Lecture 01
Logic and Mathematical Proofs

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Mathematics Review Course, Summer 2023
University of Minnesota
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Course Preview

This Course

- Review of graduate-level mathematics necessary for the 1st year sequence.
- Aimed at PhD-track. MS-track is encouraged.
- This sets the foundation (Not exhaustive).
- By the end you should feel confident tackling a variety of math situations in a short period.
- Syllabus on Github repo. Repo is the most up-to-date place for course content.
- This course is **optional**.

Preview of Course

1. Logic, Proofs, Sets, & Topology
2. Uni-variate Calculus & Multi-variate Calculus
3. Linear Algebra
4. Functions & Optimization
5. Probability & Statistics
6. Dynamic Programming

About the Instructor

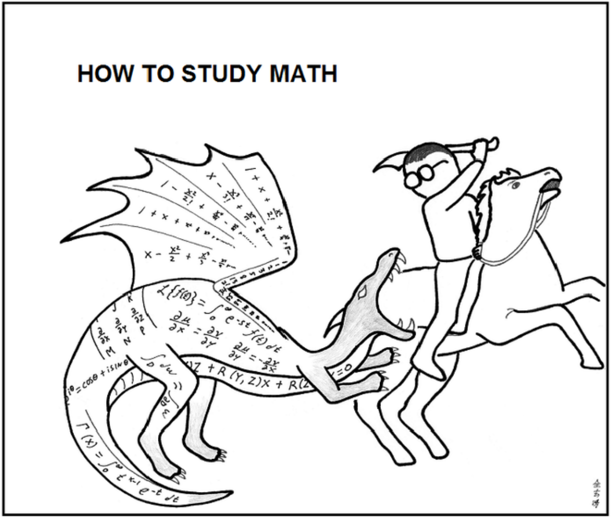
Ryan McWay

- Current: 2nd Year APEC PhD student
- Background: SLU → USF → UMich → UMN
- Research: Development, Behavior, Urban, Environment

Daily Icebreaker

- Attendance via prompt:
  - Name
  - Hometown
  - Program and track
  - Research interests
  - Daily Icebreaker: Imagine you are a professional baseball player or wrestler. What is you walk up (intro) song?

Fight with Math...



Don't just read it; fight it!

--- Paul R. Halmos

Topic: Logic

MOTIVATION

- General background
  - Logic is at the heart of reasoning and arguments.
  - Expressed in words and formalized through math, this is a foundation of theoretical arguments.
  - Deduce information correctly. Not deducing correct information.
- Why do economists' care?
  - Foundation for theory
  - Criteria to evaluate arguments
- Application in this career
  - Creating logical arguments
  - How you think about research
  - Evaluating theory and conclusions from empirical evidence

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Overview

1. Logical Statements
2. Necessary Conditions
3. Sufficient Conditions

0. Terminology

$\forall$	For all
$\exists$	Exists
$:$	Such that
$\mid$	Given
$\therefore$	Therefore
$\because$	Because
$\wedge$	And
$\vee$	Or
$\neg$	Negation
$\equiv$	Equivalent or identical.
$\implies$	Implies, then, or sufficient
$\iff$	If and only if, or necessary and sufficient
$\subset$	Strict subset
$\subseteq$	Subset
$\in$	In, or an element of the following set
$\square$	End of Proof. QED (quod erat demonstrandum $\sim$ it has been demonstrated).

0. Terminology

Αα **ALPHA** [a] *ἄλφα*	Ββ **BETA** [b] *βῆτα*	Γγ **GAMMA** [g] *γάμμα*	Δδ **DELTA** [d] *δέλτα*	Εε **EPSILON** [e] *ἒ ψιλόν*	Ζζ **ZETA** [dz] *ζῆτα*
Ηη **ETA** [ɛː] *ἦτα*	Θθ **THETA** [t^h] *θῆτα*	Ιι **IOTA** [i] *ἰῶτα*	Κκ **KAPPA** [k] *κάππα*	Λλ **LAMBDA** [l] *λάμβδα*	Μμ **MU** [m] *μῦ*
Νν **NU** [n] *νῦ*	Ξξ **XI** [ks] *ξεῖ*	Οο **OMICRON** [o] *ὂ μικρόν*	Ππ **PI** [p] *πεῖ*	Ρρ **RHO** [r] *ῥῶ*	Σσς **SIGMA** [s] *σῖγμα*
Ττ **TAU** [t] *ταῦ*	Υυ **UPSILON** [ʉ] *ὖ ψιλόν*	Φφ **PHI** [p^h] *φεῖ*	Χχ **CHI** [k^h] *χεῖ*	Ψψ **PSI** [ps] *ψεῖ*	Ωω **OMEGA** [ɔː] *ὦ μέγα*

Too Many Symbols

1. Logical Statements

- Logical Statement: Use a set of facts to infer/assume a new fact.
  - Hypothesis (If): Premise with set of facts
  - Conclusion (Then): New set of facts inferred if hypothesis is true.
  - e.g., **If** I study throughout the course, **then** I earn a higher grade.
- Family of statements:
  - Tautologies: Statement is always true $(1 = 1)$
  - Contradictions: Statement is always false $(2 = 3)$
  - Statement: $A \implies B$
  - Contrapositive: $\neg B \implies \neg A$
  - Converse: $B \implies A$
  - Inverse: $\neg A \implies \neg B$

1. Logical Statements

- Axiom: Statements assumed to be true.
  - e.g., $a = b$, $b = c \implies a = c$
- Theorem: A statement proven to be true.
- Corollary: A theorem that follows from another theorem.
- Lemma: A minor theorem used to prove another theorem.

2. Necessary Condition

- A is necessary for B
  - If B is true, A must be true: $B \implies A$.
  - If A is not true, B is not true: $\neg A \implies \neg B$
- A is needed to make the argument.

3. Sufficient Condition

- A is sufficient for B
  - If A is true, B must be true: $A \implies B$
  - If B is not true, A is not either: $\neg B \implies \neg A$
- A allows you to state B, but not necessary to make argument.

Necessary but Not Sufficient

Logic: another thing that penguins aren't very good at.

4. Necessary and Sufficient (If and Only If ∼ iff)

- If A is sufficient for B, B is necessary for A.
- If $A \implies B$ and $B \implies A$, then $A \iff B$ (iff)
  - A is necessary and sufficient for B.
  - A and B are equivalent statements.
  - A is true iff B is true: A iff B

Demonstration: Necessary and Sufficient

Question:

Is this statement true: "If I open the door, I used the key."

Answer:

Logic: Open Door (A) $\implies$ Used Key (B) Necessary: You need a key (B) to open the door (A). $B \implies A$. Sufficient: If you do not have the key ($\neg B$), then there is no way to open the door ($\neg A$). So $A \implies B$.

Demonstration: Necessary and Sufficient

Question:

Is this statement true: "If I open the door, I used the key."

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Practice: Necessary and Sufficient Conditions

1. Scoring more touchdowns than your opponent in American football means you won the game.

Practice: Necessary and Sufficient Conditions

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Answer: Show Work

Sufficient but not necessary.

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2. Obtaining a learner's permit will lead to earning a driver's license.

Practice: Necessary and Sufficient Conditions

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2. Obtaining a learner's permit will lead to earning a driver's license.

Answer: Show Work

Necessary but not sufficient.

Practice: Necessary and Sufficient Conditions

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2. Obtaining a learner's permit will lead to earning a driver's license.
3. All even whole numbers must be divisible by two.

Practice: Necessary and Sufficient Conditions

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Answer: Show Work

Both necessary and sufficient.

Topic: Proofs

Motivation

- General background
  - Method for proving or disproving a logical statement
- Why do economists' care?
  - Determine which theories are incorporated into economic theory
- Application in this career
  - Theory papers and well-developed theory sections of empirical papers.
  - Often in appendix sections to prove statements articulated as part of an argument in a paper.

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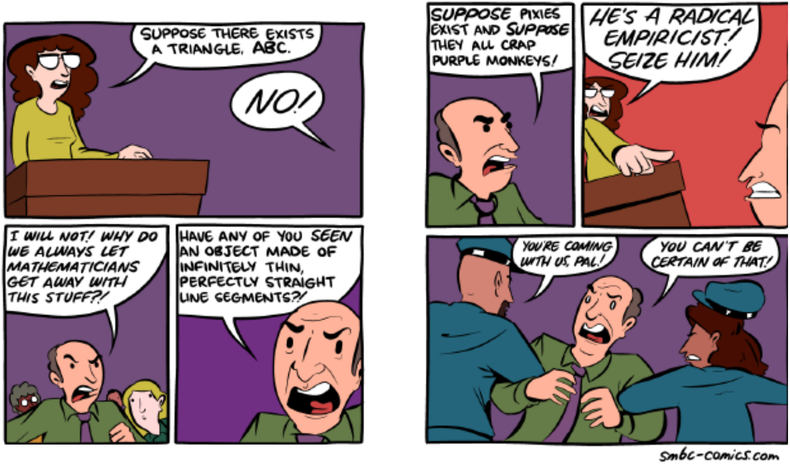
Overview

1. Truth Tables
2. Existence & Uniqueness
3. Direct Proofs
4. Proof by Contradiction
5. Proof by Induction
6. Proof by Contrapositive

Assumptions are the core of proofs...



1. Truth Table

- Shows how the truth/falsity of a compound statement depends on the truth/falsity of the simple statements from which it's constructed.
- Statements = {Known to be true, known to be false, truth unknown }
- Truth table for $(P \to Q)$:

P	Q	$P \iff Q$
T	T	T
T	F	F
F	T	F
F	F	T

Demonstration: Truth Table

Question:

Construct a truth table for $(P \to Q) \lor (Q \to P)$

Answer:

P	Q	$P \to Q$	$Q \to P$	$(P \to Q) \lor (Q \to P)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

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2. Existence and Uniqueness

- Existence: Can an answer to the truth of a statement (viz., proof) be found.
  - Sometimes you can prove no answer can exist.
- Uniqueness: An assertion that there is exactly one statement that is true for that family of statements.
  - x is the family $P(x)$.
- Ideally, you want a statement which exists and is unique.
  - Ex. If "$x + 2 = 3$, then $x = 1$" is a statement that exists and is unique.

Application: Equilibrium Exists

Existence of Equilibrium II Theorem:

Suppose that each consumer's preferences are continuous, strongly monotonic, and convex. Suppose also that, for each consumer $i, \omega_i \gg 0$. Then there exists a Walrasian equilibrium (p^*, x^*) for ε.

3. Proof by Deduction (Direct Proof)

- Show $A \implies B$
- Deductive reasoning: Use a set of premises that lead to a conclusion.
- Sometimes we need to strengthen A but adding assumptions (e.g., weak assumptions are preferred).

Demonstration: Direct Proof

Question:

Let m be an even integer and p be any integer. Then $m \times p$ is an even integer.

Answer:

Proof.

m is an even integer so $\exists$ an integer q such that $m = 2 \times q$ by the definition of an even integer. Therefore, we can make the statement:

$$m \times p = (2 \times q) \times p = 2 \times (q \times p)$$

So, $m \times p$ is an even integer. $\square$

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4. Proof by Contradiction

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- $A \implies B \equiv \neg A$ and $\neg B \equiv \neg B \implies \neg A$.
- E.g., If the conclusion is not untrue, then the premise must be untrue.

Demonstration: Proof by Contradiction

Question:

Walras' Law: $\forall x \in x(p,w)$ that maximizes consumer utility, then $x \times p = w$.

Answer:

Proof.

Suppose $\exists x \in x(p,w) : x \times p < w$ $(\neg B)$, then there must be another $y \in x(p,w)$ that is affordable and $y \succ x$ by the assumption of "local non-satiation". Therefore, since y exists and is affordable, then x does not maximize utility $(\neg A)$. $\square$

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5. Proof by Induction

Professor Schmidt demonstrates the concept of proof by induction.

5. Proof by Induction

- Inductive reasoning: Drawing conclusions by reasoning a series of specific examples generalizes.
- Often used by indexing through integers.

Demonstration: Proof by Induction

Question:

$P(n) : 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$

Answer:

Proof.

Note that $P(1)$ is true because $1 = \frac{1 \times 2}{2}$. Assume $P(n)$ is true for $k \in n$ integers: $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$. Add $(k+1)$ to both sides.

$$1 + 2 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}$$

This is $P(k+1)$, implying that $P(k)$ is true for all $P(n)$. $\square$

Demonstration: Proof by Induction

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6. Proof by Contrapositive

- Double Negation:
  $\neg B \implies \neg A \equiv \neg\neg A \implies \neg\neg B \equiv A \implies B.$
- Convenient when there is a universal quantifier ($\forall$) present included by the contrapositive.

Demonstration: Proof by Contrapositive

Question:

Suppose $x \in \mathbb{Z}$. If $7x + 9$ is even, then x is odd.

Answer:

Proof.

Suppose x is **not** odd (i.e., even) implying $x = 2a$ for some integer a. Then,

$$7x + 9 = 7(2a) + 9 = 14a + (8 + 1) = 2(7a + 4) + 1 = 2b + 1$$

if $b = 7a + 4$. Consequently, $2b + 1$ is odd for all b. Therefore $07x + 9$ is **not** even. $\square$

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Practice: Proofs

1. If x is odd, then x^2 is odd.

Practice: Proofs

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Answer: Show Work

x^2 is odd by definition of an odd number.

Practice: Proofs

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Answer: Show Work

By contrapositive, $x^2 = 6x + 5$ is odd and therefore not even.

Practice: Proofs

1. If x is odd, then x^2 is odd.
2. Suppose $x \in \mathbb{Z}$. If $x^2 = 6x + 5$ is even, then x is odd.
3. There are infinitely many prime numbers.

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Answer: Show Work

Proof by contradiction.

Review

Review of Logic

1. Logical Statement: Necessary to provide clarity to your statements
2. Necessary vs. Sufficient Conditions: Aiming to make statements that are both (iff)

Review of Proofs

1. Truth Tables
2. Four methods to prove a statement:
   - Direct proof
   - Proof by contradiction
   - Proof by induction
   - Proof by Contrapositive

Assignment

- Readings on Logic & Proofs before Lecture 02:
  - B&S Appendix A
  - Hammack Ch. 4 & 10
- Readings on Sets & Topology before Lecture 02:
  - B&S Ch. 11
  - S&B Ch. 12
- Assignment:
  - Problem Set 01 (PS01)
  - Solution set will be available following end of Lecture 02
- Struggling?
  1. Read the 'Encouraged Reading'
  2. Review 'Supplementary material'
  3. Reach out directly

N & S Conditions Question 1 Answer:

◂ Question

- Not Necessary: If you won the game (B), you may have scored other points but less touchdowns ($\neg A$).
- Sufficient: If you score more touchdowns (A) (and therefore more overall points), then you will win the game (B).

N & S Conditions Question 2 Answer:

◂ Question

- Necessary: A learner's permit (A) is required before you can get a drivers license (B).
- Not Sufficient: Not all learners ($\neg A$) successfully earn their drivers license ($\neg B$).

N & S Conditions Question 3 Answer:

◂ Question

- Necessary: To be divisible by 2 (B), you must be a even whole number (A).
- Sufficient: If you are an even whole number (A), you will have no remainder if divided by two (B).

Proofs Question 1 Answer:

◂ Question

Proof.

Suppose x is odd. Then $x = 2a + 1$ for some $a \in \mathbb{Z}$, by definition an odd number. Thus $x^2 = (2a+1)^2 = 4a^2 + 4a + 1$. This $2(2a^2 + 2a) + 1$. So $x^2 = 2b + 1$ for an integer b. Therefore, x^2 is odd, by definition of an odd number. $\square$

Proofs Question 2 Answer:

◂ Question

Proof.

Suppose x is **not** odd. Thus x is even, so $x = 2a$ for some integer a. So $x^2 - 6x - 5 = (2a)^2 - 6(2a) - 5 = 2(2a^2 - 6a - 2) + 1$. Then $x^2 - 6x + 5 = 2b + 1$ for $b = 2a^2 - 6a - 2$. Consequently, $x^2 - 6x + 5$ is odd, and therefore not even. □

Proofs Question 3 Answer:

◂ Question

Proof.

Suppose there are only finite prime numbers. Then they can be listed as $p_1, p_2, \ldots, p_n$. Then p_n is the final and largest prime number. Consider a number $a = (p_1 \cdot p_2 \cdots p_n) + 1$. a has at least one prime divisor (e.g., p_k in the list). So there is some integer c such that $(p_1 \cdot p_2 \cdots p_{k-1} p_k p_{k+1} \cdots p_n) + 1 = c \cdot p_k$. Divide both sides by p_k. Now we have $\frac{1}{p_k} = c - (p_1 \cdot p_2 \cdots p_{k-1} p_{k+1} \cdots p_n)$. The expression on the right is an integer (i.e., which prime is a part of) **but** the left is not an integer. This is a contradiction. Therefore, there must be no finite range of prime numbers. □