APEC Math Review

Lagrange

Natalia Ordaz Reynoso

Summer 2019

Constrained Optimization

- Typically, some of the constraints do matter
- We examine necessary conditions for optima under such circumstances
- Let $D = U \cap \{x \in \mathbb{R}^n | g(x) = 0, h(x) \geq 0\}$ be the constraint set
- Where $U \subset \mathbb{R}^n$ is open, $g : \mathbb{R}^n \to \mathbb{R}^k$, and $h : \mathbb{R}^n \to \mathbb{R}^l$
- functions g are equality constraints, h are inequality constraints
- This is a very general form. Write a budget constraint example

Equality Constraints and the Theorem of Lagrange

Theorem 5.1 (Theorem of Lagrange) Let $f : \mathbb{R}^n \to \mathbb{R}$ and $g_i : \mathbb{R}^n \to \mathbb{R}^k$ be C^1 functions i=1,..,k. Suppose x* is a local maximum of minimum of f on the set $D = U \cap \{x \in \mathbb{R}^n | g_i(x) = 0\}$.
Suppose also that $\rho(Dg(x*)) = k$. Then there exists a vector $\lambda* = (\lambda_1^*, ...\lambda_k^*) \in \mathbb{R}^k$ such that:

$$Df(x^*) + \sum_{i=1}^{k} \lambda_i^* Dg_i(x^*) = 0$$

- If a pair (x^*, λ^*) satisfies that $g(x^*) = 0$ and te equation above, we will say that the pair satisfies the ncessary FOC of the Theorem of Lagrange.
- Note: not sufficient!
- Rank condition is key (constraint qualification)

Lagrangean Multipliers

- The vector $\lambda * = (\lambda_1^*, ...\lambda_k^*) \in \mathbb{R}^k$ is called the vector of lagrangean multipliers corresponding to the local optimum x^*
- The i-th multiplier measures the sensitivity of the value of the objective function at x^* to a small relaxation of the i-th constraint g_i (See P 116 of Sundaram for a proof)
- In economics we call this a shadow price: the maximum amount the decision maker is willing to pay for a marginal relaxation of constraint i.

Second Order Conditions

- Consider $f : \mathbb{R}^n \to \mathbb{R}$ over $D = U \cap \{x \in \mathbb{R}^n | g(x) = 0\}$, $g : \mathbb{R}^n \to \mathbb{R}^k$, U is open. We will assume f and g are both C^2.
- Define $L(x; \lambda) = f(x) + \sum_{i=1}^k \lambda_i g_i(x)$
- Note that the second derivative $D^2 L(x; \lambda) = D^2 f(x) + \sum_{i=1}^k \lambda_i D^2 g_i(x)$ is symmetric and defines a quadratic form.

Second Order Conditions

Suppose that there are points $x^* \in D, \lambda^* \in \mathbb{R}^k$ such that $\rho(Dg(x^*)) = k$ and $Df(x^*) + \sum_{i=1}^{k} \lambda_i^* Dg_i(x^*) = 0$. Define $Z(x^*) = \{z \in \mathbb{R}^n | Dg(x^*)z = 0\}$, and let D^2L* denote the matrix $D^2L(x^*, \lambda^*)$. Then

1. If f has a local maximum on D at x^*, then D^2L* is negative semidefinite for all z
2. If f has a local minimum on D at x^* then D^2L* is positive semidefinite for all z
3. If D^2L* is negative definite for all z, x* is a strict local maximum
4. If D^2L* is positive definite for all z, x* is a strict local minimum

Note the differences with the unconstrained: we modify the second derivative by adding a correction term, and we restrict to the feasible set.

Lagreangean Method: Cookbook

Let $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^k$.
max $f(x)$ subject to $x \in D = U \cap \{x : g(x) = 0\}$

1. Set up $L(x, \lambda) = f(x) + \sum_{i=1}^{k} \lambda g_i(x)$
2. Find critical points of $L(x, \lambda)$, ie $DL(x, \lambda) = 0, x \in U$. This results in a system of (n+k) equations, and (n+k) unknowns. Let M be the set of all solutions to these equations for $x \in U$
3. Now we evaluate f at every point in the set $\{x \in \mathbb{R}^n : \exists \lambda$ such that $(x, \lambda) \in M\}$ and pick the largest one

Restatement of the Lagrangean Theorem

Suppose the following two conditions hold:

1. A global optimum x^* exists to the given problem
2. The constraint qualification is met at x^*

Then, there is a λ^* such that (x^*, λ^*) is a critical point of L

When it could fail:

- If there is an optimum but the constraint qualification is not met there
- Examples
- Exercises: Sundaram Chapter 5