Math Review
Summer 2016

Topic 4

4. Real valued functions and correspondences

4.1. Functions and correspondences

We have so far extensively used multiple definitions of a function. As we recall, functions are single-valued. For a function $f : X \rightarrow Y$, every $x \in X$ is mapped to one and only one point $y \in Y$, the point $y = f(x)$. Imagine a consumer with a *strictly quasiconcave* utility function (we define this later but for now take it to mean a nicely behaving function that gives us a unique answer when optimized) who behaves according to the *Utility Maximization expectation.* His/her market behavior is characterized by a demand function which we derived from the utility maximization process. This chosen bundle of consumption a single-valued function.

Often, however, we need to analyze behavior that is not uniquely determined. Consumers may have a place where they are indifferent between bundles at a given price for example. In this case, behavior is not single-valued.

A correspondence $f : X \rightarrow Y$ is thus a set-valued function from X to Y
— for every $x \in X, f(x)$ is a subset of Y.

In this class, we will not spend a whole lot of time on correspondences, but know that they exist and will come up often. Think of the choice structure we learned about earlier, remember that it could assign multiple elements from one set to the other? That is a correspondence (or function in the case of one element assigned)!

I feel that yesterday the class struggled a tiny bit with finding the Hessian. Let's start this session by practicing an example which will use some of the math skills we have down this far.

I have not worked through it, but it should be manageable, so I will work on it at the same as you all.

$\mathcal{Q}$: Find the Hessian of the equation $f(x,y) = (e^{x^2} + \ln y)y^2$

$\mathcal{A}$:

4.2. Homogeneous functions and Euler's Theorem

<u>Homogeneity in functions:</u> The function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is homogenous of degree k if:

$$f(\alpha x_1, \alpha x_2 + \cdots + \alpha x_n) = \alpha^k f(x_1 + x_2 + \cdots + x_n)$$

Example. The function $y = ax^k$ is homogeneous of degree k, can you tell why?

Denote: $y = y(x) = ax^k$

If this is homogenous of degree k, then we have (from above): $y(\beta x) = \beta^k y(x)$

Let's determine what $y(\beta x) =$?

$$\begin{aligned} y(\beta x) &= a(\beta x)^k \\ &= a(\beta^k)x^k \\ &= \beta^k[ax^k] \end{aligned}$$

Thus using our definition above, $y(\beta x) = \beta^k y(x)$. Thus, it is homogenous of degree k.

Q: Can you try with $y = ax_1^{k_1}x_2^{k_2}x_3^{k_3}$? It should be homogenous of degree $k_1 + k_2 + k_3$.

$\mathcal{A}$:

$$y(x) = ax_1^{k_1}x_2^{k_2}x_3^{k_3}$$

$$y(\beta x) = a(\beta x_1)^{k_1}(\beta x_2)^{k_2}(\beta x_2)^{k_2}$$

$$y(\beta x) = a\beta^{k_1}(x_1)^{k_1}\beta^{k_2}(x_2)^{k_2}\beta^{k_3}(x_2)^{k_2}$$

$$y(\beta x) = a\beta^{k_1+k_2+k_3}(x_1)^{k_1}(x_2)^{k_2}(\beta x_2)^{k_2}$$

$$y(\beta x) = \beta^{k_1+k_2+k_3}[\alpha(x_1)^{k_1}(x_2)^{k_2}(\beta x_2)^{k_2}]$$

$$y(\beta x) = \beta^{k_1+k_2+k_3}[y(x)]$$

You will learn that cost functions, denoted by $C(r,q)$, that is input prices r and output q, are homogeneous of **degree 1 in r**. Without you thinking too much into the details, how would you adapt the above definition to write this out mathematically?

$C(\lambda r, q) = \lambda C(r,q)$, where $k = 1$, (note λ concerns with r only! (compare to demand example)

I've told you the result, but let's double check with a given cost function. This cost function is from the June 2015 Micro prelim:

$$C(r,q) = (r_1^\alpha + r_2^\alpha)^{1/\alpha}\,(q_1^{\frac{3}{4}} + q_2^{\frac{1}{4}})$$

My initial fear in production was the scary functional forms. But let's practice understanding what we are looking at. We have the cost function expressed in terms of input prices r and output q. As per the functional form, we have two prices, $r_1 and\ r_2$, and two outputs $q_1 and\ q_2$.

$\mathcal{Q}$: Show that this cost function $C(r,q)$ is homogeneous of degree 1 in r.

$\mathcal{A}$: We want to show that $C(\theta r, q) = \theta C(r,q)$.

$$\begin{aligned} C(\theta r, q) &= ((\theta r_1)^\alpha + (\theta r_2)^\alpha)^{1/\alpha}\,(q_1^{\frac{3}{4}} + q_2^{\frac{1}{4}}) \\ &= (\theta^\alpha (r_1)^\alpha + \theta^\alpha (r_2)^\alpha)^{1/\alpha}\,(q_1^{\frac{3}{4}} + q_2^{\frac{1}{4}}) \\ &= \theta^{\alpha * \frac{1}{\alpha}}[(r_1)^\alpha + (r_2)^\alpha)^{1/\alpha}\,(q_1^{\frac{3}{4}} + q_2^{\frac{1}{4}})] \\ &= \theta[(r_1^\alpha + r_1^\alpha)^{\frac{1}{\alpha}}\left(q_1^{\frac{3}{4}} + q_2^{\frac{1}{4}}\right)] \\ &= \theta C(r,q) \text{ (shown)} \end{aligned}$$

One more practice. Demand if often expressed as $x(p, w)$, that is as a function of both prices, p and wealth, w.

Q: What degree of homogeneity holds for the following demand curve (in both prices and wealth)? This is a one-line answer!

$\mathcal{A}$:

$$x_2 = \frac{w}{p_2} - \left(\frac{p_1}{p_2}\right) x_1$$

$\mathcal{A}$:

$$x_2 = \frac{\theta w}{\theta p_2} - \left(\frac{\theta p_1}{\theta p_2}\right) x_1$$

Demand functions are said to be homogeneous of degree __ in prices and wealth. What that gets at is that there is no "money illusion." The same percentage increase in prices and wealth will lead to no change in consumption.

You should be able to check for the homogeneity of any given function, easy or hard. They often look harder than what they are. This is a prelim question. Try it out:

Q: *You are the proud owner of a very expensive Mercedes-Benz. It cost you a lot of money to buy that car, so now you spend all your money on only two things that give you utility: driving your car around, at a cost of* p_m *per mile driven, and buying rice to eat, which has a price of* p_r. *Your total wealth for these two activities is denoted by* w.
Your indirect utility function (<u>which is like a utility function but measured in a different way – you will learn more about this later</u>), denoted by v, *is:*

$$v\ (p, w) = -\left(\frac{1}{\lambda}\right) e^{\lambda\left(\frac{w}{p_r}\right)} - 1/\beta e^{\alpha + \beta \frac{p_m}{p_r}}$$

The indirect utility function should be homogeneous of some degree $\boldsymbol{k}$ *in prices and wealth. What is* $\boldsymbol{k}$*?*

$\mathcal{A}$:

$$v = -\left(\frac{1}{\lambda}\right) e^{\lambda\left(\frac{\theta w}{\theta p_r}\right)} - \frac{1}{\beta} e^{\alpha + \beta \frac{\theta p_m}{\theta p_r}}$$

Euler's Theorem. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function. The function f is homogenous of degree k if and only if:

$$x_1 \frac{\partial f}{\partial x_1}(x_1, \dots, x_n) + x_2 \frac{\partial f}{\partial x_2}(x_1, \dots, x_n) + \cdots x_n \frac{\partial f}{\partial x_n}(x_1, \dots, x_n) = k\, f(x_1, x_2, \dots, x_n)$$

Or more simply:

$$\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i}(x) = kf(x)$$

This can be simply expressed in vector notations as: $\boldsymbol{x} \cdot \Delta f(\boldsymbol{x}) = kf(\boldsymbol{x})$

Can we check whether the above cost function satisfies Euler's Theorem?

Since the cost function is homogeneous of degree $k = 1$, we need:

$$\sum_{i=1}^{n} r_i \frac{\partial C}{\partial r_i}(r, q) = 1 * C(r, q) = C(r, q)$$

$$\frac{\partial C}{\partial r_1}(r, q) = \frac{1}{\alpha} * \alpha r_1^{\alpha-1} (r_1^{\alpha} + r_2^{\alpha})^{\frac{1}{\alpha}-1} \left(q_1^{\frac{3}{4}} + q_2^{\frac{1}{4}}\right)$$
$$= r_1^{\alpha-1} (r_1^{\alpha} + r_2^{\alpha})^{\frac{1}{\alpha}-1} (q_1^{\frac{3}{4}} + q_2^{\frac{1}{4}})$$

$$\frac{\partial C}{\partial r_1}(r, q) = \frac{1}{\alpha} * \alpha r_1^{\alpha-1} (r_1^{\alpha} + r_2^{\alpha})^{\frac{1}{\alpha}-1} \left(q_1^{\frac{3}{4}} + q_2^{\frac{1}{4}}\right)$$
$$= r_2^{\alpha-1} (r_1^{\alpha} + r_2^{\alpha})^{\frac{1}{\alpha}-1} (q_1^{\frac{3}{4}} + q_2^{\frac{1}{4}})$$

Thus,

$$\sum_{i=1}^{n} r_i \frac{\partial C}{\partial r_i}(r, q) = r_1 (r_1^{\alpha-1} (r_1^{\alpha} + r_2^{\alpha})^{\frac{1}{\alpha}-1} \left(q_1^{\frac{3}{4}} + q_2^{\frac{1}{4}}\right))$$
$$+ r_2 (r_2^{\alpha-1} (r_1^{\alpha} + r_2^{\alpha})^{\frac{1}{\alpha}-1} \left(q_1^{\frac{3}{4}} + q_2^{\frac{1}{4}}\right))$$
$$= r_1^{\alpha} (r_1^{\alpha} + r_2^{\alpha})^{\frac{1}{\alpha}-1} \left(q_1^{\frac{3}{4}} + q_2^{\frac{1}{4}}\right) + r_2^{\alpha} (r_1^{\alpha} + r_2^{\alpha})^{\frac{1}{\alpha}-1} \left(\square_1^{\frac{3}{4}} + q_2^{\frac{1}{4}}\right)$$
$$(r_1^{\alpha} + r_2^{\alpha}) \left[(r_1^{\alpha} + r_1^{\alpha})^{\frac{1}{\alpha}-1} \left(q_1^{\frac{3}{4}} + q_2^{\frac{1}{4}}\right)\right]$$

$$= (r_1^{\alpha} + r_2^{\alpha})^{1/\alpha}\left(q_1^{\frac{3}{4}} + q_2^{\frac{1}{4}}\right) = C(r,q)$$

$\mathcal{Q}$: Let's practice and check Euler's theorem for the case of:

$$x_2 = \frac{w}{p_2} - \left(\frac{p_1}{p_2}\right)x_1$$

$\mathcal{A}$: If we are taking the derivative with respect to an arbitrary point v, then:

$$\sum_{i=1}^{n} v_i \frac{\partial f}{\partial v_i}(v_i) = 0 * f(v_i) = 0$$

$$p_1 \frac{\partial f}{\partial p_1}(.) + p_2 \frac{\partial f}{\partial p_2}(.) + w \frac{\partial f}{\partial w}(.) = 0$$

$$= p_1\left[-\frac{1}{p_2}x_1\right] + p_2\left[-\frac{w}{p_2^2} + \frac{p_1}{p_2}x_1\right] + w\left[\frac{1}{p_2}\right]$$

$$= -\frac{p_1}{p_2}x_1 - \frac{w}{p_2} + \frac{p_1}{p_2}x_1 - \frac{w}{p_2}$$

$$= 0$$

Another result that you may want to know is the following: if $f(x)$ is a continuously differentiable function that is homogenous of degree k, then f_i is homogenous of degree $k-1$ for all $i = \{1, \ldots, n\}$.

What this is telling us is that for instance in our previous exercise, since : $y = y(x) = ax^k$ was homogeneous of degree k, then y' is homogenous of degree $k-1$.

$\mathcal{Q}$: Can we check whether this is true for this given case above?

$\mathcal{A}$:

$$y = y(x) = ax^k$$
$$y' = y'(x) = a(k-1)x^{k-1}$$

Is $y'(\rho x) = \rho^{k-1}y'(x)$?- Yes.

$$y' = y'(x) = a(k-1)x^{k-1}$$
$$y'(\beta x) = a(k-1)(\beta x)^{k-1}$$
$$y'(\beta x) = \beta^{k-1}a(k-1)(x)^{k-1}$$
$$y'(\beta x) = \beta^{k-1}y'(x)$$

You should be able to check Euler's for all the different functions we tried today. Practicing now is always good, although you have plenty of time to get accustomed to this.

4.3. Concavity and quasi-concavity in functions

A summary of useful properties for multivariable functions are *(some are reminders of things we already touched on before)*:

- The function $f: \mathbb{R}^n \to \mathbb{R}$ is monotone if for all $x, y \in \mathbb{R}^n$, if $x \geq y$ then $f(x) \geq f(y)$.
- The function $f: \mathbb{R}^n \to \mathbb{R}$ is strictly monotone if for all $x, y \in \mathbb{R}^n$, if $x \geq y$ then $f(x) > f(y)$.
- The function $f: \mathbb{R}^n \to \mathbb{R}$ is concave if for all $x, y \in \mathbb{R}^n$ $\lambda \in [0,1]$ we have $f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y)$.
- The function $f: \mathbb{R}^n \to \mathbb{R}$ is convex if for all $x, y \in \mathbb{R}^n$ $\lambda \in [0,1]$we have $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$.
- The function $f: \mathbb{R}^n \to \mathbb{R}$ is quasiconcave if for all $x, y \in \mathbb{R}^n$ $\lambda \in [0,1]$we have $f(x) \geq f(y)$ imply that $f(\lambda x + (1-\lambda)y) \geq f(y)$.
- The function $f: \mathbb{R}^n \to \mathbb{R}$ is quasiconvex if for all $x, y \in \mathbb{R}^n$ $\lambda \in [0,1]$we have $f(x) \leq f(y)$ imply that $f(\lambda x + (1-\lambda)y) \leq f(y)$.

As you see, a concave function by definition is quasiconcave (not the other way though). You will learn more about this, but we often require utility functions to be quasiconcave rather than the stronger assumption of concavity. This is more micro than math, but when dealing with preferences and utility, ordering matters more than actual numbers. We care whether x is ranked higher than y, not by how much insomuch. Quasiconcavity is a good enough assumption to represent this ordering between alternatives.

4.3.1 Hessian Matrix in the mix

Now that we have functions of several variables, the second derivative check for convexity/concavity is the Hessian Matrix. In this part, we determine how to check for concavity and convexity.

Theorem. Let $f: \mathbb{R}^n \to \mathbb{R}$ be a twice continuously differentiable function. Then, f *is concave* if and only if the Hessian $D^2 f(x)$ is ***negative semidefinite*** for all $x \in \mathbb{R}^n$. The function f is *strictly concave* if and only if the Hessian is ***negative definite***. The function f is *convex* if and only if the Hessian is ***positive semidefinite*** for all $x \in \mathbb{R}^n$ and *strictly convex* if and only if the Hessian is ***positive definite***.

We start by defining what a principal minor is:

Let $\boldsymbol{A}$ be an $n \, x \, n$ square matrix.

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

We can find *principal submatrices* of $\boldsymbol{A}$ by forming the first r rows and r columns, where $r = 1,2,3, \dots, n$.

$$\boldsymbol{A_1} = [a_{11}]$$

$$\boldsymbol{A_2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Q: What is $\boldsymbol{A_3}$? Write it out.

$\boldsymbol{\mathcal{A}}$:

$$\boldsymbol{A_3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Evidently, $\boldsymbol{A_n} = \boldsymbol{A}$

In each case the *principal minor* is the determinant of the matrix. It is given by:

Determinant of $\boldsymbol{A_1}$, denoted as $det\ [a_{11}]$ **or** $|A_1|$ is just the number a_{11}.

To find the determinant of $\boldsymbol{A_2}$, we need to take the following difference:

$$|\boldsymbol{A_2}| = \boldsymbol{a_{11}a_{22} - a_{12}a_{21}}$$

For visual thinkers:

$$|\boldsymbol{A_2}| = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Solid: $+\boldsymbol{a_{11}a_{22}}$
Dashed: $-\boldsymbol{a_{12}a_{21}}$

For a $3\ x\ 3$ matrix, we have to do a little more:

$$|\boldsymbol{A_3}| = \left|\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\right|$$

$$= +a_{11}\left|\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}\right| - a_{12}\left|\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}\right| + a_{13}\left|\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}\right|$$

Q: You have all the information you need. Can we write out the final expression to obtain the determinant?

$\mathcal{A}$:

$$Det\ \boldsymbol{A_3} = a_{11}(a_{22}a_{33} - a_{23}a_{33}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}\ (a_{21}a_{32} - a_{22}a_{31})$$

Given an $n\ x\ n$ matrix $\boldsymbol{A}$, we have the following:

- The $n \times n$ matrix $\boldsymbol{A}$ is positive definite if and only if its n principal minors are all greater than 0: $detA_1 > 0, detA_2 > 0, \ldots, detA_n > 0$.

- The $n \times n$ matrix $\boldsymbol{A}$ is negative definite if and only if its n principal minors alternate in sign with the odd order ones being negative and the even order ones being positive: $det\ A_1 < 0, det\ A_2 > 0, detA_3 < 0, detA_4 > 0, \ldots.$
- The $n \times n$ matrix $\boldsymbol{A}$ is positive semidefinite if and only if its principal minors are all greater than or equal to 0: $det\ A_1 \geq 0\ and\ det\ A_2 \geq 0, \ldots, A_n \geq 0.$
- The $n \times n$ matrix $\boldsymbol{A}$ is negative semidefinite if and only if its n principal minors alternate in sign with the odd order ones being less than or equal to zero and the even order ones being greater than or equal to 0: $det\ A_1 \leq 0\ and\ det\ A_2 \geq 0, \ldots.$

Consider the following example: $f(x, y) = 2x - y - x^2 + 2xy - y^2$. Is this a concave function?

Recall two definitions:

(i) f *is concave* if and only if the Hessian $D^2 f(x)$ is ***negative semidefinite*** for all $x \in \mathbb{R}^n$.

(ii) An $n \times n$ matrix $\boldsymbol{A}$ is negative semidefinite if and only if its n principal minors alternate in sign with the odd order ones being less than or equal to zero and the even order ones being greater than or equal to 0: $det\ A_1 \leq 0\ and\ det\ A_2 \geq 0, \ldots.$

$$H = \begin{bmatrix} f_{11} & f_{21} \\ f_{12} & f_{22} \end{bmatrix}$$

$$f_1 = 2 - 2x + 2y; \quad f_{11} = -2; \ f_{12} = 2$$
$$f_2 = -1 + 2x - 2y; \ f_{22} = -2; \ f_{12} = 2$$

$$H = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$$

$|H_1| = -2$, thus ≤ 0

What is $|H_2|$?

$$|H_2| = -2(-2) - 2(2) = 0$$

$|H_2| = 0$, thus ≥ 0

Thus f is concave.

The test for quasiconcacity and quasiconvexity relies on a *bordered Hessian matrix* which is a little more complicated (but totally manageable). I will refer you to Mas-Colell's Mathematical Appendix p. 938-939 to more information.

Q: You have a utility function given by $u(x) = x_1^{\frac{1}{4}} x_2^{\frac{3}{4}}$. This is called a Cobb Douglas utility function with commodities x_1 and x_2. Is this utility concave?
(*Just a note, we define commodities to be nonnegative.*)

$\mathcal{A}$:

$$H = \begin{bmatrix} u_{11} & u_{21} \\ u_{12} & u_{22} \end{bmatrix}$$

$$u_1 = \frac{1}{4} x_1^{-\frac{3}{4}} x_2^{\frac{3}{4}}$$
$$u_{11} = -\frac{3}{16} x_1^{-\frac{7}{4}} x_2^{\frac{3}{4}}$$
$$u_{12} = \frac{3}{16} x_1^{-\frac{3}{4}} x_2^{-\frac{1}{4}}$$

$$u_2 = \frac{3}{4} x_1^{\frac{1}{4}} x_2^{-\frac{1}{4}}$$
$$u_{21} = -\frac{3}{16} x_1^{-\frac{3}{4}} x_2^{-\frac{1}{4}}$$
$$u_{22} = -\frac{3}{16} x_1^{\frac{1}{4}} x_2^{-\frac{5}{4}}$$

$$H = \begin{bmatrix} -\frac{3}{16} x_1^{-\frac{7}{4}} x_2^{\frac{3}{4}} & \frac{3}{16} x_1^{-\frac{3}{4}} x_2^{-\frac{1}{4}} \\ -\frac{3}{16} x_1^{-\frac{3}{4}} x_2^{-\frac{1}{4}} & -\frac{3}{16} x_1^{\frac{1}{4}} x_2^{-\frac{5}{4}} \end{bmatrix}$$

$|u_1| = -\frac{3}{16} x_1^{-\frac{7}{4}} x_2^{\frac{3}{4}}$, thus $\leq\ 0$

What is $|u_2|$?

$$|H_2| = (-ve) * (-ve) - (+ve)(-ve) \geq 0$$

Thus u is concave.

3.8. Implicit functions

<u>Implicit Functions</u>: Thus far, we have focused on functions whereby we have an endogenous variable y, and exogenous variables, x, on the right hand side: $y = f(x)$. These would be explicit functions.

In some cases, we have $f(x_1, x_2, \dots, x_n, y) = 0$. The endogenous variable y is an implicit function of the exogenous variables x.

Example. $y^2 - 5xy + 4x^2 = 0$

If we were to rearrange and express it as an explicit function, we have:

$$y = \frac{5x \pm \sqrt{25x^2 - 16x^2}}{2}$$

However, if we have $y^5 - 5xy + 4x^2 = 0$, we do not have a corresponding explicit function. Implicit functions are very popular in economics. For instance, think of a profit function for a firm given by $pf(x) - cx$, where $f(x)$ is the production function, p is price, and c is cost per unit. In order to find a local maximum, we need to take the first derivative and set it equal to 0. $pf'(x) - c = 0$ is an implicit function with solution $x = x(p, c)$.

The *Implicit Function Theorem* tells us when we are able to find an explicit function formula for y. Let $f(x_1, \dots, x_n, y)$ be a continuously differentiable point in the open ball, B, around the point $(\hat{x}_1, \dots, \hat{x}_n, \hat{y})$ where the point satisfies:

$f((\hat{x}_1, \dots, \hat{x}_n, \hat{y}) = c$ and $\frac{\partial f(\hat{x}_1, \dots, \hat{x}_n, \hat{y})}{\partial y} \neq 0,$

Then, there exists a continuously differentiable function $y = y(x_1, \dots, x_n)$ in the open ball B around $(\hat{x}_1, \dots, \hat{x}_n, \hat{y})$ such that:

(i) $f\big(x_1, \dots, x_n, y(x_1, \dots, x_n)\big) = c$ for all $(x_1, \dots, x_n) \in B$.

(ii) $\hat{y} = y(\hat{x}_1, \dots, \hat{x}_n)$

And the MOST used result (in my opinion) is that:

(iii) For each $i = 1,2, \dots, n$:

$$\frac{\partial y}{\partial x_i}(\hat{x}_1, \dots, \hat{x}_n) = -\frac{\dfrac{\partial f(\hat{x}_1, \dots, \hat{x}_n, \hat{y})}{\partial x_1}}{\dfrac{\partial f(\hat{x}_1, \dots, \hat{x}_n, \hat{y})}{\partial y}}$$

Example. Consider the equation $x^3 + 3y^2 + 4xz^2 - 3z^2y = 1$ in the neighborhood of $x = 0.5$ and $y = 0$. Let's apply this equation to what we know about implicit functions.

Condition 1: $0.5^3 + 4(0.5)z^2 = \frac{1}{8} + 2z^2$

$$z = \pm\sqrt{\frac{7}{16}}$$

Condition 2: We have the $\frac{\partial f(x,y,z)}{\partial z} = 8xz - 6zy$

$$\frac{\partial f\left(\frac{1}{2},0,\sqrt{\frac{7}{16}}\right)}{\partial z} = \sqrt{7} \neq 0$$

Thus, we can apply the Implicit Function theorem to the point $(\frac{1}{2}, 0, \sqrt{\frac{7}{16}})$. There exists a continuously differentiable function $z = z(x,y)$ in the open ball B around (x,y) such that we have:

$$\frac{\partial z}{\partial x}(\frac{1}{2}, 0, \sqrt{\frac{7}{16}}) = -\frac{\frac{\partial f}{dx}}{\frac{\partial f}{\partial z}} = -\frac{3x^2 + 4z}{8xz - 6zy}$$
$$\approx -\frac{40}{16\sqrt{7}}$$

$$\frac{\partial z}{\partial y}(\frac{1}{2}, 0, \sqrt{\frac{7}{16}}) = -\frac{\frac{\partial f}{dy}}{\frac{\partial f}{\partial z}} = -\frac{6y - 3z^2}{8xz - 6zy}$$
$$\approx \frac{21}{16\sqrt{7}}$$

Example. You have a utility function given by: $u^1 = u^1(x_1, x_2, x_3)$. You will be exposed to the concept of marginal rate of substitution (MRS) of good l for good k given by: $\frac{dx_k}{dx_l}$. Intuitively, the MRS is the rate at which a consumer is ready to give up one good in exchange for another good while maintaining the same level of utility. (*I will not further elaborate on this concept as you will cover this carefully in Consumer Theory).*

Applying the implicit function theorem above:

$$\frac{dx_k}{dx_l} = -\frac{\frac{\partial u^1(x_1, x_2, x_3)}{\partial x_l}}{\frac{\partial u^1(x_1, x_2, x_3)}{\partial x_k}}$$

Q: Given a utility function, denoted by $u^1 = u^1(x_1, x_2, x_3) = x_1^{\frac{1}{4}} x_2^{\frac{1}{3}} x_3^{\frac{1}{6}}$. Find $\frac{dx_2}{dx_3}$ and $\frac{dx_2}{d_{x1}}$

$\mathcal{A}$:

$$\frac{dx_2}{dx_3} = -\frac{\frac{\partial u^1(x_1, x_2, x_3)}{\partial x_3}}{\frac{\partial u^1(x_1, x_2, x_3)}{\partial x_2}}$$

$$= -\frac{\frac{1}{6} x_3^{-\frac{5}{6}} x_1^{\frac{1}{4}} x_2^{\frac{1}{3}}}{\frac{1}{3} x_2^{-\frac{2}{3}} x_1^{\frac{1}{4}} x_3^{\frac{1}{6}}}$$

$$= -\frac{x_2}{2x_3}$$