9. Optimization III

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What does the Lagrange multiplier mean?

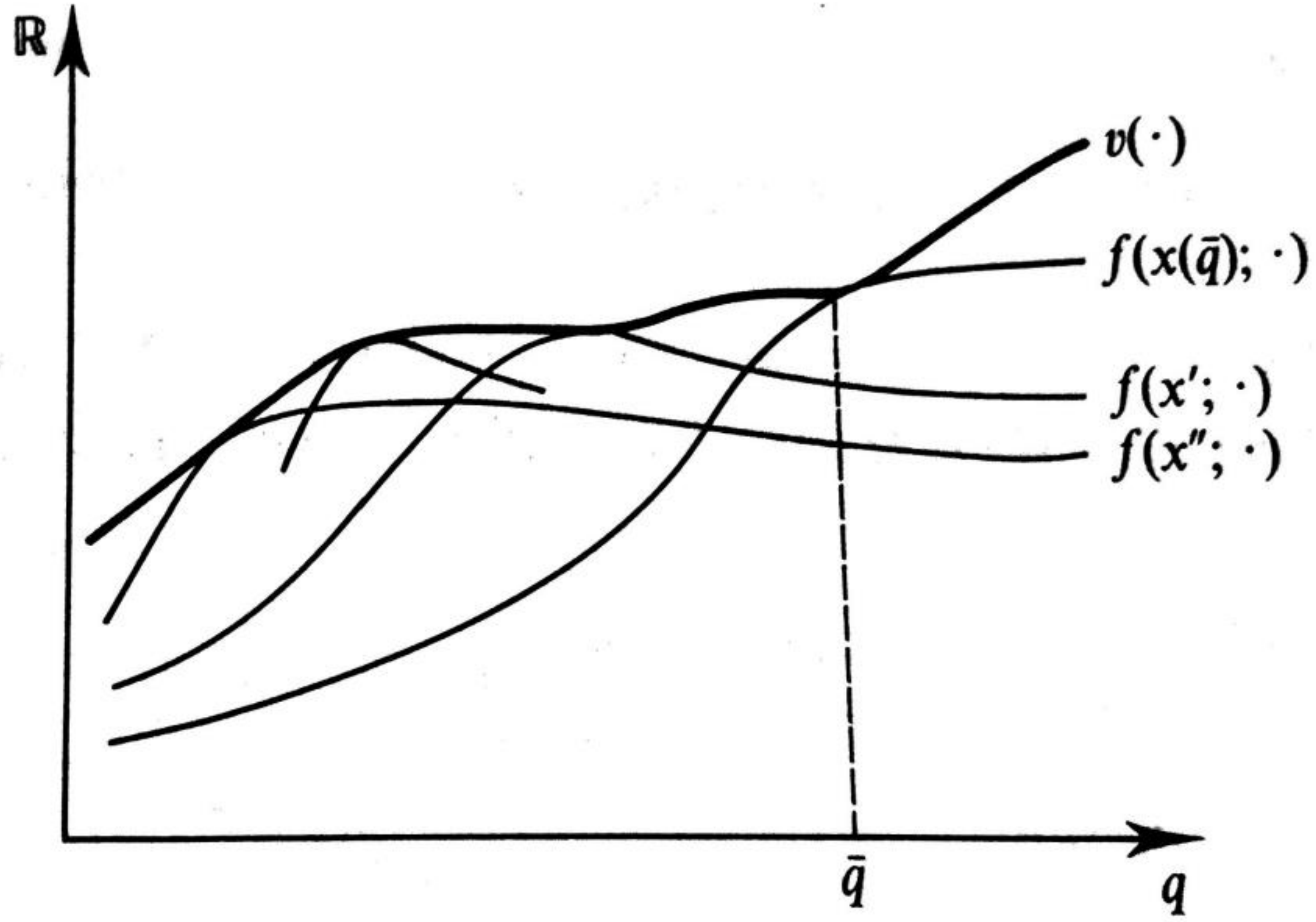
The Envelope Theorem (Unconstrained)

Let $f(\mathbf{x}; a)$ be a C^1 function of $\mathbf{x} \in \mathbb{R}^n$ and the scalar a. For each choice of the parameter a, consider the unconstrained optimization problem

$$maximize \quad f(\mathbf{x}; a) \quad w.r.t. x$$

Let $\mathbf{x}^*(a)$ be a solution of this problem. Suppose that $\mathbf{x}^*(a)$ is a C^1 function of a. Then,

$$\frac{d}{da} f(\mathbf{x}^*(a); a) = \frac{\partial}{\partial a} f(\mathbf{x}^*(a); a)$$



Source: M.W.G. p965

Example

$$maximize \quad f(x, a) = -x^2 + 2ax + 4a^2$$

F.O.C.

$$f'(x) = -2x + 2a = 0$$

$$x^* = a$$

Plugging this back into $f(x, a)$ we get a single variable function

$$f(x^*(a); a) = f(a, a) = -a^2 + 2a \cdot a + 4a^2 = 5a^2$$

So

$$\frac{df^*}{da} = 10a$$

This is equal to the partial derivative of the original function at the optimum

$$\frac{\partial f(x^*(a), a)}{\partial a} = 2x + 8a = 10a.$$

Envelope Theorem (Constrained)

Let $f, h_1, ..., h_m : \mathbb{R}^n \times \mathbb{R}^1 \to \mathbb{R}^1$ be C^1 functions. Let $\mathbf{x}^*(a) = (x_1^*(a), ..., x_n^*(a))$ denote the solution of the problem of maximizing $f(\mathbf{x}; a)$ with respect to $\mathbf{x}$ on the constraint set

$$h_1(\mathbf{x}; a) = 0, ..., h_m(\mathbf{x}; a) = 0$$

for any fixed choice of parameter a. Suppose that $\mathbf{x}^*(a)$ and the Lagrange multipliers $\mu_1(a), ..., \mu_m(a)$ are C^1 functions of a. Then

$$\frac{d}{da} f(\mathbf{x}^*(a); a) = \frac{\partial L}{\partial a}(\mathbf{x}^*, \mu(a); a)$$

Exercise

Verify that the interpretation of the Lagrange multiplier is a special case of the envelope function theorem using the example problem:

$$\max_{x_1, x_2} f(x_1, x_2) = x_1 x_2 \quad s.t \quad 2x_1 + 4x_2 = a$$

Exercise

In the utility maximizing problem, the Roy's identity says that the consumer's demand for good *i* is the ratio of the partial derivatives of the maximized utility with respect to good *i*'s price and income with a minus sign, i.e.

$$-\frac{\partial u^*(\mathbf{p}, y)/\partial p_i}{\partial u^*(\mathbf{p}, y)/\partial y} = x_i^* = \mathbf{x}_i(\mathbf{p}, y)$$

Prove it using the Envelope theorem.

Dynamic programming – Descrete time case

- Choose a time path of a control variable to maximize the sum of flow benefits plus a terminal or scrap value that result from that control variable

$$\max_{\{y_t\}} \sum_{t=0}^{T-1} u(y_t, x_t, t) + F(x_T, T)$$

- Subject to:

$$x_{t+1} = x_t + f(x_t, y_t, t)$$
$$x_0 = a$$

- Where:

t: time
T: end time
y_t: control variable at time t
x_t: state variable at time t
$u(y, x, t)$: flow benefit function
$f(x_t, y_t, t)$: dynamics function
$F(x(T), T)$: terminal/scrap value function

- Define the maximized value as:

$$J(x_0, 0) = \max_{\{y_t\}} \sum_{t=0}^{T-1} u(y_t, x_t, t) \ + F(x_T, T)$$

- The maximized value $J(x_0, 0)$ is a function starting at state x_0 and time 0.

- J is called the **value function.**

Solution approach

- Break this up into two problems:
  1. What we do at time t=0
  2. What we do from t=1 to t=T-1

$$J(x_0, 0) = \max_{y_0}\{u(y_0, x_0, 0) + \max_{\{y_t\}:t=1,2,\ldots,T-1} \sum_{t=1}^{T-1} u(y_t, x_t, t) \ + F(x_T, T)\}$$

- Along with the same constraints.
- Notice that the second part looks very similar to our original problem, but starting at t=1.

  Which means…

- If $J(x, t)$ represents the maximum value attainable form starting at state x in time t, then we can write our decomposed problem as

$$J(x_0, 0) = \max_{y_0}\{u(y_0, x_0, 0) + J(x_1, 1)\}$$

with the same constraints.

- We can further simplify by embedding the constraint that determines x_1:

$$J(x_0, 0) = \max_{y_0}\{u(y_0, x_0, 0) + J(f(x_0, y_0, 0), 1)\}$$

Bellman Equation

- Applying the same decomposition to an arbitrary starting point of x_t and time t gives us

$$J(x_t, t) = \max_{y_t}\{u(y_t, x_t, t) + J(f(x_t, y_t, t), t + 1)\}$$

- This is the famous Bellman equation.
    - His insight, the "Principle of Optimality," was evident in the steps we just went through:
        - If we follow an optimal path and stop, the remaining parts must be the optimal solution to the problem we face when we stopped.

How to solve

- 1.) Backward induction (for discrete time)
  - Solve the problem in period $t = T - 1$
    - Because it's the last period, there's no future to consider and so it's a static problem.
  - Use the answer for $J(x, T-1)$ to solve problem in $t = T - 2$
  - Repeat