Solution to assignment #2
Introduction to GR, 2020 Fall
International Centre for Theoretical Sciences
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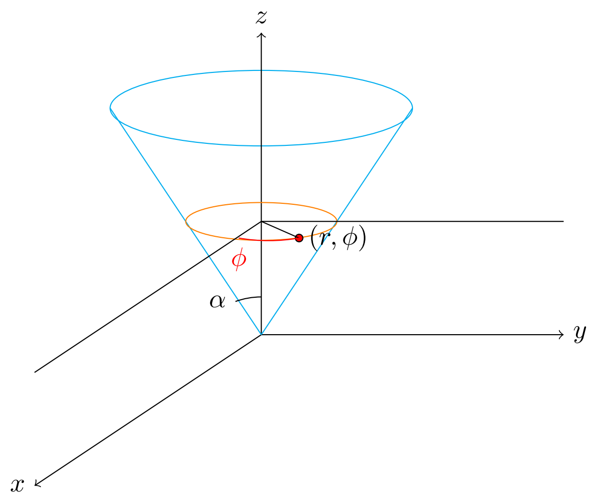
1 Parallel transport along r = constant line on the surface of a cone



We have a cone with opening angle 2α embedded in the 3-dimensional flat space. r is the distance measured from the apex. A constant r line on the surface of the cone would be parameterised by the angle ϕ. The 3-dimensional flat metric in spherical polar coordinate is given by

$$ds_{3d}^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{1}$$

θ is the angle w.r.t to the z-axis. On the surface of the cone, $\theta = \alpha =$ constant. Thus the metric on the surface of the cone becomes

$$\boxed{ds^2 = dr^2 + r^2 \sin^2\alpha d\phi^2.} \tag{2}$$

What is parallel transport of a tensor? Parallel transport is defined in the following way: A tensor field $T^{\mu\cdots}_{\nu\cdots}$ is said to be *parallel transported* along a curve $\gamma := x^\alpha(\lambda)$ if the covariant derivative of the tensor field along the curve vanishes: $D_\lambda T^{\mu\cdots}_{\nu\cdots} = u^\eta \nabla_\eta T^{\mu\cdots}_{\nu\cdots} = 0$, where $u^\eta = dx^\eta/d\lambda$.

In the given problem we are asked to compute the rotation of a vector V^μ when it is parallel transported along the $r =$ constant line from $\phi = 0$ to $\phi = 2\pi$. Thus the curve γ is represented by $x^\mu(\lambda) = (r_0, \phi(\lambda))$. The condition for parallel transport of V^μ along γ then becomes

$$u^\mu \nabla_\mu V^\nu = 0 \to \dot{\phi}\nabla_\phi V^\mu = 0 \to \nabla_\phi V^\mu = 0 \to \boxed{\partial_\phi V^\mu + \Gamma^\mu_{\phi\nu} V^\nu = 0}. \tag{3}$$

To find out the Christofell symbols we use the geodesic equations. The geodesic equations could be obtained from the Lagrangian $L = (1/2) g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$ where $\dot{x}^\mu = dx^\mu/d\tau$ (we take $\lambda = \tau$ the propertime). The geodesic e.o.m is given by

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x^\mu}}\right) - \frac{\partial L}{\partial x^\mu} = 0 \tag{4}$$

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With the Lagrangian given by

$$L = \frac{1}{2}(\dot{r}^2 + r^2 \sin^2\alpha\dot{\phi}^2) \tag{5}$$

The e.o.m for r becomes

$$\ddot{r} - r\sin^2\alpha\dot{\phi}^2 = 0, \tag{6}$$

which provides $\Gamma^r_{\phi\phi} = -r\sin^2\alpha$. Similarly e.o.m for ϕ is given by

$$\ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} = 0, \tag{7}$$

which provides $\Gamma^\phi_{r\phi} = \Gamma^r_{\phi r} = \frac{1}{r}$. Thus we have

$$\boxed{\Gamma^r_{\phi\phi} = -r\sin^2\alpha, \qquad \Gamma^\phi_{r\phi} = \Gamma^r_{\phi r} = \frac{1}{r}}. \tag{8}$$

Thus Eq. (3) becomes

$$\partial_\phi V^\mu + \Gamma^\mu_{\phi\nu}V^\nu = 0. \tag{9}$$

For $\mu = r, \phi$ this gives the following equations

$$\partial_\phi V^r + \Gamma^r_{\phi\nu}V^\nu = 0, \qquad \partial_\phi V^\phi + \Gamma^\phi_{\phi\nu}V^\nu = 0. \tag{10}$$

which using Eq. (8) becomes

$$\partial_\phi V^r + \Gamma^r_{\phi\nu}V^\nu = 0, \qquad \partial_\phi V^\phi + \Gamma^\phi_{\phi\nu}V^\nu = 0. \tag{11}$$

or after differentiating with respect to ϕ,

$$\partial^2_\phi V^r + \sin^2\alpha V^r = 0, \qquad \partial^2_\phi V^\phi + \sin^2\alpha V^\phi = 0 \tag{12}$$

Which has the following general solutions

$$\boxed{V^r = A^r_1 \sin(\sin\alpha\phi) + A^r_2\cos(\sin\alpha\phi), \quad V^\phi = A^\phi_1\sin(\sin\alpha\phi) + A^\phi_2\cos(\sin\alpha\phi)}. \tag{13}$$

Denoting the values of the components at $\phi = 0$ with suffix '0' the solutions becomes

$$\boxed{V^r = A^r_1 \sin(\sin\alpha\phi) + V^r_0\cos(\sin\alpha\phi), \quad V^\phi = A^\phi_1\sin(\sin\alpha\phi) + V^\phi_0\cos(\sin\alpha\phi)}. \tag{14}$$

Now, the constants are not all independent, there are only two independent constants. Thus the other two components could be written in terms of the A^r_0 and A^ϕ_0 using equation (11). This provides

$$A^r_1 = r\sin\alpha V^\phi_0, \quad A^\phi_1 = -V^r_0/r\sin\alpha. \tag{15}$$

Thus finally we have the solutions

$$\boxed{V^r = V^\phi_0 r\sin\alpha\sin(\phi\sin\alpha) + V^r_0\cos(\phi\sin\alpha), \quad V^\phi = -\frac{V^r_0}{r\sin\alpha}\sin(\phi\sin\alpha) + V^\phi_0\cos(\phi\sin\alpha)}. \tag{16}$$

Now we move to a orthonormal set of basis given by $E_r = \partial_r, E_\phi = (1/\sin\alpha)\partial_\phi$. In this basis, we can rewrite the components of the V as

$$\hat{V}^r = V^r, \quad \hat{V}^\phi = r\sin\alpha V^\phi. \tag{17}$$

Thus in this basis, we have the transformation

$$\boxed{\begin{bmatrix}\hat{V}^r \\ \hat{V}^\phi\end{bmatrix} = \begin{bmatrix}\cos(\phi\sin\alpha) & \sin(\phi\sin\alpha) \\ -\sin(\phi\sin\alpha) & \cos(\phi\sin\alpha)\end{bmatrix}\begin{bmatrix}\hat{V}^r_0 \\ \hat{V}^\phi_0\end{bmatrix}} \tag{18}$$

Thus the components are rotated by an angle $\beta = \phi\sin\alpha$. For $\alpha = 2\pi$ this rotation equals $\beta = 2\pi\sin\alpha$.