Math Review
Summer 2016

Topic 2

2. Basic topology of the reals

2.1. Sets, sequences and limits

<u>Sets</u>:

As you may recall, the basic set notations are:

$\cup$: Union $A \cup B = \{x: x \in A \text{ or } x \in B\}$

$$\{1,2,3\} \cup \{2,4\} = \{1,2,3,4\}$$

$\cap$: Intersection $A \cap B = \{x: x \in A \text{ and } x \in B\}$

$$\{1,2,3\} \cap \{2,4\} = \{2\}$$

C: Complement $A^C = \{x: x \notin A\}$

Everything not in A

$\backslash$: Difference $A\backslash B = \{x: x \in A \text{ and } x \notin B\}$

$$\{1,2,3\} \backslash \{2,4\} = \{1,3\}$$

$\times$: Cartesian product $A \times B = \{(a,b): a \in A \text{ and } b \in B\}$

If some are feeling rusty, the Cartesian product of two sets is the set of all possible ordered pairs whose first component is a member of the first set and whose second component is a member of the second set.

$$A = \{1,2,3\}, B = \{3,4\}$$

$$A \times B = \{\{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,3\}, \{3,4\}$$

Here, I quickly touch on the definitions for supremum and infimum.

Supremum and Infimum:

First, an **ordered set**, A, is a set in which in an order is defined. An **order** on A, denoted <, has the following properties:

(i) If $x \in A$ and $y \in A$ then one, and only one, of the following statements is true:

$$x < y$$
$$x = y$$
$$y < x$$

(ii) For $x, y, z \in A$, if $x < y$ and $y < z$, then $y < x$

Let A be an ordered set and $A_1 \subset A$.

- If there exists a $\beta \in A$ such that $x \leq \beta$ for every $x \in A_1$, then we say that $A_1$is **bounded above**, and call β an **upper bound** of A_1.
- Now, if there there exists an $\alpha \in A$ such that $x \geq \alpha$ for every $x \in A_1$, then we say that $A_1$is **bounded below** and call α the **lower bound**.

Example. Let $A = \mathbb{R}$, $A_1 = [-1,1], A_2 = (-\infty, 1], A_3 = [-1, \infty)$.

Then -2 is a lower bound for both $A_1$and A_3.

Any number you pick in the sets $A_1$and A_3, you will see that $x \geq \alpha = -2$

Q: What would be an upper bound for both A_1 and A_2?

$\mathcal{A}$: For instance, 2.

Anything different about bounds for sets A_2, A_3? As we note, the subset A_2 is not bounded below and the subset A_3 is not bounded above.

Let A be an ordered set and $A_1 \subset A$.

- Suppose $\beta \in A$ is an upper bound of A_1. If for all $\gamma < \beta$ there exists an $x \in A_1$, such that $x > \gamma$, then β is called the **least upper bound** of A_1 or the supremum of A_1. The supremum can be expressed as:

$$\beta = \sup A_1$$

This can be viewed as the smallest number among those upper bounds (*l.u.b*)

Now, not looking at specific given $\beta, like = 2,$ but more the set of β that fits our definition. The upper bound for A_1 is $[1, \infty)$.

What is $\gamma < \beta$? We are looking at all numbers in $(-\infty, 1)$. Do we have a number such that $x \in A_1$, such that $x > \gamma$? Yes, 1!

- Suppose $\alpha \in A$ is the lower bound of A_1. If for all $\delta > \alpha$ there exists an $x \in A_1$ such that $x < \delta$, then we say that α is the **greatest lower bounded** of A_1 or the infimum of A_1. The supremum can be expressed as:

$$\alpha = \inf A_1$$

This can be viewed as the largest number among those lower bounds (*g.l.b*)

Now, not looking at specific given $\alpha, like = -2,$ but more the set of α that fits our definition. The lower bound for A_1 is $(-\infty, 1]$.

What is $\delta > \alpha$? We are looking at all numbers in $(-1, \infty)$. Do we have a number such that $x \in A_1$, such that $x < \delta$? Yes, -1!

We call these $Sup\ A_1 = 1, Inf\ A_1 = -1.$

For any given subset A_1 can have at most one α and one β. If A is not bounded above then we say that $\sup A = +\infty$.

Sequences and limits:

A sequence is a function, $f(\cdot)$, defined on the set of natural numbers, $\mathbb{N}$. We have $f(n) = x_n$ for $n \in \mathbb{N}$ and usually denote the entire sequence by the symbol $\{x_n\}$, or $x_1, x_2, x_3, \ldots$. The values of $f(n)$, or x_n, are called the terms of the sequence.

A couple of examples of sequence are:

(i) $\{1, 2, 3,4, \ldots\}$ (iii) $\{3,1,4,1,5,9, \ldots\}$

(ii) $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$ (iv) $\{1,0,1,0,0,1, \ldots\}$

A sequence, $\{x_n\}$, **converges to a *limit* L**, $x_n \rightarrow L$, if given $\varepsilon > 0$, there is some N (i.e. element in the sequence), such that whenever $n > N$:

$$|x_n - L| < \varepsilon$$

We can also say that $\lim x_n = L, \lim_{n \rightarrow \infty} x_n = L, or\ x_n \rightarrow L$

Example. An example can help with the definition. Consider the sequence:
$\{1, 0.1, 0.01, 0.001, 0.0001, \ldots\}$
Is this converging or diverging?
Clearly it is getting closer and closer to 0, which is the limit for this sequence. The distance between that x_n and the limit L is less than the small number ε.

A sequence, $\{x_n\}$, **diverges** if it does not converge.

Theorem: If the sequence $\{x_n\}$ converges then the limit of $\{x_n\}$ is unique.

Consider the following properties of sequences:

Suppose that for the real number sequences x_n and y_n we have $x_n \rightarrow x$ and $y_n \rightarrow y$. Then:

(i) $\lim c x_n = cx$

(ii) $\lim(x_n + y_n) = x + y$

(iii) $\lim(x_n y_n) = xy$

(iv) $\lim \left(\frac{x_n}{y_n} = \frac{x}{y}\right)$ if for all $n \in \mathbb{N}$ we have $y_n \neq 0, y \neq 0$.

Cartesian Plane:

The set of real numbers is denoted by the symbol $\mathbb{R}$ and is defined as:

$$\mathbb{R} = \{x: -\infty \leq x \leq \infty\}$$

Considering the set product $\mathbb{R} \times \mathbb{R} = \{(x_1, x_2): x_1 \in \mathbb{R}, x_2 \in \mathbb{R}\}$. Any point in the set (any pair of numbers) can be identified with a point in the Cartesian plane.

2.2 Open and Closed Sets

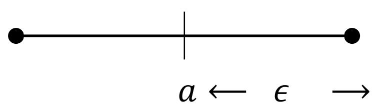
You will use the term neighborhood or a ball around a point often in Micro. This is really getting at limits. A limit is special case of an accumulation point (or cluster point): A point p is an accumulation point of a sequence, $\{x_n\}$, if for every ball around p, $(p - \epsilon, p + \epsilon)$, there are infinitely many elements of the sequence, x_n, where $x_n \in (p - \epsilon, p + \epsilon)$.

Building on the last point, a ball around point a of radius $\epsilon > 0$ is all points such that:

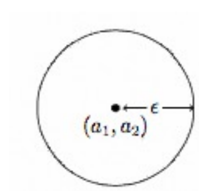
$$B(a, \epsilon) = \{|x - a| \leq \epsilon\}$$

This ball is the local neighborhood of a point a. This can be illustrated better with an figure in $\mathbb{R}^1$ and $\mathbb{R}^2$.

$\mathbb{R}^1$:



$\mathbb{R}^2$:



More generally, any n-tuple, or vector, is just an n-dimensional ordered tuple $(x_1, \ldots, x_n)$ and can be thought of as a "point" in n-dimensional Eucledian space or "n-space". A n-space is defined as the set product:

$$\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R} = \{(x_1, \ldots, x_n): x_i \in \mathbb{R}, i = 1, \ldots n\}$$

It may be useful to define the subset denoted by $\mathbb{R}^n_+$

$$\mathbb{R}^n_+ = \{(x_1, \ldots, x_n): x_i \geq 0, i = 1, \ldots n\} \subset \mathbb{R}^n$$

Metric space and sets:

A metric space is a set with a notion of distance defined among the points within the set. For any two points $x^1$and x^2in $\mathbb{R}$, denote the distance between them as

$$\mathrm{d}(x^1, x^2) = |x^1 - x^2|.$$

You may recall that in $\mathbb{R}^2$, this takes the form of:

$$\mathrm{d}(x^1, x^2) = \sqrt{(x_1^2 - x_1^1)^2 + (x_2^2 - x_2^1)^2}\,.$$

In $\mathbb{R}^n$, this takes the form of:

$$\mathrm{d}(x^1, x^2) = \sqrt{(x_1^2 - x_1^1)^2 + (x_2^2 - x_2^1)^2 + \cdots + (x_n^2 - x_n^1)^2}\,.$$

Pulling everything together, we are ready to tackle some key definitions:

Open $\varepsilon - ball.$ The open ε - ball with center x^0 and radius $\epsilon > 0$ (a real number) is the subset of points in $\mathbb{R}^n$: $B_\varepsilon(x^0) = \{x \in \mathbb{R}^n : \mathrm{d}(x^0, x) < \varepsilon\}$.

Closed $\varepsilon - ball.$ The closed ϵ - ball with center x^0 and radius $\epsilon > 0$ (a real number) is the subset of points in $\mathbb{R}^n$: $B_\varepsilon\,(x^0) = \{x \in \mathbb{R}^n : \mathrm{d}(x^0, x) \le \varepsilon\}$.

Open sets in $\mathbb{R}^n$: $S \subset \mathbb{R}^n$ is an open set if, for all $x \in S$, there exists some $\varepsilon > 0$ such that $B_\epsilon(x) \subset S$.

Open sets in $\mathbb{R}^n$: S is a closed set if its complement, S^C is an open set

2.3. Convex Set

A set is convex iff we can connect any two points in the set by a straight line that lies entirely within the set. $S \subset \mathbb{R}^n$ is a convex set if for all $x^1 \in S$ and $\square^2 \in S$, we have: we have $tx^1 + (1-t)x^2 \in S$, for all t in the interval $0 \le t \le 1$.

We go through a simple example here and we will do a slightly more applicable example from the *Production mini* at the end of this class.

Example. Let $S = [1, 13]$. Consider any two points in the set, $x^1 = 2$ and $x^2 = 8$. Define the convex combination $z = tx^1 + (1-t)x^2$. For any value of $t \in [0,1]$, for any two points in S, we have $z \in S$. If $t = \frac{1}{2}, z = \left(\frac{1}{2}\right) * 2 + \left(1 - \frac{1}{2}\right) * 8 = 5 \in S$.

Example. Let $S = [1,4] \times [1,4]$. Consider two vectors in □ ∈ ℝ, denoted by $x^1 = (x_1^1, x_2^1)$ and $x^2 = (x_1^2, x_2^2)$.

Q: How would you define this convex combination?

$$\mathcal{A}: z = tx^1 + (1-t)x^2$$
$$= (tx_1^1, tx_2^1) + \left((1-t)x_1^2, (1-t)x_2^2\right)$$
$$= (tx_1^1 + (1-t)x_1^2, tx_2^1 + (1-t)x_2^2)$$

For any value of $t \in [0,1]$, for any two points in S, we have $z \in S$.

Example. The Intersection of Convex Sets is Convex. Let S and T be convex sets in $\mathbb{R}^n$. Then $S \cap T$ is a convex set.

Proof. Let S and T be convex sets. Let x^1 and x^2 be any two points in $S \cap T$.
Because $x^1 \in S \cap T, x^1 \in S$ and $x^1 \in T$. Because $x^2 \in S \cap T, x^2 \in S$ and $x^2 \in T$.
Let $z = tx^1 + (1-t)x^2$, for $t \in [0,1]$, be any convex combination of x^1 and x^2.
Because S is a convex set, $z \in S$.
Because T is a convex set, $z \in T$.
Because $z \in S$ and $z \in T$, $z \in S \cap T$.

Recall that z, $x^1$and x^2 are all arbitrary, they could be any point!

Now, because every convex combination of any two points in $S \cap T$ is also in $S \cap T$, $S \cap T$ is a convex set.

2.4 Bounded sets in $\mathbb{R}^n$

A set in $\mathbb{R}^n$ is called bounded if it is entirely contained within some $\epsilon - ball.$ (either open or closed). That is, S is bounded if there exists some $\varepsilon > 0$, such that $S \subset B_\varepsilon(x)$ for some $x \in \mathbb{R}^n$.

2.5 Compact sets

One can think of the concept of boundedness basically means that the set has finite size. Then, a set S in $\mathbb{R}^n$ is called compact if and only if it is closed and bounded.

There are many definitions for compact set, this one is sufficient for us. You may have to learn some other definitions if you take Macro.

Let's go back to our graphs on boundedness and closed set. Can we see which one fits the definition?

2.6 Functions and properties of relations

Binary relation: A binary relation (R) is any collection of ordered pairs between the sets S and T.

Let S be the colors {red, yellow, green}, and T the set of fruits {apple, banana, pear}. The statement "is the color or" defines the relation R between the two sets. We denote this as sRt.

The following definitions are critical for your understanding of many concepts in Micro theory.

Completeness. A relation R on S is complete if, for all elements x and y in S, xRy or yRx.

Transitivity. A relation R on S is transitive if, for any three elements x, y and z in S, xRy and yRz, implies xRz.

In consumer theory, you will come across something called a preference relation, which is binary in nature. For example, if you say xPy, you mean x is preferred to $y.$

Quick Q: What is P equivalent to here from the above definition?

Professor Glewwe will cover this part in details and my example is quite simple.

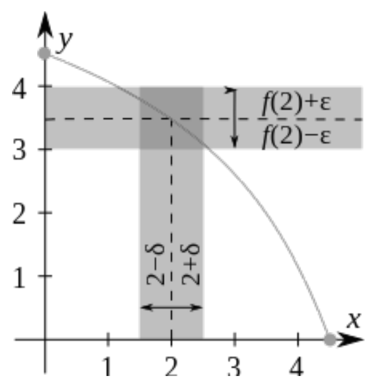
Q: You have the following set $X = \{x, y, z\}$. Your preference relationships are given by: xPy, yPz, and zPx.
Is it complete? How about transitive?
$\mathcal{A}$: For any given element a and b in X, we have aPb or bPa. So yes, completeness is satisfied.
We have xPy and yPz which imply that xPz. However, we find that zPx, so transitivity is not met.

Function: A function is a relation that associates each element of one set with a single, unique element of another set. We say that the function f is a mapping from one set D to another set R and write $f: D \rightarrow R$.

2.7 Continuity

Continuous function: A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is continuous at a point at $p \in \mathbb{R}$ if and only if, for
every $\varepsilon > 0$, there exists $\delta > 0$, such that:

$$|x-p| < \delta \; implies \; |f(x) - f(p)| < \epsilon$$



Intuitively, a function is continuous if you can trace the graph of the entire function without ever lifting your pencil from the page. For your purposes, apply the following theorem to show continuity:

Theorem: The function $f: \mathbb{R} \rightarrow \mathbb{R}$ is continuous at $p \in \mathbb{R}$ if and only, for every sequence $x_n \rightarrow p$, we have $lim\, f(x_n) = f(p)$. The function f is then continuous on R if and only if

$$x_n \rightarrow p$$

it is continuous for all $p \in \mathbb{R}$. This definition is clearer with an exercise/example:

Q: Show that that the function $f(x) = ax + b$ is continuous using the above theorem

Hint: Start by taking the limit of $f(x_n)$.

$\mathcal{A}$:

$$\begin{aligned} \lim f(x_n) &= \lim\ (ax_n + b) \\ &= \lim\ ax_n + \lim\ b \\ &= ap + b \\ &= f(p) \end{aligned}$$

■

Other properties of continuous functions are as follows:

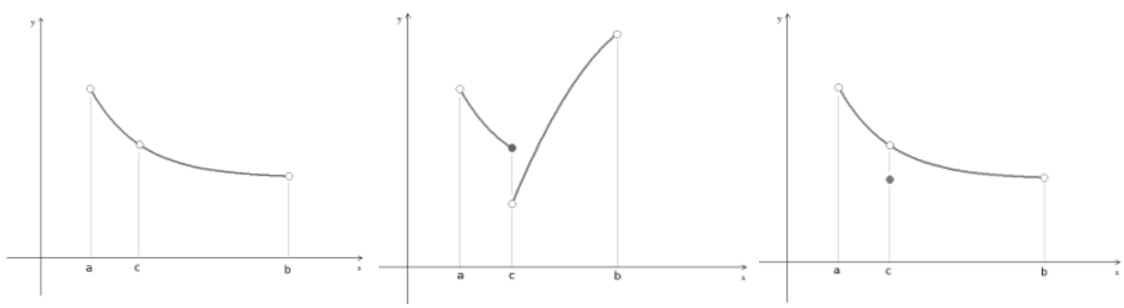
Let f and g be continuous functions in $\mathbb{R}$. Then:

(i) $f(x) + g(x)$
(ii) $f\ (x)\ \cdot\ g(x)$ where $g(x)\ \neq\ 0$
(iii) $\frac{f(x)}{g(x)}$
(iv) $(f \circ g)(x)\ =\ f\ (g(x))$ (composition of two functions)

are also *continuous.*

Q: Can we think of examples where a function, say $f(x)$ is *not continuous* a point c*?*

$\mathcal{A}$*:* $f(c)$ *is not defined ;* $\lim\limits_{x \to c} f(x)$ *does not exist ;* $\lim\ f(x) \neq \lim\ f(c)$



We now have all the tools to be able to try a more relevant example of ***convex set*** from <u>Production</u>. There are many more examples that you now equipped to try, but time is of essence here. Plus, you will the next year to see many of these yourself. The rationale of this part is show you that what we are currently reviewing translate directly into material that you see and learn in the coming year.

Couple of definitions that we will discuss to make sure we are catching notations correctly:
q: Vector of outputs
z: Vetor of inputs/factors

PPS is the Production Possibilities Set, that is the technology.

IRS is the Input Requirement Set or all combinations of input capable of producing a combination of output. It is defined as follows:

$$IRS(\boldsymbol{q}) = \{\boldsymbol{z} \in \mathbb{R}_+^N : (\boldsymbol{q}, -\boldsymbol{z}) \in \boldsymbol{PPS}\}$$

You are told that the input requirement set if *convex*.

Write out a one sentence statement to demonstrate that. No need for a proof. I would encourage you to try it out before reading through the hints. Use one hint at a time and see what you come up with!

Hint 1: If you have not already, read section 2.3 again.

Hint 2: What is S in this statement?

Hint 3: Can you introduce a couple of new variables that can help you with your statement?

Hint 4: How about $\boldsymbol{z}, \boldsymbol{z^1}$ and $\boldsymbol{z^3}$

Hint 5: Have you defined a t here yet?

Hint 5: How would you define $\boldsymbol{z^3}$ now that you have a t? Can you put it all together?

Hint 6: Now you have the answer:

For all $\boldsymbol{z} \in IRS\ (\boldsymbol{q})$, $\boldsymbol{z^1} \in IRS(\boldsymbol{q})$ and all $t \in [0,1]$, $\boldsymbol{z^3} = t\boldsymbol{z} + (1-t)\boldsymbol{z^1} \in IRS(\boldsymbol{q})$