Assignment 5: Physics of Compact Objects

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PROBLEM 1

Show that, in the white dwarf interior, the Coulomb energy per free electron is

$$\epsilon_C = -\frac{9}{10}\left(\frac{4\pi}{3}\right)^{1/3} Z^{2/3} e^2 n_e^{1/3}, \tag{1}$$

where Z is the atomic number, e is the elementary charge and n_e is the number density of free electrons.

Answer: As temperature decreases, $T \to 0$, the ions are located in a lattice that maximizes the inter-ion separation. We can consider a spherical shell of the lattice of volume $4\pi r_0^3/3 = 1/n_N$, where n_N is the number density of the nuclei. The total energy of any one sphere is the sum of potential energies due to electron-electron $(e-e)$ interactions and electron-ion $(e-i)$ interactions.

To assemble a uniform sphere of Z electrons requires energy

$$E_{e-e} = \int_0^{r_0} \frac{qdq}{r} = \int_0^{r_0} -\frac{Ze(r^3/r_0^3)d(-Ze(r^3/r_0^3))}{r} = Z^2e^2\frac{3}{r_0^6}\int_0^{r_0} r^4 dr = Z^2e^2\frac{3}{r_0^6}\frac{r_0^5}{5} = \frac{3}{5}\frac{Z^2e^2}{r_0}. \tag{2}$$

On the other hand, to assemble the electron sphere about the central nucleus of charge Ze requires energy

$$E_{e-i} = Ze\int_0^{r_0}\frac{dq}{r} = -Ze\int_0^{r_0}\frac{3drZer}{r_0^3} = -\frac{3}{2}\frac{Z^2e^2}{r_0}. \tag{3}$$

So, the total Coulomb energy of the shell is

$$E_C = E_{e-e} + E_{e-i} = \frac{3}{5}\frac{Z^2e^2}{r_0} - \frac{3}{2}\frac{Z^2e^2}{r_0} = -\frac{9}{10}\frac{Z^2e^2}{r_0}. \tag{4}$$

So the Coulomb energy per electron is

$$\epsilon_C = \frac{E_C}{Z} = -\frac{9}{10}\frac{Ze^2}{r_0} = -\frac{9}{10}\frac{Ze^2}{(3Z/4\pi n_e)^{1/3}} = -\frac{9}{10}\left(\frac{4\pi}{3}\right)^{1/3} Z^{2/3}e^2 n_e^{1/3}, \tag{5}$$

where we have used

$$\frac{4\pi}{3}r_0^3 n_e = Z. \tag{6}$$

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