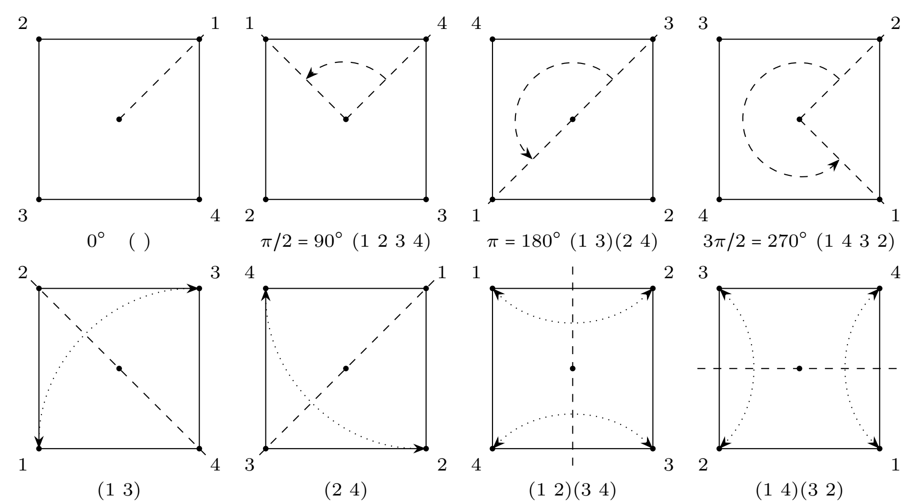
Problem 1

Consider the following elements of the group of symmetries of a square with vertices labeled $1, 2, 3, 4$: r = rotation by $\pi/2$ radians; s = reflection about the diagonal joining vertices 1 and 3.

(a) Write each one of r and s as permutations using the cycle notation.

(b) Calculate rs, r^2s, and r^3s as permutations using the cycle notation, and identify each of the results as a rotation or reflection. If reflection, specify about what axis of symmetry.

Answer 1:



(a) $r = (1\ 2\ 3\ 4)$, $s = (2\ 4)$

(b) All are reflections: vertices are V_n, midpoint of $V_m - V_n$ is M_{mn}.

$$\begin{aligned} rs &= (1\ 2\ 3\ 4)\ (2\ 4) = (1\ 2)(3\ 4) && \text{axis } M_{12} - M_{34} \\ r^2s &= (1\ 3)(2\ 4)\ (2\ 4) = (1\ 3) && \text{axis } V_2 - V_4 \\ r^3s &= (1\ 4\ 3\ 2)\ (2\ 4) = (1\ 4)(2\ 3) && \text{axis } M_{14} - M_{23} \end{aligned}$$

Problem 2:

(a) Give an example of the elements $x, y \in D_4$ that have order equal to 2, but xy has order equal to 4.

(b) Give an example of two reflections $x, y \in D_4$ such that xy is a rotation. Justify your answers.

Answer 2:

All reflections have order 2 as does r^2. And only r, r^3 have order 4.

(a) The product of two elements of order 2 must therefore be either $r^2 s$ for some reflection s or the product of two reflections. The former is ruled out because any rotation times a reflection must be a reflection. So we're looking for the product of two reflections.

$$(1\ 3)\ (1\ 2)(3\ 4) = (1\ 2\ 3\ 4) = r$$

(b) See 2a.

Problem 3:

Let G be a group and let $x, y \in G$ be two elements such that $x \neq y$ and $x, y \neq e$. Assume that y is not equal to a power of x and x is not equal to a power of y.

Prove that $H = \{e, x, y, xy\}$ is a subgroup of G if and only if x and y have order equal to 2 and $xy = yx$.

Answer 3:

$\Leftarrow$ If x and y have order 2 then they are their own inverse (as is e). And since $xy = yx$, $xyxy = xyyx = xx = e$ and so xy is its own inverse. Since every element of H has an inverse in H then all that's needed is to finish closure:

$$\begin{aligned} x(xy) = (xx)y &= y \in H \\ y(xy) = y(yx) = (yy)x &= x \in H \\ (xy)x = x(yx) = x(xy) = (xx)y &= y \in H \\ (xy)y = x(yy) &= x \in H \end{aligned}$$

$\Rightarrow$ The possible orders of elements of H are 1, 2 and 4. Only e has order 1. Neither x nor y can have order 4 since neither is a power of the other. The order of each must be 2 and each is it's own inverse. As H is a subgroup xy must have an inverse in H and the only candidate left is itself.

So x and y each have order 2 and $xy = (xy)^{-1} = y^{-1}x^{-1} = yx$.

Problem 4:

Find all the subgroups of the group

$$D_4 = \{e, r, r^2, r^3, s, rs, r^2s, r^3s\}$$

(r and s are as in problem 1).

Justify your answer (explain why the subgroups on your list are subgroups, and why there are no other subgroups). You should use the result from problem 3 as part of the justification.

Answer 4:

Since $|D_4| = 8$, the possible orders of elements are 1, 2, 4 and 8. The identity has order 1, the reflections and r^2 have order 2. The rotations r, r^3 have order 4.

The orders of the subgroups of D_4 must be 1, 2, 4 or 8.

The only subgroup of order 1 is $\{e\}$.

The only subgroup of order 8 is D_8.

The subgroups of order 2 must have both the identity and another element of order 2. The elements of order 2 are r^2 and the reflections and so the subgroups of order 2 are:

$$\{e, r^2\}, \{e, s\}, \{e, rs\}, \{e, r^2s\}, \{e, r^3s\}$$

The cyclic subgroup of order 4 is the cyclic subgroup generated by r or by r^3:

$$\langle r \rangle = \langle r^3 \rangle = \{e, r, r^2, r^3\}$$

Let H be a different subgroup of order 4. Neither r nor r^3 can be in H and so no element of H other than e is a power of another. By problem 3 the product of two is the third and those two commute.

Without r and r^3 two of the elements of H must be reflections and therefore the third must be the rotation r^2. That constrains the reflections to r^is, r^js where $r^2 = r^isr^js = r^ir^{-j}ss = r^ir^{-j}$ and so $(i - j) \in \{-2, 2\}$. Therefore H is one of

$$\{e, s, r^2s, r^2\} \qquad \{e, rs, r^3s, r^2\}$$

Problem 5:

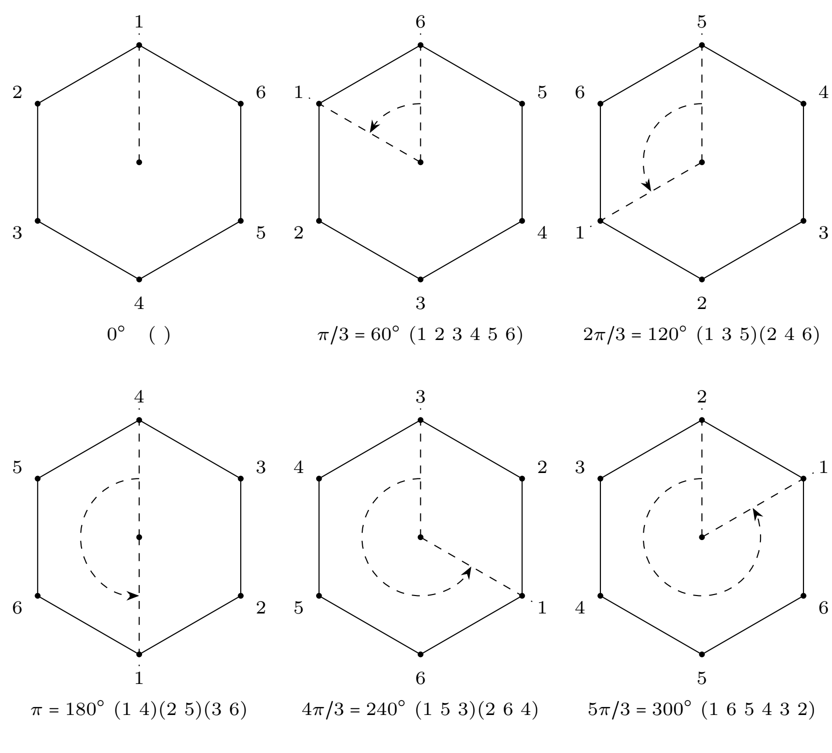
Consider the group D_6 of rotations and reflections of a regular hexagon. Label the vertices of the hexagon $1, 2, 3, 4, 5, 6$ in counterclockwise order.

(a) Write each element of D_6 as a permutation using the cycle notation and explain what each of them represents geometrically (rotation or reflection; if rotation, what is the angle of rotation; if reflection, what is the axis of symmetry).

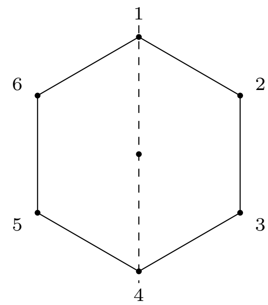
(b) Let r denote the rotation by $2\pi/6$ radians and let s denote the reflection about the diagonal joining the vertices 1 and 4. For each element from part (a), write the element as r^i or $r^i s$ for an appropriate choice of i.

Answer 5:

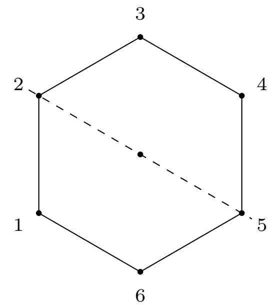
(a) Rotations:



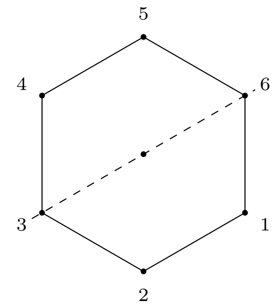
Reflections



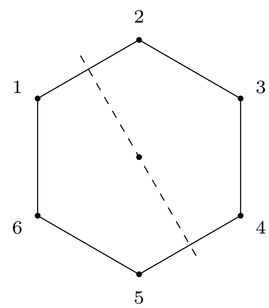
(2 6)(3 5)



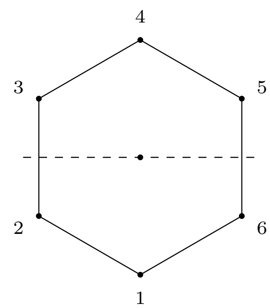
(1 3)(4 6)



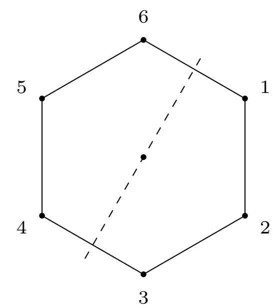
(1 5)(2 4)



(1 2)(3 6)(4 5)



(1 4)(2 3)(5 6)



(1 6)(2 5)(3 4)

(b) The rotations as above:

$$\begin{matrix} r^0 & r^1 & r^2 \\ r^3 & r^4 & r^5 \end{matrix}$$

The reflections as above:

$$\begin{matrix} r^0 s & r^2 s & r^4 s \\ rs & r^3 s & r^5 s \end{matrix}$$

Problem 6:

Consider the dihedral group described by generators and relations as follows:

$$D_n = \{e, r, \ldots, r^{n-1}, s, rs, \ldots, r^{n-1}s\}$$

with $o(r) = n$, $o(s) = 2$, and $sr = r^{-1}s$.

Using the information above, prove that each of the elements $r^j s$ for $1 \le j \le n-1$ has order equal to 2 (do not just state that these elements are reflections).

Answer 5:

Since $sr = r^{-1}s$ we have for arbitrary $j : 1 \le j \le n-1$:

$$sr^j = r^{-1}sr^{j-1} = r^{-2}sr^{j-2} = r^{-3}sr^{j-3} = \cdots r^{1-j}sr = r^{-j}s$$

and so:

$$\begin{aligned}
(r^j s)^2 &= (r^j s)(r^j s) \\
&= r^j (sr^j) s \\
&= r^j (r^{-j} s) s \\
&= (r^j r^{-j})(ss) \\
&= ee \\
&= e
\end{aligned}$$

and $o(r^j s) = 2$.