Lecture 05
Multi-variate Calculus

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Last Lecture Review

- Integration:
  - Definite Integral
  - Fundamental Theorem of Calculus
  - Integration Rules
  - Integration by Substitution
  - Integration by Parts

Review Assignment

1. Problem Set 04 solutions are available on Github.
2. Any issues or problems **You** would like to discuss?

Daily Icebreaker

- Attendance via prompt:
  - Name
  - Program and track
  - Daily icebreaker subject...

Topic: Multi-variate Calculus

MOTIVATION

- General background
  - How do handle rate of change for many variables in a single function.
  - Determine how each variable's change impacts to change of the whole.
- Why do economists' care?
  - Most calculus applications in economics involves a mult-variate scenario.
- Application in this career
  - To estimate marginal effects in econometrics.
  - To determine partial equilibrium effects.

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Overview

1. Partial Derivatives
2. Higher Order Derivatives
3. Total Differentiation
4. Multi-variable Chain Rule
5. Implicit Function Theorem
6. Multi-variable Concavity

1. Partial Derivative

- Let $f : \mathbb{R}^n \to \mathbb{R}$. Then for each variable x_i at each point $x^0 = (x_1^0, \ldots, x_n^0)$ in the domain of f when a limit exits,

$$\frac{\partial f}{\partial x_i}(x_1^0, \ldots, x_n^0) = \lim_{h \to 0} \frac{f(x_1^0, \ldots, x_i^0 + h, \ldots, x_n^0) - f(x_1^0, \ldots, x_i^0, \ldots, x_n^0)}{h}$$

- d = Single variable derivative
- ∂ = Partial derivative
- Δ = Total differential

2. Total Differentiation

- Total Derivative: A function $f : \mathbb{R}^n \to \mathbb{R}$ expressing how f changes with the **simultaneously** change in x_1 through x_n.
- Note that we can sum the partial derivatives to estimate the **total** differential effect.
- We can approximate the actual change $\Delta f = f(x^* + \Delta x) - f(x^*)$ using the total differential:

$$\Delta f = \frac{\partial f}{\partial x_1}(x^*)\Delta x_1 + \cdots + \frac{\partial f}{\partial x_n}(x^*)\Delta x_n$$

2. Total Differentiation

- The Jacobian derivative vector
- For $f : \mathbb{R}^n \to \mathbb{R}$

$$Df_{x^*} = \left(\frac{\partial F}{\partial x_1}(x^*) \cdots \frac{\partial F}{\partial x_n}(x^*) \right)$$

- For $f : \mathbb{R}^n \to \mathbb{R}^m$

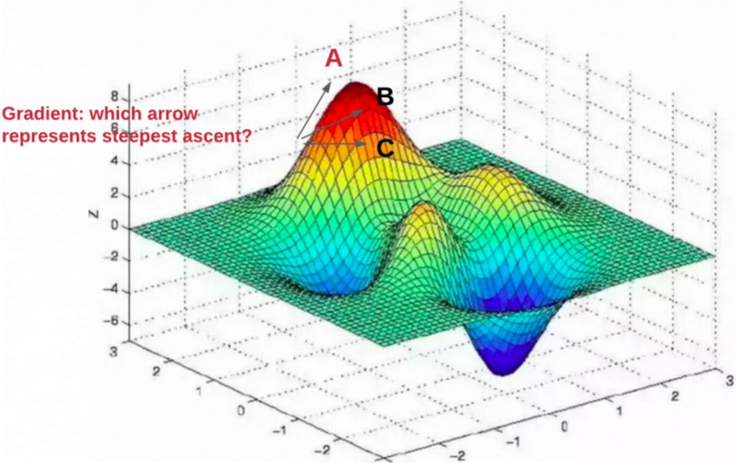
$$Df_{x^*} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x^*) & \cdots & \frac{\partial f_1}{\partial x_n}(x^*) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(x^*) & \cdots & \frac{\partial f_m}{\partial x_n}(x^*) \end{bmatrix}$$

2. Total Differentiation

- The gradient ∇ is the direction that F increases most rapidly.
- Commonly applied in machine learning.
- The gradient of x^* can be written as a column vector.

$$\nabla F(x^*) = \begin{pmatrix} \frac{\partial F}{\partial x_1}(x^*) \\ \vdots \\ \frac{\partial F}{\partial x_n}(x^*) \end{pmatrix}$$

Gradient Ascent and Descent



2. Total Differentiation

- Hessian Matrix is a **symmetric** matrix of the second order derivatives.

$$D^2 f_{x^*} \equiv \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n} & \frac{\partial^2 f}{\partial x_2 \partial x_n} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

- Young's Theorem ensures the symmetry.
- If $f : \mathbb{R}^n \to \mathbb{R}$ is $C^2 \in \mathbb{R}^n$, then $\forall x \in \mathbb{R}^n$ and each index pairs i, j:

$$\frac{\partial^2 f(x)}{\partial x_i \partial x_j} = \frac{\partial^2 f(x)}{\partial x_j \partial x_i}$$

3. Higher Order Derivatives

- Consider the partial derivative for x_1.

$$\frac{\partial f(x_1, \ldots, x_n)}{\partial x_1} = f_1(x)$$

- We can get higher order gradients for n partial derivatives of $f_1(x)$.

$$\begin{aligned}\nabla f_1(x) =& \frac{\partial^2 f(x)}{\partial x_1 \partial x_1} + \frac{\partial^2 f(x)}{\partial x_1 \partial x_2} + \cdots \frac{\partial^2 f(x)}{\partial x_1 \partial x_n} \\ =& f_{11}(x) + f_{12}(x) + \cdots + f_{1n}(x)\end{aligned}$$

4. Multi-variable Chain Rule

- If $x(t) = (x_1(t), \ldots, x_n(t)$ is a C^1 curve on an interval about t_0, and f is a C^1 function on a ball about $x(t_0)$, then $g(t) \equiv f(x_1(t), \ldots, x_n(t))$ is a C^1 function at t_0. This allows for the differentiation of t:
$$\frac{dg}{dt}(t_0) = \frac{\partial f]}{\partial x_1}(x(t_0))x_1'(t_0) + \cdots + \frac{\partial f]}{\partial x_n}(x(t_0))x_n'(t_0)$$

5. Implicit Function Theorem

- Consider a function with two variables x, y.

$$G(x, y(x)) = c)$$

- We can use the implicit function theorem with respect to x about x_0.

$$\frac{dG}{dx}(x_0, y(x_0)) \cdot \frac{dx}{dx}(x_0) + \frac{dG}{dy}(x_0, y(x_0)) \cdot \frac{dy}{dx}(x_0) = 0$$

$$\implies y'(x_0) = \frac{dy}{dx}(x_0) = -\frac{\frac{dG}{dx}(x_0, y(x_0))}{\frac{dG}{dy}(x_0, y(x_0))}$$

6. Multi-variable Concavity

- Suppose f is a convex subset $U \in \mathbb{R}^n$.
- f is **concave** iff $\forall x_1, x_2 \in U$, $g_{x_1,x_2}(t) \equiv f(tx_2 + (1-t)x_1)$ is concave on $\{t \in \mathbb{R} | tx_2 + (1-t)x_1 \in U\}$.
- E.g., if the function remains in the concave subset, then it is concave.

6. Multi-variable Concavity

- Let f be a C^2 function on an open convex subset $D \in \mathbb{R}^n$.
- Then f is **concave** on D iff the Hessian matrix $D^2 f(x)$ is **negative semidefinite** $\forall x \in D$.
- f is **convex** on D iff the Hessian matrix $D^2 f(x)$ is **positive semidefinite** $\forall x \in D$.

6. Multi-variable Concavity

- Quasi-concavity: $\forall x, y \in D \in \mathbb{R}^n$ and $\forall t \in [0,1]$
$$f(tx + (1-t)y) \geq \min\{f(x), f(y)\}$$

- With multiple variables, we can show quasi-concavity using the Hessian matrix D iff
$$f(y) \geq f(x) \implies Df(x)(y-x) \geq 0$$

Practice: Multi-variate Calculus

1.

Review of Multi-variate Calculus

1. Partial Derivatives
2. Higher Order Derivatives
3. Total Differentiation
4. Multi-variable Chain Rule
5. Implicit Function Theorem
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Assignment

- Readings on Matricies before Lecture 06:
  -
- Assignment:
  - Problem Set 05 (PS05)
  - Solution set will be available following end of Lecture 06
- Struggling?
  1. Read the 'Encouraged Reading'
  2. Review 'Supplementary material'
  3. Reach out directly