# APEC Math Review

## Part 3 Functions

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# Relations

A collection of ordered-pairs $(s, t)$ constitutes a **binary relation** $s\mathcal{R}t$ between the sets $S$ and $T$.

- A relation $\mathcal{R}$ on $S$ is **complete** if for all elements $x$ and $y$ in $S$, $x\mathcal{R}y$ or $y\mathcal{R}x$.
- A relation $\mathcal{R}$ on $S$ is **transitive** if for any three elements $x$ $y$ and $z$ in $S$, $x\mathcal{R}y$ and $y\mathcal{R}z \implies x\mathcal{R}z$.

Recall the exercise from the last lecture, a preference relation is **rational** if it is both complete and transitive.

# Correspondences and functions

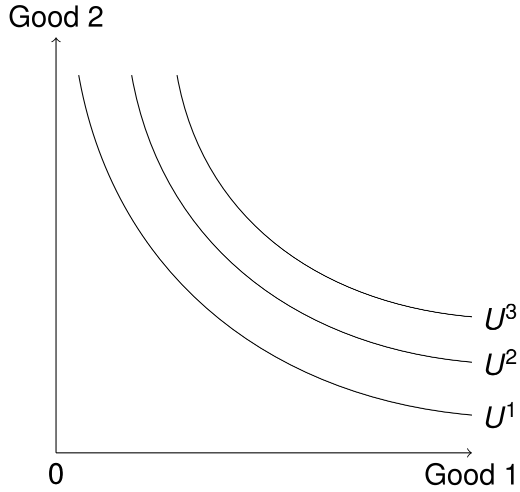
- A **correspondence** is a relation that associates each element of one set (domain) with a set of elements of another set (range)
- A **function** is a relation that associates each element of the domain with a single, unique element of the range.

$$f : D \to R$$

  - every element in the range is mapped into by some point in the domain: **onto**
  - every element in the range is assigned to at most a single point in the domain: **one-to-one**

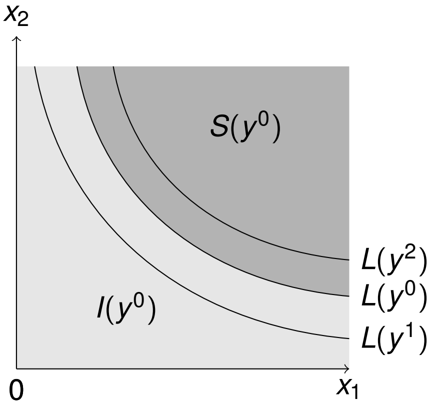
# Indifference curves

We all remember seeing this:




How do we characterize the shape of indifference curves with properties of the utility function?

# Level, superior and inferior sets




$$L(y^0) \equiv \{\mathbf{x} | \mathbf{x} \in D,\ f(\mathbf{x}) = y^0\}$$

$$S(y^0) \equiv \{\mathbf{x} | \mathbf{x} \in D,\ f(\mathbf{x}) \geq y^0\}$$

$$I(y^0) \equiv \{\mathbf{x} | \mathbf{x} \in D,\ f(\mathbf{x}) \leq y^0\}$$

# Quasiconcavity and quasiconvexity

$f : D \to R$ is **quasiconcave** iff, for all $\mathbf{x}^1$ and $\mathbf{x}^2$ in D,

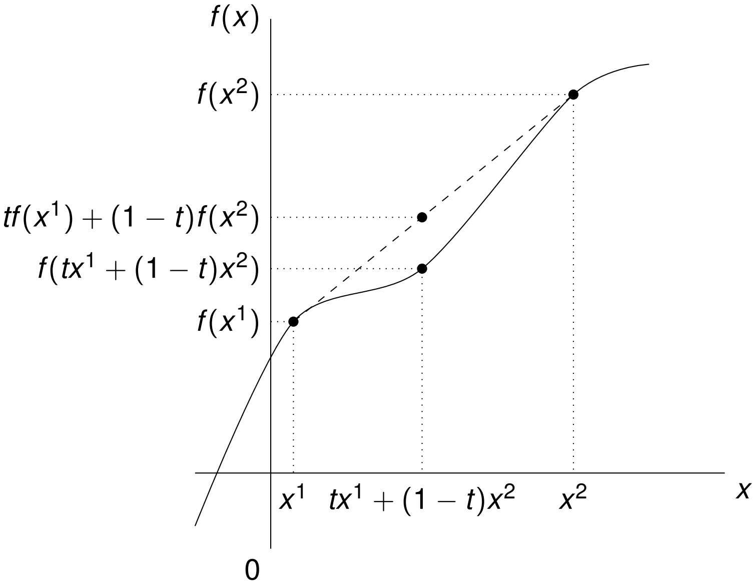
$$f(t\mathbf{x^1} + (1-t)\mathbf{x^2}) \geq min[f(\mathbf{x}^1), f(\mathbf{x}^2)] \quad \forall t \in [0,1]$$

$f : D \to R$ is **quasiconvex** iff, for all $\mathbf{x}^1$ and $\mathbf{x}^2$ in D,

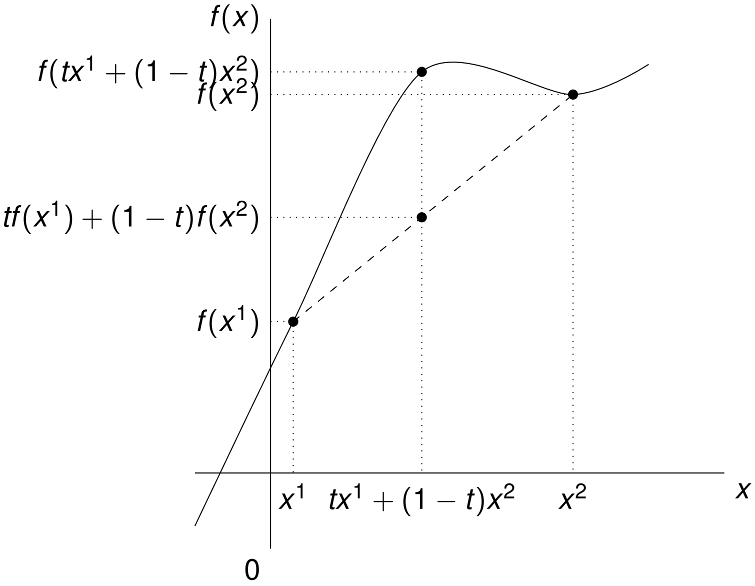
$$f(t\mathbf{x^1} + (1-t)\mathbf{x^2}) \leq max[f(\mathbf{x}^1), f(\mathbf{x}^2)] \quad \forall t \in [0,1]$$

The quansiconcavity and quasiconvexity are strict when the inequality holds for all $\mathbf{x}^1 \neq \mathbf{x}^2$.

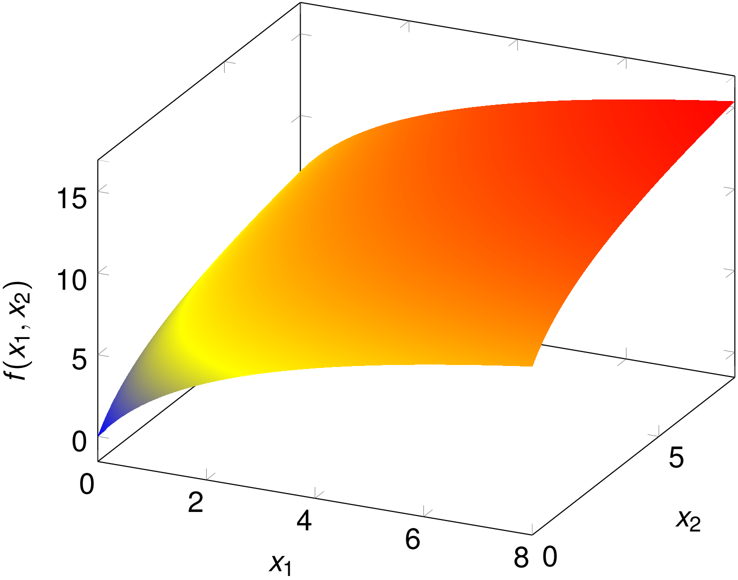
# A quasiconcave and quasiconvex function



# A quasiconcave but not quasiconvex function

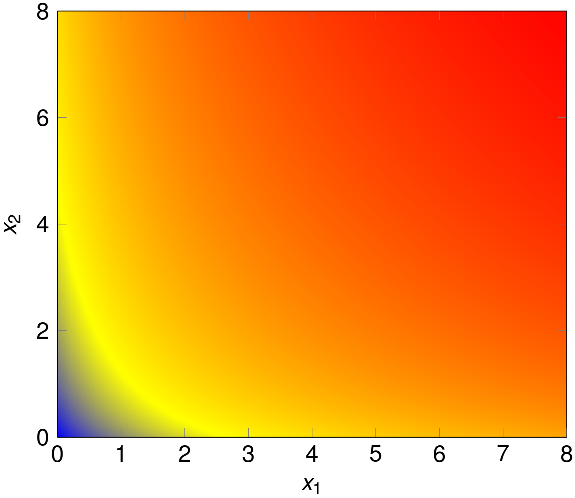


# A quasiconcave function of two variables

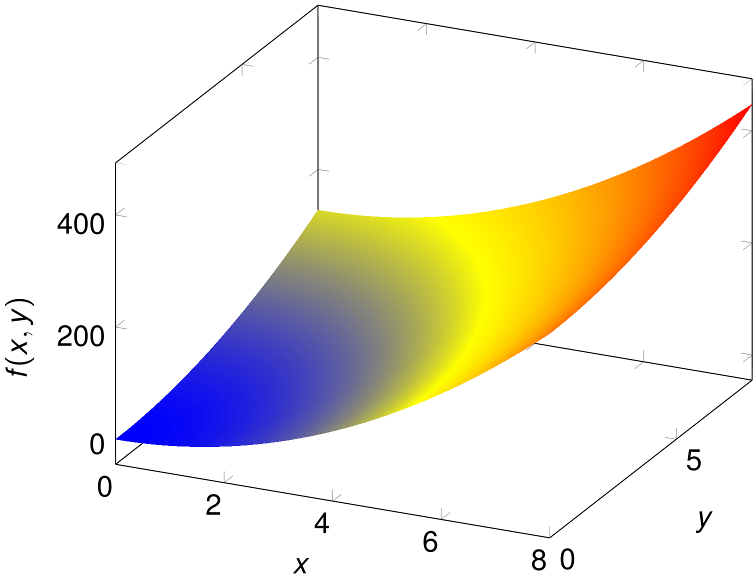


# A quasiconcave function of two variables

$f : D \to R$ is **quasiconcave** iff $S(y)$ is a **convex** set for all $y \in \mathbb{R}$.

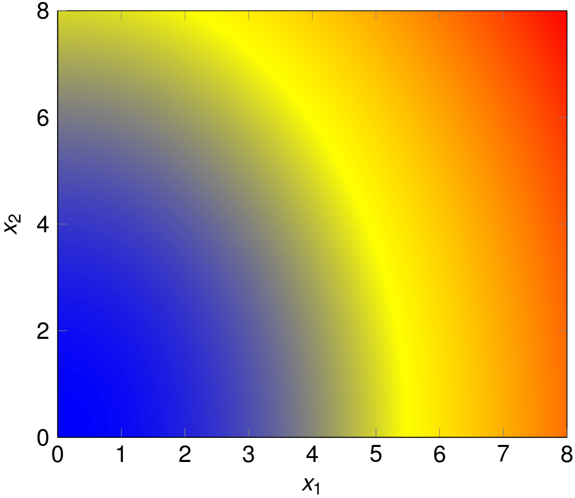


# A quasiconvex function of 2 variables



# A quasiconvex function of 2 variables

$f : D \to R$ is **quasiconvex** iff $I(y)$ is a **convex** set for all $y \in \mathbb{R}$.



# Concavity and convexity

A real-valued function $f$ defined on a convex subset $D \subset \mathbb{R}^n$ is **concave** if for all $\mathbf{x^1}, \mathbf{x^2} \in D$ and for all $t \in [0, 1]$,

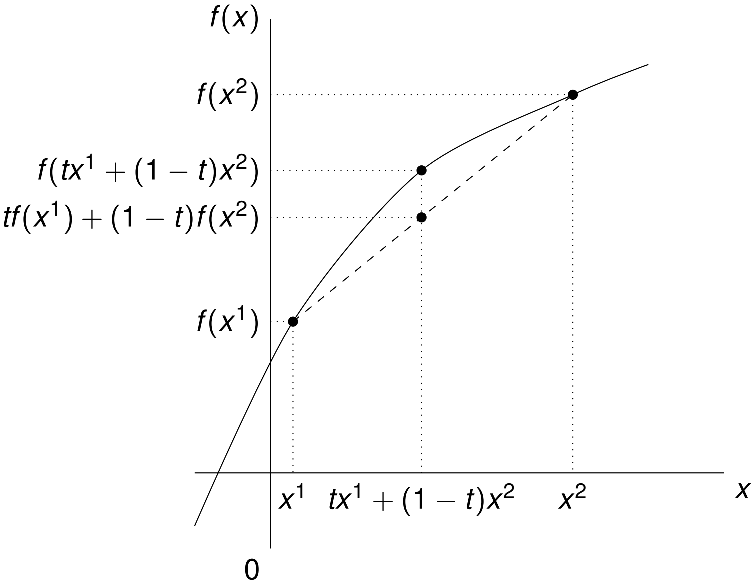
$$f(t\mathbf{x^1} + (1-t)\mathbf{x^2}) \geq tf(\mathbf{x^1}) + (1-t)f(\mathbf{x^2})$$

It is **convex** if

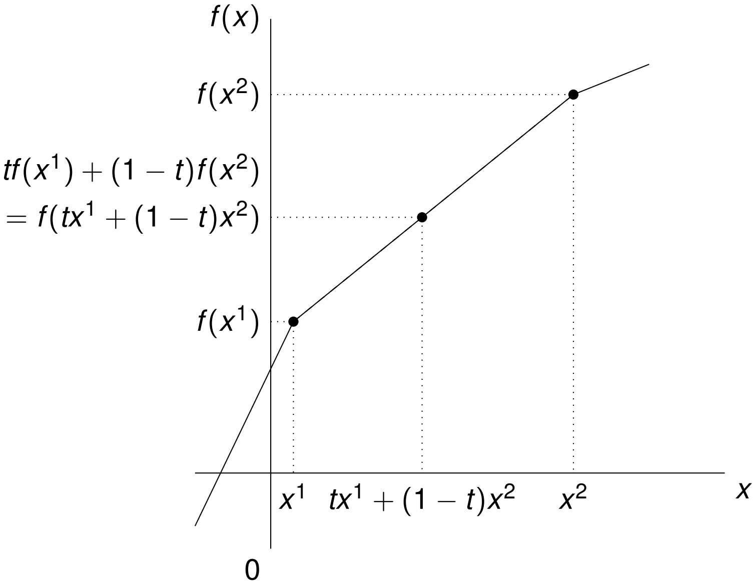
$$f(t\mathbf{x^1} + (1-t)\mathbf{x^2}) \leq tf(\mathbf{x^1}) + (1-t)f(\mathbf{x^2})$$

The concavity and convexity are **strict** when the inequality holds.

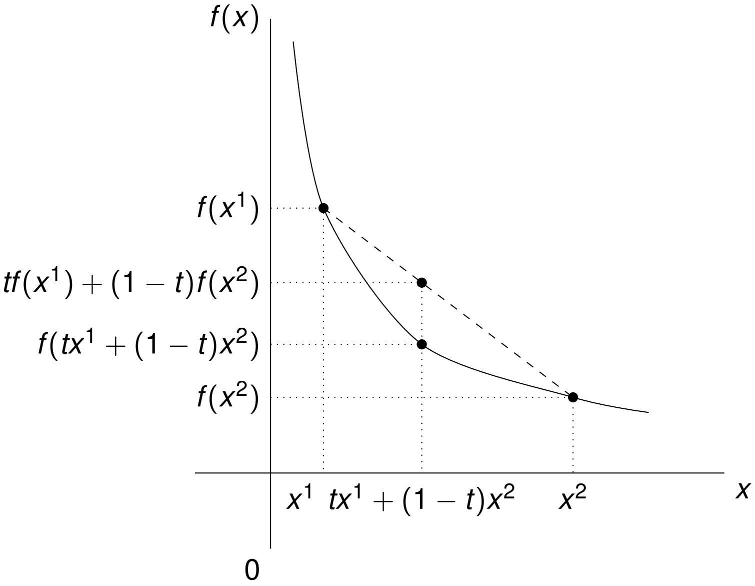
# Concave function in 2D



# Concave but not strictly concave



# Convex function in 2D



# Questions

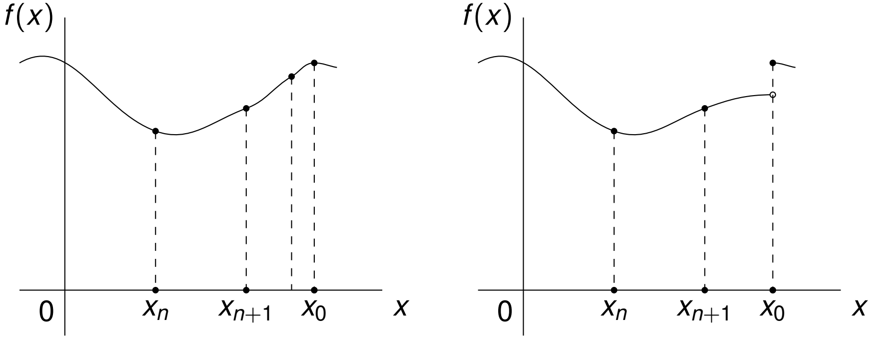
- What is the relationship between a convex function and a convex set?
  - $f$ convex $\implies$ $f$ quasiconvex $\Leftrightarrow$ inferior sets convex
- What is the relationship between a concave function and a convex set?
  - $f$ concave $\Leftrightarrow$ points on an below the graph form a convex set
- What is a concave set?
  - not defined

# Exercise

Prove that if $f$ is concave, it is quasiconcave.

# Continuity

A function $f : \mathbb{R}^m \to \mathbb{R}^n$ is **continuous** at $\mathbf{x}_0 \in \mathbb{R}^m$ if whenever $\{\mathbf{x}_n\}_{n=1}^{\infty}$ is a sequence in $\mathbb{R}^m$ which converges to $\mathbf{x}_0$, then the sequence $\{f(\mathbf{x}_n)\}_{n=1}^{\infty}$ in $\mathbb{R}^n$ converges to $f(\mathbf{x}_0)$. If $D$ is a compact subset of $\mathbb{R}^m$ , then $f(D)$ is a compact subset of $\mathbb{R}^n$.



# Continuity

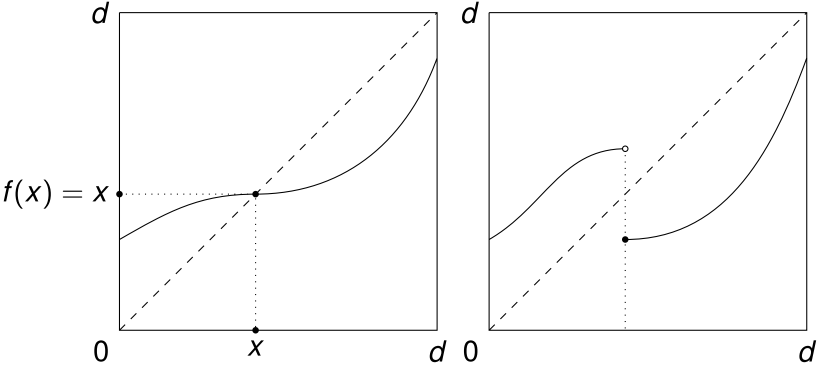
Given $\mathbf{A} \subset \mathbb{R}^n$ and the closed set $\mathbf{Y} \subset \mathbb{R}^n$, the correspondence $f : \mathbf{A} \rightarrow \mathbf{Y}$ is **upper hemicontinuous(uhc)** if it has a closed graph and the images of compact sets are bounded, that is for every compact set $\mathbf{B} \subset \mathbf{A}$ the set $f(\mathbf{B}) = \{\mathbf{y} \in \mathbf{Y} : \mathbf{y} \in f(\mathbf{x}) \textit{ for some } \mathbf{x} \in \mathbf{B}\}$ is bounded.

Given $\mathbf{A} \subset \mathbb{R}^n$ and a compact set $\mathbf{Y} \subset \mathbb{R}^n$, the correspondence $f : \mathbf{A} \rightarrow \mathbf{Y}$ is **lower hemicontinuous(lhc)** if for every sequence $\mathbf{x}^m \rightarrow \mathbf{x} \in \mathbf{A}$ with $\mathbf{x}^m \in \mathbf{A}$ for all $m$, and every $\mathbf{y} \in f(\mathbf{x})$, we can find a sequence $\mathbf{y}^m \rightarrow \mathbf{y}$ and an integer $M$ such that $\mathbf{y}^m \in f(\mathbf{x}^m)$ for $m > M$.

A correspondence is **continuous** if it is both uhc and lhc.

# Brouwer's fixed-point theorem

Suppose $D \subset \mathbb{R}^m$ is a nonempty, compact, convex set, and that $f : D \to D$ is a continuous function from $D$ to itself. Then $f(\cdot)$ has a fixed point; that is, there is an $\mathbf{x} \in D$ such that $\mathbf{x} = f(\mathbf{x})$.



# Kakutani's fixed-point theorem

Suppose $D \subset \mathbb{R}^m$ is a nonempty, compact, convex set, and that $f : D \to D$ is a continuous **upper hemicontinuous correspondence** from $D$ to itself with the property that the set $f(\mathbf{x}) \subset D$ is non-empty and convex for every $\mathbf{x} \in D$. Then $f(\cdot)$ has a fixed point; that is, there is an $\mathbf{x} \in D$ such that $\mathbf{x} = f(\mathbf{x})$.