Solution to assignment #3
Introduction to GR, 2020 Fall
International Centre for Theoretical Sciences
Instructor: Prof. Bala Iyer, Tutor: Md Arif Shaikh[1]

---

3. Prove that

$$\xi^1 = \sin\phi\partial_\theta + \cot\theta\cos\phi\partial_\phi \qquad \xi^2 = -\cos\phi\partial_\theta + \cot\theta\sin\phi\partial_\phi \tag{1}$$

are the Killing vectors of the spherically symmetric spacetime

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2}$$

Answer: Killing vectors satisfy the Killing equation

$$\nabla_\alpha\xi_\beta + \nabla_\beta\xi_\alpha = 0. \tag{3}$$

Thus we need to show that $\nabla_\alpha\xi_\beta$ is an anti-symmetric tensor. We do it for ξ^1. The contravariant components are

$$\xi^1_t = \xi^1_r = 0, \quad \xi^1_\theta = g_{\theta\theta}\xi^{1\theta} = r^2\sin\phi, \quad \xi^1_\phi = g_{\phi\phi}\xi^{1\phi} = r^2\sin^2\theta\cot\theta\cos\phi = r^2\sin\theta\cos\theta\cos\phi. \tag{4}$$

Since only ξ_θ and ξ_ϕ are non-vanishing, the relevant Christofell symbols are $\Gamma^\theta_{\alpha\beta}$ and $\Gamma^\phi_{\alpha\beta}$. These we get from the geodesic equation for θ and ϕ using $\mathcal{L}^2 = (1/2)g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta$ as the Lagrangian. For θ, we have

$$\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta\cos\theta\ddot{\phi} = 0, \tag{5}$$

which provides

$$\Gamma^\theta_{r\theta} = \frac{1}{r}, \quad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta. \tag{6}$$

Similarly for ϕ,

$$\ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} + 2\cot\theta\dot{\theta}\dot{\phi} = 0, \tag{7}$$

which provides

$$\Gamma^\phi_{r\phi} = \frac{1}{r}, \quad \Gamma^\phi_{\theta\phi} = \cot\theta. \tag{8}$$

Now that we have the required Christofell symbols, we check that $\nabla_\alpha\xi^1_\beta$ is an anti-symmetric tensor.

$$\boxed{\nabla_t\xi^1_t = \nabla_r\xi^1_r = 0}$$

trivially. and

$$\boxed{\nabla_\theta\xi^1_\theta = 0, \quad \nabla_\phi\xi^1_\phi = \partial_\phi\xi^1_\phi - \Gamma^\phi_{\phi\phi} = 0}$$

$\nabla_t\xi^1_r = -\nabla_r\xi^1_t = 0$. $\nabla_t\xi^1_\theta = 0 = -\nabla_\theta\xi^1_t$. $\nabla_t\xi^1_\phi = 0 = -\nabla_\phi\xi^1_t$.

$$\begin{aligned}\nabla_r\xi^1_\theta &= \partial_r\xi^1_\theta - \Gamma^\theta_{r\theta}\xi^1_\theta \\ &= 2r\sin\phi - \frac{1}{r}r^2\sin\phi = r\sin\phi.\end{aligned}$$

[1] arif.shaikh@icts.res.in

$$\boxed{\nabla_\theta \xi^1_r = -\Gamma^\theta_{r\theta}\xi^1_\theta = -r\sin\phi = -\nabla_r \xi^1_\theta}.$$

$$\begin{aligned}
\nabla_r \xi^1_\phi &= \partial_r \xi^1_\phi - \Gamma^\phi_{r\phi}\xi^1_\phi \\
&= 2r\sin\theta\cos\theta\cos\phi - \frac{1}{r}r^2\sin\theta\cos\theta\cos\phi \\
&= r\sin\theta\cos\theta\cos\phi.
\end{aligned}$$

$$\boxed{\nabla_\phi \xi^1_r = -\Gamma^\phi_{r\phi}\xi^1_\phi = -r\sin\theta\cos\theta\cos\phi = -\nabla_r \xi^1_\phi}.$$

$$\begin{aligned}
\nabla_\theta \xi^1_\phi &= \partial_\theta \xi^1_\phi - \Gamma^\phi_{\theta\phi}\xi^1_\phi \\
&= r^2\cos 2\theta\cos\phi - \cot\theta r^2\sin\theta\cos\theta\cos\phi. \\
&= r^2\cos\phi(\cos 2\theta - \cos^2\theta) \\
&= r^2\cos\phi(-\sin^2\theta)
\end{aligned}$$

$$\begin{aligned}
\nabla_\phi \xi^1_\theta &= \partial_\phi \xi^1_\theta - \Gamma^\phi_{\theta\phi}\xi^1_\phi \\
&= r^2\cos\phi - \cot\theta r^2\sin\theta\cos\theta\cos\phi \\
&= r^2\cos\phi(1-\cos^2\theta) \\
&= r^2\cos\phi\sin^2\theta \\
&\Rightarrow \boxed{\nabla_\phi \xi^1_\theta = -\nabla_\theta \xi^1_\phi}
\end{aligned}$$

Similarly, we can check for ξ^2.

4. A particle with electric charge e moves in a spacetime with metric $g_{\alpha\beta}$ in the presence of a vector potential A_α. The equations of motion are $u^\beta \nabla_\beta u_\alpha = eF_{\alpha\beta}u^\beta$, where u^α is the four-velocity and $F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha$. It is assumed that the spacetime possesses a Killing vector ξ^α, so that $\pounds_\xi g_{\alpha\beta} = \pounds_\xi A_\alpha = 0$. Prove that

$$(u_\alpha + eA_\alpha)\xi^\alpha \tag{9}$$

is constant on the world line of the of the charged particle.

Answer:

$$\begin{aligned}
\frac{d}{d\lambda}\left[(u_\alpha + eA_\alpha)\xi^\alpha\right] &= u^\beta \nabla_\beta \left[(u_\alpha + eA_\alpha)\xi^\alpha\right] \\
&= u^\beta(\nabla_\beta u^\alpha)\xi^\alpha + \underbrace{u^\beta u^\alpha}_{\text{symmetric}} \times \underbrace{\nabla_\beta \xi^\alpha}_{\text{anti-symmetric}} + e\xi^\alpha u^\beta \nabla_\beta A_\alpha + eA_\alpha u^\beta \nabla_\beta \xi^\alpha \\
&= eu^\beta(\nabla_\alpha A_\beta - \cancel{\nabla_\beta A_\alpha})\xi^\alpha + \cancel{e\xi^\alpha u^\beta \nabla_\beta A_\alpha} + eA_\alpha u^\beta \nabla_\beta \xi^\alpha \\
&= eu^\beta(\xi^\alpha \nabla_\alpha A_\beta + A_\alpha \nabla_\beta \xi^\alpha) \\
&= eu^\beta \pounds_\xi A_\alpha \\
&= 0
\end{aligned}$$

5. A particle moving on a circular orbit in a stationary, axially symmetric spacetime is subjected to a dissipative force which drives it to another, slightly smaller, circular orbit. During the transition, the particle loses an amount $\delta\tilde{E}$ of orbital energy (per unit rest mass) and an amount $\delta\tilde{L}$ of orbital angular momentum (per unit rest mass). Show that these quantities are related by $\delta\tilde{E} = \Omega\delta\tilde{L}$, where Ω is the particle's original angular velocity.

Hints: Express the four-velocity u^α of the particle in terms of the Killing vectors, energy angular momentum and orbital velocity. Find the variation δu^α. Use the normalization condition $u_\alpha u^\alpha = -1$.

Answer: The spacetime possesses the Killing vectors $t^\alpha = \delta^\alpha_t$ and $\phi^\alpha = \delta^\alpha_\phi$. In terms of these we can write down the four-velocity u^α as

$$u^\alpha = u^t t^\alpha + u^\phi \phi^\alpha = u^t(t^\alpha + \Omega\phi^\alpha), \qquad \Omega = \frac{u^\phi}{u^t} \tag{10}$$

as the particle moves in a circular orbit. Using the normalization condition $u_\alpha u^\alpha = -1$ we find

$$u_\alpha u^\alpha = u_t u^t + u_\phi u^t \Omega = -1 \tag{11}$$

or,

$$u^t(u_t + u_\phi \Omega) = -1 \tag{12}$$

Now the two constants of motion from the two Killing vectors are $E = -t^\alpha u_\alpha = -u_t$ and $L = \phi^\alpha u_\alpha = u_\phi$. Thus in terms of E and L, u^t becomes

$$u^t = \frac{1}{E - \Omega L} \tag{13}$$

and hence the four-velocity could be written as

$$u^\alpha = \frac{(t^\alpha + \Omega\phi^\alpha)}{(E - \Omega L)} \tag{14}$$

The variation of the four-velocity would be given by

$$\delta u^\alpha = -\frac{(t^\alpha + \Omega\phi^\alpha)}{(E - \Omega L)^2}(\delta E - \Omega\delta L) = -u^\alpha u^t(\delta E - \Omega\delta L) \tag{15}$$

contracting the above with u_α gives

$$u_\alpha \delta u^\alpha = u^t(\delta E - \Omega\delta L) \tag{16}$$

Now, since $u_\alpha u^\alpha = -1$, variation of it should vanish which implies $u_\alpha \delta u^\alpha = 0$. This provides the relation

$$\boxed{\delta E = \Omega\delta L}. \tag{17}$$