UNIVERSITY OF SOUTH CAROLINA

MATH 546 - Algebraic Structures I — Lecture Notes

1 Preliminaries

1.1 Notation

- $\ni$ is to be read as *such that*
- $\mathbb{N}$ - set of natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$
- $\mathbb{Z}$ - set of integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- $\mathbb{Q}$ - set of rational numbers $\mathbb{Q} = \{x/y \mid x \in \mathbb{Z} \wedge y \in \mathbb{N}\}$
- $\mathbb{R}$ - real numbers
- $\mathbb{C}$ - complex numbers
- $\mathbb{R}^* \equiv \mathbb{R} \setminus \{0\}$
- $a \mid b$ is to be read as *a divides b*

1.2 Definitions

- For $a, b \in \mathbb{Z}$ and $a \neq 0$: a divides b (or b is divisible by a) $\iff$ $\exists q \in \mathbb{Z} \ni aq = b$
- $p \in \mathbb{N} \setminus \{0, 1\}$ is *prime* $\iff$ $a \in \mathbb{N} \wedge a \mid p \Rightarrow a = p \wedge a = 1$
- greatest common divisor: For $a, b \in \mathbb{Z}$ (both not 0): $\gcd(a, b)$ is largest integer that divides both a and b
- $a, b \in \mathbb{Z}$ are *relatively prime* $\iff$ $\gcd(a, b) = 1$
- least common multiple: $a, b \in \mathbb{Z}^* : \operatorname{lcm}(a, b)$ is the smallest $n \in \mathrm{N}$ such that $(a \mid n) \wedge (b \mid n)$

1.3 Facts

Offered without proof:

Theorem. Fundamental Theorm of Arithmetic: every $n \in \mathbb{N}^* \setminus \{1\}$ has a unique (up to order) prime factorization

Lemma. lcm and gcd are duals:

$$\operatorname{lcm}(a,b) = \frac{ab}{\gcd(a,b)}$$

Lemma. if p is prime and $p \mid (ab)$ then $(p \mid a) \vee (p \mid b)$

Lemma. if a and b are rel prime and $a \mid bc$ then $a \mid c$

Lemma. if $d = \gcd(a,b)$ then $d = ja + kb$ for some $j, k \in \mathbb{Z}$ (linear combo)

1.4 Euclid's Algorithm

1.4.1 Finding gcd(a.b)

Assuming (wlog) $a \leq b$ set $r_0 = b$ and $r_1 = a$ and iteratively compute $r_{i+1} = (r_{i-1} \mod r_i)$. Then $\gcd(a,b) = r_i$ when $r_{i+1} = 0$.

1.4.2 gcd(a.b) is a linear combo of a and b

The expression of $\gcd(a,b) = ja + kb$ is not unique. Values for j and k can be found by working the algorithm in 1.2.1 in reverse.

The algorithm can be expressed:

$$\begin{aligned}
r_2 &= b - aq_1 = r_0 - r_1 q_1 \\
r_3 &= a - r_2 q_2 = r_1 - r_2 q_2 \\
r_4 &= r_2 - r_3 q_3 \\
&\dots \\
r_{i-1} &= r_{i-3} - r_{i-2} q_{i-2} \\
r_i &= r_{i-2} - r_{i-1} q_{i-1} \\
r_{i+1} &= r_{i-1} - r_i q_i = 0
\end{aligned}$$

Starting with the equation for r_i we substitute for the other r's and repeat until only $b = r_0$ and $a = r_1$ and q's are left. Combine terms and we have $\gcd(a, b) = ja + kb$.

2 Congruence Classes

2.1 Definitions

- For $a, n \in \mathbb{Z}$ the congruence class $[a]_n = \{b \in \mathbb{Z} \mid b \equiv a \pmod{n}\}$.
- $b \equiv a \pmod{n}\} \iff n \mid (a - b)$ i.e. a and b have the same remainder when divided by n
- $\mathbb{Z}_n = \{[a]_n\} = \{[0]_n, [1]_n \cdots [n-1]_n\}$ a.k.a. $\mathbb{Z}/n$
- and equivalence relation on a set S is a subset $R \subseteq S \times S$ such that

  $$(a, b) \in R \Longrightarrow (b, a) \in R \text{ (symmetric)}$$
  $$(a, b) \in R \Longrightarrow (a, a) \in R \text{ (reflexive)}$$
  $$(a, b) \in R \wedge (b, c) \in R \Longrightarrow (a, c) \in R \text{ (transitive)}$$

- $|S|$ is the cardinality of set S
- $[a]_n + [b]_n = [a + b]_n$ addition
- $[a]_n \cdot [b]_n = [a \cdot b]_n$ multiplication
- additive identity $[0]_n$, inverse $[-a]_n$
  $[0]_n + [a]_n = [a]_n$
  $[a]_n + [-a]_n = [0]_n$
- multiplicative identity $[1]_n$, inverse $([a]_n)^{-1}$ if it exists
  $[1]_n \cdot [a]_n = [a]_n$
  $[a]_n \cdot ([a]_n)^{-1} = [1]_n$
- $\mathbb{Z}_n^*$ is subset of $\mathbb{Z}_n$ that have multiplicative inverses viz. $\mathbb{R}^*$

2.2 Examples

- $[1]_2 = \{2k + 1 \mid k \in \mathbb{Z}\}$ i.e. odd integers
- $[0]_2 = \{2k \mid k \in \mathbb{Z}\}$ i.e. even integers
- $[3]_5 = \{\cdots, -7, -2, 3, 8, \cdots\} = \{5k + 3 \mid k \in \mathbb{Z}\}$
- $\mathbb{Z}_6^* = \{1, 5\}$
- $|\mathbb{Z}_n| = n$

2.3 Observations, Theorems and Lemmas

Lemma. $[a]_n = [b]_n \iff b \in [a]_n$

Lemma. $k \in \mathbb{Z}$ is in exactly one congruence class of $\mathbb{Z}_n$

Lemma. the congruence classes of $\mathbb{Z}_n$ partition $\mathbb{Z}$ into n partitions

Lemma. $([a]_n)^{-1}$ exists iff $\gcd(a, n) = 1$

Lemma. $([a]_n)^{-1} = [k]_n$ when $ka + \ell n = 1$ (for $k, l \in \mathbb{Z}$)

3 Groups

3.1 Definition

A group $(G; *)$ is a set G and a *binary* operation $*$ with such that:

- Closure: $\forall a, b \in G (a * b) \in G$
- Identity: $\exists\, e \in G$ such that $\forall a \in G$: $e * a = a * e = a$
- Inverse: $\forall a \in G\ \exists\, a^{-1}$ such that $a * a^{-1} = a^{-1} a = e$
- Associativity: $\forall a, b, c \in G$: $a * (b * c) = (a * b) * c$

3.2 Notes and Observations

- Commutativity is *not* a requirement for a group
- a commutative group is called an *abelian* group.
- uniqueness of the identity and of inverses is *not* part of the definition
- there *are* non-commutative groups
- associativity can be assumed for multiplication, addition, composition of functions, matrix multiplication

3.3 Examples of Groups

- $(\mathbb{Z}; +)$
- $(\mathbb{Q}^*; \cdot)$
- $(\mathbb{R}; +)$
- $(\mathbb{Z}_2; +)$
- Invertable $n \times n$ matrices under multiplication; this is a non-abelian group
- $(\mathbb{R} \setminus \{-1\}; *)$ where $a * b = a + b + ab$
  - Show associativity by expanding each side of $a * (b * c) \overset{?}{=} (a * b) * c$
  - Zero is clearly the identity.
  - Solve $a + b + ab = 0$ for b to get $a^{-1} = -a/(a + 1)$
  - Clearly $(a+b+ab) \in \mathbb{R}$: we need to show that $(a+b+ab) \neq -1$ for all a and b: solve $(a + b + ab) = -1$ for a and show $a = -1$.

3.4 Not Groups

- $(\mathbb{Z}; \cdot)$ – missing inverses
- $(\mathbb{Z} \setminus \{0\}; +)$ – no identity
- $(\mathbb{R}; -)$ – not associative

3.5 More ...

- $a, b \in (G; *)$: $(a * b)^{-1} = b^{-1} * a^{-1}$
- *Symmetric group*: for set S and $(G; \circ)$ where $G = \{f : S \mapsto S\}$ and f is a bijection (composition of functions)
- e common notation for the identity
- a^{-1} inverse of a
- $(a^{-1})^{-1} = a$
- $a^n = a * a * a \cdots * a$ (n-times)
- $a^{-n} = (a^{-1})^n$
- $a^0 = e$
- $(a^n)^m = a^{nm}$
- $\forall a, b \in G \Rightarrow \exists! x \ni a * x = b \wedge \exists! y \ni y * a = b$

4 Multiplication (Cayley) Tables

4.1 For groups

We use row index times column index (order counts when non-ablian).

For $(G; *)$ with $G = \{e, a, b, \cdots\}$

$$\begin{array}{c|ccccc}
* & e & a & b & c & \cdots \\
\hline
e & e & a & b & c & \cdots \\
a & a & & & & \\
b & b & & & b * c & \\
c & c & & & & \\
\vdots & \vdots & & & &
\end{array}$$

E.g. $\mathbb{Z}_4$:

+	[0]	[1]	[2]	[3]
[0]	[0]	[1]	[2]	[3]
[1]	[1]	[2]	[3]	[0]
[2]	[2]	[3]	[0]	[1]
[3]	[3]	[0]	[1]	[2]

4.2 Observations

Since $\forall a, b \in G \exists! x \ni a * x = b$ we have

- Each row is unique, as is each column.
- An element x appears exactly once in each row or column
- An abelian (commutative) group is symmetric on the diagonal

4.3 Examples

4.3.1 2 element group

There is just one (up to isomorphism) and it's abelian:

$*$	e	a
e	e	a
a	a	e

Isomorphic to $(\mathbb{Z}_2; +)$, $(\{1, -1\}; \cdot)$

4.3.2 3 element group

Also unique and abelian:

$*$	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

There is only one way to fill in table with each element appearing exactly once in each row and column.

4.3.3 4 element groups

There are two up to relabeling (both abelian)

$*$	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

$\mathbb{Z}_4$ and Klein-4 group respectively.

4.3.4 5 element group

Just one: $\mathbb{Z}_5$ also abelian.

+	[0]	[1]	[2]	[3]	[4]
[0]	[0]	[1]	[2]	[3]	[4]
[1]	[1]	[2]	[3]	[4]	[0]
[2]	[2]	[3]	[4]	[0]	[1]
[3]	[3]	[4]	[0]	[1]	[2]
[4]	[4]	[0]	[1]	[2]	[3]

5 Order of Groups and Elements

5.1 Definition and Notation

- The order of a group $|G|$ is the number of elements in G.
- The order of $a \in G$: $o(a)$ is the smallest positive integer n such that $a^n = e$.
- $o(a) = \infty$ if no such n.
- $\forall a \in G \setminus \{e\} : o(a) \geq 2$
- The set generated by $a \in G$ is $\langle a \rangle = \{x \in G : x = a^n, n \in \mathbb{Z}\}$
  I.e. $\langle a \rangle = \{a^0, a^1, \cdots, a^n, a^{n+1}, \cdots\}$

5.2 Examples

- $G = \mathbb{Z}_{60}$ then $|G| = 60$ and $o([8]) = n$ where $n \cdot 8 = 60k$ or $2n = 15k$ so $15 \mid 2n$. $\gcd(15,2) = 1 \Rightarrow n = 15$
- $(\mathbb{R}^*;\cdot) : o(1) = 1, o(-1) = 2, o(x \notin \{1,-1\}) = \infty$

5.3 Theorems

For finite group G:

Theorem. Let $N = |G| \in \mathbb{N}$ then $x^N = e \iff o(x) \mid N$

Proof. Let $n = o(x)$ then $0 < n \leq N$

$\Leftarrow$ $n \mid N \quad \Rightarrow \quad \exists k \ni nk = N \quad \Rightarrow \quad x^N = x^{nk} = (x^n)^k = e^k = e$

$\Rightarrow$ Suppose $x^N = e$ and n does not divide N. Then
$\exists p, r \in \mathbb{N} \ni N = qn + r$ with $0 < r < n$ and so
$e = x^N = x^{qn+r} = x^{qn}x^r = x^r$.
Therefore $o(x) = r$; a contradiction.

$\square$

Lemma. Let $N = |G| \in \mathbb{N}$ and $n = o(x)$ then $x^k = x^\ell \iff n \mid (k - l)$.

Proof. $x^k = x^\ell \iff x^k x^{-\ell} = x^\ell x^{-\ell} \iff x^{k-\ell} = e$ and use previous theorem. $\square$

Lemma. $o(a) = |\langle a \rangle|$

6 Subgroups

Definition. For group G and $H \subseteq G$. H is a *subgroup* of $G \iff H$ satisfies the requirements of a group. We say $H \leqslant G$.

Exanples:

- $2\mathbb{Z} \leqslant \mathbb{Z}$
- $G \leqslant G$ trivially
- $\{e\} \leqslant G$ trivially
- $a \in G \Rightarrow \langle a \rangle \leqslant G$

Definition. $\langle a \rangle$ for $a \in G$ is the *cyclic subgroup* of G generated by a.

Definition. A group G is *cyclic* $\iff \exists a \in G \ni \langle a \rangle = G$.

To show that H is a subroup G associativity is inherited. And so is the existence of the identity and inverses but their inclusion in H must be shown. Closure must be shown.

The one step verification:

Theorem. For $(G; *)$ and $H \subseteq G$, H is a subgroup of G ($H \leqslant G$) iff $a, b \in H \Rightarrow a * b^{-1} \in H$.

Proof. Let $a, b \in H$.

$\Rightarrow$ Since H is a group: $b \in H \Rightarrow b^{-1} \in H \Rightarrow a * b^{-1} \in H$

$\Leftarrow$ Identity: $a * a^{-1} = e \Rightarrow e \in H$.
Inverses: $a \in H \Rightarrow e * a^{-1} = a^{-1} \in H$.
Closure: $b \in H \Rightarrow b^{-1} \in H \Rightarrow a * (b^{-1})^{-1} = a * b \in H$.

□

6.1 Cyclic Subgroups

Lemma. For group G and $a \in G$ then $\langle a \rangle = \{a^0, a^1, a^2 \cdots\}$ is a subgroup of G.

Example: For $\mathbb{Z}_{10}$:
$\langle [0] \rangle = \{0\}$
$\langle [1] \rangle = \mathbb{Z}_{10}$ and
$\langle [2] \rangle = \{[0], [2], [4], [6], [8]\}$

Lemma. $\langle [a]_n \rangle = \mathbb{Z}_n \iff \gcd(a, n) = 1$

Lemma. Every cyclic subgroup is abelian.

Cyclic?:

- $(\mathbb{Z}_n; +)$: yes $\langle [1]_n \rangle$
- $(\mathbb{Z}; +)$: yes $\langle 1 \rangle$
- $\mathrm{GL}_2(\mathbb{R})$: no
- $\mathbb{R}^*$: no
- $\mathbb{Z}_8^*$: no: $\mathbb{Z}_8^* = \{[1], [3], [5], [7]\}$ and $\forall x \in \mathbb{Z}_8^* : x^2 = [1]$

Theorem. Let G be a group with $|G| = n$. G is cyclic $\iff \exists x \in G \ni o(x) = n$.

6.2 Subgroups of $\mathbb{Z}$

Theorem. Every subgroup of the additive group $(\mathbb{Z}; +)$ is cyclic.

Proof. Let H be a subgroup of $\mathbb{Z}$.

Case 1: If $H = \{0\}$, then H is cyclic, since $H = \langle 0 \rangle$.

Case 2: Suppose $H \neq \{0\}$. Then H contains some nonzero integer. Let n be the smallest positive integer in H.

Step 1: $n\mathbb{Z} \subseteq H$. Since $n \in H$ and H is a subgroup, for all integers k, $kn \in H$. Hence $n\mathbb{Z} \subseteq H$.

Step 2: $H \subseteq n\mathbb{Z}$. Take any $h \in H$. By the division algorithm, there exist integers q, r with

$$h = qn + r, \qquad 0 \le r < n.$$

Because $qn \in n\mathbb{Z} \subseteq H$ and $h \in H$, their difference

$$r = h - qn$$

also lies in H. But $r \in H$ and $r < n$. By the minimality of n, we must have $r = 0$. Thus $h = qn \in n\mathbb{Z}$.

Step 3: Combining the inclusions, we have

$$H = n\mathbb{Z} = \langle n \rangle.$$

Therefore, H is cyclic. □

Theorem. Every subgroup of a cyclic group is cyclic.

Proof. Same argument using exponents. □

Definition. The *join* of two subgroups $H \leqslant K \wedge \leqslant G$ is

$$HK = \{hk : h \in H, k \in K\}$$

.

Theorem. If G is abelian with $H \leqslant G$ and $K \leqslant G$ then $HK \leqslant G$. I.e. HK is also a subgroup of G.

Proof. For $h, h_1, h_2 \in H$ and $k, k_1, k_2 \in K$

Associativity: inherited from G.

Identity: $e \in H \wedge e \in K \Rightarrow ee = e \in HK$

Inverses: $(hk)^{-1} = k^{-1}h^{-1} = h^{-1}k^{-1} \in HK$

Closure: $$\begin{aligned}(h_1k_1)(h_2k_2) &= h_1(k_1h_2)k_2 \\ &= h_1(h_2k_1)k_2 \\ &= (h_1h_2)(k_1k_2) \\ &\in HK\end{aligned}$$

□

6.3 Subgroups of $\mathbb{Z}_n$

$\mathbb{Z}_n$ is cyclic, therefore so is $H \leqslant \mathbb{Z}_n$

Theorem. If $H \leqslant \mathbb{Z}_n$ then $k = |H|$ divides $n = |\mathbb{Z}_n|$

Proof. $\mathbb{Z}_n = \langle [1] \rangle$ and $H = \langle [h] \rangle$ for some $0 \leq h < n$ but
$n[h] = [hn] = h[n] = h[0] = [0]$ so $k = o([h])$ divides n.
(See first lemma in 5.3). $\square$

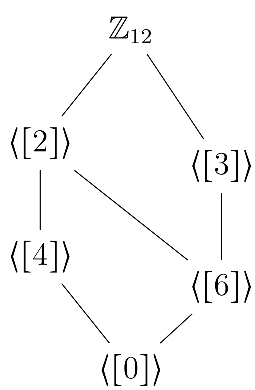
Theorem. If $d \mid n$ then $\exists H \leqslant \mathbb{Z}_n \ni |H| = d$.

Proof. Consider $H = \langle [k] \rangle$ where $kd = n$ and $0 < d < n$. Then
$H = \{[0], [k], 2[k], \cdots (d-1)[k]\}$ and $|H| = d$ $\square$

6.4 Subgroup Lattice Diagram

Also called Hasse diagram. This is a lattice with the group at the top and subgroups below with connections showing inclusion.

E.g. for $\mathbb{Z}_{12}$:



6.5 Joins

Definition. The *join* of subgroups $H \leqslant G$ and $K \leqslant G$ is the set
$HK = \{hk : h \in H \wedge k \in K\}$

Not all joins of subgroups are subgroups.

Lemma. If G is abelian then so is HK for $H \leqslant G$ and $K \leqslant G$. And HK is a subgroup.

7 Direct Product of Groups

Definition. For goups G_1, G_2 the *direct product* of $(G_1; \bullet)$ and $(G_2; *)$ is

$$G_1 \times G_2 = \{(g_1, g_2) : g_1 \in G_1 \wedge g_2 \in G_2\}$$

with

$$(a_1, a_2)(b_1, b_2) = (a_1 \bullet a_2, b_1 * b_2)$$

Think cartesian product.

Lemma. $G_1 \times G_2$ is a group.

Lemma. $o(G_1 \times G_2) = o(G_1)o(G_2)$

Lemma. $G_1 \times G_2$ is abelian iff both G_1 and G_2 are abelian.

Lemma. If $(g_1, g_2) \in G_1 \times G_2$ then $o((g_1, g_2)) = \mathrm{lcm}(o(g_1), o(g_2))$.

Theorem. If $G_1 \times G_2$ is cyclic then G_1, G_2 are both cyclic.

Proof. Let (g_1, g_2) be a generator of $G_1 \times G_2$ and consider $a_1 \in G_1$ an arbitrary element of G_1. Then $(a_1, e_2) \in G_1 \times G_2$ implies that $(a_1, e_2) = (g_1, g_2)^n = (g_1^n, g_2^n)$ and therefore $a_1 = g_1^n$ for some integer n. Since a_1 is arbitrary, $\langle g_1 \rangle = G_1$ and g_1 is a generator of G_1. Similarly for $g_2 \in G_2$. $\square$

But not conversely: consider $\mathbb{Z}_2 \times \mathbb{Z}_4$.

Lemma. If G_1, G_2 are both cyclic and $\gcd(|G_1|, |G_2|) = 1$ then $G_1 \times G_2$ is cyclic.

Lemma. If $H_1 \leqslant G_1$ and $H_2 \leqslant G_2$ then $H_1 \times H_2 \leqslant G_1 \times G_2$.

All such products of subgroups do not necessarily produce all subgoups of the product. E.g. $H = \langle (a, a) \rangle \leqslant \mathbb{Z} \times \mathbb{Z}$ is cannot be the product of subgoups of $\mathbb{Z}$

8 Lagrange's Theorem

To be proved later: needs cosets.

Theorem. If $H \leqslant G$ then $|H| \mid |G|$.

Corollary. $g \in G \Rightarrow g^{|G|} = e$

Proof. Consider $g \in G$ with $n = |G|$ and $k = |\langle g \rangle|$ then by Lagrange's theorem $k \mid n$ and therefore $n = km$ for some $0 < m \leq n$. And so $g^n = g^{km} = (g^k)^m = e^m = e$. □

Lemma. If $|G| = p$ for a prime p and $g \in G$: $o(g) = 1$ or $o(g) = p$.

Definition. Euler's Totient Function (a.k.a. Euler's Phi) for $n \in \mathbb{Z}^+$ is $\varphi(n) = |\{k \in \mathbb{Z} : 1 \leq k \leq n \wedge \gcd(k,n) = 1\}|$.

In other words $\varphi(n)$ is the number of positive integers less then n and coprime to n.

E.g.:
$\varphi(4) = 2 = |\{1,3\}|$
$\varphi(12) = 4 = |\{1,5,7,11\}|$
$\varphi(23) = 22 = |\{1,2,\cdots,21,22\}|$

Lemma. If p is prime $\varphi(p) = p - 1$

Corollary. For positive integers $a, n \ni \gcd(a,n) = 1$ then

$$a^{\varphi(n)} \equiv 1 \bmod n$$

.

Proof. $\mathbb{Z}_n^* = \{[a]_n : \gcd(a,n) = 1\}$ and so $\varphi(n) = |\mathbb{Z}_n^*|$ and therefore $[a]^{\varphi(n)} = [1]$ or $a^{\varphi(n)} \equiv 1 \bmod n$. □

9 Permutations

Definition. For $N = \{1,2,3,\cdots,n\}, S_n = \{\sigma : N \mapsto N, \sigma \text{ is a bijection}\}$. I.e. S_n is the set of all invertible functions from $\{1,2,3,\cdots,n\}$ onto itself. Each such function is a *permutation*. The elements of S_n form a group under compostion with $|S_n| = n!$.

9.1 Two-Line Notation

The two-line notation describes a permution with the top row being N and the bottom its image under the permutation. E.g. for S_3 the function σ that maps 1 to 2, 2 to 3 and 3 to 1 is

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

E.g:
$S_2 = \left\{ \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \right\}$
$S_3 = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \right\}$
The convention is that the top row is in order. Inverse of an element is found by switch top and bottom rows and reordering. The identity has top and bottom the same.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} : 1 \mapsto 2, 2 \mapsto 1, 3 \mapsto 3$$

Apply right to left (i.e. function composition):

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

9.2 Cycle Notation

This is a single line notation where each element maps to the neighbor on the right and that last maps to the first. By convention the lowest element is listed first. E.g:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 4 & 2 \end{pmatrix}$$

Any element not in the cycle maps to itself. The identity maybe written as a single cycle: (1). The inverse of a cycle is the cycle reversed.

Disjoint cycles commute. Any permutation can be written as a composition of disjoint cycles. E.g.:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 4 & 5 \end{pmatrix}$$

The order of a cycle is its length. The order of the composition of disjoint cycles is the lcm of the lengths.

E.g:
$S_2 = \{(1), (1,2)\}$ i.e. the identity and swapping 1 and 2
$S_3 = \{(1), (1,2), (1,3), (2,3), (1,2,3), (1,3,2)\}$
$S_4 = \{(1),$
$(1\ 2), (1\ 3), (1\ 4), (2\ 3), (2\ 4), (3\ 4),$
$(1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)$
$(1\ 2\ 3), (1\ 3\ 2), (1\ 2\ 4), (1\ 4\ 2),$
$(1\ 3\ 4), (1\ 4\ 3), (2\ 3\ 4), (2\ 4\ 3),$
$(1\ 2\ 3\ 4), (1\ 2\ 4\ 3), (1\ 3\ 2\ 4), (1\ 3\ 4\ 2), (1\ 4\ 2\ 3), (1\ 4\ 3\ 2)\}$

S_4 has one element of order 1. And 9 elements of order 2: 6 single cycle and 3 2-cycle elements. There are 8 single cycle elements of order 3 and 6 of order 4.

10 Transpostions and the A_n and D_n Subgroups of S_n

10.1 Transpositions

Definition. A transposition is a permutation cycle of length 2. It is the exchange of 2 elements.

Every cycle can be decomposed into a sequence of transpositions. One algorithm is to take the last element and pair it with each of the previous elements in reverse order. E.g.: $(1\ 3\ 6\ 5\ 2) \to (2\ 5)(2\ 6)(2\ 3)(2\ 1)$. So any permutation can be decomposed into a sequence of transpositions.

Definition. A permutation is even if a decomposition into transpostions has an even number of cycles. A permutation is odd if a decomposition into transpostions has an odd number of cycles.

Any decomposition of a particular permutation will always be even or odd. *Not proven here.* A cycle of length n decomposes into $n-1$ transpositions.

10.2 The Alternating Group A_n

Definition. The set of all even permutations in S_n is the *alternating (sub)group* designated A_n.

E.g: $A_3 = \{(), (1\ 2\ 3), (1\ 3\ 2)\} = \langle(1\ 2\ 3)\rangle$.

Theorem. A_n is a subgroup of S_n.

Proof. Associativity and identity are inherited. Let $\sigma \in A_n$, then $\sigma = \tau_1\tau_2\cdots\tau_n$ for n transpositions τ_i and $\sigma^{-1} = \tau_n^{-1}\tau_{n-1}^{-1}\cdots\tau_1^{-1} = \tau_n\tau_{n-1}\cdots\tau_1$. For closure, if $\alpha, \beta \in A_n$ then each has decomposition into an even number of transpositions. Composing, $\alpha\beta$ has a decomposition into α's transpositions followed by β's which is still an even number of transpositions nad $\alpha\beta \in A_n$. □

Theorem. $|S_n| = 2\,|A_n|$

Proof. Let O_n be the set of odd permutations in S_n. Let $F : A_n \mapsto O_n$ where $F(\sigma) = \sigma\ (1\ 2)$ and so $F^{-1} : O_n \mapsto A_n$ is its inverse where $F^{-1}(\alpha) = \alpha\ (1\ 2)$. $F^{-1}(F(\sigma)) = \sigma\ (1\ 2)\ (1\ 2) = \sigma$. Likewise we have $F(F^{-1}(\alpha)) = \alpha\ (1\ 2)\ (1\ 2) = \alpha$. So F is a bijection and $|A_n| = |O_n|$. Since $S_n = A_n \cup O_n$ and $A_n \cap O_n = \varnothing$ then $|S_n| = |A_n| + |O_n| = 2\,|A_n|$. □

10.3 Decompostion Types and the Order of Permutations

A permutation can be characterized by its decomposition into disjoint cycles and the lengths of those cycles.

S_4 can have permutations with single cycles of length 2, 3, or 4. And double cycles of length 2 and 2. But not triple cycles. So A_4 will contain the single cycles of lengths 1 or 3, these being the even permutations. As well ad the double cycles of lengths 2, both odd so the permutation is even:

$$A_4 = \{(),\\ (1\ 2\ 3), (1\ 2\ 4), (1\ 3\ 2), (1\ 3\ 4), (1\ 4\ 2), (1\ 4\ 3), (2\ 3\ 4), (2\ 4\ 3),\\ (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$$

The classification of types on S_n comes down to the ways n can be composed of sums of positive integers.

For S_4 we have sums: 4, 3+1, 2+2, 2+1+1, and 1+1+1+1. Ingnoring the cycles of length 1 the types are (4), (3), (2,2), and (2). The type is the number of cycles and their individual lengths.

The order of a permutation is the lcm of the lengths of its disjoint cycles. The parity of a cycle is even if its length is odd and the parity is odd if its length is even (since the number of transpositions needed to compose a cycle is one less than its length).

The even permutations of S_4 are the identity and those of type (3) or (2,2) with orders of 3 and 2 respectively. These make up the elements of A_4.

10.4 The Dihedral Group D_n

This is the group of symmetries of a regular n-gon (n-sided polygon with equal length sides).

These polygons can be inscribed in a circle with the n vertices equally spaced on the circle.

D_n is a subgroup of S_n: think of it as a restricted set of permutations of the vertices on the n-gon.

10.4.1 More formally ...

Definition. A *metric* on a space M is a function $d : M \times M \mapsto \mathbb{R}$ such that:
$\forall x, y, z \in M :$

$$d(x, x) = 0$$
$$d(x, y) = d(y, x)$$
$$d(x, y) > 0 \text{ for } x \neq y$$
$$d(x, z) \leq d(x, y) + d(x, z)$$

Definition. An *isometry* is a map from one space to another such that distances are preserved. I.e. for spaces M_1, M_2 with metrics d_1, d_2 we say $\phi : M_1 \mapsto M_2$ is an isometry iff $\forall a, b \in M_1 : d_1(a, b) = d_2(\phi(a), \phi(b))$.

Definition. A *symmetry* is an isometry of an object onto itself.

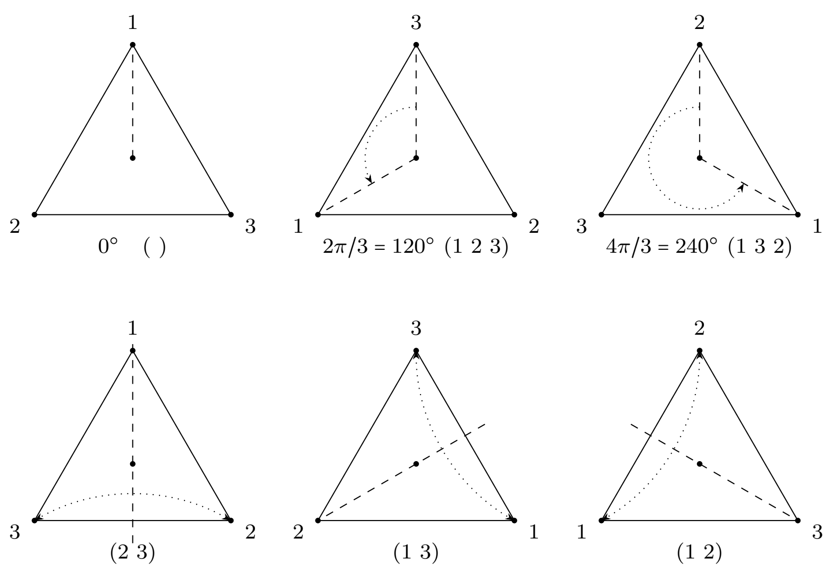
In the case of D_n, symmetries of regular n-gons, we are sticking with subsets of $\mathbb{R}^2$ with the usual Euclidean metric. The subset being the n-gon itself. The D_n group will have n rotations (including the identity of zero) and n reflections. There are $2n$ elements in D_n.

10.4.2 D_1, D_2 Trivial Symmetries of a Point and of a Line Segment

D_1 has but one element, the identity. D_2 is the group of symmetries of a line segment. It has 2 elements: the identity and the interchange of the end points: $D_2 = \{(\ ),\ (1\ 2)\} = S_2$.

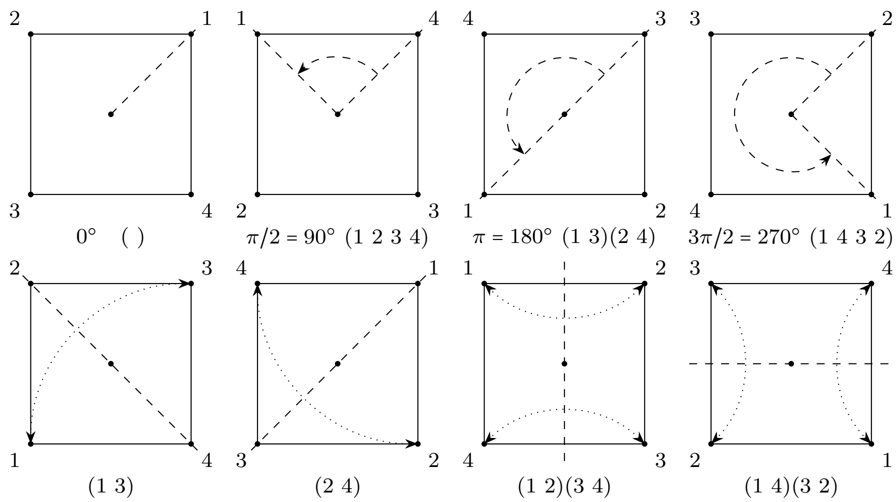
10.4.3 D_3 Symmetries of the Triangle

This is the simplest non-trivial dihedral group and it is the same as S_3. Rotations are through integral multiples of $2\pi/3 = 120°$ and reflections interchange a pair of vertices:
$D_3 = \{(),\ (1\ 2\ 3),\ (1\ 3\ 2),\ (1\ 2),\ (1\ 3),\ (2\ 3)\} = S_3$



10.4.4 D_4 Symmetries of the Square

This is smallest D_n that is a proper subgroup of S_n. Rotations are through integral multiples of $\pi/2 = 90°$ and reflections interchange 2 pairs of vertices: $D_4 = \{(),\ (1\ 2\ 3\ 4),\ (1\ 3)(2\ 4),\ (1\ 4\ 3\ 2),$
$(1\ 3),\ (2\ 4),\ (1\ 2)(3\ 4),\ (1\ 4)(2\ 3)\}$



There are a an even number of vertices and so the axes of reflection symmetry pass through opposite vertices and through the mid-points of opposite sides:

10.4.5 D_5 Symmetries of the Pentagon

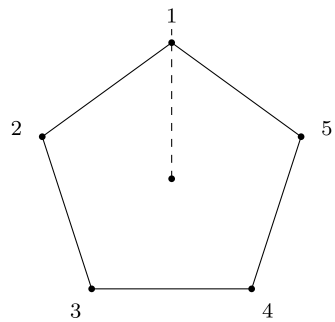
The permutations in S_5 that make up D_5 are:
$D_5 = \{(\ ),\ (1\ 2\ 3\ 4\ 5),\ (1\ 3\ 5\ 2\ 4),\ (1\ 4\ 2\ 5\ 3),\ (1\ 5\ 4\ 3\ 2)$
$(2\ 5)(3\ 4),\ (1\ 3)(4\ 5),\ (1\ 5)(2\ 4),\ (1\ 2)(3\ 5),\ (1\ 4)(2\ 3)\}$

The five rotations move the vertices and integral multiple of$2\pi/5$ around the circumcenter of the pentagon. Note that the order of the vertices is unchanged. These are in the top row above.

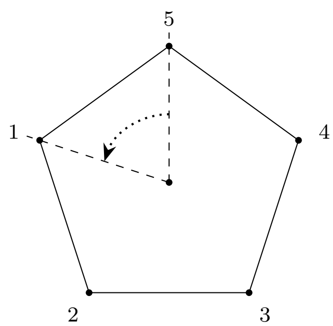
The reflections transpose the pentagon around a line of symmetry through a vertex and the mid-point opposite. This could also be considered a 3-D rotation of 180° around the axis of symmetry. Reflections interchange 2 pairs of vertices.

Rotations:

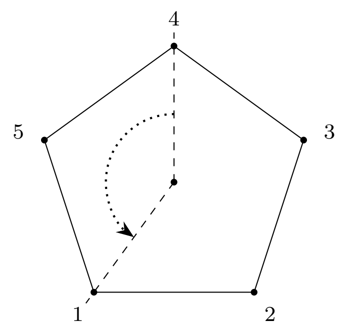
Each rotation symmetry of a pentagon take it through an integral multiple of $2\pi/5 = 72^\circ$. Below are the 5 rotation symmetries.



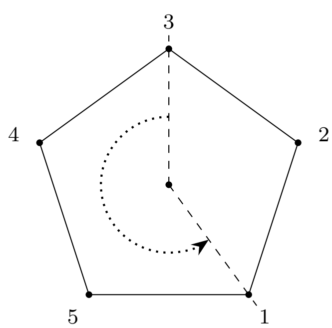
0° ( )



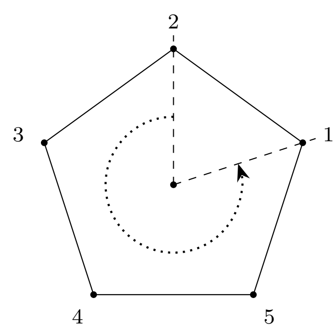
$2\pi/5 = 72^\circ$ (1 2 3 4 5)



$4\pi/5 = 144^\circ$ (1 3 5 2 4)



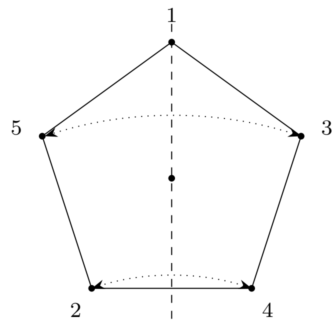
$6\pi/5 = 216^\circ$ (1 4 2 5 3)



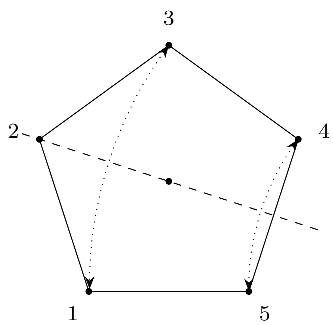
$8\pi/5 = 288^\circ$ (1 5 4 3 2)

Reflections:

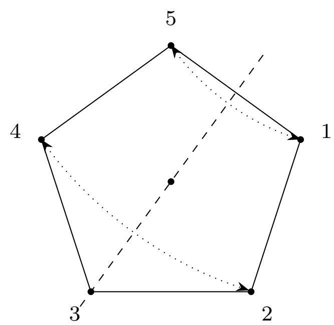
The reflection symmetries of the pentagon are defined by a line of reflection through a vertex and the mid-point of the side opposite. In the figure below each is labeled with that vertex and the corresponding permutation element of S_5. The interchanged vertices are connected by the arc.



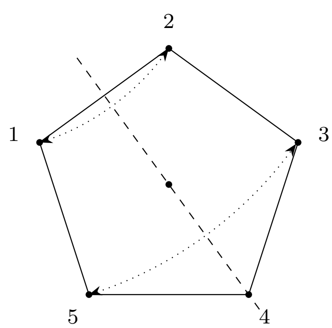
V1 (2 5)(3 4)



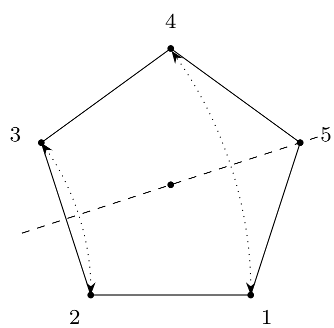
V2 (1 3)(4 5)



V3 (1 5)(2 4)



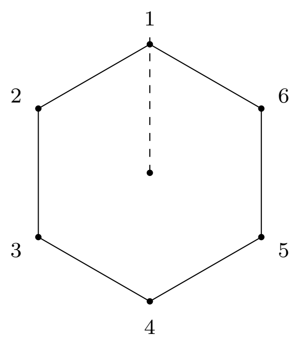
V4 (1 2)(3 5)



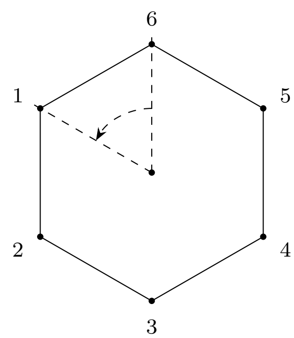
V5 (1 4)(2 3)

10.4.6 Symmetries of the Hexagon

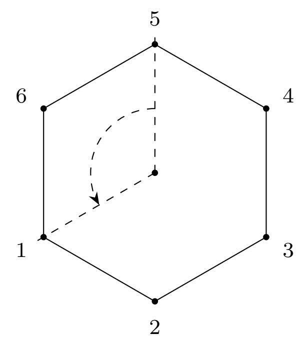
Rotations:



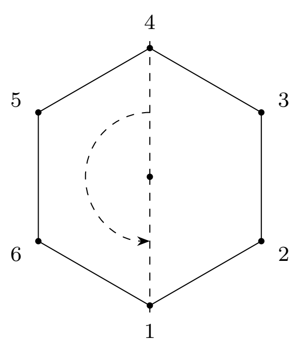
$0°$ ( )



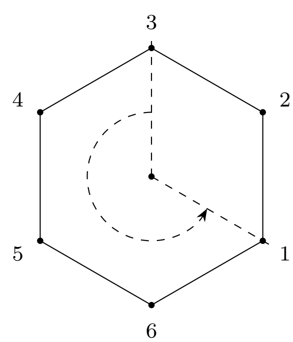
$\pi/3 = 60°$ (1 2 3 4 5 6)



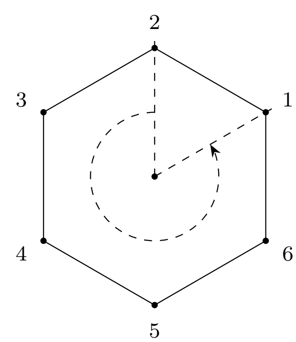
$2\pi/3 = 120°$ (1 3 5)(2 4 6)



$\pi = 180°$ (1 4)(2 5)(3 6)

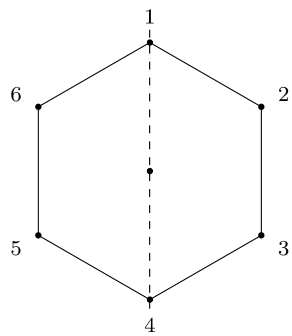


$4\pi/3 = 240°$ (1 5 3)(2 6 4)

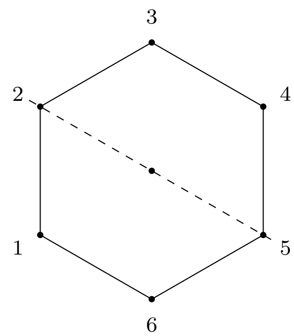


$5\pi/3 = 300°$ (1 6 5 4 3 2)

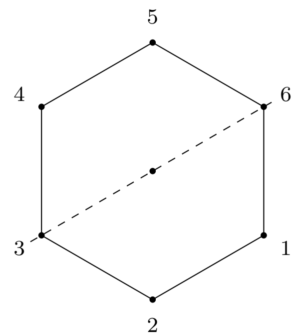
Reflections



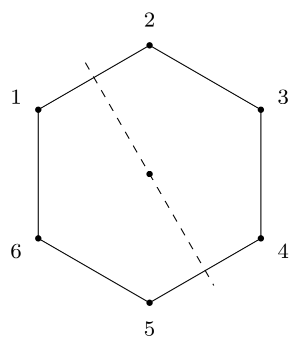
(2 6)(3 5)



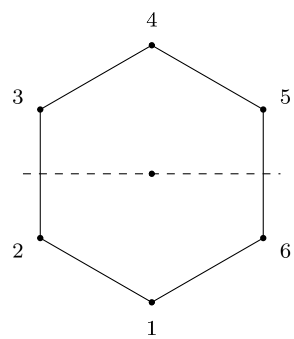
(1 3)(4 6)



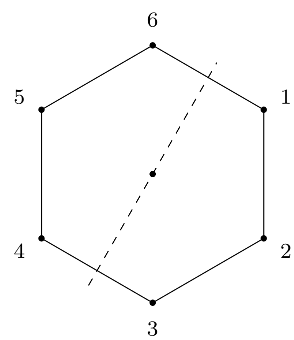
(1 5)(2 4)



(1 2)(3 6)(4 5)



(1 4)(2 3)(5 6)



(1 6)(2 5)(3 4)