FACULTY OF SCIENCE

Integral Affine Structures

Classification of Integral Affine Structures on Compact Surfaces

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Thesis presented in fulfillment of the requirements for the degree of Master of Science in Mathematics

Academic year 2025–2026

Preface

TODO: Add Preface

Summary

TODO: Add summary

List of symbols

$C^\infty(M, N)$	Smooth maps from M to N

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CHAPTER ZERO

Preliminaries

0.1 Geometric Structures on Manifolds

A lot of the content of this project extends in a nice way to more general types of geometries, namely locally homogeneous geometric structures. The idea behind these structures is that we can locally describe our "geometry" on a manifold M by charts into a space X together with a group G that tells us how to transition between charts on the overlaps.

We will require that our group G be a Lie Group acting transitively on M.

Definition 0.1. Let $U \subset X$ be an open set of our model space X. We say that a smooth map $f : U \mapsto X$ is *locally*-G if for each component $U_i \subset U$ there is a $g_i \in G$ such that $f|_{U_i} = g_i|_{U_i}$

Remark 0.2. Locally-G maps satisfy the following unique extension property. If $U \subset X$ and $f : U \mapsto X$ is a locally-G map then there is a unique $g \in G$ restricting to f.

Definition 0.3. A (G, X) atlas is a pair $(\mathcal{U}, \Phi)$ consisting of an open covering for M and a collection of charts such that the transition maps are locally-G on their respective overlaps.

Definition 0.4. A map $f : M \mapsto N$ between two (G, X)-manifolds is a (G,X)-map if for each pair of charts (ϕ, φ) then $\phi \circ f \circ \varphi^{-1}$ is locally-G on the relevant intersection.

For (G,X)-manifolds, there is an inherently nice way to "globalize" the charts via a (G,X)-map called the developing map. The study of this map will be of importance to us throughout this thesis.

Proposition 0.5. Let M be a simply connected (G, X)-manifold. Then there exists a (G, X)-map

$$dev : M \mapsto X$$

Proof. We pick a base point $x_0 \in M$ and a chart (U_0, φ_0) around x_0. Now for any $x \in M$ we define the map dev as follows. We pick a curve on M $x(t)$ satisfying $x(0) = x_0$ and $x(1) = x$. As the image of our curve is compact, we can cover the it by finitely many coordinate charts U_i with $i \in \{0, \dots, n\}$ such that $x(t) \in U_i$ for $t \in (a_i, b_i)$ with

$$a_0 < 0 < a_1 < b_0 < a_2 < b_1 < a_3 \dots < a_n < b_{n-1} < 1 < b_n$$

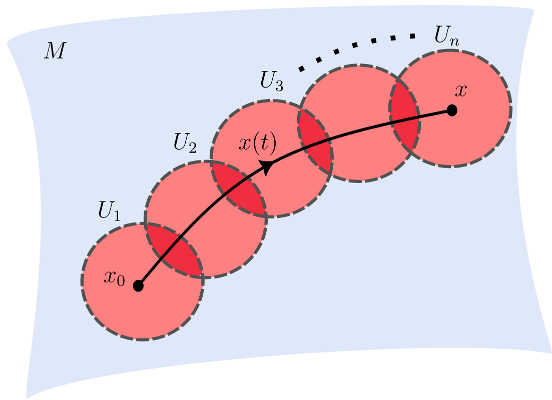

Figure 1: Developing along a curve

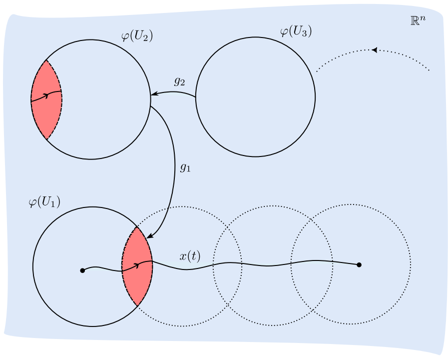

Figure 2: We successively glue our charts together using the transition maps

Now we consider where these charts overlap. Since our manifold is equipped with a (G, X)-structure our transition maps differ by an element of G. Let $g_i = \varphi_{i-1} \circ \varphi_i^{-1} \in G$ be the transition map from U_i and U_{i-1}. We then define

$$dev(x) = g_1 g_2 \dots g_n \varphi_n(x)$$

We must show that this is a well-defined map. We first show that this map does not change if we refine the cover. Suppose we insert a chart (V, ϕ) between U_i and U_{i-1}. Let

$$\varphi_{i-1} = h_{i-1} \circ \phi \text{ on } V \cap U_{i-1} \text{ and}$$
$$\phi = h_i \varphi_i \text{ on } V \cap U_{i-1}.$$

Then on $V \cap U_i \cap U_{i-1}$ we have that $\varphi_{i-1} = h_{i-1} \circ h_i \circ \varphi_i$. This gives us that $g_i = \varphi_{i-1} \circ \varphi_i^{-1} = h_{i-1} \circ h_i$ as we require that our transition maps satisfy a unique extension property.

Now when we develop along our curve with respect to this new covering, we obtain that:

$$dev(x) = g_1 \dots g_{i-1} h_{i-1} h_i g_{i+1} \dots g_n \varphi_n(x) = g_1 \dots g_{i-1} g_i g_{i+1} \dots g_n \varphi_n(x)$$

This defines dev along $x(t)$. We now need to show that it does not depend on our choice of curve. As M is simply connected, all curves with the same start point and end point will be homotopic. Indeed, by compactness we can split our homotopy into smaller homotopies such that we can break up our curve into regions

$$a_1 = 0 < a_2 < \cdots < a_n = 1$$

such that during each small homotopy the segment $x((a_i, a_{i+1}))$ lies entirely inside a coordinate patch. It then follows that dev is independent of our choice of paths completing the proof.

□

This shows that we have a well defined development map on any simply connected (G, X)-manifold. However, this map depended on the choice of base point and initial chart. We immediately get the following corrolary from this fact.

Corollary 0.6. Let M be a (G, X)-manifold and let $\tilde{M}$ be its universla cover. Then there exists a pair (dev, hol) satisfying

$$dev : \tilde{M} \mapsto X$$
$$hol : \pi_1(M) \mapsto G$$

where $hol(\gamma)$ is the element $g_1 \ldots g_n$ of G obtained from developing around a loop $\gamma \in M$.

The developing pair satisfy the following equivariance condition.

Lemma 0.7. Let $\gamma, \alpha \in \pi_1(M)$ be loops in M. Then

$$dev(\gamma \star \alpha) = hol(\gamma)dev(\alpha)$$

Proof. Let $\{(U_i, \varphi_i)\}$ be a cover for α and $\{(V_i, \phi_i)\}$ be cover for γ. Denote the transition maps on $U_i \cap U_{i+1}$ by h_i and the transition maps on $V_i \cap V_{i+1}$ by g_i. Then by definition

$$\begin{aligned} dev(\gamma \star \alpha) &= g_1 \ldots g_n h_1 \ldots h_k \varphi_k(\alpha(1)) \\ &= hol(\gamma)dev(\alpha) \end{aligned}$$

□

The role of the development pair is to "globalize" the coordinate chart of M. Indeed the following proposition makes this statment more clear.

Proposition 0.8. Let M be a (G, X)-manifold with development pair (dev, hol). Denote by Γ, the image of $\pi_1(M)$ under the holonomy map. Then there exists a (G, X)-atlas such that the transition maps lie in Γ

Proof. We pick $m \in M$ and fix a lift $\tilde{m} \in \tilde{M}$ of m. Then, as dev is a local diffeomorphism, there is an open set $\tilde{U}$ of $\tilde{m}$ such that dev is a local diffeomorphism onto X.

We pick $V \subset M$ an open neighborhood of m and take its inverse under the covering map $p^{-1}(V)$ which we may assume dev is a diffeomorphism on by intersetcting with $\tilde{U}$.

Now, define coordinate charts on M by $(V_\alpha, \psi_\alpha = dev \circ p_\alpha^{-1})$ at a point $m \in M$ where by p_α^{-1} we mean we have fixed a component of the preimage p^{-1} restricted to V_α. Now suppose that $V_i \cap V_j \neq \emptyset$ and let $\tilde{U}_i = p_i^{-1}(V_i)$ and $\tilde{U}_j = p_j^{-1}(V_j)$. Then the preimage of their intersection will differ by a deck transformation $\gamma \in \pi_1(M)$. Therefore, on the intersection $V_i \cap V_j$

$$\begin{aligned} dev \circ p_i^{-1} = dev \circ \gamma \circ p_j^{-1} \\ \Rightarrow p_i^{-1} \circ p_j = \gamma \end{aligned}$$

Now our by definition of our transition maps

$$\begin{aligned} \psi_i \circ \psi_j^{-1} = dev \circ p_i{-}1 \circ p_j \circ dev^{-1} \\ = dev \circ \gamma \circ dev^{-1} = hol(\gamma) dev \circ dev^{-1} = hol(\gamma) \in \Gamma \end{aligned}$$

and so we have thus given our coordinate charts on M in terms of the development pair. ☐

This shows that once we have fixed a development pair, our (G, X)-structure is entirely determined by this. Conversely, the following lemma shows us that Once we have fixed our (G, X)-structure, the development pair is determined up to certain factors/conjugations.

Lemma 0.9. Let M be a (G, X)-manifold with development pair (dev, hol). Suppose that (dev', hol') is another development pair for M. Then $dev' = g \circ dev$ and $hol' = g \circ hol \circ g^{-1}$ for some $g \in G$.

Proof. Our development pair was determined once we picked a base point and a chart (U, ϕ). Suppose that we pick a different base point and chart (V, φ). Now we consider how we define $dev(x)$ for $x \in \tilde{M}$. We choose a curve γ connecting x_0 and x and choose a cover U_i of γ and develop around it. Now, with respect to our new initial chart, $V \cup \{U_i\}$ is an open cover of γ. If we develop starting with this new chart, we can first glue all the U_i to U_1 as before and then glue this chain to V with an element of G by gluing U_1 to V on their overlap. Hence the entire development will have o=just changed by that element of G.

If, on the other hand, both the base point and chart have changed, we can pick a curve γ' starting at our new base point x_0' and ending at our old basepoint x_0. Developing first along this curve and then along our old curve γ we pick up some element of G from developing γ' whilst developing along γ gives the same as before. As we already showed this map is homotopy invariant this gives that $dev'(x) = g \circ dev(x)$ for some $g \in G$.

Now to show that $hol' = g \circ hol \circ g^{-1}$ we consider

$$\begin{aligned} dev'(\gamma \star \alpha) &= g \circ dev(\gamma \star \alpha) \\ &= g \circ hol(\gamma) \circ dev(\alpha) \\ &= (g \circ hol(\gamma) \circ g^{-1})(g \circ dev(\alpha)) \\ &= (g \circ hol(\gamma) \circ g^{-1}) dev'(\alpha) \end{aligned}$$

and as the development pair satisfied the equivariance property it follows that $hol' = g \circ hol \circ g^{-1}$. □

This shows that once we have fixed a development pair, any other development pair can is related by $(dev', hol') = (g \circ dev, g \circ hol \circ g^{-1})$. It is therefore, important to note that if we want to classify (G, X)-structures on a manifold M by classifying the possible development pairs, it is important to consider them up to this relation.

0.1.1 Completeness

An important property the developing map can have is that of **completeness**.

Definition 0.10. A (G, X)-manifold M is said to be complete if its developing map is a diffeomorphism onto X.

In the case that X is simply connected e.g. $X = \mathbb{R}^n$, we have that M is diffeomorphic to X/Γ where $\Gamma = hol(\pi_1(M))$.

In the case that M is a Riemannian manifold, this notion of completeness is equivalent to completeness of the Levi-Civita connection of M.

TODO : Examples of complete/ incomplete manifolds e.g. Hopf

0.2 Representation Theory and Affine Manifolds

We now turn our attention to affine manifolds before specializing to the integral affine case. An affine manifold is a (G, X)-manifold where $X = \mathbb{R}^n$ and $G = Aff(\mathbb{R}^n)$ An important tool in classifying affine structures is the representation theory of groups. Often, by studying properties of the holonomy group of an affine manifold M, we are able to obtain information about the affine structure of M. Of fundamental importance is the case where the holonomy group is nilpotent which we will shortly see. We will use these tools to determine conditions for which the affine structure is complete.

Definition 0.11. An affine transformation of $\mathbb{R}^n$ is a map

$$A : \mathbb{R}^n \mapsto \mathbb{R}^n$$

such that $A(x) = Gx + B$ where $G \in GL(\mathbb{R}^n)$ and $B \in \mathbb{R}^n$

The group $Aff(\mathbb{R}^n)$ is the group of affine transformations of $\mathbb{R}^n$ and is equal to $Aff(\mathbb{R}^n) = \mathbb{R}^n \rtimes GL(\mathbb{R}^n)$

Definition 0.12. An affine representation of a group G is a group homomorphism

$$\alpha : G \mapsto Aff(\mathbb{R}^n)$$

We have that an affine representation splits into a linear part and a translation, e.g. $\alpha(g) = \lambda(g) + \mu(g)$. The linear part makes $\mathbb{R}^n$ into a G-module with the obvious action which we will denote E.

We will be interested in some results from group cohomology. This leads us to the following definitions.

Definition 0.13 (Crossed Homomorphism)**.** A crossed homomorphism for λ, is a group homomorphism

$$\mu : G \mapsto E$$

that satisfies $\mu(gh) = \mu(g) + \lambda(g)\mu(h)$

Proposition 0.14. Let $\alpha(g) = \lambda(g) + \mu(g)$ be an affine representation on E of a group G. Then μ is a crossed homomorphism for λ

Proof. Let $g, h \in G$ and $\alpha = \mu + \lambda$ be an affine representation. Then

$$\begin{aligned} \alpha(gh) &= (\mu(gh), \lambda(gh)) \\ &= \alpha(g) \cdot \alpha(h) = (\mu(g), \lambda(g)) \cdot (\mu(h), \lambda(h)) \\ &= (\mu(g) + \lambda(g)\mu(h), \lambda(g)\lambda(h)) \\ &\Rightarrow \mu(gh) = \mu(g) + \lambda(g)\mu(h) \end{aligned}$$

and so μ is a crossed homomorphism for λ. □

Definition 0.15 (Principle Crossed Homomorphism)**.** A *principle crossed homomorphism* for λ is a crossed homomorphism of the form $\mu(g) = y - \lambda(g)y$ for some $y \in E$.

Proposition 0.16. Let $y \in E$. Then y is a *stationary point* of the action of α if and only if $\mu(g) = y - \lambda(g)y$ for all $g \in G$.

Proof. Suppose $y \in E$ is a ary point of α. Then $\alpha(g)y = y$ for every $g \in G$. This gives that

$$\begin{aligned} (\lambda(g) + \mu(g))y = \lambda(g)y + \mu(g) = y \\ \Rightarrow \mu(g) = y - \lambda(g)y \end{aligned}$$

Conversely, suppose that $\mu(g) = y - \lambda(g)y$. Then $y = \mu(g) + \lambda(g)y = \alpha(g)y$ □

Remark 0.17. If μ is a principle crossed homomorphism we will often write $\mu = D_y$

Remark 0.18. An affine representation with a fixed point is called *radiant*.

TODO: Develop some of the basics of group cohomology

Lemma 0.19. The cohomology group $H^1(G, E)$ is isomorphic to

$$H^1(G, E) \cong \frac{\{\text{crossed homomorphisms for } \lambda\}}{\{\text{principle crossed homomorphisms for } \lambda\}}$$

Proof. **TODO : Add proof e.g. construct resolution or take this as given?** □

Definition 0.20 (Radiance Obstruction)**.** Let α be an affine representation. The radiance obstruction of α is the cohomology class

$$c_\alpha = [\mu] \in H^1(G, E)$$

where μ is the translational part of α.

Lemma 0.21. The radiance obstruction vanishes, e.g. $c_\alpha = 0$ if and only if α is conjugate to its linear part by a translation, if and only if α has a stationary point.

Proof. By Lemma 0.2, $y \in E$ is stationary if and only if $\mu(g) = D_y$ is a principle crossed homomorphism for λ if and only if $c_\alpha = [\mu] = 0$.

Now suppose that $\mu = D_y$ for some $y \in E$. so that $\mu(g) = y - \lambda(g)y$. We define the translation

$$\begin{gathered} T_y : E \mapsto E \\ x \mapsto x - y \end{gathered}$$

Now, conjugating α by this translation yields

$$\begin{gathered} T_y \circ \alpha(g) \circ T_y^{-1} = T_y \circ \alpha(g)(x + y) \\ = T_y \circ (\lambda(g)x + \lambda(g)y + \mu(g)) \\ = T_y \circ (\lambda(g)x + \lambda(g)y + y - \lambda(g)y) \\ = T_y \circ (\lambda(g)x + y) = \lambda(g)x + y - y = \lambda(g)x \end{gathered}$$

Conversely, suppose that $T_y \circ \alpha(g) \circ T_y^{-1} = \lambda(g)$ for all $g \in G$ and $x \in E$. Then

$$\begin{gathered} T_y \circ \alpha(g) \circ T_y(x) = T_y \circ \alpha(g)(x + y) \\ = T_y \circ (\lambda(g)(x + y) + \mu(g)) = \lambda(g)(x + y) + \mu(g) - y = \lambda(g)x \\ \Rightarrow \mu(g) = \lambda(g)(x) - \lambda(g)(x) - \lambda(g)(y) + y \\ = y - \lambda(g)y \end{gathered}$$

as required. □

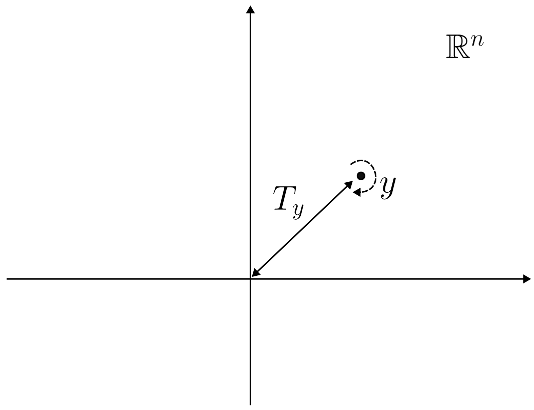

Figure 3: Conjugating a fixed point y under the affine action by a translation to the origin

Remark 0.22. This lemma will allow us to alter the development map by a translation (which alters the holonomy representation by a conjugation) to assume that our fixed point is at the origin.

Definition 0.23 (Contragradient Representation)**.** Let E be a G-module and α a representation on E. We define the contragradient representation on E^* by

$$\begin{aligned} \alpha^* : G &\mapsto GL(E^*) \\ g &\mapsto \alpha^* g \end{aligned}$$

such that $\alpha^* g(f)(x) = f(\alpha(g^{-1})x)$ for all $x \in E$ and $f \in E^*$.

Remark 0.24. This induces a representation on $\wedge^k E^*$ by
$(g \cdot \omega)(x_1, \ldots, x_n) = \omega(g^{-1} \cdot x_1, \ldots, g^{-1} \cdot x_n)$ for $\omega \in \wedge^k E^*$ and $x_i \in E$.

We will make use of the following lemma when looking at the relationship between the holonomy representation and parallel volume forms.

Lemma 0.25. Let $F : V \mapsto V$ be a linear map from a vector space V of dimension n to itself and let $\omega \in \wedge^n(V^*)$. Then

$$F^*\omega = det(A)\omega$$

where A is any matrix representing the linear transformation F.

Proof. As $dim(\wedge^n(V^*)) = 1$ and F is a linear map, it follows that our map corresponds to multiplication by some scalar $\lambda \in \mathbb{R}$. Let $\varphi : V \mapsto \mathbb{R}^n$ be an isomorphism identifying our vector space V with $\mathbb{R}$. Then we have

$$\begin{aligned} F^*\varphi^* \det &= \lambda\varphi^* \det \\ \Rightarrow (\varphi^{-1})^* F^* \varphi^* \det &= \lambda \det \\ \Rightarrow (\varphi^{-1} F \varphi)^* \det &= \lambda \det \end{aligned}$$

If we let $A = \varphi^{-1} F \varphi$ and $\{e_1, \ldots, e_n\}$ the standard basis of $\mathbb{R}^n$ then

$$A^* det(e_1 \ldots, e_n) = \lambda det(e_1, \ldots, e_n) = \lambda = det(Ae_1, \ldots, Ae_n)$$

which gives us that $det(A) = \lambda$ as required. □

This leads us to the following elementary but important property of the action of the holonomy representation on volume forms.

Corollary 0.26. Let M be an affine manifold and $\lambda \in GL(\mathbb{R}^n)$ be the linear part of the holonomy representation. Then M has a parallel volume form with respect to the affine connection if and only if $\det(\lambda(g)) = 1$ for all $g \in G$.

Proof. Suppose ω is a parallel volume form on M. Then the parallel transport of ω around a loop in M leaves ω unchanged, e.g. the linear holonomy action preserves ω. This gives us that

$$\begin{aligned} g \cdot \omega = \omega \\ \Rightarrow \omega(g^{-1} \cdot x_1, \ldots, g^{-1} \cdot x_n) = \omega(x_1, \ldots, x_n) \end{aligned}$$

for all $x_i \in M$. But by Lemma 0.2

$$\omega(g^{-1} \cdot x_1, \ldots, g^{-1} \cdot x_n) = \det(\lambda(g^{-1})) \cdot \omega(x_1, \ldots, x_n)$$

and so the volume is preserved if $\det(\lambda(g)) = 1$

Conversely, suppose that the determinant of the linear holonomy is 1 for all $g \in G$. If we pick a point $m_0 \in M$ then locally there exists a volume form ω_0 on a chart U of M with respect to the affine connection, corresponding to a constant, top degree form. We now claim that we can uniquely extend such a form. Indeed, as parallel transport provides a linear isomorphism between tangent spaces we can attempt to define a volume form on M by parallel transporting ω_0.

Pick another point $m \in M$. We claim that the parallel transport of ω_0 along a curve γ from m_0 to m does not depend on the choice of curve. If γ' is another curve then the parallel transport $(P^{\gamma'}_{m_0 \to m})^{-1} \circ P^{\gamma}_{m_0 \to m} = g$ for some g in the linear holonomy group. It follows that

$$\begin{aligned} g^* \circ \omega_0 &= ((P^{\gamma'}_{m_0 \to m})^{-1} \circ P^{\gamma}_{m_0 \to m})^* \circ \omega_0 \\ &= \det(\lambda(g)) \cdot \omega_0 = \omega_0 \\ \Rightarrow (P^{\gamma}_{m_0 \to m})^* \circ \omega_0 &= (P^{\gamma'}_{m_0 \to m})^* \circ \omega_0 \end{aligned}$$

and so the parallel transport does from m to m_0 does not depend on the choice of path. We note that this defines a continuous maps as in each affine chart the parallel transport identifies the tangent spaces via the identity map and on over laps they differ by an element of $GL(\mathbb{R}^n)$ as the derivative of an affine map extracts the linear part. □

Definition 0.27 (Radiant Manifold)**.** An affine manifold is said to be radiant if its holonomy representation has a fixed point.

Remark 0.28. By Lemma 0.21 we may conjugate our developing map by a translation to assume that the fixed point is at the origin.

When M is radiant the above assumption shows that the radiant vector field $R = \sum_i x_i \frac{\partial}{\partial x_i}$ is fixed by the affine action. There is then a unique vector field $\tilde{X}$ on the universal cover $\tilde{M}$ that is related to R by the developing map. Since this is preserved by deck transformations (by equivariance of developing map and since the representation is radiant) it descends to a vector field X on M.

Definition 0.29. The vector field X on M is called the radiant vector field of M.

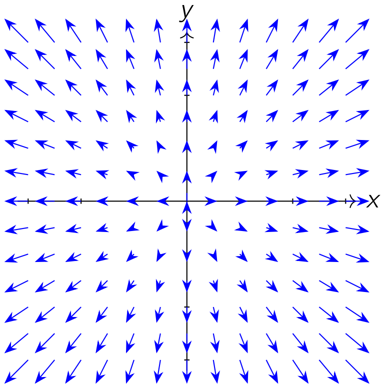

Figure 4: Radiant Vector Field

0.2.1 Nilpotent Holonomy

In this section, we will see that the condition that the holonomy group of an affine manifold M be nilpotent will be useful in determining whether the affine structure is complete. We will build up some useful results before determining some equivalent conditions that force completeness of the affine structure.

Definition 0.30 (Central Series)**.** Let G be a group. Then a central series for G is a finite collection of normal subgroups such that

$$1 = G_0 \leq G_1 \leq \cdots \leq G_s = G$$

and

$$\frac{G_{i+1}}{G_i} \leq Z(\frac{G}{G_i})$$

where $Z(\frac{G}{G_i})$ denotes the center.

Definition 0.31 (Nilpotent Group)**.** A group G is said to be nilpotent if it admits a central series.

Remark 0.32. The *nilpotent class* of G is the length of the shortest central series

For nilpotent groups, we have the following results regarding their group cohomology.

Lemma 0.33. Let G be a nilpotent group and E a G-module. If $H^0(G, E) = 0$ then also $H^i(G, E) = 0$ for all $i \geq 0$.

Proof. TODO : Add proof? - "HIRSCH, M., Flat manifolds and the cohomology of groups" □

from this we obtain the following lemma

Lemma 0.34. If G is a nilpotent group and E is a G-module, then the following statements are equivalent.

1. $H^0(G, E) = 0$
2. $H^0(G, E^*) = 0$
3. $H_0(G, E) = 0$
4. $H_0(G, E^*) = 0$

Proof. See [FGH81] □

In the case that the affine holonomy group is nilpotent, we will be able to show that completeness is equivalent to a property called unipotence of the linear holonomy. We will now outline some consequences of unipotence.

Definition 0.35 (Unipotent)**.** A G-module E is called unipotent if $(g - I)^n = 0$ for every $g \in G$ where $n = dim(E)$.

Remark 0.36. An affine representation is unipotent if its linear part defines a unipotent G-module.

Although E may not be unipotent itself, we can consider the smallest submodule (possibly the zero module) of E which is unipotent. We shall denote this submodule by E_U.

Remark 0.37. The submodule E_U is called the *Fitting submodule* of E.

Often, it will be useful to quotient out by the unipotent part of a G-module. The following lemma gives some insight on why this is often a useful idea.

Lemma 0.38. Let E_U be the Fitting submodule of E. Then

$$H^0(G, E/E_U) = 0$$

Proof. Take E_U to be the Fitting submodule and assume $E_U \neq E$ otherwise the result is trivial. As $H^0(G, E/E_U)$ is the set of ary points of the G-module E/E_U, asuuming $H^0(G, E/E_U) \neq 0$ we can take $x + E_U \in E/E_U$ not the identity so that

$$\begin{gathered} g \cdot (x + E_U) = x + E_U \text{ for all } g \in G \\ \Rightarrow (g - 1) \cdot x \in E_U \text{ for all } x \in E \end{gathered}$$

Now we consider the G-submodule spanned by x and E_U. As $x \notin E_U$ this submodule is strictly bigger than E_U. Now we have that

$$(g - 1)^n \cdot (x + e) = (g - 1)^{n-1}((g - 1)x + (g - 1)e)$$

But $(g-1)x \in E_u$ so that $(g-1)x + (g-1)e \in E_U$. But as $E_U \neq E$ it has dimension at most $n-1$ and so $(g-1)^{n-1} = 0$ restricted to E_U. Therefore this is a unipotent submodule strictly bigger than E_U which contradicts maximality. ☐

The following splitting lemma will prove important in our characterization of completeness.

Lemma 0.39. If G is a nilpotent group and E a G-module with Fitting submodule E_U, then there exists a unique submodule F such that $E = E_U \oplus F$.

Proof. Let β be the induced representation on $GL(E/E_U)$. By Lemma 0.2.1, $H^0(G, E/E_U) = 0$.

Claim. A submodule $F \subset E$ is complementary to E_U e.g. $E = E_U \oplus F$ if and only if $F = T(E/E_U)$ where $T : E/E_U \mapsto E$ is an equivariant linear map with $P \circ T = 1_{E/E_U}$ with P the canonical projection map.

Now let $S : E/E_U \mapsto E$ be a linear map satisfying $P \circ S = 1_{E/E_U}$. Then any such T as above can be uniquely written as $T = R + S$ with $R : E/E_U \mapsto E$. It is T-equivariant if and only if

$$R + S = g \circ (R + S) \circ \beta(g)^{-1}$$

$$\Rightarrow R = g \circ R \circ \beta(g)^{-1} + g \circ S \circ \beta(g)^{-1} - S \tag{1}$$

If we can show that there is a unique such T, then we have shown that E splits uniquely into $E = E_U \oplus F$. This will follow if we can show there is a unique R satisfying the above.

To do this we define a representation on $Hom(E/E_U, E)$ by

$$\gamma : G \mapsto Hom(E/E_U, E)$$
$$\gamma(g)(R) = g \circ R \circ \beta(g)^{-1}$$

and we define a crossed homomorphism for γ by $\mu(g) = g \circ S \circ \beta(g)^{-1} - S$. This defines a crossed homomorphism as

$$\begin{aligned}\mu(gh) &= g \circ h \circ S \circ \beta(h)^{-1} \circ \beta(g)^{-1} - S \\ &= \gamma(g) \circ h \circ S \circ \beta(h)^{-1} - S \\ &= \gamma(g) \circ (\mu(h) + S) - S \\ &= \gamma(g) \circ \mu(h) + g \circ S \circ \beta(g)^{-1} - S \\ &= \gamma(g)\mu(h) + \mu(g)\end{aligned}$$

As P is equivariant and $P \circ S = 1_{E/E_U}$ we get

$$\begin{aligned}P \circ \mu(g) &= P \circ g \circ S \circ \beta(g)^{-1} - P \circ S, \text{ so by equivariance} \\ &= \beta(g) \circ P \circ S \circ \beta(g)^{-1} - 1 \\ &\beta(g) \circ \beta(g)^{-1} - 1 = 0\end{aligned}$$

Now we can we write (1) as

$$R = \gamma(g) \circ R + \mu(g)$$

which by Proposition 0.16 is saying that R is a fixed point of the affine representation induced by γ and μ. We wish now to show that this affine representation has a unique fixed point. This is equivalent by Lemma 0.33 and Lemma 0.34 to $H^0(G, Hom(E/E_U)) = 0$.

To this end we suppose that $R : E/E_U \mapsto E$ is fixed under all $\gamma(g)$ for $g \in G$. Then, letting $d = dim(E_U)$ for any possibly the same collection of d elements $g_1, \ldots, g_d \in G$ we have

$$(I - g_1) \circ \cdots \circ (I - g_d)|E_U = 0$$

as they act unipotently on E_U. It now follows by equivariance that

$$\begin{aligned}
&(I - g_1) \circ \cdots \circ (I - g_d) \circ R \\
&= (I - g_1) \circ \cdots \circ (R - g_d \circ R) \\
&= (I - g_1) \circ \cdots \circ (R - R \circ \beta(g_d)) \\
&= (I - g_1) \circ \cdots \circ R \circ (1 - \beta(g_d)) \\
&\Rightarrow R \circ (1 - \beta(g_1)) \circ \cdots \circ (1 - \beta(g_d)) = 0
\end{aligned}$$

so that R vanishes on vectors of the form

$$(1 - \beta(g_1)) \circ \cdots \circ (1 - \beta(g_d))x$$

for $x \in E/E_U$. Thus if all vectors are in this span, then the only such R fixed under the action of γ is the zero map. As $(1 - \beta(g))$ is invertible on E/E_U (it acts without unipotent part), this is equivalent to showing that E/E_U is spanned by vectors of the form $(1 - \beta(g))$. By definition, this is equivalent to $H_0(G, E/E_U) = 0$ which is true by 0.2.1 completing the proof. □

This proves our result once we have cleared up that the stated claim is indeed true which we show in the following proposition.

Proposition 0.40. A submodule $F \subset E$ is complementary to E_U e.g. $E = E_U \oplus F$ if and only if $F = T(E/E_U)$ where $T : E/E_U \mapsto E$ is an equivariant linear map with $P \circ T = 1_{E/E_U}$ with P the canonical projection map.

Proof. Suppose $F = T(E/E_U)$, with T being G-equivariant so that F is a G-module. If $T(x + E_U) = e \in E$, then $P(e) = e + E_U = x + E_U \Rightarrow x \in E_U$ and so by linearity $e = 0$ and so $F \cap E_U = \{0\}$. Clearly, $E_U \oplus F \subseteq E$. For any $x \in E$, $x - T(x + E_U) \in E_U$. We can thus write $x = (x - T(x + E_U)) + T(x + E_U) \in E_U + F$.

Conversely, suppose that F is a G-submodule and $F \cap E_U = \{0\}$ and $F \oplus E_U = E$. Then its easy to see that $P|_F : F \mapsto E/E_U$ is bijective. It is G-equivariant as P is and F is a G-submodule. We can thus define T as $T = P^{-1}|_F$. □

An important consequence of this theorem is that for indecomposable affine representations of a nilpotent group the affine representation is always unipotent (**TODO: should we include this**).

0.2.2 Unipotent Representations and Completeness

The aim of this section is to show that for a compact affine manifold, unipotent holonomy group is equivalent to completeness. Of geometrical importance to this will be the concept of an *expansion* in the linear holonomy.

Definition 0.41 (Expansion)**.** Let $T : V \mapsto V$ be a linear map. Then T is said to be an expansion of V if for every eigenvalue λ of T we have $|\lambda| > 1$.

We now wish to prove the following technical lemma which holds in the absence of an expansion.

Lemma 0.42. Let G be a nilpotent group and E a G-module. Suppose that the linear holonomy group does not contain an expansion of E. Then for each $n \in \mathbb{N}_{>0}$, there is a C^r map $\varphi : E \mapsto \mathbb{R}$ which satisfies the following.

1. $\varphi > 0$ almost everywhere.
2. φ is G-invariant.
3. There exists some $a > 0$ such that
$$\varphi(e^{tx}) = e^{ta}\varphi(x)$$
for all $t \in \mathbb{R}$ and $x \in E$.

Before we can do this we will recall some results from algebraic group theory and the representation theory of Lie Algebras

Definition 0.43 (Algebraic Group)**.** An *algebraic group* is a matrix group that can be defined using polynomials.

If G is an algebraic group, then there is a unique irreducible component passing through the identity denoted by G°.

Remark 0.44. This unique irreducible component is called the *identity component* of G.

Proposition 0.45. Let G be an algebraic group. Then G° is a normal subgroup of finite index in G whose cosets are connected and irreducible components of G.

Proof. See "Linear Algebraic Groups - Humphreys (7.3)" □

TODO: Flesh these algebraic group proofs out more.

Proposition 0.46. Let G be an algebraic group and H a normal subgroup. Then

$$\overline{[G, H]} = [\overline{G}, \overline{H}]$$

where $\overline{H}$ denotes the Zariski closure.

Proof. Proposition (2.4?) Borel-Linear-Algebraic-Groups https://www.math.utah.edu/~ptrapa/math-library/borel/%20Borel-Linear-Algebraic-Groups-1991.pdf ☐

This proposition leads to the following corrolary.

Corollary 0.47. Let G be an algebraic group and N a normal subgroup that is nilpotent. Then $\overline{N}$ is also nilpotent.

Proof. By Proposition 0.46, a central series for N will pass to a central series of $\overline{N}$. ☐

The next result we will state is an important result from representation theory of Lie Algebras. It will guarantee us a basis for our representation such that the matrices of the representation are upper triangular.

Definition 0.48 (Weight Space)**.** Let $\mathfrak{g}$ be a Lie algebra and $\pi : \mathfrak{g} \mapsto \mathfrak{gl}_V$ be a representation on a vector space V. For $\lambda \in \mathfrak{g}^*$ we define the weight space of $\mathfrak{g}$ with respect to λ to be

$$V_\lambda^{\mathfrak{g}} = \{v \in V | \pi(g)v = \lambda(g)v, \quad \forall g \in \mathfrak{g}\}$$

Remark 0.49. If $V_\lambda^{\mathfrak{g}} \neq 0$ we will say that λ is a weight for π.

Lemma 0.50 (Lie's theorem)**.** Let $\mathfrak{g}$ be a solvable Lie algebra and let π be a representation of $\mathfrak{g}$ on a finite dimensional, non-zero vector space V over an algebraically closed field of characteristic zero. Then there exists a weight $\lambda \in \mathfrak{g}^*$ for π.

Proof. See the following https://math.mit.edu/classes/18.745/Notes/Lecture_5_Notes.pdf ☐

Corollary 0.51. Let V be a vector space over an algebraically closed field of characteristic 0. If $\pi : \mathfrak{g} \mapsto \mathfrak{gl}(V)$ is a finite dimensional representation of a solvable Lie algebra, then there is a basis of V such that all linear transformations in $\pi(\mathfrak{g})$ are represented by upper triangular matrices.

We now have the required machinary to start our proof.

Lemma 0.52. Let G be a nilpotent group and E a G-module. Suppose that the linear holonomy group does not contain an expansion of E. Then for each $n \in \mathbb{N}_{>0}$, there is a C^r map $\varphi : E \mapsto \mathbb{R}$ which satisfies the following.

1. $\varphi > 0$ almost everywhere.
2. φ is G-invariant.
3. There exists some $a > 0$ such that
$$\varphi(e^{tx}) = e^{ta}\varphi(x)$$
for all $t \in \mathbb{R}$ and $x \in E$.

Proof. If we can prove this for a normal subgroup G_0 of finite index, then we will be able to construct our map φ as follows. Let $g_1G_0, \ldots g_kG_0$ be the left cosets of G_0 and φ_0 a map satisfying properties 1 and 3 defined on G_0 . Now define
$$\varphi(x) = \sum_{i=1}^{k} \varphi_0(g_i x)$$

This will now be G-invariant as G_0 is a normal subgroup multiplication by an element of G will permute the cosets.

As we want to eventually apply Lies theorem, we complexify the dual space E^* as $F = \mathbb{C} \oplus E^*$. The contragradient representation of G on E^* extends to a representation on F in the obvious way. We will denote this as $\rho : G \mapsto GL(F)$. Now, $\rho(G)$ is a nilpotent subgroup of $GL(F)$ that passes through the identity.

Let H be the identity component of the algebraic closure of $\rho(G)$. As $\rho(G)$ is nilpotent, by Corollary 0.47 so is its Zariski closure and hence its identity component as subgroups of nilpotent groups are nilpotent. H will be a normal subgroup of finite index by Proposition 0.45 and so $G_0 = \rho^{-1}(H)$ will also be a normal subgroup of finite index.

We can now obtain a Lie algebra representation of the Lie algebra of H on F by
$$di : \mathfrak{h} \mapsto \mathfrak{gl}(F)$$
where i is the inclusion map. Importantly, by construction on the underlying vector spaces $\rho(G_0) \subseteq di(\mathfrak{h})$. As H is a nilpotent Lie group, its Lie algebra is nilpotent hence solvable so we can apply Lie's Theorem to obtain a decomposition $F = \oplus_{k=1}^{m} F_k$ which is $di(h)$-invariant and hence ρ-invariant by construction when restricted to G_0.

Again, by Lie's theorem there is a basis B_k of each F_k that represents the operators $di(h)|_{F_k}$ as upper triangular complex matrices
$$di_k(h) = \lambda_k(h)I + N_k(h)$$
where di_k is the restriction to F_k and $N_k(h)$ is strictly upper triangular nilpotent matrix and $\lambda_k(h)$ are the eigenvalues of $\rho(h)$. The first basis vector

in each B_k is killed by the strictly upper triangular matrix so that if $f_k \in F_k$ is the first basis vector in B_k

$$di_k(h)f_k = \lambda_k(h)f_k$$

for all $h \in H$.

Now we define the following group morphisms

$$\varphi_k : G_0 \mapsto \mathbb{R}$$
$$\varphi_k(g) = log|\lambda_k(g)|$$

for $k = 1, \ldots, m$.

This defines a map into $\mathbb{R}^m$ by

$$\varphi = (\varphi_1, \ldots, \varphi_m) : G_0 \mapsto \mathbb{R}^m$$

If G_0 does not contain any expansions of E then the image $\varphi(G_0)$ is disjoint from the set

$$P = \{y \in \mathbb{R}^m : y_k > 0, k = 1, \ldots, m\}.$$

As it is a subgroup, it follows that its linear span $span\{\varphi(G_0)\}$ is also disjoint from P as if a point in the image was in the strictly negative orthant, it's inverse would lie in P. Therefore there exists a vector in the orthogonal complement of $\varphi(G_0)$ that lies in the closure of P. We denote this by $v = (v_1, \ldots, v_m)$ and it satisfies

$$v_k \geq 0 \text{ for all } k,$$

$$\sum_{k=1}^{m} v_k = a \geq 0$$

$$\sum_{k=1}^{m} v_k \varphi_k(g) = 0 \text{ for all } g \in G_0$$

If we scale our v by a positive scalar $c > 0$ it will satisfy the same properties so that we can arrange that

$$v_k = 0 \text{ or } v_k > 2r \text{ for each k with } r \text{ coming from the lemma.}$$

Now for each k we have a vector $f_k \in Hom(\mathbb{C} \otimes E, \mathbb{C})$ satisfy $di_k(h)f_k = \lambda_k(h)f_k$. We now include E into $\mathbb{C} \otimes E$ and construct the map

$$\psi : E \mapsto \mathbb{R}$$

$$\psi(x) = \prod_{k=1}^{m} |f_k(x)|^{v_k}$$

Now ψ is C^r differentiable as $|f_k(x)|^{v_k} = (u_k(x)^2 + w_k(x)^2)^m$ with $m > r$. Properties 1) and 3) are satisfied by our construction of v_k. We now show that it is G_0-invariant.

$$\psi(g^{-1}x) = \prod_{k=1}^{m} |f_k(g^{-1}x)|^{v_k}$$

noting this is the contragradient representation and also that f_k is an eigenvector

$$\psi(g^{-1}x) = \prod_{k=1}^{m} |g \circ f_k(x)|^{v_k} = \prod_{k=1}^{m} |\lambda(g) f_k(x)|^{v_k}$$
$$= \prod_{k=1}^{m} |\lambda_k(g)|^{v_k} \prod_{k=1}^{m} |f_k(x)|^{v_k}$$

Now, recalling that $\varphi_k(g) = log|\lambda_k(g)| \Rightarrow e^{\varphi_k(g)} = |\lambda_k(g)|$ so we obtained

$$\prod_{k=1}^{m} |\lambda_k(g)|^{v_k} \prod_{k=1}^{m} |f_k(x)|^{v_k} = e^{\sum_{k=1}^{m} v_k \varphi_k(g)} \psi(x) = \psi(x)$$

completing the proof.

Next we move onto a more geomterical proof regarding expansions in the linear holonomy which will lead to some important results. $\square$

Theorem 0.53. Let M be a compact affine manifold. Let $E_0 \subset E$ be a proper linear subspace invariant under the affine holonomy and denote by Λ the linear holonomy. If the image of the linear holonomy group $\Lambda_1 \subseteq GL(E/E_0)$ is nilpotent, then some element of Λ_1 expands E/E_0.

Proof. Let $q : E \mapsto E/E_0$ be the canonical projection. Denote by R the radiant vector field on E/E_0. If $\{y_1, \ldots, y_m\}$ are corrdinates on E/E_0, then this appears as $R = \sum_i y_i \frac{\partial}{\partial y_i}$. Then in each affine chart there is a vector field X_i that is π-related to R. As M is compact, we can cover it in finitely many charts and using a partition of unity subordiante to that covering form the vector field

$$X = \sum_{\mu_i} \mu_i X_i$$

on M.

Now suppose that Λ_1 does not contain any expansions of E/E_0. Then by Lemma 0.2.2, there is a Λ_1-invariant C^1 map $\Psi : E/E_0 \mapsto \mathbb{R}$. As the flow along the radiant vector field at a point z is given by $\phi_t(z) = e^t z$, our map satisfies

$$d\Psi_z R(Z) = \frac{d}{dt} \psi(e^t z)|_{t=0} = \frac{d}{dt} e^{ta} \psi(z)|_{t=0} = a\Psi(z)$$

for some $a > 0$ and all $z \in E/E_0$.

Now the composition

$$\tilde{f} : \tilde{M} \mapsto E \mapsto E/E_0 \mapsto \mathbb{R}$$

is invariant under deck transformations. Indeed, let $\gamma \in \pi_1(M)$ then for $x \in \tilde{M}$

$$\Psi \circ \pi \circ dev(\gamma \star x) = \Psi \circ \pi \circ hol(\gamma) \circ dev(x)$$

and this is invariant as E_0 is invariant under the holonomy group.

This implies that $\tilde{f}$ covers a map $f : M \mapsto \mathbb{R}$. As by Proposition 0.1, we can give charts in terms of the development pair, in affine coordiantates we have that $f = \Psi \circ \pi$. Now

$$df_y X(y) = d(\Psi \circ \pi)_y X(y) = \frac{d}{dt} \Psi \circ \pi(\gamma(t))|_{t=0}$$
$$= \frac{d}{dt} \Psi(e^t \tilde{y})|_{t=0} = a\Psi(\tilde{y})$$

where $\tilde{y} = \pi(y)$ and $\gamma(t)$ is an integral curve of X starting at y. This gives that

$$df_y X(y) = af(y)$$

It follows that if $\alpha : I \mapsto M$ is an integral curve of X starting at $p \in M$ then

$$f(\alpha(t)) = \Psi \circ \pi(\alpha(t)) = \Psi(\overline{\alpha(t)}) = \Psi(e^t \tilde{p}) = e^{ta}\Psi(\tilde{p})$$
$$= e^{ta} f(\alpha(c))$$

Now as $\Psi > 0$ almost everywhere, there exists $x_0 \in M$ with $f(x_0) > 0$ and the integral curve α_0 through x_0 is defined for all t as M is compact. Then $\lim_{t\to\infty} f(\alpha_0(t)) = \lim_{t\to\infty} e^{ta} f(\alpha_0(c)) \to \infty$. But as f is a continuous function and M is compact it must be bounded and so we reach a contradiction. □

This proof leads to several important results. The first result follows almost immediately.

Corollary 0.54. Let M be a compact radiant manifold with nilpotent linear holonomy Λ. Then Λ contains an expansion of E.

Proof. We can take $E_0 = 0$ in Theorem 0.2.2 and the result follows immediately. □

The next corollary follows from considering the Fitting submodule of E.

Corollary 0.55. Let M be a compact affine manifold with nilpotent affine holonomy group. If $F \neq 0$ then some element of the linear holonomy group expands F.

Proof. Recall from Lemma 0.2.1, that nilpotence of the affine hoilonomy gives a unique decomposition $E = E_U \oplus F$ which is invariant under the action of the holonomy group. If we take $E_0 = 0$ in Theorem 0.2.2, then $F \cong E/E_U$ as a module of the linear holonomy group. ☐

Perhaps the most important consaequence of this result is the following theorem which is integral in providing a classification of completeness.

Theorem 0.56. Let M be a compact affine manifold with nilpotent affine holonomy group. Suppose that there is a parallel volume form on M. Then the linear holonomy is unipotent.

Proof. Again we will exploit that there is a unique decomposition $E = E_U \oplus F$ by Lemma 0.2.1. Now let g be any element of the linear holonomy group. As there is a parallel volume form by Lemma 0.25, $det(g) = 1$. But by the unique decomposition

$$1 = det(g) = det(g|_{E_U})det(g|_F)$$

But $g|_{E_U}$ is a unipotent operator and so has determinant 1. This forces $det(g|_F) = 1$ so that F cannot contain an expansion. But now Corollary 0.55 forces $F = 0$ and so $E = E_U$. ☐

This will be used in conjunction with the following theorem in order to give equivalent conditions on the completeness of an affine manifold.

Theorem 0.57. Let M be an affine manifold. If M is compact and has unipotent holonomy then M is complete.

Proof. We pick a basepoint x_0 in M. Let $p : \tilde{M} \mapsto M$ be the universal cover of M with $p(\tilde{x_0}) = x_0$. We now choose a developing map $dev : \tilde{M} \mapsto \mathbb{R}^n$ such that $dev(\tilde{x_0}) = 0$. Let hol be the holonomy representation and $hol(\pi_1(M)) = \Gamma$ be the affine holonomy group.

Now, suppose that the linear affine holonomy group is unipotent. Then by **TODO: Humphreys Linear algebraic groups** there exists a linear flag

$$0 = F_0 \subset F_1 \subset \cdots \subset F_n = \mathbb{R}^n$$

preserved by the linear affine holonomy group. Furthermore, the induced action on the quotients F_k/F_{k-1} is trivial and so the restriction to F_k preserves a linear functional $l_k : F_k \mapsto \mathbb{R}$ with kernel F_{k-1}.

This determines a family of parallel fields $\mathfrak{F}_k \subset TM$. In affine coordinates, each of these is spanned by a subset of the coordinate vector fields and so it is clear that they are integrable.Furthermore, each linear map l_i determines a parallel 1-form ω_i defined on $\mathfrak{F}_i$ and vanishing on $\mathfrak{F}_{i-1}$.

Claim: We can construct vector fields X_i on M, tangent to $\mathfrak{F}_i$ such that $X_i(l_i) = 1$ using partitions of unity. These lift to vector fields $\tilde{X}_i$ on the universal cover.

We first show surjectivity of the developing map. Let $\gamma_i(t)$ be an integral curve of $\tilde{X}_i$ starting at the point x_i. As dev is a local diffeomorphism we have that

$$\frac{d}{dt} dev \circ \gamma_i(t) = dev_* \circ \tilde{X}(\gamma_i(t))$$
$$\Rightarrow dev \circ \gamma_i(t) = dev_* \circ \tilde{X}_i(\gamma_i(t))t + dev \circ \gamma_i(0)$$

applying our linear functional l_i to this we obtained

$$l_i(dev \circ \gamma_i(t)) = l_i \circ dev_*(\tilde{X}_i(\gamma_i(t)))t + l_i(dev \circ \gamma_i(0))$$
$$= l_i(\tilde{X}_i)t + l_i(dev \circ \gamma_i(0)) = t + l_i(dev \circ \gamma_i(0)) \qquad (*)$$

Now we can start our inductive step. Taking $i = n$ we start at $\tilde{x}_0 \in \tilde{M}$ such that $dev(\tilde{x}_0) = 0$. We flow along $\tilde{X}_n$ for time $t = l_n(v)$ until the point $\tilde{x}_1 \in \tilde{M}$ with $dev(\tilde{x}_1) = v_1$. From (*) we obtain

$$l_n(v_1) = l_n(v) + 0$$
$$\Rightarrow v - v_1 \in \ker l_n = F_{n-1}$$

Now, we continue e.g. flow along $\tilde{X}_{n-1}$ for $t = l_{n-1}(v - v_1)$ ending at $\tilde{x}_2$ with $dev(\tilde{x}_2) = v_2$. Then

$$l_{n-1}(v_2) = l_{n-1}(v - v_1) + l_n(v_1)$$
$$\Rightarrow l_{n-1}(v - v_2) = 0 \Rightarrow v - v_2 \in \ker l_{n-1} = F_{n-2}$$

We continue this until we reach $F_0 = \{0\}$ showing surjectivity.

Now for injectivity we consider a path

$$\gamma_0 : [a, b] \mapsto \tilde{M}$$

which develops to a closed loop $\delta : [a, b] \mapsto \mathbb{R}^n$. We can deform γ_0 to a new path γ_1 such that it develops entirely inside F_{n-1}. We now define a new path $\gamma_t : [a, b] \mapsto \tilde{M}$ where $\gamma_t(s)$ is the image of $\gamma_0(s)$ at time $-tl_n(\delta(s))$ under the flow of $\tilde{X}_n$. Applying dev gives that

$$dev \circ \gamma_t(s) = dev(\gamma_0(s)) - tl_n(\delta(s))dev_*(\tilde{X}(\gamma_0(s)))$$
$$\Rightarrow l_n(dev \circ \gamma_t(s)) = l_n(dev(\gamma_0(s))) - tl_n(\delta(s))l_n \circ dev_*(\tilde{X}(\gamma_0(s)))$$
$$\Rightarrow l_n(dev \circ \gamma_t(s)) = l_n(dev(\gamma_0(s))) - tl_n(\delta(s))$$

At $t = 1$ this yields that $dev \circ \gamma_t(s) \in \ker l_n = F_{n-1}$. Repeating this process we eventeually get a curve that develops entirely in $F_0 = \{0\}$ showing injectivity. $\square$

0.3 Symplectic Geometry and Integral Affine Manifolds

In this section we will recall some of the basics of symplectic geometry and then formulate some results relating to integral affine manifolds.

Conclusion

TODO: Add Conclusion

Bibliography

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