APEC Math Review

Preliminaries

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"Mathematics is different from the other sciences: by its very nature, it is a deductive science. That is not to say that mathematicians do not collect facts and make observations concerning their investigations [...] However, even after these principles and conjectures are formulated, the work is far from over, for mathematicians are not satisfied until conjectures have been derived (i.e., proved) from the axioms of mathematics, from the definitions of the terms, and from results (or theorems) that have previously been proved. Thus, a mathematican statement is not a theorem until it has been carefully derived from axions, definitions, and previously proved theorems."

Sets

A set is a collection of things

- $A = \{1, 3, 5, 7\}$
- $\mathbb{Z} = \{..., -4, -3, -2, -1, 0, 1, 2, 3, 4, ...\}$
- $-3 \in \mathbb{Z}$; $0{,}333 \notin \mathbb{Z}$

Some sets have special symbols: $\mathbb{R}, \mathbb{Q}, \mathbb{Z}, \mathbb{N}$

- $B = \{x | \sqrt{x} \in \mathbb{Z}\}$
- $G = \{..., -4, -2, 0, 2, 4, ...\} = \{2n : n \in \mathbb{Z}\}$

Not just numbers $D = \{T, F\}$

- $X = \{x_1, x_2, x_3, x_4\}$

Sets

Sets can be elements of other sets

- $F = \{1, \{2,3\}, \{3,4\}, 4, 5\}$
- Note that $\{3,5\} \notin F$, $3 \notin F$

Cardinality: the number of elements of a set, denoted by $|A|$ [1]

- Empty set $\varnothing = \{\}$

Is the only set with cardinality zero, it has no elements

- $\varnothing \neq \{\varnothing\}$ Why?

[1] not to be confused with the notation for absolute value

Cartesian Product

- An ordered pair has the form: (a,b)
- It is enclosed in parentheses and the order matters
- Examples: $(23, 0) \neq (0, 23)$, $(\alpha, 5)$, $(\{1, 2\}, \beta)$
- The cartesian product of two sets is denoted A x B
- And it is defined as $AxB = \{(a, b) : a \in A, b \in B\}$

Schematic diagram on the board

Subsets

- When all the elements of one set A are contained in another set B, we say that A is a subset of B
- We denote it by $A \subseteq B$
- If not all elements of A are in B, we would write $A \nsubseteq B$
- When A is a subset of B and there is at least one element of B that is not in A, we say A is strictly contained in B, or that A is a proper subset of B, and denote it by $A \subset B$

Explain: $\{a, c, e\} \subseteq \{a, b, c, d, e, f\}$

- $\{a, c, e\} \subseteq \{a, c, e\}$
- $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$
- $\mathbb{R}x\mathbb{N} \subseteq \mathbb{R}x\mathbb{R}$
- The empty set is a subset of every set

Power Sets

(Definition 1.4 from Hammack) If A is a set, the power set of A is another set, denoted $\wp(A)$ and defined to be the set of all subsets of A.

$$\wp(A) = \{X : X \subseteq A\}$$

- $A = \{a, b, c\}$
- If A is finite, $|\wp(A)| = 2^{|A|}$

Union, Intersection, Difference

(Definition 1.5 from Hammack) Let A and B be sets

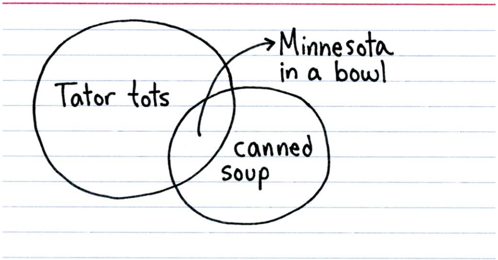
- The union of A and B is the set $A \cup B = \{x : x \in A \text{ or } x \in B\}$
- The intersection of A and B is the set $A \cap B = \{x : x \in A \text{ and } x \in B\}$
- The difference of A and B is the set $A - B = \{x : x \in A \text{ and } x \notin B\}$
- You will sometimes see $A \backslash B$ instead of A - B

Complement

(Definition 1.6 from Hammack, with different notation) Let A be a set with a universal set U. The complement of a set A, denoted A^c is the set $A^c = U \backslash A$

- The difference of A and B is the set $A \backslash B = A \cap B^c$
- Venn Diagrams can be useful to think about these!
- Which brings me to... this

Jessica Hagy- My daily recommendation



Indexed sets

- Indexed sets are sets that have subscripts, such as A_1, A_2, A_3
- This notation is very convenient and will be very useful in economics

You can write:

- $A_1 \cup A_2 \cup A_3 \cup ... \cup A_n = \{x : x \in A_i \text{for at least one set } A_i, for 1 \leq i \leq n$
- $\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup A_3 \cup ... \cup A_n$

And the analogous for the intersection.

- You can also write the previous example as: $\bigcup_{i \in \{1,2,...,n\}} A_i$
- You would this call $I = \{1, 2, ..., n\}$ the index set
- Note that index sets don't have to be integers, they are just labels. They could be anything.

Logic

Now we move away from sets, and dive into logic!

- It is the common language we will be using, and it is very precise and unambiguous.
- Logic is a systematic way of thinking that allows us to deduce new information from old information.
- Deducing information correctly, not deducing correct information (Why?)
- This is not only important for proofs and mathcamp but for the way you think about answering questions: how conduct your research

Statements

- All proofs and mathematical reasoning is based on propositions or statements
- Statements are sentences that are either true or false
  1. I was born in Mexico City
  2. Math camp is fun
  3. $a, b \in \mathbb{R}$, $|a+b| \leq |a| + |b|$
  4. I am me and you are me and we are all together

There are different kinds of statements

- Tautologies: statements that are always true (1=1)
- Contradictions: statements that are always false (2=3)
- Statements that are true or false depending on the context ($x^2 = 1$)
  $\rightarrow$ the context must be properly established or defined

Famous Propositions / Statements

- Fermat's last Theorem: There are no three positive integers a, b, c that satisfy the equation: $a^n + b^n = c^n$ with $n > 2$

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- Euclid's theorem: There are infinitely many prime numbers
- Several proofs already exist! (My favorite is still the OG, we will study it later!)

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- Statements P and Q are said to be logically equivalent if P is true exactly when Q is true.
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- Eg: x is the female Cofounder of JPAL $\equiv$ x is Esther Duflo

Statements and their combinations

There are many ways to create new statements from statements P, Q

- Negation: (Not P) $\neg P$
- Conjunction: (P and Q), $P \wedge Q$
- Disjunction: (P or Q), $P \vee Q$
- They are all related through the DeMorgan's laws:

$$\neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q)$$
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- Can you draw some Venn Diagrams that show DeMorgan's Laws?
- Truth tables (board)

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Hint: try P=$x \in A$

DeMorgan Would be Proud

Implications

let P and Q be statements. The following are equivalent:

- $P \Rightarrow Q$
- if P then Q, or
- P implies Q
- Q whenever P
- P is referred to as the hypothesis and Q as the conclusion.
- The proposition $P \Rightarrow Q$ is considered true when the hypothesis does not hold

Implications

If I win the lottery, I will quit my PhD

- I have not won the lottery, so the fact that I have not quit my PhD does not mean the proposition is false
- Conditional statements, or implications, are only of interest when P is true.
- When we encounter these, we take P as true, and set to see if Q is implied by that.
- This statement is only false when P is true and Q is false

Necessary v. Sufficient

We will take this opportunity to emphasize the relationship between necessary and sufficient statements.

The proposition $P \Rightarrow Q$ has a few other grammatical equivalences

- P is sufficient for Q
- Q is necessary for P
- P only if Q

Eg: (You have successfully defended your dissertation) $\Rightarrow$ (You have a PhD)

Talk through each of the grammatical equivalences

Contrapositive

- The contrapositive of a statement $P \Rightarrow Q$ is $\neg Q \Rightarrow \neg P$
- In the previous example: You do not have a PhD $\Rightarrow$ you have not successfully defended your dissertation
- A statement and its contrapositive are logically equivalent.
- sometimes it is easier to prove the contrapositive of a statement than the statement itself.

Converse

- The converse of statement $P \Rightarrow Q$ is $Q \Rightarrow P$
- A proposition (or conditional statement) and its converse represent entirely different things
- Try with the following examples: is $P \Rightarrow Q$ true? Is $Q \Rightarrow P$ true?
  1. P= a is even, Q= a is divisible by 2
  2. P=You have successfully defended your dissertation, Q= You have a PhD
  3. P= x is a prime number, Q= x=7
  4. P= Natalia is Mexican, Q= Natalia likes dogs

If and only if

Another way to construct propositions from conditional statements is logical equivalence, double implication, or biconditional. These are also known as "if and only if conditions", or "necessary and sufficient conditions"

$$P \iff Q$$

- What this means is $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$
- Looking at our grammatical equivalences for $\Rightarrow$, we can see that the first of the two parentheses above means that P is sufficient for Q, and the second one can be thought of as P is necessary for Q.
- This is why we read $P \iff Q$ as P is necessary and sufficient for Q.
- To prove these kinds of statements we would prove one direction of the implication first, and then the other.

Quantifiers

These are the symbols and expressions that give context to our statements. Examples on the board

$$x + y = 0$$

- You should begin by stating what your symbols are
- $\forall$
- $\exists$
- Order matters!
- Negating a statement with quantifiers (board)
- Note: Given a set S, a quantified statement of the form $\forall x \in S, P(x)$ is understood to be true if P(x) is true for all x in S.

Negating

- Given a statement R, it's negation is denoted $\neg$R or $\sim$R
- To negate a statement, we can use DeMorgan's Laws

$$\neg(P \wedge Q) = (\neg P) \vee (\neg Q)$$

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Examples:

- You can solve it by factoring or with the quadratic formula
- The numbers x and y are both odd
- $\neg(\forall x \in S, P(x))$
- $\neg(\exists x \in S, P(x))$
- If x is odd then x^2 is odd
- Try the statement above but for a particular constant a.

Examples

Write the mathematical expression for the following English statements

- Every integer that is not odd is even
- There is an integer that is not even
- Every integer is even
- There is an integer n for which $n^2 = 2$
- For every real number x, there is a real number y for which $y^3 = x$
- For every real number x, there is a real number y for which $y^2 = x$
- Given any two rational numbers a and b, it follows that $\sqrt{ab}$ is rational.
- Given any two rational numbers a and b, it follows that ab is rational.

Can you tell which ones are true?

Exercises for this section

From Hammack: Either odd or even exercises of sections 2.6, 2.7, 2.9 and 2.10