UNIVERSITY OF SOUTH CAROLINA

MATH-546 Algebraic Structures I Homework 9

Problem 1:

Let $F : \mathbb{Z}_{10} \to \mathbb{Z}_{16}$ be the function defined by

$$F([x]_{10}) = [8x]_{16}.$$

(Note: this is a well-defined function; you will not be required to verify this as a question for exam 3, but checking well-definedness might be a question for the final exam.)

a. Find the kernel and the image of F.

b. Let K be the subgroup of $\mathbb{Z}_{16}$ generated by $[2]_{16}$. Find the preimage of K under F.

Answer 1:

a. $\operatorname{Ker} F = \{[2i]_{10} \mid i \in \{0 \dots 4\}\}$. $\operatorname{Im} F = \{[0]_{16}, [8]_{16}\}$

b. The preimage of $\langle [2]_{16} \rangle$ is $\mathbb{Z}_{10}$. Every element of $\mathbb{Z}_{10}$ maps to either $[0]_{16}$ or $[8]_{16}$, both of which are in $\langle [2]_{16} \rangle$.

Problem 2:

Give an example of two groups G_1, G_2 and a homomorphism $F : G_1 \to G_2$ such that G_1 has order at least two (it is not just the identity) and

a. G_1 is abelian but G_2 is not abelian.

b. G_1 is cyclic but G_2 is not cyclic.

Answer 2:

Consider $F : \mathbb{Z}_4 \to D_4$ where $F(x) = r^x$. $\mathbb{Z}_4$ is both cyclic and abelian but D_4 is neither. But the image of F is the subgroup of rotations, $\langle r \rangle \leqslant D_4$, which is both.

Problem 3:

Give an example of two groups G_1 and G_2, and a surjective homomorphism $F : G_1 \to G_2$ such that G_2 has order at least two (it is not just the identity), and

a. G_2 is abelian but G_1 is not abelian.

b. G_2 is cyclic but G_1 is not cyclic.

Answer 3:

Consider $F : S_n \to \mathbb{Z}_2$ where F is the parity operator:

$$F(p) = \begin{cases} 0 & \text{if } p \text{ is even} \\ 1 & \text{if } p \text{ is odd} \end{cases}$$

$\mathbb{Z}_2$ is both cyclic and abelian and S_n is neither for $n \geq 3$.
Another example is $A_4 \to A_4/K_4 \cong \mathbb{Z}_3$ with $F(a) = aK_4$. I.e the group of cosets of K_4.

Problem 4:

Consider the following groups:

$$\mathbb{Z}_{24}, \quad \mathbb{Z}_{12} \times \mathbb{Z}_2, \quad \mathbb{Z}_8 \times \mathbb{Z}_3, \quad \mathbb{Z}_4 \times \mathbb{Z}_6.$$

For each pair of two groups out of the four listed above (there are 6 such pairs), decide whether the two groups are isomorphic or not, and justify your answer.

Answer 4:

All the groups have the same order. The cyclic groups are $\mathbb{Z}_{24}$ and $\mathbb{Z}_8 \times \mathbb{Z}_3$. These must be isomorphic. Neither can be isomorphic to the non-cyclic groups. The leaves the question of the congruency of the two non-cyclic groups $\mathbb{Z}_{12} \times \mathbb{Z}_2$ and $\mathbb{Z}_4 \times \mathbb{Z}_6$.

Consider that $\mathbb{Z}_{12} \cong \mathbb{Z}_4 \times \mathbb{Z}_3$ since both are cyclic and $\mathbb{Z}_6 \cong \mathbb{Z}_2 \times \mathbb{Z}_3$ for the same reason. Then

$$\begin{aligned}
\mathbb{Z}_{12} \times \mathbb{Z}_2 &\cong (\mathbb{Z}_4 \times \mathbb{Z}_3) \times \mathbb{Z}_2 \\
&\cong \mathbb{Z}_4 \times (\mathbb{Z}_3 \times \mathbb{Z}_2) \\
&\cong \mathbb{Z}_4 \times \mathbb{Z}_6
\end{aligned}$$

Problem 5:

For each of the following, decide whether the specified function is a homomorphism between the two specified groups. Also decide whether the function is one-to-one and whether it is onto. Prove your answers.

a. $F : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$, defined by $F((x, y)) = x - y$.

b. $F : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$, defined by $F((x, y)) = (x, x - 5y)$.

Answer 5:

a. Homomorphism: yes:

$$\begin{aligned} F((a,b) + (c,d)) &= F((a+c),(b+d)) \\ &= (a+c) - (b+d) \\ &= (a-b) + (c-d) \\ &= F((a,b)) + F((c,d)) \end{aligned}$$

Onto: yes: $F(\mathbb{Z} \times \{0\}) = \mathbb{Z}$
1-to-1: no: $F((1,0)) = F((0,-1)) = 1$

b. Homomorphism: yes:

$$\begin{aligned} F((a,b) + (c,d)) &= F((a+c),(b+d)) \\ &= ((a+c),(a+c) - 5(b+d)) \\ &= (a, a-5b) + (c, c-5d) \\ &= F((a,b)) + F((c,d)) \end{aligned}$$

1-to-1: yes:

$$\begin{aligned} (F((a,b)) = F((c,d)) &\Rightarrow (a, a-5b) = (c, c-5d) \\ &\Rightarrow (a = c) \wedge (a - 5b = a - 5d) \\ &\Rightarrow (a = c) \wedge (5b = 5d) \\ &\Rightarrow (a = c) \wedge (b = d) \end{aligned}$$

Onto: no: Counter example: $(1,2) = (1, 1-5b) \in F(\mathbb{Z} \times \mathbb{Z}) \Rightarrow$
$\exists b \in \mathbb{Z} \ni 2 = 5b$ an impossibility

Problem 6:

Assume that $F : \mathbb{Z}_5 \to S_4$ is a group homomorphism.

a. Prove that $F([1]_5)$ must be equal to e.

b. Prove that $F([x]_5)$ must be equal to e for all $x \in \mathbb{Z}_5$.

Answer 6:

Consider $\langle F([1]_5)\rangle$. This must be a cyclic subgroup of S_4. The order of $\langle F([1]_5)\rangle$ must divide 5 since $o([1]_5) = 5$ and so $F([1]_5)^5 = F([0]_5) = e$. Therefore $o(F([1]_5))$ must be either 1 or 5. It can't be 5 since S_4 has no subgroups of order 5. I.e. 5 does not divide 24.

a. $o(\langle F([1]_5)\rangle) = 1 \Rightarrow F([1]_5) = e$

b. $F([x]_5) = F(x[1]_5) = F([1]_5)^x = e^x = e$

Theoretical Question 1:

If $F : G_1 \to G_2$ is an isomorphism, prove that $F^{-1} : G_2 \to G_1$ is also an isomorphism.

Answer 1:

Since F is bijective so is F^{-1}:

$$\forall g_2 \in G_2 \ \exists! \ g_1 \in G_1 \ni F(g_1) = g_2 \wedge F^{-1}(g_2) = F^{-1}(F(g_1)) = g_1$$

F is a homomorphism, therefore:

$$\forall x, y \in G_1 : F(xy) = F(x)F(y)$$

So for arbitrary $a_2, b_2 \in G_2 : \exists ! \, a_1, b_1 \in G_1 \ni F(a_1) = a_2, F(b_1) = b_2$. Therefore:

$$\begin{aligned}
F^{-1}(a_2)F^{-1}(b_2) &= a_1 b_1 \\
&= F^{-1}(F(a_1 b_1)) \\
&= F^{-1}(F(a_1)F(b_1)) \\
&= F^{-1}(a_2 b_2) \qquad \square
\end{aligned}$$

Theoretical Question 2:

Let $F : G_1 \to G_2$ be a group homomorphism. Let H be a subgroup of G_1 and let K be a subgroup of G_2.

a. Prove that $F(H) := \{F(x) \mid x \in H\}$ is a subgroup of G_2.

b. Prove that $F^{-1}(K) := \{x \in G_1 \mid F(x) \in K\}$ is a subgroup of G_1.

Answer 2:

a. Let e be the identity of G_1 and g, h arbitrary elements of H:

Associativity: Inherited

Identity: $F(eh) = F(e)F(h) = F(h)$
therefore $F(e)$ is the identity in $F(H)$.

Inverses: $F(e) = F(gg^{-1}) = F(g)F(g^{-1})$
therefore $F(g^{-1})$ is the inverse of $F(g) \in H$.

Closure: $F(g)F(h) = F(gh) \in F(H)$

b. Let $g, h \in F^{-1}(K)$ and e_1 the identity in G_1 and e_2 the identity in G_2.

Associativity: Inherited

Identity: $F(e_1 h) = F(e_1)F(h) = F(h)$
therefore $e_1 \in F^{-1}(e_2) \subseteq F^{-1}(K)$

Inverses: $e_2 = F(e_1) = F(hh^{-1}) = F(h)F(h^{-1})$
so $: F(h^{-1}) = (F(h))^{-1} \in K$ and $h^{-1} \in F^{-1}(K)$

Closure: $F(gh) = F(g)F(h)$ so by closure of K,
$F(gh) \in K$ and $gh \in F^{-1}(K)$

Theoretical Question 3:

Let $F : G_1 \to G_2$ be a group homomorphism. Prove that $\ker(F)$ is a normal subgroup of G_1.

Answer 3:

Let g, h be arbitrary elements of $G_1, \ker(F)$ respectively and e_1, e_2 the identities of G_1, G_2 respectively. By TQ#2 $F^{-1}(\{e_2\}) = \ker(F)$ is a subgroup of G_1 and

$$\begin{aligned}
F(ghg^{-1}) &= F(g)F(h)F(g^{-1}) \\
&= F(g)e_2F(g^{-1}) \\
&= F(g)F(g^{-1}) \\
&= F(gg^{-1}) \\
&= F(e_1) \\
&= e_2 \\
\therefore \quad ghg^{-1} &\in \ker F
\end{aligned}$$

and $\ker F$ is a normal subgroup of G_1.

Theoretical Question 4:

Let $F : G_1 \to G_2$ be a group homomorphism. Prove that F is one-to-one if and only if $\ker(F) = \{e_{G_1}\}$.

Answer 4:

- $\Leftarrow$ If F is 1-to-1 then only one element in G_1 maps to the identity of G_2 and that must be e_{G_1}.

- $\Rightarrow$ *By contradiction*: Suppose $\ker(F) = \{e_{G_1}\}$ and F is not 1-to-1. Then for some $a, b \in G_1 : (a \neq b) \wedge (F(a) = F(b))$.

$$\begin{aligned} F(a) &= F(b) \\ F(a)F(a^{-1}) &= F(b)F(a^{-1}) \\ F(aa^{-1}) &= F(ba^{-1}) \\ e_{G_2} &= F(ba^{-1}) \\ \therefore \quad ba^{-1} &\in \ker(F) \end{aligned}$$

  But by the uniqueness of inverses $ba^{-1} \neq e_{G_1}$ and $\ker(F) \neq \{e_{G_1}\}$

Theoretical Question 5:

Let $F : G_1 \to G_2$ be a surjective homomorphism (surjective means onto). Prove that:

a. If G_1 is abelian then so is G_2.

b. If G_1 is cyclic then so is G_2.

Answer 5:

Since F is surjective

$$\forall a_2, b_2 \in G_2 : \exists a_1, b_1 \in G_1 \ni a_2 = F(a_1), b_2 = F(b_1)$$

a. So let a_2, b_2 be arbitrary elements of G_2 and a_1, b_1 be elements of G_1 such that $a_2 = F(a_1), b_2 = F(b_1)$.

$$a_2 b_2 = F(a_1)F(b_1) = F(a_1 b_1) = F(b_1 a_1) = F(b_1)F(a_1) = b_2 a_2$$

b. Since G_1 is cyclic there is a generator: g such that every element $a \in G_1$ is a power of g. I.e. $a \in G_1 \Rightarrow \exists n \in \mathbb{Z} \ni a = g^n$. So for $b_2 \in G_2 \, \exists \, b_1 \in G_1 \ni F(b_1) = b_2$. So for some $k \in \mathbb{Z}$:

$$\begin{aligned} b_2 &= F(b_1) \\ &= F(g^k) \\ &= F(g)^k \end{aligned}$$

therefore $\langle F(g) \rangle = G_2$ and G_2 is cyclic.