Mathematics Review Course
Summer 2023
Problem Set 01
Solutions

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Necessary & Sufficient Conditions

1. [Khan Academy] Identify any necessary or sufficient conditions to falsify the statement, "My car needs gas in order to run. Therefore, if I put some gas in the tank, my car will run."

Solution: Necessary: The car running is the goal, and gas in the requirement. Run (B) $\rightarrow$ Gas (A).
Sufficient: Gas in the car is not sufficient for the car to run. You first must turn the car on by starting the ignition. Gas (A) $\not\rightarrow$ Run (B).

2. [Khan Academy] Identify any necessary or sufficient conditions to falsify the statement, "Plagiarizing would have given Michael a high score on his history paper. Since Michael just received the highest score possible on the history paper, we can be reasonably confident that he plagiarized it."

Solution: Necessary: Michael might have gotten the highest score in other ways than plagiarizing, such as studying extremely hard. Highest Score (B) $\not\rightarrow$ Plagiarizing (A).
Sufficient: Plagiarizing would have sufficiently provided Michael with the highest score. Plagiarizing (A) $\rightarrow$ Highest Score (B).

3. [Kaplan LSAT] Identify any necessary or sufficient conditions to falsify the statement, "If the lawnmower starts, then the key must be in the ignition."

Solution: Necessary: To start a lawnmower, you must start the ignition. Started (B) $\rightarrow$ Ignition (A).
Sufficient: Whenever you turn a key in a lawnmower, you will start the cutting mechanism. Ignition (A) $\rightarrow$ Started (B).

4. [Hayden Economics] Identify any necessary or sufficient conditions to falsify the statement, "Those who passed the course must have had a Grade A in the class."

Solution: Necessary: Students who get a C, B, or A grade will pass a class. So not all students who pass with have a grade A. Pass (B) $\not\rightarrow$ Grade A.
Sufficient: All grade A students pass a course. Grade A (A) $\rightarrow$ Pass (B)

5. Identify any necessary or sufficient conditions to falsify the statement, "When a market clears, it must be in a Walrasian equilibrium."

Solution: Necessary: For Walrasian equilibrium, market supply equals market demand meaning there is no excess demand. That means that in equilibrium, the market must clear (i.e., all buyers find a seller – no one is left wanting after the transaction period.) Equilibrium (B) $\rightarrow$ Market clears (A).

Sufficient: When a market clears, all buyers find a seller. That means the quantity of goods demanded is the same level as quantity of goods supplied. Therefore, the market is in a Walrasian equilibrium. Market clears (A) $\rightarrow$ Equilibrium (B).

Mathematical Proofs

6. [Hammack] If $n \in \mathbb{Z}$, then $n^2 + 3n + 4$ is even.

Solution:

Proof. This is a direct proof. Suppose $n \in \mathbb{Z}$. We can consider two possible cases. **Case 1:** Suppose n is even. Then $n = 2a$ for some $a \in \mathbb{Z}$. Therefore $n^2 + 3n + 4 = (2a)^2 + 3(2a) + 4 = 2(2a^2 + 3 + 2)$. So $n^2 + 3n + 4 = 2b$ where $b = 2a^2 + 3 + 2$ and is therefore even. **Case 2:** Suppose n is odd. Then $n = 2a + 1 \forall a \in \mathbb{Z}$. Therefore $n^2 + 3n + 4 = (2a + 1)^2 + 3(2a + 1) + 4 = 2(2a^2 + 5a + 4)$. So $n^2 + 3n + 4 = 2b$ where $b = 2a^2 + 5a + 4$ and is therefore even. In either case, $n^2 + 3n + 4$ is even. $\square$

7. [Hammack] Suppose $x, y \in \mathbb{Z}$. If $x^2(y + 3)$ is even, then x is even or y is odd. Note: It may be helpful to use De Morgan's Law – $\left[\bigcup_{i=1}^{k} A_i\right]^c = \bigcap_{i=1}^{k} A_i^c$.

Solution:

Proof. This is a proof by contrapositive. Suppose it is not the case that x is even or that y is odd. Using De Morgan's Law (covered in Lecture 2), this means x is not even and y is not odd. This is to say x is odd and y is even. Thus, there are integers a and b for which $x = 2a + 1$ and $y = 2b$. Consequently, $x^2(y + 3) = (2a + 1)^2(2b + 3) = 2(4a^2b + 4ab + b + 6a^2 + 6a + 1) + 1$. This shows $x^2(y + 3) = 2c + 1$ where $c = 4a^2b + 4ab + b + 6a^2 + 6a + 1$. Therefore, $x^2(y + 3)$ is not even, and it must be that either x is even or y is odd. $\square$

8. [Hammack] Prove that $\sqrt{3}$ is irrational.

Solution:

Proof. This is a proof by contradiction. For the sake of contradiction, suppose that $\sqrt{3}$ is not irrational. Therefore it is is rational, so there exists integers a and b such that $\sqrt{3} = \frac{a}{b}$. Let us assume that this fraction is reduced, so a and b have no common factor. Notice that $\sqrt{3}^2 = \left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2} = 3$ or $3b^2 = a^2 \implies 3|a^2$ (where $|$ means 'divisable by' in this context). Now we need to show that if $a \in \mathbb{Z}$ and $3|a^2$, then $3|a$ (i.e., we need a proof inside a proof). We will use a proof by contrapositive to prove this conditional statement. Suppose that $3 \not| a$ (e.g., 3 is not divisible by a). Then there is a remainder of either 1 or 2 when divided by a. **Case 1:** There is a remainder of 1. Then $a = 3m + 1$ for some integer m. Then, $a^2 = 9m^2 + 6m + 1 = 3(3m^2 + 2m) + 1$. This means 3 divides into a^2 with a remainder of 1. Thus $3 \not| a^2$. **Case 2:** There is a remainder of 2. Then $a = 3m + 2$ for some integer m. Then $a^2 = 9m^2 + 12m + 4 = 3(3m^2 + 4m + 1) + 1$. This means 3 divides into a^2 with a remainder of 1. Thus $3 \not| a^2$. In either case, $3 \not| a^2 \implies 3 \not| a$. So by contrapositive, if $3|a^2$ then $3|a$. Back to the original proof by contradicition, using this implication $3|a$ when $a = 3d$ for some integer d. Now we can restate $3b^2 = a^2 = 9d^2$ re-written as $b^2 = 3d^2$. This means $3|b^2 \implies 3|b$. We can conclude that $3|a \wedge 3|b$. This contradicts the fact that fraction $\frac{a}{b}$ is reduced. Therefore $\sqrt{3}$ must be irrational as it does not reduce. $\square$

9. [Ling Yao; 2020] A rational preference has two properties. 1. Completeness: $\forall x, y$ in a set of possible alternatives, either $x \succsim y$ or $y \succsim x$, or both. 2. Transitivity: $\forall x, y, z$ in a set of possible alternatives, $x \succsim y$ and $y \succsim z \implies x \succsim z$ Prove that $x \succ y \succsim z \implies x \succ z$. **Note:** $\succ$ is a strong preference similar to $>$, and $\succsim$ is a weak preference similar to $\geq$. $\sim$ is an indifferent preference similar to $\equiv$.

Solution:

Proof. This is a proof by contradiction. Let us assume rational preferences. Suppose $\exists x, y, z \in X$ and that the preference relations $x \succ y$ and $y \succsim z$ hold. By the property of completeness there exists either or both of the following cases: $x \succsim z$, $z \succsim x$. For the sake of contradiction, let us assume that $x \succsim x$ holds true. by the property of transitivity, this implies that if $y \succsim z$ then $\exists y \succsim x$. This is a contradiction from our given premise that $x \succ y$ and therefore cannot be true. From the property of transitivity for $\succ$: $x \succ y$, $y \succsim z \implies x \succsim z$ but $\neg z \succsim x$. This exists only if $x \succ z$. Therefore $x \succ y$, $y \succsim z \implies x \succ z$. $\square$