Math Review
Summer 2017

Topic 8

8. Optimization

8.1 Unconstrained maximization

We start this lesson by considering the simplest of optimization problems, those without conditions, or what we refer to as **unconstrained optimization problems**. Unconstrained optimization is equivalent to finding the max/min of a function. In one-variable calculus we were able to find these points by finding the critical points of a function and looking at the sign of the second derivative at that point. In multivariable calculus the results are much the same. To identify maxima and minima we again need to be able to find the critical points of f:The point $\hat{x} \in \mathbb{R}^n$ is a critical point of $f: \mathbb{R}^n \to \mathbb{R}$ if $Df(\hat{x}) = 0$.

Recall 2 things from our previous classes:

(1) The local maximum, x^*, can be either a boundary max or an interior max. If x^* is an interior max, then x^* is a critical point.
(2) If the function f has a critical point in $C \subset \mathbb{R}^n$ then the Hessian matrix can be used to identify whether the critical point is a maximum or minimum.

Let's remember those cases:

$Df(x^*)$	$D^2f(x^*)$	Max/Min

- The $n \times n$ matrix $\boldsymbol{A}$ is positive definite if and only if its n principal minors are all greater than 0: $detA_1 > 0, detA_2 > 0, \dots, detA_n > 0$.
- The $n{\times}n$ matrix $\boldsymbol{A}$ is negative definite if and only if its n principal minors alternate in sign with the odd order ones being negative and the even order ones being positive: $det\ A_1 < 0, det\ A_2 > 0, detA_3 < 0, detA_4 > 0, \dots$.
- The $n \times n$ matrix $\boldsymbol{A}$ is positive semidefinite if and only if its principal minors are all greater than or equal to 0: $det\ A_1 \geq 0\ and\ det\ A_2 \geq 0, \dots, A_n \geq 0$.

- The $n \times n$ matrix $\boldsymbol{A}$ is negative semidefinite if and only if its n principal minors alternate in sign with the odd order ones being less than or equal to zero and the even order ones being greater than or equal to 0: $det\ A_1\ \leq\ 0\ and\ det\ A_2\ \geq\ 0, \ldots.$

Example. Consider this simple example.

$$f(x, y) = x^4 + x^2 - 6xy + 3y^2$$

Let's find the critical points and classify them as local max, local min or saddle points.

Q: Solve the following problem

Let $f(x, y, z) = e^{x-y} + e^{y-x} + e^{x^2} + z^2$. Find the critical points, is that a minimum, maximum or a saddle point?

8.2 Kuhn Tucker and constrained maximization

8.2.1 General forms of optimization

While you will work through a few unconstrained max/min problems, most economic decisions are the result of an optimization problem subject to one or a series of constraints:

- Consumers make decisions on what to buy constrained by the fact that their choice must be affordable.
- Firms make production decisions to maximize their profits subject to the constraint that they have limited production capacity.
- Households make decisions on how much to work/play with the constraint that there are only so many hours in the day.
- Firms minimize costs subject to the constraint that they have orders to fulfill.

All of these problem fall under the category of constrained optimization and luckily for us, there is a uniform process that we can use to solve these problems.

Constrained optimization problems are of the general form:

We call $f(x_1, x_2, \ldots, x_n)$ the objective function and $g_1, g_2, \ldots, g_k$ and $h_1, h_2, \ldots h_m$ are the constraint functions. The constraints $g_i(x_1, x_2, \ldots, x_n) \leq b_i$ for $i = 1, 2, \ldots, k$ are the inequality constraints and the $h_j(x_1, x_2, \ldots, x_n) = c_j$ for $j = 1, 2, \ldots, m$ are the equality constraints.

A standard consumer problem may take the following form. A consumer with budget I wishes to purchase the goods $x_1, x_2, \dots x_n$ with prices $p_1, p_2, \dots p_n$ so as to maximize utility, $u(x_1, x_2, \dots x_n)$. The consumer's problem is then:

$$\max_{\{x_1, x_2, \dots, x_n\}} u(x_1, x_2, \dots x_n)$$

subject to:

$$p_1 x_1 + p_2 x_2 + \cdots + p_n x_n \leq I$$

$$\text{and } x_i \geq 0 \ \forall i = 1,2,3, \dots, n$$

The inequality constraints $x_i \geq 0$ are usually referred to as non-negativity constraints as they restrict the consumer from purchasing a negative amount of some good i.

8.2.2. The Lagrangian/Langragean (*toMAYto-toMAHto*)

Let's start simple and build up to more involved optimization problem.

You are given this maximization problem framed as:

$$\max_{\{x_1, x_2\}} f(x_1, x_2)$$

subject to:

$$h(x_1, x_2) = c$$

We want to choose the values of (x_1, x_2) that maximize f and that are in the set $\{(x_1, x_2) : h(x_1, x_2) = c\}$.

There are two ways of solving the above maximization problem with given constraints. The first via *substitution*, the second via a *Lagrangean*. When dealing with constrained optimization, you often want to form a Lagrangean denoted by $\mathcal{L}$. You may read about the substitution method but most of Micro will really revolve around building the $\mathcal{L}$. At first this method may seem more difficult, but which in fact can be very useful (and sometimes easier). I will go through the formal definition, but the application is even simpler.

Lagrange's Theorem:

We need require that $\frac{\partial g_j}{\partial x_i} \neq 0, \forall i, \forall j$ from our definition of the Langrange function. This restriction is referred to as the constraint qualification. For a linear constraint this qualification is always satisfied. How do we know if the constraint qualification is met? Recall that a critical point is where the Jacobian equals zero which is equivalent to $\frac{\partial g_j}{\partial x_i} = 0 \; \forall i, j$. Therefore, the constraint qualification is not met if a critical point of the constraint lies in the constraint set. (You will see algebraically why below.)

In a nutshell here, you want to form $\mathcal{L}(\mathrm{x}_1, \mathrm{x}_2, \lambda)$

The first order conditions are:

These first order conditions can then be used to solve for x_1, x_2, λ

What can we note about λ^*?

Working with the first and second equation often will eliminate the λ^*, which is beauty of this method to problem solve. λ^* can then be solved for by plugging the results back into the last ugly equation (L).

Let's check how good our skill of applying definitions work: You are given a problem framed as follows:

$$\max_{\{x_1,x_2\}} -ax_1^2 - bx_2^2$$
$$\text{subject to:}$$
$$x_1 + x_2 = 1$$

Form the Lagrangean. Check for the constraint qualifications and find the solutions.

Now try this one on your own:

$$\max_{x_1, x_2} x_1^2 x_2$$
$$\text{subject to}$$
$$2x_1^2 + x_2^2 = 3$$

Form the Lagrangean and find the solutions.

Economic interpretation of the Lagrangian:

In economics, the values of λ^* often have important economic interpretations. Consider a budget constraint given by $p_1x_1 + p_2x_2 = w$, that is price multiplied by commodities add up to your wealth. Now think of a more general constraint of the form $g_j(x_j) = b_j$. If the right hand side b_j of constraint j is increased by Δ, say by one unit, then the optimum objective value increases by approximately $\lambda_i^*\Delta$ (or simply λ_i^* if Δ is one unit) . We won't expand too much on that. Often the λ_i^* simply drops out, but it is valuable to know where it all is coming from. In our consumer problem, with the budget constraint, parameters like wealth are usually fixed, but really, all of our solutions are dependent on that value. In reality, we can denote $x_1^*(w), x_2^*(w), \lambda^*(w)$ as the solutions. As wealth changes, these will change. You will find that Prof. Glewwe calls this the shadow value of wealth in the consumer problem.

Try at home:

Solve for x_1 and x_2

$\min_{x_1,x_2} x_1^2 + x_2^2$, subject to $x_1x_2 = 1$

$\min_{x_1,x_2} x_1x_2$, subject to $x_1^2 + x_2^2 = 1$

$\max_{x_1,x_2} x_1x_2^2$, subject to $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$

How about a general consumer problem? Solve for x, y

$\max_{x,y} kx^\alpha y^\beta$, subject to $p_x x + p_y y = m$

Now a general producer problem, solve for K, L

$\max_{K,L} wL + rK$, subject to $AK^\alpha L^\beta = q$

Constrained optimization with multiple equality constraints

Let $f(x_1, x_2, \ldots, x_n)$ be a continuously differentiable objective function and $g_1(x_1, x_2, \ldots, x_n), g_2(x_1, x_2, \ldots, x_n), \ldots g_k(x_1, x_2, \ldots, x_n)$ be the equality constraints. Then we wish to solve the following problem:

$$\max_{\{x_1, x_2, \ldots, x_n\}} f(x_1, x_2, \ldots, x_n)$$

subject to:

$$\begin{gathered} g_1(x_1, x_2, \ldots, x_n) = b_1 \\ \vdots \\ g_k(x_1, x_2, \ldots, x_n) = b_k \end{gathered}$$

Suppose $(x_1^*, x_2^*, \ldots, x_n^*)$ is a solution to this problem. If the rank of the Jacobian matrix dG is k, then there exists $\lambda_1^*, \lambda_2^*, \ldots, \lambda_k^*$ such that $(x_1^*, x_2^*, \ldots, x_n^*; \lambda_1^*, \lambda_2^*, \ldots, \lambda_k^*) = (x^*, \lambda^*)$ is a critical point of the langragian given by:

$$\mathcal{L}(x, \lambda) = f(x) - \lambda_1(g_1(x) - b_1) - \lambda_2(g_2(x) - b_2) - \ldots - \lambda_k(g_k(x) - b_k)$$

Rank $g(x_1^*, x_2^*, \ldots, x_n^*) = \begin{bmatrix} \frac{\partial g_1}{\partial x_1}(x^*) & \frac{\partial g_1}{\partial x_2}(x^*) & \cdots & \frac{\partial g_1}{\partial x_n}(x^*) \\ \frac{\partial g_2}{\partial x_1}(x^*) & \frac{\partial g_2}{\partial x_2}(x^*) & \cdots & \frac{\partial g_2}{\partial x_n}(x^*) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_l}{\partial x_1}(x^*) & \frac{\partial g_l}{\partial x_2}(x^*) & \cdots & \frac{\partial g_l}{\partial x_n}(x^*) \end{bmatrix} = l$

For example, let's solve:

$$\max_{x,y,z} yz + xz$$

subject to:

$$y^2 + z^2 = 1 \text{ and}$$

$$xz = 3$$

8.2.2. Kuhn-Tucker and inequality conditions

We can really spend a whole day on Kuhn-Tucker conditions – but we won't! Let's go through the essential material you will need to know in order to be able to keep up with Micro theory. **I do not expect you to understand this fully today, it took me many weeks to get it when I was first exposed**. However, let's try our best to get the most out of this section. Let's try not to get too stuck on details – if not today, you WILL get it over the next few times.

In economics it is much more common to start with inequality constraints of the form $g\ (x, y)\ \le\ c$, for instance in our consumer problem, we would have, $p_1x_1 + p_2x_2 \le w$. The constraint is said to be **binding** if at the optimum $g\ (x^*, y^*)\ =\ c$, and it is said to be slack otherwise. Luckily for us, with a small tweak, the Lagrangean can still be used with the same FOC's except now we have **three "Kuhn-Tucker" necessary conditions** for each inequality constraint.

Proof of Kuhn-Tucker can be found on pp. 959-960 Mas-Colell.

A few things to note:

- The first condition is just a restatement of the constraint.
- The second condition says that λ^* is always non-negative.
- The third condition says that either λ^*or $c - g\ (x^*, y^*)$ must be zero.
- If $\lambda^* = 0$, intuitively, we are not giving any weight to the constraint. The problem can really turn into an unconstrained optimization problem.
- If $\lambda^* > 0$, then the constraint must be binding then the problem turns into the standard Lagrangean considered above.

Formally,

Theorem. Kuhn - Tucker Necessary Conditions for Optima of Real Valued Functions Subject to Inequality Constraints

All in all, the Kuhn-Tucker condition provide a neat mathematical way of turning the problem into either an unconstrained problem or a constrained one. Unfortunately you **typically have to check both cases**, unless you know for sure it's one or the other. Let's work through an example together to get at what the Kuhn-Tucker does.

Example.

$$\max_{x_1,x_2} u(x_1, x_2) = 4x_1 + 3x_2$$

subject to:

$$g(x_1, x_2) = 2x_1 + 3x_2 \leq 10$$

$x_1, x_2 \geq 0$, non-negativity constraints

It would be cruel of me to have you solve a full blown Kuhn-Tucker problem today. I do want you to be able to read them and understand what you are looking at. These are general examples from the Production mini.

This is a profit maximization problem framed as:

$$\max_{q\geq 0, z\geq 0} \pi(p,r) = p.q - r.z$$
$$\text{subject to:}$$
$$D_o(q,z) \leq 1$$

I will go over what the different components mean briefly.

What is the Lagrangean?

For any given output m and any given input n, please find the first order Kuhn-tucker conditions.

Let's have you solve one of Prof. Terry Hurley famous profit maximization problems. If you can do this – you are at least 20% ready to take a Production final ☺

Consider this profit maximization problem with two outputs and two inputs constrained by an input distance function given by $D_1(q, z)$.

$$\max_{q\geq 0, z\geq 0} \pi(p, r) = p_1 q_1 + p_2 q_2 - r_1 z_1 - r_2 z_2$$

subject to:

$$D_I(q, z) = \frac{\sqrt{z_1 z_2}}{q_1^2 + q_2^2} \geq 1$$

Form the Lagrangean for this problem:

Write out the first order conditions (with Kuhn-Tucker):

Example. Let's practice an example of writing out the Kuhn-Tucker conditions for a minimization problem.

Consider this cost minimization problem two inputs constrained by an input distance function given by $D_1(q, z)$.

$$\min_{z\geq 0} C = r_1 z_1 - r_2 z_2$$

subject to:

$$D_I(q, z) = \frac{\sqrt{z_1 z_2}}{q_1^2 + q_2^2} \geq 1$$

Write down the Lagrangian, Kuhn-Tucker conditions and solve for $z_1(r, q)$ and $z_2(r, q)$ **assuming interior solutions afterwards.**

Final example. Maximize $U = U(x, y)$ Subject to $B \geq P_x x + P_y y$ and $C \geq c_x x + c_y y$.
$x \geq 0$ and $y \geq 0$.

Write out the Langragean for this problem:

Write out the Kuhn-tucker conditions.