Clarification [Lagrange multiplier in the mix]

Like I convinced you:

Theorem: $\frac{\partial f(x,y)}{\partial c} = \mu$

This implies that μ tells you that a 1-unit increase in c results in μ change in your objective function.

The solution of our optimization problem

$$\max_{x,y} f(x,y), \;\; s.t. \, h(x,y) = c$$

which is (x^*, y^*, μ^*) depends on the constraint c, so this solution is an implicit function of c: $x^* = x^*(c), y^* = y^*(c), \mu^* = \mu^*(c)$.

Your Lagrangian $\mathcal{L}(x,y,\mu) = f(x,y) + \mu\,[c - h(x,y)]$

If you *were* to take the derivative of this with respect of c:

$$\frac{\partial \mathcal{L}(x,y,\mu)}{\partial c} = \frac{\partial f(x,y)}{dc} + \frac{\partial}{\partial c}\,\mu\,[c - h(x,y)] = 0$$

i.e. $\frac{\partial f(x,y)}{dc} = \frac{\partial}{\partial c}\,\mu\,[c - h(x,y)]$

Although as an elementary way you can remember it as part of your constraint dropping off from your Lagrangian if you were to take the derivative with respect to c, that is NOT a proof – just a way to remember things (that's how I remember it for my sake, but I should probably not have shared it as part of the class). I should have been clearer that it was NOT a proof of the statement.

Proof.

$\mathcal{L}(x,y,\mu) = f(x(c),y(c)) + \mu(c)\,[c - h(x(c),y(c))]$

Take derivative with respect to x and set it equal to 0.

$\frac{\partial \mathcal{L}(x,y,\mu)}{\partial x} = \frac{\partial f(x(c),y(c))}{\partial x}$-$\mu(c)h\big(x(c),y(c)\big) = 0 \rightarrow \therefore \frac{\partial f(x(c),y(c))}{\partial x} = \mu(c)\frac{\partial h\left(x(c),y(c)\right)}{\partial x}$ **(1)**

Take derivative with respect to y and set it equal to 0.

$\frac{\partial \mathcal{L}(x,y,\mu)}{\partial y} = \frac{\partial f(x(c),y(c)}{\partial y}$-$\mu(c)h\big(x(c),y(c)\big) = 0 \rightarrow \therefore \frac{\partial f(x(c),y(c))}{\partial y} = \mu(c)\frac{\partial h\left(x(c),y(c)\right)}{\partial y}$ **(2)**

Since: $h(x(c),y(c)) = c$, differentiating that with respect of c yields:

$\frac{\partial h\left(x(c),y(c)\right)}{\partial x}.\frac{\partial x(c)}{\partial c} + \frac{\partial h\left(x(c),y(c)\right)}{\partial y}.\frac{\partial y(c)}{\partial c} = 1$ **(3)**

Take the derivative of your objective function with respect to c.

Now, $\frac{\partial f(x(c),y(c))}{\partial c} = \frac{\partial f\left(x(c),y(c)\right)}{\partial x}.\frac{\partial x(c)}{\partial c} + \frac{\partial f\left(x(c),y(c)\right)}{\partial y}.\frac{\partial y(c)}{\partial c}$

Plug in your 1st two results **(1)** and **(2)**

$$\frac{\partial f(x(c),y(c))}{\partial c} = \mu(c)\frac{\partial h\big(x(c),y(c)\big)}{\partial x}.\frac{\partial x(c)}{\partial c} + \mu(c)\frac{\partial h\big(x(c),y(c)\big)}{\partial y}.\frac{\partial y(c)}{\partial c}$$

$$= \mu(c)[\frac{\partial h\big(x(c),y(c)\big)}{\partial x}.\frac{\partial x(c)}{\partial c} + \frac{\partial h\big(x(c),y(c)\big)}{\partial y}.\frac{\partial y(c)}{\partial c}]$$

You may recognize now, the expression in the bracket is just the derivative of the constraint from **(3)**

$$= \mu(c)[1]$$
$$= \mu(c)$$

QED