An Applied Course in Microeconomic Theory

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Appendix A

A Mathematical Vaccination

A.1 Introduction

Economic theory gets done using mathematics. In fact, at the higher level theoretical economics may be considered to be a branch of applied mathematics, and a goodly number of the top mathematical economists are trained as mathematicians. For better or worse, this is simply the state of the world in our discipline. If one wishes to read certain of the top journals, it is important to be able to understand sophisticated mathematics. This is not to say that the balance between theoretical and applied work is as it should be. Indeed, I would say that the trend is toward less rather than more emphasis on purely mathematical economics and more on applied or empirical research.

At the Ph.D. level, microeconomic theory simply cannot be done without a heavy dose of mathematical sophistication. Readers of this book may not yet have been exposed to the more esoteric ideas that become routine to the more advanced student. Thus, an appendix whose purpose is to introduce a variety of mathematical concepts without attempting to cover them completely may prove to be useful. I have chosen to call the appendix "A Mathematical Vaccination," and perhaps I should explain this somewhat curious choice. In short, I believe that a leading barrier to an understanding of mathematics is that the first time one sees it, the notation, the concepts, and the names of things are foreign and new. There are probably few people who really understand the idea of quasi-concavity the first time they see it, and there is no end to the list of mathematical notions about which this statement could be said with some confidence. How can there be more irrational numbers than rational numbers, even though there are an infinite number of each? Struggling to understand these ideas can require so much energy that very little is left over with which to ponder the economics that is going on, seemingly in the distant background.

The point is that a graduate program in Economics or in Applied Economics will require a thorough understanding of certain mathematical tools, and that a student will see them many times. *The first time is the worst.* My intention is to present a little bit about a fairly large number of mathematical ideas. Even if this appendix is not able to foster a complete understanding (a preposterous goal, anyway, given that entire courses and, indeed, entire careers, are devoted to each of the topics I will consider), I hope that it vaccinates readers by showing them definitions, symbols, and even some proofs for the first time. The next infection, which will occur in this course or in others, will be far less harmful, even pleasant.

Before getting underway, I want to record some of my views concerning methodology. There are those who believe that economics is by now *too* mathematical for its own good. The techniques we use, it is sometimes claimed, have become so important that they limit the problems that we take up, and restrict the way that economists think about social behavior. This may be true. There are some who take a more extreme view, claiming that economics would be improved if it were stripped entirely of its mathematical accoutrements, and if we went back to composing essays that communicate our ideas to one another. I cannot agree with this veiw. I believe it is useful to think of mathematics as the craft that underlies the art (or science) that we call economics. A photographer, as artist, must be in command of the craft of photography (this includes an understanding of light, of composition, of the working of the camera and the chemistry of film, etc.). In much the same way, a scholar must master a craft upon which creative research is to depend. It is true that a language-based craft, bereft of symbols, mathematical proofs, and so on, could be used as the universal craft of economics. A lot of economics does still get done in this way. And make no mistake: writing well is still as important as it has ever been. But in my view jettisoning mathematics would mean giving up the precision of communication that math provides, and that on the whole this would be bad. It would mean that arguments would be more vague and opaque, when the very thing that is needed is the opposite.

This is a brief summary of my views, but I hope that the student is skeptical. More than this, I hope you are able to take a course in the philosophy of economics, where matters such as this are taken up in much greater detail. In the meantime, however, the references contain a list of accessible readings that I encourage you to find. If nothing else, perhaps these writings will provide some comfort during the dark days of the semester when the idea that mathematics is not helpful might be especially appealing.

A.2 Mathematical Logic

First, some notation and definitions must be set down. Capital letters from early in the alphabet, such as A and B, will denote objects or properties or *statements*. (Later on, they will refer to sets.) If A is the statement, "Person

Symbol	Meaning
$\wedge$	"and"
$\vee$	"or"
$\neg$	"not"
$\forall$	"for all"
$\exists$	"there exists"
$\nexists$	"there does not exist
$\Longrightarrow$	"implies"
$\Longleftrightarrow$	"implies and is implied by" (or "if and only if," often written "iff")

Table A.1: Common logic symbols

X owns a Jaguar automobile," then A may be either true or false. Table A.2 contains some of the symbols that will be employed periodically.

Consider the following two logical statements:

$$A = \text{"Lives in St. Paul"}$$
$$B = \text{"Lives in Minnesota"}$$

The expression $[\,A \Longrightarrow B\,]$ is read "A implies B." It has at least three alternative interpretations, all equivalent and all true if $[\,A \Longrightarrow B\,]$ is true: "A is *sufficient* for B;" "B is *necessary* for A;" "if A, then B." A bit of reflection will be enough to convince you that the statement $[\,A \Longrightarrow B\,]$ is a true statement: one cannot live in St. Paul (here I implicitly mean this St. Paul) without living in Minnesota.

Any given statement, whether true or false, is accompanied by a family of related statements that are connected to the original statement logically. These statements, their names and their forms, are as follows.

Statement	$A \Longrightarrow B$
Contrapositive	$\neg B \Longrightarrow \neg A$
Converse	$B \Longrightarrow A$
Inverse	$\neg A \Longrightarrow \neg B$

The truth or falsehood of a statement and the truth or falsehood of its contrapositive always agree. The two statements are logically equivalent That is, we may write

$$[\,A \Longrightarrow B\,] \Longleftrightarrow [\,\neg B \Longrightarrow \neg A\,].$$

This statement is read, "[A implies B] is true if and only if [not B implies not A] is true." For our example, it is true that living in St. Paul is sufficient for living in Minnesota: [$A \Longrightarrow B$] is true. Thus, it is also true that not living in Minnesota is sufficient for not living in St. Paul. Is the statement $[\,B \Longrightarrow A\,]$ true? No, because one can live in Minnesota without living in St. Paul. However, a statement's converse and its inverse are logically equivalent. If one is true the other must be true.

When are two statements or properties implied by each other? When are they, in a certain sense, the same statement or property? Consider the following:

$$\begin{aligned} C &= \text{“Stands 72 inches tall”} \\ D &= \text{“Stands 6 feet tall”} \end{aligned}$$

It is easy to see that C can be true only if D is true, and vice versa. Thus, $[\,C \iff D\,]$ is a true statement. In general, the expression $[\,C \iff D\,]$ is read, “C if and only if D.”

A.2.1 Three methods of proof

The essential difference between beginning mathematics and more advanced mathematics, in my view, is that the former emphasizes learning rules and formulas while the latter emphasizes proof. At the Ph.D. level in Economics, math comes into play through the use of proofs. Some of the work in this course will also involve understanding and possibly reproducing proofs, though not at the Ph.D. level.

Formal mathematical results—theorems, lemmas, propositions, and the like—begin with a set of assumptions, axioms, and other statements that are taken to be true. The proof of a result proceeds from this foundation, taking a series of logical steps that are supported by sound mathematical reasoning, to a point at which the author of the proof, perhaps a bit weary, puts down her pen and proclaims the proof complete. The idea here is not to teach you all about proving mathematical statements, but rather to give you a glimpse of the various methods of proof. Three will be introduced here: the direct or constructive proof, proof by contradiction, and proof by induction.

Example A.1 (Direct proof)**.** Consider the following result. A function is homogeneous of degree r if for all $t > 0$, for all x, $f(tx) = t^r f(x)$.

Theorem A.1. Suppose $f(x)$, a function mapping the real numbers into the real numbers, is differentiable and homogeneous of degree (hod) r. Then its first derivative $f'(x)$ is homogenous of degree $r - 1$.

Proof. Because f is hod r, by definition, it must be true that, for all $t > 0$,

$$f(tx) = t^r f(x). \tag{A.1}$$

The differentiability of f ensures that we can differentiate (A.1) with respect to x:

$$\begin{aligned} tf'(tx) &= t^r f'(x), \quad \text{or} \\ f'(tx) &= t^{r-1} f'(x), \end{aligned}$$

which is the definition of homogeneity of degree $r - 1$. This completes the proof. □

Example A.2 (Proof by contradiction)**.** Consider the following familiar result and its proof. A rational number is one that can be expressed as the irreducible ratio of two integers.

Theorem A.2. The square root of 2 is irrational.

Proof. By way of contradiction, suppose that $\sqrt{2}$ were rational. Then there exist two integers, m and n, that contain no common factors, with $\sqrt{2} = m/n$ or $2 = (m/n)^2$. But then $2n^2 = m^2$, so m^2 is even because it is twice n^2. If m^2 is even, though, m is even so m^2 must be divisible by 4, which means that $m^2/2$ is even. Thus n^2 is even and we know that m and n are both divisible by 2, contradicting the claim that m and n contain no common factors. This completes the proof. □

A similarly simple example of proof by contradiction is the proof supporting the claim that $\log_2 3$ is irrational[1]

Example A.3 (Proof by induction)**.** Some results have a recursive element that permits a particular direct form of proof to work. The natural numbers, denoted $\mathcal{N}$, are the positive integers.

Theorem A.3. For an integer $n \in \mathcal{N}$, we have $\sum_{i=1}^{n} i = n(n+1)/2$.

Proof. For any $n \in \mathcal{N}$, let $P(n)$ be the statement

$$P(n) : 1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

First we check whether $P(1)$ is true, by computing

$$1 = \frac{1(1+1)}{2} = \frac{2}{2} = 1.$$

Thus, $P(1)$ is true. Now, assume that $P(k)$ is true for $k > 1$. We must show that $P(k+1)$ is true. Because by assumption that $P(k)$ is true, we know that

$$P(k) := 1 + 2 + \cdots + k = \frac{k(k+1)}{2}.$$

Add $k+1$ to both sides to get

[1]Suppose not. Then there are integers m and n so that $\log_2 3 = m/n$, or $2^{m/n} = 3$. But this would mean that $(2^{m/n})^n = 3^n$ or $2^m = 3^n$. But 2 raised to any power is even and 3 raised to any power is odd, a contradiction.

$$
\begin{aligned}
P(k+1) := 1+2+\cdots+k+k+1 &= \frac{k(k+1)}{2} + k + 1 \\
&= \left(\frac{k}{2}+1\right)(k+1) \\
&= \frac{(k+2)(k+1)}{2}.
\end{aligned}
$$

We conclude that $P(k+1)$ is true. This completes the proof. $\square$

Consider the statement, "I am lying." This statement can be neither true nor false. In the higher realms of mathematical logic, one learns that the question of proof is harder even than it first appears. For many centuries, mathematicians believed that the deductive method—begin with a set of axioms or assumptions, and from them use deductive logic to derive theorems—could be used to establish all interesting results. In a famous 1931 paper, however, Kurt Gödel proved that for many branches of mathematics, no internally logical system can be developed within which proofs can be reliably established. Gödel's proof is too advanced for our purposes, but it bears a resemblance to a more accessible and curious result known as Russell's Paradox. Bertrand Russell was a philosopher and mathematician who, with Alfred North Whitehead, wrote a landmark three-volume treatise in mathematical logical entitled *Principia Mathematica*. A goal of the book was to establish a system within which all mathematical results could be proved true or false. Russell was bothered by his paradox, and with good reason. Gödel's theorem, which showed that no such system can exist, used methods that bear a resemblance to the paradox.

Russell's Paradox. Let $\mathcal{S}$ be a set. $\mathcal{S}$ is **proper** if it does not contain itself. $\mathcal{S}$ is **improper** if it does contain itself. Let $\mathcal{R}$ be the set of all proper sets. Does it contain itself? That is, is $\mathcal{R}$ proper? If so, it contains itself, so it is improper. If not, is does not contain itself, so it is proper.

A.3 Set theory

We need first the notion of a set.

Definition A.1. A **set** S is a collection of objects, which are called members of this set.

From what (possibly larger) collection are the elements of S drawn? From the **universal set**, $\mathbf{U}$, within which members of S reside. Given a set S whose elements s are in $\mathbf{U}$, we say that $s \in S$ if s is an **element of** S. The notation for a set S is

$$S = \{s \in \mathbf{U} : \mathcal{P}\},$$

where $\mathcal{P}$ denotes a property.

Example A.4. Suppose $\mathbf{U}$ is the set of all people in the world and $\mathcal{P}$ is the statement

$$\mathcal{P} : s \text{ is a native of Minnesota.}$$

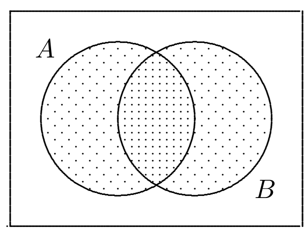
Using this definition of $\mathcal{P}$, S is the set of all Minnesota natives.

Example A.5. A set can be defined by simply listing its members, without reference to a rule specifying them. The first four positive integers make up a set: $S = \{1, 2, 3, 4\}$. Note that the set $\{s\}$ is different from s.

A.3.1 Union, intersection, complement

Definition A.2. Given two sets A and B, their **union**, denoted $A \cup B$, is the set

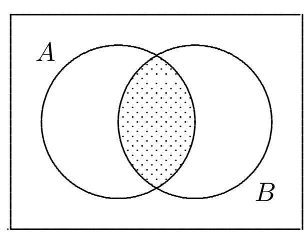
$$A \cup B = \{x \in \mathbf{U} : x \in A \text{ or } x \in B\}.$$



Union: $A \cup B$

Definition A.3. Given two sets A and B, their **intersection**, denoted $A \cap B$, is the set

$$A \cap B = \{x \in \mathbf{U} : x \in A \text{ and } x \in B\}.$$



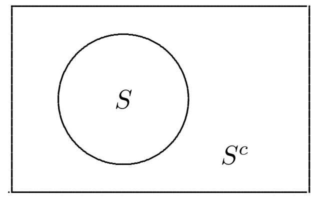
Intersection: $A \cap B$

Example A.6. Suppose $A = \{x \in \mathcal{R} : 0 \leq x \leq 2\}$ and $B = \{x \in \mathcal{R} : -1 \leq x \leq 1\}$. Then $A \cup B = [-1, 2]$ and $A \cap B = [0, 1]$, where $[a, b]$ is the interval in the real numbers from a to b that contains the endpoints.

Definition A.4. Given a set S in $\mathbf{U}$, the **complement of** S **in** $\mathbf{U}$ is

$$S^c = \{x \in \mathbf{U} : x \notin S\},$$

where the symbol $\notin$ means that x is not in S.



Complement of S

Example A.7. Suppose $\mathbf{U}$ is the real numbers and $S = [0, 5]$. The complement of S is $S^c = (-\infty, 0) \cup (5, \infty)$, where (a, b) is the interval in the real numbers from a to b that does not contain its endpoints.

Definition A.5. The set with no elements, denoted $\emptyset$, is the **empty set**. Note that this set depends on the universal set $\mathbf{U}$.

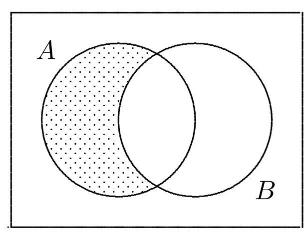
Definition A.6. The two sets A and B are **equivalent**, denoted $A = B$, if for all $x \in \mathbf{U}$

$$[x \in A] \Longleftrightarrow [x \in B].$$

A.3.2 Set difference, subsets, and partitions

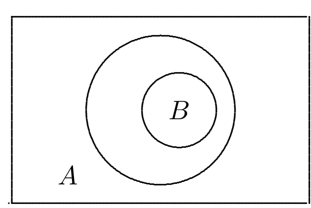
Definition A.7. Given two sets A and B in $\mathbf{U}$, their **difference**, denoted $A \backslash B$, is

$$A \backslash B = \{x \in \mathbf{U} : x \in A \text{ and } x \notin B\}.$$



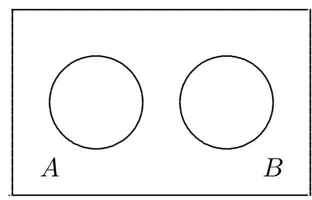
Set Difference: $A \backslash B$

Definition A.8. If A and B are sets in $\mathbf{U}$, then B is a **subset** of A, denoted $B \subset A$, if $[x \in B] \Longrightarrow [x \in A]$. B is a **proper subset** of A if $B \subset A$ and $B \neq A$.



B a subset of A, $B \subset A$

Definition A.9. Two sets A and B in $\mathbf{U}$ are **disjoint** if there is no $x \in \mathbf{U}$ with $[x \in A]$ and $[x \in B]$, or if $A \cap B = \emptyset$.

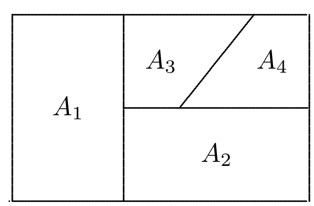


Disjoint sets: $A \cap B = \emptyset$

Definition A.10. Given a set $S \subset \mathbf{U}$, a **partition of** S is a collection of disjoint sets $A_1, A_2, \ldots, A_k$ such that

$$\bigcup_{i=1}^{k} A_i = S.$$

Here, disjointness means that for all i and j, $A_i \cap A_j = \emptyset$.



Partition of S

The union and intersection operators are commutative. For any sets A and B,

$$A \cup B = B \cup A \quad \text{and} \quad A \cap B = B \cap A.$$

The are also associative. For any sets A, B, and C,

$$(A \cup B) \cup C = A \cup (B \cup C) \quad \text{and} \quad (A \cap B) \cap C = A \cap (B \cap C).$$

As an exercise, I recommend that you prove these results.

A.3.3 de Morgan's laws and Cartesian product

Definition A.11 (de Morgan's laws.)**.** Given a collection $A_1, A_2, \ldots, A_k$ of subsets of $\mathbf{U}$, the following are true:

$$\left[\bigcup_{i=1}^{k} A_i\right]^c = \bigcap_{i=1}^{k} A_i^c \quad \text{and} \quad \left[\bigcap_{i=1}^{k} A_i\right]^c = \bigcup_{i=1}^{k} A_i^c$$

To understand the first of these expressions, consider the following three sets, all subsets of the real numbers: $A_1 = [0, 1]$, $A_2 = [.5, 3]$, and $A_3 = [4, 5]$. The union of these three sets is $\cup_i A_i = [0, 3] \cup [4, 5]$; the complement of this union is

$$\left[\bigcup_{i=1}^{3} A_i\right]^c = (-\infty, 0) \cup (3, 4) \cup [5, \infty).$$

The complements of the three sets are

$$\begin{aligned} A_1^c &= (-\infty, 0) \cup (1, \infty), \\ A_2^c &= (-\infty, .5) \cup (3, \infty), \quad \text{and} \\ A_3^c &= (-\infty, 4) \cup (5, \infty). \end{aligned}$$

The intersection of the three complements is

$$\bigcap_{i=1}^{3} A_i^c = (-\infty, 0) \cup (3, 4) \cup [5, \infty),$$

which is what we expected.

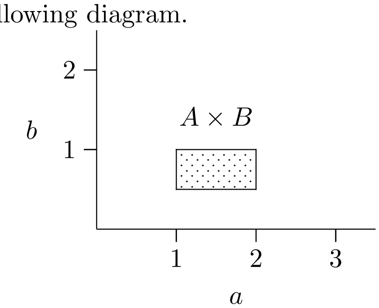
Definition A.12. For two sets A and B in $\mathbf{U}$, their **Cartesian product** is

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}.$$

For example, if $A = [1, 2]$ and $B = [.5, 1]$, then their Cartesian product is

$$A \times B = \{(a, b) \in \mathcal{R}^2 : a \in [1, 2] \text{ and } b \in [.5, 1]\},$$

as illustrated in the following diagram.



A.3.4 Cardinality and Countability

The *cardinality* of a set A, denoted $\#|A|$, is the number of elements it contains. In some cases it is easy to discern a set's cardinality. If $A = \{a, b, c\}$, for example, one can see immediately that $\#|A| = 3$. In other cases, as in the case of $A = \{1, 2, 3, \dots\}$, counting the elements is not an option. Yet it turns out that there are cases in which it is important to be able to compare the cardinality of infinite sets. There are three categories of cardinality: finite, countably infinite, and uncountable. A finite set is the easy case.

Definition A.13. A set A is **finite** if $\# \mid A \mid < \infty$.

For infinite sets, we need to be able to determine whether the number of elements in a set is equivalent to the number of positive integers or counting integers, $\mathcal{N} = \{1, 2, 3, \dots\}$.

Definition A.14. A **countable set** is (a finite set or) any infinite set that can be placed in one-to-one correspondence with $\mathcal{N}$.

Example A.8. Consider the set of all integers, $\mathcal{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$. This set is countable. The following rule places it in one-to-one correspondence with the counting integers:

$$\begin{array}{cccccccc} \in \mathcal{N} & 1 & 2 & 3 & 4 & 5 & 6 & \dots \\ & - & \text{—} & - & \text{—} & - & \text{—} & \\ \in \mathcal{Z} & 0 & -1 & 1 & -2 & 2 & -3 & \dots \end{array}$$

The essential point here is that the algorithm assigns to each element of $\mathcal{N}$ exactly one element of $\mathcal{Z}$. If one were to select the number -2388, for example, the algorithm could be used to determine to which positive integer it is assigned. Intuition may suggest that there are twice as many integers as positive integers, but this is not true! There are the same number of each.

More surprising, perhaps, is the fact that the rational numbers are countable. The rationals have been mentioned before, but have not yet been defined.

Definition A.15. The **rational numbers** consist of all real numbers that can be expressed as the ratio of two irreducible integers:

$$\mathcal{Q} = \{x \in \mathcal{R} : x = p/q \text{ with } p, q \text{ irreducible and with } p, q \in \mathcal{Z}\}$$

The number of rationals is of course enormous. They are *dense* in the reals: between any two rationals, one can find another rational. Indeed, between any two rationals, say 0.0004 and 0.00041, one can find any infinity of rationals. But with some ingenuity one can find an algorithm to place even the rationals into a one-to-one correspondence with the positive integers, so $\mathcal{Q}$ is countable.

The third level of cardinality is characterized by the uncountable sets, of which a leading example is the set of irrational numbers. An irrational number is a real number that cannot be expressed as the ratio of two integers or, and this turns out to be an equivalent statement, whose digital representation never repeats. We've seen that the square root of 2 is irrational. So are e, $\cos \pi/4$, and $\ln 4$. The irrationals, it turns out, cannot be placed in one-to-one correspondence with the positive integers. Even though the rationals are dense in $\mathcal{R}$, they leave holes and *the holes are more numerous than the rationals!* The following theorem was published in 1891 by Georg Cantor, the father of set theory.

Theorem A.4. The irrational numbers are uncountable.

Proof. Suppose not. Then the irrationals can be listed, in decimal representation, as $x_1, x_2, \ldots$. Consider the elements of such a list, for example:

$$
\begin{aligned}
x_1 &= \mathbf{1}.41421356237309505\ldots \\
x_2 &= 2.\mathbf{7}1828182845904524\ldots \\
x_3 &= 0.6\mathbf{9}314718055994531\ldots \\
x_4 &= 0.87\mathbf{7}58256189037271\ldots \\
x_5 &= 2.080\mathbf{0}8382305190411\ldots \\
x_6 &= 6.7082\mathbf{0}393249936909\ldots \\
&\vdots
\end{aligned}
$$

Now create a number, x_0, from this list as follows. The ith digit of x_0 will be a 7, unless the ith digit of x_i is a 7, in which case the ith digit of x_0 will be a 6. Then we may write x_0 as

$$x_0 = 7.67677\ldots.$$

Can x_0 be in the list? No, because it was created in such a way that it is different from every number in the list, in at least one digit. This is a contradiction to the claim that every irrational number is in the list. We conclude that the irrationals are uncountable, completing the proof. □

The real numbers consist of the union of the rationals and the irrationals.

A.3.5 Convex sets

The notion of convexity of a set is important in much of economics.

Definition A.16. Consider a set $X \subset \mathcal{R}^n$, the n-dimensional real numbers. X is **convex** if for all $t \in [0, 1]$ and all $x, y \in X$, the element $x_t = tx + (1-t)y$ is in X.

A.4 The Real Numbers

Now it is time to look more carefully at the real numbers, which have been used as examples in much of what we've already seen. Where exactly does $\mathcal{R}$ come from? Here is a recipe by which the reals can be built up:

1. Take the set of counting integers $\mathcal{N}$.
2. Add to it all negative integers and all non-integral rationals, positive and negative.
3. Finally, add all non-rational limits of sequences of rational numbers.

The resulting set is the real numbers. What is meant by "non-rational limits of sequences of rationals?"

Example A.9. Take an increasing sequence of rational numbers $\{x_n\}_{n=1}^{\infty}$, with $x_n^2 < 2$; for example $\{1, 1.4, 1.41, 1.414, 1.4142, \dots\}$. The limit of this sequence is $\sqrt{2}$, which is not in $\mathcal{Q}$, and thus occupies a hole in the rational numbers.

The set of characteristics that the reals must possess is given as follows.

I. **Addition axioms.**

(a) *Commutativity:* $x + y = y + x, \forall x, y.$

(b) *Associativity:* $(x + y) + z = x + (y + x), \forall x, y, z.$

(c) *Existence of zero:* $x + 0 = x, \forall x.$

(d) *Existence of negative inverses:* $\forall x, \exists w$such that $x + w = 0$.

II. **Multiplication axioms.**

(a) *Commutativity:* $x \cdot y = y \cdot x, \forall x, y.$

(b) *Associativity:* $(x \cdot y) \cdot z = x \cdot (y \cdot z), \forall x, y, z.$

(c) *Existence of unity:* $\forall x, x \cdot 1 = x.$

(d) *Distributive law:* $x \cdot (y + z) = x \cdot y + x \cdot z, \forall x, y, z.$

III. **Order axioms.**

(a) *Transitivity:* If $x \leq y$ and $y \leq z$, then $x \leq z$, $\forall x, y, z.$

(b) *Reflexivity:* $\forall x, y, z, x \leq y$ and $y \leq z \Longrightarrow x \leq z.$

(c) *Trichotomy:* $\forall x, y,$ either $x \leq y$ or $y \leq x.$

(d) $\forall x, y,$ $x \leq 0$ and $y \leq 0 \Longrightarrow 0 \leq x \cdot y.$

Any "system" obeying axioms I.–III. is an *ordered field*. The axioms allow us to introduce the notions of *magnitude*, $|x|$, and *distance* between x and y, $d(x, y) = \sqrt{(x - y)^2} = |x - y|$. Magnitude obeys the triangle inequality, $|x + y| \leq |x| + |y|$.

This set of axioms does not uniquely identify the reals, though, because it turns out that the rational numbers obey I.–III. We need one more axiom:

IV. **Completeness:** If $\{x_n\}_{n=1}^{\infty}$ is an increasing sequence that is bounded above, then it converges to a number.

The rationals, as we have seen, do not satisfy this axiom.

Theorem A.5. The only set of numbers that satisfies axioms I.–IV. is the real numbers.

A.4.1 Greatest lower bound and least upper bound

It is sometimes useful to have a precise idea of the largest or smallest number in a set.

Definition A.17. Given a set $A \subset \mathcal{R}$, the number b is an **upper bound** for A if $\forall a \in A$, $b \geq a$.

Definition A.18. Given a set $A \subset \mathcal{R}$, b is the **least upper bound** (l.u.b.) for A if (i) b is an upper bound for A and (ii) for all upper bounds b', $b \leq b'$.

The l.u.b. of A is sometimes called the *supremum* of A, written $\sup(A)$.

Example A.10. Let $A = (0, 5)$. The supremum of A is $\sup(A) = 5$.

Example A.11. Let $A = [0, 5]$. The supremum of A is $\sup(A) = 5$.

Note that some sets, for example $A = \mathcal{N}$, have no upper bound. Also note that the supremum of a set can be different than the maximum value of a set. The set $A = (0, 5)$ has no maximum value (why?), but it does have a supremum.

Definition A.19. Given a set $A \subset \mathcal{R}$, b is a lower bound for A if $\forall a \in A$, $b \leq a$.

Definition A.20. Given a set $A \subset \mathcal{R}$, b is the **greatest lower bound** (g.l.b.) for A if (i) b is a lower bound for A and (ii) for all lower bounds b', $b \geq b'$.

The g.l.b of A is sometimes called the *infimum* of A, written $\inf(A)$.

A.4.2 The n-dimensional reals

The n-dimensional reals, written $\mathcal{R}^n$, are made up of the Cartesian product of $\mathcal{R}$ with itself n times:

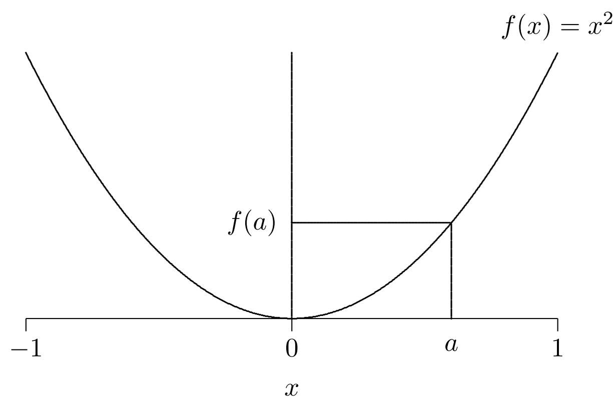
$$\begin{aligned}
\mathcal{R}^2 &= \text{ the Cartesian product of } \mathcal{R} \text{ with itself} \\
\mathcal{R}^3 &= \text{ the Cartesian product of } \mathcal{R}^2 \text{ with } \mathcal{R} \\
\mathcal{R}^4 &= \text{ the Cartesian product of } \mathcal{R}^3 \text{ with } \mathcal{R} \\
&\vdots \\
\mathcal{R}^n &= \text{ the Cartesian product of } \mathcal{R}^{n-1} \text{ with } \mathcal{R}.
\end{aligned}$$

A.5 Functions

In principle, a function is a simple thing. Of course, some functions are quite complicated and working with them can be even moreso. The language that you will see in connection with functions, though, is what I would like to demystify slightly. Economists concern themselves with characteristics of functions that you probably did not see in your undergraduate calculus courses. We begin with the most basic definition, and we will usually refer only to functions defined on $\mathcal{R}^n$, where often n will be one.

Definition A.21. Given two sets A and B, a **function** $f : A \longrightarrow B$ is a rule that assigns to each $a \in A$ a unique element $b \in B$, written $b = f(a)$.

In the following figure is depicted the familiar representation of the function $y : \mathcal{R} \longrightarrow \mathcal{R}$ with $y = f(x) = x^2$.



There is an important difference between the notation for a function with which you are most familiar and the notation introduced in the definition above. When one writes $y = f(x)$, it is unclear just where the function is defined and where its values live. The notation $f : A \longrightarrow B$ conveys this information. The function is defined on the set A, its *domain*. It maps elements of A into the set B, its *range*. In the example from the figure, the domain of f is $\mathcal{R}$ and the range could be defined as any subset of $\mathcal{R}$ that contains all of $\mathcal{R}_+$. A completely specified function should contain information both about the domain and range and about the rule itself. The following, for example, is a complete specification of a function: $f : \mathcal{R} \longrightarrow \mathcal{R}$ with $y = f(x) = x^3$.

The domain must be specified in such a way that the function is defined on the entire domain. (The range can be "too big" for the function, but not "too small.") The function $y = f(x) = \ln x$, for example, makes no sense anywhere but on the strictly positive reals: $f(0)$ does not map into the reals and neither does $f(x)$ for any $x < 0$. One would not, then, want to write that $f : \mathcal{R} \longrightarrow \mathcal{R}$ is defined by $y = \ln x$.

A.5.1 Bijections

Mathematicians have terms to describe a function that has particular qualities in regard to its use of its domain and range.

Definition A.22. A function $f : A \longrightarrow B$ is **one-to-one** if for each $a_1, a_2 \in A$ with $a_1 \neq a_2$, $f(a_1) \neq f(a_2)$. That is, one and only one member of A maps into any given member of B.

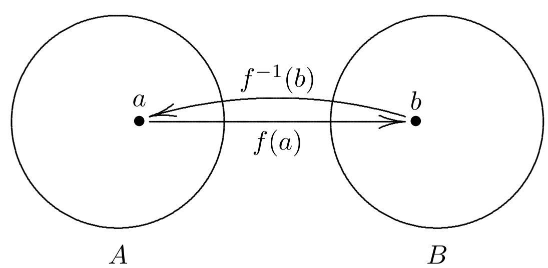
Definition A.23. A function $f : A \longrightarrow B$ is **onto** if for each $b \in B$, there is at least one $a \in A$ such that $f(a) = b$. That is, every member of A is mapped into by some member of B.

Note that the choice of A and B is an essential part of the definition of f.

Consider our function $f(x) = x^2$. If this function is defined by $f : \mathcal{R} \longrightarrow \mathcal{R}$, it is neither one-to-one (for every $b \in \mathcal{R}_{++}$, two elements of A map into b) nor onto (no elements of $\mathcal{R}$ map into $\mathcal{R}_{-}$).[2] If the function is defined by $f : \mathcal{R} \longrightarrow \mathcal{R}_{+}$, then it is onto but not one-to-one. If it is defined by $f : \mathcal{R}_{+} \longrightarrow \mathcal{R}_{+}$, then it is both one-to-one and onto.

Definition A.24. If $f : A \longrightarrow B$ is both one-to-one and onto, then f is a **bijection**.

The following diagram illustrates for an abstract case.



Theorem A.6. If $f : A \longrightarrow B$ is a bijection, then there exists a function $f^{-1} : B \longrightarrow A$, the **inverse function** of f, such that

$$\begin{aligned} f(f^{-1}(b)) &= b \quad \forall\, b \in B \quad \text{and} \\ f^{-1}(f(a)) &= a \quad \forall\, a \in A. \end{aligned}$$

Example A.12. If $f : \mathcal{R}_{+} \longrightarrow \mathcal{R}_{+}$ is given by $y = f(x) = x^2$, then $f^{-1}(y) = \sqrt{y}$ is the inverse function, where $f^{-1} : \mathcal{R}_{+} \longrightarrow \mathcal{R}_{+}$.

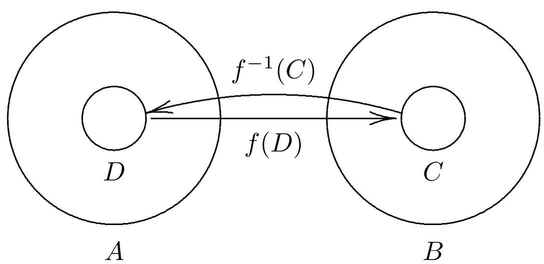
Example A.13. If $f : \mathcal{R}_{++} \longrightarrow \mathcal{R}$ is given by $y = f(x) = \ln x$, then $f^{-1}(y) = e^y$ is the inverse function, where $f^{-1} : \mathcal{R} \longrightarrow \mathcal{R}_{++}$.

[2]The notation for a subset of $\mathcal{R}^n$ whose components are all strictly greater than zero is $\mathcal{R}^n_{++}$. The notation for a subset of $\mathcal{R}^n$ whose components are all strictly negative is $\mathcal{R}^n_{--}$.

The behavior of a function with respect to its domain and range will sometimes be described using the terms *image* and *pre-image*. The image of a function at a given point or set in its domain is simply the set of points in the range mapped into by the given point or set. The pre-image of a point or set in the range is the point or set in the domain that map(s) into the given point or set.

Definition A.25. For $f : A \longrightarrow B$ and $D \subset A$, let $f(D) = \{b \in B : f(d) = b \text{ for some } d \in D\}$, and for $C \subset B$, let $f^{-1}(C) = \{a \in A : f(a) \in C\}$. The set $f(D)$ is the image of D under f and the set $f^{-1}(C)$ is the pre-image of C under f.

If f is a bijection, then for any point $a \in A$, the pre-image of the image of a is a itself. Likewise, for any set $D \subset A$, with image $C \subset B$, the pre-image of C is D itself, as the following figure illustrates.



A.5.2 Composite functions

Consider two functions $f : A \longrightarrow B$ and $g : B \longrightarrow C$.

Definition A.26. The composition $g \circ f : A \longrightarrow C$ is the function $g \circ f(a) = g(f(a))$. This notation is read "g of f of x".

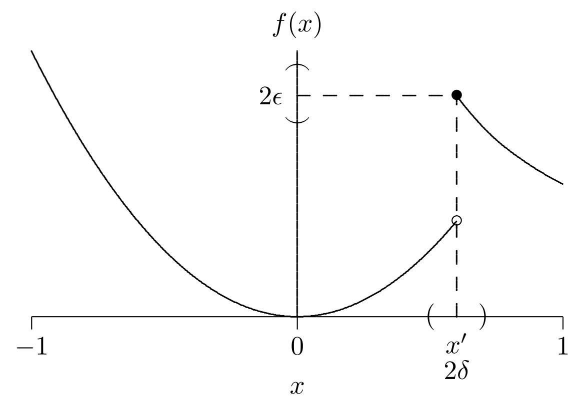
Example A.14. Let $f : [0, 1] \longrightarrow \mathcal{R}$ be defined by $f(x) = 3x$ and let $g : \mathcal{R} \longrightarrow \mathcal{R}$ be defined by $g(y) = y^2 + \sqrt{y}$. Then the composite function is $g(f(x)) = 9x^2 + \sqrt{3x}$.

A.5.3 Continuity

We all know that a function $f : \mathcal{R} \longrightarrow \mathcal{R}$ is continuous if it can be drawn on a sheet of paper without lifing the pen from the paper. In some cases this definition is not sufficiently precise. There are strange functions, such as $f : \mathcal{R}_{++} \longrightarrow \mathcal{R}$ given by $y = f(x) = \sin(1/x)$, that behave in such a way that it is difficult to tell whether the function is continuous as x approaches zero. A more formal notion of continuity is often useful.

Definition A.27. For $A \subset \mathcal{R}^n$, the function $f : A \longrightarrow \mathcal{R}^m$ is **continuous at** $x^0 \in A$ if for all $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in A$, $[\,||x - x^0|| < \delta\,] \Longrightarrow [\,||f(x) - f(x^0)|| < \epsilon\,]$.

The following diagram shows a function that is not continuous at x'.



How can our definition be interpreted at x'? Take ϵ, as in the figure, where 2ϵ is the length of the open interval around $f(x')$. Is there a δ around x' sufficiently small that for all x less than δ from x', $f(x)$ is less than ϵ from $f(x)$? No. Thus, f is not not continuous at x'.

Definition A.28. For $A \subset \mathcal{R}^n$, the function $f : A \longrightarrow \mathcal{R}^m$ is **continuous** if it is continuous for all $x \in A$.

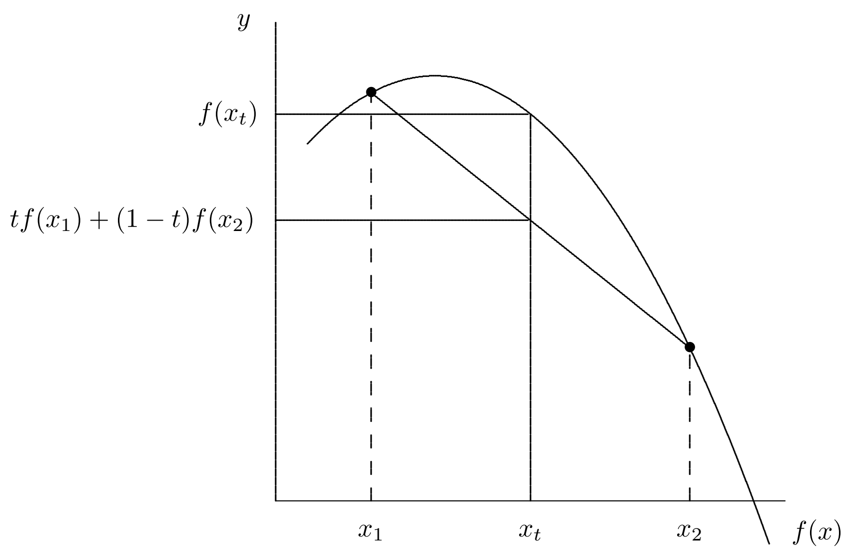
The following facts may be interesting. If $f : A \longrightarrow B$ is a continuous function, then the images of compact sets in A are compact in B and the images of open sets in A are open in B. (See the definition of compactness below.)

A.5.4 Concavity

Definition A.29. For $A \subset \mathcal{R}^n$, a function $f : A \longrightarrow \mathcal{R}$ is **concave** if for all x_1, x_2 in A, for all $t \in [0,1]$, $f(tx_1 + (1-t)x_2) \geq tf(x) + (1-t)f(y)$. f is **strictly concave** if the inequality is strict for $t \in (0,1)$.

Definition A.30. For $A \subset \mathcal{R}^n$, a function $f : A \longrightarrow \mathcal{R}$ is **convex** if for all x_1, x_2 in A, for all $t \in [0,1]$, $f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$. f is **strictly convex** if the inequality is strict for $t \in (0,1)$.

A concave function can be increasing or decreasing, but it "curves down," as in the following figure, where $x_t = tx_1 + (1-t)x_2$. A convex function "curves up." The function $y = f(x) = x^2$ is convex. Concavity and convexity are important characteristics for optimization techniques, as we will see.



Note that the notion of a convex function is quite different than the notion of a convex set.

A more esoteric concept, but one that we will also use, is that of quasi-concavity. Here I give only a definition; elsewhere the notion is explained more fully.

Definition A.31. For $A \subset \mathcal{R}^n$, a function $f : A \longrightarrow \mathcal{R}$ is **quasi-concave** if for all $y \in \mathcal{R}$, the set $A_y = \{x \in A : f(x) \geq y\}$ is convex in $\mathcal{R}$. f is **strictly quasi-concave** if A_y is strictly convex.

Definition A.32. For $A \subset \mathcal{R}^n$, a function $f : A \longrightarrow \mathcal{R}$ is **quasi-convex** if for all $y \in \mathcal{R}$, the set $A_y = \{x \in A : f(x) \leq y\}$ is convex in $\mathcal{R}$. f is **strictly quasi-convex** if A_y is strictly convex.

A.6 Relations

Even if you haven't seen it before, the concept of relations is essential to the study of consumer behavior. Preferences were probably represented exclusively by utility functions in your intermediate microeconomic theory course, but underlying the utility function is a binary preference ordering based upon relations.

Two given entities x and y may be "related" in many ways. Consider the following list:

$$\begin{aligned}
&x = y \\
&x < y \\
&x \text{ is "better than" } y \\
&x \text{ is "further than" } y \\
&x \text{ is "less damp than" } y.
\end{aligned}$$

Generically, let the notation xRy mean that x is related to y under the relation R.

Example A.15. Suppose that R means "greater than" for $x, y \in \mathcal{R}$. Then xRy is equivalent to $x > y$.

Example A.16. Suppose that R means "lives in the same county as." Then xRy means that x lives in the same county as y.

The following three properties of relations play an important role in the theory of the consumer.

Definition A.33. Given a set of alternatives X, we say that the relation R is **reflexive** if xRx for any $x \in X$. R is **symmetric** if $xRy \Longrightarrow yRx$ for any $x, y \in X$. R is **transitive** if $[xRy \text{ and } yRz] \Longrightarrow xRz$ for any $x, y, z \in X$. If R is reflexive, transitive, and symmetric, then it is an **equivalence relation**. A set $A \subset X$ whose members are all related to each other by the equivalence relation is called an **equivalence class.**

An equivalence relation can be used to divide an underlying set into subsets whose members are the same according to some criterion of interest. For example, if the underlying set is $X = \mathcal{R}$, then the relation "=" is an equivalence relation, though an extremely simply one: only x is related to itself. A more interesting example is the indifference relation on $\mathcal{R}^n_+$, whose equivalence classes are made up of indifference curves. Other examples, on the obvious underlying sets respecitvely, are

"is the same age as"
"comes from the same county as"
"receives the same amount of rainfall as".

In the last case, one can recall contour maps showing areas with equal average annual rainfall, represented by curves on the map.

Given a utility function $U : \mathcal{R}^2_+ \longrightarrow \mathcal{R}$, an equivalence class is given by $U^c = \{x \in \mathcal{R}^2_+ : U(x) = c\}$. The indifference relation, usually denoted "$x \sim y$," is read "x gives the same level of utility as y," or "the consumer is indifferent between bundles x and y."

To check whether indifference is really an equivalence class, we check whether it satisfies the three criteria: reflexivity, symmetry, and transitivity.

1. *Reflexivity:* Is $U(x) = U(x)$? Yes.
2. *Symmetry:* If $U(x) = U(y)$, must $U(y) = U(x)$? Yes.
3. *Transitivity:* If $U(x) = U(y)$ and $U(y) = U(z)$, must $U(x) = U(z)$? Yes.

A relation R can have other interesting properties, including the following.

1. *Completeness:* for all $x, y \in X$, xRy or yRx.
2. *Irreflexivity:* For all $x \in X$, $\neg xRx$.
3. *Totality:* For all $x, y \in X$ with $x \neq y$, xRy or yRx.
4. *Asymmetry:* For all $x, y \in X, [xRy] \Longrightarrow \neg[yRx]$.

Consider the order operators on the reals. The operator "equal to" behaves like indifference in consumer theory. The operator "greater than or equal to" behaves like preference in consumer theory. The operator "strictly greater than" behaves like strict preference in consumer theory.

The weak preference operator "$\succcurlyeq$" (we say that $x \succcurlyeq y$ if the consumer prefers x to y) is complete, reflexive, and transitive. The indiffrence operator $\sim$ is reflexive, symmetric, and transitive. The strict preference operator $\succ$ (we say that $x \succ y$ if the consumer strictly prefers x to y) is irreflexive, asymmetric, and transitive (and total). Sometimes, given an underlying preference ordering $\succcurlyeq$, people call $\sim$ the *symmetric part* and $\succ$ the *asymmetric part* of $\succcurlyeq$.

A.7 Linear Algebra

Linear algebra is an extremely useful set of techniques and methods. It is the dominant language of statistics and econometrics at the graduate level. This language, and the corresponding tools, are also useful for microeconomic theory. We will concern ourselves with the notation of linear algebra, as well as the manipulation of matrices. We will begin with some definitions and basic matrix operations.

A.7.1 Matrix addition, multiplication, and transpose

A matrix is a rectangular array of numbers,

$$\mathbf{A}_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

where m is the number of rows and n the number of columns in $\mathbf{A}$. Two matrices can be added together if they are of the same *dimension*, $m \times n$. In the case of 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$, the sum $\mathbf{A}+\mathbf{B}$ is

$$\mathbf{A}+\mathbf{B} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}.$$

If the matrices have different dimensions, the addition operation is undefined.

A *scalar* is a 1×1 matrix; a number. Scalar multiplication is the operation of multiplying a matrix by a scalar:

$$c\mathbf{A} = c\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{bmatrix}.$$

A *vector* is a one-dimensional matrix and can be either a column or a row vector. We will typically denote vectors by bold lower-case letters such as $\mathbf{a}$. Unless stated otherwise, a vector is a column of numbers. These are represented, respectively, by

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}, \quad \text{and} \quad \mathbf{a}' = \begin{bmatrix} a_1 & a_2 & \dots & a_m \end{bmatrix},$$

where the " $'$ " symbol refers to the *transpose* operator, in which rows are made columns. (See more on this operator below.)

Matrix multiplication is more complicated. It is carried out by multiplying rows in the first matrix by columns in the second. The simplest case is the multiplication of two vectors. If we have vectors $\mathbf{a}$ and $\mathbf{b}$, both of the same dimension m, their product is

$$\mathbf{a}'\mathbf{b} = \begin{bmatrix} a_1 & a_2 & \dots & a_m \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = a_1b_1 + a_2b_2 + \cdots + a_mb_m = \sum_{i=1}^{m} a_ib_i.$$

The product of two vectors is called their *dot product* or *inner product*, also denoted $\mathbf{a} \cdot \mathbf{b}$, and is undefined if the two vectors are not of the same dimension. More generally, matrix multiplication requres a particular kind of *conformability*. In the expression $\mathbf{AB}$, the number of columns in $\mathbf{A}$ must equal the number of rows in $\mathbf{B}$. For example, if

$$\mathbf{A}_{3\times 2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \quad \text{and} \quad \mathbf{B}_{2\times 2} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix},$$

then their product is

$$\mathbf{AB}_{3\times 2} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \\ a_{31}b_{11} + a_{32}b_{21} & a_{31}b_{12} + a_{32}b_{22} \end{bmatrix}.$$

The order of multiplication matters. If $\mathbf{A}$ is 3×5 and $\mathbf{B}$ is 5×4, then the operation $\mathbf{AB}$ is well defined (and results in a product matrix $\mathbf{AB}_{3\times 4}$), but $\mathbf{BA}$ is not. If $\mathbf{B}$ were instead 5×3, both operations would be well defined but $\mathbf{AB}$ would be of dinemsion 3×3 while $\mathbf{BA}$ would be 5×5.

Given a matrix $\mathbf{A}_{m\times n}$, its *transpose* is the matrix obtained by interchanging the rows and columns of $\mathbf{A}$:

$$\mathbf{A}' = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix},$$

The transpose of the product of two matrices is the product of their transposes:

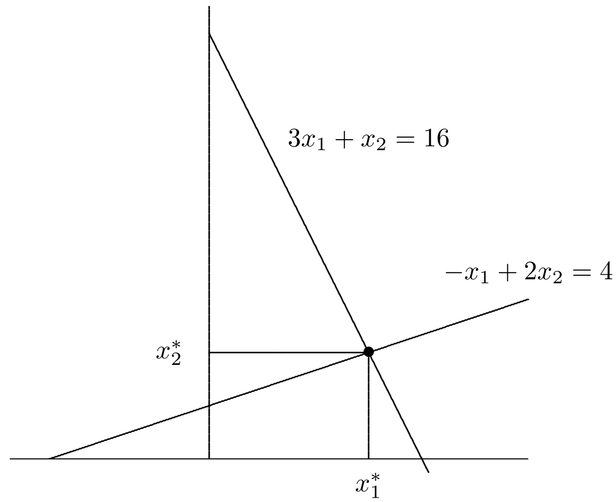
$$(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'.$$

A.7.2 Determinants and sets of linear equations

An important use to which linear algebra can be put is the solution of sets of linear equations. Consider the following system of two equations in two unknowns:

$$3x_1 + x_2 = 16 \tag{A.2}$$
$$-x_1 + 2x_2 = 4. \tag{A.3}$$

We would like to solve them simultaneously for x_1^* and x_2^*, which is the vector at which the two lines cross in $\mathcal{R}^2$.



One method of solution is substitution. Solve equation A.2 for $x_1 = 2x_2 - 4$ and substitute this expression into A.2 where x_1 appears:

$$3(2x_2 - 4) + x_2 = 16,$$

which can be written $7x_2 = 28$, or $x_2^* = 4$. Plug this into the expression for x_1 to obtain $x_1^* = 4$.

The system of equations could also be written in matrix form as

$$\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}_{2\times 2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{2\times 1} = \begin{bmatrix} 16 \\ 4 \end{bmatrix}_{2\times 1}, \tag{A.4}$$

or as

$$\mathbf{Ax} = \mathbf{b},$$

where $\mathbf{A}$ is the 2×2 matrix in (A.4).

If the matrices in (A.4) were scalar, so that $ax = b$, and if $a \neq 0$, solving for x would be simple: divide both sides by a to get $x = b \cdot (1/a)$. We want to find an object that is analogous to $(1/a)$ or a^{-1}. That object is the *inverse matrix* $\mathbf{A}^{-1}$. This matrix would have the property that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$, where $\mathbf{I}$ is the n-dimensional *identity matrix*, a square matrix with n rows and columns that has ones on the main diagonal and zeros everywhere else:

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ & & \dots & & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}.$$

An identity matrix is analogous to the number one: any matrix pre- or postmultiplied by $\mathbf{I}$ is unchanged. If $\mathbf{A}$ is of dimension $m \times n$, then

$$\mathbf{I}_m\mathbf{A} = \mathbf{A}\mathbf{I}_n = \mathbf{A}.$$

A matrix has an inverse only if it is both square and nonsingular. A matrix is nonsingular if its *determinant* is nonzero, or equivalently if none of its rows (or columns) is a linear combination of other rows (or columns). In order to understand the determinant, it is easiest to define the formula for 2×2 and 3×3 matrices. Consider the matrix

$$\mathbf{A}_{2\times 2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

The determinant of $\mathbf{A}$, denoted $\det(\mathbf{A})$ or $|\mathbf{A}|$, is given by

$$\det(\mathbf{A}) = a_{11}a_{22} - a_{12}a_{21}.$$

For $\mathbf{A}$ given in (A.4), the determinant is $\det(\mathbf{A}) = 3 \cdot 2 - 1 \cdot (-1) = 7$. The following matrix is singular: the second column is exactly twice the first column, which means that the columns are *linearly dependent*:

$$\mathbf{B} = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}.$$

The determinant of $\mathbf{B}$ is $\det(\mathbf{B}) = 2 \cdot (-2) - 4 \cdot (-1) = 0$.

The determinant of a 3×3 matrix,

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

is

$$\det(\mathbf{A}) = a_{11} \cdot (a_{22}a_{33} - a_{23}a_{32}) - a_{12} \cdot (a_{21}a_{33} - a_{23}a_{31}) + a_{13} \cdot (a_{21}a_{32} - a_{22}a_{31}).$$

We can generalize this to reveal the pattern as the size of the matrix increases. If $\mathbf{A}$ is an $n \times n$ matrix, let $\mathbf{A}_{ij}$ be the $(n-1) \times (n-1)$ submatrix obtained by deleting the ith row and the jth column of $\mathbf{A}$. The scalar $M_{ij} = \det(\mathbf{A}_{ij})$ is called the (i, j)th *minor* of $\mathbf{A}$ and the scalar $C_{ij} = (-1)^{i+j}M_{ij}$ is called the (i, j)th *cofactor* of $\mathbf{A}$. A minor and a cofactor are distinguished only by the fact that the cofactor is signed; if $i + j$ is even they are the same.

The determinant of a 2×2 matrix may be written

$$\begin{aligned} \det(\mathbf{A}_{2\times 2}) &= a_{11}M_{11} - a_{12}M_{12} \\ &= a_{11}C_{11} + a_{12}C_{12}, \end{aligned}$$

and the determinant of a 3×3 matrix may be written

$$\begin{aligned}\det(\mathbf{A}_{3\times 3}) &= a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13} \\ &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}.\end{aligned}$$

The determinant of an $n \times n$ matrix is

$$\begin{aligned}\det(\mathbf{A}_{n\times n}) &= a_{11}M_{11} - a_{12}M_{12} + \cdots + (-1)^{n+1}a_{1n}M_{1n} \\ &= a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}.\end{aligned}$$

Fortunately, computers are very good at computing inverses of even quite large matrices.

A.7.3 Inverse matrices

Now we are prepared to define the inverse of a matrix. First, though, we must set out an intermediate step.

Definition A.34. Given a matrix $\mathbf{A}_{n\times n}$, the **adjoint** of $\mathbf{A}$, written $\mathrm{adj}(\mathbf{A})$, is the $n \times n$ matrix whose (i, j)th entry is C_{ji}, the (j, i)th cofactor of $\mathbf{A}$.

The inverse of a square matrix may be derived by combining its determinant and its adjoint.

Theorem A.7. The inverse of a matrix $\mathbf{A}_{n\times n}$ is given by

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \cdot \mathrm{adj}(\mathbf{A}).$$

Example A.17. Consider the 2×2 matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}.$$

The determinant of $\mathbf{A}$ is $|\mathbf{A}| = -5$. The adjoint is

$$\mathrm{adj}(\mathbf{A}) = \begin{bmatrix} 1 & -2 \\ -4 & 3 \end{bmatrix}.$$

Thus, the inverse of $\mathbf{A}$ is

$$\mathbf{A}^{-1} = \frac{-1}{5}\begin{bmatrix} 1 & -2 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} & \frac{2}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix}.$$

Check for yourself to make sure that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

Example A.18. Consider now the 3×3 matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 1 & 3 \\ 1 & 6 & 5 \end{bmatrix}.$$

The determinant is $|\mathbf{A}| = 15$. The $(2,3)$ cofactor of $\mathbf{A}$ is $\mathbf{C}_{23} = 2 \cdot 6 - 3 \cdot 1 = 9$. You can find the others for yourself, as practice. The inverse of $\mathbf{A}$ is

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\mathrm{adj}(\mathbf{A}) = \frac{1}{15}\begin{bmatrix} & -13 & 9 & 5 \\ & -17 & 6 & 10 \\ 23 & -9 & & -10 \end{bmatrix}.$$

A.7.4 Cramer's rule

There is another technique, based on determinants, for solving the $n \times n$ system $\mathbf{Ax} = \mathbf{b}$. Let $\mathbf{B}_i$ denote the $n \times n$ matrix obtained by replacing the ith column of $\mathbf{A}$ by the vector $\mathbf{b}$. If $n = 3$ we have, for example,

$$\mathbf{B}_1 = \begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix}.$$

Theorem A.8 (Cramer's rule)**.** Given an $n \times n$ system $\mathbf{Ax} = \mathbf{b}$ with $\det(\mathbf{A}) \neq 0$, the unique solution x_i is

$$x_i^* = \frac{\det(\mathbf{B}_i)}{\det(\mathbf{A})}.$$

Returning to the example in equation (A-3), we can illustrate Cramer's rule. We have

$$\mathbf{B}_1 = \begin{bmatrix} 16 & 1 \\ 4 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{B}_2 = \begin{bmatrix} 3 & 16 \\ -1 & 4 \end{bmatrix}.$$

Thus, $\det(\mathbf{B}_1) = 16 \cdot 2 - 4 \cdot 1 = 28$ and $\det(\mathbf{B}_2) = 3 \cdot 4 - 16 \cdot (-1) = 28$. The determinant of $\mathbf{A}$ is $\det\mathbf{A} = 7$, so

$$x_1^* = \frac{28}{7} = 4 \quad \text{and} \quad x_2^* = \frac{28}{7} = 4,$$

as we expected.

A.8 Topology

For this section, we will confine our attention to $\mathcal{R}^n$, which is only one case of an ordered field. (The set of all continuous functions on the interval $[0,1]$, for example, is also an ordered field.) We begin with the notion of an open set.

Definition A.35. Given $x, y \in \mathcal{R}^n$, the *distance* between x and y is

$$d(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.$$

The distance between x and y is sometimes written $||x - y||$ and the magnitude of x is sometimes written $||x||$.

Note that in $\mathcal{R}^2$, this is the formula for the length of the hypotenuse of a right triangle.

Definition A.36. Given $x \in \mathcal{R}$ and an arbitrarily small number $\epsilon > 0$, an **ϵ-ball around** x is

$$B(x, \epsilon) = \{y \in \mathcal{R}^n : d(x, y) < \epsilon\}.$$

Definition A.37. A set $A \in \mathcal{R}^n$ is **open** if $\forall x \in A$, there exists an $\epsilon > 0$ with $B(x, \epsilon) \subset A$.

Example A.19. Consider the set $A = (0, 1) \in \mathcal{R}$. This set is open because for any element in the set, including $a = 0.9999$, one can find an ϵ sufficiently small so that the ϵ-ball around a remains in A. (Take $\epsilon = 0.000001$.)

Example A.20. Consider the set $A = [0, 1] \in \mathcal{R}$. This set is not open. For $a = 1$, there is no ϵ small enough to keep the ϵ-ball around a in the set. The ball will always include some elements in A and some not in A.

Example A.21. Consider the set $A = \{x \in \mathcal{R}^2 : |x| \leq 1\}$. This set is not open. In the cases we will consider, from $\mathcal{R}^n$, an open set is one that does not contain its boundary.

Definition A.38. Given a set $A \subset \mathcal{R}^n$, the **interior of** A is

$$\text{int}(A) = \{x \in \mathcal{R}^n : \exists\, \epsilon > 0 \text{ with } B(x, \epsilon) \subset A\}.$$

Example A.22. For $A = \{x \in \mathcal{R}^2 : ||x|| \leq 1\}$, the interior is the open ball around zero with radius one.

Example A.23. For $A = \{(x, y) \in \mathcal{R}^2_+ : x \cdot y \geq 1\}$, the interior is the set of all points in the northeast quadrant strictly above the rectangular hyperbola given by $x \cdot y = 1$. And what is the subscript "+" on $\mathcal{R}^2$ here? It means that we consider only elements of $\mathcal{R}^2$ each of whose components is greater than or equal to zero.

The definition of a closed set is derived from our primitive definition of an open set.

Definition A.39. A set $A \subset \mathcal{R}^n$ is **closed** if its complement in $\mathcal{R}^n$, A^c, is open.

Example A.24. Consider the set $A = (-\infty, 0] \cup [5, \infty)$. This set is closed because its complement, $A^c = (0, 5)$, is open.

Example A.25. Consider the set $A = \{x \in \mathcal{R}^n : ||x|| \leq 1\}$. This set is closed.

Example A.26. Consider the set $A = \{x \in \mathcal{R}^2 : x_2 \geq 0 \text{ and } x_2 < x_1^2\}$. This set is neither open nor closed.

The notion of the boundary of a set can be used to define open and closed sets in a slightly more intuitive way.

Definition A.40. Given a set $A \subset \mathcal{R}^n$, its **boundary**, denoted $\text{bd}(A)$, is

$$\text{bd}(A) = \{x \in \mathcal{R}^n : \forall \epsilon > 0, B(x, \epsilon) \cap A \neq \emptyset \text{ and } B(x, \epsilon) \cap A^c \neq \emptyset\}.$$

An open set in $\mathcal{R}^n$, put simply, is one that does not contain any part of its boundary. A closed set is one that contains all of its boundary. A set that contains only part of its boundary is neither open nor closed.

We will see that in optimization, closed sets play a very important role. So do compact sets.

Definition A.41. Given a set $A \subset \mathcal{R}^n$, A is **bounded** if there is an integer $M > 0$, possibly large, with

$$A \subset B(0, M) = \{x \in \mathcal{R}^n : ||x|| < M\}.$$

Example A.27. For $N < \infty$, consider the set $A = \{x \in \mathcal{R}^n_+ : x_1 + x_2 \leq N\}$. This set is bounded (take $M > N$).

Example A.28. Consider the set $A = \{x \in \mathcal{R}^n_+ : x_2 > 1/x_1\}$. This set is unbounded.

Optimization is often a well-defined operation only if the constraint set is both closed and bounded. There is a name for this joint property as well.

Definition A.42. A set $A \subset \mathcal{R}^n$ is **compact** if it is closed and bounded.

A compact set contains its boundary and does not “blow up.”

A.9 Methodological Readings

In economics, perhaps too often, the focus is so firmly upon learning and doing the mathematics that we spend little time considering whether it is *good* for economics to be so mathematical. There are people who think about this, though. The following readings may be interesting.

Krugman, Paul, “Two Cheers for Formalism,” *The Economic Journal*, 108 (1998), 1829–1836. Krugman argues that mathematics in economics—formalism—is fine. See also the other articles in the symposium on formalism in economics in the same issue of the journal.

McKloskey, Donald, *The Rhetoric of Economics*, (Madison, WI: University of Wisconsin Press 1985). McCloskey argues that not only economics but also physics, and mathematics itself, are really not as precise as some would have you believe: it's all about persuasion.

Morishima, Michio, “General Equilibrium Theory in the Twenty-First Century,” *The Economic Journal*, 101 (1991), 69–74. Morishima worries that economics has become too mathematical. See also the other articles in this special centennial issue of the journal, regarding the next 100 years of economics.

von Neumann, John, and Oskar Morgenstern, *Theory of Games and Economic Behavior*, (Princeton, NJ: Princeton University Press, 1944). The first chapter of this book, “The Mathematical Method in Economics,” presents what I still regard as one of the best defenses of mathematics in economics.

Weintraub, E. Roy, *General Equilibrium Analysis: Studies in Appraisal*, (Cambridge: Cambridge University Press, 1985). A wonderful defense of the value of abstract and highly mathematical models in economics.