Frobenius's Theorem

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A vector ξ^α is said to be a Killing vector if the Lie derivative of the metric $g_{\alpha\beta}$ along the given vector vanishes. This gives the Killing equation

$$\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0, \tag{1}$$

which basically implies that the tensor $\nabla_\alpha \xi_\beta$ must be anti-symmetric. A spacetime is said to be stationary if it possesses a time-like Killing vector. It is static if this time-like Killing vector is hypersurface orthogonal. Now a vector u^α is called hypersurface orthogonal if there exists a hypersurface whose normal n^α is everywhere proportional to u^α. The necessary and sufficient condition for a vector to be hypersurface orthogonal is given by the Frobenius's theorem which demands that the completely anti-symmetric tensor $u_{[\alpha;\beta}u_{\gamma]}$ should vanish. So the condition becomes

$$u_{[\alpha;\beta}u_{\gamma]} = 0. \tag{2}$$

1. Consider the Schwarzschild metric in the ingoing Eddington-Finkelstein coordinates

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2 d\Omega^2 \tag{3}$$

   - Show that $\xi^\alpha \partial_\alpha = \partial_v$ is a Killing vector. (Show that it satisfies the Killing equation (1)).
   - Show that ξ^α is time-like for $r > 2M$.
   - Show that ξ^α is hypersurface orthogonal, i.e., it satisfies equation (2). This would imply that the spacetime is a static.

2. Repeat the previous exercise for the Kerr metric in Boyer-Lindquist coordinates. Take the Killing vector $\xi^\alpha \partial_\alpha = \partial_t$. Is it hypersurface surface orthogonal?

3. Show that the quantity $\xi_\alpha u^\alpha$ is conserved, where ξ^α is a Killing vector and u^α is the four-velocity. Find the conserved quantities for Schwarzschild and Kerr metric the Killing vectors ∂_t and ∂_ϕ.