# Lecture 07
# Linear Algebra

Ryan McWay[†]

[†] Applied Economics,
University of Minnesota

Mathematics Review Course, Summer 2023
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# Last Lecture Review

- Matrices:
  - Matrix Operators
  - Rank & Trace
  - The Determinant
  - Positive and Negative Definite Matrices
  - Linear Independence

# Review Assignment

1. Problem Set 06 solutions are available on Github.
2. Any issues or problems **You** would like to discuss?

# Daily Icebreaker

- Attendance via prompt:
  - Name
  - Daily Icebreaker: Your parents inform you that you are an adopted superhero. What super-human ability do you possess?

# Topic: Linear Algebra

# Motivation

- General background
  - The algebra of manipulating matrices.
  - A compact, efficient method for working with systems of equations (e.g., a spreadsheet).
- Why do economists' care?
  - This is the foundation of how economists manipulate data.
- Application in this career
  - Important applications in econometrics.
  - Useful whenever you are creating or modifying an estimator.

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# Overview

1. Systems of Linear Equations
2. Gaussian Elimination
3. Linear Operators
4. Existence of a Solution
5. Cramer's Rule
6. Eigenvalues and the Characteristic Equation
7. Leading Principle Minors
8. Regression as a Matrix
9. Centering Matrix
10. Residual Maker

# 1. Systems of Linear Equations

- Linear Functions: $y = Ax$ with elements $y_i$ of $y$ such that $y_i = a_i^T x$.

$$a_i = \frac{\partial y_i}{\partial x} = \frac{\partial}{\partial x}(a_i^T x)$$

- Linear System: $m$ equations for $n$ unknown variables.

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

# 2. Gaussian Elimination

- Augmented Matrix

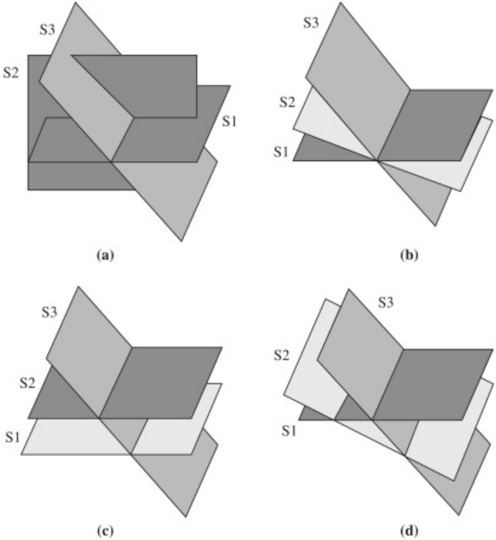
$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n}|b_1 \\ a_{21} & a_{22} & \cdots & a_{2n}|b_2 \\ \vdots & \vdots & \ddots & \vdots|\vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn}|b_m \end{pmatrix}$$

# 2. Gaussian Elimination

- Reduced Row Echelon Form
  - Interchange any two rows.
  - Change row by adding a multiple of another row.
  - Multiply each element in a row by a non-zero scalar.

$$\begin{pmatrix} 1 & 0 & \cdots & 0|b_1 \\ 0 & 1 & \cdots & 0|b_2 \\ \vdots & \vdots & \ddots & \vdots|\vdots \\ 0 & 0 & \cdots & 1|b_m \end{pmatrix}$$

# Reduced Row Echelon Form



# Demonstration: Row Echelon Form

*Question:*

Reduced REF for $\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 2 \\ 1 & -3 & -1 \end{bmatrix}$

*Answer:*

$$R_2 \leftarrow R_2 - 3R_1$$
$$R_3 \leftarrow R_3 - R_1$$
$$R_2 \leftarrow R_2 + R_3$$
$$R_1 \leftarrow R_1 + R_2$$
$$R_3 \leftarrow R_3 + 2R_2$$
$$R_3 \leftarrow \frac{-1}{17} R_3$$
$$R_1 \leftarrow R_1 + 5R_3$$
$$R_2 \leftarrow R_2 + 7R_3$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

# Demonstration: Row Echelon Form

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*Answer:*

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$$\rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

# Practice: Row Echelon Form

1. Reduced REF for $\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 9 \end{array}\right)$.

# Practice: Row Echelon Form

*1.* Reduced REF for $\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 9 \end{array}\right)$.

*Answer:* Show Work

$$\left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right)$$

# Practice: Row Echelon Form

1. Reduced REF for $\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 9 \end{array}\right)$.
2. Reduced REF for $\left(\begin{array}{ccc|c} -1 & 1 & 0 & 1 \\ -2 & -3 & -1 & -2 \\ -3 & -1 & -2 & -1 \end{array}\right)$.

# Practice: Row Echelon Form

1. Reduced REF for $\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 9 \end{array}\right)$.
2. Reduced REF for $\left(\begin{array}{ccc|c} -1 & 1 & 0 & 1 \\ -2 & -3 & -1 & -2 \\ -3 & -1 & -2 & -1 \end{array}\right)$.

*Answer:* Show Work

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{-1}{3} \\ 0 & 1 & 0 & \frac{2}{3} \\ 0 & 0 & 1 & \frac{2}{3} \end{array}\right)$$

# Practice: Row Echelon Form

1. Reduced REF for $\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 9 \end{array}\right)$.
2. Reduced REF for $\left(\begin{array}{ccc|c} -1 & 1 & 0 & 1 \\ -2 & -3 & -1 & -2 \\ -3 & -1 & -2 & -1 \end{array}\right)$.
3. Reduced REF for $\left(\begin{array}{ccc|c} 3 & -3 & -2 & -1 \\ 0 & 2 & -3 & -3 \\ 3 & 3 & 2 & -3 \end{array}\right)$.

# Practice: Row Echelon Form

1. Reduced REF for $\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 9 \end{array}\right)$.
2. Reduced REF for $\left(\begin{array}{ccc|c} -1 & 1 & 0 & 1 \\ -2 & -3 & -1 & -2 \\ -3 & -1 & -2 & -1 \end{array}\right)$.
3. Reduced REF for $\left(\begin{array}{ccc|c} 3 & -3 & -2 & -1 \\ 0 & 2 & -3 & -3 \\ 3 & 3 & 2 & -3 \end{array}\right)$.

*Answer:* Show Work

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{-2}{3} \\ 0 & 1 & 0 & \frac{-9}{13} \\ 0 & 0 & 1 & \frac{7}{13} \end{array}\right)$$

# 3. Linear Operators

- Product Rule:

$$\frac{\partial a^T b}{\partial x} = \frac{\partial a^T}{\partial x} b + \frac{\partial b^T}{\partial x} a$$

- Quadratic Form:

$$x^T A x = \sum_{i=1}^{n} \sum_{i=1}^{n} x_i x_j a_{ij}$$

# 4. Existence of Solution

- Important to know that a solution for a system exists.
- A system of linear equations with coefficient matrix $A$ and an augmented matrix $\hat{A}$ has a solution iff:

$$rank\hat{A} = rankA$$

- There are infinite solutions if $\#rows(A) < \#cols(A)$.
  - More unknown variables than observations.
- Non-singular Square Matrix: Ensure only one solutions exists iff

$$\#rows(A) = \#cols(A) = rank(A)$$

# 5. Cramer's Rule

- A unique solution $x = (x_1, \cdots, x_n)$ for $n \times n$ system $Ax = b$ is

$$x_i = \frac{detB_i}{detA} \forall i = 1, \dots, n$$

- Where $B_i$ replaces the $i$'th column of $A$ with the $b$ vector.
- Cramer's rule is an alternative to Gaussian elimination. It is an analog to 'partialing' out the effect for only 1 dimension (i.e., variable) in a system of equations.

# 5. Cramer's Rule

**Cramer's Rule**

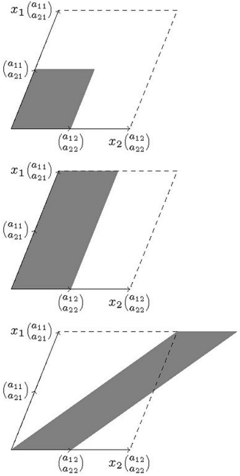
$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

$$\text{Let } D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

If $D \neq 0$ then

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{D} \qquad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{D} \qquad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{D}$$

# CRAMER'S RULE = STRETCH SCALAR



# Demonstration: Cramer's Rule

*Question:*

Solve for $x$ given

$$\begin{aligned} 12x + 3y =& 15 \\ 2x - 3y =& 13 \end{aligned}$$

.

*Answer:*

$$x = \frac{D_x}{D} = \frac{\begin{vmatrix} 15 & 3 \\ 13 & -3 \end{vmatrix}}{\begin{vmatrix} 12 & 3 \\ 2 & -3 \end{vmatrix}} = \frac{-45 - 39}{-36 - 6} = \frac{-84}{-42} = 2$$

# Demonstration: Cramer's Rule

*Question:*

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# Practice: Cramer's Rule

1. Solve for $x, y$ given $\left(\begin{array}{cc|c} -2 & 1 & 3 \\ 8 & 4 & 9 \end{array}\right)$.

# Practice: Cramer's Rule

*1.* Solve for $x, y$ given $\left(\begin{array}{cc|c} -2 & 1 & 3 \\ 8 & 4 & 9 \end{array}\right)$.

*Answer:* Show Work

$$x = \frac{-3}{16}$$
$$y = \frac{21}{8}$$

# PRACTICE: CRAMER'S RULE

1. Solve for $x, y$ given $\left(\begin{array}{cc|c} -2 & 1 & 3 \\ 8 & 4 & 9 \end{array}\right)$.
2. Solve for $x, y, z$ given $\left(\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 3 & -2 & 1 & -5 \\ 1 & 3 & -2 & 14 \end{array}\right)$.

# Practice: Cramer's Rule

1. Solve for $x, y$ given $\left(\begin{array}{cc|c} -2 & 1 & 3 \\ 8 & 4 & 9 \end{array}\right)$.
2. Solve for $x, y, z$ given $\left(\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 3 & -2 & 1 & -5 \\ 1 & 3 & -2 & 14 \end{array}\right)$.

*Answer:* Show Work

$$\begin{aligned} x &= 1 \\ y &= 3 \\ z &= -2 \end{aligned}$$

# Practice: Cramer's Rule

1. Solve for $x, y$ given $\left(\begin{array}{cc|c} -2 & 1 & 3 \\ 8 & 4 & 9 \end{array}\right)$.
2. Solve for $x, y, z$ given $\left(\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 3 & -2 & 1 & -5 \\ 1 & 3 & -2 & 14 \end{array}\right)$.
3. Solve for $x, y, z$ given $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 11 \\ 1 & -2 & 1 & 0 \end{array}\right)$.

# Practice: Cramer's Rule

1. Solve for $x, y$ given $\left(\begin{array}{cc|c} -2 & 1 & 3 \\ 8 & 4 & 9 \end{array}\right)$.
2. Solve for $x, y, z$ given $\left(\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 3 & -2 & 1 & -5 \\ 1 & 3 & -2 & 14 \end{array}\right)$.
3. Solve for $x, y, z$ given $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 11 \\ 1 & -2 & 1 & 0 \end{array}\right)$.

*Answer:* Show Work

$$\begin{aligned} x &= 1 \\ y &= 2 \\ z &= 3 \end{aligned}$$

# 6. Eigenvalues and the Characteristic Equation

- Characteristic Equation

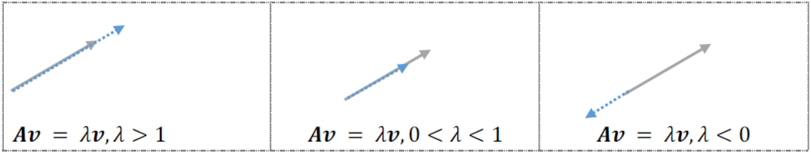
$$Ac = \lambda c$$

- Characteristic vectors: $(c, \lambda)$
- Eigenvectors: $c$
- Eigenvalues: $\lambda$

$$Ac = \lambda Ic \iff (A - \lambda I)c = 0$$

- Homogeneous system has non-zero solution if it is singular and has zero determinant: $det(A - \lambda I) = 0$

# Eigenvectors Scaled by Eigenvalues



# Demonstration: Eigenvalues

*Question:*

Find the Eigenvalues and Eigenvector for $A = \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}$.

*Answer:*

For the Eigenvalues:

$$\begin{aligned} det(A - \lambda I) &= 0 \\ det(\begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) &= 0 \\ det(\begin{bmatrix} -5-\lambda & 2 \\ -7 & 4-\lambda \end{bmatrix}) &= 0 \\ \lambda^2 + \lambda - 6 &= 0 \\ &= \{2, -3\} \end{aligned}$$

# Demonstration: Eigenvalues

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# Demonstration: Eigenvalues

*Question:*

Find the Eigenvalues and Eigenvector for $A = \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}$.

*Answer:*

For the Eigenvectors:

$$\left( \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -7 & 2 \\ -7 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & \frac{2}{7} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\textbf{Solution:} \quad = s \begin{bmatrix} \frac{2}{7} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 7 \end{bmatrix}$$

# Demonstration: Eigenvalues

*Question:*

Find the Eigenvalues and Eigenvector for $A = \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}$.

*Answer:*

For the Eigenvectors:

$$\begin{aligned}
\left( \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} -7 & 2 \\ -7 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\rightarrow \begin{bmatrix} 1 & \frac{2}{7} \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\textbf{Solution:} \quad &= s \begin{bmatrix} \frac{2}{7} & 1 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 7 \end{bmatrix}
\end{aligned}$$

# Practice: Eigenvalues

1. Find the Eigenvalues for $\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$.

# Practice: Eigenvalues

*1.* Find the Eigenvalues for $\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$.

*Answer:* Show Work

$$\lambda = \{5, -2\}$$

# Practice: Eigenvalues

1. Find the Eigenvalues for $\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$.
2. Find the Eigenvalues for $\begin{bmatrix} 1 & -3 & -7 \\ 0 & -2 & -6 \\ 0 & 2 & 5 \end{bmatrix}$.

# PRACTICE: EIGENVALUES

1. Find the Eigenvalues for $\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$.
2. Find the Eigenvalues for $\begin{bmatrix} 1 & -3 & -7 \\ 0 & -2 & -6 \\ 0 & 2 & 5 \end{bmatrix}$.

*Answer:* Show Work

$$\lambda = \{1, 2\}$$

# Practice: Eigenvalues

1. Find the Eigenvalues for $\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$.
2. Find the Eigenvalues for $\begin{bmatrix} 1 & -3 & -7 \\ 0 & -2 & -6 \\ 0 & 2 & 5 \end{bmatrix}$.
3. Find the Eigenvalues for $\begin{bmatrix} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{bmatrix}$.

# Practice: Eigenvalues

1. Find the Eigenvalues for $\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$.
2. Find the Eigenvalues for $\begin{bmatrix} 1 & -3 & -7 \\ 0 & -2 & -6 \\ 0 & 2 & 5 \end{bmatrix}$.
3. Find the Eigenvalues for $\begin{bmatrix} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{bmatrix}$.

*Answer:* Show Work

$$\lambda = \{5, 10\}$$

# 7. Leading Principle Minors

- A way to test for matrix definiteness.
- Leading Principal Sub-matrix: Let $A$ be a $N \times N$ matrix. The $k$'th order principal sub-matrix of $A$ obtained by deleting the last $N - K$ rows and the last $N - K$ columns.
- Leading Principal Minor: The determinant of the $K$'th order leading principal sub-matrix.
- Ex., for a $3 \times 3$ matrix, the leading principal minors are:

$$|a_{11}|, \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

# 7. Leading Principle Minors

- Positive Definite: Iff all $N$ leading principal minors are $\geq 0$.
- Negative Definite: Iff the $N$ leading principal minors alternate signs

$$det(A_1) < 0, det(A_2) > 0, det(A_3) < 0 \text{ , etc.}$$

- Indefinite: Leading principal minors follow any other order.

# 8. Regression as a Matrix

- A linear OLS model:

$$Y_{N\times 1} = X_{N\times K}\beta_{K\times 1} + e_{N\times 1}$$

- With the goal of selecting $\hat{\beta}$ that minimizes squared predicted errors.

$$\hat{e}^T\hat{e} = (Y - X\hat{\beta})^T(Y - X\hat{\beta})$$

# 8. Regression as a Matrix

- So, taking the first derivative we can get:

$$X^T X\hat{\beta} - X^T Y = 0$$
$$\Longrightarrow$$
$$-X^T(Y - X\hat{\beta}) = -X^T\hat{e} = 0$$

- And the solution to OLS is:

$$\hat{\beta} = (X^T X)^{-1} X^T Y$$

# 9. Centering Matrix

- It is common in statistics to transform data to be 'deviations from the mean'.
- This can be done by creating the "centering matrix".
- First, create a multiplier $\frac{1}{N}ii^T$.

$$\frac{1}{N}ii^T = \frac{1}{N}\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}\begin{pmatrix}1 & 1 & \cdots & 1\end{pmatrix} = \begin{pmatrix}\frac{1}{N} & \frac{1}{N} & \cdots & \frac{1}{N}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{1}{N} & \frac{1}{N} & \cdots & \frac{1}{N}\end{pmatrix}$$

- Then define vectors of means.

$$i\bar{x} = \frac{1}{N}ii^T x = \begin{pmatrix}\frac{1}{N} & \frac{1}{N} & \cdots & \frac{1}{N}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{1}{N} & \frac{1}{N} & \cdots & \frac{1}{N}\end{pmatrix}\begin{pmatrix}x_1\\x_2\\\vdots\\x_N\end{pmatrix} = \begin{pmatrix}\bar{x}\\\bar{x}\\\vdots\\\bar{x}\end{pmatrix}$$

# 9. Centering Matrix

- The vector of derivatives can be expressed as:

$$x - i\bar{x} = \begin{pmatrix} x_1 - \bar{x} \\ x_2 - \bar{x} \\ \vdots \\ x_N - \bar{x} \end{pmatrix} = x - \frac{1}{N}ii^T x = (I - \frac{1}{N}ii^T)x = M^0 x$$

- Summing these derivatives can be written as follows:

$$\sum_{i=1}^{N}(x_i - x)^2 = (x - i\bar{x})^T(x - i\bar{x}) = (M^0 x)^T(M^0 x) = x^T M^{0T} M^0 x$$

# 10. Residual Maker

- Define
$$M = I - X(X^T X)^{-1} X^T$$

- Then
$$\hat{e} = Y - X\hat{\beta} = (I - X(X^T X)^{-1} X^T)Y = MY$$

- Residual Maker: $M$

# Review

# Review of Linear Algebra

1. Systems of Linear Equations
2. Gaussian Elimination
3. Linear Operators
4. Existence of a Solution
5. Cramer's Rule
6. Eigenvalues and the Characteristic Equation
7. Leading Principle Minors
8. Regression as a Matrix
9. Centering Matrix
10. Residual Maker

# Assignment

- Readings on Numbers & Functions before Lecture 08:
  - Hammack Ch. 12
  - MWG Appendix M.B, M.C, M.F., & M.I.,
- Assignment:
  - Problem Set 07 (PS07)
  - Solution set will be available following end of Lecture 08
- Struggling?
  1. Read the ‘Encouraged Reading’
  2. Review ‘Supplementary material’
  3. Reach out directly

# Row Echelon Form Question 1 Answer:

◂ Question

$$R_2 \leftarrow R_2 - 4R_1$$
$$R_2 \leftarrow \frac{-1}{3} R_2$$
$$R_1 \leftarrow R_1 - 2R_2$$

# Row Echelon Form Question 2 Answer:

◂ Question

$$
\begin{aligned}
R_1 &\leftarrow - R_1 \\
R_2 &\leftarrow R_2 + 2R_1 \\
R_3 &\leftarrow R_3 + 3R_1 \\
R_2 &\leftarrow R_2 - \frac{3}{2}R_3 \\
R_1 &\leftarrow R_1 + R_2 \\
R_3 &\leftarrow R_3 + 4R_2 \\
R_3 &\leftarrow \frac{1}{6}R_3 \\
R_1 &\leftarrow R_1 - 2R_3 \\
R_2 &\leftarrow R_2 - 2R_3
\end{aligned}
$$

# Row Echelon Form Question 3 Answer:

◂ Question

$$R_1 \leftarrow R_1 - \frac{2}{3}R_3$$
$$R_3 \leftarrow R_3 - 3R_1$$
$$R_2 \leftarrow \frac{1}{2}R_2$$
$$R_1 \leftarrow R_1 + 5R_2$$
$$R_3 \leftarrow R_3 - 18R_2$$
$$R_3 \leftarrow \frac{1}{39}R_3$$
$$R_1 \leftarrow R_1 + \frac{65}{6}R_3$$
$$R_2 \leftarrow R_2 + \frac{3}{2}R_3$$

# Cramer's Rule Question 1 Answer:

◂ Question

$$
\begin{aligned}
D &= -16 \\
D_x &= 3 \\
D_y &= -42 \\
x &= \frac{-3}{16} \\
y &= \frac{21}{8}
\end{aligned}
$$

# Cramer's Rule Question 2 Answer:

Question

$$
\begin{aligned}
D &= -3 \\
D_x &= -3 \\
D_y &= -9 \\
D_z &= 6 \\
x &= 1 \\
y &= 3 \\
z &= -2
\end{aligned}
$$

# Cramer's Rule Question 3 Answer:

◂ Question

$$
\begin{aligned}
D &= 9 \\
D_x &= 9 \\
D_y &= 18 \\
D_z &= 27 \\
x &= 1 \\
y &= 2 \\
z &= 3
\end{aligned}
$$

# Eigenvalues Question 1 Answer:

◂ Question

$$\begin{aligned} \lambda^2 - 3\lambda - 10 &= 0 \\ \lambda &= \{5, -2\} \end{aligned}$$

# Eigenvalues Question 2 Answer:

◂ Question

$$
\begin{aligned}
-\lambda^3 + 4\lambda^2 - 5\lambda + 2 =& 0 \\
-(\lambda - 1)^2(\lambda - 2) =& 0 \\
\lambda =& \{1, 2\}
\end{aligned}
$$

# Eigenvalues Question 3 Answer:

◂ Question

$$\begin{aligned}(\lambda - 5)(\lambda^2 - 20\lambda + 100) =& 0 \\ \lambda =& \{5, 10\}\end{aligned}$$