Lecture 02
Set Theory and Topology

Ryan McWay[†]

[†] *Applied Economics,*
University of Minnesota

Mathematics Review Course, Summer 2023
University of Minnesota
August 8th, 2023

Last Lecture Review

- Logic:
  - Logical statements
  - Necessary vs. sufficient
- Proofs:
  - Proof by Deduction/Construction (Direct Proofs)
  - Proof by Contradiction
  - Proof by Induction

Review Assignment

1. Problem Set 01 solutions are available on Github.
2. Any issues or problems **You** would like to discuss?

Daily Icebreaker

- Attendance via prompt:
  - Name
  - Program and track
  - Daily icebreaker subject...

Topic: Set Theory

Motivation

- General background
  - How collections of mathematical objects are organized.
  - A foundation for all of math.
- Why do economists' care?
  - Need to have strong understanding of the basics.
- Application in this career
  - Rarely.
  - Sometimes useful when considering proofs.

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Overview

1. Sets
2. Set Operators
3. Set Space
4. de Morgans' Law & Cartesian Product
5. Cardinality & Countability
6. Convex Sets
7. Open & Closed Sets
8. Bounded & Compact Sets

1. Sets

- Sets
  - A collection of objects (elements or members)
  - $S = \{s \in U : P\}$ for the universal set U such that is satisfies properties P.
- Elements
  - The components within a set.
  - An element can be a complex object; such as another set.
- Empty Set
  - $\emptyset = \{s \notin U\}$ contains nothing.
  - $\emptyset \neq \{\emptyset\}$

1. Sets

$\mathbb{R}$	Real Numbers: $\{x : -\infty \leq x \leq \infty\}$
$\mathbb{R} \times \mathbb{R}$	Cartesian Plane
$\mathbb{N}$	Natural Numbers
$\mathbb{W}$	Whole Numbers: $\mathbb{N} \wedge 0$
$\mathbb{Z}$	Integers
$\mathbb{Q}$	Rational Numbers
$\mathbb{P}$	Irrational Numbers

2. Set Operators

- Complement: $A^c \equiv \{x \in U : x \notin A\}$
- Intersection: $A \cap B \equiv \{x \in U : x \in A \wedge x \in B\}$
- Union: $A \cup B \equiv \{x \in U : x \in A \vee x \in B\}$
- Set Difference (Partition): $A \setminus B \equiv \{x \in U : x \in A \wedge x \notin B\}$
- Disjoint Set: $A \cap B = \emptyset$
- Subset: $B \subset A$ if $[x \in B] \implies [x \in A]$
- Proper (Strict) Subset: $B \subset A \wedge B \neq A$.
- Power Set: $\surd(A) \equiv \{X : X \subseteq A\}$
- Indexed Set: $A_1, A_2, \ldots, A_i$

2. Set Operators

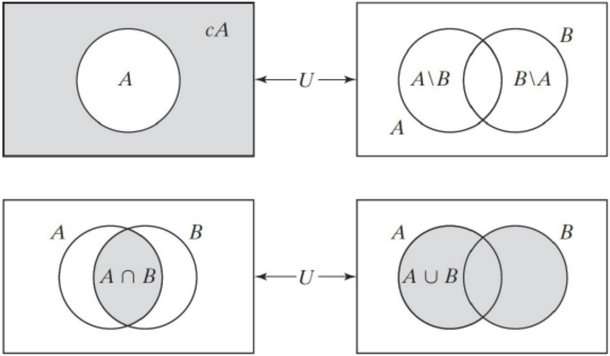

Figure A1. 1. Venn diagrams.

Source: Jehle & Reny (2011)

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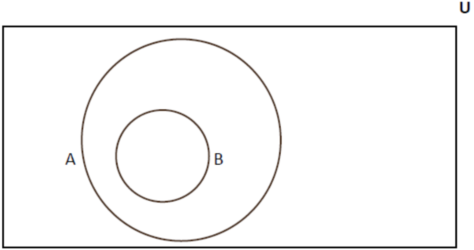

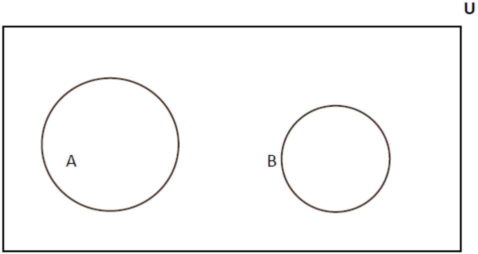

3. Set Space

- Set Product: A set of ordered pairs
$$S \times T \equiv \{(s,t) | s \in S, t \in T\}$$

- N-dimensional Euclidean Space
$$\mathbb{R}^n \equiv \mathbb{R} \times \mathbb{R} \times \cdots \mathbb{R} \equiv \{(x_1, \ldots, x_n) | x_i \in \mathbb{R}, \forall i = 1, \ldots, n\}$$

- Cartesian Plane
$$\mathbb{R}^2 \equiv \mathbb{R} \times \mathbb{R} \equiv \{(x_1, x_2) | x_1, x_2 \in \mathbb{R}\}$$

4. de Morgan's Law and Cartesian Product

- de Morgan's Law: Assume A_i are subsets

$$\left[\bigcup_{i=1}^{k} A_i\right]^c = \bigcap_{i=1}^{k} A_i^c$$

- Cartesian Product: For 2 sets A and B, the cartesian product is:

$$A \times B \equiv \{(a, b) : a \in A, b \in B\}$$

5. Cardinality and Countability

- Cardinality: $|A|$ is the number of elements in the set.
  - Types: Finite, countably infinite, and uncountable
- Countable (Finite) Set: Any infinite set that can be placed in a 1 to 1 correspondence with $\mathbb{N}$.

6. Convex Sets

- $S \subset \mathbb{R}^n$ is a convex set $\forall x_1, x_2 \in S$, if we have $tx_1 + (1-t)x_2 \in S$.

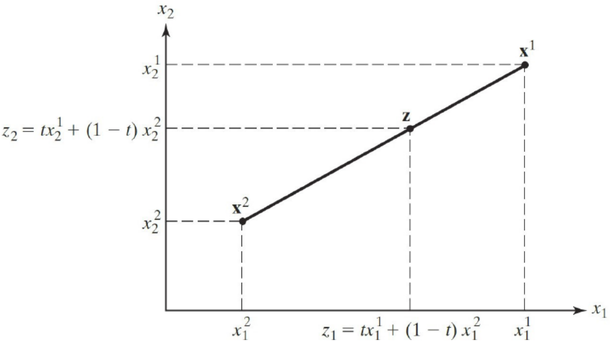

Figure A1.4. Some convex combinations in $\mathbb{R}^2$.

Source: Jehle & Reny (2011)

7. Open and Closed Sets

- Metric Space (e.g., point distance): $d(x_1, x_2) = |x_1 - x_2|$
- An open ε-ball with center x_0 and radius $\varepsilon > 0$ is a subset of points in $\mathbb{R}^n$:

$$B_\varepsilon(x_0) \equiv \{x \in \mathbb{R}^n \mid |x - x_0| < \varepsilon\}$$

- A closed ε-ball is similar, but includes the circumference (e.g., edge of the ball)

$$B_\varepsilon(x_0) \equiv \{x \in \mathbb{R}^n \mid |x - x_0| \leq \varepsilon\}$$

7. Open and Closed Sets

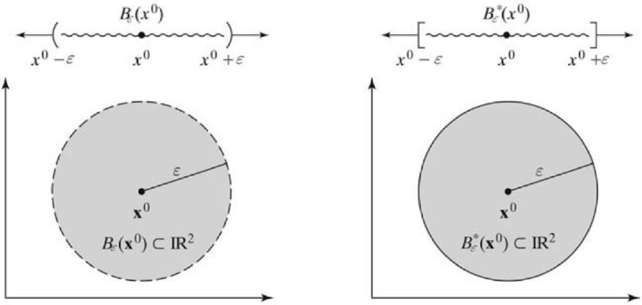

Figure A1. 10. Balls in $\mathbb{R}$ and $\mathbb{R}^2$.

Source: Jehle & Reny (2011)

8. Bounded and Compact Sets

- Bounded: A set $S \subset \mathbb{R}^n$ is bounded if it is entirely contained within some ε-ball (either closed or open)
- Compact: A set $S \subset \mathbb{R}^n$ is compact if it is both closed and bounded.
- We like working with compacted sets.

Topic: Topology

Motivation

- General background
  - Understanding of spatial relationships and how the parts are integrated into the whole.
- Why do economists' care?
  - Used in proofs.
  - Several theorems used as lemmas invoked in proofs.
- Application in this career
  - Welfare theorem
  - Consumer behavior
  - Macroeconomics and time series

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Overview

1. Supremum & Infimum
2. Sequences and Limits
3. Separating Hyperplane Theorem

1. Supremum and Infimum

- Ordered Set: When elements have a defined order ($<$).
- To be ordered:
  - For $x, y \in A$, only one of the following statements can be true: (1) $x < y$, $x = y$, $x > y$.
  - For $x, y, z \in A$, if $x < y \wedge y < z \implies x < z$.
- A subset (A_1) of an ordered set may be bounded from above and below if:
  - Upper Bound: $\{\beta \in A : x \leq \beta \forall x \in A_1\}$
  - Lower Bound: $\{\beta \in A : x \geq \beta \forall x \in A_1\}$

1. Supremum and Infimum

- Supremum: Least upper bound (1 value)

  $$\beta = \sup A_1 | \beta \in A : y < \beta \exists x \in A_1 : x > y$$

- Infimum: Greatest lower bound (1 value)

  $$\alpha = \inf A_1 | \alpha \in A : \delta > \alpha \exists x \in A_1 : x < \delta$$

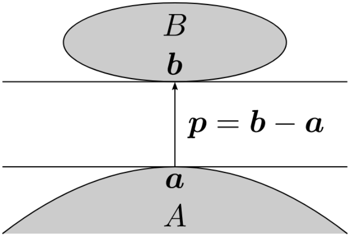
2. Sequences and Limits

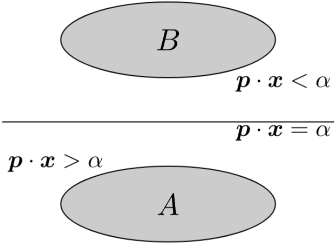
- Sequence: Function $f(\cdot)$ defined on a set of natural numbers, $\mathbb{N}$.
- Limit: A sequence $\{x_n\}$ converges to a limit, $x_n \to L$ or $\lim_{n\to\infty} x_n = L$, if given $\varepsilon > 0$ there is an element N such that whenever $n > N : |x_n - L| < \varepsilon$.
- A sequence diverges when it does not converge to a limit.

Theorem:
If the sequence $\{x_n\}$ converges, then the limit of $\{x_n\}$ is unique (e.g., single valued).

3. Separating Hyperplane Theorem

- There exists a line dividing an n-dimensional space.
- Given $p \in \mathbb{R}^n : p \neq 0$ and $c \in \mathbb{R}$, the hyperplane generated is the set $H_{p,c} = \{z \in \mathbb{R}^n | p \cdot z = c\}$.





Review of Sets

1. Sets are the foundation of organizing objects in math.
2. de Margan's Law
3. Cartesian Product
4. Convex Sets
5. Bounded Sets
6. Compact Sets

Review of Topology

1. Supremum and Infimum
2. Seperating Hyperplane Theorem

Assignment

- Readings on Derivatives before Lecture 03:
  -
- Assignment:
  - Problem Set 02 (PS02)
  - Solution set will be available following end of Lecture 03
- Struggling?
  1. Read the 'Encouraged Reading'
  2. Review 'Supplementary material'
  3. Reach out directly