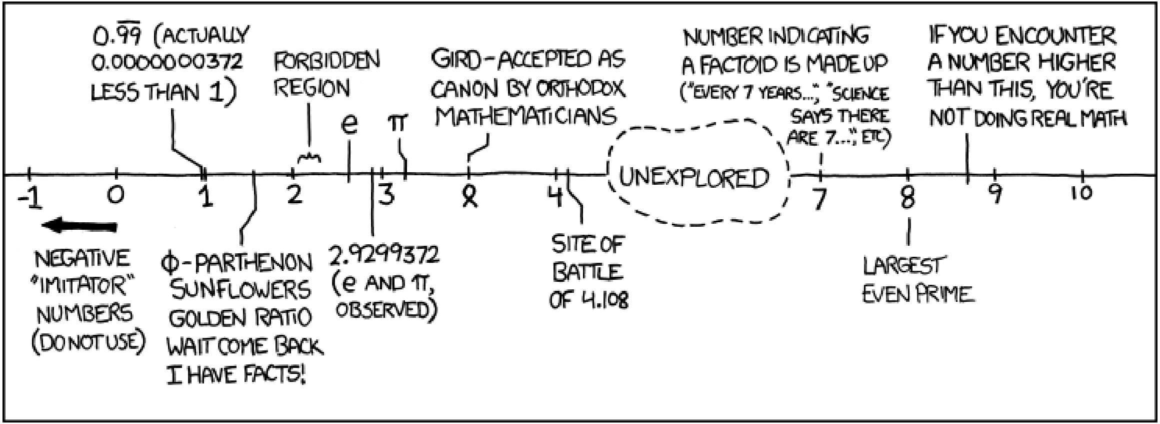
APEC Math Review

Numbers

Natalia Ordaz Reynoso

Summer 2019

Numbers! Real Numbers!



Real numbers

Algebraic and order properties of the real numbers: There are two binary operations: addition and multiplication, which satisfy the following properties

- (A1) $\forall a, b \in \mathbb{R}, a + b = b + a$
- (A2)$\forall a, b, c \in \mathbb{R}(a + b) + c = a + (b + c)$
- (A3) There is an element 0 in $\mathbb{R}$ such that $a + 0 = a, \forall a \in \mathbb{R}$
- (A4) $\forall a \in \mathbb{R}, \exists - a \in \mathbb{R} : a + (-a) = 0$
- (M1-M4)
- (D) $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ and $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$

Theorems: uniqueness of 0, 1, reciprocals

- Theorem: a) if z and y are real numbers and z+a=a, then z=0. b) If u and b, different from 0 are real numbers such that $u \cdot b = b$, then u=1 c) if a is a real number, $a \cdot 0 = 0$
- Theorem: a) If a is a non-zero real number, and b is a real number such that $a \cdot b = 1$, then $b = 1/a$. b) If $a \cdot b = 0$ then either a=0 or b=0

We will skip most order and algebraic properties of the real numbers (trichotomy, positive numbers, inequalities, square numbers are positive). All of these can be derived from the ones we saw before. If you need a refresher, go to B & S Chapter 2, sections 2.1 and 2.2

Some fun proofs in R

- Show that if a and b are positive real numbers, the arithmetic-geometric mean inequality holds:
$$\sqrt{ab} < 1/2(a+b)$$
- Theorem: (Bernoulli's inequality)if $x > -1$ then $\forall n \in \mathbb{N}, (1+x)^n \geq 1 + nx$

Absolute Value and the Real Line

Definition: a real number's absolute value is denoted —a— and is given by:

- $|a| = a$ if $a > 0$
- $|a| = 0$ if a=0
- $|a| = -a$ if $a < 0$

Theorem: For all a, b, c real numbers

1. $|ab| = |a||b|$
2. $|a|^2 = |a^2|$
3. if c is not negative, $|a| \leq c \iff -c \leq a \leq c$
4. $-|a| \leq a \leq |a|$

Triangle Inequality

Theorem: If a, b in the real numbers, then

$$|a + b| \leq |a| + |b|$$

Corollary:

1. $||a| - |b|| \leq |a - b|$
2. $|a - b| \leq |a| + |b|$

Neighborhoods

Definition: let $a, \epsilon \in \mathbb{R}$, $\epsilon > 0$. Then the ϵ-neighborhood of a is the set
$V_\epsilon(a) := \{x \in \mathbb{R} : |x - a| < \epsilon\}$
Example:

- Let $I := \{x : 0 \leq x \leq 1\}$ Then $\forall \epsilon > 0$, $V_\epsilon(0)$ has elements in I and elements not in I, so $V_\epsilon(0) \nsubseteq I$

Completeness

- So far we have said that $\mathbb{R}$ is an ordered field
- So are the rational numbers...
- Now we will show that $\mathbb{R}$ is complete ($\mathbb{Q}$ is not).
- We will begin with the notion of supremum

Definition: Let S be a non-empty set of $\mathbb{R}$.

1. The set S is said to be bounded from above if there is a number $u \in \mathbb{R}$ such that $s \leq u$ $\forall s \in S$. Each one of these numbers u is called an upper bound of S
2. The set S is said to be bounded from below if there is a number $w \in \mathbb{R}$ such that $s \geq w$ $\forall s \in S$. Each one of these numbers w is called an lower bound of S
3. A set is bounded if it has both an upper and a lower bound. Otherwise it is said that the set is unbounded.

Supremum and infimum

- Note that if a set has an upper bound, it has an infinite number of upper bounds. The same for a lower bound

Definition: let S be a non empty subset of the real numbers.

1. If S is bounded from above, then a number u is said to be the supremum of S if it satisfies the following two conditions: a) u is an upper bound of S, b) if v is any upper bound of S, then $u \leq v$
2. if S is bounded from below, then a number w is said to be the infimum of S if it satisfies the following two conditions: a) w is a lowe bound of S, b) if t is any lower bound of S, then $t \leq w$

Note: can you prove that supremum and infimum are unique?

Supremum and Infimum

- Not all subsets of $\mathbb{R}$ have supremum and infimum
- Lemma: A number u is the supremum of a non-empty set $S \subseteq \mathbb{R}$ if and only if it satisfies the following two conditions:
  1. $s \leq u \forall s \in S$
  2. if $v < u$, then $\exists s' \in S$ such that $v < s'$

Supremum and Completeness

Theorem (supremum axiom): Every non-empty subset of $\mathbb{R}$ that is bounded above has a supremum in $\mathbb{R}$

Applications of the supremum

Determining suprema and infima of sets is compatible with the algebraic properties of the real numbers

- Theorem: Let $S \subseteq \mathbb{R}, S \neq \emptyset$ that is bounded from above. Let $a \in \mathbb{R}$. Define the set $a + S := \{a + s : s \in S\}$. Then sup (a+S)=a+sup(S)
- Theorem: Let A, B be nonempty subsets of $\mathbb{R}$ such that $a \leq b\ \forall a \in A$ and $b \in B$. Then sup(A) $\leq$ inf(B)
- The same holds for functions: $f(x) \leq g(x) \in D$ then $supf(D) \leq supg(D)$

Note: third bullet point does not imply anything about the relationship between sup f(D) and inf g(D). Why? What would I need to have such a relationship?

Archimedian Property

Theorem: if $x \in \mathbb{R}$ then $\exists n_x \in \mathbb{N}$ such that $x < n_x$

Archimedian Property

Theorem: if $x \in \mathbb{R}$ then $\exists n_x \in \mathbb{N}$ such that $x < n_x$

Proof:

Archimedian Property

Theorem: if $x \in \mathbb{R}$ then $\exists n_x \in \mathbb{N}$ such that $x < n_x$

Proof: If this is false, then $n \leq x \forall n \in \mathbb{N}$

Archimedian Property

Theorem: if $x \in \mathbb{R}$ then $\exists n_x \in \mathbb{N}$ such that $x < n_x$

Proof: If this is false, then $n \leq x \forall n \in \mathbb{N}$
This means that x is an upper bound of $\mathbb{N}$. So the natural numbers have a supremum in the real numbers. Denote it u.

Archimedian Property

Theorem: if $x \in \mathbb{R}$ then $\exists n_x \in \mathbb{N}$ such that $x < n_x$

Proof: If this is false, then $n \leq x \forall n \in \mathbb{N}$
This means that x is an upper bound of $\mathbb{N}$. So the natural numbers have a supremum in the real numbers. Denote it u.
Then u-1 is not an upper bound of $\mathbb{N}$. So $\exists m \in \mathbb{N} : u - 1 < m$.

Archimedian Property

Theorem: if $x \in \mathbb{R}$ then $\exists n_x \in \mathbb{N}$ such that $x < n_x$

Proof: If this is false, then $n \leq x \forall n \in \mathbb{N}$
This means that x is an upper bound of $\mathbb{N}$. So the natural numbers have a supremum in the real numbers. Denote it u.
Then u-1 is not an upper bound of $\mathbb{N}$. So $\exists m \in \mathbb{N} : u - 1 < m$.
Then $u < m + 1$! This contradicts the fact that u is an upper bound of the set of natural numbers

Completeness...

Why did I say that the rational numbers are not a complete, ordered field?

Completeness...

Why did I say that the rational numbers are not a complete, ordered field?

Last: innumerability of $\mathbb{R}$.
Theorem: The set $\mathbb{R}$ is not countable.

- Exercises from sections 2.2-2.4 of B&S

The study of geometric properties and spatial relations unaffected by the continuous change of shape or size of figures.

- Definition: a neighborhood of a point x $\in \mathbb{R}$ is any set V that contains an ϵ-neighborhood $V_\epsilon := (x-\epsilon, x+\epsilon)$ of x for some positive epsilon
- Definition:
  1. A subset G of $\mathbb{R}$ is open in $\mathbb{R}$ if $\forall x \in G, \exists V$ of x such that $V \subseteq G$
  2. A subset F of $\mathbb{R}$ is closed in $\mathbb{R}$ if its complement C(F) is open in $\mathbb{R}$

Examples: $\mathbb{R}$ is open, $G := \{x \in \mathbb{R} : 0 < x < 1\}$ is also open $(\epsilon = min\{x, 1-x\}$, any open interval is open. No closed intervals are open sets (why?), $G := \{x \in \mathbb{R} : 0 < x \leq 1\}$ is neither open nor closed. $\emptyset$ is open in the reals. Why?

Properties of Open and Closed Sets

- The union of open sets in $\mathbb{R}$ is open.
- The finite intersection of open sets in $\mathbb{R}$ is an open set.
- The intersection of closed sets in $\mathbb{R}$ is closed.
- The finite union of closed sets in $\mathbb{R}$ is closed.

Examples: $G_n := (0, 1 + 1/n)$, $n \in \mathbb{N}$, $F_n := [1/n, 1]$, $n \in \mathbb{N}$

Compact Sets

Theorem (Heine-Borel) $K \subseteq \mathbb{R}$ is compact if and only if it is closed and bounded

- this is not the definition, but it is logically equivalent.
- Theorem: if K is a compact subset of $\mathbb{R}$, and if $f : K \to \mathbb{R}$ is continuous in K, then f(K) is compact
- We are skipping A LOT of good stuff, because it is not as useful for your 1st year. But I encourage you to go deeper into sequences, because it is fun.

Metric Spaces

What is a metric space?

Metric Spaces

What is a metric space?

- A metric of a set S is a function $d : SxS \rightarrow \mathbb{R}$ That satisfies the following properties
  1. $d(x, y) \geq 0 \; \forall x, y \in S$ (positivity)
  2. $d(x, y) = 0 \iff x = y$ (definitiveness)
  3. $d(x, y) = d(y, x) \; \forall x, y \in S$ (symmetry)
  4. $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality)
- A Metric is a measure of "distance"

A metric space (S,d) is a set S with a metric d in S

Examples of Metric Spaces

- $d(x, y) := |x - y|$, $\mathbb{R}$
- Pythagorean distance in $\mathbb{R}^2$: $d(P_1, P_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
- Also in $\mathbb{R}^2$, $d_1(P_1, P_2) := |x_1 - x_2| + |y_1 - y_2|$
- Again in $\mathbb{R}^2$, $d_\infty(P_1, P_2) := sup\{|x_1 - x_2|, |y_1 - y_2|\}$
- $C = [0, 1]$, $d_1(f, g) = \int_0^1 |f - g|$
- or even the discrete metric

If metric d with set S form a metric space, then d will form a metric space with any non-empty subset of S

Sets in metric spaces

- You can define neighborhoods in metric spaces!
- You can define open and closed sets in metric spaces!
- Compactness is also preserved in metric spaces!

Exercises of Sections 11.1 and 11.4 (you can skip those with sequences if you find any!)