Problem Set 4

APEC Math Review

August 2020

1. (Simon & Blume Exercise 9.13) Use Cramer's rule to solve for the following system of equation:
$$5x_1 + x_2 = 3$$
$$2x_1 - x_2 = 4$$

2. Prove that if matrix $\mathbf{A}$ is positive definite and $\mathbf{B}$ is a nonsingnular matrix, then $\mathbf{B}'\mathbf{A}\mathbf{B}$ is positive definite. (Hint: start by defining a vector $\mathbf{y} = \mathbf{B}\mathbf{x}$ for any $\mathbf{x} \neq \mathbf{0}$.)

3. (Davidson 4.6 Exercise 6)Let
$$\mathbf{X} = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 3 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$
   (a) Show that $\mathbf{X}'\mathbf{X}$ is positive definite.
   (b) Calculate $\mathbf{X}'\mathbf{X}^{-1}$. Is it also positive definite?

4. (Greene, Chapter 3 Exercise 2) Show that the OLS estimator is indeed least squares. Take an arbitary $K \times 1$ vector $\mathbf{c}$ that is different from $\hat{\boldsymbol{\beta}}$. Show that the difference between two sums of squared residual
$$(\mathbf{Y} - \mathbf{X}\mathbf{c})'(\mathbf{Y} - \mathbf{X}\mathbf{c}) - (\mathbf{Y} - \mathbf{X}\hat{\boldsymbol{\beta}})'(\mathbf{Y} - \mathbf{X}\hat{\boldsymbol{\beta}})$$

2. Write $\mathbf{c}$ as $\mathbf{b}$ + ($\mathbf{c}$ - $\mathbf{b}$). Then, the sum of squared residuals based on $\mathbf{c}$ is
($\mathbf{y}$ - $\mathbf{Xc}$)′($\mathbf{y}$ - $\mathbf{Xc}$) = [$\mathbf{y}$ - $\mathbf{X}$($\mathbf{b}$ + ($\mathbf{c}$ - $\mathbf{b}$))] ′[$\mathbf{y}$ - $\mathbf{X}$($\mathbf{b}$ + ($\mathbf{c}$ - $\mathbf{b}$))] = [($\mathbf{y}$ - $\mathbf{Xb}$) + $\mathbf{X}$($\mathbf{c}$ - $\mathbf{b}$)] ′[($\mathbf{y}$ -
= ($\mathbf{y}$ - $\mathbf{Xb}$) ′($\mathbf{y}$ - $\mathbf{Xb}$) + ($\mathbf{c}$ - $\mathbf{b}$) ′$\mathbf{X}'\mathbf{X}$($\mathbf{c}$ - $\mathbf{b}$) + 2($\mathbf{c}$ - $\mathbf{b}$) ′$\mathbf{X}'$($\mathbf{y}$ -
But, the third term is zero, as 2($\mathbf{c}$ - $\mathbf{b}$) ′$\mathbf{X}'$($\mathbf{y}$ - $\mathbf{Xb}$) = 2($\mathbf{c}$ - $\mathbf{b}$)$\mathbf{X}'\mathbf{e}$ = $\mathbf{0}$. Therefore,
($\mathbf{y}$ - $\mathbf{Xc}$) ′($\mathbf{y}$ - $\mathbf{Xc}$) = $\mathbf{e}'\mathbf{e}$ + ($\mathbf{c}$ - $\mathbf{b}$) ′$\mathbf{X}'\mathbf{X}$($\mathbf{c}$ - $\mathbf{b}$)
or ($\mathbf{y}$ - $\mathbf{Xc}$) ′($\mathbf{y}$ - $\mathbf{Xc}$) - $\mathbf{e}'\mathbf{e}$ = ($\mathbf{c}$ - $\mathbf{b}$) ′$\mathbf{X}'\mathbf{X}$($\mathbf{c}$ - $\mathbf{b}$).
The right hand side can be written as $\mathbf{d}'\mathbf{d}$ where $\mathbf{d}$ = $\mathbf{X}$($\mathbf{c}$ - $\mathbf{b}$), so it is necessarily positi
we knew at the outset, least squares is least squares.

is strictly positive.