Lecture Review
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Linear Algebra
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Review
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Lecture 07
Linear Algebra

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University of Minnesota

Mathematics Review Course, Summer 2023
University of Minnesota
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LAST LECTURE REVIEW

▶ Matrices:
    ▶ Matrix Operators
    ▶ Rank Trace
    ▶ The Determinant
    ▶ Positive and Negative Definite Matrices
    ▶ Linear Independence

REVIEW ASSIGNMENT

1. Problem Set 06 solutions are available on Github.

2. Any issues or problems **You** would like to discuss?

DAILY ICEBREAKER

- ► Attendance via prompt:
    - ► Name
    - ► Daily Icebreaker: Your parents inform you that you are an adopted superhero. What super-human ability do you possess?

Topic: Linear Algebra

MOTIVATION

▶ General background
  ▶ The algebra of manipulating matrices.
  ▶ A compact, efficient method for working with systems of equations (e.g., a spreadsheet).

▶ Why do economists' care?
  ▶ This is the foundation of how economists manipulate data.

▶ Application in this career
  ▶ Important applications in econometrics.
  ▶ Useful whenever you are creating or modifying an estimator.

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OVERVIEW

1. Systems of Linear Equations

2. Gaussian Elimination

3. Linear Operators

4. Existence of a Solution

5. Cramer's Rule

6. Eigenvalues and the Characteristic Equation

7. Leading Principle Minors

8. Regression as a Matrix

9. Centering Matrix

10. Residual Maker

1. SYSTEMS OF LINEAR EQUATIONS

▶ Linear Functions: $y = Ax$ with elements y_i of y such that $y_i = a_i^T x$.

$$a_i = \frac{\partial y_i}{\partial x} = \frac{\partial}{\partial x}(a_i^T x)$$

▶ Linear System: m equations for n unknown variables.

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

2. GAUSSIAN ELIMINATION

▶ Augmented Matrix

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n}|b_1 \\ a_{21} & a_{22} & \cdots & a_{2n}|b_2 \\ \vdots & \vdots & \ddots & \vdots|\vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn}|b_m \end{pmatrix}$$

2. GAUSSIAN ELIMINATION

▶ Reduced Row Echelon Form
  ▶ Interchange any two rows.
  ▶ Change row by adding a multiple of another row.
  ▶ Multiply each element in a row by a non-zero scalar.

$$\begin{pmatrix} 1 & 0 & \cdots & 0 | b_1 \\ 0 & 1 & \cdots & 0 | b_2 \\ \vdots & \vdots & \ddots & \vdots | \vdots \\ 0 & 0 & \cdots & 1 | b_m \end{pmatrix}$$

REDUCED ROW ECHELON FORM
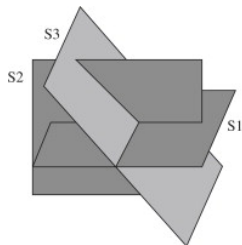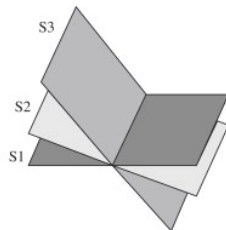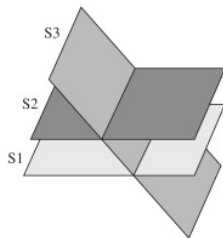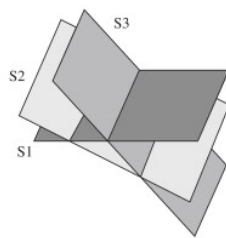
DEMONSTRATION: ROW ECHELON FORM

Question:

Reduced REF for $\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 2 \\ 1 & -3 & -1 \end{bmatrix}$

Answer:

$R_2 \leftarrow R_2 - 3R_1$
$R_3 \leftarrow R_3 - R_1$
$R_2 \leftarrow R_2 + R_3$
$R_1 \leftarrow R_1 + R_2$
$R_3 \leftarrow R_3 + 2R_2$
$R_3 \leftarrow \dfrac{-1}{17}R_3$
$R_1 \leftarrow R_1 + 5R_3$
$R_2 \leftarrow R_2 + 7R_3$

$\rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

DEMONSTRATION: ROW ECHELON FORM

Question:

Reduced REF for $\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 2 \\ 1 & -3 & -1 \end{bmatrix}$

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$$R_3 \leftarrow \frac{-1}{17}R_3$$
$$R_1 \leftarrow R_1 + 5R_3$$
$$R_2 \leftarrow R_2 + 7R_3$$

PRACTICE: ROW ECHELON FORM

1. Reduced REF for $\begin{pmatrix} 1 & 2 & | & 3 \\ 4 & 5 & | & 9 \end{pmatrix}$.

PRACTICE: ROW ECHELON FORM

1. Reduced REF for $\begin{pmatrix} 1 & 2 & | & 3 \\ 4 & 5 & | & 9 \end{pmatrix}$.

Answer: ◂ Show Work

$$\begin{pmatrix} 1 & 0 & | & -7 \\ 0 & 1 & | & 5 \end{pmatrix}$$

PRACTICE: ROW ECHELON FORM

1. Reduced REF for $\begin{pmatrix} 1 & 2 & | & 3 \\ 4 & 5 & | & 9 \end{pmatrix}$.

2. Reduced REF for $\begin{pmatrix} -1 & 1 & 0 & | & 1 \\ -2 & -3 & -1 & | & -2 \\ -3 & -1 & -2 & | & -1 \end{pmatrix}$.

PRACTICE: ROW ECHELON FORM

1. Reduced REF for $\begin{pmatrix} 1 & 2 & \big| & 3 \\ 4 & 5 & \big| & 9 \end{pmatrix}$.

2. Reduced REF for $\begin{pmatrix} -1 & 1 & 0 & \big| & 1 \\ -2 & -3 & -1 & \big| & -2 \\ -3 & -1 & -2 & \big| & -1 \end{pmatrix}$.

Answer: ◂ Show Work

$$\begin{pmatrix} 1 & 0 & 0 & \big| & \frac{-1}{3} \\ 0 & 1 & 0 & \big| & \frac{2}{3} \\ 0 & 0 & 1 & \big| & \frac{2}{3} \end{pmatrix}$$

PRACTICE: ROW ECHELON FORM

1. Reduced REF for $\begin{pmatrix} 1 & 2 & | & 3 \\ 4 & 5 & | & 9 \end{pmatrix}$.

2. Reduced REF for $\begin{pmatrix} -1 & 1 & 0 & | & 1 \\ -2 & -3 & -1 & | & -2 \\ -3 & -1 & -2 & | & -1 \end{pmatrix}$.

3. Reduced REF for $\begin{pmatrix} 3 & -3 & -2 & | & -1 \\ 0 & 2 & -3 & | & -3 \\ 3 & 3 & 2 & | & -3 \end{pmatrix}$.

PRACTICE: ROW ECHELON FORM

1. Reduced REF for $\begin{pmatrix} 1 & 2 & | & 3 \\ 4 & 5 & | & 9 \end{pmatrix}$.

2. Reduced REF for $\begin{pmatrix} -1 & 1 & 0 & | & 1 \\ -2 & -3 & -1 & | & -2 \\ -3 & -1 & -2 & | & -1 \end{pmatrix}$.

3. Reduced REF for $\begin{pmatrix} 3 & -3 & -2 & | & -1 \\ 0 & 2 & -3 & | & -3 \\ 3 & 3 & 2 & | & -3 \end{pmatrix}$.

Answer: ◂ *Show Work*

$$\begin{pmatrix} 1 & 0 & 0 & | & \frac{-2}{3} \\ 0 & 1 & 0 & | & \frac{-9}{13} \\ 0 & 0 & 1 & | & \frac{7}{13} \end{pmatrix}$$

3. LINEAR OPERATORS

▶ Product Rule:
$$\frac{\partial a^T b}{\partial x} = \frac{\partial a^T}{\partial x} b + \frac{\partial b^T}{\partial x} a$$

▶ Quadratic Form:
$$x^T A x = \sum_{i=1}^{n} \sum_{i=1}^{n} x_i x_j a_{ij}$$

4. EXISTENCE OF SOLUTION

- ▶ Important to know that a solution for a system exists.
- ▶ A system of linear equations with coefficient matrix A and an augmented matrix $\hat{A}$ has a solution iff:

  $$rank\hat{A} = rankA$$

- ▶ There are infinite solutions if $\#rows(A) < \#cols(A)$.
  - ▶ More unknown variables than observations.
- ▶ Non-singular Square Matrix: Ensure only one solutions exists iff

  $$\#rows(A) = \#cols(A) = rank(A)$$

5. CRAMER'S RULE

▶ A unique solution $x = (x_1, \cdots, x_n)$ for $n \times n$ system $Ax = b$ is

$$x_i = \frac{det B_i}{det A} \forall i = 1, \ldots, n$$

▶ Where B_i replaces the i'th column of A with the b vector.

5. CRAMER'S RULE

Cramer's Rule

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

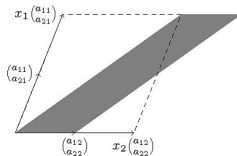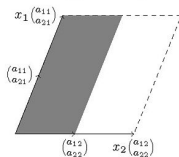$$\text{Let } D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$
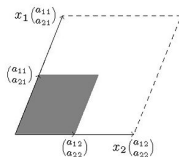
If D ≠ 0 then

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{D} \qquad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{D} \qquad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{D}$$

CRAMER'S RULE = STRETCH SCALAR

DEMONSTRATION: CRAMER'S RULE

Question:

Solve for x given

$$12x + 3y = 15$$
$$2x - 3y = 13$$

.

Answer:

$$x = \frac{D_x}{D} = \frac{\begin{vmatrix} 15 & 2 \\ 13 & -3 \end{vmatrix}}{\begin{vmatrix} 12 & 3 \\ 2 & -3 \end{vmatrix}} = \frac{-45 - 39}{-36 - 6} = \frac{-84}{-42} = 2$$

DEMONSTRATION: CRAMER'S RULE

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Solve for x given

$$12x + 3y = 15$$
$$2x - 3y = 13$$

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Answer:

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PRACTICE: CRAMER'S RULE

1. Solve for x, y given $\left(\begin{array}{cc|c} -2 & 1 & 3 \\ 8 & 4 & 9 \end{array}\right)$.

PRACTICE: CRAMER'S RULE

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Answer: ◂ *Show Work*

$$x = \frac{-3}{16}$$
$$y = \frac{21}{8}$$

PRACTICE: CRAMER'S RULE

1. Solve for x, y given $\begin{pmatrix} -2 & 1 & \bigm| & 3 \\ 8 & 4 & \bigm| & 9 \end{pmatrix}$.

2. Solve for x, y, z given $\begin{pmatrix} 1 & 1 & -1 & \bigm| & 6 \\ 3 & -2 & 1 & \bigm| & -5 \\ 1 & 3 & -2 & \bigm| & 14 \end{pmatrix}$.

PRACTICE: CRAMER'S RULE

1. Solve for x, y given $\left(\begin{array}{cc|c} -2 & 1 & 3 \\ 8 & 4 & 9 \end{array}\right)$.

2. Solve for x, y, z given $\left(\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 3 & -2 & 1 & -5 \\ 1 & 3 & -2 & 14 \end{array}\right)$.

Answer: ◂ Show Work

$$x = 1$$
$$y = 3$$
$$z = -2$$

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PRACTICE: CRAMER'S RULE

1. Solve for x, y given $\left(\begin{array}{cc|c} -2 & 1 & 3 \\ 8 & 4 & 9 \end{array}\right)$.

2. Solve for x, y, z given $\left(\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 3 & -2 & 1 & -5 \\ 1 & 3 & -2 & 14 \end{array}\right)$.

3. Solve for x, y, z given $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 11 \\ 1 & -2 & 1 & 0 \end{array}\right)$.

PRACTICE: CRAMER'S RULE

1. Solve for x, y given $\left(\begin{array}{cc|c} -2 & 1 & 3 \\ 8 & 4 & 9 \end{array}\right)$.

2. Solve for x, y, z given $\left(\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 3 & -2 & 1 & -5 \\ 1 & 3 & -2 & 14 \end{array}\right)$.

3. Solve for x, y, z given $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 11 \\ 1 & -2 & 1 & 0 \end{array}\right)$.

Answer: ◂ Show Work

$$x = 1$$
$$y = 2$$
$$z = 3$$

6. EIGENVALUES AND THE CHARACTERISTIC EQUATION

▶ Characteristic Equation

$$Ac = \lambda c$$
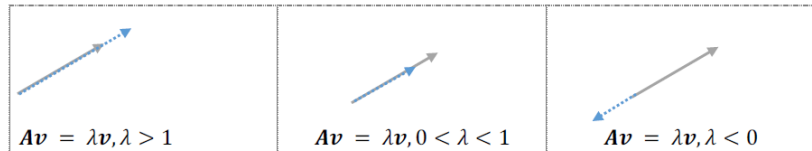
▶ Characteristic vectors: (c, λ)

▶ Eigenvectors: c

▶ Eigenvalues: λ

$$Ac = \lambda Ic \iff (A - \lambda I)c = 0$$

▶ Homogeneous system has non-zero solution if it is singular
and has zero determinant: $det(A - \lambda I) = 0$

EIGENVECTORS SCALED BY EIGENVALUES

$$Av = \lambda v, \lambda > 1 \qquad Av = \lambda v, 0 < \lambda < 1 \qquad Av = \lambda v, \lambda < 0$$

DEMONSTRATION: EIGENVALUES

Question:

Find the Eigenvalues and Eigenvector for $A = \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}$.

Answer:
For the Eigenvalues:

$$det(A - \lambda I) = 0$$
$$det(\begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = 0$$
$$det(\begin{bmatrix} -5 - \lambda & 2 \\ -7 & 4 - \lambda \end{bmatrix}) = 0$$
$$\lambda^2 + \lambda - 6 = 0$$
$$= \{2, -3\}$$

DEMONSTRATION: EIGENVALUES

Question:

Find the Eigenvalues and Eigenvector for $A = \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}$.

Answer:

For the Eigenvalues:

$$det(A - \lambda I) = 0$$
$$det(\begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = 0$$
$$det(\begin{bmatrix} -5 - \lambda & 2 \\ -7 & 4 - \lambda \end{bmatrix}) = 0$$
$$\lambda^2 + \lambda - 6 = 0$$
$$= \{2, -3\}$$

DEMONSTRATION: EIGENVALUES

Question:

Find the Eigenvalues and Eigenvector for $A = \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}$.

Answer:

For the Eigenvectors:

$$\left(\begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -7 & 2 \\ -7 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & \frac{2}{7} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solution: $= s \begin{bmatrix} \frac{2}{7} & 1 \end{bmatrix}$

$$= \begin{bmatrix} 2 & 7 \end{bmatrix}$$

DEMONSTRATION: EIGENVALUES

Question:

Find the Eigenvalues and Eigenvector for $A = \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}$.

Answer:

For the Eigenvectors:

$$\left(\begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -7 & 2 \\ -7 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & \frac{2}{7} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solution: $= s \begin{bmatrix} \frac{2}{7} & 1 \end{bmatrix}$

$$= \begin{bmatrix} 2 & 7 \end{bmatrix}$$

PRACTICE: EIGENVALUES

1. Find the Eigenvalues for $\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$.

PRACTICE: EIGENVALUES

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Answer: ◂ Show Work

$$\lambda = \{5, -2\}$$

PRACTICE: EIGENVALUES

1. Find the Eigenvalues for $\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$.

2. Find the Eigenvalues for $\begin{bmatrix} 1 & -3 & -7 \\ 0 & -2 & -6 \\ 0 & 2 & 5 \end{bmatrix}$.

PRACTICE: EIGENVALUES

1. Find the Eigenvalues for $\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$.

2. Find the Eigenvalues for $\begin{bmatrix} 1 & -3 & -7 \\ 0 & -2 & -6 \\ 0 & 2 & 5 \end{bmatrix}$.

Answer: ◂ Show Work

$$\lambda = \{1, 2\}$$

PRACTICE: EIGENVALUES

1. Find the Eigenvalues for $\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$.

2. Find the Eigenvalues for $\begin{bmatrix} 1 & -3 & -7 \\ 0 & -2 & -6 \\ 0 & 2 & 5 \end{bmatrix}$.

3. Find the Eigenvalues for $\begin{bmatrix} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{bmatrix}$.

PRACTICE: EIGENVALUES

1. Find the Eigenvalues for $\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$.

2. Find the Eigenvalues for $\begin{bmatrix} 1 & -3 & -7 \\ 0 & -2 & -6 \\ 0 & 2 & 5 \end{bmatrix}$.

3. Find the Eigenvalues for $\begin{bmatrix} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{bmatrix}$.

Answer: ◄ Show Work

$$\lambda = \{5, 10\}$$

7. LEADING PRINCIPLE MINORS

▶ A way to test for matrix definiteness.

▶ Leading Principal Sub-matrix: Let A be a $N \times N$ matrix. The k'th order principal sub-matrix of A obtained by deleting the last $N - K$ rows and the last $N - K$ columns.

▶ Leading Principal Minor: The determinant of the K'th order leading principal sub-matrix.

▶ Ex., for a 3×3 matrix, the leading principal minors are:

$$|a_{11}|, \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

7. Leading Principle Minors

▶ Positive Definite: Iff all N leading principal minors are ≥ 0.

▶ Negative Definite: Iff the N leading principal minors alternate signs

$$det(A_1) < 0, det(A_2) > 0, det(A_3) < 0 \text{ , etc.}$$

▶ Indefinite: Leading principal minors follow any other order.

8. REGRESSION AS A MATRIX

▶ A linear OLS model:

$$Y_{N \times 1} = X_{N \times K} \beta_{K \times 1} + e_{N \times 1}$$

▶ With the goal of selecting $\hat{\beta}$ that minimizes squared predicted errors.

$$\hat{e}^T \hat{e} = (Y - X\hat{\beta})^T (Y - X\hat{\beta})$$

8. Regression as a Matrix

▶ So, taking the first derivative we can get:

$$X^T X \hat{\beta} - X^T Y = 0$$
$$\implies$$
$$-X^T(Y - X\hat{\beta}) = -X^T \hat{e} = 0$$

▶ And the solution to OLS is:
$$\hat{\beta} = (X^T X)^{-1} X^T Y$$

9. Centering Matrix

- ► It is common in statistics to transform data to be 'deviations from the mean'.
- ► This can be done by creating the "centering matrix".
- ► First, create a multiplier $\frac{1}{N}ii^T$.

$$\frac{1}{N}ii^T = \frac{1}{N}\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{N} & \frac{1}{N} & \cdots & \frac{1}{N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{N} & \frac{1}{N} & \cdots & \frac{1}{N} \end{pmatrix}$$

- ► Then define vectors of means.

$$i\bar{x} = \frac{1}{N}ii^T x = \begin{pmatrix} \frac{1}{N} & \frac{1}{N} & \cdots & \frac{1}{N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{N} & \frac{1}{N} & \cdots & \frac{1}{N} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} \bar{x} \\ \bar{x} \\ \vdots \\ \bar{x} \end{pmatrix}$$

9. CENTERING MATRIX

▶ The vector of derivatives can be expressed as:

$$x - i\bar{x} = \begin{pmatrix} x_1 - \bar{x} \\ x_2 - \bar{x} \\ \vdots \\ x_N - \bar{x} \end{pmatrix} = x - \frac{1}{N}ii^T x = (I - \frac{1}{N}ii^T)x = M^0 x$$

▶ Summing these derivatives can be written as follows:

$$\sum_{i=1}^{N}(x_i - x)^2 = (x - i\bar{x})^T(x - i\bar{x}) = (M^0 x)^T(M^0 x) = x^T M^{0T} M^0 x$$

10. RESIDUAL MAKER

▶ Define
$$M = I - X(X^TX)^{-1}X^T$$

▶ Then
$$\hat{e} = Y - X\hat{\beta} = (I - X(X^TX)^{-1}X^T)Y = MY$$

▶ Residual Maker: M

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REVIEW OF LINEAR ALGEBRA

1. Systems of Linear Equations
2. Gaussian Elimination
3. Linear Operators
4. Existence of a Solution
5. Cramer's Rule
6. Eigenvalues and the Characteristic Equation
7. Leading Principle Minors
8. Regression as a Matrix
9. Centering Matrix
10. Residual Maker

ASSIGNMENT

▶ Readings on Numbers & Functions before Lecture 08:
  ▶ Hammack Ch. 12
  ▶ MWG Appendix M.B, M.C, M.F., & M.I.,
▶ Assignment:
  ▶ Problem Set 07 (PS07)
  ▶ Solution set will be available following end of Lecture 08
▶ Struggling?
  1. Read the 'Encouraged Reading'
  2. Review 'Supplementary material'
  3. Reach out directly

ROW ECHELON FORM QUESTION 1 ANSWER:

$$R_2 \leftarrow R_2 - 4R_1$$
$$R_1 \leftarrow R_1 - 2R_2$$

Row Echelon Form Question 2 Answer:

$$R_1 \leftarrow -R_1$$
$$R_2 \leftarrow R_2 + 2R_1$$
$$R_3 \leftarrow R_3 + 3R_1$$
$$R_2 \leftarrow R_2 - \frac{3}{2}R_3$$
$$R_1 \leftarrow R_1 + R_2$$
$$R_3 \leftarrow R_3 + 4R_2$$
$$R_3 \leftarrow \frac{1}{6}R_3$$
$$R_1 \leftarrow R_1 - 2R_3$$
$$R_2 \leftarrow R_2 - 2R_3$$

ROW ECHELON FORM QUESTION 3 ANSWER:

◀ QUESTION

$$R_1 \leftarrow R_1 - \frac{2}{3}R_3$$

$$R_3 \leftarrow R_3 - 3R_1$$

$$R_2 \leftarrow \frac{1}{2}R_2$$

$$R_1 \leftarrow R_1 + 5R_2$$

$$R_3 \leftarrow R_3 - 18R_2$$

$$R_3 \leftarrow \frac{1}{39}R_3$$

$$R_1 \leftarrow R_1 + \frac{65}{6}R_3$$

$$R_2 \leftarrow R_2 + \frac{3}{2}R_3$$

CRAMER'S RULE QUESTION 1 ANSWER:

◄ QUESTION

$$D = -16$$
$$D_x = 3$$
$$D_y = -42$$
$$x = \frac{-3}{16}$$
$$y = \frac{21}{8}$$

CRAMER'S RULE QUESTION 2 ANSWER:

◄ QUESTION

$$D = -3$$
$$D_x = -3$$
$$D_y = -9$$
$$D_z = 6$$
$$x = 1$$
$$y = 3$$
$$z = -2$$

CRAMER'S RULE QUESTION 3 ANSWER:

◂ QUESTION

$$D = 9$$
$$D_x = 9$$
$$D_y = 18$$
$$D_z = 27$$
$$x = 1$$
$$y = 2$$
$$z = 3$$

EIGENVALUES QUESTION 1 ANSWER:

◂ QUESTION

$$\lambda^2 - 3\lambda - 10 = 0$$
$$\lambda = \{5, -2\}$$

EIGENVALUES QUESTION 2 ANSWER:

$$-\lambda^3 + 4\lambda^2 - 5\lambda + 2 = 0$$
$$-(\lambda - 1)^2(\lambda - 2) = 0$$
$$\lambda = \{1, 2\}$$

EIGENVALUES QUESTION 3 ANSWER:

◄ QUESTION

$$(\lambda - 5)(\lambda^2 - 20\lambda + 100 = 0$$
$$\lambda = \{5, 10\}$$