LOGIC AND PROOFS

1. Definitions and notations

$\forall$	For all...
$\exists$	There exists...
$\nexists$	There does not exist...
$\therefore$	Therefore...
$\because$	Because...
$\neg$	Negation
$\equiv$	Identical to or the same as… For example, we write $f \equiv g$ if $f(x) = g(x)$ for all x
$\Longrightarrow$	A $\Longrightarrow$ B means: “A implies B, “If A then B or "A is sufficient condition for B”
$\Leftrightarrow$	A $\Leftrightarrow$B means “A if and only if B”,“A is equivalent to B” or “A is a necessary and sufficient condition for B”
$A \subset B$	"B strictly contains A" or “A is a proper subset of B”
$A \subseteq B$	“B contains A” or “A is a subset of B”
$\in$ ($\notin$)	In... (Not in...) or an element of... (Not an element of...)
$\blacksquare$	Bonus: End of proof, Q.E.D.

Note: QED is the abbreviation for **quod erat demonstrandum**, translating "Which was to be demonstrated" or "which is what had to be proved".

The last 3 notations have to do with sets. We shall discuss sets in more detail tomorrow. A set is defined by its elements A

Capital letters from the few first letters of the alphabet usually are commonly used to denote properties or object of statements.

A = “ student attending the 2022 Math Review”
B = “student at the University of Minnesota”
C = “student in the department of Applied Economics”

Which statements are true if the following statement is true?

$$A \Rightarrow B$$

Answer:

a. "A implies B
b. "A is sufficient for B"
c. "B is necessary for A"
d. "if A, then B"

A statement is followed by a family of statements. Their names are:

Statement	A	$\Rightarrow$	B
Contrapositive	¬B	$\Rightarrow$	¬A
Converse	B	$\Rightarrow$	A
Inverse	¬A	$\Rightarrow$	¬B

Fill in the table below using the notations in the table above for the 3 statements A, B and C about students in the Math Review class.

	A	B	C
A			
B		$\Leftrightarrow$	
C			

When are two statements or properties implied by each other? Give examples.

More vocabulary

Axiom: statements we assume to be true

e.g. a = b, b = c ⇒ a = c

Theorem: a statement that has been proven to be true.

Corollary: a theorem that follows on from another theorem.

Lemma: a less important theorem that is used to prove another theorem.

3 methods of proof.

1. Direct proof

Example 1. Prove that if m is an even integer and n is any integer, the m*n is an even integer.

Proof:

m is an even integer so ∃ an integer q such that m = 2 ∗ q by the definition of even integer.

m ∗ p = (2 ∗ q) ∗ p = 2 ∗ (q ∗ p) , so m ∗ p is an even integer

Example 2. Prove that if a function mapping real numbers to real numbers and that it is homogenous of degree r, its first derivative is a function of degree r-1.

Note: A function, f(x), is homogeneous of degree r if for any t,

$$f(tx) = t^r f(x)$$

Proof:

2. Proof by contradiction
   Example: Prove that $\sqrt{2}$ is irrational.

   Note : An irrational number is a number that can be expressed as a irreducible ratio of 2 integers.

   Proof. By way of contradiction, suppose that √2 were rational. Then there exist two integers, m and n, that contain no common factors, with √2 = $\frac{m}{n}$ or 2 = $\left(\frac{m}{n}\right)^2$. But then $2n^2 \ = \ m^2$, so m^2 is even because it is twice n^2 . If m^2 is even, though, m is even so m^2 must be divisible by 4, which means that $\frac{m^2}{2}$ is even. Thus n^2 is even and we know that m and n are both divisible by 2, contradicting the claim that m and n contain no common factors. This completes the proof.

3. Proof by induction
   Example: Prove that $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$

   First, consider the case where $n = 1$. Then we have:

$$1 = \frac{1(1+1)}{2} = 1$$

   The statement holds.

   Now consider $n = k \geq 1$ We have the following:

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$$

   Now we show that if this is true, then $n = k + 1$, that is:

$$1 + 2 + 3 + \cdots + k + (\boldsymbol{k} + \mathbf{1}) = \frac{(k+1)\big((k+1)+1\big)}{2} = \frac{(k+1)(k+2)}{2}$$

   We have shown that it is true for n = 1 and true for any integer k+1 if it is true for k, implying that:

   $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

Practice problems

1. Prove that $\log_2 3$ is irrational
2. Prove: The sum of two even integers is always even
3. Given that : $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$ prove that if $\sigma \to 1, u(c) \to \log(c)$