UNIVERSITY OF SOUTH CAROLINA

MATH-546 Algebraic Structures I Homework X

Problem 1:

Let

$$G_1 = \{f_{m,b} : \mathbb{R} \to \mathbb{R} \mid f_{m,b}(x) = mx + b,\ m \neq 0\}$$

with composition of functions as operation, and

$$G_2 = \left\{ \begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix} \mid m \neq 0 \right\}$$

with matrix multiplication as operation. We already know that G_1 and G_2 are groups. Prove that $G_1 \cong G_2$.

Answer 1:

The mapping

$$F(f_{m.b}) = \begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix}$$

is obviously bijective, so to prove isomorphism we show

$$F(f_{m_1,b_1} \circ f_{m_2,b_2}) = F(f_{m_1,b_2}) F(f_{m_2,b_2})$$

$$\begin{aligned}
(f_{m_1,b_1} \circ f_{m_2,b_2})(x) &= f_{m_1,b_1}(f_{m_2,b_2}(x)) \\
&= f_{m_1,b_1}(m_2 x + b_2) \\
&= m_1(m_2 x + b_2) + b_1 \\
&= f_{m_1 m_2, m_1 b_2 + b_1}(x) \\
\therefore \quad F(f_{m_1,b_1} \circ f_{m_2,b_2}) &= \begin{bmatrix} m_1 m_2 & m_1 b_2 + b_1 \\ 0 & 1 \end{bmatrix}
\end{aligned}$$

and

$$\begin{aligned}
F(f_{m_1,b_2}) F(f_{m_2,b_2}) &= \begin{bmatrix} m_1 & b_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m_2 & b_2 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} m_1 m_2 & m_1 b_2 + b_1 \\ 0 & 1 \end{bmatrix} \\
&= F(f_{m_1,b_1} \circ f_{m_2,b_2})
\end{aligned}$$

$\square$

Problem 2:

Let $C = \{1, -1\}$ with multiplication as operation and let

$$G = C \times \mathbb{R}^+,$$

where $\mathbb{R}^+$ is the group of positive real numbers with multiplication as operation. Prove that $\mathbb{R}^* \cong G$.

Answer 2:

Let F be a map $F : G \mapsto \mathbb{R}^*$ by $F((c,r)) = cr$. This is clearly bijective with $F^{-1}(r) = (\text{sgn}(r), |r|)$.

Consider $(c_1, r_1), (c_2, r_2) \in G$. Then

$$\begin{aligned} F((c_1, r_1)(c_2, r_2)) &= F((c_1c_2, r_1r_2)) \\ &= (c_1c_2)(r_1r_2) \\ &= (c_1r_1)(c_2r_2) \\ &= F((c_1, r_1))F((c_2, r_2)) \end{aligned}$$

□

Problem 3:

Let

$$G = \{x \in \mathbb{R} \mid x > 0 \text{ and } x \neq 1\},$$

with operation $*$ defined as

$$a * b = a^{\ln b}$$

Prove that $G \cong \mathbb{R}^*$. *Hint:* In order to define $F : G \to \mathbb{R}^*$, think of a function that sends the identity of G to the identity of $\mathbb{R}^*$.

Answer 3:

From the definition of the operation we have

$$a * b = a^{\ln b} = (\exp(\ln a))^{\ln b} = \exp(\ln a \ln b)$$

and from that we have the identity in G being $e = \exp(1)$. So $F(g) = \ln g$ with inverse $F^{-1}(r) = \exp(r)$ is a bijection with the problematic $F(1) = 0$ already excluded. With

$$\begin{aligned} F(a * b) &= F(\exp(\ln a \ln b)) \\ &= \ln(\exp(\ln a \ln b)) \\ &= \ln a \ln b \\ &= F(a)F(b) \end{aligned}$$

□

Problem 4:

Let G be a group and define

$$F_1 : G \to G \text{ by } F_1(x) = x^2, \quad F_2 : G \to G \text{ by } F_2(x) = x^{-1}.$$

a. Prove that F_1 is a homomorphism if and only if G is abelian.

b. Prove that F_2 is a homomorphism if and only if G is abelian.

c. Give an example of a group G for which the function F_1 is an isomorphism. Explain.

Answer 4:

$\forall a, b \in G$:

a. $\Rightarrow$

$$\begin{aligned} F_1(ab) &= (ab)^2 \\ &= abab \\ &= aabb && \text{since } G \text{ is abelian} \\ &= (aa)(bb) \\ &= F_1(a)F_1(b) && \square \end{aligned}$$

$\Leftarrow$ F_1 is a homomorphism therefore

$$\begin{aligned} F_1(ab) &= F_1(a)F_1(b) \\ abab &= aabb \\ a^{-1}ababb^{-1} &= a^{-1}aabbb^{-1} \\ ba &= ab \qquad \square \end{aligned}$$

b. $\Rightarrow$

$$\begin{aligned} F_2(ab) &= (ab)^{-1} \\ &= b^{-1}a^{-1} \\ &= a^{-1}b^{-1} && \text{since } G \text{ is abelian} \\ &= F_2(a)F_2(b) && \square \end{aligned}$$

$\Leftarrow$ F_2 is a homomorphism therefore

$$\begin{aligned} F_2(a^{-1}b^{-1}) &= F_2(a^{-1})F_2(b^{-1}) \\ (a^{-1}b^{-1})^{-1} &= ab \\ ba &= ab \qquad \square \end{aligned}$$

c. Let $G = \mathbb{R}^+$ with multiplication. Clearly G is an abelian group and F_1 a bijection. And

$$\begin{aligned} F_1(ab) &= (ab)^2 \\ &= a^2b^2 \\ &= F_1(a)F_1(b) \end{aligned}$$

and F_1 is a homomorphism.

Problem 5:

Let $G = \mathbb{R}$ with operation $*$ defined as

$$a * b = a + b - 1.$$

Prove that G is isomorphic to $\mathbb{R}$ (the real numbers with addition).

Answer 5:

Consider $F : G \mapsto \mathbb{R}$ with $F(g) = g - 1$ and $F^{-1}(r) = r + 1$. This is clearly a bijection and

$$F(a * b) = (a + b - 1) - 1 = (a - 1) + (b - 1) = F(a) + F(b)$$

so F is an isomorphism.

Problem 6:

Let $G = \mathbb{R} \setminus \{-1\}$ with operation $*$ defined as

$$a * b = a + b + ab.$$

Prove that G is isomorphic to $\mathbb{R}^*$ (the nonzero real numbers with multiplication).

Answer 6:

Problem 7:

For each of the following, decide whether the given function is a homomorphism. Justify.

a. $F : \mathbb{C}^* \to \mathbb{R}^*, \quad F(a + bi) = a^2 + b^2.$

b. $G : \mathbb{R} \to \mathrm{GL}_2(\mathbb{R}), \quad G(a) = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}.$

c. $F : S_3 \to S_3, \quad F(\sigma) = \sigma^2.$

d. $F : GL_2(\mathbb{R}) \to \mathbb{R}^*, \quad F\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ab.$

Answer 7:

a. Yes: $a + bi = re^{i\theta} \mapsto r^2$ and $F(r_1 e^{i\theta} r_2 e^{i\phi}) = F(r_1 r_2 e^{i(\theta+\phi)}) = r_1^2 r_2^2 = F(r_1 e^{i\theta}) F(r_2 e^{i\phi})$

b. Yes: $G(a)G(b) = G(a + b)$:

$$\begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ a+b & 1 \end{bmatrix}$$

c. No: $S_3 \cong A_3$ and using A_3 to simplify notation $F(rs) = e$ but $F(r)F(s) = r^2$

d. No: The identity $I \in GL_2(\mathbb{R})$ but $F(I) = 0 \notin \mathbb{R}^*$

Problem 8:

a. Prove that $\mathbb{R}$ is not isomorphic to $\mathbb{R}^*$.

b. Prove that $\mathbb{R}^*$ is not isomorphic to $\mathbb{C}^*$.

Answer 8:

In each case there is a mismatch of possible orders of elements:

a. For $x \in \mathbb{R}, o(x) \in \{1, \infty\}$
   For $x \in \mathbb{R}^*, o(x) \in \{1, 2, \infty\}$ since $o(-1) = 2$

b. For $x \in \mathbb{R}^*, o(x) \in \{1, 2, \infty\}$
   But in $\mathbb{C}^*$ we have $o(i) = 4$.