Problem Set 3

APEC Math Review

August 2020

1. (Simon & Blume Exercise 17.5) Suppose a firm has a Cobb-Douglas production function $Q = x^a y^b$ and that it faces output prices p and input prices w and r, respectively. Solve the first-order conditions for a profit maximizing input bundle. Use the second order conditions to determined the values of the parameters a, b, p, w and r for which this solution is a global max (hint: use the leading principle minors to check the definiteness of the Hessian).

17.5 $\Pi(x, y) = px^a y^b - wx - ry.$

$\Pi_x = pax^{a-1}y^b - w = 0,\ \Pi_y = pbx^a y^{b-1} - r = 0.$

$$\frac{pax^{a-1}y^b}{pbx^a y^{b-1}} = \frac{w}{r} \quad \text{implies} \quad \frac{y}{x} = \frac{bw}{ar}; \qquad \text{so} \qquad y = \frac{bw}{ar}x.$$

Plug this into $\Pi_x = 0$ to solve for x (and then y).

$$H = \begin{pmatrix} pa(a-1)x^{a-2}y^b & pabx^{a-1}y^{b-1} \\ pabx^{a-1}y^{b-1} & pb(b-1)x^a y^{b-2} \end{pmatrix}.$$

$|H_1| = pa(a-1)x^{a-2}y^b < 0$ if and only if $0 < a < 1$. $|H_2| = p^2[a(a-1)b(b-1) - a^2b^2]x^{2a-2}y^{2b-2} > 0$ if and only if $ab[(a-1)(b-$

$1) - ab] = ab[1 - (a+b)] > 0$. If $0 < a < 1$, $0 < b < 1$, and $a + b < 1$, the solution to the first order conditions is a global max.

2. (Polasky, APEC 8003 Problem set 1, 2019) In Newtonia all individuals, $i = 1, 2, ..., N$, are identical. They all earn income (Y) and can spend their income on public parks (p_i) and Volvo stationwagons (v_i). Suppose the price of each good is 1 so that the budget constraint is: $Y = p_i + v_i$.

Suppose that each person's utility function is $U(P, v_i) = P^\alpha v_i^{(1-\alpha)}$, where $P = \sum_{i=1}^{N} p_i$.Find the utility maximizing amount of spending on Volvos and public parks by each individual.

a. Each person's utility function is $U(P, v_i) = P^\alpha v_i^{1-\alpha}$, and their budget constraint is $Y = p_i + v_i$. The Lagrangian for this problem is:

$$Max\ P^\alpha v_i^{1-\alpha} + \lambda(Y - p_i - v_i)$$

The necessary conditions for an optimal interior solution are:

$$\alpha P^{\alpha-1} v_i^{1-\alpha} = \lambda$$

$$(1-\alpha) P^\alpha v_i^{-\alpha} = \lambda$$

Set these two equations equal to each other:

$$\alpha P^{\alpha-1} v_i^{1-\alpha} = (1-\alpha) P^\alpha v_i^{-\alpha}$$

$$\alpha v_i = (1-\alpha) P$$

Using the budget constraint to solve for v_i:

$$\alpha(Y - p_i) = (1-\alpha) P$$

Summing this expression over N identical people yields:

$$N\alpha Y - \alpha P = N(1-\alpha) P$$

Solving for P:

$$P = \frac{N\alpha Y}{N(1-\alpha) + \alpha}$$

Using this to solve for p_i and v_i:

$$p_i = \frac{\alpha Y}{N(1-\alpha) + \alpha} \qquad v_i = \frac{N(1-\alpha) Y}{N(1-\alpha) + \alpha}$$

Note that as N gets large, the fraction of the budget devoted to parks goes to 0.

3. (Simon & Blume Exercise 18.17) Minimize $x^2 - 2y$ subject to the constraints $x^2 + y^2 \leq 1$, $x \geq 0$,$y \geq 0$.

18.17 The Lagrangian is

$$L = x^2 - 2y - \lambda(-x^2 - y^2 + 1) - \nu_1 x - \nu_2 y.$$

The first order conditions are

$$L_x = 2x + 2\lambda x - \nu_1 = 0$$

$$L_y = -2 + 2\lambda y - \nu_2 = 0$$

$$\lambda(x^2 + y^2 - 1) = 0$$

$$\nu_1 x = 0$$

$$\nu_2 y = 0$$

$$\lambda, \quad \nu_1, \quad \nu_2, \quad x, \quad y, \quad 1 - x^2 - y^2 \geq 0.$$

If $\lambda = 0$ or $y = 0$, then $\nu_2 < 0$, so in any solution $\lambda > 0$, $y > 0$ and $\nu_2 = 0$. Multiplying the $L_x = 0$ condition by x gives $2x^2(1 + \lambda) = 0$, so $x = 0$, and consequently $y = 1$. Then $\lambda = 1$ and $\nu_1 = 0$.

4. Set up a utility maximizing problem, show that the optimized utility is increasing in income.