4. Derivatives II

Mean Value Theorem

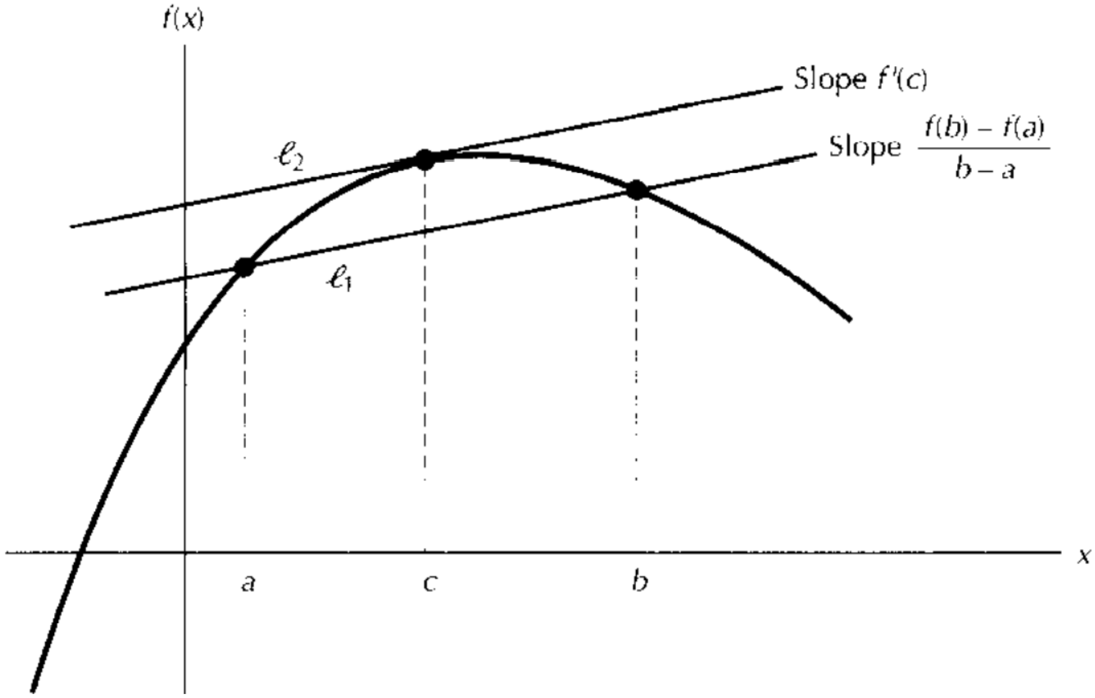
The theorem (Symon and Blume page 824) is as follows:

> **Theorem 30.3 (Mean Value Theorem)** Let $f: U \to \mathbf{R}^1$ be a C^1 function on a (connected) interval U in $\mathbf{R}^1$. For any points $a, b \in U$, there is a point c between a and b so that
>
> $$f(b) - f(a) = f'(c)(b - a). \tag{2}$$

This can be written as:

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

That is, there is a point c at which the slope is equal to that of the line connecting a and b.

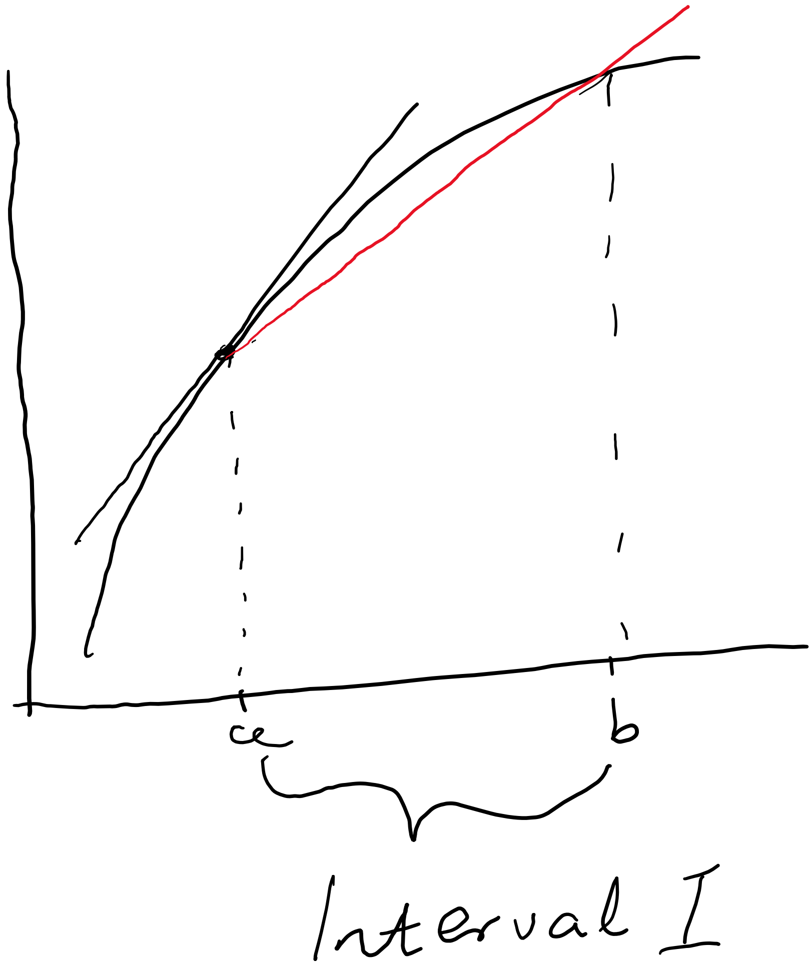


What can be a possible economic implication of point c?

Calculus criteria for convexity and concavity

If we have an interval from a to b, denoted by I, then a function is concave if:

$f'(a)(b-a) \geq f(b) - f(a)$ for all $a, b \in I$



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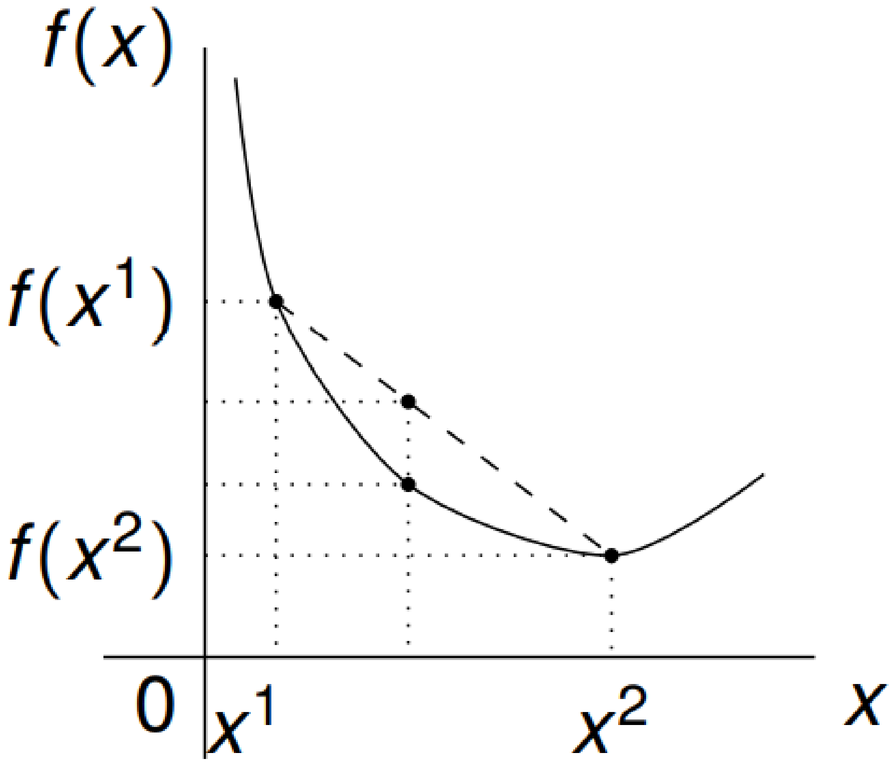


Second order derivatives and concavity

A function is concave if $f''(x) \geq 0$

A function is convex if $f''(x) \leq 0$

Critical points

These are points at which $f'(x) = 0\ or\ undefined$.

A point is a local maximum/minimun if $f'(0) = 0$

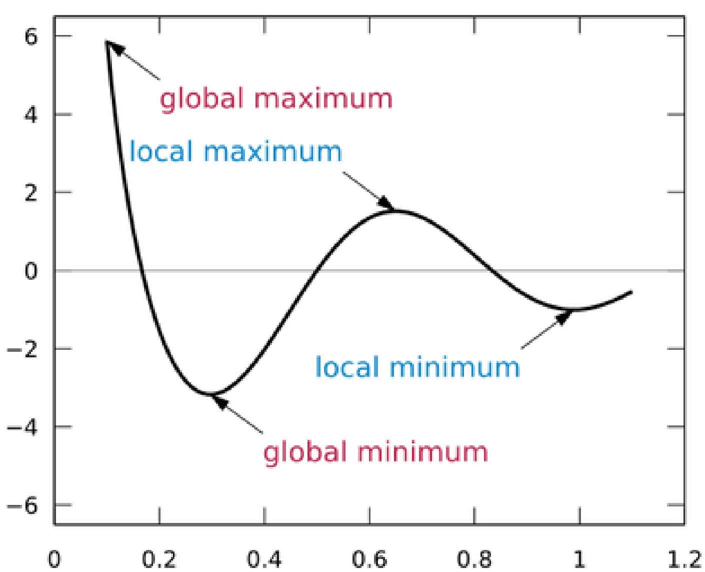
A point is a global maximum/minimum if $f(a) > or < f(x) \quad for\ all\ x \neq a$

Fill the blanks below:

If $f'(x_0) = 0$ and $f''(x_0) < 0$, then x_0 is ______

If $f'(x_0) = 0$ and $f''(x_0) > 0$, then x_0 is ______

If $f'(x_0) = 0$ and $f''(x_0) = 0$, then x_0 is ______



Example:

A monopoly operates in a market with a demand curve described by $P = 10 - Q$ and faces marginal costs given by $C = 0.25Q$. P is the price that consumers are willing to pay and Q is the quantity produced by the monopolist. What is the quantity to be produced by the monopolist?

Integrals/Antiderivatives

F(x) is the antiderivative of f(x) if f(x) is the derivative of F(x).

It is written as:

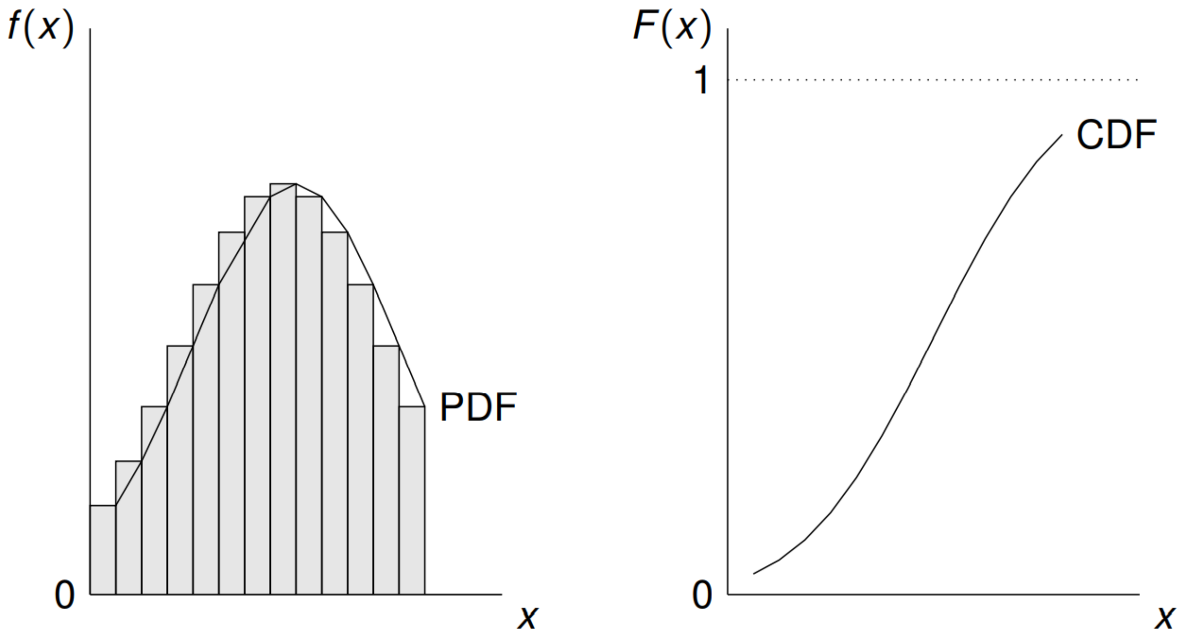
$$F(x) = \int f(x)dx$$

For numbers a and b, the **definite integral** of $f(x)$ from a to b is $F(b) - F(a)$, where $F(x)$ is an antiderivative of f.

$$\int_a^b f(x)dx = F(b) - F(a), \textit{ where } F' = f$$

If we divide the interval (a, b) to N subintervals and denote each end point as x_i. The **Reimann Sum** is

$$\lim_{\Delta \to 0} \sum_{i=1}^{N} f(x_i)\Delta = \int_a^b f(x)dx$$



When else do we normally use integrals?

Integrals of common functions:

$$\int af(x)\,dx = a\int f(x)\,dx \qquad \int (f+g)\,dx = \int f\,dx + \int g\,dx$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \qquad \int \frac{1}{x}\,dx = \ln x + C$$

$$\int e^x\,dx = e^x + C \qquad \int e^{f(x)} f'(x)\,dx = e^{f(x)} + C$$

$$\int (f(x))^n f'(x)\,dx = \frac{1}{n+1}(f(x))^{n+1} + C \quad (n \neq -1)$$

$$\int \frac{1}{f(x)} f'(x)\,dx = \ln f(x) + C$$

Examples

From the monopolist example above, use integration to calculate:

(i) The consumer surplus
(ii) DWL

Calculate

$$\int (4x^2 + x^1 - \tfrac{3}{x})\,dx$$

Integration by parts

It follows from the product rule of differentiation that:

$$\int u(x) \cdot v'(x)\, dx = u(x) \cdot v(x) - \int u'(x) \cdot v(x)\, dx$$

It is also expressed as: $udv = duv - vdu \quad or \quad \int udv = uv - \int vdu$

Example

Find $\int \ln(x) dx$

Example

Find $\int xe^{2x} dx$