CSCI203

# Algorithms and Data Structures

## Hashing

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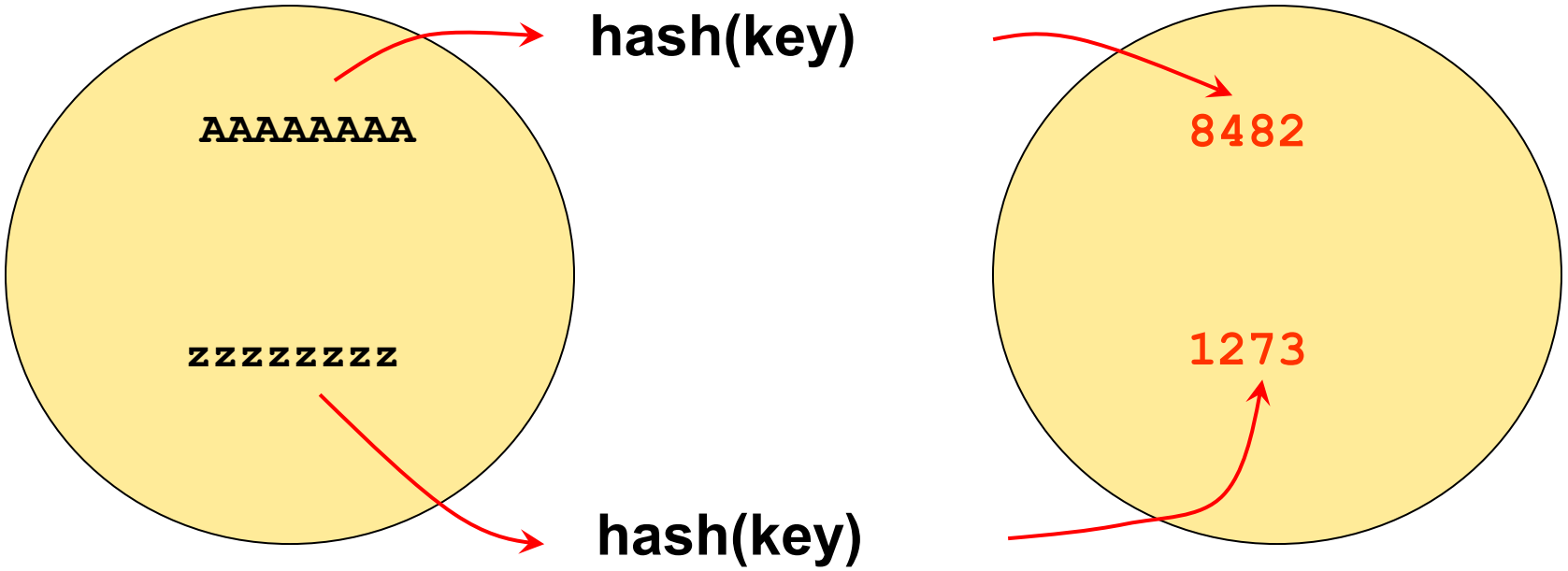
# Hash method works something like…

**Convert a String key into an integer that will be in the range of 0 through the maximum capacity-1**

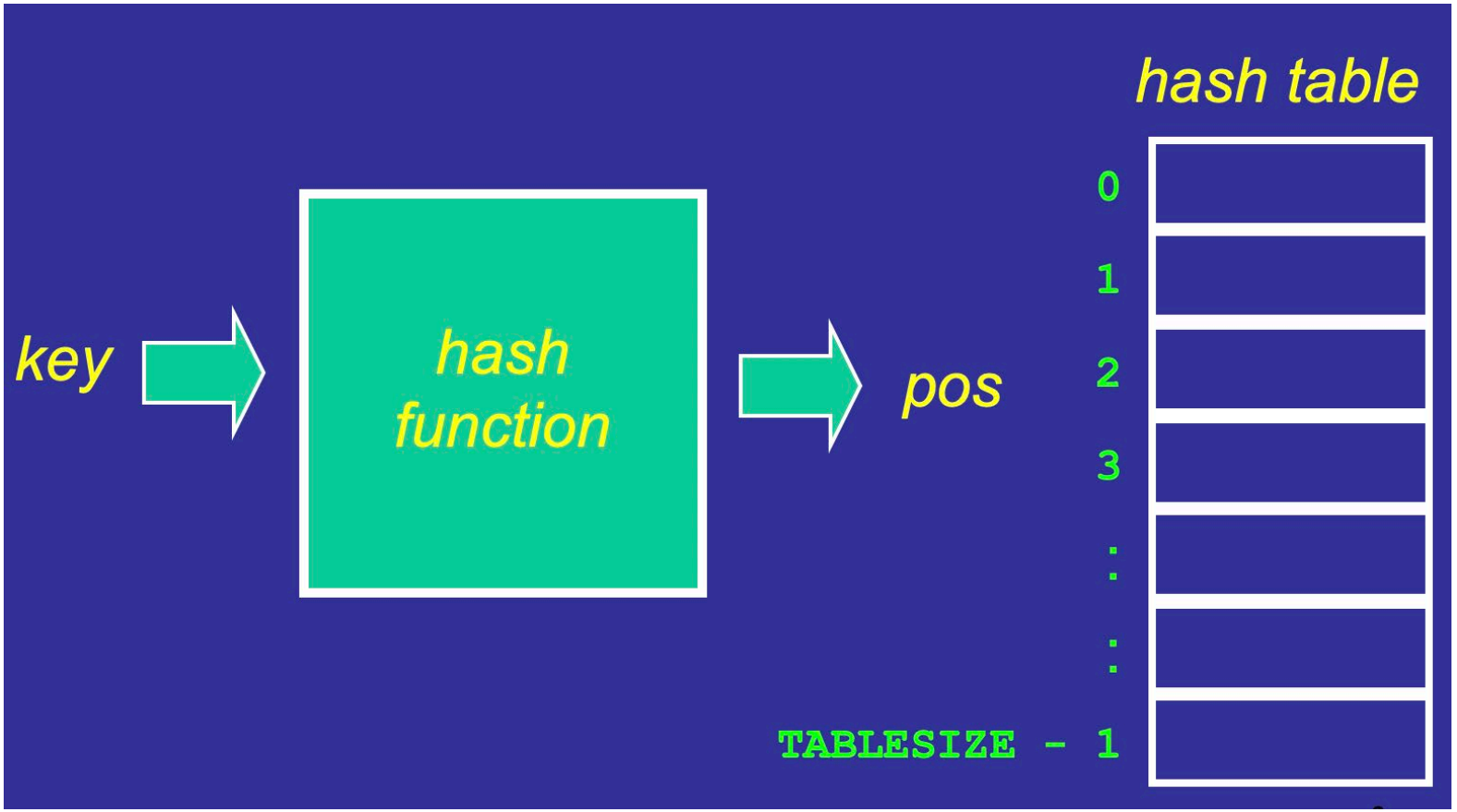
***Assume the array capacity is 9997***



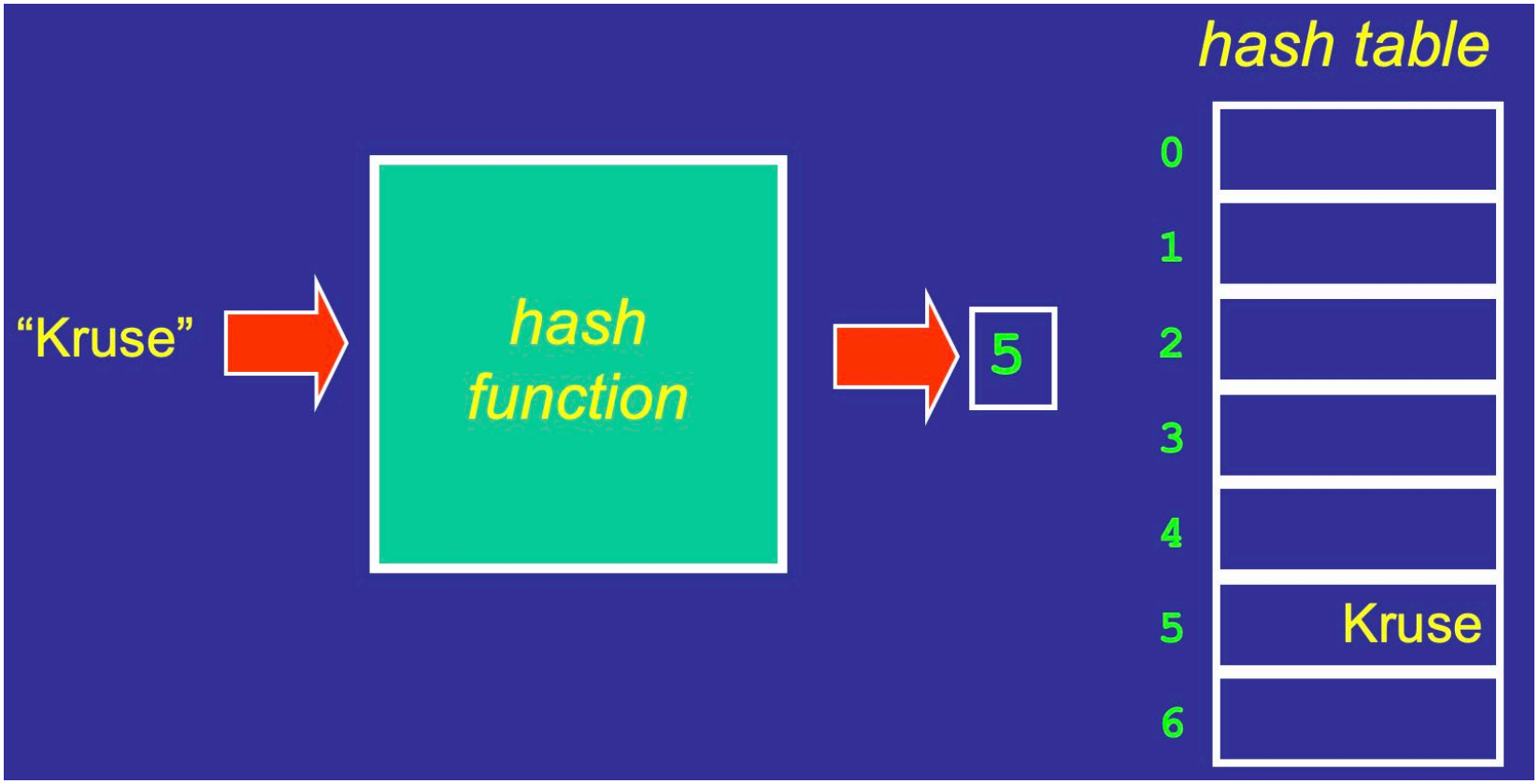

**A string of 8 chars**

**Range: 0 ... 9996**

# Hashing



# Hashing



# Example: use ASCII code

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Mia | M | 77 | i | 105 | a | 97 | 279 | 4 |
| Tim | T | 84 | i | 105 | m | 109 | 298 | 1 |
| Bea | B | 66 | e | 101 | a | 97 | 264 | 0 |
| Zoe | Z | 90 | o | 111 | e | 101 | 302 | 5 |
| Jan | J | 74 | a | 97 | n | 110 | 281 | 6 |
| Ada | A | 65 | d | 100 | a | 97 | 262 | 9 |
| Leo | L | 76 | e | 101 | o | 111 | 288 | 2 |
| Sam | S | 83 | a | 97 | m | 109 | 289 | 3 |
| Lou | L | 76 | o | 111 | u | 117 | 304 | 7 |
| Max | M | 77 | a | 97 | x | 120 | 294 | 8 |
| Ted | T | 84 | e | 101 | d | 100 | 285 | 10 |

| Bea | Tim | Leo | Sam | Mia | Zoe | Jan | Lou | Max | Ada | Ted |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |

# Hashing

- Each item has a unique key.
- Use a large array called a Hash Table.
- Use a Hash Function.

# Operations

- Initialize
  - all locations in Hash Table are empty.
- Insert
- Search
- Delete

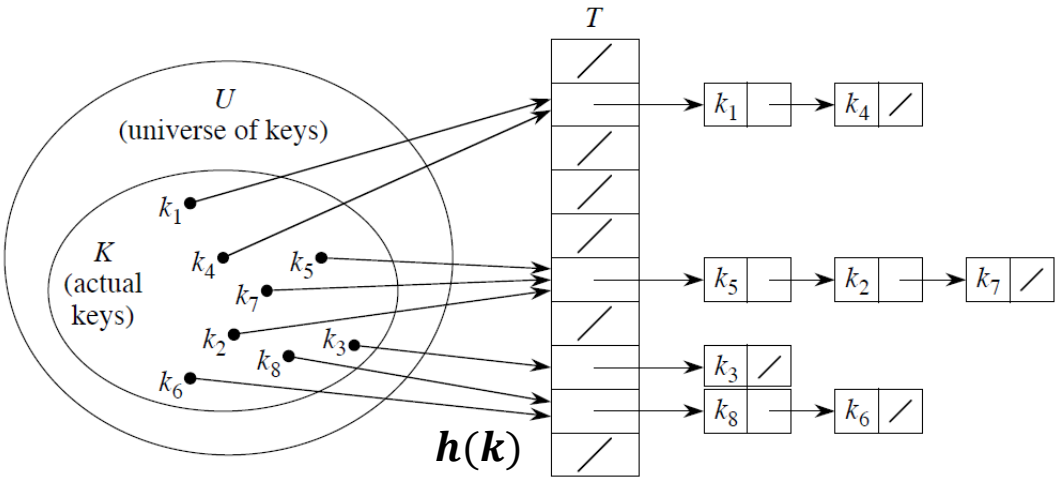
# Hash Function

- Maps keys to positions in the Hash Table.
- Be easy to calculate.
- Use all of the key.
- Spread the keys uniformly.

# Hashing

- Ideally, if we have $n$ keys with associated values, we would like $m \in \Theta(n)$.
  - $m = 2n, m = 3n.$
- This presents a problem:
  - Although $m > n$, the number of keys we are storing, it is far smaller that the number of possible keys.
  - There will always be circumstances where $key_1 \neq key_2$ but $h(key_1) = h(key_2)$.
  - This leads to a collision:
    - Two different keys with the same hash value;
    - Two different keys with the same location in the table.
  - How do we fix this?

# Hashing with Chaining



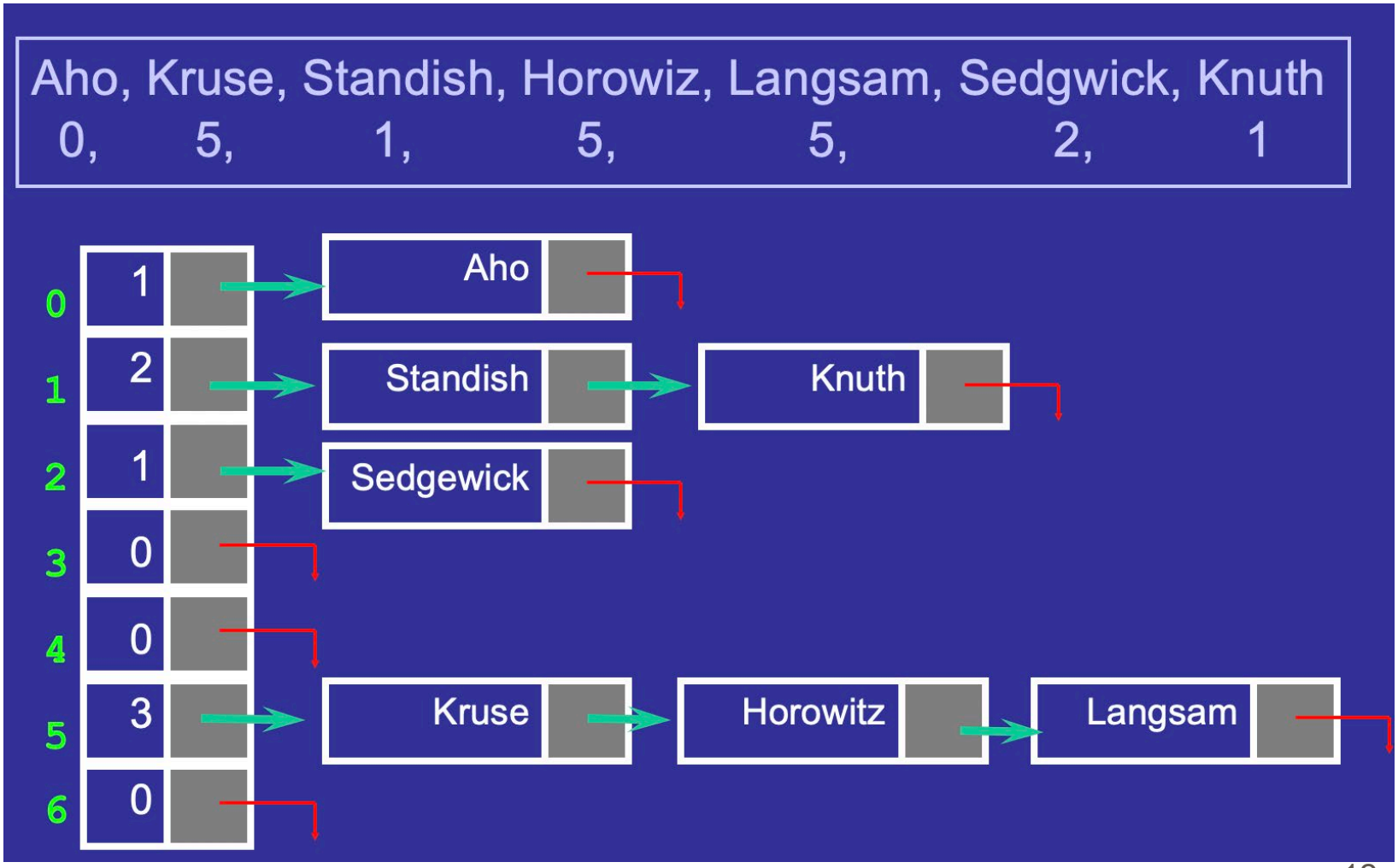

$m$ number of slots in T

- We call $h$ s hash function
- $h : U \rightarrow \{0, 1, \ldots, m - 1\}$, so that $h(k)$ is a legal slot number in $T$ .
- We say that k hashes to slot $h(k)$.
- Collision
  - When two or more keys hash to the same slot

# Chaining

- Uses a Linked List at each position in the Hash Table.

# Chaining

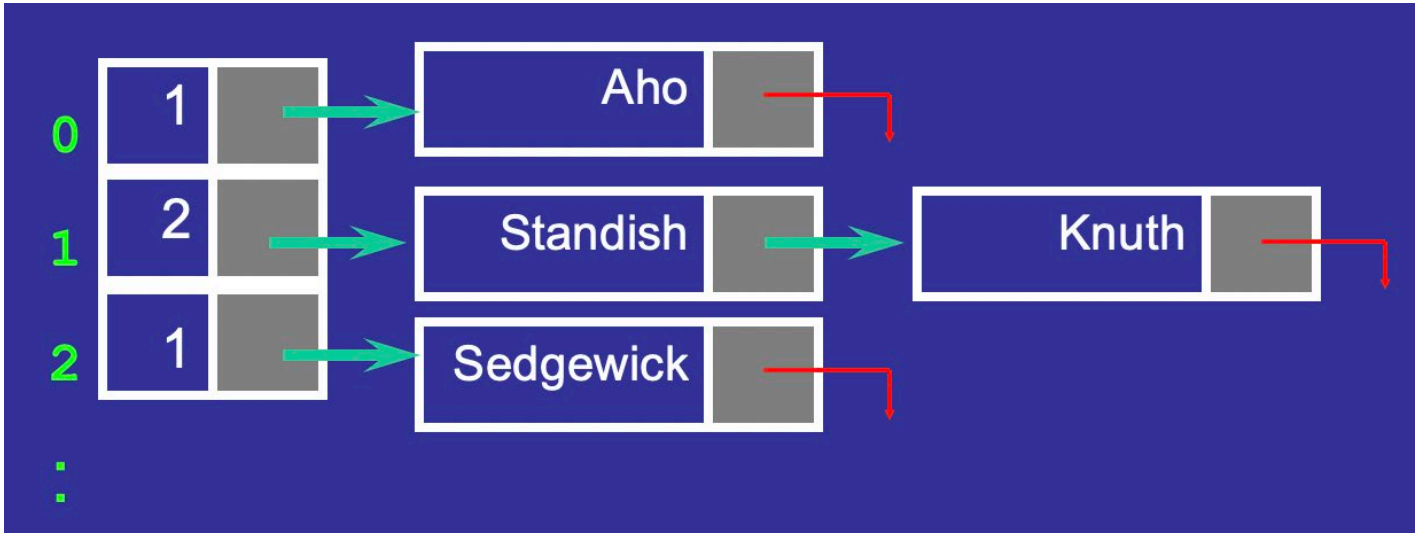

Aho, Kruse, Standish, Horowiz, Langsam, Sedgwick, Knuth
0, 5, 1, 5, 5, 2, 1
0
1
2
3
4
5
6
1
2
1
0
0
3
0
Aho
Standish
Knuth
Sedgewick
Kruse
Horowitz
Langsam

# Insert with Chaining

- Apply hash function to get a position in the array.
- Insert key into the Linked List at this position in the array.

# Insert with Chaining

```
module InsertChaining(item)
{
    posHash = hash(key of item)
    insert (hashTable[posHash], item);
}
```

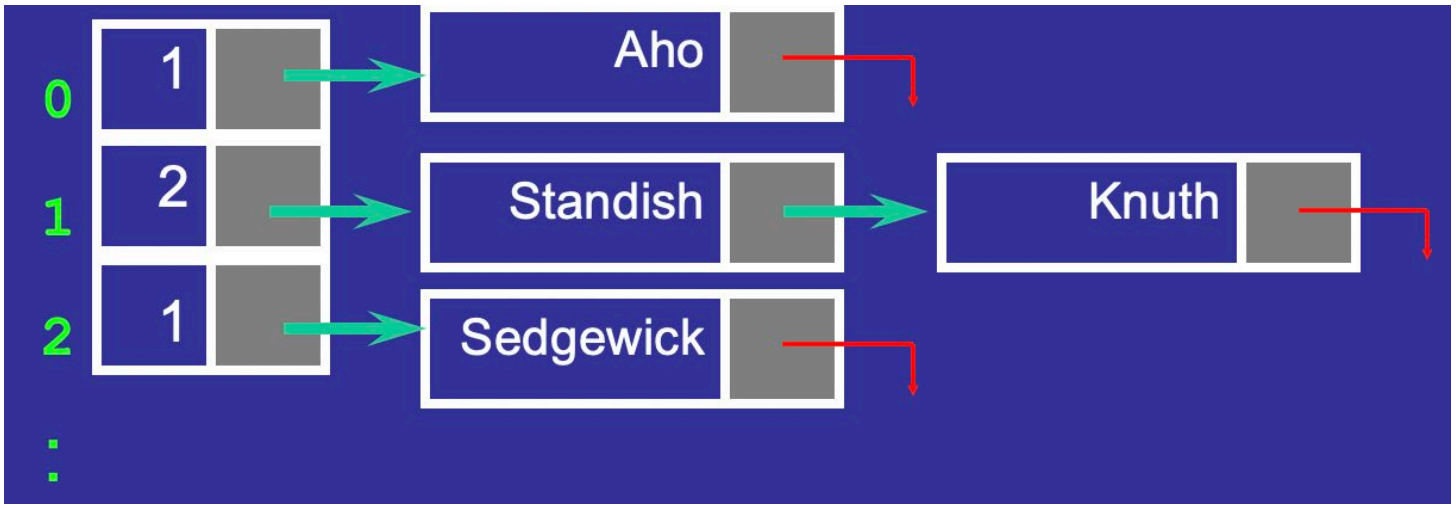


# Search with Chaining

- Apply hash function to get a position in the array.
- Search the Linked List at this position in the array.

# Search with Chaining

```
/* module returns NULL if not found, or the address of the
 *  node if found */
module SearchChaining(item){
   posHash = hash(key of item)
   Node* found;
   found = searchList (hashTable[posHash], item);
   return found;
}
```

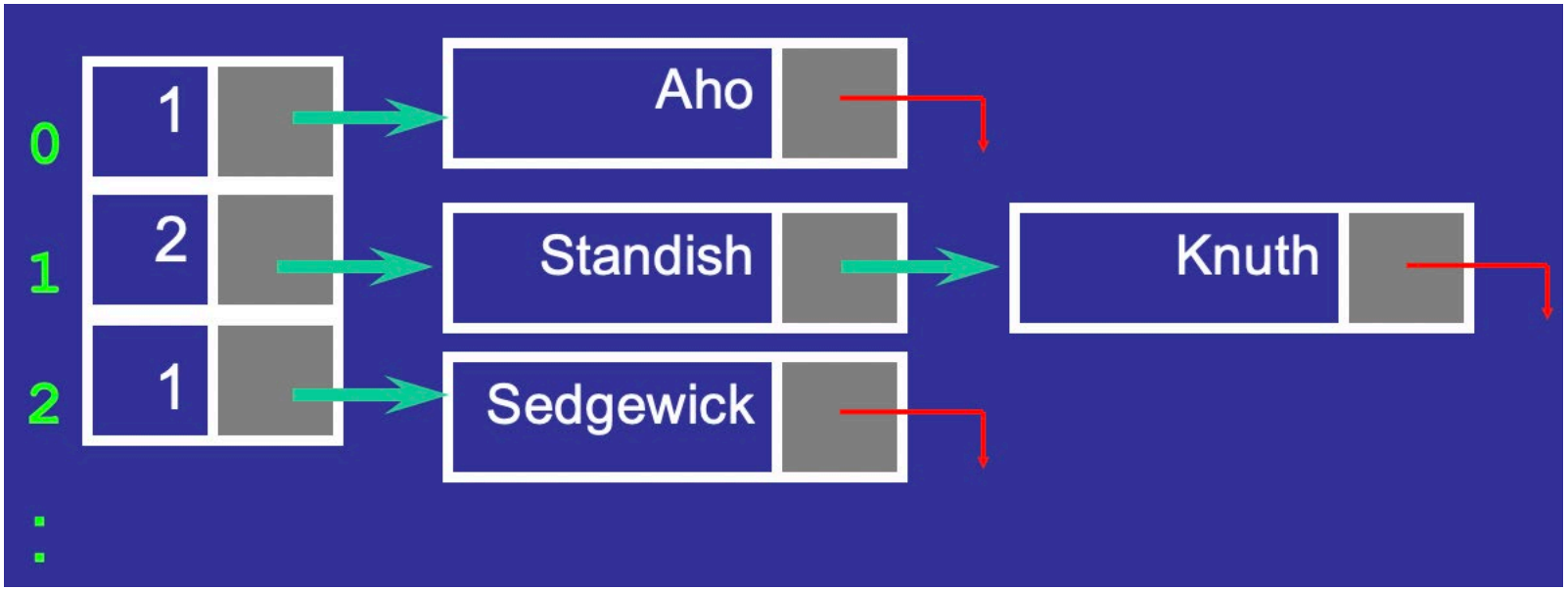


# Delete with Chaining

- Apply hash function to get a position in the array.
- Delete the node in the Linked List at this position in the array.

# Delete with Chaining

```
/* module uses the Linked list delete function to delete an item
 *inside that list, it does nothing if that item isn't there.  */
module DeleteChaining(item){
   posHash = hash(key of item)
   deleteList (hashTable[posHash], item);
}
```



# Advantages of Chaining

- Insertions and Deletions are easy and quick.
- Allows more records to be stored.
- Naturally resizable, allows a varying number of records to be stored.

# Disadvantages of Chaining

- Uses more space.
- More complex to implement.
  - A linked list at every element in the array

# Collision Frequency

- Birthdays *or* the von Mises paradox
  - There are 365 days in a normal year
    - Birthdays on the same day unlikely?
  - How many people do I need before "it's an even bet" (*ie* the probability is > 50%) that two have the same birthday?
  - View
    - the days of the year as the slots in a hash table
    - the "birthday function" as mapping people to slots
  - Answering von Mises' question answers the question about the probability of collisions in a hash table

# Distinct Birthdays

- Let $Q(n)$ = probability that $n$ people have distinct birthdays
- $Q(1) = 1$
- With two people, the 2nd has only 364 "free" birthdays

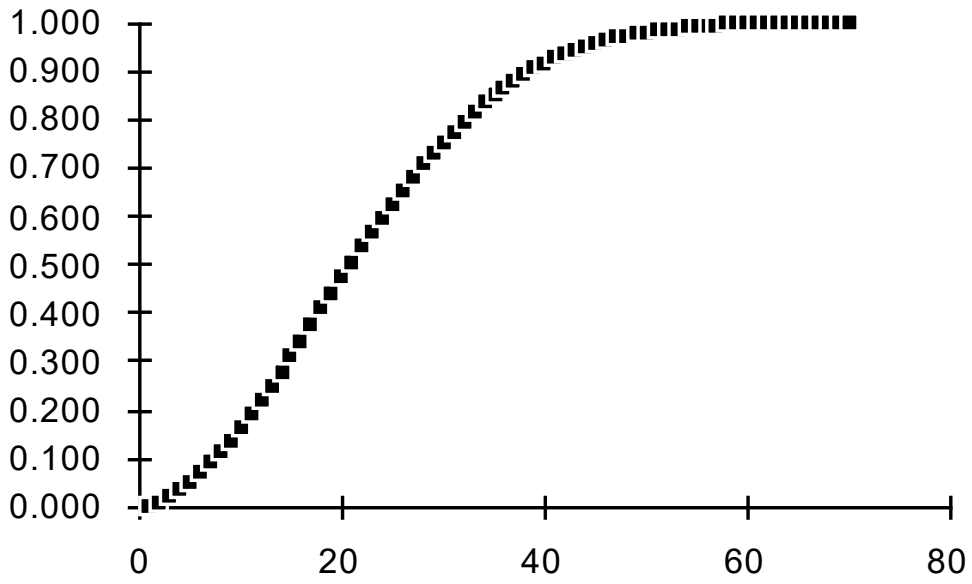
$$Q(2) = Q(1) * \frac{364}{365}$$

- The 3rd has only 363, and so on:

$$Q(n) = Q(1) * \frac{364}{365} * \frac{364}{365} * \ldots * \frac{365-n+1}{365}$$

# Coincident Birthdays

- Probability of having two identical birthdays
- $P(n) = 1 - Q(n)$
- $P(23) = 0.507$




- With 23 entries, table is only 23/365 = 6.3% full!

# Hash Tables - Load factor

- Collisions are very probable!
- Table load factor

$$\alpha = \frac{n}{m}$$

**$n$ = number of items**
**$m$ = number of slots**

must be kept low

- Detailed analyses of the average chain length (or number of comparisons/search) are available
- Separate chaining
  - linked lists attached to each slot

gives best performance

  - but uses more space!

# Hash Tables - General Design

- Choose the table size
  - Large tables reduce the probability of collisions!
  - Table size, $m$
  - $n$ items
  - Collision probability $\alpha = n / m$
- Choose a table organisation
  - Does the collection keep growing?
    - Linked lists (..... but consider a tree!)
  - Size relatively static?
    - Overflow area *or*
    - Re-hash

# Hash Tables - General Design

- Choose a hash function
  - A simple (and fast) one may well be fine ...
  - Read your text for some ideas!
- Check the hash function against your data
  - Fixed data
    - Try various $h$, $m$ until the maximum collision chain is acceptable
    - Known performance
  - Changing data
    - Choose some representative data
    - Try various $h$, $m$ until collision chain is OK
    - Usually predictable performance

# Hashing Functions

- The following are simple approaches which often work reasonably well:
- The Division method:
  - $h(k) = k \bmod m$
  - Good if $m$ is prime and is not close to a power of $2$ or a power of $10$.
- The Multiplication method:
  - $h(k) = a \times k \bmod 2^w >> (w - r)$
  - $a$ is an arbitrary $w - bit$ integer; (a should be odd and not close to a power of 2)
  - $w$ is the word length of the machine you are using;
  - >> is the right-shift operator;
  - $m = 2^r$.

# Hashing Functions…

- Universal Hashing:
  - $h(k) = (a \times k + b \bmod p) \bmod m$
  - $p$ is a prime number, bigger than $|U|$, the number of all possible keys.
    - Yes, p is BIG!
  - $a$ and $b$ are random integers between $0$ and $p - 1$.
- This is an excellent hash function.
- The worst-case probability of two keys colliding is $1/m$.
- This means that, even if $a$ and $b$ are poorly chosen, this hash function will always work well.
- The problems of finding a large prime and performing arithmetic on big integers will be left for now.

# Worst Case

- What if $h(key)$ has the same value for all the keys in our set?
- Our hash table has just become a complicated way of storing a single linked list!
- Access to a given $key$: $value$ pair is now $O(n)$.
- So, should we give up on hashing?
  - No!
  - In practice this does not happen.

# Picking up $m$

- $m$ is the number of slots in the dictionary, to be $\Theta(n)$, where $n$ is the number of entries in the dictionary.
- Operations on a dictionary are $O(1 + n/m)$, so if n grows too large we get less and less efficient.
- The problem we face is that, often, we do not know how many records $n$ we will need to store.
  - If m is too small, the dictionary becomes inefficient.
  - If m is too large, we waste storage (memory or disc).
- How do we get the right value for $m$?
- Let's say we want $m \geq n$ at all times.

# Lucky Guess?

- If we have no knowledge of the ultimate size of n, what can we do?
  - Guess.
  - Pick $m$ based on an optimistic assessment of the likely size of $n$.
  - No idea?
    - Pick your favourite small number.
    - m = 8, say.
  - Now what?
    - What if $n$ turns out to be greater than $8$?
  - Make m bigger.
    - How much bigger?

# Changing $m$

- When $n$ becomes large, change $m$
- If we change $m$ we have problems:
  - Our hash array is too small.
  - Our hashed keys will be wrong.
    - They depend on the value of $m$.
- Does this mean that we have to recreate the hash table from scratch?
  - It sure does.
  - Isn't this a BAD THING™?

# Growing a Hash Table

- What exactly has to happen if we change $m$?
  - Let's say the new table size is $m'$.
- We now need a new array with $m'$ elements.
  - We also need to move all of the existing elements from the old table to the new one.
- Build a new hash function $h'$.
  - Remember, the hash function depends on $m'$.
- Insert the existing data into the new table.
  - This involves re-hashing every key.
- So, the first question is:
  - How much do we grow $m$?

# $m' = ?$

- Each time we grow the table we perform $\Theta(m + n + m')$ operations.
  - This is $\Theta(n)$.
- Let's look at some options:
  - $m' = m + 1$.
  - What is the cost of $n$ insertions?
    - $\Theta(1)$ for the first m insertions.
    - $\Theta(m')$ for each insertion after that.
  - Overall $\Theta(n^2)$

# $m' = ?$

- $m' = 2m$
  - $\Theta(1)$ for the first m insertions.
  - $\Theta(m)$ for the next insertion.
  - $\Theta(1)$ for the next $m - 1$ insertions.
  - $\Theta(2m)$ for the next insertion.
  - $\Theta(1)$ for the next $2m - 1$ insertions.
- Overall $\Theta(n + (n/2) + (n/4) + \cdots) = \Theta(2n) = \Theta(n)$
- The cost of expanding the table gets spread over the extra elements we are making room for.
- This is known as Amortized cost.
- Note: an amortized cost of $\Theta(1)$ per operation does not mean that every operation has this cost.
  - Just that this is the average cost per operation.

# Amortized Cost

- We say an operation has a cost of "$T(k)$ Amortized" if $k$ operations take a total of $k \times T(k)$ time.
- Table doubling takes $\Theta(n)$ operations for $n$ insertions so the amortized cost is $\Theta(1)$.
- This is, actually, a GOOD THING™.
- Note: we can use table doubling to implement any solution where we do not know the size of the data structure in advance and it grows in a "well behaved" way.
- Table doubling minimizes the cost associated with dynamic data structures.

# Deletions

- What about deletions?
  - Each deletion is still $\Theta(1)$.
  - They simply increase the number of operations (insertions and deletions) we can perform between doublings.
- What if it's all deletions?
  - In this case the table becomes progressively less and less full.
  - Solution: Shrink the table.
- How, exactly?

# Shrinking a Hash Table

- What should our strategy for reducing the size of the table be?
- How about "if $n < m/2$ make $m' = m/2$"?
- What if the next operation is an insertion?
  - Double the table size!
  - Then a deletion?
  - Halve the table!
  - Insertion?
  - Double…
- We now have $\Theta(n)$ operations for each change in the data.
- Instead use "if $n < m/4$ make $m' = m/2$".

# Hashing With Chaining Considered Bad

- There is still one small issue with this method.
  - We have a hybrid data structure—an array of linked lists.
- A second approach uses just a simple array.
- Clearly, we still have a potential problem with collision
  - two keys which hash to the same value.
- We resolve this with a technique known as Open Addressing.

# Open Addressing

- An Alternative to Chaining
- We wish to hash $n$ items into an array with $m$ slots.
- We may only store one item per slot.
- Clearly, $m \geq n$.
- We insert an item into the table using an iterative technique known as probing.

# Probing

- This process works as follows: (for insertion)

```
Set hash function to starting value, h0
repeat
    calculate probe = hash(key)
    if table(probe) contains data then
            go to the next hash function
    else
    store the item in table(probe)
    fi
until we have stored the item
```

- This means we must have a sequence of hash functions, h0, h1, h2 …
- … or a hash function which produces a sequence of values.

# The Hash Function

- Our new hash function requires two arguments:
  - The key;
  - The iteration count.
- Thus: $probe = OpenHash(key, count)$
- Here:
  - key is a valid element of U, the universe of keys;
  - count is a non-negative integer.
  - As usual, $0 \leq probe < m - 1$.

# The Hash Function…

- In addition, we want our hash function to have the following property:
- For any arbitrary key $k$ the sequence of $m$ probes:
  - $h(k,0), h(k,1), h(k,2), \ldots, h(k,m-1)$;
- Must be a permutation of the integers:
  - $0, 1, 2, \ldots, m-1$.
- This property guarantees that we must eventually find a vacant slot to insert the item into.
- Clearly, the sequence of probes must be different for different keys.
- We can see this with an example.

# Example: Insertion with Open Addressing

- Consider the following table:

| k | h(0,k) | h(1,k) | h(2,k) | h(3,k) | h(4,k) | h(5,k) | h(6,k) | h(7,k) | h(8,k) | h(9,k) |
|---|---|---|---|---|---|---|---|---|---|---|
| 899 | 9 | 8 | 5 | 6 | 0 | 7 | 8 | 2 | 4 | 1 |
| 950 | 5 | 7 | 4 | 9 | 2 | 3 | 1 | 6 | 8 | 0 |
| 12 | 3 | 8 | 7 | 2 | 5 | 9 | 1 | 6 | 0 | 4 |
| 367 | 7 | 1 | 2 | 3 | 4 | 5 | 6 | 8 | 9 | 0 |
| 359 | 2 | 1 | 9 | 5 | 6 | 7 | 3 | 8 | 0 | 4 |
| 980 | 4 | 7 | 1 | 8 | 9 | 3 | 0 | 5 | 2 | 6 |
| 229 | 0 | 8 | 2 | 7 | 1 | 6 | 3 | 9 | 4 | 5 |
| 598 | 8 | 6 | 3 | 5 | 0 | 7 | 9 | 1 | 4 | 2 |
| 838 | 6 | 2 | 6 | 7 | 1 | 3 | 8 | 2 | 0 | 2 |
| 549 | 9 | 8 | 4 | 6 | 7 | 5 | 0 | 1 | 2 | 3 |

- Let us insert the keys into our hash table in order

# Example: Insertion with Open Addressing…

| k | h(0,k) | h(1,k) | h(2,k) | h(3,k) | h(4,k) | h(5,k) | h(6,k) | h(7,k) | h(8,k) | h(9,k) |
|---|---|---|---|---|---|---|---|---|---|---|
| 899 | 9 | 8 | 5 | 6 | 0 | 7 | 8 | 2 | 4 | 1 |
| 950 | 5 | 7 | 4 | 9 | 2 | 3 | 1 | 6 | 8 | 0 |
| 12 | 3 | 8 | 7 | 2 | 5 | 9 | 1 | 6 | 0 | 4 |
| 367 | 7 | 1 | 2 | 3 | 4 | 5 | 6 | 8 | 9 | 0 |
| 359 | 2 | 1 | 9 | 5 | 6 | 7 | 3 | 8 | 0 | 4 |
| 980 | 3 | 7 | 1 | 8 | 9 | 4 | 0 | 5 | 2 | 6 |
| 229 | 0 | 8 | 2 | 7 | 1 | 6 | 3 | 9 | 4 | 5 |
| 598 | 8 | 6 | 3 | 5 | 0 | 7 | 9 | 1 | 4 | 2 |
| 838 | 6 | 2 | 4 | 7 | 1 | 3 | 8 | 2 | 0 | 2 |
| 549 | 9 | 8 | 4 | 6 | 7 | 5 | 0 | 1 | 2 | 3 |

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 229 | 980 | 359 | 12 | 549 | 950 | 838 | 367 | 598 | 899 |

# Search with Open Addressing

- The procedure used to search using open addressing is similar to insertion.

```
count=0
repeat
    probe=hash(key, count)
    if table(probe)==key then
        return item
    else
        count++
    fi
until table(probe)==empty or count==n
return not found
```

- This is pretty straightforward.

# Deletion with Open Addressing

- When we get to deletion we have a new problem.

```
count=0
repeat
    probe=hash(key, count)
    if table(probe)==key then
        delete item
        return
    else
        count++
    fi
until table(probe)==empty or count==n
return not found
```

- How, exactly, do we delete the item?

# Deletion…

- If we simply replace the item with our empty value we will have an issue:
  - What if the key we next search for is after the probe corresponding to the deleted key's location.
  - If, in our previous example, we delete 899, where h(899,0)=9, and then search for 549, where the sequence of hash values are 9, 8, 4…
    - We test D(9) and discover it has the value empty.
    - We conclude that 549 is not in the table.
    - Wrong! It is in D(4).
- To fix this we need a second special value, deleted.

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 229 | 980 | 359 | 12 | 549 | 950 | 838 | 367 | 598 | 899 |

# Deletion…

- Our deletion process becomes:

```
count=0
repeat
    probe=hash(key, count)
    if table(probe)==key then
        table(probe)==deleted
        return
    else
        count++
    fi
until table(probe)==empty or count==n
return not found
```

- This fixes search but introduces a problem with insertion.

# Insertion Revisited

- We note that we can insert a new item into the dictionary in two circumstances:

```
D(i)==empty
D(i)==deleted
```

- We modify our insert process as follows:

```
count=0
repeat
     probe=hash(key, count)
     if table(probe)==empty or table(probe)==deleted then
          store item in table(probe)
          return
     else
          count++
     fi
until count==n
return no room
```

- Now we can insert into the first vacant slot, empty or deleted, that we find in the table.

# Search Revisited

- Because empty and deleted are different, we do not have to modify our search procedure.
- The search will skip over deleted records because they do not match the key but will still terminate when it reaches an empty record.

# Open Addressing Hash Functions

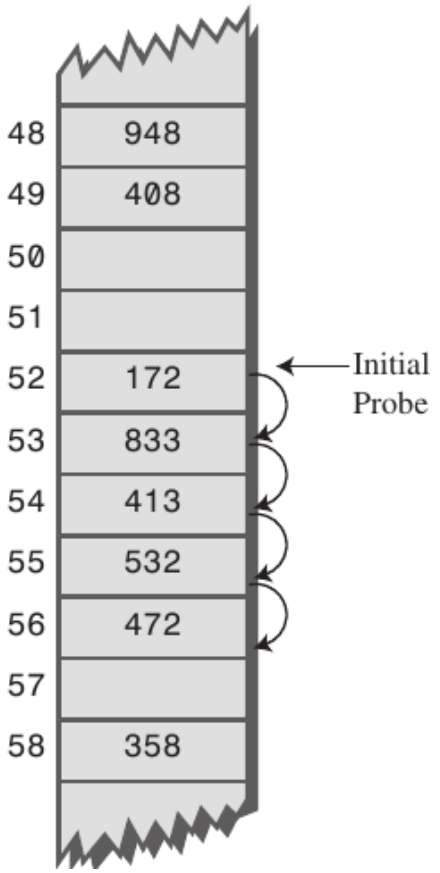
- One question remains.
- Can we find a function $h(k, i)$ which is:
  - Easy to compute;
  - Produces a permutation of $\{0, 1, \ldots, m-1\}$ as $i$ varies over $\{0, 1, \ldots, m-1\}$?
- Let us examine two possible strategies.

# Strategy I : Linear Probing

- In this approach we simply take a standard hash function, $h(k)$ and compute the probe $p(k, i)$ as follows:
  - $p(k, i) = (h(k) + i)\ mod\ m$
- In other words, we simply look at sequential entries in the dictionary starting at the entry corresponding to $h(k)$.
  - This is certainly easy to compute.
  - It does satisfy the permutation.
- There are $\Theta(m)$ distinct probing sequences

# Strategy I ; Linear Probing

- Is it any good?
  - No! It produces sets of consecutive occupied slots.
  - Primary Clustering – tendency to create long runs of filled slots near the hash position of keys
- The bigger the cluster, the more likely it is to be hit..
  - …and it gets even bigger!



# Strategy II: Double Hashing

- In this strategy we have two standard hash functions, $h_1(k)$ and $h_2(k)$.
- We compute $p(k)$, our probe value as follows:
  - $p(k,i) = (h1(k) + i \times h2(k)) \, mod \, m.$
- Do we still satisfy our requirements?
  - This is still easy to compute.
  - Do we always get a permutation?
    - No.
    - Unless we are clever in how we define $h_2$.

# Choose $h_2$

- We need $h_2(k)$ to be relatively prime to $m$.
  - i.e. $h_2(k)$ and $m$ must have no common factors except $1$.
- This is easy in many cases.
- If we select $m$ to be a power of $2$; say $m = 2^r$ then all we need is for $h_2(k)$ to always be an odd number.
- For example, if we have a standard hash function $h'(k)$, we can create $h_2(k)$ as follows:
  - $h_2(k) = (2h'(k) + 1)\ mod\ m$
- There are $\Theta(m^2)$ probing sequences
  - Each possible $(h_1(k), h_2(k))$ pair yields a distinct probe sequence

# Table Doubling

- Once again, we need to expand the dictionary whenever it becomes too full.
- What does "too full" mean in this case?
- We define the occupancy of a table, $\alpha$, to be the ratio of $n$, the number of entries to $m$, the number of slots.
  - $\alpha = n/m$
  - $0 \leq \alpha \leq 1$
- We can show that the average cost of an operation on a table with occupancy $\alpha$ is in $\Theta(1/(1-\alpha))$.
- In practice we want this value to be reasonably close to 1 so we double as soon as $\alpha$ exceeds $0.5$ or thereabouts.
- This keeps operations between $\Theta(1)$ and $\Theta(2)$.

# An Important Note on $\alpha$

- When calculating the occupancy value, $\alpha$, we must count slots with a value of deleted as containing data.
- This is because some operations, notably searching, treat deleted records as still containing data.
- Slots containing deleted may be removed in two ways:
  - Being overwritten with valid data as a result of an insert operation;
  - Being cleaned up when the table is expanded.
- If we did not count deleted records in calculating $\alpha$ we could have a notionally empty table in which every slot was deleted.
- Search (and delete) in this table would be $\Theta(m)$, not $\Theta(1)$, as we might expect.

# Chaining vs. Open Addressing

- So, which is the better scheme?
- Open Addressing:
  - Uses less memory—no need for pointers;
  - Is faster—provided $\alpha$ is kept below 0.5;
  - Is a little harder to implement and understand.
  - Is clean—one data structure, the array.
- Chaining:
  - Uses more memory;
  - Is faster—if we are not careful with open addressing.
  - Is a little easier to implement and understand.
  - Is a bit messy—arrays of linked lists.

# Looking for Text (in all the right places

- Consider the problem of String Searching:
  - Given a text, $t$, is the subtext, $s$, present in it?
- This problem occurs in many real-life applications:
  - grep;
  - find in a text editor;
  - Genome matching;
  - Google search.
- There are a wide number of techniques to achieve this.
- Let us look at a couple of examples.

# The Naïve Approach: Linear Search

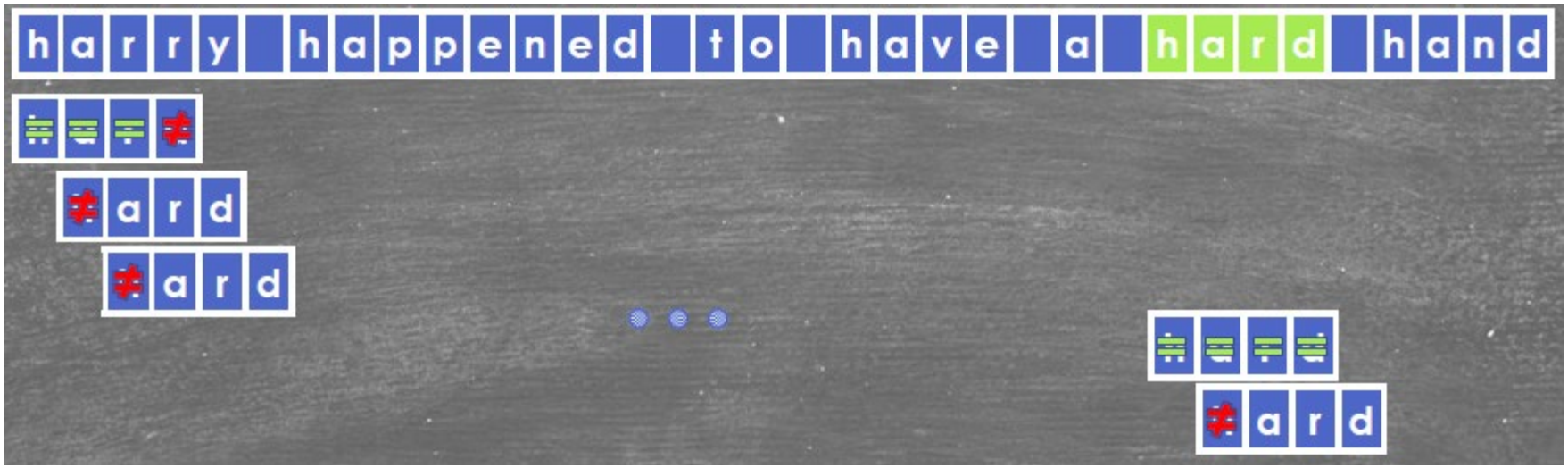
- The simplest possible approach is linear search:
  - Try to match $s$ starting at each location in $t$.

```
for i in 0… length(t) – length(s)
    j=0
    while j < length(s) do
        if (s(j) != t(i+j)) break
        j++
    od
    if j == length(s) print(" string found starting at location " i)
rof
```

- We can see this with an example.

# Linear Search: An Example

- Let $t$ be the string "*harry happened to have a hard hand*".
- Let $s$ be the string "*hard*".
- The search proceeds as follows:



# Linear Search ≠ Linear Time Search

- The outer loop in our algorithm is repeated $|t| - |s|$ times.
- Typically the string $t$ is much longer than the string $s$, so this is $\Theta(t)$.
- The inner loop is repeated up to $|s|$ times for each time round the outer loop.
  - This is $\Theta(s)$
- The total number of comparisons is $\Theta(|s| \times |t|)$.
- Is this the best we can do?
- The best we can possibly do is $\Theta(|s| + |t|)$;
  - we have to at least look at each string!
- Can we actually achieve this goal of a linear time algorithm?

# Linear Time Search

- To do this we will use hashing.
- We compare the hash of string $s$ with the hash of each substring of $t$ with the same length:

```
hash_s = hash(s)
for i in 0…length(t)-length(s)
    hash_t = hash(t[i..i+length(s)-1])
    if hash_s == hash_t then
        brute-force compare s and the substring
        if they match print(" s found starting at location " i)
    fi
rof
```

# Linear Time Search

- This algorithm takes linear time, provided:
  - The hash function only collides rarely;
  - The hash function takes constant time to compute;
    - Independent of the length of string $s$!
- Surely, the second requirement is impossible.
- To hash a string of length $|s|$ must take $\Theta(|s|)$ operations.
  - Yes?
  - No!
- Not if we are clever.

# Clever Hashing

- We note that we need $\Theta(|s|)$ time to compute $h(s)$.
- We also need $\Theta(|s|)$ time to compute $h(t[0..|s|-1])$, the initial substring of $t$.
- The trick is to compute the hash of each successive substring of $t$ in constant time.
- If we look closely at these substrings, we see an interesting feature:
  - Successive substrings differ only by two characters.
- The first character of the first substring; **harr**
- The last character of the next substring. **arry**

# Rolling Hash

- Maybe we can define a hash function which, given $h(\text{“}harr\text{”})$ can compute $h(\text{“}arry\text{”})$ in constant time.
- Let us define a rolling hash function, $r()$, so that:
  - $h(\text{“}arry\text{”}) = r(h(\text{“}harr\text{”}), \text{”}h\text{”}, \text{”}y\text{”})$
- We compute the hash of the next substring by removing the first and appending the new last characters;
- In this case we remove “$h$” and append “$y$”.
- If we can compute a rolling hash in constant time than we can do string matching in linear time.
- How?

# Karp-Rabin String Search

- The Karp-Rabin algorithm looks like this:

```
hash_s=hash(s)
hash_t=hash(t[0..length(s)-1])
for i in 0…length(t)-length(s)
    if hash_s == hash_t then
        brute-force compare s and the substring
        if they match print(" string found starting at location
        " i)
    fi
    hash_t=roll(hash_t,t[i],t[i+length(s)])
rof
```

- The function $roll(h, p, s)$ computes the rolling hash of the next substring given the hash of the existing substring, $h$, with the prefix $p$ removed and the suffix, $s$, appended.
- We need only find a suitable function $roll()$.

# How We Roll

- One popular way to compute $roll()$ is to use something called the Rabin fingerprint.
- We start by treating each symbol in the alphabet as an integer – use the ASCII code for example.
- We then find a random prime number > the size of the alphabet—let's pick $257$.
- We now compute $h(\text{“}harr\text{”})$ as:
  - $257^3 * 104 + 257^2 * 97 + 257^1 * 114 + 257^0 * 114$
  - $= 1{,}771{,}793{,}837$

- Note: $\text{“}h\text{”} = 104$, $\text{“}a\text{”} = 97$ and $\text{“}r\text{”} = 114$.

# The Next Hash

- Given that $h(\text{“}harr\text{”}) = 1{,}771{,}793{,}837$ how do we get $h(\text{“}arry\text{”})$?

- Simply compute $r(h, p, s) = 257 * (h - 257^3 * p) + s$
  $= 257.(1{,}771{,}793{,}837 - 257^3 * 104) + 121$
- In this case the result is $1{,}654{,}094{,}526$ which is exactly the same as $h(\text{“}arry\text{”})$
  $257^3 * 97 + 257^2 * 114 + 257^1 * 114 + 257^0 * 121$

- Note: if these values become too large, we can reduce them modulo $m$, where $m$ is a convenient value—say $2^{15}$or $2^{31}$.

# Efficient?

- We can compute our hash values for $s$ and the initial substring of $t$ using compact evaluation.

$$p^{k-1} * c_1 + p^{k-2} * c_2 + \ \ldots \ + \ p * c_{k-1} + c_k$$

- This requires a lot of multiplication!
- It can be re-written as…

$$h = \ c_k + p(c_{k-1} + p(c_{k-2} + \cdots + p(c_3 + \ p(c_2 + pc_1))..))$$

- Where $k = |s|$ and $c_i$ is the $i^{th}$ character of $s$.
- This requires $|s| - 1$ multiplications and $|s| - 1$ additions.

# Efficiency!

- If we precompute $q = p^{k-1}$ we can find the next hash value, $h$' as:
- $h' = p * (h - q * c_i) + c_j$ where we remove character $i$ and add character $j$.
- This requires only $1$ multiplication and $1$ addition.
  - Constant time.
- Thus we have $\Theta(|s|)$ operations to perform the initial hashes and $|t| - |s| \ * \Theta(1)$ operations to do the rehashing.
- Overall: our algorithm operates in $\Theta(|s| + |t|)$ time.
- We win!

# Related References

- Introduction to the Design and Analysis of Algorithms, A. Levitin, 3rd Ed., Pearson 2011.
  - Chapters 7.3
- Introduction to Algorithms, T. H. Cormen, 3rd Ed, MIT Press 2009.
  - Chapters 11