# Lecture 08
# Numbers and Functions

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Mathematics Review Course, Summer 2023
University of Minnesota
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# Last Lecture Review

- Linear Algebra:
  - Gaussian Elimination
  - Linear Operators
  - Existence of a Solution
  - Cramer's Rule
  - Eigenvalues
  - Regression as a Matrix

# Review Assignment

1. Problem Set 07 solutions are available on Github.
2. Any issues or problems **You** would like to discuss?

# Daily Icebreaker

- Attendance via prompt:
  - Name
  - Daily Icebreaker: You just won a quiz on a radio show for a trip to a free concert. Which band/artist are you going to see?

# Topic: Numbers

# MOTIVATION

- General background
  - The terminology of mathematics
  - Formalized by the branch of math called 'Number Theory'.
- Why do economists' care?
  - Economists express values, sets, and concepts using numerical (quantitative) values.
- Application in this career
  - Throughout your whole experience.

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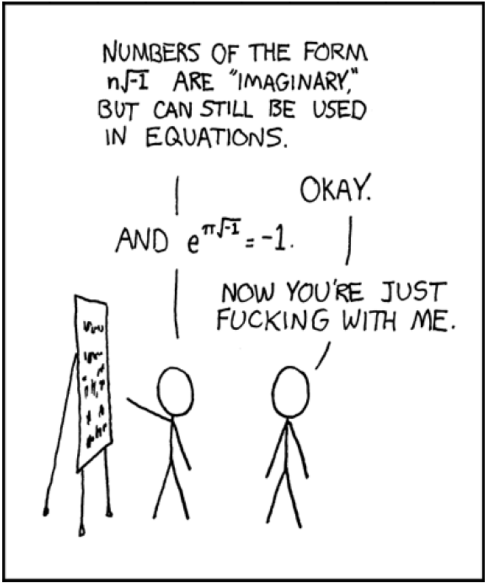
# Overview

1. Common Number Sets
2. Real Numbers
3. Absolute Value and Number Line
4. Triangle Inequality
5. Neighborhoods

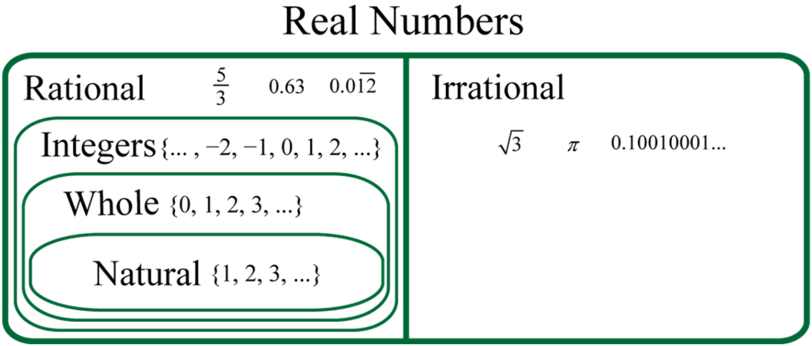
# NUMBERS


NUMBERS OF THE FORM $n\sqrt{-1}$ ARE "IMAGINARY," BUT CAN STILL BE USED IN EQUATIONS.
OKAY.
AND $e^{\pi\sqrt{-1}} = -1$.
NOW YOU'RE JUST FUCKING WITH ME.

# 1. Common Number Sets

- Natural Numbers: $\mathbb{N} = \{1, 2, 3, \cdots\}$.
- Integers: $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.
- Rational Numbers: $\mathbb{Q} = \{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\}$.
- Real Numbers: $\mathbb{R} = \{$ all decimals $\}$.
- Complex Numbers: $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}, i = \sqrt{-1}\}$.

# 1. Common Number Sets

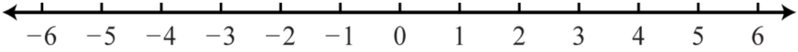


# 2. Real Numbers

- Two binary operators for the Reals
  - Addition: $a + b$
  - Multiplication: $a \cdot b$
- Properties:
  - Commutative: $\forall a, b \in \mathbb{R}, a + b = b + a$.
  - Commutative: $\forall a, b, c \in \mathbb{R}, (a + b) + c = a + (b + c)$.
  - Zero Exists: $\forall a \in \mathbb{R}, a + 0 = a$.
  - Negation Exists: $\forall a \in \mathbb{R}, \exists(-a) \in \mathbb{R} : a + (-a) = 0$.
  - Distributive: $\forall a, b, c \in \mathbb{R}, a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.
- Reciprocal: $\forall a \in \mathbb{R}, \frac{1}{a}$

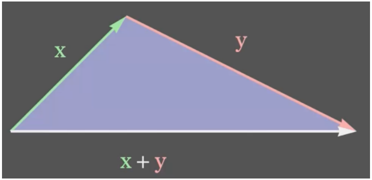
# 3. Absolute Value and Number Line

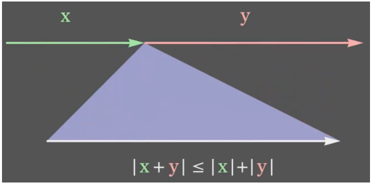
- Absolute Value: $|\pm a| = a$.
- Properties:
  - $\forall a > 0 \in \mathbb{R}, |a| = a$.
  - $\forall a = 0 \in \mathbb{R}, |a| = 0$.
  - $\forall a < 0 \in \mathbb{R}, |a| = -a$.
  - $|ab| = |a||b|$.
  - $|a|^2 = |a^2|$.
  - If $|a| \leq c \iff -c \leq a \leq c$.



# 4. Triangle Inequality

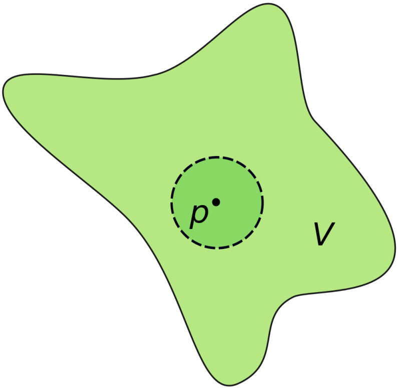
- $\forall a, b \in \mathbb{R} \rightarrow |a+b| \leq |a| + |b|$.
- Corollaries:
  - $||a| - |b|| \leq |a-b|$.
  - $|a-b| \leq |a| + |b|$.
- Visual aid [Click Me].



# 5. Neighborhoods

- Let $a, \varepsilon \in \mathbb{R}, \varepsilon > 0$.
- Let the $\varepsilon$-neighborhood of $a$ be the set $V_\varepsilon(a) := \{x \in \mathbb{R} : |x - a| < \varepsilon\}$.
- I.e., $x$ is a value within the neighborhood of $a$ such that it is within $\varepsilon$ distance from $a$.



# *Topic: Functions*

# Motivation

- General background
  - How we map sets onto other sets.
  - The most common way to represent relationships between variables.
- Why do economists' care?
  - This is the primary way that we represent preferences, utility, and production.
  - Functions are at the core of how theorems are represented.
- Application in this career
  - Both throughout the development and presentation of proofs, as well as in estimating equations from data.

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# Overview

1. Relations
2. Correspondences and Functions
3. Injective & Surjective Functions
4. Composition of Functions
5. Inverse Functions
6. Image and Pre-image
7. Homogeneity
8. Level, Superior & Inferior Sets
9. Euler's Theorem
10. Quasiconcavity & Quasiconvexity
11. Concavity & Convexity
12. Continuity
13. Upper- and Lower-Hemicontinuity
14. Brouwer's Fixed-point Theorem
15. Kakutani's Fixed-point Theorem

# 1. Relations

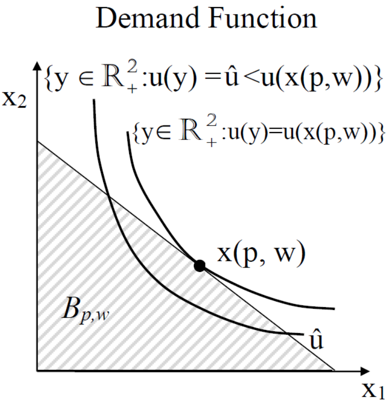
- A collection of ordered-pairs $(s, t)$ has a binary relation $s\mathcal{R}t$ between sets $S$ and $T$.
  - Reflexive: $\forall x \in S, x\mathcal{R}x$.
  - Symmetric: $\forall x, y \in S, x\mathcal{R}y \implies y\mathcal{R}x$.
  - Complete: $\forall x, y \in S \to x\mathcal{R}y \vee y\mathcal{R}x$.
  - Transitive: $\forall x, y, z \in \mathcal{R}, x\mathcal{R}y \wedge y\mathcal{R}z \implies x\mathcal{R}z$.
- Equivalent Relation $(=)$: Is reflexive, symmetric, and transitive.
- Common Relations:
  - Equal: $=$
  - Equivalent: $\equiv$
  - Better than: $>$
  - Less than: $<$

# 2. Correspondences and Functions

- Correspondence: A relation that associates each element of one set (the domain) to the elements of another set (the range).
- Function: A relation that associates each element in the domain to a single, unique element of the range.
- Onto: Every element in the range is mapped into a point in the domain.
- One-to-one: Every element in the range is assigned only a single point in the domain.

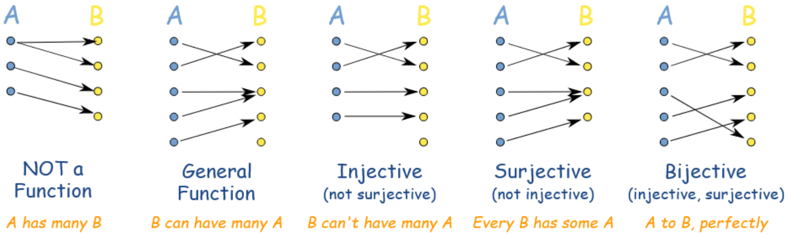
$$f : D \to R$$

# Application: Utility Functions (Correspondences)



# 3. Injective & Surjective Functions

- Function $f : A \to B$:
  - Injective (one-to-one): $\forall a_1, a_2 \in A, a_1 \neq a_2 \implies f(a_1) \neq f(a_2)$.
  - Surjective (onto $B$): $\forall b \in B, \exists a \in A : f(a) = b$.
  - Bijective: Both injective and surjective.



# 4. Composition of Functions

- $f : A \to B$ and $g : B \to C$, then
$$g \circ f(x) = g(f(x)) : A \to C$$

- Follows from associative property of functions:
$$(h \circ g) \circ f = h \circ (g \circ f)$$

- If $f$ and $g$ are surjective, then $g \circ f$ is surjective.

# 5. Inverse Functions

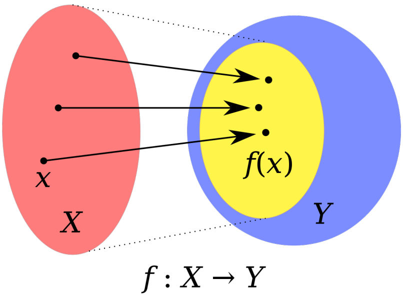
- If $f : A \to B$ is bijective, then the inverse function is $f^{-1} : B \to A$.

$$\begin{aligned} f^{-1} \circ f(x) &= x \\ f \circ f^{-1}(x) &= x \end{aligned}$$

- $f : A \to B$ is bijective iff the inverse $f^{-1}$ is a function $f : B \to A$.

# 6. Image and Pre-image

- Let $f : A \to B$.
  - Image: If $X \subseteq A$ is set $f(X) = \{f(x) : x \in X\} \subseteq B$.
  - Pre-image: If $Y \subseteq B$ is set $f^{-1}(Y) = \{x \in A : f(x) \in Y\} \subseteq A$.



# 7. Homogeneity

- Consider $f(x_1, x_2, \dots, x_N)$ is defined for all $(x_1, x_2, \dots, x_N) \geq 0$.
- Homogeneous: A function $f(x_1, x_2, \dots, x_N)$ is **homogeneous of degree** $r \in \mathbb{Z}$ if $\forall x > 0$:

$$f(t(x_1), t(x_2), \dots, t(x_N)) = t^r f(x_1, x_2, \dots, x_N)$$

- For $f$ homogeneous of degree $r$, then the partial derivative $(x_1, x_2, \dots, x_N)/\partial x_n$ is homogeneous of degree $r - 1$.

# Demonstration: Homogeneity

*Question:*

Determine degree of homogeneity of $f(x_1, x_2) = x_1^{\alpha} x_2^{\beta}$.

*Answer:*

Consider $\gamma > 0$.

$$\begin{aligned} f(\gamma x_1, \gamma x_2) &= (\gamma x_1)^{\alpha} (\gamma x_2)^{\beta} \\ &= \gamma^{\alpha+\beta} x_1^{\alpha} x_2^{\beta} \\ &= \gamma^{\alpha+\beta} f(x_1, x_2) \end{aligned}$$

So, homogeneous of degree $\alpha + \beta$.

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So, homogeneous of degree $\alpha + \beta$.

# Practice: Homogeneity

1. What degree of homogeneity is $f(x_1, x_2) = 30x_1^{1/2}x_2^{3/2} - 2x_1^3x_2^{-1}$?

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*Answer:* Show Work

Homogeneous of degree two.

# Practice: Homogeneity

1. What degree of homogeneity is $f(x_1, x_2) = 30x_1^{1/2}x_2^{3/2} - 2x_1^3x_2^{-1}$?
2. What degree of homogeneity is $f(x, y) = x^{1/2}y^{1/4} + x^2y^{-5/4}$?

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*Answer:* Show Work

Homogeneous of degree $\frac{3}{4}$.

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2. What degree of homogeneity is $f(x, y) = x^{1/2}y^{1/4} + x^2y^{-5/4}$?
3. What degree of homogeneity is the cost function $c(r_1, r_2, q) = r_1^{\alpha}r_2^{\beta}q^2$ in terms of $r_1, r_2$?

# Practice: Homogeneity

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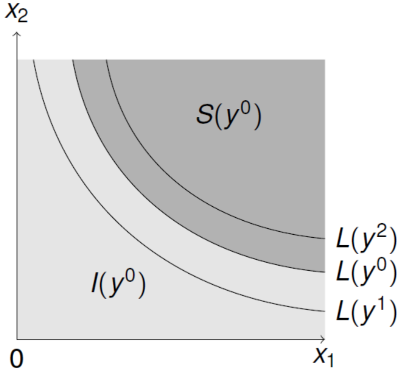
*Answer:* Show Work

Homogeneous of degree $\alpha + \beta$. For a cost function, we know it must be H.D.1, and therefore $\alpha + \beta = 1$.

# 8. Level, Superior, & Inferior Sets

- Level set: $L(y^0) \equiv \{x | x \in D, f(x) = y^0\}$.
- Superior set: $S(y^0) \equiv \{x | x \in D, f(x) \geq y^0\}$.
- Inferior set: $I(y^0) \equiv \{x | x \in D, f(x) \leq y^0\}$.



# 9. Euler's Theorem

- Let $f(x_1, x_2, \ldots, x_N)$ be homogeneous of degree $r$, and differentiable.

$$\nabla f(\bar{x}) \cdot \bar{x} = \sum_{n=1}^{N} \frac{\partial f(\bar{x}_1, \ldots, \bar{x}_N)}{\partial x_n} \bar{x}_n = r f(\bar{x}_1, \ldots, \bar{x}_N)$$

- **Application:** In production theory, Euler's theorem states that a production function homogeneous of degree 1 (CRS) with factors paid their marginal product will have no surplus or deficit in total product.

# Demonstration: Euler's Theorem

*Question:*

Verify using the Cobb-Douglas function $f(x_1, x_2) = Ax_1^{\alpha}x_2^{\beta}$ where $\alpha + \beta = 1$.

*Answer:*

$$\begin{aligned} f(\cdot) =& A(\gamma x_1)^{\alpha}(\gamma x_2)^{\beta} \\ =& A\gamma^{\alpha+\beta}x_1^{\alpha}x_2^{\beta} \\ \because& \alpha + \beta = 1 \\ =& \gamma A x_1^{\alpha}x_2^{\beta} \\ \gamma f(\cdot) =& \gamma\left(Ax_1^{\alpha}x_2^{\beta}\right) \end{aligned}$$

# Demonstration: Euler's Theorem

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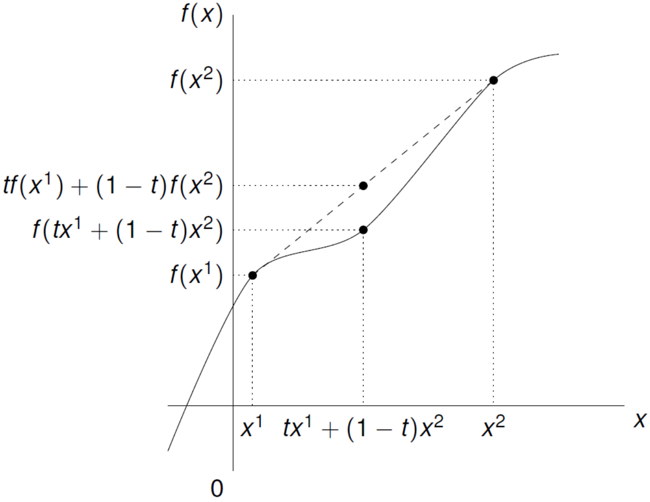
\gamma f(\cdot) =& \gamma \left( Ax_1^{\alpha}x_2^{\beta} \right)
\end{aligned}
$$

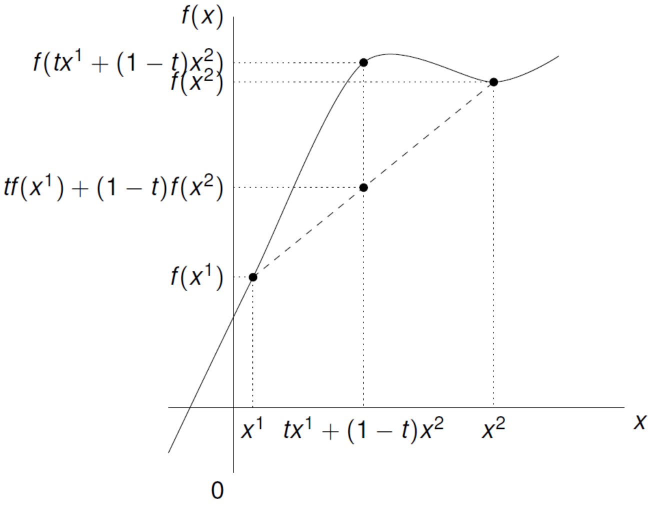
# 10. Quasiconcavity & Quasiconvexity

- A concave function is the **negative** of a convex function.
- Quasiconcavity: $\forall x_1, x_2 \in D, f : D \rightarrow R$ iff
  $$f(tx_1 + (1-t)x_2) \geq min[f(x_1), f(x_2)] \forall t \in [0,1]$$
- Quasiconvexity: $\forall x_1, x_2 \in D, f : D \rightarrow R$ iff
  $$f(tx_1 + (1-t)x_2) \leq max[f(x_1), f(x_2)] \forall t \in [0,1]$$
- These become **strict** when the inequalities hold for all $x_1 \neq x_2$.

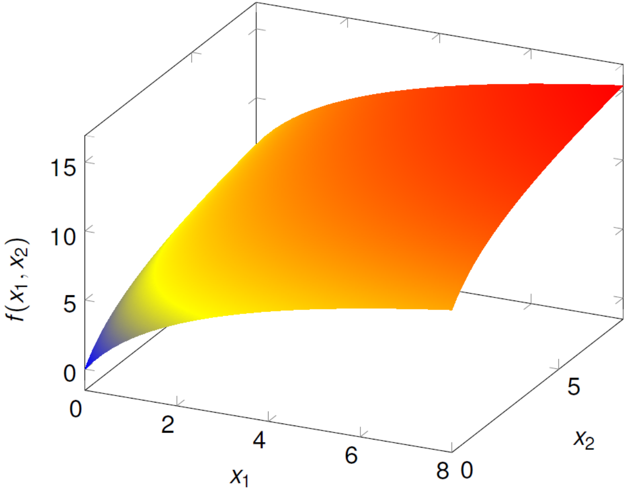
# Quasiconcave And Quasiconvex



# Quasiconcave **But Not** Quasiconvex

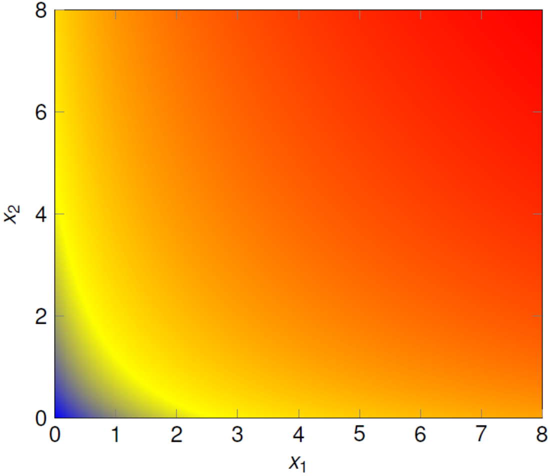


# Quasiconcave in Two Dimensions

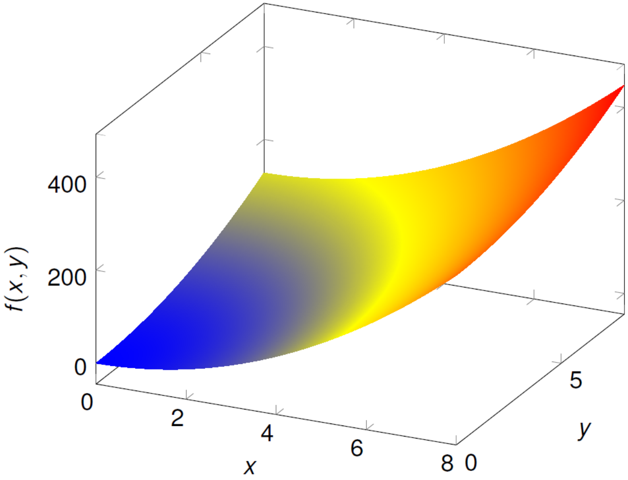


# Quasiconcave in Two Dimensions

- $f : D \to R$ is **quasiconcave** iff $S(y)$ is a **convex** set for all $y \in \mathbb{R}$.



# Quasiconvex in Two Dimensions



# Quasiconvex in Two Dimensions

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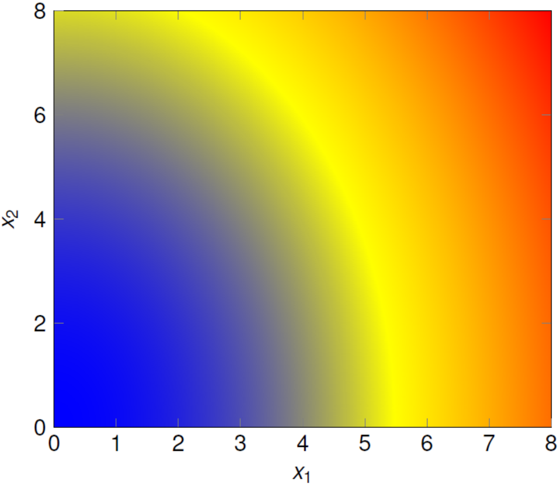


# 11. Concavity & Convexity

- $f$ is defined on a convex subset $D \subset \mathbb{R}^n \forall x_1, x_2 \in D, \forall t \in [0,1]$:
- Concave:

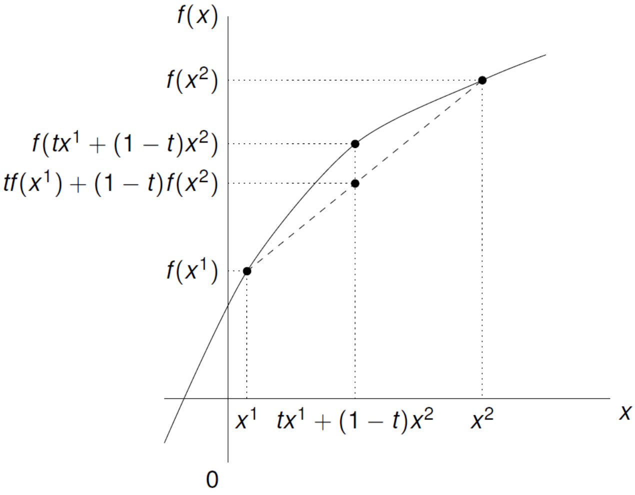
$$f(tx_1 + (1-t)x_2) \geq tf(x_1) + (1-t)f(x_2)$$

- Convex:

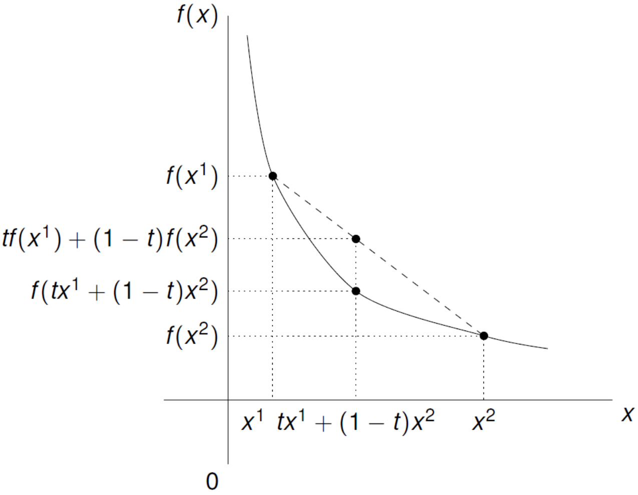
$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$$

- Strict concavity or convexity when the inequality holds.

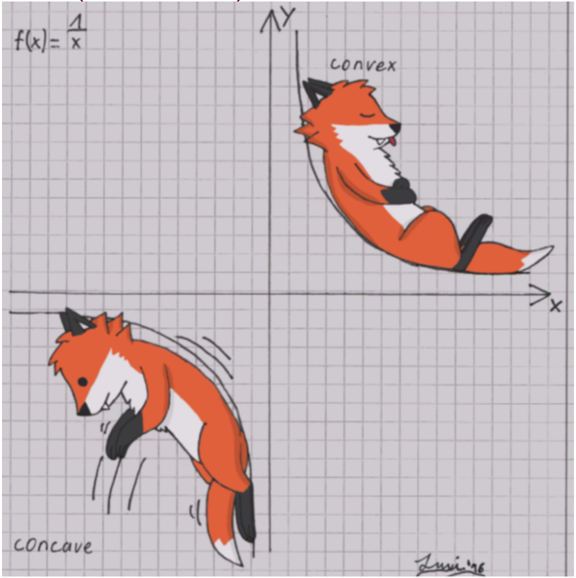
# Concavity



# Convexity



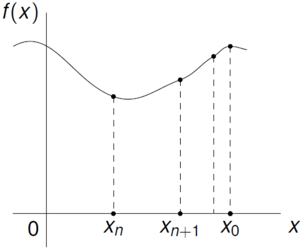
# Concave Up (Convex) & Concave Down

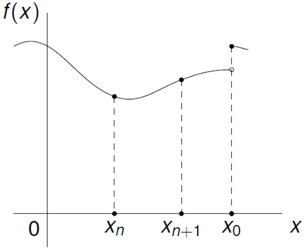


# 12. Continuity

- Continuous: $f : \mathbb{R}^m \to \mathbb{R}^n$ at $x_0 \in \mathbb{R}^m$ if whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence in $\mathbb{R}^m$ which converges to $x_0$, **then** the sequence $\{f(x_n)\}_{n=1}^{\infty}$ in $\mathbb{R}^n$ converges to $f(x_0)$.

$$\forall \varepsilon > 0, \exists \delta > 0 : \forall x \in A, [||x - x_0|| < \delta] \implies [||f(x) - f(x_0)|| < \varepsilon]$$



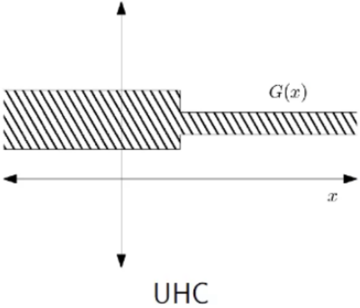
# 13. Upper- and Lower-Hemicontinuity

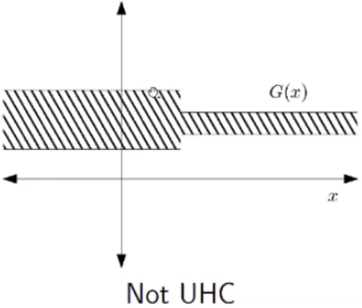
- $A \subset \mathbb{R}^n$ and a closed set $Y \subset \mathbb{R}^n$.
  - Upper Hemicontinuous: Correspondence $f : A \to Y$ if it has a closed graph and the images of compact sets are bounded.

  $\forall B \subset A, f(B) = \{y \in Y : y \in f(x) \exists x \in B\}$ is bounded.

  - Lower Hemicontinous: Correspondence $f : A \to Y$ if for every sequence $x^m \to x \in A$ with $x^m \in A \forall m$, and every $y \in f(x)$, we can find a sequence $y^m \to y$ and an integer $M : y^m \in f(x^m) \forall m > M$.
- Continuous: Both upper- and lower-hemicontinuous.

# 13. Upper-Hemicontinuity



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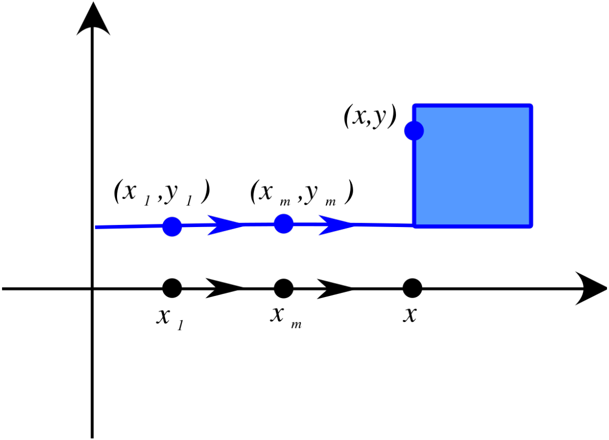

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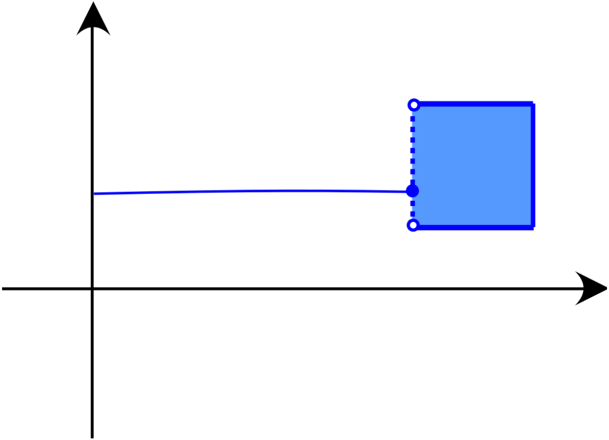
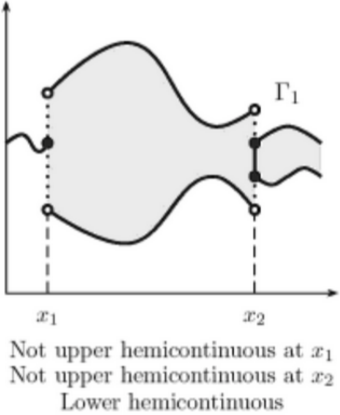
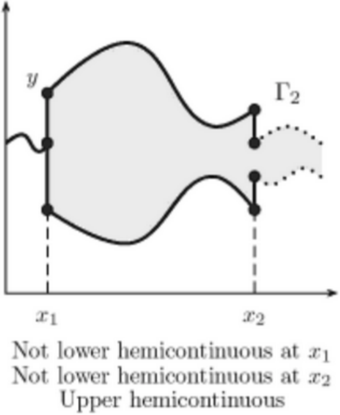
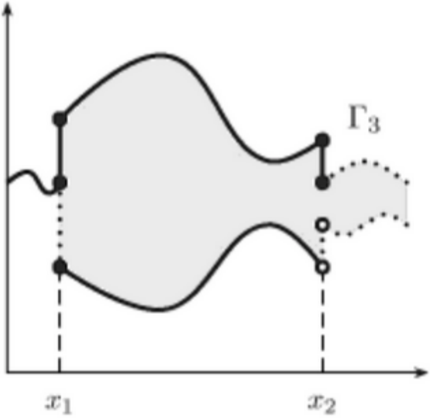


# 13. Upper- and Lower-Hemicontinuity



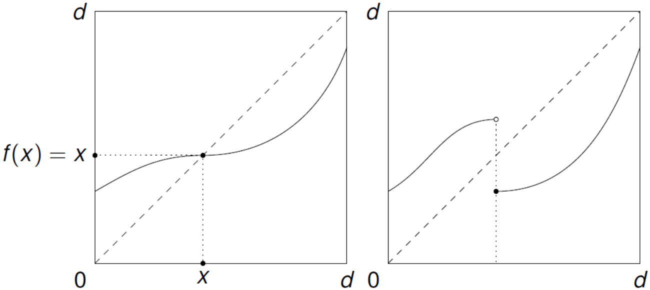
# 13. Not Upper- and Not Lower-Hemicontinuous




Not upper hemicontinuous at $x_1$ and $x_2$
Not lower hemicontinuous at $x_1$ and $x_2$

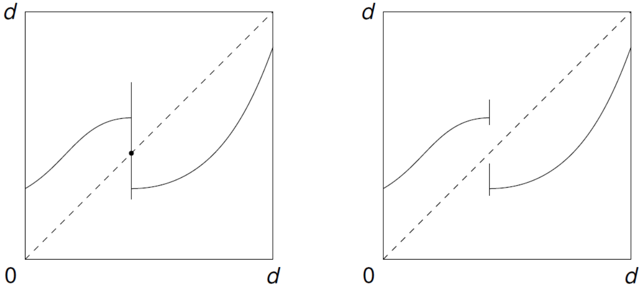
# 14. Brouwer's Fixed-point Theorem

- Suppose $D \subset \mathbb{R}^m$ is non-empty, compact, convex set, and $f : D \to D$ is a **continuous function**.
- Then $f(\cdot)$ has a fixed point, e.g., there is an $x \in D : x = f(x)$.



# 15. Kakutani's Fixed-point Theorem

- Suppose $D \subset \mathbb{R}^m$ is non-empty, compact, convex set, and $f : D \to D$ is a **upper-hemicontinuous correspondence** with the property $f(x) \subset D$ is non-empty and convex for all $x \in D$.
- Then $f(\cdot)$ has a fixed point, e.g., there is an $x \in D : x = f(x)$.



# Review

# Review of Numbers

1. Real Numbers
2. Common Number Sets
3. Absolute Value and Number Line
4. Triangle Inequality
5. Neighborhoods

# Review of Functions

1. Relations
2. Correspondences and Functions
3. Injective & Surjective Functions
4. Composition of Functions
5. Inverse Functions
6. Image and Pre-image
7. Homogeneity
8. Level, Superior & Inferior Sets
9. Euler's Theorem
10. Quasiconcavity & Quasiconvexity
11. Concavity & Convexity
12. Continuity
13. Upper- and Lower-Hemicontinuity
14. Brouwer's Fixed-point Theorem
15. Kakutani's Fixed-point Theorem

# Assignment

- Readings on Optimization before Lecture 09:
  - MWG Appendix M.J., M.K., & M.L.
  - S&B Ch.17, 18, & 19
- Assignment:
  - Problem Set 08 (PS08)
  - Solution set will be available following end of Lecture 09
- Struggling?
  1. Read the 'Encouraged Reading'
  2. Review 'Supplementary material'
  3. Reach out directly

# Homogeneity Question 1 Answer:

◂ Question

$$
\begin{aligned}
f =& \gamma^{1/2+3/2}(\cdot) - \gamma^{2}(\cdot) \\
=& \gamma^{2}(\cdot)
\end{aligned}
$$

# Homogeneity Question 2 Answer:

◂ Question

$$
\begin{aligned}
f =& \gamma^{1/2+1/4}(\cdot) + \gamma^{2-5/4}(\cdot) \\
=& \gamma^{3/4}(\cdot)
\end{aligned}
$$

# Homogeneity Question 3 Answer:

◂ Question

$$c = \gamma^{\alpha+\beta}(\cdot)$$