

Lecture 01

Logic and Mathematical Proofs

Ryan McWay[†]

[†] *Applied Economics,
University of Minnesota*

Mathematics Review Course, Summer 2023
University of Minnesota
August 7th, 2023

Course Preview

THIS COURSE

- ▶ Review of graduate-level mathematics necessary for the 1st year sequence.
- ▶ Aimed at PhD-track. MS-track is encouraged.
- ▶ This sets the foundation (Not exhaustive).
- ▶ By the end you should feel confident tackling a variety of math situations in a short period.
- ▶ Syllabus on **Github repo**. Repo is the most up-to-date place for course content.
- ▶ This course is **optional**.

PREVIEW OF COURSE

1. Logic, Proofs, Sets, & Topology
2. Uni-variate Calculus & Multi-variate Calculus
3. Linear Algebra
4. Functions & Optimization
5. Probability & Statistics
6. Dynamic Programming

ABOUT THE INSTRUCTOR



Ryan McWay

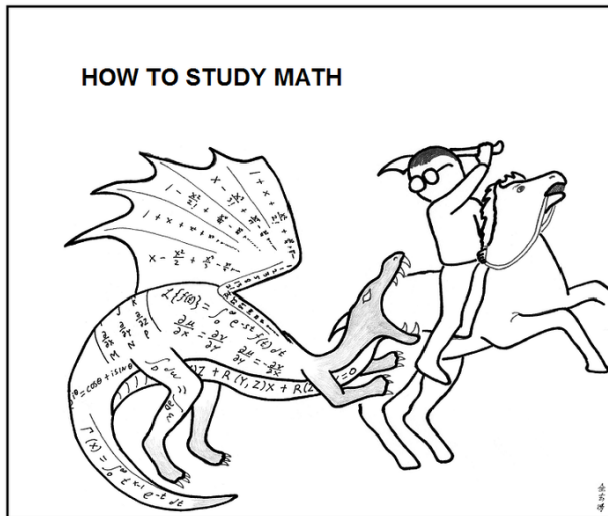
- ▶ Current: 2nd Year
APEC PhD student
- ▶ Background: SLU →
USF → UMich → UMN
- ▶ Research: Development,
Behavior, Urban,
Environment

DAILY ICEBREAKER

- ▶ Attendance via prompt:
 - ▶ Name
 - ▶ Hometown
 - ▶ Program and track
 - ▶ Research interests
 - ▶ Daily Icebreaker: Imagine you are a professional baseball player or wrestler. What is you walk up (intro) song?



FIGHT WITH MATH...



Don't just read it; fight it!

— Paul R. Halmos

Topic: Logic

MOTIVATION

- ▶ General background
 - ▶ Logic is at the heart of reasoning and arguments.
 - ▶ Expressed in words and formalized through math, this is a foundation of theoretical arguments.
 - ▶ Deduce information correctly. Not deducing correct information.
- ▶ Why do economists' care?
 - ▶ Foundation for theory
 - ▶ Criteria to evaluate arguments
- ▶ Application in this career
 - ▶ Creating logical arguments
 - ▶ How you think about research
 - ▶ Evaluating theory and conclusions from empirical evidence

MOTIVATION

- ▶ General background
 - ▶ Logic is at the heart of reasoning and arguments.
 - ▶ Expressed in words and formalized through math, this is a foundation of theoretical arguments.
 - ▶ Deduce information correctly. Not deducing correct information.
- ▶ Why do economists' care?
 - ▶ Foundation for theory
 - ▶ Criteria to evaluate arguments
- ▶ Application in this career
 - ▶ Creating logical arguments
 - ▶ How you think about research
 - ▶ Evaluating theory and conclusions from empirical evidence

MOTIVATION

- ▶ General background
 - ▶ Logic is at the heart of reasoning and arguments.
 - ▶ Expressed in words and formalized through math, this is a foundation of theoretical arguments.
 - ▶ Deduce information correctly. Not deducing correct information.
- ▶ Why do economists' care?
 - ▶ Foundation for theory
 - ▶ Criteria to evaluate arguments
- ▶ Application in this career
 - ▶ Creating logical arguments
 - ▶ How you think about research
 - ▶ Evaluating theory and conclusions from empirical evidence

OVERVIEW

1. Logical Statements
2. Necessary Conditions
3. Sufficient Conditions

0. TERMINOLOGY

\forall	For all
\exists	Exists
$:$	Such that
$ $	Given
\therefore	Therefore
\because	Because
\wedge	And
\vee	Or
\neg	Negation
\equiv	Equivalent or identical.
\implies	Implies, then, or sufficient
\iff	If and only if, or necessary and sufficient
\subset	Strict subset
\subseteq	Subset
\in	In, or an element of the following set
\square	End of Proof. QED (quod erat demonstrandum ~ it has been demonstrated).

0. TERMINOLOGY

Αα

ALPHA [a]
ἄλφα

Ββ

BETA [b]
βῆτα

Γγ

GAMMA [g]
γάμμα

Δδ

DELTA [d]
δέλτα

Εε

EPSILON [e]
ἒ ψιλόν

Ζζ

ZETA [dz]
ζῆτα

Ηη

ETA [ɛː]
ἦτα

Θθ

THETA [tʰ]
θῆτα

Ιι

IOTA [i]
ιώτα

Κκ

KAPPA [k]
κάππα

Λλ

LAMBDA [l]
λάμβδα

Μμ

MU [m]
μῦ

Νν

NU [n]
νῦ

Ξξ

XI [ks]
ξεῖ

Οο

OMICRON [o]
ὀ μικρόν

Ππ

PI [p]
πεῖ

Ρρ

RHO [r]
ῥώ

Σσς

SIGMA [s]
σίγμα

Ττ

TAU [t]
ταῦ

Υυ

UPSILON [u]
ὕ ψιλόν

Φφ

PHI [pʰ]
φεῖ

Χχ

CHI [kʰ]
χεῖ

Ψψ

PSI [ps]
ψεῖ

Ωω

OMEGA [ɔː]
ὦ μέγα

1. LOGICAL STATEMENTS

- ▶ Logical Statement: Use a set of facts to infer/assume a new fact.
 - ▶ Hypothesis (If): Premise with set of facts
 - ▶ Conclusion (Then): New set of facts inferred if hypothesis is true.
 - ▶ e.g., **If** I study throughout the course, **then** I earn a higher grade.
- ▶ Family of statements:
 - ▶ Tautologies: Statement is always true ($1 = 1$)
 - ▶ Contradictions: Statement is always false ($2 = 3$)
 - ▶ Statement: $A \implies B$
 - ▶ Contrapositive: $\neg B \implies \neg A$
 - ▶ Converse: $B \implies A$
 - ▶ Inverse: $\neg A \implies \neg B$

1. LOGICAL STATEMENTS

- ▶ Axiom: Statements assumed to be true.
 - ▶ e.g., $a = b, b = c \implies a = c$
- ▶ Theorem: A statement proven to be true.
- ▶ Corollary: A theorem that follows from another theorem.
- ▶ Lemma: A minor theorem used to prove another theorem.

2. NECESSARY CONDITION

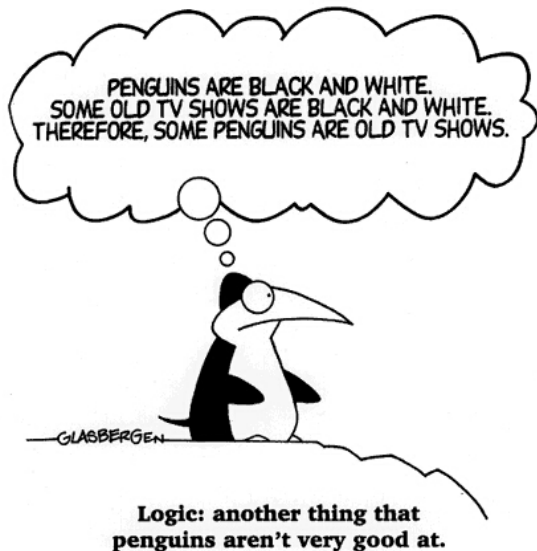
- ▶ A is necessary for B
 - ▶ If B is true, A must be true: $B \implies A$.
 - ▶ If A is not true, B is not true: $\neg A \implies \neg B$
- ▶ A is needed to make the argument.



3. SUFFICIENT CONDITION

- ▶ A is sufficient for B
 - ▶ If A is true, B must be true: $A \implies B$
 - ▶ If B is not true, A is not either: $\neg B \implies \neg A$
- ▶ A allows you to state B , but not necessary to make argument.

NECESSARY BUT NOT SUFFICIENT



4. NECESSARY AND SUFFICIENT (IF AND ONLY IF \sim IFF)

- ▶ If A is sufficient for B , B is necessary for A .
- ▶ If $A \implies B$ and $B \implies A$, then $A \iff B$ (iff)
 - ▶ A is necessary and sufficient for B .
 - ▶ A and B are equivalent statements.
 - ▶ A is true iff B is true: A iff B

DEMONSTRATION: NECESSARY AND SUFFICIENT

Question:

Is this statement true: “If I open the door, I used the key.”

Answer:

Logic: Open Door (A) \implies Used Key (B) Necessary: You need a key (B) to open the door (A). $B \implies A$. Sufficient: If you do not have the key ($\neg B$), then there is no way to open the door ($\neg A$). So $A \implies B$.

DEMONSTRATION: NECESSARY AND SUFFICIENT

Question:

Is this statement true: “If I open the door, I used the key.”

Answer:

Logic: Open Door (A) \implies Used Key (B) Necessary: You need a key (B) to open the door (A). $B \implies A$. Sufficient: If you do not have the key ($\neg B$), then there is no way to open the door ($\neg A$). So $A \implies B$.

PRACTICE: NECESSARY AND SUFFICIENT CONDITIONS

1. Scoring more touchdowns than your opponent in American football means you won the game.

PRACTICE: NECESSARY AND SUFFICIENT CONDITIONS

1. Scoring more touchdowns than your opponent in American football means you won the game.

Answer: [◀ Show Work](#)

Sufficient but not necessary.

PRACTICE: NECESSARY AND SUFFICIENT CONDITIONS

1. Scoring more touchdowns than your opponent in American football means you won the game.
2. Obtaining a learner's permit will lead to earning a driver's license.

PRACTICE: NECESSARY AND SUFFICIENT CONDITIONS

1. Scoring more touchdowns than your opponent in American football means you won the game.
2. Obtaining a learner's permit will lead to earning a driver's license.

Answer: ◀ Show Work

Necessary but not sufficient.

PRACTICE: NECESSARY AND SUFFICIENT CONDITIONS

1. Scoring more touchdowns than your opponent in American football means you won the game.
2. Obtaining a learner's permit will lead to earning a driver's license.
3. All even positive whole numbers must be divisible by two.

PRACTICE: NECESSARY AND SUFFICIENT CONDITIONS

1. Scoring more touchdowns than your opponent in American football means you won the game.
2. Obtaining a learner's permit will lead to earning a driver's license.
3. All even positive whole numbers must be divisible by two.

Answer: ◀ Show Work

Both necessary and sufficient.

Topic: Proofs

MOTIVATION

- ▶ General background
 - ▶ Method for proving or disproving a logical statement
- ▶ Why do economists' care?
 - ▶ Determine which theories are incorporated into economic theory
- ▶ Application in this career
 - ▶ Theory papers and well-developed theory sections of empirical papers.
 - ▶ Often in appendix sections to prove statements articulated as part of an argument in a paper.

MOTIVATION

- ▶ General background
 - ▶ Method for proving or disproving a logical statement
- ▶ Why do economists' care?
 - ▶ Determine which theories are incorporated into economic theory
- ▶ Application in this career
 - ▶ Theory papers and well-developed theory sections of empirical papers.
 - ▶ Often in appendix sections to prove statements articulated as part of an argument in a paper.

MOTIVATION

- ▶ General background
 - ▶ Method for proving or disproving a logical statement
- ▶ Why do economists' care?
 - ▶ Determine which theories are incorporated into economic theory
- ▶ Application in this career
 - ▶ Theory papers and well-developed theory sections of empirical papers.
 - ▶ Often in appendix sections to prove statements articulated as part of an argument in a paper.

OVERVIEW

1. Truth Tables
2. Existence & Uniqueness
3. Direct Proofs
4. Proof by Contradiction
5. Proof by Induction
6. Proof by Contrapositive

ASSUMPTIONS ARE THE CORE OF PROOFS...



1. TRUTH TABLE

- Shows how the truth/falsity of a compound statement depends on the truth/falsity of the simple statements from which it's constructed.
- Statements = {Known to be true, known to be false, truth unknown }
- Truth table for $(P \rightarrow Q)$:

P	Q	$P \iff Q$
T	T	T
T	F	F
F	T	F
F	F	T

DEMONSTRATION: TRUTH TABLE

Question:

Construct a truth table for $(P \rightarrow Q) \vee (Q \rightarrow P)$

Answer:

P	Q	$P \rightarrow Q$	$Q \rightarrow P$	$(P \rightarrow Q) \vee (Q \rightarrow P)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

DEMONSTRATION: TRUTH TABLE

Question:

Construct a truth table for $(P \rightarrow Q) \vee (Q \rightarrow P)$

Answer:

P	Q	$P \rightarrow Q$	$Q \rightarrow P$	$(P \rightarrow Q) \vee (Q \rightarrow P)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

2. EXISTENCE AND UNIQUENESS

- ▶ Existence: Can an answer to the truth of a statement (viz., proof) be found.
 - ▶ Sometimes you can prove no answer can exist.
- ▶ Uniqueness: An assertion that there is exactly one statement that is true for that family of statements.
 - ▶ x is the family $P(x)$.
- ▶ Ideally, you want a statement which exists and is unique.
 - ▶ Ex. If “ $x + 2 = 3$, then $x = 1$ ” is a statement that exists and is unique.

APPLICATION: EQUILIBRIUM EXISTS

Existence of Equilibrium II Theorem:

Suppose that each consumer's preferences are continuous, strongly monotonic, and convex. Suppose also that, for each consumer i , $\omega_i \gg 0$. Then there exists a Walrasian equilibrium (p^*, x^*) for ε .

3. PROOF BY DEDUCTION (DIRECT PROOF)

- ▶ Show $A \implies B$
- ▶ Deductive reasoning: Use a set of premises that lead to a conclusion.
- ▶ Sometimes we need to strengthen A but adding assumptions (e.g., weak assumptions are preferred).

DEMONSTRATION: DIRECT PROOF

Question:

Let m be an even integer and p be any integer. Then $m \times p$ is an even integer.

Answer:

Proof.

m is an even integer so \exists an integer q such that $m = 2 \times q$ by the definition of an even integer. Therefore, we can make the statement:

$$m \times p = (2 \times q) \times p = 2 \times (q \times p)$$

So, $m \times p$ is an even integer. □

DEMONSTRATION: DIRECT PROOF

Question:

Let m be an even integer and p be any integer. Then $m \times p$ is an even integer.

Answer:

Proof.

m is an even integer so \exists an integer q such that $m = 2 \times q$ by the definition of an even integer. Therefore, we can make the statement:

$$m \times p = (2 \times q) \times p = 2 \times (q \times p)$$

So, $m \times p$ is an even integer. □

4. PROOF BY CONTRADICTION

- ▶ $A \implies B \equiv \neg A \text{ and } \neg B \equiv \neg B \implies \neg A$.
- ▶ E.g., If the conclusion is not untrue, then the premise must be untrue.

DEMONSTRATION: PROOF BY CONTRADICTION

Question:

Walras' Law: $\forall x \in x(p, w)$ that maximizes consumer utility, then $x \times p = w$.

Answer:

Proof.

Suppose $\exists x \in x(p, w) : x \times p < w$ ($\neg B$), then there must be another $y \in x(p, w)$ that is affordable and $y \succ x$ by the assumption of “local non-satiation”. Therefore, since y exists and is affordable, then x does not maximize utility ($\neg A$). \square

DEMONSTRATION: PROOF BY CONTRADICTION

Question:

Walras' Law: $\forall x \in x(p, w)$ that maximizes consumer utility, then $x \times p = w$.

Answer:

Proof.

Suppose $\exists x \in x(p, w) : x \times p < w$ ($\neg B$), then there must be another $y \in x(p, w)$ that is affordable and $y \succ x$ by the assumption of “local non-satiation”. Therefore, since y exists and is affordable, then x does not maximize utility ($\neg A$). \square

5. PROOF BY INDUCTION



Professor Schmidt demonstrates
the concept of proof by induction.

5. PROOF BY INDUCTION

- ▶ Inductive reasoning: Drawing conclusions by reasoning a series of specific examples generalizes.
- ▶ Often used by indexing through integers.

DEMONSTRATION: PROOF BY INDUCTION

Question:

$$P(n) : 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Answer:

Proof.

Note that $P(1)$ is true because $1 = \frac{1 \times 2}{2}$. Assume $P(n)$ is true for $k \in \mathbb{N}$ integers: $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$. Add $(k+1)$ to both sides.

$$1 + 2 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}$$

This is $P(k+1)$, implying that $P(k)$ is true for all $P(n)$. □

DEMONSTRATION: PROOF BY INDUCTION

Question:

$$P(n) : 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Answer:

Proof.

Note that $P(1)$ is true because $1 = \frac{1 \times 2}{2}$. Assume $P(n)$ is true for $k \in n$ integers: $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$. Add $(k+1)$ to both sides.

$$1 + 2 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}$$

This is $P(k+1)$, implying that $P(k)$ is true for all $P(n)$. □

6. PROOF BY CONTRAPOSITIVE

- Double Negation:

$$\neg B \implies \neg A \equiv \neg\neg A \implies \neg\neg B \equiv A \implies B.$$

- Convenient when there is a universal quantifier (\forall) present included by the contrapositive.

DEMONSTRATION: PROOF BY CONTRAPOSITIVE

Question:

Suppose $x \in \mathbb{Z}$. If $7x + 9$ is even, then x is odd.

Answer:

Proof.

Suppose x is **not** odd (i.e., even) implying $x = 2a$ for some integer a . Then,

$$7x + 9 = 7(2a) + 9 = 14a + (8 + 1) = 2(7a + 4) + 1 = 2b + 1$$

if $b = 7a + 4$. Consequently, $2b + 1$ is odd for all b . Therefore $7x + 9$ is **not** even. □

DEMONSTRATION: PROOF BY CONTRAPOSITIVE

Question:

Suppose $x \in \mathbb{Z}$. If $7x + 9$ is even, then x is odd.

Answer:

Proof.

Suppose x is **not** odd (i.e., even) implying $x = 2a$ for some integer a . Then,

$$7x + 9 = 7(2a) + 9 = 14a + (8 + 1) = 2(7a + 4) + 1 = 2b + 1$$

if $b = 7a + 4$. Consequently, $2b + 1$ is odd for all b . Therefore $7x + 9$ is **not** even. □

PRACTICE: PROOFS

1. If x is odd, then x^2 is odd.

PRACTICE: PROOFS

1. If x is odd, then x^2 is odd.

Answer: [◀ Show Work](#)

x^2 is odd by definition of an odd number.

PRACTICE: PROOFS

1. If x is odd, then x^2 is odd.
2. Suppose $x \in \mathbb{Z}$. If $x^2 = 6x + 5$ is even, then x is odd.

PRACTICE: PROOFS

1. If x is odd, then x^2 is odd.
2. Suppose $x \in \mathbb{Z}$. If $x^2 = 6x + 5$ is even, then x is odd.

Answer: [◀ Show Work](#)

By contrapositive, $x^2 = 6x + 5$ is odd and therefore not even.

PRACTICE: PROOFS

x is odd, then x^2 is odd. Suppose $x \in \mathbb{Z}$. If $x^2 = 6x + 5$ is even, then x is odd. There are infinitely many prime numbers.

PRACTICE: PROOFS

x is odd, then x^2 is odd. Suppose $x \in \mathbb{Z}$. If $x^2 = 6x + 5$ is even, then x is odd. There are infinitely many prime numbers.

Answer: [◀ Show Work](#)

Proof by contradiction.

Review

REVIEW OF LOGIC

1. Logical Statement: Necessary to provide clarity to your statements
2. Necessary vs. Sufficient Conditions: Aiming to make statements that are both (iff)

REVIEW OF PROOFS

1. Truth Tables
2. Four methods to prove a statement:
 - ▶ Direct proof
 - ▶ Proof by contradiction
 - ▶ Proof by induction
 - ▶ Proof by Contrapositive

ASSIGNMENT

- ▶ Readings on Sets & Topology before Lecture 02:
 - ▶
- ▶ Assignment:
 - ▶ Problem Set 01 (PS01)
 - ▶ Solution set will be available following end of Lecture 02
- ▶ Struggling?
 1. Read the ‘Encouraged Reading’
 2. Review ‘Supplementary material’
 3. Reach out directly

N & S CONDITIONS QUESTION 1 ANSWER:

◀ QUESTION

- ▶ Not Necessary: If you won the game (B), you may have scored other points but less touchdowns ($\neg A$).
- ▶ Sufficient: If you score more touchdowns (A) (and therefore more overall points), then you will win the game (B).

N & S CONDITIONS QUESTION 2 ANSWER:

◀ QUESTION

- ▶ Necessary: A learner's permit (A) is required before you can get a drivers license (B).
- ▶ Not Sufficient: Not all learners ($\neg A$) successfully earn their drivers license ($\neg B$).

N & S CONDITIONS QUESTION 3 ANSWER:

◀ QUESTION

- ▶ Necessary: To be divisible by 2 (B), you must be an even whole number (A).
- ▶ Sufficient: If you are an even whole number (A), you will have no remainder if divided by two (B).

PROOFS QUESTION 1 ANSWER:

◀ QUESTION

Proof.

Suppose x is odd. Then $x = 2a + 1$ for some $a \in \mathbb{Z}$, by definition an odd number. Thus $x^2 = (2a + 1)^2 = 4a^2 + 4a + 1$. This is $2(2a^2 + 2a) + 1$. So $x^2 = 2b + 1$ for an integer b . Therefore, x^2 is odd, by definition of an odd number. \square

PROOFS QUESTION 2 ANSWER:

◀ QUESTION

Proof.

Suppose x is **not** odd. Thus x is even, so $x = 2a$ for some integer a . So $x^2 - 6x + 5 = (2a)^2 - 6(2a) + 5 = 2(2a^2 - 6a + 2) + 1$. Then $x^2 - 6x + 5 = 2b + 1$ for $b = 2a^2 - 6a + 2$. Consequently, $x^2 - 6x + 5$ is odd, and therefore not even. □

PROOFS QUESTION 3 ANSWER:

◀ QUESTION

Proof.

Suppose there are only finite prime numbers. Then they can be listed as p_1, p_2, \dots, p_n . Then p_n is the final and largest prime number. Consider a number $a = (p_1 \cdot p_2 \cdots p_n) + 1$. a has at least one prime divisor (e.g., p_k in the list). So there is some integer c such that $(p_1 \cdot p_2 \cdots p_{k-1} p_k p_{k+1} \cdots p_n) + 1 = c \cdot p_k$. Divide both sides by p_k . Now we have $\frac{1}{p_k} = c - (p_1 \cdot p_2 \cdots p_{k-1} p_{k+1} \cdots p_n)$. The expression on the right is an integer (i.e., which prime is a part of) **but** the left is not an integer. This is a contradiction. Therefore, there must be no finite range of prime numbers. \square