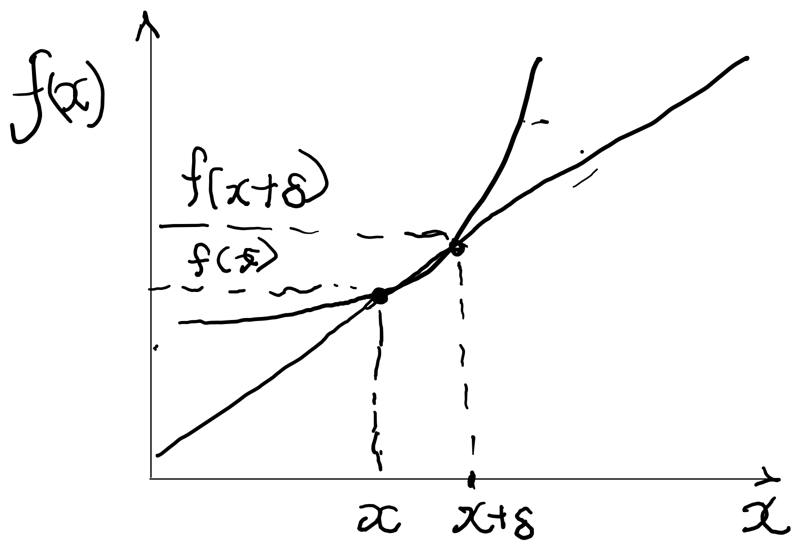
3. Calculus: Derivatives of single variable functions

The derivative provides us the :

- rate of change of a function w.r.t the variable
- slope of a function at any point
- marginal effects

These terms are used interchangeably. We make use of the following limit to derive derivatives:



$$\frac{df(x)}{dx} = \lim_{\delta \to 0} \frac{f(x+\delta)}{\delta}$$

Exercise

Using this limit, find derivative w.r.t x for these functions.

- $f(x) = x^2$
- $f(x) = x^3$
- $f(x) = x^n$

Commonly used derivatives

$$\frac{d}{dx}x^n = nx^{n-1}$$

$$\frac{d}{dx}(a^x) = a^x \ln(a)$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x}, x > 0$$

$$\frac{\mathrm{d}}{\mathrm{dx}}(\log_{\mathrm{a}}(\mathrm{x})) = \frac{1}{\mathrm{xlna}}, \mathrm{x} > 0$$

Basic Rules of differentiation

1. For constant : $\frac{\partial \alpha}{\partial \mathrm{x}} = 0$
2. For sums: $\frac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g'(x)$
3. Product rule $\frac{d}{dx}f(x)g(x) = g(x)\left(\frac{df(x)}{dx}\right) + f(x)\left(\frac{dg(x)}{dx}\right)$
4. Quotient rule $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\left(g(x)\left(\frac{df(x)}{dx}\right) - f(x)\left(\frac{dg(x)}{dx}\right)\right)}{g^2(x)}$

Exercise

find $\frac{dy}{dx}$ $given\ that\ \ y =$

$a)\ xe^{3x}$ $\quad b)\ e^{x^2(3x-2)}$ $\quad c)\ \ln(x^4+2)^2$

$d)\ \frac{x}{e^x}$ $\quad e)\ \frac{x}{lnx}$ $\quad f)\ \frac{lnx}{x}$ $\quad g)\ \left(\frac{x+6}{x+5}\right)^{\frac{1}{4}}$

$h)\ f(x)g(x)h(x)$

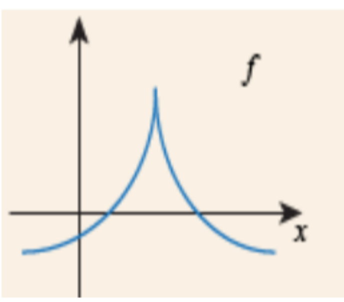
Differentiability

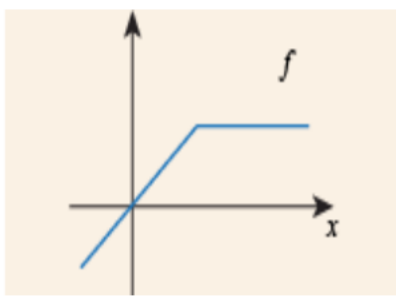
If a function is differentiable, it is continuous. That is

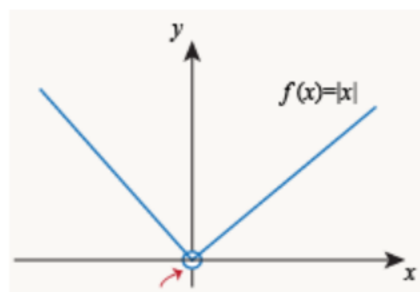
$differentiable \Rightarrow continuous$

However, $differentiable \Leftarrow continuous$

Functions with kinks and sharp ends are not differentiable.







Second Derivatives

The second derivative $f''(x)\ or\ \frac{d^2x}{dx^2}$ is the derivative of the (first) derivative:

$$f''(x) = [f'(x)]' = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

- If f' is continuous, then we say f is continuously differentiable, denote as C^1 .
- If $f''$is continuous, then we say f is twice continuously differentiable, denote as C^2.

Implicit functions

Example:

$e^y + xy - e = 0$

Take the derivative of x on both sides,

$\frac{d}{dx}(e^y + xy - e) = e^y * \frac{dy}{dx} + y + x\frac{dy}{dx} = 0$

Rearranging, we have: $\frac{dy}{dx} = -\frac{y}{x + e^y},\ (x + e^y) \neq 0$

Find the derivative of y with respect to x for the following problem:

$$x^2y^3 - xy = 10$$

Exercise

1. For parts a. and b., consider the following 2×2 competitive exchange economy, with consumers $j = 1, 2$ and goods x and y. The consumers' preferences are given by

$$U_1(x_1, y_1) = \ln x_1 + 3\ln y_1 \qquad \text{and} \qquad U_2(x_2, y_2) = 3\ln x_2 + \ln y_2.$$

Endowments are $\omega_1 = (4, 4)$ and $\omega_2 = (4, 4)$.

Find the contract curve. Hint MRS1 = MRS2

L'Hopital's Rule

This rule is handy when we wish to find the result when two functions form a fraction.

$$\lim_{x \to a} . \frac{f(x)}{g(x)} = \lim_{x \to a} . \frac{f'(x)}{g'(x)}$$

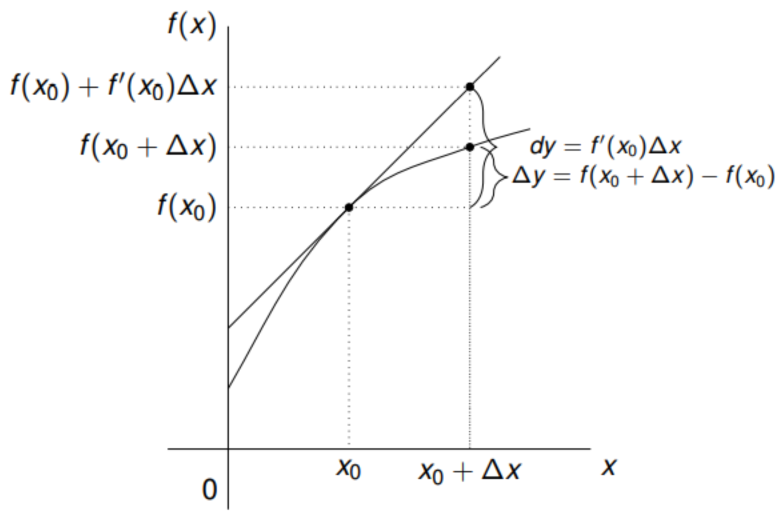
This law applies when a approaches zero or when the functions $f(x)$ *and* $g(x)$approach zero or infinity when a approaches a.

Exercise: Show that

Show that the constant elasticity of substitution (CES) function $Y = A(\alpha K^{\gamma} + (1-\alpha)L^{\gamma})^{\frac{1}{\gamma}}$ is Cobb-Douglas function $Y = AK^{\alpha}L^{1-\alpha}$ when γ → 0.

Hint: First, take log on both sides

Approximation by differentials



$$\Delta y \approx dy = f'(x_0)\Delta x$$

Taylor series

A better approximation that what we have above is :

$$f(x_0+\Delta x) = f(x_0)+f'(x_0)\Delta x+\frac{1}{2!}f''(x_0)(\Delta x)^2+\ldots+\frac{1}{k!}f^{(k)}(x_0)(\Delta x)^k + R_k(\Delta x, x_0),$$

where $R_k(\Delta x, x_0) = \frac{f^{(k+1)}(c^*)}{(k+1)!}(\Delta x)^{k+1}$, $c^* \in (x_0, x_0 + \Delta x)$ and $\frac{R_k(\Delta x, x_0)}{(\Delta x)^k} \to 0$ as $\Delta x \to 0$.

This is the Tayor series. Note that the function must be differentiable to order k+1.

Application on translog function.

Henningsen and Henningsen, 2011 (page 57) proved that the CES function can be transformed into a series using Tylor series, and the result is:

$$\begin{aligned} ln\, Y &= ln\, A + \alpha ln\, K + (1-\alpha) ln\, L + \frac{1}{2}\gamma\alpha(1-\alpha)(ln\, K - ln\, L)^2 \\ &= \beta_0 + \beta_1 ln\, K + \beta_2 ln\, L + \beta_3 ln^2 K + \beta_4 ln^2 L + \beta_5 ln\, K\, ln\, L \end{aligned}$$

Application: Log differences

We can use Taylor series to prove that when x is small

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \ = O(x^2) \approx x$$

If y* is c percent greater than y, then $y* = \left(1 + \frac{c}{100}\right) y$

Taking logs gives : ln(y*) = $lny + \ln\left(1 + \frac{c}{100}\right)$

Therefore: $ln(y*) - ln(y) = \frac{c}{100}$

What is the interpretation of the coefficient β in the regression:

$$\log(y) = \alpha + \beta x + \epsilon$$

Exercise

Prove that

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \ = O(x^2) \approx x$$