UNIVERSITY OF SOUTH CAROLINA

MATH 546 - Algebraic Structures I Lecture Notes

1 Preliminaries

1.1 Notation

- $\ni$ is to be read as *such that*
- $\mathbb{N}$ - set of natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$
- $\mathbb{Z}$ - set of integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- $\mathbb{Q}$ - set of rational numbers $\mathbb{Q} = \{x/y \mid x \in \mathbb{Z} \wedge y \in \mathbb{N}\}$
- $\mathbb{R}$ - real numbers
- $\mathbb{C}$ - complex numbers
- $\mathbb{R}^* \equiv \mathbb{R} \setminus \{0\}$
- $a \mid b$ is to be read as *a divides b*

1.2 Definitions

- For $a, b \in \mathbb{Z}$ and $a \neq 0$: a divides b (or b is divisible by a) $\iff \exists q \in \mathbb{Z} \ni aq = b$
- $p \in \mathbb{N} \setminus \{0, 1\}$ is *prime* $\iff a \in \mathbb{N} \wedge a \mid p \Rightarrow a = p \wedge a = 1$
- greatest common divisor: For $a, b \in \mathbb{Z}$ (both not 0): $\gcd(a, b)$ is largest integer that divides both a and b
- $a, b \in \mathbb{Z}$ are *relatively prime* $\iff \gcd(a, b) = 1$
- least common multiple: $a, b \in \mathbb{Z}^* : \operatorname{lcm}(a, b)$ is the smallest $n \in \mathrm{N}$ such that $(a \mid n) \wedge (b \mid n)$

1.3 Facts

Offered without proof:

Theorem. Fundamental Theorm of Arithmetic: every $n \in \mathbb{N}^* \setminus \{1\}$ has a unique (up to order) prime factorization

Lemma. lcm and gcd are duals:

$$\mathrm{lcm}(a,b) = \frac{ab}{\gcd(a,b)}$$

Lemma. if p is prime and $p \mid (ab)$ then $(p \mid a) \vee (p \mid b)$

Lemma. if a and b are rel prime and $a \mid bc$ then $a \mid c$

Lemma. if $d = \gcd(a,b)$ then $d = ja + kb$ for some $j,k \in \mathbb{Z}$ (linear combo)

1.4 Euclid's Algorithm

1.4.1 Finding gcd(a.b)

Assuming (wlog) $a \leq b$ set $r_0 = b$ and $r_1 = a$ and iteratively compute $r_{i+1} = (r_{i-1} \mod r_i)$. Then $\gcd(a,b) = r_i$ when $r_{i+1} = 0$.

1.4.2 gcd(a.b) is a linear combo of a and b

The expression of $\gcd(a,b) = ja + kb$ is not unique. Values for j and k can be found by working the algorithm in 1.2.1 in reverse.

The algorithm can be expressed:

$$\begin{aligned}
r_2 &= b - aq_1 = r_0 - r_1 q_1 \\
r_3 &= a - r_2 q_2 = r_1 - r_2 q_2 \\
r_4 &= r_2 - r_3 q_3 \\
&\dots \\
r_{i-1} &= r_{i-3} - r_{i-2} q_{i-2} \\
r_i &= r_{i-2} - r_{i-1} q_{i-1} \\
r_{i+1} &= r_{i-1} - r_i q_i = 0
\end{aligned}$$

Starting with the equation for r_i we substitute for the other r's and repeat until only $b = r_0$ and $a = r_1$ and q's are left. Combine terms and we have $\gcd(a,b) = ja + kb$.

2 Congruence Classes

2.1 Definitions

- For $a, n \in \mathbb{Z}$ the congruence class $[a]_n = \{b \in \mathbb{Z} \mid b \equiv a \pmod{n}\}$.
- $b \equiv a \pmod{n}\} \iff n \mid (a-b)$ i.e. a and b have the same remainder when divided by n
- $\mathbb{Z}_n = \{[a]_n\} = \{[0]_n, [1]_n \cdots [n-1]_n\}$ a.k.a. $\mathbb{Z}/n$
- and equivalence relation on a set S is a subset $R \subseteq S \times S$ such that

$$(a,b) \in R \Longrightarrow (b,a) \in R \text{ (symmetric)}$$
$$(a,b) \in R \Longrightarrow (a,a) \in R \text{ (reflexive)}$$
$$(a,b) \in R \wedge (b,c) \in R \Longrightarrow (a,c) \in R \text{ (transitive)}$$

- $|S|$ is the cardinality of set S
- $[a]_n + [b]_n = [a+b]_n$ addition
- $[a]_n \cdot [b]_n = [a \cdot b]_n$ multiplication
- additive identity $[0]_n$, inverse $[-a]_n$
  $[0]_n + [a]_n = [a]_n$
  $[a]_n + [-a]_n = [0]_n$
- multiplicative identity $[1]_n$, inverse $([a]_n)^{-1}$ if it exists
  $[1]_n \cdot [a]_n = [a]_n$
  $[a]_n \cdot ([a]_n)^{-1} = [1]_n$
- $\mathbb{Z}_n^*$ is subset of $\mathbb{Z}_n$ that have multiplicative inverses viz. $\mathbb{R}^*$

2.2 Examples

- $[1]_2 = \{2k+1 \mid k \in \mathbb{Z}\}$ i.e. odd integers
- $[0]_2 = \{2k \mid k \in \mathbb{Z}\}$ i.e. even integers
- $[3]_5 = \{\cdots, -7, -2, 3, 8, \cdots\} = \{5k+3 \mid k \in \mathbb{Z}\}$
- $\mathbb{Z}_6^* = \{1, 5\}$
- $|\mathbb{Z}_n| = n$

2.3 Observations, Theorems and Lemmas

Lemma. $[a]_n = [b]_n \iff b \in [a]_n$

Lemma. $k \in \mathbb{Z}$ is in exactly one congruence class of $\mathbb{Z}_n$

Lemma. the congruence classes of $\mathbb{Z}_n$ partition $\mathbb{Z}$ into n partitions

Lemma. $([a]_n)^{-1}$ exists iff $\gcd(a, n) = 1$

Lemma. $([a]_n)^{-1} = [k]_n$ when $ka + \ell n = 1$ (for $k, l \in \mathbb{Z}$)

3 Groups

3.1 Definition

A group $(G; *)$ is a set G and a *binary* operation $*$ with such that:

- Closure: $\forall a, b \in G (a * b) \in G$
- Identity: $\exists\, e \in G$ such that $\forall a \in G$: $e * a = a * e = a$
- Inverse: $\forall a \in G\ \exists\, a^{-1}$ such that $a * a^{-1} = a^{-1}a = e$
- Associativity: $\forall a, b, c \in G$: $a * (b * c) = (a * b) * c$

3.2 Notes and Observations

- Commutativity is *not* a requirement for a group
- a commutative group is called an *abelian* group.
- uniqueness of the identity and of inverses is *not* part of the definition
- there *are* non-commutative groups
- associativity can be assumed for multiplication, addition, composition of functions, matrix multiplication

3.3 Examples of Groups

- $(\mathbb{Z};+)$
- $(\mathbb{Q}^*;\cdot)$
- $(\mathbb{R};+)$
- $(\mathbb{Z}_2;+)$
- Invertable $n \times n$ matrices under multiplication; this is a non-abelian group
- $(\mathbb{R} \smallsetminus \{-1\};*)$ where $a * b = a + b + ab$
  - Show associativity by expanding each side of $a * (b * c) \stackrel{?}{=} (a * b) * c$
  - Zero is clearly the identity.
  - Solve $a + b + ab = 0$ for b to get $a^{-1} = -a/(a+1)$
  - Clearly $(a+b+ab) \in \mathbb{R}$: we need to show that $(a+b+ab) \neq -1$ for all a and b: solve $(a+b+ab) = -1$ for a and show $a = -1$.

3.4 Not Groups

- $(\mathbb{Z};\cdot)$ – missing inverses
- $(\mathbb{Z} \smallsetminus \{0\};+)$ – no identity
- $(\mathbb{R};-)$ – not associative

3.5 More ...

- $a, b \in (G; *)$: $(a * b)^{-1} = b^{-1} * a^{-1}$
- *Symmetric group*: for set S and $(G; \circ)$ where $G = \{f : S \mapsto S\}$ and f is a bijection (composition of functions)
- e common notation for the identity
- a^{-1} inverse of a
- $(a^{-1})^{-1} = a$
- $a^n = a * a * a \cdots * a$ (n-times)
- $a^{-n} = (a^{-1})^n$
- $a^0 = e$
- $(a^n)^m = a^{nm}$
- $\forall a, b \in G \Rightarrow \exists! x \ni a * x = b \wedge \exists! y \ni y * a = b$

4 Multiplication (Cayley) Tables

4.1 For groups

We use row index times column index (order counts when non-ablian).

For $(G; *)$ with $G = \{e, a, b, \cdots\}$

$*$	e	a	b	c	$\cdots$
e	e	a	b	c	$\cdots$
a	a				
b	b			$b * c$	
c	c				
$\vdots$	$\vdots$				

E.g. $\mathbb{Z}_4$:

+	[0]	[1]	[2]	[3]
[0]	[0]	[1]	[2]	[3]
[1]	[1]	[2]	[3]	[0]
[2]	[2]	[3]	[0]	[1]
[3]	[3]	[0]	[1]	[2]

4.2 Observations

Since $\forall a, b \in G \exists! x \ni a * x = b$ we have

- Each row is unique, as is each column.
- An element x appears exactly once in each row or column
- An abelian (commutative) group is symmetric on the diagonal

4.3 Examples

4.3.1 2 element group

There is just one (up to isomorphism) and it's abelian:

$*$	e	a
e	e	a
a	a	e

Isomorphic to $(\mathbb{Z}_2; +)$, $(\{1, -1\}; \cdot)$

4.3.2 3 element group

Also unique and abelian:

$*$	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

There is only one way to fill in table with each element appearing exactly once in each row and column.

4.3.3 4 element groups

There are two up to relabeling (both abelian)

$*$	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

$\mathbb{Z}_4$ and Klein-4 group respectively.

4.3.4 5 element group

Just one: $\mathbb{Z}_5$ also abelian.

+	[0]	[1]	[2]	[3]	[4]
[0]	[0]	[1]	[2]	[3]	[4]
[1]	[1]	[2]	[3]	[4]	[0]
[2]	[2]	[3]	[4]	[0]	[1]
[3]	[3]	[4]	[0]	[1]	[2]
[4]	[4]	[0]	[1]	[2]	[3]

5 Order of Groups and Elements

5.1 Definition and Notation

- The order of a group $|G|$ is the number of elements in G.
- The order of $a \in G$: $o(a)$ is the smallest positive integer n such that $a^n = e$.
- $o(a) = \infty$ if no such n.
- $\forall a \in G \setminus \{e\} : o(a) \geq 2$
- The set generated by $a \in G$ is $\langle a \rangle = \{x \in G : x = a^n, n \in \mathbb{Z}\}$
  I.e. $\langle a \rangle = \{a^0, a^1, \cdots, a^n, a^{n+1}, \cdots\}$

5.2 Examples

- $G = \mathbb{Z}_{60}$ then $|G| = 60$ and $o([8]) = n$ where $n \cdot 8 = 60k$ or $2n = 15k$ so $15 \mid 2n$. $\gcd(15, 2) = 1 \Rightarrow n = 15$
- $(\mathbb{R}^*; \cdot) : o(1) = 1, o(-1) = 2, o(x \notin \{1, -1\}) = \infty$

5.3 Theorems

For finite group G:

Theorem. Let $N = |G| \in \mathbb{N}$ then $x^N = e \iff o(x) \mid N$

Proof. Let $n = o(x)$ then $0 < n \leq N$

$\Leftarrow$ $n \mid N \quad \Rightarrow \quad \exists k \ni nk = N \quad \Rightarrow \quad x^N = x^{nk} = (x^n)^k = e^k = e$

$\Rightarrow$ Suppose $x^N = e$ and n does not divide N. Then
$\exists p, r \in \mathbb{N} \ni N = qn + r$ with $0 < r < n$ and so
$e = x^N = x^{qn+r} = x^{qn}x^r = x^r$.
Therefore $o(x) = r$; a contradiction.

□

Lemma. Let $N = |G| \in \mathbb{N}$ and $n = o(x)$ then $x^k = x^\ell \iff n \mid (k - l)$.

Proof. $x^k = x^\ell \iff x^k x^{-\ell} = x^\ell x^{-\ell} \iff x^{k-\ell} = e$ and use previous theorem. □

Lemma. $o(a) = |\langle a \rangle|$

6 Subgroups

Definition. For group G and $H \subseteq G$. H is a *subgroup* of $G \iff H$ satisfies the requirements of a group. We say $H \leqslant G$.

Exanples:

- $2\mathbb{Z} \leqslant \mathbb{Z}$
- $G \leqslant G$ trivially
- $\{e\} \leqslant G$ trivially
- $a \in G \Rightarrow \langle a \rangle \leqslant G$

Definition. $\langle a \rangle$ for $a \in G$ is the *cyclic subgroup* of G generated by a.

Definition. A group G is *cyclic* $\iff \exists a \in G \ni \langle a \rangle = G$.

To show that H is a subroup G associativity is inherited. And so is the existence of the identity and inverses but their inclusion in H must be shown. Closure must be shown.

The one step verification:

Theorem. For $(G; *)$ and $H \subseteq G$, H is a subgroup of G $(H \leqslant G)$ iff $a, b \in H \Rightarrow a * b^{-1} \in H$.

Proof. Let $a, b \in H$.

$\Rightarrow$ Since H is a group: $b \in H \Rightarrow b^{-1} \in H \Rightarrow a * b^{-1} \in H$

$\Leftarrow$ Identity: $a * a^{-1} = e \Rightarrow e \in H$.
Inverses: $a \in H \Rightarrow e * a^{-1} = a^{-1} \in H$.
Closure: $b \in H \Rightarrow b^{-1} \in H \Rightarrow a * (b^{-1})^{-1} = a * b \in H$.

□

6.1 Cyclic Subgroups

Lemma. For group G and $a \in G$ then $\langle a \rangle = \{a^0, a^1, a^2 \cdots\}$ is a subgroup of G.

Example: For $\mathbb{Z}_{10}$:
$\langle [0] \rangle = \{0\}$
$\langle [1] \rangle = \mathbb{Z}_{10}$ and
$\langle [2] \rangle = \{[0], [2], [4], [6], [8]\}$

Lemma. $\langle [a]_n \rangle = \mathbb{Z}_n \iff \gcd(a, n) = 1$

Lemma. Every cyclic subgroup is abelian.

Cyclic?:

> $(\mathbb{Z}_n; +)$: yes $\langle [1]_n \rangle$
>
> $(\mathbb{Z}; +)$: yes $\langle 1 \rangle$
>
> $\mathrm{GL}_2(\mathbb{R})$: no
>
> $\mathbb{R}^*$: no
>
> $\mathbb{Z}_8^*$: no: $\mathbb{Z}_8^* = \{[1], [3], [5], [7]\}$ and $\forall x \in \mathbb{Z}_8^* : x^2 = [1]$

Theorem. Let G be a group with $|G| = n$. G is cyclic $\iff \exists x \in G \ni o(x) = n$.

6.2 Subgroups of $\mathbb{Z}$

Theorem. Every subgroup of the additive group $(\mathbb{Z}; +)$ is cyclic.

Proof. Let H be a subgroup of $\mathbb{Z}$.

Case 1: If $H = \{0\}$, then H is cyclic, since $H = \langle 0 \rangle$.

Case 2: Suppose $H \neq \{0\}$. Then H contains some nonzero integer. Let n be the smallest positive integer in H.

Step 1: $n\mathbb{Z} \subseteq H$. Since $n \in H$ and H is a subgroup, for all integers k, $kn \in H$. Hence $n\mathbb{Z} \subseteq H$.

Step 2: $H \subseteq n\mathbb{Z}$. Take any $h \in H$. By the division algorithm, there exist integers q, r with

$$h = qn + r, \qquad 0 \leq r < n.$$

Because $qn \in n\mathbb{Z} \subseteq H$ and $h \in H$, their difference

$$r = h - qn$$

also lies in H. But $r \in H$ and $r < n$. By the minimality of n, we must have $r = 0$. Thus $h = qn \in n\mathbb{Z}$.

Step 3: Combining the inclusions, we have

$$H = n\mathbb{Z} = \langle n \rangle.$$

Therefore, H is cyclic. □

Theorem. Every subgroup of a cyclic group is cyclic.

Proof. Same argument using exponents. □

Definition. The *join* of two subgroups $H \leqslant K \wedge \leqslant G$ is

$$HK = \{hk : h \in H, k \in K\}$$

.

Theorem. If G is abelian with $H \leqslant G$ and $K \leqslant G$ then $HK \leqslant G$. I.e. HK is also a subgroup of G.

Proof. For $h, h_1, h_2 \in H$ and $k, k_1, k_2 \in K$

Associativity: inherited from G.

Identity: $e \in H \wedge e \in K \Rightarrow ee = e \in HK$

Inverses: $(hk)^{-1} = k^{-1}h^{-1} = h^{-1}k^{-1} \in HK$

Closure:
$$\begin{aligned}(h_1k_1)(h_2k_2) &= h_1(k_1h_2)k_2\\ &= h_1(h_2k_1)k_2\\ &= (h_1h_2)(k_1k_2)\\ &\in HK\end{aligned}$$

□

6.3 Subgroups of $\mathbb{Z}_n$

$\mathbb{Z}_n$ is cyclic, therefore so is $H \leqslant \mathbb{Z}_n$

Theorem. If $H \leqslant \mathbb{Z}_n$ then $k = |H|$ divides $n = |\mathbb{Z}_n|$

Proof. $\mathbb{Z}_n = \langle [1] \rangle$ and $H = \langle [h] \rangle$ for some $0 \le h < n$ but
$n[h] = [hn] = h[n] = h[0] = [0]$ so $k = o([h])$ divides n.
(See first lemma in 5.3). $\square$

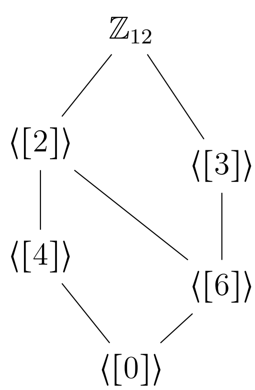
Theorem. If $d \mid n$ then $\exists H \leqslant \mathbb{Z}_n \ni |H| = d$.

Proof. Consider $H = \langle [k] \rangle$ where $kd = n$ and $0 < d < n$. Then
$H = \{[0], [k], 2[k], \cdots (d-1)[k]\}$ and $|H| = d$ $\square$

6.4 Subgroup Lattice Diagram

Also called Hasse diagram. This is a lattice with the group at the top and subgroups below with connections showing inclusion.

E.g. for $\mathbb{Z}_{12}$:



6.5 Joins

Definition. The *join* of subgroups $H \leqslant G$ and $K \leqslant G$ is the set
$HK = \{hk : h \in H \wedge k \in K\}$

Not all joins of subgroups are subgroups.

Lemma. If G is abelian then so is HK for $H \leqslant G$ and $K \leqslant G$. And HK is a subgroup.

7 Direct Product of Groups

Definition. For goups G_1, G_2 the *direct product* of $(G_1; \bullet)$ and $(G_2; *)$ is

$$G_1 \times G_2 = \{(g_1, g_2) : g_1 \in G_1 \wedge g_2 \in G_2\}$$

with

$$(a_1, a_2)(b_1, b_2) = (a_1 \bullet a_2, b_1 * b_2)$$

Think cartesian product.

Lemma. $G_1 \times G_2$ is a group.

Lemma. $o(G_1 \times G_2) = o(G_1)o(G_2)$

Lemma. $G_1 \times G_2$ is abelian iff both G_1 and G_2 are abelian.

Lemma. If $(g_1, g_2) \in G_1 \times G_2$ then $o((g_1, g_2)) = \mathrm{lcm}(o(g_1), o(g_2))$.

Theorem. If $G_1 \times G_2$ is cyclic then G_1, G_2 are both cyclic.

Proof. Let (g_1, g_2) be a generator of $G_1 \times G_2$ and consider $a_1 \in G_1$ an arbitrary element of G_1. Then $(a_1, e_2) \in G_1 \times G_2$ implies that $(a_1, e_2) = (g_1, g_2)^n = (g_1^n, g_2^n)$ and therefore $a_1 = g_1^n$ for some integer n. Since a_1 is arbitrary, $\langle g_1 \rangle = G_1$ and g_1 is a generator of G_1. Similarly for $g_2 \in G_2$. $\square$

But not conversely: consider $\mathbb{Z}_2 \times \mathbb{Z}_4$.

Lemma. If G_1, G_2 are both cyclic and $\gcd(|G_1|, |G_2|) = 1$ then $G_1 \times G_2$ is cyclic.

Lemma. If $H_1 \leqslant G_1$ and $H_2 \leqslant G_2$ then $H_1 \times H_2 \leqslant G_1 \times G_2$.

All such products of subgroups do not necessarily produce all subgoups of the product. E.g. $H = \langle (a, a) \rangle \leqslant \mathbb{Z} \times \mathbb{Z}$ is cannot be the product of subgoups of $\mathbb{Z}$

8 Lagrange's Theorem

To be proved later: needs cosets.

Theorem. If $H \leqslant G$ then $|H| \mid |G|$.

Corollary. $g \in G \Rightarrow g^{|G|} = e$

Proof. Consider $g \in G$ with $n = |G|$ and $k = |\langle g \rangle|$ then by Lagrange's theorem $k \mid n$ and therefore $n = km$ for some $0 < m \leq n$. And so $g^n = g^{km} = (g^k)^m = e^m = e$. $\square$

Lemma. If $|G| = p$ for a prime p and $g \in G$: $o(g) = 1$ or $o(g) = p$.

Definition. Euler's Totient Function (a.k.a. Euler's Phi) for $n \in \mathbb{Z}^+$ is $\varphi(n) = |\{k \in \mathbb{Z} : 1 \leq k \leq n \wedge \gcd(k, n) = 1\}|$.

In other words $\varphi(n)$ is the number of positive integers less then n and coprime to n.

E.g.:
$\varphi(4) = 2 = |\{1, 3\}|$
$\varphi(12) = 4 = |\{1, 5, 7, 11\}|$
$\varphi(23) = 22 = |\{1, 2, \cdots, 21, 22\}|$

Lemma. If p is prime $\varphi(p) = p - 1$

Corollary. For positive integers $a, n \ni \gcd(a, n) = 1$ then

$$a^{\varphi(n)} \equiv 1 \bmod n$$

.

Proof. $\mathbb{Z}_n^* = \{[a]_n : \gcd(a, n) = 1\}$ and so $\varphi(n) = |\mathbb{Z}_n^*|$ and therefore $[a]^{\varphi(n)} = [1]$ or $a^{\varphi(n)} \equiv 1 \bmod n$. $\square$

9 Permutations

Definition. For $N = \{1, 2, 3, \cdots, n\}, S_n = \{\sigma : N \mapsto N, \sigma \text{ is a bijection}\}$. I.e. S_n is the set of all invertible functions from $\{1, 2, 3, \cdots, n\}$ onto itself. Each such function is a *permutation*. The elements of S_n form a group under compostion with $|S_n| = n!$.

9.1 Two-Line Notation

The two-line notation describes a permution with the top row being N and the bottom its image under the permutation. E.g. for S_3 the function σ that maps 1 to 2, 2 to 3 and 3 to 1 is

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

E.g:
$S_2 = \left\{ \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \right\}$
$S_3 = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \right\}$
The convention is that the top row is in order. Inverse of an element is found by switch top and bottom rows and reordering. The identity has top and bottom the same.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} : 1 \mapsto 2, 2 \mapsto 1, 3 \mapsto 3$$

Apply right to left (i.e. function composition):

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

9.2 Cycle Notation

This is a single line notation where each element maps to the neighbor on the right and that last maps to the first. By convention the lowest element is listed first. E.g:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 4 & 2 \end{pmatrix}$$

Any element not in the cycle maps to itself. The identity maybe written as a single cycle: (1). The inverse of a cycle is the cycle reversed.

Disjoint cycles commute. Any permutation can be written as a composition of disjoint cycles. E.g.:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 4 & 5 \end{pmatrix}$$

The order of a cycle is its length. The order of the composition of disjoint cycles is the lcm of the lengths.

E.g:
$S_2 = \{(1), (1,2)\}$ i.e. the identity and swapping 1 and 2
$S_3 = \{(1), (1,2), (1,3), (2,3), (1,2,3), (1,3,2)\}$
$S_4 = \{(1),$
$(1\ 2), (1\ 3), (1\ 4), (2\ 3), (2\ 4), (3\ 4),$
$(1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)$
$(1\ 2\ 3), (1\ 3\ 2), (1\ 2\ 4), (1\ 4\ 2),$
$(1\ 3\ 4), (1\ 4\ 3), (2\ 3\ 4), (2\ 4\ 3),$
$(1\ 2\ 3\ 4), (1\ 2\ 4\ 3), (1\ 3\ 2\ 4), (1\ 3\ 4\ 2), (1\ 4\ 2\ 3), (1\ 4\ 3\ 2)\}$

S_4 has one element of order 1. And 9 elements of order 2: 6 single cycle and 3 2-cycle elements. There are 8 single cycle elements of order 3 and 6 of order 4.