

Math Review

Summer 2016

Topic 2

2. Basic topology of the reals

2.1. Sets, sequences and limits

Sets:

As you may recall, the basic set notations are:

U:	Union	$A \cup B = \{x: x \in A \text{ or } x \in B\}$
\cap :	Intersection	$A \cap B = \{x: x \in A \text{ and } x \in B\}$
C:	Complement	$A^c = \{x: x \notin A\}$
\setminus :	Difference	$A \setminus B = \{x: x \in A \text{ and } x \notin B\}$
\times :	Cartesian product	$A \times B = \{(a, b): a \in A \text{ and } b \in B\}$

Here, I quickly touch on the definitions for supremum and infimum.

Supremum and Infimum:

First, an **ordered set**, A , is a set in which an order is defined. An **order** on A , denoted $<$, has the following properties:

(i) If $x \in A$ and $y \in A$ then one, and only one, of the following statements is true:

$$x < y$$

$$x = y$$

$$y < x$$

(ii) For $x, y, z \in A$, if $x < y$ and $y < z$, then $x < z$

Let A be an ordered set and $A_1 \subset A$.

Upper bound:

Lower bound:

Example. Let $A = \mathbb{R}$, $A_1 = [-1, 1]$, $A_2 = (-\infty, 1]$, $A_3 = [-1, \infty)$.

Then -2 is a lower bound for both A_1 and A_3 .

Q: What would be an **upper bound** for both A_1 and A_2 ?

A:

Do we note anything different about the bounds for subset A_2 and the subset A_3 ?

Let A be an ordered set and $A_1 \subset A$.

- Suppose $\beta \in A$ is an upper bound of A_1 . If for all $\gamma < \beta$ there exists an $x \in A_1$, such that $x > \gamma$, then β is called the **least upper bound** of A_1 or the supremum of A_1 . The supremum can be expressed as:

$$\beta = \sup A_1$$

- Suppose $\alpha \in A$ is the lower bound of A_1 . If for all $\delta > \alpha$ there exists an $x \in A_1$ such that $x < \delta$, then we say that α is the **greatest lower bounded** of A_1 or the infimum of A_1 . The infimum can be expressed as:

$$\alpha = \inf A_1$$

For any given subset A_1 can have at most one α and one β . If A is not bounded above then we say that $\sup A = +\infty$.

Let's think about $\sup A_1$ and $\inf A_1$ from one of the examples above.

Sequences and limits:

A sequence is a function, $f(\cdot)$, defined on the set of natural numbers, \mathbb{N} . We have $f(n) = x_n$ for $n \in \mathbb{N}$ and usually denote the entire sequence by the symbol $\{x_n\}$, or x_1, x_2, x_3, \dots . The values of $f(n)$, or x_n , are called the terms of the sequence.

Can we think of a couple of examples of a sequence?

Converging sequence- Definition:

Theorem: If the sequence $\{x_n\}$ converges then the limit of $\{x_n\}$ is unique.

Consider the following properties of sequences:

Suppose that for the real number sequences x_n and y_n we have $x_n \rightarrow x$ and $y_n \rightarrow y$. Then:

$$(i) \quad \lim cx_n = cx$$

$$(ii) \quad \lim(x_n + y_n) = x + y$$

$$(iii) \quad \lim(x_n y_n) = xy$$

$$(iv) \quad \lim \left(\frac{x_n}{y_n} = \frac{x}{y} \right) \text{ if for all } n \in \mathbb{N} \text{ we have } y_n \neq 0, y \neq 0.$$

Cartesian Plane:

The set of real numbers is denoted by the symbol \mathbb{R} and is defined as:

$$\mathbb{R} = \{x: -\infty \leq x \leq \infty\}$$

Defining the set product $\mathbb{R} \times \mathbb{R}$:

2.2 Open and Closed Sets

You will use the term neighborhood or a ball around a point often in Micro. This is really getting at limits. A limit is special case of an accumulation point (or cluster point): A point p is an accumulation point of a sequence, $\{x_n\}$, if for every ball around p , $(p - \epsilon, p + \epsilon)$, there are infinitely many elements of the sequence, x_n , where $x_n \in (p - \epsilon, p + \epsilon)$.

What is a ball around a point?

More generally, any n-tuple, or vector, is just an n-dimensional ordered tuple (x_1, \dots, x_n) and can be thought of as a "point" in n-dimensional Euclidian space or "n-space". A n-space is defined as the set product:

$$\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \dots \times \mathbb{R} = \{(x_1, \dots, x_n): x_i \in \mathbb{R}, i = 1, \dots, n\}$$

How do we define \mathbb{R}_+^n ?

Metric space and sets:

A metric space is a set with a notion of distance defined among the points within the set. For any two points x^1 and x^2 in \mathbb{R} , denote the distance between them as

$$d(x^1, x^2) = |x^1 - x^2|.$$

You may recall that in \mathbb{R}^2 , this takes the form of:

$$d(x^1, x^2) = \sqrt{(x_1^2 - x_1^1)^2 + (x_2^2 - x_2^1)^2}.$$

In \mathbb{R}^n , this takes the form of:

$$d(x^1, x^2) = \sqrt{(x_1^2 - x_1^1)^2 + (x_2^2 - x_2^1)^2 + \cdots + (x_n^2 - x_n^1)^2}.$$

Pulling everything together, we are ready to tackle some key definitions:

Open ϵ - ball.

Closed ϵ - ball.

Open sets in \mathbb{R}^n : $S \subset \mathbb{R}^n$ is an open set if, for all $x \in S$, there exists some $\epsilon > 0$ such that $B_\epsilon(x) \subset S$.

Closed sets in \mathbb{R}^n : S is a closed set if its complement, S^c is an open set

2.3. Convex Set

What is a convex set?

Example. Let $S = [1, 13]$. Consider any two points in the set, $x^1 = 2$ and $x^2 = 8$. Define the convex combination $z = tx^1 + (1 - t)x^2$. For any value of $t \in [0, 1]$, for any two points in S , we have $z \in S$. If $t = \frac{1}{2}$, $z = \left(\frac{1}{2}\right) * 2 + \left(1 - \frac{1}{2}\right) * 8 = 5 \in S$.

Example. Consider two vectors in $S \in \mathbb{R}$, denoted by $x^1 = (x_1^1, x_2^1)$ and $x^2 = (x_1^2, x_2^2)$.

Q: How would you define this convex combination?

A:

Example. The Intersection of Convex Sets is Convex. Let S and T be convex sets in \mathbb{R}^n . Then show that $S \cap T$ is a convex set.

2.4 Bounded sets in \mathbb{R}^n

A set in \mathbb{R}^n is called bounded if it is entirely contained within some ϵ – *ball*. (either open or closed). That is, S is bounded if there exists some $\epsilon > 0$, such that $S \subset B_\epsilon(x)$ for some $x \in \mathbb{R}^n$.

2.5 Compact sets

One can think of the concept of boundedness basically means that the set has finite size. Then, a set S in \mathbb{R}^n is called compact if and only if it is closed and bounded.

2.6 Functions and properties of relations

Binary relation: A binary relation (R) is any collection of ordered pairs between the sets S and T .

Let S be the colors {red, yellow, green}, and T the set of fruits {apple, banana, pear}. The statement “is the color of” defines the relation R between the two sets. We denote this as sRt .

The following definitions are critical for your understanding of many concepts in Micro theory.

Completeness. A relation R on S is complete if, for all elements x and y in S , xRy or yRx .

Transitivity. A relation R on S is transitive if, for any three elements x, y and z in S , xRy and yRz , implies xRz .

In consumer theory, you will come across something called a preference relation, which is binary in nature. For example, if you say xPy , you mean x is preferred to y .

Quick Q : What is P equivalent to here from the above definition?

Professor Glewwe will cover this part in details and this one is quite simple.

Q: You have the following set $X = \{x, y, z\}$. Your preference relationships are given by: xPy , yPz , and zPx . Is it complete? How about transitive?

A:

Function: A function is a relation that associates each element of one set with a single, unique element of another set. We say that the function f is a mapping from one set D to another set R and write $f: D \rightarrow \mathbb{R}$.

2.7 Continuity

What is a continuous function?

Intuitively, a function is continuous if you can trace the graph of the entire function without ever lifting your pencil from the page. For your purposes, apply the following theorem to show continuity:

Theorem: The function $f: \mathbb{R} \rightarrow \mathbb{R}$ is continuous at $p \in \mathbb{R}$ if and only, for every sequence $x_n \rightarrow p$, we have $\lim f(x_n) = f(p)$. The function f is then continuous on R if and only if

$$x_n \rightarrow p$$

it is continuous for all $p \in \mathbb{R}$.

This definition is clearer with an exercise/example:

Q: Show that the function $f(x) = ax + b$ is continuous using the above theorem

Hint: Start by taking the limit of $f(x_n)$.

A:

What are some other properties of continuous functions?

Q: Can we think of examples where a function, say $f(x)$ is *not continuous* at a point c ?

A:

We now have all the tools to be able to try a more relevant example of **convex set** from Production. There are many more examples that you now equipped to try, but time is of essence here. Plus, you will the next year to see many of these yourself.

The rationale of this part is show you that what we are currently reviewing translate directly into material that you see and learn in the coming year.

Couple of definitions that we will discuss to make sure we are catching notations correctly:

q : Vector of outputs

z : Vetor of inputs/factors

PPS is the Production Possibilities Set, that is, the technology.

IRS is the Input Requirement Set or all combinations of input capable of producing a combination of output. It is defined as follows:

$$IRS(q) = \{z \in \mathbb{R}_+^N : (q, -z) \in PPS\}$$

You are told that the input requirement set if *convex* in **z** .

Let's write out a statement or jot down to the best of our ability things that help us show that this set is convex. No need for a proof. I would encourage you to try it out before I start going through the steps!

Exercise for home:

Let a collection of L commodities vectors x be given as:

$$X = R_+^L = \{x_l \geq 0 \text{ for all } l = 1, 2, \dots, L\}$$

You can view L commodities as food, clothing, entertainment etc. Each x will have multiple elements, for instance food may contain dairy, candy, fresh fruits, and so on.

We can call an element x^1 of this set as a “bundle.”

1. One feature of this consumption set is that it is convex. Define a new bundle such that it is also an element of this set.

2. Let price be p and wealth be w . Then a person's budget is given by $p \cdot x \leq w$.

Let the budget set be defined as: $B = \{x \in R_L : p \cdot x \leq w\}$. Show that this budget set is convex. (*Hint: use the fact that you know that the consumption set is convex to begin with*).