UNIVERSITY OF SOUTH CAROLINA

MATH-546 Algebraic Structures I Homework 2

Problem 1:

Let G be the set of all 2×2 matrices of the form $\begin{bmatrix} 1 & 0 \\ a & b \end{bmatrix}$ with $b \neq 0$. Prove that G is a group with matrix multiplication as the operation. *(Reminder: you may take it for granted that matrix multiplication is associative.)*

Answer 1:

Associativity: Given

Identity:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in G$$

Closure:

$$\begin{bmatrix} 1 & 0 \\ a & b \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ a + bc & bd \end{bmatrix} \in G$$

($bd \neq 0$ since $b \neq 0$ and $d \neq 0$)

Inverses: If $\begin{bmatrix} 1 & 0 \\ a & b \end{bmatrix}$ has an inverse it is of the form $\begin{bmatrix} 1 & 0 \\ c & d \end{bmatrix}$ with $d \neq 0$ and $\begin{bmatrix} 1 & 0 \\ a & b \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ a & b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and so

$$0 = a + bc$$
$$0 = c + ad$$
$$d = 1/b$$

Solving, we get $\begin{bmatrix} 1 & 0 \\ a & b \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -a/b & 1/b \end{bmatrix} \in G$ with $1/b \neq 0$.

Problem 2:

Let G be the set of all the functions of the form $f_{m,b} : \mathbb{R} \to \mathbb{R}$, where $f_{m,b}(x) = mx + b$ and m, b are real numbers with $m \neq 0$. Prove that G is a group with composition of functions as the operation.

Answer 2:

- Associativity: given (composition of functions)
- Closure: Let $f_{j,k}$ and $f_{p,q}$ be arbitrary elements of G:

$$\begin{aligned}
(f_{j,k} \circ f_{p,q})(x) &= f_{j,k}(f_{p,q}(x)) \\
&= f_{j,k}(px + q) \\
&= j(px + q) + k \\
&= (jp)x + (jq + k) \\
&= f_{jp,jq+k}(x) \\
f_{jp,jq+k} &\in G \qquad \text{since } jp \neq 0
\end{aligned}$$

- Identity: $f_{1,0}(x) = 1x + 0 = x$ so $f_{1,0}$ is the identity and $f_{1,0} \in G$
- Inverses: Let $f_{j,k} = (f_{p,q})^{-1}$ where $f_{p,q} \in G$ then

$$\begin{aligned}
(f_{j,k} \circ f_{p,q})(x) &= (jp)x + (jq + k) \\
&= 1x + 0 \\
j &= 1/p \\
k &= -q/p
\end{aligned}$$

  and $f_{1/p,-q/p} \circ f_{p,q} = f_{1,0}$ by construction with $f_{1/p,-q/p} \in G$ since $1/p \neq 0$. And

$$\begin{aligned}
(f_{p,q} \circ f_{1/p,-q/p})(x) &= f_{p,q}(x/p - q/p) \\
&= p(x/p - q/p) + q \\
&= x - q + q \\
&= x \\
f_{p,q} \circ f_{1/p,-q/p} &= f_{1,0}
\end{aligned}$$

Problem 3:

Let $(G, *)$ be a group and let $a, b \in G$ be arbitrary elements. Prove that
$$a * b = b * a \iff (a * b)^{-1} = (a^{-1} * b^{-1}).$$

Answer 3:

$\Rightarrow$ Let e be the identity

$$\begin{aligned}(a * b) &= (b * a)\\ (a * b)^{-1} &= (b * a)^{-1}\\ &= (a^{-1} * b^{-1})\end{aligned}$$

$\Leftarrow$ using $(a^{-1})^{-1} = a$

$$\begin{aligned}(a * b)^{-1} &= (a^{-1} * b^{-1})\\ ((a * b)^{-1})^{-1} &= (a^{-1} * b^{-1})^{-1}\\ (a * b) &= ((b^{-1})^{-1} * (a^{-1})^{-1})\\ &= b * a\end{aligned}$$

$\Leftarrow$ without using $(a^{-1})^{-1} = a$

$$\begin{aligned}(a * b)^{-1} &= (a^{-1} * b^{-1})\\ (a * b) * (a * b)^{-1} &= (a * b) * (a^{-1} * b^{-1})\\ e &= (a * b) * (a^{-1} * b^{-1})\\ (b * a) &= (a * b) * (a^{-1} * b^{-1}) * (b * a)\\ &= (a * b) * a^{-1} * (b^{-1} * b) * a\\ &= (a * b) * (a^{-1} * a)\\ &= (a * b)\\ a * b &= b * a\end{aligned}$$

Problem 4:

Let $(G, *)$ be a group. Assume that $a * a = e$ for all $a \in G$. Prove that G must be an abelian group.

Answer 4:

$\forall a, b \in G:$

$$\begin{aligned} a * b &= (a * b)^{-1} \\ &= (b^{-1} * a^{-1}) \\ &= b * a \end{aligned}$$

Problem 5:

Let $G = GL_2(\mathbb{R})$ be the group of 2×2 invertible matrices, with matrix multiplication as the operation.

a. Give an example of two elements $A, B \in G$ such that $AB \neq BA$.

b. For the A, B in your example for part (a), calculate ABA^{-1}. Is ABA^{-1} equal to B or not?

Answer 5:

a.

$$
\begin{aligned}
A &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \\
A^{-1} &= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad B^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \\
AB &= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \qquad \text{and} \qquad BA = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}
\end{aligned}
$$

b.

$$
\begin{aligned}
ABA^{-1} &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \neq B
\end{aligned}
$$

Alternatively if $ABA^{-1} = B$ then $ABA^{-1}A = BA$ and $AB = BA$ so A and B commute which is not the case here.

Theoretical Question 1:

Assume that $(G, *)$ is a group. Prove that the identity element is unique.

Answer:

That there exists an identity is in the definition of a group. So all we need show is that any two identities are the same element of G.

If e and e' are identities then $e = e' * e = e'$ so e' and e must be the same element of G.

Theoretical Question 2:

Assume that $(G, *)$ is a group, and let $a \in G$ be a fixed element. Prove that the inverse of a is unique.

Answer:

That there exists an inverse is in the definition of a group. So all we need show is that any two inverses are the same element of G.

Let b and c be inverses of a with e the identity of G, then

$$\begin{aligned} a * b &= e \\ c * a * b &= c \\ e * b &= c \\ b &= c \end{aligned}$$

Theoretical Question 3:

Let $(G, *)$ be a group, and let $a, b \in G$ be fixed arbitrary elements. Prove that there is a unique $x \in G$ such that $a * x = b$.

Answer:

Consider $x = a^{-1} * b$. $x \in G$ by closure.

$$\begin{aligned} a * x &= a * a^{-1} * b \\ &= e * b \\ &= b \end{aligned}$$

If $y \in G$ such that $a * y = b$ then by transitivity $a * x = a * y$ and by multiplying on the left by a^{-1} we get $x = y$ and so they must be the same element of G.