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Classifying Integral Affine Structures on Compact Surfaces

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Overview

Affine Manifolds

Definition

An *affine manifold* is a differentiable manifold such that the transition maps lie in the affine group $\mathsf{Aff}(\mathbb{R}^n)$

Remark

The group Aff$(\mathbb{R}^n) := $ *GL*$(\mathbb{R}^n) \ltimes \mathbb{R}^n$. *An element* $g \in$ *Aff*$(\mathbb{R}^n)$ *acts on* $x \in \mathbb{R}^n$ *by* $g(x) = Ax + b$ *where* $A \in$ *GL*$(\mathbb{R}^n)$ *and* $b \in \mathbb{R}^n$.

(Integral) Affine Manifolds

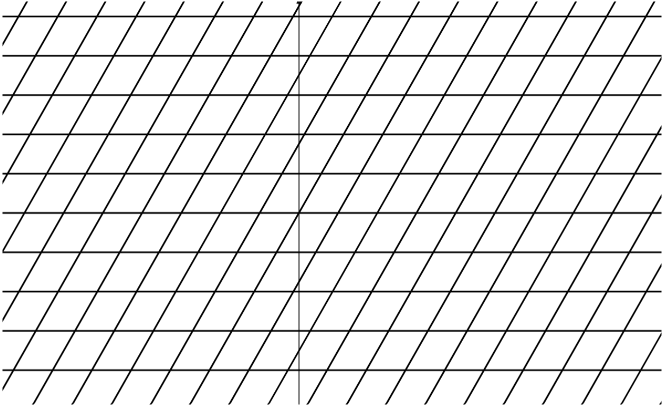

Figure: Affine shear transformation

The Affine Connection

Proposition

An affine manifold M comes equipped with a flat, torsion free connection ∇. In affine coordinates this appears as the standard Euclidean connection.

Remark

Conversely, a manifold M equipped with a flat, torsion free connection can be endowed with an affine structure.

The Development Pair

Lemma (C. Ehresmann)

Let M be an affine manifold and denote by $\tilde{M}$ its universal cover. Then there exists a pair (dev, hol) called the *development pair* where

$$\text{dev} : \tilde{M} \to \mathbb{R}^n,$$

$$\text{hol} : \pi_1(M) \to \text{Aff}(\mathbb{R}^n)$$

The linear part of hol is the holonomy of the affine connection ∇.

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Integral Affine Manifolds

Definition

An *integral affine manifold* is an affine manifold such that the transition maps lie in the integral affine group $\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n)$ - e.g. the linear part is in $\mathrm{GL}(\mathbb{Z}^n)$

Remark

An integral affine structure on a manifold M^n is equivalent to a rank-n lattice in T^*M locally spanned by closed 1-forms

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Complete Affine Manifolds

Definition

An affine manifold is said to be complete if the development map is a global diffeomorphism. This is equivalent to completeness of the connection ∇.

Remark

If a discrete group G acts freely and properly on $\mathbb{R}^n$ by affine transformations then $\mathbb{R}^n/G$ will inherit a complete affine structure.

Classifying Complete Integral Affine Structures

- Completeness forces injectivity of holonomy representation
- Classifying complete structures $\longleftrightarrow$ classifying injective holonomy representations inducing free and proper action on $\mathbb{R}^n$.

Possible Compact Affine Surfaces

Theorem (Milnor)

If M is a compact, orientable surface of genus $g \geq 2$, then it does not possess a flat affine connection.

Corollary

The only possible orientable compact surfaces admitting an (integral) affine structure are sphere S^2 *and the torus* T^2

Possible Compact Affine Surfaces

Lemma

S^2 does not admit an (integral) affine structure

Corollary

The only compact integral affine surfaces are the torus and Klein bottle.

Nilpotent Holonomy

Theorem (David Fried, William Goldman, Morris W. Hirsch)

Let M be a compact affine manifold with nilpotent holonomy group. Then M is complete if and only if it possesses a parallel volume form.

Corollary

A compact integral affine surface with nilpotent holonomy is complete.

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Nilpotent Holonomy

- $\pi_1(T^2) = \mathbb{Z} \times \mathbb{Z}$
- $\pi_1(K^2) = \langle a, b \mid aba = b \rangle$

$$\begin{tikzcd}
& \mathbb{R}^2 \arrow[dl] \arrow[dd] \\
T^2 \arrow[dr] & \\
& K^2
\end{tikzcd}$$

Nilpotent Holonomy

Corollary

Any integral affine structure on the 2-torus or the Klein bottle is complete.

Classification of Integral Affine Structures on $\mathbb{T}^2$ - $\pi_1(T^2) = \mathbb{Z} \times \mathbb{Z}$

Theorem (K.N. Mishachev.3)

The possible generators for the affine holonomy group are

1. $(I, \begin{pmatrix} a \\ c \end{pmatrix})$ and $(I, \begin{pmatrix} b \\ d \end{pmatrix})$ with $\det(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) \neq 0$ with $a, b, c, d \in \mathbb{R}$
2. $(I, \begin{pmatrix} a \\ 0 \end{pmatrix})$ and $(\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ b \end{pmatrix})$ with $n \in \mathbb{N}$ and $a, b > 0$

Two integral affine structures of the first type are isomorphic if and only if

$$X = GX'H$$

where

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad X' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix},$$

and G, H are some matrices from $\mathrm{GL}(2, \mathbb{Z})$.

Classification of Integral Affine Structures on $\mathbb{K}^2$ - $\pi_1(K^2) = \langle a, b \mid aba = b \rangle$

Let $A_1 = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ and $B_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $B_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $B_3 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$

Theorem (Daniele Sepe)

The possible generators for the affine holonomy group are

1. $(I, x\mathbf{e}_2)$ and $(B_1, y\mathbf{e}_1)$ with $x, y > 0$.
2. $(A_1, x\mathbf{e}_2)$ and $(B_1, y\mathbf{e}_1 + x\mathbf{e}_2)$ with $x, y > 0$.
3. $(I, x(\mathbf{e}_1 - \mathbf{e}_2))$ and $(B_2, y\mathbf{e}_2)$ with $x, y > 0$.
4. $(A_2, x\mathbf{e}_2)$ and $(B_3, y\mathbf{e}_1 + \frac{n-1}{n}x\mathbf{e}_2)$ with $2y \neq \frac{n-1}{n}x$ and $x > 0$.

Lagrangian Fibrations

- Whats next? - Lagrangian Fibrations over Compact Surfaces