Lecture 09
Optimization Day 1

Ryan McWay[†]

[†] Applied Economics,
University of Minnesota

Mathematics Review Course, Summer 2023
University of Minnesota
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Last Lecture Review

- Numbers:
  - Triangle Inequality
  - Neighborhoods
- Functions:
  - Homogeneity
  - Euler's Theorem
  - Quasiconcavity & Quasiconvexity
  - Concavity & Convexity
  - Continuity
  - Upper- and Lower-Hemicontinuity
  - Brouwer's and Kakutani's Fixed-point Theorem

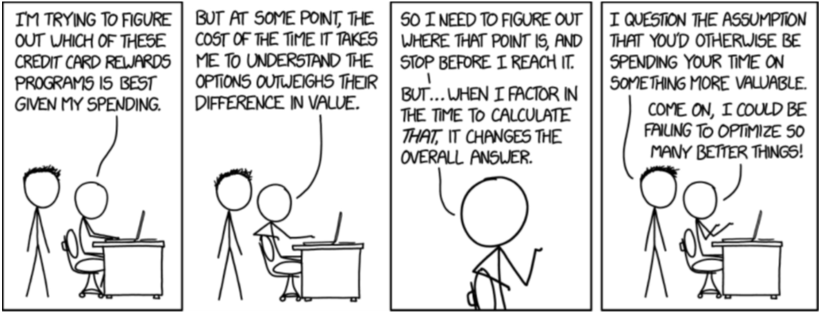
Review Assignment

1. Problem Set 08 solutions are available on Github.
2. Any issues or problems **You** would like to discuss?

Daily Icebreaker

- Attendance via prompt:
  - Name
  - Daily Icebreaker: What is one of the top adventures/ activities you want to complete from your "bucket list"?

Optimization



Unconstrained Optimization

Motivation

- General background
  - Simplest form of optimization.
  - Solution involves finding critical points of the function.
- Why do economists' care?
  - Used in simple applications for testing first order conditions and second order conditions.
- Application in this career
  - Unconstrained optimization is primarily used to determine FOC and SOC.

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Overview

1. First Order Conditions
2. Second Order Conditions
3. Global Min and Max

1. First Order Conditions

- Let $F : U \to \mathbb{R}$ be a differentiable defined on subset U of $\mathbb{R}^n$.
- If $x^* \in \mathbb{R}^n$ is a local minimum or local maximum of $F(\cdot)$ and x^* is an interior point of U, then:

$$\nabla F(x^*) = 0 \text{ or } \frac{\partial F(x^*)}{\partial x_n} \forall n$$

- The FOC can be summarized by the Jacobian matrix.

$$J = \nabla f(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix}$$

Application: OLS

$$\min_{\beta_0,\beta_1} \sum_{i=1}^{N} (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2$$

The FOC for $\hat{\beta}_1$:

$$\frac{\partial W}{\partial \hat{\beta}_1} = \sum_{i=1}^{N} -2x_i(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) = 0$$

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{N}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{N}(x_i - \bar{x})^2}$$

Demonstration: 1st Order Conditions (FOC)

Question:

In a monopoly, the demand curve is $P = 10 - Q$ and faces marginal costs of $C = 0.25Q$. P is the price and Q is the quantity. What is the quantity produced?

Answer:

$$\begin{aligned}
\pi(P, Q) &= PQ - C \\
\pi &= (10 - Q)Q - 0.25Q \\
\frac{d\pi}{dQ} &= 10 - 2Q - 0.25 = 0 \\
Q^* &= \frac{9.75}{2}
\end{aligned}$$

Demonstration: 1st Order Conditions (FOC)

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Practice: FOC

1. Minimize $f(x) = 3x^2 - 8x + 1$.

PRACTICE: FOC

1. Minimize $f(x) = 3x^2 - 8x + 1$.

Answer: Show Work

$$x^* = \frac{4}{3}$$

Practice: FOC

1. Minimize $f(x) = 3x^2 - 8x + 1$.
2. Minimize $g(y) = (1-y)^2 - 2y$.

Practice: FOC

1. Minimize $f(x) = 3x^2 - 8x + 1$.
2. Minimize $g(y) = (1 - y)^2 - 2y$.

Answer: Show Work

$$y^* = 2$$

Practice: FOC

1. Minimize $f(x) = 3x^2 - 8x + 1$.
2. Minimize $g(y) = (1 - y)^2 - 2y$.
3. What is monopoly quantity supplied Q^* for the profit function $\pi(Q) = (7 - 0.5Q)Q - 0.5Q$?

PRACTICE: FOC

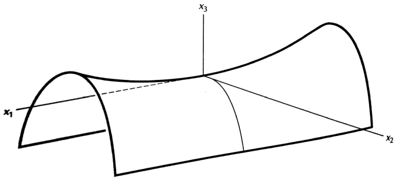
1. Minimize $f(x) = 3x^2 - 8x + 1$.
2. Minimize $g(y) = (1 - y)^2 - 2y$.
3. What is monopoly quantity supplied Q^* for the profit function $\pi(Q) = (7 - 0.5Q)Q - 0.5Q$?

Answer: Show Work

$$Q^* = 6.5$$

2. Second Order Conditions

- Let $F : U \to \mathbb{R}$ be C^2 whose domain is an open set $U \in \mathbb{R}^n$.
- Suppose $\nabla F(x^*) = 0$ (Note this is at x^* rather than any x).
  - If $D^2 f(x^*)$ is negative (positive) definite, then x^* is a strict local max (min).
  - If $D^2 f(x^*)$ is indefinite, then x^* is neither a local max nor min.
- Suppose x^* is a local max (min) of F.
  - Then, $\nabla F(x^*) = 0$ and the symmetric $n \times n$ matrix $D^2 f(x^*)$ is negative (positive) **semi-definite**.



The graph of the indefinite form $Q_3(x_1, x_2) = x_1^2 - x_2^2$.

2. Second Order Conditions

- The SOC is captured by the Hessian matrix.

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

Demonstration: 2nd Order Conditions (SOC)

Question:

Check the SOC for a local max of
$U(x_1, x_2) = x_1^{0.5} x_2^{0.5} : x_1, x_2 > 0.$

Answer:

FOC:

$$U_{x_1} = 0.5 \left(\frac{x_2}{x_1}\right)^{0.5} = 0 \implies x_1^* = \infty$$

$$U_{x_2} = 0.5 \left(\frac{x_1}{x_2}\right)^{0.5} = 0 \implies x_2^* = \infty$$

SOC: $$H = \begin{bmatrix} -0.25 x_1^{-1.5} x_2^{0.5} & 0.25(x_1 x_2)^{-0.5} \\ 0.25(x_1 x_2)^{-0.5} & -0.25 x_1^{0.5} x_2^{-1.5} \end{bmatrix}$$

Yes, $D^2 f(x^*)$ is negative definite $\rightarrow$ local max.

Demonstration: 2nd Order Conditions (SOC)

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Check the SOC for a local max of
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Answer:

FOC:

$$U_{x_1} = 0.5 \left(\frac{x_2}{x_1}\right)^{0.5} = 0 \implies x_1^* = \infty$$
$$U_{x_2} = 0.5 \left(\frac{x_1}{x_2}\right)^{0.5} = 0 \implies x_2^* = \infty$$

SOC: $$H = \begin{bmatrix} -0.25 x_1^{-1.5} x_2^{0.5} & 0.25(x_1 x_2)^{-0.5} \\ 0.25(x_1 x_2)^{-0.5} & -0.25 x_1^{0.5} x_2^{-1.5} \end{bmatrix}$$

Yes, $D^2 f(x^*)$ is negative definite $\rightarrow$ local max.

3. Global Min and Max

- Any point x^* of a concave (convex) function $f(\cdot)$ satisfying $\nabla F(x^*) = 0$.
- Note this may be a boundary point (corner solution) or an interior point (critical point).

$Df(x^*)$	$D^2f(x^*)$	Max/Min
$= 0$	Negative Semi-definite	Local Max
$= 0$	Positive Semi-definite	Local Min
$= 0$	Indefinite	Saddle point or Inflexion

(Equality) Constrained Optimization

Motivation

- General background
  - Optimization of a function considering a constraint from other function(s).
  - Leads to ‘interior solutions’.
- Why do economists’ care?
  - This is a typical case for optimization.
- Application in this career
  - Used throughout microeconomics.

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Overview

1. Lagrangian Method
2. Second Order Conditions
3. Bordered Hessian

1. Lagrangian Method

- Let f and h be C^1 functions.
- Suppose $x^* = (x_1^*, x_2^*)$ is a **solution** to the problem when unconstrained:

$$\max f(x_1, x_2)$$
$$\text{s.t. } h(x_1, x_2) = c$$

- Consider (x_1^*, x_2^*) are **not** critical points of h.
- Then μ^* is a real number such that (x_1^*, x_2^*, μ^*) **is** a critical point of the following Lagrangian function.

$$L(x_1, x_2, \mu) \equiv f(x_1, x_2) - \mu[h(x_1, x_2) - c]$$

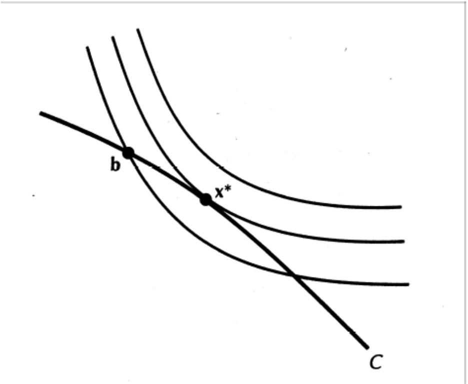
- That is...

$$\frac{\partial L}{\partial x_1} = 0,\ \frac{\partial L}{\partial x_2} = 0,\ \frac{\partial L}{\partial \mu} = 0$$

1. Lagrangian Method

- Intuition:

$$\mu = -\frac{\frac{\partial f}{\partial x_1}(x^*)}{\frac{\partial f}{\partial x_2}(x^*)} = -\frac{\frac{\partial h}{\partial x_1}(x^*)}{\frac{\partial h}{\partial x_2}(x^*)}$$



1. Lagrangian Method

- So, we have two equations represented by $\nabla f(x^*) = \mu^* \nabla h(x^*)$.

$$\begin{aligned} \frac{\partial f}{\partial x_1}(x^*) - \mu^* \frac{\partial h}{\partial x_1}(x^*) &= 0 \\ \frac{\partial f}{\partial x_2}(x^*) - \mu^* \frac{\partial h}{\partial x_2}(x^*) &= 0 \end{aligned}$$

1. Lagrangian Method

- Consider there may be **many** constraints in the constraint set C_h with local max/min $x^* \in C_h$.

$$C_h = \{x = (x_1, \ldots, x_n) : (h_1(x = a_1), \ldots, h_m(x = a_m)\}$$

- If x^* is not the critical point of $h = (h_1, \ldots, h_m)$ (i.e., $rank(Dh(x^*) < m)$ – unique value for all constraints).
- Then there are $\mu_1^*, \ldots, \mu_m^*$ real numbers such that $(x_1^*, \ldots, x_n^*, \mu_1^*, \ldots, \mu_m^*)$ is the critical point of the Lagrangian function:

$$L(x^*, \mu^*) \equiv f(x) - \mu_1[h(x) - a_1] - \cdots - \mu_m[h(x) - a_m]$$

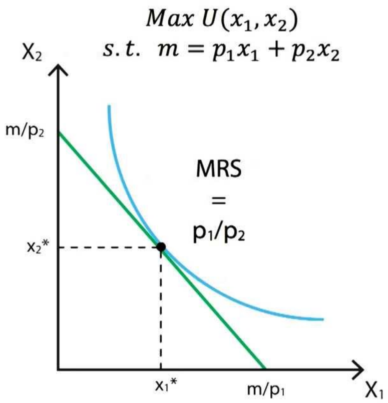
- Alternatively, all partials for x_n and μ_m are set to zero.

1. Lagrangian Method

- How to determine the signs of the Lagrange multiplier and the constraint?

For a Minimum	For a Maximum
$-\mu(c - h(\cdot))$	$-\mu(h(\cdot) - c)$
$+\mu(h(\cdot) - c)$	$+\mu(c - h(\cdot))$
Constrained above c	Constrained below c
c is a floor	c is a ceiling

Application: Consumer Utility Maximization



Demonstration: Lagrangian

Question:

Classic utility maximization: $u(x_1, x_2) = x_1^\alpha x_2^\beta$ s.t. $w \geq p_1 x_1 + p_2 x_2$.

Answer:

$$\mathcal{L} = x_1^\alpha x_2^\beta + \lambda(w - p_1 x_1 - p_2 x_2)$$

$$\begin{aligned}
\mathcal{L}_{x_1} =& \alpha x_1^{\alpha-1} x_2^\beta + \lambda(-p_1) = 0 \\
\mathcal{L}_{x_2} =& \beta x_1^\alpha x_2^{\beta-1} + \lambda(-p_2) = 0 \\
\mathcal{L}_\lambda =& w - p_1 x_1 - p_2 x_2 = 0
\end{aligned}$$

Demonstration: Lagrangian

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\mathcal{L}_\lambda =& w - p_1 x_1 - p_2 x_2 = 0
\end{aligned}$$

Demonstration: Lagrangian

Answer:

$$\lambda = \frac{\alpha}{p_1} x_1^{\alpha-1} x_2^{\beta}$$

$$\lambda = \frac{\beta}{p_2} x_1^{\alpha} x_2^{\beta-1}$$

$$\frac{\alpha}{p_1} x_1^{\alpha-1} x_2^{\beta} = \frac{\beta}{p_2} x_1^{\alpha} x_2^{\beta-1}$$

$$x_1^{\alpha-1} x_1^{-\alpha} = \frac{\beta}{\alpha} \frac{p_1}{p_2} x_2^{\beta-1} x_2^{-\beta}$$

$$x_1^{-1} = \frac{\beta}{\alpha} \frac{p_1}{p_2} x_2^{-1}$$

$$x_1 = \frac{\alpha}{\beta} \frac{p_2}{p_1} x_2$$

$$x_2 = \frac{\beta}{\alpha} \frac{p_1}{p_2} x_1$$

Demonstration: Lagrangian

Answer:

Plug back into the constraint …

$$
\begin{aligned}
w =& p_1 x_1 + p_2 \frac{p_1}{p_2} \frac{\beta}{\alpha} x_1 \\
w =& x_1 p_1 (1 + \frac{\beta}{\alpha}) \\
x_1^* =& \frac{w}{p_1(1 + \frac{\beta}{\alpha})}
\end{aligned}
$$

By analog to save space …

$$
x_2^* = \frac{w}{p_2(1 + \frac{\alpha}{\beta})}
$$

Practice: Lagrangian

1. Max $ax_1^2 + bx_2^2$ s.t. $x_1 + x_2 = 1$.

Practice: Lagrangian

1. Max $ax_1^2 + bx_2^2$ s.t. $x_1 + x_2 = 1$.

Answer: Show Work

$$x_1^* = \frac{1}{1 + \frac{a}{b}}$$
$$x_2^* = \frac{1}{1 + \frac{b}{a}}$$

Practice: Lagrangian

1. Max $ax_1^2 + bx_2^2$ s.t. $x_1 + x_2 = 1$.
2. Min $x_1^2 x_2$ s.t. $2x_1^2 + x_2^2 = 3$.

Practice: Lagrangian

1. Max $ax_1^2 + bx_2^2$ s.t. $x_1 + x_2 = 1$.
2. Min $x_1^2 x_2$ s.t. $2x_1^2 + x_2^2 = 3$.

Answer: Show Work

$$x_1^* = 1$$
$$x_2^* = 1$$

Practice: Lagrangian

1. Max $ax_1^2 + bx_2^2$ s.t. $x_1 + x_2 = 1$.
2. Min. $x_1^2 x_2$ s.t. $2x_1^2 + x_2^2 = 3$.
3. Max $\pi(q_1, q_2) = p_1 q_1 + p_2 q_2$ s.t. $B \geq 0.3q_1 + 0.5q_2$.

Practice: Lagrangian

1. Max $ax_1^2 + bx_2^2$ s.t. $x_1 + x_2 = 1$.
2. Min. $x_1^2 x_2$ s.t. $2x_1^2 + x_2^2 = 3$.
3. Max $\pi(q_1, q_2) = p_1 q_1 + p_2 q_2$ s.t. $B \geq 0.3q_1 + 0.5q_2$.

Answer: Show Work

$$q_1^* = \frac{B}{0.3(1+p_2)}$$
$$q_2^* = \frac{B}{0.5(1+p_1)}$$

2. Second Order Conditions

- To ensure a maximum, we need to know that the second differential of the objective function f at the critical point is **decreasing along the constraint**.
- Let $y = f(x_1, x_2(x_1))$ be the value of the objective function subject to the constraint.
- By the implicit function theorem:

$$\frac{dx_2}{dx_1} = \frac{\partial h/\partial x_1}{\partial h/\partial x_2}$$

- By chain rule:

$$\frac{dy}{dx_1} = \frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dx_1} = \frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2}\frac{\partial h/\partial x_1}{\partial h/\partial x_2}$$

3. Bordered Hessian

- This determines the sufficient SOC for a local max/min.
- The sufficient condition for a critical point is:

$$\frac{d^2y}{dx_1^2} < 0$$

- It can be shown that

$$\frac{d^2y}{dx_1^2} = \frac{-1}{(\partial h/\partial x_2)^2} B$$

- Where B is the **bordered Hessian** of L.
- The determinant of the bordered Hessian tests for quasiconcavity or convexity to ensure a local max/min
- Negative definite assuming $\nabla f(x) \neq 0 \forall x$ and $f(\cdot)$ is strictly quasiconcave.

3. Bordered Hessian

- Called a 'bordered' hessian because we replace the first row and first column with the FOC with respect to the **constraint**.
- Local minima (x_1^*, x_2^*) if $B(x_1^*, x_2^*, \lambda^*) < 0$.
- Local maximum (x_1^*, x_2^*) if $B(x_1^*, x_2^*, \lambda^*) > 0$.

$$B(x_1, x_2, \lambda) = det \left| \begin{bmatrix} 0 & \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} \\ \frac{\partial h}{\partial x_1} & \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} \\ \frac{\partial h}{\partial x_2} & \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} \end{bmatrix} \right| > 0$$

Review

Unconstrained Optimization

1. First Order Conditions
2. Second Order Conditions
3. Global Min and Max

(Equality) Constrained Optimization

1. Lagrangian Method
2. Second Order Conditions
3. Bordered Hessian

Assignment

- Assignment:
  - Problem Set 09 (PS09)
  - Solution set will be available following end of Lecture 10
- Struggling?
  1. Read the ‘Encouraged Reading’
  2. Review ‘Supplementary material’
  3. Reach out directly

FOC Question 1 Answer:

◂ Question

$$f_x = 6x - 8 = 0$$
$$x^* = \frac{8}{6}$$

FOC Question 2 Answer:

◂ Question

$$
\begin{aligned}
f_y =& 2(1-x)(-1) - 2 \\
f_y =& 2x - 4 = 0 \\
y^* =& 2
\end{aligned}
$$

FOC Question 3 Answer:

◂ Question

$$\pi_Q = 7 - \frac{2}{2}Q - 0.5 = 0$$
$$Q^* = 6.5$$

Lagrangian Question 1 Answer:

◂ Question

$$\mathcal{L} = ax_1^2 + bx_2^2 + \lambda(1 - x_1 - x_2)$$

$$x_1 = \frac{b}{a}x_2$$
$$x_2 = \frac{a}{b}x_1$$

Plug back into the constraint.

Lagrangian Question 2 Answer:

◂ Question

$$\mathcal{L} = x_1^2 x_2 + \lambda(2x_1^2 + x_2^2 - 3)$$

$$x_1 = x_2$$

Plug back into the constraint.

Lagrangian Question 3 Answer:

◂ Question

$$\mathcal{L} = p_1 q_1 + p_2 q_2 + \lambda(B - 0.3q_1 - 0.5q_2)$$

$$q_1 = \frac{0.5}{0.3}\frac{p_1}{p_2}q_2$$
$$q_2 = \frac{0.3}{0.5}\frac{p_2}{p_1}q_1$$

Plug back into the constraint.