Assignment #4
Introduction to GR, 2020 Fall
International Centre for Theoretical Sciences
Instructor: Prof. Bala Iyer, Tutor: Md Arif Shaikh[1]
Due on November 1, 2020 11:59 PM.

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1. From d'Inverno's book, 10.3, 10.5, 10.6, 10.7, 10.8, 10.9, 10.10.
2. From d'Inverno's book, 12.2, 12.4, 12.5, 12.6, 12.9, 12.10, 12.11, 12.12.
3. From d'Inverno's book, 14.1, 14.2, 14.6, 14.9, 14.13, 14.14, 14.15.
4. (a) Consider a spacetime with a Killing vector ξ. Show that if the energy-momentum tensor $T^{\mu\nu}$ satisfies the conservation equation

$$\nabla_\mu T^{\mu\nu} = 0, \tag{1}$$

then the four-vector $Q^\mu := T^{\mu\nu}\xi_\nu$ also satisfies a conservation law $\nabla_\mu Q^\mu = 0$.

(b) Now let us consider a flow consisting of perfect fluid and governed by adiabatic equation of state. Using first law of thermodynamics and specific enthalpy $h := (p+\varepsilon)/\rho$, where p is pressure density, ε is the internal energy density and ρ is the fluid density, show that the conservation law in Eq. (1) provides

$$u^\mu \nabla_\mu (h u_\nu) = \frac{\nabla_\nu p}{\rho} \tag{2}$$

(c) Using Eq. (2), find the expression for $\pounds_u(hu_\mu)$. Contracting it with ξ, find the expression for $\pounds_u(hu_\mu\xi^\mu)$ in terms of $\pounds_\xi p$ and $\pounds_\xi h$.

(d) Let's also assume that the flow is invariant under the same symmetry group of the spacetime which provieds the Killing vector ξ. Mathematically this gives the condition

$$\pounds_\xi(B) = 0, \tag{3}$$

where B is any tensor field associated with the flow, e.g., pressure, density etc. Show that this implies that $hu_\mu\xi^\mu$ is conserved along the flow lines.

(e) Find the conserved quantity along the flow lines when the spacetime is stationary and the flow is stationary. This is the *general relativistic Bernoulli constant*.

(f) Find the Newtonian limit and check that the quantity reduces to the familiar expression of Bernoulli equation.

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