6. Functions and correspondences

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Relations

- Two entities, x and y can be related in many ways.
- We can use the notation xRy, where R is the relation
- R could be:
  - =
  - >
  - <
  - “better than”
  - “further than”
  - “as good as”

Relations are:

- Reflexice if : xRx for all x
- Symetric if : $xRy \Rightarrow yRx$
- Transitive if : $xRy\ and\ yRz \Rightarrow xRz$

- If R is reflexive, symmetric and transitive, then it is an equivalence relation

Examples

- If the underlying set is X=R, “=“ is an equivalent relation
- Others can be:
  - “is the same age as”
  - “comes from the same country as”
  - “is the same altitude as”

Examples

- Given a utility function $U : R_2^+ \rightarrow R$, an equivalence class is given by $U^c = \{x \in R_2^+ : U(x) = c\}$. The indifference relation, usually denoted "x ∼ y," is read "x gives the same level of utility as y," or "the consumer is indifferent between bundles x and y."

The weak preference operator "$\succcurlyeq$" (we say that $x \succcurlyeq y$ if the consumer prefers x to y) is complete, reflexive, and transitive. The indiffrence operator $\sim$ is reflexive, symmetric, and transitive. The strict preference operator $\succ$ (we say that $x \succ y$ if the consumer strictly prefers x to y) is irreflexive, asymmetric, and transitive (and total). Sometimes, given an underlying preference ordering $\succcurlyeq$, people call $\sim$ the *symmetric part* and $\succ$ the *asymmetric part* of $\succcurlyeq$.

Other properties of relations

1. *Completeness:* for all $x, y \in X$, xRy or yRx.
2. *Irreflexivity:* For all $x \in X$, $\neg xRx$.
3. *Totality:* For all $x, y \in X$ with $x \neq y$, xRy or yRx.
4. *Asymmetry:* For all $x, y \in X$, $[xRy] \implies \neg[yRx]$.

Correspondences and Functions

- A **correspondence** is a relation that associates each element of one set (domain) with a set of elements of another set (range)
- A **function** is a relation that associates each element of the domain with a single, unique element of the range.

$$f : D \to R$$

  - every element in the range is mapped into by some point in the domain: **onto**
  - every element in the range is assigned to at most a single point in the domain: **one-to-one**

Quasiconcavity and Quasiconvexity

$f : D \rightarrow R$ is **quasiconcave** iff, for all $\mathbf{x}^1$ and $\mathbf{x}^2$ in D,

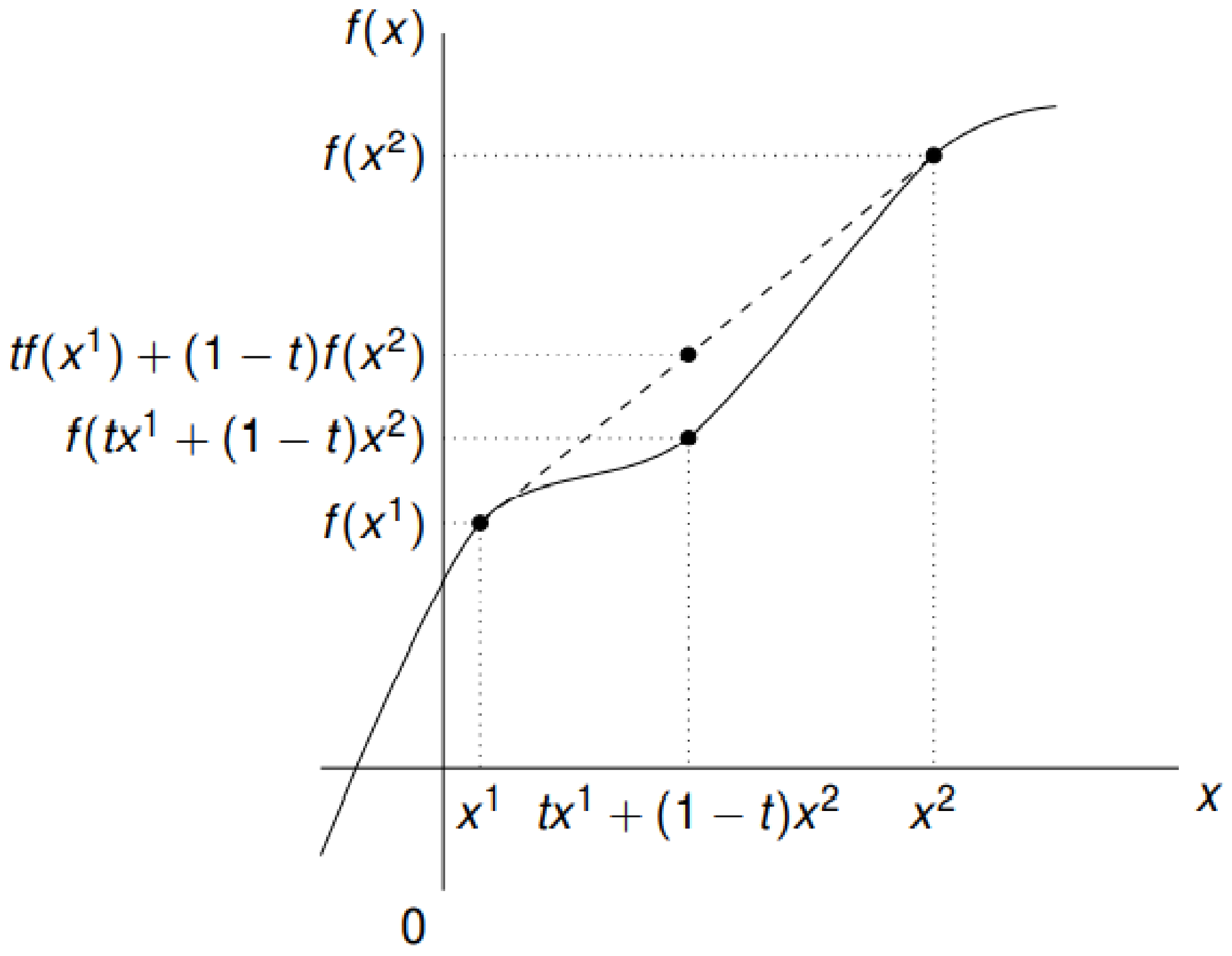
$$f(t\mathbf{x}^1 + (1-t)\mathbf{x}^2) \geq min[f(\mathbf{x}^1), f(\mathbf{x}^2)] \quad \forall t \in [0,1]$$

$f : D \rightarrow R$ is **quasiconvex** iff, for all $\mathbf{x}^1$ and $\mathbf{x}^2$ in D,

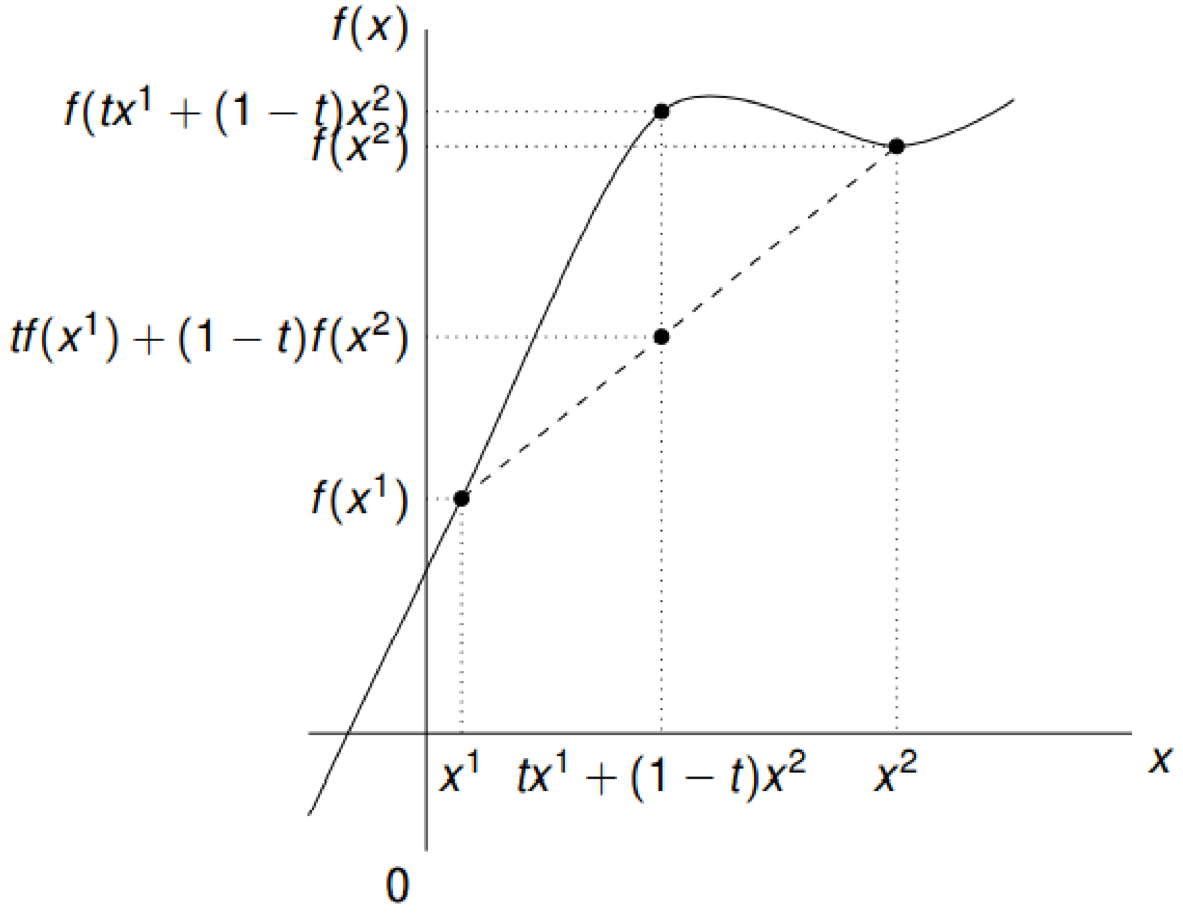
$$f(t\mathbf{x}^1 + (1-t)\mathbf{x}^2) \leq max[f(\mathbf{x}^1), f(\mathbf{x}^2)] \quad \forall t \in [0,1]$$

The quansiconcavity and quasiconvexity are strict when the inequality holds for all $\mathbf{x}^1 \neq \mathbf{x}^2$.

Quasiconcave and quasiconvex



Quasiconcave but not quasiconvex



Concavity and convexity

A real-valued function f defined on a convex subset $D \subset \mathbb{R}^n$ is **concave** if for all $\mathbf{x}^1, \mathbf{x}^2 \in D$ and for all $t \in [0, 1]$,

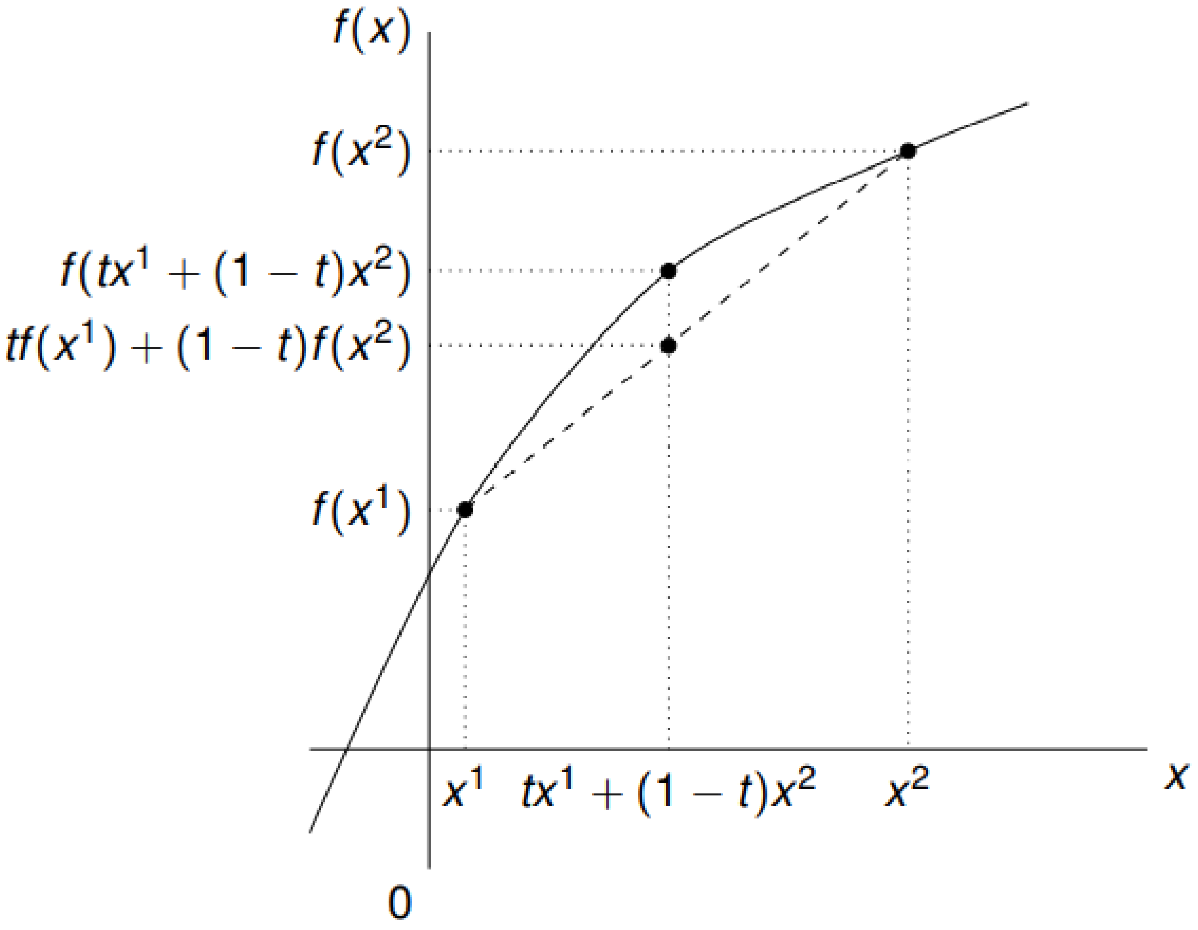
$$f(t\mathbf{x}^1 + (1-t)\mathbf{x}^2) \geq tf(\mathbf{x}^1) + (1-t)f(\mathbf{x}^2)$$

It is **convex** if

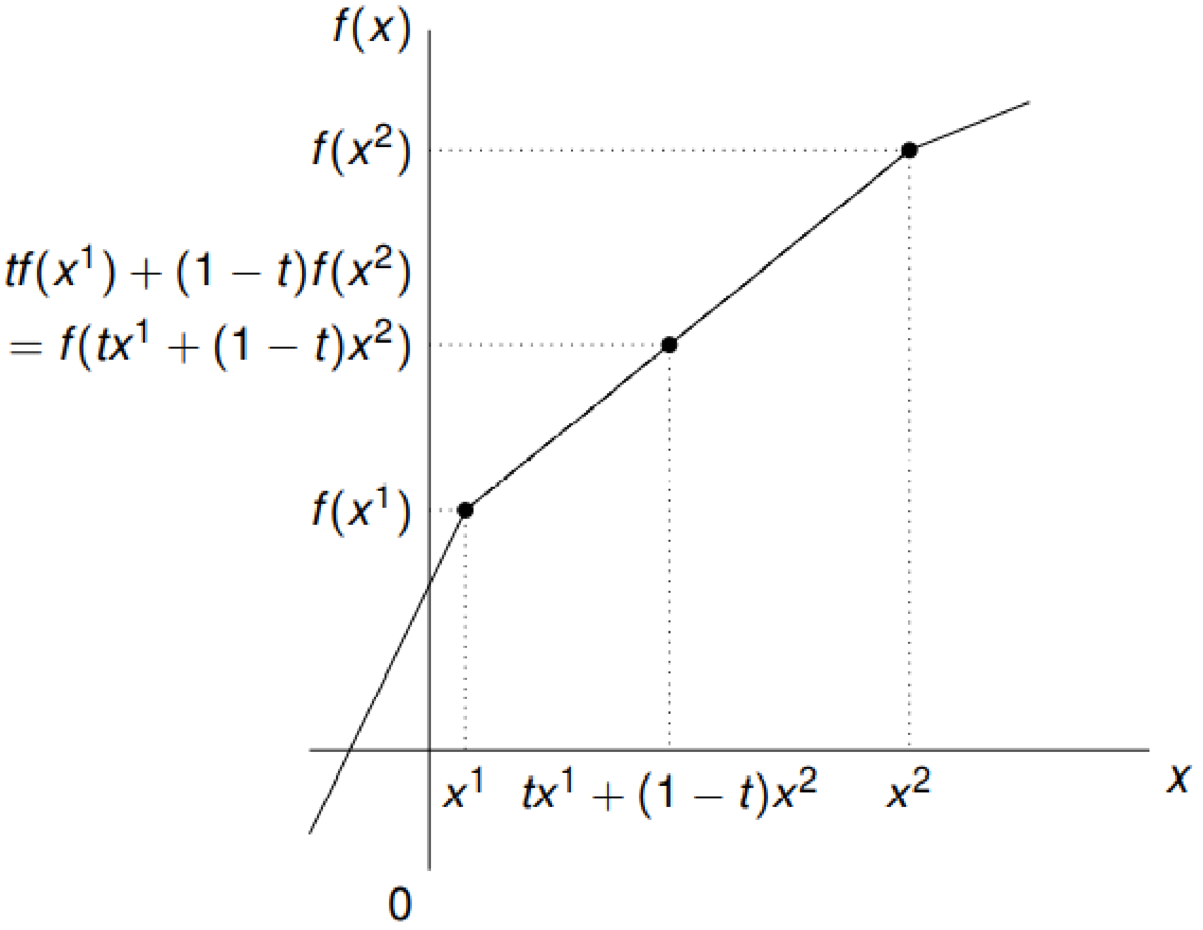
$$f(t\mathbf{x}^1 + (1-t)\mathbf{x}^2) \leq tf(\mathbf{x}^1) + (1-t)f(\mathbf{x}^2)$$

The concavity and convexity are **strict** when the inequality holds.

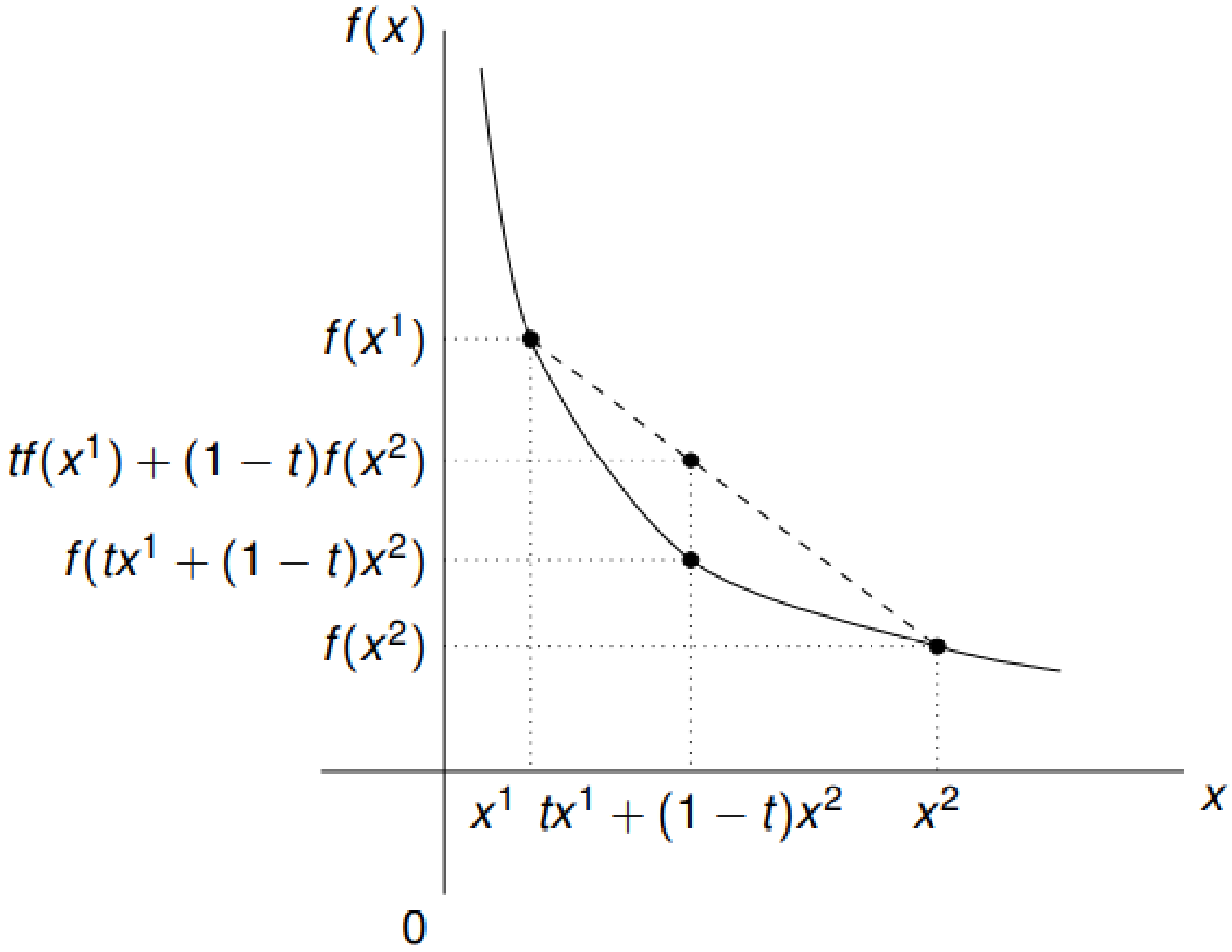
Concave function (single variable)



Concave but not strictly concave



Convex function (single variable)



Exercise

- What is the relationship between convex function and convex set?
- What is the relationship between concave function and concave set?
- What is a concave set?

Exercise

- Prove that if f is concave, it is quasiconcave

Continuity

Definition A.27. For $A \subset \mathcal{R}^n$, the function $f : A \longrightarrow \mathcal{R}^m$ is **continuous at** $x^0 \in A$ if for all $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in A$, $[||x - x^0|| < \delta] \Longrightarrow [||f(x) - f(x^0)|| < \epsilon]$.

Continuity for correspondences

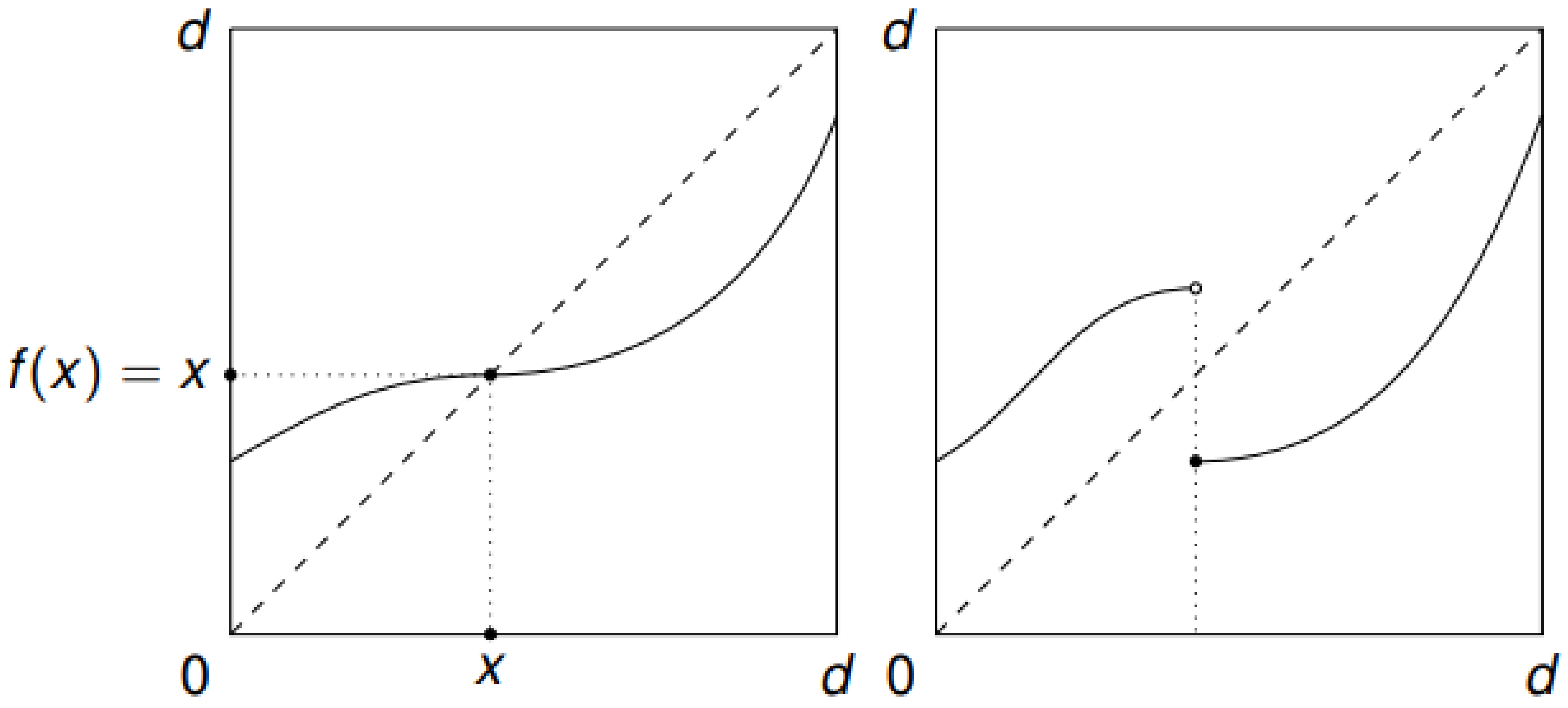
Given $\mathbf{A} \subset \mathbb{R}^n$ and the closed set $\mathbf{Y} \subset \mathbb{R}^n$, the correspondence $f : \mathbf{A} \to \mathbf{Y}$ is **upper hemicontinuous(uhc)** if it has a closed graph and the images of compact sets are bounded, that is for every compact set $\mathbf{B} \subset \mathbf{A}$ the set $f(\mathbf{B}) = \{\mathbf{y} \in \mathbf{Y} : \mathbf{y} \in f(\mathbf{x})\ \textit{for some}\ \mathbf{x} \in \mathbf{B}\}$ is bounded.

Given $\mathbf{A} \subset \mathbb{R}^n$ and a compact set $\mathbf{Y} \subset \mathbb{R}^n$, the correspondence $f : \mathbf{A} \to \mathbf{Y}$ is **lower hemicontinuous(lhc)** if for every sequence $\mathbf{x}^m \to \mathbf{x} \in \mathbf{A}$ with $\mathbf{x}^m \in \mathbf{A}$ for all m, and every $\mathbf{y} \in f(\mathbf{x})$, we can find a sequence $\mathbf{y}^m \to \mathbf{y}$ and an integer M such that $\mathbf{y}^m \in f(\mathbf{x}^m)$ for $m > M$.

A correspondence is **continuous** if it is both uhc and lhc.

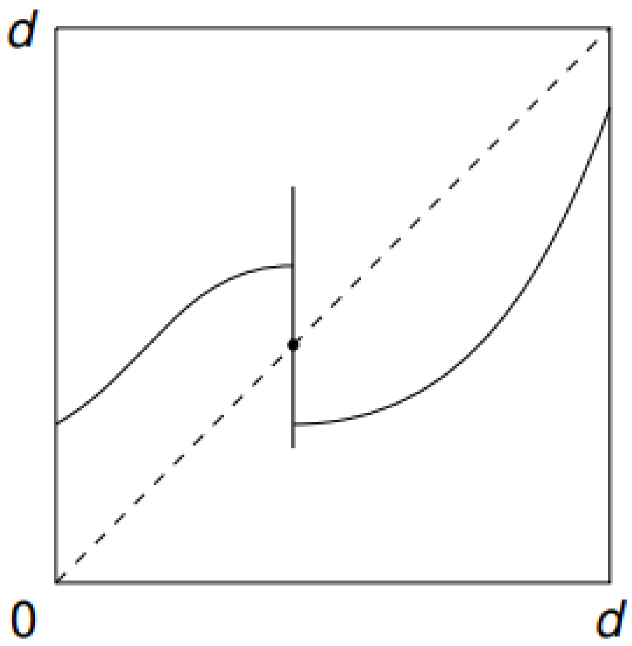
Brouwer's foxed point theorem

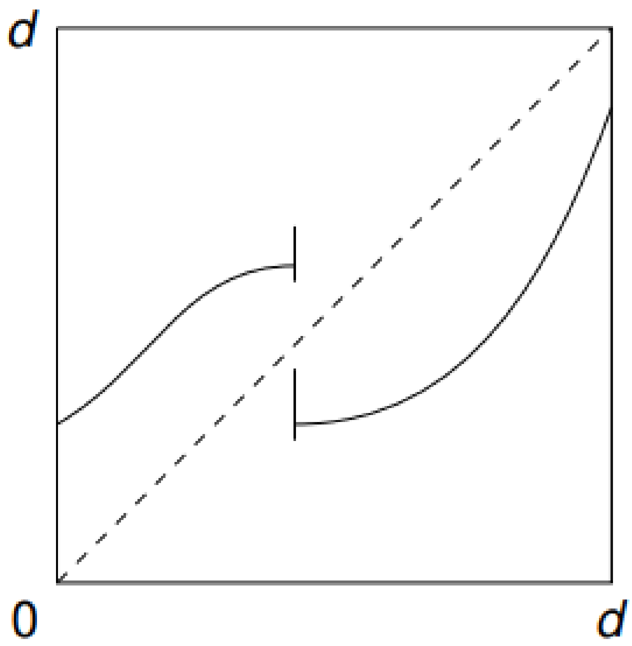
Suppose $D \subset \mathbb{R}^m$ is a nonempty, compact, convex set, and that $f : D \to D$ is a continuous function from D to itself. Then $f(\cdot)$ has a fixed point; that is, there is an $\mathbf{x} \in D$ such that $\mathbf{x} = f(\mathbf{x})$.



Kakutani's fixed point theorem

Suppose $D \subset \mathbb{R}^m$ is a nonempty, compact, convex set, and that $f : D \to D$ is a continuous **upper hemicontinuous correspondence** from D to itself with the property that the set $f(\mathbf{x}) \subset D$ is non-empty and convex for every $\mathbf{x} \in D$. Then $f(\cdot)$ has a fixed point; that is, there is an $\mathbf{x} \in D$ such that $\mathbf{x} = f(\mathbf{x})$.





Exercise

Consider the following utility functions:

a. $U(x, y) = ln(x) + ln(y)$

b. $U(x, y) = min(x, y)$

c. $U(x, y) = x^2 + y^2$

For each function, plot the set of indifference curves and find and plot the demand functions for the commodity x and y as a function of p. Are the preferences represented by these utility functions convex? Strictly convex? Monotone? Strictly monotone? Continuous?