Lecture 09
Optimization Day 1

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Mathematics Review Course, Summer 2023
University of Minnesota
August 17th, 2023

Last Lecture Review

- Numbers:
  - Triangle Inequality
  - Neighborhoods
- Functions:
  - Homogeneity
  - Euler's Theorem
  - Quasiconcavity & Quasiconvexity
  - Concavity & Convexity
  - Continuity
  - Upper- and Lower-Hemicontinuity
  - Brouwer's Fixed-point Theorem
  - Kakutani's Fixed-point Theorem

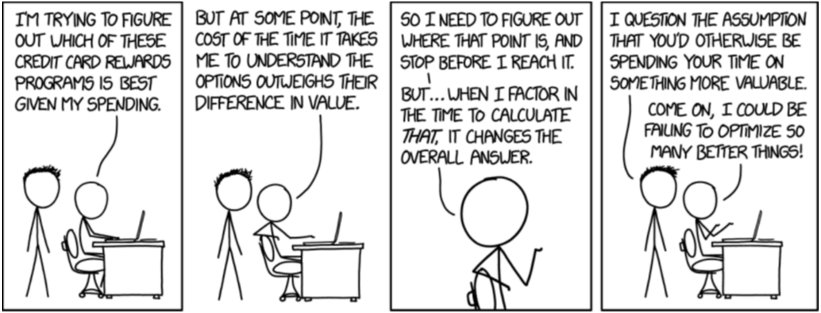
Review Assignment

1. Problem Set 08 solutions are available on Github.
2. Any issues or problems **You** would like to discuss?

Daily Icebreaker

- Attendance via prompt:
  - Name
  - Program and track
  - Daily icebreaker subject...

Optimization



Unconstrained Optimization

Motivation

- General background
  - Simplest form of optimization.
  - Solution involves finding critical points of the function.
- Why do economists' care?
  - Used in simple applications for testing first order conditions and second order conditions.
- Application in this career
  - Unconstrained optimization is primarily used to determine FOC and SOC.

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Overview

1. First Order Conditions
2. Second Order Conditions
3. Global Min and Max

1. First Order Conditions

- Let $F : U \to \mathbb{R}$ be a differentiable defined on subset U of $\mathbb{R}^n$.
- If $x^* \in \mathbb{R}^n$ is a local minimum or local maximum of $F(\cdot)$ and x^* is an interior point of U, then:

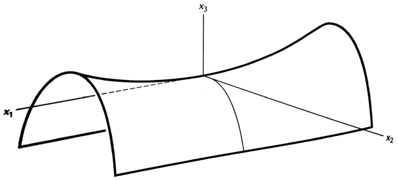
$$\nabla F(x^*) = 0 \text{ or } \frac{\partial F(x^*)}{\partial x_n} \forall n$$

- The FOC can be summarized by the Jacobian matrix.

$$J = \nabla f(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix}$$

2. Second Order Conditions

- Let $F : U \to \mathbb{R}$ be C^2 whose domain is an open set $U \in \mathbb{R}^n$.
- Suppose $\nabla F(x^*) = 0$.
  - If $D^2 f(x^*)$ is negative (positive) definite, then x^* is a strict local max (min).
  - If $D^2 f(x^*)$ is indefinite, then x^* is neither a local max nor min.
- Suppose x^* is a local max (min) of F.
  - Then, $\nabla F(x^*) = 0$ and the symmetric $n \times n$ matrix $D^2 f(x^*)$ is negative (positive) **semi-definite**.



The graph of the indefinite form $Q_3(x_1, x_2) = x_1^2 - x_2^2$.

2. Second Order Conditions

- The SOC is captured by the Hessian matrix.

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

3. Global Min and Max

- Any point x^* of a concave (convex) function $f(\cdot)$ satisfying $\nabla F(x^*) = 0$.
- Note this may be a boundary point (corner solution) or an interior point (critical point).

$Df(x^*)$	$D^2f(x^*)$	Max/Min
= 0	Negative Semi-definite	Local Max
= 0	Positive Semi-definite	Local Min
= 0	Indefinite	Saddle point or Inflexion

Practice: Unconstrained Optimization

1.

(Equality) Constrained Optimization

Motivation

- General background
  - Optimization of a function considering a constraint from other function(s).
  - Leads to ‘interior solutions’.
- Why do economists’ care?
  - This is the most typical case for optimization.
- Application in this career
  - Used throughout microeconomics.

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Overview

1. Lagrangian Method
2. Second Order Conditions
3. Bordered Hessian

1. Lagrangian Method

- Let f and h be C^1 functions.
- Suppose $x^* = (x_1^*, x_2^*)$ is a **solution** to the problem:

$$\max f(x_1, x_2)$$
$$\text{s.t. } h(x_1, x_2) = c$$

- Consider (x_1^*, x_2^*) are **not** critical points of h.
- Then μ^* is a real number such that (x_1^*, x_2^*, μ^*) **is** a critical point of the following Lagrangian function.

$$L(x_1, x_2, \mu) \equiv f(x_1, x_2) - \mu[h(x_1, x_2) - c]$$

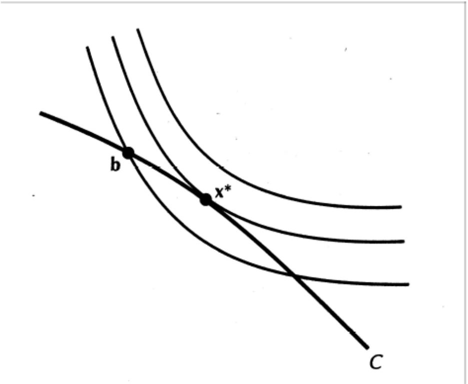
- That is...

$$\frac{\partial L}{\partial x_1} = 0, \ \frac{\partial L}{\partial x_2} = 0, \ \frac{\partial L}{\partial \mu} = 0$$

1. Lagrangian Method

- Intuition:

$$\mu = -\frac{\frac{\partial f}{\partial x_1}(x^*)}{\frac{\partial f}{\partial x_2}(x^*)} = -\frac{\frac{\partial h}{\partial x_1}(x^*)}{\frac{\partial h}{\partial x_2}(x^*)}$$



1. Lagrangian Method

- So, we have two equations represented by $\nabla f(x^*) = \mu^* \nabla h(x^*)$.

$$\begin{aligned}\frac{\partial f}{\partial x_1}(x^*) - \mu^* \frac{\partial h}{\partial x_1}(x^*) &= 0 \\ \frac{\partial f}{\partial x_2}(x^*) - \mu^* \frac{\partial h}{\partial x_2}(x^*) &= 0\end{aligned}$$

1. Lagrangian Method

- Consider there may be **many** constraints in the constraint set C_h with local max/min $x^* \in C_h$.

  $C_h = \{x = (x_1, \ldots, x_n) : (h_1(x = a_1), \ldots, h_m(x = a_m)\}$

- If x^* is not the critical point of $h = (h_1, \ldots, h_m)$ (i.e., $rank(Dh(x^*) < m)$ – unique value for all constraints), then there are $\mu_1^*, \ldots, \mu_m^*$ real numbers such that $(x_1^*, \ldots, x_n^*, \mu_1^*, \ldots, \mu_m^*)$ is the critical point of the Lagrangian function:

  $L(x^*, \mu^*) \equiv f(x) - \mu_1[h(x) - a_1] - \cdots - \mu_m[h(x) - a_m]$

- Alternatively, all partials for x_n and μ_m are set to zero.

2. Second Order Conditions

- To ensure a maximum, we need to know that the second differential of the objective function f at the critical point is **decreasing along the constraint**.
- Let $y = f(x_1, x_2(x_1))$ be the value of the objective function subject to the constraint.
- By the implicit function theorem:

$$\frac{dx_2}{dx_1} = \frac{\partial h/\partial x_1}{\partial h/\partial x_2}$$

- By chain rule:

$$\frac{dy}{dx_1} = \frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial x_1} = \frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2}\frac{\partial h/\partial x_1}{\partial h/\partial x_2}$$

3. Bordered

- The sufficient condition for a critical point is:
$$\frac{d^2y}{dx_1^2} < 0$$
- It can be shown that
$$\frac{d^2y}{dx_1^2} = \frac{-1}{(\partial h/\partial x_2)^2}\bar{D}$$
- Where $\bar{D}$ is the **bordered Hessian** of L.
- The determinant of the bordered Hessian tests for quasiconcavity or convexity to ensure a local max/min
- Negative definite assuming $\nabla f(x) \neq 0 \forall x$ and $f(\cdot)$ is strictly quasiconcave.

3. Bordered

- Local minima (x_1^*, x_2^*) if $B(x_1^*, x_2^*, \lambda^*) < 0$. Local maximum (x_1^*, x_2^*) if $B(x_1^*, x_2^*, \lambda^*) > 0$.

$$B(x_1, x_2, \lambda) = \left| \begin{bmatrix} 0 & \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} \\ \frac{\partial h}{\partial x_1} & \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} \\ \frac{\partial h}{\partial x_2} & \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} \end{bmatrix} \right| > 0$$

Practice: (Equality) Constrained Optimization

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Unconstrained Optimization

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(Equality) Constrained Optimization

1. Lagrangian Method
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Assignment

- Assignment:
  - Problem Set 09 (PS09)
  - Solution set will be available following end of Lecture 10
- Struggling?
  1. Read the 'Encouraged Reading'
  2. Review 'Supplementary material'
  3. Reach out directly