# Math Review
Summer 2017

*Topic 8*

## 8. Optimization

### 8.1 Unconstrained maximization

We start this lesson by considering the simplest of optimization problems, those without conditions, or what we refer to as **unconstrained optimization problems**. Unconstrained optimization is equivalent to finding the max/min of a function. In one-variable calculus we were able to find these points by finding the critical points of a function and looking at the sign of the second derivative at that point. In multivariable calculus the results are much the same. To identify maxima and minima we again need to be able to find the critical points of $f$:The point $\hat{x} \in \mathbb{R}^n$ is a critical point of $f: \mathbb{R}^n \to \mathbb{R}$ if $Df(\hat{x}) = 0$.

Recall 2 things from our previous classes:

(1) The local maximum, $x^*$, can be either a boundary max or an interior max. If $x^*$ is an interior max, then $x^*$ is a critical point.
(2) If the function $f$ has a critical point in $C \subset \mathbb{R}^n$ then the Hessian matrix can be used to identify whether the critical point is a maximum or minimum.

Let's remember those cases:

| $Df(x^*)$ | $D^2f(x^*)$ | Max/Min |
|---|---|---|
| $= 0$ | Negative semidefinite | Local max |
| $= 0$ | Positive semidefinite | Local min |
| $= 0$ | Neither | Saddle point or inflexion |

- The $n \times n$ matrix $\boldsymbol{A}$ is positive definite if and only if its $n$ principal minors are all greater than 0: $detA_1 > 0, detA_2 > 0, \dots, detA_n > 0$.
- The $n\times n$ matrix $\boldsymbol{A}$ is negative definite if and only if its $n$ principal minors alternate in sign with the odd order ones being negative and the even order ones being positive: $det\ A_1 < 0, det\ A_2 > 0, detA_3 < 0, detA_4 > 0, \dots$.
- The $n \times n$ matrix $\boldsymbol{A}$ is positive semidefinite if and only if its principal minors are all greater than or equal to 0: $det\ A_1 \geq 0\ and\ det\ A_2 \geq 0, \dots, A_n \geq 0$.

- The $n \times n$ matrix $\boldsymbol{A}$ is negative semidefinite if and only if its $n$ principal minors alternate in sign with the odd order ones being less than or equal to zero and the even order ones being greater than or equal to 0: $det\ A_1\ \leq\ 0\ and\ det\ A_2\ \geq\ 0, \ldots.$

*Example.* Consider this simple example.

$$f(x, y) = x^4 + x^2 - 6xy + 3y^2$$

Let's find the critical points and classify them as local max, local min or saddle points.

This type of question can be solved in a matrix setting, but you can have a preference.

Let $J = \begin{bmatrix} f_x & f_y \end{bmatrix} = [4x^3 + 2x - 6y \quad - 6x + 6y] = [0 \quad 0]$

Thus we have, $-6x + 6y = 0 \rightarrow y = x$

This is a system, so this solution is part of the equations, thus, we can simply plug this in our 1st equation derived.

$$4x^3 + 2x - 6y = 0$$
$$4x^3 + 2x - 6(x) = 0$$

$$4x^3 - 4x = 0$$
$$x(x^2 - 1) = 0$$

$(x + 1)(x - 1) = 0 \rightarrow x = 1, x = -1$ and also $x = 0$

Thus, we can denote our optimized solutions as: $x^* = \pm 1, 0, y^* = x^*$

Since we are in the multivariable world, let's figure out which points are max or min through the Hessian.

$$H = \begin{bmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{bmatrix} = \begin{bmatrix} 12x^2 + 2 & -6 \\ -6 & 6 \end{bmatrix}$$

This Hessian has to be defined for each and every set of critical points, that is

(i) (-1,-1)

$$H = \begin{bmatrix} 14 & -6 \\ -6 & 6 \end{bmatrix}$$

$$detA_1 > 0$$
$$detA_2 = 14(6) - (-6)(-6) = 48 > 0$$

This is a local minimum

(ii) (1,1)

$$H = \begin{bmatrix} 14 & -6 \\ -6 & 6 \end{bmatrix}$$

$$detA_1 > 0$$
$$detA_2 > 0$$

This is a local minimum

(iii) (0,0)

$$H = \begin{bmatrix} 2 & -6 \\ -6 & 6 \end{bmatrix}$$

$$detA_1 > 0$$
$$detA_2 = 2(6) - (-6)(-6) = -24 < 0$$

Recalling our definitions, this is neither positive nor negative semidefinite, hence a saddle point.

Extra: With a little more algebra, we can think of generalized solutions for our system. We know that $detA_1 \geq 0$ each time. For sure, we only have minimums we are dealing with.

$detA_1 = 12x^2 + 2 \geq 0$ for all $x \in \mathbb{R}$
$detA_2 = 6(12x^2 + 2) - 36 \geq 0$

This can be easily solved as $12x^2 - 4 \geq 0$

$$x \geq \sqrt{\frac{1}{3}}, x \leq -\sqrt{\frac{1}{3}}$$

Therefore, in the domain $\left\{(x, y): x \geq \sqrt{\frac{1}{3}}\right\}$, the critical point $(1,1)$ is a global min and in the domain $\left\{(x, y): x \leq -\sqrt{\frac{1}{3}}\right\}$, the critical point $(-1, -1)$ is a global min.

$Q$: Solve the following problem

Let $f(x,y,z) = e^{x-y} + e^{y-x} + e^{x^2} + z^2$. Find the critical points, is that a minimum, maximum or a saddle point?

$$f_x = e^{x-y} - e^{y-x} + 2xe^{x^2}$$
$$f_y = -e^{x-y} + e^{y-x}$$
$$f_z = 2z$$

We set these equal to 0.

$$z = 0$$
$$y = 0$$
$$x = 0$$

Therefore, (0, 0, 0) is the only critical point.

The Hessian is given by:

$$Hf(x,y,z) = \begin{bmatrix} e^{x-y} + e^{y-x} + 4x^2e^{x^2} + 2e^{x^2} & -e^{x-y} - e^{y-x} & 0 \\ -e^{x-y} - e^{y-x} & e^{x-y} + e^{y-x} & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$Hf(0,0,0) = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$H_1 = 3 \geq 0$$
$$H_2 = 6 - 4 = 2 \geq 0$$
$$H_3 = 4 \geq 0$$

Since this is positive definite, it yields a local minimum.

## 8.2 Kuhn Tucker and constrained maximization

### 8.2.1 General forms of optimization

While you will work through a few unconstrained max/min problems, most economic decisions are the result of an optimization problem subject to one or a series of constraints:

- Consumers make decisions on what to buy constrained by the fact that their choice must be affordable.
- Firms make production decisions to maximize their profits subject to the constraint that they have limited production capacity.
- Households make decisions on how much to work/play with the constraint that there are only so many hours in the day.
- Firms minimize costs subject to the constraint that they have orders to fulfill.

All of these problem fall under the category of constrained optimization and luckily for us, there is a uniform process that we can use to solve these problems.

Constrained optimization problems are of the general form:

$$\max_{\{x_1,x_2,\dots,x_n\}} f(x_1, x_2, \dots, x_n)$$

subject to:

$$\begin{gathered} g_1(x_1, x_2, \dots, x_n) \leq b_1 \\ \vdots \\ g_k(x_1, x_2, \dots, x_n) \leq b_k \\ \\ h_1(x_1, x_2, \dots, x_n) = c_1 \\ \vdots \\ h_m(x_1, x_2, \dots, x_n) = c_m \end{gathered}$$

We call $f(x_1, x_2, \dots, x_n)$ the objective function and $g_1, g_2, \dots, g_k$ and $h_1, h_2, \dots h_m$ are the constraint functions. The constraints $g_i(x_1, x_2, \dots, x_n) \leq b_i$ for $i = 1, 2, \dots, k$ are the inequality constraints and the $h_j(x_1, x_2, \dots, x_n) = c_j$ for $j = 1, 2, \dots, m$ are the equality constraints.

A standard consumer problem may take the following form. A consumer with budget $I$ wishes to purchase the goods $x_1, x_2, \dots x_n$ with prices $p_1, p_2, \dots p_n$ so as to maximize utility, $u(x_1, x_2, \dots x_n)$. The consumer's problem is then:

$$\max_{\{x_1,x_2,\dots,x_n\}} u(x_1, x_2, \dots x_n)$$

subject to:

$$p_1x_1 + p_2x_2 + \cdots + p_nx_n \leq I$$
$$\text{and } x_i \geq 0 \ \forall i = 1,2,3, \dots, n$$

The inequality constraints $x_i \geq 0$ are usually referred to as non-negativity constraints as they restrict the consumer from purchasing a negative amount of some good $i$.

8.2.2. The Lagrangian/Langragean (*toMAYto-toMAHto*)

Let's start simple and build up to more involved optimization problem.

You are given this maximization problem framed as:

$$\max_{\{x_1,x_2\}} f(x_1, x_2)$$
subject to:
$$h(x_1, x_2) = c$$

We want to choose the values of $(x_1, x_2)$ that maximize $f$ and that are in the set $\{(x_1, x_2) : h(x_1, x_2) = c\}$.

There are two ways of solving the above maximization problem with given constraints. The first via *substitution*, the second via a *Lagrangean*. When dealing with constrained optimization, you often want to form a Lagrangean denoted by $\mathcal{L}$. You may read about the substitution method but most of Micro will really revolve around building the $\mathcal{L}$. At first this method may seem more difficult, but which in fact can be very useful (and sometimes easier). I will go through the formal definition, but the application is even simpler.

*Lagrange's Theorem:* Let $f(x)$ and $g_j(x), j = 1, \ldots, m$ be a continuously differentiable real-valued functions over some domain $D \to \mathbb{R}^n$. Let $x^*$ be an interior point of $D$ and suppose that $x^*$ is an optimum (maximum or minimum) of $f$ subject to the constraints, $g_j(x) = 0$. If the gradient vectors $\nabla g_j(x^*), j = 1, \ldots, m$, are linearly independient, then there exists $m$ unique numbers $\lambda_j^*, j = 1, \ldots, m$, such that:

$$\frac{\partial L(x^*, \lambda^*)}{\partial x_i} = \frac{f(x^*)}{\partial x_i} + \sum_{j=1}^{m} \lambda_j^* \frac{\partial g_j(x^*)}{\partial x_i} = 0$$

For $i = 1,2, \ldots, n$. $L(x, \lambda) = f(x) + \sum_{j=1}^{m} \lambda_j g_j(x^*)$

We need require that $\frac{\partial g_j}{\partial x_i} \neq 0, \forall i, \forall j$ from our definition of the Langrange function. This restriction is referred to as the constraint qualification. For a linear constraint this qualification is always satisfied. How do we know if the constraint qualification is met? Recall that a critical point is where the Jacobian equals zero which is equivalent to $\frac{\partial g_j}{\partial x_i} = 0 \ \forall i, j$. Therefore, the constraint qualification is not met if a critical point of the constraint lies in the constraint set. (You will see algebraically why below. )

In a nutshell here, you want to form

$\mathcal{L}(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda(c - h(x_1, x_2))$ from our example above.

Or equivalently:

$$\mathcal{L}(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda(h(x_1, x_2) - c)$$

The first order conditions are:

$$\frac{\partial \mathcal{L}(x_1,x_2)}{\partial x_1} = \frac{\partial f}{\partial x_1}(x_1^*, x_2^*) - \lambda^* \frac{\partial h}{\partial x_1}(x_1^*, x_2^*) = 0 \qquad (1)$$
$$\frac{\partial \mathcal{L}(x_1,x_2)}{\partial x_2} = \frac{\partial f}{\partial x_2}(x_1^*, x_2^*) - \lambda^* \frac{\partial h}{\partial x_2}(x_1^*, x_2^*) = 0 \qquad (2)$$
$$\frac{\partial \mathcal{L}(x_1,x_2)}{\partial \lambda} = h(x_1, x_2) - c = 0 \qquad (3)$$

These first order conditions can then be used to solve for $x_1, x_2, \lambda$

It may be valuable to note that: $\lambda^* = \frac{\frac{\partial f}{\partial x_1}(x_1^*,x_2^*)}{\frac{\partial h}{\partial x_1}(x_1^*,x_2^*)} = \frac{\frac{\partial f}{\partial x_2}(x_1^*,x_2^*)}{\frac{\partial h}{\partial x_2}(x_1^*,x_2^*)}$. - (L)

Working with the first and second equation often will eliminate the $\lambda^*$, which is beauty of this method to problem solve. $\lambda^*$ can then be solved for by plugging the results back into the last ugly equation (L).

Let's check how good our skill of applying definitions work: You are given a problem framed as follows:

$$\max_{\{x_1,x_2\}} -ax_1^2 - bx_2^2$$
subject to:
$$x_1 + x_2 = 1$$

Form the Lagrangean:

$$\mathcal{L}(x_1, x_2, \lambda) = -ax_1^2 - bx_2^2 - \lambda(x_1 + x_2 - 1)$$

This can be equivalently expressed as:

$$\mathcal{L}(x_1, x_2, \lambda) = -ax_1^2 - bx_2^2 + \lambda(1 - x_1 - x_2)$$

Can we check for the constraint qualification?

Denote the constraint as $h(x_1, x_2) = x_1 + x_2$. The first order conditions for the critical points are:

$h_{x_1} = 1\,; h_{x_2} = 1$, which implies that the point (1, 1) is a critical point of $h(x_1, x_2)$. But note that $h(1,1) \neq 1$ so the only critical point (1, 1) is not in the constraint set and $h(x_1, x_2)$ satisfies the constraint qualification.

The first order conditions (FOC) are given as:

$$\frac{\partial L(x_1,x_2)}{\partial x_1} = -2ax_1 - \lambda = 0 \quad (1)$$
$$\frac{\partial \square(x_1,x_2)}{\partial x_2} = -2bx_2 - \lambda = 0 \quad (2)$$
$$\frac{\partial L(x_1,x_2)}{\partial \lambda} = x_1 + x_2 - 1 = 0 \quad (3)$$

We are now ready to solve for $x_1, x_2, \lambda$

From (1) and (2) we have

$$\frac{-2ax_1}{-2bx_2} = \frac{\lambda}{\lambda}$$
$$ax_1 = bx_2$$
$$x_1 = \frac{bx_2}{a}$$

We can plug in $x_1$ in our equation (3)

$$x_1 + x_2 - 1 = 0$$

$$\frac{bx_2}{a} + x_2 = 1$$

$$bx_2 + ax_2 = a$$
$$x_2(a+b) = a$$
$$x_2 = \frac{a}{a+b}$$
$$x_1 = \frac{bx_2}{a} = \frac{b}{a} * \frac{a}{a+b} = \frac{b}{a+b}$$

Often, we are not too worried about the value of $\lambda$, but for completion sake, $\lambda = \frac{2ab}{a+b}$

Now try this one on your own:

$$\max_{x_1,x_2} x_1^2 x_2$$
$$\text{subject to}$$
$$2x_1^2 + x_2^2 = 3$$

$$L = x_1^2 x_2 - \lambda(2x_1^2 + x_2^2 - 3)$$

Note, this is equivalent to:
$$L = x_1^2 x_2 + \lambda(3 - 2x_1^2 - x_2^2)$$

Different professors may use different ways of writing their $\mathcal{L}$.

Checking for the constraint qualification should be straightforward.

The first order conditions are given by:

$$\frac{\partial L(x_1,x_2)}{\partial x_1} = 2x_1x_2 - \lambda 4x_1 = 0 \quad (1)$$
$$\frac{\partial L(x_1,x_2)}{\partial x_2} = x_1^2 - \lambda 2x_2 = 0 \quad (2)$$
$$\frac{\partial L(x_1,x_2)}{\partial \lambda} = 2x_1^2 + x_2^2 - 3 = 0 \quad (3)$$

Using (1) and (2) we have:

$$\frac{2x_1x_2}{x_1^2} = \frac{\lambda 4x_1}{\lambda 2x_2}$$
$$\frac{2x_2}{x_1} = \frac{2x_1}{x_2}$$
$$x_1^2 = x_2^2$$

Plugging this last result in (3), we have:

$$2x_1^2 + x_1^2 - 3 = 0$$
$$x_1^2 = 1$$
$$x_1 = \pm 1$$

For these values of $x_1, x_2$, we have $2x_1x_2 = \lambda 4x_1$from (2). This implies $\lambda = \pm\frac{1}{2}$

It may feel nice to stop here, but there may be other solutions. We have:

$$2x_1x_2 - \lambda 4x_1 = 0$$

$$x_1(x_2 - 2\lambda) = 0, \text{ we have } x_1 = 0, x_2 = 2\lambda$$
$$\text{If } x_1 = 0, \rightarrow\ 2x_1^2 + x_2^2 - 3 = 0, \text{ means that } x_2^2 = 3.$$

Then, $x_2 = \pm\sqrt{3}$

We have found that $x_1 = 0$ and we also have, $x_1^2 - \lambda 2x_2 = 0$
For this to hold, $\lambda = 0$.

Let's recap all the solutions:

$$\left(1,1,\frac{1}{2}\right), \left(-1,-1,-\frac{1}{2}\right), \left(1,-1,-\frac{1}{2}\right), \left(-1,1,\frac{1}{2}\right), (0,\sqrt{3},0) \, and \ (0,-\sqrt{3},0)$$

Note that, we have to carry out a second derivative test to check which of these yield the maximum solution. If that does not work, we can plug in the values of $x_1$ and $x_2$ and check which yield the highest 'output.'

*Economic interpretation of the Lagrangian:*

In economics, the values of $\lambda^*$ often have important economic interpretations. Consider a budget constraint given by $p_1x_1 + p_2x_2 = w$, that is price multiplied by commodities add up to your wealth. Now think of a more general constraint of the form $g_j(x_j) = b_j$. If the right hand side $b_j$ of constraint $j$ is increased by $\Delta$, say by one unit, then the optimum objective value increases by approximately $\lambda_i^*\Delta$ (or simply $\lambda_i^*$ if $\Delta$ is one unit) . We won't expand too much on that. Often the $\lambda_i^*$ simply drops out, but it is valuable to know where it all is coming from. In our consumer problem, with the budget constraint, parameters like wealth are usually fixed, but really, all of our solutions are dependent on that value. In reality, we can denote $x_1^*(w), x_2^*(w), \lambda^*(w)$ as the solutions. As wealth changes, these will change. You will find that Prof. Glewwe calls this the shadow value of wealth in the consumer problem.

*Try at home:*

Solve for $x_1$ and $x_2$

$\min_{x_1,x_2} x_1^2 + x_2^2$, subject to $x_1x_2 = 1$

$\min_{x_1,x_2} x_1x_2$, subject to $x_1^2 + x_2^2 = 1$

$\max_{x_1,x_2} x_1x_2^2$ , subject to $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$

How about a general consumer problem? Solve for $x, y$

$\max_{x,y} kx^{\alpha}y^{\beta}$ , subject to $p_x x + p_y y = m$

Now a general producer problem, solve for $K, L$

$\max_{K,L} wL + rK$ , subject to $AK^{\alpha}L^{\beta} = q$

Constrained optimization with multiple equality constraints

Let $f(x_1, x_2, \ldots, x_n)$ be a continuously differentiable objective function and $g_1(x_1, x_2, \ldots, x_n), g_2(x_1, x_2, \ldots, x_n), \ldots g_k(x_1, x_2, \ldots, x_n)$ be the equality constraints. Then we wish to solve the following problem:

$$\max_{\{x_1, x_2, \ldots, x_n\}} f(x_1, x_2, \ldots, x_n)$$
subject to:

$$g_1(x_1, x_2, \ldots, x_n) = b_1$$
$$\vdots$$
$$g_k(x_1, x_2, \ldots, x_n) = b_k$$

Suppose $(x_1^*, x_2^*, \ldots, x_n^*)$ is a solution to this problem. If the rank of the Jacobian matrix $dG$ is $k$, then there exists $\lambda_1^*, \lambda_2^*, \ldots, \lambda_k^*$ such that $(x_1^*, x_2^*, \ldots, x_n^*; \lambda_1^*, \lambda_2^*, \ldots, \lambda_k^*) = (x^*, \lambda^*)$ is a critical point of the langragian given by:

$$\mathcal{L}(x, \lambda) = f(x) - \lambda_1(g_1(x) - b_1) - \lambda_2(g_2(x) - b_2) - \ldots - \lambda_k(g_k(x) - b_k)$$

$$\text{Rank } g(x_1^*, x_2^*, \ldots, x_n^*) = \begin{bmatrix} \frac{\partial g_1}{\partial x_1}(x^*) & \frac{\partial g_1}{\partial x_2}(x^*) & \cdots & \frac{\partial g_1}{\partial x_n}(x^*) \\ \frac{\partial g_2}{\partial x_1}(x^*) & \frac{\partial g_2}{\partial x_2}(x^*) & \ldots & \frac{\partial g_2}{\partial x_n}(x^*) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_l}{\partial x_1}(x^*) & \frac{\partial g_l}{\partial x_2}(x^*) & \cdots & \frac{\partial g_l}{\partial x_n}(x^*) \end{bmatrix} = l$$

For example, let's solve:

$$\max_{x,y,z} yz + xz$$

subject to:
$y^2 + z^2 = 1$ and

$$xz = 3$$

First, we check for the constraint qualification. The Jacobian of matrix of the constraints:

$$J = \begin{bmatrix} 0 & 2y & 2z \\ z & 0 & x \end{bmatrix}$$

The rank of this matrix is not 2 if $z = 0$ and $y = 0$ or $x = 0$, which would violate the constraint set.

We can now frame the Langragian:

$$\max_{x,y,z} yz + xz - \lambda_1(y^2 + z^2 - 1) - \lambda_2(xz - 3)$$

The FOC are given as:

$$\frac{\partial \mathcal{L}}{\partial x} = z - z\lambda_2 = 0\ ;\ \frac{\partial \mathcal{L}}{\partial y} = z - 2y\lambda_1 = 0;\ \frac{\partial \mathcal{L}}{\partial z} = y + x - 2\lambda_1 z - \lambda_2 x = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_1} = -y^2 - z^2 + 1$$
$$\frac{\partial \mathcal{L}}{\partial \lambda_2} = -xz + 3$$

From the first first order condition we have $\lambda_2 = 1$ and from the second first-order condition
we have $\lambda_1 = \frac{z}{2y}$. Plugging these values into the third first order condition we obtain:

$y + x - 2\lambda_1 z - \lambda_2 x = 0$
$y + x - 2.\frac{z}{2y}.z - x = 0$

$$y - \frac{z^2}{y} = 0$$
$$y^2 - z^2 = 0$$

$$y = \pm z$$

We know, $y^2 + z^2 = 1 \rightarrow y^2 + y^2 = 1 \rightarrow 2y^2 = 1 \rightarrow y = \pm\sqrt{\frac{1}{2}}$

We can use the second constraint: $xz = 3 \rightarrow x(\pm\sqrt{\frac{1}{2}}) = 3 \rightarrow x = \pm 3\sqrt{2}$

As a next step, we can plug in these values in the objective function to see which one(s) yield a maximum.

### 8.2.2. Kuhn-Tucker and inequality conditions

We can really spend a whole day on Kuhn-Tucker conditions – but we won't! Let's go through the essential material you will need to know in order to be able to keep up with Micro theory. **I do not expect you to understand this fully today, it took me many weeks to get it when I was first exposed**. However, let's try our best to get the most out of this section. Let's try not to get too stuck on details – if not today, you WILL get it over the next few times.

In economics it is much more common to start with inequality constraints of the form $g\ (x, y) \ \leq \ c$, for instance in our consumer problem, we would have, $p_1x_1 + p_2x_2 \leq w$. The constraint is said to be **binding** if at the optimum $g\ (x^*, y^*) \ = \ c$, and it is said to be slack otherwise. Luckily for us, with a small tweak, the Lagrangean can still be used with the same FOC's except now we have **three "Kuhn-Tucker" necessary conditions** for each inequality constraint.

$$\frac{\partial \mathcal{L}}{\partial \lambda} = c - g\ (x^*, y^*) \geq 0$$
$$\lambda^* \geq 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda}\lambda^* = \lambda^*[c - g\ (x^*, y^*)] = 0$$

Proof of Kuhn-Tucker can be found on pp. 959-960 Mas-Colell.

A few things to note:

- The first condition is just a restatement of the constraint.
- The second condition says that $\lambda^*$ is always non-negative.
- The third condition says that either $\lambda^*$or $c - g\ (x^*, y^*)$ must be zero.
- If $\lambda^* = 0$, intuitively, we are not giving any weight to the constraint. The problem can really turn into an unconstrained optimization problem.
- If $\lambda^* > 0$, then the constraint must be binding then the problem turns into the standard Lagrangean considered above.

Formally,

*Theorem.* Kuhn - Tucker Necessary Conditions for Optima of Real Valued Functions Subject to Inequality Constraints

Lef $f(x)$ and $g^j(x), j = 1,\ldots,m$ , be continuously differentiable real -valued function over
some domain $D \to R^n$. Let $x^*$ be an interior point of $D$ and suppose that $x^*$is an optimum
(maximum or minimum) of $f$ subject to the constraints, $g^j(x^*) \geq 0, j = 1,\ldots,m$

If the gradient vectors $\nabla g^j(x^*)$associated with all binding constraints are linearly independent,
then there exists a unique vector $\Lambda^*$ such that $(x^*, \Lambda^*)$ satisfies the Kuhn - Tucker conditions:

$$\frac{\partial \mathcal{L}(x^*, \Lambda^*)}{\partial x_i} = \frac{\partial f(x^*)}{\partial x_i} + \sum_{i=1}^{m} \lambda_j^* \frac{\partial g^j(x^*)}{\partial x_i} = 0, j = 1,\ldots,m$$

$\lambda_j^* g^j(x^*) = 0,\ j = 1,\ldots,m$

$g^j(x^*) \geq 0,\ j = 1,\ldots,m$

Further, the vector $\Lambda^*$ is nonnegative if $x^*$is a maximum, and a nonpositive if it is a minimum.

All in all, the Kuhn-Tucker condition provide a neat mathematical way of turning the problem into either an unconstrained problem or a constrained one. Unfortunately you **typically have to check both cases**, unless you know for sure it's one or the other. Let's work through an example together to get at what the Kuhn-Tucker does.

*Example.*

$$\max_{x_1, x_2} u(x_1, x_2) = 4x_1 + 3x_2$$

subject to:

$$g(x_1, x_2) = 2x_1 + 3x_2 \leq 10$$

$x_1, x_2 \geq 0$, non-negativity constraints

We can form $\mathcal{L} = 4x_1 + 3x_2 + \mu(10 - 2\mathrm{x}_1 - x_2)$

The necessary conditions for a point to be a maximum are:

$\frac{\partial \mathcal{L}}{\partial x_1} = 4 - 2\mu^* \leq 0 \qquad x_1^* \geq 0 \qquad x_1^*(4 - 2\mu^*) = 0 \qquad$ (i)
$\frac{\partial \mathcal{L}}{\partial x_2} = 3 - \mu^* \leq 0 \qquad x_2^* \geq 0 \qquad x_2^*(3 - \mu^*) = 0 \qquad$ (ii)
$\frac{\partial \mathcal{L}}{\partial \mu} = 10 - 2x_1^* - x_2^* \geq 0 \qquad \mu^* \geq 0 \qquad \mu^*(10 - 2x_1^* - x_2^*) = 0 \qquad$ (iii)

Note that for consistency, we can rewrite $\frac{\partial \mathcal{L}}{\partial \mu} = 2x_1^* + x_2^* - 10 \leq 0$.

We solve this set of inequalities and equations to find the points which may be maxima.

* One more thing before we proceed, in case this were a minimization problem, all the signs on the *derivatives of the Lagrangean* would be inversed.

Now let's get to solving, this is intuitive, but can be quite tedious.

From (i), $x_1^*(4 - 2\mu^*) = 0 \rightarrow x_1 = 0$ or $\mu^* = 2$

Suppose $\mu^* = 2$, then, from (ii), $3 - \mu^* \leq 0$. This however implies that: $3 - 2 \leq 0$ which is NOT true. Them it must be that $\boldsymbol{x_1^* = 0}$.

With this info, let's proceed further,

From (ii) $x_2^*(3 - \mu^*) = 0 \rightarrow x_2^* = 0$ or $\mu^* = 3$

Suppose $x_1^* = 0$ and $x_2^* = 0$, together, that would imply that:

$$\mu^*(10 - 2(0) - 0) = 0$$

Thus, $10\mu^* = 0 \rightarrow \mu^* = 0$, but that would contradict the first condition on (i).

i.e. if we have $4 - 2\mu \leq 0$ and if $\mu^* = 0$, we would have: $4 \leq 0$, which is NOT true.
Thus, $x_2^*$ cannot be equal to 0, and we conclude $\boldsymbol{\mu^* = 3}$.

Doing good so far!

From (iii) we have $\mu^*(10 - 2x_1^* - x_2^*) = 0$
Thus, with our findings thus far, $3(10 - x_2) = 0 \rightarrow x_2 = 10$

And we've done it, we have solved this system and found only one solution that is:

$\boldsymbol{x_1^* = 0, x_2^* = 10, \mu^* = 3.}$

It would be cruel of me to have you solve a full blown Kuhn-Tucker problem today. I do want you to be able to read them and understand what you are looking at. These are general examples from the Production mini.

This is a profit maximization problem framed as:

$$\max_{q\geq 0, z\geq 0} \pi(p,r) = p.q - r.z$$
subject to:
$$D_o(q,z) \leq 1$$

I will go over what the different components mean briefly.

We can set up the Lagrangean as:

$$\mathcal{L} = p.q - r.z + \lambda_0(1 - D_o(q,z))$$

For any given output $m$ and any given input $n$, please find the first order Kuhn-tucker conditions.

$$\frac{\partial \mathcal{L}}{\partial q_m} = p_m - \lambda_0^* \frac{\partial D_o(q,z)}{\partial q_m} \leq 0 \qquad \frac{\partial \mathcal{L}}{\partial q_m} q_m^* = 0 \qquad q_m^* \geq 0$$

$$\frac{\partial \mathcal{L}}{\partial z_n} = -r_n - \lambda_0^* \frac{\partial D_o(q,z)}{\partial z_n} \leq 0 \qquad \frac{\partial \mathcal{L}}{\partial z_n} z_n^* = 0 \qquad z_n^* \geq 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_0} = 1 - D_o(q,z) \geq 0 \qquad \frac{\partial \mathcal{L}}{\partial \lambda_0} \lambda_0^* = 0 \qquad \lambda_0^* \geq 0$$

Let's have you solve one of Prof. Terry Hurley famous profit maximization problems. If you can do this – you are at least 20% ready to take a Production final ☺

Consider this profit maximization problem with two outputs and two inputs constrained by an input distance function given by $D_1(q,z)$.

$$\max_{q\geq 0, z\geq 0} \pi(p,r) = p_1q_1 + p_2q_2 - r_1z_1 - r_2z_2$$
subject to:
$$D_I(q,z) = \frac{\sqrt{z_1z_2}}{q_1^2 + q_2^2} \geq 1$$

Form the Lagrangean for this problem:

$$\mathcal{L} = p_1q_1 + p_2q_2 - r_1z_1 - r_2z_2 + \lambda_I \frac{\sqrt{z_1z_2}}{(q_1^2 + q_2^2)} - 1)$$

Write out the first order conditions (with Kuhn-Tucker):

$\frac{\partial \mathcal{L}}{\partial q_1} = p_1 - \lambda_I^* \frac{\partial D_I(q,z)}{\partial q_1} \leq 0$ $\quad$ $\frac{\partial \mathcal{L}}{\partial q_1} q_1^* = 0$ $\quad$ $q_1^* \geq 0$

$\frac{\partial \mathcal{L}}{\partial q_2} = p_2 - \lambda_I^* \frac{\partial D_I(q,z)}{\partial q_2} \leq 0$ $\quad$ $\frac{\partial \mathcal{L}}{\partial q_2} q_2^* = 0$ $\quad$ $q_2^* \geq 0$

$\frac{\partial \mathcal{L}}{\partial z_1} = -r_1 - \lambda_I^* \frac{\partial D_I(q,z)}{\partial z_1} \leq 0$ $\quad$ $\frac{\partial \mathcal{L}}{\partial z_1} z_1^* = 0$ $\quad$ $z_1^* \geq 0$

$\frac{\partial \mathcal{L}}{\partial z_2} = -r_2 - \lambda_I^* \frac{\partial D_I(q,z)}{\partial z_2} \leq 0$ $\quad$ $\frac{\partial \mathcal{L}}{\partial z_2} z_2^* = 0$ $\quad$ $z_2^* \geq 0$

$\frac{\partial \mathcal{L}}{\partial \lambda_1} = D_I(q,z) - 1 \geq 0$ $\quad$ $\frac{\partial \mathcal{L}}{\partial \lambda_I} \lambda_I^* = 0$ $\quad$ $\lambda_I^* \geq 0$

*Example.* Let's practice an example of writing out the Kuhn-Tucker conditions for a minimization problem.

Consider this cost minimization problem two inputs constrained by an input distance function given by $D_1(q,z)$.

$$\min_{z \geq 0} C = r_1z_1 - r_2z_2$$

subject to:

$$D_I(q,z) = \frac{\sqrt{z_1z_2}}{q_1^2 + q_2^2} \geq 1$$

Write down the Lagrangian, Kuhn-Tucker conditions and solve for $z_1(r,q)$ and $z_2(r,q)$ assuming interior solutions afterwards.

Lagrangian:

$$\mathcal{L} = r_1z_1 - r_2z_2 - \lambda_I(1 - \frac{\sqrt{z_1z_2}}{q_1^2 + q_2^2})$$

*Notice, now the term with the multiplier is* $\leq 0$ *versus our maximization problems.*

Kuhn-Tucker conditions:

$$\frac{\partial \mathcal{L}}{\partial z_1} = r_1 - \lambda_I^* \frac{\partial D_I(q,z)}{\partial z_1} \geq 0 \qquad \frac{\partial \mathcal{L}}{\partial z_1} z_1^* = 0 \qquad z_1^* \geq 0$$

$$\frac{\partial \mathcal{L}}{\partial z_2} = r_2 - \lambda_I^* \frac{\partial D_I(q,z)}{\partial z_2} \geq 0 \qquad \frac{\partial \mathcal{L}}{\partial z_2} z_2^* = 0 \qquad z_2^* \geq 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_1} = 1 - D_I(q,z) \leq 0 \qquad \frac{\partial \mathcal{L}}{\partial \lambda_I} \lambda_I^* = 0 \qquad \lambda_I^* \geq 0$$

You can now assume interior solution. Find the solutions and derive the cost function.

$$\frac{\partial \mathcal{L}}{\partial z_1} = r_1 - \lambda_I^* \frac{\sqrt{z_2^*}}{2\sqrt{z_1^*}(q_1^{*2} + q_2^{*2})} = 0$$

$$\frac{\partial \mathcal{L}}{\partial z_2} = r_2 - \lambda_I^* \frac{\sqrt{z_1^*}}{2\sqrt{z_2^*}(q_1^{*2} + q_2^{*2})} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_1} = 1 - \frac{\sqrt{z_1^* z_2^*}}{q_1^{2*} + q_2^{2*}} = 0$$

From the first two:

$$z_2^* = \frac{r_1}{r_2} z_1^*$$

We can substitute this into the constraint:

$$\frac{\sqrt{z_1^* z_2^*}}{q_1^2 + q_2^2} = 1$$

$$\frac{\sqrt{z_1^*}}{q_1^{2*} + q_2^{2*}} . \sqrt{\frac{r_1}{r_2} z_1^*} = 1$$

$$\sqrt{\frac{r_1}{r_2} z_1^*} = \frac{q_1^{2*} + q_2^{2*}}{\sqrt{z_1^*}}$$

Squaring on both sides:

$$\frac{r_1}{r_2}z_1^* = \frac{(q_1^{2*} + q_2^{2*})^2}{z_1^*}$$

$$z_1^{2*} = \frac{(q_1^{2*} + q_2^{2*})^2}{r_1}r_2$$

$$z_1^* = \frac{(q_1^{2*} + q_2^{2*}}{\sqrt{r_1}}\sqrt{r_2}$$

Symmetrically,

$$z_2^* = \frac{(q_1^{2*} + q_2^{2*}}{\sqrt{r_2}}\sqrt{r_1}$$

Thus, the cost function is: $r_1 z_1 + r_2 z_2 = r_1 \frac{(q_1^{2*}+q_2^{2*}}{\sqrt{r_1}}\sqrt{r_2} + r_2 \frac{(q_1^{2*}+q_2^{2*}}{\sqrt{r_2}}\sqrt{r_1}$

$$c(r,q) = 2\sqrt{r_1 r_2}(q_1^2 + q_2^2)$$

Final example. Maximize $U = U(x,y)$ Subject to $B \geq P_x x + P_y y$ and $C \geq c_x x + c_y y$.
$x \geq 0$ and $y \geq 0$.

Write out the Langragean for this problem:

$\mathcal{L} = U(x,y) + \lambda_1(B - P_x x - P_y y) + \lambda_2(C - c_x x - c_y y)$

Write out the Kuhn-tucker conditions.

$\mathcal{L}_x = U_x - \lambda_1 p_x - \lambda_2 c_x \leq 0 \quad ; x \geq 0 \quad ; x.(U_x - \lambda_1 p_x - \lambda_2 c_x) = 0$
$\mathcal{L}_y = U_y - \lambda_1 p_y - \lambda_2 c_y \leq 0 \quad ; y \geq 0 \quad ; y.(U_y - \lambda_1 p_y - \lambda_2 c_y) = 0$
$\mathcal{L}_{\lambda_1} = B - P_x x - P_y y \geq 0 \quad ; \lambda_1 \geq 0; \lambda_1.(B - P_x x - P_y y) = 0$
$\mathcal{L}_y = C - c_x x - c_y y \geq 0 \quad ; \lambda_2 \geq 0; \lambda_2.(C - c_x x - c_y y) = 0$