Mathematics Review Course
Summer 2023
Problem Set 02
Solutions

Ryan McWay

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Sets

1. [S&B] Let A be the set of even integers, and B the set of odd integers. Describe both $A \cap B$ and $A \cup B$.

Solution: By definition, odd and even value integers never overlap. This includes 0 which is considered an even integer. Therefore $A \cap B = \emptyset$ and $A \cup B = \mathbb{Z}$.

2. Show that the Cartesian product $A \times B \times C \neq (A \times B) \times C$. Let $A = \{1, 2\}, B = \{3, 4\}, C = \{5, 6\}$.

Solution: Note the three following general forms of the Cartesian product.

$$\begin{aligned} A \times B =& \{(a, b) | a \in A \wedge b \in B\} \\ (A \times B) \times C =& \{(x, c) | x \in A \times B \wedge c \in C\} \\ (A \times B) \times C =& \{((a, b), c) | (a \in A \wedge b \in B) \wedge c \in C\} \end{aligned}$$

Then we can determine that:

$$\begin{aligned} AB \times C =& \{(1, 3, 5), (1, 3, 6), (1, 4, 5), (1, 4, 6), \\ & (2, 3, 5), (2, 3, 6), (2, 4, 5), (2, 4, 6)\} \\ (A \times B) \times C =& \{((1, 3), 5), ((1, 3), 6), ((1, 4), 5), ((1, 4), 6), \\ & ((2, 3), 5), ((2, 3), 6), ((2, 4), 5), ((2, 4), 6)\} \end{aligned}$$

While almost identical, but in $(A \times B) \times C$ we will return a couplet of elements within the elements in the set. This is a subtle, but important distinction in the arrangment of the construction of a Cartesian product.

3. Attempt a proof for De Morgan's Law: $\left[\bigcup_{i=1}^{k} A_i\right]^c = \bigcap_{i=1}^{k} A_i^c$. Note: You will want to prove this by considering two sets A and B and proving both directions (1) $(A \cap B)^c \subseteq A^c \cup B^c$ and (2) $A^c \cup B^c \subseteq (A \cap B)^c$.

Solution: Consider two sets A and B with complements A^c and B^c. We will need to prove both directions for De Morgan's Law. That is (1) $(A \cap B)^c \subseteq A^c \cup B^c$ and (2) $A^c \cup B^c \subseteq (A \cap B)^c$. For (1), this is a direct proof. Let $x \in (A \cap B)^c$. Then $x \notin A \cap B$. Since $A \cap B = \{y | y \in A \wedge y \in B\}$ it follows that $x \in A, B$. The corollary then is that $x \in A^c \implies x \in A^c \cup B^c$. By analog, this follows for $x \notin B \implies x \in B^c \implies x \in A^c \cup B^c$. Therefore, $\forall x : x \in (A \cap B)^c \implies x \in A^c \cup B^c$. Restated, this means $(A \cap B)^c \subseteq A^c \cup B^c$. For (2), this is a proof by contradiction. Let $x \in A^c \cup B^c$. Suppose for contradiction that $x \notin (A \cap B)^c$. By corollary, $x \in (A \cap B)$. And it follows $x \in A$ and $x \in B$ which implies $x \notin A^c$ and $x \notin B^c$. This would imply that $x \notin A^c \cup B^c$ which is a contradiction to the premise $x \in A^c \cup B^c$. Therefore $x \in (A \cap B)^c$ must be true. And therefore $\forall x : x \in A^c \cup B^c \implies x \in (A \cap B)^c$. Restated, this means $A^c \cup B^c \subseteq (A \cap B)^c$.

Because $(A \cap B)^c \subseteq A^c \cup B^c$ and $A^c \cup B^c \subseteq (A \cap B)^c$ is true, it must be that $(A \cap B)^c = A^c \cup B^c$ – proving De Morgan's Law.

4. Let $S = (1, 2, 3, 4, 5)$. Show that the set $\{s \in \mathbb{R}^5 | s \cdot r \leq 25\}$ is a convex set.

Solution: The set S is convex if $s, q \in S : ts + (1-t)q \in S \forall t \in [0, 1]$. Let $s, q \in S$ and $t \in [0, 1]$.

$$\begin{aligned}
&(ts + (1-t)q) \cdot v \\
&= (ts) \cdot v + (1-t)q \cdot v \\
&\leq t \times 25 + (1-t) \times 25 \\
&= 25 \in S
\end{aligned}$$

5. Let c_1 and c_2 be convex sets. And let $c = c_1 \cap c_n$. Show that c must also be a convex set.

Solution: Let $x_1, x_2 \in C$. Then we know they are convex if:

$$S = \{x | x = tx_1 + (1-t)x_2, t \in (0, 1)\}$$

Note that if $x_1, x_2 \in C$, then by the construction of C it must be that $x_1, x_2 \in C_1$ and $x_1, x_2 \in C_2$. Therefore $S \subset C_1$ and $S \subset C_2$. So it follows that $S \subset C_1 \cap C_2 \implies S \subset C$. Since S is convex, it must be that C is also convex.

Topology

6. Prove that $\inf S = \{x \in \mathbb{R} : 0 < x < 1\} = 0$.

Solution:

Proof. This is a proof by contradiction. Suppose $\exists a : 0 - \varepsilon$ is in the set S above for some arbitrarily small $\varepsilon > 0$. Note that but construction of the set 0 is the lower bound. Then we can show that:

$$\begin{gathered}
a = 0 - \varepsilon < 0 = \inf S \\
a < 0
\end{gathered}$$

But, since by construction of the set no value in the set can be less than 0. Therefore $a \in S^c$ and is not apart of the set S which is a contradiction to our premise. Therefore, $\inf S = 0$. ☐

7. [UC Davis] Show that if functions $f, g : A \to \mathbb{R}$ are bounded functions such that $|f(x) - f(y)| \leq |g(x) - g(y)| \forall x, y \in A$ then $\sup_A f - \inf_A f \leq \sup_A g - \inf_A g$.

Solution:

Proof. The conditions imply that $\forall x, y \in A$ we can state:

$$f(x) - f(y) \leq |g(x) - g(y)| = \max[g(x), g(y)] - \min[g(x), g(y)] \leq \sup_A g - \inf_A g$$

From this statement, we can imply:

$$\sup\{f(x) - f(y) : x, y \in A\} \leq \sup_A g - \inf_A g$$

Note that supremums of sets have a distributive property. This means that $\sup(A + B) = \sup A + \sup B$ and more importantly $\sup(A - B) = \sup A - \inf B$. So we know that:

$$\sup\{f(x) - f(y) : x, y \in A\} = \sup_A f - \inf_A f$$

Taken together, this shows that $\sup_A f - \inf_A f \leq \sup_A g - \inf_A g$. □

8. [S&B] Prove that $|xy| = |x| \cdot |y| \forall x, y$.

Solution:

Proof. This is a direct proof. Suppose $x, y \geq 0$ such that xy and $|x||y| = xy = |x||y|$. That is you will get positive values by multiplication regardless of the absolute value signs. By corollary, it is also that if $x, y \leq 0 \implies xy = |xy|$. In the case when $x \geq 0$ but $y \leq 0$, then $xy \leq 0$ and $|xy| = -(xy)$. Then we can state $|x||y| = x(-y) = -(xy) = |xy|$. The corollary follows for $x \leq 0$ and $y \geq 0$. Therefore, we have shown for all cases (i.e., ranges) of x, y the separability of multiplication in absolute values. This is to say $|xy| = |x| \cdot |y| \forall x, y$. □

9. [UC Davis] Show that $\lim \left(\frac{2n+1}{5n+4}\right) \to \frac{2}{5}$.

Solution:

Proof. This is a direct proof. Let $\frac{2n}{5n+4}$ be the sequence a_n. Let $\frac{1}{5n+4}$ be the sequence b_n. And let $\frac{2n+1}{5n+4}$ be the sequence c_n. We can restate the limit as $\lim(c_n) = lim(a_n + b_n) = lim(a_n) + lim(b_n)$. Now we can determine the limit of the two sequences separately. For the sequence b_n, let $\varepsilon > 0$. Then there exists some $N \in \mathbb{N} : N > \frac{1}{\varepsilon}$. Then $\forall n > N$, $\frac{1}{5n+4} < \frac{1}{5N+4} < \frac{1}{N} < \varepsilon$. Therefore, the $\lim \left(\frac{1}{5n+4}\right) \to 0$. For the sequence a_n, again consider the same ε and N. We can show that the sequence $a_n - a$, where $a = \frac{2}{5}$ is the proposed limit:

$$\left|\frac{2n}{5n+4} - \frac{2}{5}\right| < \varepsilon$$
$$\implies$$
$$\left|\frac{2n}{5n+4} - \frac{2}{5}\right| = \left|\frac{-8}{5(5n+4)}\right|$$

From here, we just need to show that both $\left|\frac{-8}{5(5n+4)}\right| < \varepsilon$ and $\left|\frac{8}{5(5n+4)}\right| < \varepsilon$ is true. Using the same ε and N, we can show:

$$\frac{8}{5(5n+4)} < \frac{8}{5(5(1/\varepsilon) + 4)} = \frac{8\varepsilon}{25 + 20\varepsilon} < \frac{8\varepsilon}{25} < \varepsilon$$

Therefore, $\frac{8}{5(5n+4)} < \varepsilon$. And since both a_n and b_n converge, we can combine them to show that $\lim \left(\frac{2n+1}{5n+4}\right) \to \frac{2}{5}$. □

10. [S&B] Prove that a set of real numbers $\mathbb{R}$ can have at most one least upper bound (i.e., sup).

Solution:

Proof. This is a direct proof. Suppose there is more than one least upper bound. So, suppose that b and c are least upper bounds of the set $S \in \mathbb{R}$. Since b is a least upper bound and c is a least upper bound, it follows that $b \leq c$ and that $c \leq b$. Since both statements are true, only one possibility remains. That is that $b = c$. Since they are the same value, only one least upper bound must exist for any set of real numbers. $\square$