Math Review
Summer 2017

Topic 8 – Part II

8. Optimization Part II.

We have covered multiples examples of setting up Lagrangian and woking with Kuhn-Tucker conditions in Part I of Optimization. Now we will cover some miscellaneous examples of optimization and get you some practice.

Exercise: This is a standard problem from production that you are NOW equipped to solve. We will walk through the steps together and figure out how to approach a problem given to us.

Consider the following producer's revenue maximization problem assuming competitive output prices $p.$ z are the inputs, $z = 1, . . , N$ and q the outputs, $q = 1 \dots M.$

$$\max_{q \geq 0} p.q$$

subject to

$$\frac{\left[\sum_{m\prime=1}^{M} q_{m\prime}^{\rho}\right]^{1/\rho}}{\left[\sum_{n=1}^{N} z_n^{\alpha}\right]^{1/\beta}} = 1$$

1. Write down the Lagrangian for this problem.
2. Derive the first-order conditions for a given output $m.$
3. Find q_m. Note that you may need to take the FOC for another random output q_1 in order to derive q_m as a function of prices and inputs.
3. Write down the revenue function using your optimal solution by multiply prices by optimal outputs, i.e. $R(p, z) = \sum_{m=1}^{M} p_m q_m$
4. Show that this revenue function is homogeneous of degree α/β in inputs, $z.$

<u>Optimizing with expected values</u>

You will run into some optimization with expected values in consumer theory and game theory. We will do 1 example together and move on. Consider this part of a question:

Suppose there are 2 firms in a market, Firm 1 and Firm 2. We want to solve for the Bayesian Nash equilibrium in a duopoly setting where firm 2 has entered the market and believes that there is a certain probability that firm 1 has high costs and a certain probability that firm 1 has low costs. We want to solve for the expected level of profit for firm 2. The marginal cost for Firm 2 is: 12.

Assume the inverse market demand curve is given by $P(Q) = 24 - Q$, where P is price and $Q = q_1 + q_2$ is market quantity produced.

We can write out Firm's 2 problem as:

$$\max E(24 - 12 - q_1 - q_2)q_2$$

The FOC is given as:

$$12 - \frac{\partial E(q_1)}{\partial q_2} - 2q_2 = 0$$

Suppose, you further know that Firm 1 has costs of $c_i(q_1) = c_i q_1$, where q_1 is the quantity produced by firm 1, and marginal cost, c_i, equals c_H or c_L. Assume that high marginal cost are $c_H = 12$ and low marginal cost are $c_L = 6$. Firm 2 can assume high costs occur with probability 0.5 and low costs occur with probability 0.5. Solve for q_2.

We want to be able to find a workable solution for the expected value term.

Firm 1 with high cost:

$$\max(24 - 12 - q_1^H - q_2)q_1^H$$
$$\therefore 12 - 2q_1^H - q_2 = 0$$
$$q_1^H = \frac{12 - q_2}{2}$$

Firm 1 with low cost:

$$\max(24 - 6 - q_1^L - q_2)q_1^L$$
$$\therefore 18 - 2q_1^L - q_2 = 0$$
$$q_1^L = \frac{18 - q_2}{2}$$

Naturally, from what we know,

$$E(q_1) = \frac{1}{2}q_1^H + \frac{1}{2}q_2^L$$
$$= \frac{1}{2}\left(\frac{12 - q_2}{2}\right) + \frac{1}{2}\left(\frac{18 - q_2}{2}\right)$$
$$= \frac{15 - q_2}{2}$$

From Firm's 2 FOC, we have:

$$12 - \frac{\partial E(q_1)}{\partial q_2} - 2q_2 = 0$$

$$12 - \frac{\partial}{\partial q_2}(\frac{15 - q_2}{2}) - 2q_2 = 0$$
$$q_2 = 2$$

Besides knowing how to take derivatives with expected values – which is pretty straightforward, the goal is to be able to read paragraphs of information and set the problem up. Often, that is the hardest part, outweighing the grinding through math.

Practice setting up a problem under uncertainty

Consider a risk averse agent. He faces a health risk of D (he/she has to go to the hospital and pay D, and he/she'll be fine) with probability π. He/she can buy insurance at price q (that is he/she can buy at cost q a contract that pays 1 dollar if he/she has to go to the Hospital). This person starts with W, which is the initial wealth.

Set up the expected utility problem for this person. How many units of the contract will the agent buy is the price is $q = \pi$.

Think about an agent who buys x units of insurance. If they get ill, the amount of money that they will have is $W - D - qx + x$, where W is initial wealth. If they do not get ill, their wealth will be $W - qx$. The expected utility of such an agent is:

$$U = \pi u(W - D - qx + x) + (1 - \pi)u(W - qx)$$

where *u(.) is some arbitrary decreasing marginal returns utility function.* The agent wishes to choose x to maximize their expected utility. We therefore take the derivative of the above function with respect to x and set it equal to zero.

$$\frac{\partial U}{\partial x} = \pi(1 - q)u'(W - D - qx + x) - (1 - \pi)qu'(W - qx) = 0$$
$$\rightarrow \pi(1 - q)u'(W - D - qx + x) = (1 - \pi)qu'(W - qx)$$

The question tells us that $q = \pi$, so we have:

$$\pi(1 - \pi)u'(W - D - \pi x + x) = (1 - \pi)\pi u'(W - \pi x)$$
$$u'(W - D - \pi x + x) = u'(W - \pi x)$$

We would like to conclude from this that $W - D - \pi x + x = W - \pi x$, but we need to make sure that if the slope of the function is equal, then the argument of the function is also equal. This is only true if the slope of the function is not the same for any two points. We need to assume that the second derivative is less than 0 (always).

$$W - D - \pi x + x = W - \pi x - D + x = 0$$
$$D = x$$

This risk averse agent will therefore buy enough insurance to completely cover their risk.

Optimization with 'odd' forms:

Complements

You are given a utility function:

$$u(x, y) = \min(2x, y)$$
subject to
$$p_1 x + p_2 y = m$$

We start by noting that this utility function is not differentiable at the kink $2x = y$. The optimal allocation (x, y) must satisfy $2x = y$ since it would be the cheapest way to achieve.

Then, we can plug y into the budget constraint:

$$p_1 x + p_2 2x = m$$
$$x(p_1 + 2p_2) = m$$
$$x = \frac{m}{p_1 + 2p_2}$$

And accordingly,
$y = 2x = \frac{2m}{p_1+2p_2}$.

The optimal solution is given by: $(x^*, y^*) = (\frac{m}{p_1+2p_2}, \frac{2m}{p_1+2p_2})$

Q: This should be very similar, just to give you a tiny bit of practice. You have a utility function given by: $u(x_1, x_2) = \min\{\alpha x_1, \beta x_2\}$. The price of good 1 is p_1 and price of good 2 is p_2, with wealth, w. Solve for this consumer problem, find x_1 and x_2.

$\mathcal{A}$:

$$u(x_1, x_2) = \min\{\alpha x_1, \beta x_2\}$$
subject to
$$p_1 x_1 + p_2 x_2 = w$$

We set $\alpha x_1 = \beta x_2 \rightarrow x_1 = \frac{\beta x_2}{\alpha}$

Plugging into the constraint:

$$p_1 \frac{\beta x_2}{\alpha} + p_2 x_2 = w$$
$$x_2(p_1 \frac{\beta}{\alpha} + p_2) = w$$
$$x_2 = \frac{w}{p_1 \frac{\beta}{\alpha} + p_2}$$

Then, $x_1 = \frac{\beta}{\alpha} x_2 \rightarrow \frac{\beta}{\alpha} . \frac{w}{p_1 \frac{\beta}{\alpha} + p_2}$

Substitutes

You are given this maximization problem denoted as:

$$\max_{c,s} u(c,s) = 3c + 2s$$
s.t
$$p_c c + p_s s = m$$

$$\mathcal{L} = 3c + 2s + \lambda(m - p_c c + p_s s)$$

The FOC are given as:

[c] $3 = \lambda p_c$
[s] $2 = \lambda p_s$
[λ] $m = p_c c + p_s s$

From the first and second conditions, we have:

$$\frac{p_s}{p_c} = \frac{2}{3}$$

If we are to think of what this looks like, this budget constraint has the exact same slope as the utility function. In other words, Jack may decide any point on the utility function and get the same happiness from skiing and camping. However, this price ratio can change, so what happens then? This is what we want to present as our solution.

What if $\frac{p_s}{p_c} > \frac{2}{3}$? The budget constraint gets steeper. Have c on the y-axis, s on the x-axis. What if $\frac{p_s}{p_c} < \frac{2}{3}$? We have to present all solutions:

$$x_s(p_s, p_c, m) = \begin{cases} \frac{m}{p_s} & if \frac{p_s}{p_c} < \frac{2}{3} \\ \left[0, \frac{m}{p_s}\right] & if \frac{p_s}{p_c} = \frac{2}{3} \\ 0 & if \frac{p_s}{p_c} > \frac{2}{3} \end{cases}$$

$$x_c(p_s, p_c, m) = \begin{cases} 0 & if\ \frac{p_s}{p_c} < \frac{2}{3} \\ \left[0, \frac{m}{p_c}\right] & if\ \frac{p_s}{p_c} = \frac{2}{3} \\ \frac{m}{p_c} & if\ \frac{p_s}{p_c} > \frac{2}{3} \end{cases}$$

Q: Can you try with the following example?

You are given this maximization problem denoted as:

$$u(x_1, x_2) = 3x_1 + 5x_2$$

s.t

$$p_1 x_1 + p_2 x_2 = m$$

8.3. Envelope Theorem

Sometimes, the objective function f is also a function of some of the exogenous parameters, for example, i.e. $f(x;\ a)$. The envelope theorem is a general principle describing how the value of an optimization problem changes as the parameters of the problem change. In other words, we are looking at how the maximal value of a function depends on some parameters. You will most likely see envelope theorem applied in 'dual' problem.

Let's motivate this with an example:

Example. Let $f\ (x;\ a)\ =\ -x^2\ +\ 2ax\ +\ 4a^2$ be a function in one variable x that depends on a parameter a. For a given value of a, the stationary points of f is given by

$$\frac{\partial f}{\partial x} = -2x + 2a = 0 \Leftrightarrow x = a$$

The optimal value function $f\ *\ (a)\ =\ f\ (x\ *\ (a);\ a)\ =\ -a^2\ +\ 2a^2\ +\ 4a^2\ =\ 5a^2$ gives the corresponding maximum value.

Now, the derivative of the value function is given by

$$\frac{\partial f\ *}{\partial a} = \frac{\partial}{\partial a} f\ (x\ *\ (a);\ a) = \frac{\partial}{\partial a} 5a^2\ = 10a$$

On the other hand, we see that

$f\ (x;\ a) = -x^2\ +\ 2ax\ +\ 4a^2\ gives \frac{\partial f}{\partial a}\ =\ 2x\ +\ 8a$. This evaluated at $x = a$, gives us:

$\frac{\partial f}{\partial a}\ =\ 2a +\ 8a = 10a.$

The fact that these computations give the same result is not a coincidence, but a consequence of the envelope theorem for unconstrained optimization problems.

Envelope theorem for unconstrained maxima

Theorem: Let $f\ (x;\ a)$ be a function in n variables $x_1, \ldots, x_n$ that depends on a parameter a. For each value of a, let $x^*(a)$ be a maximum or minimum point for $f\ (x;\ a)$. Then

$\frac{\partial}{\partial a} f\ (x\ *\ (a);\ a) = \frac{\partial f}{\partial a}$ for $x = x\ *(a)$

Envelope theorem for constrained maxima

Theorem: Let $f, h_1, h_2 \dots, h_k$ be continuously differentiable functions. Let $x^*(a) = (x_1^*(a), x_2^*(a), \dots, x_3^*(a))$ be the solution to:

$$\max_{\{x_1, x_2, \dots, x_n\}} f(x_1, x_2, \dots, x_n; a)$$
subject to:
$$h_1(x_1, x_2, \dots, x_n; a) = 0$$
$$h_2(x_1, x_2, \dots, x_n; a) = 0$$
$$\vdots$$
$$h_k(x_1, x_2, \dots, x_n; a) = 0$$

for any fixed choice of the exogenous parameter a.

Suppose that the Jacobian matrix associated with the equality constraints has maximum rank. If $x^*(a), \lambda_1(a), \dots \lambda_k(a)$ are all continuously differentiable functions of a, then:

$$\frac{\partial f(x^*(a); a)}{\partial a} = \frac{\partial \mathcal{L}}{\partial a}(x^*(a), \lambda(a); a)$$

You will learn eventually about Hotelling's Lemma and Shephard's lemma and their applications to economic theory/results. The Envelope Theorem is what pulls a lot of these together.

Example and exercise. Consider this 'dual' cost minimization problem given a fixed level of utility.

$$\min_{x,y} p_x x + p_y y$$

Subject to

$$u_o = xy$$

The Lagrangian, $\mathcal{L} = p_x x + p_y y + \lambda(u_o - xy)$

$$\frac{\partial \mathcal{L}}{\partial x} = p_x - \lambda y = 0$$
$$\frac{\partial \mathcal{L}}{\partial x} = p_y - \lambda x = 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = u_o - xy = 0$$

Solving for x, y and λ, we have:

$$x^H = \left(\frac{p_y u_0}{p_x}\right)^{\frac{1}{2}}$$
$$y^H = \left(\frac{p_x u_0}{p_y}\right)^{\frac{1}{2}}$$
$$\lambda^H = \left(\frac{p_x p_y}{u_0}\right)^{\frac{1}{2}}$$

Your TA may ask you to check for sufficient conditions for a minimum and you may have to construct a 3x3 Hessian. I didn't have to do one, and I hope you do not too. Just know it exists and is do-able. We can substitute x^H and y^H into the objective function to yield the minimum value for cost:

$$p_x x^H + p_y y^H = p_x \left(\frac{p_y u_0}{p_x}\right)^{\frac{1}{2}} + p_y \left(\frac{p_x u_0}{p_y}\right)^{\frac{1}{2}}$$
$$= (p_x p_y u_o)^{\frac{1}{2}} + (p_x p_y u_o)^{\frac{1}{2}}$$
$$= 2(p_x p_y u_o)^{\frac{1}{2}} = E(p, u_0)$$

Q: Can you check for the Envelope Theorem holds?

$$\frac{\partial E(p, u_0)}{\partial x} = x^H$$
$$\frac{\partial E(p, u_0)}{\partial y} = y^H$$
$$\frac{\partial E(p, u_0)}{\partial u_0} = \lambda^H$$

This is what we call the Shephard's lemma which uses the Envelope Theorem to work. The reasoning might feel a little circular, but this is a powerful tool that you will often use. You do not have to go through yet another optimization problem if you have the cost function given. Simply use Envelope Theorem and you have your dual results.

Bordered Hessian

The bordered Hessian is a second-order condition for local maxima and minima in Lagrange problems.

Suppose we have a function of two variables $f(x_1, x_2)$. Assume that $\nabla f(x) \neq 0$ for every x. $f(.)$ is strictly quasiconcave if the determinant of the bordered Hessian of the matrix is negative definite:

$$\begin{vmatrix} f_{11}(x_1, x_2) & f_{12}(x_1, x_2) & f_1(x_1, x_2) \\ f_{21}(x_1, x_2) & f_{22}(x_1, x_2) & f_2(x_1, x_2) \\ f_1(x_1, x_2) & f_2(x_1, x_2) & 0 \end{vmatrix} > 0$$

You can also check that:

$$2f_1(x_1, x_2)f_2(x_1, x_2)f_{12}(x_1, x_2) - [f_1(x_1, x_2)]^2 f_{22}(x_1, x_2) - [f_2(x_1, x_2)]^2 f_{11}(x_1, x_2) > 0$$

$f(.)$ is quasi-concave if $2f_1(x_1, x_2)f_2(x_1, x_2)f_{12}(x_1, x_2) - [f_1(x_1, x_2)]^2 f_{22}(x_1, x_2) - [f_2(x_1, x_2)]^2 f_{11}(x_1, x_2) \geq 0$

The bordered Hessian to test quasi-concavity/convexity for n variables:

$$|H| = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} & f_1 \\ f_{21} & f_{22} & \cdots & f_{2n} & f_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} & f_n \\ f_1 & f_2 & \cdots & f_n & 0 \end{bmatrix}$$

This can also be expressed differently with the 0 at the top.

Bordered Hessian in Lagrange problems

The bordered Hessian is a second-order condition for local maxima and minima in Lagrange problems. We consider the simplest case, where the objective function $f(x)$ is a function in two variables and there is one constraint of the form $g(x) = b$. In this case, the bordered Hessian is the determinant:

$$|B| = \begin{vmatrix} 0 & g_1{}' & g_2{}' \\ g_1{}' & \mathcal{L}'_{11} & \mathcal{L}'_{12} \\ g_2{}' & \mathcal{L}'_{21} & \mathcal{L}'_{22} \end{vmatrix}$$

Theorem. Consider the following local Lagrange problem: Find local maxima/minima for $f(x_1, x_2)$ subject to $g(x_1, x_2) = b$. Assume that $x* = (x_1^*, x_2^*)$ satisfy the constraint $g(x_1^*, x_2^*) = b$ and that $(x_1^*, x_2^*, \lambda^*)$ satisfy the first order conditions for some Lagrange multiplier λ^*. Then we have:

1 If the bordered Hessian $B(x_1^*, x_2^*, \lambda^*) < 0$, then (x_1^*, x_2^*) is a local minima for $f(x)$ subject to $g(x) = b$.
2 If the bordered Hessian $B(x_1^*, x_2^*, \lambda^*) > 0$, then (x_1^*, x_2^*) is a local maxima for $f(x)$ subject to $g(x) = b$.

Find local maxima/minima for $f\ (x_1, x_2) = \ x_1 \ + \ 3x^2$ subject to the constraint $g(x_1, x_2) = x_1^2 + x_2^2 = 10.$ Find the bordered Hessian for the Lagrange problem and determine whether your solutions are minimum/maximum or none.

The Lagrangian is $L \ = \ x_1 \ + \ 3x^2 \ - \ \lambda(x_1^2 + x_2^2 - 10.)$. The solutions should be: $(x_1, x_2, \lambda) \ = \ (1, 3, 1/2)$ and $(x_1, x_2, \lambda) \ = \ (-1, -3, -1/2)$.

We compute the bordered Hessian as

$$|B| = \begin{vmatrix} 0 & g_1{}' & g_2{}' \\ g_1{}' & \mathcal{L}_{11}' & \mathcal{L}_{12}' \\ g_2{}' & \mathcal{L}_{21}' & \mathcal{L}_{22}' \end{vmatrix}$$

$$|B| = \begin{vmatrix} 0 & 2x_1 & 2x_2 \\ 2x_1 & -2\lambda & 0 \\ 2x_2 & 0 & -2\lambda \end{vmatrix}$$
$$= -2x_1(-4x_1\lambda) + \ 2x_2(4x_2\lambda) = 8\lambda(x_1^2 + x_2^2)$$

and since $x_1^2 + x_2^2 = 10$ by the constraint, we get $B \ = \ 80\lambda$. We solved the first order conditions and the constraint earlier, and found the two solutions $(x_1, x_2, \lambda) \ = \ (1, 3, 1/2)$ and $(x_1, x_2, \lambda) \ = \ (-1, -3, -1/2)$. So the bordered Hessian is B = 40 in $x \ = \ (1, 3)$, and B = −40 in x = (−1, −3). Using the following theorem, we see that (1, 3) is a local maximum and that (−1, −3) is a local minimum for $f\ (x_1, x_2)$ subject to $x_1^2 + x_2^2 = 10$.