APEC Math Review

Matrices

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Notation

- An nxm matrix A is an array:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{32} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix}$$

- Where a_{ij} is a real number for every $i = 1, ...n$, $j = 1, ...m$
- We can also write the matrix in a more compact notation: $A = (a_{ij})$

Matrix Operations

- If A and B are two matrices, their sum is done element by element, so they must have the same dimensions
- the i,j entry of A+B is $a_{i,j} + b_{i,j}$
- If A is an nxm matrix and B is an mxk matrix, their product AB is the nxk matrix whose (i,j) entry is the inner product of the ith row of A and the jth column of B. Example on the board.
- In general, $AB \neq BA$

Matrix Operations

Theorem 1.39: The matrix sum and product have the following properties:

1. $A + B = B + A$
2. Addition is associative
3. Multiplication is associative
4. Multiplication distributes over addition

Transpose

- The transpose of a matrix A, denoted A' or A^T, is the matrix whose (i,j)-th entry is $a_{j,i}$
- If A is n x m, A' is m x n
- Example on the board
- The transpose has the following properties:
  1. (A+B)'=A' + B'
  2. (AB)' = B'A' (note that LHS product is well defined IFF RHS product is well defined)
- Note: it is *customary* to denote vectors as columns, and refer to x' when talking about rows.

Some Special Matrices

- Square matrix: is nxn, it's $a_{i,i}$ elements are called the diagonal elements, and the others are called the off-diagonale
- Symmetric matrix: square matrix that satisfies $a_{i,j} = a_{j,i}$ $\forall i, j$, IFF A=A'
- Diagonal matrix: square matrix with only zeros in the off-diagonal entries
- Identity matrix: aquare diagonal entry, where all $a_{i,i} = 1$. Has the property that $AI = IA = A$ for any matrix for which the product is well defined
- Lower-triangular matrix: square matrix which has the property that all entries above the diagonal are zero
- Upper-triangular matrix: analogous, note relationship with the transpose of a LTM

Rank

- Let a finite collection of vectors $x_1, ...x_k$ in $\mathbb{R}^n$ be given. The vectors are said to be lineraly dependent i there are real numbers $\alpha_1, ..., \alpha_k$ with $\alpha_i \neq 0 \exists i$, such that:

$$\alpha_1 x_1 + ... + \alpha_k x_k = 0$$

- Otherwise, the vectors are said to be linearly independent
- Examples on the board
- Let A be an n x m matrix. Each row defines a vector in $\mathbb{R}^m$.
- The row rank of A, denoted $\rho^r(A)$ is the maximum number of linearly independent rows of A

Rank

- Similarly, each of the columns of A is a vector in $\mathbb{R}^n$
- The column rank of A, denoted $\rho^c(A)$ is the maximum number of linearly independent columns of A
- Because A is n x m, we must have $\rho^r \leq n$, $\rho^c \leq m$
- Theorem 1.40: The row rank of any nxm matrix coincides with its column rank.
- So we can call it just rank and denote it $\rho(A)$
- Corollary: $\rho(A) = \rho(A')$

Operations and Rank

- Theorem 1.42: Let A be a given nxm matrix. If B is an nxm matrix obtained from A by:
  1. interchanging two rows of A
  2. multiplying each entry in a given row by a nonzero constant, or
  3. replacing a given row by itself plus a scalar multiple of some other row

  Then $\rho(A) = \rho(B)$
- Same for columns

Operations and Rank

let A be nxm, B nxk

- $\rho(AB) = min\{\rho(A), \rho(B)\}$
- Let P, Q be square matrices of orders m and n respectively, that are both full rank
- Then, $\rho(PA) = \rho(AQ) = \rho(PAQ) = \rho(A)$

The Determinant

Let A be a square matrix of order n.

- The determinant is a function that assigns every A a single real number, denoted by $|A|$
- How do we calculate the determinant?
- In reality, the determinant adds and subtracts odd and even permutations. But I will show you how to calculate it only.
- Define $C_{i,j}(A) = (-1)^{i+j}|A(ij)|$, the i,jth cofactor of A
- Where $A(i,j)$ is the (n-1)x(n-1) matrix obtained by deleting row i and column j
- Then $|A| = \sum_k = 1^n a_{ik} C_{ik}$ for any row i
- This is a recursive method

Theorem 1.44 Properties of The Determinant

Let A be a square matrix of order n

1. If B is obtained from A by interchanging two rows: $|B| = -|A|$
2. If B is obtained from A by multiplying each enter of some row of A by a non zero constant α, then $|B| = \alpha|A|$
3. If B is obtained from A by replacing row i of A by row i plus α times row j, then $|A| = |B|$
4. if A has a row of zeros, then $|A| = 0$
5. If A is a lower(upper)-triangular matrix of order n, then the determinant of A is the product of the diagonal terms.

Also for columns.

Submatrices

- From A, by deleting some rows and some columns
- Theorem 1.45 Let B be an nxm matrix. Let k be the order of the largest square submatrix of B whose determinant is non zero. Then, $\rho(B) = k$.
- In particular, the rows of B are lineraly independent if and only if B contains some square submatrix of order n whose determinant is non zero
- A special case of this result is that square matrices have full rank if and only if they have a non-zero determinant.

The Inverse

Let an nxn matrix A be given. The inverse of A, denoted A^{-1} is defined to be an nxn matrix such that $AA^{-1} = I$

- A has an inverse if and only if A has rank n, or equivalently that $|A| \neq 0$
- If A^{-1} exists, it is unique

Some properties:

- $(A^{-1})^{-1} = A$
- The inverse of the transpose is the transpose of the inverse
- $(AB)^{-1} = B^{-1}A^{-1}$
- $|A^{-1}| = 1/|A|$
- The inverse of a lower (upper) traingular matrix is also a lower (upper) triangular matrix

Quadratic Forms and Definiteness

$$x'Ax$$

(Where A is symmetric)
A quadratic form is said to be:

- Positive definite if we have $x'Ax > 0 \ \forall x \in \mathbb{R}^n, x \neq 0$
- Positive semidefinite if we have $x'Ax \geq 0 \ \forall x \in \mathbb{R}^n, x \neq 0$
- Negative definite if we have $x'Ax < 0 \ \forall x \in \mathbb{R}^n, x \neq 0$
- Negative semidefinite if we have $x'Ax \leq 0 \ \forall x \in \mathbb{R}^n, x \neq 0$

Examples: S pg 51
Note: matrices "sufficiently close" to positive or negative definite matrices are also positive or negative definite

Identifying Definiteness and Semidefiniteness

Let A_k denote the k x k submatrix of A that is obtained when only the first k rows and columns are retained. (kth naturally ordered principal minor of A)

Theorem 1.61: An nxn symmetric matrix A is:

1. negative definite IFF $(-1)^k|A_k| > 0$ for all $k \in \{1, ..., n\}$
2. positive definite IFF $|A_k| > 0$ for all $k \in \{1, ..., n\}$

   Moreover, a positive semidefinite quadratic form A is positive definite IFF $|A| \neq 0$, while a negative semidefinite quadratic form is negative definite IFF $|A| \neq 0$.