Day 5 Notes

Friday, August 13, 2021 9:05 AM

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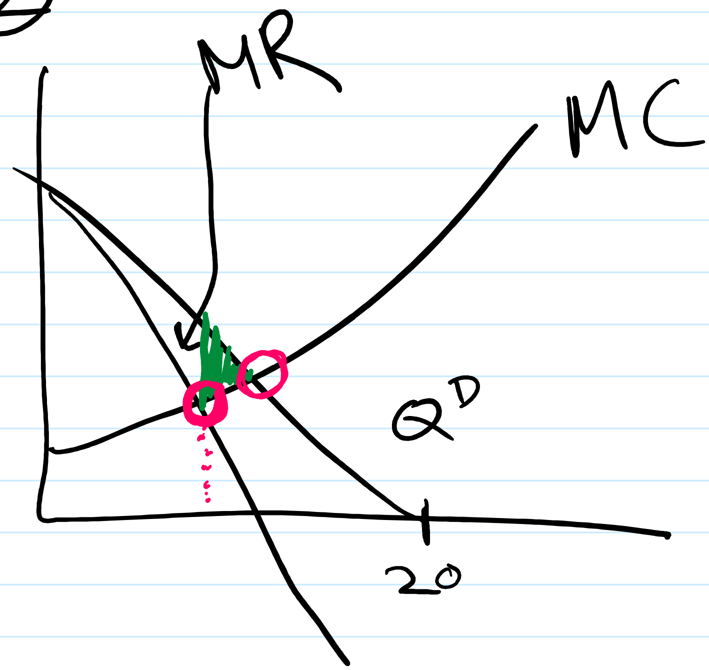


Marg Rev:

Revenue = $P \cdot Q$

$Q = 20 - 6P$

inverse demand:

$$\frac{Q - 20}{-6} = P$$

$$P(Q) = \frac{20}{6} - \frac{Q}{6}$$

Rev: $Q \cdot \left(\frac{20}{6} - \frac{Q}{6}\right)$

Marg. Rev: $\frac{d}{dQ}(\text{Rev})$

$$\frac{d}{dQ}$$

$$= \frac{20}{6} - \frac{2Q}{6}$$

$$MC = \frac{d}{dQ}TC = \frac{3Q^2}{3} = Q^2$$

Lower endpoint:

$$MR = MC$$

$$\frac{20}{6} - \frac{2Q}{6} = Q^2$$

$$Q^2 + \frac{Q}{3} - \frac{20}{6} = 0$$

Roots:

$$Q = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-\frac{1}{3} \pm \sqrt{\left(\frac{1}{3}\right)^2 - 4 \cdot 1 \cdot \left(-\frac{20}{6}\right)}}{2}$$

$$= \frac{-\frac{1}{3} \pm \sqrt{\frac{1}{9} + \frac{80}{6}}}{2}$$

$$= \frac{-\frac{1}{3} \pm \sqrt{13.4}}{2}$$

$$\overline{\qquad} \quad 2$$

$$= \frac{-\frac{1}{3} \pm 3.66}{2}$$

$$Q = \frac{-\frac{1}{3} + 3.66}{2}$$

$$\approx 1.7 = Q^{monop}$$

Also solve $MC =$ Inverse demand to get Q^{comp}

Next: integrate

$$\int_{Q^{monop}}^{Q^{comp}} \left[\left(\frac{20}{6} - \frac{Q}{6}\right) - Q^2\right] dQ$$

$$= \frac{20}{6}Q - \frac{Q^2}{12} - \frac{Q^3}{3}\Bigg]_{Q^{monop}}^{Q^{comp}}$$

Euclidean space

A point in n-space is a vector.

$(1, 2, 3, 4, 5)$

$$\begin{bmatrix} f(x) \\ g(x) \\ h(x) \end{bmatrix} \quad \begin{bmatrix} f & g & h \end{bmatrix}$$

In single variable calculus, $f(x)$, $x \in \mathbb{R}$.

Now: $f(x)$, $x \in \mathbb{R}^N$.

x is a vector.

$\vec{x}$ vector

Dot product

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = x_1 y_1 + x_2 y_2 + x_3 y_3$$

$$= x^T y$$

Matrix mult:

$$AB = \begin{bmatrix} (\text{row}_i \text{ of } A) \cdot (\text{col}_j \text{ of } B) \\ \\ \end{bmatrix}$$

$$f : \mathbb{R}^n \to \mathbb{R}$$

$$u : \mathbb{R}^{k \cdot m} \to \underline{\mathbb{R}^m}$$

$$\begin{bmatrix} u_1(x^1) \\ u_2(x^2) \\ \vdots \\ u_m(x^m \end{bmatrix} \qquad (x^1 \in \mathbb{R}^k)$$

$[u_m(x \ldots]$

$$f(x_1, x_2) = x_1^2 + 3x_1x_2 - x_2^2$$

$$\frac{\partial f(x_1, x_2)}{\partial x_1} = \underline{2x_1 + 3x_2}$$

$$\frac{\partial f}{\partial x_2} = -2x_2 + 3x_1$$

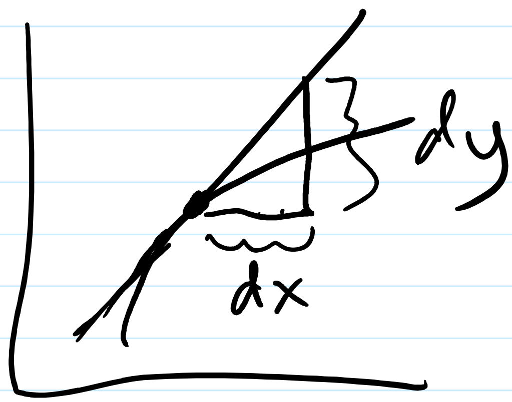
$$y = f(x)$$

$$dy \equiv f'(x)dx$$ dx is a free variable

<u>Now</u>

$$z = f(\overbrace{x_1, x_2, \ldots, x_n}^{X})$$

$$dz \equiv f_1(x)dx_1 + f_2(x)dx_2 + \cdots + f_n(x)dx_n$$

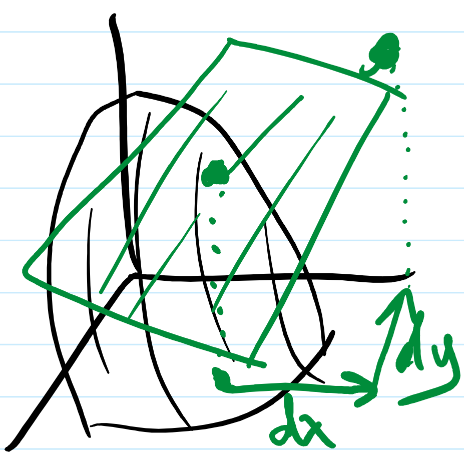


dz is a linear approx.

$ax+by=c$ line

$ax+by+cz=d$ plane

dz gives the tangent plane



dz gives the formula and when you choose dx, dy.

This is a linear approx. of the 3D fn.

N>3: hyperplane

Jacobian

Single-valued fn: row vector.
Multi-valued fn: matrix

Let $f: \mathbb{R}^n \to \mathbb{R}^m$

$$J = Df = \begin{bmatrix} f_1^1 & f_2^1 & \cdots & f_n^1 \\ \vdots & & \ddots & \vdots \\ f_1^m & f_2^m & \cdots & f_n^m \end{bmatrix}$$

$f = (f^1, f^2, f^3, \ldots, f^m)$

f_2^3 is the partial derivative of f^3 w.r.t. x_2

If $f: \mathbb{R}^n \to \mathbb{R}$ then

$$J = Df = \begin{bmatrix} f_1 & f_2 & \cdots & f_n \end{bmatrix}$$

Ex1 Let $f(x,y) = (x^2+y, y^3)$

Ex Let $f(x,y) = (x^2+y, y^3)$

f^1 f^2

2 values

$$J = \begin{bmatrix} \frac{\partial f^1}{\partial x} & \frac{\partial f^1}{\partial y} \\ \frac{\partial f^2}{\partial x} & \frac{\partial f^2}{\partial y} \end{bmatrix}$$

$$= \begin{bmatrix} 2x & 1 \\ 0 & 3y^2 \end{bmatrix}$$

Gradient Vector

1) Only defined for single-value functions

2) It is the transpose of the Jacobian. So it a column vector.

$$\nabla f$$

Let $z = f(x)$. Then $dz = \nabla f \cdot dx$

which means: $dz = f_1 dx_1 + f_2 dx_2 + \dots + f_n dx_n$

Ex) $f(x,y) = x^2 + y$

$$\nabla f = \begin{bmatrix} 2x \\ 1 \end{bmatrix}$$

Total derivative

$f(u,v,w)$ but $u(t)$
$v(t)$
$w(t)$

the $\frac{df}{dt}$ is the total derivative w.r.t. t.

$$\frac{df}{dt} = \frac{\partial f}{\partial u} \cdot \frac{du}{dt} + \frac{\partial f}{\partial v} \cdot \frac{dv}{dt} + \frac{\partial f}{\partial w} \cdot \frac{dw}{dt}$$

Multivariate Chain Rule

Ex) $w = x^3 yz + xy + z + 3$

Total differential:

$$dw = (3x^2yz + y)\,dx + (x^3z + x)\,dy + (x^3y + 1)\,dz$$

Let $x = 2t + 1$
$y = 4t^2$
$z = \ln t$

$$\frac{dw}{dt} = (3x^2yz + y)\cdot 2 + (x^3z + x)\cdot 8t + (x^3y + 1)\frac{1}{t}$$

$\rightarrow$ plug in $x(t), y(t), z(t)$

d or ∂?

$\downarrow$

total deriv.

$f(\underline{x}, \underline{y}, \underline{t}) \qquad x(t), y(t)$

Then $\dfrac{df}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$

$$\frac{}{dt} \quad \frac{}{\partial x\, dt} \quad \frac{}{\partial y\, dt}$$

$$+\underbrace{\frac{\partial f}{\partial t}}_{\neq \frac{df}{dt}}$$

Higher order derivatives

$f: \mathbb{R}^{10} \to \mathbb{R}$

$$\underbrace{\frac{\partial f}{\partial x_1}} = \underbrace{f_1}$$

$$\frac{\partial}{\partial x_6}\left(\frac{\partial f}{\partial x_1}\right) = \underbrace{\frac{\partial^2 f}{\partial x_6 \partial x_1}} = f_{16}$$

In economics:

$$\frac{\partial}{\partial x_6}\left(\frac{\partial f}{\partial x_1}\right) = \frac{\partial^2 f}{\partial x_1 \partial x_6}$$

$$\frac{\partial}{\partial x_3}\left(\frac{\partial}{\partial x_6}\left(\frac{\partial f}{\partial x_1}\right)\right) = \frac{\partial^3 f}{\partial x_1 \partial x_6 \partial x_3}$$

f_{16} $\qquad$ f_{163}

Young's Theorem:

in most cases:

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{\partial^2 f}{\partial x_2 \partial x_1}$$

ie. $f_{12} = f_{21}$

Hessian

$f: \mathbb{R}^n \to \mathbb{R}$

then $D^2 f = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & & \ddots & \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix}$

Row index: first deriv.

Col index: 2nd deriv.

Hessian is:

1) only defined for single-valued fns

2) Square

3) Symmetric

so $[D^2 f]_{ij} = [D^2 f]_{ji}$

because of Young's Theorem.

Ex $y = e^{x_1} + 2x_2 + 4x_1 x_2^2$

Find Hessian:

$H = D^2 y$ is $\underline{2 \times 2}$

bec. there are 2 indep. variables.

$y_1 = e^{x_1} + 4x_2^2$

$y_2 = 2 + 8x_1 x_2$

$y_{11} = e^{x_1} \qquad y_{12} = 8x_2$

$y_{21} = 8x_2 \quad y_{22} = 8x_1$

So $D^2 y = \underbrace{\begin{bmatrix} e^{x_1} & 8x_2 \\ 8x_2 & 8x_1 \end{bmatrix}}_{\text{symmetric}}$

Young's Theorem does not require f_{12} and f_{21} to be continuous on $\mathbb{R}^n$.

They need to be continuous in a <u>neighborhood</u> of x, ie., an open ball of <u>any</u> positive radius.

$$H = \begin{bmatrix} 2e^{-y} & -2xe^{-y} \\ -2xe^{-y} & e^{-y}(2 - 4y + x^2 + y^2) \end{bmatrix}$$

<u>Implicit function theorem</u>

Implicit function theorem

$G(x,y)=0$ defines y as a function of x implicitly.

$$\frac{x^y}{\ln y^2}\cdot \arctan x = 0$$

We can't find $y=f(x)$ but we can find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = -\frac{\frac{\partial G}{\partial x}}{\frac{\partial G}{\partial y}}$$ where $G_y \neq 0$

Differential

$$dG = G_x\,dx + G_y\,dy$$

$$\frac{dG}{dx} = G_x + G_y \frac{dy}{dx}$$

$\boxed{G(x,y)=0}$ so $dG = 0$

so $0 = G_x + G_y \frac{dy}{dx}$

$$-G_y \frac{dy}{dx} = G_x$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{-G_x}{G_y}}$$

Addendum: This is actually a precise derivation because
If y=f(x) then dy=f'(x)dx by definition and
Thus dy/dx=f'(x)dx/dx = f'(x)

ΔG from x

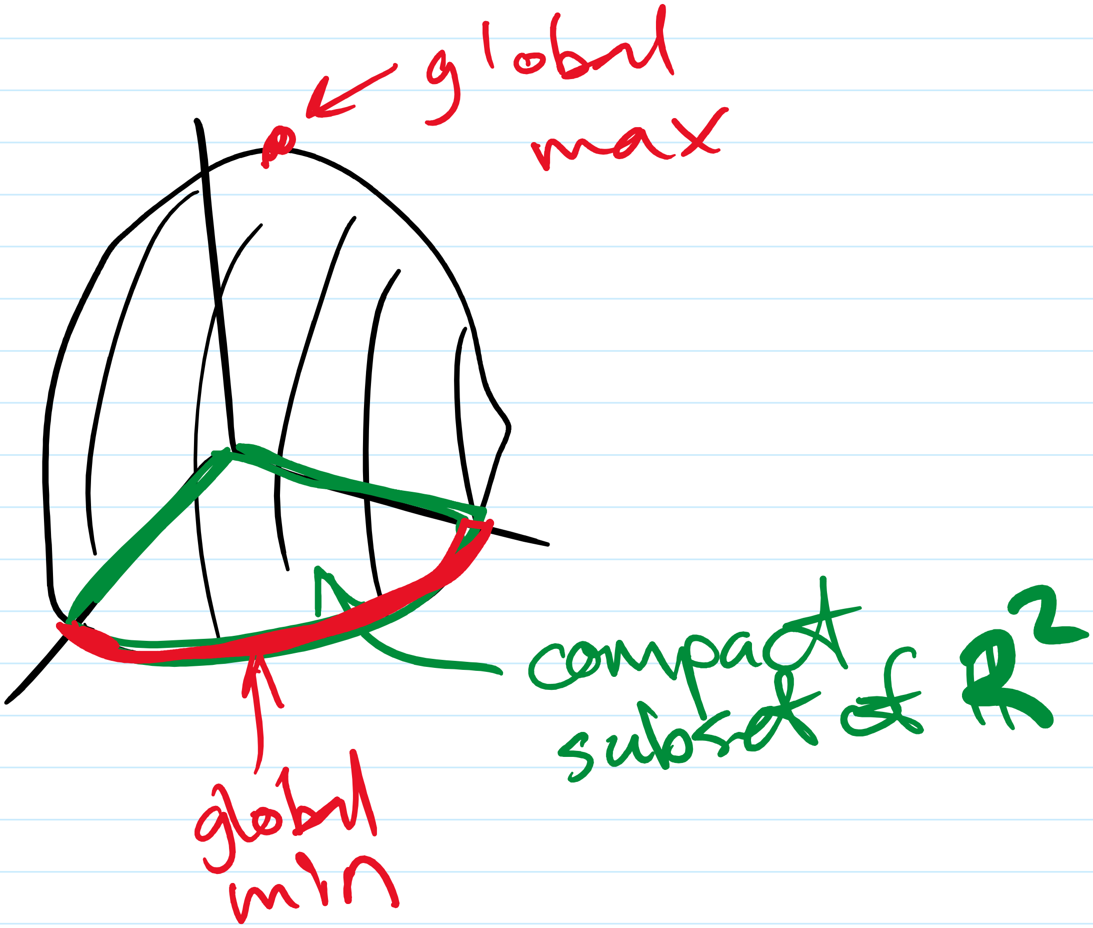
$= -\Delta G$ from y

Weierstrauss

1) continuous

2) compact domain

↳ closed and bounded



Fixed Point Theorems

A fixed point of

A fixed point of the function $\boxed{f: X \to X}$ is an element $x^* \in X$ such that $f(x^*) = x^*$.

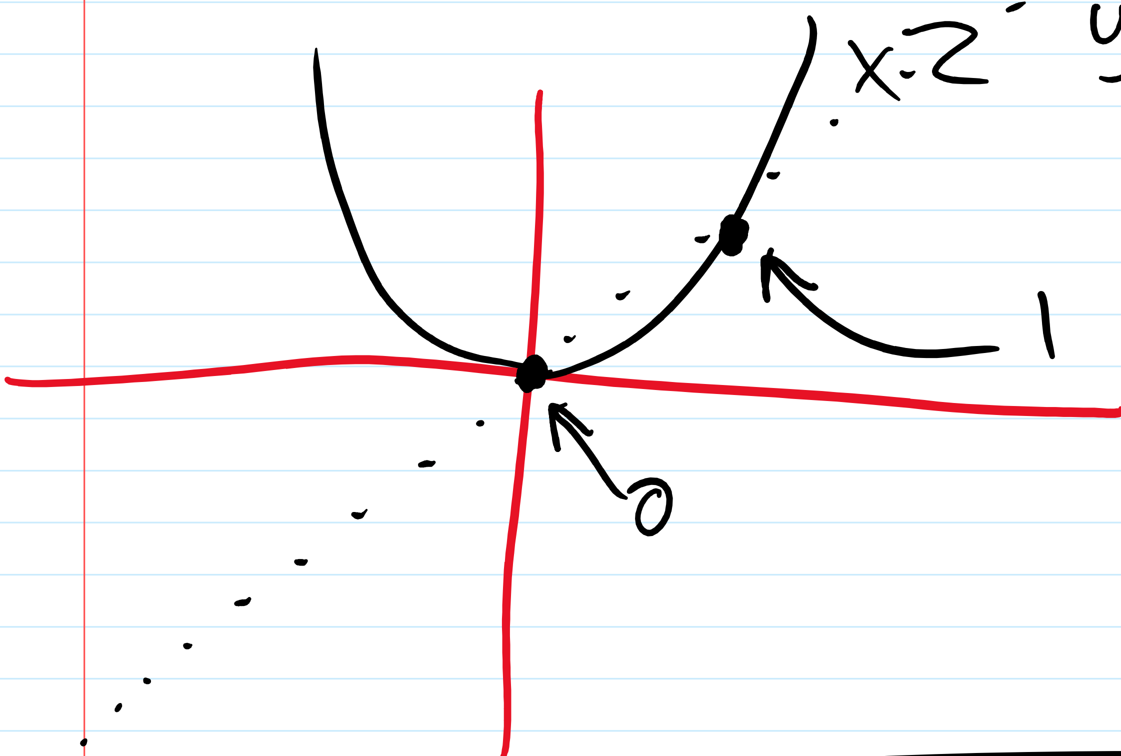
$$f(x) = x^2$$

$f(0) = 0$ so 0 is a fixed point

$f(1) = 1$ so 1 is another fixed point

$f(2) = 4$ so 2 is not a F.P.

$f(-1) = 1$ so -1 is not

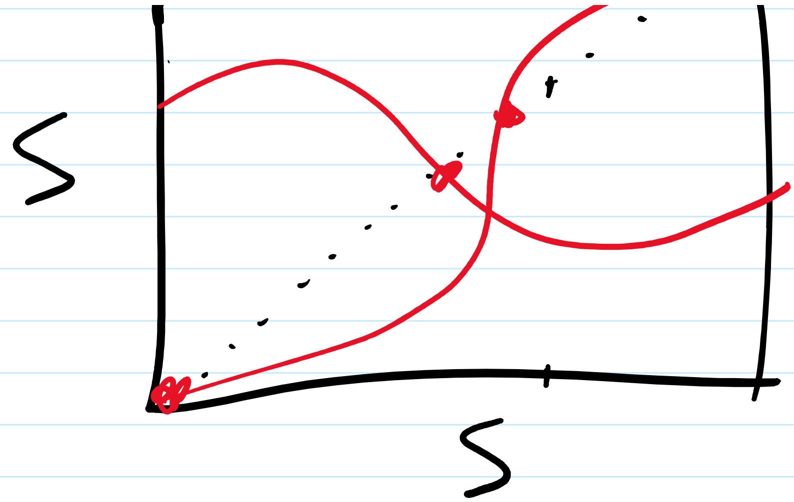


Brouwer's FPT:

~~If~~ $f : S \to S$ is <u>continuous</u> on S, ~~it~~ has a fixed point.

$S \subseteq \mathbb{R}^n$ such that S is compact.

Addendum: S should be both <u>compact</u> AND <u>convex</u>



Addendum: S should be both compact AND convex

Convexity of fns in $\mathbb{R}^n$

A fn is convex if

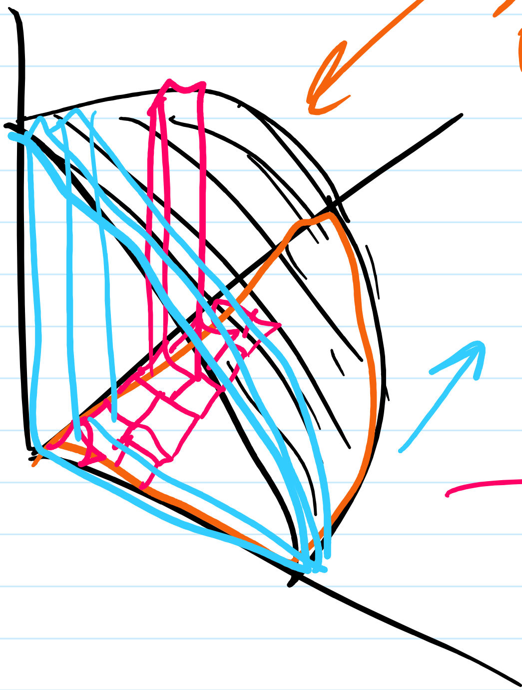
$$\forall x_1, x_2 \quad f(tx_1 + (1-t)x_2) \le tf(x_1) + (1-t)f(x_2)$$

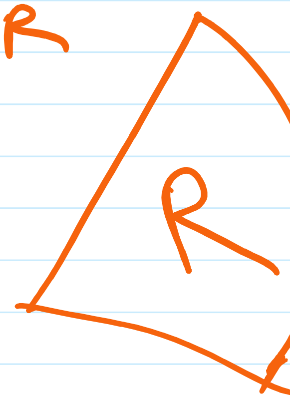
same as before, but now x_1, x_2 are vectors.

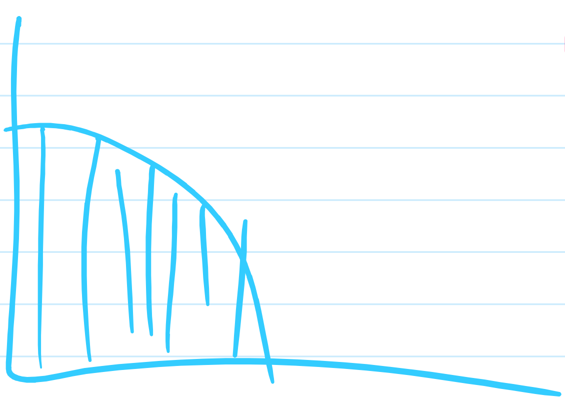
Iterated integrals

$\iint f(x,y)$

$$\iint_R f(x,y)$$







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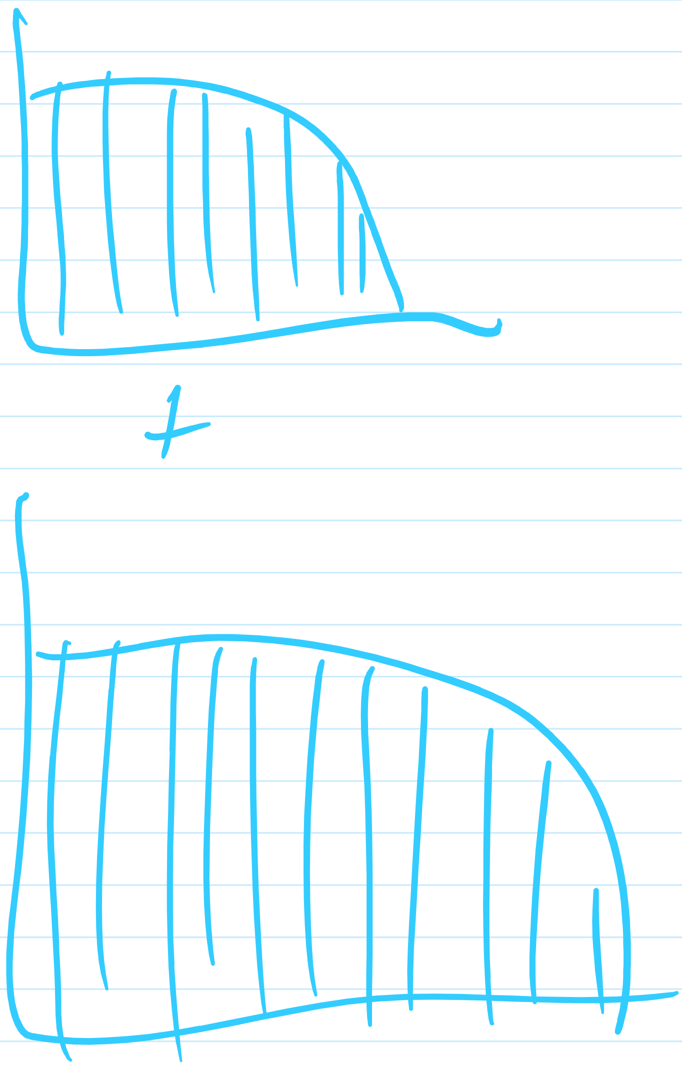

Fubini's Theorem

If f is continuous on R which is a rectangle:

$$R = \{(x,y) : a \le x \le b,\ c \le y \le d\}$$

then

$$\iint_R f(x,y) = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

$$= \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$$

iterated integral.

Common in probability and stats.
