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Knots and Fractals

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KU Leuven

Academic year 2024 – 2025

Outline

Outline

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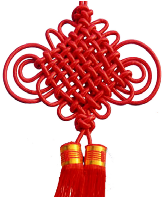
What is a knot



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What is a knot

- To study knots rigorously, we need to connect the two ends of the rope.

Definition In mathematics, a knot is an embedding of the circle $\mathbf{S}^1$ into 3-dimensional Euclidean space, $\mathbf{R}^3$.

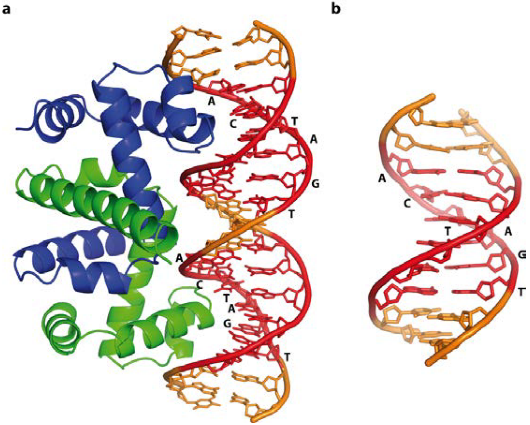
Motivation

- Peter Guthrie Tait and Lord Kelvin (19th century): During the research on composition of atoms, they conjectured that atoms are made out of vortex ring of ether.

images/simple knots.png

Motivation

- The structure of protein and DNA



Motivation

- The knot structure of material

images/material.jpg

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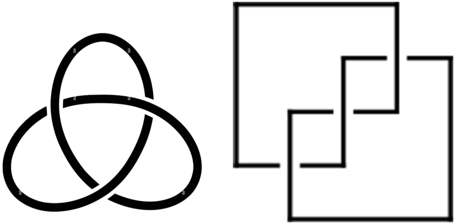
Diagram

- How to represent a knot? Regular projection.
- A knot projection is called a regular projection if no three points on the knot project to the same point, and no vertex projects to the same point as any other point on the knot.

Diagram

- A knot diagram is the regular projection of a knot to the plane with broken lines indicating where one part of the knot undercrosses the other part.



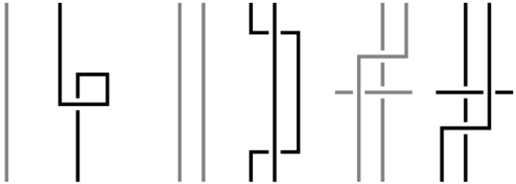
Reidemeister move

- Equivalence of knot?

images/equivalence.png

Reidemeister move

- A Reidemeister move is an operation that can be performed on the diagram of a knot whithout altering the corresponding knot.



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Alternating Knots

In these types of knots, if you follow any strand in any direction you switch between upside and downside at any crossing.

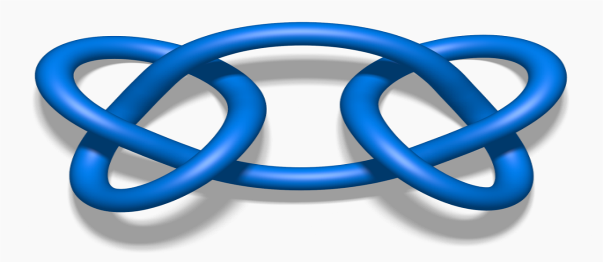

Figure: An alternating knot

(https://math.stackexchange.com/users/24947/adam-lowrance), n.d.

Rational Knots

Rational knots are obtained by closing off the edges of rational tangles. A tangle is a region in a knot that is separated from the knot by a circle and has four outgoing strands crossing the circle.

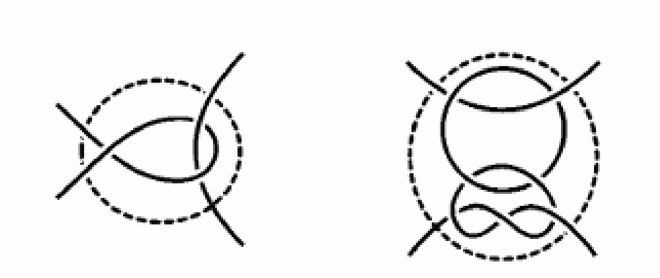

Figure: Tangles

Adams, 2004

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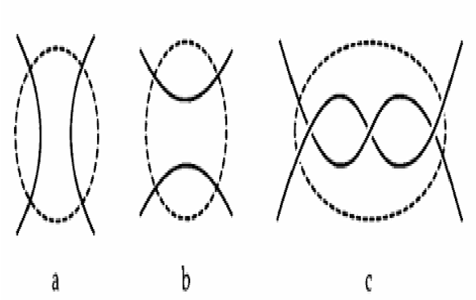


Figure: ∞, 0 and 3 tangles, they are fundamental

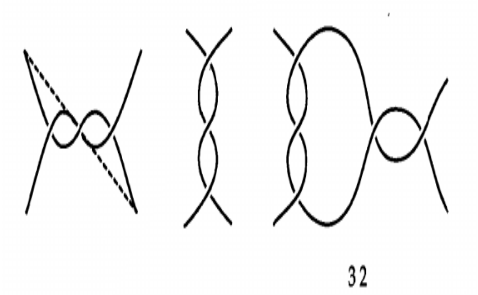
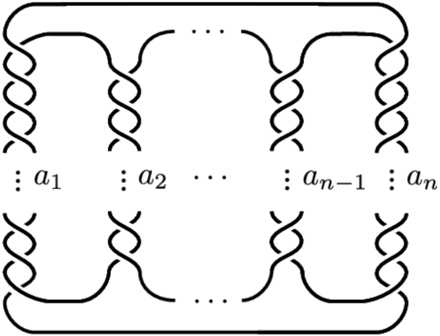


Figure: Generating Tangles

Adams, 2004

Pretzel Knots

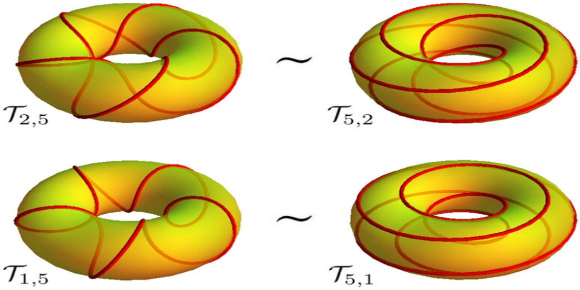
Pretzel knots are rational knots that are obtained by adding up rational tangles. If every term of a rational knot klm (for instant 311) is multiplied with 0 and the result is a pretzel knot that is denoted by k,l,m (3,1,1)



Komendarczyk and Michaelides, 2016

Torus Knots

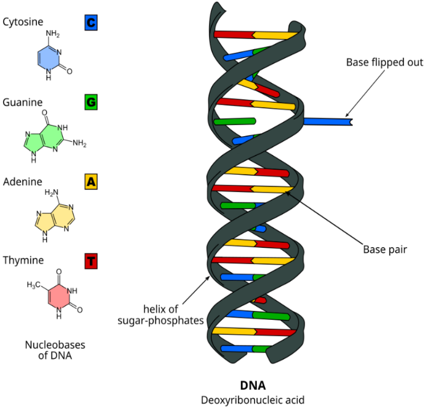
These knots are wrapped around unknotted tori and do not cross themselves as they move around their tori. There are 'short' paths called meridian and 'long' paths called longitude. A (p,q)-torus knot crosses the longitude p times and crosses meridians q times.



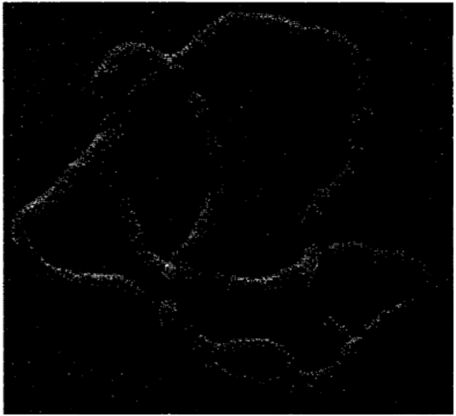
Oberti and Ricca, 2017

Knots and Biology

en.wikipedia.org, n.d.



Adams, 2004



- Tw(R): Twist of the ribbon. Average of crossings over the axis
- Wr(R): Writhe of the ribbon. Average of crossings of the axis from every projection it is
  $$\frac{\int signed{-}crossover{-}number.dA}{\int dA}$$
- Lk(R): Linking number
- Lk(R) = Tw(R) + Wr(R)

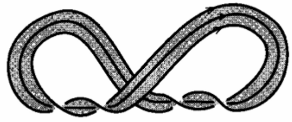
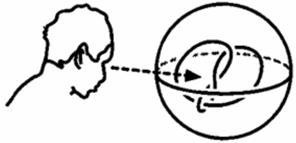


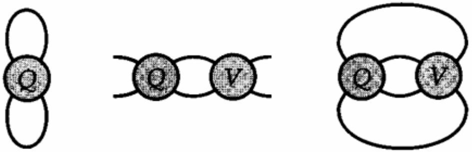
Figure: Axis

Adams, 2004

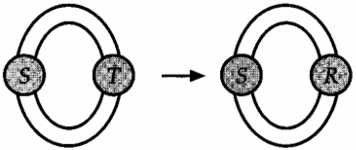


Adams, 2004

- S: Substrat tangle
- T: Site tangle
- R: Recombination tangle
- N(Q): Converting tangle to a knot or a link
- $N(S + T) = N(1)$ (unknot)
- $N(S + R) = N(2)$ (Hopf link)
- $N(S + R + R) = N(211)$ (figure-eight knot)
- $N(S + R + R + R) = N(11111)$ (Whitehead link)



Adams, 2004



Adams, 2004

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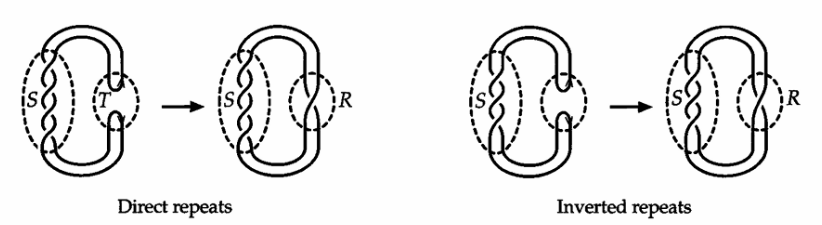


Figure: Effects of Int enzyme

Adams, 2004

Knots and Chemistry

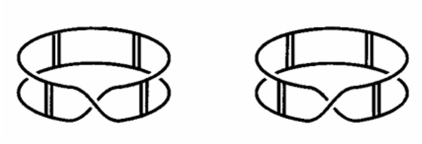
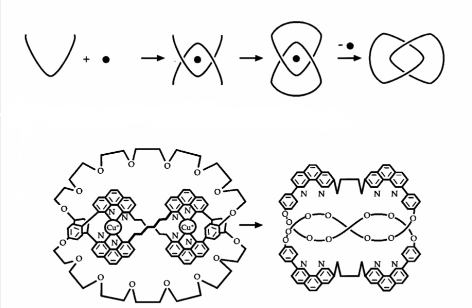


Figure: Homeomorphic but not isotopic molecules

Adams, 2004



Adams, 2004

Knots and Topology

Poincaré Conjecture: Every three-dimensional topological manifold which is closed, connected, and has trivial fundamental group is homeomorphic to the three-dimensional sphere.

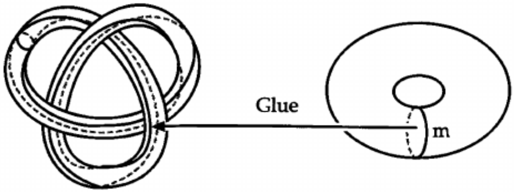


Figure: Dehn Surgery

Adams, 2004

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Open Problems

- Could there be a constant c such that for any knot K and for any two projections P 1 and P2 of K, each with no more than n crossings, one can get from one projection to the other by Reidemeister moves without ever having more than n + c crossings at any intermediate stage?
- Find all of the 14-crossing prime knots.
- Show that the number of distinct prime (n + 1)-crossing knots is greater than the number of distinct prime n-crossing knots, for each positive integer n.
- Is it true that a knot with unknotting number n cannot be a composite knot with n + 1 factor knots?
- Show that the crossing number of a composite knot is the sum of the crossing numbers of the factor knots, that is, $c(K_1K_2) = c(K_1) + c(K_2)$

Adams, 2004

Outline

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Knots Inside the Menger Sponge

What is a knot in the Menger Sponge?

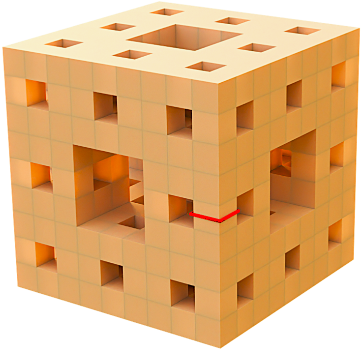
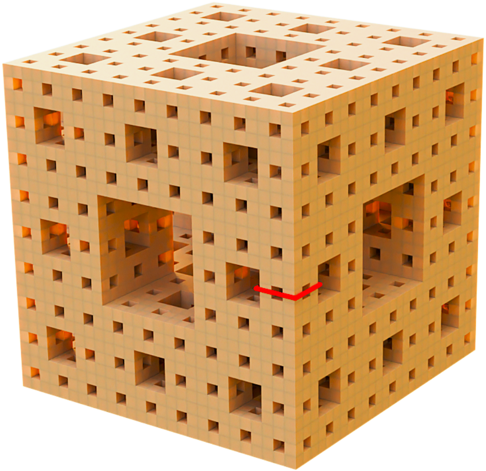


Figure: The Unknot?

Not a Knot!



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Knot or Not?

- Does the following knot live on the fractal?

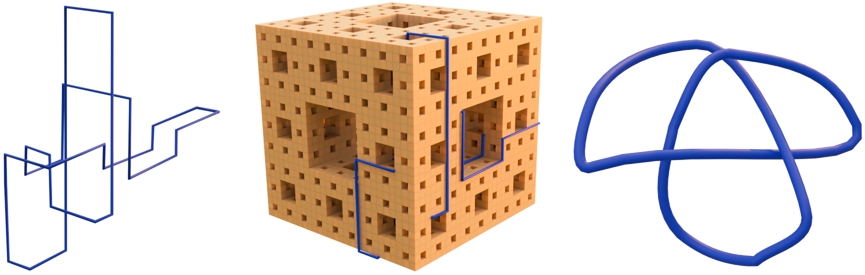

Figure: Enter Caption

- What possible knots can we find in this fractal?

The Positive Answer

This is the question asked by highschoolers Niko Voth (top right), Joshua Broden (bottom right) and Noah Nazareth under the supervision of Malors

- In 2024 they were able to show that you could find every possible knot embedded in the Menger Sponge!
- Published a paper on ArXiv.

Figure: **Quanta**

The Cantor Set

- Split the interval $[0, 1]$ into thirds and remove the middle third
- Repeat on each remaining interval

A point $x \in [0, 1]$ is in the Cantor set if it does not have a 1 in its ternary expansion.

Figure: Cantor Set - **cantor**

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Cantor Dust

Cantor dust is a 2 dimensional fractal and is the cartesian product of 2 cantor sets

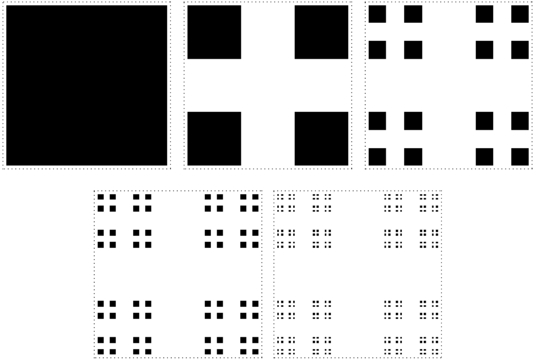

Figure: Enter Caption

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Relation With Sierpienski Carpet

Cantor dust is a 2 dimensional fractal and is the cartesian product of 2 cantor sets

Lemma

(x, y) lies on the Sierpinski carpet if and only if x and y share no digit 1 in their ternary expansion.

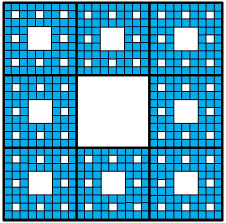

Figure: **sierpinski2016**

Which Points Lie on the Menger Sponge?

Lemma

Let (x, y) be a Cantor Dust point, then $(x, y, z) \in M$ for all $0 \leq z \leq 1$.

Proof.

Consider the first 2 stages where regions not in black have a hole behind them.

Figure: Stage 0 and Stage 1

□

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Which Points Lie on the Menger Sponge?

Proof (Cont.)

- In the next iteration of the sponge we need only consider the black regions as it is never possible to fill in a hole.
- However, each black region occurs at a square of dimension $\frac{1}{3} \times \frac{1}{3}$ and so can be viewed as Stage 0 when zoomed in.
- Repeating this argument inductively we see that the black areas precisely align with points of the Cantor Dust fractal.

$\square$

Arc Presentation of Knot

Definition

An Arc Presentation in grid form is an ordered list of n unordered pairs $\{a_1, b_1\}, \ldots, \{a_n, b_n\}$, such that $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are permutations of $1, \ldots, n$ and a_i, b_i, for $i \in \{1, \ldots., n\}$

Example

An Arc Presentation of the Eight Knot 4_1 is

$$\{3,5\}, \{6,4\}, \{5,2\}, \{1,3\}, \{2,6\}, \{4,1\}.$$

Arc Presentation and Arc Index

An arc presentation of a link L is an embedding of L in finitely many pages of the open-book decomposition so that each of these pages meets L in a single simple arc

MichaelImages/arc_index_fig1.png

Figure: Arc Presentation Trefoil

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Arc Presentation and Arc Index

Definition

The minimum number of pages required to represent a knot is called its arc index

Example

The arc index of a (p, q) torus knot is $p + q$. This was shown using techniques from contact geometry.

Knot Game

Before we start the proof we will play a quick game! The goal is to guess the correct knot inside the sponge.

Knots in The Menger Sponge

We now give the proof that every knot can be found in a finite iteration of the merger sponge.

- Given a knot K we let $\{a_1, b_1\}, \ldots, \{a_n, b_n\}$ be its arc presentation.
- Take an iteration of the Cantor set C, with at least n endpoints and pick n of these points $p_1, \ldots p_k \in C$.
- We now re-write the arc presentation as $\{p_{a_1}, p_{b_1}\} \ldots \{p_{a_n}, p_{b_n}\}$
- This does not change the knot!

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Knots in The Menger Sponge

- We notice if $x_0 \in C$ then $(x_0, y, 0)$ lies entirely on the front face of the sponge for $0 \leq y \leq 1$
- Similarly if $y_0 \in C$ $(x, y_0, 0)$ lies entirely on the front face of the sponge for $0 \leq x \leq 1$
- We conclude the entire knot diagram induced by the given arc presentation lies on the front face of the sponge

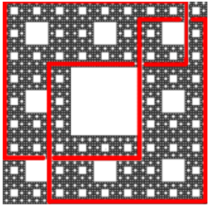

Figure: Enter Caption

Push Through the Sponge!

The last big idea is to push the knot through the sponge

- We draw the vertical lines on the front face and the horizontal lines on the back face.
- We connect them through the sponge.
- This is possible as the points (x_0, y_0) were cantor dust points!
- The knot is our original knot as the projection onto the front face recovers our knot diagram.

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Outline

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Difficulties

Menger Sponge	**Sierpinski Tetrahedron**
Easily understandable representation	Difficulties caused by tetrahedrons
arc representation	?

Difficulties

Menger Sponge	Sierpinski Tetrahedron
Easily understandable representation	Difficulties caused by tetrahedrons
arc representation	?

Solution:

- Combinatorial representation of the Sierpinski Tetrahedron
- Pretzel knots

Combinatorial representation of the Tetrahedron

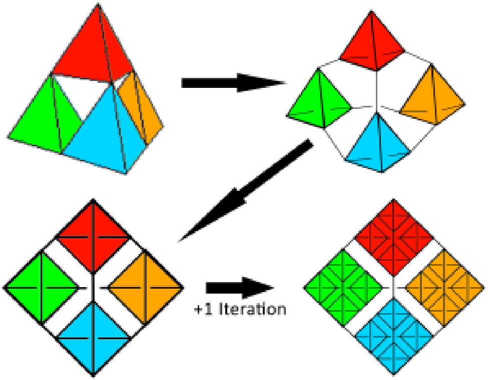

Figure: Combinatorial representation of the Tetrahedron Broden et al., 2024

Combinatorial representation of the Tetrahedron

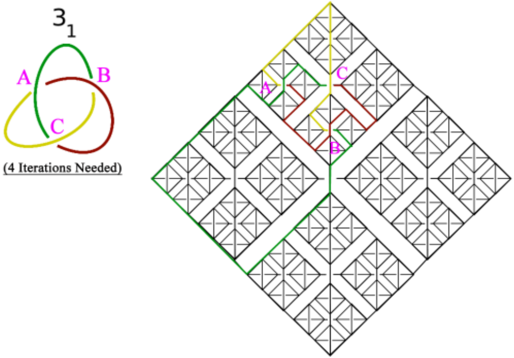

Figure: Example 3^1 knot Broden et al., 2024

Combinatorial representation of the Tetrahedron

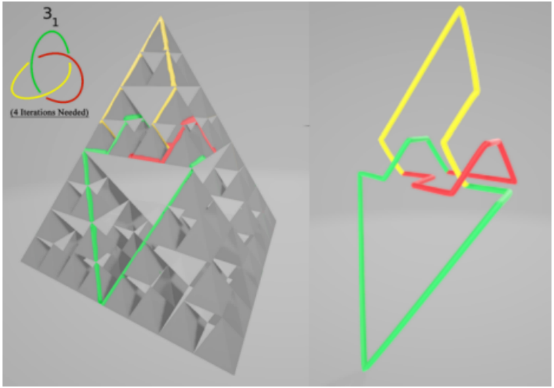

Figure: Example 3^1 knot Broden et al., 2024

Sierpinski Tetrahedron

Theorem

All Pretzel Knots are inside a finite iteration of the Sierpinski Tetrahedron. Broden et al., 2024

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Comparison between Fractals

Theorem

Let K be a not, then we have

$$M(K) \leq S(K),$$

whenever $S(K)$ is defined. Broden et al., 2024

Comparison between Fractals

Theorem

Let K be a not, then we have

$$M(K) \leq S(K),$$

whenever $S(K)$ is defined. Broden et al., 2024

Proof.

Shortly: The i^{th} iteration of the Sierpinski Tetrahedron can be embedded into the Menger Sponge. $\square$

Comparison between Fractals

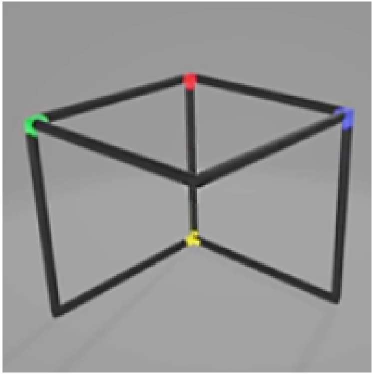

Figure: S_0 in M_0

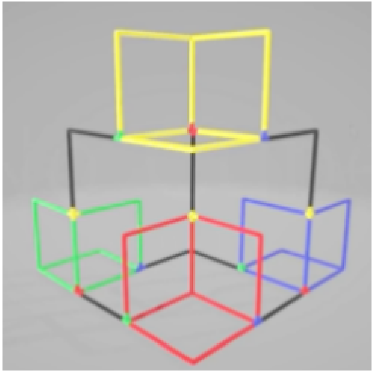

Figure: S_1 in M_1

Introductory Frame

Introductory text.

Frame Title

Text.

Closing Frame

Closing text.

References I

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Fantastic Knots and where to find them