Day 6 Notes

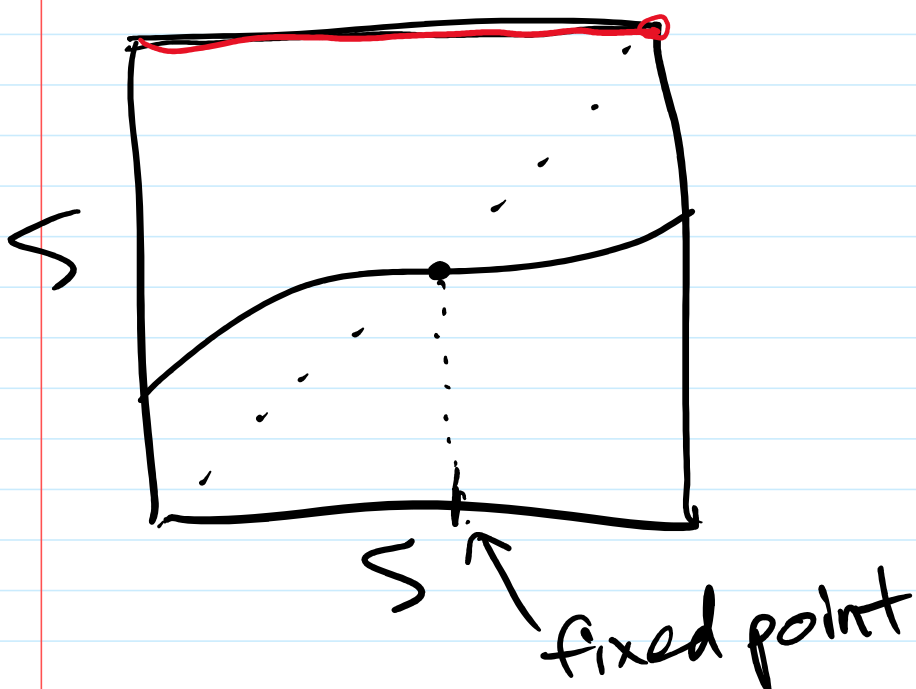
Monday, August 16, 2021 9:05 AM

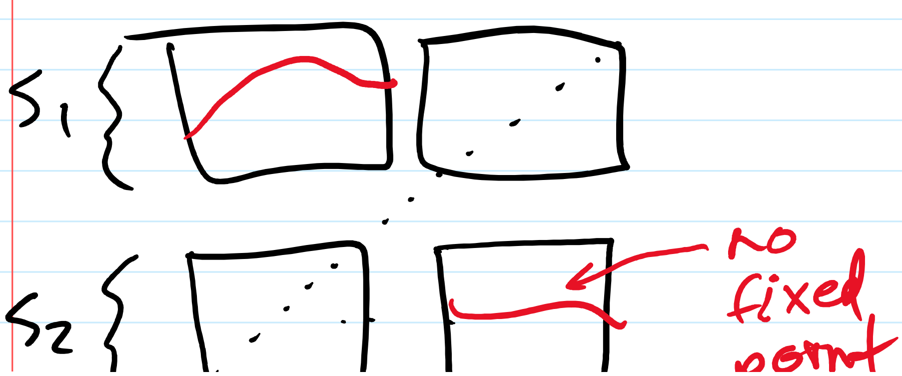
Brouwer's F.P.T.

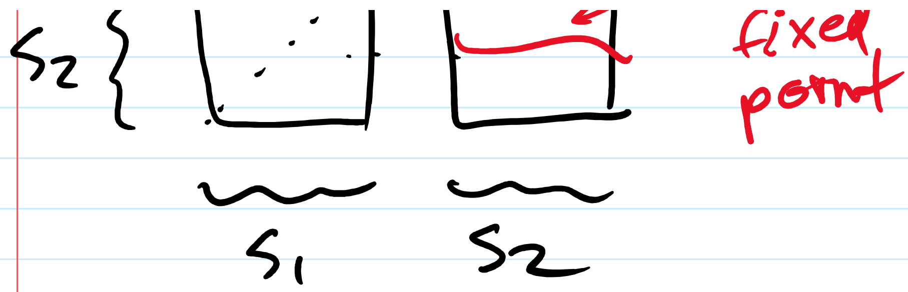
$f: S \to S \quad S \subseteq \mathbb{R}^n$

S compact and convex

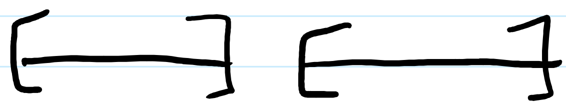
and f continuous.







$S = S_1 \cup S_2$



$f(x,y) = (e^{x^2} + \ln y)\, y^2$

$f_x = y^2 e^{x^2} \cdot 2x$

$f_y = e^{x^2} \cdot 2y + \frac{1}{y} y^2 + 2y \ln y$

$f_{xx} = y^2 \left(2x e^{x^2} \cdot 2x + 2e^{x^2} \right)$

$f_{yy} = e^{x^2} \cdot 2 + 1 + 2 \ln y + \frac{2y}{y}$

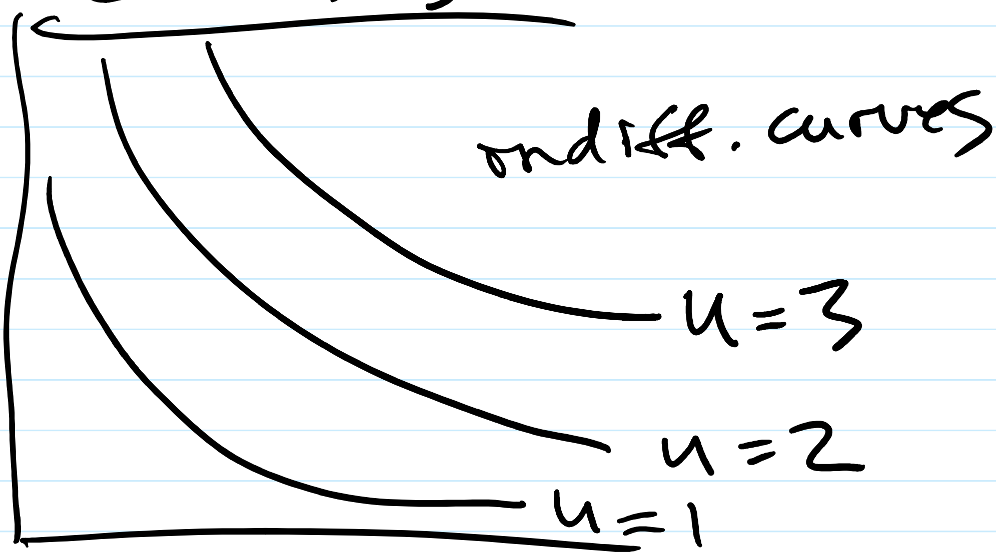
$f_{xy} = 2y e^{x^2} \cdot 2x$

$f_{yx} = e^{x^2} \cdot 2y \cdot 2x$

so $H = \Big[\, y^2(4x^2 e^{x^2} + 2e^{x^2}) \quad 4xye^{x^2}$

$$\text{so } H = \begin{bmatrix} y^2(4x^2e^{x^2}+2e^{x^2}) & 4xye^{x^2} \\ 4xye^{x^2} & 2e^{x^2}+2\ln y+3 \end{bmatrix}$$

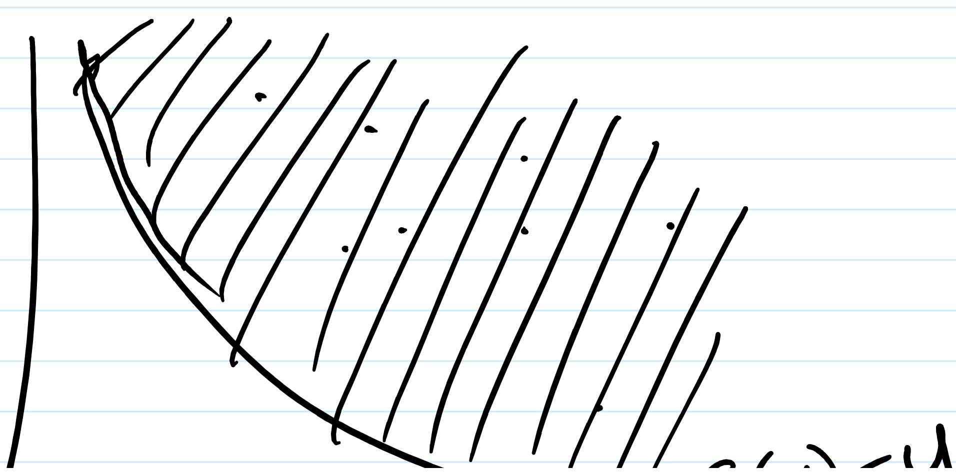
Level sets

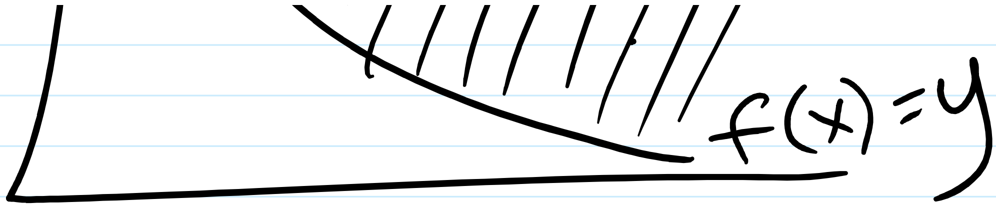


$$\{x : f(x) = y\}$$

Upper contour set

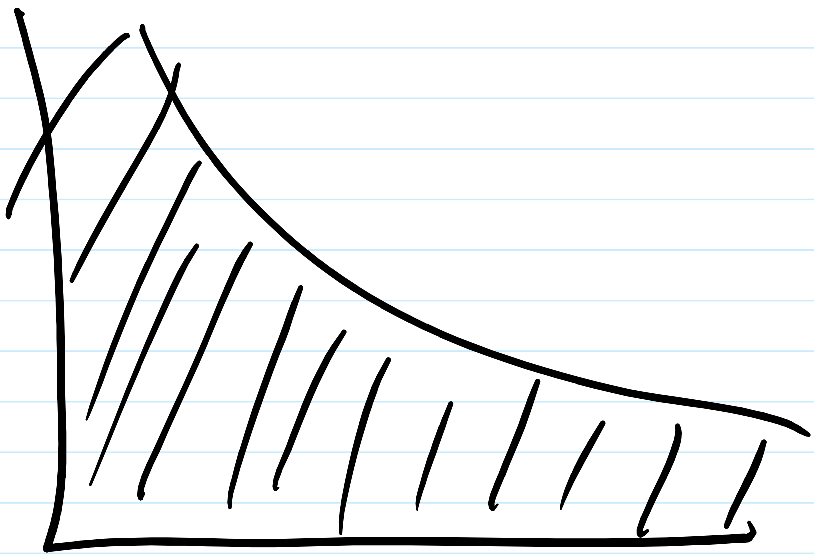
$$\{x : f(x) \geq y\}$$





Lower contour set

$\{x : f(x) \leq y\}$



Homogeneous fn

HD(K): homogeneous of degree K

A fn is HD(K) if ~~for~~ any constant a :

$f(ax) = a^k f(x)$

Constant Returns to Scale

A CRS prod. fn is HD(1) in all inputs.

$F(K, L) = K + L$

$F(2K, 2L) = 2K + 2L = 2F(K, L)$
so F is HD(1).

Ex| $y = bx^k$

$y(x) = bx^k$

$y(\alpha x) = b(\alpha x)^k = b\alpha^k x^k$

$= \alpha^k \cdot bx^k = \alpha^k y(x)$

Ex| $y = \boxed{a x_1^{k_1} x_2^{k_2} x_3^{k_3}}$

Ex] $y = \boxed{a x_1^{k_1} x_2^{k_2} x_3^{k_3}}$

what is the HD( )?

Cobb Douglas

HD(K) where

K is the sum of exponents.

$Y = AK^{\alpha} L^{1-\alpha}$ is always HD(1) or CRS

$$C(\lambda r; q) = \lambda C(r, q)$$

$$C(r; q) = (r_1^{\alpha} + r_2^{\alpha})^{1/\alpha} (q_1^{3/4} + q_2^{1/2})$$

$$C(tr; q) = ((tr_1)^{\alpha} + (tr_2)^{\alpha})^{1/\alpha} \cdot (q_1^{3/4} + q_2^{1/2})$$

$$= (t^{\alpha}[r_1^{\alpha} + r_2^{\alpha}])^{1/\alpha} \cdot g(q)$$

$$= \left(t^{\alpha}\left[r_1^{\alpha} + r_2^{\alpha}\right]\right) \cdot g(q)$$

$$= (t^{\alpha})^{\frac{1}{\alpha}} \cdot (r_1^{\alpha} + r_2^{\alpha})^{\frac{1}{\alpha}} \cdot g(q)$$

$$= t(r_1^{\alpha} + r_2^{\alpha})^{\frac{1}{\alpha}} \cdot g(q)$$

$$= t\,C(r, q)$$

So this is HD(1) in r.

Demand functions

$$x_2^*(p, w) = \frac{w}{p_2} \cdot \frac{\alpha_2}{\alpha_1 + \alpha_2}$$

$$x_2(\lambda p, \lambda w) = \frac{\lambda w}{\lambda p_2} \cdot \frac{\alpha_2}{\alpha_1 + \alpha_2}$$

$$= \frac{w}{p_2} \cdot \frac{\alpha_2}{\alpha_1 + \alpha_2}$$

$$= \lambda^0 \cdot x_2(p, w)$$

→ So this is HD(0) in p, w.

All demand fns are HD(0)

Ex $\frac{\cancel{\lambda} w}{\cancel{\lambda} pr}$ $\frac{\cancel{\lambda} pm}{\cancel{\lambda} pr}$

→ HD(0)

Euler's theorem

If f is HD(k) then

$$kf(x) = \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i}(x)$$

$$= x \cdot \nabla f$$

dot product

gradient vector

Suppose f is $HD(K)$.

Then $F(tx) = t^K F(x)$

Differentiate both sides wrt t.

$$\frac{\partial F}{\partial x_1} \cdot \boxed{\frac{\partial}{\partial t}(t x_1)} + \frac{\partial F}{\partial x_2} \cdot \frac{\partial}{\partial t}(t x_2) + \cdots$$

evaluate at $t=1$

$$= \boxed{K t^{K-1} F(x)}$$

$$\Rightarrow \frac{\partial F}{\partial x_1} \cdot x_1 + \frac{\partial F}{\partial x_2} \cdot x_2 + \cdots = K F(x)$$

∎

Ex| Demand fn

$$x_2 = \frac{w}{p_2} \cdot \frac{\alpha_2}{\alpha_1 + \alpha_2}$$

$$\underbrace{\frac{\partial x_2}{\partial p_1} \cdot p_1}_{0} + \frac{\partial x_2}{\partial p_2} \cdot p_2 + \frac{\partial x_2}{\partial w} \cdot w = 0$$

0

$$\frac{w\alpha_2}{\alpha_1+\alpha_2}\cdot -1 \cdot p_2^{-2}\cdot p_2$$

$$= -w\cdot\frac{\alpha_2}{\alpha_1+\alpha_2}\cdot\frac{1}{p_2}$$

$$\frac{1}{p_2}\cdot\frac{\alpha_2}{\alpha_1+\alpha_2}\cdot w$$

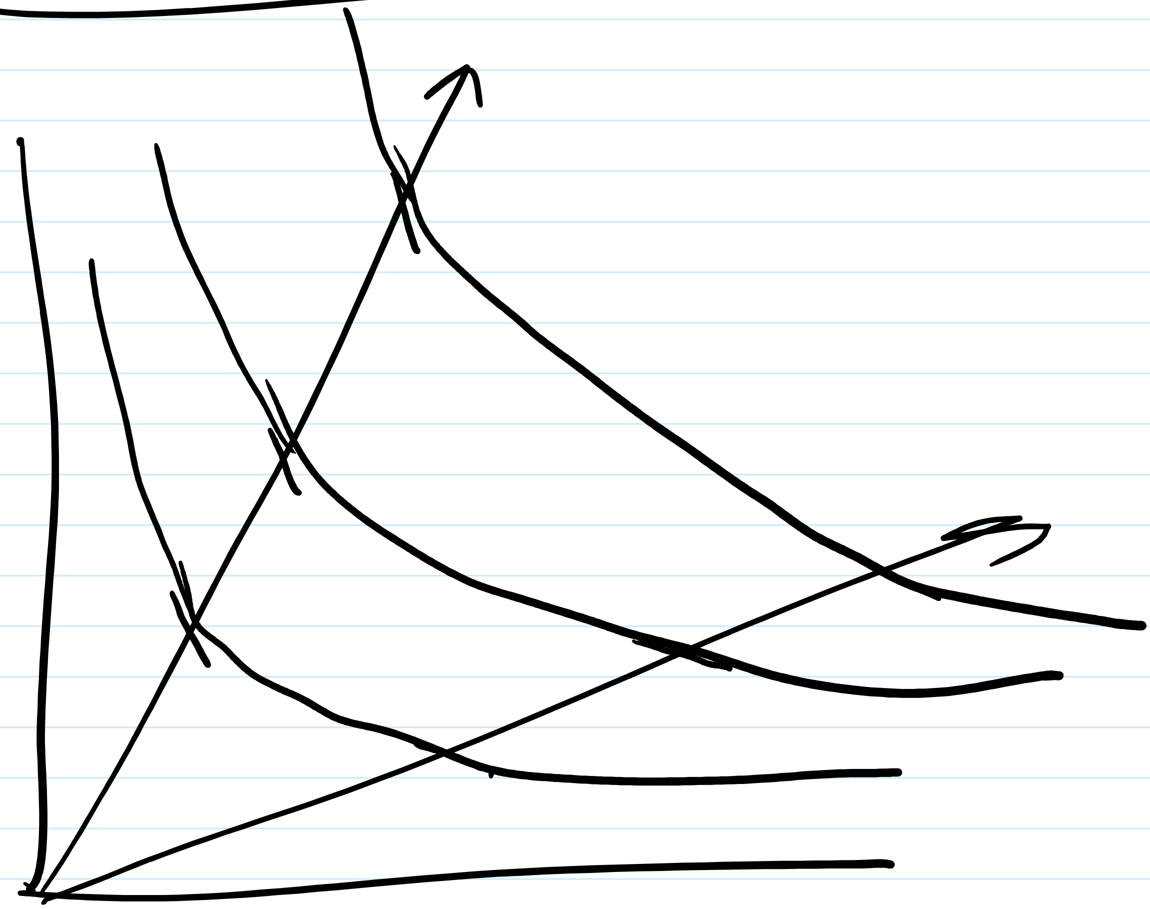
So $\frac{\partial x_2}{\partial p_2}\cdot p_2 = -\frac{\partial x_2}{\partial w}\cdot w$

So their sum is 0.

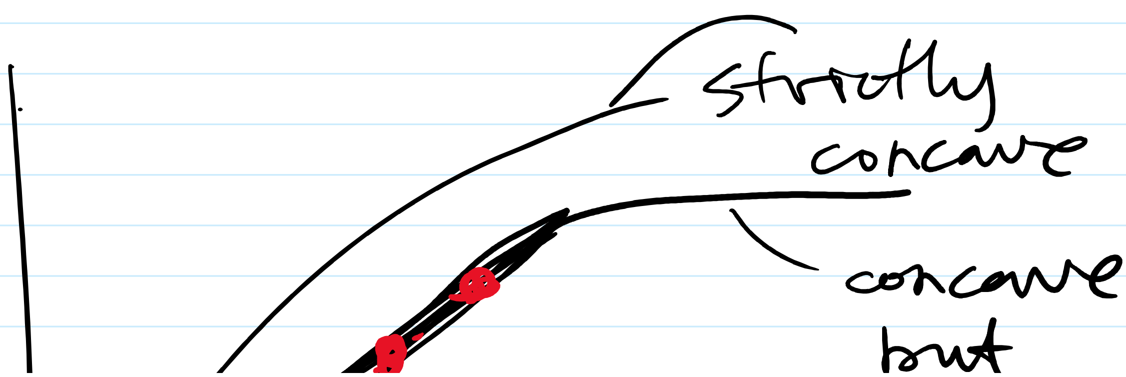
Recording was Paused for the

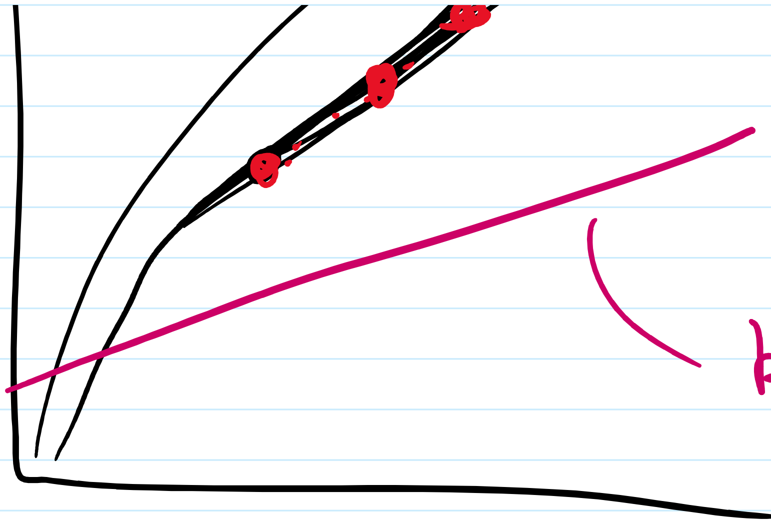
next section

Homothetic



Level sets are parallel.

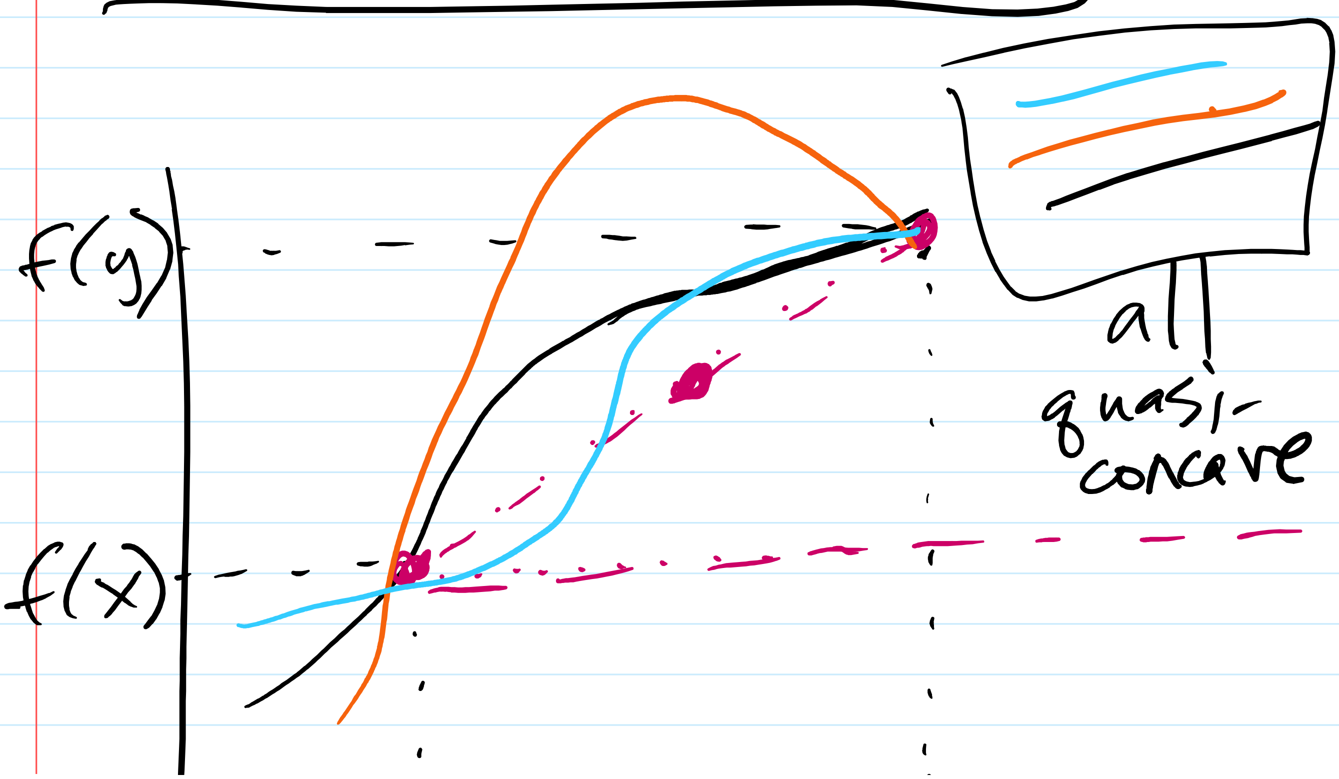


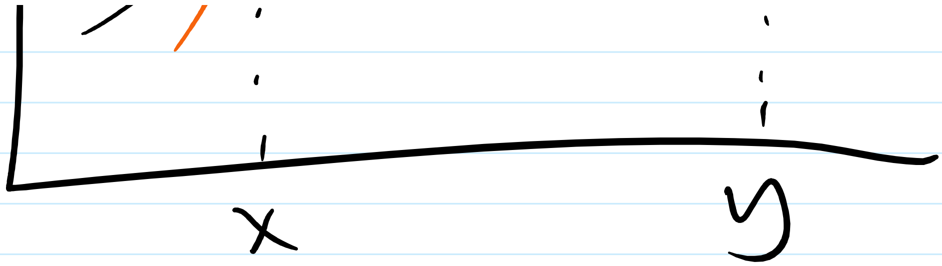


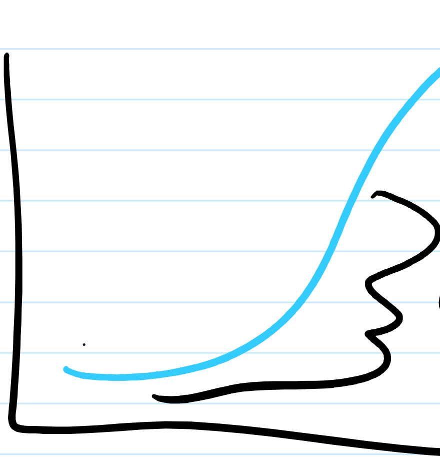
concave
but
not
strictly

both concave
and
convex
(but not
strictly)
→ only linear
functions

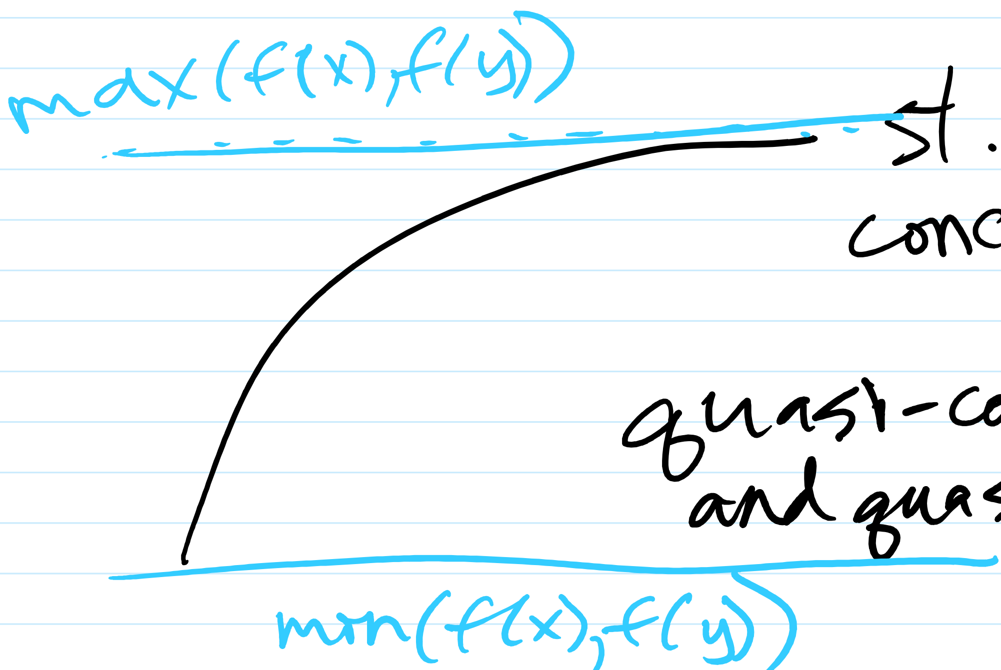
Quasiconcave







convex but quasi-concave

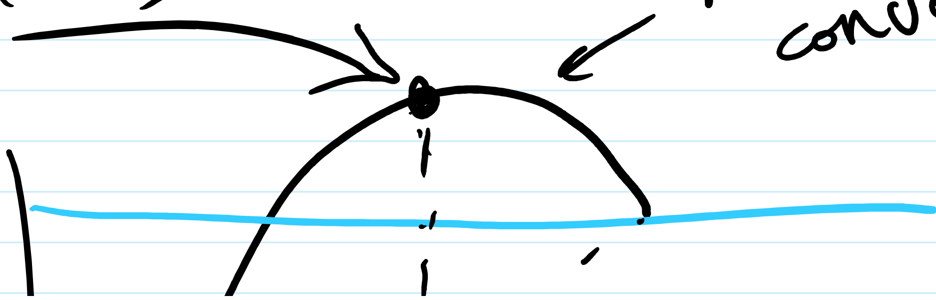


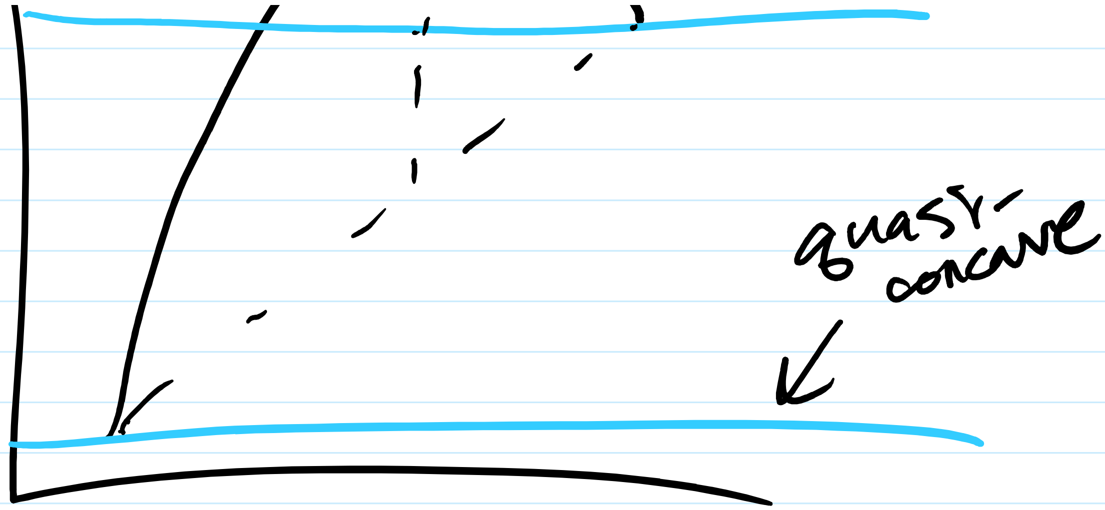
st. concave

quasi-concave and quasi convex

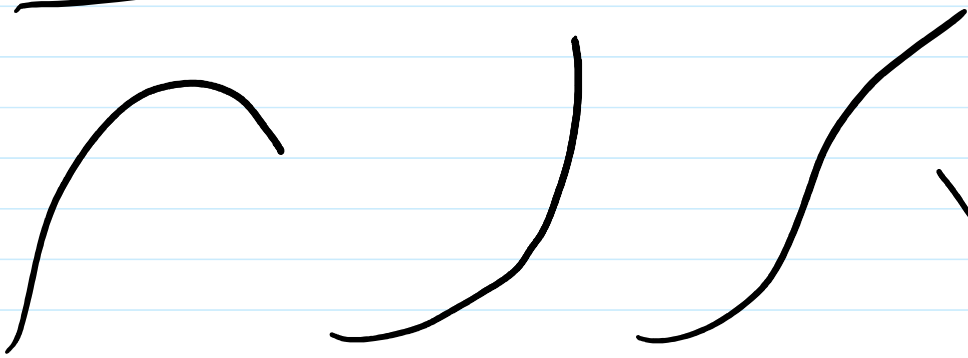
$f((\alpha x+(1-\alpha)y)) > \max(f(x),f(y))$

not quasi convex

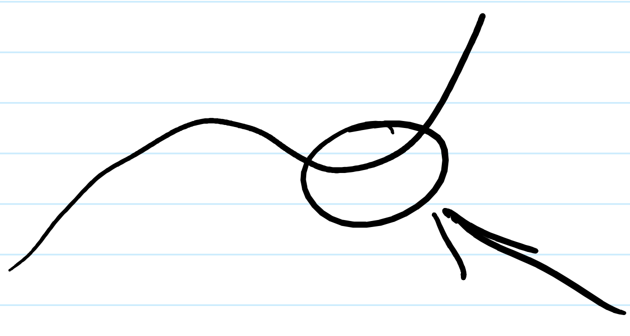




Quasiconcave



Yes

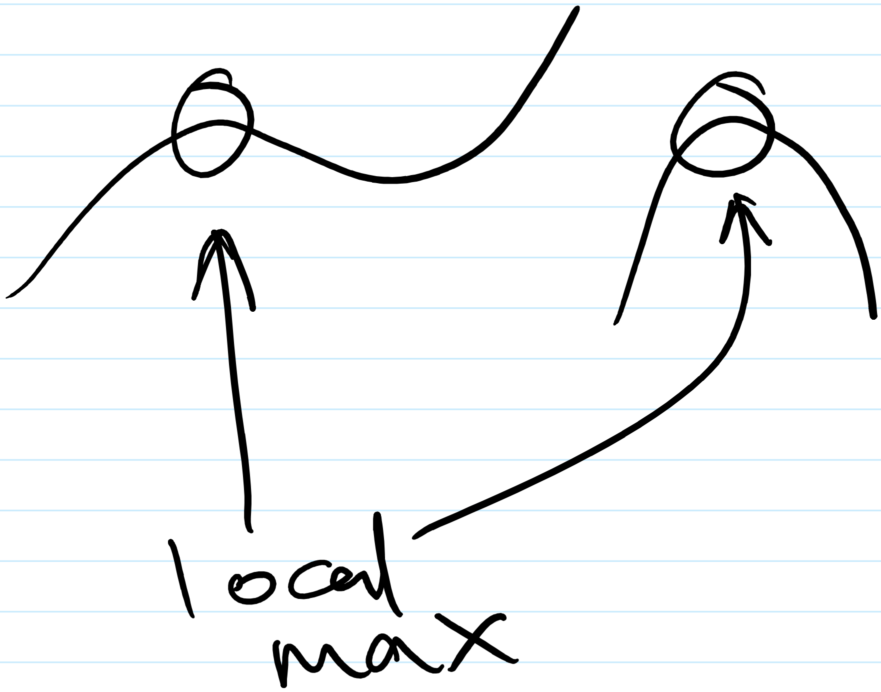


NO

local min

Quasiconvex

Yes



No

<u>Interior</u> local min/max

$\Longrightarrow$ not quasi-concave/convex

One implication:

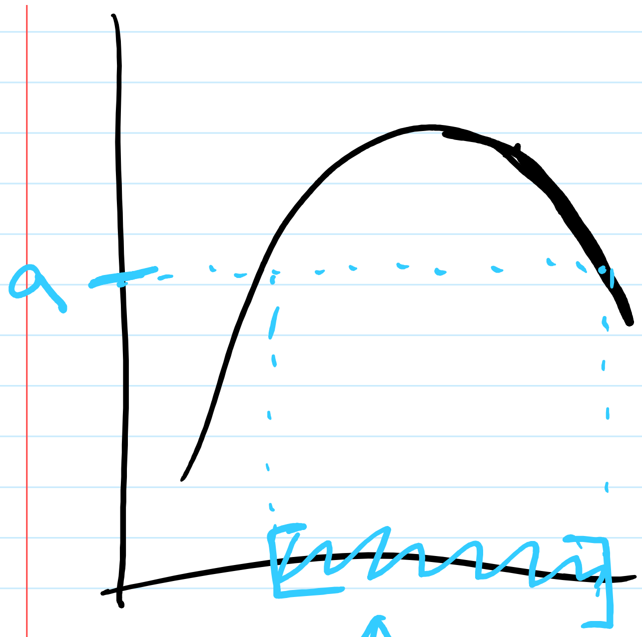
A fn is quasi-concave

and quasi-convex ~~iff~~

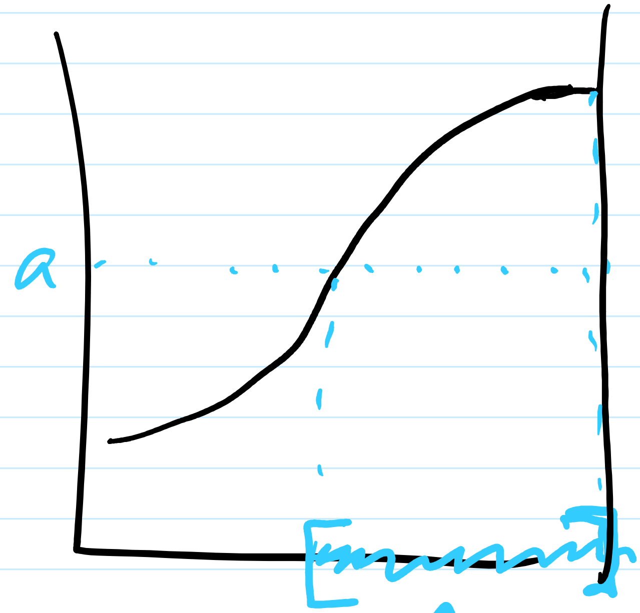
~~it~~ is monotonic.

EX

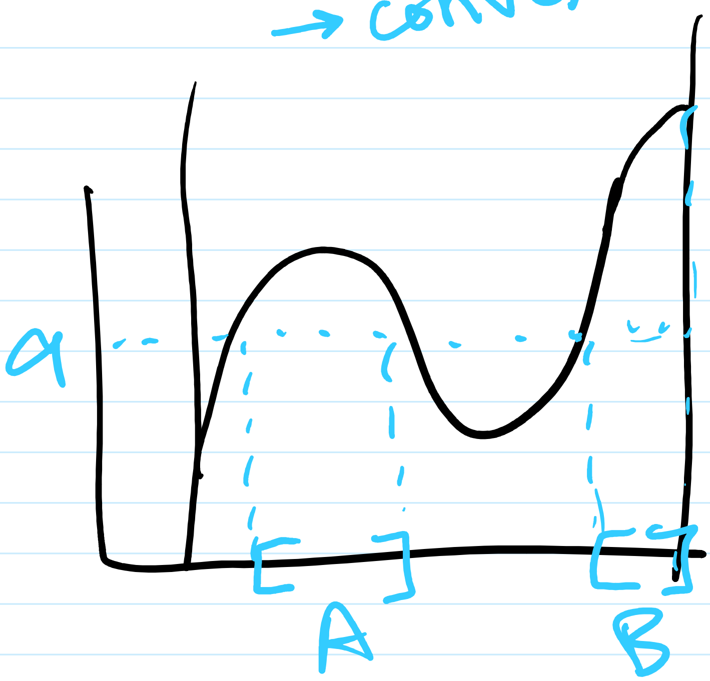
A fn is quasiconcave iff all of its upper contour sets are convex.



upper contour set → convex



U.C.S → convex



upper. cont. set

$= A \cup B$

$= A \cup B$

$\Rightarrow$ not convex

$\Rightarrow$ not quasiconcave

Consumer has a quasiconcave utility function.

$\Longrightarrow$ All the upper contour sets are convex

$\Longrightarrow$ The consumer's preferences are convex.

Concavity in $\mathbb{R}^n$

Recall:

A fn $f: \mathbb{R} \to \mathbb{R}$ is convex iff $f''(x) > 0$.

In $\mathbb{R}^n$:

A fn $f : \mathbb{R}^n \to \mathbb{R}$ is strictly convex iff its <u>Hessian</u> is positive definite.

A symmetric matrix (e.g. the Hessian) is positive definite iff

$$\forall x \neq 0, \quad \underbrace{x' H x}_{\text{quadratic form}} > 0$$

$x \in \mathbb{R}^n$

raw vector

col vector

quadratic form

$$x'Hx = \sum_i \sum_j x_i x_j \underbrace{H_{ij}}_{\substack{i,j^{th} \\ \text{element} \\ \text{of } H}}$$

Ex: $H = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x'Hx = 2x_1^2 + \underbrace{6x_1x_2}_{2\cdot(3x_1x_2)} + 4x_2^2$$

$x'Hx > 0$ pos. definite $\Leftrightarrow$ strictly convex

≥ 0 pos. semidef. — convex

< 0 neg. def. — st. concave

≤ 0 neg. semidef — concave

Recording resumes here

$f(x_1, x_2 \ldots, x_n, y) = C$

Ex) $u(x_1, x_1, x_3)$

Want to know MRS_{23}.

= slope of the indiff.
(in $\mathbb{R}^2$).

Hold u constant at $\bar{u}$,
to define a level set
of u.

Let's use $u(x_1, x_2, x_3)$

$$= x_1^{1/4} x_2^{1/3} x_3^{1/6}.$$

Find $\dfrac{dx_2}{dx_3} = MRS_{23}$ holding utility constant.

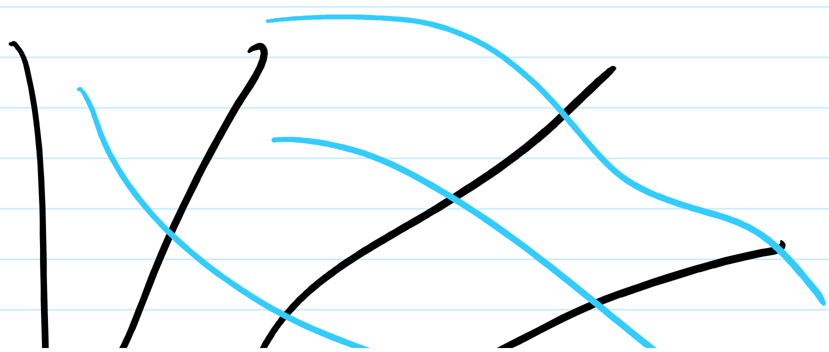
$$\frac{dx_2}{dx_3} = -\frac{\frac{\partial u}{\partial x_3}}{\frac{\partial u}{\partial x_2}}$$

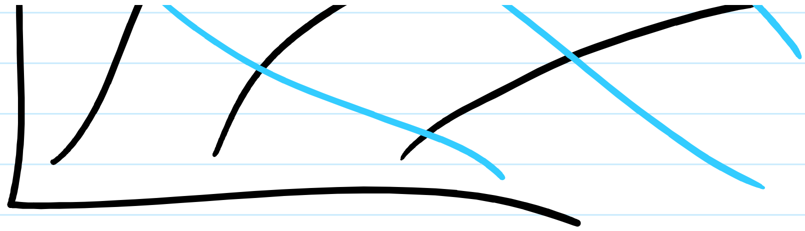
$$= -\frac{1/6\, x_1^{1/4} x_2^{1/3} x_3^{-5/6}}{1/3\, x_1^{1/4} x_2^{-2/3} x_3^{1/6}}$$

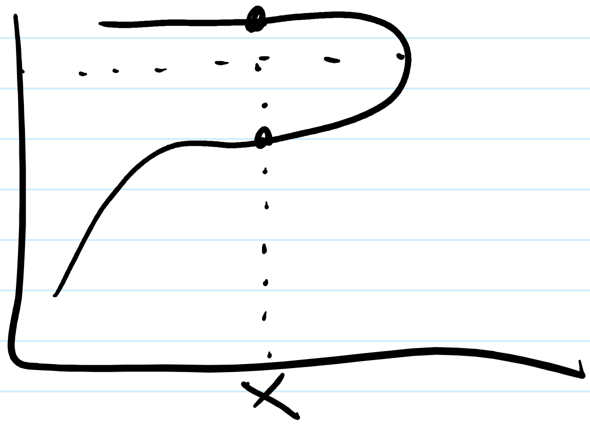
$$= -\frac{1}{2} \cdot \frac{x_2}{x_3}$$

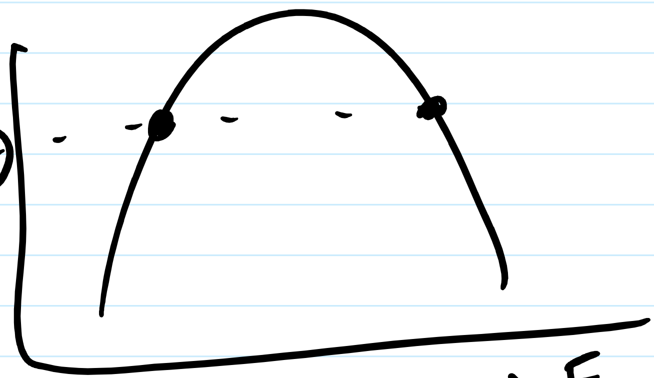
Inverse fn

→ strictly monotonic



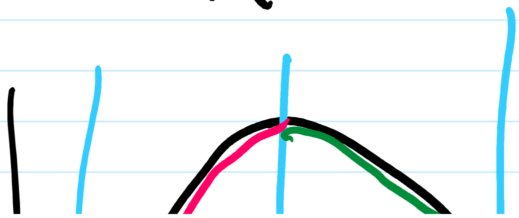


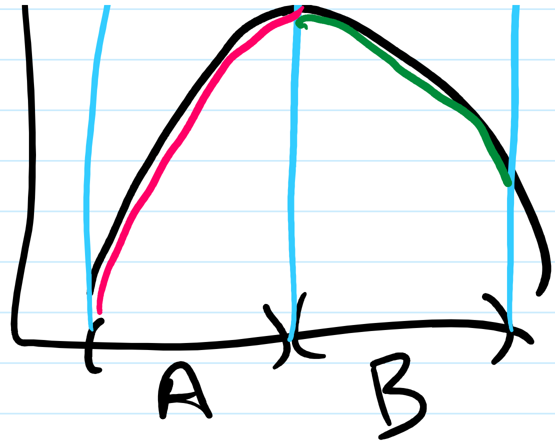




$f^{-1}(y)$ D.N.E.

But the inverse of a non-monotic fn may exist if you restrict the domain





$f_A(x)$ $f_B(x)$

$f_A : A \to \mathbb{R}$ $f_B : B \to \mathbb{R}$

f_A^{-1} exists f_B^{-1} exists

Correspondences

$$f(x) = y$$

$$F(x) = \overbrace{\{y_1, y_2, y_3\}}^{\text{a \underline{set} not a vector}}$$

<u>Not</u> a multi-valued fn or vector field.

$F(x_1) = \{1, 2, 3\}$

$F(x_2) = \{0, 0, 1, 5, 10\}$

$F(x_3) = \emptyset$

$$\boxed{F : X \Rightarrow \mathbb{R}}$$

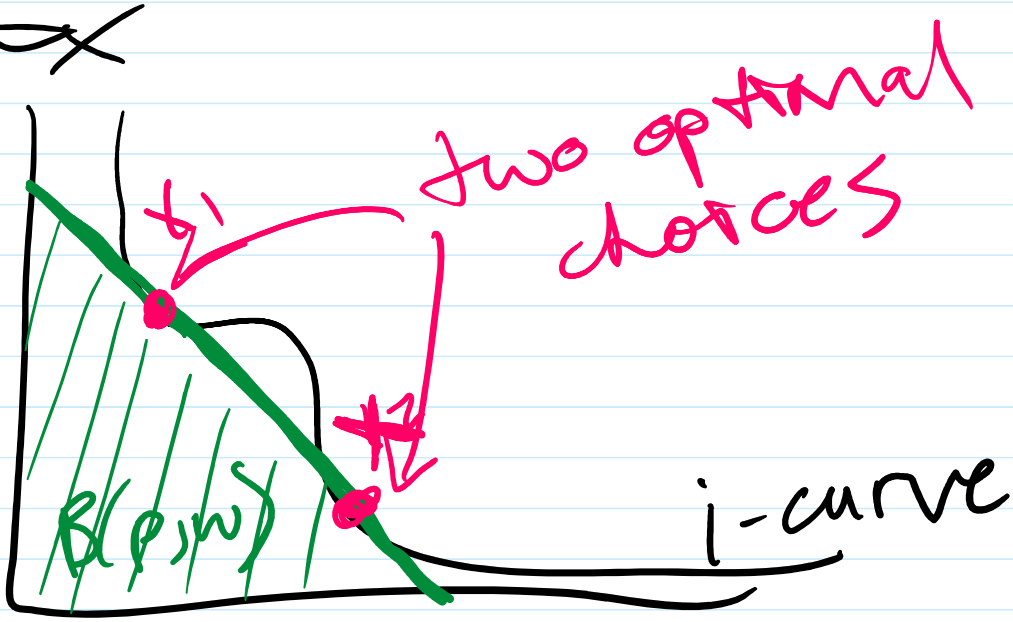
$\mathcal{P}(X)$ powerset : set of all subsets of X.

$F : X \Rightarrow \mathbb{R}$

$\equiv$

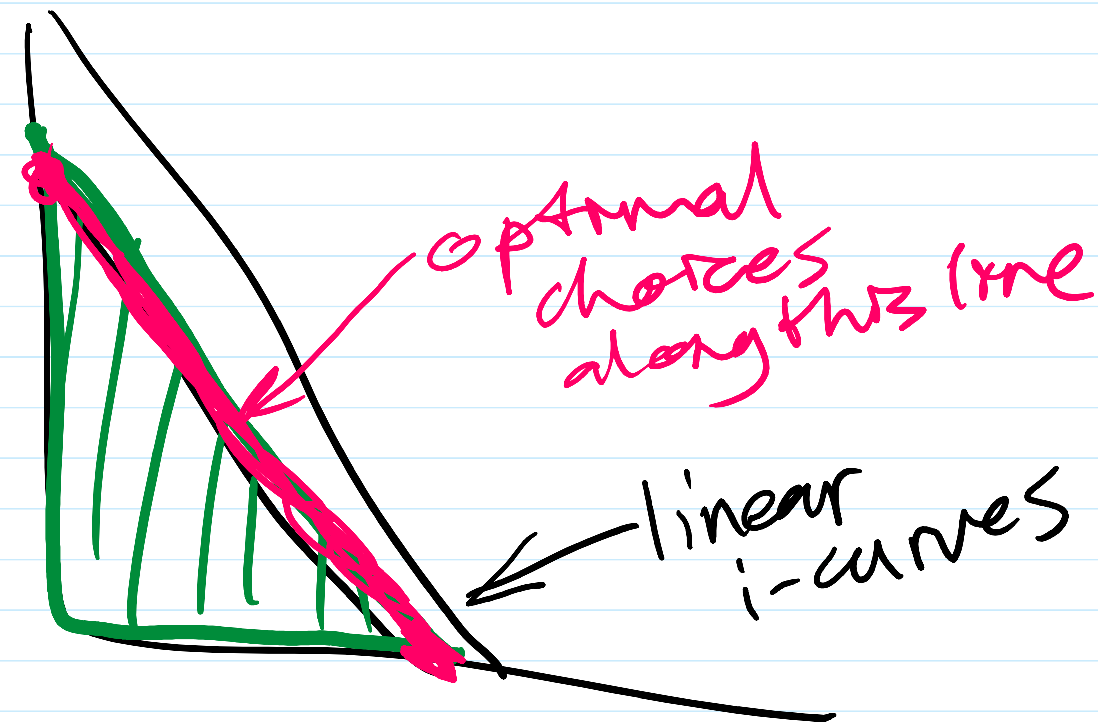
$F : X \to \mathcal{P}(\mathbb{R})$

Ex

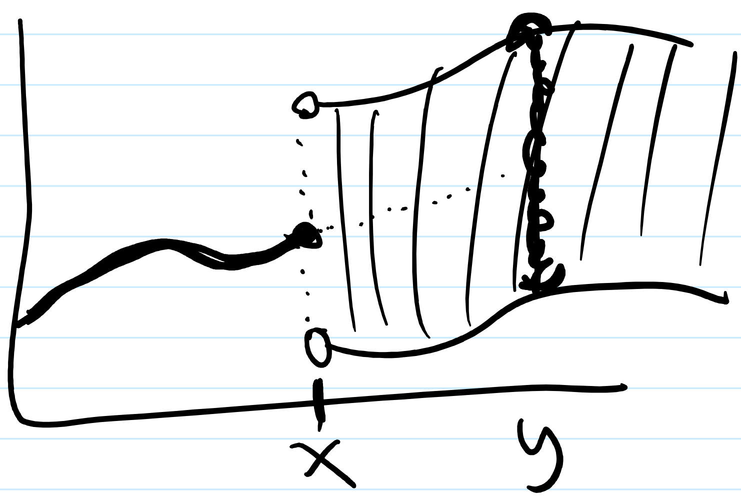


Demand correspondence

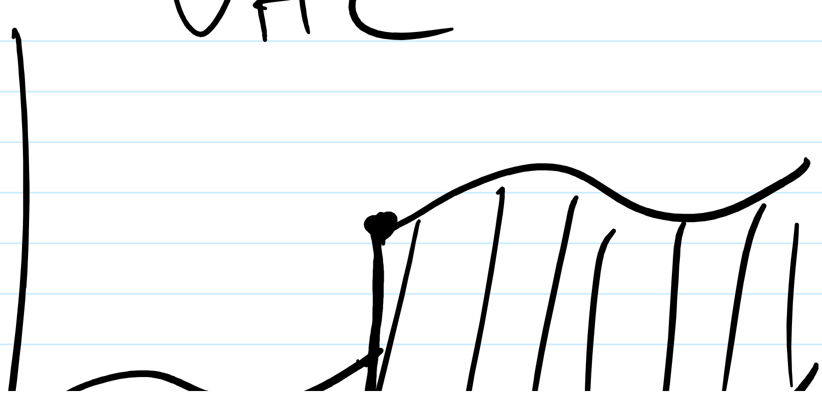
$$\underbrace{x^*(p,w)}_{\text{maximizer}} = \{x_1, x_2\}$$

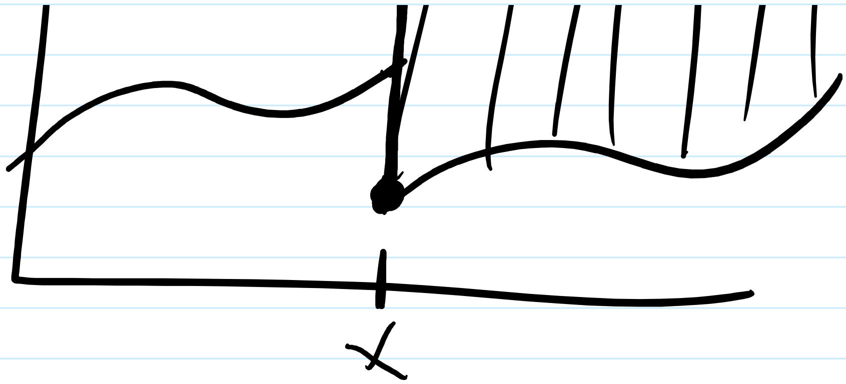


LHC



UHC





1) There are multiple equivalent definitions, using diff. characterizations of open and closed set.

2) A correspondence is called <u>continuous</u> if it is both LHC and UHC.

3) Both LHC and UHC are equivalent to continuity for single-valued functions.

Kakutani's F.P.T.

$\hat{x}$ is a fixed point of ϕ if $\hat{x} \in \phi(\hat{x})$