Assignment #3
Introduction to GR, 2020 Fall
International Centre for Theoretical Sciences
Instructor: Prof. Bala Iyer, Tutor: Md Arif Shaikh[1]
Due on October 18, 2020 11:59 PM.

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1. Problem 7.1, 7.3, 7.6, 7.7, 7.10, 7.11, 7.14 from Exercise of Chapter 7 of D'Inverno's book.

2. (a) Let $x^\alpha(\lambda)$ describe a timelike geodesic parametrized by a nonaffine parameter λ, and let $t^\alpha = dx^\alpha/d\lambda$ be the geodesic's tangent vector. Calculate how $\varepsilon \equiv -t_\alpha t^\alpha$ changes as a function of λ.
   (b) Let ξ^α be a Killing vector. Calculate how $p \equiv \xi_\alpha t^\alpha$ changes as a function of λ on the same geodesic.
   (c) Let b^α be such that in a spacetime with metric $g_{\alpha\beta}$, $\pounds_b g_{\alpha\beta} = 2c g_{\alpha\beta}$, where c is a constant. Let $x^\alpha(\tau)$ describe a timelike geodesic parametrized by proper time τ, and let $u^\alpha = dx^\alpha/d\tau$ be the four-velocity. Calculate how $q \equiv b_\alpha u^\alpha$ changes with τ.

3. Prove that
$$\xi_1^\alpha = \sin\phi\partial_\theta + \cot\theta\cos\phi\partial_\phi \qquad \xi_2^\alpha = -\cos\phi\partial_\theta + \cot\theta\sin\phi\partial_\phi \tag{1}$$
are the Killing vectors of the spherically symmetric spacetime
$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2}$$

4. A particle with electric charge e moves in a spacetime with metric $g_{\alpha\beta}$ in the presence of a vector potential A_α. The equations of motion are $u^\beta\nabla_\beta u_\alpha = eF_{\alpha\beta}u^\beta$, where u^α is the four-velocity and $F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha$. It is assumed that the spacetime possesses a Killing vector ξ^α, so that $\pounds_\xi g_{\alpha\beta} = \pounds_\xi A_\alpha = 0$. Prove that
$$(u_\alpha + eA_\alpha)\xi^\alpha \tag{3}$$
is constant on the world line of the of the charged particle.

5. A particle moving on a circular orbit in a stationary, axially symmetric spacetime is subjected to a dissipative force which drives it to another, slightly smaller, circular orbit. During the transition, the particle loses an amount $\delta\tilde{E}$ of orbital energy (per unit rest mass) and an amount $\delta\tilde{L}$ of orbital angular momentum (per unit rest mass). Show that these quantities are related by $\delta\tilde{E} = \Omega\delta\tilde{L}$, where Ω is the particle's original angular velocity.

   Hints: Express the four-velocity u^α of the particle in terms of the Killing vectors, energy angular momentum and orbital velocity. Find the variation δu^α. Use the normalization condition $u_\alpha u^\alpha = -1$.

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