Assignment #6
Introduction to GR, 2020 Fall
International Centre for Theoretical Sciences
Instructor: Prof. Bala Iyer, Tutor: Md Arif Shaikh[1]
Due on Dec 9, 2020 11:59 PM.

---

1. (a) Show that under an infinitesimal coordinate transformation $x^{\text{new}\mu} = x^{\text{old}\mu} + \xi^\mu$, $h_{\mu\nu}$ transforms as

$$h^{\text{new}}_{\mu\nu} = h^{\text{old}}_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu \tag{1}$$

and the trace reversed metric perturbation transforms as

$$\bar{h}^{\text{new}}_{\mu\nu} = \bar{h}^{\text{old}}_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu + \eta_{\mu\nu}\partial_\alpha \xi^\alpha \tag{2}$$

   (b) Assuming solution of the form $\bar{h}_{\mu\nu} = A_{\mu\nu}\exp(ik_\alpha x^\alpha)$ and $\xi_\mu = B_\mu \exp(ik_\alpha x^\alpha)$, express eq. 2 in terms of $A_{\mu\nu}$ and B_μ. Find the Lorenz gauge condition on $A_{\mu\nu}$. Show that $A^{\text{new}}_{\mu\nu}$ satisfies the gauge condition if $A^{\text{old}}_{\mu\nu}$ does.

   (c) Find the constrain on B_μ in order to make $A_{\mu\nu}$ traceless.

   (d) Show that the condition $A_{\mu\nu}u^\nu = 0$, imposes only three constraints on B_μ. (u^μ is a consant time-like four-velocity). Do this by showing that the combination $k^\mu(A_{\mu\nu}u^\nu)$ vanishes for any B_μ.

   (e) Use the results of the previous two questions to solve for B^μ as a function of k^μ, $A^{\text{old}}_{\mu\nu}$ and u^μ.

   (f) Show that it is possible to choose ξ^α in eq. 1 to make any superposition of plane waves satisfy the equations $A_{\mu\nu}u^\mu = 0$ and $A^\mu_\mu = 0$, so that these are generally applicable to gravitational waves of any sort.

   (g) Show that we cannot achieve $A_{\mu\nu}u^\mu = 0$ and $A^\mu_\mu = 0$ for a static solution, i.e. one for which $\omega = 0$.

2. The equation $(\nabla^2 + \Omega^2)B_{\mu\nu} = -16\pi S_{\mu\nu}$ in the vacuum region outside the source – i.e. where $S_{\mu\nu} = 0$ – can be solved by separation of variables. Assume a solution for $B_{\mu\nu}$ of the form $\sum_{lm} A^{lm}_{\mu\nu} f_l(r) Y_{lm}(\theta,\phi)/\sqrt{r}$, where Y_{lm} is the spherical harmonic.

   (a) Show that $f_l(r)$ satisfies the equation

$$f_l^{''} + \frac{1}{r}f_l^{'} + (\Omega^2 - \frac{(l+\frac{1}{2})^2}{r^2})f_l = 0 \tag{3}$$

   (b) Show that the most general spherically symmetric solution is given by

$$B_{\mu\nu} = \frac{A_{\mu\nu}}{r}e^{i\Omega r} + \frac{Z_{\mu\nu}}{r}e^{-i\Omega r}. \tag{4}$$

3. Show that

$$\oint \vec{n}.\nabla B_{\mu\nu} dS \approx -4\pi A_{\mu\nu}. \tag{5}$$

(Assume that the source is nonzero only inside a sphere of radius $\epsilon \ll 2\pi/\Omega$.)

[1] arif.shaikh@icts.res.in