Solution to assignment #3
Introduction to GR, 2020 Fall
International Centre for Theoretical Sciences
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3. Prove that

$$\xi_1^\alpha = \sin\phi\partial_\theta + \cot\theta\cos\phi\partial_\phi \qquad \xi_2^\alpha = -\cos\phi\partial_\theta + \cot\theta\sin\phi\partial_\phi \tag{1}$$

are the Killing vectors of the spherically symmetric spacetime

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2}$$

4. A particle with electric charge e moves in a spacetime with metric $g_{\alpha\beta}$ in the presence of a vector potential A_α. The equations of motion are $u^\beta\nabla_\beta u_\alpha = eF_{\alpha\beta}u^\beta$, where u^α is the four-velocity and $F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha$. It is assumed that the spacetime possesses a Killing vector ξ^α, so that $\pounds_\xi g_{\alpha\beta} = \pounds_\xi A_\alpha = 0$. Prove that

$$(u_\alpha + eA_\alpha)\xi^\alpha \tag{3}$$

is constant on the world line of the of the charged particle.

Answer:

$$\begin{aligned}
\frac{d}{d\lambda}\left[(u_\alpha + eA_\alpha)\xi^\alpha\right] &= u^\beta\nabla_\beta\left[(u_\alpha + eA_\alpha)\xi^\alpha\right] \\
&= u^\beta(\nabla_\beta u^\alpha)\xi^\alpha + \underbrace{u^\beta u^\alpha}_{\text{symmetric}} \times \underbrace{\nabla_\beta\xi^\alpha}_{\text{anti-symmetric}} + e\xi^\alpha u^\beta\nabla_\beta A_\alpha + eA_\alpha u^\beta\nabla_\beta\xi^\alpha \\
&= eu^\beta(\nabla_\alpha A_\beta - \cancel{\nabla_\beta A_\alpha})\xi^\alpha + \cancel{e\xi^\alpha u^\beta\nabla_\beta A_\alpha} + eA_\alpha u^\beta\nabla_\beta\xi^\alpha \\
&= eu^\beta(\xi^\alpha\nabla_\alpha A_\beta + A_\alpha\nabla_\beta\xi^\alpha) \\
&= eu^\beta\pounds_\xi A_\alpha \\
&= 0
\end{aligned}$$

5. A particle moving on a circular orbit in a stationary, axially symmetric spacetime is subjected to a dissipative force which drives it to another, slightly smaller, circular orbit. During the transition, the particle loses an amount $\delta\tilde{E}$ of orbital energy (per unit rest mass) and an amount $\delta\tilde{L}$ of orbital angular momentum (per unit rest mass). Show that these quantities are related by $\delta\tilde{E} = \Omega\delta\tilde{L}$, where Ω is the particle's original angular velocity.

Hints: Express the four-velocity u^α of the particle in terms of the Killing vectors, energy angular momentum and orbital velocity. Find the variation δu^α. Use the normalization condition $u_\alpha u^\alpha = -1$.

Answer: The spacetime possesses the Killing vectors $t^\alpha = \delta^\alpha_t$ and $\phi^\alpha = \delta^\alpha_\phi$. In terms of these we can write down the four-velocity u^α as

$$u^\alpha = u^t t^\alpha + u^\phi\phi^\alpha = u^t(t^\alpha + \Omega\phi^\alpha), \qquad \Omega = \frac{u^\phi}{u^t} \tag{4}$$

as the particle moves in a circular orbit. Using the normalization condition $u_\alpha u^\alpha = -1$ we find

$$u_\alpha u^\alpha = u_t u^t + u_\phi u^t\Omega = -1 \tag{5}$$

or,

$$u^t(u_t + u_\phi\Omega) = -1 \tag{6}$$

Now the two constants of motion from the two Killing vectors are $E = -t^\alpha u_\alpha = -u_t$ and $L = \phi^\alpha u_\alpha = u_\phi$. Thus in terms of E and L, u^t becomes

$$u^t = \frac{1}{E - \Omega L} \tag{7}$$

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and hence the four-velocity could be written as

$$u^\alpha = \frac{(t^\alpha + \Omega\phi^\alpha)}{(E - \Omega L)} \tag{8}$$

The variation of the four-velocity would be given by

$$\delta u^\alpha = -\frac{(t^\alpha + \Omega\phi^\alpha)}{(E - \Omega L)^2}(\delta E - \Omega\delta L) = -u^\alpha u^t(\delta E - \Omega\delta L) \tag{9}$$

contracting the above with u_α gives

$$u_\alpha \delta u^\alpha = u^t(\delta E - \Omega\delta L) \tag{10}$$

Now, since $u_\alpha u^\alpha = -1$, variation of it should vanish which implies $u_\alpha \delta u^\alpha = 0$. This provides the relation

$$\boxed{\delta E = \Omega\delta L}. \tag{11}$$