# Math Review
# Summer 2016

## *Topic 1*

## 1. Introduction to mathematical notations and logic

Goals:
- Able to read proofs and definition comfortably
- Understand what a definition means and does not mean

Now you should be able to understand this:
Sample example from Consumer Theory:

$\mathfrak{B}$ is a family of nonempty subsets of $X$. Every element of $\mathfrak{B}$ is a set $B \in X$. $C()$ is choice rule that assigns a nonempty set of chosen elements in B, i.e. $C(B) \subset B$.

Definition: The choice structure $(\mathfrak{B}, \mathrm{C}())$ satisfies the *weak axiom of revealed preference*:

If for some $B \in \mathfrak{B}$ with $x, y \in B$, we have $x \in C(B)$, then for any $B' \in \mathfrak{B}$ with $x, y \in B'$ and $y \in C(B)$, we must also have $x \in C(B')$.

We will come back to this in a bit after going through some notations.

**Demonstration:** $\boldsymbol{X = \{x, y, z\}, \mathfrak{B} = \{\{x, y\}, \{y, z\}, \{x, y, z\}\}, C(\{x, y\}) = \{y\}}$,
$\boldsymbol{C(\{x, y, z\}) = \{y\}}$ is OK
**OR** $\boldsymbol{C(\{x, y, z\}) = \{z, y\}}$ is ok too.
**How about** $\boldsymbol{C(\{x, y, z\}) = \{z, y\}}$ No, not ok.

Which of these violate the axiom listed here?

1. The choice structure is $(\mathfrak{B}, \mathrm{C}_1())$ where the choice rule $\mathrm{C}_1()$ is $\mathrm{C}_1(\{\mathrm{x}, \mathrm{y}\}) = \{\mathrm{x}\}$ and $\mathrm{C}_1(\{\mathrm{x}, \mathrm{y}, \mathrm{z}\}) = \{\mathrm{x}\}$ *(does not violate)*

2. 1. The choice structure is $(\mathfrak{B}, \mathrm{C}_2())$ where the choice rule $\mathrm{C}_2()$ is $\mathrm{C}_2(\{\mathrm{x}, \mathrm{y}\}) = \{\mathrm{x}\}$ and $\mathrm{C}_2(\{\mathrm{x}, \mathrm{y}, \mathrm{z}\}) = \{\mathrm{x}, \mathrm{y}\}$ *(violates)*

3. 1. The choice structure is $(\mathfrak{B}, \mathrm{C}_3())$ where the choice rule , $C_3(\{x, y\}) = \{x\}$

, $C_3(\{y,z\}) = \{y\}, C_3(\{x,z\}) = \{z\}$ *(does not violate)*

## 1.1. Mathematical notation

We start by getting acquainted (or re-acquainted for some of you) to the basic mathematical notations that you will see in economics.

| | |
|---|---|
| $\forall$ | For all... |
| $\exists$ | There exists... |
| $\nexists$ | There does not exist... |
| $\therefore$ | Therefore... |
| $\because$ | Because... |
| $\neg$ | Negation |
| $\equiv$ | Identical to or the same as… For example, we write $f \equiv g$ if $f(x) = g(x)$ for all x |
| $\Longrightarrow$ | A $\Longrightarrow$ B means: “A implies B, “If A then B or "A is sufficient condition for B” |
| $\Leftrightarrow$ | A $\Leftrightarrow$B means “A if and only if B”,“A is equivalent to B” or “A is a necessary and sufficient condition for B” |
| $A \subset B$ | "B strictly contains A" or “A is a proper subset of B” |
| $A \subseteq B$ | “B contains A” or “A is a subset of B” |
| $\in$ ($\notin$) | In... (Not in...) or an element of... (Not an element of...) |
| $\blacksquare$ | Bonus: End of proof, Q.E.D. |

What QED means? Latin phrase quod erat demonstrandum, meaning "which is what had to be proved".

The last three notations deal with sets. Formally, a set is a collection of well-defined and distinct objects (usually numbers). For example, the set A is completely determined by the elements in A, where:

$$A = \{x : x \in A\}.$$

We will touch more on sets in the next section.

We will deal with sets quite a bit today, you can think of it as a collection of things, you call members. Well-defined means that you know what is a member and most importantly, who is NOT. Set theory says that it does not matter how the members of the set are arranged. Therefore, {2, 3, 4, 6, 8, 10} is identical to {10, 2, 4, 8, 6, 3}. It does not affect the well-defined nature of a set.

### 1.2. Numbers

The different sets of numbers in mathematics are:

*Natural numbers*: $\mathbb{N} = \{1, 2, 3, \dots\}$
*Integers:* $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$
*Rational numbers*: $\mathbb{Q} = \{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\}$

$Q$: What do you think is missing in this definition of rational numbers? It has something to do with q.
$\mathcal{A}$: $q \neq 0$

*Real numbers*: $\mathbb{R} = \{all\ decimals\}$
*Complex numbers*: $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}, i = \sqrt{-1}\}$

#### 1.2.1. Intervals in ℝ

These are the four sets of intervals in the real line:

*Closed interval*: $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$
*Open interval*: $(a, b) = \{x \in \mathbb{R} : a < x < b\}$
*Right-half closed or left-half open* $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$
*Other*: $[a, \infty) = \{x \in \mathbb{R} : a \leq x\}$

where ∞ denotes infinity. We also have −∞ for negative infinity.

### 1.3. Necessity and sufficiency

Before, jumping into proofs, we establish what we really mean by necessity and sufficiency. Necessary and sufficient have two very different meanings.

- If you advance that "**A is necessary for B**," this is what is entailed:
  - "A is implied by B" ($A \Longleftarrow B$)

- For B to be true requires A to be true <u>or</u> equally, A is required to have B. Every time you have B you will have A, without exception

*Example.* Let A be the set "x is an integer less than 9" and let B be the set "x is an integer less than 7". Then A is implied by B, because "x is an integer less than 9" is implied by the statement "x is an integer less than 7".

$$A = \{\ldots,5,6,7,8\}$$
$$B = \{\ldots,5,6\}$$

- If you advance that "**A is sufficient for B**," this is what is entailed:
  - "A implies B" ($A \Longrightarrow B$)
  - Whenever A holds, B must hold. Every time you have A you will have B

*Example.* If Sally gets an 100% in all her graded assignments (**A**), she gets a pass in the class (**B**). Getting 100% in all assignments is a sufficient condition to pass the class. But Sally may very well get an 88% in Homework#7 and still get an A in the class.

<u>Contrapositive form</u>:

Suppose we know that $A \Longleftarrow B$ is true. Then, as A is necessary for B, when A is not true, then B cannot be true.

$Q$: Look back at your table of notations, how can you write this contrapositive form for A and B?

$\mathcal{A}$: $\neg A \Longrightarrow \neg B$

*Example.* Let A="x is an integer less than 9" and B= "x is an integer less than 7". As we saw earlier, A is implied by B. Now, we form the contrapositives, so ¬A= "x is not an integer less than 9" and ¬B= "x is not an integer less than 7". This implies ¬A ⇒¬B is a true statement.

$$\neg A = \{9, 10, 11, \ldots\}$$
$$\neg B = \{7, 8, 9, 10, 11, \ldots\}$$

### 1.4. Theorems and proofs

A mathematical proof is used to show the validity of a specified statement. A proof uses logic and deductive reasoning to show that the statement is <u>always</u> true. Proofs are usually statements take the form "if A then B." There are three types of proofs that are frequently used. I have them down here by their popularity (in my opinion) in the first year micro series.

### 1.4.1. Proof by contradiction

This is a very powerful form of proof. In a proof by contradiction you show that "if not B then not A." Logically, this is what it means:

$$A \Longrightarrow B$$
$$\equiv$$
$$\neg\, A \text{ } and \text{ } \neg B$$
$$\equiv$$
$$\neg\, B \Longrightarrow \neg A$$

All these three statements are *all equivalent*.

A good proof by contradiction has the following steps:

*Step 1*: Assume B is false
*Step 2*: Show that A must also be false.

We start with a simple math example, and later we will go through a slightly more involved example from micro theory after completing Topic 2.

*Example*. Prove that $\sqrt{2}$ is irrational.

We could jump to Steps 1 and 2 but let's be a little more careful.

Define related concepts: Rational numbers are all numbers of the form $\frac{p}{q}$, where $p$ and $q$ are integers and $q \neq 0$.

Think of some different examples: Rational numbers take the form of $\frac{1}{2}, -\frac{5}{3}, 2, 0, \frac{50}{10}$

Re-define the concept of interest: A number which is not rational is said to be irrational.

Proof:

*Step 1*: Assume to the contrary that $\sqrt{2}$ is rational. Thus we can write

$$\sqrt{2} = \frac{p}{q}$$

Moreover, let $p$ and $q$ have no common divisor > 1, that is, $\frac{p}{q}$ is in the lowest terms. Then we have:

$$\left(\sqrt{2}\right)^2 = \frac{p^2}{q^2}$$

which implies:

$$2q^2 = p^2$$

*Step 2*: Because $p, q \in \mathbb{Z}$ we have $p^2, q^2 \in \mathbb{Z}$. An integer $k$ is even if $2n = k$ so, by definition, $p^2$ must be an even integer. The square of an odd number is always odd (square of even is even) so because $p^2$ is even it must also hold that $p$ is even. Because $p$ is even there exists $n \in \mathbb{Z}$ such that $2n = p$

Thus:

$$2q^2 = (2n)^2 = 4n^2$$
$$q^2 = 2n^2$$

You can apply the same logic as above to show that $q$ must also be even. But if both $p$ and $q$ are even then the fraction $\frac{p}{q}$ cannot be in lowest terms since both integers are divisible by 2, a contradiction. Therefore $\sqrt{2}$ is has to be irrational. ■

### 1.4.2. Proof by construction

In proof by construction you use true statements to construct the actual statement that you wish to prove. Suppose we have the theorem "A ⇒ B". Here, A is called the premise and B the conclusion. In a constructive proof we assume that A is true, deduce various consequences of that, and use them to show that B must also hold. This proof technique is a little less structured, as it is more dependent on the nature of the statement you are trying to prove.

Proof by construction follows these two steps:

*Step 1*: State what you wish to show (i.e. your claim)
*Step 2*: Use valid logic and parameters to construct the statement.
*Step 3*: Conclusion. This is optional, you can re-state the goal if desired.

*Example*: Prove that if $a$ and $b$ are consecutive integers, then the sum $a + b$ is odd.

Proof.
*Step 1*: Assume that $a$ and $b$ are consecutive integers.

*Step 2*: Because $a$ and $b$ are consecutive we know that $b = a + 1$. Thus, the sum $a + b$ may be re-written as $2a + 1$. Thus, there exists a number $k$ such that $a + b = 2k + 1$,
*Step 3*: So the sum $a + b$ is odd. ■

*Q*: How would you approach this simple proof as a proof by contradiction?

*A*: Assume that $a$ and $b$ are consecutive integers.
Assume also that the sum $a + b$ is ***not*** odd.
Because the sum $a + b$ is not odd, there exists no number $k$ such that $a + b = 2k + 1$:
$a + b \neq 2k + 1$
However, the integers $a$ and $b$ are consecutive, so we may write the sum $a + b = a + a + 1$ as $2a + 1$. Thus, we have derived that $a + b \neq 2k + 1$ for any integer $k$ and also that $a + b = 2a + 1$. This is a contradiction. If we hold that $a$ and $b$ are consecutive integers then we know that the sum $a + b$ must be odd. ■

### 1.4.3. Proof by Induction

Proof by induction is another great method in which we use recursion to demonstrate an infinite number of facts in a finite amount of space. In other words, you wish to show that some statement, $S$, is true for all $n$, $S_n$. To prove this general statement with induction we follow two steps:

> *Step 1*: Show that a propositional form is true for some basis case. It is typical to begin by showing that either $S_0$ is true or $S_1$ is true for example.
> *Step 2*: Assume that $S_k$ is true for some $k$. This assumption is called the inductive hypothesis. Prove that $S_{k+1}$ is also true, using the assumption that $S_k$ is true.

I will skip this proof by induction and leave it as a reading material for you *(if we have time we will reference this again when going through tips for writing proofs)*

*Example.* Prove that $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

<u>Proof</u>:
*Step 1*: Consider the case where $n = 1$. Then we have:

$$1 = \frac{1(1+1)}{2} = 1$$

and the statement holds.

*Step 2.a*: Now assume that for some $n = k \geq 1$. We have the following hold:

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$$

*Step 2.b*: We now wish to show that is true for $n = k + 1$, that is:

$$1 + 2 + 3 + \cdots + k + (\boldsymbol{k} + \mathbf{1}) = \frac{(k+1)\big((k+1)+1\big)}{2},$$

$$ie. \frac{(k+1)(k+2)}{2} = ?$$

We know from our inductive hypothesis that $1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$. Plugging this into the equation into the above, we have:

$$\frac{k(k+1)}{2} + (k+1) = \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$$

Thus we have shown that the statement also holds for $k + 1$ which implies that $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

■

General tips for approaching and writing proofs:

- As much as possible, use complete sentences when writing your proof. When writing a proof for a homework, exam or prelim, be as legible as possible. This holds for all parts of submitted work, and especially for proofs.
- Always remember to define any variables you introduce.
- It's a good practice to say what type of proof you are using (e.g. Proof by contradiction) to help your reader.
- Overly wordy proofs may result in more likelihood for errors – keep things concise and simple.
- Avoid the use of words such as *obviously*, *clearly*, *as we know*, etc. State what is clear and obvious to you as it may not be for the reader. You might see these words in your micro notes, but I would personally stay clear of these.
- If asked to prove $A \Leftrightarrow B$, that is "A if and only if B" then you must remember to complete both directions of the proof. You must prove "*if A then B*" and "*if B then A*."

1. Prove: The sum of two even integers is always even. (*hint: use definition of even numbers*)

Proof: Let $x$ and $y$ be any two even integers, so there exist integers $a$ and $b$ such that $x = 2a$ and $y = 2b$. Then, $x + y = 2a + 2b = 2(a + b)$, which is even.

2. Prove: Suppose $a$ and $b$ are integers and $a \neq 0$. If $a$ does not divide $b$, then the equation $ax^2 + bx + b - a = 0$ has no positive integer solution. (*hint: use quadratic formula*)

Proof. Suppose $x > 0$ is an integer such that $ax^2 + bx + b - a = 0$.
Then, the quadratic formula tells us

$$x = \frac{-b \pm \sqrt{b^2 - 4\,(a(b-a))}}{2a}$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ab + 4a^2}}{2a}$$
$$x = \frac{-b \pm \sqrt{(b-2a)^2}}{2a}$$
$$x = \frac{-b \pm (b-2a)}{2a}$$

Because $x > 0$, we must have $x = \frac{-b-(b-2a)}{2a} = \frac{-2b-2a}{2a}$

$$ax = a - b$$
$$b = a(1-x)$$

Thus, $b = a(1 - x)$ and $a$ divides $b$.