Solution to assignment #4
Introduction to GR, 2020 Fall
International Centre for Theoretical Sciences
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4. (a) Consider a spacetime with a Killing vector ξ. Show that if the energy-momentum tensor $T^{\mu\nu}$ satisfies the conservation equation
$$\nabla_\mu T^{\mu\nu} = 0, \tag{1}$$
then the four-vector $Q^\mu := T^{\mu\nu}\xi_\nu$ also satisfies a conservation law $\nabla_\mu Q^\mu = 0$.

(b) Now let us consider a flow consisting of perfect fluid and governed by adiabatic equation of state. Using first law of thermodynamics and specific enthalpy $h := (p+\varepsilon)/\rho$, where p is pressure density, ε is the internal energy density and ρ is the fluid density, show that the conservation law in Eq. (1) provides
$$u^\mu \nabla_\mu (h u_\nu) = \frac{\nabla_\nu p}{\rho} \tag{2}$$

(c) Using Eq. (2), find the expression for $\pounds_u(hu_\mu)$. Contracting it with ξ, find the expression for $\pounds_u(hu_\mu\xi^\mu)$ in terms of $\pounds_\xi p$ and $\pounds_\xi h$.

(d) Let's also assume that the flow is invariant under the same symmetry group of the spacetime which provieds the Killing vector ξ. Mathematically this gives the condition
$$\pounds_\xi(B) = 0, \tag{3}$$
where B is any tensor field associated with the flow, e.g., pressure, density etc. Show that this implies that $hu_\mu\xi^\mu$ is conserved along the flow lines.

(e) Find the conserved quantity along the flow lines when the spacetime is stationary and the flow is stationary. This is the *general relativistic Bernoulli constant*.

(f) Find the Newtonian limit and check that the quantity reduces to the familiar expression of Bernoulli equation.

Answer:

(a) $\nabla_\mu (T^{\mu\nu}\xi_\nu) = \underbrace{\nabla_\mu T^{\mu\nu}}_{=0 \text{ due to conservation law}} \times \xi_\nu + \underbrace{T^{\mu\nu}}_{\text{symmetric}} \times \underbrace{\nabla_\mu \xi_\nu}_{\text{anti-symmetric}} = 0.$

(b) The energy-momentum tensor for perfect fluid is given by $T^{\mu\nu} = (p+\varepsilon)v^\mu v^\nu + p g^{\mu\nu}$.

$$\begin{aligned}
\nabla_\mu T^{\mu\nu} &= \nabla_\mu[(p+\varepsilon)v^\mu v^\nu] + g^{\mu\nu}\nabla_\mu p \quad \text{metric is divergenceless} \\
&= \nabla_\mu (h v^\nu \times \rho v^\mu) + g^{\mu\nu}\nabla_\mu p \\
&= \nabla_\mu (h v^\nu)\rho v^\mu + g^{\mu\nu}\nabla_\mu p \quad \text{using} \quad \nabla_\mu(\rho v^\mu) = 0 \\
&= 0.
\end{aligned}$$

This gives
$$\boxed{v^\mu \nabla_\mu (h v^\nu) = -\frac{g^{\mu\nu}\nabla_\mu p}{\rho} = -\frac{\nabla^\nu p}{\rho}.} \tag{4}$$

(c)
$$\begin{aligned}
\pounds_v(hv_\mu) &= v^\nu \nabla_\nu (h v_\mu) + h v_\nu \nabla_\mu v^\nu \\
&= -\frac{\nabla_\mu p}{\rho}
\end{aligned}$$

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The second term in first line vanishes due to the $v^\mu v_\mu = -1$. Contracting with ξ gives

$$\xi^\mu \pounds_v (h v_\mu) = -\frac{1}{\rho}\xi^\mu \nabla_\mu p$$

Therefore,

$$\begin{aligned}
\pounds_v(\xi^\mu h v_\mu) &= \xi^\mu \pounds_v (h v_\mu) + h v_\mu \pounds_v \xi^\mu \\
&= -\frac{\xi^\mu}{\rho}\nabla_\mu p + h v_\mu v^\nu \nabla_\nu \xi^\mu - h v_\mu \xi^\nu \nabla_\nu v^\mu \\
&= -\frac{\xi^\mu}{\rho}\nabla_\mu p \\
&= -\frac{\pounds_\xi p}{\rho}.
\end{aligned}$$

In the second line, the second term vanishes because $\nabla_\nu \xi^\mu$ is anti-symmetric but $v_\mu v^\nu$ is symmetric. The third term vanishes using $v^\mu v_\mu = -1$.

(d) Using $\pounds_\xi p = 0$ gives $\pounds_v(\xi^\mu h v_\mu) = 0$.

$$\pounds_v(\xi^\mu h v_\mu) = v^\nu \nabla_\nu (\xi^\mu h v_\mu) = 0$$

which implies that $\xi^\mu h v_\mu$ is constant along the flow line.

(e) In stationary spacetime we have the stationary Killing vector $\xi^\mu = \delta^\mu_t$. This provides the constant $h v_t$ along the flow lines.

(f)

$$\begin{aligned}
h v_t &= \frac{\varepsilon + p}{\rho} v_t \\
&= \frac{\rho(c^2 + \epsilon) + p}{\rho} v_t \\
&= (c^2 + \epsilon + \frac{p}{\rho}) v_t \quad \epsilon \ll c^2, \text{in non-relativistic limit} \\
&= (c^2 + \frac{p}{\rho})(-1 - \frac{\phi}{c^2} - \frac{1}{2}\frac{v_i v^i}{c^2}) \\
&= -c^2 - \phi - \frac{1}{2} v_i v^i - \frac{p}{\rho} + \mathcal{O}(1/c^2)
\end{aligned}$$

ϵ is the thermal energy and ϕ is the potential energy. This gives the newtonian limit of Bernoulli constant $\boxed{\phi + (1/2) v_i v^i + p/\rho}$.