Lecture 03
Derivatives

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Last Lecture Review

- Set Theory:
  - Set Operators
  - De Morgan's Law
  - Cartesian Product
  - Convex Sets
  - Bounded & Compact Sets
- Topology:
  - Supremum and Infimum and Limits
  - Separating Hyperplane Theorem

Review Assignment

1. Problem Set 02 solutions are available on Github.
2. Any issues or problems **You** would like to discuss?

Daily Icebreaker

- Attendance via prompt:
  - Name
  - Daily Icebreaker: You are a late night show host. Who is the first celebrity you would invite to interview?

Topic: Derivatives

Motivation

- General background
  - Understanding a rate of change.
  - A core component of calculus alongside integration.
- Why do economists' care?
  - How we determine a marginal effect (e.g., coefficient of interest).
  - Heavily used throughout theory.
- Application in this career
  - The main math tool you will use throughout the microeconomic theory (alongside optimization).

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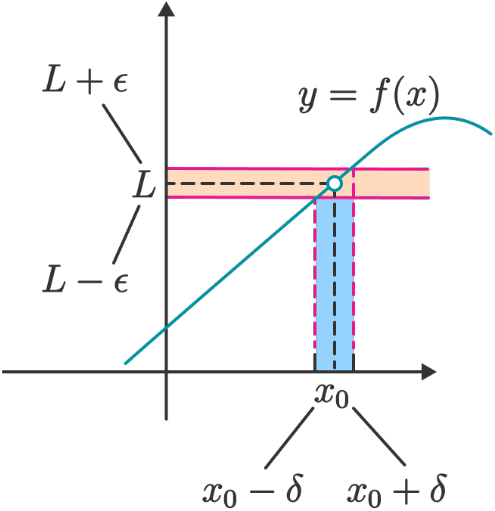
Overview

1. Continutity & Differentiability
2. First Derivative
3. Second Derivative
4. Derivative Rules
5. Implicit Function
6. l'Hopital's Rule
7. Taylor Series Approximation
8. Mean Value Theorem
9. Convexity

1. Continuity and Differentiability

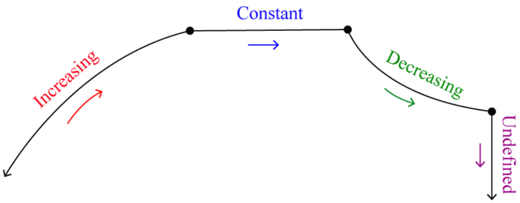
- Continuous: A function $f : \mathbb{R} \to \mathbb{R}$ is continuous at point $p \in \mathbb{R} \iff \forall \varepsilon > 0 \exists \delta > 0 : |x - p| < \delta \implies |f(x) - f(p)| < \varepsilon$.
  - E.g., All x uniquely maps to $f(x)$ at all x.
- Differentiable: A function f is differentiable at x if and only if (iff) a limit exists. The entire function is differentiable if it is differentiable for all points of $x \in \mathbb{R}$.
- Differentiable $\implies$ continuous.
- Continuous $\not\Longrightarrow$ differentiable.
- $C^1 = f'$ is continuously differentiable.
- $C^2 = f''$ is twice continuously differentiable.

Continuity



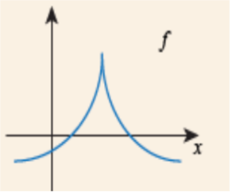
Slope Change

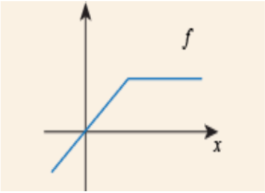
- Increasing: $f'(x) > 0 \forall x \in [a, b]$.
- Decreasing: $f'(x) < 0 \forall x \in [a, b]$.
- Monotonically Increasing: $f'(x) \geq 0 \forall x \in \mathbb{R}$.
- Strictly Increasing: $f'(x) > 0 \forall x \in \mathbb{R}$.

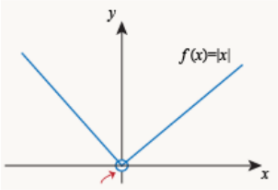


Continuous but **Not** Differentiable

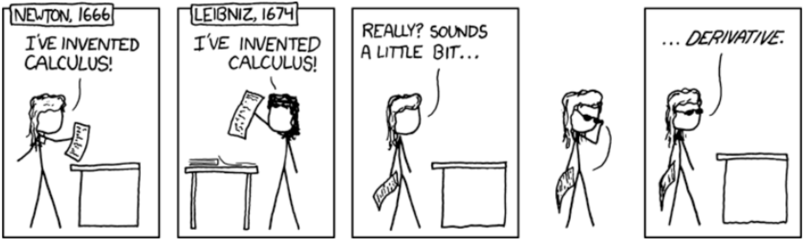
- Sharp points.
- Edges.
- Jumps/holes.





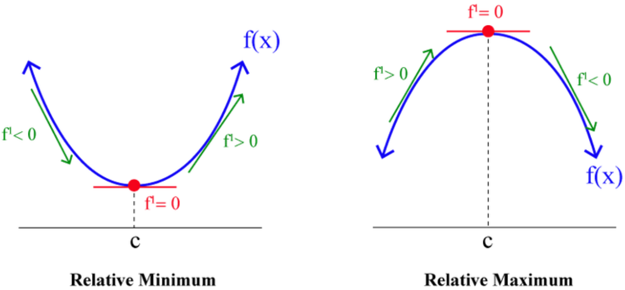


2. Newton or Leibniz



2. First Derivative

$$f'(x_0) \equiv \frac{df}{dx}(x_0) \equiv \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$



Common First Derivatives

- Constant: $a' = 0$
- Base Variable: $(x^a)' = ax^{a-1}$
- Base Constant: $(a^x)' = a^x ln(a)$
- Exponent Variable: $(e^x)' = e^x$
- Logarithmic: $ln(x)' = \frac{1}{x}$

3. Second Derivative

$$f''(x_0) \equiv \frac{d}{dx}\left(\frac{df}{dx}\right)(x_0) \equiv \frac{d^2 f}{dx^2}(x_0)$$

- Can be taken at higher orders, but rarely applied within first year coursework.

4. Derivative Rules

- Sum Rule

$$[f(x) \pm g(x)]' \equiv f'(x) \pm g'(x)$$

- Power Rule

$$[\alpha x^n]' \equiv n\alpha x^{n-1}$$

- Product Rule

$$[f(x)g(x)]' \equiv f'(x)g(x) + f(x)g'(x)$$

- Quotient Rule

$$\left[\frac{f(x)}{g(x)}\right]' \equiv \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

Demonstration: Product Rule

Question:
Find $\frac{df(x)}{dx}$ for $f(x) = (3x - 2x^2)(5 + 4x)$.

Answer:
Note the two functions. $f(x) = (3x - 2x^2)$ and $g(x) = (5 + 4x)$.
By the product rule:

$$\begin{aligned} f'(x) =& (3x - 2x^2)(4) + (3 - 4x)(5 + 4x) \\ =& -24x^2 + 4x + 15 \end{aligned}$$

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Demonstration: Quotient Rule

Question:
Find $\frac{dh(z)}{dz}$ for $h(z) = \frac{4\sqrt{z}}{z^2-2}$.

Answer:
Again, identify the two functions $f(z) = (4\sqrt{z})$ and $g(z) = (z^2 - 2)$. Then by the rule we have:

$$h'(z) = \frac{4(1/2)x^{\frac{-1}{2}}(x^2-2) - 4x^{\frac{1}{2}}(2x)}{(x^2-2)^2}$$
$$= \frac{-6x^{\frac{3}{2}} - 4x^{\frac{-1}{2}}}{(x^2-2)^2}$$

Note: Quotients can be done as products.

Demonstration: Quotient Rule

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Note: Quotients can be done as products.

4. Derivative Rules

- Inverse Rule
  - When $f(x)$ is monotone, differentiable, $f'(x) \neq 0$, and $f^{-1\prime}(x)$ is differentiable.

$$[f^{-1}(x)]' \equiv \frac{1}{f'(x)}$$

- Chain Rule

$$\frac{d}{dx}h(g(x)) \equiv h'(g(x))g'(x)$$

Demonstration: Chain Rule

Question:
Find $\frac{df(x)}{dx}$ for $g(x) = ln(x^{-4} + x^4)$.

Answer:

$$g'(x) = \frac{1}{x^{-4} + x^4}(-4x^{-5} + 4x^3) = \frac{-4x^{-5} + 4x^3}{x^{-4} + x^4}$$

Demonstration: Chain Rule

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Practice: First Derivatives

1. $f(x) = xe^{3x}$

Practice: First Derivatives

1. $f(x) = xe^{3x}$

Answer: Show Work

$$f'(x) = (1+3x)x^{3x}$$

Practice: First Derivatives

1. $f(x) = xe^{3x}$
2. $f(x) = ln(x^4 + 2)^2$

Practice: First Derivatives

1. $f(x) = xe^{3x}$
2. $f(x) = ln(x^4 + 2)^2$

Answer: Show Work

$$f'(x) = \frac{8x^3 ln(x^4 + 2)}{x^4 + 2}$$

Practice: First Derivatives

1. $f(x) = xe^{3x}$
2. $f(x) = ln(x^4 + 2)^2$
3. $f(x) = \left(\frac{x+4}{x-3}\right)^{2/3}$

Practice: First Derivatives

1. $f(x) = xe^{3x}$
2. $f(x) = ln(x^4 + 2)^2$
3. $f(x) = \left(\frac{x+4}{x-3}\right)^{2/3}$

Answer: Show Work

$$f'(x) = \frac{-14}{3(x+4)(x-3)^{5/3}}$$

5. Implicit Function

- Implicit Function Theorem requires invoking the Jacobian matrix for partial derivatives. This involves knowledge of matrices and multivariate calculus covered later in the course.
- Sometimes y cannot be expressed as an explicit function of x.
- But we still can calculate $\frac{dy}{dx}$ ... implicitly.

Demonstration: Implicit Functions

Question:

Find $\frac{dy}{dx}$ for $y = 5x^2 - 9e^y$.

Answer:

$$\begin{aligned}\frac{dy}{dx}(1) =& \frac{dy}{dx}5x^2 - \frac{dy}{dx}(9e^y)\\ \frac{dy}{dx} =& 10x - (9e^y)\frac{dy}{dx}\\ \frac{dy}{dx}(1+9e^y) =& 10x\\ \frac{dy}{dx} =& \frac{10x}{1+9e^y}\end{aligned}$$

Demonstration: Implicit Functions

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\frac{dy}{dx}(1 + 9e^y) =& 10x \\
\frac{dy}{dx} =& \frac{10x}{1 + 9e^y}
\end{aligned}$$

Practice: Implicit Functions

1. Find $\frac{dy}{dx}$ for $x^2y^3 - xy = 10$.

Practice: Implicit Functions

1. Find $\frac{dy}{dx}$ for $x^2y^3 - xy = 10$.

Answer: Show Work

$$\frac{dy}{dx} = \frac{-2xy^3 + y}{3x^2y^2 - x}$$

Practice: Implicit Functions

1. Find $\frac{dy}{dx}$ for $x^2y^3 - xy = 10$.
2. Find $\frac{dy}{dx}$ for $e^y + xy - e = 0$.

PRACTICE: IMPLICIT FUNCTIONS

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2. Find $\frac{dy}{dx}$ for $e^y + xy - e = 0$.

Answer: Show Work

$$\frac{dy}{dx} = \frac{-y}{e^y + x}$$

Practice: Implicit Functions

1. Find $\frac{dy}{dx}$ for $x^2y^3 - xy = 10$.
2. Find $\frac{dy}{dx}$ for $e^y + xy - e = 0$.
3. Find $\frac{dy}{dx}$ for $\frac{x^2y}{yx^2} = 4y$

PRACTICE: IMPLICIT FUNCTIONS

1. Find $\frac{dy}{dx}$ for $x^2y^3 - xy = 10$.
2. Find $\frac{dy}{dx}$ for $e^y + xy - e = 0$.
3. Find $\frac{dy}{dx}$ for $\frac{x^2y}{yx^2} = 4y$

Answer: Show Work

$$\frac{dy}{dx} = \frac{4y^2x^4 + 2yx - 2y^2x^3}{yx^4 + x^2}$$

6. l'Hopital's Rule

- Consider you are taking a limit (derivative) with two functions in the numerator and denominator of a fraction, respectively.
- Applies when:
  - $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$
  - $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$
- Both $f(x)$ and $g(x)$ need to be differentiable over the interval $I : a \in I$.
- In both scenarios, we assume that the denominator does not equal 0 or ∞.

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(x)}{g'(x)}$$

Application: Constant Elasticity of Substitution (CES)

To show that the CES $Y = A(\alpha K^\gamma + (1-\alpha)L^\gamma)^{\frac{1}{\gamma}}$ is a Cobb-Douglas function $Y = AK^\alpha L^{1-\alpha}$ when $\gamma \to 0$.

Proof.

First take the log of both sides.

$$ln(Y) = ln(A) + \frac{1}{\gamma} ln(\alpha K^\gamma + (1-\alpha)L^\gamma)$$

Then by l'Hopital's Rule, □

Application: Constant Elasticity of Substitution (CES)

Proof.

$$
\begin{aligned}
\lim_{\gamma\to 0}\frac{ln(\alpha K^\gamma+(1-\alpha)L^\gamma)}{\gamma} =& \lim_{\gamma\to 0}\frac{\frac{dln(\alpha K^\gamma+(1-\alpha)L^\gamma)}{d\gamma}}{\frac{d\gamma}{d\gamma}}\\
=&\frac{\alpha K^\gamma ln(K)+(1-\alpha)L^\gamma ln(L)}{\alpha K^\gamma+(1-\alpha)L^\gamma}\\
=&\alpha ln(K)+(1-\alpha)ln(L)\\
\therefore \lim_{\gamma\to 0} ln(Y) =& ln(A)+\alpha ln(K)+(1-\alpha)ln(L)
\end{aligned}
$$

This is the Cobb-Douglas function. □

7. Taylor Series Approximation

- Taylor Series:

$$
\begin{aligned}
f(x) =& f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \cdots \\
=& \sum_{k=0}^{n} \frac{f^k(a)}{k!}(x-a)^k
\end{aligned}
$$

- Use Taylor Series to approximate with a remainder $R(\Delta x, x_0)$:

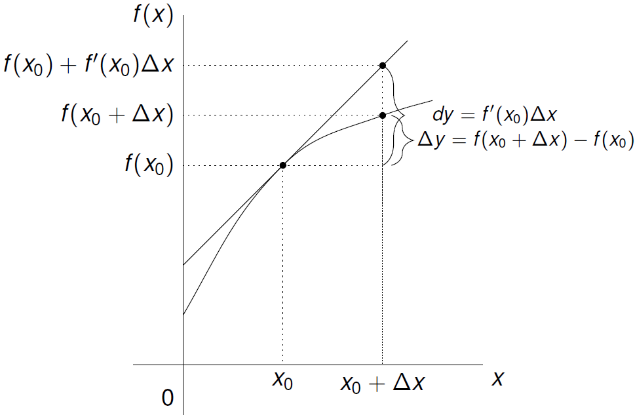
$$
\begin{aligned}
f(x_0 + \Delta x) \approx& f(x_0) + f'(x_0)\Delta x + R(\Delta x, x_0) \\
R(\Delta x, x_0) =& f(x_0 + \Delta x) - f(x_0) - f'(x_0)\Delta x
\end{aligned}
$$

7. Taylor Series Approximation

- We can approximate to the $(k+1)$ order of derivatives.

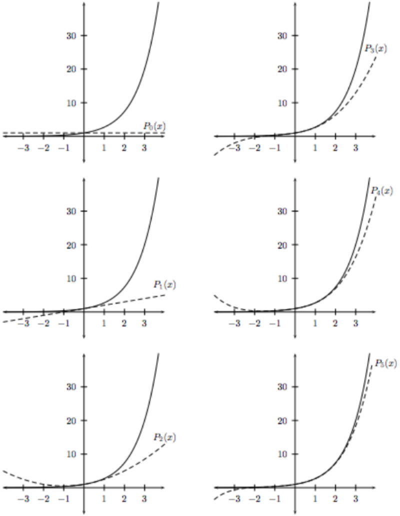
$$
\begin{aligned}
f(x_0+\Delta x) =& f(x_0) + f'(x_0)\Delta x + \frac{1}{2!}f''(x_0)(\Delta x)^2 + \dots \\
&+ \frac{1}{k!}f^k(x_0)(\Delta x)^k + R_k(\Delta x, x_0) \\
R_k(\Delta x, x_0) =& \frac{f^{(k+1)}(c^*)}{(k+1)!}(\Delta x)^{k+1} \ , \ c^* \in (x_0, x_0+\Delta x) \\
&\lim_{\Delta x \to 0} \frac{R_k(\Delta x, x_0)}{(\Delta x)^k} \to 0
\end{aligned}
$$

7. Taylor Series Approximation



$$\Delta y \approx dy = f'(x_0)\Delta x$$

7. Taylor Series Approximation



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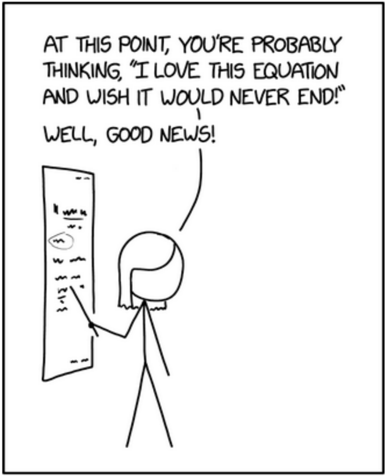
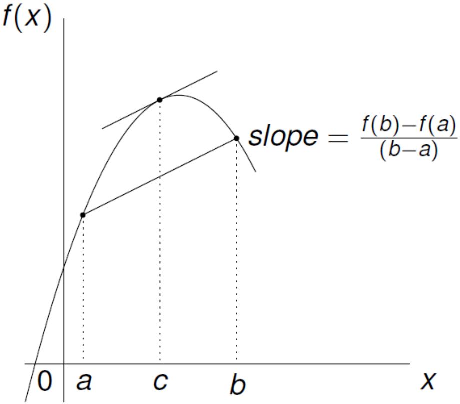


TAYLOR SERIES EXPANSION IS THE WORST.

8. Mean Value Theorem

- Let $f : U \to \mathbb{R}$ be a C^1 function over the interval $U \subset \mathbb{R}$.

$$\forall a, b \in U \exists c : a \leq c \leq b : f'(c) = \frac{f(b) - f(a)}{b - a}$$

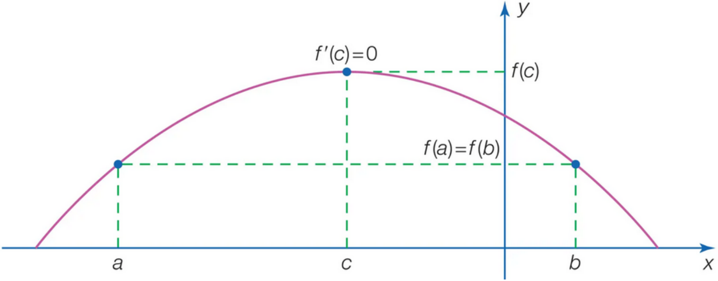


Rolle's Theorem

- A special case of the mean value theorem.
- E.g., If a continuous curve passes through the same y-value twice, and has a unique tangent line (i.e., a derivative) for all points in the interval, **then** a tangent parallel to the x-axis (i.e., critical value) exists in the interval.

Rolles Theorem:
If a function f is continuous on the the interval $[a,b]$ and differentiable on the interval (a,b) such that $f(a) = f(b)$, then $f'(x) = 0$ for some $x|a \leq x \leq b$.

Rolle's Theorem



9. Convexity & Critical Points

Weierstrass Theorem:
A continuous function $f(\cdot)$ over a closed and bounded interval $[a, b]$ attains both a local maximum and minimum.

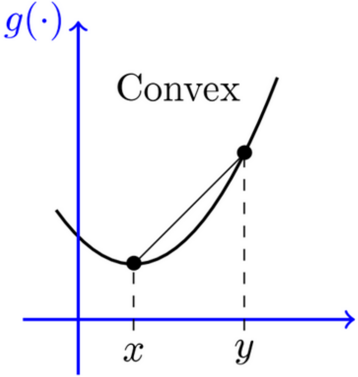
- Concave function:

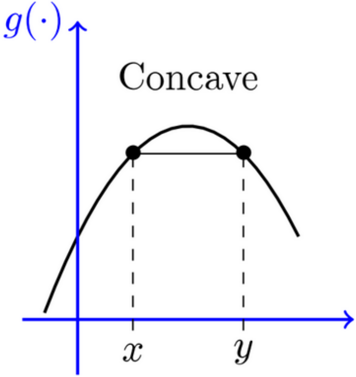
$$\forall x, y \in I : f(y) - f(x) \leq f'(x)(y-1) \vee f''(x) \leq 0$$

- Convex function:

$$\forall x, y \in I : f(y) - f(x) \geq f'(x)(y-1) \vee f''(x) \geq 0$$

Concave Up and Concave Down

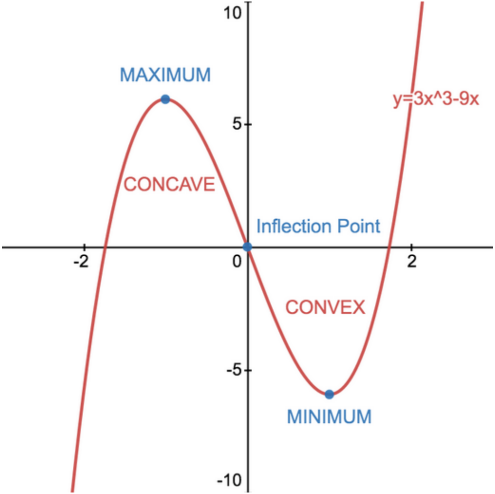




9. Convexity & Critical Points

- Critical Points: Values of x where $f'(x) = 0$ or is undefined.
- Local Max/Min (over interval I):
  $x_0, x \in I : f(x_0) \geq (\leq) f(x) \forall x$.
- Global Max/Min (over domain f):
  $x_0, x \in f : f(x_0) \geq (\leq) f(x) \forall x$.

Maximums and Minimums



Demonstration: Critical Values

Question:

What are the critical values for $f(x) = x^4 + 3x^2 + 10$

Answer:

$$
\begin{aligned}
f'(x) =& 4x^3 + (3)(2)x \\
=& 4x^3 6x = 0 \\
x^* =& \{0\}
\end{aligned}
$$

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$$= 4x^3 6x = 0$$
$$x^* = \{0\}$$

Practice: Critical Values

1. Critical values for $f(x) = 8x^3 + 81x^2 - 42x - 8$

PRACTICE: CRITICAL VALUES

1. Critical values for $f(x) = 8x^3 + 81x^2 - 42x - 8$

Answer: Show Work

$$x^* = \{-7, \frac{1}{4}\}$$

Practice: Critical Values

1. Critical values for $f(x) = 8x^3 + 81x^2 - 42x - 8$
2. Critical values for $g(w) = 2w^3 - 7w^2 - 3w - 2$

Practice: Critical Values

1. Critical values for $f(x) = 8x^3 + 81x^2 - 42x - 8$
2. Critical values for $g(w) = 2w^3 - 7w^2 - 3w - 2$

Answer: Show Work

$$w^* = \{\frac{7 \pm \sqrt{67}}{6}\}$$

Practice: Critical Values

1. Critical values for $f(x) = 8x^3 + 81x^2 - 42x - 8$
2. Critical values for $g(w) = 2w^3 - 7w^2 - 3w - 2$
3. Critical values for $r(y) = (y^2 - 6y)^{1/5}$

Practice: Critical Values

1. Critical values for $f(x) = 8x^3 + 81x^2 - 42x - 8$
2. Critical values for $g(w) = 2w^3 - 7w^2 - 3w - 2$
3. Critical values for $r(y) = (y^2 - 6y)^{1/5}$

Answer: Show Work

$$y^* = \{0, 3, 6\}$$

Review

Review of Derivatives

1. Continutity & Differentiability
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4. Derivative Rules
5. Implicit Function
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Assignment

- Readings on Integration before Lecture 04:
  - S&B Ch. 2, 3, & 4
- Assignment:
  - Problem Set 03 (PS03)
  - Solution set will be available following end of Lecture 04
- Struggling?
  1. Read the 'Encouraged Reading'
  2. Review 'Supplementary material'
  3. Reach out directly

Derivative Question 1 Answer:

◂ Question

$$f'(x) = (1)x^{3x} + x(3e^{3x}) = (1 + 3x)x^{3x}$$

Derivative Question 2 Answer:

◂ Question

$$f'(x) = 2ln(x^4+2)\frac{1}{x^4+2}4x^3$$
$$= \frac{8x^3 ln(x^4+2)}{x^4+2}$$

Derivative Question 3 Answer:

Question

Two Notes: I treat the quotient using the product rule: $(x+4)(x-3)^{-1}$. And I am able to flip fractions to force exponents to be positive.

$$\begin{aligned}
f'(x) =& \frac{2}{3}\left(\frac{x+4}{x-3}\right)^{-1/3}\left((1)(x-3)^{-1} + (x+4)(-1)(x-3)^{-2}(1)\right) \\
=& \frac{2}{3}\left(\frac{x+4}{x-3}\right)^{-1/3}\left(\frac{x-3-x-4}{(x-3)^2}\right) \\
=& \frac{2}{3}\left(\frac{x-3}{x+4}\right)^{1/3}\left(\frac{-7}{(x-3)^2}\right) \\
=& \frac{2}{3}\frac{1}{x+4}\frac{-7}{(x-3)^{5/3}} \\
=& \frac{-14}{3(x+4)(x-3)^{5/3}}
\end{aligned}$$

Implicit Functions Question 1 Answer:

◂ Question

$$
\begin{aligned}
2xy^3 + 3x^2y^2\frac{dy}{dx} - y - x\frac{dy}{dx} &= 0 \\
(2xy^3 - y) + (3x^2y^2 - x)\frac{dy}{dx} &= 0 \\
\frac{dy}{dx} &= \frac{-2xy^3 + y}{3x^2y^2 - x}
\end{aligned}
$$

Implicit Functions Question 2 Answer:

Question

$$\begin{aligned} e^y\frac{dy}{dx} + y + x\frac{dy}{dx} - 0 =& 0 \\ y + (e^y + x)\frac{dy}{dx} =& 0 \\ \frac{dy}{dx} =& \frac{-y}{e^y + x} \end{aligned}$$

IMPLICIT FUNCTIONS QUESTION 3 ANSWER:

◂ QUESTION

$$
\begin{aligned}
(2xy + x^2(1)\frac{dy}{dx})(yx^2)^{-1} + (-1)(yx^2)^{-2}(x^2\frac{dy}{dx} + 2yx) &= 4 \\
\frac{(2xy + x^2\frac{dy}{dx})(yx^2)}{(yx^2)^2} - \frac{x^2\frac{dy}{dx} + 2yx}{(yx^2)^2} &= 4 \\
2y^2x^3 + yx^4\frac{dy}{dx} - x^2\frac{dy}{dx} - 2yx &= 4(yx^2)^2 \\
\frac{dy}{dx}(yx^4 - x^2) = 4y^2x^4 + 2yx - 2y^2x^3 \\
\frac{dy}{dx} = \frac{4y^2x^4 + 2yx - 2y^2x^3}{yx^4 + x^2}
\end{aligned}
$$

Critical Values Question 1 Answer:

◂ Question

$$
\begin{aligned}
f'(x) =& 8(3)x + 81(2)x - 42 \\
=& 24x^2 + 162x - 42 = 0 \\
=& 6(x+7)(4x-1) = 0 \\
x^* =& \{-7, \frac{1}{4}\}
\end{aligned}
$$

Critical Values Question 2 Answer:

◂ Question

$$
\begin{aligned}
g'(w) =& 2(3)w^2 - 7(2)w - 3 \\
=& 6w^2 - 14w - 3 = 0 \\
\because \text{Quad. Formula: } & \frac{b \pm \sqrt{b^2 - 4ac}}{2a} \\
w^* =& \frac{14 \pm \sqrt{268}}{12} \\
w^* =& \{\frac{7 \pm \sqrt{67}}{6}\}
\end{aligned}
$$

Critical Values Question 3 Answer:

◂ Question

$$
\begin{aligned}
r'(y) =& \frac{1}{5}(y^2 - 6y)^{-4/5}(2y-6) \\
=& \frac{2y-6}{5(y^2-6y)^{\frac{4}{5}}} \\
& \because \text{CV when } y = 0 \\
\implies & 2y - 6 = 0 \rightarrow y = 3 \\
\implies & y^2 - 6y = 0 \rightarrow y = \{0, 6\} \\
y^* =& \{0, 3, 6\}
\end{aligned}
$$