Course Preview
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Logic
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Proofs
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Review
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Lecture 01
Logic and Mathematical Proofs

Ryan McWay[†]

[†] *Applied Economics,*
University of Minnesota

Mathematics Review Course, Summer 2023
University of Minnesota
August 7th, 2023

Course Preview

THIS COURSE

▶ Review of graduate-level mathematics necessary for the 1st year sequence.

▶ Aimed at PhD-track. MS-track is encouraged.

▶ This sets the foundation (Not exhaustive).

▶ By the end you should feel confident tackling a variety of math situations in a short period.

▶ Syllabus on Github repo. Repo is the most up-to-date place for course content.

▶ This course is **optional**.

PREVIEW OF COURSE

1. Logic, Proofs, Sets, & Topology

2. Uni-variate Calculus & Multi-variate Calculus

3. Linear Algebra

4. Functions & Optimization

5. Probability & Statistics

6. Dynamic Programming

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ABOUT THE INSTRUCTOR

Ryan McWay

▶ Current: 2nd Year APEC PhD student

▶ Background: SLU → USF → UMich → UMN

▶ Research: Development, Behavior, Urban, Environment

DAILY ICEBREAKER

► Attendance via prompt:
  ► Name
  ► Hometown
  ► Program and track
  ► Research interests
  ► Daily Icebreaker: Imagine you are a professional baseball player or wrestler. What is you walk up (intro) song?

FIGHT WITH MATH...
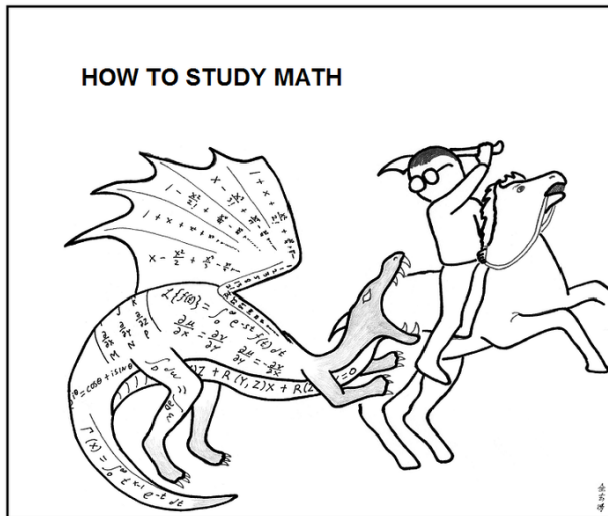
Don't just read it; fight it!

--- Paul R. Halmos

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Topic: Logic

MOTIVATION

▶ General background
  ▶ Logic is at the heart of reasoning and arguments.
  ▶ Expressed in words and formalized through math, this is a foundation of theoretical arguments.
  ▶ Deduce information correctly. Not deducing correct information.

▶ Why do economists' care?
  ▶ Foundation for theory
  ▶ Criteria to evaluate arguments

▶ Application in this career
  ▶ Creating logical arguments
  ▶ How you think about research
  ▶ Evaluating theory and conclusions from empirical evidence

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OVERVIEW

1. Logical Statements

2. Necessary Conditions

3. Sufficient Conditions

0. TERMINOLOGY

$\forall$	For all
$\exists$	Exists
:	Such that
|	Given
$\therefore$	Therefore
$\because$	Because
$\wedge$	And
$\vee$	Or
$\neg$	Negation
$\equiv$	Equivalent or identical.
$\implies$	Implies, then, or sufficient
$\iff$	If and only if, or necessary and sufficient
$\subset$	Strict subset
$\subseteq$	Subset
$\in$	In, or an element of the following set
$\square$	End of Proof. QED (quod erat demonstrandum $\sim$ it has been demonstrated).

0. TERMINOLOGY

$A\alpha$	$B\beta$	$\Gamma\gamma$	$\Delta\delta$	$E\varepsilon$	$Z\zeta$
ALPHA [a]	**BETA** [b]	**GAMMA** [g]	**DELTA** [d]	**EPSILON** [e]	**ZETA** [dz]
ἄλφα	*βῆτα*	*γάμμα*	*δέλτα*	*ἒ ψιλόν*	*ζῆτα*
$H\eta$	$\Theta\theta$	$I\iota$	$K\kappa$	$\Lambda\lambda$	$M\mu$
ETA [ɛː]	**THETA** [tʰ]	**IOTA** [i̯]	**KAPPA** [k]	**LAMBDA** [l]	**MU** [m]
ἦτα	*θῆτα*	*ἰῶτα*	*κάππα*	*λάμβδα*	*μῦ*
$N\nu$	$\Xi\xi$	Oo	$\Pi\pi$	$P\rho$	$\Sigma\sigma\varsigma$
NU [n]	**XI** [ks]	**OMICRON** [o]	**PI** [p]	**RHO** [r]	**SIGMA** [s]
νῦ	*ξεῖ*	*ὂ μικρόν*	*πεῖ*	*ῥῶ*	*σῖγμα*
$T\tau$	$\Upsilon\upsilon$	$\Phi\varphi$	$X\chi$	$\Psi\psi$	$\Omega\omega$
TAU [t]	**UPSILON** [ʉ]	**PHI** [pʰ]	**CHI** [kʰ]	**PSI** [ps]	**OMEGA** [ɔː]
ταῦ	*ὒ ψιλόν*	*φεῖ*	*χεῖ*	*ψεῖ*	*ὦ μέγα*

1. LOGICAL STATEMENTS

▶ Logical Statement: Use a set of facts to infer/assume a new fact.

  ▶ Hypothesis (If): Premise with set of facts
  ▶ Conclusion (Then): New set of facts inferred if hypothesis is true.
  ▶ e.g., **If** I study throughout the course, **then** I earn a higher grade.

▶ Family of statements:

  ▶ Tautologies: Statement is always true $(1 = 1)$
  ▶ Contradictions: Statement is always false $(2 = 3)$
  ▶ Statement: $A \implies B$
  ▶ Contrapositive: $\neg B \implies \neg A$
  ▶ Converse: $B \implies A$
  ▶ Inverse: $\neg A \implies \neg B$

1. Logical Statements

- ▶ Axiom: Statements assumed to be true.
  - ▶ e.g., $a = b$, $b = c \implies a = c$
- ▶ Theorem: A statement proven to be true.
- ▶ Corollary: A theorem that follows from another theorem.
- ▶ Lemma: A minor theorem used to prove another theorem.

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2. NECESSARY CONDITION

- ▶ *A* is necessary for *B*
  - ▶ If *B* is true, *A* must be true: $B \implies A$.
  - ▶ If *A* is not true, *B* is not true: $\neg A \implies \neg B$
- ▶ *A* is needed to make the argument.

3. SUFFICIENT CONDITION

- ▶ *A* is sufficient for *B*
  - ▶ If *A* is true, *B* must be true: $A \implies B$
  - ▶ If *B* is not true, *A* is not either: $\neg B \implies \neg A$
- ▶ *A* allows you to state *B*, but not necessary to make argument.

NECESSARY BUT NOT SUFFICIENT

4. NECESSARY AND SUFFICIENT
(IF AND ONLY IF ∼ IFF)

▶ If A is sufficient for B, B is necessary for A.
▶ If $A \implies B$ and $B \implies A$, then $A \iff B$ (iff)
  ▶ A is necessary and sufficient for B.
  ▶ A and B are equivalent statements.
  ▶ A is true iff B is true: A iff B

DEMONSTRATION: NECESSARY AND SUFFICIENT

Question:

Is this statement true: "If I open the door, I used the key."

Answer:

Logic: Open Door (A) $\implies$ Used Key (B) Necessary: You need a key (B) to open the door (A). $B \implies A$. Sufficient: If you do not have the key ($\neg B$), then there is no way to open the door ($\neg A$). So $A \implies B$.

DEMONSTRATION: NECESSARY AND SUFFICIENT

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PRACTICE: NECESSARY AND SUFFICIENT CONDITIONS

1. Scoring more touchdowns than your opponent in American football means you won the game.

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PRACTICE: NECESSARY AND SUFFICIENT CONDITIONS

1. Scoring more touchdowns than your opponent in American football means you won the game.

Answer: ◂ *Show Work*

Sufficient but not necessary.

PRACTICE: NECESSARY AND SUFFICIENT CONDITIONS

1. Scoring more touchdowns than your opponent in American football means you won the game.

2. Obtaining a learner's permit will lead to earning a driver's license.

PRACTICE: NECESSARY AND SUFFICIENT CONDITIONS

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2. Obtaining a learner's permit will lead to earning a driver's license.

Answer: ◄ Show Work

Necessary but not sufficient.

PRACTICE: NECESSARY AND SUFFICIENT CONDITIONS

1. Scoring more touchdowns than your opponent in American football means you won the game.

2. Obtaining a learner's permit will lead to earning a driver's license.

3. All even positive whole numbers must be divisible by two.

PRACTICE: NECESSARY AND SUFFICIENT CONDITIONS

1. Scoring more touchdowns than your opponent in American football means you won the game.

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Answer: (◂ Show Work)

Both necessary and sufficient.

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Topic: Proofs

MOTIVATION

▶ General background
  ▶ Method for proving or disproving a logical statement

▶ Why do economists' care?
  ▶ Determine which theories are incorporated into economic theory

▶ Application in this career
  ▶ Theory papers and well-developed theory sections of empirical papers.
  ▶ Often in appendix sections to prove statements articulated as part of an argument in a paper.

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OVERVIEW

1. Truth Tables

2. Existence & Uniqueness

3. Direct Proofs

4. Proof by Contradiction

5. Proof by Induction

6. Proof by Contrapositive

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Proofs
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ASSUMPTIONS ARE THE CORE OF PROOFS…

1. TRUTH TABLE

► Shows how the truth/falsity of a compound statement depends on the truth/falsity of the simple statements from which it's constructed.

► Statements = {Known to be true, known to be false, truth unknown }

► Truth table for $(P \to Q)$:

P	Q	$P \iff Q$
T	T	T
T	F	F
F	T	F
F	F	T

DEMONSTRATION: TRUTH TABLE

Question:

Construct a truth table for $(P \rightarrow Q) \vee (Q \rightarrow P)$

Answer:

P	Q	$P \rightarrow Q$	$Q \rightarrow P$	$(P \rightarrow Q) \vee (Q \rightarrow P)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

DEMONSTRATION: TRUTH TABLE

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Answer:

P	Q	$P \to Q$	$Q \to P$	$(P \to Q) \vee (Q \to P)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

2. EXISTENCE AND UNIQUENESS

▶ Existence: Can an answer to the truth of a statement (viz., proof) be found.
  ▶ Sometimes you can prove no answer can exist.
▶ Uniqueness: An assertion that there is exactly one statement that is true for that family of statements.
  ▶ x is the family $P(x)$.
▶ Ideally, you want a statement which exists and is unique.
  ▶ Ex. If "$x + 2 = 3$, then $x = 1$" is a statement that exists and is unique.

APPLICATION: EQUILIBRIUM EXISTS

Existence of Equilibrium II Theorem:

Suppose that each consumer's preferences are continuous, strongly monotonic, and convex. Suppose also that, for each consumer $i, \omega_i \gg 0$. Then there exists a Walrasian equilibrium (p^*, x^*) for ε.

3. PROOF BY DEDUCTION (DIRECT PROOF)

▶ Show $A \implies B$

▶ Deductive reasoning: Use a set of premises that lead to a conclusion.

▶ Sometimes we need to strengthen A but adding assumptions (e.g., weak assumptions are preferred).

DEMONSTRATION: DIRECT PROOF

Question:

Let m be an even integer and p be any integer. Then $m \times p$ is an even integer.

Answer:

Proof.

m is an even integer so $\exists$ an integer q such that $m = 2 \times q$ by the definition of an even integer. Therefore, we can make the statement:

$$m \times p = (2 \times q) \times p = 2 \times (q \times p)$$

So, $m \times p$ is an even integer. $\square$

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So, $m \times p$ is an even integer. $\hspace{2cm}$ □

4. PROOF BY CONTRADICTION

▶ $A \implies B \equiv \neg A$ and $\neg B \equiv \neg B \implies \neg A$.

▶ E.g., If the conclusion is not untrue, then the premise must be untrue.

DEMONSTRATION: PROOF BY CONTRADICTION

Question:

Walras' Law: $\forall x \in x(p, w)$ that maximizes consumer utility, then $x \times p = w$.

Answer:

Proof.

Suppose $\exists x \in x(p, w) : x \times p < w$ ($\neg B$), then there must be another $y \in x(p, w)$ that is affordable and $y \succ x$ by the assumption of "local non-satiation". Therefore, since y exists and is affordable, then x does not maximize utility ($\neg A$). $\qquad\square$

DEMONSTRATION: PROOF BY CONTRADICTION

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5. PROOF BY INDUCTION

Professor Schmidt demonstrates
the concept of proof by induction.

5. PROOF BY INDUCTION

▶ Inductive reasoning: Drawing conclusions by reasoning a series of specific examples generalizes.

▶ Often used by indexing through integers.

DEMONSTRATION: PROOF BY INDUCTION

Question:

$P(n): 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$

Answer:

Proof.

Note that $P(1)$ is true because $1 = \frac{1 \times 2}{2}$. Assume $P(n)$ is true for $k \in n$ integers: $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$. Add $(k+1)$ to both sides.

$$1 + 2 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}$$

This is $P(k+1)$, implying that $P(k)$ is true for all $P(n)$. □

DEMONSTRATION: PROOF BY INDUCTION

Question:
$P(n) : 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$

Answer:

Proof.
Note that $P(1)$ is true because $1 = \frac{1 \times 2}{2}$. Assume $P(n)$ is true for $k \in n$ integers: $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$. Add $(k+1)$ to both sides.

$$1 + 2 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}$$

This is $P(k+1)$, implying that $P(k)$ is true for all $P(n)$. $\qquad \square$

6. PROOF BY CONTRAPOSITIVE

▶ Double Negation:
$\neg B \implies \neg A \equiv \neg\neg A \implies \neg\neg B \equiv A \implies B.$

▶ Convenient when there is a universal quantifier ($\forall$) present included by the contrapositive.

DEMONSTRATION: PROOF BY CONTRAPOSITIVE

Question:

Suppose $x \in \mathbb{Z}$. If $7x + 9$ is even, then x is odd.

Answer:

Proof.

Suppose x is **not** odd (i.e., even) implying $x = 2a$ for some integer a. Then,

$$7x + 9 = 7(2a) + 9 = 14a + (8 + 1) = 2(7a + 4) + 1 = 2b + 1$$

if $b = 7a + 4$. Consequently, $2b + 1$ is odd for all b. Therefore $07x + 9$ is **not** even. $\qquad\square$

DEMONSTRATION: PROOF BY CONTRAPOSITIVE

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PRACTICE: PROOFS

1. If x is odd, then x^2 is odd.

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Answer: ◂ Show Work
x^2 is odd by definition of an odd number.

PRACTICE: PROOFS

1. If x is odd, then x^2 is odd.

2. Suppose $x \in \mathbb{Z}$. If $x^2 = 6x + 5$ is even, then x is odd.

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PRACTICE: PROOFS

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2. Suppose $x \in \mathbb{Z}$. If $x^2 = 6x + 5$ is even, then x is odd.

Answer: ◂ Show Work

By contrapositive, $x^2 = 6x + 5$ is odd and therefore not even.

PRACTICE: PROOFS

1. If x is odd, then x^2 is odd.
2. Suppose $x \in \mathbb{Z}$. If $x^2 = 6x + 5$ is even, then x is odd.
3. There are infinitely many prime numbers.

PRACTICE: PROOFS

1. If x is odd, then x^2 is odd.

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3. There are infinitely many prime numbers.

Answer: ◀ Show Work

Proof by contradiction.

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Review

REVIEW OF LOGIC

1. Logical Statement: Necessary to provide clarity to your statements

2. Necessary vs. Sufficient Conditions: Aiming to make statements that are both (iff)

REVIEW OF PROOFS

1. Truth Tables

2. Four methods to prove a statement:
- ▶ Direct proof
- ▶ Proof by contradiction
- ▶ Proof by induction
- ▶ Proof by Contrapositive

ASSIGNMENT

- ▶ Readings on Logic & Proofs before Lecture 02:
  - ▶ B&S Appendix A
  - ▶ Hammack Ch. 4 & 10
- ▶ Readings on Sets & Topology before Lecture 02:
  - ▶ B&S Ch. 11
  - ▶ S&B Ch. 12
- ▶ Assignment:
  - ▶ Problem Set 01 (PS01)
  - ▶ Solution set will be available following end of Lecture 02
- ▶ Struggling?
  - *1.* Read the 'Encouraged Reading'
  - *2.* Review 'Supplementary material'
  - *3.* Reach out directly

N & S Conditions Question 1 Answer:

- ▶ Not Necessary: If you won the game (B), you may have scored other points but less touchdowns ($\neg A$).
- ▶ Sufficient: If you score more touchdowns (A) (and therefore more overall points), then you will win the game (B).

N & S Conditions Question 2 Answer:

- ▶ Necessary: A learner's permit (A) is required before you can get a drivers license (B).
- ▶ Not Sufficient: Not all learners ($\neg A$) successfully earn their drivers license ($\neg B$).

N & S Conditions Question 3 Answer:

▶ Necessary: To be divisible by 2 (B), you must be a even whole number (A).

▶ Sufficient: If you are an even whole number (A), you will have no remainder if divided by two (B).

PROOFS QUESTION 1 ANSWER:

◂ QUESTION

Proof.

Suppose x is odd. Then $x = 2a + 1$ for some $a \in \mathbb{Z}$, by definition an odd number. Thus $x^2 = (2a + 1)^2 = 4a^2 + 4a + 1$. This $2(2a^2 + 2a) + 1$. So $x^2 = 2b + 1$ for an integer b. Therefore, x^2 is odd, by definition of an odd number. $\qquad\square$

PROOFS QUESTION 2 ANSWER:

◂ QUESTION

Proof.

Suppose x is **not** odd. Thus x is even, so $x = 2a$ for some integer a. So $x^2 - 6x + 5 = (2a)^2 - 6(2a) + 5 = 2(2a^2 - 6a + 2) + 1$. Then $x^2 - 6x + 5 = 2b + 1$ for $b = 2a^2 - 6a + 2$. Consequently, $x^2 - 6x + 5$ is odd, and therefore not even. □

PROOFS QUESTION 3 ANSWER:

◂ QUESTION

Proof.

Suppose there are only finite prime numbers. Then they can be listed as $p_1, p_2, \ldots, p_n$. Then p_n is the final and largest prime number. Consider a number $a = (p_1 \cdot p_2 \cdots p_n) + 1$. a has at least one prime divisor (e.g., p_k in the list). So there is some integer c such that $(p_1 \cdot p_2 \cdots p_{k-1} p_k p_{k+1} \cdots p_n) + 1 = c \cdot p_k$. Divide both sides by p_k. Now we have $\frac{1}{p_k} = c - (p_1 \cdot p_2 \cdots p_{k-1} p_{k+1} \cdots p_n)$. The expression on the right is an integer (i.e., which prime is a part of) **but** the left is not an integer. This is a contradiction. Therefore, there must be no finite range of prime numbers. □