KU LEUVEN

# Knots and Fractals

Caglar Yusuf Berkay Lyu Zhendong Michael McGloin Bence Sógor

KU Leuven

Academic year 2024 – 2025

# Outline

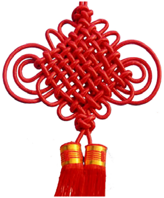




KU LEUVEN

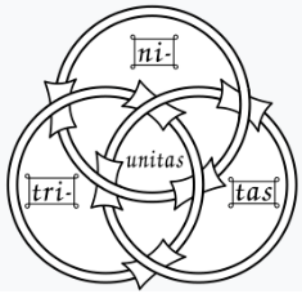
# Outline

# What is a knot



## What is a knot

- Composite knots



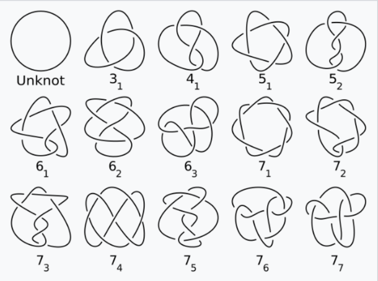
## What is a knot

- To study knots rigorously, we need to connect the two ends of the rope.

Definition In mathematics, a knot is an embedding of the circle $\mathbf{S}^1$ into 3-dimensional Euclidean space, $\mathbf{R}^3$.

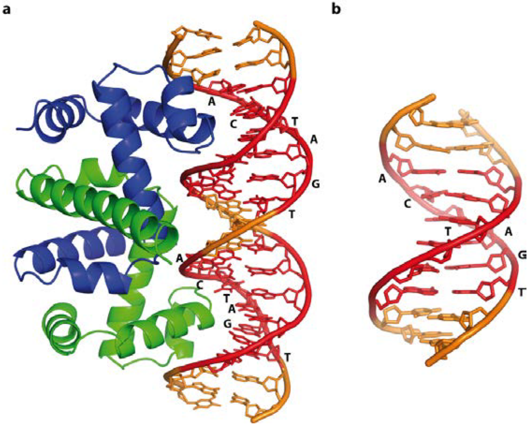
# Motivation

- Peter Guthrie Tait and Lord Kelvin (19th century): During the research on composition of atoms, they conjectured that atoms are made out of vortex ring of ether.



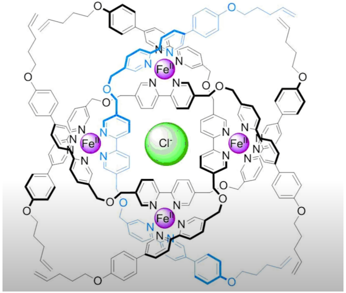
# Motivation

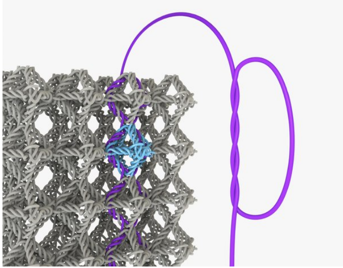
- The structure of protein and DNA



# Motivation

- The knot structure of chemical compound and material



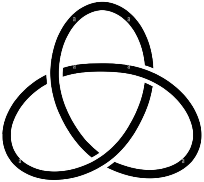


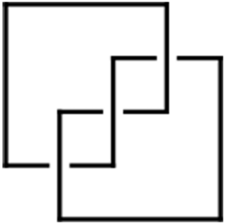
# Diagram

- How to represent a knot? Regular projection.
- A knot projection is called a regular projection if no three points on the knot project to the same point, and no vertex projects to the same point as any other point on the knot.

# Diagram

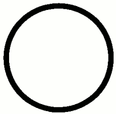
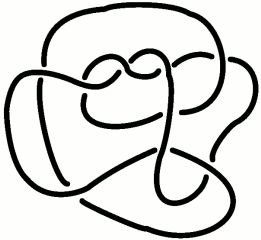
- A knot diagram is the regular projection of a knot to the plane with broken lines indicating where one part of the knot undercrosses the other part.





## Equivalence of knots

- Are they equivalent?

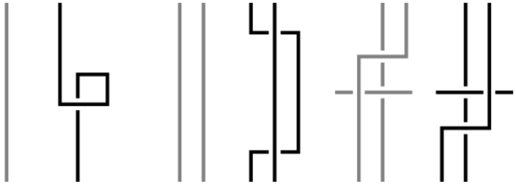
## Reidemeister move

- Equivalence of knot?

- Formal definition: two knots $\mathbf{K}_1$ and $\mathbf{K}_2$ are equivalent if there is an orientation-preserving homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ with $h(\mathbf{K}_1) = h(\mathbf{K}_2)$.
- Reidemeister's Theorem: If two knots are equivalent, their diagrams are related by a sequence of Reidemeister moves.

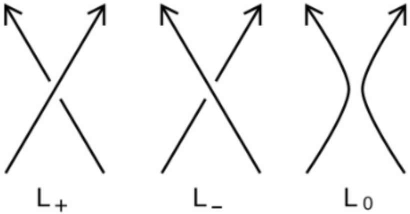
## Reidemeister move

- A Reidemeister move is an operation that can be performed on the diagram of a knot whithout altering the corresponding knot.



# Invariant polynomial

- 1923(James Waddell Alexander): Alexander polynomial $\nabla$. Reformulated by John




Conway in 1969.

$$\nabla(O) = 1$$

$$\nabla(L^+) - \nabla(L^-) = z\nabla(L_0)$$

## Invariant polynomial

- 1984(Vaughan Jones): Jones polynomial $V$. Jones won the field's medal for this discovery in 1990.

$$V(L_0) = 1$$

$$t^{-1}V(L_+) - tV(L_-) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})V(L_0)$$

- 1985(multiple authors): HOMFLY-PT polynomial $P$: A generalization of the above invariants.

$$P(L_0) = 1$$

$$tP(L_+) + t^{-1}P(L_-) + mP(L_0) = 0$$

# Outline

# Alternating Knots

In these types of knots, if you follow any strand in any direction you switch between upside and downside at any crossing.

Figure: An alternating knot

Fielden et al., 2017

## Rational Knots

Rational knots are obtained by closing off the edges of rational tangles. A tangle is a region in a knot that is separated from the knot by a circle and has four outgoing strands crossing the circle.

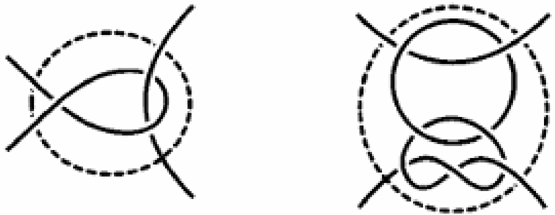


Figure: Tangles

Adams, 2004



KU LEUVEN

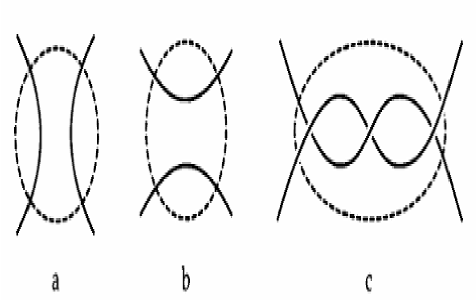



Figure: $\infty$, 0 and 3 tangles, they are fundamental

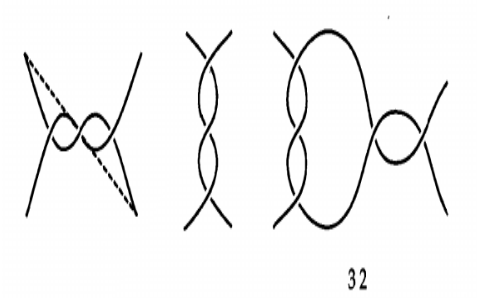
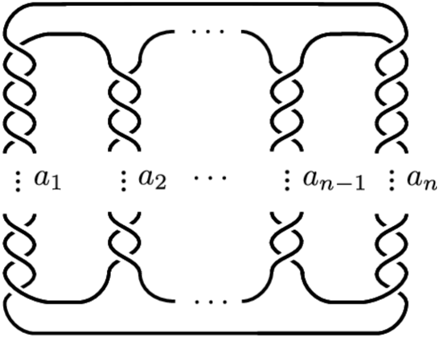



Figure: Generating Tangles

Adams, 2004

# Pretzel Knots

Pretzel knots are rational knots that are obtained by adding up rational tangles. If every term of a rational knot klm (for instant 311) is multiplied with 0 and then added altogether, the result is a pretzel knot that is denoted by k,l,m (3,1,1)



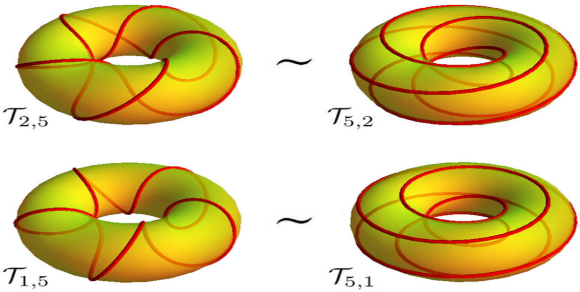

Komendarczyk and Michaelides, 2016



KU LEUVEN

## Torus Knots

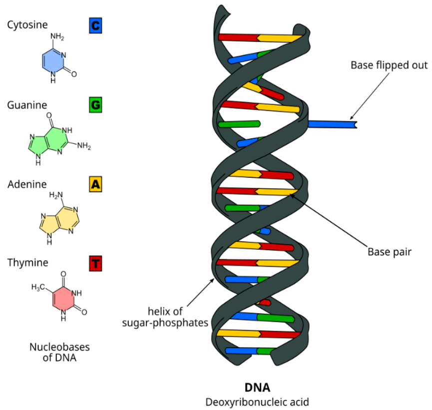
These knots are wrapped around unknotted tori and do not cross themselves as they move around their tori. There are 'short' paths called meridian and 'long' paths called longitude. A (p,q)-torus knot travels the longitude p times and travels meridians q times.



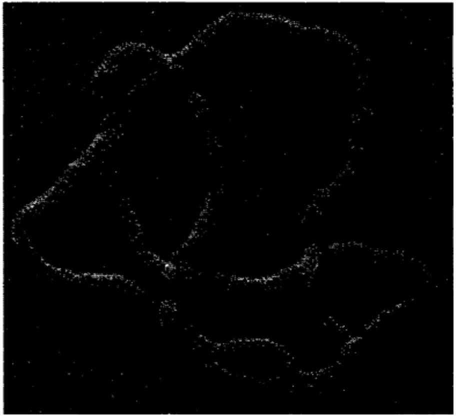

Oberti and Ricca, 2017

# Knots and Biology

en.wikipedia.org, n.d.




Adams, 2004





KU LEUVEN

- Tw(R): Twist of the ribbon. Average of crossings over the axis
- Wr(R): Writhe of the ribbon. Average of crossings of the axis from every projection it is

$$\frac{\int signed{-}crossover{-}number.dA}{\int dA}$$

- Lk(R): Linking number
- Lk(R) = Tw(R) + Wr(R)

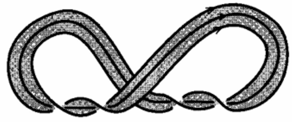
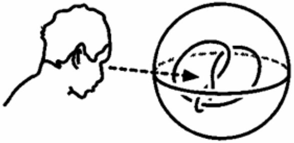


Figure: Axis

Adams, 2004

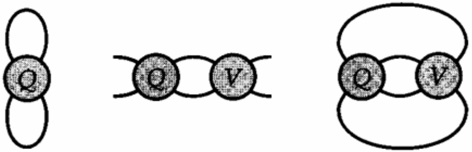


Adams, 2004

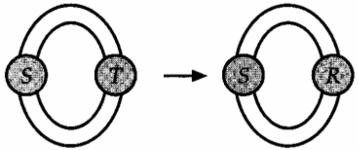


KU LEUVEN

- S: Substrat tangle
- T: Site tangle
- R: Recombination tangle
- N(Q): Converting tangle to a knot or a link
- N(S + T) = N(1) (unknot)
- N(S + R) = N(2) (Hopf link)
- N(S + R + R) = N(211) (figure-eight knot)
- N(S + R + R + R) = N(11111) (Whitehead link)




Adams, 2004




Adams, 2004

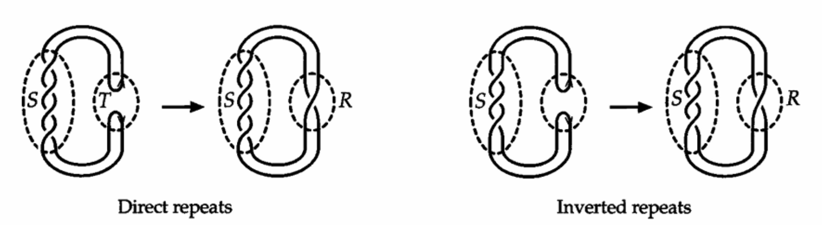



Figure: Effects of Int enzyme

Adams, 2004

# Knots and Chemistry

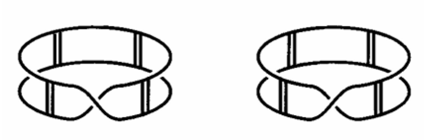
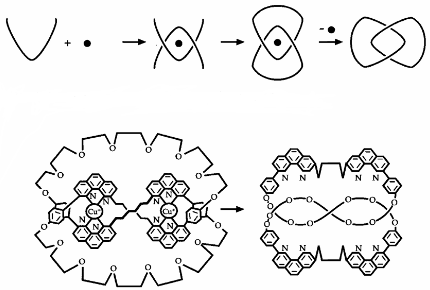


Figure: Homeomorphic but not isotopic molecules

Adams, 2004



Adams, 2004

# Knots and Topology

Poincaré Conjecture: Every three-dimensional topological manifold which is closed, connected, and has trivial fundamental group is homeomorphic to the three-dimensional sphere.

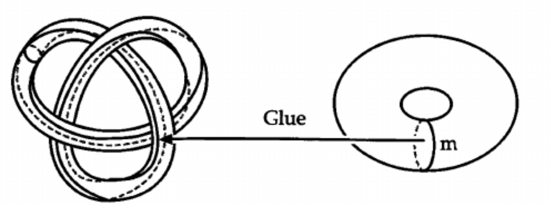



Figure: Dehn Surgery

Adams, 2004

## Open Problems

- Could there be a constant c such that for any knot K and for any two projections P 1 and P2 of K, each with no more than n crossings, one can get from one projection to the other by Reidemeister moves without ever having more than n + c crossings at any intermediate stage?
- Find all of the 14-crossing prime knots.
- Show that the number of distinct prime (n + 1)-crossing knots is greater than the number of distinct prime n-crossing knots, for each positive integer n.
- Is it true that a knot with unknotting number n cannot be a composite knot with n + 1 factor knots?
- Show that the crossing number of a composite knot is the sum of the crossing numbers of the factor knots, that is, $c(K_1 K_2) = c(K_1) + c(K_2)$

Adams, 2004



KU LEUVEN

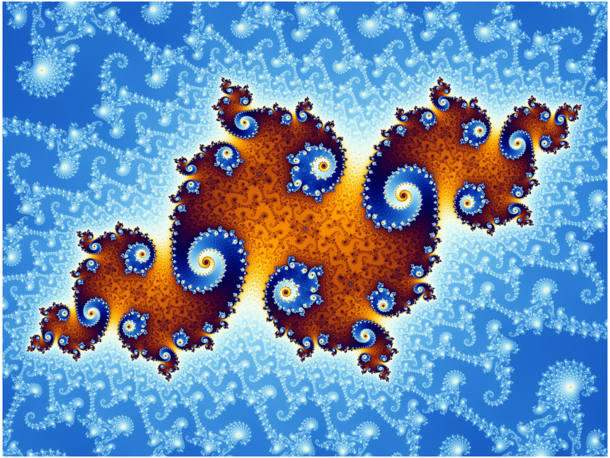
# Outline





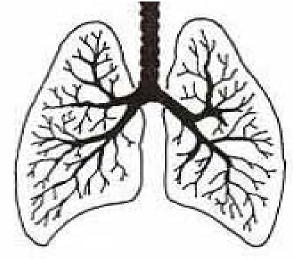
KU LEUVEN

# Fractals



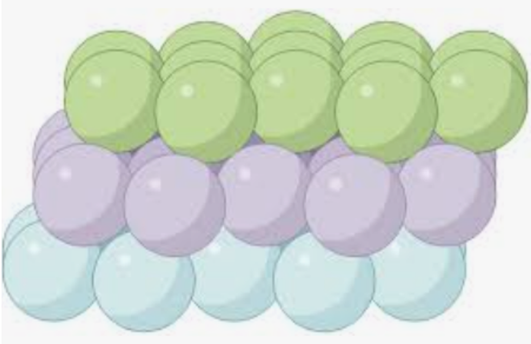
# Motivation

- Structure of Coastline
- Lungs

## Some applications

- Computer simulation: Fractal landscape
- Sphere packing: Fractal dimension



# Geometric Fractals

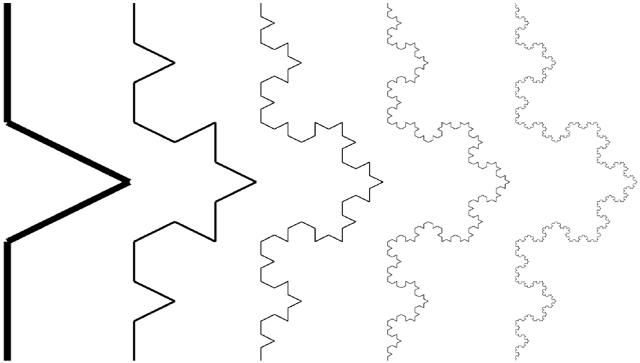


Figure: Koch curve (**frac**)

# Algebraic Fractals

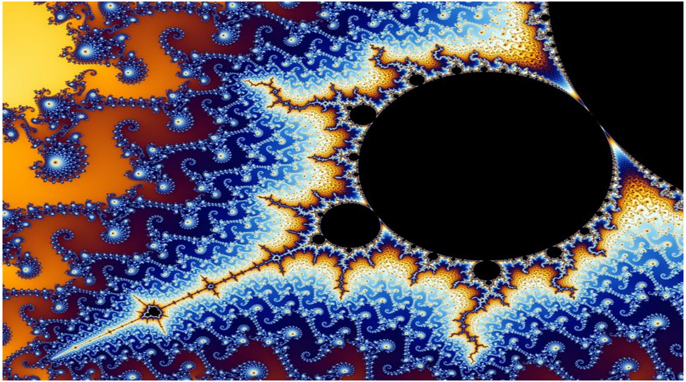


Figure: Section of a Mandelbrot set (**wiki:xxx**)



KU LEUVEN

# Fractal Dimension

- Key Question: How does the object scale?

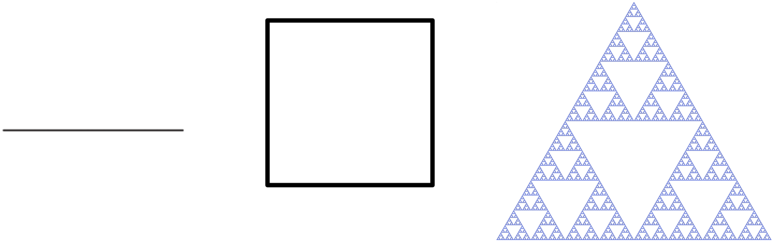


Figure: Wikipedia, accessed 3-April-2025

# Box-Counting Dimension

- Place our fractal in grid of "boxes"
- How does the number of boxes change as they get smaller?

$$N \approx N_0 s^{-d} \implies \dim_{box}(S) = \lim_{s \to 0} \frac{log(N(s))}{log(s^{-1})}$$

Figure: Dimension of UK coastline is approx 1.25, (Mandelbrot, 1983, Wikipedia, 2025)

# Application

- Medicine
  - Identifying tumors in brain MR images Iftekharuddin et al., 2003
  - Diabetic retinopathy, etc. : blood vessel's diameter Uahabi and Atounti, 2015
- Economics
  - Market properties, price fluctuation, money flow. Takayasu and Takayasu, 2009
- Geology and Ecology
  - Landscape patterns, vegetation structure, animal habitats. Loehle and Li, 1996
- Computer science
  - Image encryption. Sangavi and Thangavel, 2019
  - Computer graphics. Sala, 2021
- Architecture
  - Combining aesthetics and functionality. Lorenz, 2002


KU LEUVEN

# Outline





KU LEUVEN

# The Menger Sponge, Menger, 1926

- Universal - Contains every curve

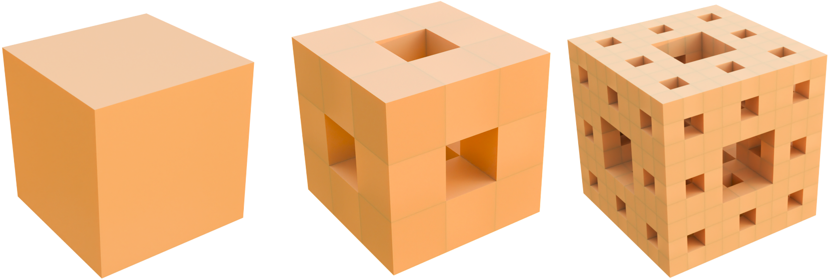


Figure: Michael McGloin (2025)

# Knots Inside the Menger Sponge

What is a knot in the Menger Sponge?

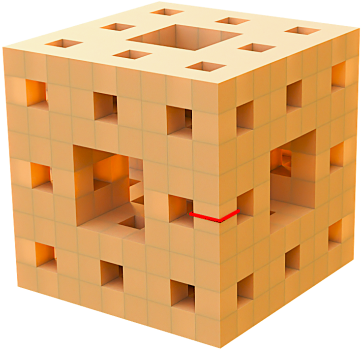
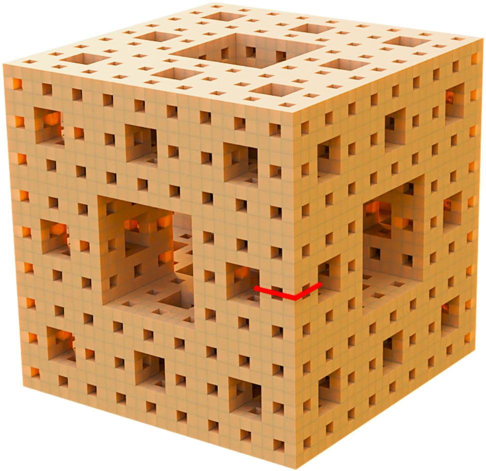


Figure: The Unknot? - Michael McGloin (2025)

# Not a Knot!



## Knot or Not?

- Does the following knot live on the fractal?

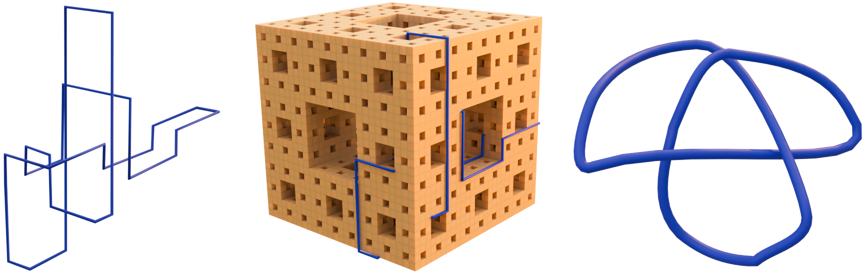


Figure: Trefoil Knot in Sponge - Michael McGloin (2025)

- What possible knots can we find in this fractal?

## The Positive Answer

- This is the question asked by a group of highschoolers.
- In 2024 they were able to show that you could find every possible knot embedded in the Menger Sponge!
- Published a paper on ArXiv.


Figure: Niko Voth (top right), Joshua Broden (bottom right) and Noah Nazareth under the supervision of Malors. Barber, 2024

## The Cantor Set

- Split the interval $[0, 1]$ into thirds and remove the middle third
- Repeat on each remaining interval

Figure: Cantor Set - Commons, 2025



KU LEUVEN

# Cantor Dust

Cantor dust is a 2 dimensional fractal and is the cartesian product of 2 cantor sets

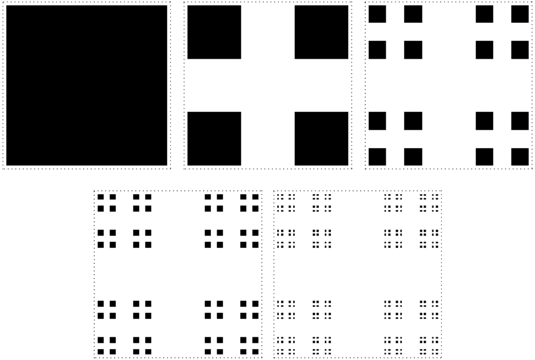


Figure: Dickau, 2014

## Relation With Sierpienski Carpet

**Lemma**

*$(x, y)$ lies on the Sierpinski carpet if and only if $x$ and $y$ share no digit $1$ in their ternary expansion.*

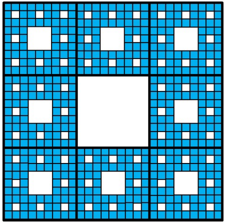


Figure: Sierpinski Carpet, Illustrative-Mathematics, 2016

# Which Points Lie on the Menger Sponge?

**Lemma**

*Let $(x, y)$ be a Cantor Dust point, then $(x, y, z) \in M$ for all $0 \leq z \leq 1$.*

**Proof.**

Consider the first 2 stages where regions not in black have a hole behind them.

Figure: Stage 0 and Stage 1 Broden et al., 2024

$\square$

# Cantor Dust!

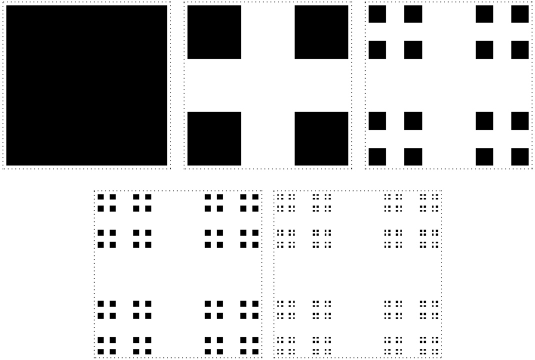
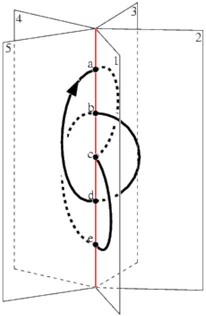


Figure: Dickau, 2014

## Arc Presentation and Arc Index

An arc presentation of a knot K is an embedding of K in finitely many pages of the open-book decomposition so that each of these pages meets K in a single simple arc




$$K = \{3,5\},\{2,4\},\{1,3\},\{2,5\},\{1,4\}$$

Figure: Arc Presentation Trefoil, Cromwell, 1996

# Arc Presentation and Arc Index

## Definition

The minimum number of pages required to represent a knot is called its arc index

## Example

The arc index of a $(p, q)$ torus knot is $p + q$. This was shown using techniques from contact geometry. (Etnyre and Honda, 2000)



KU LEUVEN

# Knots in The Menger Sponge

We now give the proof that every knot can be found in a finite iteration of the merger sponge.

- Let $K = \{a_1, b_1\}, \ldots, \{a_n, b_n\}$ be its arc presentation.
- Take $n$ endpoints, $p_1, \ldots p_n$ of Cantor Set.
- We now re-write $K = \{p_{a_1}, p_{b_1}\} \ldots \{p_{a_n}, p_{b_n}\}$

## Knots in The Menger Sponge

- We notice if $x_0 \in C$ then vertical line through $x_0$ on sponge.
- Same with vertical lines for $y_0$
- Can draw our knot diagram on the face!

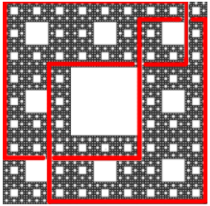


Figure: Broden et al., 2024



KU LEUVEN

## Push Through the Sponge!

The last big idea is to push the knot through the sponge

- Vertical lines on front face.
- Horizontal Lines on back face.
- We connect them through the sponge.
- The knot is our original knot as the projection onto the front face recovers our knot diagram.

# Knot Game

Before we start the proof we will play a quick game! The goal is to guess the correct knot inside the sponge.

# Outline

# Difficulties

| Menger Sponge | Sierpinski Tetrahedron |
|---|---|
| Easily understandable representation | Difficulties caused by tetrahedrons |
| arc representation | ? |

## Difficulties

| Menger Sponge | Sierpinski Tetrahedron |
|---|---|
| Easily understandable representation | Difficulties caused by tetrahedrons |
| arc representation | ? |

Solution:

- Combinatorial representation of the Sierpinski Tetrahedron
- Pretzel knots

# Combinatorial representation of the Tetrahedron

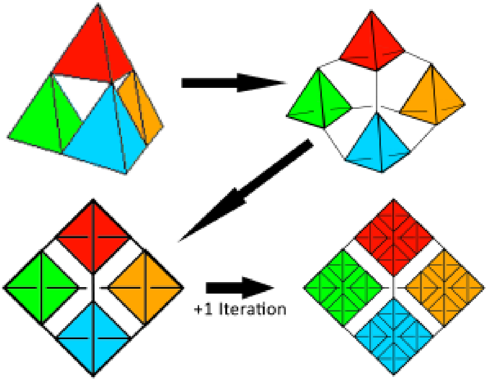



Figure: Combinatorial representation of the Tetrahedron Broden et al., 2024


KU LEUVEN

# Combinatorial representation of the Tetrahedron

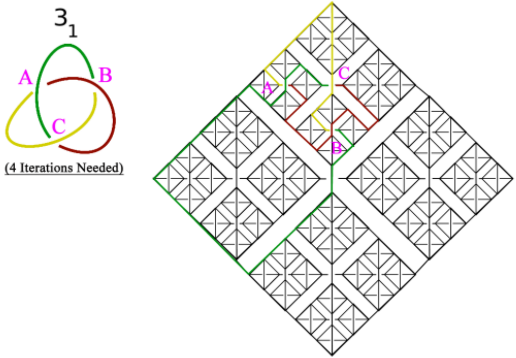



Figure: Example $3^1$ knot Broden et al., 2024

# Combinatorial representation of the Tetrahedron

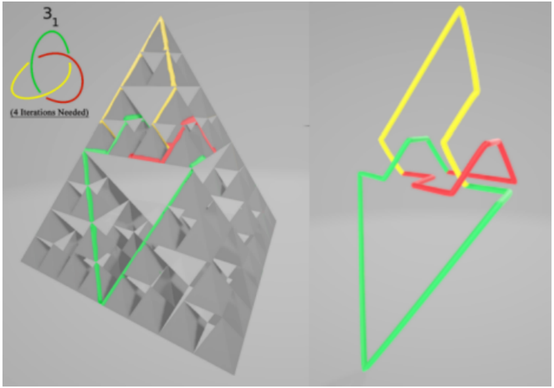



Figure: Example $3^1$ knot Broden et al., 2024

## Sierpinski Tetrahedron

**Theorem**

*All Pretzel Knots are inside a finite iteration of the Sierpinski Tetrahedron. Broden et al., 2024*

## Sierpinski Tetrahedron

**Theorem**

*All Pretzel Knots are inside a finite iteration of the Sierpinski Tetrahedron. Broden et al., 2024*

**Proof.**

**Lemma**

*Every isolated helix can be put inside a finite iteration of the tetrahedron. Broden et al., 2024*

## Sierpinski Tetrahedron

**Theorem**

*All Pretzel Knots are inside a finite iteration of the Sierpinski Tetrahedron. Broden et al., 2024*

**Proof.**

**Lemma**

*Every isolated helix can be put inside a finite iteration of the tetrahedron. Broden et al., 2024*

To finish the proof they showed, that the helices can be connected as required. □



KU LEUVEN

## Sierpinski Tetrahedron

**Theorem**

*All Pretzel Knots are inside a finite iteration of the Sierpinski Tetrahedron. Broden et al., 2024*

**Proof.**

**Lemma**

*Every isolated helix can be put inside a finite iteration of the tetrahedron. Broden et al., 2024*

To finish the proof they showed, that the helices can be connected as required. □

- It is easy to follow that a certain Pretzel knot in which iteration can be embedded.

## Sierpinski Tetrahedron

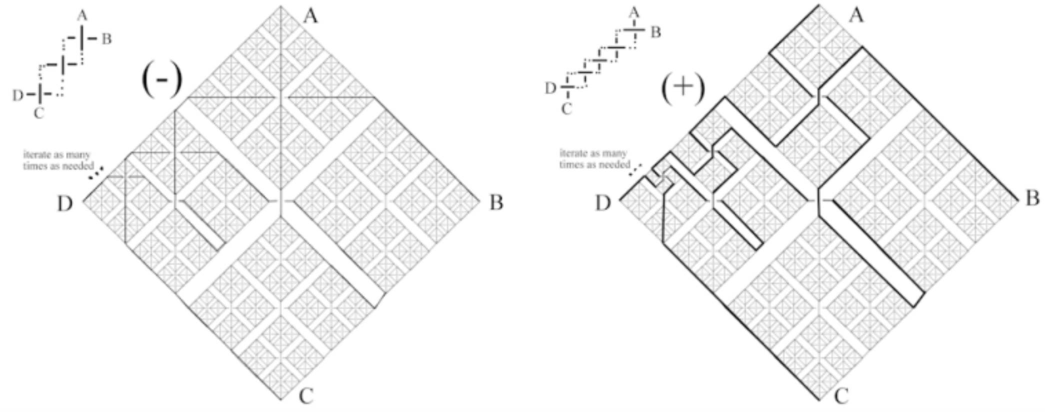



Figure: Helices in a certain iteration. Broden et al., 2024

# Sierpinski Tetrahedron

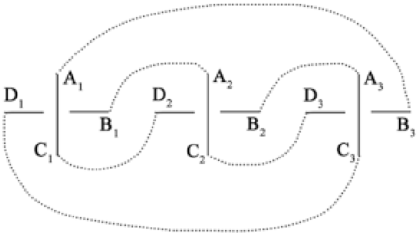



Figure: Every Pretzel Knot can be simplified to the above form. Broden et al., 2024

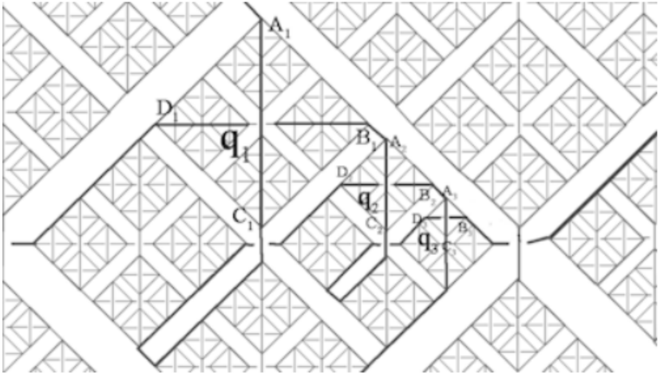



Figure: It is possible to connect the helices in the required order. Broden et al., 2024

## Comparison between Fractals

Denote with $M_n$ and $S_n$ the one-skeleton of the $n^{th}$ iteration of the Menger sponge and the Sierpinski Tetrahedron.
Let $M(K) = \min n$, such that $K \subset M_n$. Similarly $S(K) = \min n, s.t. K \subset S_n$.


KU LEUVEN

## Comparison between Fractals

Denote with $M_n$ and $S_n$ the one-skeleton of the $n^{th}$ iteration of the Menger sponge and the Sierpinski Tetrahedron.
Let $M(K) = \min n$, such that $K \subset M_n$. Similarly $S(K) = \min n, s.t. K \subset S_n$.

**Theorem**

*Let K be a not, then we have*

$$M(K) \leq S(K),$$

*whenever $S(K)$ is defined. Broden et al., 2024*



KU LEUVEN

## Comparison between Fractals

Denote with $M_n$ and $S_n$ the one-skeleton of the $n^{th}$ iteration of the Menger sponge and the Sierpinski Tetrahedron.
Let $M(K) = \min n$, such that $K \subset M_n$. Similarly $S(K) = \min n, s.t. K \subset S_n$.

**Theorem**

*Let K be a not, then we have*

$$M(K) \leq S(K),$$

*whenever $S(K)$ is defined. Broden et al., 2024*

**Proof.**

Shortly: The one-skeleton of the $i^{th}$ iteration of the Sierpinski Tetrahedron can be embedded into the one-skeleton of the $i^{th}$ iteration of the Menger Sponge. This embedding preserves the knot type. $\square$

# Comparison between Fractals

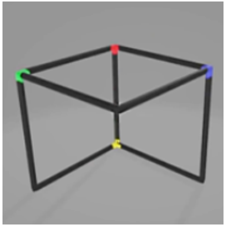


Figure: $S_0$ in $M_0$

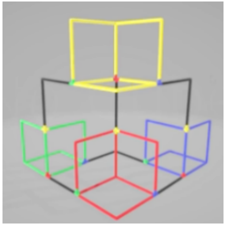


Figure: $S_1$ in $M_1$

## Comparison between Fractals

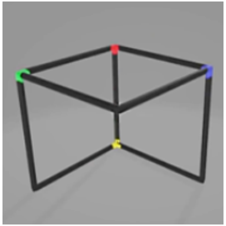


Figure: $S_0$ in $M_0$

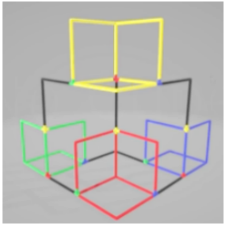


Figure: $S_1$ in $M_1$

What next?

# Comparison between Fractals

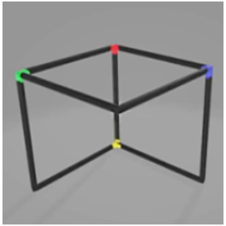


Figure: $S_0$ in $M_0$

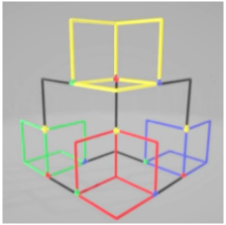


Figure: $S_1$ in $M_1$

What next?

- To get $S_2$ in $M_2$: replace every $S_0$ in $S_1$ by $S_1$.

# Comparison between Fractals

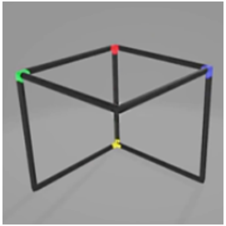


Figure: $S_0$ in $M_0$

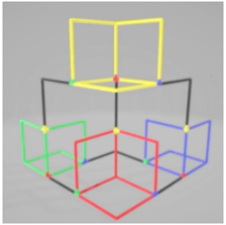


Figure: $S_1$ in $M_1$

What next?

- To get $S_2$ in $M_2$: replace every $S_0$ in $S_1$ by $S_1$.
- To get $S_{n+1}$ in $M_{n+1}$: replace every $S_0$ in $S_n$ by $S_1$.



KU LEUVEN

## Comparison between Fractals

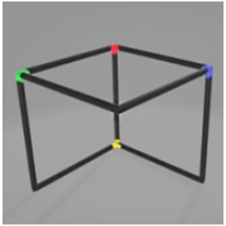


Figure: $S_0$ in $M_0$

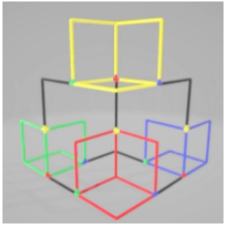


Figure: $S_1$ in $M_1$

What next?

- To get $S_2$ in $M_2$: replace every $S_0$ in $S_1$ by $S_1$.
- To get $S_{n+1}$ in $M_{n+1}$: replace every $S_0$ in $S_n$ by $S_1$.
- Substitutions are in disjoint regions, so the embedding preserves the knot type.

## Open questions Broden et al., 2024

- Every knot can be embedded into the Sierpinski tetrahedron.

# Open questions Broden et al., 2024

- Every knot can be embedded into the Sierpinski tetrahedron.

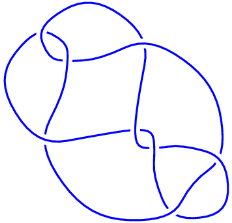


Figure: $8_{12}$ knot is not a Pretzel knot Livingston and Moore, 2025

## Open questions Broden et al., 2024

- Every knot can be embedded into the Sierpinski tetrahedron.

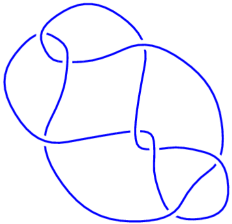


Figure: $8_{12}$ knot is not a Pretzel knot Livingston and Moore, 2025

- Is there a fractal defined by iterative steps that doesn't admit all of the knots? If it is so, what does this tell us about the fractal?

## Conclusion

- Knot theory
- Fractals
- Intersection: Knots inside Fractals
- Open questions

## References I

Adams, C. (2004). *The knot book: An elementary introduction to the mathematical theory of knots*. American Mathematical Society. https://books.google.be/books?id=RqiMCgAAQBAJ

Barber, G. (2024). Teen mathematicians tie knots through a mind-blowing fractal [Accessed: 2025-03-26]. https://www.quantamagazine.org/teen-mathematicians-tie-knots-through-a-mind-blowing-fractal-20241126/

Broden, J., Espinosa, M., Nazareth, N., & Voth, N. (2024). Knots inside fractals. https://arxiv.org/abs/2409.03639

Commons, W. (2025). File:cantor set in seven iterations.svg — wikimedia commons, the free media repository [[Online; accessed 26-March-2025]]. https://commons.wikimedia.org/w/index.php?title=File:Cantor_set_in_seven_iterations.svg&oldid=979565506



KU LEUVEN

## References II


Cromwell, P. (1996). Arc presentations of knots and links. *Banach Center Publications, 42.* https://doi.org/10.4064/-42-1-57-64

Dickau, R. (2014). Cantor dust [Accessed: 2025-04-03]. https://www.robertdickau.com/cantor.html

en.wikipedia.org. (n.d.). Dna [Accessed on 21.03.2025]. https://commons.wikimedia.org/wiki/File:Dna-base-flipping.svg

Etnyre, J. B., & Honda, K. (2000). Knots and contact geometry. https://arxiv.org/abs/math/0006112

Fielden, S. D. P., Leigh, D. A., & Woltering, S. L. (2017). Molecular knots. *Angewandte Chemie International Edition*, *56*(37), 11166–11194. https://doi.org/https://doi.org/10.1002/anie.201702531

Iftekharuddin, K. M., Jia, W., & Marsh, R. (2003). Fractal analysis of tumor in brain mr images. *Machine Vision and Applications*, *13*, 352–362.

## References III


Illustrative-Mathematics. (2016). Sierpinski carpet [Accessed: 2025-03-26]. https://tasks.illustrativemathematics.org/content-standards/tasks/1523

Komendarczyk, R., & Michaelides, A. (2016).Ropelength and finite type invariants. https://doi.org/10.48550/arXiv.1604.03870

Livingston, C., & Moore, A. H. (2025, March). Knotinfo: Table of knot invariants [Accessed on 21.03.2025].

Loehle, C., & Li, B.-L. (1996).Statistical properties of ecological and geologic fractals. *Ecological Modelling*, *85*(2), 271–284. https://doi.org/https://doi.org/10.1016/0304-3800(94)00177-4

Lorenz, W. E. (2002). *Fractals and fractal architecture*. na.

Mandelbrot, B. (1983). *The fractal geometry of nature*. W. H. Freeman; Co.

Menger, K. (1926). *"allgemeine räume und cartesische räume. i."*. Springer-Verlag.

## References IV


Oberti, C., & Ricca, R. (2017). Induction effects of torus knots and unknots. *Journal of Physics A: Mathematical and Theoretical*, *50*. https://doi.org/10.1088/1751-8121/aa80b0

Sala, N. (2021). Fractal geometry in computer graphics and in virtual reality.. *Chaos & Complexity Letters*, *15*.

Sangavi, V., & Thangavel, P. (2019). An image encryption algorithm based on fractal geometry. *Procedia Computer Science*, *165*, 462–469.

Takayasu, M., & Takayasu, H. (2009). Fractals and economics. *Complex systems in finance and econometrics*, 444–463.

Uahabi, K. L., & Atounti, M. (2015). Applications of fractals in medicine. *Annals of the University of Craiova-Mathematics and Computer Science Series*, *42*(1), 167–174.

## References V


Wikipedia. (2025). "minkowski–bouligand dimension - wikipedia the free encyclopedia" ["[Online; accessed 3-April-2025]"]. https://en.wikipedia.org/w/index.php?title=Minkowski%E2%80%93Bouligand_dimension&oldid=1280632501

Wikipedia. (accessed 3-April-2025). "sierpiński triangle - wikipedia the free encyclopedia". https://en.wikipedia.org/w/index.php?title=Sierpi%C5%84ski_triangle&oldid=1281040352

Wikipedia contributors. (2024). Contact geometry — Wikipedia, the free encyclopedia [[Online; accessed 31-March-2025]]. https://en.wikipedia.org/w/index.php?title=Contact_geometry&oldid=1262008950

# Fantastic Knots and Where to Find Them

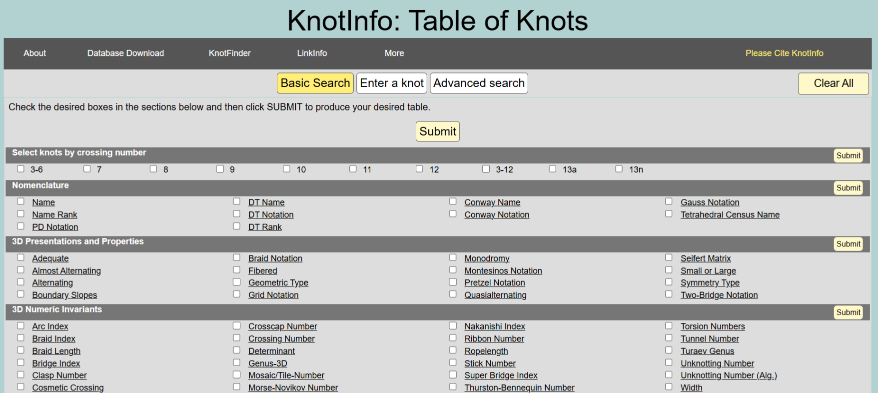



Figure: Knot table Livingston and Moore, 2025



KU LEUVEN

# Contact geometry

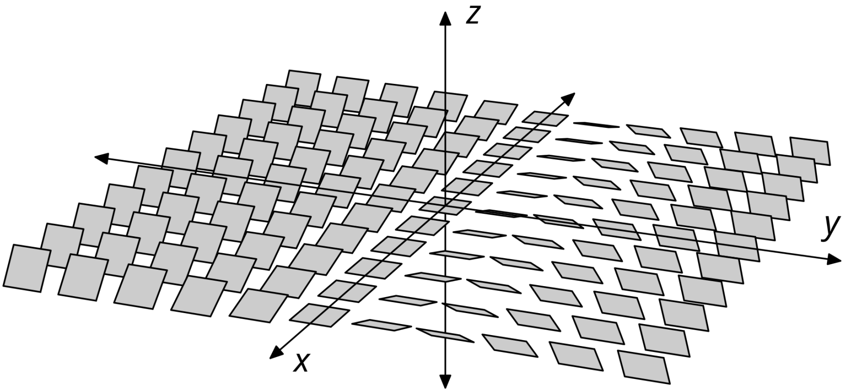



Figure: The standard contact structure on $\mathbb{R}^3$. Wikipedia contributors, 2024