Math Review
Summer 2017

Topic 1

1. Introduction to mathematical notations and logic

1.1. Mathematical notation

We start by getting acquainted (or re-acquainted for some of you) to the basic mathematical notations that you will see in economics.

$\forall$	For all...
$\exists$	There exists...
$\nexists$	There does not exist...
$\therefore$	Therefore...
$\because$	Because...
$\neg$	Negation
$\equiv$	Identical to or the same as… For example, we write $f \equiv g$ if $f(x) = g(x)$ for all x
$\Longrightarrow$	A $\Longrightarrow$ B means: "A implies B, "If A then B or "A is sufficient condition for B"
$\Leftrightarrow$	A $\Leftrightarrow$B means "A if and only if B","A is equivalent to B" or "A is a necessary and sufficient condition for B"
$A \subset B$	"B strictly contains A" or "A is a proper subset of B"
$A \subseteq B$	"B contains A" or "A is a subset of B"
$\in$ ($\notin$)	In... (Not in...) or an element of... (Not an element of...)
$\blacksquare$	Bonus: End of proof, Q.E.D.

Anybody knows what Q.E.D means?

The last three notations deal with sets. Formally, a set is a collection of well-defined and distinct objects (usually numbers). For example, the set A is completely determined by the elements in A, where:

$$A = \{x : x \ \in A\}.$$

We will touch more on sets in the next section.

1.2. Numbers

The different sets of numbers in mathematics are:

Natural numbers: $\mathbb{N} = \{1, 2, 3, \ldots\}$
Integers: $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
Rational numbers: $\mathbb{Q} = \{\frac{p}{q} : p, q \in \mathbb{Z}\}$

Q: What do you think is missing in this definition of rational numbers? It has something to do with q.

$\mathcal{A}$:

Real numbers: $\mathbb{R} = \{all\ decimals\}$
Complex numbers: $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}, i = \sqrt{-1}\}$

1.2.1. Intervals in $\mathbb{R}$

These are the four sets of intervals in the real line:

Closed interval: $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$
Open interval: $(a, b) = \{x \in \mathbb{R} : a < x < b\}$
Right-half closed or left-half open $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$
Other: $[a, \infty) = \{x \in \mathbb{R} : a \leq x\}$

where ∞ denotes infinity. We also have $-\infty$ for negative infinity.

1.3. Necessity and sufficiency

Before, jumping into proofs, we establish what we really mean by necessity and sufficiency. Necessary and sufficient have two very different meanings.

- If you advance that "**A is necessary for B**," what does that entail?

Example. Let A be the set "x is an integer less than 9" and let B be the set "x is an integer less than 7". Then A is implied by B, because "x is an integer less than 9" is implied by the statement "x is an integer less than 7".

- If you advance that "**A is sufficient for B**," this is what is entailed:
    - "A implies B" $(A \Longrightarrow B)$
    - Whenever A holds, B must hold.

Example. If Sally gets a 100% in all her graded assignments (**A**), she gets a pass in the class (**B**). Getting 100% in all assignments is a sufficient condition to pass the class. But Sally may very well get an 88% in Homework#7 and still get an A in the class.

Contrapositive form:

Suppose we know that $A \Longleftarrow B$ is true. Then, as A is necessary for B, when A is not true, then B cannot be true.

$\mathcal{Q}$: Look back at your table of notations, how can you write this contrapositive form for A and B?

$\mathcal{A}$:

Can we walk through the example from above in the contrapositive form?

1.4. Theorems and proofs

A mathematical proof is used to show the validity of a specified statement. A proof uses logic and deductive reasoning to show that the statement is always true. Proofs are usually statements take the form "*if A then B*." There are three types of proofs that are frequently used. I have them down here by their popularity (in my opinion) in the first year micro series.

1.4.1. Proof by contradiction

This is a very powerful form of proof. In a proof by contradiction you show that "if not B then not A." Logically, this is what it means:

$$A \Longrightarrow B$$
$$\equiv$$
$$\neg\, A \text{ and } \neg B$$

$$\equiv$$
$$\neg B \Longrightarrow \neg A$$

All these three statements are *all equivalent*.

A good proof by contradiction has the following steps:

Step 1: Assume B is false
Step 2: Show that A must also be false.

We start with a simple math example, and later we will go through a slightly more involved example from micro theory after completing Topic 2.

Example. Prove that $\sqrt{2}$ is irrational.

We could jump to Steps 1 and 2 but let's be a little more careful.

Define related concepts: What form do rational numbers take?

Think of some different examples? Any cases I am forgetting?

Anything we can redefine to get started on a proof by contradiction? It is often helpful to reframe some concepts.

Proof:

1.4.2. Proof by construction

In proof by construction you use true statements to construct the actual statement that you wish to prove. Suppose we have the theorem "A $\Rightarrow$ B". Here, A is called the premise and B the conclusion. In a constructive proof we assume that A is true, deduce various consequences of that, and use them to show that B must also hold. This proof technique is a little less structured, as it is more dependent on the nature of the statement you are trying to prove.

Proof by construction follows these two steps:

Step 1: State what you wish to show (i.e. your claim)
Step 2: Use valid logic and parameters to construct the statement.
Step 3: Conclusion. This is optional, you can re-state the goal if desired.

Example: Prove that if a and b are consecutive integers, then the sum $a + b$ is odd.

<u>Proof</u>.

$\mathcal{Q}$: How would you approach this simple proof as a proof by contradiction?

$\mathcal{A}$:

1.4.3. Proof by Induction

Proof by induction is another great method in which we use recursion to demonstrate an infinite number of facts in a finite amount of space. In other words, you wish to show that some statement, S, is true for all n, S_n. To prove this general statement with induction we follow two steps:

> *Step 1*: Show that a propositional form is true for some basis case. It is typical to begin by showing that either S_0 is true or S_1 is true for example.
> *Step 2*: Assume that S_k is true for some k. This assumption is called the inductive hypothesis. Prove that S_{k+1} is also true, using the assumption that S_k is true.

Example. Prove that $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

Proof:

General tips for approaching and writing proofs:

- As much as possible, use complete sentences when writing your proof. When writing a proof for a homework, exam or prelim, be as legible as possible. This holds for all parts of submitted work, and especially for proofs.
- Always remember to define any variables you introduce.
- It's a good practice to say what type of proof you are using (e.g. Proof by contradiction) to help your reader.
- Overly wordy proofs may result in more likelihood for errors – keep things concise and simple.
- Avoid the use of words such as *obviously*, *clearly*, *as we know*, etc. State what is clear and obvious to you as it may not be for the reader. You might see these words in your micro notes, but I would personally stay clear of these.
- If asked to prove $A \Leftrightarrow B$, that is "A if and only if B" then you must remember to complete both directions of the proof. You must prove "*if A then B*" and "*if B then A*."

Exercise for home (these are simple enough – you may only need 2-3 lines):

Prove: The sum of two even integers is always even. (*hint: use definition of even numbers*)

Prove: Suppose a and b are integers and $a \neq 0$. If a does not divide b, then the equation $ax^2 + bx + b - a = 0$ has no positive integer solution. (*hint: use quadratic formula*)