Lecture 02
Set Theory and Topology

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Last Lecture Review

- Logic:
  - Logical statements
  - Necessary vs. sufficient
- Proofs:
  - Proof by Deduction/Construction (Direct Proofs)
  - Proof by Contrapositive
  - Proof by Contradiction
  - Proof by Induction

Review Assignment

1. Problem Set 01 solutions are available on Github.
2. Any issues or problems **You** would like to discuss?

Daily Icebreaker

- Attendance via prompt:
  - Name
  - Program and track
  - Daily Icebreaker: The zombie apocalypse is tomorrow. What is your strategy to survive?

Topic: Set Theory

Motivation

- General background
  - How collections of mathematical objects are organized.
  - A foundation for all of math.
- Why do economists' care?
  - Need to have strong understanding of the basics.
  - How we categorize in economics.
- Application in this career
  - Rarely directly.
  - Sometimes useful when considering proofs.

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Overview

1. Sets
2. Set Operators
3. Set Space
4. de Morgans' Law & Cartesian Product
5. Cardinality & Countability
6. Convex Sets
7. Open & Closed Sets
8. Bounded & Compact Sets

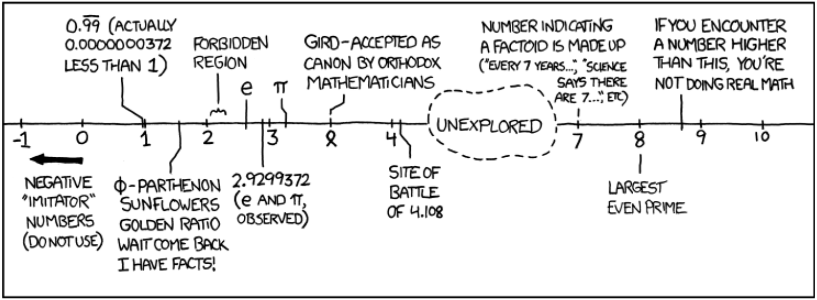
1. Sets

- Sets
  - A collection of objects (elements or members)
  - $S = \{s \in U : P\}$ for the universal set U such that is satisfies properties P.
- Elements
  - The components within a set.
  - An element can be a complex object; such as another set.
- Empty Set
  - $\emptyset = \{s \notin U\}$ contains nothing.
  - $\emptyset \neq \{\emptyset\}$

1. Sets

$\mathbb{R}$	Real Numbers: $\{x : -\infty \leq x \leq \infty\}$
$\mathbb{R} \times \mathbb{R}$	Cartesian Plane
$\mathbb{N}$	Natural Numbers
$\mathbb{W}$	Whole Numbers: $\mathbb{N} \wedge 0$
$\mathbb{Z}$	Integers
$\mathbb{Q}$	Rational Numbers
$\mathbb{P}$	Irrational Numbers

The Number Line



2. Set Operators

- Complement: $A^c \equiv \{x \in U : x \notin A\}$
- Intersection: $A \cap B \equiv \{x \in U : x \in A \wedge x \in B\}$
- Union: $A \cup B \equiv \{x \in U : x \in A \vee x \in B\}$
- Set Difference (Partition): $A \setminus B \equiv \{x \in U : x \in A \wedge x \notin B\}$
- Disjoint Set: $A \cap B = \emptyset$
- Subset: $B \subset A$ if $[x \in B] \implies [x \in A]$
- Proper (Strict) Subset: $B \subset A \wedge B \neq A$.
- Power Set (All subsets of a set): $\mathcal{P}(A) \equiv \{X : X \subseteq A\}$
- Indexed Set: $A_1, A_2, \ldots, A_i$

2. Set Operators

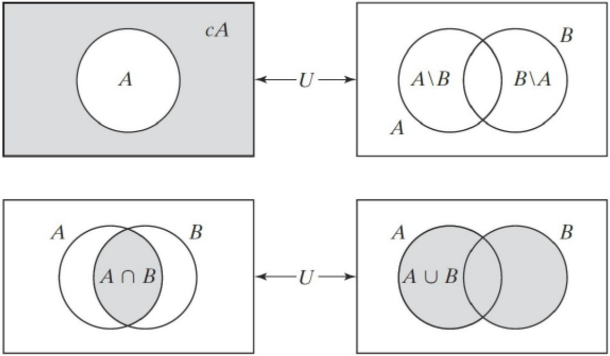
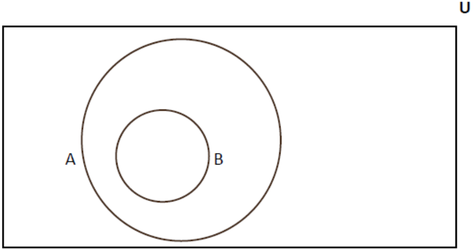
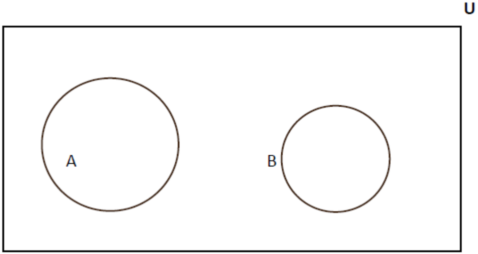


Figure A1. 1. Venn diagrams.

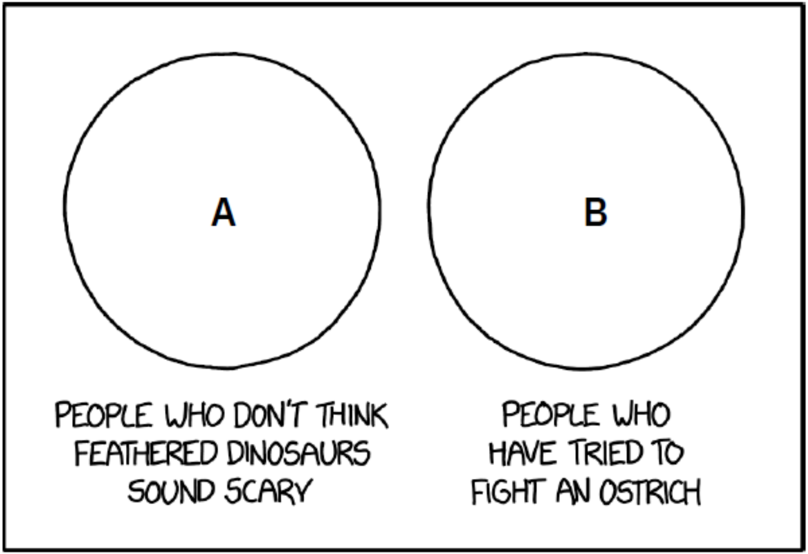
Source: Jehle & Reny (2011)

2. Set Operators





Disjoint Sets



3. Set Space

- Set Product: A set of ordered pairs

$$S \times T \equiv \{(s,t) | s \in S, t \in T\}$$

- N-dimensional Euclidean Space

$$\mathbb{R}^n \equiv \mathbb{R} \times \mathbb{R} \times \cdots \mathbb{R} \equiv \{(x_1, \ldots, x_n) | x_i \in \mathbb{R}, \forall i = 1, \ldots, n\}$$

- Cartesian Plane

$$\mathbb{R}^2 \equiv \mathbb{R} \times \mathbb{R} \equiv \{(x_1, x_2) | x_1, x_2 \in \mathbb{R}\}$$

4. de Morgan's Law and Cartesian Product

- de Morgan's Law: Assume A_i are subsets

$$\left[\bigcup_{i=1}^{k} A_i\right]^c = \bigcap_{i=1}^{k} A_i^c$$

- Cartesian Product: For 2 sets A and B, the Cartesian product is:

$$A \times B \equiv \{(a, b) : a \in A, b \in B\}$$

5. Cardinality and Countability

- Cardinality: $|A|$ is the number of elements in the set.
  - Types: Finite, countably infinite, and uncountable
- Countable (Finite) Set: Any infinite set that can be placed in a 1 to 1 correspondence with $\mathbb{N}$.

6. Convex Sets

- Convex Set: $S \subset \mathbb{R}^n$ is a convex set $\forall x_1, x_2 \in S, \forall t \in (0,1)$, if we have $tx_1 + (1-t)x_2 \in S$.

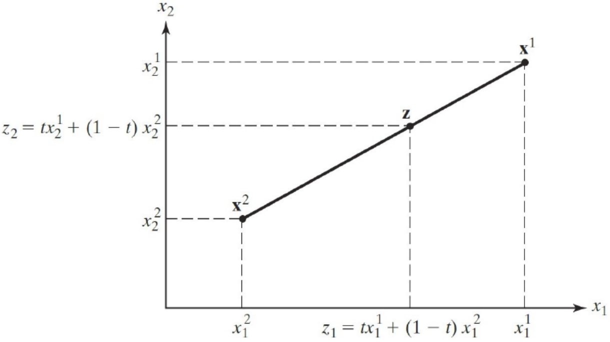


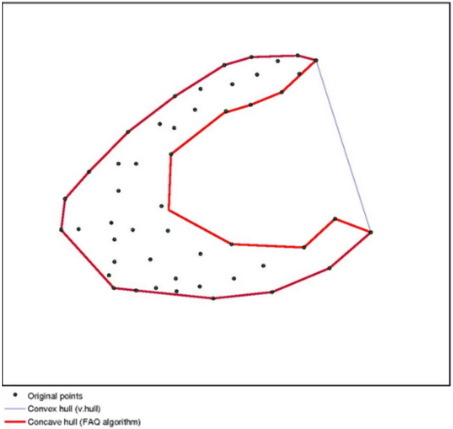
Figure A1.4. Some convex combinations in $\mathbb{R}^2$.

Source: Jehle & Reny (2011)

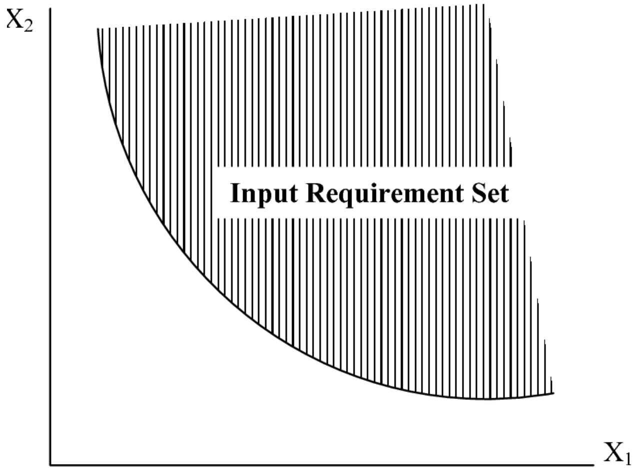
6. Convex Sets

- Convex Hull: For set $B \subset \mathbb{R}^N$, convex hull is:

$$CoB = \{\sum_{j=1}^{J} \alpha_j x_j : x1, \ldots, x_j \in B \forall j < J \wedge (\alpha_1, \ldots, \alpha_J) \geq 0, \sum_{j+1}^{J} \alpha_j = 1\}$$



Application: Input Requirement Set



Demonstration: Convex Sets

Question:

Let $C = (3, 2)$. Show that the set $S = \{u \in \mathbb{R}^2 | u \cdot v \leq 9\}$ is a convex set.

Answer:

The set S is convex if $u, w \in C : tu + (1-t)w \in C \forall t \in [0, 1]$. Let $u, w \in S$ and $t \in [0, 1]$.

$$
\begin{aligned}
&(tu + (1-t)w) \cdot v \\
&= (tu) \cdot v + (1-t)w \cdot v \\
&\leq t \times 9 + (1-t) \times 9 \\
&= 9 \in S
\end{aligned}
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DEMONSTRATION: CONVEX SETS

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7. Open and Closed Sets

- Metric Space (e.g., point distance): $d(x_1, x_2) = |x_1 - x_2|$
- An open ε-ball with center x_0 and radius $\varepsilon > 0$ is a subset of points in $\mathbb{R}^n$:

$$B_\varepsilon(x_0) \equiv \{x \in \mathbb{R}^n | |x - x_0| < \varepsilon\}$$

- A closed ε-ball is similar, but includes the circumference (e.g., edge of the ball)

$$B_\varepsilon(x_0) \equiv \{x \in \mathbb{R}^n | |x - x_0| \leq \varepsilon\}$$

7. Open and Closed Sets

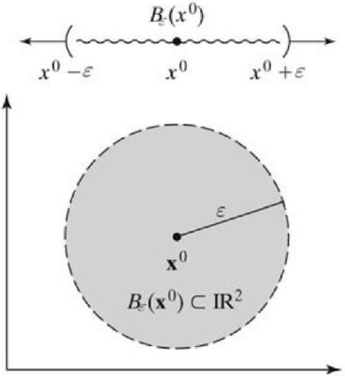


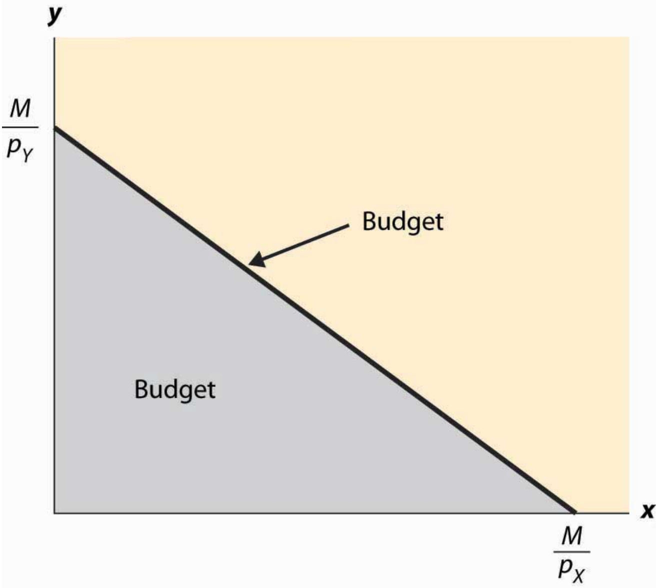
Figure A1. 10. Balls in $\mathbb{R}$ and $\mathbb{R}^2$.

Source: Jehle & Reny (2011)

8. Bounded and Compact Sets

- Bounded: A set $S \subset \mathbb{R}^n$ is bounded if it is entirely contained within some ε-ball (either closed or open)
- Compact: A set $S \subset \mathbb{R}^n$ is compact if it is both closed and bounded.
- We like working with compact sets.

Application: Budget Set



Practice: Sets

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Answer: Show Work

$A \cap B = \{1, 2, 3\}$

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Answer: Show Work

$$[tx_1 + (1-t)x_2] + [ty_1 + (1-t)y_2]$$
$$= x + y \in K + L$$

Topic: Topology

Motivation

- General background
  - Understanding of spatial relationships and how the parts are integrated into the whole.
- Why do economists' care?
  - Used in proofs.
  - Several theorems used as lemmas invoked in proofs.
- Application in this career
  - Welfare theorem
  - Consumer behavior
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Overview

1. Supremum & Infimum
2. Sequences and Limits
3. Separating Hyperplane Theorem

1. Supremum and Infimum

- Ordered Set: When elements have a defined order ($<$).
- To be ordered:
  - For $x, y \in A$, only one of the following statements can be true: (1) $x < y$, (2) $x = y$, or (3) $x > y$.
  - For $x, y, z \in A$, if $x < y \wedge y < z \implies x < z$.
- A subset (A_1) of an ordered set may be bounded from above and below if:
  - Upper Bound: $\{\beta \in A : x \leq \beta \forall x \in A_1\}$
  - Lower Bound: $\{\beta \in A : x \geq \beta \forall x \in A_1\}$

1. Supremum and Infimum

- Supremum: Least upper bound (1 value)

$$\beta = \sup A_1 | \beta \in A : y < \beta \exists x \in A_1 : x > y$$

- Infimum: Greatest lower bound (1 value)

$$\alpha = \inf A_1 | \alpha \in A : \delta > \alpha \exists x \in A_1 : x < \delta$$

Demonstration: Supremum & Infimum

Question:

Prove $sup\{\frac{n}{n+1}|n \in \mathbb{N}\} = 1$

Answer:

1 is the upper bound by $n+1 \geq n \implies 1 \geq \frac{n}{n+1}$. Let $\varepsilon > 0$ be arbitrarily small. Then $\exists n : \frac{n}{n+1} > 1 - \varepsilon$.

$$\begin{aligned} \varepsilon >& 1 - \frac{n}{n+1} \\ \varepsilon >& \frac{1}{n+1} \\ n >& \frac{1}{\varepsilon} - 1 \end{aligned}$$

Note we can go in reverse order to show

$$\frac{n}{n+1} > 1 - \varepsilon$$

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Note we can go in reverse order to show

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2. Sequences and Limits

- Sequence: Function $f(\cdot)$ defined on a set of natural numbers, $\mathbb{N}$.
- Limit: A sequence $\{x_n\}$ converges to a limit, $x_n \to L$ or $\lim_{n\to\infty} x_n = L$, if given $\varepsilon > 0$ there is an element N such that whenever $n > N : |x_n - L| < \varepsilon$.
- A sequence diverges when it does not converge to a limit.

Theorem:
If the sequence $\{x_n\}$ converges, then the limit of $\{x_n\}$ is unique (e.g., single valued).

Demonstration: Limits

Question:

Show $\lim \frac{1}{n} \to 0$.

Answer:

Let $\varepsilon > 0$ which is arbitrarily small. Note that for some $a \in \mathbb{N} : \frac{1}{a} < \varepsilon$. So, if $n > a$, then:

$$|\frac{1}{n} - 0| = \frac{1}{n} < \frac{1}{a} < \varepsilon$$

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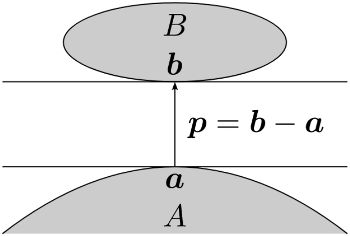
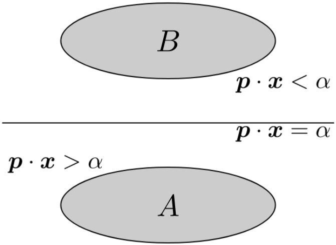
3. Separating Hyperplane Theorem

- There exists a line dividing an n-dimensional space.
- Given $p \in \mathbb{R}^n : p \neq 0$ and $c \in \mathbb{R}$, the hyperplane generated is the set $H_{p,c} = \{z \in \mathbb{R}^n | p \cdot z = c\}$.





Practice: Topology

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Answer: Show Work

$\sup = 11$

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1. What is the $\sup\{a+b : a \in (0,2), b \in (3,9)\}$?
2. Solve $\lim_{n\to\infty} \frac{4n^3+3n}{n^3-6}$.

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Answer: Show Work

$\lim_{n\to\infty} \frac{4n^3+3n}{n^3-6} \to 4$

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3. Let $x_n \geq 0$. Show that if the sequence $x_n \to 0$, then $\sqrt{x_n} \to 0$.

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3. Let $x_n \geq 0$. Show that if the sequence $x_n \to 0$, then $\sqrt{x_n} \to 0$.

Answer: Show Work

We can re-write $lim(x_n) = lim(\sqrt{x_n}\sqrt{x_n})$. Use this fact to show $(\lim(\sqrt{x_n}))^2 = 0$ implying the answer.

Review

Review of Sets

1. Sets are the foundation of organizing objects in math.
2. de Morgan's Law
3. Cartesian Product
4. Convex Sets
5. Bounded Sets
6. Compact Sets

Review of Topology

1. Supremum and Infimum
2. Limits
3. Separating Hyperplane Theorem

Assignment

- Readings on Derivatives before Lecture 03:
  - MWG Appendix M.A.
  - B&S Ch. 6
- Assignment:
  - Problem Set 02 (PS02)
  - Solution set will be available following end of Lecture 03
- Struggling?
  1. Read the ‘Encouraged Reading’
  2. Review ‘Supplementary material’
  3. Reach out directly

Sets Question 1 Answer:

Question

$$
\begin{aligned}
A =& \{1, 2, 3, 6, 9, 18\} \\
B =& \{1, 2, 3, 4, 5\} \\
A \cap B =& \{1, 2, 3\}
\end{aligned}
$$

Sets Question 2 Answer:

◂ Question

- No extra work.

$$
\begin{aligned}
&A \times B \times C = \\
&\{(a_1, b_1, c_1), (a_1, b_1, c_2), (a_1, b_1, c_3) \\
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\end{aligned}
$$

Sets Question 3 Answer:

◂ Question

Let $u_1, u_2 \in K + L$ so that $x_1, x_2 \in K$ and $y_1, y_2 \in L$ and let $t \in [0,1]$. Then:

$$
\begin{aligned}
&tu_1 + (1-t)u_2 = t(x_1 + y_1) + (1-t)(x_2 + y_2) \\
&= [tx_1 + (1-t)x_2] + [ty_1 + (1-t)y_2] \\
&= x + y \in K + L
\end{aligned}
$$

Topology Question 1 Answer:

Question

Note that the set is $a + b$. We can use the distributive property to show that $sup(a + b) = \sup a + \sup b$. Then we just need to know the least upper bound for a, b. Note these values are over an open interval $(\cdot)$ rather than a closed interval $[\cdot]$. So $\sup a + \sup b = 2 + 9 = 11$

Topology Question 2 Answer:

◂ Question

Multiply by $\frac{1}{n^3}$. Then distribute the limit and determine what happens at $n \to \infty$.

$$\lim \frac{1}{n^3} \cdot \frac{4n^3+3n}{n^3-6} = \lim \frac{\frac{4n^3}{n^3}+\frac{3n}{n^3}}{\frac{n^3}{n^3}-\frac{6}{n^3}} = \lim \frac{4+\frac{3}{n^2}}{1-\frac{6}{n^3}} = \frac{4+\lim \frac{3}{n^2}}{1-\lim \frac{6}{n^3}} \to \frac{4+0}{1-0} = 4$$

Topology Question 3 Answer:

Question

Note that $\lim(x_n) = 0$ is given. Let $lim(x_n) = lim(\sqrt{x_n}\sqrt{x_n}) = 0$.
Again, note that for some convergent sequences a_n, b_n we have $lim(a_n) = a$ and $lim(b_n) = b$ implying that $lim(a_nb_n) = ab$.
Applied to this scenario,
$lim(\sqrt{x_n}\sqrt{x_n}) = lim(\sqrt{x_n})lim(\sqrt{x_n}) = (lim(\sqrt{x_n}))^2 = 0 = 0 \cdot 0$.
$\therefore lim(\sqrt{x_n}) \to 0$.