# Lecture 04
# Integration

Ryan McWay[†]

[†] *Applied Economics,*
*University of Minnesota*

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# Last Lecture Review

- Derivatives:
  - Continutity & Differentiability
  - First & Second Derivatives
  - Derivative Rules
  - Implicit Function
  - l'Hopital's Rule
  - Taylor Series Approximation
  - Mean Value Theorem
  - Critical Points

# Review Assignment

1. Problem Set 03 solutions are available on Github.
2. Any issues or problems **You** would like to discuss?

# Daily Icebreaker

- Attendance via prompt:
  - Name
  - Daily Icebreaker: What ‘pet peeve’ do you have about something that is mundane or against a regularly accepted social norm?

# *Topic: Integration*

# Motivation

- General background
  - The total area under the curve.
  - Understanding the accumulation of the parts as a whole.
  - A core component of calculus alongside derivatives.
- Why do economists' care?
  - A tool to aggregate effects and approximate sums.
- Application in this career
  - Evaluating surplus or total welfare.

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# Overview

1. Definite Integral
2. Reimann Sum
3. Fundamental Theorem of Calculus
4. Integration Rules
5. Integration by Substitution
6. Integration by Parts
7. Leibnz's Rule

# 1. Definite Integral

- Consider the anti-derivative $F(x)$.

$$\int_a^b f(x)dx = \lim_{x\to\Delta x} \sum_{x=a}^{b} f(x)\Delta x = F(b) - F(a) : F'(x) = f(x)$$

# Demonstration: Definite Integral

*Question:*

$\int_1^4 (2x+3)dx$

*Answer:*

$$
\begin{aligned}
&= \int_1^4 (2x)dx + \int_1^4 (3)dx \\
&= \frac{2}{2}x^2 + 3x\Big|_1^4 \\
&= (4^2 + 3(4)) - (1^2 + 3(1)) \\
&= (16 + 12) - (1 + 3) \\
&= 24
\end{aligned}
$$

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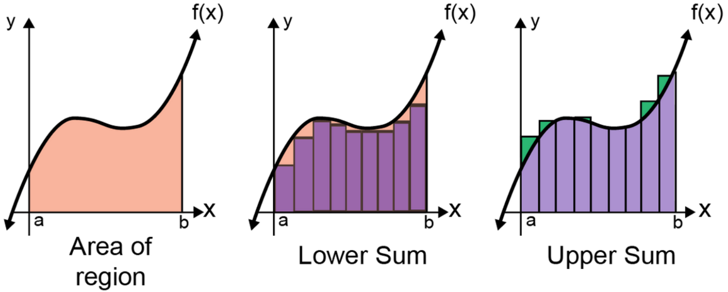
&= 24
\end{aligned}$$

# 2. Reimann Sum

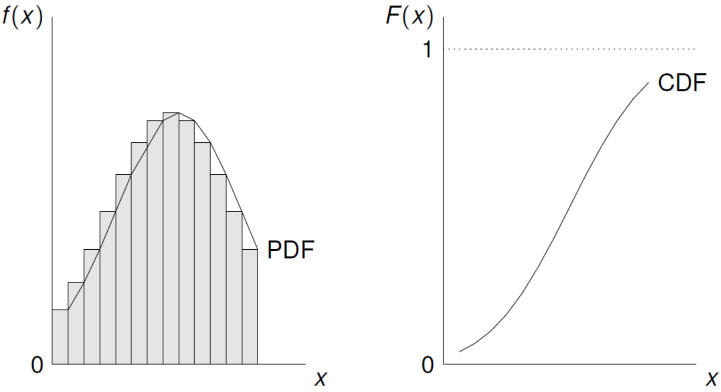
- Subdivide the interval $(a, b)$ into $N$ sub-intervals with endpoints $x_i$.

$$\lim_{\Delta \to 0} \sum_{i=1}^{N} f(x_i)\Delta = \int_a^b f(x)dx$$

# 2. Reimann Sum



# Application: Probability and Cumulative Density Functions

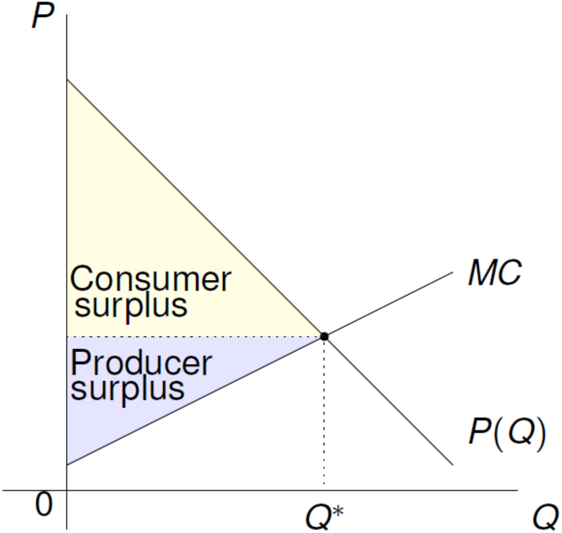


# 3. Fundamental Theorem of Calculus

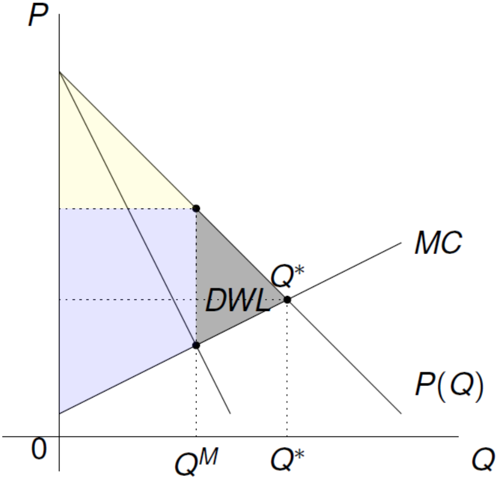
- Theorem that connects differentiation to integration.
- Let $f$ be a continuous function on the open interval $[a, b]$. If $f(x) = F'(x)$, then:

$$\int_a^b f(x)dx = F(b) - F(a)$$

# Application: Social Surplus



# Application: Dead-weight Loss (DWL)



# 4. Integration Rules

- Constant:

$$\int a dx = ax + C$$

- Constant Multiplication:

$$\int cf(x)dx = c\int f(x)dx$$

- Reciprocal:

$$\int \frac{1}{x}dx = ln(x) + C$$

- Exponential:

$$\int e^x dx = e^x + C$$

$$\int a^x dx = \frac{a^x}{ln(a)} + C$$

# Demonstration: Exponential

*Question:*

$\int (5^x) dx$

*Answer:*

$\frac{5^x}{ln(5)} + C$

# Demonstration: Exponential

*Question:*

$\int (5^x)dx$

*Answer:*

$\frac{5^x}{ln(5)} + C$

# 4. Integration Rules

- Logarithm:

$$\int ln(x)dx = xln(x) - x + C$$

- Power Rule:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

- Sum/Difference Rule:

$$\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx$$

# Demonstration: Power Rule

*Question:*

$\int (x^5 + 3x^3 + 2x)dx$

*Answer:*

$\frac{1}{6}x^6 + \frac{3}{4}x^4 + x^2 + C$

# Demonstration: Power Rule

*Question:*

$\int (x^5 + 3x^3 + 2x)dx$

*Answer:*

$\frac{1}{6}x^6 + \frac{3}{4}x^4 + x^2 + C$

# Demonstration: Sum/Difference Rule

*Question:*

$\int (4x^2 + x - \frac{3}{x})dx$

*Answer:*

$\frac{4}{3}x^3 + \frac{1}{2}x^2 - 3ln(x) + C$

# Demonstration: Sum/Difference Rule

*Question:*

$\int(4x^2 + x - \frac{3}{x})dx$

*Answer:*

$\frac{4}{3}x^3 + \frac{1}{2}x^2 - 3ln(x) + C$

# Practice: Integration

1. $\int w^{-2} + 10w^{-5} - 8dw$

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*Answer:* Show Work

$$-w^{-1} - \frac{5}{2}w^{-4} - 8w + C$$

# Practice: Integration

1. $\int w^{-2} + 10w^{-5} - 8dw$
2. $\int \frac{4}{x^2} + 2 - \frac{1}{8x^3}dx$

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2. $\int \frac{4}{x^2} + 2 - \frac{1}{8x^3}dx$

*Answer:* Show Work

$$-4x^{-1} + 2x + \frac{1}{16}x^{-2} + C$$

# Practice: Integration

1. $\int w^{-2} + 10w^{-5} - 8dw$
2. $\int \frac{4}{x^2} + 2 - \frac{1}{8x^3}dx$
3. $\int t^3 - \frac{e^{-t}-4}{e^{-t}}dt$

# Practice: Integration

1. $\int w^{-2} + 10w^{-5} - 8dw$
2. $\int \frac{4}{x^2} + 2 - \frac{1}{8x^3}dx$
3. $\int t^3 - \frac{e^{-t}-4}{e^{-t}}dt$

*Answer:* Show Work

$$\frac{1}{4}t^4 - t + 4e^t + C$$

# 5. Integration by Substitution

- Reverse chain rule from differentiation.
- Commonly referred to as $u$ substitution.

$$\int f(g(x))g'(x)dx$$

$$\int f(u)du$$

# Demonstration: Int. by Sub.

*Question:*

$\int x^2(3-10x^3)^4 dx$

*Answer:*

$$
\begin{aligned}
u &= 3 - 10x^3 \\
du &= -30x^2 dx \\
\implies dx &= \frac{-1}{30x^2} du \\
\int x^2(3-10x^3)^4 dx &= \frac{-1}{30} \int u^4 du \\
&= \frac{-1}{30} \cdot \frac{1}{5} u^5 + C \\
&= \frac{-1}{150}(3-10x^3)^5 + C
\end{aligned}
$$

# Demonstration: Int. by Sub.

*Question:*

$\int x^2(3-10x^3)^4 dx$

*Answer:*

$$\begin{aligned} u =& 3-10x^3 \\ du =& -30x^2 dx \\ \implies dx =& \frac{-1}{30x^2} du \\ \int x^2(3-10x^3)^4 dx =& \frac{-1}{30}\int u^4 du \\ =& \frac{-1}{30}\cdot\frac{1}{5}u^5 + C \\ =& \frac{-1}{150}(3-10x^3)^5 + C \end{aligned}$$

# Practice: Int. by Sub.

1. $\int \frac{1}{x^2+x}dx$

# PRACTICE: INT. BY SUB.

1. $\int \frac{1}{x^2+x}dx$

*Answer:* Show Work

$$-ln(\frac{1}{x}+1)+C$$

# Practice: Int. by Sub.

1. $\int \frac{1}{x^2+x} dx$
2. $\int 3(8y-1)e^{4y^2-y} dy$

# Practice: Int. by Sub.

1. $\int \frac{1}{x^2+x} dx$
2. $\int 3(8y-1)e^{4y^2-y} dy$

*Answer:* Show Work

$$3e^{4y^2-y} + C$$

# PRACTICE: INT. BY SUB.

1. $\int \frac{1}{x^2+x} dx$
2. $\int 3(8y-1)e^{4y^2-y} dy$
3. $\int \frac{2t^3+1}{(t^4+2t)^3} dt$

# Practice: Int. by Sub.

1. $\int \frac{1}{x^2+x} dx$
2. $\int 3(8y-1)e^{4y^2-y} dy$
3. $\int \frac{2t^3+1}{(t^4+2t)^3} dt$

*Answer:* Show Work

$$\frac{-1}{4}(t^4+2t)^{-2}+C$$

# 6. Integration by Parts

- Reverse product rule from differentiation.
- Rarely used in economic applications, but important to know.

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

$$\int udv = u \cdot v - \int vdu$$

# DEMONSTRATION: INT. BY PARTS

*Question:*

$\int ln(x)dx$

*Answer:*

$$u = ln(x) \ , \ dv = 1$$
$$du = \frac{1}{x} \ , \ v = x$$
$$\int ln(x)dx = ln(x)x - \int x \cdot \frac{1}{x} dx$$
$$= xln(x) - x + C$$
$$= x(ln(x) - 1) + C$$

# Demonstration: Int. by Parts

*Question:*

$\int ln(x)dx$

*Answer:*

$$
\begin{aligned}
u =& ln(x) \text{ , } dv = 1 \\
du =& \frac{1}{x} \text{ , } v = x \\
\int ln(x)dx =& ln(x)x - \int x \cdot \frac{1}{x}dx \\
=& xln(x) - x + C \\
=& x(ln(x) - 1) + C
\end{aligned}
$$

# Practice: Int. by Parts

1. $\int (xe^{2x})dx$

# Practice: Int. by Parts

1. $\int (xe^{2x})dx$

*Answer:* Show Work

$$\frac{(2x-1)e^{2x}}{4} + C$$

# Practice: Int. by Parts

1. $\int (xe^{2x})dx$
2. $\int (2+5x)e^{\frac{1}{3}x}dx$

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1. $\int (xe^{2x})dx$
2. $\int (2+5x)e^{\frac{1}{3}x}dx$

*Answer:* Show Work

$$(15x-39)e^{\frac{1}{3}x}+C$$

# Practice: Int. by Parts

1. $\int (xe^{2x})dx$
2. $\int (2+5x)e^{\frac{1}{3}x}dx$
3. $\int x^2 e^x dx$

# Practice: Int. by Parts

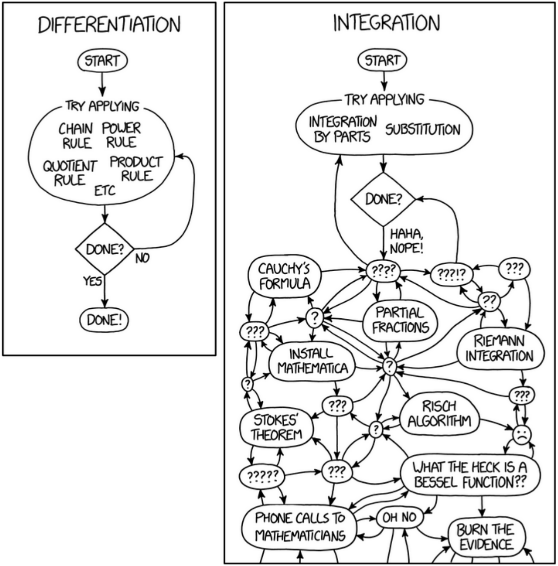
1. $\int (xe^{2x})dx$
2. $\int (2+5x)e^{\frac{1}{3}x}dx$
3. $\int x^2e^x dx$

*Answer:* Show Work

Int. by Parts **Twice** ...

$$e^x(x^2 - 2x + 2) + C$$

# It Can Get . . . Complicated

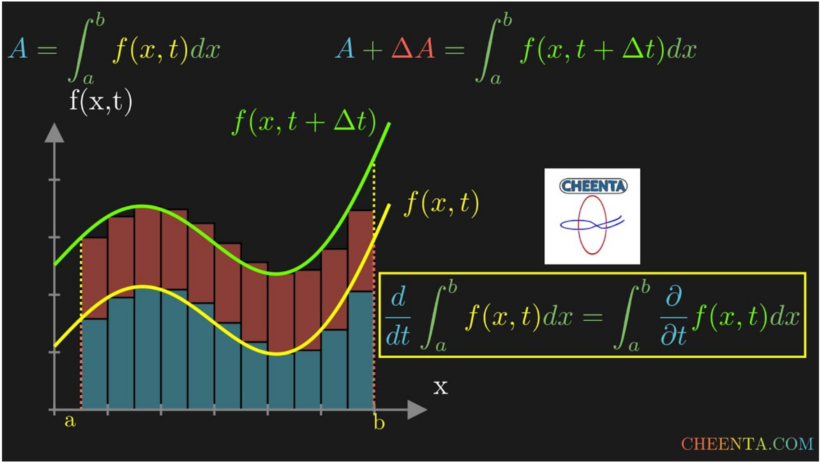


# 7. Leibnz's Rule

- A general rule for differentiating integrals.

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)dx = \frac{db(t)}{dt}f(b(t),t) - \frac{da(t)}{dt}f(a(t),t) = \int_{a(t)}^{b(t)} \frac{\partial f(x,t)}{\partial t}dx$$

# 7. Leibnz's Rule



# Review

# Review of Integrals

1. Definite Integral
2. Reimann Sum
3. Fundamental Theorem of Calculus
4. Integration Rules
5. Integration by Substitution
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7. Leibnz's Rule

# Assignment

- Readings on Multi-variate Calculus before Lecture 05:
  - S&B Ch. 14, 15, & 20
- Assignment:
  - Problem Set 04 (PS04)
  - Solution set will be available following end of Lecture 05
- Struggling?
  1. Read the ‘Encouraged Reading’
  2. Review ‘Supplementary material’
  3. Reach out directly

# Integration Question 1 Answer:

◂ Question

$$-w^{-1} - \frac{5}{2}w^{-4} - 8w + C$$

# Integration Question 2 Answer:

◂ Question

$$-4x^{-1} + 2x + \frac{1}{16}x^{-2} + C$$

# Integration Question 3 Answer:

Question

$$\int t^3 - \frac{e^{-t} - 4}{e^{-t}} dt = \int t^3 - 1 + 4e^t dt$$
$$= \frac{1}{4}t^4 - t + 4e^t + C$$

# Int. by Sub. Question 1 Answer:

◂ Question

$$\text{Re-write: } \int \frac{1}{(\frac{1}{x}+1)x^2}dx$$

$$\begin{aligned} u =& \frac{1}{x}+1 \\ du =& \frac{-1}{x^2}dx \\ \int \frac{1}{(\frac{1}{x}+1)x^2}dx =& -\int \frac{1}{u}du \\ =& -ln(u)+C \\ =& -ln(\frac{1}{x}+1)+C \end{aligned}$$

# Int. by Sub. Question 2 Answer:

◂ Question

$$\begin{aligned} u &= 4y^2 - y \\ du &= (8y-1)dy \\ \int 3(8y-1)e^{4y^2-y}dy &= 3\int e^u du \\ &= 3e^u + C \\ &= 3e^{4y^2-y} + C \end{aligned}$$

# Int. by Sub. Question 3 Answer:

◂ Question

$$
\begin{aligned}
u =& t^4 + 2t \\
du =& (4t^3 + 2)dt = 2(2t^3 + 1)dt \\
\int \frac{2t^3 + 1}{(t^4 + 2t)^3} =& \frac{1}{2} \int \frac{1}{u^3} du \\
=& \frac{-1}{4} (t^4 + 2t)^{-2} + C
\end{aligned}
$$

# INT. BY PARTS QUESTION 1 ANSWER:

◂ QUESTION

$$
\begin{aligned}
u =& x \ , \ dv = e^{2x} \\
du =& 1 \ , \ v = \frac{e^{2x}}{2} \\
\int (xe^{2x}) dx =& \frac{xe^{2x}}{2} - \int \frac{e^{2x}}{2} dx \\
=& \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + C \\
=& \frac{(2x-1)e^{2x}}{4} + C
\end{aligned}
$$

# Int. by Parts Question 2 Answer:

Question

$$
\begin{aligned}
u =& 2 + 5x \ , \ dv = e^{\frac{1}{3}x} dx \\
du =& 5dx \ , \ v = 3e^{\frac{1}{3}x} \\
\int (2+5x) e^{\frac{1}{3}x} dx =& 3e^{\frac{1}{3}x}(2+5x) - 15 \int e^{\frac{1}{3}x} dx \\
=& 3e^{\frac{1}{3}x}(2+5x) - 45e^{\frac{1}{3}x} + C \\
=& (15x - 39)e^{\frac{1}{3}x} + C
\end{aligned}
$$

# Int. by Parts Question 3 Answer:

Question

$$
\begin{aligned}
u =& x^2 \ , \ dv = e^x dx \\
du =& 2x dx \ , \ v = e^x \\
\text{1st answer: } =& x^2 e^x - 2 \int x e^x dx \\
u =& x \ , \ dv = e^x dx \\
du =& dx \ , \ v = e^x \\
\text{2nd answer: } =& x^2 e^x - 2(x e^x - e^x) + C \\
=& e^x (x^2 - 2x + 2) + C
\end{aligned}
$$