Day 7 Notes

Tuesday, August 17, 2021 9:01 AM

a_{ij} = the element in the i^{th} row and j^{th} column.

$m \times 1$ matrix : column vector

$1 \times n$ matrix : row vector

$$\begin{bmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{bmatrix}$$ main diagonal

Upper or lower triangular then $\det A$ = product of diagonal entries

LU factorization

$AA = A$ idempotent

Ex

1) $A+B$

$$= \begin{pmatrix} 2 & 8 \\ 0 & 7 \\ 8 & 5 \end{pmatrix} + \begin{pmatrix} 11 & -2 \\ 4 & 9 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2+11 & 8-2 \\ 0+4 & 7+9 \\ 8+0 & 5-1 \end{pmatrix} = \begin{pmatrix} 13 & 6 \\ 4 & 16 \\ 8 & 4 \end{pmatrix}$$

2) DNE

3)

4) DNE

5) $E+D$ $\begin{pmatrix} 6 & 2 & 0 \\ 10 & 9 & 3 \end{pmatrix} + \begin{pmatrix} 3 & 9 & 9 \\ 6 & 7 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 0+3 & 2+9 & 0+9 \\ 10+6 & 9+7 & 3+1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 11 & 9 \\ 16 & 16 & 4 \end{pmatrix}$$

Matrix multiplication

$$\underset{m\times p}{A}\,\underset{p\times n}{B} = \underset{m\times n}{AB}$$

\# cols of A = \# rows of B

"conformable"

$$AB = \begin{bmatrix} (\text{1st row of } A)\cdot(\text{1st col of } B) & & \\ (\text{2nd row of } A)\cdot(\text{1st col of } B) & & \\ & \ddots & \\ & & (AB)_{ij} = (i^{th}\text{ row of } A)\cdot(j^{th}\text{ col of } B) \end{bmatrix}$$

AB: each col of AB is a linear combination of the columns of A, using entries as coefficients.

$$\begin{array}{cc} A & B \\ \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} & \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ 5 & 3 \end{bmatrix} \end{array}$$

n-dim column vectors: $n\times 1$

Then $AB = \underbrace{\begin{bmatrix} \underset{\substack{\uparrow \\ \text{vector} \\ n\times 1}}{1a_1+2a_2+5a_3} & \underset{\substack{\uparrow \\ \text{vector} \\ n\times 1}}{2a_1+0a_2+3a_3} \end{bmatrix}}_{AB \text{ is } 2\times n}$

Ex $A = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 1 & 5 \end{pmatrix}$ 2×3

$B = \begin{pmatrix} 3 & 6 & 4 \\ 2 & 5 & 8 \\ 7 & 1 & 9 \end{pmatrix}$ 3×3

$$AB = \begin{pmatrix} 0\cdot 3+1\cdot 2+2\cdot 7 & 0\cdot 6+1\cdot 5+2\cdot 1 & 0\cdot 4+1\cdot 8+2\cdot 9 \\ 2\cdot 3+1\cdot 2+5\cdot 7 & 2\cdot 6+1\cdot 5+5\cdot 1 & 2\cdot 4+1\cdot 8+5\cdot 9 \end{pmatrix}$$

$$= \begin{pmatrix} 16 & 7 & 26 \\ 43 & 22 & 61 \end{pmatrix}$$

2×3

AB : 3×4, 4×6 — AB is 3×6

$\cancel{AC}$: 3×4, 6×3

$\cancel{BA}$: 4×6, 3×4

BC : 4×6, 6×3 — BC is 4×3

$\cancel{CB}$: 6×3, 4×6

CA : 6×3, 3×4 — CA is 6×4

Distributive

$A(B+C) = AB + AC$

$(A+B)C = AC + BC$

Note:

$A(B+C) \neq (B+C)A$

Transpose

$A^T = A'$

$(A')' = A$

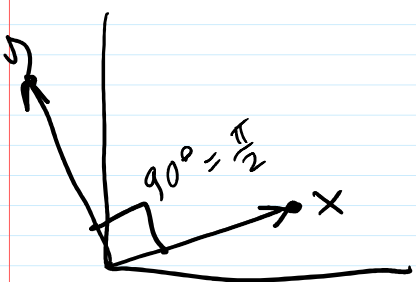
$(AB)' = B'A'$

Orthogonal matrix

Def'n: orthogonal

Two vectors $x, y \in \mathbb{R}^n$ are orthogonal if their inner product/dot product is zero.

$x \cdot y = 0$ iff y, x orthogonal



orthogonal matrix:

$$AA^T = I \quad \text{or} \quad A^T = A^{-1}$$

this means each column is orthogonal to all the others.

$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \cdots \end{bmatrix}$$

$$AA^T = \begin{bmatrix} a_1 \cdot a_1 & a_1 \cdot a_2 & \\ a_2 \cdot a_1 & a_2 \cdot a_2 & \\ \boxed{a_i \cdot a_j = 0} & & \ddots \end{bmatrix}$$

$$\begin{bmatrix} a_1 \cdot a_1 & & 0 \\ & a_2 \cdot a_2 & \\ 0 & & \ddots \end{bmatrix}$$

orthogonal matrices generally assume each column has a length of 1.

$$\|a_1\| = \sqrt{a_{11}^2 + a_{12}^2 + a_{13}^2}$$

---

symmetric if $A' = A$

$a_{ij} = a_{ji}$

Hessian is symmetric

$A'A$ is symmetric

Trace

$$Tr\begin{pmatrix} 1 & 2 & 3 \\ 10 & 20 & 30 \\ 100 & 200 & 300 \end{pmatrix}$$

$$= 1 + 20 + 300 = 321$$

Determinant

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$= ad - bc$$

Ex $c = \begin{pmatrix} 4 & 6 \\ 3 & 8 \end{pmatrix}$

$$|c| = 4 \cdot 8 - 3 \cdot 6 = 32 - 18$$
$$= 14$$

$$D = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

$|D| = 1\cdot 4 - 2\cdot 2 = 0$

col 2 = 2 × col 1

$\Rightarrow$ det = 0

Formula for determinant of $n\times n$ matrix: sum of products of n numbers, each product taking one element from each column.

Det $\neq 0$ iff the matrix is invertible.

Cofactor expansion

cofactor of A is

$$\underbrace{(-1)^{i+j}}_{\text{sign}} \cdot \text{Det}(\underbrace{A_{-ij}}_{\text{A with row } i \text{ and col } j \text{ deleted}})$$

Ex) $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 1 \\ 1 & 0 & 2 \end{pmatrix}$

Det A = ?

1) choose a row or column
2) go down the row, a multiply each element w/ its respective cofactor
3) Add them up

Expand along row 3

$\text{Det}(A) = 1\cdot C_{31} + 0\cdot C_{32} + 2\cdot C_{33}$

$$C_{31} = (-1)^{3+1}\cdot \begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix}$$

$$C_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix}$$

$$C_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix}$$

$$\text{Det}(A) = 1 \cdot (-1)^4 \cdot (2 \cdot 1 - 3 \cdot 4)$$

$$+ 0 \cdot (\ldots) \rightarrow 0$$

$$+ 2 \cdot (-1)^6 \cdot (1 \cdot 4 - 2 \cdot 0)$$

$$= (2 - 12) + 2 \cdot (4)$$

$$= -10 + 8 = -2$$

$$A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix}$$

$$B = \begin{bmatrix} \vec{a}_1 & 5\vec{a}_2 & \vec{a}_3 \end{bmatrix}$$

$$\text{Det } B = 5 \cdot \text{Det } A$$

$$B = \begin{bmatrix} \vec{a}_1 & \vec{a}_3 & \vec{a}_2 \\ & & \end{bmatrix}$$

$$\operatorname{Det} B = -\operatorname{Det} A$$

$$\operatorname{Det} AB = \operatorname{Det} A \operatorname{Det} B$$

Inverse Matrices

If A is square and $\exists B$ such that $AB = I$, the we call B the inverse of A, denoted A^{-1}.

$$AB = BA = I$$

All right inverses are also left inverses.

If A^{-1} does not exist, A is called <u>singular</u>.

If A^{-1} exists, A is called nonsingular or invertible.

nonsingular or invertible.

If AB is invertible

then $(AB)^{-1} = B^{-1}A^{-1}$.

→ be sure that A and

B are $n \times n$.

$$\text{Det}(A^{-1}) = \underbrace{\frac{1}{\text{Det}A}}_{\substack{\text{does not exist} \\ \text{if Det}A = 0 \\ \text{which means} \\ A^{-1}\text{ DNE}}}$$

Econometrics

Linear regression

$$\hat{\beta} = (X'X)^{-1}X'y$$

X^3 → all the observations on X^3

$$X = \begin{bmatrix} 1 & x_1^1 & x_1^2 & x_1^3 & \dots \\ 1 & x_2^1 & x_2^2 & x_2^3 & \dots \\ 1 & \vdots & & & \\ \vdots & \vdots & & & \end{bmatrix}$$

$$\begin{bmatrix} 1 & \vdots & & \\ 1 & \vdots & & \\ & X_N^1 & X_N^2 & X_N^3 \end{bmatrix}$$

Each row: one person's observations on each variable.

$(X'X)$ is singular if 2 variables are linearly dependent

Age < 20

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Age ≥ 20

col 1 = col 2 + col 3

Mathematical expression of inverse:

$$A^{-1} = \frac{1}{\mathrm{Det}(A)} \cdot C^T$$

where C is the matrix of cofactors of A

$$\begin{bmatrix} \underbrace{C_{11}} & C_{12} & C_{13} & \cdots \\ C_{21} & C_{22} & & \\ \vdots & & \ddots & \\ \vdots & & & \end{bmatrix}$$

2x2

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$C_{11} = (-1)^{1+1}|d| = d$$

$$C_{12} = (-1)^{1+2}|c| = -c$$

$$C_{21} = (-1)^{2+1}|b| = -b$$

$$C_{22} = (-1)^{2+2}|a| = a$$

$$C = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

So $C^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Proof: $(AB)^{-1} = B^{-1}A^{-1}$

$$B^{-1}A^{-1}AB = B^{-1}(A^{-1}A)B$$
$$= B^{-1}IB$$
$$= B^{-1}B$$
$$= I$$

So $(B^{-1}A^{-1})(AB) = I$

$$\therefore\ B^{-1}A^{-1} = (AB)^{-1} \quad \blacksquare$$

Finding an inverse using elementary row operations.

(i) multiply a row by a constant

$$E_i(\alpha): \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = E_3(\alpha)$$

$$A = \begin{pmatrix} - r_1 - \\ - r_2 - \\ - r_3 - \\ - r_4 - \end{pmatrix}$$

$$\underbrace{E_3(\alpha) \cdot A}_{\text{matrix multiplication}} = \begin{bmatrix} r_1 \\ r_2 \\ \alpha r_3 \\ r_4 \end{bmatrix}$$

(ii) Interchange 2 rows

$$P_{ij} = \begin{array}{r} \\ \\ \text{row}_i \to \\ \\ \text{row}_j \to \\ \\ \\ \\ \end{array} \begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 0 & & 1 & & & \\ & & & 1 & & & & \\ & & 1 & & 0 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{bmatrix}$$

Ex] $\underbrace{P_{13}}\cdot\underbrace{A}$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$= \begin{pmatrix} 0&0&1 \\ 0&1&0 \\ 1&0&0 \end{pmatrix} \begin{pmatrix} 1&2&3 \\ 4&5&6 \\ 7&8&9 \end{pmatrix}$$

$$= \begin{pmatrix} 7&8&9 \\ 4&5&6 \\ 1&2&3 \end{pmatrix}$$

(iii) add a multiple of one row to another

$$D_{ij}(\alpha) = \begin{bmatrix} 1&0&0&0&0 \\ 0&1&&&\\ 0&&1&&\\ \alpha&&&1&\\ 0&&&&1 \end{bmatrix}$$

add $\alpha \cdot$ row 4 to row 1

Inverse

$$\underbrace{D_{23}(5)}_{\text{op4}} \underbrace{E_4(\tfrac{1}{2})}_{\text{op3}} \underbrace{P_{14}}_{\text{op2}} \underbrace{E_2(3)}_{\text{op 1}} A = I$$

$$\boxed{\dot{D}\dot{E}\dot{P}\dot{E}\ I} \rightarrow DEPE = A^{-1}$$

Then we know

$(DEPE)A = I$

so $(DEPE) = A^{-1}$

$\boxed{\text{Ex}}$ Let $B = \begin{pmatrix} 1&2&3 \\ 2&4&5 \\ 3&5&6 \end{pmatrix}$

Find B^{-1}.

Augmented Matrix

$= [B \vdots I]$

Augmented Matrix

$= [B \,\vdots\, I]$

$$= \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 4 & 5 & 0 & 1 & 0 \\ 3 & 5 & 6 & 0 & 0 & 1 \end{array}\right]$$

→ goal: transform this into $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$\overset{-3R_1+R_3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 4 & 5 & 0 & 1 & 0 \\ 0 & -1 & -3 & -3 & 0 & 1 \end{array}\right]$$

$$\overset{-2R_1+R_2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & 1 & 0 \\ 0 & -1 & -3 & -3 & 0 & 1 \end{array}\right]$$

col 1 complete

$$\overset{R_2 \leftrightarrow R_3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -3 & -3 & 0 & 1 \\ 0 & 0 & -1 & -2 & 1 & 0 \end{array}\right]$$

pivots

Row echelon form

→ good enough to solve a linear system of equations

$$\overset{-1 \cdot R_2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3 & 3 & 0 & -1 \\ & & & & & \end{array}\right]$$

$$\overset{-1 \cdot R_3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3 & 3 & 0 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right]$$

next step: eliminate numbers above this pivot

$$-2R_2+R_1 \sim \left[\begin{array}{ccc:ccc} 1 & 0 & -3 & -5 & 0 & 2 \\ 0 & 1 & 3 & 3 & 0 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right]$$

$$-3R_3+R_2 \sim \left[\begin{array}{ccc:ccc} 1 & 0 & -3 & -5 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right]$$

$$3R_3+R_1 \sim \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & -3 & 2 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right]$$

GOAL inverse

Check:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}\begin{pmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1\cdot 1 + -3\cdot 2 + 2\cdot 3 & 1\cdot -3 + 2\cdot 3 - 1\cdot 3 & 1\cdot 2 + 2\cdot -1 + 3\cdot 0 \\ 2\cdot 1 + 4(-3) + 5\cdot 2 & 2(-3) + 4\cdot 3 + 5(-1) & 2\cdot 2 + 4(-1) + 5(0) \\ 3\cdot 1 + 5\cdot(-3) + 6(2) & 3\cdot(-3) + 5\cdot 3 - 6\cdot 1 & 3\cdot 2 + 5(-1) + 6\cdot 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \checkmark$$

So ~~A~~ IS the inverse.

Systems of Equations

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

$$\equiv \quad Ax = b$$

where $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \rightarrow$ coefficients

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \rightarrow$ solve for x

$b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \rightarrow$ known

---

Before: $[A \mid I]$

Transform A to I using elementary row operations.

Now: $[A \mid b]$

Transform A into row echelon form and then use it to solve the equation by back subst.
