

Lecture 09

Optimization Day 1

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Mathematics Review Course, Summer 2023
University of Minnesota
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LAST LECTURE REVIEW

- ▶ Numbers:
 - ▶ Triangle Inequality
 - ▶ Neighborhoods
- ▶ Functions:
 - ▶ Homogeneity
 - ▶ Euler's Theorem
 - ▶ Quasiconcavity & Quasiconvexity
 - ▶ Concavity & Convexity
 - ▶ Continuity
 - ▶ Upper- and Lower-Hemicontinuity
 - ▶ Brouwer's and Kakutani's Fixed-point Theorem

REVIEW ASSIGNMENT

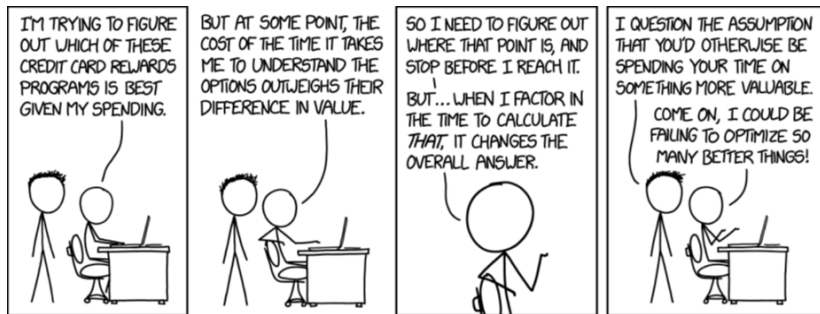
1. Problem Set 08 solutions are available on Github.
2. Any issues or problems **You** would like to discuss?

DAILY ICEBREAKER

- ▶ Attendance via prompt:
 - ▶ Name
 - ▶ Daily Icebreaker: What is one of the top adventures/activities you want to complete from your “bucket list”?



OPTIMIZATION



Unconstrained Optimization

MOTIVATION

- ▶ General background
 - ▶ Simplest form of optimization.
 - ▶ Solution involves finding critical points of the function.
- ▶ Why do economists' care?
 - ▶ Used in simple applications for testing first order conditions and second order conditions.
- ▶ Application in this career
 - ▶ Unconstrained optimization is primarily used to determine FOC and SOC.

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OVERVIEW

1. First Order Conditions
2. Second Order Conditions
3. Global Min and Max

1. FIRST ORDER CONDITIONS

- ▶ Let $F : U \rightarrow \mathbb{R}$ be a differentiable defined on subset U of \mathbb{R}^n .
- ▶ If $x^* \in \mathbb{R}^n$ is a local minimum or local maximum of $F(\cdot)$ and x^* is an interior point of U , then:

$$\nabla F(x^*) = 0 \text{ or } \frac{\partial F(x^*)}{\partial x_n} \forall n$$

- ▶ The FOC can be summarized by the Jacobian matrix.

$$J = \nabla f(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix}$$

APPLICATION: OLS

$$\min_{\beta_0, \beta_1} \sum_{i=1}^N (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2$$

The FOC for $\hat{\beta}_1$:

$$\begin{aligned} \frac{\partial W}{\partial \hat{\beta}_1} &= \sum_{i=1}^N -2x_i(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) = 0 \\ \hat{\beta}_1 &= \frac{\sum_{i=1}^N (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^N (x_i - \bar{x})^2} \end{aligned}$$

DEMONSTRATION: 1ST ORDER CONDITIONS (FOC)

Question:

In a monopoly, the demand curve is $P = 10 - Q$ and faces marginal costs of $C = 0.25Q$. P is the price and Q is the quantity. What is the quantity produced?

Answer:

$$\begin{aligned}\pi(P, Q) &= PQ - C \\ \pi &= (10 - Q)Q - 0.25Q \\ \frac{d\pi}{dQ} &= 10 - 2Q - 0.25 = 0 \\ Q^* &= \sqrt{9.75}\end{aligned}$$

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PRACTICE: FOC

1. Minimize $f(x) = 3x^2 - 8x + 1$.

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Answer: [◀ Show Work](#)

$$x^* = \frac{4}{3}$$

PRACTICE: FOC

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Answer: [◀ Show Work](#)

$$y^* = 2$$

PRACTICE: FOC

1. Minimize $f(x) = 3x^2 - 8x + 1$.
2. Minimize $g(y) = (1 - y)^2 - 2y$.
3. What is monopoly quantity supplied Q^* for the profit function $\pi(Q) = (7 - 0.5Q)Q - 0.5Q$?

PRACTICE: FOC

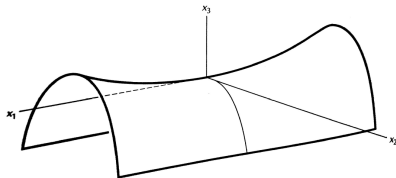
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Answer: [◀ Show Work](#)

$$Q^* = 6.5$$

2. SECOND ORDER CONDITIONS

- ▶ Let $F : U \rightarrow \mathbb{R}$ be C^2 whose domain is an open set $U \in \mathbb{R}^n$.
- ▶ Suppose $\nabla F(x^*) = 0$.
 - ▶ If $D^2f(x^*)$ is negative (positive) definite, then x^* is a strict local max (min).
 - ▶ If $D^2f(x^*)$ is indefinite, then x^* is neither a local max nor min.
- ▶ Suppose x^* is a local max (min) of F .
 - ▶ Then, $\nabla F(x^*) = 0$ and the symmetric $n \times n$ matrix $D^2f(x^*)$ is negative (positive) **semi-definite**.



The graph of the indefinite form $Q_3(x_1, x_2) = x_1^2 - x_2^2$.

2. SECOND ORDER CONDITIONS

- The SOC is captured by the Hessian matrix.

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

DEMONSTRATION: 2ND ORDER CONDITIONS (SOC)

Question:

Check the SOC for a local max of
 $U(x_1, x_2) = x_1^{0.5} x_2^{0.5} : x_1, x_2 > 0$.

Answer:

FOC:

$$U_{x_1} = 0.5 \left(\frac{x_2}{x_1} \right)^{0.5}$$

$$U_{x_2} = 0.5 \left(\frac{x_1}{x_2} \right)^{0.5}$$

SOC:
$$H = \begin{bmatrix} -0.25x_1^{-1.5}x_2^{0.5} & 0.25(x_1x_2)^{-0.5} \\ 0.25(x_1x_2)^{-0.5} & -0.25x_1^{0.5}x_2^{-1.5} \end{bmatrix}$$

Yes, $D^2f(x^*)$ is negative definite \rightarrow local max.

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$$H = \begin{bmatrix} -0.25x_1^{-1.5}x_2^{0.5} & 0.25(x_1x_2)^{-0.5} \\ 0.25(x_1x_2)^{-0.5} & -0.25x_1^{0.5}x_2^{-1.5} \end{bmatrix}$$

Yes, $D^2f(x^*)$ is negative definite \rightarrow local max.

3. GLOBAL MIN AND MAX

- ▶ Any point x^* of a concave (convex) function $f(\cdot)$ satisfying $\nabla F(x^*) = 0$.
- ▶ Note this may be a boundary point (corner solution) or an interior point (critical point).

$Df(x^*)$	$D^2f(x^*)$	Max/Min
$= 0$	Negative Semi-definite	Local Max
$= 0$	Positive Semi-definite	Local Min
$= 0$	Indefinite	Saddle point or Inflexion

(Equality) Constrained Optimization

MOTIVATION

- ▶ General background
 - ▶ Optimization of a function considering a constraint from other function(s).
 - ▶ Leads to ‘interior solutions’.
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OVERVIEW

1. Lagrangian Method
2. Second Order Conditions
3. Bordered Hessian

1. LAGRANGIAN METHOD

- ▶ Let f and h be C^1 functions.
- ▶ Suppose $x^* = (x_1^*, x_2^*)$ is a **solution** to the problem:

$$\begin{aligned} \max & f(x_1, x_2) \\ \text{s.t. } & h(x_1, x_2) = c \end{aligned}$$

- ▶ Consider (x_1^*, x_2^*) are **not** critical points of h .
- ▶ Then μ^* is a real number such that (x_1^*, x_2^*, μ^*) **is** a critical point of the following Lagrangian function.

$$L(x_1, x_2, \mu) \equiv f(x_1, x_2) - \mu[h(x_1, x_2) - c]$$

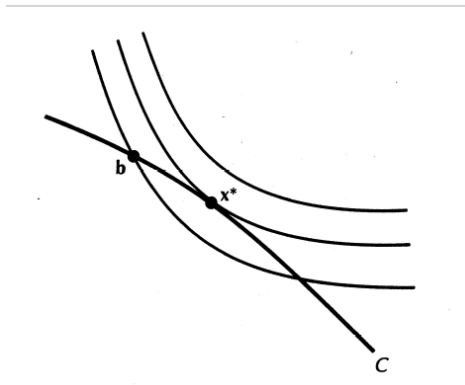
- ▶ That is...

$$\frac{\partial L}{\partial x_1} = 0, \quad \frac{\partial L}{\partial x_2} = 0, \quad \frac{\partial L}{\partial \mu} = 0$$

1. LAGRANGIAN METHOD

► Intuition:

$$\mu = -\frac{\frac{\partial f}{\partial x_1}(x^*)}{\frac{\partial f}{\partial x_2}(x^*)} = -\frac{\frac{\partial h}{\partial x_1}(x^*)}{\frac{\partial h}{\partial x_2}(x^*)}$$



1. LAGRANGIAN METHOD

- So, we have two equations represented by $\nabla f(x^*) = \mu^* \nabla h(x^*)$.

$$\frac{\partial f}{\partial x_1}(x^*) - \mu^* \frac{\partial h}{\partial x_1}(x^*) = 0$$

$$\frac{\partial f}{\partial x_2}(x^*) - \mu^* \frac{\partial h}{\partial x_2}(x^*) = 0$$

1. LAGRANGIAN METHOD

- Consider there may be **many** constraints in the constraint set C_h with local max/min $x^* \in C_h$.

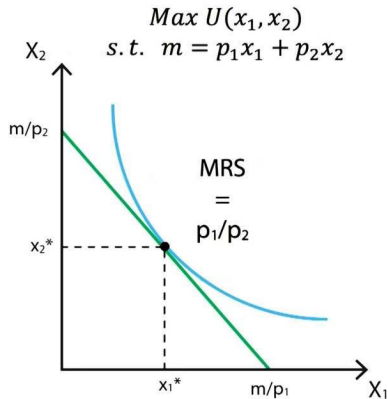
$$C_h = \{x = (x_1, \dots, x_n) : (h_1(x) = a_1), \dots, h_m(x) = a_m)\}$$

- If x^* is not the critical point of $h = (h_1, \dots, h_m)$ (i.e., $\text{rank}(Dh(x^*)) < m$) – unique value for all constraints).
- Then there are μ_1^*, \dots, μ_m^* real numbers such that $(x_1^*, \dots, x_n^*, \mu_1^*, \dots, \mu_m^*)$ is the critical point of the Lagrangian function:

$$L(x^*, \mu^*) \equiv f(x) - \mu_1[h(x) - a_1] - \dots - \mu_m[h(x) - a_m]$$

- Alternatively, all partials for x_n and μ_m are set to zero.

APPLICATION: CONSUMER UTILITY MAXIMIZATION



DEMONSTRATION: LAGRANGIAN

Question:

Classic utility maximization: $u(x_1, x_2) = x_1^\alpha x_2^\beta$ s.t.
 $w \geq p_1 x_1 + p_2 x_2$.

Answer:

$$\mathcal{L} = x_1^\alpha x_2^\beta + \lambda(w - p_1 x_1 - p_2 x_2)$$

$$\mathcal{L}_x = \alpha x_1^{\alpha-1} x_2^\beta - \lambda p_1 = 0$$

$$\mathcal{L}_y = \beta x_1^\alpha x_2^{\beta-1} - \lambda p_2 = 0$$

$$\mathcal{L}_\lambda = w - p_1 x_1 - p_2 x_2 = 0$$

DEMONSTRATION: LAGRANGIAN

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 $w \geq p_1 x_1 + p_2 x_2$.

Answer:

$$\mathcal{L} = x_1^\alpha x_2^\beta + \lambda(w - p_1 x_1 - p_2 x_2)$$

$$\mathcal{L}_x = \alpha x_1^{\alpha-1} x_2^\beta + \lambda(-p_1) = 0$$

$$\mathcal{L}_y = \beta x_1^\alpha x_2^{\beta-1} + \lambda(-p_2) = 0$$

$$\mathcal{L}_\lambda = w - p_1 x_1 - p_2 x_2 = 0$$

DEMONSTRATION: LAGRANGIAN

Answer:

$$\lambda = \frac{\alpha}{p_1} x_1^{\alpha-1} x_2^{\beta}$$

$$\lambda = \frac{\beta}{p_2} x_1^{\alpha} x_2^{\beta-1}$$

$$\frac{\alpha}{p_1} x_1^{\alpha-1} x_2^{\beta} = \frac{\beta}{p_2} x_1^{\alpha} x_2^{\beta-1}$$

$$x_1^{\alpha-1} x_1^{-\alpha} = \frac{\beta}{\alpha} \frac{p_1}{p_2} x_2^{\beta-1} x_2^{-\beta}$$

$$x_1^{-1} = \frac{\beta}{\alpha} \frac{p_1}{p_2} x_2^{-1}$$

$$x_1 = \frac{\alpha}{\beta} \frac{p_2}{p_1} x_2$$

$$x_2 = \frac{\beta}{\alpha} \frac{p_1}{p_2} x_1$$

DEMONSTRATION: LAGRANGIAN

Answer:

Plug back into the constraint ...

$$w = p_1 x_1 + p_2 \frac{p_1}{p_2} \frac{\beta}{\alpha} x_1$$

$$w = x_1 p_1 \left(1 + \frac{\beta}{\alpha}\right)$$

$$x_1^* = \frac{w}{p_1 \left(1 + \frac{\beta}{\alpha}\right)}$$

By analog to save space ...

$$x_2^* = \frac{w}{p_2 \left(1 + \frac{\alpha}{\beta}\right)}$$

PRACTICE: LAGRANGIAN

1. Max $ax_1^2 + bx_2^2$ s.t. $x_1 + x_2 = 1$.

PRACTICE: LAGRANGIAN

1. Max $ax_1^2 + bx_2^2$ s.t. $x_1 + x_2 = 1$.

Answer: [◀ Show Work](#)

$$x_1^* = \frac{1}{1 + \frac{a}{b}}$$

$$x_2^* = \frac{1}{1 + \frac{b}{a}}$$

PRACTICE: LAGRANGIAN

1. Max $ax_1^2 + bx_2^2$ s.t. $x_1 + x_2 = 1$.
2. Min $x_1^2x_2$ s.t. $2x_1^2 + x_2^2 = 3$.

PRACTICE: LAGRANGIAN

1. Max $ax_1^2 + bx_2^2$ s.t. $x_1 + x_2 = 1$.
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Answer: [◀ Show Work](#)

$$x_1^* = 1$$

$$x_2^* = 1$$

PRACTICE: LAGRANGIAN

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2. Min. $x_1^2x_2$ s.t. $2x_1^2 + x_2^2 = 3$.
3. Max $\pi(q_1, q_2) = p_1q_1 + p_2q_2$ s.t. $B \geq 0.3q_1 + 0.5q_2$.

PRACTICE: LAGRANGIAN

1. Max $ax_1^2 + bx_2^2$ s.t. $x_1 + x_2 = 1$.
2. Min. $x_1^2x_2$ s.t. $2x_1^2 + x_2^2 = 3$.
3. Max $\pi(q_1, q_2) = p_1q_1 + p_2q_2$ s.t. $B \geq 0.3q_1 + 0.5q_2$.

Answer: [◀ Show Work](#)

$$q_1^* = \frac{B}{0.3(1 + p_2)}$$
$$q_2^* = \frac{B}{0.5(1 + p_1)}$$

2. SECOND ORDER CONDITIONS

- ▶ To ensure a maximum, we need to know that the second differential of the objective function f at the critical point is **decreasing along the constraint**.
- ▶ Let $y = f(x_1, x_2(x_1))$ be the value of the objective function subject to the constraint.

- ▶ By the implicit function theorem:

$$\frac{dx_2}{dx_1} = \frac{\partial h / \partial x_1}{\partial h / \partial x_2}$$

- ▶ By chain rule:

$$\frac{dy}{dx_1} = \frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2} \frac{dx_2}{dx_1} = \frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2} \frac{\partial h / \partial x_1}{\partial h / \partial x_2}$$

3. BORDERED HESSIAN

- ▶ The sufficient condition for a critical point is:

$$\frac{d^2y}{dx_1^2} < 0$$

- ▶ It can be shown that

$$\frac{d^2y}{dx_1^2} = \frac{-1}{(\partial h / \partial x_2)^2} \bar{D}$$

- ▶ Where \bar{D} is the **bordered Hessian** of L .
- ▶ The determinant of the bordered Hessian tests for quasiconcavity or convexity to ensure a local max/min
- ▶ Negative definite assuming $\nabla f(x) \neq 0 \forall x$ and $f(\cdot)$ is strictly quasiconcave.

3. BORDERED HESSIAN

- ▶ Local minima (x_1^*, x_2^*) if $B(x_1^*, x_2^*, \lambda^*) < 0$.
- ▶ Local maximum (x_1^*, x_2^*) if $B(x_1^*, x_2^*, \lambda^*) > 0$.

$$B(x_1, x_2, \lambda) = \det \left| \begin{bmatrix} 0 & \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} \\ \frac{\partial h}{\partial x_1} & \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} \\ \frac{\partial h}{\partial x_2} & \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} \end{bmatrix} \right| > 0$$

Review

UNCONSTRAINED OPTIMIZATION

1. First Order Conditions
2. Second Order Conditions
3. Global Min and Max

(EQUALITY) CONSTRAINED OPTIMIZATION

1. Lagrangian Method
2. Second Order Conditions
3. Bordered Hessian

ASSIGNMENT

- ▶ Assignment:
 - ▶ Problem Set 09 (PS09)
 - ▶ Solution set will be available following end of Lecture 10
- ▶ Struggling?
 1. Read the ‘Encouraged Reading’
 2. Review ‘Supplementary material’
 3. Reach out directly

FOC QUESTION 1 ANSWER:

[◀ QUESTION](#)

$$f_x = 6x - 8 = 0$$

$$x^* = \frac{8}{6}$$

FOC QUESTION 2 ANSWER:

◀ QUESTION

$$f_y = 2(1 - x)(-1) - 2$$

$$f_y = 2x - 4 = 0$$

$$y^* = 2$$

FOC QUESTION 3 ANSWER:

◀ QUESTION

$$\pi_Q = 7 - \frac{2}{2}Q - 0.5 = 0$$

$$Q^* = 6.5$$

LAGRANGIAN QUESTION 1 ANSWER:

◀ QUESTION

$$\mathcal{L} = ax_1^2 + bx_2^2 + \lambda(1 - x_1 - x_2)$$

$$x_1 = \frac{b}{a}x_2$$

$$x_2 = \frac{a}{b}x_1$$

Plug back into the constraint.

LAGRANGIAN QUESTION 2 ANSWER:

◀ QUESTION

$$\mathcal{L} = x_1^2 x_2 + \lambda(2x_1^2 + x_2^2 - 3)$$

$$x_1 = x_2$$

Plug back into the constraint.

LAGRANGIAN QUESTION 3 ANSWER:

◀ QUESTION

$$\mathcal{L} = p_1 q_1 + p_2 q_2 + \lambda(B - 0.3q_1 - 0.5q_2)$$

$$q_1 = \frac{0.5}{0.3} \frac{p_1}{p_2} q_2$$

$$q_2 = \frac{0.3}{0.5} \frac{p_2}{p_1} q_1$$

Plug back into the constraint.