ICTS Graduate Course PHY-404.5: Physics of Compact Objects
Tutorials

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(Dated: January 16, 2021)

Assignment 1: Stellar Structure

1. Assuming the equations of hydrostatic, radiative and thermal equilibria, derive the following equations describing the structure of homologous stars.

$$\frac{dp}{dm} = -\frac{m}{x^4}, \quad \frac{dx}{dm} = \frac{t^b}{x^2 p^a}, \quad \frac{dt}{dm} = -\frac{p^{an}\, l}{x^4 t^{3+s+bn}}, \quad \frac{dl}{dm} = A p^{a\lambda}\, t^{\nu - b\lambda}, \tag{0.1}$$

where the dimensionless quantities p, x, t and l are defined in terms of the central pressure P_c, central temperature T_c and the scaling variables M, R and L.

$$P = P_c p, \quad T = T_c t, \quad r = Rx, \quad L_r := L(r) = Ll, M_r := M(r) = Mm, \tag{0.2}$$

and assuming the following power-law approximations for the density ρ, opacity κ and energy density density ϵ:

$$\rho = \rho_0 P^a T^{-b}, \quad \kappa = \kappa_0 \rho^n T^{-s}, \quad \epsilon = \epsilon_0 \rho^\lambda T^\nu. \tag{0.3}$$

The scaling variables M, R and L are defined by the following conditions

$$\frac{GM^2}{4\pi R^4 P_c} = 1, \quad \frac{M}{4\pi R^3 \rho_0} \frac{T_c^b}{P_c^a} = 1, \quad \frac{3\kappa_0 \rho_0^n}{64\pi^2 ac} \frac{P_c^{an} ML}{T_c^{4+s+bn} R^4} = 1, \tag{0.4}$$

while A is given by

$$A = \frac{3\kappa_0 \epsilon_0 \rho_0^{n+\lambda}}{16\pi Gac} \frac{P_c^{a(n+\lambda)+1}}{T_c^{4+s+b(n+\lambda)-\nu}}. \tag{0.5}$$

2. Numerically solve the set of equations Eq. (0.1) assuming the boundary conditions: $p(0) = t(0) = 1, x(0) = l(0) = 0$ and the following power-law indices:

   - *Ideal gas equation of state*: $a = b = 1, \rho_0 = \mu/\mathcal{R}$ where $\mu \simeq 0.5X^{-0.57}$ is the mean molecular weight, $\mathcal{R} = 8.3 \times 10^7$ ergs/mol/K is the gas constant and X is the mass fraction of hydrogen (assume $X \simeq 0.75$).
   - *Electron scattering opacity*: $n = s = 0, \kappa_0 = 0.2(1 + X)$ cm^2/g.
   - *Energy generation by $p - p$ chain*: $\lambda = 1, \nu = 4, \epsilon_0 \sim 10^{-30} X^2$ ergs cm^3 g^{-2} s^{-1} K^{-4}.

   You can start by solving the dimensionless equations Eq.(0.1) by setting $A = A_0$, some constant. (Numerical experimentation shows that $p(m)$ goes to zero only for the choice $A_0 \lesssim 0.55$). Plot $p(m), x(m), l(m)$ and $t(m)$. Identify the value of m at which the $p(m)$ goes to zero, denoted by $m_\star$; i.e. $p(m_\star) = 0$. Physically we require that $t(m)$ should go to zero at the same value of m; i.e., $t(m_\star) = 0$. This is satisfied only for a specific value of A, say $A_\star$. Using $A = A_\star$, re-plot $p(m), x(m), l(m)$ and $t(m)$. These relations are independent of the mass of the star.

3. Setting $A = A_\star$ in Eq.(0.5), we can express P_c fully in terms of T_c. This allows us to express R, T_c and L in terms of M using Eqs.(0.4). Now, the only free parameter is $M := M_\star/m_\star$, where $M_\star$ is the actual mass of 4the star. Now plot the stellar structure quantities $M(r), \rho(r), P(r)$ and $L(r)$ for a star of mass $M_\star = 1 M_\odot$ in terms of the physical radial coordinate $r/R_\odot$.

4. Repeat the calculation using stars of different masses. Plot the radius $r_\star$, central temperature T_c, and surface luminosity $L_\star$ of the stars as a function of their masses.

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