# APEC Math Review

## Convex Sets

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# Convex Sets

Definition: A set $S \subset \mathbb{R}^N$ is convex if $\forall x, y \in S, \forall \alpha \in (0,1), z = \alpha x + (1-\alpha)y \in S$

- If it contains two vectors, it also contains the whole segment that connects them
- Draw convex and non-convex sets in $\mathbb{R}$, $\mathbb{R}^N$
- Unions of convex sets need not be convex. Intersections always are.

# Convex Hull

Definition: given a set $B \subset \mathbb{R}^N$, the convex hull of B, denoted $CoB$, is the smallest convex set that contains B, that is the intersection of all convex sets that contain B.

- If a set its convex, it is equal to its convex hull.
- Equivalent to the set of all convex combinations of elements of B

$CoB = \{\Sigma_{j=1}^{J} \alpha_j x_j$ : for some $x_1, x_2, ..., x_J, x_j \in B \forall j < J$ and some $(\alpha_1, \alpha_2, ..., \alpha_J) \geq 0$ with $\Sigma_{j+1}^{J} \alpha_j = 1\}$

# Extreme Points

Definition: The vector $x \in B$ is an extreme point of some convex set $B \subset \mathbb{R}^N$ if it cannot be expressed as $x = \alpha y + (1 - \alpha)z$ for any $y, z \in B$ and $\alpha \in (0, 1)$

- Theorem: Let $B \subset \mathbb{R}^N$. If B is convex and compact, then any $x \in B$ can be expressed as a convex combination of at most N+1 extreme points of B

# Hyperplane

Definition: Given $p \in \mathbb{R}^N$ with $p \neq 0, c \in \mathbb{R}$. The hyperplane generated by p and C is the set:

$$H_{p,c} = \{z \in \mathbb{R}^N : p \cdot z = c\}$$

The sets $\{z \in \mathbb{R}^N : p \cdot z \geq c\}$ and $\{z \in \mathbb{R}^N : p \cdot z \leq c\}$ are called respectively the half-space above and half-space below the hyperplane $H_{p,c}$.

# Separating Hyperplane Theorem

Let $B \subset \mathbb{R}^N$ be convex and closed, and $x \notin B$. Then $\exists p \in \mathbb{R}^N$, $p \neq 0$, and a value $c \in \mathbb{R}$ such that $p \cdot x > c$ and $p \cdot y < c$ for every y in B

More generally, suppose convex sets A,B $\subset \mathbb{R}^N$ are disjoint. Then there is $p \in \mathbb{R}^N, p \neq 0$ and a real number c, such that $p \cdot x \geq c$ for every x in A and $p \cdot y \leq c$ for every y in B

In other words, there is a hyperplane that separates the two sets, leaving one set in one side and the other set in the other side

# Supporting Hyperplane Theorem

Suppose $B \subset \mathbb{R}^N$ is convex, and that x is not an element of the interior of B. Then there is $p \in \mathbb{R}^N, p \neq 0$ such that $p \cdot x \geq p \cdot y \forall y \in B$