Lecture 01
Logic and Mathematical Proofs

Ryan McWay[†]

[†] *Applied Economics,*
University of Minnesota

Mathematics Review Course, Summer 2023
University of Minnesota
August 7th, 2023

This Course

- Review of graduate-level mathematics necessary for the 1st year sequence.
- Aimed at PhD-track. MS-track is encouraged.
- This sets the foundation (Not exhaustive).
- By the end you should feel confident tackling a variety of math situations in a short period.
- Syllabus on Github repo. Repo is the most up-to-date place for course content.
- This course is **optional**.

Preview of Course

1. Logic, Proofs, Sets, & Topology
2. Uni-variate Calculus & Multi-variate Calculus
3. Linear Algebra
4. Functions & Optimization
5. Probability & Statistics
6. Dynamic Programming

About the Instructor

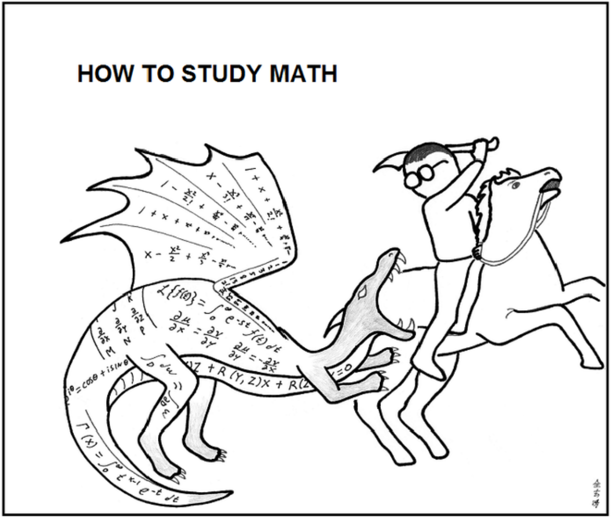
Ryan McWay

- Current: 2nd Year APEC PhD student
- Background: SLU → USF → UMich → UMN
- Research: Development, Behavior, Urban, Environment

Daily Icebreaker

- Attendance via prompt:
  - Name
  - Hometown
  - Program and track
  - Research interests
  - Daily icebreaker subject...

FIGHT WITH MATH...



Don't just read it; fight it!

--- Paul R. Halmos

Topic: Logic

Motivation

- General background
  - Logic is at the heart of reasoning and arguments.
  - Expressed in words and formalized through math, this is a foundation of theoretical arguments.
  - Deduce information correctly. Not deducing correct information.
- Why do economists' care?
  - Foundation for theory
  - Criteria to evaluate arguments
- Application in this career
  - Creating logical arguments
  - How you think about research
  - Evaluating theory and conclusions from empirical evidence

Motivation

- General background
  - Logic is at the heart of reasoning and arguments.
  - Expressed in words and formalized through math, this is a foundation of theoretical arguments.
  - Deduce information correctly. Not deducing correct information.
- Why do economists' care?
  - Foundation for theory
  - Criteria to evaluate arguments
- Application in this career
  - Creating logical arguments
  - How you think about research
  - Evaluating theory and conclusions from empirical evidence

Motivation

- General background
  - Logic is at the heart of reasoning and arguments.
  - Expressed in words and formalized through math, this is a foundation of theoretical arguments.
  - Deduce information correctly. Not deducing correct information.
- Why do economists' care?
  - Foundation for theory
  - Criteria to evaluate arguments
- Application in this career
  - Creating logical arguments
  - How you think about research
  - Evaluating theory and conclusions from empirical evidence

Overview

1. Logical Statements
2. Necessary Conditions
3. Sufficient Conditions

0. Terminology

$\forall$	For all
$\exists$	Exists
$\therefore$	Therefore
$\because$	Because
$\wedge$	And
$\vee$	Or
$\neg$	Negation
$\equiv$	Equivalent or identical.
$\implies$	Implies, then, or sufficient
$\iff$	If and only if, or necessary and sufficient
$\subset$	Strict subset
$\subseteq$	Subset
$\in$	In, or an element of the following set
$\square$	End of Proof. QED (quod erat demonstrandum $\sim$ it has been demonstrated).

1. Logical Statements

- Logical Statement: Use a set of facts to infer/assume a new fact.
  - Hypothesis (If): Premise with set of facts
  - Conclusion (Then): New set of facts inferred if hypothesis is true.
  - e.g., **If** I study throughout the course, **then** I earn a higher grade.
- Family of statements:
  - Tautologies: Statement is always true $(1 = 1)$
  - Contradictions: Statement is always false $(2 = 3)$
  - Statement: $A \implies B$
  - Contrapositive: $\neg B \implies \neg A$
  - Converse: $B \implies A$
  - Inverse: $\neg A \implies \neg B$

1. Logical Statements

- Axiom: Statements assumed to be true.
  - e.g., $a = b, b = c \implies a = c$
- Theorem: A statement proven to be true.
- Corollary: A theorem that follows from another theorem.
- Lemma: A minor theorem used to prove another theorem.

2. Necessary Condition

- A is necessary for B
  - If B is true, A must be true: $B \implies A$.
  - If A is not true, B is not true: $\neg A \implies \neg B$
- A is needed to make the argument.

3. Sufficient Condition

- A is sufficient for B
  - If A is true, B must be true: $A \implies B$
  - If B is not true, A is not either: $\neg B \implies \neg A$
- A allows you to state B, but not necessary to make argument.

4. Necessary and Sufficient (If and Only If $\sim$ iff)

- If A is sufficient for B, B is necessary for A.
- If $A \implies B$ and $B \implies A$, then $A \iff B$ (iff)
  - A is necessary and sufficient for B.
  - A and B are equivalent statements.
  - A is true iff B is true: A iff B

Topic: Proofs

Motivation

- General background
  - Method for proving or disproving a logical statement
- Why do economists' care?
  - Determine which theories are incorporated into economic theory
- Application in this career
  - Theory papers and well-developed theory sections of empirical papers.
  - Often in appendix sections to prove statements articulated as part of an argument in a paper.

Motivation

- General background
  - Method for proving or disproving a logical statement
- Why do economists' care?
  - Determine which theories are incorporated into economic theory
- Application in this career
  - Theory papers and well-developed theory sections of empirical papers.
  - Often in appendix sections to prove statements articulated as part of an argument in a paper.

Motivation

- General background
  - Method for proving or disproving a logical statement
- Why do economists' care?
  - Determine which theories are incorporated into economic theory
- Application in this career
  - Theory papers and well-developed theory sections of empirical papers.
  - Often in appendix sections to prove statements articulated as part of an argument in a paper.

Overview

1. Direct Proofs
2. Proof by Contradiction
3. Proof by Induction

1. Proof by Deduction (Direct Proof)

- Show $A \implies B$
- Deductive reasoning: Use a set of premises that lead to a conclusion.
- Ex. Let m be an even integer and p be any integer. Then $m \times p$ is an even integer.

Proof.

m is an even integer so $\exists$ an integer q such that $m = 2 \times q$ by the definition of an even integer. Therefore, we can make the statement:

$$m \times p = (2 \times q) \times p = 2 \times (q \times p)$$

So, $m \times p$ is an even integer. □

2. Proof by Contradiction

- $A \implies B \equiv \neg A$ and $\neg B \equiv \neg B \implies \neg A$.
- Ex. Walras' Law: $\forall x \in x(p, w)$ that maximizes consumer utility, then $x \times p = w$.

Proof.

Suppose $\exists x \in x(p, w) : x \times p < w$ $(\neg B)$, then there must be another $y \in x(p, w)$ that is affordable and $y \succ x$ by the assumption of "local non-satiation". Therefore, since y exists and is affordable, then x does not maximize utility $(\neg A)$. □

3. Proof by Induction

- Inductive reasoning: Drawing conclusions by reasoning a series of specific examples generalizes.
- Often used by indexing through integers.
- Ex. $P(n) : 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$

Proof.

Note that $P(1)$ is true because $1 = \frac{1\times 2}{2}$. Assume $P(n)$ is true for $k \in n$ integers: $1+2+\cdots+k = \frac{k(k+1)}{2}$. Add $(k+1)$ to both sides.

$$1 + 2 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}$$

This is $P(k+1)$, implying that $P(k)$ is true for all $P(n)$. $\square$

Review of Logic

1. Logical Statement: Necessary to provide clarity to your statements
2. Necessary vs. Sufficient Conditions: Aiming to make statements that are both (iff)

Review of Proofs

1. Three methods to prove a statement:
   - Direct proof
   - Proof by contradiction
   - Proof by induction

Assignment

- Readings on Sets & Topology before Lecture 02:
  -
- Assignment:
  - Problem Set 01 (PS1)
  - Answer key will be available following end of Lecture 02
- Struggling?
  1. Read the 'Encouraged Reading'
  2. Review 'Supplementary material'
  3. Reach out directly