Lecture 05
Multi-variate Calculus

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Mathematics Review Course, Summer 2023
University of Minnesota
August 11th, 2023

Last Lecture Review

- Integration:
  - Definite Integral
  - Fundamental Theorem of Calculus
  - Integration Rules
  - Integration by Substitution
  - Integration by Parts

Review Assignment

1. Problem Set 04 solutions are available on Github.
2. Any issues or problems **You** would like to discuss?

Daily Icebreaker

- Attendance via prompt:
  - Name
  - Daily Icebreaker: Despite your years of sedentary lifestyle studying in the library, you are gifted with physical prowess from a fairy. Which Olympic sport do you participate in?

Topic: Multi-variate Calculus

Motivation

- General background
  - How do handle rate of change for many variables in a single function.
  - Determine how each variable's change impacts to change of the whole.
- Why do economists' care?
  - Most calculus applications in economics involves a multi-variate scenario.
- Application in this career
  - To estimate marginal effects in econometrics.
  - To determine partial equilibrium effects in microeconomics.

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Overview

1. Partial Derivatives
2. Total Differentiation
3. Higher Order Derivatives
4. Multi-variable Chain Rule
5. Implicit Function Theorem
6. Multi-variable Concavity

1. Partial Derivative

- Let $f : \mathbb{R}^n \to \mathbb{R}$. Then for each variable x_i at each point $x^0 = (x_1^0, \ldots, x_n^0)$ in the domain of f when a limit exits,

$$\frac{\partial f}{\partial x_i}(x_1^0, \ldots, x_n^0) = \lim_{h \to 0} \frac{f(x_1^0, \ldots, x_i^0 + h, \ldots, x_n^0) - f(x_1^0, \ldots, x_i^0, \ldots, x_n^0)}{h}$$

- d = Single variable derivative
- ∂ = Partial derivative
- Δ = Total differential

Demonstration: Partial Derivative

Question:

$f(x_1, x_2) = x_1^2 + 3x_1x_2 - x_2^2$

Answer:

$$\frac{\partial f}{\partial x_1} = f_{x_1} = 2x_1 + 3x_2$$
$$\frac{\partial f}{\partial x_2} = f_{x_2} = 3x_1 - 2x_2$$

Demonstration: Partial Derivative

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Practice: Partial Derv.

1. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for $f(x,y) = 2xy + y^2 - ln(x)$.

Practice: Partial Derv.

1. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for $f(x, y) = 2xy + y^2 - ln(x)$.

Answer: Show Work

$$f_x = 2y - \frac{1}{x}$$
$$f_y = 2x + 2y$$

Practice: Partial Derv.

1. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for $f(x, y) = 2xy + y^2 - ln(x)$.
2. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for $f(x, y) = 5^{xy} - e^{xy} + \frac{2+x}{xy}$.

Practice: Partial Derv.

1. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for $f(x, y) = 2xy + y^2 - ln(x)$.
2. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for $f(x, y) = 5^{xy} - e^{xy} + \frac{2+x}{xy}$.

Answer: Show Work

$$f_x = y(ln(5)5^{xy} - e^{xy}) - \frac{2y}{(xy)^2}$$
$$f_y = x(ln(5)5^{xy} - e^{xy}) - \frac{x(2+x)}{(xy)^2}$$

Practice: Partial Derv.

1. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for $f(x,y) = 2xy + y^2 - ln(x)$.
2. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for $f(x,y) = 5^{xy} - e^{xy} + \frac{2+x}{xy}$.
3. Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$ for $f(x,y,z) = x^2y^{-1}z^{3/2} + e^y ln(z) - \frac{x+y}{z}$

Practice: Partial Derv.

1. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for $f(x,y) = 2xy + y^2 - ln(x)$.
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Answer: Show Work

$$f_x = 2xy^{-1}z^{3/2} - \frac{1}{z}$$
$$f_y = -y^{-2}x^2z^{3/2} + e^y ln(z) - \frac{1}{z}$$
$$f_y = \frac{3}{2}x^2y^{-1}z^{1/2} + \frac{e^y}{z} + \frac{x+y}{z^2}$$

2. Total Differentiation

- Total Derivative: A function $f : \mathbb{R}^n \to \mathbb{R}$ expressing how f changes with the **simultaneous** change in x_1 through x_n.
- Note that we can sum the partial derivatives to estimate the **total** differential effect.
- We can approximate the actual change $\Delta f = f(x^* + \Delta x) - f(x^*)$ using the total differential:
$$\Delta f = \frac{\partial f}{\partial x_1}(x^*)\Delta x_1 + \cdots + \frac{\partial f}{\partial x_n}(x^*)\Delta x_n$$

Application: Slutsky Matrix

Substitution Matrix:
Substitution effect for each pair of elements (i.e., goods/commodities).

$$S(p,w) = \begin{bmatrix} s_{11}(p,w) & \cdots & s_{1L}(p,w) \\ \vdots & \ddots & \vdots \\ s_{L1}(p,w) & \cdots & s_{LL}(p,w) \end{bmatrix}$$

$$s_{l,k}(p,w) = \frac{\partial x_l}{\partial p_k} + \frac{\partial x_l}{\partial w} x_k$$

Demonstration: Total Differentiation

Question:

$u(t, r, s) = \frac{t^3 r^6}{s^2}$

Answer:

$$du = u_t dt + u_r dr + u_s ds$$

$$du = \frac{3t^2 r^6}{s^2} dt + \frac{6t^3 r^5}{s^2} dr - \frac{2t^3 r^6}{s^3} ds$$

Demonstration: Total Differentiation

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$$du = \frac{3t^2 r^6}{s^2} dt + \frac{6t^3 r^5}{s^2} dr - \frac{2t^3 r^6}{s^3} ds$$

Practice: Total Diff.

1. Differentiate $f(x, y, z) = ln\left(\frac{xy^2}{z^3}\right)$

Practice: Total Diff.

1. Differentiate $f(x, y, z) = ln\left(\frac{xy^2}{z^3}\right)$

Answer: Show Work

$$df = \frac{1}{x}dx + \frac{2}{y}dy - \frac{3}{z}dz$$

Practice: Total Diff.

1. Differentiate $f(x, y, z) = ln\left(\frac{xy^2}{z^3}\right)$
2. Differentiate $w(x, y, z) = \frac{x^4z^8}{y}$

PRACTICE: TOTAL DIFF.

1. Differentiate $f(x, y, z) = ln\left(\frac{xy^2}{z^3}\right)$
2. Differentiate $w(x, y, z) = \frac{x^4z^8}{y}$

Answer: Show Work

$$dw = \frac{4x^3z^8}{y}dx - \frac{x^4z^8}{y^2}dy + \frac{8z^7x^4}{y}dz$$

Practice: Total Diff.

1. Differentiate $f(x, y, z) = ln\left(\frac{xy^2}{z^3}\right)$
2. Differentiate $w(x, y, z) = \frac{x^4 z^8}{y}$
3. Differentiate $g(a, b, c, d) = \frac{e^a b^2}{a} - \frac{d^{1/2} ln(c)}{b}$

Practice: Total Diff.

1. Differentiate $f(x,y,z) = ln\left(\frac{xy^2}{z^3}\right)$
2. Differentiate $w(x,y,z) = \frac{x^4z^8}{y}$
3. Differentiate $g(a,b,c,d) = \frac{e^ab^2}{a} - \frac{d^{1/2}ln(c)}{b}$

Answer: Show Work

$$
\begin{aligned}
dg =&(\frac{b^2e^a}{a})(1-a)da \\
&+(\frac{2be^a}{a} + \frac{d^{1/2}ln(c)}{b^2})db \\
&-\left(\frac{d^{1/2}}{cd}\right)dc - \left(\frac{ln(c)}{2bd^{1/2}}\right)dd
\end{aligned}
$$

2. Total Differentiation

- The Jacobian derivative vector (or matrix).
- For $f : \mathbb{R}^n \to \mathbb{R}$

$$Df_{x^*} = \left(\frac{\partial f}{\partial x_1}(x^*) \cdots \frac{\partial f}{\partial x_n}(x^*) \right)$$

- For $f : \mathbb{R}^n \to \mathbb{R}^m$

$$Df_{x^*} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x^*) & \cdots & \frac{\partial f_1}{\partial x_n}(x^*) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(x^*) & \cdots & \frac{\partial f_m}{\partial x_n}(x^*) \end{bmatrix}$$

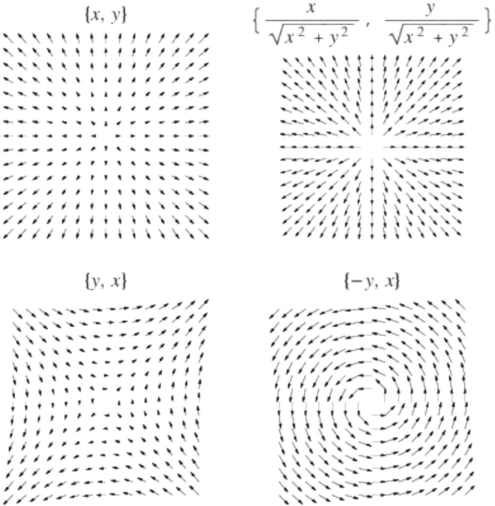
2. Total Differentiation

- The gradient ∇ is the direction that F increases most rapidly.
- E.g., a 'directional derivative'.
- Commonly throughout econometrics (and popular in machine learning).
- The gradient of x^* can be written as a column vector.

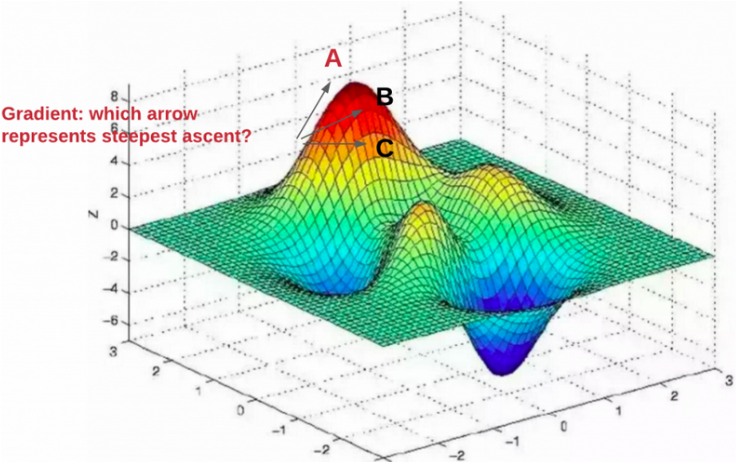
$$\nabla F(x^*) = \begin{pmatrix} \frac{\partial F}{\partial x_1}(x^*) \\ \vdots \\ \frac{\partial F}{\partial x_n}(x^*) \end{pmatrix}$$

- Jacobian is a collection (e.g., matrix) of gradients:
  - Gradient: 1 function, many parameters.
  - Jacobian: Many functions f_m, many parameters.

VECTOR FIELDS



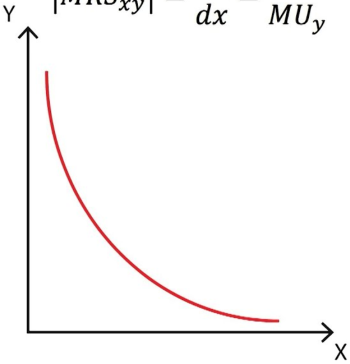
Gradient Ascent and Descent



Gradients and Machine Learning

Application: Marginal Rates of Substitution

$$|MRS_{xy}| = \frac{dy}{dx} = \frac{MU_x}{MU_y}$$



$$U = x \cdot y$$

$$U'_x = y > 0$$
$$U'_y = x > 0$$

$$|MRS_{xy}| = \frac{y\downarrow}{x\uparrow}\Big\downarrow$$

Demonstration: Gradient

Question:

$f(x, y, z) = ze^{-xy}$

Answer:

$$
\begin{aligned}
f_x &= -yze^{-xy} \\
f_y &= -xze^{-xy} \\
f_z &= e^{-xy} \\
\nabla f &= \langle -yze^{-xy}, -xze^{-xy}, e^{-xy} \rangle
\end{aligned}
$$

Demonstration: Gradient

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Practice: Gradients

1. Find ∇ for $f(x, y) = ln(\sqrt{x^2 + y^2})$

Practice: Gradients

1. Find ∇ for $f(x, y) = ln(\sqrt{x^2 + y^2})$

Answer: Show Work

$$\nabla f = \langle \frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \rangle$$

Practice: Gradients

1. Find ∇ for $f(x,y) = ln(\sqrt{x^2+y^2})$
2. Find ∇ for $g(x,y,z) = \frac{e^x 3^z}{y^2}$.

Practice: Gradients

1. Find ∇ for $f(x, y) = ln(\sqrt{x^2 + y^2})$
2. Find ∇ for $g(x, y, z) = \frac{e^x 3^z}{y^2}$.

Answer: Show Work

$$\nabla g = \langle \frac{e^x 3^z}{y^2}, \frac{-2e^x 3^z}{y^3}, \frac{ln(3)e^x 3^z}{y^2} \rangle$$

Practice: Gradients

1. Find ∇ for $f(x, y) = ln(\sqrt{x^2 + y^2})$
2. Find ∇ for $g(x, y, z) = \frac{e^x 3^z}{y^2}$.
3. Find ∇ for $f(x, y, z) = z^2 e^{x^2+4y} + ln(\frac{xy}{z})$.

Practice: Gradients

1. Find ∇ for $f(x,y) = ln(\sqrt{x^2+y^2})$
2. Find ∇ for $g(x,y,z) = \frac{e^x 3^z}{y^2}$.
3. Find ∇ for $f(x,y,z) = z^2 e^{x^2+4y} + ln(\frac{xy}{z})$.

Answer: Show Work

$$\nabla f = \langle 2xz^2 e^{x^2+4y} + \frac{1}{x}, 4z^2 e^{x^2+4y} + \frac{1}{y}, 2ze^{x^2+4y} - \frac{1}{z} \rangle$$

3. Higher Order Derivatives

- Consider the partial derivative for x_1.
$$\frac{\partial f(x_1,\ldots,x_n)}{\partial x_1} = f_1(x)$$

- We can get higher order gradients for n partial derivatives of $f_1(x)$.

$$\begin{aligned}\nabla f_1(x) =& \frac{\partial^2 f(x)}{\partial x_1 \partial x_1} + \frac{\partial^2 f(x)}{\partial x_1 \partial x_2} + \cdots \frac{\partial^2 f(x)}{\partial x_1 \partial x_n}\\ =& f_{11}(x) + f_{12}(x) + \cdots + f_{1n}(x)\end{aligned}$$

3. Higher Order Derivatives

- Hessian Matrix is a **symmetric** matrix of the second order derivatives.

$$D^2 f_{x^*} \equiv \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n} & \frac{\partial^2 f}{\partial x_2 \partial x_n} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

- Young's Theorem ensures the symmetry.

Young's Theorem

If $f : \mathbb{R}^n \to \mathbb{R}$ is $C^2 \in \mathbb{R}^n$, then $\forall x \in \mathbb{R}^n$ and each index pairs i, j:

$$\frac{\partial^2 f(x)}{\partial x_i \partial x_j} = \frac{\partial^2 f(x)}{\partial x_j \partial x_i}$$

Demonstration: Hessian Matrix

Question:

Compute Hessian for $f(x,y) = x^3 - 2xy - y^6$.

Answer:

$$f_x = 3x^2 - 2y$$
$$f_y = -2x - 6y^5$$

$$\mathbf{H}f(x,y) = \begin{bmatrix} 6x & -2 \\ -2 & -30y^4 \end{bmatrix}$$

DEMONSTRATION: HESSIAN MATRIX

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Compute Hessian for $f(x,y) = x^3 - 2xy - y^6$.

Answer:

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$$\mathbf{H}f(x,y) = \begin{bmatrix} 6x & -2 \\ -2 & -30y^4 \end{bmatrix}$$

4. Multi-variable Chain Rule

- If $x(t) = (x_1(t), \ldots, x_n(t))$ is a C^1 curve on an interval about t_0, and f is a C^1 function on a ball about $x(t_0)$,
- Then $g(t) \equiv f(x_1(t), \ldots, x_n(t))$ is a C^1 function at t_0.
- This allows for the differentiation of t:
$$\frac{df}{dt}(t_0) = \frac{\partial f}{\partial x_1}(x(t_0))x_1'(t_0) + \cdots + \frac{\partial f}{\partial x_n}(x(t_0))x_n'(t_0)$$
- Suppose $z = f(x(t), y(t))$. Then
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Application: Cournot & Engel Aggregation

Cournot Aggregation:

Budget share of goods weighted by own and cross price elasticises.

$$s_x \varepsilon_{x,p_x} + s_y \varepsilon_{y,p_x} = -s_x$$
$$s_x = \frac{p_x x}{m}$$

Engel Aggregation:

Budget share of goods interacted with their income elasticises.

$$\sum_{i=1}^{n} s_i \varepsilon(x_i, m) = 1$$

- For two goods:

$$\begin{aligned} s_1 \varepsilon(x_1, m) + s_2 \varepsilon(x_2, m) =& 1 \\ \frac{p_1 x_1}{m} \varepsilon(x_1, m) + \frac{p_1 x_1}{m} \varepsilon(x_2, m) \end{aligned}$$

Demonstration: Chain Rule

Question:

Differentiate $\frac{\partial z}{\partial t}(z(x(t), y(t) : x = t^2, y = 2t) = x^2y - y^2$.

Answer:

$$\begin{aligned}\frac{\partial z}{\partial t} =& (2xy)(2t) + 2x^2 - 2(2)y \\ =& (2t^2 \cdot 2t)(2t) + ((t^2)^2 - 2(2t))(2) \\ =& 8t^4 + 2t^4 - 8t \\ =& 10t^4 - 8t\end{aligned}$$

Demonstration: Chain Rule

Question:

Differentiate $\frac{\partial z}{\partial t}(z(x(t), y(t) : x = t^2, y = 2t) = x^2y - y^2$.

Answer:

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5. Implicit Function Theorem

- Consider a function with two variables x, y.

$$G(x, y(x)) = c)$$

- We can use the implicit function theorem with respect to x about x_0.

$$\frac{dG}{dx}(x_0, y(x_0)) \cdot \frac{dx}{dx}(x_0) + \frac{dG}{dy}(x_0, y(x_0)) \cdot \frac{dy}{dx}(x_0) = 0$$
$$\implies y'(x_0) = \frac{dy}{dx}(x_0) = -\frac{\frac{dG}{dx}(x_0, y(x_0))}{\frac{dG}{dy}(x_0, y(x_0))}$$

5. Implicit Function Theorem

- Implicit Partial Differentiation:

$$\frac{\partial F}{\partial s}(F(x(s), y(s)), z(s)) = \frac{\partial F}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial s}$$

Demonstration: Implicit Partial Diff.

Question:

$\frac{\partial}{\partial t}(f(x, y(t), t) = xy^2 + x^2y = 3t$

Answer:

$$\begin{aligned}
\frac{\partial}{\partial t}(xy^2) + \frac{\partial}{\partial t}(x^2y) =& \frac{\partial}{\partial t}(3t) \\
2xyf_t + x^2f_t =& 3 \\
f_t(2xy + x^2) =& 3 \\
f_t =& \frac{3}{2xy + x^2}
\end{aligned}$$

Demonstration: Implicit Partial Diff.

Question:

$\frac{\partial}{\partial t}(f(x, y(t), t) = xy^2 + x^2y = 3t$

Answer:

$$
\begin{aligned}
\frac{\partial}{\partial t}(xy^2) + \frac{\partial}{\partial t}(x^2y) =& \frac{\partial}{\partial t}(3t) \\
2xyf_t + x^2f_t =& 3 \\
f_t(2xy + x^2) =& 3 \\
f_t =& \frac{3}{2xy + x^2}
\end{aligned}
$$

Practice: Implicit Partial Diff.

1. $\frac{\partial}{\partial x} f(x, y(x)) = e^x y + ln(y) = x$

Practice: Implicit Partial Diff.

1. $\frac{\partial}{\partial x} f(x, y(x)) = e^x y + ln(y) = x$

Answer: Show Work

$$f_x = \frac{1 - e^x y}{e^x + \frac{1}{y}}$$

Practice: Implicit Partial Diff.

1. $\frac{\partial}{\partial x} f(x, y(x)) = e^x y + ln(y) = x$
2. $\frac{\partial}{\partial x} f(x(z), y, z) = 3xyz + x^2 y + \frac{z}{x}$

Practice: Implicit Partial Diff.

1. $\frac{\partial}{\partial x} f(x, y(x)) = e^x y + ln(y) = x$
2. $\frac{\partial}{\partial x} f(x(z), y, z) = 3xyz + x^2 y + \frac{z}{x}$

Answer: Show Work

$$f_z = \frac{-3xy - \frac{1}{x}}{3yz + 2xy - zx^{-2}}$$

Practice: Implicit Partial Diff.

1. $\frac{\partial}{\partial x} f(x, y(x)) = e^x y + ln(y) = x$
2. Question 2
3. $\frac{\partial}{\partial y} f(x(y), y) = e^{x+y} - e^{y^2 x^{-1}}$

Practice: Implicit Partial Diff.

1. $\frac{\partial}{\partial x} f(x, y(x)) = e^x y + ln(y) = x$
2. Question 2
3. $\frac{\partial}{\partial y} f(x(y), y) = e^{x+y} - e^{y^2 x^{-1}}$

Answer: Show Work

$$f_y = \frac{e^{y^2 e^{-1}}}{x(2x^{x+y+xy} + \frac{y^2}{x^2} e^{y^2 x^{-1}})}$$

6. Multi-variable Concavity

- Suppose f is a **convex** subset $U \in \mathbb{R}^n$.
- The f is **concave** iff $\forall x_1, x_2 \in U$, $g_{x_1,x_2}(t) \equiv f(tx_2 + (1-t)x_1)$ is concave on $\{t \in \mathbb{R} | tx_2 + (1-t)x_1 \in U\}$.
- E.g., if the function remains in the convex subset, then it is concave.

6. Multi-variable Concavity

- Let f be a C^2 function on an open **convex** subset $D \in \mathbb{R}^n$.
- Then f is **concave** on D iff the Hessian matrix $D^2 f(x)$ is **negative semidefinite** $\forall x \in D$.
- f is **convex** on D iff the Hessian matrix $D^2 f(x)$ is **positive semidefinite** $\forall x \in D$.

Demonstration: Negative Semidefinite

Question:

Is this Hessian Negative semidefinite (e.g., Any value of $diag(\mathbf{H}(\cdot)) \leq 0$.

$$\mathbf{H}f(x, y, z) = \begin{bmatrix} 8x & -3 & -4xz \\ -3 & -5y^2 & 0 \\ -4xz & 0 & 2z \end{bmatrix}$$

Answer:

Yes, because $-5y^2 \leq 0$.

Demonstration: Negative Semidefinite

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Is this Hessian Negative semidefinite (e.g., Any value of $diag(\mathbf{H}(\cdot)) \leq 0$.

$$\mathbf{H}f(x,y,z) = \begin{bmatrix} 8x & -3 & -4xz \\ -3 & -5y^2 & 0 \\ -4xz & 0 & 2z \end{bmatrix}$$

Answer:

Yes, because $-5y^2 \leq 0$.

6. Multi-variable Concavity

- Quasi-concavity: $\forall x, y \in D \in \mathbb{R}^n$ and $\forall t \in [0, 1]$
$$f(tx + (1 - t)y) \geq \min\{f(x), f(y)\}$$

- With multiple variables, we can show quasi-concavity using the Hessian matrix D iff
$$f(y) \geq f(x) \implies Df(x)(y - x) \geq 0$$

Review

Review of Multi-variate Calculus

1. Partial Derivatives
2. Total Differentiation
3. Higher Order Derivatives
4. Multi-variable Chain Rule
5. Implicit Function Theorem
6. Multi-variable Concavity

Assignment

- Readings on Matrices before Lecture 06:
  - MWG Appendix M.D. & M.M.
- Assignment:
  - Problem Set 05 (PS05)
  - Solution set will be available this weekend.
- Struggling?
  1. Read the 'Encouraged Reading'
  2. Review 'Supplementary material'
  3. Reach out directly

Partial Derv. Question 1 Answer:

Question

$$
\begin{aligned}
f_x &= 2y - \frac{1}{x} \\
f_y &= 2x + 2y
\end{aligned}
$$

Partial Derv. Question 2 Answer:

◂ Question

$$
\begin{aligned}
f_x =& yln(5)5^{xy} - ye^{xy} + (1)(xy)^{-1} + (2+x)(-1)(xy)^{-2}(y) \\
f_x =& y(ln(5)5^{xy} - e^{xy}) - \frac{2y}{(xy)^2}
\end{aligned}
$$

$$
\begin{aligned}
f_y =& xln(5)5^{xy} - xe^{xy} + (2+x)(-1)(xy)^{-2}(x) \\
f_y =& x(ln(5)5^{xy} - e^{xy}) - \frac{x(2+x)}{(xy)^2}
\end{aligned}
$$

Partial Derv. Question 3 Answer:

◂ Question

$$
\begin{aligned}
f_x =& 2xy^{-1}z^{3/2} - \frac{1}{z} \\
f_y =& -y^{-2}x^2z^{3/2} + e^y ln(z) - \frac{1}{z} \\
f_y =& \frac{3}{2}x^2y^{-1}z^{1/2} + \frac{e^y}{z} + \frac{x+y}{z^2}
\end{aligned}
$$

Total Diff. Question 1 Answer:

◂ Question

$$
\begin{aligned}
df =& \frac{1}{x} + \frac{1}{y^2}(2y) + \frac{1}{z^3}(3z) \\
df =& \frac{1}{x}dx + \frac{2}{y}dy - \frac{3}{z}dz
\end{aligned}
$$

Total Diff. Question 2 Answer:

◂ Question

$$dw = \frac{4x^3z^8}{y}dx - \frac{x^4z^8}{y^2}dy + \frac{8z^7x^4}{y}dz$$

Total Diff. Question 3 Answer:

◂ Question

$$
\begin{aligned}
dg =& (\frac{b^2e^a}{a})(1-a)da \\
&+(\frac{2be^a}{a}+\frac{d^{1/2}ln(c)}{b^2})db \\
&-\left(\frac{d^{1/2}}{cd}\right)dc-\left(\frac{ln(c)}{2bd^{1/2}}\right)dd
\end{aligned}
$$

Gradients Question 1 Answer:

Question

$$\nabla f = \langle \frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \rangle$$

Gradients Question 2 Answer:

◂ Question

$$\nabla g = \langle \frac{e^x 3^z}{y^2}, \frac{-2e^x 3^z}{y^3}, \frac{ln(3) e^x 3^z}{y^2} \rangle$$

Gradients Question 3 Answer:

◂ Question

$$\nabla f = \langle 2xz^2e^{x^2+4y} + \frac{1}{x}, 4z^2e^{x^2+4y} + \frac{1}{y}, 2ze^{x^2+4y} - \frac{1}{z}\rangle$$

Implicit Partial Diff. Question 1 Answer:

◂ Question

$$
\begin{aligned}
e^x y + e^x f_x + \frac{1}{y} f_x =& 1 \\
f_x =& \frac{1 - e^x y}{e^x + \frac{1}{y}}
\end{aligned}
$$

Implicit Partial Diff. Question 2 Answer:

Question

$$3yzf_z+3xy+2xyf_z+\frac{1}{x}-x^{-2}zf_z=0$$
$$f_z=\frac{-3xy-\frac{1}{x}}{3yz+2xy-zx^{-2}}$$

Implicit Partial Diff. Question 3 Answer:

◂ Question

$$
\begin{aligned}
e^x e^y e^{xy} f_y + e^x e^y e^{xy} f_y =& x^{-1} e^{y^2 x^{-1}} 2y - y^2 e^{y^2 x^{-1}} x^{-2} f_y \\
f_y (2x^{x+y+xy} + \frac{y^2}{x^2} e^{y^2 x^{-1}}) =& e^{y^2 x^{-1}} \\
f_y =& \frac{e^{y^2 e^{-1}}}{x(2x^{x+y+xy} + \frac{y^2}{x^2} e^{y^2 x^{-1}})}
\end{aligned}
$$