white dwarfs

Course on Compact Objects: Assignment 4

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Problem 1

Show that the pressure exerted by a gas of particle with isotropic momentum distribution $n(p)$ is given by

$$P = \frac{1}{3}\int_0^\infty p v_p n(p) dp$$

where v_p is the velocity associated with momentum p.

Let us consider the pressure due to a particle moving at an angle θ with the z direction. The component of velocity along z would be given by $v\cos\theta$

$$v_z = v\cos\theta$$

and therefore,

$$p_z = p\cos\theta.$$

The rate of change of momentum is given by

$$\frac{p_z}{\Delta z/v_z} = \frac{p_z v_z}{\Delta z} = \frac{pv\cos^2\theta}{\Delta z}$$

So pressure per particle moving at an angle θ is

$$\frac{pv\cos^2\theta}{\Delta z\Delta x\Delta y} = \frac{pv\cos^2\theta}{V}.$$

The total pressure for all particles moving at an angle θ would be given by

$$P = \int_0^\infty \frac{pv\cos^2\theta}{V} N(p,\theta) dp$$

where $N(p, \theta)$ is the number of particles with momentum between p and $p + dp$ and with angle θ to $\theta + d\theta$. This number would be

$$N(p, \theta) = \frac{d^3p}{h^3} g_s f(p) = g_s f(p) \frac{2\pi p^2 \sin\theta d\theta dp}{h^3},$$

where g_s is the degeracy of the state and $f(p)$ is the probabilty that the state would be occupied.

Thus,

$$P = \int_0^\infty \frac{pv\cos^2\theta}{V} g_s f(p) \frac{2\pi p^2 \sin\theta d\theta dp}{h^3} dp.$$

To get the average total pressure we integrate over θ

$$P = \int_0^\pi d\theta \cos^2\theta \sin\theta \int_0^\infty \frac{pv}{V} \frac{g_s f(p) 2\pi p^2}{h^3} dp = \frac{1}{3} \int_0^\infty \frac{pv}{V} \frac{g_s f(p) 4\pi p^2}{h^3} dp = \frac{1}{3} \int_0^\infty pvn(p)d$$

where $n(p) = g_s f(p) 4\pi p^2 / V$.

Problem 2

Argue why we are justified in using a 'cold' degenerate equation of state to describe a white dwarf with a temperature $T \sim 10^4$K (Hint: Show that the degeneracy parameter $\mu/kT \gg 0$, where μ is the chemical potential and k the Boltzmann constant. The density of the white dwarf is $\sim 10^6 g/cm^3$ and the chemical potential $\sim$ the Fermi energy. Assume that $\mu_e = 2$).

We have to basically show that the thermal energy of the electron gas is negligible compared to Fermi energy

```julia
using Unitful, UnitfulAtomic
```

```
kB = k
```

```julia
kB = u"k_au"
```

```
Temp = 10000.0 K
```

```julia
Temp = 1e4 * u"K"
```

```
ThermalEnergy = 2.0709735e-12 erg
```

```julia
ThermalEnergy = uconvert(u"erg", (3/2) * kB * Temp)
```

```
FermiEnergy (generic function with 1 method)
```

```julia
function FermiEnergy(ne, me, h)
    pF = h * (3 * ne / (8 * pi))^(1/3)
    sqrt(pF^2 * c^2 + me^2 * c^4)
end
```

```
h = 6.283185307179586 ħ
```

```
h = 2 * pi * u"ħ_au"
```

```
μe = 2
```

```
μe = 2
```

```
amu = 1.66054e-24 g
```

```
amu = 1.66054e-24 * u"g"
```

```
ne = 3.011068688498922e29 cm^-3
```

```
ne = 1e6 * u"g/cm^3" / (μe * amu)
```

```
me = 9.109000000000001e-31 kg
```

```
me = 9.109 * 1e-31 * u"kg"
```

```
c = c
```

```
c = u"c"
```

```
Fenergy = 1.0488022231371987e-6 erg
```

```
Fenergy = uconvert(u"erg", FermiEnergy(ne, me, h))
```

```
506429.5719559901
```

```
Fenergy/ThermalEnergy
```

Problem 3

Derive Lane-Emden equation

Lane-Emden equation

$$\frac{dm(r)}{dr} = 4\pi r^2 \rho(r), \frac{dp(r)}{dr} = -\frac{Gm(r)\rho(r)}{r^2}$$

. These two equations can be combined to give

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho(r)}\frac{dp(r)}{dr}\right) = -4\pi G\rho(r)$$

This can be further written in terms of a dimensionless form using

$$\rho = \rho_c \theta^n, r = a\xi, \Gamma = 1 + \frac{1}{n}$$

Then we get the following equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n$$

This the Lane-Emden equation.

Problem 4

Solve Lane-Emden Equation

To solve the lane-emden equation we write it as two first order odes by defining $\psi = \frac{d\theta}{d\xi}$

$$\frac{d\theta}{d\xi} = \psi, \frac{d\psi}{d\xi} = -\frac{2\psi}{\xi} - \theta^n$$

With the following initial values

$$\theta(0) = 1, \psi(0) = 0$$

```
using DifferentialEquations
```

```
using StaticArrays
```

```
laneEmden (generic function with 1 method)
```

```
function laneEmden(u, n, ξ)
    θ, ψ = u
    dθ = ψ
    dψ = - (2 * ψ / ξ) - θ^n
    @SVector [dθ, dψ]
end
```

```
u0 = ▸StaticArrays.SArray{Tuple{2},Float64,1,2}: [1.0,  0.0]
```

```
u0 = @SVector [1., 0.]
```

```
ξspan1 = ▸(1.0e-10,  10)
```

```
ξspan1 = (1e-10, 10)
```

```
problem1 =
�[36mODEProblem�[0m with uType �[36mStaticArrays.SArray{Tuple{2},Float64,1,2}�[0m and tTyp
timespan: (1.0e-10, 10.0)
u0: [1.0, 0.0]
```

```
problem1 = ODEProblem(laneEmden, u0, ξspan1, 1.)
```

sol1 =

timestamp	value1	value2

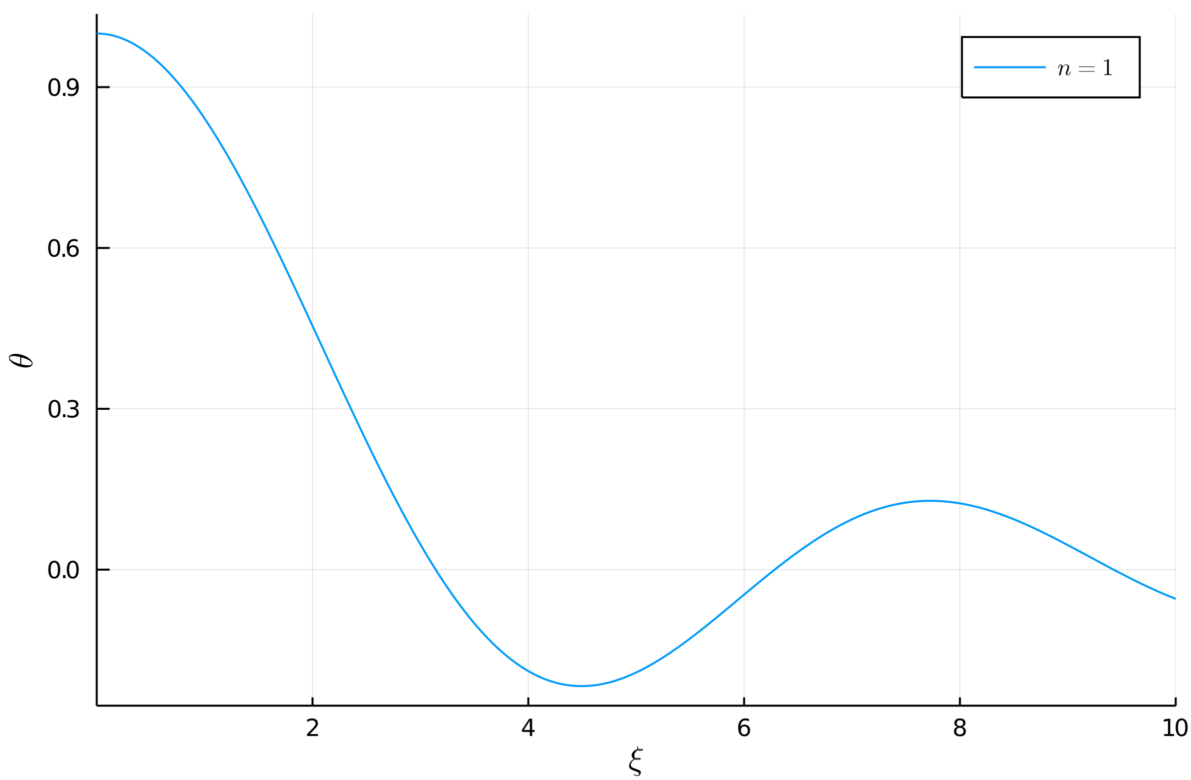
	timestamp	value1	value2
1	1.0e-10	1.0	0.0
2	3.05653e-5	1.0	-9.0812e-6
3	4.38094e-5	1.0	-1.40642e-5
4	0.000107578	1.0	-3.57691e-5
5	0.000187092	1.0	-6.2334e-5
6	0.000467431	1.0	-0.000155805
7	0.000989918	1.0	-0.000329971
8	0.00256049	0.999999	-0.000853496
9	0.00736913	0.999991	-0.00245636

```
sol1 = solve(problem1)
```

▸ Plots.GRBackend()

```
using Plots; gr()
```

```
using LaTeXStrings
```




```
plot(sol1, vars = (0, 1), xlabel = L"\xi", ylabel = L"\theta", label = L"n = 1")
```

Solutions for different integer n

```
Enter cell code...
```

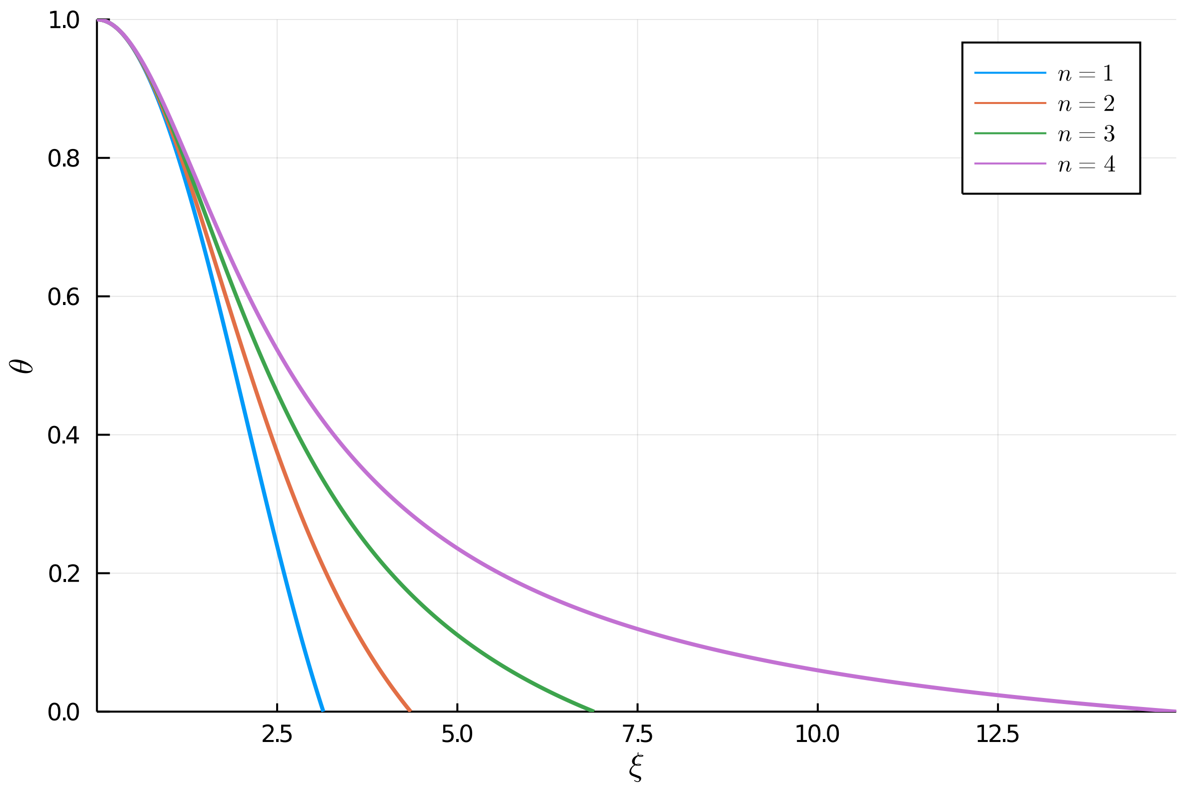
```
condition(u, ξ, integrator) =  u[1];
```

```
affect!(integrator) = terminate!(integrator);
```

```
cb = ContinuousCallback(condition, affect!);
```

```
ξspan = ▸(1.0e-10,  20)
```

```
ξspan = (1e-10, 20)
```




```
begin
    p = plot();
    for n in 1:4
        problem = ODEProblem(laneEmden, u0, ξspan, n)
        sol = solve(problem, callback = cb)
        plot!(p, sol, vars = (0, 1), lw = 2, label = L"n = %$n")
    end
    plot!(xlabel = L"\xi", ylabel = L"\theta", ylim = (0, 1))
    p
end
```

for fractional n the solver would throw error as negative value of θ would give complex value for θ^n. To avoid this problem we can just change θ in our set of odes to abs(θ).

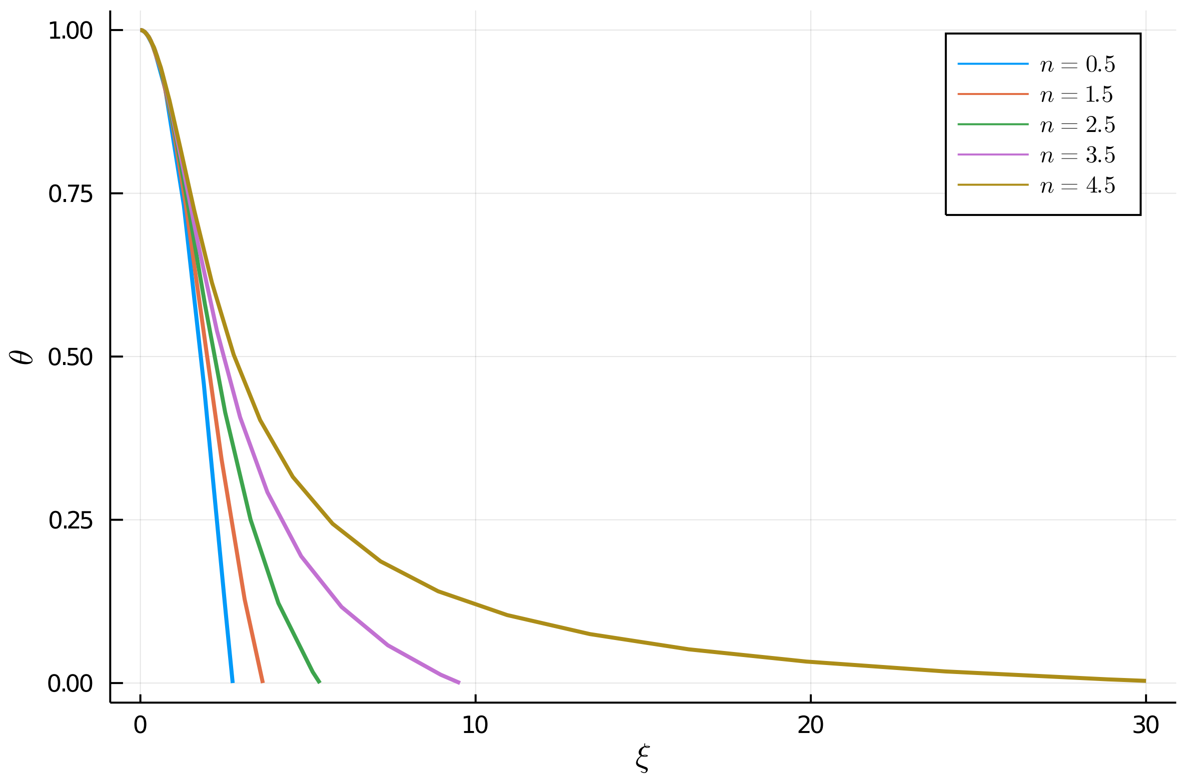
```
laneEmden2 (generic function with 1 method)
```

```
function laneEmden2(u, n, ξ)
    θ, ψ = u
    dθ = ψ
    dψ = - (2 * ψ / ξ) - abs(θ)^n
    @SVector [dθ, dψ]
end
```

```julia
begin
    pfrac = plot();
    ns = 0.5:1.0:4.5
    ξspan2 = [1.0e-10, 30.0]
    ξ1s = []
    ψs = []
    for n in ns
        problem = ODEProblem(laneEmden2, u0, ξspan2, n)
        sol = solve(problem, callback = cb)
        push!(ξ1s, sol.t[end])
        push!(ψs, sol[2, :][end])
        plot!(pfrac, sol.t, sol[1, :], lw = 2, xlabel = L"\xi", ylabel = L"\theta",
label = L"n = %$n")
    end
end
```

```
Any[2.75439,  3.65392,  5.3553,  9.53613,  30.0]
```

```julia
ξ1s
```




```julia
pfrac
```

```julia
using DataFrames
```

data =

	n	ξ1	ψ1
1	0.5	2.75439	-0.493092
2	1.5	3.65392	-0.203681
3	2.5	5.3553	-0.0762582

	n	ξ1	ψ1
4	3.5	9.53613	-0.0207888
5	4.5	30.0	-0.00193084

```
data = DataFrame(n = ns, ξ1 = ξ1s, ψ1 = ψs)
```

Problem 5

Radius and Mass of white dwarf

Mass of the star is given by

$$M = \int_0^\infty 4\pi\rho r^2 dr$$

now,

$$r = \xi\left(\frac{4\pi G{\rho_c}^{1-\frac{1}{n}}}{K(n+1)}\right)^{-\frac{1}{2}}$$

$$\rho = \rho_c\theta^n$$

$$\theta^n = -\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)$$

Thus,

$$M = 4\pi\rho_c\left(\frac{4\pi G{\rho_c}^{1-\frac{1}{n}}}{K(n+1)}\right)^{-3/2}\int_0^{\xi_1} d\left(\xi^2\frac{d\theta}{d\xi}\right) = 4\pi\rho_c\left(\frac{4\pi G{\rho_c}^{1-\frac{1}{n}}}{K(n+1)}\right)^{-3/2}\left(\xi^2\frac{d\theta}{d\xi}\right)_{\xi_1}$$

or

$$M = 4\pi\rho_c^{(3-n)/2n}\left(\frac{4\pi G}{K(n+1)}\right)^{-3/2}\left(\xi^2\frac{d\theta}{d\xi}\right)_{\xi_1}$$

Now, the radius would be given by value of r at $\xi = \xi_1$ which is

$$R_\star = \xi_1\left(\frac{4\pi G{\rho_c}^{1-\frac{1}{n}}}{K(n+1)}\right)^{-\frac{1}{2}}$$

From this we can express ρ_c in terms of $R_\star$ and ξ_1 as

$$\rho_c = \left(\frac{K(n+1)\xi_1^2}{4\pi G R_\star^2}\right)^{\frac{n}{n-1}}$$

So,

$$\boxed{M = 4\pi R_\star^{(3-n)/(1-n)}\left(\frac{K(n+1)}{4\pi G}\right)^{n/(n-1)} \xi_1^{-(3-n)/(1-n)}\xi_1^2 \frac{d\theta}{d\xi}\bigg|_{\xi_1}}$$

Problem 6

Compute mass and radius of white dwarfs

From problem 4, we get that for $n = 3/2$, $\xi_1 \approx 3.655$. Now given that

$$\rho_c = 10^6 - 10^9 g/cm^3, K \approx 10^{13}\mu_e^{-5/3}, \mu_e = 2.$$

```
MassWhiteDwarf (generic function with 1 method)
```

```
function MassWhiteDwarf(R, K, G, ξ1, ψ1, n)
    4 * pi * R^((3. -n)/(1. - n))*((K*(n+1))/(4*pi*G))^(n/(n-1))*ξ1^(-(3. - n)/(1. -
n))*ξ1^2 * abs(ψ1)
end
```

```
Rstar (generic function with 1 method)
```

```
function Rstar(G, ξ1, ρc, K, n)
    return ξ1 * (4 * pi * G * ρc^(1.0 - (1.0 / n))/(K * (n + 1)))^(-1.0 / 2.0)
end
```

```
ξ1 = 3.6539245994932976
```

```
ξ1 = data.ξ1[2]
```

```
ψ1 = -0.20368099291868041
```

```
ψ1 = data.ψ1[2]
```

```
using UnitfulAstro
```

```
G = GM⊙ M⊙^-1
```

```
G = u"GMsun"/u"Msun"
```

```
μ = 2.0
```

```
μ = 2.0
```

```
n = 1.5
```

```
n = 3/2
```

```
K = 3.149802624737183e12 dyn cm^3 g^-3752999689475413/2251799813685248
```

```
K = 1e13 * μ^(-5.0/3.0) * u"dyn/cm^2"/(u"g/cm^3")^(1+(1/n))
```

ρcs =
▸Unitful.Quantity{Float64,M L^-3,Unitful.FreeUnits{(g, cm^-3),M L^-3,nothing}}[1.0e6 g cm^

```
ρcs = [1e6, 1e7, 1e8, 1e9] * u"g/cm^3"
```

1.0e7 g cm^-3

```
ρcs[2]
```

r1 = 11196.014751530298 km

```
r1 = uconvert(u"km", Rstar(G, ξ1, ρcs[1], K, n))
```

mass1 = 2.922862216500595e26 dyn^3 cm^9 M⊙^3 GM⊙^-3 g^-5 km^-3

```
mass1 = MassWhiteDwarf(r1, K, G, ξ1, ψ1, n)
```

0.49440774095900636 M⊙

```
uconvert(u"Msun", mass1)
```

```
begin
    masses = []
    radii = []

    for ρc in ρcs
        radius = uconvert(u"km", Rstar(G, ξ1, ρc, K, n))
        mass = uconvert(u"Msun", MassWhiteDwarf(radius, K, G, ξ1, ψ1, n))
        push!(masses, mass)
        push!(radii, radius)
    end
end
```

massRadius =

	central_density	radius	mass
1	1.0e6 g cm^-3	11196.0 km	0.494408 M⊙
2	1.0e7 g cm^-3	7627.76 km	1.56345 M⊙
3	1.0e8 g cm^-3	5196.73 km	4.94408 M⊙
4	1.0e9 g cm^-3	3540.49 km	15.6345 M⊙

```
massRadius = DataFrame(central_density = ρcs, radius = radii, mass = masses)
```