UNIVERSITY OF SOUTH CAROLINA

MATH-546 Algebraic Structures I Homework 10

Problem 1:

Let $G = \mathbb{Z}_6$. Label the elements as follows: $g_1 = [0],\ g_2 = [1],\ g_3 = [2],\ g_4 = [3],\ g_5 = [4],\ g_6 = [5]$.

In the proof of Cayley's theorem, we constructed a function $F : G \to S_6$. Using the method from that proof, what is $F([2]_6)$? What is $F([5]_6)$? Give your answers as permutations in standard notation.

Answer 1:

Cayley table for $\mathbb{Z}_6$ mapped to g_i. We can read the permutation off the subscripts for each row:

$$\begin{array}{r|cccccc} + & g_1 & g_2 & g_3 & g_4 & g_5 & g_6 \\ \hline g_1 & g_1 & g_2 & g_3 & g_4 & g_5 & g_6 \\ g_2 & g_2 & g_3 & g_4 & g_5 & g_6 & g_1 \\ [2]_6 = g_3 & g_3 & g_4 & g_5 & g_6 & g_1 & g_2 \\ g_4 & g_4 & g_5 & g_6 & g_1 & g_2 & g_3 \\ g_5 & g_5 & g_6 & g_1 & g_2 & g_3 & g_4 \\ [5]_6 = g_6 & g_6 & g_1 & g_2 & g_3 & g_4 & g_5 \end{array}$$

$$F([2]_6) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 1 & 2 \end{pmatrix} = (1\ 3\ 5)\,(2\ 4\ 6)$$

$$F([5]_6) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 3 & 4 & 5 \end{pmatrix} = (1\ 6\ 5\ 4\ 3\ 2)$$

Problem 2:

Let $G = \mathbb{Z}_4 \times \mathbb{Z}_2$ and let $H = \{\sigma_1^i \sigma_2^j \mid i, j \in \mathbb{Z}\}$ where $\sigma_1 = (1\ 2)$ and $\sigma_2 = (3\ 4\ 5\ 6)$ are elements of S_6.

a. Prove that $|H| = 8$ and that H is a subgroup of S_6.

b. Prove that $G \cong H$.

Answer 2:

Since disjoint cycles commute:

a. $o(\sigma_1) = 2$ and $o(\sigma_2) = 4$ so $o(\sigma_1\sigma_2) = 8$
Identity: $e = \sigma_1^0\sigma_2^0 \in H$
Inverse: $(\sigma_1^i\sigma_2^j)^{-1} = \sigma_2^{-j}\sigma_1^{-i} = \sigma_1^{-i}\sigma_2^{-j} \in H$
Closure: $(\sigma_1^i\sigma_2^j)(\sigma_1^k\sigma_2^\ell) = \sigma_1^i(\sigma_2^j\sigma_1^k)\sigma_2^\ell = \sigma_1^i\sigma_1^k\sigma_2^j\sigma_2^\ell = \sigma_1^{i+k}\sigma_2^{j+\ell} \in H$

b. *Sketch:* Consider $F(([j]_4, [i]_2)) = \sigma_1^i\sigma_2^j$. It's 1-to-1, onto and easily shown to be a homomorphism.

Problem 3:

Find the smallest n such that S_n has a subgroup isomorphic to $\mathbb{Z}_{10}$. Justify your answer.

Answer 3:

S_7: an isomorphic subgroup must have elements with orders 2 and 5 and be cyclic and be abelian. Therefore disjoint cycles of length 2 and 5: to generate a subgroup of order 10.

$$\mathbb{Z}_{10} \cong \langle (1\ 2)(3\ 4\ 5\ 6\ 7) \rangle$$

Problem 4:

Find a subgroup of S_6 that is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$. Prove the isomorphism.

Answer 4:

Sketch: Let $\sigma_1 = (1\ 2\ 3)$ and $\sigma_2 = (4\ 5\ 6)$ with $F(([i]_3, [j]_3)) = \sigma_1^i \sigma_2^j$. Our subgroup is $H = \operatorname{Im} F = \{\sigma_1^i \sigma_2^j : \ i, j \in \mathbb{Z}\}$. This is onto by construction and 1-to-1 since elements each have order 3. F is a homomorphism mapping addition in $\mathbb{Z}_3$ to addition of exponents.

Problem 5:

Let

$$G = D_4 = \{e, r, r^2, r^3, s, rs, r^2s, r^3s\}$$

be the group of symmetries of a square, and let $H = \{e, r^2\}$. How many distinct cosets are there with respect to H? List all the cosets; for each coset, list its elements.

Answer 5:

There are $|G|/|H| = 4$ distinct cosets:

$$\begin{aligned}
eH = r^2H &= \{e, r^2\} \\
sH = r^2sH &= \{s, r^2s\} \\
rH = r^3H &= \{r, r^3\} \\
rsH = r^3sH &= \{rs, r^3s\}
\end{aligned}$$

Problem 6:

Let $G = S_3$ and $H = \{e, (1\ 2)\}$. We saw in class that there are three distinct cosets with respect to H. Decide whether a union of two of these cosets is a subgroup of S_3 or not. Justify your answer.

Answer 6:

It's not possible. The cosets of H each have 2 elements and are disjoint. Therefore the union of any 2 cosets has 4 elements but $|S_3| = 6$ so there are no subgroups of order 4 since 4 does not divide 6.

Problem 7:

Let G be a group and H a subgroup of G. Recall that H is called a normal subgroup if the following requirement holds:

$$\forall g \in G \wedge \forall h \in H : \; ghg^{-1} \in H.$$

Prove that H is a normal subgroup if and only if $aH = Ha$ for all $a \in G$ (right cosets equal left cosets).

Answer 7:

If $\forall a \in G : \; aH = Ha$ then $\forall g \in G \; \exists\, h, h' \in H \ni gh = h'g$ and so $\forall g \in G : \; ghg^{-1} = h'gg^{-1} = h' \in H$ and H is normal.

If H is normal then $\forall g \in G \wedge \forall h \in H$:

$$\begin{aligned} & ghg^{-1} \in H \\ & ghg^{-1}g \in Hg \\ & gh \in Hg \\ \therefore \quad & gH = Hg \end{aligned}$$

Problem 8:

Let G be a finite group and H a subgroup of G with $|H| = |G|/2$. Prove that H must be a normal subgroup.

Answer 8:

Since $|H| = |G|/2$ there are only two cosets and so if $a \in H$ then $aH = Ha$ by closure of H. If $a \notin H$ then $aH = G \setminus H$ likewise $Ha = G \setminus H$ and

$$\forall\, a \in G : aH = Ha$$

By problem 7 H is normal.