APEC Math Review

Part 7 Linear Algebra

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2. Square matrices

Square matrices

Linear systems can also be solved with matrix inversion if the number of unknowns is the same as the number of equations, that is, the coefficient matrix is square and nonsingular.

Inversion

Let $\mathbf{B} = \mathbf{A}^{-1}$ be the inverse of a full-rank $k \times k$ matrix $\mathbf{A}$. The matrices satisfy

$$\mathbf{AB} = \mathbf{I}_k$$

If an $n \times n$ matrix $\mathbf{A}$ is invertible, then it is nonsingular, and the unique solution to the linear system $\mathbf{Ax} = \mathbf{b}$ is $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.

Inversion

The following statements about a $n \times n$ square matrix $\mathbf{A}$ are equivalent

- $\mathbf{A}$ is invertible
- $\mathbf{A}$ is nonsingular
- A has maximal rank n (full rank)
- every system $\mathbf{Ax} = \mathbf{b}$ has one and only one solution for every $\mathbf{b}$

Determinant

A $n \times n$ square matrix $\mathbf{A}$ is nonsingular if and only if its **determinant** is nonzero.

$$
\begin{aligned}
det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \; or \; \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\
= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\
- a_{13}a_{22}a_{31} - a_{21}a_{12}a_{33} - a_{11}a_{23}a_{32}
\end{aligned}
$$

- $det\,\mathbf{A}^T = det\,\mathbf{A}$
- $det\,(\mathbf{A} \cdot \mathbf{B}) = det\,\mathbf{A} \cdot det\,\mathbf{B}$
- $det\,(\mathbf{A} + \mathbf{B}) \neq det\,\mathbf{A} + det\,\mathbf{B}$

What will happen if $\mathbf{A}$ is not full rank?

Inversion

Let $\mathbf{A}$ and $\mathbf{B}$ be square invertible matrices. Then,

1. *det* $\mathbf{A}$ and *det* $\mathbf{B}$ are nonzero
2. $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$
3. $(\mathbf{A}^{\mathbf{T}})^{-1} = (\mathbf{A}^{-1})^{\mathbf{T}}$
4. $\mathbf{AB}$ is invertible, and $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

Inversion

Let **A** be a nonsingular matrix,

$$\mathbf{A}^{-1} = \frac{1}{det\mathbf{A}} \cdot adj\,\mathbf{A},$$

where $adj\,\mathbf{A}$ is a $n \times n$ square matrix in which the element on the ith row and j the column is

$$(-1)^{i+j} \times det(submatrix\ of\ \mathbf{A}\ without\ the\ ith\ row\ and\ the\ jth\ column)$$

Exercise: matrix inversion

Invert the following matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 1 & 0 & 8 \end{pmatrix}$$

Cramer's rule

The unique solution $\mathbf{X} = (x_1, \cdots, x_n)$ of the $n \times n$ system $\mathbf{Ax} = \mathbf{b}$ is

$$x_i = \frac{det\,\mathbf{B}_i}{det\,\mathbf{A}} \quad for\ i = 1, \cdots, n,$$

where $\mathbf{B}_i$ is the matrix $\mathbf{A}$ with the right-hand side $\mathbf{b}$ replacing the ith column of $\mathbf{A}$.

Exercise: Cramer's rule

Use the Cramer's rule to calculate x_3 in the following system:

$$\begin{pmatrix} 1 & 1 & 1 \\ 12 & 2 & -3 \\ 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ -4 \end{pmatrix}$$

Trace

The trace of a $k \times k$ square matrix $\mathbf{A}$ is the sum of its diagonal elements

$$tr(\mathbf{A}) = \sum_{i=1}^{k} a_{ii}$$

Properties

- $tr(c\mathbf{A}) = c\, tr(\mathbf{A})$
- $tr(\mathbf{A}') = tr(\mathbf{A})$
- $tr(\mathbf{A} + \mathbf{B}) = tr(\mathbf{A}) + tr(\mathbf{B})$
- $tr(\mathbf{I}_k) = k$
- for $k \times r$ $\mathbf{A}$ and $r \times k$ $\mathbf{B}$, $tr(\mathbf{AB}) = tr(\mathbf{BA})$

Characteristic equation

For a square matrix $\mathbf{A}$, we can define a set of equations

$$\mathbf{Ac} = \lambda\mathbf{c}$$

The pairs of solutions $(\mathbf{c}, \lambda)$ are the **characteristic vectors (eigenvectors) c** and **characteristic roots (eigenvalues)** λ**.** The equations imply that

$$\mathbf{Ac} = \lambda\mathbf{Ic} \Leftrightarrow (\mathbf{A} - \lambda\mathbf{I})\mathbf{c} = \mathbf{0}$$

This homogeneous system has a nonzero solution only if $(\mathbf{A} - \lambda\mathbf{I})$ is singular and has zero determinant.

$$|\mathbf{A} - \lambda\mathbf{I}| = 0$$

This polynomial of λ is the characteristic equation.

Exercise: characteristic equation

Solve for the characteristic roots for the following matrix.

$$\mathbf{A} = \begin{pmatrix} -1 & 3 \\ 2 & 0 \end{pmatrix}$$

Characteristic vectors

With the characteristic roots, we can solve for the characteristic vectors corresponding to each characteristic roots using

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{c} = \mathbf{0}$$

$(\mathbf{A} - \lambda \mathbf{I})$ is singular so there exists a non-zero solution. In fact, there are infinite nonzero solutions, just pick one that is nonzero. Normalize it so that $\mathbf{c}'\mathbf{c} = \mathbf{1}$.

Solve for the two characteristic vectors in the last example.

Some useful results

- A $k \times k$ matrix has k distinct characteristic vectors that are orthogonal to each other.
- The rank of a symmetric matrix is the number of nonzero characteristic roots it contains.
- The trace of a matrix equals the sum of its characteristic roots.
- The determinant of a matrix equals the product of its characteristic roots.
- If $\mathbf{A}^{-1}$ exists,then the characteristic roots of $\mathbf{A}^{-1}$ are the reciprocals of those of $\mathbf{A}$ and the characteristic vectors are the same.

Decomposition

Define

$$\mathbf{C} = \begin{pmatrix} \mathbf{c_1} & \mathbf{c_2} & \cdots & \mathbf{c_k} \end{pmatrix}$$

$$\boldsymbol{\Lambda} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k \end{pmatrix}$$

For each k

$$\mathbf{A}\mathbf{c_k} = \lambda_\mathbf{k}\mathbf{c_k}$$

So

$$\mathbf{AC} = \mathbf{C}\boldsymbol{\Lambda}$$

Decomposition

Because the vectors are orthogonal and $\mathbf{c_i'c_i} = \mathbf{1}$

$$\mathbf{C'C} = \mathbf{I}$$

So

$$\mathbf{C}' = \mathbf{C}^{-\mathbf{1}},$$

$$\mathbf{CC}' = \mathbf{CC}^{-\mathbf{1}} = \mathbf{I}.$$

The diagonalization of a matrix **A** is

$$\mathbf{C'AC} = \mathbf{C'C\Lambda} = \mathbf{I\Lambda} = \mathbf{\Lambda}$$

The spectral decomposition of **A** is

$$\mathbf{A} = \mathbf{C\Lambda C}' = \sum_{i=1}^{k} \lambda_i \mathbf{c_i c_i'}$$

Definite matrices

How do we know if a symmetric matrix $\mathbf{A}$ is positive or negative (semi)definite or indefinite?

We know a symmetric matrix can be decomposed into

$$\mathbf{A} = \mathbf{C}\boldsymbol{\Lambda}\mathbf{C}'$$

Therefore,

$$\mathbf{x}'\mathbf{A}\mathbf{x} = \mathbf{x}'\mathbf{C}\boldsymbol{\Lambda}\mathbf{C}'\mathbf{x} \quad \textit{for any } \mathbf{x}$$

Let $\mathbf{y} = \mathbf{C}'\mathbf{x}$. Then

$$\mathbf{x}'\mathbf{A}\mathbf{x} = \mathbf{y}'\boldsymbol{\Lambda}\mathbf{y} = \sum_{i=1}^{N} \lambda_i y_i^2$$

Definite matrices

- $\mathbf{A}$ is positive (negative) definite if and only if all the characteristic roots of $\mathbf{A}$ are positive (negative).
- $\mathbf{A}$ is positive (negative) semidefinite if and only if all the characteristic roots are ≥ 0 (≤ 0).
- $\mathbf{A}$ is indefinite if and only if $\mathbf{A}$ has both positive and negative characteristic roots.

Some results related to definiteness

If $\mathbf{X}$ is any $N \times K$ matrix, the symmetric $K \times K$ matrix $\mathbf{X}'\mathbf{X}$ is positive definite, if and only if $N \geq K$ and $rank(\mathbf{X}) = K$.

To see this, define a $N \times 1$ vector

$$\mathbf{p} = \mathbf{Xz}$$

for any $K \times 1$ vector $\mathbf{z} \neq \mathbf{0}$. Then

$$\mathbf{z}'\mathbf{XXz} = \mathbf{p}'\mathbf{p} = \sum_{i=1}^{N} p_i^2 \geq 0$$

$$rank(\mathbf{X}'\mathbf{X} = K) \Leftrightarrow rank(\mathbf{X} = K)$$

$\Leftrightarrow$ There can exist no $\mathbf{z} \neq \mathbf{0}$ such that $\mathbf{p} = \mathbf{Xz} = \mathbf{0}$ (Recall the rank condition for a homogeneous system to have nonzero solution).
$\Leftrightarrow \sum_{i=1}^{N} p_i^2 \neq 0$
$\Leftrightarrow \mathbf{z}'\mathbf{XXz} = \mathbf{p}'\mathbf{p} > 0$ (matrix $\mathbf{X}'\mathbf{X}$ is positive definite)

Exercise

Using a similar proof as in the last slide, show that if $\mathbf{X}$ is any $N \times K$ matrix, the symmetric $N \times N$ matrix $\mathbf{XX}'$ is positive semidefinite if and only if $N \geq K$ and $rank(\mathbf{X}) = K$.

Another way to test definiteness: leading principle minors

Let **A** be an $N \times N$ matrix. The Kth order principal submatrix of **A** obtained by deleting the last $N - K$ rows and the last $N - K$ columns is called the Kth order **leading principal submatrix.** Its determinant is call the Kth order **leading principal minor**.

E.g. for a 3×3 matrix, the three leading principal minors are

$$\begin{vmatrix} a_{11} \end{vmatrix}, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

Another way to test definiteness: leading principle minors

Let **A** be an $N \times N$ matrix.

- **A** is positive definite (semidefinite) if and only if all its N leading principal minors > 0 (≥ 0).
- **A** is negative definite if and only if its N leading principle minors alternate in sign as follows:

$$|\mathbf{A_1}| < 0, \quad |\mathbf{A_2}| > 0, \quad |\mathbf{A_3}| > 0, \quad \textit{etc.}$$

  (Negative semidefinte if the inequalities are weak.)
- **A** is indefinite if the leading principal minors follow other patterns.

Exercise

(Davidson 4.6 Exercise 6)
Let

$$\mathbf{X} = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 3 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

1. Show that $\mathbf{X}'\mathbf{X}$ is positive definite.
2. Calculate $\mathbf{X}'\mathbf{X}^{-1}$. Is it also positive definite?