Linear Algebra

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Systems of linear equations

$$a_{11}\, x_1 + a_{12} x_2 + \ \dots \dots a_{1n}\, x_n = b_1$$
$$a_{21}\, x_1 + a_{22} x_2 + \ \dots \dots a_{2n}\, x_n = b_2$$
$$\ddots$$
$$a_{m1}\, x_1 + a_{m2} x_2 + \ \dots \dots a_{mn}\, x_n = b_m$$

For example

$$
\begin{aligned}
x_1 - 0.4x_2 - 0.3x_3 &= 130 \\
-0.2x_1 + 0.88x_2 - 0.14x_3 &= 74 \\
-0.5x_1 - 0.2x_2 + 0.95x_3 &= 95
\end{aligned}
$$

Gaussian Elimination

- We solve by a series of substitution and elimination.

$$
\begin{aligned}
x_1 - 0.4x_2 - 0.3x_3 &= 130 \\
-0.2x_1 + 0.88x_2 - 0.14x_3 &= 74 \\
-0.5x_1 - 0.2x_2 + 0.95x_3 &= 95
\end{aligned}
$$

$$\Downarrow$$

$$
\begin{aligned}
x_1 - 0.4x_2 - 0.3\ x_3 &= 130 \\
x_2 - 0.25x_3 &= 125 \\
x_3 &= 300
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
x_1 &= 300 \\
x_2 &= 200 \\
x_3 &= 300,
\end{aligned}
$$

Elementary row operations

Alternatively, we can use elementary row operations to solve the linear system.
First, obtain the **augmented matrix**

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{12} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m2} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

Next, do the following **elementary row operations** until the matrix is in the **row echelon form**.

- interchange two rows
- change a row by adding to it a multiple of another row
- multiply each element in a row by the same nonzero number

For example

$$\left(\begin{array}{ccc|c} 1 & -0.4 & -0.3 & 130 \\ -0.2 & 0.88 & -0.14 & 74 \\ -0.5 & -0.2 & 0.95 & 95 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -0.4 & -0.3 & 130 \\ 0 & 0.8 & -0.2 & 100 \\ 0 & 0 & 0.7 & 210 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -0.4 & -0.3 & 130 \\ 0 & 1 & -0.25 & 125 \\ 0 & 0 & 1 & 300 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 300 \\ 0 & 1 & 0 & 200 \\ 0 & 0 & 1 & 300 \end{array}\right)$$

How do we know whether a system has solutions or not?

Rank of a matrix

- If the rows and columns of a matrix A_{mn} are linearly independent and non-zero, the rank of the matrix is :

$$rank = \min(m, n)$$

Rank of a matrix

The **rank** of a matrix is the number of nonzero rows in its row echelon form.

- $rank\mathbf{A} = rank\mathbf{A}' \leq min(\#rows, \#columns)$
- $rank\mathbf{AB} \leq min(rank\mathbf{A}, rank\mathbf{B})$
- $rank\mathbf{A} = rank\mathbf{A}'\mathbf{A} = rank\mathbf{AA}'$
- A matrix is **full rank** if the rank equals to the number of columns.

Let $\mathbf{A}$ be the coefficient matrix and $\hat{\mathbf{A}}$ be the corresponding augmented matrix,

- $rank\mathbf{A} \leq$ number of rows of $\mathbf{A}$
- $rank\mathbf{A} \leq$ number of columns of $\mathbf{A}$
- $rank\hat{\mathbf{A}} > rank\mathbf{A}$

Existence of a solution

A system of linear equations with coefficient matrix **A** and augmented matrix $\hat{\mathbf{A}}$ has a solution iif

$$rank\hat{\mathbf{A}} = rank\mathbf{A}$$

That is, no augmented matrix like this form

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ 0 & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & b_m \end{array}\right)$$

Example

(Simon & Blume Exercise 7.18)
For what values of the parameter a does the following system of equations have a solution?

$$\begin{aligned} 6x + y &= 7 \\ 3x + y &= 4 \\ -6x - 2y &= a \end{aligned}$$

Existence of a solution

A linear system has infinitely many solutions if

$$\textit{number of rows of } \mathbf{A} < \textit{number of columns of } \mathbf{A}.$$

A linear system has one and only one solution for every choice of right-hand side $b_1, \cdots, b_m$ iif

$$\textit{number of rows of } \mathbf{A} = \textit{number of columns of } \mathbf{A} = \textit{rank}\, \mathbf{A}$$

Such a coefficient matrix $\mathbf{A}$ is called a **nonsingular square matrix**.

Linear independence

A **homogeneous** system, which has $b_i = 0$ for all i,

$$\mathbf{A}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \mathbf{0}$$

is guaranteed to have at least one solution: $x_i = 0$ for all i.

But iif there is a nonzero solution (which means there is infinitely more), each column vectors $\mathbf{a_1}, \cdots, \mathbf{a_n}$ in A are **linearly dependent** . This means at least one of the column vectors can be written as **linear combination** of the others

A set of vector is **linearly independent** if and only if the only solution is the zero solution.

Square matrices

- We can solve for a system of equations if the number of variables is equal to the number of equations by inverting matrices.

Let $\mathbf{B} = \mathbf{A}^{-1}$ be the inverse of a full-rank $k \times k$ matrix $\mathbf{A}$. The matrices satisfy

$$\mathbf{AB} = \mathbf{I}_k$$

If an $n \times n$ matrix $\mathbf{A}$ is invertible, then it is nonsingular, and the unique solution to the linear system $\mathbf{Ax} = \mathbf{b}$ is $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.

Inversion

The following statements about a $n \times n$ square matrix $\mathbf{A}$ are equivalent

- $\mathbf{A}$ is invertible
- $\mathbf{A}$ is nonsingular
- A has maximal rank n (full rank)
- every system $\mathbf{Ax} = \mathbf{b}$ has one and only one solution for every $\mathbf{b}$

Exercise

- What is the determinant for thigs Matrix?

$$det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \; or \; \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Square matrices

- A square matrix is non-singular iif it determinant is zero

Properties

$$\det \mathbf{A}^T = \det \mathbf{A}$$

$$\det(\mathbf{A} \cdot \mathbf{B}) = \det \mathbf{A} \cdot \det \mathbf{B}$$

$$\det(\mathbf{A} + \mathbf{B}) \neq \det \mathbf{A} + \det \mathbf{B}$$

Determinants

- What about if the matrix is not full rank?

Inversion

Let **A** be a nonsingular matrix,

$$\mathbf{A}^{-1} = \frac{1}{det\mathbf{A}} \cdot adj\,\mathbf{A},$$

where *adj* **A** is a $n \times n$ square matrix in which the element on the *i*th row and *j* the column is

$$(-1)^{i+j} \times det(submatrix\ of\ \mathbf{A}\ without\ the\ ith\ row\ and\ the\ jth\ column)$$

Example

Invert the following matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 1 & 0 & 8 \end{pmatrix}$$

Example

- Invert

$$A = \begin{bmatrix} 1 & 4 & -2 \\ 0 & -1 & 2 \\ 3 & 5 & -3 \end{bmatrix}$$

Cramer’s rule

The unique solution $\mathbf{x} = (x_1, \cdots, x_n)$ of the $n \times n$ system
$\mathbf{Ax} = \mathbf{b}$ is

$$x_i = \frac{det\,\mathbf{B}_i}{det\,\mathbf{A}} \quad for\ i = 1, \cdots, n,$$

where $\mathbf{B}_i$ is the matrix $\mathbf{A}$ with the right-hand side $\mathbf{b}$ replacing the ith column of $\mathbf{A}$.

Exercise

- Use Cramer's rule to solve for thigs system:

Use the Cramer's rule to calculate x_3 in the following system:

$$\begin{pmatrix} 1 & 1 & 1 \\ 12 & 2 & -3 \\ 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ -4 \end{pmatrix}$$