APEC Math Review

Part 4 One-variable Calculus

Ling Yao

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Derivatives

First derivative

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = f'(x_0) \; or \; \frac{df}{dx}(x_0)$$

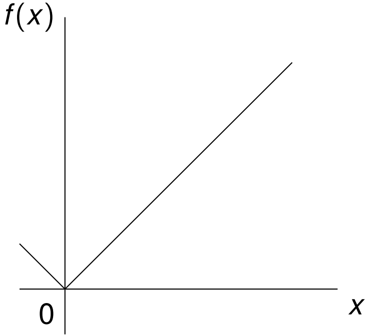
Second derivative

$$f''(x_0) \; or \; \frac{d}{dx}(\frac{df}{dx})(x_0) = \frac{d^2 f}{dx^2}(x_0)$$

Commonly used derivatives

- $a' = 0$
- $(x^a)' = ax^{a-1}$
- $(a^x)' = a^x ln(a)$
- $(e^x)' = e^x$
- $(ln\, x)' = \frac{1}{x}$

Continuity and differentialbility



A continuous but not differentialble function

- differentiable $\Rightarrow$ continuous
- continuous $\not\Rightarrow$ differentiable
- If f' is continuous, then we say f is continuously differentiable, denote as C^1.
- If f'' is continuous, then we say f is twice continuously differentiable, denote as C^2.

Rules

- For sums: $[f(x) \pm g(x)]' = f'(x) \pm g'(x)$
- Product rule: $[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$
- Quotient rule: $[\frac{f(x)}{g(x)}]' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$
- Inverse rule: $[f^{-1}(x)]' = \frac{1}{f'(x)}$ if $f(x)$ is monotone, differntiable, $f'(x) \neq 0$ and $f^{-1\prime}(x)$ is differetiable
- Chain rule: $\frac{d}{dx} h(g(x)) = h'(g(x))g'(x)$

Exercise: Chain rule

$$ln\, y = \beta_0 + \beta_1 ln\, x + \epsilon$$

Prove that $\beta_1 = \frac{dy}{dx} * \frac{x}{y}$.

Implicit function

Sometimes y cannot be expressed as a explicit function of x, but we can still calculate $\frac{dy}{dx}$. For example:

$$e^y + xy - e = 0$$

Take the derivative of x on both sides,

$$\frac{d}{dx}(e^y + xy - e) = e^y * \frac{dy}{dx} + y + x\frac{dy}{dx} = 0$$

Rearrange we have:

$$\frac{dy}{dx} = -\frac{y}{x + e^y} \quad (x + e^y \neq 0)$$

l'Hopital's rule

(l'Hopital's Rule for zero over zero): Suppose that $\lim_{x\to a} f(x) = 0$, $\lim_{x\to a} g(x) = 0$, and that functions f and g are differentiable on an open interval I containing a. Assume also that $g'(x) \neq 0$ in I if $x \neq a$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

(l'Hopital's Rule for infinity over infinity): Assume that functions f and g are differentiable for all x larger than some fixed number. If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Application: l'Hopital's rule

Show that the constant elasticity of substitution (CES) function $Y = A(\alpha K^{\gamma} + (1-\alpha)L^{\gamma})^{1/\gamma}$ is Cobb-Douglas function $Y = AK^{\alpha}L^{1-\alpha}$ when $\gamma \to 0$.

Proof: First, take log on both sides:

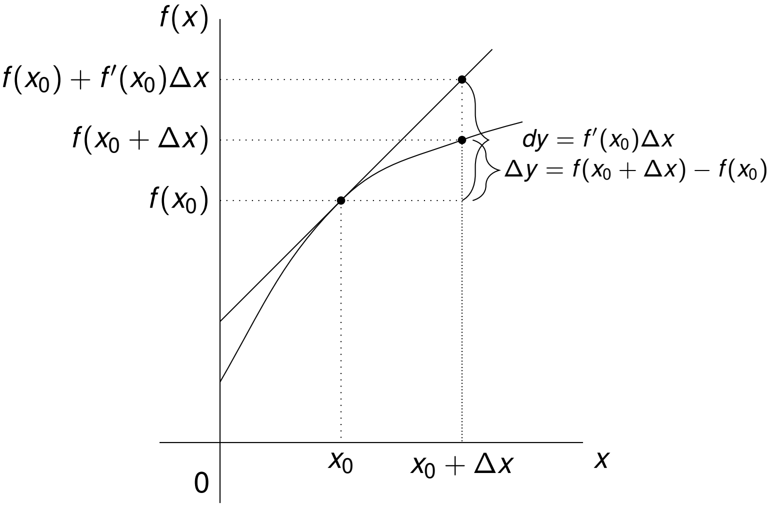
$$\mathit{ln}\, Y = \mathit{ln}\, A + \frac{1}{\gamma}\mathit{ln}(\alpha K^{\gamma} + (1-\alpha)L^{\gamma})$$

By the l'Hopital's rule,

$$\begin{aligned}
\lim_{\gamma \to 0} \frac{\mathit{ln}(\alpha K^{\gamma} + (1-\alpha)L^{\gamma})}{\gamma} &= \lim_{\gamma \to 0} \frac{\frac{d\, \mathit{ln}(\alpha K^{\gamma} + (1-\alpha)L^{\gamma})}{d\gamma}}{\frac{d\gamma}{d\gamma}} \\
&= \lim_{\gamma \to 0} \frac{\alpha K^{\gamma}\, \mathit{ln}\, K + (1-\alpha)L^{\gamma}\, \mathit{ln}\, L}{\alpha K^{\gamma} + (1-\alpha)L^{\gamma}} \\
&= \alpha\, \mathit{ln}\, K + (1-\alpha)\mathit{ln}\, L
\end{aligned}$$

So $\lim_{\gamma \to 0} \mathit{ln}\, Y = \mathit{ln}\, A + \alpha\, \mathit{ln}\, K + (1-\alpha)\mathit{ln}\, L$.

Approximation by differentials



$$\Delta y \approx dy = f'(x_0)\Delta x$$

Taylor series approximation

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x$$

We can denote the remainder as:

$$R(\Delta x, x_0) = f(x0 + \Delta x) - f(x_0) + f'(x_0)\Delta x$$

If the function has $(k+1)$ orders of derivatives, we can further approximate:

$$f(x_0+\Delta x) = f(x_0)+f'(x_0)\Delta x+\frac{1}{2!}f''(x_0)(\Delta x)^2+...+\frac{1}{k!}f^{(k)}(x_0)(\Delta x)^k + R_k(\Delta x, x_0),$$

where $R_k(\Delta x, x_0) = \frac{f^{(k+1)}(c^*)}{(k+1)!}(\Delta x)^{k+1}$, $c^* \in (x_0, x_0 + \Delta x)$ and $\frac{R_k(\Delta x, x_0)}{(\Delta x)^k} \to 0$ as $\Delta x \to 0$.

Application: translog function

Suppose we want to estimate a CES production function of 2 inputs:

$$Y = A(\alpha K^{\gamma} + (1-\alpha)L^{\gamma})^{1/\gamma}$$

Hard to estimate because it is nonlinear.
We can use the Taylor series approximation to transfer it into a linear model.
First, take log on both sides:

$$\ln Y = \ln A + \frac{1}{\gamma}\ln(\alpha K^{\gamma} + (1-\alpha)L^{\gamma})$$

The Taylor series approximation of $\ln Y$ around $\gamma \to 0$ is

$$\lim_{\gamma\to 0} \ln Y + \lim_{\gamma\to 0} \frac{d \ln Y}{d\,\gamma} \times (\gamma - 0) + \textit{higher orders}$$

Application: translog function

By the l'Hopital's rule:

$$\lim_{\gamma \to 0} \mathit{ln}\, Y = \mathit{ln}\, A + \alpha \mathit{ln}\, K + (1-\alpha)\mathit{ln}\, L$$

$$\lim_{\gamma \to 0} \frac{d\, \mathit{ln}\, Y}{d\, \gamma} \times (\gamma - 0) = \frac{1}{2}\gamma\alpha(1-\alpha)(\mathit{ln}\, K - \mathit{ln}\, L)^2$$

So we can approximate the CES function with the following translog funciton:

$$\begin{aligned} \mathit{ln}\, Y &= \mathit{ln}\, A + \alpha \mathit{ln}\, K + (1-\alpha)\mathit{ln}\, L + \frac{1}{2}\gamma\alpha(1-\alpha)(\mathit{ln}\, K - \mathit{ln}\, L)^2 \\ &= \beta_0 + \beta_1 \mathit{ln}\, K + \beta_2 \mathit{ln}\, L + \beta_3 \mathit{ln}^2 K + \beta_4 \mathit{ln}^2 L + \beta_5 \mathit{ln}\, K\, \mathit{ln}\, L \end{aligned}$$

[See (Henningsen & Henningsen, 2011, page 57) for a full proof.]

Application: log differences

Using Taylor series expansion we can show that for small x:

$$log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = x + O(x^2) \approx x$$

The symbol $O(x^2)$ means that the remainder is bounded by Ax^2 as $x \to 0$ for some $A < \infty$.

If y^* is c% greater than y then $y^* = (1 + c/100)y$. Taking natural logarithms,

$$log\, y^* = log\, y + log(1 + c/100)$$

$$\Leftrightarrow log\, y^* - log\, y = log(1 + c/100) \approx \frac{c}{100}$$

What is the interpretation of β_1 in $log\, y = \beta_0 + \beta_1 x + \epsilon$?

Exercise: Taylor series expansion

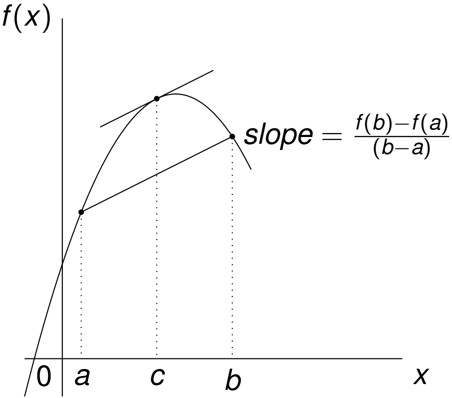
Show that

$$log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

Mean value theorem

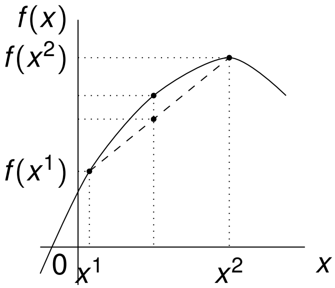
Let $f : U \to \mathbb{R}$ be a C^1 function on a interval $U \subset \mathbb{R}$. For any point $a, b \in U$, there is a point c between a and b such that

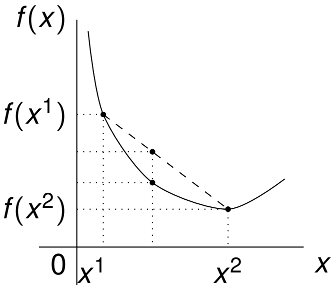
$$f(b) - f(a) = f'(c)(b - a)$$



Calculus criteria for convexity

If f is a C^1 function on an interval I in $\mathbb{R}$.





f is **concave** on I iif

$$f(x^2) - f(x^1) \leq f'(x^1)(x^2 - x^1)$$

f is **convex** on f iif

$$f(x^2) - f(x^1) \geq f'(x^1)(x^2 - x^1)$$

for all $x^1, x^2 \in I$.

Calculus criteria for convexity

Proof:
Suppose f is concave. Let $t \in (0, 1]$, by the definition of concavity,

$$tf(x_2) + (1-t)f(x^1) \leq f(tx^2 + (1-t)x^1)$$

$$\Leftrightarrow tf(x_2) - tf(x^1) \leq f(tx^2 + (1-t)x^1) - f(x^1)$$

$$\begin{aligned}\Leftrightarrow f(x^2) - f(x^1) &\leq \frac{f(tx^2 + (1-t)x^1) - f(x^1)}{t} \\ &= \frac{f(x^1 + t(x^2 - x^1)) - f(x^1)}{t(x^2 - x^1)}(x^2 - x^1)\end{aligned}$$

$$\lim_{t \to 0} \frac{f(x^1 + t(x^2 - x^1)) - f(x^1)}{t(x^2 - x^1)} = f'(x^1)$$

$$\Rightarrow f(x^2) - f(x^1) \leq f'(x^1)(x^2 - x^1)$$

Calculus criteria for convexity

...continued

Suppose $f(x^2) - f(x^1) \leq f'(x^1)(x^2 - x^1)$ for all $x^1,\ x^2 \in I$. Then,

$$f(x^2) - f\underbrace{((1-t)x^1 + tx^2)}_{\text{new } x^1} \leq f'\underbrace{((1-t)x^1 + tx^2)}_{\text{new } x^1}(x^2 - \underbrace{(1-t)x^1 - tx^2}_{\text{new } x^1})$$

$$= (1-t)(x^2 - x^1)f'((1-t)x^1 + tx^2) \tag{1}$$

Similarly, $f(x^1) - f(x^2) \leq f'(x^2)(x^1 - x^2)$ for all $x^1,\ x^2 \in I$.

$$f(x^1) - f\underbrace{((1-t)x^1 + tx^2)}_{\text{new } x^2} \leq f'\underbrace{((1-t)x^1 + tx^2)}_{\text{new } x^2}(x^1 - \underbrace{(1-t)x^1 - tx^2}_{\text{new } x^2})$$

$$= -t(x^2 - x^1)f'((1-t)x^1 + tx^2) \tag{2}$$

$(1) \times t + (2) \times (1-t)$:

$$tf(x_2) + (1-t)f(x^1) \leq f(tx^2 + (1-t)x^1) \qquad \square$$

Second derivatives and convexity

A differentiable function f for which $f''(x) \leq 0$ on an interval I is concave.
A differentiable function f for which $f''(x) \geq 0$ on an interval I is convex.

Critical points

- Critical points: points where $f'(x) = 0$ or f' is undefined
- f has **local min (max)** at x_0 if $f(x_0) \leq (\geq) f(x)$ for all x in some interval
- f has **global min (max)** at x_0 if $f(x_0) \leq (\geq) f(x)$ for all x in the domain of f.
- If $f'(x_0) = 0$ and $f''(x_0) < 0$, then x_0 is ______
- If $f'(x_0) = 0$ and $f''(x_0) > 0$, then x_0 is ______
- If $f'(x_0) = 0$ and $f''(x_0) = 0$, then x_0 is ______

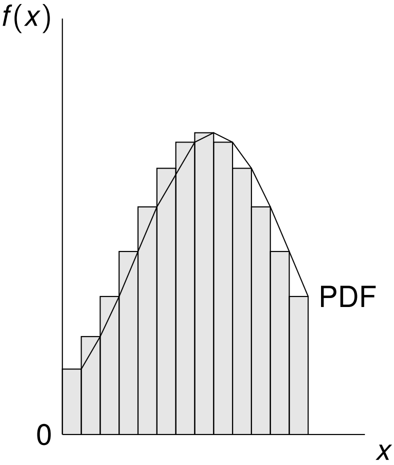
For numbers a and b, the **definite integral** of $f(x)$ from a to b is $F(b) - F(a)$, where $F(x)$ is an antiderivative of f.

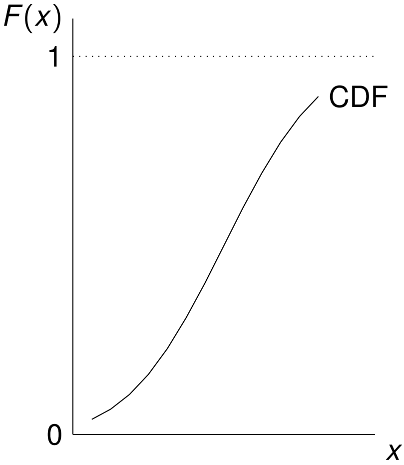
$$\int_a^b f(x)dx = F(b) - F(a), \; where\, F' = f$$

If we divide the interval (a, b) to N subintervals and denote each end point as x_i. The **Reimann Sum** is

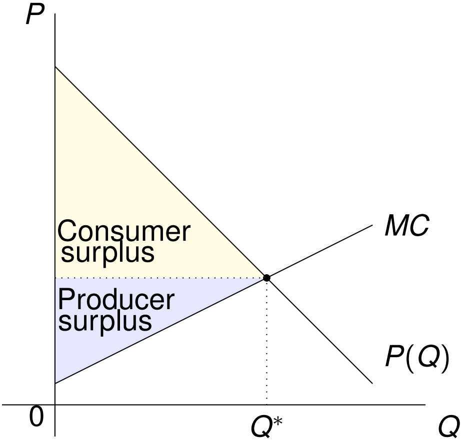
$$\lim_{\Delta \to 0} \sum_{i=1}^{N} f(x_i)\Delta = \int_a^b f(x)dx$$

Application: Probability and cumulative density functions





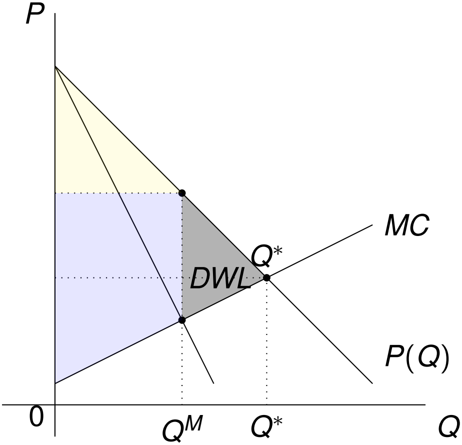
Application: Social surplus



Total surplus = consumer surplus + producer surplus

$$W(Q^*) = \int_0^{Q^*} [P(Q) - MC(Q)]dQ$$

Application: Dead-weight loss from monopoly



Deadweight loss

$$DWL = \int_{Q^M}^{Q^*} [P(Q) - MC(Q)]dQ$$