Lecture 10
Optimization Day 2

Ryan McWay[†]

[†] *Applied Economics, University of Minnesota*

Mathematics Review Course, Summer 2023
University of Minnesota
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Last Lecture Review

- Unconstrained Optimization:
  - First Order Conditions
  - Second Order Conditions
- (Equality) Constrained Optimization:
  - Lagrangian Method
  - Bordered Hessian

Review Assignment

1. Problem Set 09 solutions are available on Github.
2. Any issues or problems **You** would like to discuss?

Daily Icebreaker

- Attendance via prompt:
  - Name
  - Daily Icebreaker: When you go to the movie (or sit on your couch at home), what candy/treat do you sneak in to eat while you watch?

(Inequality) Constrained Optimization

Motivation

- General background
  - Optimization of a function considering a constraint from other function(s).
  - May lead to a potential 'corner solution'.
- Why do economists' care?
  - This is the most typical case for optimization.
- Application in this career
  - Used throughout microeconomics.

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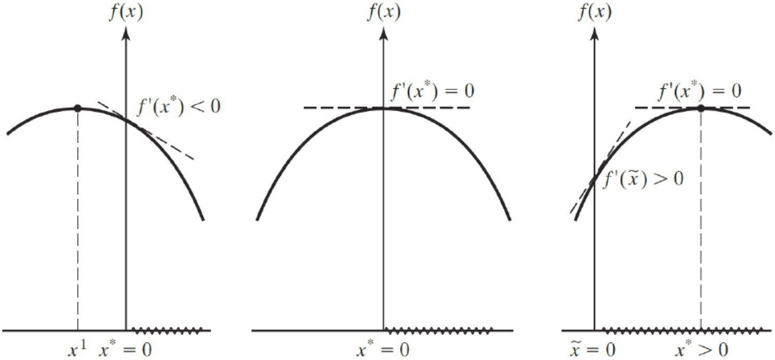
Overview

1. Minimization
2. Maximization
3. Kuhn Tucker Conditions
4. Corner Solutions
5. Quasiconcavity and Optimization

INEQUALITY



(a) Case 1 (b) Case 2 (c) Case 3

1. Minimization

- Necessary conditions for optimal in real-valued functions subject to non-negative constraints:
  - Let $f(x)$ be continuously differentiable.
  - If x^* minimizes $f(x)$ subject to $x \geq 0$, then x^* satisfies:

1. $\frac{\partial f(x)}{\partial x_i} \geq 0 \forall i = 1, \ldots, n.$
2. $x_i^* \left(\frac{\partial f(x)}{\partial x_i} \right) = 0 \forall i = 1, \ldots, n.$
3. $x_i^* \geq 0 \forall i = 1, \ldots, n.$

2. Maximization

- Necessary conditions for optimal in real-valued functions subject to non-negative constraints:
  - Let $f(x)$ be continuously differentiable.
  - If x^* maximizes $f(x)$ subject to $x \geq 0$, then x^* satisfies:

1. $\frac{\partial f(x)}{\partial x_i} \leq 0 \forall i = 1, \ldots, n.$
2. $x_i^* \left(\frac{\partial f(x)}{\partial x_i} \right) = 0 \forall i = 1, \ldots, n.$
3. $x_i^* \geq 0 \forall i = 1, \ldots, n.$

3. Kuhn Tucker Conditions

$$\max_{x \in \mathbb{R}^n_+} f(x) \text{ s.t. } g(x) \leq b, x \geq 0$$

$$L = f(x) - \lambda_1[g_1(x) - b_1] - \cdots - \lambda_k[g_k(x) - b_k]$$

- The constraints are 'binding' if at the optimum $g(x^*, y^*) = c$, and is said to 'slack' otherwise.
- The **three** "Kuhn-Tucker" necessary conditions (FOC) are...

1.	$\frac{\partial L}{\partial x_i^*} \leq 0$	$\frac{\partial L}{\partial \lambda_j^*} \geq 0,$	FOC w/ inequalities
2.	$x_i^* \frac{\partial L}{\partial x_i^*} = 0,$	$\lambda_j^* \frac{\partial L}{\partial \lambda_j^*} = 0,$	Complimentary slackness
3.	$x_i^* \geq 0,$	$\lambda_j^* \geq 0,$	Non-negative condition
	$\forall i = 1, \ldots, n$	$\forall j = 1, \ldots, k$	

3. Kuhn Tucker Conditions

- $\lambda_j^* \frac{\partial L}{\partial \lambda_j^*} = 0$ implies that **at least one** of the λ_j^* and $\frac{\partial L}{\partial \lambda_j^*}$ must be zero.
- If the constraint is non-binding, then $\lambda_j^* = 0$ and we give **no** weight to that constraint (i.e., unconstrained).

$$\frac{\partial L}{\partial \lambda_j^*} \equiv b_j - g_j(x) > 0$$

- If $\lambda_j^* > 0$ then the constraint must be binding (i.e., constrained).

$$b_j = g_j(x)$$

- Kuhn-Tucker determine allows **any** problem to be solved as constrained or unconstrained.
- Hence, you check **both cases**.

3. Kuhn Tucker Conditions

- You need to test out all the cases to see if the constraint is binding $\lambda > 0$ and if any of the parameters (inputs) are optimally set to zero $x = 0 \vee y = 0$.
  - $\lambda = 0$, $\lambda > 0$
  - $x = 0$, $y > 0$
  - $x > 0$, $y = 0$
  - $x > 0$, $y > 0$
- Finally, you need to check that your solution is consistent to the K-T conditions and does not lead to any contradictions.

Demonstration: Kuhn Tucker Conditions

Question:

Max $f(x,y) = xy$ s.t. $x + y^2 \leq 2$.

Answer:

$$\mathcal{L} = xy + \lambda(2 - x - y^2)$$

First Order Conditions:

$$\begin{aligned} \mathcal{L}_x =& y - \lambda \leq 0, x \geq 0 \\ \mathcal{L}_y =& x - 2y\lambda \leq 0, y \geq 0 \\ \mathcal{L}_\lambda =& 2 - x + y^2 \geq 0, \lambda \geq 0 \end{aligned}$$

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Demonstration: Kuhn Tucker Conditions

Answer:
Non-Negative:

$$x, y, \lambda \geq 0$$

Necessary Conditions:

$$\begin{aligned} x\mathcal{L}_x =& x(y-\lambda) = 0 \\ y\mathcal{L}_y =& y(x-2y\lambda) = 0 \\ \lambda\mathcal{L}_\lambda =& \lambda(2-x+y^2) = 0 \end{aligned}$$

Demonstration: Kuhn Tucker Conditions

Answer:
Consider $\lambda = 0$. Then by $\mathcal{L}_x, \mathcal{L}_y$ we can say $y = 0, x = 0$. This is the feasible option $f(0,0) = 0(0) = 0$.
Also, consider $\lambda > 0$. Now the constraint applies. And we have the identity $x = 2 - y^2$. What we would do is now go through three possible cases:

- $x > 0, y = 0$
- $x = 0, y > 0$
- $x > 0, y > 0$

To save space, let us just consider the final case $x > 0, y > 0$.

Demonstration: Kuhn Tucker Conditions

Answer:

Then by $\mathcal{L}_x \implies y = \lambda$. And $\mathcal{L}_y \implies x - 2y^2 = 0$. This gives the identities:

$$y = \left(\frac{x}{2}\right)^{1/2}$$
$$x = 2y^2$$

If we plug this back into the constraint (which we can use now that $\lambda > 0$, we get the critical point $(\frac{4}{3}, \frac{\sqrt{2}}{2})$. Now, to determine the max we compare $f_1(0,0)$ to $f_2(\frac{4}{3}, \frac{\sqrt{2}}{2})$. Since $f_1 < f_2$ we can conclude that the interior solution $(\frac{4}{3}, \frac{\sqrt{2}}{2})$ is the global maximum, while $(0,0)$ is a local maximum.

Practice: Kuhn Tucker

1. Max $U = ln(x) + y$ s.t. $y + x \leq 2$.

Practice: Kuhn Tucker

1. Max $U = ln(x) + y$ s.t. $y + x \leq 2$.

Answer: Show Work

$$x^* = 1 \quad y^* = 1 \quad \lambda^* = 1$$

Practice: Kuhn Tucker

1. Max $U = ln(x) + y$ s.t. $y + x \leq 2$.
2. [Hard.] Max $-(x-5)^2 - (y-5)^2$ s.t. $x^2 + y \leq 9$.

Practice: Kuhn Tucker

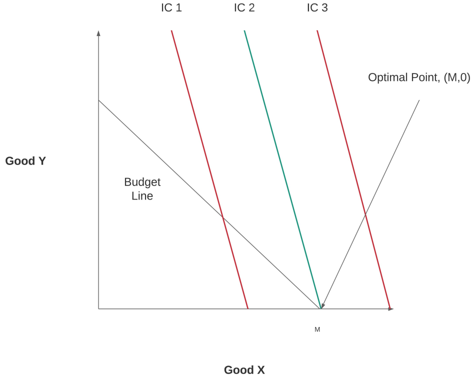
1. Max $U = ln(x) + y$ s.t. $y + x \leq 2$.
2. [Hard.] Max $-(x-5)^2 - (y-5)^2$ s.t. $x^2 + y \leq 9$.

Answer: Show Work

$$x^* = \frac{\sqrt{11}+1}{2} \quad y^* = \frac{12-\sqrt{11}}{2} \quad \lambda^* = \sqrt{11} - 2$$

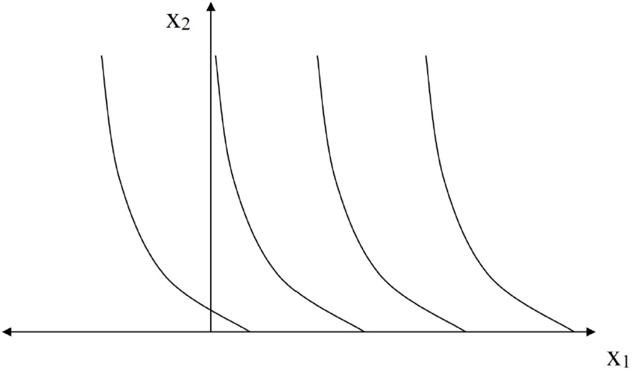
4. Corner Solutions

- The boundary condition
- The optimal value is **not** tangential to the constraint. But rather at the 'corner' where you choose to set one of the other inputs (parameters) to zero and at the boundary of the other inputs.



Application: Quasilinear Preferences

- Quasilinear preferences (A strong preference for x_2)



5. Concavity, Convexity, and Optimization

- Convex Maximization Problem: With convex constraint sets and concave objective functions, the FOC are **both necessary and sufficient** to identify global maxima.
- Convex Minimization Problem: With convex constraint sets and convex objective functions, the FOC are **both necessary and sufficient** to identify global maxima.
- Both provide 'uniqueness' in the solution.
- So, for optimization we need:
  - Continuity on the domain.
  - Differentiablity
  - Concavity/convexity of the C^2 function is completely characterized by the second derivative.
- Quasi-concavity does **not** imply continuity...

Comparative Statics & Envelope Theorem

Motivation

- General background
  - Allows you determine how an optimum changes are the parameters values change.
- Why do economists' care?
  - Used primarily in macroeconomics, but in microeconomics as well.
- Application in this career
  - Used to measure policy alternatives but changing the initial conditions.

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Overview

1. The Multiplier
2. Comparative Statics
3. Unconstrained Envelope Theorem
4. Constrained Envelope Theorem

1. The Multiplier

- Consider the maximization problem:

$$\max f(x, y) \text{ s.t. } h(x, y) = a$$

- Let the solution be $(x^*(a), y^*(a))$ with the corresponding **multiplier** $\mu^*(a)$.
- Suppose x^*, y^*, and μ^* are C^1 functions of a. Then,

$$\mu^*(a) = \frac{d}{da} f(x^*(a), y^*(a))$$

- Or, for multiple variables (n) and multiple constraints (m)

$$\mu_j^*(a_1, \cdots, a_m) = \frac{\partial}{\partial a_j} f(x_1^*(a_1, \cdots, a_m), \ldots, x_n^*(a_1, \cdots, a_m)) \forall j = 1, \ldots, m$$

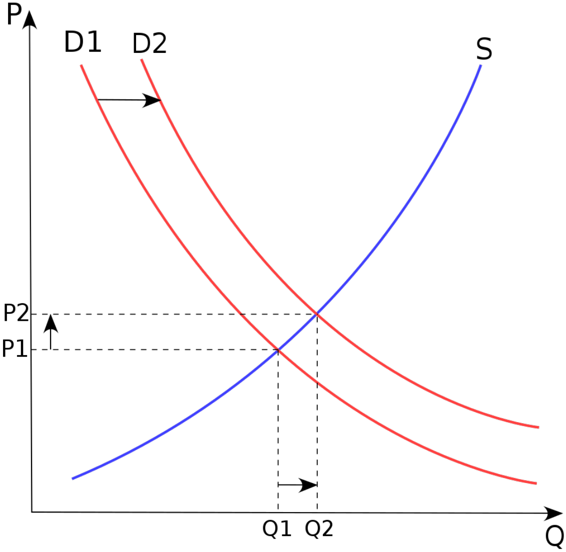
- So μ_j^* measures the sensitivity of the objective function to the constraint.
- E.g., The marginal change in the objective function for a marginal relaxation of the constraint.

2. Comparative Statics

- Examining the change in optimization after changing an 'exogenous' parameter (a).
- In essence, the difference between two equilibrium states.
- We can use the implicit function theorem to determine a comparative static derivative.
- Adding constraints, we can apply the Envelope Theorem to generalize the following formula.

$$\begin{aligned} f(x,a) &= 0 \\ Bdx + Cda &= 0 \\ \frac{dx}{da} &= -B^{-1}C \end{aligned}$$

2. Comparative Statics



Demonstration: Comparative Statics

Question:

Market Model: Suppose we determined the equilibrium price as $P^* = \frac{a+c}{b+d}$. How does P^* change as the non-negatives a, b, c, d change?

Answer:

$$\frac{\partial P^*}{\partial a} = \frac{1}{b+d} > 0$$
$$\frac{\partial P^*}{\partial b} = \frac{-(a+c)}{(b+d)^2} < 0$$
$$\frac{\partial P^*}{\partial c} = \frac{1}{b+d} > 0$$
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Practice: Comparative Statics

1. What is the comparative static if we increase taxes t given national income $Y^* = \frac{a-bd+I+G}{1-b(1-t)}$?

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Answer: Show Work

$$\frac{\partial Y^*}{\partial t} = \frac{-b(a-bd+I+G)}{(1-b(1-t))^2} < 0 \; | a+I+G > bd$$

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Answer: Show Work

$$\frac{\partial C^*}{\partial t} = \frac{-b(I+G+bd-(a+2b(1-t)(I+G)}{(1-b(1-t))^2} < 0 \ |(\cdot) > 0$$

Practice: Comparative Statics

1. What is the comparative static if we increase government spending G given $Y^* = \frac{a-bd+I+G}{1-b(1-t)}$?
2. What is the comparative static if we increase taxes t given national consumption $C^* = \frac{a-bd+b(1-t)(I+G)}{1-b(1-t)}$?
3. What is the comparative static if we increase government spending G given tax revenues are $T^* = \frac{d(1-b)+t(a+I+G)}{1-b(1-t)}$?

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3. What is the comparative static if we increase government spending G given tax revenues are $T^* = \frac{d(1-b)+t(a+I+G)}{1-b(1-t)}$?

Answer: Show Work

$$\frac{\partial T^*}{\partial G} = \frac{t}{1-b(1-t)} > 0 \; | 1 > b(1-t)$$

3. Unconstrained Envelope Theorem

- The idea is that we can use the implicit function theorem to substitute into the objective function so we only take the derivative to one function (simplify life).
- Let $f(x,a)$ be a C^1 function of $x \in \mathbb{R}^n$ with scalar a.
- For each possible parameter a, consider the **unconstrained** optimization problem:

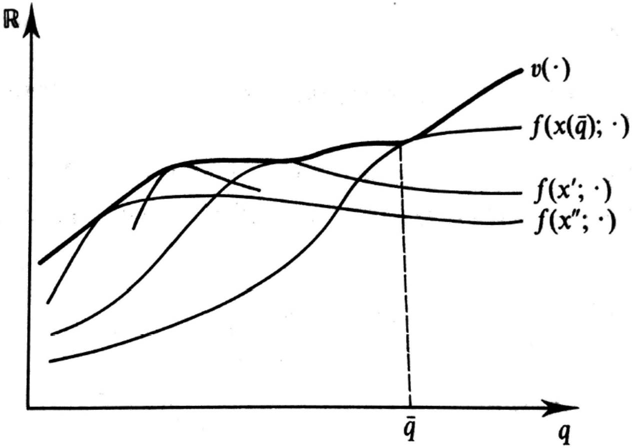
$$\max f(x,a) \text{ w.r.t. } x$$

- Let $x^*(a)$ be the solution.
- Suppose that $x^*(a)$ is a C^1 function of a.
- Then,

$$\frac{d}{da} f(x^*, a) = \frac{\partial}{\partial a} f(x^*(a), a)$$

3. Unconstrained Envelope Theorem

- Intuition



Demonstration: Unconstrained E.T.

Question:

Max $f(x(\theta), \theta) = 2\theta x - x^2 + \theta - \theta^2$.

Answer:

$$
\begin{aligned}
f_x &= 2\theta - 2x = 2(\theta - x) = 0 \implies x = \theta \\
f_\theta &= 2x + 1 - 2\theta \\
\because x &= \theta \implies \\
f_\theta &= 2\theta + 1 - 2\theta = 1 \\
\rightarrow \theta^* &= 1 \wedge x = \theta^* = 1
\end{aligned}
$$

Demonstration: Unconstrained E.T.

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Max $f(x(\theta), \theta) = 2\theta x - x^2 + \theta - \theta^2.$

Answer:

$$\begin{aligned} f_x =& 2\theta - 2x = 2(\theta - x) = 0 \implies x = \theta \\ f_\theta =& 2x + 1 - 2\theta \\ \because x =& \theta \implies \\ f_\theta =& 2\theta + 1 - 2\theta = 1 \\ \rightarrow \theta^* = 1 \wedge& x = \theta^* = 1 \end{aligned}$$

4. Constrained Envelope Theorem

- Let $f, h_1, \ldots, h_m : \mathbb{R}^n \times \mathbb{R}^1 \to \mathbb{R}^1$ be C^1 functions.
- Let $x^*(a) = (x_1^*(a), \ldots, x_n^*(a))$ be the solution maximizing $f(x, a)$ with respect to x given the following constraint set:

$$h_1(x, a) = 0, \ldots, h_m(x, a) = 0$$

- Suppose that $x^*(a)$ and the Lagrange multipliers $\mu_1(a), \ldots, \mu_m(a)$ are C^1 functions of a.
- Then,

$$\frac{d}{da} f(x^*(a), a) = \frac{\partial L}{\partial a} f(x^*, \mu(a), a)$$

- The Lagrange multiplier is a **special case** of the envelope theorem.

Review

(Inequality) Constrained Optimization

1. Kuhn Tucker Conditions
2. Corner Solutions

Comparative Statics & Envelope Theorem

1. The Multiplier
2. Comparative Statics
3. Envelope Theorem

Assignment

- Readings on Probability before Lecture 11:
  - Hansen Metrics Ch. 1 & 2
- Assignment:
  - Problem Set 10 (PS10)
  - Solution set will be available this weekend.
- Struggling?
  1. Read the 'Encouraged Reading'
  2. Review 'Supplementary material'
  3. Reach out directly

Kuhn Tucker Question 1 Answer:

◂ Question

$$
\begin{aligned}
\mathcal{L}_x =& \frac{1}{x} - \lambda \leq 0, x \geq 0, x \cdot \mathcal{L}_x = 0 \\
\mathcal{L}_y =& 1 - \lambda \leq 0, y \geq 0, y \cdot \mathcal{L}_y = 0 \\
\mathcal{L}_\lambda =& 2 - x - y \geq 0, \lambda \geq 0, \lambda \cdot \mathcal{L}_\lambda = 0
\end{aligned}
$$

Case:

$\lambda > 0, x > 0, y > 0$

Kuhn Tucker Question 2 Answer:

◂ Question

I have this written in my tablet. I can go through this. Do the FOC. Then start with $\lambda = 0$. Then consider $\lambda > 0$ and examine the non-negativity of x, y. There should be four local max, only one of which is the global max.

Comparative Static Question 1 Answer:

◂ Question

- Can write this on the board.

Comparative Static Question 2 Answer:

Question

- Can write this on the board.

Comparative Static Question 3 Answer:

◂ Question