Lecture 06
Matrices

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Last Lecture Review

- Multi-variate Calculus:
  - Partial Derivatives
  - Total Differentiation
  - Multi-variable Chain Rule
  - Implicit Function Theorem
  - Multi-variable Concavity

Review Assignment

1. Problem Set 05 solutions are available on Github.
2. Any issues or problems **You** would like to discuss?

Daily Icebreaker

- Attendance via prompt:
  - Name
  - Program and track
  - Daily icebreaker subject...

Topic: Matrices

Motivation

- General background
  - A structured way to capture a series of related numbers.
  - This is the original conception of a 'spreadsheet' like you might find in Excel.
- Why do economists' care?
  - We would with tabular datsets primarily.
  - This is the main way with conceptualize information and manipulate it in practice.
- Application in this career
  - Matrices are throughout applied work as they are the foundation of both data storage, as well as how statistical software performs operations.

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Overview

1. Foundations of Matrices
2. Matrix Operators
3. Rank
4. Special Matrices
5. The Determinant
6. Trace
7. Matrix Decomposition
8. Positive and Negative Definite Matrices
9. Linear Independence
10. Chain Rule for Vectors

1. Foundations of Matrices

- Scalar: Single number. e.g., [5]
- Vector: Either a $k \times 1$ column or $1 \times k$ row.

$$a = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{pmatrix}$$

- Matrix: A $k \times r$ rectangular array.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kr} \end{bmatrix}$$

2. Matrix Operators

- Transpose: Flip a matrix about the diagonal.

$$a^T = \begin{pmatrix} a_1 & a_2 & \cdots & a_k \end{pmatrix}$$

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{k1} \\ a_{12} & a_{22} & \cdots & a_{k2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1r} & a_{2r} & \cdots & a_{kr} \end{bmatrix}$$

2. Matrix Operators

- Partition: Divide a matrix into column or row vectors or into smaller matrices.

$$A = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & \cdots & a_r \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{k1} \end{bmatrix} \cdots \begin{bmatrix} a_{1r} \\ a_{2r} \\ \vdots \\ a_{kr} \end{bmatrix}$$

2. Matrix Operators

- Addition:
  - Commutative: $A + B = B + A$
  - Associate: $A + (B + C) = (A + B) + C$
  - Distributive: $(A + B)^T = A^T + B^T$

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1r} + b_{1r} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2r} + b_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} + b_{k1} & a_{k2} + b_{k2} & \cdots & a_{kr} + b_{kr} \end{bmatrix}$$

2. Matrix Operators

- Multiplication by constant

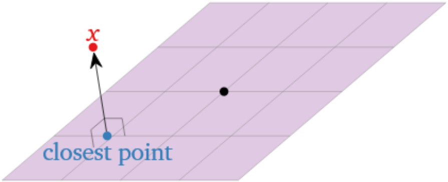
$$A \cdot c = \begin{bmatrix} a_{11}c & a_{12}c & \cdots & a_{1r}c \\ a_{21}c & a_{22}c & \cdots & a_{2r}c \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1}c & a_{k2}c & \cdots & a_{kr}c \end{bmatrix}$$

- Inner product of two $k \times 1$ vectors

$$a^T \cdot b = a_1 \cdot b_1 + a_2 \cdot b_2 + \cdots + a_k \cdot b_k = \sum_{j=1}^{k} a_j b_j$$

2. Matrix Operators

- Orthogonality: A perpendicular (i.e., right angle) in n dimensions.
- Orthogonal vectors are $a^T \cdot b = 0$



2. Matrix Operators

- Cross Product: Need to make sure they are conformable.
  - Conformable: $[kr] \times [r \times s] = [r \times s]$
  - Multiplication is not commutative $A \times B \neq B \times A$

$$A \times B = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_k^T \end{bmatrix} \times \begin{bmatrix} b_1 & b_2 & \cdots & b_s \end{bmatrix}$$

$$= \begin{bmatrix} a_1^T b_1 & a_1^T b_2 & \cdots & a_1^T b_s \\ a_2^T b_1 & a_2^T b_2 & \cdots & a_2^T b_s \\ \vdots & \vdots & \ddots & \vdots \\ a_k^T b_1 & a_k^T b_2 & \cdots & a_k^T b_s \end{bmatrix}$$

2. Matrix Operators

- Inversion: Solve when when matrix is square and non-singular ($\#rows(A) = \#cols(A)$).
- Let $B = A^{-1}$ be the inverse of full-rank $k \times k$ matrix A.
- This satisfies $AB = I_k$.
- Suppose $n \times n$ matrix A is invertible. Then it is non-singular and has a unique solution:

$$\begin{aligned} Ax =& b \\ x =& A^{-1}b \end{aligned}$$

- This is important for determining coefficients in OLS.

3. Rank

- Rank: Number of non-zero rows in the row echelon form.
- $rank = \min(m, n)$
  - Full Rank: # of rows = # of columns
  - $rankA \leq \#rows(A)$
  - $rankA \leq \#cols(A)$
  - $rank\hat{A} \geq rankA$

4. Trace

- Trace: Sum of the diagonal elements of $k \times k$ matrix A
  - $tr(cA) = ctr(A)$
  - $tr(A^T) = tr(A)$
  - $tr(A + B) = tr(A) + tr(B)$
  - $tr(I_k) = k$
  - If conformable, $tr(AB) = tr(BA)$

$$tr(A) = \sum_{i=1}^{k} a_{ii}$$

5. Special Matrices

- Square Matrix: $k = r$
  - Square matrices are symmetric $A = A^T$
  - Called a **diagonal** if all off diagonal elements are zero.
  - Called an **upper diagonal** (or lower) if all elements below (above) the diagonal are zero.
  - Idempotent: $B^2 = BB = B$
- Identity Matrix: Diagonal matrix with only 1's as values in diagonal.

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

- Zero Matrix: A null matrix with only zeros.
  - E.g., $[0]$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

6. The Determinant

- A matrix A is non-singular iff its determinant is non-zero.

$$det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\begin{aligned} det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} =& a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ & - a_{13}a_{22}a_{31} - a_{21}a_{12}a_{33} - a_{11}a_{23}a_{32} \end{aligned}$$

- $detA^T = detA$
- $det(A \cdot B) = detA \cdot detB$
- $det(A + B) \neq detA + detB$

6. The Determinant

- Minor of Matrix: A determinant of a smaller square matrix cut from A by removing one or more rows and columns.

$$
\begin{aligned}
|A| &= a_{11}|M_{11}| - a_{12}|M_{12}| + a_{13}|M_{13}| \\
&= a_{11}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} - a_{12}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + a_{13}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \\
&= a_{11}\cdot\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\cdot\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\cdot\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}
\end{aligned}
$$

- Adjacent of Matrix: $adjA = (-1)^{i+j} \times det(\text{minor of } i, j)$.
- A non-singular matrix has the inversion:

$$A^{-1} = \frac{1}{detA} \cdot adjA$$

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

7. Matrix Decomposition

- Eigenvectors $= c_i$.
- Eigenvalues $= \lambda_i$

$$C = \begin{pmatrix} c_1 & c_2 & \cdots & c_k \end{pmatrix}$$

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k \end{pmatrix}$$

$$Ac_k = \lambda_k c_k$$

$$AC = C\Lambda$$

7. Matrix Decomposition

- The vectors are orthogonal.
- $c_i^T c_i = 1 \implies C^T C = I \implies CC^T = CC^{-1} = I.$
- The diagonalization of A is
$$C^T A C = C^T C \Lambda = I\Lambda = \Lambda$$
- The spectral decomposition
$$A = C\Lambda C^T = \sum_{i=1}^{k} \lambda_i c_i c_i^T$$

8. Positive and Negative Definite Matrices

- Positive Definite: Iff for $k \times k$ real symmetric matrix A, $\forall c \neq 0 \in \mathbb{R}^k$, $c^T A c > 0$.
- Negative Definite: Iff for $k \times k$ real symmetric matrix A, $\forall c \neq 0 \in \mathbb{R}^k$, $c^T A c < 0$.
- Semi-definite: A weak inequality $\geq$, $\leq$ in either case.
- Negative Semi-definite: All diagonal elements must be ≤ 0.

9. Linear Independence

- Homogeneous System: Guaranteed to have at least one solution $x_i = 0 \forall i$ when $b_i = 0 \forall i$

$$A \begin{pmatrix} x_1 \\ \cdots \\ x_n \end{pmatrix} = 0$$

- Linearly Dependent: Iff there is a non-zero solution.
  - Means one of the column vectors a_n can be written as a linear combination of the other vectors.
  - Implies infinite solutions.
  - Short Rank ($\#rows(A) < \#cols(A)$ must be linearly dependent.
- Linearly Independent: Iff the only solution is the zero solution.
- Singular: When a square matrix has a non-zero solution.

10. Chain Rule for Vectors

- Let x, y, z be vectors such that z is a function of y, and y is a function of x.
- We can apply the chain rule noting that with vectors we must chain the results **from the left**:

$$\frac{\partial z}{\partial x} = \frac{\partial y}{\partial x}\frac{\partial z}{\partial y} = \frac{\partial z}{\partial y}\frac{\partial y}{\partial x}$$

10. Chain Rule for Vectors

$$\frac{\partial z}{\partial x} = \begin{bmatrix} \sum_{q=1}^{r} \frac{\partial z_1}{\partial y_q}\frac{\partial y_q}{\partial x_1} & \cdots & \sum_{q=1}^{r} \frac{\partial z_1}{\partial y_q}\frac{\partial y_q}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \sum_{q=1}^{r} \frac{\partial z_m}{\partial y_q}\frac{\partial y_q}{\partial x_1} & \cdots & \sum_{q=1}^{r} \frac{\partial z_m}{\partial y_q}\frac{\partial y_q}{\partial x_n} \end{bmatrix}$$

Practice: Matrices

1.

Review of Matrices

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Assignment

- Readings on Linear Algebra before Lecture 07:
  -
- Assignment:
  - Problem Set 06 (PS06)
  - Solution set will be available following end of Lecture 07
- Struggling?
  1. Read the 'Encouraged Reading'
  2. Review 'Supplementary material'
  3. Reach out directly