Assignment #2
Introduction to GR, 2020 Fall
International Centre for Theoretical Sciences
Instructor: Prof. Bala Iyer, Tutor: Md Arif Shaikh[1]
Due on October 04, 2020 11:59 PM.

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1. If s is an affine parameter, show that, under the transformation $s \to \bar{s} = \bar{s}(s)$, the parameter $\bar{s}$ will be affine only if $\bar{s} = \alpha s + \beta$, where α and β are constants.

2. Show that if $t^\alpha = dx^\alpha/d\lambda$ obeys the geodesic equation in the form $Dt^\alpha/d\lambda = kt^\alpha$, then $u^\alpha = dx^\alpha/d\lambda^\star$ satisfies $Du^\alpha/d\lambda^\star = 0$ if $\lambda^\star$ and λ are related by $d\lambda^\star/d\lambda = \exp(\int k(\lambda)d\lambda)$.

3. Prove that if a manifold is affine flat then the connection is necessarily integrable and symmetric.

4. Find the geodesic equation for $\mathbb{R}^3$ in cylindrical polars.

5. Prove that the covariant derivative of the metric tensor vanishes.

6. Establish the theorem that any two-dimensional Riemann manifold is conformally flat in the case of a metric of signature 0, i.e., at any point the metric can be reduced to the diagonal form $(+1 - 1)$ say.

7. Show that the Einstein tensor G_{ab} vanishes if and only if the Ricci tensor R_{ab} vanishes.

8. Let's take the coordinates $x^a = (t, r, \theta, \phi)$ and the line element

$$ds^2 = e^\nu dt^2 - e^\lambda dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2, \tag{1}$$

where $\nu = \nu(t, r)$ and $\lambda = \lambda(t, r)$ are arbitrary functions of t and r.

   (a) Find g_{ab}, g and g^{ab}.
   (b) Calculate the connections using the metric elements.
   (c) Calculate the Riemann tensor.
   (d) Calculate the Ricci tensor, Ricci scalar and the Einstein tensor.

9. The surface of a two-dimensional cone is embedded in three-dimensional flat space. The cone has an opening angle of 2α. Points on the cone which all have the same distance r from the apex define a circle, and ϕ is the angle that runs along the circle.

   (a) Write down the metric of the cone in terms of r and ϕ.
   (b) Find the coordinate transformations $x(r, \phi)$ and $y(r, \phi)$ that brings the metric into the form $ds^2 = dx^2 + dy^2$. Do these coordinates cover the entire two-dimensional plane?
   (c) Prove that any vector parallel transported along a circle of constant r on the surface of the cone ends up rotated by an angle β after a complete trip. Express β in terms of α.

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