

# Finely Stratified Rerandomization Designs

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## Abstract

We study estimation and inference on causal parameters under finely stratified rerandomization designs, which use baseline covariates to match units into groups (e.g. matched pairs), then rerandomize within-group treatment assignments until a balance criterion is satisfied. We show that finely stratified rerandomization does partially linear regression adjustment “by design,” providing nonparametric control over the stratified covariates and linear control over the rerandomized covariates. We introduce several new rerandomization designs, allowing for imbalance metrics based on nonlinear estimators. We also propose a novel minimax scheme that uses pilot data or prior information to minimize the computational cost of rerandomization, subject to a strict bound on statistical efficiency. While the asymptotic distribution of generalized method of moments (GMM) estimators under stratified rerandomization is generically non-normal, we show how to restore asymptotic normality using ex-post linear adjustment tailored to the stratification. This enables simple asymptotically exact inference on superpopulation parameters, as well as efficient conservative inference on finite population parameters.

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# 1 Introduction

Stratified randomization is commonly used to increase statistical precision in experimental research.<sup>1</sup> Recent theoretical work (e.g. [Bai et al. \(2021\)](#)) has shown that fine stratification, which randomizes within small groups of units tightly matched on baseline covariate information, makes unadjusted estimators like difference of means semiparametrically efficient.<sup>2</sup> In finite samples, however, the performance of such designs can deteriorate rapidly with the dimension of the stratification variables due to a curse of dimensionality in matching.<sup>3</sup> This motivates the search for alternative designs that insist upon nonparametric balance for a few important covariates, but only attempt to balance linear functions of the remaining variables. In this paper, we study finely stratified rerandomization designs, which first tightly match the units into groups using a small set of important covariates, then rerandomize within-groups treatment assignments until a balance criterion on the remaining covariates is satisfied.

Our first contribution is to derive the asymptotic distribution of generalized method of moments (GMM) estimators under stratified rerandomization, allowing for estimation of generic causal parameters defined by moment equalities. We consider both superpopulation and finite population parameters, the latter of which may be more appropriate for experiments run in a convenience sample ([Abadie et al. \(2014\)](#)). As in previous work on rerandomization (e.g. [Li et al. \(2018\)](#)), the asymptotic distribution of GMM estimators is an independent sum of a normal and a truncated normal term. We show that, modulo this residual truncated term, the asymptotic variance of unadjusted estimation under stratified rerandomization is the same as that of semiparametrically adjusted GMM (e.g. [Graham \(2011\)](#)) under an iid design. Intuitively, stratified rerandomization implements partially linear regression adjustment “by design.”

Our second contribution is to introduce several novel forms of rerandomization based on nonlinear balance criteria. For example, we allow acceptance or rejection of an allocation based on the difference of covariate density estimates within each treatment arm, attempting to balance nonlinear features of the covariate distribution. Similarly, we propose a design that rerandomizes until a nonlinear estimate of the propensity score is approximately constant, effectively forcing the covariates to have no predictive power for treatment assignments. In both cases, these nonlinear rerandomization schemes are asymptotically equivalent to standard rerandomization based on a difference of covariate means, but with an implicit choice of covariates and acceptance region, which we characterize.

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<sup>1</sup>For example, [Cytrynbaum \(2023\)](#) reports a survey of 50 experimental papers in the AER and AEJ from 2018-2023, where 57% used some form of stratified randomization.

<sup>2</sup>See [Cytrynbaum \(2024\)](#), [Armstrong \(2022\)](#), and [Bai et al. \(2024\)](#) for more detailed discussion.

<sup>3</sup>Under regularity conditions, the convergence rate of finite sample variance to asymptotic variance is  $O(n^{-2/(d+1)})$  for dimension  $d$  covariates, see [Cytrynbaum \(2024\)](#).

Our third contribution is to study optimization of the balance criterion itself. We propose a novel minimax approach that allows the researcher to specify prior information about the relationship between covariates and outcomes, then rerandomizes until the worst case correlation between treatments and covariates consistent with this prior is small. We prove that this design minimizes the (asymptotic) computational cost of rerandomization, subject to a strict bound on statistical efficiency over the set of DGP’s consistent with the prior. If the prior information set contains the truth, this design strictly bounds the asymptotic variance within a small additive factor of the optimal semiparametrically adjusted variance. Extending this result, we show that if the prior information set is a confidence region estimated from pilot data, then this minimax design bounds the asymptotic variance in the main experiment with high probability.

Our fourth contribution is to provide simple t-statistic and Wald based inference methods for general causal parameters under stratified rerandomization designs. To do this, we first characterize and provide a feasible implementation of the optimal ex-post linear adjustment for GMM estimation under stratified rerandomization.<sup>4</sup> Crucially, optimal ex-post adjustment makes the asymptotic distribution insensitive to the rerandomization acceptance criterion, removing the truncated normal term from the limiting distribution and restoring asymptotic normality. For superpopulation parameters, our inference methods are asymptotically exact. For finite population parameters, our inference is conservative due to non-identification of the asymptotic variance, but still exploits the efficiency gains from both stratified rerandomization and ex-post optimal adjustment.

## 1.1 Related Literature

This paper builds on the literature on fine stratification in econometrics as well as the literature on rerandomization in statistics. Stratified randomization has a long history in statistics, see [Cochran \(1977\)](#) for a survey. Recent work on fine stratification in econometrics includes [Bai et al. \(2021\)](#), [Bai \(2022\)](#), [Cytrynbaum \(2024\)](#), [Armstrong \(2022\)](#), and [Bai et al. \(2024\)](#). Some important theoretical contributions to the literature on rerandomization include [Morgan and Rubin \(2012\)](#) and [Li et al. \(2018\)](#), [Wang et al. \(2021\)](#), and [Wang and Li \(2022\)](#). We build on both of these literatures, studying the consequence of rerandomizing treatments within data-adaptive fine strata. We show that finely stratified rerandomization does semiparametric (partially linear) regression adjustment “by design,” providing nonparametric control over a few important variables and linear control over the rest.

For our main asymptotic theory (Section 3), the most closely related previous work is [Wang et al. \(2021\)](#) and [Bai et al. \(2024\)](#). [Wang et al. \(2021\)](#) study estimation of the

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<sup>4</sup>This extends recent work on optimal adjustment under pure stratified randomization for ATE estimation, e.g. see [Cytrynbaum \(2023\)](#), [Bai et al. \(2023\)](#), or [Liu and Yang \(2020\)](#).

sample average treatment effect (SATE) under stratified rerandomization, with quadratic imbalance metrics based on the Mahalanobis norm. We study rerandomization within data-adaptive fine strata, providing asymptotic theory for generic superpopulation and finite population causal parameters defined by moment equalities. We also allow for essentially arbitrary rerandomization acceptance criteria, not necessarily based on quadratic forms. [Bai et al. \(2024\)](#) study estimation of superpopulation parameters defined by moment equalities under pure stratified randomization. We extend these results to stratified rerandomization as well as generic finite population parameters, providing “SATE-like” versions of the parameters in [Bai et al. \(2024\)](#).<sup>5</sup> In concurrent work, [Wang and Li \(2024\)](#) study GMM estimation of univariate superpopulation parameters under stratified rerandomization with fixed, discrete strata. We study significantly more general forms of stratification and rerandomization criteria than considered in their work, allowing for both finite and superpopulation parameters of arbitrary dimension and fine stratification with continuous covariates.

For nonlinear rerandomization (Section 4), the closest related results are [Ding and Zhao \(2024\)](#) and [Li et al. \(2021\)](#). [Ding and Zhao \(2024\)](#) rerandomize based on the p-value of a logistic regression coefficient, while we rerandomize until a general smooth propensity estimate is close to constant. To the best of our knowledge, we present the first asymptotic theory for rerandomization based on the difference of nonlinear (e.g. density) estimates. For acceptance region optimization (Section 5), the closest related results are [Schindl and Branson \(2024\)](#), who study the optimal choice of norm for quadratic rerandomization, while [Liu et al. \(2023\)](#) chooses a specific quadratic rerandomization using a Bayesian criterion, in both cases for rerandomization without stratification. We provide a novel minimax approach that accepts or rejects based on the value of a convex penalty function, tailored to prior information provided by the researcher. Our work on optimal adjustment (Section 6) extends recent work on adjustment for stratified designs, e.g. [Liu and Yang \(2020\)](#), [Cytrynbaum \(2023\)](#), [Bai et al. \(2023\)](#), to stratified rerandomization and GMM parameters. Finally our inference methods (Section 7) build on previous work by [Abadie and Imbens \(2008\)](#), [Bai et al. \(2021\)](#), and [Cytrynbaum \(2024\)](#). To the best of our knowledge we provide the first asymptotically exact inference for causal GMM parameters under stratified rerandomization, as well as conservative inference for their finite population analogues.

## 2 Framework and Designs

Consider data  $W_i = (R_i, S_i(1), S_i(0))$  with  $(W_i)_{i=1}^n \stackrel{\text{iid}}{\sim} F$ . The  $S_i(d) \in \mathbb{R}^{d_s}$  denote potential outcome vectors for a binary treatment  $d \in \{0, 1\}$ , while  $R_i$  denote other pre-treatment

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<sup>5</sup>These parameters can be seen as causal versions of the conditional estimand defined in [Abadie et al. \(2014\)](#).

variables, such as covariates. For treatment assignments  $D_i \in \{0, 1\}$ , the realized outcome  $S_i = S_i(D_i) = D_i S_i(1) + (1 - D_i) S_i(0)$ . In what follows, for any array  $(a_i)_{i=1}^n$  we denote  $E_n[a_i] = n^{-1} \sum_{i=1}^n a_i$ , with  $\bar{a}_1 = E_n[a_i D_i] / E_n[D_i]$  and  $\bar{a}_0 = E_n[a_i (1 - D_i)] / E_n[(1 - D_i)]$ . Next, we define stratified rerandomization designs.

**Definition 2.1** (Stratified Rerandomization). Let treatment proportions  $p = l/k$  and suppose that  $n$  is divisible by  $k$  for notational simplicity.

- (1) (Stratification). Partition the experimental units into  $n/k$  disjoint groups  $s$  with  $\{1, \dots, n\} = \bigcup_s s$  disjointly and  $|s| = k$ . Let  $\psi = \psi(R)$  with  $\psi \in \mathbb{R}^{d_\psi}$  denote a vector of stratification variables, which may be continuous or discrete. Suppose the groups satisfy the homogeneity condition<sup>6</sup>

$$\frac{1}{n} \sum_s \sum_{i,j \in s} |\psi_i - \psi_j|_2^2 = o_p(1). \quad (2.1)$$

Require that the groups only depend on the stratification variables  $\psi_{1:n}$  and data-independent randomness  $\pi_n$ , so that  $s = s(\psi_{1:n}, \pi_n)$  for each  $s$ .

- (2) (Randomization). Independently for each  $|s| = k$ , draw treatment variables  $(D_i)_{i \in s}$  by setting  $D_i = 1$  for exactly  $l$  out of  $k$  units, uniformly at random.
- (3) (Check Balance). For rerandomization covariates  $h = h(R)$ , consider an imbalance metric  $\mathcal{I}_n = \sqrt{n}(\bar{h}_1 - \bar{h}_0) + o_p(1)$ .<sup>7</sup> For an acceptance region  $A \subseteq \mathbb{R}^{d_h}$ , check if the balance criterion  $\mathcal{I}_n \in A$  is satisfied. If so, accept  $D_{1:n}$ . If not, repeat from the beginning of (2).

Intuitively, steps (1) and (2) describe a data-adaptive “matched k-tuples” design, while step (3) rerandomizes within k-tuples until the balance criterion is satisfied. Equation 2.1 is a tight-matching condition, requiring that the groups are clustered locally in  $\psi$  space. Cytrynbaum (2024) provides algorithms to match units into groups that satisfy this condition for any fixed  $k$ .

**Example 2.2** (Matched Pairs Rerandomization). For  $k = 2$ , the optimal matched pairs in Equation 2.1 can be found by Derigs (1988) algorithm. Suppose we have done so, and consider rerandomizing until the imbalance criterion  $n(\bar{X}_1 - \bar{X}_0)' \Sigma_n (\bar{X}_1 - \bar{X}_0) \leq \epsilon^2$  is satisfied for positive-definite  $\Sigma_n \xrightarrow{p} \Sigma$ .<sup>8</sup> Let  $\mathcal{I}_n \equiv \Sigma_n^{1/2} \sqrt{n}(\bar{X}_1 - \bar{X}_0) = \sqrt{n}(\bar{h}_1 - \bar{h}_0) + o_p(1)$  for modified covariates  $h = \Sigma^{1/2} X$ . This quadratic acceptance criterion is equivalent to

<sup>6</sup>The matching condition in Equation 2.1 was introduced by Bai et al. (2021) for matched pairs randomization ( $k = 2$ ). See Bai (2022) and Cytrynbaum (2024) for generalizations.

<sup>7</sup>In particular, we require  $\mathcal{I}_n = \sqrt{n}(\bar{h}_1 - \bar{h}_0) + o_p(1)$  under “pure” stratified randomization, the design in steps (1) and (2) only, studied e.g. in Cytrynbaum (2024). We give several examples below.

<sup>8</sup>Several recent papers in the statistics literature have considered such criteria. See e.g. Morgan and Rubin (2012), Li et al. (2018), Wang et al. (2021) among others.

$\mathcal{I}_n \in A$  for acceptance region  $A = \{x : |x|_2 \leq \epsilon\}$ . We study the efficiency consequences of different covariates and acceptance regions in detail in Sections 3 and 5 below.

**Example 2.3** (Stratification). Stratification without rerandomization can be obtained by setting  $A = \mathbb{R}^{d_h}$  in Definition 2.1. Treatment effect estimation under such designs was studied in Bai (2022), Cytrynbaum (2024), and Bai et al. (2024). Definition 2.1 allows for fine stratification (also known as matched k-tuples), with the number of data-dependent groups  $s = s(\psi_{1:n}, \pi_n)$  growing with  $n$ . It also allows for coarse stratification with fixed strata  $T \in \{1, \dots, m\}$  and fixed  $m$ , as in Bugni et al. (2018), which can be obtained in this framework by setting  $\psi = T$  and matching units into groups  $s$  at random within each  $\{i : T_i = k\}$ .

**Example 2.4** (Complete Randomization). For  $p = l/k$ , we say that  $D_{1:n}$  are completely randomized with probability  $p$  if  $P(D_{1:n} = d_{1:n}) = 1/\binom{n}{np}$  for all  $d_{1:n}$  with  $\sum_i d_i = np$ .<sup>9</sup> If so, we denote  $D_{1:n} \sim \text{CR}(p)$ . Cytrynbaum (2024) shows that  $\text{CR}(p)$  randomization can be obtained by setting  $\psi = 1$  and  $A = \mathbb{R}^{d_h}$  in Definition 2.1, matching units into groups at random. Intuitively, random matched k-tuples is equivalent to complete randomization.

**Causal Estimands.** Next, we introduce a generic family of causal estimands defined by moment equalities. Let  $g(D, R, S, \theta) \in \mathbb{R}^{d_g}$  be a score function for generalized method of moments (GMM) estimation. Recall  $W = (R, S(1), S(0))$  and for  $D|W \sim \text{Bernoulli}(p)$  define  $\phi(W, \theta) = E[g(D, R, S, \theta)|W] = pg(1, R, S(1), \theta) + (1 - p)g(0, R, S(0), \theta)$ . By construction, we have  $E[\phi(W, \theta)] = 0 \iff E[g(D, R, S, \theta)] = 0$ . The function  $\phi(W, \theta)$  provides a convenient parameterization to introduce our causal estimands.

**Definition 2.5** (Causal Estimands). The *superpopulation* estimand  $\theta_0$  is the unique solution to  $E[\phi(W, \theta)] = 0$ . The *finite population* estimand  $\theta_n$  is the unique solution to  $E_n[\phi(W_i, \theta)] = 0$ .

In what follows, we study GMM estimation of both  $\theta_0$  and  $\theta_n$  under stratified rerandomization designs, showing an asymptotic equivalence between stratified rerandomization and partially linear covariate adjustment. In particular, this framework allows us to introduce several useful finite population estimands  $\theta_n$  that do not appear to have been considered previously in the literature. Note that GMM estimation of  $\theta_0$  under pure stratification was studied in Bai et al. (2024) for the exactly identified case. Our finite population parameter  $\theta_n$  can be viewed as a causal version of the finite population estimand defined in Abadie et al. (2014).<sup>10</sup>

**Example 2.6** (ATE). Define the Horvitz-Thompson weights  $H = \frac{D-p}{p-p^2}$  and let  $g(D, Y, \theta) = HY - \theta$ , so that  $\phi(W, \theta) = E[HY|W] - \theta = Y(1) - Y(0) - \theta$ . Then  $\theta_0 = E[Y(1) - Y(0)] =$

<sup>9</sup>For notational simplicity, we may assume that  $n = lk$  for some  $l \in \mathbb{N}$ .

<sup>10</sup>See also the related finite population estimands studied under iid sampling and assignment in Xu (2021) and Takehi and Otsu (2024).

ATE, the average treatment effect, and  $\theta_n = E_n[Y_i(1) - Y_i(0)] = \text{SATE}$ , the sample average treatment effect.

For a more interesting example, consider the best parametric predictor of treatment effect heterogeneity in experiments with noncompliance.

**Example 2.7** (LATE Heterogeneity). Let  $D(z)$  be potential treatments for a binary instrument  $z \in \{0, 1\}$ . Let  $Y(d)$  be the potential outcomes, with realized outcome  $Y = Y(D(Z))$ . Suppose  $D(1) \geq D(0)$ , and define compliance indicator  $C = \mathbb{1}(D(1) > D(0))$ , assuming  $E[C] > 0$ . [Imbens and Angrist \(1994\)](#) define the local average treatment effect  $\text{LATE} = E[Y(1) - Y(0) | C = 1]$ . Let  $H = (Z - p)/(p - p^2)$  and consider the score function  $g(Z, D, Y, X, \theta) = (HY - HD \cdot f(X, \theta)) \nabla_\theta f(X, \theta)$ . Using standard LATE manipulations,

$$\phi(W, \theta) = E[g(Z, D, Y, X, \theta) | W] = C \cdot (Y(1) - Y(0) - f(X, \theta)) \nabla_\theta f(X, \theta).$$

Then  $E[\phi(W, \theta)] = 0$  is the first order condition of a treatment effect prediction problem in the complier population. In particular, for  $\tau \equiv Y(1) - Y(0)$ , the parameter  $\theta_0$  is the best parametric predictor of treatment effects for compliers:

$$\theta_0 = \underset{\theta}{\operatorname{argmin}} E[(\tau - f(X, \theta))^2 | C = 1].$$

For example, if  $Y$  is binary then  $Y(1) - Y(0) \in \{-1, 0, 1\}$ , so a scaled link function model  $f(X, \theta) = 2L(X'\theta) - 1$  may be appropriate. We can easily estimate marginal effects by adding  $m(X_i, \theta, \beta) = \beta - (\partial/\partial\theta')f(X_i, \theta)$  to the score function.

**Example 2.8** (Finite Population Heterogeneity). Continuing Example 2.7, note that for  $\tau_i = Y_i(1) - Y_i(0)$  the corresponding finite population parameter is

$$\theta_n = \underset{\theta}{\operatorname{argmin}} E_n[(\tau_i - f(X_i, \theta))^2 | C_i = 1]. \quad (2.2)$$

We can view  $\theta_n$  as a “SATE-like” version of  $\theta_0$ , the best parametric predictor of treatment effects in the *within-sample* complier population.  $\theta_n$  may be a more appropriate target for experiments run in a convenience sample. If  $f(X, \theta) = X'\theta$  linear, then  $\theta_n = \underset{\theta}{\operatorname{argmin}} E_n[(\tau_i - X_i'\theta)^2 | C_i = 1]$  is the within-sample best linear predictor. In the case of perfect compliance  $C_i = 1$  for all  $i$ , this is  $\theta_n = \underset{\theta}{\operatorname{argmin}} E_n[(\tau_i - X_i'\theta)^2]$ , a finite-sample version of the best linear predictor of the conditional average treatment effect (CATE). The case  $X = 1$  recovers  $\theta_n = E_n[Y_i(1) - Y_i(0) | C_i = 1]$ , the finite-population LATE, studied e.g. in [Ren \(2023\)](#). Our inference methods in Section 7 produce tighter confidence intervals for these finite population parameters than  $\theta_0$ , since we only need to account for the uncertainty due to random assignment, with no sampling uncertainty.

**GMM Estimation.** Let positive-definite weighting matrix  $M_n \in \mathbb{R}^{d_g \times d_g}$  with  $M_n \xrightarrow{p}$



$M \succ 0$ . For sample moment  $\hat{g}(\theta) \equiv E_n[g(D_i, R_i, S_i, \theta)]$ , the GMM estimator<sup>11</sup> is

$$\hat{\theta} = \underset{\theta \in \Theta}{\operatorname{argmin}} \hat{g}(\theta)' M'_n \hat{g}(\theta). \quad (2.3)$$

In the exactly identified case,  $\hat{\theta}$  solves  $\hat{g}(\hat{\theta}) = 0$ . In the next section, we study generalized method of moments (GMM) estimation of the causal parameters  $\theta_0$  and  $\theta_n$  under stratified rerandomization.

**Remark 2.9** (M-estimation). Our results below also extend to M-estimators of the form  $\hat{\theta} = \operatorname{argmax}_{\theta} E_n[m(D_i, R_i, S_i, \theta)]$ , even when  $\hat{\theta}$  cannot be formulated as a GMM estimator e.g. due to the existence of local maxima. For example, this happens in some nonconvex problems in density estimation (Newey and McFadden (1994)). We briefly discuss this extension in Section 8.4.

### 3 Asymptotics for GMM Estimation

In this section, we characterize the asymptotic distribution of the GMM estimator  $\hat{\theta}$  under stratified rerandomization designs, as in Definition 2.1. We show that the variance under stratified rerandomization is proportional to the residuals of a partially linear regression model, up to an extra term that reflects slackness in the rerandomization criterion. In this sense, stratified rerandomization does partially linear regression adjustment “by design.” First, we state some technical conditions that are needed for the following results.

**Assumption 3.1** (Acceptance Region). *Suppose  $A \subseteq \mathbb{R}^{d_h}$  has non-empty interior and  $\operatorname{Leb}(\partial A) = 0$ ,<sup>12</sup> and require  $E[\operatorname{Var}(h|\psi)] \succ 0$  and  $E[|\psi|_2^2 + |h|_2^2] < \infty$ .*

Next we state the technical conditions needed for GMM estimation. Define the matrix  $G = E[(\partial/\partial\theta')\phi(W, \theta)]|_{\theta=\theta_0} \in \mathbb{R}^{d_g \times d_\theta}$  and let  $g_d(W, \theta) = g(d, R, S(d), \theta)$  for  $d \in \{0, 1\}$ . Recall the Frobenius norm  $|B|_F^2 = \sum_{ij} B_{ij}^2$  for any matrix  $B$ .

**Assumption 3.2** (GMM). *The following conditions hold for  $d \in \{0, 1\}$ :*

- (a) (Identification). *The matrix  $G$  is full rank, and  $g_0(\theta) = 0$  iff  $\theta = \theta_0$ .*
- (b) *We have  $E[g_d(W, \theta_0)^2] < \infty$  and  $E[\sup_{\theta \in \Theta} |g_d(W, \theta)|_2] < \infty$ . Also  $\theta \rightarrow g_d(W, \theta)$  is continuous almost surely, and  $\Theta$  is compact.*<sup>13</sup>

<sup>11</sup>In our examples, we will mainly be concerned with the exactly identified case. However, the theory for the over identified case is almost identical, so we include this as well.

<sup>12</sup>Note that  $\partial A$  denotes the boundary of  $A$ , the limit points of both  $A$  and  $A^c$ .

<sup>13</sup>We can formally resolve measurability issues with the sup expressions by either (1) explicitly working with outer probability (e.g. van der Vaart and Wellner (1996)) or (2) requiring that  $\{g_d(\cdot, \theta), \theta \in \Theta\}$  is universally separable for  $d = 0, 1$  (Pollard (1984), p.38). To focus on the practical design issues, we avoid this formalism, implicitly assuming that all quantities are appropriately measurable.



(c) *There exists a neighborhood  $\theta_0 \in U \subseteq \Theta$  such that  $G_d(W, \theta) \equiv \partial/\partial\theta' g_d(W, \theta)$  exists and is continuous. Also  $E[\sup_{\theta \in U} |\partial/\partial\theta' g_d(W, \theta)|_F] < \infty$ .*

Compactness could likely be relaxed using concavity assumptions or a VC class condition, but we do not pursue this here. In what follows it will be conceptually useful to reparameterize the score function.

**Orthogonal Expansion.** Recall  $\phi(W, \theta) = E[g(D, R, S, \theta)|W]$  for  $W = (R, S(1), S(0))$ . Define the assignment influence component  $a(W, \theta) \equiv \text{Var}(D)(g_1(W, \theta) - g_0(W, \theta))$ . For Horvitz-Thompson weights  $H = (D - p)/(p - p^2)$ , a simple calculation shows that we can expand

$$g(D, R, S, \theta) = \phi(W, \theta) + Ha(W, \theta). \quad (3.1)$$

Our work below shows that  $a(W, \theta)$  parameterizes estimator variance due to random assignment, while  $\phi(W, \theta)$  parameterizes the variance due to random sampling. We work directly with this expansion in what follows.

**Example 3.3 (SATE).** Continuing Example 2.6 above, let  $\bar{Y} = (1 - p)Y(1) + pY(0)$ , a convex combination that summarizes each unit's potential outcome level. Then for the score  $g(D, Y, \theta) = HY - \theta$ , we have  $a(W, \theta) = \bar{Y}$ . A simple calculation shows that for  $\hat{\theta} = E_n[H_i Y_i]$  and  $\theta_n = E_n[Y_i(1) - Y_i(0)]$ , we have

$$\hat{\theta} - \theta_n = E_n[H_i a(W_i)] = \frac{\text{Cov}_n(D_i, \bar{Y}_i)}{\text{Var}_n(D_i)}. \quad (3.2)$$

Intuitively, the term  $E_n[H_i a(W_i)]$  from Equation 3.1 isolates the estimator variance due to chance in-sample correlations between the assignments  $D_i$  and outcome levels  $\bar{Y}_i$ . By contrast,  $\phi(W, \theta) = Y(1) - Y(0) - \theta$  does not depend on assignments  $D_i$ , and  $\text{Var}(\phi(W, \theta)) = \text{Var}(Y(1) - Y(0))$  isolates the estimator variance due to random sampling of units and heterogeneity of individual treatment effects.

### 3.1 Finite Population Estimand

Our first theorem studies GMM estimation of the finite population estimand  $\theta_n$ , which solves  $E_n[\phi(W_i, \theta_n)] = 0$ . We extend these results to  $\theta_0$  in Corollary 3.8 below. To state the theorem, define the GMM linearization matrix  $\Pi = -(G'MG)^{-1}G'M \in \mathbb{R}^{d_\theta \times d_g}$ . Note that in the exactly identified case  $d_g = d_\theta$ , we just have  $\Pi = -G^{-1}$ . For brevity, we also denote  $v_D = \text{Var}(D) = p - p^2$ .

Before stating the main result, we first derive the influence function for GMM estimation of  $\theta_n$  under stratified rerandomization.

**Lemma 3.4 (Linearization).** *Suppose  $D_{1:n}$  as in Definition 2.1 and require Assumption 3.1, 3.2. Then  $\sqrt{n}(\hat{\theta} - \theta_n) = \sqrt{n}E_n[H_i \Pi a(W_i, \theta_0)] + o_p(1)$ .*

Lemma 3.4 generalizes Equation 3.2 above, showing that

$$\hat{\theta} - \theta_n = \frac{\text{Cov}_n(D_i, \Pi a(W_i, \theta_0))}{\text{Var}_n(D_i)} + o_p(n^{-1/2}).$$

This implies that to first order, the errors in estimating  $\theta_n$  are driven by the random in-sample correlations between treatment assignments  $D_i$  and the assignment influence function  $\Pi a(W_i, \theta_0)$ . Our main theorem shows that, by balancing  $\psi$  and  $h$ , stratified rerandomization reduces these correlations, improving precision.

**Theorem 3.5** (GMM). *Suppose  $D_{1:n}$  as in Definition 2.1. Require Assumption 3.1, 3.2. Then  $\sqrt{n}(\hat{\theta} - \theta_n)|W_{1:n} \Rightarrow \mathcal{N}(0, V_a) + R_A$ , independent RV's with*

$$V_a = \min_{\gamma \in \mathbb{R}^{d_h \times d_\theta}} v_D^{-1} E[\text{Var}(\Pi a(W, \theta_0) - \gamma' h | \psi)]. \quad (3.3)$$

Let  $\gamma_0$  be optimal in Equation 3.3. The term  $R_A$  is a truncated Gaussian

$$R_A \sim \gamma_0' Z_h | Z_h \in A, \quad Z_h \sim \mathcal{N}(0, v_D^{-1} E[\text{Var}(h | \psi)]). \quad (3.4)$$

Note that the variance matrix  $V_a \in \mathbb{R}^{d_\theta \times d_\theta}$ , so the minimum should be interpreted in the positive semidefinite sense. In particular, we say  $V(\gamma_0) = \min_\gamma V(\gamma)$  if  $V(\gamma_0) \preceq V(\gamma)$  for all  $\gamma \in \mathbb{R}^{d_h \times d_\theta}$ . Theorem 3.5 shows that  $\sqrt{n}(\hat{\theta} - \theta_n)$  is asymptotically distributed as an independent sum of a normal  $\mathcal{N}(0, V_a)$  and truncated normal  $R_A$ . The normal term  $\mathcal{N}(0, V_a)$  only depends on the “treatment assignment” component of the influence function,  $\Pi a(W, \theta_0)$ . The variance is attenuated nonparametrically by the stratification variables  $\psi$  and linearly by rerandomization covariates  $h$ .

**Residual Imbalance.** The truncated Gaussian term  $R_A \sim \gamma_0' Z_h | Z_h \in A$  arises from leftover covariate imbalances due to slackness in the rerandomization acceptance criterion,  $\sqrt{n}(\bar{h}_1 - \bar{h}_0) \in A$ , since  $A \neq \{0\}$ . If the acceptance region  $A$  is symmetric about zero, i.e.  $x \in A \iff -x \in A$ , then  $E[R_A] = 0$ , so the GMM estimator  $\hat{\theta}$  is first-order asymptotically unbiased. In principle,  $R_A$  could be made negligible relative to  $\mathcal{N}(0, V_a)$  in large samples by choosing a small enough acceptance region  $A$ . For example, if  $A = B(0, \epsilon)$  then  $R_{B(0, \epsilon)} \sim \{\gamma_0' Z_h | |Z_h|_2 \leq \epsilon\} \xrightarrow{p} 0$  as  $\epsilon \rightarrow 0$ . However, in finite samples and for small enough  $\epsilon$ , this acceptance region may be infeasible. We study a minimax style criterion to choose an efficient acceptance region  $A$  in Section 5 below.

To isolate the precision gains due to rerandomization, the following corollary specializes Theorem 3.5 to the case of stratification without rerandomization ( $A = \mathbb{R}^{d_h}$ ), as well as complete randomization, as defined in Examples 2.3 and 2.4.

**Corollary 3.6** (Pure Stratification). *Suppose  $D_{1:n}$  as in Definition 2.1 with  $A = \mathbb{R}^{d_h}$ . Require Assumption 3.1. Then  $\sqrt{n}(\hat{\theta} - \theta_n)|W_{1:n} \Rightarrow \mathcal{N}(0, V)$  with  $V = v_D^{-1} E[\text{Var}(\Pi a(W, \theta_0) | \psi)]$ .*

In particular, if  $D_{1:n} \sim \text{CR}(p)$  then  $V = v_D^{-1} \text{Var}(\Pi a(W, \theta_0))$ .

Corollary 3.6 shows that fine stratification reduces the variance of GMM estimation to  $V = v_D^{-1} E[\text{Var}(\Pi a(W, \theta_0) | \psi)] \leq v_D^{-1} \text{Var}(\Pi a(W, \theta_0))$ , a nonparametric improvement. Rerandomization as in Definition 2.1 provides a further linear variance reduction to  $V_a = \min_{\gamma \in \mathbb{R}^{d_h \times d_\theta}} E[\text{Var}(\Pi a(W, \theta_0) - \gamma' h | \psi)]$ , up to the residual imbalance term  $R_A$ .

**Remark 3.7** (Design-Based Asymptotics). Our results above show that  $\sqrt{n}(\hat{\theta} - \theta_n) | W_{1:n} \Rightarrow \mathcal{N}(0, V_a) + R_A$ , conditional on the sampled data  $W_{1:n} = (R_i, S_i(1), S_i(0))_{i=1}^n$ .<sup>14</sup> This result is “design-based” in the sense that the variance in the limiting distribution arises solely due to randomness of the treatment assignments  $D_{1:n}$ . However, we impose structure on the sequence of populations  $W_{1:n}$  ex-ante, assuming each population is drawn from a fixed measure,  $W_i \sim F$ . This allows us to provide simple variance expressions that connect our results with the superpopulation-based literature on GMM and partially linear adjustment in econometrics. By contrast, the “sequence of finite populations model” often used in the statistics literature (e.g. Li et al. (2018)) begins with an arbitrary sequence of finite populations  $(W_{i,n})_{i=1}^n$ , imposing the minimal structure needed for certain moments to converge ex-post. It would be interesting to extend our results to this setting, but we leave this to future work.

## 3.2 Superpopulation Estimand

The next result extends Theorem 3.5 to the superpopulation estimand  $\theta_0$ , which uniquely solves  $E[\phi(W, \theta_0)] = 0$ .

**Corollary 3.8** (Superpopulation Estimand). *Suppose  $D_{1:n}$  is as in Definition 2.1. Require Assumption 3.1, 3.2.*

- (a) We have  $\sqrt{n}(\hat{\theta} - \theta_0) \Rightarrow \mathcal{N}(0, V_\phi) + \mathcal{N}(0, V_a) + R_A$ , independent RV’s with  $V_\phi = \text{Var}(\Pi \phi(W, \theta_0))$  and  $V_a, R_A$  exactly as in Theorem 3.5.
- (b) (Pure Stratification). If  $A = \mathbb{R}^{d_h}$ , this is  $\sqrt{n}(\hat{\theta} - \theta_0) \Rightarrow \mathcal{N}(0, V)$  with

$$V = \text{Var}(\Pi \phi(W, \theta_0)) + v_D^{-1} E[\text{Var}(\Pi a(W, \theta_0) | \psi)].$$

Comparing Corollary 3.8 with the results above, we see that targeting  $\theta_0$  instead of  $\theta_n$  adds an extra independent Gaussian term  $\mathcal{N}(0, V_\phi)$  to the asymptotic distribution. Intuitively,  $V_\phi$  arises due to iid random sampling of  $\Pi \phi(W, \theta_0)$ . Notice that stratification and rerandomization only affect the assignment influence function component  $\Pi a(W, \theta_0)$ , while the sampling influence component  $\Pi \phi(W, \theta_0)$  is irreducible. In this sense, the statistical consequences of different designs and adjustment strategies all happen at the level

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<sup>14</sup>See Proposition 8.16 in the appendix for a formal statement.

of the finite population estimand  $\theta_n$ , while targeting the superpopulation estimand  $\theta_0$  just adds extra irreducible noise. For pure stratification, [Bai et al. \(2024\)](#) were the first to derive an analogue of part (b) of Corollary 3.8 in the exactly identified case, under different GMM regularity conditions than we use here.

**Example 3.9 (SATE).** Continuing Example 2.6, we had  $\phi(W, \theta) = Y(1) - Y(0) - \theta$ , so  $G = 1$  and  $\Pi = 1$ . As above,  $a(W, \theta) = (1 - p)Y(1) + pY(0) \equiv \bar{Y}$ . The GMM estimator  $\hat{\theta} = \bar{Y}_1 - \bar{Y}_0$  is just difference of means. Then by Theorem 3.5 and Corollary 3.8, we have  $\sqrt{n}(\hat{\theta} - \text{SATE})|W_{1:n} \Rightarrow \mathcal{N}(0, V_a) + R_A$  and  $\sqrt{n}(\hat{\theta} - \text{ATE}) \Rightarrow \mathcal{N}(0, V_\phi + V_a) + R_A$  with

$$V_\phi = \text{Var}(Y(1) - Y(0)) \quad V_a = \min_{\gamma \in \mathbb{R}^{d_h}} v_D^{-1} E[\text{Var}(\bar{Y} - \gamma' h | \psi)]. \quad (3.5)$$

The term  $V_\phi$ , which only appears when estimating the superpopulation estimand  $\theta_0$ , reflects sampling variance due to treatment effect heterogeneity. The term  $V_a$  is the variance due to random assignment, caused by random in-sample correlations between treatments  $D$  and outcome levels  $\bar{Y}$ . Covariate-adaptive randomization and adjustment can be used to reduce  $V_a$ , while  $V_\phi$  is an irreducible sampling variance.

**Remark 3.10.** [Wang et al. \(2021\)](#) study SATE estimation under stratified rerandomization in the sequence of finite populations framework. Relative to [Wang et al. \(2021\)](#), by imposing the tight-matching condition 2.1 we are able to derive a simple closed form for the asymptotic variance in terms of the measure  $W \sim F$ , showing an equivalence with partially linear regression adjustment.

**Example 3.11 (Treatment Effect Heterogeneity).** Continuing Example 2.7, consider the case with perfect compliance  $D = Z$  and  $f(X, \theta) = X'\theta$ . Then we can use the slightly modified score  $g(D, X, Y, \theta) = (HY - X'\theta)X$ . Then for  $\tau = Y(1) - Y(0)$  we have  $\phi(W, \theta_0) = (\tau - X'\theta_0)X$ , and the parameters  $\theta_n, \theta_0$  are the best linear predictors of treatment effect heterogeneity

$$\theta_n = \underset{\theta}{\text{argmin}} E_n[(\tau_i - X_i'\theta)^2], \quad \theta_0 = \underset{\theta}{\text{argmin}} E[(\tau - X'\theta)^2].$$

It's also easy to see that  $a(W, \theta_0) = \bar{Y}X$  and  $\Pi = E[XX']^{-1}$ . Then for  $e = \tau - X'\theta_0$ , the variance matrices in Corollary 3.8 are

$$V_\phi = E[XX']^{-1} E[e^2 XX'] E[XX']^{-1} \quad V_a = \min_{\gamma \in \mathbb{R}^{d_h \times d_x}} v_D^{-1} E[\text{Var}(\bar{Y}\Pi X - \gamma' h | \psi)].$$

The expression for  $V_a$  shows that if we want to precisely estimate treatment effect heterogeneity, it is important to stratify and rerandomize not only the variables that predict outcome levels  $\bar{Y}$ , but also their interactions with the heterogeneity variable  $X$ .

### 3.3 Equivalence with Partially Linear Adjustment

Example 3.9 showed that, up to the rerandomization imbalance  $R_A$ , the unadjusted estimator  $\hat{\theta} = \bar{Y}_1 - \bar{Y}_0$  has asymptotic variance  $V_a = \min_{\gamma \in \mathbb{R}^{d_h}} v_D^{-1} E[\text{Var}(\bar{Y} - \gamma' h | \psi)]$ . This can be rewritten in terms of the residuals of a partially linear regression of  $\bar{Y}$  on  $\psi$  and  $h$ :

$$V_a = \min_{\substack{\gamma \in \mathbb{R}^{d_h} \\ t \in L_2(\psi)}} v_D^{-1} \text{Var}(\bar{Y} - \gamma' h - t(\psi)). \quad (3.6)$$

More generally, Theorem 3.5 shows that under stratified rerandomization designs, the usual GMM estimator  $\hat{\theta}$  behaves like semiparametrically adjusted GMM in the iid setting. Formally, let  $\mathcal{L}(\psi) = L_2^{d_\theta}(\psi)$  be the  $d_\theta$ -fold Cartesian product of  $L_2(\psi)$ , the space of square-integrable functions. Then the variance due to random assignment  $V_a$  in Theorem 3.5 is can be written in terms of the residuals of the influence function  $\Pi a(W, \theta_0)$  in a partially linear regression on  $\psi$  and  $h$ :

$$V_a = \min_{\substack{\gamma \in \mathbb{R}^{d_h \times d_\theta} \\ t \in \mathcal{L}(\psi)}} v_D^{-1} \text{Var}(\Pi a(W, \theta_0) - \gamma' h - t(\psi)). \quad (3.7)$$

Intuitively, stratified rerandomization does partially linear regression adjustment “by design,” providing nonparametric control over  $\psi$  and linear control over  $h$ . For a more explicit equivalence statement, define  $m(\psi, h) = \gamma'_0 h + t_0(\psi)$  to be the partially linear function achieving the optimum in Equation 3.7. Define the oracle semiparametrically adjusted GMM estimator

$$\hat{\theta}^* = \hat{\theta} - E_n[H_i m(\psi_i, h_i)]. \quad (3.8)$$

For example, for the SATE estimation problem one can show that  $\hat{\theta}^*$  is just an oracle version of the usual augmented inverse propensity weighting (AIPW) estimator (Robins and Rotnitzky (1995)), with partially linear regression models in each arm.<sup>15</sup>

**Theorem 3.12** (Partially Linear Adjustment). *Suppose that  $D_{1:n} \sim \text{CR}(p)$ . The oracle partially linearly adjusted GMM estimator  $\sqrt{n}(\hat{\theta}^* - \theta_n) | W_{1:n} \Rightarrow \mathcal{N}(0, V_a)$ , with variance  $V_a$  as defined in Theorem 3.5.*

Under a completely randomized design, we require ex-post semiparametric adjustment to achieve  $V_a$ . Under stratified rerandomization, however, the simple GMM estimator  $\hat{\theta}$  automatically achieves  $V_a$ , up to the leftover imbalance term  $R_A$ .

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<sup>15</sup>Feasible partially linear adjustment in an iid mean estimation problem with missing data was studied in Wang et al. (2004). See also the related semiparametric adjustment for GMM parameters in Graham (2011).

## 4 Nonlinear Rerandomization

In this section, we study several novel “nonlinear” rerandomization criteria, proving that in many cases such criteria are asymptotically equivalent to linear rerandomization (Definition 2.1), with an implicit choice of rerandomization covariates  $h$  and acceptance region  $A$ . This shows that our asymptotics and inference methods apply to a broad class of asymptotically linear rerandomization schemes.

### 4.1 GMM Rerandomization

First, we generalize the imbalance metric  $\mathcal{I}_n$  introduced in Definition 2.1, allowing rejection of a treatment allocation  $D_{1:n}$  based on potentially nonlinear features of the in-sample distribution of treatments and covariates  $(D_i, X_i)_{i=1}^n$ . We can define a large class of nonlinear imbalance metrics by letting  $m(X_i, \beta)$  be a score function and considering within-arm GMM estimators  $\hat{\beta}_1$  and  $\hat{\beta}_0$  defined by

$$E_n[D_i m(X_i, \hat{\beta}_1)] = 0, \quad E_n[(1 - D_i) m(X_i, \hat{\beta}_0)] = 0. \quad (4.1)$$

We propose to rerandomize until the within-arm parameter estimates are approximately equal,  $\sqrt{n}(\hat{\beta}_1 - \hat{\beta}_0) \approx 0$ .

**Definition 4.1** (GMM Rerandomization). Define  $\mathcal{I}_n^m = \sqrt{n}(\hat{\beta}_1 - \hat{\beta}_0)$  as above, where  $m(X, \beta)$  is a score satisfying Assumption 3.2. Suppose  $d_\beta = d_m$  (exact identification) and let  $A$  be a symmetric acceptance region. Do the following: (1) form groups as in Definition 2.1. (2) Draw  $D_{1:n}$  by stratified randomization. (3) If imbalance  $\mathcal{I}_n^m = \sqrt{n}(\hat{\beta}_1 - \hat{\beta}_0) \in A$ , accept  $D_{1:n}$ . Otherwise, repeat from (2).

Intuitively, the generalized imbalance metric  $\mathcal{I}_n^m$  allows us to randomize until possibly nonlinear features of the covariates are balanced between the treatment and control groups. Observe that if  $m(X_i, \beta) = X_i - \beta$ , then  $\hat{\beta}_d = \bar{X}_d$  for  $d = 0, 1$  and  $\mathcal{I}_n^m = \mathcal{I}_n$ , so linear rerandomization is a special case.

**Example 4.2** (Density Rerandomization). Let  $f(X, \beta)$  be a parametric density model for covariates  $X$ , which may be misspecified. After drawing  $D_{1:n}$  by stratified randomization, consider forming (quasi) maximum likelihood estimators  $\hat{\beta}_1 \in \arg\max_\beta E_n[D_i \log f(X_i, \beta)]$  and  $\hat{\beta}_0 \in \arg\max_\beta E_n[(1 - D_i) \log f(X_i, \beta)]$  for the density of covariates assigned to each treatment arm, rerandomizing until the estimated parameters  $\sqrt{n}|\hat{\beta}_1 - \hat{\beta}_0|_2 \leq \epsilon$ . Under regularity conditions,<sup>16</sup>  $\hat{\beta}_d$  are GMM estimators as in Equation 4.1 with score function  $m(X_i, \beta) = \nabla_\beta \log f(X_i, \beta)$ , so this procedure is a GMM rerandomization with acceptance region  $A = \{x : |x|_2 \leq \epsilon\}$ .

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<sup>16</sup>For example, if  $\beta \rightarrow \log f(X, \beta)$  is a.s. strictly concave, the key identification condition in Assumption 3.2 will be satisfied.

Let  $\beta^*$  be the unique solution to  $E[m(X, \beta^*)] = 0$  and define  $G_m = E[(\partial/\partial\beta')m(X_i, \beta^*)]$ . Our next result shows that GMM rerandomization with acceptance criterion  $\mathcal{I}_n^m \in A$  is equivalent to linear rerandomization (Definition 2.1) with an implicit choice of rerandomization covariates  $h_i = m(X_i, \beta^*)$  and linearly transformed acceptance region.

**Theorem 4.3** (GMM Rerandomization). *Suppose  $D_{1:n}$  is as in Definition 4.1 and Assumption 3.2 holds. Then  $\sqrt{n}(\hat{\theta} - \theta_n)|W_{1:n} \Rightarrow \mathcal{N}(0, V_a) + R$ , independent RV's with*

$$V_a = \min_{\gamma \in \mathbb{R}^{d_m \times d_\theta}} v_D^{-1} E[\text{Var}(\Pi a(W, \theta_0) - \gamma' m(X_i, \beta^*) | \psi)]. \quad (4.2)$$

The residual  $R \sim \gamma'_0 Z_m | Z_m \in G_m A$  for  $Z_m \sim \mathcal{N}(0, v_D^{-1} E[\text{Var}(m(X_i, \beta^*) | \psi)])$ , where  $\gamma_0$  is optimal in Equation 4.2.

Theorem 4.3 shows that by rerandomizing until  $\sqrt{n}(\hat{\beta}_1 - \hat{\beta}_0) \in A$ , we implicitly balance the influence function  $-G_m^{-1}m(X_i, \beta^*)$  for the difference of GMM estimators in Equation 4.1. This suggests an equivalent, but computationally much simpler design with only one round of nonlinear estimation. In particular, let  $\hat{\beta}$  solve  $E_n[m(X_i, \hat{\beta})] = 0$  be the pooled GMM estimator and set rerandomization covariates  $\hat{h}_i = m(X_i, \hat{\beta})$ , rerandomizing until  $\sqrt{n}E_n[H_i \hat{h}_i] \in G_m A$ . The next result shows that this design, which generalizes Definition 2.1 to allow for estimated covariates, is asymptotically equivalent to the GMM rerandomization in Definition 4.1.

**Corollary 4.4.** *Suppose Assumption 3.1, 3.2 hold and let  $m(X, \beta)$  be as in Definition 4.1. Let  $D_{1:n}$  be rerandomized as in Definition 2.1 with  $\hat{h}_i = m(X_i, \hat{\beta})$  and acceptance region  $G_m A$ . Then  $\sqrt{n}(\hat{\theta} - \theta_n)|W_{1:n} \Rightarrow \mathcal{N}(0, V_a) + R$ , with both variables identical to those in Theorem 4.3.*

Corollary 4.4 is a useful tool for showing the equivalence of computationally intensive designs based on nonlinear estimation with simpler linear schemes. For instance, Example 4.5 shows that density rerandomization over an exponential family with sufficient statistic  $r(X_i)$  is asymptotically equivalent to linear rerandomization with  $h_i = r(X_i)$ .

**Example 4.5** (Density Rerandomization). Continuing Example 4.2, for  $x \in \mathcal{X}$  and a sufficient statistic  $r(x) \in \mathbb{R}^{d_r}$ , define the exponential family  $f(x, \beta) = \exp(\beta' r(x) - t(\beta))$ , with  $t(\beta) = \log \int_{\mathcal{X}} \exp(\beta' r(x)) d\nu(x)$  for some measure  $\nu$  on  $\mathcal{X}$ . If the sufficient statistics  $(r_j(x))_{j=1}^k$  are  $\nu$ -a.s. linearly independent, then  $\beta \rightarrow \log f(x, \beta)$  is strictly concave for all  $x$ .<sup>17</sup> Then  $E[m(X, \beta)] = 0$  has a unique solution for score  $m(X, \beta) = \nabla_\beta \log f(X, \beta)$ , showing that quasi-MLE in this family can be formulated as a GMM problem. By Corollary 4.4, density rerandomization using  $f(x, \beta)$  is asymptotically equivalent to linear rerandomization with  $\hat{h}_i = \nabla_\beta \log f(X_i, \hat{\beta}) = r(X_i) - t(\hat{\beta})$ . Since  $E_n[H_i t(\hat{\beta})] =$

<sup>17</sup>This holds since the log partition function  $t(\beta)$  is strictly convex for  $\beta$  s.t.  $t(\beta) < \infty$  in this case. See e.g. Wainwright and Jordan (2008) Chapter 3 for an introduction to the properties of the log partition function  $t(\beta)$ .



$t(\hat{\beta})E_n[H_i] = 0$ , this is in turn equivalent to linear rerandomization with  $h_i = r(X_i)$ , directly balancing the sufficient statistics for the family. For example, if  $x \in \{\pm 1\}^k$  are binary variables, consider density estimation in the graphical model<sup>18</sup>

$$f(x, \beta) = \exp \left( \sum_j x_j \beta_j + \sum_{j < l} x_j x_l \beta_{jl} - t(\beta) \right).$$

The sufficient statistic is  $r(x) = ((x_j)_j, (x_j x_l)_{j < l})$ , and the parameters  $\beta_{jl}$  model correlation between the binary variables  $x_j$  and  $x_l$ . By the discussion above, a design that rerandomizes based on the difference of quasi-MLE density estimates in this family<sup>19</sup> is asymptotically equivalent to the much simpler linear rerandomization in Definition 2.1 with covariates  $h_i = ((x_j)_j, (x_j x_l)_{j < l})$ .

## 4.2 Propensity Score Rerandomization

To motivate a propensity score based rerandomization procedure, note that under stratified randomization we have  $E[D_i|X_i] = p$  for all units. In finite samples, however, the *realized propensity*  $\hat{p}(B) = E_n[D_i|X_i \in B]$  may significantly diverge from  $p$  in certain regions  $B \subseteq \mathbb{R}^{d_x}$  of the covariate space. This implies that covariates are predictive of treatment assignments post-randomization, a form of “in-sample confounding,” which vanishes as  $n \rightarrow \infty$  but affects precision. To prevent this, we could, for instance, reject allocations where  $|\hat{p}(B) - p| > \epsilon$  for some collection of sets  $B$ . To make this idea tractable without fully discretizing, consider a parametric propensity model  $p(X, \beta) = L(X'\beta)$  and define the MLE estimator

$$\hat{\beta} \in \operatorname{argmax}_{\beta \in \mathbb{R}^{d_\beta}} E_n[D_i \log L(X'_i \beta) + (1 - D_i) \log(1 - L(X'_i \beta))]. \quad (4.3)$$

We can measure the average gap between the estimated and true propensity score using

$$\mathcal{J}_n = n E_n[(p - L(X'_i \hat{\beta}))^2]. \quad (4.4)$$

Intuitively, if  $\mathcal{J}_n$  is large, then the covariates  $X$  are predictive of treatment status in some parts of the covariate space. To avoid this, we propose rerandomizing until the imbalance metric  $\mathcal{J}_n$  is below a threshold:

**Definition 4.6** (Propensity Rerandomization). Do the following: (1) form groups as in Definition 2.1. (2) Draw  $D_{1:n}$  and estimate the propensity model in Equation 4.3. (3) If imbalance  $\mathcal{J}_n \leq \epsilon$ , accept. Otherwise, repeat from (2).

<sup>18</sup>This is the Ising model from statistical physics. Categorical variables with  $l \geq 2$  levels and higher order interactions can be added. See [Wainwright and Jordan \(2008\)](#) for MLE algorithms in this family.

<sup>19</sup>This is well-motivated when  $\psi$  is expected to be more important than  $(x_j)_j$ . We don’t want to stratify on both, since this could radically decrease match quality on  $\psi$ .

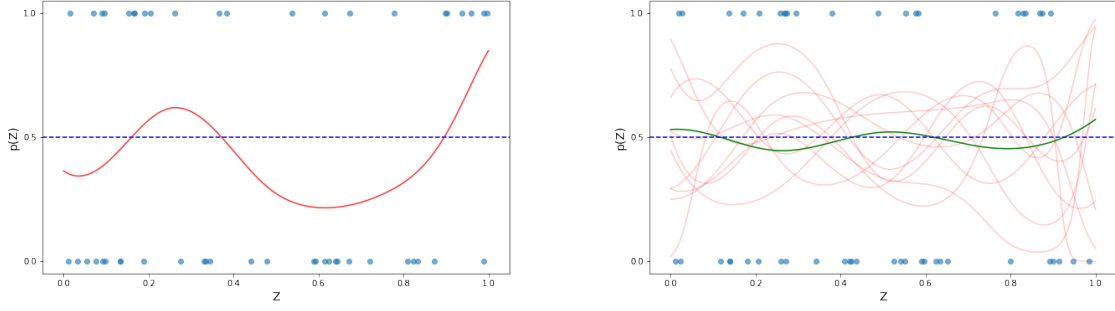


Figure 1: Propensity rerandomization (Definition 4.6) for  $Z \sim \text{Unif}[0, 1]$  and  $X = B(Z)$  a B-spline basis. LHS:  $D_{1:n}$  and estimated propensity with  $L(\hat{\beta}'X) \ll 1/2$ , for  $Z \in [0.5, 0.9]$ . RHS: Accepted allocation  $D_{1:n}$  with  $\mathcal{J}_n \leq \epsilon$

Our next result shows that propensity rerandomization as in Definition 4.6 is equivalent to a simpler linear rerandomization design, with an implicit choice of ellipsoidal acceptance region. We require some extra regularity conditions on the link function  $L$ , which for brevity we state in Appendix 8.5.

**Theorem 4.7** (Propensity Rerandomization). *Suppose  $D_{1:n}$  is as in Definition 4.6. Require Assumptions 3.2, 8.12. Then  $\sqrt{n}(\hat{\theta} - \theta_n) | W_{1:n} \Rightarrow \mathcal{N}(0, V_a) + R$ .*

$$V_a = \min_{\gamma \in \mathbb{R}^{d_h \times d_\theta}} v_D^{-1} E[\text{Var}(\Pi a(W, \theta_0) - \gamma' h | \psi)].$$

The residual  $R \sim \gamma_0' Z_h | Z_h' \text{Var}(h)^{-1} Z_h \leq \epsilon$  for  $Z_h \sim \mathcal{N}(0, v_D^{-1} E[\text{Var}(h | \psi)])$  and  $\gamma_0$  optimal in the equation above.

Theorem 4.7 shows that for any sufficiently regular link function, propensity rerandomization is asymptotically equivalent to the quadratic rerandomization design in Example 2.2, with acceptance criterion  $n(\bar{h}_1 - \bar{h}_0)' \text{Var}_n(h_i)^{-1} (\bar{h}_1 - \bar{h}_0) \leq \epsilon$ . Equivalently, propensity rerandomization behaves like linear rerandomization with  $\mathcal{I}_n = \sqrt{n}(\bar{h}_1 - \bar{h}_0)$  and ellipsoidal acceptance region  $A = \text{Var}(h)^{1/2} B(0, \epsilon)$ .<sup>20</sup>

**Implicit Acceptance Regions.** Both nonlinear designs in this section turned out to be equivalent to the standard rerandomization scheme in Definition 2.1, with a specific, implicit choice of rerandomization moments and acceptance region determined by the choice of score  $m$  and marginal covariate distribution. However, this implicit choice is not likely to be optimal, since the residual term in the asymptotic error distribution  $R_A \sim \gamma_0' Z_h | Z_h \in A$  depends on both the covariates  $Z_h \sim \mathcal{N}(0, v_D^{-1} E[\text{Var}(h | \psi)])$ , and the partially linear coefficient  $\gamma_0$ . This coefficient is determined by the *joint* distribution of the assignment influence function  $\Pi a(W, \theta_0)$  and covariates  $(\psi, h)$ . In the next section, we

<sup>20</sup>A related result was found by Ding and Zhao (2024), who study rerandomizing until the p-value of a logistic regression coefficient is above a threshold.

show how to use prior information about this joint distribution to optimize the acceptance region and bound the variance of  $R_A$ .

## 5 Optimizing Acceptance Regions

In this section, we study efficient choice of the acceptance region  $A \subseteq \mathbb{R}^{d_h}$ . We propose a novel minimax rerandomization criterion that minimizes asymptotic computational cost subject to a strict lower bound on statistical efficiency.

For intuition, we first restrict to the case of estimating  $\theta_n = \text{SATE}$ , generalizing in what follows. Example 3.9 showed that  $\sqrt{n}(\hat{\theta} - \text{SATE})|W_{1:n} \Rightarrow \mathcal{N}(0, V(\gamma_0)) + \gamma'_0 Z_{hA}$ , independent RV's with  $Z_{hA} = Z_h|Z_h \in A$  and  $V(\gamma_0)$  that does not depend on  $A$ . The coefficient  $\gamma_0$  is identified by the partially linear regression<sup>21</sup>

$$\bar{Y} = \gamma'_0 h + t_0(\psi) + e, \quad E[e|\psi] = 0, \quad E[eh] = 0.$$

The term  $R_A = \gamma'_0 Z_{hA}$  arises from fluctuations of  $\bar{Y}$  predictable by  $h$  that are not balanced by rerandomization due to slackness in the acceptance region,  $A \neq \{0\}$ . More precisely,  $\gamma'_0 Z_{hA}$  is the rerandomized limiting distribution of  $\gamma'_0 \mathcal{I}_n = \gamma'_0 \sqrt{n}(\bar{h}_1 - \bar{h}_0)$ , the projection of covariate imbalances in  $h$  along the direction  $\gamma_0$ . This suggests an oracle acceptance criterion that rerandomizes until  $|\gamma'_0 \mathcal{I}_n| \leq \epsilon$ , with acceptance region  $A = \{x : |\gamma'_0 x| \leq \epsilon\}$ . Of course, this is infeasible since  $\gamma_0$  is unknown when designing the experiment.

### 5.1 Minimax Rerandomization

Since  $\gamma_0$  is unknown at design-time, we instead take a minimax approach that incorporates prior information about the coefficient  $\gamma_0$ . For information set  $B \subseteq \mathbb{R}^{d_h}$  specified by the researcher, consider rerandomizing until the worst case in-sample correlation between treatments and covariates is small enough,

$$\sup_{\gamma \in B} |\gamma' \mathcal{I}_n| = \sup_{\gamma \in B} \left| \frac{\text{Cov}_n(\gamma' h_i, D_i)}{\text{Var}_n(D_i)} \right| \leq \epsilon. \quad (5.1)$$

Equivalently, we rerandomize until  $p_B(\mathcal{I}_n) \leq \epsilon$  for the convex penalty function  $p_B(x) = \sup_{\gamma \in B} |\gamma' x|$ . This significantly generalizes the commonly used quadratic penalty function  $p(x) = x' \Sigma x$  (e.g. Schindl and Branson (2024)). Our next result shows that Equation 5.1 is a linear rerandomization, characterizing the implicit acceptance region  $A$ .

**Proposition 5.1** (Acceptance Region). *The criterion  $p_B(\mathcal{I}_n) \leq \epsilon \iff \mathcal{I}_n \in A_0$  for  $A_0 = \epsilon B^\circ$  with  $B^\circ = \{x : \sup_{\gamma \in B} |\gamma' x| \leq 1\} \subseteq \mathbb{R}^{d_h}$ , the absolute polar of  $B$ . The set  $A_0$*

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<sup>21</sup>This expansion is without loss of generality. We do not impose well-specification  $E[e|\psi, h] = 0$ .

is symmetric and convex. If  $B$  is bounded,  $A_0$  is closed and has non-empty interior.<sup>22</sup>

Note that since  $A_0$  is symmetric, the discussion after Theorem 3.5 implies that the asymptotic distribution of  $\widehat{\theta}$  under the design in Equation 5.1 is centered at zero. We let  $B$  be totally bounded in what follows. The proposition shows that in this case  $A_0$  has non-empty interior, satisfying the conditions of Assumption 3.1.

We showed above that  $\sqrt{n}(\widehat{\theta} - \text{SATE})|W_{1:n} \Rightarrow L_0$  for  $L_0 = \mathcal{N}(0, V(\gamma_0)) + \gamma'_0 Z_{hA}$ . Since  $\gamma_0$  is unknown at design time, define a family  $\{L_{\gamma,A} : \gamma \in \mathbb{R}^{d_h}, A \subseteq \mathbb{R}^{d_h}\}$  of possible limiting distributions of  $\widehat{\theta}$ , with each  $L_{\gamma,A} = \mathcal{N}(0, V(\gamma)) + \gamma' Z_{hA}$  a sum of independent RV's as above. For any distribution in this family, the conditional asymptotic bias of  $\widehat{\theta}$  given realized covariate imbalances  $Z_{hA}$  is  $\text{bias}(L_{\gamma,A}|Z_{hA}) \equiv E[L_{\gamma,A}|Z_{hA}]$ . Our main result shows that the polar acceptance region  $A_0 = \epsilon B^\circ$  maximizes rerandomization acceptance probability  $P(Z_h \in A)$ , subject to a strict constraint on asymptotic conditional bias, uniformly over all limiting distributions consistent with the prior information  $\gamma_0 \in B$ .

**Theorem 5.2** (Minimax). *The acceptance region  $A_0 = \epsilon B^\circ$  solves<sup>23</sup>*

$$A_0 = \underset{A \subseteq \mathbb{R}^{d_h}}{\text{argmax}} P(Z_h \in A) \quad \text{s.t.} \quad \sup_{\gamma \in B} |\text{bias}(L_{\gamma,A}|Z_{hA})| \leq \epsilon. \quad (5.2)$$

*In particular, if  $\gamma_0 \in B$  (well-specification) then  $|\text{bias}(L_0|Z_{hA_0})| \leq \epsilon$  and  $\text{Var}(L_0) \leq V_a + \epsilon^2$ , where  $V_a$  is the partially linear variance in Equation 3.6.*

The expected number of independent draws of  $Z_h$  until  $Z_h \in A$  is  $P(Z_h \in A)^{-1}$ . Then, equivalently, Equation 5.2 shows that the acceptance region  $A_0$  minimizes the (asymptotic) computational cost of rerandomization, subject to a strict guarantee on statistical performance. From the final statement of the theorem, if the prior information set  $B$  is well-specified ( $\gamma_0 \in B$ ), setting  $A_0 = \epsilon B^\circ$  bounds the magnitude of the conditional asymptotic bias  $E[L_0|Z_{hA_0}]$  of the GMM estimator  $\widehat{\theta}$  below  $\epsilon$ . In particular, this implies that the variance  $\text{Var}(L_0)$  of the asymptotic distribution  $\sqrt{n}(\widehat{\theta} - \theta_n) \Rightarrow L_0 = \mathcal{N}(0, V_a) + \gamma'_0 Z_{hA_0}$  is within  $\epsilon^2$  of the optimal partially linear variance  $V_a$  in Equation 3.7.

**Remark 5.3** (Integral Probability Metric). Before continuing, we briefly note an interesting interpretation of the design in Equation 5.1. For distributions  $P, Q$  and a function class  $\mathcal{F}$ , the integral probability metric is  $\rho(P, Q; \mathcal{F}) \equiv \sup_{f \in \mathcal{F}} |E_P[f(X)] - E_Q[f(X)]|$ .<sup>24</sup> Let  $\mathcal{F}_B = \{\gamma' h : \gamma \in B\}$  and define empirical distributions  $\widehat{P}_d = (h_i | D_i = d)$  for  $d = 0, 1$ . We have

$$\sup_{\gamma \in B} |\gamma' \mathcal{I}_n| \leq \epsilon \iff \sqrt{n} \rho(\widehat{P}_1, \widehat{P}_0; \mathcal{F}_B) \leq \epsilon.$$

<sup>22</sup>Also if  $\text{int } B \neq \emptyset$  then  $A_0$  is bounded. See Aliprantis and Border (2006) for more on polar sets.

<sup>23</sup>Implicitly, we maximize only over Borel-measurable sets  $A \in \mathcal{B}(\mathbb{R}^{d_h})$ . The solution  $A_0$  is unique up to the equivalence class  $\{A \in \mathcal{B}(\mathbb{R}^{d_h}) : \text{Leb}(A \Delta A_0) = 0\}$ , where  $\Delta$  denotes symmetric difference.

<sup>24</sup>The pseudometric  $\rho$  is also referred to as the maximum mean discrepancy. This is a commonly used statistic in two-sample testing, see e.g. Gretton et al. (2008).

The minimax design rerandomizes until covariates  $h$  are balanced according to  $\rho(\hat{P}_1, \hat{P}_0; \mathcal{F}_B)$ , a distance between covariate distributions that is only sensitive to the projections  $\gamma'h$  that actually matter for estimating  $\theta_n = \text{SATE}$ . By doing so, we maximize acceptance probability subject to the statistical guarantee in Theorem 5.2.

## 5.2 Specifying Prior Information

Without pilot data, we must use introspection to choose the prior information set  $B$  containing  $\gamma_0$ . Intuitively,  $\gamma_0$  parameterizes how much the average outcome level  $\bar{Y}$  can change for a unit change in  $h$ , holding  $\psi$  fixed. If  $t_0(\psi) = t'\psi$  happens to be linear, then  $\bar{Y} = c + \gamma'_0 h + t'\psi + e$  and  $\gamma_0$  is an OLS coefficient. The following examples provide some reasonable prior information specifications and their associated acceptance regions, using a general characterization of acceptance regions  $A_0 = \epsilon B^\circ$  in Lemma 5.6 below.

**Example 5.4 (Rectangle).** One specification is to assume  $\gamma_{0j} \in [l_j, u_j]$  for each  $1 \leq j \leq d_h$ , setting  $B = \prod_{j=1}^{d_h} [l_j, u_j]$ . This allows for sign constraints, e.g.  $0 \leq \gamma_{0j} \leq m$  for some  $j$  and  $-m \leq \gamma_{0j} \leq 0$  for others. Lemma 5.6 below shows that if  $B = \prod_{j=1}^{d_h} [l_j, u_j]$ , then  $A_0 = \epsilon B^\circ = \{x : |x'l + x'u| + \sum_j |x_j|u_j - |x_j|l_j \leq 2\epsilon\}$ , where  $l = (l_j)_j$  and  $u = (u_j)_j$ . An example is shown in Figure 2. Note that the region  $A_0$  is conservative in the direction of the set  $B = [1, 2] \times [1, 3/2]$ , rejecting covariate imbalances that are too closely aligned with adverse coefficient values  $\gamma_0 \in B$ . The region  $A_0$  is more lenient in directions approximately orthogonal to  $B$ .

**Example 5.5 (Ellipse).** Another natural specification is to set  $B = \bar{\gamma} + B_2(0, m)$ , for an uncertainty parameter  $m$  and guess  $\gamma_0 \approx \bar{\gamma}$ . By Lemma 5.6,  $A_0 = \{x : |x'\bar{\gamma}| + m|x|_2 \leq \epsilon\}$ . More generally, if  $B = \bar{\gamma} + \Sigma B_2(0, 1)$  for a positive-definite matrix  $\Sigma$ , then  $A_0 = \{x : |x'\bar{\gamma}| + |\Sigma x|_2 \leq \epsilon\}$ . One application of this specification is when  $B$  is a Wald confidence region constructed using pilot data, as discussed below.

The following lemma provides a useful characterization of the acceptance region  $A_0 = \epsilon B^\circ$  from Theorem 5.2 for a large family of prior information set specifications. To state the lemma, recall that  $|x|_p = (\sum_j |x_j|^p)^{1/p}$  for  $p \in [1, \infty)$  and  $|x|_\infty = \max_j |x_j|$ . For  $p \in [1, \infty]$ , denote  $B_p(0, 1) = \{x : |x|_p \leq 1\}$ .

**Lemma 5.6 (Acceptance Regions).** *For  $p \in [1, \infty]$ , let  $1/p + 1/q = 1$ , setting  $q = 1$  if  $p = \infty$  and vice-versa. Suppose  $B = b + \Sigma B_p(0, 1)$ , for  $b \in \mathbb{R}^{d_h}$  and  $\Sigma$  invertible. Then  $A_0 = \{x : |x'b| + |\Sigma'x|_q \leq \epsilon\}$ .*

## 5.3 Using Pilot Data

Next, we discuss an alternative strategy that uses pilot data to specify the set  $B$ . Suppose we have access to  $\mathcal{D}_{\text{pilot}} \perp\!\!\!\perp (W_{1:n}, D_{1:n})$  of size  $m$ . Suppose  $\sqrt{m}(\hat{\gamma}_{\text{pilot}} - \gamma_0) \approx \mathcal{N}(0, \hat{\Sigma}_{\text{pilot}})$

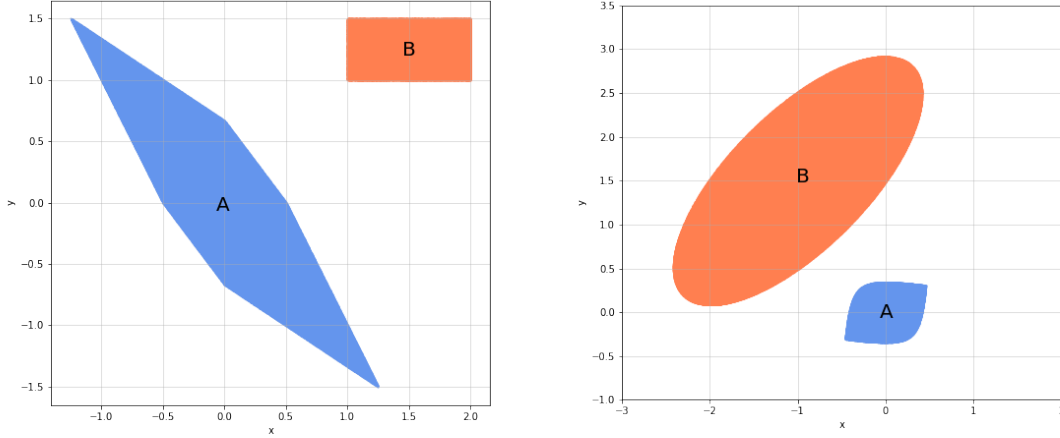


Figure 2: Prior information  $B$  and  $A_0 = \epsilon B^\circ$  for Examples 5.4 and 5.5.

for some pilot estimator  $\hat{\gamma}_{pilot}$ , discussed below. Consider forming the Wald region  $\hat{B}_{pilot} = \{\gamma : m(\hat{\gamma}_{pilot} - \gamma)' \hat{\Sigma}_{pilot}^{-1} (\hat{\gamma}_{pilot} - \gamma) \leq c_\alpha\}$  using critical value  $P(\chi_{d_h}^2 \leq c_\alpha) = 1 - \alpha$  for  $\alpha \in (0, 1)$ . Equivalently, one can write this Wald region as

$$\hat{B}_{pilot} = \hat{\gamma} + c_\alpha^{1/2} m^{-1/2} \cdot \hat{\Sigma}_{pilot}^{1/2} B_2(0, 1). \quad (5.3)$$

By Example 5.5, using  $\hat{B}_{pilot}$  as a prior information set gives acceptance region

$$\hat{A}_{pilot} = \epsilon \hat{B}_{pilot}^\circ = \{x : |x' \hat{\gamma}_{pilot}| + \frac{c_\alpha^{1/2} |\hat{\Sigma}_{pilot}^{1/2} x|_2}{m^{1/2}} \leq \epsilon\}. \quad (5.4)$$

Note that the acceptance region  $\hat{A}_{pilot}$  expands as the pilot size  $m$  is larger. This reflects smaller uncertainty about the true parameter  $\gamma_0$ , and thus less adversarial worst case imbalance  $\sup_{\gamma \in \hat{B}_{pilot}} |\gamma' \mathcal{I}_n|$ . Conversely,  $\hat{A}_{pilot}$  shrinks as the confidence parameter  $\alpha$  and the scale of the variance estimate  $\hat{\Sigma}_{pilot}$  increases, reflecting greater uncertainty and a more conservative approach to covariate balances. Our next result shows that rerandomization with acceptance region  $\hat{A}_{pilot}$  controls the variance of the residual imbalance  $R_A = \gamma_0' Z | Z \in \hat{A}_{pilot}$  with high probability, marginally over the realizations of the pilot data. The result is an immediate consequence of Theorem 3.5 and Theorem 5.2.

**Corollary 5.7** (Pilot Data). *Suppose  $P(\gamma_0 \in \hat{B}_{pilot}) \geq 1 - \alpha$ , for  $\mathcal{D}_{pilot} \perp\!\!\!\perp (W_{1:n}, D_{1:n})$ . Let  $D_{1:n}$  as in Definition 2.1 with  $A = \hat{A}_{pilot} = \epsilon \hat{B}_{pilot}^\circ$ . If Assumptions 3.1, 3.2 hold, then  $\sqrt{n}(\hat{\theta} - \theta_n) | \mathcal{D}_{pilot} \Rightarrow v_D^{-1} \mathcal{N}(0, \text{Var}(e)) + R_A$ , where  $\text{Var}(R_A | \mathcal{D}_{pilot}) \leq \epsilon^2$  with probability  $\geq 1 - \alpha$ .*

Formally, the pilot estimate of  $\gamma_0$  and Wald region could be constructed as in Robinson (1988). In practice, a simple approach suggested by the theory is to let  $\hat{\gamma}_{pilot}, \hat{\Sigma}_{pilot}$  be

point and variance estimators from the regression  $Y_T \sim 1 + h + \psi$ , for the “tyranny of the minority” (Lin (2013)) outcomes  $Y_T = (1 - p)DY/p + p(1 - D)Y/(1 - p)$ , noting that  $E[Y_T|W] = (1 - p)Y(1) + pY(0) = \bar{Y}$ .

**General Parameters.** For completeness, we extend the preceding work to general parameters  $\theta_n$  as in Definition 2.5. Let  $\Pi a(W, \theta_0)$  be the assignment influence function. As in Equation 3.7, consider the partially linear decomposition

$$\Pi a(W, \theta_0) = \gamma'_0 h + t_0(\psi) + e, \quad E[e|\psi] = 0, \quad E[eh] = 0.$$

Note that  $e \in \mathbb{R}^{d_\theta}$  and  $E[e|\psi] = 0$  is interpreted componentwise. Consider prior information sets  $B_j$  for each  $\gamma_0^j$  with  $1 \leq j \leq d_\theta$ , where  $\gamma_0^j \in \mathbb{R}^{d_h}$  is the  $j$ th column of  $\gamma_0$ . The final result of this section bounds the asymptotic imbalance term  $R_A$  if all these prior information sets are well specified.

**Theorem 5.8.** *Let  $D_{1:n}$  as in Definition 2.1 with  $A = \cap_{j=1}^{d_\theta} \epsilon B_j^\circ$ . Then  $\sqrt{n}(\hat{\theta} - \theta_n)|W_{1:n} \Rightarrow \mathcal{N}(0, V_a) + R_A$ , as defined in Theorem 3.5. If  $\gamma_0^j \in B_j \forall j$ , then  $\max_{j=1}^{d_\theta} \text{Var}((R_A)_{jj}) \leq \epsilon^2$ .*

Note that by construction the conservative acceptance region  $A = \cap_{j=1}^{d_\theta} \epsilon B_j^\circ$  is symmetric and convex.

## 6 Restoring Normality

In this section, we study optimal linearly adjusted GMM estimation under stratified rerandomization. We show that, to first order, optimal ex-post linear adjustment completely removes the impact of the acceptance region  $A$  and imbalance term  $R_A$ , restoring asymptotic normality. This enables standard t-statistic and Wald-test based inference on the parameters  $\theta_n$  and  $\theta_0$ , provided in Section 7 below.

Let  $w$  denote the covariates used for ex-post adjustment and suppose  $E[|w|_2^2] < \infty$ .

**Definition 6.1** (Adjusted GMM). Suppose that  $\hat{\alpha} \xrightarrow{p} \alpha \in \mathbb{R}^{d_w \times d_g}$ . For  $H_i = \frac{D_i - p}{p - p^2}$ . Define the linearly adjusted GMM estimator  $\hat{\theta}_{adj} = \hat{\theta} - E_n[H_i \hat{\alpha}' w_i]$ . We refer to  $\hat{\alpha}$  as the *adjustment coefficient matrix*.

First, we extend Corollary 3.6 to provide asymptotics for the adjusted GMM estimator under pure stratification ( $A = \mathbb{R}^{d_h}$ ).

**Proposition 6.2** (Linear Adjustment). *Suppose  $D_{1:n}$  as in Definition 2.1 with  $A = \mathbb{R}^{d_h}$ . Require Assumption 3.2. Then we have  $\sqrt{n}(\hat{\theta}_{adj} - \theta_n)|W_{1:n} \Rightarrow \mathcal{N}(0, V(\alpha))$  with  $V(\alpha) = v_D^{-1} E[\text{Var}(\Pi a(W, \theta_0) - \alpha' w | \psi)]$  and  $\sqrt{n}(\hat{\theta}_{adj} - \theta_0) \Rightarrow \mathcal{N}(0, V_\phi + V(\alpha))$ .*



A version of this result was given in [Cytrynbaum \(2023\)](#) for the special case  $\theta_0 = \text{ATE}$ . Motivated by Proposition 6.2, we define the optimal linear adjustment coefficient as the minimizer of the asymptotic variance  $V(\alpha)$ , in the positive semidefinite sense.

**Optimal Adjustment Coefficient.** Define the coefficient

$$\alpha_0 \in \underset{\alpha \in \mathbb{R}^{d_w \times d_g}}{\operatorname{argmin}} E[\operatorname{Var}(\Pi a(W, \theta_0) - \alpha' w | \psi)]. \quad (6.1)$$

Note that if  $w = h$  then  $\alpha_0 = \gamma_0$ , as in Theorem 3.5. If  $E[\operatorname{Var}(w | \psi)] \succ 0$ , then the unique minimizer of Equation 6.1 is the partially linear regression coefficient  $\alpha_0 = E[\operatorname{Var}(w | \psi)]^{-1} E[\operatorname{Cov}(w, \Pi a(W, \theta_0) | \psi)]$ . Observe that the optimal adjustment coefficient  $\alpha_0$  varies with the stratification variables  $\psi$ , as observed in [Cytrynbaum \(2024\)](#) and [Bai et al. \(2023\)](#) for ATE estimation. The main result of this section shows that adjustment by a consistent estimate of  $\alpha_0$  restores asymptotic normality.

**Theorem 6.3** (Restoring Normality). *Suppose  $D_{1:n}$  is rerandomized as in Definition 2.1. Require Assumption 3.1, 3.2. Let  $h \subseteq w$  and suppose  $\hat{\alpha} \xrightarrow{p} \alpha_0$ . Then  $\sqrt{n}(\hat{\theta}_{\text{adj}} - \theta_0) | W_{1:n} \Rightarrow \mathcal{N}(0, V_a^{\text{adj}})$  and  $\sqrt{n}(\hat{\theta}_{\text{adj}} - \theta_0) \Rightarrow N(0, V_\phi + V_a^{\text{adj}})$ .*

$$V_\phi = \operatorname{Var}(\Pi \phi(W, \theta_0)) \quad V_a^{\text{adj}} = \min_{\alpha \in \mathbb{R}^{d_w \times d_g}} v_D^{-1} E[\operatorname{Var}(\Pi a(W, \theta_0) - \alpha' w | \psi)].$$

**Two-step Adjustment.** For nonlinear models, the coefficient  $\alpha_0$  may depend on the unknown parameter  $\theta_0$ . This suggests a two-step adjustment strategy, where we

- (1) Use the unadjusted GMM estimator  $\hat{\theta}$  to consistently estimate  $\hat{\alpha} \xrightarrow{p} \alpha_0$ .
- (2) Report the adjusted estimator  $\hat{\theta}_{\text{adj}} = \hat{\theta} - E_n[H_i \hat{\alpha}' w_i]$ .

Similar to two-step efficient GMM, this process could be iterated until convergence to improve finite sample properties. One feasible estimator  $\hat{\alpha} \xrightarrow{p} \alpha_0$  is given in the following theorem. To state the result, define the within-group partialled covariates  $\check{w}_i = w_i - \sum_{j \in s(i)} w_j$ , where group  $s(i)$  contains unit  $i$  in Definition 2.1. Let  $\hat{\Pi} \xrightarrow{p} \Pi$  consistently estimate the linearization matrix and denote the score evaluation  $\hat{g}_i \equiv g(D_i, R_i, S_i, \hat{\theta})$ .

**Theorem 6.4** (Feasible Adjustment). *Suppose  $D_{1:n}$  is as in Definition 2.1. Require Assumption 3.1, 3.2. Assume that  $E[\operatorname{Var}(w | \psi)] \succ 0$ . Define  $\hat{\alpha} = v_D E_n[\check{w}_i \check{w}_i']^{-1} E_n[H_i \check{w}_i \hat{g}_i'] \hat{\Pi}'$ . Then  $\hat{\alpha} = \alpha_0 + o_p(1)$ .*

In some cases,  $\alpha_0$  is independent of  $\theta_0$ . For example, if  $a(W, \theta) = a_1(\psi, \theta) + a_2(W)$  then  $\alpha_0 = E[\operatorname{Var}(w | \psi)]^{-1} E[\operatorname{Cov}(w, \Pi a_2(W) | \psi)]$  does not depend on  $\theta_0$ . In such cases, one-step optimal adjustment is possible.

**Corollary 6.5** (One-step Adjustment). *Suppose  $a(W, \theta) = a_1(\psi, \theta) + a_2(W)$ . Then for any  $\theta \in \Theta$ , substituting  $g_i = g(D_i, R_i, S_i, \theta)$  for  $\hat{g}_i$  in  $\hat{\alpha}$  above, we have  $\hat{\alpha} = \alpha_0 + o_p(1)$ .*

One-step adjustment is possible in many linear GMM problems, including the best linear predictor of treatment effects parameter in Example 3.11.

**Example 6.6** (Treatment Effect Heterogeneity). Continuing Example 3.11 with score  $g(Y, D, X, \theta) = (HY - X'\theta)X$  and  $\theta_n = \operatorname{argmin}_\theta E_n[(\tau_i - X_i'\theta)^2]$ , recall that  $a(W, \theta_0) = \bar{Y}X$  and  $\Pi = E[XX']^{-1}$ . Letting  $\theta = 0$  gives  $g(Y, D, X, 0) = HYX$ . By Corollary 6.5, we have  $\hat{\alpha} = \alpha_0 + o_p(1)$  for the adjustment coefficient matrix

$$\hat{\alpha} = v_D E_n[\ddot{w}_i \ddot{w}_i']^{-1} E_n[H_i^2 Y_i \ddot{w}_i X_i'] E_n[X_i X_i']^{-1}. \quad (6.2)$$

This allows us to estimate treatment effect heterogeneity relative to a low-dimensional vector of important covariates  $X$ , adjusting optimally for a larger set of covariates  $w$  exposed in order to improve precision, as well as restore asymptotic normality when  $A \neq \mathbb{R}^{d_h}$ . In the case  $X = 1$  (SATE estimation),  $\hat{\alpha}$  is equivalent to the “tyranny-of-the-minority” style estimator proposed in Cytrynbaum (2023).

## 7 Inference

In this section, we provide novel methods for inference on general causal parameters under stratified rerandomization designs. We make crucial use of asymptotic normality of the optimally adjusted estimator  $\hat{\theta}_{adj}$ , shown in Theorem 6.3. For the superpopulation parameter  $\theta_0$ , we provide asymptotically exact inference methods. The asymptotic variance for estimating the finite population parameter  $\theta_n$  is generally not identified. In this case, we provide conservative variance estimation that still reflects the precision gains due to stratification and rerandomization.

### 7.1 Asymptotically Exact Inference

To define our variance estimator, we begin with some definitions. Let  $\mathcal{S}_n$  denote the set of groups constructed in Definition 2.1. For each  $s \in \mathcal{S}_n$  define the centroid  $\bar{\psi}_s = |s|^{-1} \sum_{i \in s} \psi_i$ . Let  $\nu : \mathcal{S}_n \rightarrow \mathcal{S}_n$  be a bijective matching between groups satisfying  $\nu(s) \neq s$ ,  $\nu^2 = \operatorname{Id}$ , and the homogeneity condition

$$\frac{1}{n} \sum_{s \in \mathcal{S}_n} |\bar{\psi}_s - \bar{\psi}_{\nu(s)}|_2^2 = o_p(1). \quad (7.1)$$

In practice,  $\nu$  is obtained by simply matching the group centroids  $\bar{\psi}_s$  into pairs using the Derigs (1988) non-bipartite matching algorithm. Let  $\mathcal{S}_n'' = \{s \cup \nu(s) : s \in \mathcal{S}_n\}$  be the unions of paired groups formed by this matching. Denote  $a(s) = \sum_{i \in s} D_i$  and  $k(s) = |s|$ . Define the adjusted moment  $\hat{m}_i \equiv \hat{\Pi} \hat{g}_i - H_i \hat{\alpha}' w_i$ , where  $\hat{g}_i \equiv g(D_i, X_i, Y_i, \hat{\theta}_{adj})$ . Suppose that  $\hat{\Pi} \xrightarrow{p} \Pi$  and  $\hat{\alpha} \xrightarrow{p} \alpha_0$  for the optimal adjustment coefficient in Equation 6.1. For

instance, we can use the consistent estimator provided by Theorem 6.4. Finally, define the variance estimator components

$$\begin{aligned}\widehat{v}_1 &= n^{-1} \sum_{s \in \mathcal{S}_n'} \frac{1}{a(s) - 1} \sum_{i \neq j \in s} \widehat{m}_i \widehat{m}_j' D_i D_j / p \\ \widehat{v}_0 &= n^{-1} \sum_{s \in \mathcal{S}_n'} \frac{1}{(k - a)(s) - 1} \sum_{i \neq j \in s} \widehat{m}_i \widehat{m}_j' (1 - D_i)(1 - D_j) / (1 - p) \\ \widehat{v}_{10} &= n^{-1} \sum_{s \in \mathcal{S}_n} \frac{k}{a(k - a)(s)} \sum_{i, j \in s} \widehat{m}_i \widehat{m}_j' D_i (1 - D_j).\end{aligned}$$

Using these terms, construct the variance estimator

$$\widehat{V} = \text{Var}_n(\widehat{m}_i) - v_D(\widehat{v}_1 + \widehat{v}_0 - \widehat{v}_{10} - \widehat{v}_{10}'). \quad (7.2)$$

We require a slight strengthening of our GMM assumptions 3.2.

**Assumption 7.1.** *There exists  $\theta_0 \in U \subseteq \Theta$  open s.t.  $E[\sup_{\theta \in U} |\partial/\partial\theta' g_d(W, \theta)|_F^2] < \infty$ .*

Under this condition, we can state our first inference result, showing consistent estimation of the asymptotic variance matrix in Theorem 6.3.

**Theorem 7.2 (Inference).** *Suppose  $D_{1:n}$  is as in Definition 2.1, and impose Assumptions 3.1, 3.2, 7.1. Then  $\widehat{V} \xrightarrow{p} V_\phi + V_a^{adj}$ .*

By Theorem 6.3,  $\sqrt{n}(\widehat{\theta}_{adj} - \theta_0) \Rightarrow N(0, V_\phi + V_a^{adj})$ . Then the variance estimation result above allows for joint inference on  $\theta_0$  using e.g. standard Wald-test or t-statistic based confidence regions.

## 7.2 Inference on the Finite Population Parameter

In this section, we provide asymptotically conservative inference on linear contrasts of the finite population parameter  $c'\theta_n$ .

As noted above, the asymptotic variance  $V_a^{adj}$  in Theorem 6.3 for estimating the finite population parameter  $\theta_n$  is generically not identified. This happens because it depends on terms of the form  $\text{Var}(a|\psi) \propto \text{Var}(g_1|\psi) + \text{Var}(g_0|\psi) - 2\text{Cov}(g_1, g_0|\psi)$ , with  $g_d = g(d, X, S(d), \theta_0)$ . However,  $S(1)$  and  $S(0)$  are never simultaneously observed (Neyman (1990)), so  $\text{Cov}(g_1, g_0|\psi)$  is generically not identified. We work with linear contrasts  $c'\theta_n$  since this allows us to tighten our upper bounds on the (non-identified) variance. To do so, let  $\widehat{u}_1 = E_n[\frac{D_i}{p} \widehat{m}_i \widehat{m}_i'] - \widehat{v}_1$  and  $\widehat{u}_0 = E_n[\frac{1-D_i}{1-p} \widehat{m}_i \widehat{m}_i'] - \widehat{v}_0$  using the estimator components above and consider the variance estimator

$$\widehat{V}_a(c) = v_D([c'\widehat{u}_1 c]^{1/2} + [c'\widehat{u}_0 c]^{1/2})^2. \quad (7.3)$$

By Theorem 6.3, we have  $\sqrt{n}(c'\hat{\theta}_{adj} - c'\theta_n)|W_{1:n} \Rightarrow \mathcal{N}(0, c'V_a^{adj}c)$ . Our next result shows how to consistently estimate an upper bound on this asymptotic variance.

**Theorem 7.3** (Inference). *Suppose  $D_{1:n}$  as in Definition 2.1 and impose Assumptions 3.1, 3.2, 7.1. Then  $\hat{V}_a(c) \xrightarrow{P} \bar{V}_a(c) \geq c'V_a^{adj}c$ .*

The variance upper bound  $\bar{V}_a(c) \geq c'(V_\phi + V_a^{adj})c$ , so the confidence intervals derived from this approach are always weakly shorter than those using the variance estimator in Equation 7.2. See Section 8.8 in the appendix for an explicit comparison. The upper bound  $\bar{V}_a(c)$  incorporates the efficiency gains from stratification, rerandomization, and adjustment. However, this upper bound is generally not sharp (Aronow et al. (2014)). We leave sharp upper bounds on the asymptotic variance matrix  $V_a^{adj}$  to future work.

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## 8 Proofs

### 8.1 Rerandomization Distribution

In what follows, we carefully distinguish between the the law of the data  $(W_{1:n}, D_{1:n})$  under “pure” stratified randomization, which we denote by  $P$ , and the law under rerandomized stratification, which we denote by  $Q$ . First, we formally define pure stratification.

**Definition 8.1** (Pure Stratification). For  $(W_i)_{i=1}^n \stackrel{\text{iid}}{\sim} F$ , let  $P$  denote the law of  $(W_{1:n}, D_{1:n})$  under the design in steps (1) and (2) of Definition 2.1, as studied in Cytrynbaum (2024).

Next, we slightly generalize the rerandomization designs introduced in Definition 2.1, which will be useful for our study of nonlinear rerandomization in Section 4. We let  $Q$  denote the law of  $(W_{1:n}, D_{1:n})$  under this design.

**Definition 8.2** (Rerandomization). Consider the following:

- (a) (Acceptance Regions). Suppose  $\mathcal{I}_n = \sqrt{n}\hat{\Delta}_h + o_p(1)$  for  $\hat{\Delta}_h = E_n[H_i h_i]$  with  $H_i = (D_i - p)/(p - p^2)$  and  $\tau_n = \tau + o_p(1)$  for  $\tau \in \mathbb{R}^{d_\tau}$  under  $P$ . Define sample acceptance region  $T_n = \{x : b(x, \tau_n) \leq 0\}$  and population region  $T = \{x : b(x, \tau) \leq 0\}$  for  $b(x, y)$  a measurable function. We accept  $D_{1:n}$  if  $\mathcal{I}_n \in T_n$ .
- (b) (Rerandomization Distribution). Let  $\mathcal{F}_n = \sigma(W_{1:n}, \pi_n)$ , where  $\pi_n \perp\!\!\!\perp W_{1:n}$  is possibly used to break ties in matching (Equation 2.1). For any event  $B$  and  $P$  as in Definition 8.1, define the rerandomization distribution

$$Q(B|\mathcal{F}_n) = P(B|\mathcal{F}_n, \mathcal{I}_n \in T_n), \quad Q(B) = E[Q(B|\mathcal{F}_n)]. \quad (8.1)$$

- (c) (Assumptions). Assume  $P(b(Z_h, \tau) = 0) = 0$  for  $Z_h \sim \mathcal{N}(0, E[\text{Var}(h|\psi)])$ . Require  $P(Z_h \in T) > 0$ . Suppose  $E[|\phi|_2^2 + |h|_2^2] < \infty$ .

Our work below shows that rerandomization as in Definition 2.1 of the main text specializes Definition 8.2 to  $b(x, y) = b(x) = d(x, A) - d(x, A^c)$  for distance function  $d(x, A) = \inf_{z \in \mathbb{R}^{d_h}} |x - z|_2$ .

The following essential lemma shows that the high level properties (e.g. convergence in probability) of  $P$  are inherited by the rerandomized version  $Q$ . The proof is given in Section 8.9 below.

**Lemma 8.3** (Dominance). *Let  $(B_n)_{n \geq 1}$  and  $(R_n)_{n \geq 1}$  events and random variables. Suppose that the rerandomization measure  $Q$  is as in Definition 8.2.*

- (a) *If  $B_n \in \mathcal{F}_n$  then  $P(B_n) = Q(B_n)$ . In particular, if a random variable  $R_n$  is  $\mathcal{F}_n$ -measurable then  $R_n = o_p(1)/O_p(1)$  under  $P \iff R_n = o_p(1)/O_p(1)$  under  $Q$ .*

(b)  $Q(B_n) = o(1)$  if  $P(B_n) = o(1)$ . If  $R_n = o_p(1)/O_p(1)$  under  $P$  then  $R_n = o_p(1)/O_p(1)$  under  $Q$ .

Equipped with this lemma, we will take the following approach: (1) show linearization of the GMM estimator  $\hat{\theta}$  about  $\theta_n$  and  $\theta_0$  under  $P$ , (2) invoke Lemma 8.3 to show these properties still hold under  $Q$ , then (3) prove distributional convergence of the simpler linearized quantities directly under  $Q$ . GMM linearization (1) is discussed in Section 8.3. For (3), the next section derives the conditional asymptotic distribution of quantities of the form  $\sqrt{n}E_n[H_i a(W_i)]$  under the rerandomization measure  $Q$ .

## 8.2 Rerandomization Asymptotics

Before studying rerandomization, we first establish a CLT for pure stratified designs, conditional on the data  $W_{1:n}$ .

**Theorem 8.4** (CLT). *Suppose  $E[|a(W)|_2^2] < \infty$ . Define  $\mathcal{F}_n = \sigma(W_{1:n}, \pi_n)$ . Let  $D_{1:n}$  as in Definition 8.1. Then  $X_n \equiv \sqrt{n}E_n[H_i a(W_i)]$  has  $X_n|\mathcal{F}_n \Rightarrow \mathcal{N}(0, V)$ . In particular, for each  $t \in \mathbb{R}^{d_a}$  we have  $E[e^{it'X_n}|\mathcal{F}_n] = \phi(t) + o_p(1)$  with  $\phi(t) = e^{-t'Vt/2}$  and  $V = v_D^{-1}E[\text{Var}(a|\psi)]$ .*

*Proof.* First consider the case  $d_g = 1$ . Define  $u_i = a_i - E[a_i|\psi_i]$ . By Lemma A.3 in Cytrynbaum (2024), since  $E[a_i^2] < \infty$  we have  $\sqrt{n}E_n[(D_i - p)E[a_i|\psi_i]] = o_p(1)$ . Then it suffices to study  $\sqrt{n}E_n[(D_i - p)u_i]$ . To do so, we will use a martingale difference sequence (MDS) CLT. Fix an ordering  $l = 1, \dots, n/k$  of  $s(l) \in \mathcal{S}_n$ , noting that  $|\mathcal{S}_n| \leq n/k$ . Define  $D_{s(l)} = (D_i)_{i \in s(l)}$ . Define  $\mathcal{H}_{0,n} = \mathcal{F}_n$  and  $\mathcal{H}_{j,n} = \sigma(\mathcal{F}_n, D_{s(l)}, l \in [j])$  for  $j \geq 1$ . Define  $D_{l,n} = n^{-1/2} \sum_{i \in s(l)} (D_i - p)u_i$  and  $S_{j,n} = \sum_{i=1}^j D_{i,n}$ .

(1) We claim that  $(S_{j,n}, \mathcal{H}_{j,n})_{j \geq 1}$  is an MDS. Adaptation is clear from our definitions.

$$\begin{aligned} E[(D_i - p)\mathbf{1}(i \in s(j))|\mathcal{H}_{j-1,n}] &= E[(D_i - p)\mathbf{1}(i \in s(j))|\mathcal{F}_n, (D_{s(l)})_{l=1}^{j-1}] \\ &= E[(D_i - p)\mathbf{1}(i \in s(j))|\mathcal{F}_n] = E[(D_i - p)|\mathcal{F}_n]\mathbf{1}(i \in s(j)) = 0. \end{aligned}$$

The second equality since  $D_{s(j)} \perp\!\!\!\perp (D_{s(l)})_{l \neq j}|\mathcal{F}_n$ . Then we compute  $E[Z_{j,n}|\mathcal{H}_{j-1,n}] = n^{-1/2} \sum_{i \in s(l)} u_i E[(D_i - p)|\mathcal{H}_{j-1,n}] = 0$ . This shows the MDS property.

(2). Next, we compute the variance process. By the same argument in (1), we have

$$\sigma_n^2 \equiv \sum_{j=1}^{n/k} E[Z_{j,n}^2|\mathcal{H}_{j-1,n}] = n^{-1} \sum_{j=1}^{n/k} \left( \sum_{r \neq t \in s(j)} u_r u_t \text{Cov}(D_s, D_t|\mathcal{F}_n) + \sum_{i \in s(j)} u_i^2 \text{Var}(D_i|\mathcal{F}_n) \right)$$

By Lemma C.10 of Cytrynbaum (2024), we have  $\text{Cov}(D_s, D_t|\mathcal{F}_n)\mathbf{1}(s, t \in s(l)) = -l(k -$

$l)/k^2(k-1) \equiv c$  and  $\text{Var}(D_i|\mathcal{F}_n) = p - p^2$ . Then we may expand  $\sigma_n^2$  as

$$cn^{-1} \sum_{j=1}^{n/k} \sum_{r \neq t \in s(j)} u_r u_t + (p - p^2) E_n[u_i^2] \equiv cn^{-1} \sum_{j=1}^{n/k} v_j + (p - p^2) E_n[u_i^2] \equiv T_{n1} + T_{n2}.$$

First consider  $T_{n1}$ . Our plan is to apply the WLLN in Lemma C.7 of [Cytrynbaum \(2024\)](#) to show  $T_{n1} = o_p(1)$ . Define  $\mathcal{F}_n^\psi = \sigma(\psi_{1:n}, \pi_n)$  so that  $\mathcal{S}_n \in \mathcal{F}_n^\psi$ . For  $r \neq t$  we have  $E[u_r u_t | \psi_{1:n}, \pi_n] = E[u_r E[u_t | \psi_{1:n}, u_r, \pi_n] | \psi_{1:n}, \pi_n] = E[u_r E[u_t | \psi_t] | \psi_{1:n}, \pi_n] = 0$ . The second equality follows by applying  $(A, B) \perp\!\!\!\perp C \implies A \perp\!\!\!\perp C | B$  with  $A = u_t$ ,  $B = \psi_t$  and  $C = (\psi_{-t}, u_r, \pi_n)$ . Then  $E[v_j | \mathcal{F}_n^\psi] = 0$  for  $j \in [n/k]$ . Next, observe that for any positive constants  $(a_k)_{k=1}^m$  we have  $\sum_k a_k \mathbb{1}(\sum_k a_k > c) \leq m \sum_k a_k \mathbb{1}(a_k > c/m)$  and  $ab \mathbb{1}(ab > c) \leq a^2 \mathbb{1}(a^2 > c) + b^2 \mathbb{1}(b^2 > c)$ . Then for  $c_n \rightarrow \infty$  with  $c_n = o(\sqrt{n})$  we have

$$\begin{aligned} |v_j| \mathbb{1}(|v_j| > c_n) &\leq \sum_{r \neq t \in s(j)} |u_r u_t| \mathbb{1} \left( \sum_{r \neq t \in s(j)} |u_r u_t| > c_n \right) \\ &\leq k^2 \sum_{r \neq t \in s(j)} |u_r u_t| \mathbb{1}(|u_r u_t| > c_n/k^2) \leq 2k^3 \sum_{r \in s(j)} u_r^2 \mathbb{1}(u_r^2 > c_n/k^2). \end{aligned}$$

Then we have

$$n^{-1} E \left[ \sum_{j=1}^{n/k} E[|v_j| \mathbb{1}(|v_j| > c_n) | \mathcal{F}_n^\psi] \right] \leq 2k^3 E_n [E[u_i^2 \mathbb{1}(u_i^2 > c_n/k^2) | \psi_{1:n}, \pi_n]] \equiv A_n.$$

Then  $E[A_n] = 2k^3 E[E_n[E[u_i^2 \mathbb{1}(u_i^2 > c_n/k^2) | \psi_i]]] = 2k^3 E[u_i^2 \mathbb{1}(u_i^2 > c_n/k^2)] \rightarrow 0$  as  $n \rightarrow \infty$ . The first equality is by the conditional independence argument above, the second equality is tower law, and the limit by dominated convergence since  $E[u_i^2] \leq E[a_i^2] < \infty$  by the contraction property of conditional expectation. Then  $A_n = o_p(1)$  by Markov inequality. The conclusion  $cn^{-1} \sum_{j=1}^{n/k} v_j = o_p(1)$  now follows by Lemma C.7 of [Cytrynbaum \(2024\)](#). For  $T_{n2}$ , we have  $E_n[u_i^2] \xrightarrow{p} E[u_i^2] = E[\text{Var}(a|\psi)]$  by vanilla WLLN. Then we have shown  $\sigma_n^2 \xrightarrow{p} (p - p^2) E[\text{Var}(a|\psi)]$ .

(3) Finally, we show the Lindberg condition  $\sum_{j=1}^{n/k} E[Z_{j,n}^2 \mathbb{1}(|Z_{j,n}| > \epsilon) | \mathcal{H}_{0,n}] = o_p(1)$ .

$$\begin{aligned} Z_{j,n}^2 \mathbb{1}(|Z_{j,n}| > \epsilon) &= Z_{j,n}^2 \mathbb{1}(Z_{j,n}^2 > \epsilon^2) \leq n^{-1} \sum_{r,t \in s(j)} |u_r u_t| \mathbb{1} \left( n^{-1} \sum_{r,t \in s(j)} |u_r u_t| > \epsilon^2 \right) \\ &\leq k^2 n^{-1} \sum_{r,t \in s(j)} |u_r u_t| \mathbb{1}(|u_r u_t| > n\epsilon^2/k^2) \leq k^3 n^{-1} \sum_{r \in s(j)} u_r^2 \mathbb{1}(u_r^2 > n\epsilon^2/k^2). \end{aligned}$$

Then using the inequality above we compute

$$\begin{aligned} E \left[ \sum_{j=1}^{n/k} E[Z_{j,n}^2 \mathbf{1}(|Z_{j,n}| > \epsilon) | \mathcal{H}_{0,n}] \right] &\leq k^3 E \left[ n^{-1} \sum_{j=1}^{n/k} \sum_{r \in s(j)} E[u_r^2 \mathbf{1}(u_r^2 > n\epsilon^2/k^2) | \mathcal{F}_n^\psi] \right] \\ &= k^3 E \left[ E_n \left[ E[u_i^2 \mathbf{1}(u_i^2 > n\epsilon^2/k^2) | \psi_i] \right] \right] = k^3 E \left[ u_i^2 \mathbf{1}(u_i^2 > n\epsilon^2/k^2) \right] = o(1). \end{aligned}$$

The first equality by the conditional independence argument above. The second equality by dominated convergence. Then  $\sum_{j=1}^{n/k} E[Z_{j,n}^2 \mathbf{1}(|Z_{j,n}| > \epsilon) | \mathcal{H}_{0,n}] = o_p(1)$  by Markov. This finishes the proof of the Lindberg condition. Since  $\mathcal{H}_{0,n} = \mathcal{F}_n$ , by Theorem C.4 in [Cytrynbaum \(2024\)](#), we have shown that  $E[e^{it\sqrt{n}E_n[(D_i-p)a_i]} | \mathcal{F}_n] = \phi(t) + o_p(1)$  for  $\phi(t) = e^{-t^2V/2}$  with  $V = (p-p^2)E[\text{Var}(a|\psi)]$ .

Finally, consider  $\dim(a) \geq 1$ . Fix  $t \in \mathbb{R}^d$  and let  $\bar{a}(W_i) = t'a(W_i) \in \mathbb{R}$ . Then we have  $X_n(t) \equiv X'_n t = E_n[(D_i - p)a(W_i)]'t = E_n[(D_i - p)a(W_i)'t] = E_n[(D_i - p)\bar{a}(W_i)]$ . By the previous result  $E[e^{iX_n(t)} | \mathcal{F}_n] \xrightarrow{p} e^{-v(t)/2}$  with variance  $v(t) = E[\text{Var}(\bar{a}|\psi)] = E[\text{Var}(t'a|\psi)] = t'E[\text{Var}(a|\psi)]t = t'Vt$ . Then we have shown  $E[e^{it'X_n} | \mathcal{F}_n] = e^{-t'Vt/2} + o_p(1)$  as claimed.  $\square$

Next, we provide asymptotic theory for stratified rerandomization. The following definition generalizes Definition 2.1 in Section 1.

**Lemma 8.5.** *Let Definition 8.2 hold. Let  $\hat{\Delta}_a = E_n[H_i a_i]$  and  $\rho = (a, h)$ . Fix  $t \in \mathbb{R}^{d_a}$ . Let  $(Z_a, Z_h) \sim \mathcal{N}(0, \Sigma)$  for  $\Sigma = v_D^{-1}E[\text{Var}(\rho|\psi)]$ . Then under  $P$  in Definition 8.1*

$$E \left[ e^{it'\sqrt{n}\hat{\Delta}_a} \mathbf{1}(\mathcal{I}_n \in T_n) | \mathcal{F}_n \right] = E \left[ e^{it'Z_a} \mathbf{1}(Z_h \in T) \right] + o_p(1).$$

*Proof.* (1). Define  $B_n = (\sqrt{n}\hat{\Delta}_a, \mathcal{I}_n, \tau_n)$ . Fix  $t = (t_1, t_2, t_3) \in \mathbb{R}^{d_g+d_h+d_\tau}$  and consider the characteristic function

$$\begin{aligned} \phi_{B_n}(t) &= E[e^{it'_1\sqrt{n}\hat{\Delta}_a+it'_2\mathcal{I}_n+it'_3\tau_n} | \mathcal{F}_n] = e^{it'_3\tau} E[e^{it'_1\sqrt{n}\hat{\Delta}_a+it'_2\mathcal{I}_n} | \mathcal{F}_n] + o_p(1) \\ &= e^{it'_3\tau} E[e^{it'_1\sqrt{n}\hat{\Delta}_a+it'_2\sqrt{n}\hat{\Delta}_h} | \mathcal{F}_n] + o_p(1) = e^{it'_3\tau} e^{-t'\Sigma t/2} + o_p(1) = \phi_B(t) + o_p(1). \end{aligned}$$

For the second equality, note that  $e^{it'_3\tau_n} \xrightarrow{p} e^{it'_3\tau}$  by continuous mapping. Then  $R_n = e^{it'_1\sqrt{n}\hat{\Delta}_a+it'_2\sqrt{n}\hat{\Delta}_h}(e^{it'_3\tau_n} - e^{it'_3\tau}) = o_p(1)$ . Clearly  $|R_n| \leq 2$ , so  $E[|R_n| | \mathcal{F}_n] = o_p(1)$  by Lemma 8.19. The third equality is identical, noting that  $e^{it'_2\mathcal{I}_n} \xrightarrow{p} e^{it'_2\sqrt{n}\hat{\Delta}_h}$  again by continuous mapping. The fourth equality is Theorem 8.4 applied to  $\sqrt{n}E_n[H_i \rho_i]$ . The final expression is the characteristic function of  $B = (Z_a, Z_h, \tau)$  with  $(Z_a, Z_h) \sim \mathcal{N}(0, \Sigma)$ . Then we have shown that  $B_n | \mathcal{F}_n \Rightarrow B$  in the sense of Proposition 8.16. Fix  $t \in \mathbb{R}$  and define  $G(z_1, z_2, x) = e^{it'z_1} \mathbf{1}(b(z_2, x) \leq 0)$  and note that

$$G(B_n) = e^{it'\sqrt{n}\hat{\Delta}_a} \mathbf{1}(b(\mathcal{I}_n, \tau_n) \leq 0) = e^{it'\sqrt{n}\hat{\Delta}_a} \mathbf{1}(\mathcal{I}_n \in T_n).$$

Define  $E_G = \{w : G(\cdot)$  not continuous at  $w\}$ . By Proposition 8.16, if  $P(B \in E_G) = 0$  then  $E[G(B_n)|\mathcal{F}_n] = E[G(B)] + o_p(1) = E[G(Z_a, Z_h, \tau)] + o_p(1)$ , which is the required claim.

To finish the proof, we show that  $P(B \in E_G) = 0$ . Write  $G(z_1, z_2, x) = f(z_1)g(z_2, x)$  for  $f(z_1) = e^{it'z_1}$  and  $g(z_2, x) = \mathbb{1}(b(z_2, x) \leq 0)$  and define discontinuity point sets  $E_f$  and  $E_g$  as for  $E_G$  above. By continuity of multiplication for bounded functions, if  $z_1 \in E_f^c$  and  $(z_2, x) \in E_g^c$  then  $(z_1, z_2, x) \in E_G^c$ . By contrapositive,

$$E_G \subseteq (E_f \times \mathbb{R}^{d_h+d_\tau}) \cup (\mathbb{R} \times E_g).$$

Clearly  $E_f = \emptyset$ , so  $P(B \in E_G) = P((Z_h, \tau) \in E_g)$ . Let  $E_g^1 = \{z_h : (z_h, \tau) \in E_g\}$ . We have  $(Z_h, \tau) \in \mathbb{R}^{d_h} \times \{\tau\}$ . Then  $P((Z_h, \tau) \in E_g) = P(Z_h \in E_g^1)$ . Since  $z_h \rightarrow b(z_h, \tau)$  is continuous,  $\{z_h : b(z_h, \tau) > 0\}$  is open. Let  $z_h \in \{z_h : b(z_h, \tau) > 0\}$ . Then for small enough  $r$ , if  $z' \in B(z_h, r)$  then  $b(z', \tau) > 0$  and  $g(z', \tau) = 0$ , so  $g(z', \tau) - g(z_h, \tau) = 0$ , so  $z_h$  is a continuity point. A similar argument applied to  $z_h \in \{z_h : b(z_h, \tau) < 0\}$  shows that the discontinuity points  $E_g^1 \subseteq \{z_h : b(z_h, \tau) = 0\}$ .  $\square$

**Theorem 8.6** (Asymptotic Distribution). *Let Definition 8.2 hold. Suppose that  $(Z_a, Z_h) \sim v_D^{-1}E[\text{Var}((a, h)|\psi)]$ . Then under  $Q$  in Definition 8.2 the following hold:*

(a) *We have  $\sqrt{n}E_n[H_ia(W_i)]|\mathcal{F}_n \Rightarrow Z_a|Z_h \in T = \mathcal{N}(0, V_a) + R$ , independent RV's s.t.*

$$V_a = v_D^{-1}E[\text{Var}(a(W) - \gamma'_0 h|\psi)] = \min_{\gamma \in \mathbb{R}^{d_h \times d_\theta}} v_D^{-1}E[\text{Var}(a(W) - \gamma' h|\psi)].$$

*The residual term  $R \sim \gamma'_0 Z_h | Z_h \in T$ .*

(b) *Let  $X_n = E_n[\phi(W_i)] + E_n[H_ia(W_i)]$ . Then we have*

$$\sqrt{n}(X_n - E[\phi(W)]) \Rightarrow Z_\phi + Z_a|Z_h \in T = \mathcal{N}(0, V_\phi) + \mathcal{N}(0, V_a) + R.$$

*The RV's are independent with  $V_\phi = \text{Var}(\phi(W))$ .*

*Proof.* First, we prove (a). Let  $\hat{\Delta}_a = E_n[H_ia(W_i)]$ . Let  $t \in \mathbb{R}^{d_a}$ . By definition of  $Q$

$$E_Q \left[ e^{it' \sqrt{n} \hat{\Delta}_a} | \mathcal{F}_n \right] = E \left[ e^{it' \sqrt{n} \hat{\Delta}_a} | \mathcal{I}_n \in T_n, \mathcal{F}_n \right] = \frac{E \left[ e^{it' \sqrt{n} \hat{\Delta}_a} \mathbb{1}(\mathcal{I}_n \in T_n) | \mathcal{F}_n \right]}{P(\mathcal{I}_n \in T_n | \mathcal{F}_n)} \equiv \frac{a_n}{b_n}.$$

Define  $a_\infty = E \left[ e^{it' Z_a} \mathbb{1}(Z_h \in T) \right]$  and  $b_\infty = P(Z_h \in T)$ . By Lemma 8.5,  $a_n \xrightarrow{p} a_\infty$  and  $b_n \xrightarrow{p} b_\infty$ , with  $b_\infty > 0$  by assumption in Definition 8.2. Then we have  $b_n^{-1} = O_p(1)$ . Then  $|a_n/b_n - a_\infty/b_\infty|$  may be expanded as  $\left| \frac{a_n b_\infty - a_\infty b_n}{b_n b_\infty} \right| = O_p(1) |(a_n - a_\infty)b_\infty + a_\infty(b_\infty - b_n)| \lesssim_P$

$|a_n - a_\infty| + |b_\infty - b_n| = o_p(1)$ . The final equality by Lemma 8.5. Then we have shown

$$E_Q [e^{itA_n} | \mathcal{F}_n] = \frac{a_\infty}{b_\infty} + o_p(1) = \frac{E [e^{it'Z_a} \mathbf{1}(Z_h \in T)]}{P(Z_h \in T)} = E[e^{it'Z_a} | Z_h \in T] + o_p(1).$$

This proves the first statement. Next, we characterize the law of  $Z_a | Z_h \in T$ . Define  $\phi(t) \equiv E [e^{it'Z_a} | Z_h \in T]$ . Let  $\gamma_0 \in \mathbb{R}^{d_h \times d_g}$  satisfy the normal equations  $E[\text{Var}(h|\psi)]\gamma_0 = E[\text{Cov}(h, a|\psi)]$ . Such a  $\gamma_0$  exists and satisfies the stated inequality by Lemma 8.17. Letting  $\tilde{Z}_a = Z_a - \gamma_0' Z_h$ , by Lemma 8.17  $\tilde{Z}_a \perp\!\!\!\perp Z_h$  and  $\tilde{Z}_a$  is Gaussian. Then  $\tilde{Z}_a \perp\!\!\!\perp (Z_h, \mathbf{1}(Z_h \in T))$ . Recall that  $A \perp\!\!\!\perp (S, T) \implies A \perp\!\!\!\perp S | T$ . Using this fact, we have  $\tilde{Z}_a \perp\!\!\!\perp Z_h | Z_h \in T$ . Then for any  $t \in \mathbb{R}^{d_g}$

$$\begin{aligned} \phi(t) &= E[e^{it'Z_a} | Z_h \in T] = E[e^{it'\tilde{Z}_a} e^{it'\gamma_0'Z_h} | Z_h \in T] \\ &= E[e^{it'\tilde{Z}_a} | Z_h \in T] E[e^{it'\gamma_0'Z_h} | Z_h \in T] = E[e^{it'\tilde{Z}_a}] E[e^{it'\gamma_0'Z_h} | Z_h \in T]. \end{aligned}$$

By Proposition 8.16, we have shown  $Z_a | Z_h \in T \stackrel{d}{=} \tilde{Z}_a + [\gamma_0' Z_h | Z_h \in T]$ , where the RHS is a sum of independent random variables with the given distributions. Clearly  $E[\tilde{Z}_a] = 0$  and  $\text{Var}(\tilde{Z}_a) = v_D^{-1} E[\text{Var}(a - \gamma_0' h | \psi)]$ . This finishes the proof of (a).

Next we prove (b). We may expand  $\sqrt{n}(X_n - E[\phi(W)]) = \sqrt{n}(E_n[\phi(W_i)] - E[\phi(W)]) + \sqrt{n}\hat{\Delta}_a \equiv A_n + B_n$ . We have  $A_n \Rightarrow \mathcal{N}(0, V_\phi)$  with  $V_\phi = \text{Var}(\phi(W))$  by vanilla CLT. Then let  $t \in \mathbb{R}^{d_a}$  and calculate

$$E_Q [e^{it'X_n}] = E_Q [e^{it'A_n} E_Q [e^{it'B_n} | \mathcal{F}_n]] = \phi(t) E_Q [e^{it'A_n}] + o(1) = \phi(t) e^{-t'V_\phi t/2} + o(1).$$

The first equality since  $A_n \in \mathcal{F}_n$ . The second equality since

$$\left| E_Q [e^{it'A_n} (E_Q [e^{it'B_n} | \mathcal{F}_n] - \phi(t))] \right| \leq E_Q [|E_Q [e^{it'B_n} | \mathcal{F}_n] - \phi(t)|] = o(1).$$

To see this, note that the integrand is  $o_p(1)$  by our work above. It is also bounded so it converges to zero in  $L_1(Q)$  by Lemma 8.19. The final equality since  $A_n \in \mathcal{F}_n = \sigma(W_{1:n}, \pi_n)$  and the marginal distribution of  $(W_{1:n}, \pi_n)$  is identical under  $P$  and  $Q$  by definition. Then  $E_Q [e^{it'A_n}] = E_P [e^{it'A_n}] = e^{-t'V_\phi t/2} + o(1)$  by vanilla CLT. Then we have shown

$$E_Q [e^{it'X_n}] = e^{-t'(V_\phi + V_a)t/2} E[e^{it'\gamma_0'Z_h} | Z_h \in B] + o(1).$$

This finishes the proof of (b).  $\square$

**Lemma 8.7** (Linearization). *Suppose Definition 8.2 and Assumption 3.2 hold. Let  $\Pi = -(G'MG)^{-1}G'M$ . Then  $\sqrt{n}(\hat{\theta} - \theta_n) = \sqrt{n}E_n[H_i\Pi a(W_i, \theta_0)] + o_p(1)$  and  $\sqrt{n}(\hat{\theta} - \theta_0) = \sqrt{n}E_n[\Pi\phi(W_i, \theta_0) + H_i\Pi a(W_i, \theta_0)] + o_p(1)$ .*

See Section 8.3 below for the proof of this lemma.

*Proof of Theorem 3.5.* We claim that the conditions of Definition 8.2 hold. This will allow us to apply our general rerandomization asymptotics in Theorem 8.6 and linearization in Lemma 8.7. To check part (a), define  $b(x, y) = b(x) = d(x, A) - d(x, A^c)$ , where  $d(x, A) = \inf_{s \in \mathbb{R}^{d_h}} |x - s|_2$ . It's well known that  $x \rightarrow d(x, S)$  is continuous for any set  $S$ , so  $b$  is continuous. The sample and population regions  $T_n = T = \{x : b(x) \leq 0\}$ . If  $b(x) \leq 0$  then  $d(x, A) = 0$ , so  $x \in A \cup \partial A \subseteq A$  by closedness. If  $b(x) > 0$  then  $x \notin A$ . This shows  $T_n = A$ , so  $\{\mathcal{I}_n \in T_n\} = \{\mathcal{I}_n \in A\}$ . Then our criterion is of the form in Definition 8.2. For part (b),  $P(b(Z_h) = 0) = P(Z_h \in \partial A) = 0$  since  $\text{Leb}(\partial A) = 0$  and by absolute continuity of  $Z_h$  relative to Lebesgue measure  $\text{Leb}$ . We also have  $P(Z_h \in T) = P(Z_h \in A) > 0$  since  $Z_h$  is full measure by  $E[\text{Var}(h|\psi)] \succ 0$  and since  $A$  has non-empty interior.

This proves the claim. Then by Lemma 8.7,  $\sqrt{n}(\hat{\theta} - \theta_n) = \sqrt{n}E_n[H_i \Pi a(W_i, \theta_0)] + o_p(1)$ . The result now follows immediately by Slutsky and Theorem 8.6(a), letting  $a \rightarrow \Pi a$ . Likewise, Corollary 3.8 follows from Theorem 8.6(b), letting  $\phi \rightarrow \Pi \phi$ .  $\square$

*Proof of Corollary 3.6.* By Theorem 3.5, since  $A = \mathbb{R}^{d_h}$  we have  $\sqrt{n}(\hat{\theta} - \theta_n)|W_{1:n} \Rightarrow \mathcal{N}(0, V_a) + R$ , independent RV's with  $V_a = v_D^{-1}E[\text{Var}(\Pi a(W, \theta_0) - \gamma'_0 h|\psi)]$  and  $R \sim \gamma'_0 Z_h$  for  $Z_h \sim \mathcal{N}(0, v_D^{-1}E[\text{Var}(h|\psi)])$ . Then  $\mathcal{N}(0, V_a) + R \sim \mathcal{N}(0, V)$  with  $V = V_a + \text{Var}(\gamma'_0 Z_h) = v_D^{-1}E[\text{Var}(\Pi a(W, \theta_0) - \gamma'_0 h + \gamma'_0 h|\psi)] - 2v_D^{-1}E[\text{Cov}(\Pi a(W, \theta_0) - \gamma'_0 h, \gamma'_0 h|\psi)] = v_D^{-1}E[\text{Var}(\Pi a(W, \theta_0)|\psi)]$ . The covariance term is zero by Lemma 8.17. The second statement follows by setting  $\psi = 1$ .  $\square$

### 8.3 GMM Linearization

This section collects proofs needed for the key linearization result in Lemma 8.7. First, define the following curves and objective functions

$$\begin{aligned} g_0(\theta) &= E[\phi(W_i, \theta)], & g_n(\theta) &= E_n[\phi(W_i, \theta)], & \hat{g}(\theta) &= E_n[\phi(W_i, \theta)] + E_n[H_i a(W_i, \theta)], \\ H_0(\theta) &= g_0(\theta)' M g_0(\theta), & H_n(\theta) &= g_n(\theta)' M g_n(\theta), & \hat{H}(\theta) &= \hat{g}(\theta)' M_n \hat{g}(\theta) \end{aligned}$$

Define  $\hat{G}(\theta) = (\partial/\partial\theta')\hat{g}(\theta)$  and  $G_n(\theta) = (\partial/\partial\theta')g_n(\theta)$  and  $G_0(\theta) = (\partial/\partial\theta')g_0(\theta)$ . Define  $G = G_0(\theta_0)$ . For each  $d \in \{0, 1\}$ , define  $g_d(W, \theta) = g(d, X, S(d), \theta)$ .

**Lemma 8.8** (ULLN). *Working under  $P$  in Definition 8.1:*

- (a) *If Assumption 3.2(b) holds,  $\|\hat{g} - g_0\|_{\infty, \Theta} = o_p(1)$ ,  $\|g_n - g_0\|_{\infty, \Theta} = o_p(1)$ , and  $g_0(\theta)$  is continuous. If also  $M_n \xrightarrow{P} M$  then  $|H_n - H_0|_{\infty, \Theta} = o_p(1)$  and  $|\hat{H} - H_0|_{\infty, \Theta} = o_p(1)$ .*
- (b) *If Assumption 3.2(c) holds, then there is an open ball  $U \subseteq \Theta$  with  $\theta_0 \in U$  and  $\|\hat{G}_n - G_0\|_{\infty, U} = o_p(1)$  and  $\|G_n - G_0\|_{\infty, U} = o_p(1)$ . Also,  $G_0(\theta)$  is continuous on  $U$  for  $G_0(\theta) = \partial/\partial\theta' E[\phi(W, \theta)]$ .*

*Proof.* Consider (a). First we show  $\|\hat{g} - g_0\|_{\infty, \Theta} = o_p(1)$ , modifying the approach used in the iid setting in Tauchen (1985). It suffices to prove the statement componentwise. Then



without loss assume  $d_g = 1$  and fix  $\epsilon > 0$ . Note also that  $\phi, a$  are linear combinations of  $g_d$  for  $d \in \{0, 1\}$ , so  $\phi$  and  $a$  inherit the properties in Assumption 3.2. We have  $(\hat{g} - g_n)(\theta) = E_n[H_i a(W_i, \theta)]$ . For each  $\theta \in K$  define  $U_{\theta m} = B(\theta, m^{-1})$  and  $\bar{v}_{\theta m}(D_i, W_i) = \sup_{\bar{\theta} \in U_{\theta m}} H_i a(W_i, \bar{\theta})$ . Then  $\bar{v}_{\theta m}(D_i, W_i)$  may be expanded

$$\begin{aligned} \sup_{\bar{\theta} \in U_{\theta m}} H_i a(W_i, \bar{\theta}) &= \frac{D_i}{p} \sup_{\bar{\theta} \in U_{\theta m}} a(W_i, \bar{\theta}) + \frac{1 - D_i}{1 - p} \sup_{\bar{\theta} \in U_{\theta m}} -a(W_i, \bar{\theta}) \\ &= \sup_{\bar{\theta} \in U_{\theta m}} a(W_i, \bar{\theta}) + \sup_{\bar{\theta} \in U_{\theta m}} -a(W_i, \bar{\theta}) \\ &+ H_i((1 - p) \sup_{\bar{\theta} \in U_{\theta m}} a(W_i, \bar{\theta}) + p \inf_{\bar{\theta} \in U_{\theta m}} a(W_i, \bar{\theta})) \equiv f_{\theta m}(W_i) + H_i r_{\theta m}(W_i). \end{aligned}$$

In particular,  $E[\bar{v}_{\theta m}(X_i)] = E[f_{\theta m}(W_i)]$ . Note both expectations exist by the envelope condition in Assumption 3.2. By continuity at  $\theta$ ,  $f_{\theta m}(W_i) \rightarrow a(W_i, \theta) - a(W_i, \theta) = 0$  as  $m \rightarrow \infty$ . Also  $|f_{\theta m}(W_i)| \lesssim \sup_{\bar{\theta} \in U_{\theta m}} |a(W_i, \bar{\theta})| \leq \sup_{\theta \in \Theta} |a(W_i, \theta)|$ . Then by our envelope assumption  $\sup_m f_{\theta m}(W_i) \in L_1(P)$ , so  $\lim_m E[\bar{v}_{\theta m}(D_i, W_i)] = \lim_m E[f_{\theta m}(W_i)] = 0$  by dominated convergence. For each  $\theta$ , let  $m(\theta)$  s.t.  $E[f_{\theta m(\theta)}(W_i)] \leq \epsilon$ . Then  $\{U_{\theta m(\theta)} : \theta \in \Theta\}$  is an open cover of  $\Theta$ , so by compactness it admits a finite subcover  $\{U_{\theta_l, m(\theta_l)}\}_{l=1}^{L(\epsilon)} \equiv \{U_l\}_{l=1}^{L(\epsilon)}$ . Next, for each  $(\theta, m)$  we claim  $E_n[\bar{v}_{\theta m}(D_i, W_i)] = E[f_{\theta m}(W_i)] + o_p(1)$ . We have  $E_n[f_{\theta m}(W_i)] = E[f_{\theta m}(W_i)] + o_p(1)$  by WLLN since  $E[f_{\theta m}(W_i)] < \infty$  as just shown. Similarly, we have

$$|r_{\theta m}(W_i)| = |(1 - p) \sup_{\bar{\theta} \in U_{\theta m}} a(W_i, \bar{\theta}) + p \inf_{\bar{\theta} \in U_{\theta m}} a(W_i, \bar{\theta})| \leq \sup_{\bar{\theta} \in U_{\theta m}} |a(W_i, \bar{\theta})| \in L_1(P).$$

Then  $E_n[H_i r_{\theta m}(W_i)] = o_p(1)$  by Lemma A.2 in Cytrynbaum (2024). This proves the claim. Define  $f_l(W)$  and  $r_l(W)$  to be the functions above evaluated at  $(\theta_l, m(\theta_l))$ . Putting this all together, we have

$$\begin{aligned} \sup_{\theta \in K} E_n[H_i a(W_i, \theta)] &\leq \max_{l=1}^{L(\epsilon)} \sup_{\theta \in U_l} E_n[H_i a(W_i, \theta)] \leq \max_{l=1}^{L(\epsilon)} E_n[v_{\theta_l m(\theta_l)}(D_i, W_i)] \\ &= \max_{l=1}^{L(\epsilon)} (E[f_{\theta_l m(\theta_l)}(W_i)] + T_{nl}) \leq \epsilon + \max_{l=1}^{L(\epsilon)} T_{nl} = \epsilon + o_p(1). \end{aligned}$$

By symmetry, also  $\sup_{\theta \in K} -E_n[H_i a(W_i, \theta)] \leq \epsilon + o_p(1)$ . Then  $\sup_{\theta \in K} |E_n[H_i a(W_i, \theta)]| \leq 2\epsilon + o_p(1)$ . Since  $\epsilon > 0$  was arbitrary, this finishes the proof of (1).

Next we show  $\|g_n - g_0\|_{\infty, \Theta} = o_p(1)$ . We have  $(g_n - g_0)(\theta) = E_n[\phi(W_i, \theta)] - E[\phi(W, \theta)]$ . Under our assumptions,  $|E_n[\phi(W_i, \theta)] - E[\phi(W, \theta)]|_{\infty, \Theta} = o_p(1)$  and  $g_0(\theta) = E[\phi(W, \theta)]$  is continuous by Lemma 2.4 of Newey and McFadden (1994). This proves the second claim.

For the statement about objective functions, observe that

$$\begin{aligned}
|\widehat{H}(\theta) - H_n(\theta)| &= |\widehat{g}(\theta)' M_n \widehat{g}(\theta) - g_n(\theta)' M g_n(\theta)| \leq |(\widehat{g} - g_n)(\theta)' M_n \widehat{g}(\theta)| \\
&+ |g_n(\theta)' (M_n - M) \widehat{g}(\theta)| + |g_n(\theta)' M (\widehat{g} - g_n)(\theta)| \leq |\widehat{g} - g_n|_2(\theta) \|M_n\|_2 |\widehat{g}(\theta)|_2 \\
&+ |g_n(\theta)|_2 \|M_n - M\|_2 |\widehat{g}(\theta)|_2 + |g_n(\theta)|_2 \|M\|_2 |\widehat{g} - g_n|_2(\theta) \lesssim |\widehat{g} - g_n|_{\infty, \Theta} \|M_n\|_2 |\widehat{g}|_{\infty, \Theta} \\
&+ |g_n|_{\infty, \Theta} \|M_n - M\|_2 |\widehat{g}|_{\infty, \Theta} + |g_n|_{\infty, \Theta} \|M\|_2 |\widehat{g} - g_n|_{\infty, \Theta}.
\end{aligned}$$

The first inequality by telescoping, then Cauchy-Schwarz, then using equivalence of finite-dimensional vector space norms and  $\sup_{\theta} a(\theta)b(\theta) \leq \sup_{\theta} a(\theta) \sup_{\theta} b(\theta)$  for positive  $a, b$ . We have  $|g_n|_{\infty, \Theta}, |\widehat{g}|_{\infty, \Theta} = o_p(1) + |g_0|_{\infty, \Theta} = O_p(1)$  since  $|g_0|_{\infty, \Theta} \leq E[\sup_{\theta \in \Theta} \phi(W, \theta)] < \infty$ . Also  $\|M_n\|_2 = O_p(1)$  and  $\|M_n - M\|_2 = o_p(1)$  by continuous mapping. Taking  $\sup_{\theta \in \Theta}$  on both sides gives the result. The proof that  $|H_n - H_0|_{\infty, K} = o_p(1)$  is identical. By triangle inequality, this proves the claim.

Next consider (2). Let  $U_1 \subseteq \tilde{U}$  an open set  $\theta_0 \in U_1$  such that the closed  $1/m'$  enlargement  $\tilde{U}_1^{1/m'} \subseteq \tilde{U}$  for some  $m' \geq 1$ . Set  $\tilde{\Theta} = \tilde{U}_1^{1/m'}$ , which is compact. As in the proof of (1), let  $U_{\theta m} = B(\theta, m^{-1})$  for  $m \geq m'$ . The conclusion now follows from the exact argument in (1), applied to the alternate moment functions  $\tilde{g}_z(W_i, \theta) \equiv \partial/\partial\theta' g_z(W_i, \theta)$ . In particular, uniform convergence holds on any open set  $U \subseteq \tilde{\Theta} \subseteq \tilde{U}$ . The final statement about  $G_0(\theta)$  follows by dominated convergence.  $\square$

**Lemma 8.9** (Consistency). *Under the distribution  $P$  in Definition 8.1, if Assumption 3.2 holds then  $\widehat{\theta} - \theta_0 = o_p(1)$  and  $\theta_n - \theta_0 = o_p(1)$ .*

*Proof.* By definition,  $\widehat{\theta} = \operatorname{argmin}_{\theta \in \Theta} \widehat{H}(\theta)$ . Moreover,  $g_n(\theta_n) = 0$  so  $H_n(\theta_n) = 0$  and  $\theta_n \in \operatorname{argmin}_{\theta \in \Theta} H_n(\theta)$ . For (2), since  $g_0(\theta_0) = 0$  uniquely and  $\operatorname{rank}(M) = d_g$ , then  $H_0(\theta)$  is uniquely minimized at  $\theta_0$ . Then by uniform convergence of  $\widehat{H}, H_n$  to  $H_0$ , extremum consistency (e.g. Theorem 2.1 in Newey and McFadden (1994)) implies that  $\theta_n \xrightarrow{p} \theta_0$  and  $\widehat{\theta} \xrightarrow{p} \theta_0$ .  $\square$

*Proof of Lemma 8.7.* By Lemma 8.3, it suffices to show the result under  $P$  in Definition 8.1. Since  $\widehat{\theta} = \operatorname{argmin}_{\theta \in \Theta} \widehat{H}(\theta)$ , we have  $\nabla_{\theta} \widehat{H}(\widehat{\theta}) = 0$ , which is  $\widehat{G}(\widehat{\theta})' M_n \widehat{g}(\widehat{\theta}) = 0$ . By differentiability in Assumption 3.2 and applying Taylor's Theorem componentwise, for each  $k \in [d_g]$  and some  $\tilde{\theta}_k \in [\theta_0, \widehat{\theta}]$  we have

$$\widehat{g}(\widehat{\theta}) = \widehat{g}(\theta_0) + \frac{\partial \widehat{g}_k}{\partial \theta'}(\tilde{\theta}_k)_{k=1}^{d_g} (\widehat{\theta} - \theta_0).$$

Then we may expand

$$\begin{aligned}
0 &= \widehat{G}(\widehat{\theta})' M_n [\widehat{g}(\theta_0) + \frac{\partial \widehat{g}_k}{\partial \theta'}(\tilde{\theta}_k)_{k=1}^{d_g} (\widehat{\theta} - \theta_0)] \\
\widehat{\theta} - \theta_0 &= -(\widehat{G}(\widehat{\theta})' M_n \frac{\partial \widehat{g}_k}{\partial \theta'}(\tilde{\theta}_k)_{k=1}^{d_g})^{-1} \widehat{G}(\widehat{\theta})' M_n \widehat{g}(\theta_0).
\end{aligned}$$

On the event  $S_n = \{\hat{\theta} \in U\}$ ,  $\tilde{\theta}_k \in U$  for each  $k$ . Then  $\mathbb{1}(S_n) \|\frac{\partial \hat{g}_k}{\partial \theta'}(\tilde{\theta}_k)_{k=1}^{d_g} - \frac{\partial g_{0k}}{\partial \theta'}(\tilde{\theta}_k)_{k=1}^{d_g}\|_F^2 \leq \sum_{k=1}^{d_g} \sup_{\theta \in U} \|\frac{\partial \hat{g}_k}{\partial \theta'}(\theta) - \frac{\partial g_{0k}}{\partial \theta'}(\theta)\|_2^2 \leq d_g \sup_{\theta \in U} \|\hat{G}(\theta) - G_0(\theta)\|_F^2 = o_p(1)$  by Lemma 8.8. Similarly,  $\mathbb{1}(S_n) \|\hat{G}(\hat{\theta}) - G_0(\hat{\theta})\|_F^2 \leq \sup_{\theta \in U} \|\hat{G}(\theta) - G_0(\theta)\|_F^2 = o_p(1)$ . Moreover, since  $\hat{\theta} \xrightarrow{p} \theta_0$  and  $\tilde{\theta}_k \in [\theta_0, \hat{\theta}] \forall k$ , we have  $\mathbb{1}(S_n) \|G_0(\hat{\theta}) - G(\theta_0)\|_F^2 = o_p(1)$  and  $\mathbb{1}(S_n) \|\frac{\partial \hat{g}_k}{\partial \theta'}(\tilde{\theta}_k)_{k=1}^{d_g} - G(\theta_0)\|_F^2 = o_p(1)$ , using continuous mapping and continuity of  $\theta \rightarrow G_0(\theta)$  on  $U$ , shown in Lemma 8.8. Since  $P(S_n) \rightarrow 1$ , we have shown  $\|\hat{G}(\hat{\theta}) - G(\theta_0)\|_F^2 = o_p(1)$  and  $\|\frac{\partial \hat{g}_k}{\partial \theta'}(\tilde{\theta}_k)_{k=1}^{d_g} - G(\theta_0)\|_F^2 = o_p(1)$ . Since  $\hat{g}(\theta_0) = O_p(n^{-1/2})$  by Theorem 8.4, by the work above and continuous mapping theorem we have

$$\begin{aligned} \sqrt{n}(\hat{\theta} - \theta_0) &= -(\hat{G}(\hat{\theta})' M_n \frac{\partial \hat{g}_k}{\partial \theta'}(\tilde{\theta}_k)_{k=1}^{d_g})^{-1} \hat{G}(\hat{\theta})' M_n \sqrt{n} \hat{g}(\theta_0) \\ &= -(G' M G)^{-1} G' M \sqrt{n} \hat{g}(\theta_0) + o_p(1) = \Pi \sqrt{n} \hat{g}(\theta_0) + o_p(1). \end{aligned}$$

This proves the second statement of Lemma 8.7. For the first statement, substituting  $\theta_n, H_n, G_n$  for  $\hat{\theta}, \hat{H}, \hat{G}$  in the above argument, we have  $\sqrt{n}(\theta_n - \theta_0) = \Pi \sqrt{n} g_n(\theta_0) + o_p(1)$ . Then we have  $\sqrt{n}(\hat{\theta} - \theta_n) = \sqrt{n}(\hat{\theta} - \theta_0 + \theta_0 - \theta_n) = \Pi \sqrt{n}(\hat{g}(\theta_0) - g_n(\theta_0)) + o_p(1) = \Pi \sqrt{n} E_n[H_i a(W_i, \theta_0)] + o_p(1)$ . This finishes the proof.  $\square$

## 8.4 Linearization for M-Estimation

In this section, we extend our key result to M-estimation  $\hat{\theta} = \operatorname{argmax}_{\theta \in \Theta} E_n[m(D_i, R_i, S_i, \theta)]$ . M-estimation is often equivalent to GMM with score  $\nabla_{\theta} m(D, R, S, \theta)$ , e.g. if  $\theta \rightarrow m(\cdot, \theta)$  is strictly concave. However, this equivalence fails when  $E[m(D, R, S, \theta)]$  has local maxima, violating GMM identification (Assumption 3.2). E.g. see Newey and McFadden (1994) for examples. To handle such cases, in this section we analyze M-estimation under weaker conditions. Let  $m_d(W, \theta) = m(d, R, S(d), \theta)$  and define  $\varphi_m(W, \theta) = E[m(D, R, S, \theta) | W] = p m_1(W, \theta) + (1 - p) m_0(W, \theta)$  and  $\theta_n = \operatorname{argmax}_{\theta \in \Theta} E_n[\varphi_m(W_i, \theta)]$ . Define  $g(D, R, S, \theta) = \nabla_{\theta} m(D, R, S, \theta)$  and let  $\phi, a$  as in the main text, e.g.  $\phi(W, \theta) = \nabla_{\theta} E[m(D, R, S, \theta) | W]$ .

**Assumption 8.10** (M-estimation). *The following conditions hold for  $d \in \{0, 1\}$ :*

- (a) (Consistency).  $\theta_0 = \operatorname{argmax}_{\theta \in \Theta} E[\varphi_m(W, \theta)]$  uniquely and  $E[\sup_{\theta \in \Theta} |m_d(W, \theta)|_2] < \infty$ . Also  $\theta \rightarrow m_d(W, \theta)$  is continuous almost surely and  $\Theta$  is compact.
- (b) (CLT). Let  $g_d(W, \theta) = \nabla_{\theta} m_d(W, \theta)$ . We have  $E[g_d(W, \theta_0)^2] < \infty$ . There exists a neighborhood  $\theta_0 \in U \subseteq \Theta$  such that  $G_d(W, \theta) \equiv \partial / \partial \theta' g_d(W, \theta) = (\partial^2 / \partial \theta \partial \theta') m_d(W, \theta)$  exists and is continuous. Also  $E[\sup_{\theta \in U} |G_d(W, \theta)|_F] < \infty$ .

The next result extends our key lemma to this setting. Combined with the results of Section 8.2, this suffices to show that the main results of Sections 3-7 also apply to M-estimators with multiple local maxima.

**Lemma 8.11** (Linearization). *Suppose Definition 8.2 and Assumption 8.10 hold for the M-estimator  $\hat{\theta}$ . Let  $G = E[(\partial^2/\partial\theta\partial\theta')m(W, \theta_0)]$  and set  $\Pi = -G^{-1}$ . Then  $\sqrt{n}(\hat{\theta} - \theta_n) = \sqrt{n}E_n[H_i\Pi a(W_i, \theta_0)] + o_p(1)$  and  $\sqrt{n}(\hat{\theta} - \theta_0) = \sqrt{n}E_n[\Pi\phi(W_i, \theta_0) + H_i\Pi a(W_i, \theta_0)] + o_p(1)$ .*

*Proof.* By Lemma 8.3, it suffices to show the result under the distribution  $P$ . We have  $|E_n[m(D_i, R_i, S_i, \theta)] - E[\varphi_m(W, \theta)]|_{\infty, \Theta} = o_p(1)$ ,  $\theta \rightarrow E[\varphi_m(W, \theta)]$  continuous, and 8.8 and also  $|E_n[\varphi_m(W_i, \theta)] - E[\varphi_m(W, \theta)]|_{\infty, \Theta} = o_p(1)$ , all by Lemma 8.8. Then by extremum consistency, we have  $\theta_n \xrightarrow{P} \theta_0$  and  $\hat{\theta} \xrightarrow{P} \theta_0$ . By Lemma 8.8 again, there is an open ball  $U \subseteq \Theta$  with  $\theta_0 \in U$  and  $\|\hat{G}_n - G_0\|_{\infty, U} = o_p(1)$  and  $\|G_n - G_0\|_{\infty, U} = o_p(1)$  for  $\hat{G}_n(\theta) = (\partial^2/\partial\theta\partial\theta')E_n[m(D_i, R_i, S_i, \theta)]$ ,  $G_n(\theta) = (\partial^2/\partial\theta\partial\theta')E_n[\varphi_m(W_i, \theta)]$ , and  $G_0(\theta) = (\partial^2/\partial\theta\partial\theta')E[\varphi_m(W, \theta)]$ . Also,  $G_0(\theta)$  is continuous on  $U$ . Defining  $\hat{g}(\theta) = E_n[(\partial/\partial\theta)m(D_i, R_i, S_i, \theta)]$  and  $g_n(\theta) = E_n[\varphi_m(W_i, \theta)]$ , by optimality we have  $\hat{g}(\hat{\theta}) = 0$  and  $g_n(\theta_n) = 0$ . Then result now follows exactly by the proof of Lemma 8.7, with a slightly simpler first order condition.  $\square$

## 8.5 Nonlinear Rerandomization

*Proof of Theorem 4.3.* We first prove a slightly more general result, allowing for over-identified GMM estimation with positive definite weighting matrix  $\Delta_n \xrightarrow{P} \Delta$ . For  $|x|_{2,A}^2 = x'Ax$ , define

$$\hat{\beta}_d \in \operatorname{argmin}_{\beta \in \mathbb{R}^{d\beta}} |E_n[\mathbf{1}(D_i = d)m(X_i, \beta)]|_{2, \Delta_n}^2.$$

Define  $g^1(D, X, \beta) = Dm(X, \beta)$  and  $g^0(D, X, \beta) = (1 - D)m(X, \beta)$ . Under the expansion in Equation 3.1, we have  $\phi^1(X, \beta) = pg^1(1, X, \beta) = pm(X, \beta)$  and  $a^1(X, \beta) = v_D g^1(1, X, \beta) = v_D m(X, \beta)$ . Similarly,  $\phi^0(X, \beta) = (1 - p)g^0(0, X, \beta) = (1 - p)m(X, \beta)$  and  $a^0(X, \beta) = -v_D g^0(0, X, \beta) = -v_D m(X, \beta)$ . Note that  $E[g^1(D, X, \beta)] = pE[m(X, \beta)]$  and  $E[g^0(D, X, \beta)] = (1 - p)E[m(X, \beta)]$ , so the GMM parameters  $\beta_1 = \beta_0 = \beta^*$ , where  $\beta^*$  uniquely solves  $E[m(X, \beta^*)] = 0$ . Let  $G_m = E[(\partial/\partial\beta')m(X, \beta^*)]$ , which is full rank by assumption. Then  $G^1 = E[(\partial/\partial\beta')g^1(D, X, \beta^*)] = pE[(\partial/\partial\beta')m(X, \beta^*)] = pG_m$  and  $\Pi^1 = -((G^1)' \Delta G^1)^{-1}(G^1)' \Delta = -p^{-1}(G_m' \Delta G_m)^{-1}G_m' \Delta \equiv p^{-1}\Pi_m$ . By symmetry, we have  $\Pi^0 = (1 - p)^{-1}\Pi_m$ . Observe that

$$\begin{aligned} (\Pi^1 \phi^1 - \Pi^0 \phi^0)(X, \beta) &= p^{-1}\Pi_m pm(X, \beta) - (1 - p)^{-1}\Pi_m(1 - p)m(X, \beta) = 0, \\ (\Pi^1 a^1 - \Pi^0 a^0)(X, \beta) &= p^{-1}\Pi_m v_D m(X, \beta) - (1 - p)^{-1}\Pi_m v_D (-m(X, \beta)) \\ &= (1 - p)\Pi_m m(X, \beta) + p\Pi_m m(X, \beta) = \Pi_m m(X, \beta). \end{aligned}$$

Then applying Lemma 8.7 to GMM estimation using  $g^1$  and  $g^0$ , under the measure  $P$  in Definition 8.1 we have

$$\begin{aligned}\sqrt{n}(\hat{\beta}_1 - \hat{\beta}_0) &= \sqrt{n}(\hat{\beta}_1 - \beta^* - (\hat{\beta}_0 - \beta^*)) = \sqrt{n}\Pi^1 E_n[\phi^1(X_i, \beta^*) + H_i a^1(X_i, \beta^*)] \\ &\quad - \sqrt{n}\Pi^0 E_n[\phi^0(X_i, \beta^*) + H_i a^0(X_i, \beta^*)] + o_p(1) = \sqrt{n}\Pi_m E_n[H_i m(X, \beta^*)] + o_p(1).\end{aligned}$$

Then Definition 4.1 is an example of Definition 2.1 with  $\mathcal{I}_n = \sqrt{n}E_n[H_i h_i] + o_p(1)$  for  $h_i = \Pi_m m(X_i, \beta^*)$ . Then Theorem 3.5 holds with  $h_i = \Pi_m m(X_i, \beta^*)$ . Consider the exactly identified case, so  $\Pi_m = -G_m^{-1}$  and  $h_i = -G_m^{-1}m(X_i, \beta^*)$ . Then by Theorem 3.5,  $\sqrt{n}(\hat{\theta} - \theta_n)|W_{1:n} \Rightarrow \mathcal{N}(0, V_a) + R_A$ . Denote  $\Pi a = \Pi a(W, \theta_0)$  and  $m = m(X, \beta^*)$ . Then the rerandomization coefficient  $\gamma_0$  is

$$\begin{aligned}\gamma_0 &= E[\text{Var}(h|\psi)]^{-1} E[\text{Cov}(h, \Pi a|\psi)] = -E[\text{Var}(G_m^{-1}m|\psi)]^{-1} E[\text{Cov}(G_m^{-1}m, \Pi a|\psi)] \\ &= -E[G_m^{-1} \text{Var}(m|\psi)(G_m^{-1})']^{-1} E[G_m^{-1} \text{Cov}(m, \Pi a|\psi)] = -G'_m E[\text{Var}(m|\psi)]^{-1} E[\text{Cov}(m, \Pi a|\psi)].\end{aligned}$$

Then  $V_a = v_D^{-1} E[\text{Var}(\Pi a - \gamma'_0(-G_m^{-1}m)|\psi)] = v_D^{-1} E[\text{Var}(\Pi a - \gamma'_0 m|\psi)]$ , where

$$\gamma_0 = \underset{\gamma \in \mathbb{R}^{d_\beta \times d_\theta}}{\text{argmin}} v_D^{-1} E[\text{Var}(\Pi a - \gamma' m|\psi)].$$

From above, we have  $\gamma_0 = -G'_m \gamma_0$ . Then the residual term

$$\begin{aligned}R_A &\sim \gamma'_0 Z_h | Z_h \in A \sim -\gamma'_0 G_m Z_h | Z_h \in A \sim -\gamma'_0 G_m Z_h | (-G_m^{-1})(-G_m) Z_h \in A \\ &\sim \gamma'_0 Z_m | -G_m^{-1} Z_m \in A \sim \gamma'_0 Z_m | Z_m \in -G_m A.\end{aligned}$$

The variable  $Z_h \sim \mathcal{N}(0, v_D^{-1} E[\text{Var}(h|\psi)])$ , so  $Z_m = G_m Z_h \sim \mathcal{N}(0, v_D^{-1} G_m E[\text{Var}(h|\psi)] G'_m) \sim \mathcal{N}(0, v_D^{-1} E[\text{Var}(G_m h|\psi)]) \sim \mathcal{N}(0, v_D^{-1} E[\text{Var}(m|\psi)])$  since  $G_m h = G_m G_m^{-1} m = m(X, \beta^*)$ . Summarizing, we have shown  $V_a = v_D^{-1} E[\text{Var}(\Pi a - \gamma'_0 m|\psi)]$  and  $R_A \sim \gamma'_0 Z_m | Z_m \in G_m A$  for  $Z_m \sim \mathcal{N}(0, v_D^{-1} E[\text{Var}(m|\psi)])$ .

For the corollary, consider letting  $\hat{\beta} \in \underset{\beta \in \mathbb{R}^{d_\beta}}{\text{argmin}} \|E_n[m(X_i, \beta)]\|_{2, \Delta_n}^2$ . Relative to the expansion in Equation 3.1,  $a_m(X_i, \beta) = 0$  and  $\phi_m(X_i, \beta) = m(X_i, \beta)$ , with linearization matrix  $\Pi_m$  as above. Then by Lemma 8.7  $\sqrt{n}(\hat{\beta} - \beta^*) = \Pi_m E_n[m(X_i, \beta^*)] + o_p(1) = O_p(1)$ . Consider setting  $h_i = m(X_i, \hat{\beta})$ . By the mean value theorem,  $m(X_i, \hat{\beta}) - m(X_i, \beta^*) = \frac{\partial m(X_i, \tilde{\beta}_i)}{\partial \beta}(\hat{\beta} - \beta^*)$ , where the  $\tilde{\beta}_i \in [\beta^*, \hat{\beta}]$  may change by row. Then we have

$$\sqrt{n}E_n[H_i m(X_i, \hat{\beta})] - \sqrt{n}E_n[H_i m(X_i, \beta^*)] = E_n[H_i(\partial/\partial \beta')m(X_i, \tilde{\beta}_i)]\sqrt{n}(\hat{\beta} - \beta^*).$$

We claim that  $E_n[H_i(\partial/\partial \beta')m(X_i, \tilde{\beta}_i)] = o_p(1)$ . Let  $U$  open s.t.  $E[\sup_{\beta \in U} |m(X_i, \beta)|_F] < \infty$  and define  $S_n = \{\hat{\beta} \in U\}$ . Then by consistency  $E_n[H_i(\partial/\partial \beta')m(X_i, \tilde{\beta}_i)]\mathbf{1}(S_n^c) = o_p(1)$ . Define  $v_{ijk}^n = \mathbf{1}(S_n)((\partial/\partial \beta')m(X_i, \tilde{\beta}_i))_{jk}$ . By the definition of  $\hat{\beta}$ , clearly  $v_{ijk}^n \in \mathcal{F}_n =$

$\sigma(W_{1:n}, \pi_n)$ . Moreover, we have  $|v_{ijk}^n| \leq \sup_{\beta \in U} |(\partial/\partial\beta')m(X_i, \beta)|_F \in L_1$  by definition of  $S_n$  and  $\tilde{\beta}_i \in [\beta^*, \hat{\beta}]$  for each  $n$ , so by domination  $(v_{ijk}^n)_n$  is uniformly integrable, so  $E_n[H_i v_{ijk}^n] = o_p(1)$  by Lemma A.2 of [Cytrynbaum \(2024\)](#). This proves the claim, showing that  $\mathcal{I}_n = \sqrt{n}E_n[H_i m(X_i, \hat{\beta})] = \sqrt{n}E_n[H_i m(X_i, \beta^*)] + o_p(1)$ . The result now follows from Theorem 3.5.  $\square$

**Assumption 8.12** (Propensity Rerandomization). *Impose the following conditions.*

- (a) Let  $L$  be twice differentiable, with  $|L'|_\infty, |L''|_\infty < \infty$ . For each  $p \in (0, 1)$ , there is a unique  $c$  with  $L(c) = p$ . Also,  $|L'(c)| > 0$ .
- (b) The score  $m(D_i, X_i, \beta) = D_i \frac{L'(X'_i \beta) X_i}{L(X'_i \beta)} - (1 - D_i) \frac{L'(X'_i \beta) X_i}{1 - L(X'_i \beta)}$  satisfies condition 3.2. The solution to Equation 4.3 exists.
- (c) Covariates  $X = (1, h)$  for  $E[|h|_2^2] < \infty$ . Also,  $E[\text{Var}(h|\psi)], \text{Var}(h)$  are full rank.

*Proof of Theorem 4.7.* By assumption,  $\hat{\beta}$  is a GMM estimator for  $m(D_i, X_i, \beta) = D_i \frac{L'(X'_i \beta) X_i}{L(X'_i \beta)} - (1 - D_i) \frac{L'(X'_i \beta) X_i}{1 - L(X'_i \beta)}$ . Let  $c$  such that  $L(c) = p$ . Then  $\beta^* = (c, 0)$  has  $E[m(D, X, \beta^*)] = E[H_i L'(c) X_i] = 0$ . Relative to the decomposition in Equation 3.1, we have  $\phi(X, \beta) = p \frac{L'(X'_i \beta) X_i}{L(X'_i \beta)} - (1 - p) \frac{L'(X'_i \beta) X_i}{1 - L(X'_i \beta)}$  and  $a(X, \beta) = v_D \left( \frac{L'(X'_i \beta) X_i}{L(X'_i \beta)} + \frac{L'(X'_i \beta) X_i}{1 - L(X'_i \beta)} \right)$ . Since  $L(X'_i \beta^*) = L(c) = p$ , apparently we have  $\phi(X, \beta^*) = 0$  and  $a(X, \beta^*) = L'(c) X_i$ . It's easy to see  $\text{Var}(h) \succ 0$  implies  $E[XX'] \succ 0$  for  $X = (1, h)$ . A calculation shows that  $G_m = E[\frac{\partial}{\partial\beta'} \phi(X, \beta^*)] = -L'(c)^2 E[X_i X'_i]$ , so  $\Pi_m = -G_m^{-1} = \frac{1}{L'(c)^2} E[X_i X'_i]^{-1}$ . By Lemma 8.7, we have shown

$$\begin{aligned} \sqrt{n}(\hat{\beta} - \beta^*) &= \sqrt{n} \Pi_m E_n[\phi(X_i, \beta^*) + H_i a(X_i, \beta^*)] + o_p(1) \\ &= \frac{\sqrt{n}}{L'(c)} E[X_i X'_i]^{-1} E_n[H_i X_i] + o_p(1). \end{aligned}$$

Consider rerandomizing until  $\mathcal{J}_n = n E_n[(p - L(X'_i \hat{\beta}))^2] \leq \epsilon^2$ . Then for  $\beta^*$  s.t.  $L(x' \beta^*) = p$ , the above quantity is  $n E_n[(L(X'_i \hat{\beta}) - L(X'_i \beta^*))^2]$ . By Taylor's Theorem,  $L(X'_i \hat{\beta}) - L(X'_i \beta^*) = L'(\xi_i)(X'_i \hat{\beta} - X'_i \beta^*) = L'(\xi_i) X'_i (\hat{\beta} - \beta^*)$  for some  $\xi_i \in [X'_i \beta^*, X'_i \hat{\beta}]$ . Then we have

$$\mathcal{J}_n = n(\hat{\beta} - \beta^*)' E_n[X_i X'_i L'(\xi_i)^2] (\hat{\beta} - \beta^*).$$

Claim that  $E_n[X_i X'_i L'(\xi_i)^2] = E_n[X_i X'_i L'(X'_i \beta^*)^2] + o_p(1)$ . If so, then  $E_n[X_i X'_i L'(\xi_i)^2] = L'(c)^2 E_n[X_i X'_i] + o_p(1) = L'(c)^2 E[X_i X'_i] + o_p(1)$ . To see this, note that  $|L'(X'_i \beta^*)^2 - L'(\xi_i)^2| = |L'(X'_i \beta^*) - L'(\xi_i)| |L'(X'_i \beta^*) + L'(\xi_i)| \leq 2|L'|_\infty |L''|_\infty |X'_i \beta^* - \xi_i|_2 \lesssim |X'_i \beta^* - X'_i \hat{\beta}|_2 \leq |X_i|_2 |\beta^* - \hat{\beta}|_2$ . Then we have

$$\begin{aligned} |E_n[X_i X'_i L'(\xi_i)^2] - E_n[X_i X'_i L'(X'_i \beta^*)^2]|_2 &\leq E_n[|X_i|_2^2 |L'(X'_i \beta^*)^2 - L'(\xi_i)^2|] \\ &\lesssim E_n[|X_i|_2^3] |\beta^* - \hat{\beta}|_2 = o_p(1) \end{aligned}$$

The last equality if  $E_n[|X_i|_2^3] = o_p(n^{1/2})$ . Note that  $E_n[|X_i|_2^3] \leq E_n[|X_i|_2^2] \max_{i=1}^n |X_i|_2 = O_p(1)o_p(n^{1/2})$  since  $E[|X_i|_2^2] < \infty$  by assumption, using Lemma C.8 of [Cytrynbaum \(2024\)](#). Then using the claim,  $\sqrt{n}(\hat{\beta} - \beta^*) = O_p(1)$ , and the linear expansion of  $\sqrt{n}(\hat{\beta} - \beta^*)$  above, we have shown  $\mathcal{J}_n = L'(c)^2 n(\hat{\beta} - \beta^*)' E[X_i X_i'] (\hat{\beta} - \beta^*) + o_p(1)$ , which is

$$\begin{aligned} &= L'(c)^2 (L'(c)^{-1} E[X_i X_i']^{-1} \sqrt{n} E_n[H_i X_i])' E[X_i X_i'] (L'(c)^{-1} E[X_i X_i']^{-1} \sqrt{n} E_n[H_i X_i]) + o_p(1) \\ &= \sqrt{n} E_n[H_i X_i]' E[X_i X_i']^{-1} \sqrt{n} E_n[H_i X_i] + o_p(1). \end{aligned}$$

Note  $E_n[H_i] = O_p(n^{-1})$  by stratification. Since  $X = (1, h)$ ,  $\sqrt{n} E_n[H_i X_i]' = (0, \sqrt{n} E_n[H_i h_i]') + O_p(n^{-1/2})$ . Also, by block inversion  $(E[X_i X_i']^{-1})_{hh} = \text{Var}(h_i)^{-1}$ . For some  $\xi_n = o_p(1)$

$$\begin{aligned} \mathcal{J}_n &= (0, \sqrt{n} E_n[H_i h_i]') E[X_i X_i']^{-1} (0, \sqrt{n} E_n[H_i h_i]')' + o_p(1) \\ &= \sqrt{n} E_n[H_i h_i]' (E[X_i X_i']^{-1})_{hh} \sqrt{n} E_n[H_i h_i] + o_p(1) \\ &= \sqrt{n} E_n[H_i h_i]' \text{Var}(h_i)^{-1} \sqrt{n} E_n[H_i h_i] + \xi_n. \end{aligned}$$

Define the function  $b(x, y) = x' \text{Var}(h)^{-1} x + y - \epsilon$ . Then  $\mathcal{J}_n \leq \epsilon \iff b(\mathcal{I}_n, \xi_n) \leq 0$  for  $\mathcal{I}_n = \sqrt{n} E_n[H_i h_i]$  and  $\xi_n \xrightarrow{p} 0$ . Clearly,  $x \rightarrow b(x, 0)$  is continuous. Also note  $E[|h|_2^2] < \infty$  by assumption. Finally, for  $Z_h \sim \mathcal{N}(0, E[\text{Var}(h|\psi)])$ , have  $P(b(Z_h, 0) = 0) = P(Z_h' \text{Var}(h)^{-1} Z_h = \epsilon^2) = 0$  since  $E[\text{Var}(h|\psi)]$  is full rank. Then this rerandomization satisfies all the conditions in Definition 8.2. By Lemma 8.7, the GMM estimator  $\sqrt{n}(\hat{\theta} - \theta_0) = \sqrt{n} E_n[H_i \Pi a(W_i, \theta_0)] + o_p(1)$  under this rerandomization. By Theorem 8.6, have  $\sqrt{n} E_n[H_i \Pi a(W_i)] | \mathcal{F}_n \Rightarrow \mathcal{N}(0, V_a) + R$  with  $R \sim \gamma_0' Z_h | Z_h \in T \sim \gamma_0' Z_h | Z_h' \text{Var}(h)^{-1} Z_h \leq \epsilon$  for acceptance region  $T = \{x : b(x, 0) \leq 0\} = \{x : x' \text{Var}(h)^{-1} x \leq \epsilon\}$  and

$$V_a = \min_{\gamma \in \mathbb{R}^{d_h \times d_\theta}} v_D^{-1} E[\text{Var}(\Pi a(W) - \gamma' h | \psi)].$$

This finishes the proof. □

## 8.6 Covariate Adjustment

*Proof of Theorem 3.12.* By Lemma 8.7,  $\sqrt{n}(\hat{\theta}^* - \theta_n)$  may be expanded as

$$\begin{aligned} \sqrt{n}(\hat{\theta} - \theta_n - E_n[H_i m(\psi_i, h_i)]) &= \sqrt{n} E_n[H_i (\Pi a(W_i, \theta_0) - m(\psi_i, h_i))] + o_p(1) \\ &\equiv \sqrt{n} E_n[H_i \beta(W_i, \theta_0)] + o_p(1). \end{aligned}$$

By Theorem 8.6,  $\sqrt{n} E_n[H_i \beta(W_i, \theta_0)] | \mathcal{F}_n \Rightarrow \mathcal{N}(0, V)$  with  $V = v_D^{-1} \text{Var}(\beta(W, \theta_0))$ . Since  $\beta(W, \theta_0) = \Pi a(W, \theta_0) - \gamma_0' h - t_0(\psi)$  for  $(\gamma_0, t_0)$  solving Equation 3.7, this completes the proof. □

*Proof of Proposition 6.2.* Since  $\hat{\theta}_{adj} = \hat{\theta} - E_n[H_i \hat{\alpha}' w_i]$  for  $\hat{\alpha} \xrightarrow{p} \alpha$  and  $E_n[H_i w_i] = O_p(n^{-1/2})$



by Theorem 8.4, then  $\widehat{\theta}_{adj} = \widehat{\theta} - E_n[H_i \alpha' w_i] + o_p(n^{-1/2}) = E_n[H_i(\Pi a(W_i, \theta_0) - \alpha' w_i)] + o_p(n^{-1/2})$ , the final equality by Lemma 8.7. The first statement now follows from Slutsky and Theorem 8.4. The second statement follows by the same argument used in the proof of Corollary 3.8.  $\square$

*Proof of Theorem 6.3.* By the same argument in the proof of Proposition 6.2, we have  $\widehat{\theta}_{adj} = E_n[H_i(\Pi a(W_i, \theta_0) - \alpha'_0 w_i)] + o_p(n^{-1/2})$ . Then by Theorem 8.6,  $\sqrt{n}(\widehat{\theta}_{adj} - \theta_n) | \mathcal{F}_n \Rightarrow \mathcal{N}(0, V) + R$ , independent with

$$V = v_D^{-1} E[\text{Var}(\Pi a(W) - \alpha'_0 w - \beta'_0 h | \psi)] = \min_{\beta \in \mathbb{R}^{d_h \times d_\theta}} v_D^{-1} E[\text{Var}(\Pi a(W) - \alpha'_0 w - \beta' h | \psi)].$$

The residual term  $R \sim \beta'_0 Z_h | Z_h \in A$ . Then it suffices to show that  $\beta_0 = 0$ . Define  $a_{\Pi\alpha} = \Pi a(W, \theta_0) - \alpha'_0 w$ . By Lemma 8.17, it further suffices to show  $\beta_0 = 0$  solves  $E[\text{Var}(h | \psi)] \beta_0 = E[\text{Cov}(h, a_{\Pi\alpha} | \psi)]$ , i.e. that  $E[\text{Cov}(h, a_{\Pi\alpha} | \psi)] = 0$ . To do so, note that  $E[\text{Cov}(h, a_{\Pi\alpha} | \psi)] = E[\text{Cov}(h, (\Pi a - \alpha'_0 w) | \psi)] = E[\text{Cov}(h, \Pi a | \psi)] - E[\text{Cov}(h, w | \psi)] \alpha_0$ . By assumption,  $E[\text{Var}(w | \psi)] \alpha_0 = E[\text{Cov}(w, \Pi a | \psi)]$ . Since  $h \subseteq w$ , we have

$$\begin{aligned} E[\text{Cov}(h, w | \psi)] \alpha_0 &= (E[\text{Var}(w | \psi)])_{hw} \alpha_0 = (E[\text{Var}(w | \psi)] \alpha_0)_{h\theta} \\ &= (E[\text{Cov}(w, \Pi a | \psi)])_{h\theta} = E[\text{Cov}(h, \Pi a | \psi)] \end{aligned}$$

This shows that  $[\text{Cov}(h, a_{\Pi\alpha} | \psi)] = 0$ , so  $\beta_0 = 0$  is a solution, proving the claim. This finishes the proof of the statement for  $\theta_n$ . The result for  $\theta_0$  follows trivially, as in Corollary 3.8.  $\square$

*Proof of Theorem 6.4.* By Lemma 8.3, it suffices to show the result under  $P$  in Definition 8.1. By Lemma 8.13,  $E_n[\check{w}_i \check{w}_i'] = k^{-1}(k-1)E[\text{Var}(w | \psi)] + o_p(1)$ . Then if  $E[\text{Var}(w | \psi)] \succ 0$ , we have  $E_n[\check{w}_i \check{w}_i']^{-1} \xrightarrow{P} k(k-1)^{-1}E[\text{Var}(w | \psi)]^{-1}$  by continuous mapping. We have  $\widehat{\Pi} \xrightarrow{P} \Pi$  by assumption. Then it suffices to show  $E_n[\check{w}_i(D_i - p)\widehat{g}_i'] = k^{-1}(k-1)E[\text{Cov}(w, a | \psi)] + o_p(1)$ . First, claim  $E_n[\check{w}_i(D_i - p)\widehat{g}_i'] = E_n[\check{w}_i(D_i - p)g_i(\theta_0)'] + o_p(1)$ , for  $g_i(\theta) \equiv g(D_i, X_i, S_i, \theta)$ . By Taylor's theorem,  $|g_i(\widehat{\theta}) - g_i(\theta_0)|_2 \leq |\frac{\partial g_i}{\partial \theta'}(\tilde{\theta}_i)|_2 |\widehat{\theta} - \theta_0|_2$ , where  $\tilde{\theta}_i$  may change by row. Then  $|E_n[\check{w}_i(D_i - p)(g_i(\widehat{\theta}) - g_i(\theta_0))']|_2 \leq E_n[|\check{w}_i|_2 |g_i(\widehat{\theta}) - g_i(\theta_0)|_2] \leq |\widehat{\theta} - \theta_0|_2 E_n[|\check{w}_i|_2 |\frac{\partial g_i}{\partial \theta'}(\tilde{\theta}_i)|_2] \leq |\widehat{\theta} - \theta_0|_2 (E_n[|\check{w}_i|_2^2] + E_n[|\frac{\partial g_i}{\partial \theta'}(\tilde{\theta}_i)|_2^2])$  by Young's inequality. We showed  $E_n[|\frac{\partial g_i}{\partial \theta'}(\tilde{\theta}_i)|_2^2] = O_p(1)$  in the proof of Lemma 8.15. Similarly,  $E_n[|\check{w}_i|_2^2] \leq E_n[|w_i|_2^2] = O_p(1)$  by the bound in Lemma 8.13. Since  $|\widehat{\theta} - \theta_0|_2 = o_p(1)$  by Theorem 3.5, this proves the claim.

Next, claim  $E_n[\check{w}_i(D_i - p)g_i(\theta_0)'] = E_n[\check{w}_i a(W_i, \theta_0)'] + o_p(1)$ . By definition, we have  $E_n[\check{w}_i(D_i - p)g_i(\theta_0)'] = E_n[\check{w}_i(D_i - p)\phi(W_i, \theta_0)'] + \text{Var}(D)^{-1} E_n[(D_i - p)^2 \check{w}_i a(W_i, \theta_0)'] \equiv A_n + B_n$ . Expanding  $(D_i - p)^2$ ,  $B_n = \text{Var}(D)^{-1} E_n[(\text{Var}(D) + (D_i - p)(1 - 2p))\check{w}_i a(W_i, \theta_0)'] = E_n[\check{w}_i a(W_i, \theta_0)'] + \frac{1-2p}{\text{Var}(D)} E_n[(D_i - p)\check{w}_i a(W_i, \theta_0)']$ . Since  $\phi = pg_1 + (1-p)g_0$  and  $a = \text{Var}(D)(g_1 - g_0)$ , apparently it suffices to show  $E_n[(D_i - p)\check{w}_i g_d(W_i, \theta_0)'] = o_p(1)$  for



each  $d = 0, 1$ . Since  $E[|g_d(W_i, \theta_0)|_2^2] < \infty$ , this follows from Lemma 8.13. Finally,  $E_n[\check{w}_i a(W_i, \theta_0)] = k^{-1}(k-1)E[\text{Cov}(w_i, a(W_i, \theta_0)|\psi_i)] + o_p(1)$  since  $E[|w|_2^2 + |g_d|_2^2] < \infty$  and by applying Lemma 8.13 componentwise. This finishes the proof.  $\square$

**Lemma 8.13.** *Let  $E[w_i^2 + v_i^2] < \infty$  with  $w_i, v_i \in \sigma(W_i)$ . Then under  $P$  in Definition 8.1,  $E_n[(D_i - p)\check{w}_i\check{v}_i] = o_p(1)$  and  $E_n[(D_i - p)\check{w}_i v_i] = o_p(1)$ . Also  $E_n[\check{w}_i\check{v}_i] = \frac{k-1}{k}E[\text{Cov}(w, v|\psi)] + o_p(1)$ .*

*Proof.* First, note  $|s|^{-1} \sum_{i \in s} \check{w}_i^2 = |s|^{-1} \sum_{i \in s} (w_i - |s|^{-1} \sum_{j \in s} w_j)^2 = \text{Var}_s(w_i) \leq E_s[w_i^2] = |s|^{-1} \sum_{i \in s} w_i^2$ . Then in particular  $\sum_{i \in s} \check{w}_i^2 \leq \sum_{i \in s} w_i^2$  and  $E_n[\check{w}_i^2] \leq E_n[w_i^2]$ . Write  $E_n[(D_i - p)\check{w}_i\check{v}_i] = n^{-1} \sum_s u_s$  for  $u_s = \sum_{i \in s} (D_i - p)\check{w}_i\check{v}_i$ . Let  $\mathcal{F}_n = \sigma(W_{1:n}, \pi_n)$ . Then  $\mathcal{S}_n \in \mathcal{F}_n$ ,  $E[u_s|\mathcal{F}_n] = 0$  and  $u_s \perp\!\!\!\perp u_{s'}|\mathcal{F}_n$  for  $s \neq s'$  by Lemma C.10 and Lemma C.9 of Cytrynbaum (2024). By Lemma C.7 of Cytrynbaum (2024), it suffices to show  $n^{-1} \sum_s E[|u_s| \mathbb{1}(|u_s| > c_n) | \mathcal{F}_n] = o_p(1)$  for some  $c_n = o(\sqrt{n})$  with  $c_n \rightarrow \infty$ . Note that  $|u_s| \leq \sum_{i \in s} |\check{w}_i\check{v}_i| \leq \sum_{i \in s} \check{w}_i^2 + \sum_{i \in s} \check{v}_i^2 \leq \sum_{i \in s} w_i^2 + \sum_{i \in s} v_i^2$  by Young's inequality and the bound above. Note that for any positive constants  $(a_k)_{k=1}^m$  we have  $\sum_k a_k \mathbb{1}(\sum_k a_k > c) \leq m \sum_k a_k \mathbb{1}(a_k > c/m)$ . Applying this fact and the upper bounds gives

$$\begin{aligned} n^{-1} \sum_s E[|u_s| \mathbb{1}(|u_s| > c_n) | \mathcal{F}_n] &\leq n^{-1} \sum_s E \left[ \sum_{i \in s} (w_i^2 + v_i^2) \mathbb{1}(\sum_{i \in s} (w_i^2 + v_i^2) > c_n) | \mathcal{F}_n \right] \\ &\leq 2kn^{-1} \sum_s \sum_{i \in s} w_i^2 \mathbb{1}(w_i^2 > c_n/2k) + 2kn^{-1} \sum_s \sum_{i \in s} v_i^2 \mathbb{1}(v_i^2 > c_n/2k) \end{aligned}$$

The final quantity is  $2kE_n[w_i^2 \mathbb{1}(w_i^2 > c_n/2k)] + 2kE_n[v_i^2 \mathbb{1}(v_i^2 > c_n/2k)] = o_p(1)$ . This follows by Markov inequality since  $E[E_n[w_i^2 \mathbb{1}(w_i^2 > c_n/2k)]] = E[w_i^2 \mathbb{1}(w_i^2 > c_n/2k)] \rightarrow 0$  for any  $c_n \rightarrow \infty$  by dominated convergence. This proves the first statement, and the second statement follows by setting  $\check{v}_i \rightarrow v_i$  above. For the final statement, calculate

$$\sum_{i \in s} \check{w}_i \check{v}_i = \sum_{i \in s} (w_i - k^{-1} \sum_{j \in s} w_j)(v_i - k^{-1} \sum_{j \in s} v_j) = k^{-1}(k-1) \sum_{i \in s} w_i v_i - k^{-1} \sum_{i \neq j \in s} v_i w_j$$

Clearly  $n^{-1}k^{-1}(k-1) \sum_s \sum_{i \in s} w_i v_i = k^{-1}(k-1)E_n[w_i v_i] = k^{-1}(k-1)E[w_i v_i] + o_p(1)$ . Then it suffices to show  $(kn)^{-1} \sum_s \sum_{i \neq j \in s} v_i w_j = k^{-1}(k-1)E[E[w_i|\psi_i]E[v_i|\psi_i]] + o_p(1)$ . If so,  $E_n[\check{w}_i\check{v}_i] = k^{-1}(k-1)(E[w_i v_i] - E[E[w_i|\psi_i]E[v_i|\psi_i]]) + o_p(1) = k^{-1}(k-1)E[\text{Cov}(w_i, v_i|\psi_i)] + o_p(1)$  as claimed. The analysis of the term  $\hat{v}_{10}$  in Lemma A.6 of Cytrynbaum (2024) shows

$$\begin{aligned} n^{-1} \sum_s \sum_{i \neq j \in s} v_i w_j &= n^{-1} \sum_s \sum_{i \neq j \in s} E[v_i|\psi_i]E[w_j|\psi_j] + o_p(1) \\ &= (k-1)E_n[E[v_i|\psi_i]E[w_i|\psi_i]] + o_p(1) = (k-1)E[E[v_i|\psi_i]E[w_i|\psi_i]] + o_p(1). \end{aligned}$$

By above work, this finishes our proof of the claim.  $\square$

## 8.7 Acceptance Region Optimization

*Proof of Proposition 5.1.* First we prove part (a). Define the function  $f(a) = \sup_{b \in B} |b'a|$ . As the sup of linear functions,  $f$  is convex (e.g. Rockafellar (1996)). Then the sublevel set  $A \equiv \{a : f(a) \leq 1\}$  is convex. Note that  $f(a) = f(-a)$ , so  $A$  is symmetric. For the main statement of the theorem, let  $a_n = \sqrt{n}E_n[H_i h_i]$ . Clearly,  $f$  is positive homogeneous, i.e.  $f(\lambda a) = \lambda f(a)$  for  $\lambda \geq 0$ . Then note that the LHS event occurs iff  $f(a_n) \leq \epsilon \iff f(a_n/\epsilon) \leq 1 \iff a_n/\epsilon \in A \iff a_n \in \epsilon \cdot A$ . This proves the main statement. Suppose  $B$  is bounded. Then by Cauchy-Schwarz  $f(a) \leq |a|_2 \sup_{b \in B} |b|_2 < \infty$  for any  $a \in \mathbb{R}^{d_h}$ . Then  $f$  is a proper function, so  $f$  is continuous by Corollary 10.1.1. of Rockafellar (1996). Then  $A = f^{-1}([0, 1])$  is closed. Moreover, the open set  $f^{-1}((1/3, 2/3)) \subseteq f^{-1}([0, 1]) = A$ , so  $A$  has non-empty interior. Suppose that  $B$  is open. Then  $B$  contains an open ball  $B(x, \delta)$  for some  $x \in \mathbb{R}^{d_h}$  and  $\delta > 0$ . Fix  $a \in \mathbb{R}^{d_h}$  and define  $b(a) = x + \text{sgn}(a'x) \frac{\delta}{2|a|} a$ . By assumption,  $b(a) \in B$ . Then  $f(a) = \sup_{b \in B} |b'a| \geq |b(a)'a| = |a'x + \text{sgn}(a'x)(\delta/2)|a|| = |a'x| + (\delta/2)|a| \geq (\delta/2)|a|$ . Then  $f(a) = \sup_{b \in B} |a'b| \geq (\delta/2)|a|$ , so  $A \subseteq B(0, 2/\delta)$ .  $\square$

*Proof of Theorem 5.2.* First we show the set  $A_0$  is feasible in Equation 5.2. We have  $L_{\gamma,A} = T_\gamma + \gamma' Z_{hA}$ , where  $T_\gamma \sim \mathcal{N}(0, V(\gamma))$  and  $T_\gamma \perp\!\!\!\perp Z_{hA}$ . Then  $\text{bias}(L_{\gamma,A}|Z_h) = E[L_{\gamma,A}|Z_h] = E[T_\gamma|Z_h] + \gamma' Z_{hA} = \gamma' Z_{hA}$ . For  $A_0 = \epsilon B^\circ$ , we have  $\sup_{\gamma \in B} |\text{bias}(L_{\gamma,A}|Z_h)| = \sup_{\gamma \in B} |\gamma' Z_{hA}|$ . Note  $Z_{hA} \in \epsilon B^\circ$ , so  $Z_{hA}/\epsilon \in B^\circ$ . Then we have

$$\sup_{\gamma \in B} |\gamma' Z_{hA}| \leq \epsilon \cdot \sup_{b \in B^\circ} \sup_{\gamma \in B} |\gamma' b| \leq \epsilon \cdot 1.$$

The final inequality by definition of  $B^\circ$ . This shows that  $A_0$  is feasible. We claim  $A_0$  is optimal. Suppose for contradiction that there exists  $A \subseteq \mathbb{R}^{d_h}$  with  $\text{Leb}(A \triangle A_0) \neq 0$  and  $P(Z_h \in A) > P(Z_h \in A_0)$ . Clearly  $A \not\subseteq A_0$ . Then  $\text{Leb}(A \setminus A_0) > 0$ , so  $P(Z_h \in A \setminus A_0) > 0$  by absolute continuity. For any  $x \in A \setminus A_0 \subseteq (\epsilon B^\circ)^c$ , we must have  $\sup_{\gamma \in B} |\gamma' x| > \epsilon$ .  $B$  is totally bounded by assumption, so as observed in the previous paragraph  $\sup_{\gamma \in B} |Z_{hA}| = p_B(Z_{hA})$ , for  $p_B$  continuous. Then the event  $\{\sup_{\gamma \in B} |Z_{hA}| > \epsilon\} = \{p_B(Z_{hA}) > \epsilon\}$  is measurable. Then  $\{\sup_{\gamma \in B} \text{bias}(L_{\gamma,A}|Z_h) > \epsilon\} = \{\sup_{\gamma \in B} |Z_{hA}| > \epsilon\} \supseteq \{Z_{hA} \in A \setminus A_0\}$ . Then  $P(\sup_{\gamma \in B} \text{bias}(L_{\gamma,A}|Z_h) > \epsilon) \geq P(Z_h \in A \setminus A_0) > 0$ , which contradicts feasibility of  $A$ , proving the claim.  $\square$

*Proof of Lemma 5.6.* For  $B = x + \Sigma B_p$  we compute the upper bound.

$$\begin{aligned} \sup_{b \in B} |a'b| &= \sup_{u \in \Sigma B_p} |a'x + a'u| \leq |a'x| + \sup_{u \in \Sigma B_p} |a'\Sigma \Sigma^{-1}u| \\ &= |a'x| + \sup_{v \in B_p} |(\Sigma'a)'v| = |a'x| + |\Sigma'a|_q. \end{aligned}$$

Before proceeding, we claim that for any  $z \in \mathbb{R}^{d_h}$ , we have  $\max_{v \in B_p} v'z = \max_{v \in B_p} |v'z|$ . Clearly  $\max_{v \in B_p} v'z \leq \max_{v \in B_p} |v'z|$ . Since  $B_p$  is compact and  $v \rightarrow v'z$  continuous,

$v^* \in \operatorname{argmax}_{v \in B_p} |v'z|$  exists. Then  $\max_{v \in B_p} |v'z| = |z'v^*| = z'v^* \operatorname{sgn}(z'v^*) = z'w$  for  $w = v^* \operatorname{sgn}(z'v^*) \in B_p$  since  $v^* \in B_p$ . Then  $\max_{v \in B_p} |v'z| = z'w \leq \max_{w \in B_p} z'w$ . This proves the claim. Next, define  $b(a) = x + \operatorname{sgn}(a'x)\Sigma v(a)$  with  $v(a) \in \operatorname{argmax}_{v \in B_p} v'\Sigma'a$ , which exists by compactness and continuity. Note  $b(a) \in B$  by construction. We may calculate  $|a'b(a)| = |a'x + \operatorname{sgn}(a'x)a'\Sigma v(a)|$ . By the claim,  $a'\Sigma v(a) \geq 0$ . Then by matching signs,  $|a'x + \operatorname{sgn}(a'x)a'\Sigma v(a)| = |a'x| + |\operatorname{sgn}(a'x)a'\Sigma v(a)| = |a'x| + |a'\Sigma v(a)|$ . By the claim again, this is  $|a'x| + a'\Sigma v(a) = |a'x| + \max_{v \in B_p} |a'\Sigma v| = |a'x| + |\Sigma'a|_q$ . Combining with the upper bound above, we have shown that  $\sup_{b \in B} |a'b| = |a'x| + |\Sigma'a|_q$ .  $\square$

## 8.8 Inference

*Proof of Theorem 7.2.* By Lemma 8.3, it suffices to show the result under  $P$  in Definition 8.1. Define  $m_i = \Pi g_i(\theta_0) - H_i \alpha'_0 w_i$ , the population version of  $\widehat{m}_i$ . Also define  $m_{1i} = \Pi g_{1i}(\theta_0) - \alpha'_0 w_i/p$  and  $m_{0i} = \Pi g_{0i}(\theta_0) + \alpha'_0 w_i/(1-p)$ . We may expand

$$\begin{aligned}\phi_b &\equiv pm_{1i} + (1-p)m_{0i} = pg_{1i} + (1-p)g_{0i} = \Pi\phi(W, \theta_0), \\ a_b &\equiv v_D(m_{1i} - m_{0i}) = \Pi a(W, \theta_0) - \alpha'_0 w_i.\end{aligned}$$

By Theorem 6.3, we need to estimate  $V = \operatorname{Var}(\phi_b) + v_D^{-1}E[\operatorname{Var}(a_b|\psi)] = \operatorname{Var}(\phi_b) + v_D^{-1}E[a_b a'_b] - v_D^{-1}E[E[a_b|\psi]E[a_b|\psi]'] \equiv V_1 - V_2$ . We expand  $V_1 = \operatorname{Var}(\phi_b) + v_D^{-1}E[a_b a'_b]$  as

$$\begin{aligned}V_1 &= \operatorname{Var}(pm_{1i} + (1-p)m_{0i}) + v_D E[(m_{1i} - m_{0i})(m_{1i} - m_{0i})'] \\ &= E[(pm_{1i} + (1-p)m_{0i})(pm_{1i} + (1-p)m_{0i})'] + v_D E[(m_{1i} - m_{0i})(m_{1i} - m_{0i})'] \\ &= (p^2 + v_D)E[m_{1i}m'_{1i}] + ((1-p)^2 + v_D)E[m_{0i}m'_{0i}] \\ &= pE[m_{1i}m'_{1i}] + (1-p)E[m_{0i}m'_{0i}] = \operatorname{Var}_n(\widehat{m}_i) + o_p(1).\end{aligned}$$

The second equality since  $E[\phi_b] = 0$ , and the final equality by Lemma 8.14. By Lemma 8.15, we also have

$$\begin{aligned}V_2 &= v_D^{-1}E[E[a_b|\psi]E[a_b|\psi]'] = v_D(E[E[m_{1i}|\psi]E[m_{1i}|\psi]'] + E[E[m_{0i}|\psi]E[m_{0i}|\psi]']) \\ &\quad - v_D(E[E[m_{1i}|\psi]E[m_{0i}|\psi]'] + E[E[m_{0i}|\psi]E[m_{1i}|\psi]']) \\ &= v_D(\widehat{v}_1 + \widehat{v}_0 - \widehat{v}_{10} - \widehat{v}'_{10}) + o_p(1).\end{aligned}$$

This finishes the proof.  $\square$

*Proof of Theorem 7.3.* By Lemma 8.3, it suffices to show the result under  $P$  in Definition 8.1. With notation as in the proof of Theorem 7.2, by Theorem 6.3,  $c'(\widehat{\theta} - \theta_0) \Rightarrow \mathcal{N}(0, V_a(c))$  with variance  $V_a(c) = v_D^{-1}c'E[\operatorname{Var}(a_b|\psi)]c$  for  $a_b = v_D(m_{1i} - m_{0i})$ . Then

$V_a(c) = v_D c' E[\text{Var}(m_{1i} - m_{0i}|\psi)]c$  may be expanded as

$$v_D \cdot c' (E[\text{Var}(m_{1i}|\psi)] + E[\text{Var}(m_{0i}|\psi)] - 2E[\text{Cov}(m_{1i}, m_{0i}|\psi)])c.$$

Note that by Cauchy-Schwarz and Jensen we have the bound

$$\begin{aligned} -2c' E[\text{Cov}(m_{1i}, m_{0i}|\psi)]c &\leq 2|E[\text{Cov}(c'm_{1i}, c'm_{0i}|\psi)]| \\ &\leq 2E[\text{Var}(c'm_{1i}|\psi)]^{1/2} E[\text{Var}(c'm_{0i}|\psi)]^{1/2} \leq 2(E[\text{Var}(c'm_{1i}|\psi)]E[\text{Var}(c'm_{0i}|\psi)])^{1/2} \\ &= 2(c' E[\text{Var}(m_{1i}|\psi)]c \cdot c' E[\text{Var}(m_{0i}|\psi)]c)^{1/2}. \end{aligned}$$

Then we bound

$$V_a(c) \leq \bar{V}_a(c) \equiv v_D [(c' E[\text{Var}(m_{1i}|\psi)]c)^{1/2} + (c' E[\text{Var}(m_{0i}|\psi)]c)^{1/2}]^2.$$

Note  $E[\text{Var}(m_{1i}|\psi)] = E[m_{1i}m'_{1i}] - E[E[m_{1i}|\psi_i]E[m_{1i}|\psi_i]'] = E_n[\frac{D_i}{p}\hat{m}_i\hat{m}'_i] - \hat{v}_1 + o_p(1)$  by Lemma 8.14 and Lemma 8.15. Similarly,  $E[\text{Var}(m_{0i}|\psi)] = E_n[\frac{1-D_i}{1-p}\hat{m}_i\hat{m}'_i] - \hat{v}_0 + o_p(1)$ . Then for  $\hat{u}_1 = E_n[\frac{D_i}{p}\hat{m}_i\hat{m}'_i] - \hat{v}_1$  and  $\hat{u}_0 = E_n[\frac{1-D_i}{1-p}\hat{m}_i\hat{m}'_i] - \hat{v}_0$  by continuous mapping

$$\hat{V}_a(c) = v_D ([c'\hat{u}_1c]^{1/2} + [c'\hat{u}_0c]^{1/2})^2 \xrightarrow{p} \bar{V}_a(c) \geq V_a(c).$$

This finishes the proof.  $\square$

**Comparison of Variances.** The superpopulation variance is

$$\begin{aligned} V(c) &= \text{Var}(c'\phi_b) + v_D \cdot (E[\text{Var}(c'm_{1i}|\psi)] + E[\text{Var}(c'm_{0i}|\psi)] - 2E[\text{Cov}(c'm_{1i}, c'm_{0i}|\psi)]) \\ &= p^2 \text{Var}(c'm_{1i}) + (1-p)^2 \text{Var}(c'm_{0i}) + v_D \cdot (E[\text{Var}(c'm_{1i}|\psi)] + E[\text{Var}(c'm_{0i}|\psi)]). \end{aligned}$$

Then the variance gap  $V(c) - V_a(c)$  is

$$\begin{aligned} &p^2 \text{Var}(c'm_{1i}) + (1-p)^2 \text{Var}(c'm_{0i}) - 2v_D (E[\text{Var}(m_{1i}|\psi)] \cdot E[\text{Var}(m_{0i}|\psi)])^{1/2} \\ &= p^2 \text{Var}(E[c'm_{1i}|\psi_i]) + (1-p)^2 \text{Var}(E[c'm_{0i}|\psi_i]) \\ &\quad + (pE[\text{Var}(c'm_{1i}|\psi)]^{1/2} - (1-p)E[\text{Var}(c'm_{0i}|\psi)]^{1/2})^2 \geq 0. \end{aligned}$$

**Lemma 8.14.** *Impose Assumptions 3.1, 3.2, 7.1. Then under  $P$  in Definition 8.1, the following hold:*

- (a)  $E_n[\frac{D_i}{p}\hat{m}_i\hat{m}'_i] = E[m_{1i}m'_{1i}] + o_p(1)$  and  $E_n[\frac{1-D_i}{1-p}\hat{m}_i\hat{m}'_i] = E[m_{0i}m'_{0i}] + o_p(1)$ .
- (b)  $\text{Var}_n(\hat{m}_i) = pE[m_{1i}m'_{1i}] + (1-p)E[m_{0i}m'_{0i}] + o_p(1)$ .

*Proof.* For (a), consider the first statement. We may expand this as

$$E_n[(D_i/p)\hat{m}_i\hat{m}'_i] = E_n[(D_i/p)\hat{m}_i(\hat{m}_i - m_i)'] + E_n[(D_i/p)(\hat{m}_i - m_i)m'_i] + E_n[(D_i/p)m_im'_i].$$

For  $g_i = g_i(\theta_0)$ , we have  $|\widehat{m}_i - m_i|_2 = |\widehat{\Pi}\widehat{g}_i - \Pi g_i - H_i(\widehat{\alpha} - \alpha)'w_i|_2 \lesssim |\widehat{\Pi} - \Pi|_2|\widehat{g}_i|_2 + |\Pi|_2|\widehat{g}_i - g_i|_2 + |\widehat{\alpha} - \alpha_0|_2|w_i|_2$ . Then the first term above has

$$|E_n[(D_i/p)\widehat{m}_i(\widehat{m}_i - m_i)']| \leq |\widehat{\Pi} - \Pi|_2 E_n[|\widehat{m}_i|_2|\widehat{g}_i|_2] + |\Pi|_2 E_n[|\widehat{m}_i|_2|\widehat{g}_i - g_i|_2] + |\widehat{\alpha} - \alpha_0|_2 E_n[|\widehat{m}_i|_2|w_i|_2].$$

We claim this term is  $o_p(1)$ . Note that  $|\widehat{\Pi} - \Pi|_2 = o_p(1)$  and  $|\widehat{\alpha} - \alpha_0|_2 = o_p(1)$  by assumption. Then applying Cauchy-Schwarz, it suffices to show  $E_n[|\widehat{m}_i|_2^2 + |\widehat{g}_i|_2^2 + |w_i|_2^2] = O_p(1)$  and  $E_n[|\widehat{g}_i - g_i|_2^2] = o_p(1)$ . First, note  $E_n[|w_i|_2^2] = O_p(1)$  since  $E[|w|_2^2] < \infty$ . Next, note  $E_n[|\widehat{m}_i|_2^2] = E_n[|\widehat{\Pi}\widehat{g}_i - H_i\widehat{\alpha}'w_i|_2^2] \leq 2E_n[|\widehat{\Pi}\widehat{g}_i|_2^2] + 2E_n[|\widehat{\alpha}'w_i|_2^2] \leq 2|\widehat{\Pi}|_2^2 E_n[|\widehat{g}_i|_2^2] + 2|\widehat{\alpha}|_2^2 E_n[|w_i|_2^2]$ , so clearly it suffices to show  $E_n[|\widehat{g}_i|_2^2] = O_p(1)$  to handle this term.

We start by showing that  $E_n[|\widehat{g}_i - g_i|_2^2] = o_p(1)$ . By the mean value theorem  $g_i(\widehat{\theta}) - g_i(\theta_0) = \frac{\partial g_i}{\partial \theta'}(\tilde{\theta}_i)(\widehat{\theta} - \theta_0)$ , where  $\tilde{\theta}_i \in [\theta_0, \widehat{\theta}]$  may change by row. Then we have  $E_n[|g_i(\widehat{\theta}) - g_i(\theta_0)|_2^2] \leq |\widehat{\theta} - \theta_0|_2^2 E_n[|\frac{\partial g_i}{\partial \theta'}(\tilde{\theta}_i)|_2^2]$ , so it suffices to show  $E_n[|\frac{\partial g_i}{\partial \theta'}(\tilde{\theta}_i)|_2^2] = O_p(1)$ . Since  $g_i(\theta) = D_i g_{1i}(\theta) + (1 - D_i)g_{0i}(\theta)$  for all  $\theta$ ,  $|\frac{\partial g_i}{\partial \theta'}(\tilde{\theta}_i)|_2^2 \leq 2|\frac{\partial g_{1i}}{\partial \theta'}(\tilde{\theta}_i)|_2^2 + 2|\frac{\partial g_{0i}}{\partial \theta'}(\tilde{\theta}_i)|_2^2$ . Define the event  $S_n = \{\widehat{\theta} \in U\}$ . Then on  $S_n$  we have

$$\begin{aligned} |\frac{\partial g_{1i}}{\partial \theta'}(\tilde{\theta}_i)|_2^2 + |\frac{\partial g_{0i}}{\partial \theta'}(\tilde{\theta}_i)|_2^2 &\leq |\frac{\partial g_{1i}}{\partial \theta'}(\tilde{\theta}_i)|_F^2 + |\frac{\partial g_{0i}}{\partial \theta'}(\tilde{\theta}_i)|_F^2 = \sum_{d=0,1} \sum_{k=1}^{d_g} |\nabla g_{di}^k(\tilde{\theta}_{ik})|_2^2 \\ &\leq \sum_{d=0,1} \sum_{k=1}^{d_g} \sup_{\theta \in U} |\nabla g_{di}^k(\theta)|_2^2 \equiv \bar{U}_i. \end{aligned}$$

Then  $E_n[|\frac{\partial g_i}{\partial \theta'}(\tilde{\theta}_i)|_2^2] \mathbb{1}(S_n) \leq E_n[\bar{U}_i] \mathbb{1}(S_n) = O_p(1)$  since  $E[\sup_{\theta \in U} |\nabla g_{di}^k(\theta)|_2^2] < \infty$  by assumption. Then  $E_n[|\frac{\partial g_i}{\partial \theta'}(\tilde{\theta}_i)|_2^2] = O_p(1)$  since  $P(S_n^c) \rightarrow 0$ . This finishes the proof of  $E_n[|\widehat{g}_i - g_i|_2^2] = o_p(1)$ . Finally, the claim  $E_n[|\widehat{g}_i|_2^2] = O_p(1)$  is clear since  $E_n[|\widehat{g}_i|_2^2] \leq 2E_n[|\widehat{g}_i - g_i|_2^2] + 2E_n[|g_i|_2^2] = o_p(1) + O_p(1)$  by the preceding claim.

Then we have shown  $|E_n[(D_i/p)\widehat{m}_i(\widehat{m}_i - m_i)']| = o_p(1)$  and  $E_n[(D_i/p)(\widehat{m}_i - m_i)m_i'] = o_p(1)$  by an identical argument. This shows that  $E_n[(D_i/p)\widehat{m}_i\widehat{m}_i'] = E_n[(D_i/p)m_i m_i'] + o_p(1)$ . Next, we have  $E_n[(D_i/p)m_i m_i'] = E_n[(D_i/p)m_{1i}m_{1i}'] = E_n[m_{1i}m_{1i}'] + o_p(1) = E[m_{1i}m_{1i}'] + o_p(1)$ . The first equality is by definition of  $m_i, m_{1i}$ . The second equality by Lemma A.2 of [Cytrynbaum \(2024\)](#) and the third equality by vanilla WLLN, using  $E[|m_i|_2^2] < \infty$ . This finishes our proof of the first statement of (a), and the second statement follows by symmetry.

For (b), note  $E_n[\widehat{m}_i\widehat{m}_i'] = pE_n[\frac{D_i}{p}\widehat{m}_i\widehat{m}_i'] + (1-p)E_n[\frac{1-D_i}{1-p}\widehat{m}_i\widehat{m}_i'] = pE[\frac{D_i}{p}m_{1i}m_{1i}'] + (1-p)E[m_{0i}m_{0i}'] + o_p(1)$  by part (a) of the lemma. Moreover,  $E_n[\widehat{m}_i] = E_n[\widehat{\Pi}\widehat{g}_i - H_i\widehat{\alpha}'w_i] = \widehat{\Pi}E_n[\widehat{g}_i] + o_p(1)$ . Note that  $E_n[\widehat{g}_i] = \widehat{g}(\widehat{\theta})$  and  $\widehat{g}(\widehat{\theta}) - \widehat{g}(\theta_0) = g_0(\widehat{\theta}) - g_0(\theta_0) + o_p(1) = o_p(1)$ . The first equality since  $|\widehat{g} - g_0|_{\Theta, \infty} = o_p(1)$  and the second by continuous mapping, using Lemma 8.8. Then  $\text{Var}_n(\widehat{m}_i) = pE[m_{1i}(\theta_0)m_{1i}(\theta_0)'] + (1-p)E[m_{0i}(\theta_0)m_{0i}(\theta_0)'] + o_p(1)$ ,

finishing the proof.  $\square$

**Lemma 8.15.** *Require Assumptions 3.1, 3.2, 7.1. Then under  $P$  in Definition 8.1, the variables in the statement of Theorem 7.2 have  $\widehat{v}_{10} \xrightarrow{P} E[E[m_{1i}(\theta_0)|\psi]E[m_{0i}(\theta_0)|\psi']]$ ,  $\widehat{v}_1 \xrightarrow{P} E[E[m_{1i}(\theta_0)|\psi]E[m_{1i}(\theta_0)|\psi']]$ , and  $\widehat{v}_0 \xrightarrow{P} E[E[m_{0i}(\theta_0)|\psi]E[m_{0i}(\theta_0)|\psi']]$ .*

*Proof.* Let  $\widehat{v}_1^o$  denote the oracle version of  $\widehat{v}_1$ , substituting  $m_i = \Pi g_i(\theta_0) - H_i \alpha'_0 w_i$  for  $\widehat{m}_i$ , and similarly for  $\widehat{v}_0^o, \widehat{v}_{10}^o$ . In Lemma A.6 of [Cytrynbaum \(2024\)](#), set  $A_i = m_{1i}$  and  $B_i = m_{1i}$ . Applying the lemma componentwise,  $\widehat{v}_1^o \xrightarrow{P} E[E[m_{1i}(\theta_0)|\psi]E[m_{1i}(\theta_0)|\psi']]$ ,  $\widehat{v}_0^o \xrightarrow{P} E[E[m_{0i}(\theta_0)|\psi]E[m_{0i}(\theta_0)|\psi']]$ , and  $\widehat{v}_{10}^o \xrightarrow{P} E[E[m_{1i}(\theta_0)|\psi]E[m_{0i}(\theta_0)|\psi']]$ . Then it suffices to show that  $\widehat{v}_1 - \widehat{v}_1^o = o_p(1)$ ,  $\widehat{v}_0 - \widehat{v}_0^o = o_p(1)$ , and  $\widehat{v}_{10} - \widehat{v}_{10}^o = o_p(1)$ . For the first statement, expand

$$\widehat{v}_1 - \widehat{v}_1^o = (np)^{-1} \sum_{s \in \mathcal{S}_n^\nu} \frac{1}{a(s) - 1} \sum_{i \neq j \in s} D_i D_j (\widehat{m}_i \widehat{m}_j' - m_i m_j')$$

Expand  $\widehat{m}_i \widehat{m}_j' - m_i m_j' = \widehat{m}_i (\widehat{m}_j' - m_j') + (\widehat{m}_i - m_i) m_j' \equiv A_{ij} + B_{ij}$ . Using triangle inequality,  $a(s) - 1 \geq 1$  and  $p > 0$ , we calculate  $\widehat{v}_1^o - \widehat{v}_1 \lesssim n^{-1} \sum_{s \in \mathcal{S}_n^\nu} \sum_{i, j \in s} |A_{ij}|_2 + |B_{ij}|_2 \equiv A_n + B_n$ . First consider  $B_n$ . Using that  $|xy'|_2 \leq |x|_2 |y|_2$ , we have

$$\begin{aligned} |B_{ij}|_2 &\leq |\widehat{m}_i - m_i|_2 |m_j|_2 = |\widehat{\Pi} \widehat{g}_i - \Pi g_i - H_i (\widehat{\alpha} - \alpha)' w_i|_2 |m_j|_2 \\ &\leq |\widehat{\Pi} - \Pi|_2 |\widehat{g}_i|_2 |m_j|_2 + |\Pi|_2 |\widehat{g}_i - g_i|_2 |m_j|_2 + |\widehat{\alpha} - \alpha|_2 |w_i|_2 |m_j|_2. \end{aligned}$$

Then  $B_n = n^{-1} \sum_{s \in \mathcal{S}_n^\nu} \sum_{i, j \in s} |\widehat{\Pi} - \Pi|_2 |\widehat{g}_i|_2 |m_j|_2 + |\Pi|_2 |\widehat{g}_i - g_i|_2 |m_j|_2 + |\widehat{\alpha} - \alpha|_2 |w_i|_2 |m_j|_2 \equiv B_{n1} + B_{n2} + B_{n3}$ . Consider  $B_{n1}$ . This is

$$\begin{aligned} B_{n1} &= |\widehat{\Pi} - \Pi|_2 \cdot n^{-1} \sum_{s \in \mathcal{S}_n^\nu} \sum_{i, j \in s} |\widehat{g}_i|_2 |m_j|_2 \leq |\widehat{\Pi} - \Pi|_2 \cdot (2n)^{-1} \sum_{s \in \mathcal{S}_n^\nu} \sum_{i, j \in s} |\widehat{g}_i|_2^2 + |m_j|_2^2 \\ &\leq |\widehat{\Pi} - \Pi|_2 \cdot (2n)^{-1} \sum_{s \in \mathcal{S}_n^\nu} |s| \sum_{i \in s} |\widehat{g}_i|_2^2 + |m_i|_2^2 \lesssim |\widehat{\Pi} - \Pi|_2 E_n[|\widehat{g}_i|_2^2 + |m_i|_2^2]. \end{aligned}$$

By an identical argument  $B_{n3} \lesssim |\widehat{\alpha} - \alpha|_2 E_n[|w_i|_2^2 + |m_i|_2^2]$ . Then to show  $B_{n1} + B_{n3} = o_p(1)$ , suffices to show  $E_n[|w_i|_2^2 + |m_i|_2^2 + |\widehat{g}_i|_2^2] = O_p(1)$ . That  $E_n[|w_i|_2^2 + |\widehat{g}_i|_2^2] = O_p(1)$  was shown in the proof of Lemma 8.14. Note  $E_n[|m_i|_2^2] = E_n[|\Pi g_i - H_i \alpha'_0 w_i|_2^2] \leq 2E_n[|\Pi g_i|_2^2] + 2E_n[|\alpha'_0 w_i|_2^2] \leq 2|\Pi|_2^2 E_n[|g_i|_2^2] + 2|\alpha_0|_2^2 E_n[|w_i|_2^2] = O_p(1)$  since  $E[|g_i|_2^2] < \infty$  by assumption. Then  $B_{n1} + B_{n3} = o_p(1)$ . Finally, consider  $B_{n2}$ . By the mean value theorem  $g_i(\widehat{\theta}) - g_i(\theta_0) = \frac{\partial g_i}{\partial \theta'}(\tilde{\theta}_i)(\widehat{\theta} - \theta_0)$ , where  $\tilde{\theta}_i \in [\theta_0, \widehat{\theta}]$  may change by row. Then we have

$$\begin{aligned} B_{n2} &= n^{-1} \sum_{s \in \mathcal{S}_n^\nu} \sum_{i, j \in s} |\Pi|_2 |\widehat{g}_i - g_i|_2 |m_j|_2 \leq |\widehat{\theta} - \theta_0|_2 |\Pi|_2 \cdot n^{-1} \sum_{s \in \mathcal{S}_n^\nu} \sum_{i, j \in s} \left| \frac{\partial g_i}{\partial \theta'}(\tilde{\theta}_i) \right|_2 |m_j|_2 \\ &\lesssim |\widehat{\theta} - \theta_0|_2 |\Pi|_2 E_n \left[ \left| \frac{\partial g_i}{\partial \theta'}(\tilde{\theta}_i) \right|_2^2 + |m_i|_2^2 \right] = o_p(1). \end{aligned}$$

The final equality follows since  $E_n[|\frac{\partial g_i}{\partial \theta'}(\tilde{\theta}_i)|_2^2] = O_p(1)$ , as shown in the proof of Lemma 8.14. Then we have shown  $B_n = o_p(1)$ , and  $A_n = o_p(1)$  is identical. This completes the proof that  $\hat{v}_1 - \hat{v}_1^o = o_p(1)$ , and the proof of  $\hat{v}_0 - \hat{v}_0^o = o_p(1)$ , and  $\hat{v}_{10} - \hat{v}_{10}^o = o_p(1)$  are identical.  $\square$

## 8.9 Lemmas

**Proposition 8.16** (Lévy). *Consider probability spaces  $(\Omega_n, \mathcal{G}_n, P_n)$  and  $\sigma$ -algebras  $\mathcal{F}_n \subseteq \mathcal{G}_n$ . We say  $A_n \in \mathbb{R}^d$  has  $A_n|\mathcal{F}_n \Rightarrow A$  if  $\phi_n(t) \equiv E[e^{it'A_n}|\mathcal{F}_n] = E[e^{it'A}|\mathcal{F}_n] + o_p(1)$  for each  $t \in \mathbb{R}^d$ . If  $g : \mathbb{R}^d \rightarrow \mathbb{C}$  is bounded, measurable, and  $P(A \in \{a : g(\cdot) \text{ discontinuous at } a\}) = 0$  then we have*

$$E[g(A_n)|\mathcal{F}_n] = E[g(A)] + o_p(1). \quad (8.2)$$

See [Cytrynbaum \(2021\)](#) for the proof.

**Lemma 8.17.** *The following statements hold*

- (a) *There exists  $\gamma_0 \in \mathbb{R}^{d_h \times d_a}$  solving  $E[\text{Var}(h|\psi)]\gamma_0 = E[\text{Cov}(h, a|\psi)]$ . For any solution, we have  $E[\text{Var}(a - \gamma'_0 h|\psi)] \preceq E[\text{Var}(a - \gamma' h|\psi)]$  for all  $\gamma \in \mathbb{R}^{d_h \times d_a}$ .*
- (b) *Let  $Z = (Z_a, Z_h)$  a random variable with  $\text{Var}(Z) = E[\text{Var}((a, h)|\psi)] \equiv \Sigma$  and define  $\tilde{Z}_a = Z_a - \gamma'_0 Z_h$ . Then  $\text{Cov}(\tilde{Z}_a, Z_h) = 0$ . In particular, if  $(Z_a, Z_h)$  are jointly Gaussian, then  $\tilde{Z}_a$  is Gaussian with  $\tilde{Z}_a \perp\!\!\!\perp Z_h$ .*

*Proof.* In the notation of (b), it suffices to show  $\Sigma_{hh}\gamma_0 = \Sigma_{ha}$ . If  $\text{rank}(\Sigma_{hh}) = 0$  then  $Z_h = c_h$  a.s. for constant  $c_h$  and  $\Sigma_{ha} = \text{Cov}(Z_h, Z_a) = 0$ . Then any  $\gamma \in \mathbb{R}^{d_h \times d_a}$  is a solution. Then suppose  $\text{rank}(\Sigma_{hh}) = r \geq 1$ . Let  $\Sigma_{hh} = U\Lambda U'$  be the compact SVD with  $U \in \mathbb{R}^{d_h \times r}$  and  $\text{rank}(\Lambda) = r$ , and  $U'U = I_r$ . We claim  $Z_h = UU'Z_h$  a.s. Calculate  $\text{Var}((UU' - I)Z_h) = (UU' - I)U\Lambda U'(UU' - I) = 0$ . Note that  $\Sigma_{hh}\gamma = \Sigma_{ha} \iff \text{Var}(Z_h)\gamma = \text{Cov}(Z_h, Z_a) \iff \text{Var}(UU'Z_h)\gamma = \text{Cov}(UU'Z_h, Z_a) \iff U[\text{Var}(U'Z_h)U'\gamma - \text{Cov}(U'Z_h, Z_a)] = 0$ . Define  $\bar{Z}_h = U'Z_h$  and note  $\text{Var}(\bar{Z}_h) = U'U\Lambda U'U = \Lambda \succ 0$ . Then let  $\bar{\gamma} = \text{Var}(\bar{Z}_h)^{-1} \text{Cov}(\bar{Z}_h, a)$  so that  $\text{Var}(\bar{Z}_h)\bar{\gamma} - \text{Cov}(\bar{Z}_h, Z_a) = 0$ . Then it suffices to find  $\gamma$  such that  $U'\gamma = \bar{\gamma}$ . Since  $U' : \mathbb{R}^{d_h} \rightarrow \mathbb{R}^r$  is onto, there exists  $\gamma^k$  with  $U'\gamma^k = \bar{\gamma}^k$ . Then let  $\gamma_0^k \in [\gamma^k + \ker(U')]$  and set  $\gamma_0 = (\gamma_0^k : k = 1, \dots, d_a)$ , so that  $U'\gamma_0 = \bar{\gamma}$ . Then  $\Sigma_{hh}\gamma_0 = \Sigma_{ha}$  by work above. For the optimality statement, calculate

$$\begin{aligned} E[\text{Var}(a - \gamma' h|\psi)] &= \Sigma_{aa} - \Sigma_{ah}\gamma - \gamma'\Sigma_{ha} + \gamma'\Sigma_{hh}\gamma = \Sigma_{aa} - \Sigma_{ah}(\gamma - \gamma_0 + \gamma_0) \\ &\quad - (\gamma - \gamma_0 + \gamma_0)'\Sigma_{ha} + \gamma'\Sigma_{hh}\gamma = \Sigma_{aa} - 2\gamma_0'\Sigma_{hh}\gamma_0 - (\gamma - \gamma_0)'\Sigma_{ha} - \Sigma_{ah}(\gamma - \gamma_0) \\ &\quad + \gamma'\Sigma_{hh}\gamma \propto -(\gamma - \gamma_0)'\Sigma_{hh}\gamma_0 - \gamma_0'\Sigma_{hh}(\gamma - \gamma_0) + \gamma'\Sigma_{hh}\gamma = -(\gamma - \gamma_0)'\Sigma_{hh}\gamma_0 \\ &\quad - \gamma_0'\Sigma_{hh}(\gamma - \gamma_0) + \gamma'\Sigma_{hh}\gamma + (\gamma - \gamma_0 + \gamma_0)'\Sigma_{hh}(\gamma - \gamma_0 + \gamma_0) \\ &= \gamma_0'\Sigma_{hh}\gamma_0 + (\gamma - \gamma_0)'\Sigma_{hh}(\gamma - \gamma_0). \end{aligned}$$

Then  $E[\text{Var}(a - \gamma'h|\psi)] - E[\text{Var}(a - \gamma_0'h|\psi)] = (\gamma - \gamma_0)' \Sigma_{hh} (\gamma - \gamma_0)$  and for any  $a \in \mathbb{R}^d$  we have  $a'(\gamma - \gamma_0)' \Sigma_{hh} (\gamma - \gamma_0)a \geq 0$  since  $\Sigma_{hh} \succeq 0$ . This proves the claim. Finally, we have  $\text{Cov}(\tilde{Z}_a, Z_h) = \text{Cov}(Z_a - \gamma_0' Z_h, Z_h) = \Sigma_{ah} - \gamma_0' \Sigma_{hh} = 0$ . The final statement follows from well-known facts about the normal distribution.  $\square$

**Lemma 8.18** (SVD). *Suppose  $\Sigma \in \mathbb{R}^{m \times m}$  is symmetric PSD with  $\text{rank}(\Sigma) = r$ . Then  $\Sigma = U\Lambda U'$  for  $U \in \mathbb{R}^{m \times r}$  with  $U'U = I_r$  and  $\Lambda$  diagonal.*

*Proof.* Since  $\Sigma$  is symmetric PSD, there exists  $B'B = \Sigma$  for  $\text{rank}(B) = r$ . Let  $VAU'$  be the compact SVD of  $B$ , with  $A$  diagonal. Then  $\Sigma = B'B = UA^2U' \equiv U\Lambda U'$  with  $U'U = I_r$ .  $\square$

**Lemma 8.19.** *Consider probability spaces  $(\Omega_n, \mathcal{G}_n, P_n)$  and  $\sigma$ -algebras  $\mathcal{F}_n \subseteq \mathcal{G}_n$ . Suppose  $0 \leq A_n \leq B < \infty$  and  $A_n = o_p(1)$ . Then  $E[A_n|\mathcal{F}_n] = o_p(1)$ .*

*Proof.* For any  $\epsilon > 0$ , note that  $E[A_n|\mathcal{F}_n] = E[A_n \mathbf{1}(A_n \leq \epsilon)|\mathcal{F}_n] + E[A_n \mathbf{1}(A_n > \epsilon)|\mathcal{F}_n] \leq \epsilon + BP(A_n > \epsilon|\mathcal{F}_n)$ . We have  $E[P(A_n > \epsilon|\mathcal{F}_n)] = P(A_n > \epsilon) = o(1)$  by tower law and assumption. Then  $P(A_n > \epsilon|\mathcal{F}_n) = o_p(1)$  by Markov inequality. Then we have shown  $E[A_n|\mathcal{F}_n] \leq \epsilon + T_n(\epsilon)$  with  $T_n(\epsilon) = o_p(1)$ . Fix  $\delta > 0$  and let  $\epsilon = \delta/2$ . Then  $P(E[A_n|\mathcal{F}_n] > \delta) \leq P(\delta/2 + T_n(\delta/2) > \delta) = P(T_n(\delta/2) > \delta/2) = o(1)$  since  $T_n(\delta/2) = o_p(1)$ . Since  $\delta$  was arbitrary, we have shown that  $E[A_n|\mathcal{F}_n] = o_p(1)$ .  $\square$

**Lemma 8.20.**  $A_n = O_p(1) \iff A_n = o_p(c_n)$  for every sequence  $c_n \rightarrow \infty$ .

*Proof.* It suffices to consider  $A_n \geq 0$ . The forward direction is clear. For the backward direction, suppose for contradiction that there exists  $\epsilon > 0$  such that  $\sup_{n \geq 1} P(A_n > M) > \epsilon$  for all  $M$ . Then find  $n_k$  such that  $P(A_{n_k} > k) > \epsilon$  for each  $k \geq 1$ . We claim  $n_k \rightarrow \infty$ . Suppose not and  $\liminf_k n_k \leq N < \infty$ . Then let  $k(j) \rightarrow \infty$  such that  $n_{k(j)} \leq N$  for all  $j$ . Choose  $M' < \infty$  such that  $P(A_n > M') < \epsilon$  for all  $n = 1, \dots, N$ . Then for  $k(j) > M'$  we have  $P(A_{n_{k(j)}} > k(j)) \leq P(A_{n_{k(j)}} > M') < \epsilon$ , which is a contradiction. Then apparently  $\lim_k n_k = +\infty$ . Define  $Z_j = \{i : i \geq j\}$ . Regard the sequence  $n_k$  as map  $n : \mathbb{N} \rightarrow \mathbb{N}$ . For  $m \in \text{Image}(n)$ , define  $n^\dagger(m) = \min n^{-1}(m)$ . It's easy to see that  $n^\dagger(m_k) \rightarrow \infty$  for  $\{m_k\}_k \subseteq \text{Image}(n)$  with  $m_k \rightarrow \infty$ . Then write

$$\sup_{k \geq j} P(A_{n_k} > k) = \sup_{m \in n(Z_j)} \sup_{a \in n^{-1}(m)} P(A_m > a) \leq \sup_{m \in n(Z_j)} P(A_m > n^\dagger(m))$$

Note  $A_{m_k}/n^\dagger(m_k) = o_p(1)$  by assumption for any  $\{m_k\}_k \subseteq \text{Image}(n)$  with  $m_k \rightarrow \infty$ . Then we have

$$\limsup_k P(A_{n_k} > k) = \limsup_j \sup_{k \geq j} P(A_{n_k} > k) = \lim_j \sup_{m \in n(Z_j)} P(A_m > n^\dagger(m)) = o(1).$$

This is a contradiction, which completes the proof.  $\square$



*Proof of Lemma 8.3.* The first set of statements since  $Q = P$  on  $\mathcal{F}_n$  by definition. Let  $c = P(Z_h \in T)$ , with  $c > 0$  by assumption. Define  $S_n = \{P(\mathcal{I}_n \in T_n | \mathcal{F}_n) \geq c/2\}$ . Then by Lemma 8.5,  $P(\mathcal{I}_n \in T_n | \mathcal{F}_n) \xrightarrow{P} P(Z_h \in T) = c$ , so  $P(S_n) \rightarrow 1$ . We have the upper bound

$$\begin{aligned} \mathbb{1}(S_n)Q(B_n | \mathcal{F}_n) &= \mathbb{1}(S_n)P(B_n | \mathcal{I}_n \in T_n, \mathcal{F}_n) = \mathbb{1}(S_n) \frac{P(B_n, \mathcal{I}_n \in T_n | \mathcal{F}_n)}{P(\mathcal{I}_n \in T_n | \mathcal{F}_n)} \\ &\leq (c/2)^{-1} \mathbb{1}(S_n)P(B_n, \mathcal{I}_n \in T_n | \mathcal{F}_n) \leq (c/2)^{-1} P(B_n | \mathcal{F}_n). \end{aligned}$$

The first equality by definition of  $Q$ . The first inequality by the definition of  $S_n$ . The final inequality by additivity of measures. Then for  $r_n \equiv (1 - \mathbb{1}(S_n))Q(B_n | \mathcal{F}_n)$ , we have  $Q(B_n | \mathcal{F}_n) = \mathbb{1}(S_n)Q(B_n | \mathcal{F}_n) + r_n$ . Note that  $|r_n| \leq 1$  and  $r_n \xrightarrow{P} 0$ , so  $E_Q[r_n] = o(1)$  by modes of convergence. Then expand  $Q(B_n)$  as

$$\begin{aligned} E_Q[Q(B_n | \mathcal{F}_n)] &= E_Q[\mathbb{1}(S_n)Q(B_n | \mathcal{F}_n)] + E_Q[r_n] \leq (c/2)^{-1} E_Q[P(B_n | \mathcal{F}_n)] + o(1) \\ &= (c/2)^{-1} E_P[P(B_n | \mathcal{F}_n)] + o(1) = (c/2)^{-1} P(B_n) + o(1). \end{aligned}$$

The second equality follows from part (a), and the final equality by tower law. The  $o_p(1)$  results follow by setting  $B_n = \{R_n > \epsilon\}$ . The  $O_p(1)$  results follow by the  $o_p(1)$  statement and Lemma 8.20.  $\square$