $\frac{dy}{dx}$ for $e^y + xy - e = 0$.

Practice: Implicit Functions

1. Find $\frac{dy}{dx}$ for $x^2y^3 - xy = 10$.
2. Find $\frac{dy}{dx}$ for $e^y + xy - e = 0$.

Answer: Show Work

$$\frac{dy}{dx} = \frac{-y}{e^y + x}$$

Practice: Implicit Functions

1. Find $\frac{dy}{dx}$ for $x^2y^3 - xy = 10$.
2. Find $\frac{dy}{dx}$ for $e^y + xy - e = 0$.
3. Find $\frac{dy}{dx}$ for $\frac{xy+y^2}{x^2-xy} = 4y$

Practice: Implicit Functions

1. Find $\frac{dy}{dx}$ for $x^2y^3 - xy = 10$.
2. Find $\frac{dy}{dx}$ for $e^y + xy - e = 0$.
3. Find $\frac{dy}{dx}$ for $\frac{xy+y^2}{x^2-xy} = 4y$

Answer: Show Work

$$\frac{dy}{dx} = \frac{y(8x - 4y - 1)}{x + 2y - 4x^2 + 8xy}$$

6. l'Hopital's Rule

- Consider you are taking a limit (derivative) with two functions in the numerator and denominator of a fraction, respectively.
- Applies when:
  - $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$
  - $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$
- Both $f(x)$ and $g(x)$ need to be differentiable over the interval $I : a \in I$.
- In both scenarios, we assume that the denominator does not equal 0 or ∞.

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(x)}{g'(x)}$$

Application: Constant Elasticity of Substitution (CES)

To show that the CES $Y = A(\alpha K^{\gamma} + (1-\alpha)L^{\gamma})^{\frac{1}{\gamma}}$ is a Cobb-Douglas function $Y = AK^{\alpha}L^{1-\alpha}$ when $\gamma \to 0$.

Proof.

First take the log of both sides.

$$ln(Y) = ln(A) + \frac{1}{\gamma} ln(\alpha K^{\gamma} + (1-\alpha)L^{\gamma})$$

Then by l'Hopital's Rule, □

Application: Constant Elasticity of Substitution (CES)

Proof.

$$
\begin{aligned}
\lim_{\gamma \to 0} \frac{ln(\alpha K^\gamma + (1-\alpha)L^\gamma)}{\gamma} =& \lim_{\gamma \to 0} \frac{\frac{dln(\alpha K^\gamma + (1-\alpha)L^\gamma)}{d\gamma}}{\frac{d\gamma}{d\gamma}} \\
=& \frac{\alpha K^\gamma ln(K) + (1-\alpha)L^\gamma ln(L)}{\alpha K^\gamma + (1-\alpha)L^\gamma} \\
=& \alpha ln(K) + (1-\alpha)ln(L) \\
\therefore \lim_{\gamma \to 0} ln(Y) =& ln(A) + \alpha ln(K) + (1-\alpha)ln(L)
\end{aligned}
$$

This is the Cobb-Douglas function. □

7. Taylor Series Approximation

- Taylor Series:

$$
\begin{aligned}
f(x) =& f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \cdots \\
=& \sum_{k=0}^{n} \frac{f^k(a)}{k!}(x-a)^k
\end{aligned}
$$

- Use Taylor Series to approximate with a remainder $R(\Delta x, x_0)$:

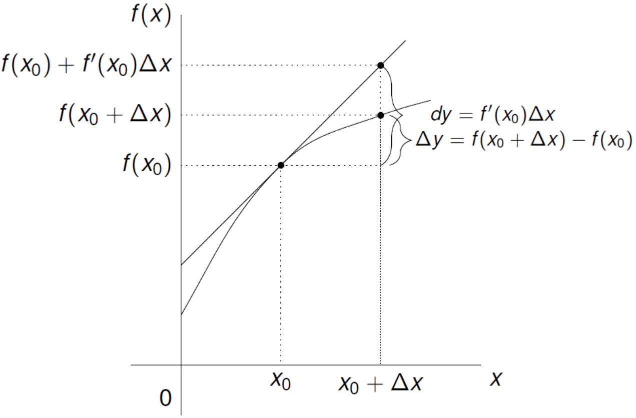
$$
\begin{aligned}
f(x_0 + \Delta x) \approx& f(x_0) + f'(x_0)\Delta x + R(\Delta x, x_0) \\
R(\Delta x, x_0) =& f(x_0 + \Delta x) - f(x_0) - f'(x_0)\Delta x
\end{aligned}
$$

7. Taylor Series Approximation

- We can approximate to the $(k+1)$ order of derivatives.

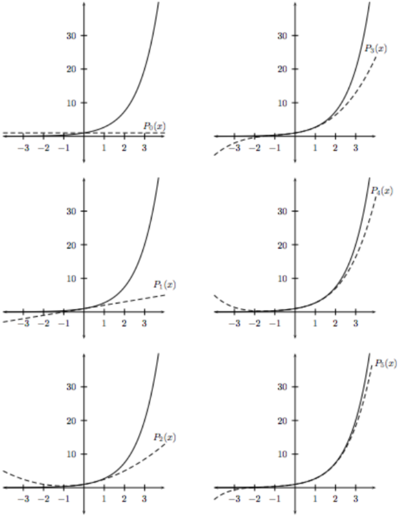
$$\begin{aligned} f(x_0+\Delta x) =& f(x_0) + f'(x_0)\Delta x + \frac{1}{2!} f''(x_0)(\Delta x)^2 + \dots \\ & + \frac{1}{k!} f^k(x_0)(\Delta x)^k + R_k(\Delta x, x_0) \\ R_k(\Delta x, x_0) =& \frac{f^{(k+1)}(c^*)}{(k+1)!}(\Delta x)^{k+1} \ , \ c^* \in (x_0, x_0+\Delta x) \\ & \lim_{\Delta x \to 0} \frac{R_k(\Delta x, x_0)}{(\Delta x)^k} \to 0 \end{aligned}$$

7. Taylor Series Approximation

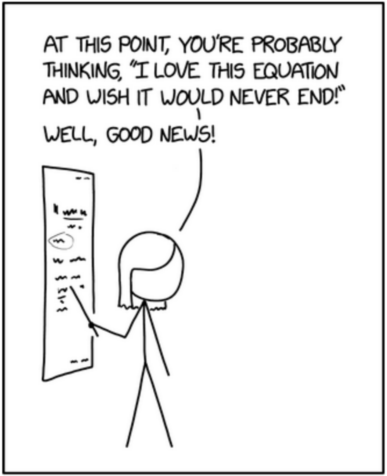


$$\Delta y \approx dy = f'(x_0)\Delta x$$

7. Taylor Series Approximation



7. Taylor Series Approximation

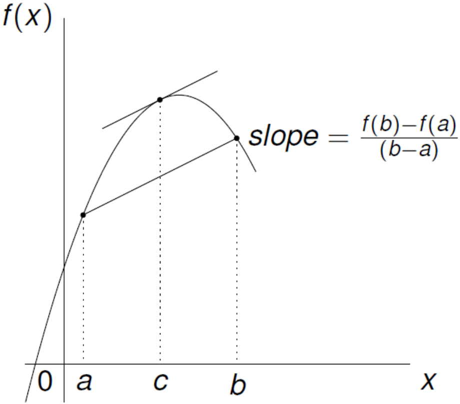


TAYLOR SERIES EXPANSION IS THE WORST.

8. Mean Value Theorem

- Let $f : U \to \mathbb{R}$ be a C^1 function over the interval $U \subset \mathbb{R}$.

$$\forall a, b \in U \exists c : a \leq c \leq b : f'(c) = \frac{f(b) - f(a)}{b - a}$$

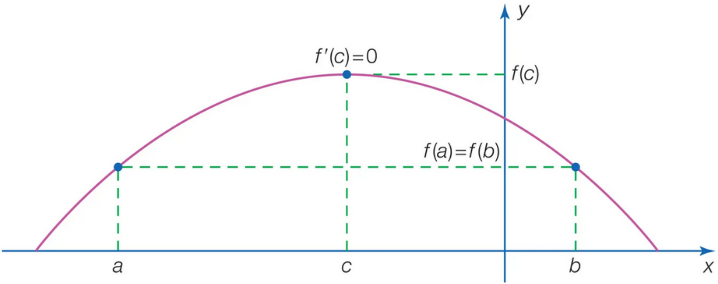


Rolle's Theorem

- A special case of the mean value theorem.
- E.g., If a continuous curve passes through the same y-value twice, and has a unique tangent line (i.e., a derivative) for all points in the interval, **then** a tangent parallel to the x-axis (i.e., critical value) exists in the interval.

Rolles Theorem:
If a function f is continuous on the the interval $[a, b]$ and differentiable on the interval (a, b) such that $f(a) = f(b)$, then $f'(x) = 0$ for some $x | a \leq x \leq b$.

ROLLE'S THEOREM



9. Critical Points

Weierstrass Theorem:
A continuous function $f(\cdot)$ over a closed and bounded interval $[a, b]$ attains both a local maximum and minimum.

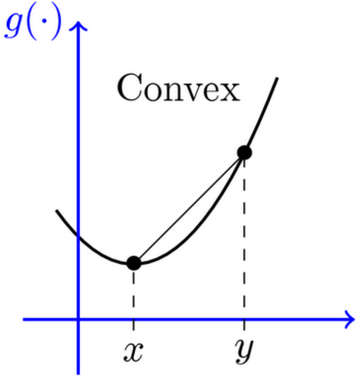
- Concave function:

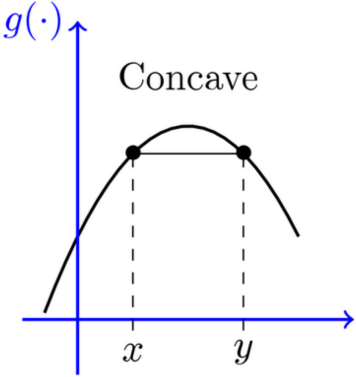
$$\forall x, y \in I : f(y) - f(x) \leq f'(x)(y-1) \vee f''(x) \leq 0$$

- Convex function:

$$\forall x, y \in I : f(y) - f(x) \geq f'(x)(y-1) \vee f''(x) \geq 0$$

Concave Up and Concave Down

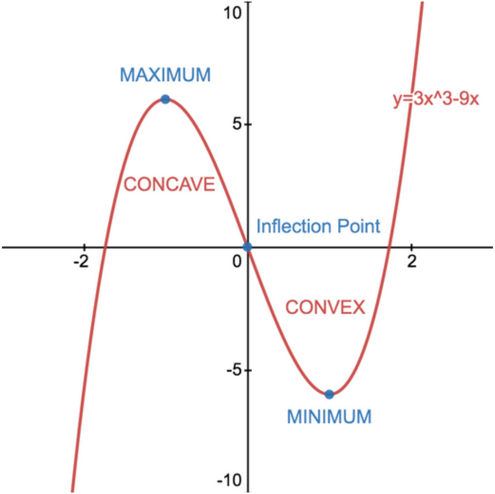




9. Convexity & Critical Points

- Critical Points: Values of x where $f'(x) = 0$ or is undefined.
- Local Max/Min (over interval I):
  $x_0, x \in I : f(x_0) \geq (\leq) f(x) \forall x$.
- Global Max/Min (over domain f):
  $x_0, x \in f : f(x_0) \geq (\leq) f(x) \forall x$.

MAXIMUMS AND MINIMUMS



Demonstration: Critical Points

Question:

What are the critical values for $f(x) = x^4 + 3x^2 + 10$

Answer:

$$\begin{aligned} f'(x) =& 4x^3 + (3)(2)x \\ =& 4x^3 + 6x = 0 \\ x^* =& \{0\} \end{aligned}$$

Demonstration: Critical Points

Question:

What are the critical values for $f(x) = x^4 + 3x^2 + 10$

Answer:

$$\begin{aligned} f'(x) =& 4x^3 + (3)(2)x \\ =& 4x^3 + 6x = 0 \\ x^* =& \{0\} \end{aligned}$$

Practice: Critical Points

1. Critical points for $f(x) = 8x^3 + 81x^2 - 42x - 8$

Practice: Critical Points

1. Critical points for $f(x) = 8x^3 + 81x^2 - 42x - 8$

Answer: Show Work

$$x^* = \{-7, \frac{1}{4}\}$$

Practice: Critical Points

1. Critical points for $f(x) = 8x^3 + 81x^2 - 42x - 8$
2. Critical points for $g(w) = 2w^3 - 7w^2 - 3w - 2$

Practice: Critical Points

1. Critical points for $f(x) = 8x^3 + 81x^2 - 42x - 8$
2. Critical points for $g(w) = 2w^3 - 7w^2 - 3w - 2$

Answer: Show Work

$$w^* = \{\frac{7 \pm \sqrt{67}}{6}\}$$

Practice: Critical Points

1. Critical points for $f(x) = 8x^3 + 81x^2 - 42x - 8$
2. Critical points for $g(w) = 2w^3 - 7w^2 - 3w - 2$
3. Critical points for $r(y) = (y^2 - 6y)^{1/5}$

PRACTICE: CRITICAL POINTS

1. Critical points for $f(x) = 8x^3 + 81x^2 - 42x - 8$
2. Critical points for $g(w) = 2w^3 - 7w^2 - 3w - 2$
3. Critical points for $r(y) = (y^2 - 6y)^{1/5}$

Answer: Show Work

$$y^* = \{0, 3, 6\}$$

Review

Review of Derivatives

1. Continutity & Differentiability
2. First Derivative
3. Second Derivative
4. Derivative Rules
5. Implicit Function
6. l'Hopital's Rule
7. Taylor Series Approximation
8. Mean Value Theorem
9. Critical Points

Assignment

- Readings on Integration before Lecture 04:
  - S&B Ch. 2, 3, & 4
- Assignment:
  - Problem Set 03 (PS03)
  - Solution set will be available following end of Lecture 04
- Struggling?
  1. Read the 'Encouraged Reading'
  2. Review 'Supplementary material'
  3. Reach out directly

Derivative Question 1 Answer:

Question

$$f'(x) = (1)x^{3x} + x(3e^{3x}) = (1+3x)x^{3x}$$

Derivative Question 2 Answer:

Question

$$f'(x) = 2ln(x^4+2)\frac{1}{x^4+2}4x^3$$
$$= \frac{8x^3 ln(x^4+2)}{x^4+2}$$

Derivative Question 3 Answer:

Question

Two Notes: I treat the quotient using the product rule: $(x+4)(x-3)^{-1}$. And I am able to flip fractions to force exponents to be positive.

$$\begin{aligned}
f'(x) =& \frac{2}{3}\left(\frac{x+4}{x-3}\right)^{-1/3}((1)(x-3)^{-1}+(x+4)(-1)(x-3)^{-2}(1) \\
=& \frac{2}{3}\left(\frac{x+4}{x-3}\right)^{-1/3}\left(\frac{x-3-x-4}{(x-3)^2}\right) \\
=& \frac{2}{3}\left(\frac{x-3}{x+4}\right)^{1/3}\left(\frac{-7}{(x-3)^2}\right) \\
=& \frac{2}{3}\frac{1}{(x+4)^{1/3}}\frac{-7}{(x-3)^{5/3}} \\
=& \frac{-14}{3(x+4)^{1/3}(x-3)^{5/3}}
\end{aligned}$$

Implicit Functions Question 1 Answer:

◂ Question

$$
\begin{aligned}
2xy^3 + 3x^2y^2\frac{dy}{dx} - y - x\frac{dy}{dx} &= 0 \\
(2xy^3 - y) + (3x^2y^2 - x)\frac{dy}{dx} &= 0 \\
\frac{dy}{dx} &= \frac{-2xy^3 + y}{3x^2y^2 - x}
\end{aligned}
$$

Implicit Functions Question 2 Answer:

◂ Question

$$
\begin{aligned}
e^y \frac{dy}{dx} + y + x\frac{dy}{dx} - 0 &= 0 \\
y + (e^y + x)\frac{dy}{dx} &= 0 \\
\frac{dy}{dx} &= \frac{-y}{e^y + x}
\end{aligned}
$$

Implicit Functions Question 3 Answer:

◂ Question

$$\text{Re-write } xy + y^2 = 4y(x^2 - xy)$$

$$\frac{dy}{dx}(xy + y^2 = 4x^2y - 4xy^2)$$

$$\frac{dy}{dx}(x + 2y - 4x^2 + 8xy) = 8xy - 4y^2 - y \frac{dy}{dx} = \frac{y(8x - 4y - 1}{x + 2y - 4x^2 + 8}$$

Critical Points Question 1 Answer:

◂ Question

$$
\begin{aligned}
f'(x) =& 8(3)x + 81(2)x - 42 \\
=& 24x^2 + 162x - 42 = 0 \\
=& 6(x+7)(4x-1) = 0 \\
x^* =& \{-7, \frac{1}{4}\}
\end{aligned}
$$

Critical Points Question 2 Answer:

◂ Question

$$
\begin{aligned}
g'(w) =& 2(3)w^2 - 7(2)w - 3 \\
=& 6w^2 - 14w - 3 = 0 \\
\because \text{Quad. Formula: } & \frac{b \pm \sqrt{b^2 - 4ac}}{2a} \\
w^* =& \frac{14 \pm \sqrt{268}}{12} \\
w^* =& \{\frac{7 \pm \sqrt{67}}{6}\}
\end{aligned}
$$

Critical Points Question 3 Answer:

◂ Question

$$
\begin{aligned}
r'(y) =& \frac{1}{5}(y^2 - 6y)^{-4/5}(2y - 6) \\
=& \frac{2y - 6}{5(y^2 - 6y)^{\frac{4}{5}}} \\
& \because \text{CV when } y = 0 \\
\implies & 2y - 6 = 0 \rightarrow y = 3 \\
\implies & y^2 - 6y = 0 \rightarrow y = \{0, 6\} \\
y^* =& \{0, 3, 6\}
\end{aligned}
$$