Assorted linear algebra topics

I. More on linear systems

II. More on determinants

III. Geometry of vector spaces

IV. others?

More on systems

Prop. 1 of vector spaces is they are closed.

span$(x_1, x_2, \cdots, x_n)$ is the set of all vectors that can be represented as linear combos of $x_1, x_2, \cdots, x_n$

Def'n: Column Space of a matrix A, Colsp(A), is the space spanned by the colums of A, viewed as vectors.

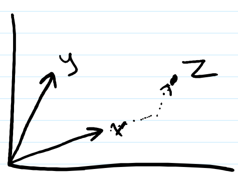
$\underbrace{A}_{\text{square } n\times n \text{ matrix}} x = b$

$Ax = b$ has:

- a unique solution if the colsp(A) spans $\mathbb{R}^n$.

RREF $= I$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



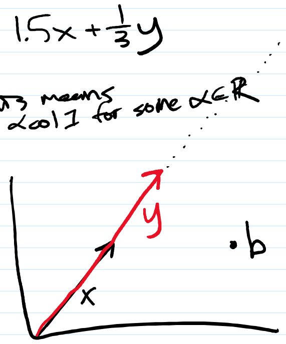
$z = 1.5x + \frac{1}{3}y$

- if the colsp(A) does not span $\mathbb{R}^n$, then either:

in $\mathbb{R}^2$ this means col 2 $= \alpha$ col 1 for some $\alpha \in \mathbb{R}$

i) No solution → if $b \notin$ colsp(A) "inconsistent"

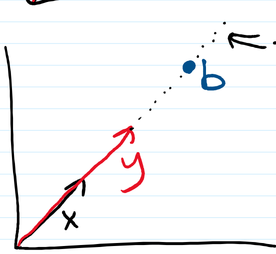
RREF $\left(\begin{array}{ccc|c} 1 & 0 & \cdot & c_1 \\ 0 & b & \cdot & c_2 \\ 0 & 0 & 0 & c_3 \end{array}\right)$ $c_3 \neq 0$



ii) Infinite solutions → if $b \in$ colsp(A)

RREF

$$\left\{\left(\begin{array}{ccc|c} 1 & 0 & 0 & c_1 \\ 0 & 1 & 0 & c_2 \\ 0 & 0 & 0 & 0 \end{array}\right)\right\}$$

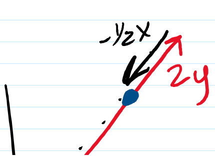


← this line is the colsp(A)

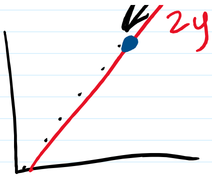
$b = x + y$ but $y = 1.5x$

so $b = 2.5x$

or $b = 2y - \frac{1}{2}x = (3 - \frac{1}{2})x = 2.5x$



so $(1,1)$ is a solution



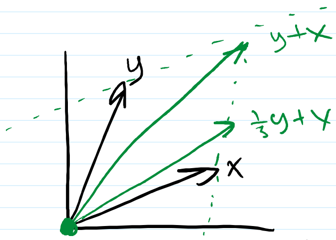
so $(1,1)$ is a solution
and $(2,-1/2)$ is a solution

In 3D, linearly dependent columns lie on the same plane.

Homogeneous systems

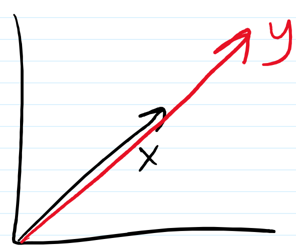
$Ax = 0$

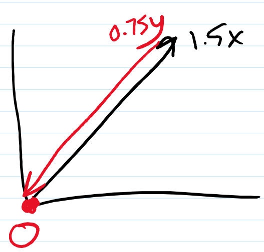
- only the trivial solution if A is nonsingular, i.e. colsp(A) spans $\mathbb{R}^n$.



$ax+by=0$?

- It has infinite solutions if A is singular, i.e. the columns are linearly dependent





Solution to a linear system

If you know A^{-1}, then the solution to $Ax=b$ is: $A^{-1}Ax = A^{-1}b$

$$Ix = A^{-1}b$$

$$\boxed{x = A^{-1}b}$$

Determinants

Q. Are determinants

a) some magic formula that someone made up

b) simple result of Gaussian

someone made up

b) simple result of Gaussian elimination

2x2 case

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\underbrace{-\frac{c}{a}R_1+R_2} \quad \begin{bmatrix} a & b \\ 0 & -\frac{c}{a}b+d \end{bmatrix} \rightarrow \text{row echelon form}$$

pivot 1 : a

pivot 2 : $-\frac{c}{a} \cdot b + d$

A nonsingular requires all pivots nonzero.

Also know $P_1 \cdot P_2 = 0$ iff at least one of them is zero.

So, ... the system has a unique solution iff the product of the pivots is nonzero.

$$P_1 P_2 = a \cdot \left(-\frac{cb}{a} + d\right)$$

$$= \underbrace{ad - bc}_{\text{determinant}}$$

3x3 case

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

$$\begin{array}{l} \underbrace{-\frac{b_1}{a_1}R_1+R_2} \\ -\frac{c_1}{a_1}R_1+R_3 \end{array} \quad \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & -\frac{b_1}{a_1}a_2+b_2 & -\frac{b_1}{a_1}a_3+b_3 \\ 0 & \underbrace{-\frac{c_1}{a_1}a_2+c_2}_{C^2} & \underbrace{-\frac{c_1}{a_1}a_3+c_3}_{C^3} \end{bmatrix}$$

$$\boxed{\begin{aligned} P_1 &= a_1 \\ P_2 &= -\frac{b_1}{a_1}a_2+b_2 \\ C^2 &= -\frac{c_1}{a_1}a_2+c_2 \end{aligned}}$$

$$\boxed{C^2 = \frac{-c_1}{a_1}a_2 + c_2}$$

$$\underbrace{-\frac{C^2}{P_2}R_2 + R_3}\begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & \frac{-b_1}{a_1}a_2 + b_2 & \frac{-b_1}{a_1}a_3 + b_3 \\ 0 & 0 & -\left(\frac{C^2}{P_2}\right)\left(\frac{-b_1}{a_1}a_3 + b_3\right) + C^3 \end{bmatrix}$$

$P_1 = a_1$

$P_2 = \frac{-b_1}{a_1}a_2 + b_2$

$P_3 = \nearrow$

$$P_1 \cdot P_2 \cdot P_3 = \cancel{a_1}\left(\frac{a_1 b_1 - b_1 a_2}{\cancel{a_1}}\right)\left[(-1)\left[\frac{\frac{-c_1}{a_1}a_2 + c_2}{\frac{-b_1}{a_1}a_2 + b_2}\right]\left(\frac{-b_1}{a_1}a_3 + b_3\right) + \left(\frac{-c_1}{a_1}a_3 + c_3\right)\right]$$

$$= \cancel{(a_1 b_1 - b_1 a_2)}\left[\frac{\frac{a_1c_2 - c_1a_2}{\cancel{a_1}}}{\frac{\cancel{a_1b_2 - b_1a_2}}{\cancel{a_1}}}(-1)\cdot\left(\frac{a_1b_3 - b_1a_3}{a_1}\right) + \frac{}{\cancel{}}\left(\frac{a_1c_3 - c_1a_3}{a_1}\right)\right]$$

$$= (-1)(a_1c_2 - c_1a_2)\left(\frac{a_1b_3 - b_1a_3}{a_1}\right) + (a_1b_2 - b_1a_2)\left(\frac{a_1c_3 - c_1a_3}{a_1}\right)$$

$$= (-1)\left[\frac{1}{\cancel{a_1}}\cdot\left(a_1^2b_3c_2 - a_1c_2b_1a_3 - c_1a_2a_1b_3 + \cancel{c_1a_2b_1a_3}\right)\right]$$

$$+ \frac{1}{\cancel{a_1}}\left(a_1b_1a_1c_3 - a_1b_2c_1a_3 - b_1a_2a_1c_3 + \cancel{b_1a_2c_1a_3}\right)$$

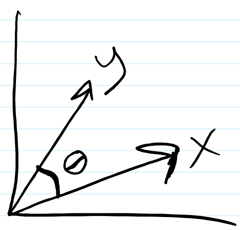
$$= \boxed{\begin{aligned} &-a_1b_3c_2 + a_3b_1c_2 + a_2b_3c_1 \\ &+ a_1b_2c_3 - a_3b_2c_1 - a_2b_1c_3 \end{aligned}}$$

$= \mathrm{Det}(A)$

$= \mathrm{Det}(A)$

Geometry of vector spaces

vectors are points in n-space.



How to find angle θ?

Def'n: norm is a function assigning a positive number to each vector in a vector space. With these properties:

1) $\|aX\| = |a|\,\|X\|$
2) $\|X\| = 0$ iff X is the zero vector
3) triangle inequality $\|a+b\| \leq \|a\| + \|b\|$

Euclidean norm

$X \in \mathbb{R}^n$

$$\|X\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

$$= \sqrt{X \cdot X}$$

Each normed vector space induces a measure of distance (a metric)

$$d(x,y) = \|X - Y\|.$$

Inner product

$\langle \cdot , \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ (could be complex)

where: $\langle u, v \rangle = \langle v, u \rangle$ "commutative"

$\langle au + bv, w \rangle = a\langle u, w \rangle + b \langle v, w \rangle$ "multilinearity"

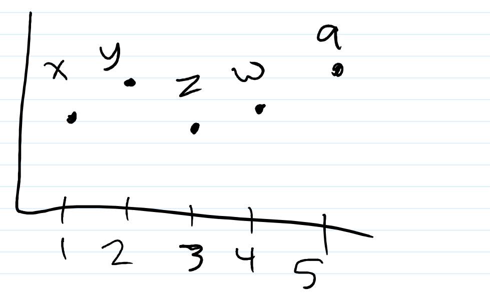
(aside: vector space of continuous fns on a closed interval)

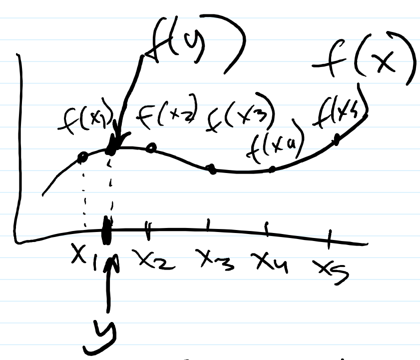
→ infinite dimensional

$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \rightarrow$ 3 dimensions

$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

$\begin{bmatrix} x & y & z & w & a \end{bmatrix} \rightarrow$ 5 dimensions





$\overbrace{[\text{5 integers}]}^{\text{5 elements}}$ $\mathbb{R}^5$

$\overbrace{[\infty \text{ real numbers on the interval } [a,b]]}^{\infty \text{ elements}}$ contin. fns

$\langle au+bv, w\rangle = a\langle u,w\rangle + b\langle v,w\rangle$ "multilinearity"

$\langle u,u\rangle > 0$ for nonzero u "positive definite"

[on the interval $[a,b]$] fns

Example: Dot product

$$\langle u,v\rangle = u\cdot v = u_1v_1+u_2v_2+\cdots+u_nv_n = u'v$$

So: $\|u\| = \sqrt{u\cdot u} = \sqrt{\langle u,u\rangle}$

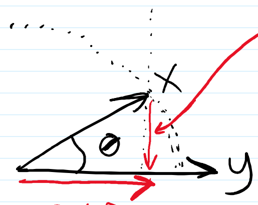
Ex Continuous fns on $[a,b]$

common inner product:

$$\langle f,g\rangle = \int_a^b f(x)g(x)\,dx \quad \text{OR} \quad \int_a^b f(x)g(x)w(x)\,dx$$

weighted inner product

Back to angles



$x - P_y(x)$ → this is orthogonal to the projection

$P_y(x)$ → projection of x onto y

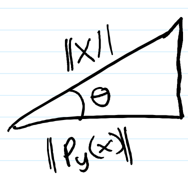
y : x-axis
x : hypotenuse of right triangle

Two vectors are orthogonal if $\langle u,v\rangle = 0$.
Intuitively, they're at right angles.

Multiply:

$$\boxed{\|y\|\cdot\|P_y(x)\| = \langle x,y\rangle}$$

θ:



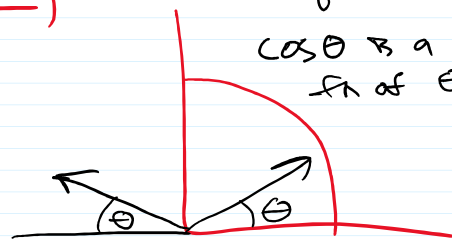
$\cos\theta = \dfrac{\text{adjacent}}{\text{hypothenuse}}$

$= \dfrac{\|P_y(x)\|}{\|x\|}$

$$\Rightarrow \boxed{\|y\|\,\|x\|\cos\theta = \langle x,y\rangle}$$

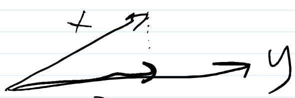
$$\Rightarrow \boxed{\|y\|\,\|x\|\cos\theta = \langle x, y\rangle}$$

In quadrant 1
$\cos\theta$ is a 1-1 fn of θ



these are basically equivalent

Projection



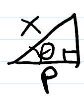
$$P_y(x) = \frac{y'x}{y'y}\,y$$

$$= \frac{\langle y, x\rangle}{\|y\|^2}\cdot y$$

$$= \frac{\|y\|\,\|x\|\cos\theta}{\|y\|^2}\cdot y$$

$$= \frac{\|x\|\cos\theta}{\|y\|}\cdot y \qquad \|x\|\cos\theta = \text{length of } P_y(x)$$

$$= \underbrace{\|x\|\cos\theta}_{\text{length of projection}} \cdot \underbrace{\frac{y}{\|y\|}}_{\text{unit vector in the direction of } y}$$



$\frac{p}{x} = \cos\theta$
$\Rightarrow p = x\cos\theta$

$= P_y(x)$ a vector

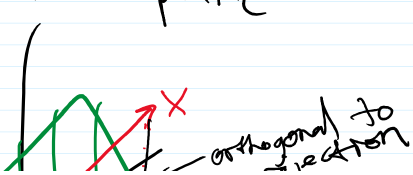
$$\boxed{P_y(x) = y(y'y)^{-1}y'x}$$

X, Y are matrices
Then $P_X(Y) = X\underbrace{(X'X)^{-1}X'Y}_{\hat\beta}$

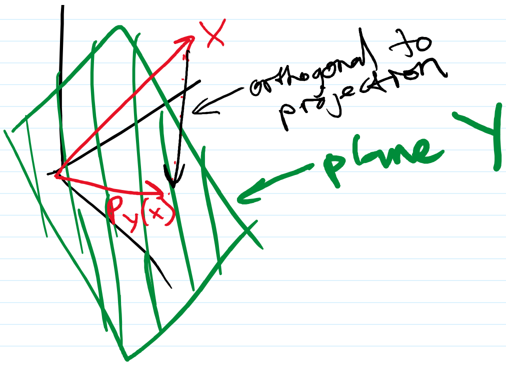
Now consider an orthogonal projection onto a plane

$\underbrace{X(X'X)^{-1}X'}_{\text{projection matrix } P_X}$

$P_X A$ gives the projection of A onto the col sp. of X, for any vector or matrix A.



$P_y(X)$ is the closest vector to X in



$P_Y(X)$ is the closest vector to X in the whole plane Y.

colsp. of ... any vector or matrix A.

$$\underset{n\times m}{A} = \begin{bmatrix} a_1 & a_2 & \cdots & a_m \end{bmatrix}$$

$$P_X A = \begin{bmatrix} P_X a_1 & P_X a_2 & \cdots & P_X a_m \end{bmatrix}$$

Instead of Y, consider any subspace of R^n.

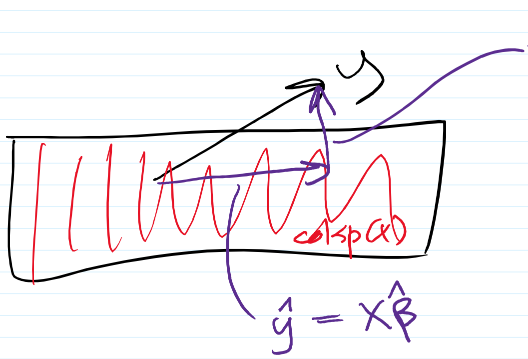
For example: linear regression

$$Y = X\beta + \varepsilon$$

data matrix $N \times K$

Cols of X span a subspace of R^N called the column space of X.

Regression: Project Y onto colsp(X)



Fact: $X'e = 0$

i.e. e is orthogonal to each column of X.

Application: random variables

Random variables are vectors in a vector space.

Let X, Y be R.V.'s. (continuous)

Means: μ_x, μ_y

Std Dev. σ_x, σ_y

Correlation: ρ

joint pdf

Std Dev. σ_x, σ_y

Correlation: ρ

Covariance: $\mathrm{cov}(X,Y)$

Inner product $\langle X, Y \rangle$ $\quad E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\,\underbrace{\overbrace{f(x,y)}^{\text{joint pdf}}}_{\text{weight of the inner product}}\,dx\,dy$

$\mathrm{Cov}(X,Y) = E[(X-\mu_x)(Y-\mu_y)]$ = Inner product of two <u>demeaned</u> random variables.

$\mathrm{Var}(X) = E[(X-\mu_x)^2]$ = Inner prod. of a demeaned RV. with itself.

$= \|X\|^2$

$$\boxed{\begin{aligned} \|X\| &= \sqrt{\langle X, X\rangle} \\ \|X\|^2 &= \langle X, X\rangle \end{aligned}}$$

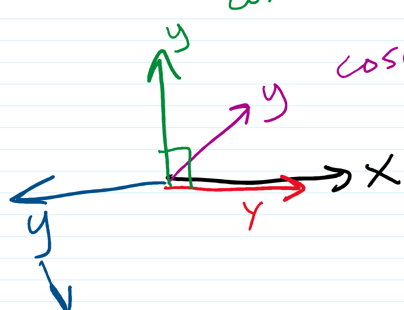
$\sigma_x = \sqrt{\mathrm{Var}X} = \sqrt{\|X\|^2} = \|X-\mu_x\|$: <u>length</u> of a demeaned RV

$$\rho = \frac{\mathrm{cov}(X,Y)}{\sigma_x \sigma_y} = \frac{\langle X-\mu_x, Y-\mu_y\rangle}{\|X-\mu_x\|\,\|Y-\mu_y\|} = \cos\theta$$

$\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2} \rightarrow$ uncorrelated

$\cos\theta \in (0,1) \Rightarrow \theta \in (0, \frac{\pi}{2})$

$\rightarrow$ correlated



$\cos\theta = 1 \Rightarrow \theta = 0$

$\rightarrow$ perfectly correlated

$\rho = -1$

perfect negative correlation