Lecture 08
Numbers and Functions

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Mathematics Review Course, Summer 2023
University of Minnesota
August 16th, 2023

Last Lecture Review

- Linear Algebra:
  - Gaussian Elimination
  - Linear Operators
  - Existence of a Solution
  - Cramer's Rule
  - Eigenvalues
  - Regression as a Matrix

Review Assignment

1. Problem Set 07 solutions are available on Github.
2. Any issues or problems **You** would like to discuss?

Daily Icebreaker

- Attendance via prompt:
  - Name
  - Program and track
  - Daily icebreaker subject...

Topic: Numbers

Motivation

- General background
  - The terminology of mathematics
  - Formalized by the branch of math called ‘Number Theory’.
- Why do economists’ care?
  - Economists express values, sets, and concepts using numerical (quantitative) values.
- Application in this career
  - Throughout your whole experience.

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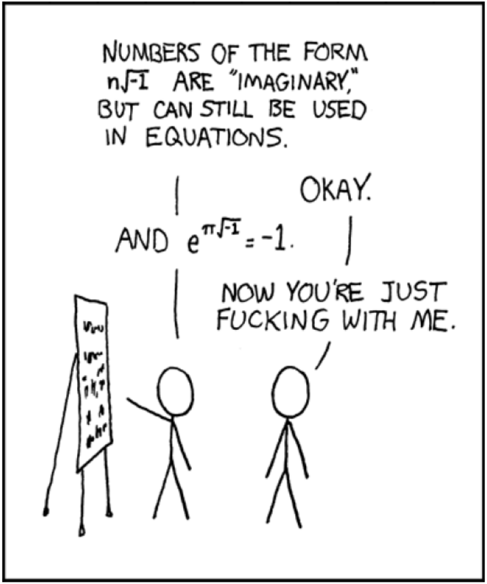
Overview

1. Common Number Sets
2. Real Numbers
3. Absolute Value and Number Line
4. Triangle Inequality
5. Neighborhoods

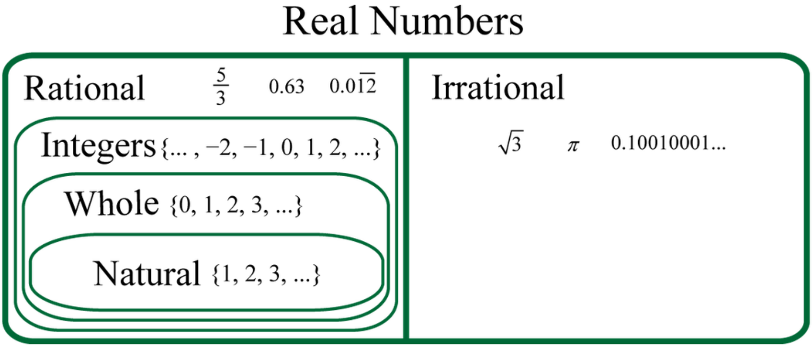
Numbers


NUMBERS OF THE FORM $n\sqrt{-1}$ ARE "IMAGINARY," BUT CAN STILL BE USED IN EQUATIONS.
OKAY.
AND $e^{\pi\sqrt{-1}} = -1$.
NOW YOU'RE JUST FUCKING WITH ME.

1. Common Number Sets

- Natural Numbers: $\mathbb{N} = \{1, 2, 3, \cdots\}$.
- Integers: $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3\}$.
- Rational Numbers: $\mathbb{Q} = \{\frac{p}{q} p, q \in \mathbb{Z}, q \neq 0\}$.
- Real Numbers: $\mathbb{R} = \{$ all decimals $\}$.
- Complex Numbers: $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}, i = \sqrt{-1}\}$.

1. Common Number Sets

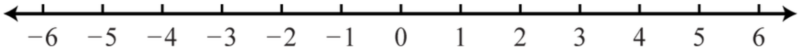


2. Real Numbers

- Two binary operators for the Reals
  - Addition: $a + b$
  - Multiplication: $a \cdot b$
- Properties:
  - Commutative: $\forall a, b \in \mathbb{R}, a + b = b + a$.
  - Commutative: $\forall a, b, c \in \mathbb{R}, (a + b) + c = a + (b + c)$.
  - Zero Exists: $\forall a \in \mathbb{R}, a + 0 = a$.
  - Negation Exists: $\forall a \in \mathbb{R}, \exists(-a) \in \mathbb{R} : a + (-a) = 0$.
  - Distributive: $\forall a, b, c \in \mathbb{R}, a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.
- Reciprocal: $\forall a \in \mathbb{R}, \frac{1}{a}$

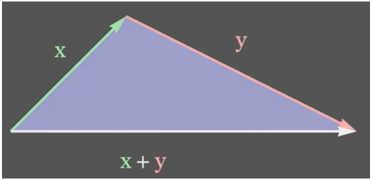
3. Absolute Value and Number Line

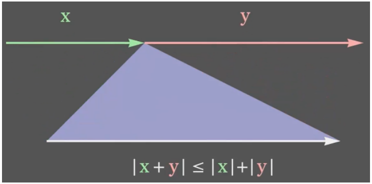
- Absolute Value: $|\pm a| = a.$
- Properties:
  - $\forall a > 0 \in \mathbb{R}, |a| = a.$
  - $\forall a = 0 \in \mathbb{R}, |a| = 0.$
  - $\forall a < 0 \in \mathbb{R}, |a| = -a.$
  - $|ab| = |a||b|.$
  - $|a|^2 = |a^2|.$
  - If $|a| \leq c \iff -c \leq a \leq c.$



4. Triangle Inequality

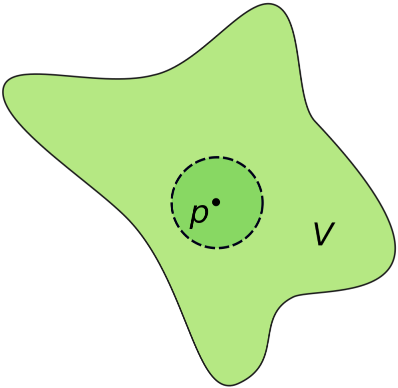
- $\forall a, b \in \mathbb{R} \rightarrow |a+b| \leq |a| + |b|$.
- Corollaries:
  - $||a| - |b|| \leq |a-b|$.
  - $|a-b| \leq |a| + |b|$.
- Visual aid [Click Me].





5. Neighborhoods

- Let $a, \varepsilon \in \mathbb{R}, \varepsilon > 0$.
- Let the ε-neighborhood of a be the set $V_\varepsilon(a) := \{x \in \mathbb{R} : |x - a| < \varepsilon\}$.
- I.e., x is a value within the neighborhood of a such that it is within ε distance from a.



Practice: Numbers

1.

Topic: Functions

Motivation

- General background
  - How we map sets onto other sets.
  - The most common to represent relationships between variables.
- Why do economists' care?
  - This is primary way that we represent preferences, utility, and production.
  - Functions are at the core of how theorems are represented.
- Application in this career
  - Both throughout the development and presentation of proofs, as well as in estimating equations from data.

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Overview

1. Relations
2. Correspondences and Functions
3. Injective & Surjective Functions
4. Composition of Functions
5. Inverse Functions
6. Image and Pre-image
7. Homogeneity
8. Level, Superior & Inferior Sets
9. Euler's Theorem
10. Quasiconcavity & Quasiconvexity
11. Concavity & Convexity
12. Continuity
13. Upper- and Lower-Hemicontinuity
14. Brouwer's Fixed-point Theorem
15. Kakutani's Fixed-point Theorem

1. Relations

- A collection of ordered-pairs (s, t) has a binary relation $s\mathcal{R}t$ between sets S and T.
  - Reflexive: $\forall x \in S, x\mathcal{R}x$.
  - Symmetric: $\forall x, y \in S, x\mathcal{R}y \implies y\mathcal{R}x$.
  - Complete: $\forall x, y \in S \to x\mathcal{R}y \vee y\mathcal{R}x$.
  - Transitive: $\forall x, y, z \in \mathcal{R}, x\mathcal{R}y \wedge y\mathcal{R}z \implies x\mathcal{R}z$.
- Equivalent Relation $(=)$: Is reflexive, symmetric, and transitive.
- Common Relations:
  - Equal: $=$
  - Equivalent: $\equiv$
  - Better than: $>$
  - Less than: $<$

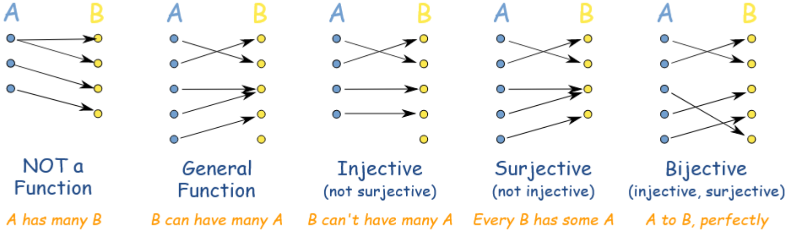
2. Correspondences and Functions

- Correspondence: A relation that associates each element of one set (the domain) to the elements of another set (the range).
- Function: A relation that associates each element in the domain to a single, unique element of the range.
- Onto: Every element in the range is mapped into a point in the domain.
- One-to-one: Every element in the range is assigned only a single point in the domain.

$$f : D \to R$$

3. Injective & Surjective Functions

- Function $f : A \to B$:
  - Injective (one-to-one):
    $\forall a_1, a_2 \in A, a_1 \neq a_2 \implies f(a_1) \neq f(a_2)$.
  - Surjective (onto B): $\forall b \in B, \exists a \in A : f(a) = b$.
  - Bijective: Both injective and surjective.



4. Composition of Functions

- $f : A \to B$ and $g : B \to C$, then
$$g \circ f(x) = g(f(x)) : A \to C$$

- Follows from associative property of functions:
$$(h \circ g) \circ f = h \circ (g \circ f)$$

- If f and g are surjective, then $g \circ f$ is surjective.

5. Inverse Functions

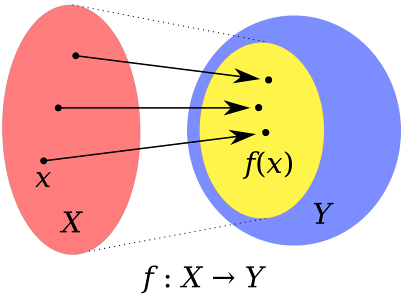
- If $f : A \to B$ is bijective, then the inverse function is $f^{-1} : B \to A$.

$$f^{-1} \circ f(x) = x$$
$$f \circ f^{-1}(x) = x$$

- $f : A \to B$ is bijective iff the inverse f^{-1} is a function $f : B \to A$.

6. Image and Pre-image

- Let $f : A \to B$.
  - Image: If $X \subseteq A$ is set $f(X) = \{f(x) : x \in X\} \subseteq B$.
  - Pre-image: If $Y \subseteq B$ is set $f^{-1}(Y) = \{x \in A : f(x) \in Y\} \subseteq A$.



7. Homogeneity

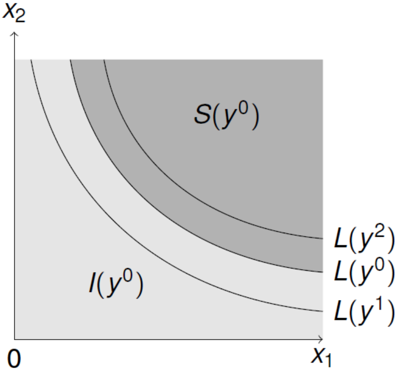
- Consider $f(x_1, x_2, \ldots, x_N)$ is defined for all $(x_1, x_2, \ldots, x_N) \geq 0$.
- Homogeneous: A function $f(x_1, x_2, \ldots, x_N)$ is **homogeneous of degree** $r \in \mathbb{Z}$ if $\forall x > 0$:

$$f(t(x_1), t(x_2), \ldots, t(x_N)) = t^r f(x_1, x_2, \ldots, x_N)$$

- For f homogeneous of degree r, then the partial derivative $(x_1, x_2, \ldots, x_N)/\partial x_n$ is homogeneous of degree $r - 1$.

8. Level, Superior, & Inferior Sets

- Level set: $L(y^0) \equiv \{x | x \in D, f(x) = y^0\}$.
- Superior set: $S(y^0) \equiv \{x | x \in D, f(x) \geq y^0\}$.
- Inferior set: $I(y^0) \equiv \{x | x \in D, f(x) \leq y^0\}$.



9. Euler's Theorem

- Let $f(x_1, x_2, \ldots, x_N)$ be homogeneous of degree r, and differentiable.

$$\nabla f(\bar{x}) \cdot \bar{x} = \sum_{n=1}^{N} \frac{\partial f(\bar{x}_1, \ldots, \bar{x}_N)}{\partial x_n} \bar{x}_n = r f(\bar{x}_1, \ldots, \bar{x}_N)$$

- **Application:** In production theory, Euler's theorem states that a production function homogeneous of degree 1 (CRS) with factors paid their marginal product will have no surplus or deficit in total product.

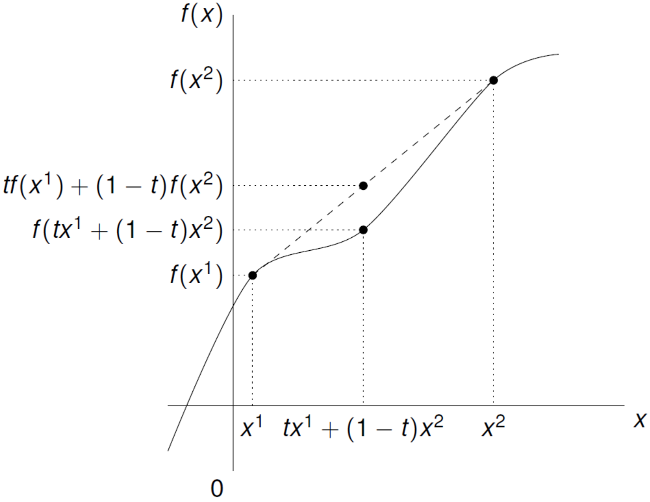
10. Quasiconcavity & Quasiconvexity

- Quasiconcavity: $\forall x_1, x_2 \in D, f : D \to R$ iff
  $$f(tx_1 + (1-t)x_2) \geq min[f(x_1), f(x_2)] \forall t \in [0,1]$$

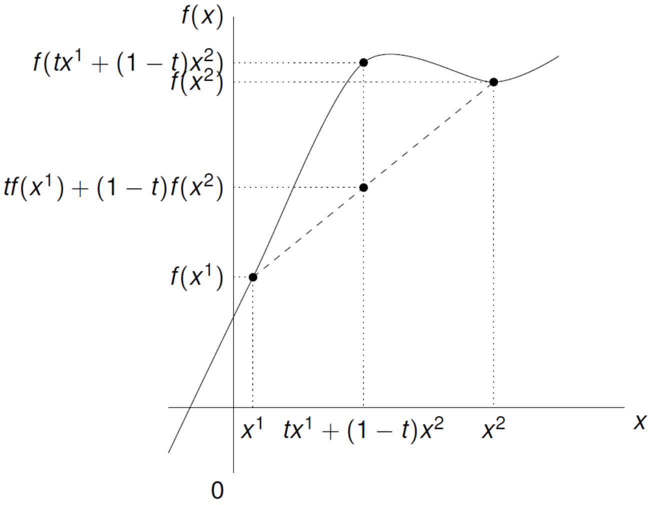
- Quasiconvexity: $\forall x_1, x_2 \in D, f : D \to R$ iff
  $$f(tx_1 + (1-t)x_2) \leq max[f(x_1), f(x_2)] \forall t \in [0,1]$$

- These become **strict** when the inequalities hold for all $x_1 \neq x_2$.

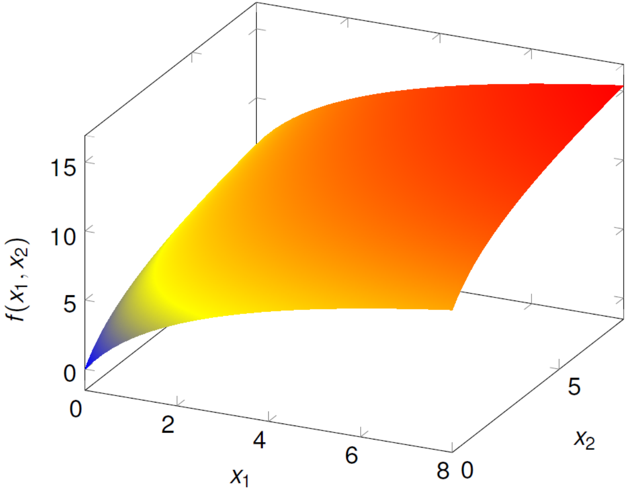
Quasiconcave And Quasiconvex



QUASICONCAVE BUT NOT QUASICONVEX

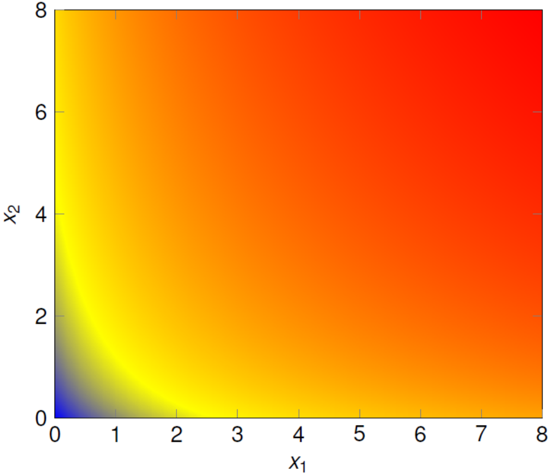


Quasiconcave in Two Dimensions

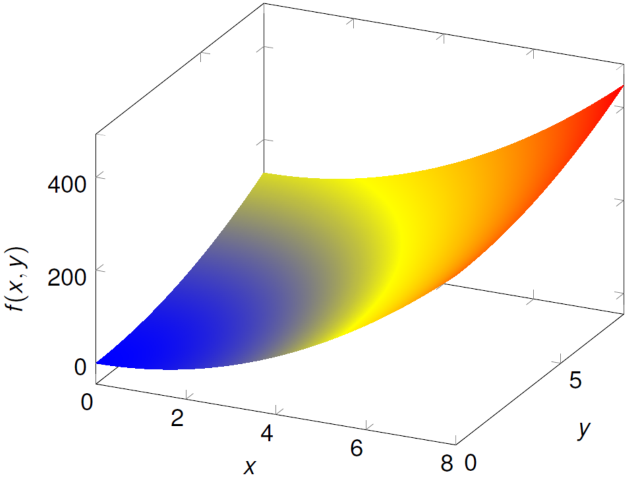


Quasiconcave in Two Dimensions

- $f : D \to R$ is quasiconcave iff $S(y)$ is a **convex** set for all $y \in \mathbb{R}$.

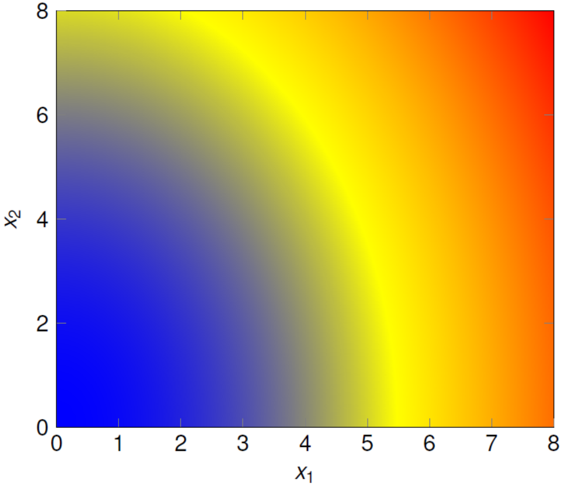


Quasiconvex in Two Dimensions



Quasiconvex in Two Dimensions

- $f : D \to R$ is quasiconvex iff $I(y)$ is a **convex** set for all $y \in \mathbb{R}$.



11. Concavity & Convexity

- f is defined on a convex subset $D \subset \mathbb{R}^n \forall x_1, x_2 \in D, \forall t \in [0,1]$:
- Concave:

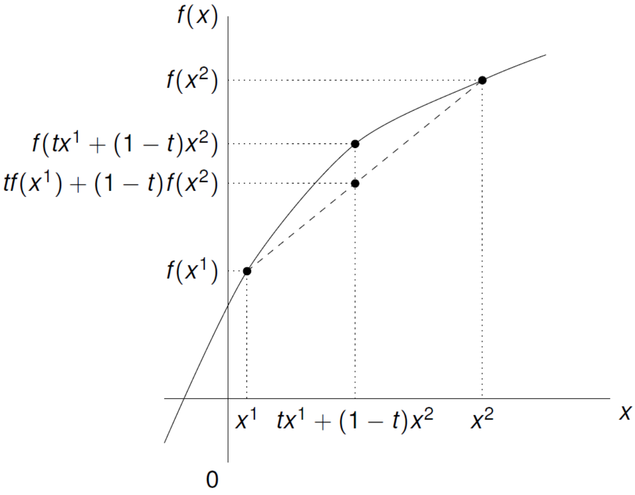
$$f(tx_1 + (1-t)x_2) \geq tf(x_1) + (1-t)f(x_2)$$

- Convex:

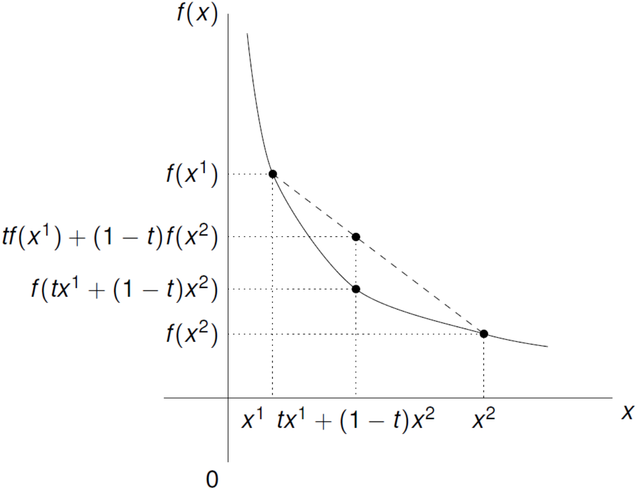
$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$$

- Strict concavity or convexity when the inequality holds.

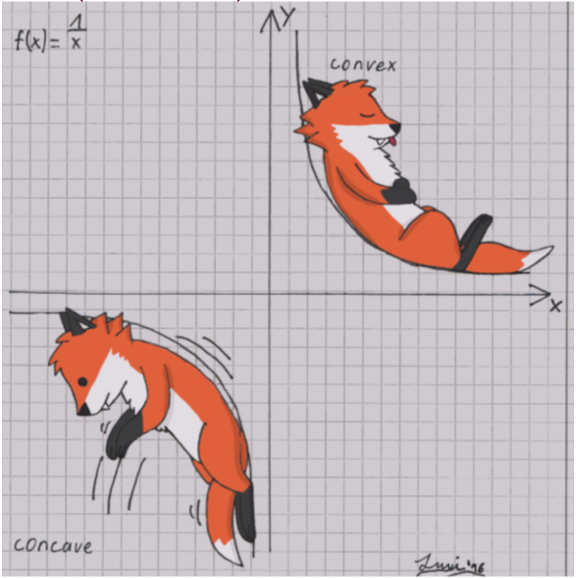
Concavity



Convexity



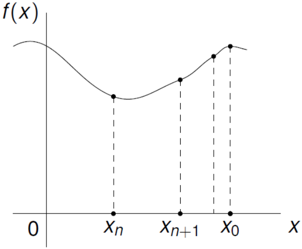
Concave Up (Convex) & Concave Down

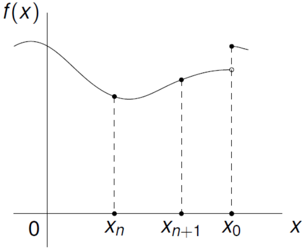


12. Continuity

- Continuous: $f : \mathbb{R}^m \to \mathbb{R}^n$ at $x_0 \in \mathbb{R}^m$ if whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence in $\mathbb{R}^m$ which converges to x_0, **then** the sequence $\{f(x_n)\}_{n=1}^{\infty}$ in $\mathbb{R}^n$ converges to $f(x_0)$.

$$\forall \varepsilon > 0, \exists \delta > 0 : \forall x \in A, [||x - x_0|| < \delta] \implies [||f(x) - f(x_0)|| < \varepsilon]$$





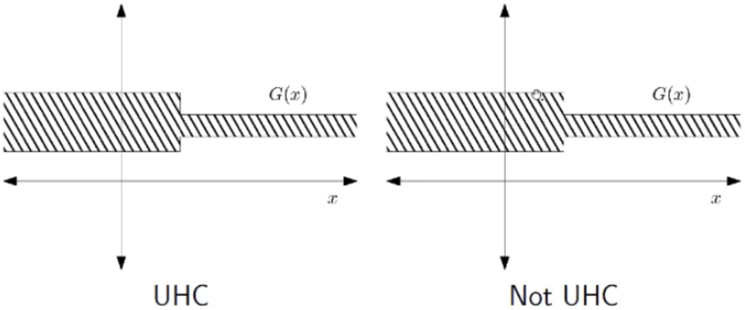
13. Upper- and Lower-Hemicontinuity

- $A \subset \mathbb{R}^n$ and a closed set $Y \subset \mathbb{R}^n$.
  - Upper Hemicontinuous: Correspondence $f : A \to Y$ if it has a closed graph and the images of compact sets are bounded.

  $\forall B \subset A, f(B) = \{y \in Y : y \in f(x) \exists x \in B\}$ is bounded.

  - Lower Hemicontinous: Correspondence $f : A \to Y$ if for every sequence $x^m \to x \in A$ with $x^m \in A \forall m$, and every $y \in f(x)$, we can find a sequence $y^m \to y$ and an integer $M : y^m \in f(x^m) \forall m > M$.
- Continuous: Both upper- and lower-hemicontinuous.

13. Upper-Hemicontinuity



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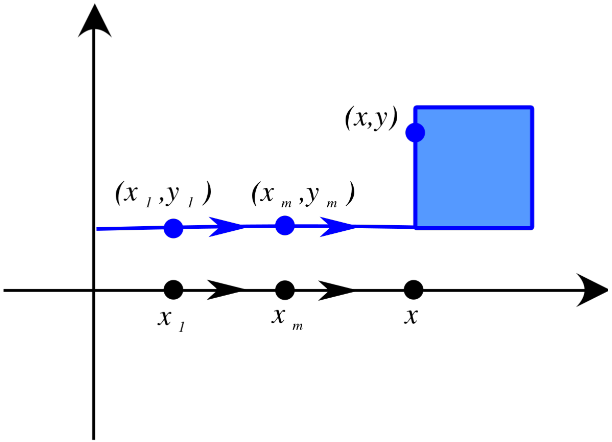
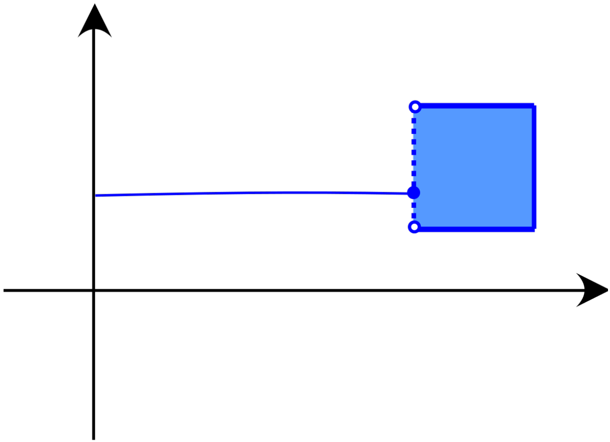


13. Lower-Hemicontinuity



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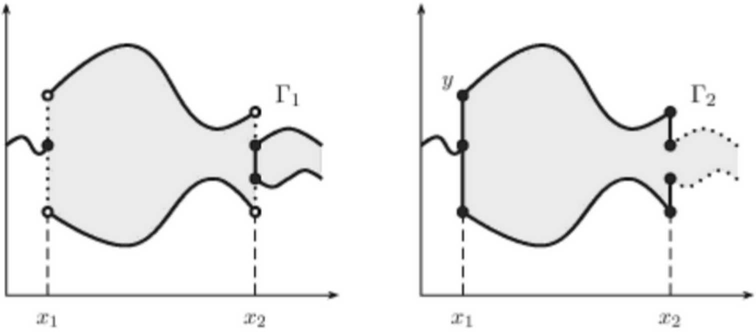
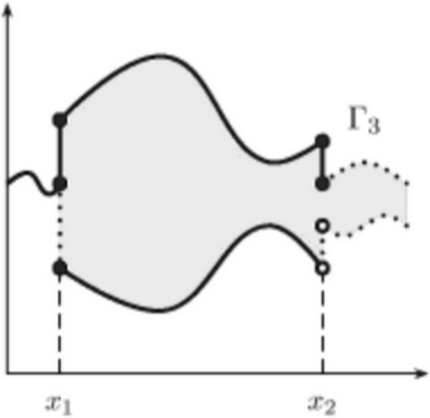


Not upper hemicontinuous at x_1
Not upper hemicontinuous at x_2
Lower hemicontinuous

Not lower hemicontinuous at x_1
Not lower hemicontinuous at x_2
Upper hemicontinuous

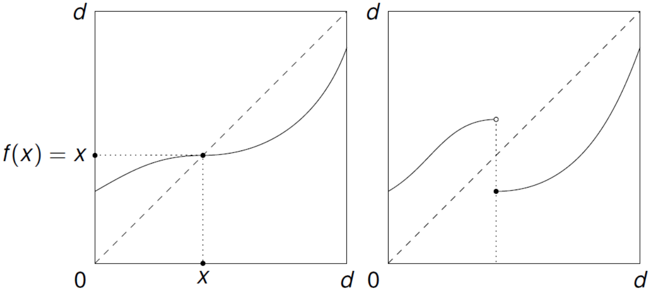
13. Not Upper- and Lower-Hemicontinuity



Not upper hemicontinuous at x_1 and x_2
Not lower hemicontinuous at x_1 and x_2

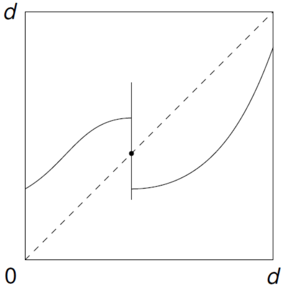
14. Brouwer's Fixed-point Theorem

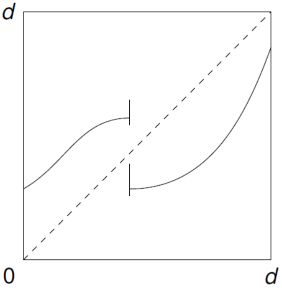
- Suppose $D \subset \mathbb{R}^m$ is non-empty, compact, convex set, and $f : D \to D$ is a **continuous function**.
- Then $f(\cdot)$ has a fixed point, e.g., there is an $x \in D : x = f(x)$.



15. Kakutani's Fixed-point Theorem

- Suppose $D \subset \mathbb{R}^m$ is non-empty, compact, convex set, and $f : D \to D$ is a **upper-hemicontinuous correspondence** with the property $f(x) \subset D$ is non-empty and convex for all $x \in D$.
- Then $f(\cdot)$ has a fixed point, e.g., there is an $x \in D : x = f(x)$.





Practice: Functions

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Review of Numbers

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Assignment

- Readings on Optimization before Lecture 09:
  -
- Assignment:
  - Problem Set 08 (PS08)
  - Solution set will be available following end of Lecture 09
- Struggling?
  1. Read the 'Encouraged Reading'
  2. Review 'Supplementary material'
  3. Reach out directly