ICTS Graduate Course PHY-404.5: Physics of Compact Objects
Tutorials

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Assignment 1: Stellar Structure

1. Assuming the equations of hydrostatic, radiative and thermal equilibria, derive the following equations describing the structure of homologous stars.

$$\frac{dp}{dm} = -\frac{m}{x^4}, \quad \frac{dx}{dm} = \frac{t^b}{x^2 p^a}, \quad \frac{dt}{dm} = -\frac{p^{an}\, l}{x^4 t^{3+s+bn}}, \quad \frac{dl}{dm} = A p^{a\lambda}\, t^{\nu - b\lambda}, \tag{0.1}$$

where the dimensionless quantities p, x, t and l are defined in terms of the central pressure P_c, central temperature T_c and the scaling variables M, R and L.

$$P = P_c p, \quad T = T_c t, \quad r = Rx, \quad L_r := L(r) = Ll, M_r := M(r) = Mm, \tag{0.2}$$

and assuming the following power-law approximations for the density ρ, opacity κ and energy density density ϵ:

$$\rho = \rho_0 P^a T^{-b}, \quad \kappa = \kappa_0 \rho^n T^{-s}, \quad \epsilon = \epsilon_0 \rho^\lambda T^\nu. \tag{0.3}$$

The scaling variables M, R and L are defined by the following conditions

$$\frac{GM^2}{4\pi R^4 P_c} = 1, \quad \frac{M}{4\pi R^3 \rho_0} \frac{T_c^b}{P_c^a} = 1, \quad \frac{3\kappa_0 \rho_0^n}{64\pi^2 ac} \frac{P_c^{an} ML}{T_c^{4+s+bn} R^4} = 1, \tag{0.4}$$

while A is given by

$$A = \frac{3\kappa_0 \epsilon_0 \rho_0^{n+\lambda}}{16\pi Gac} \frac{P_c^{a(n+\lambda)+1}}{T_c^{4+s+b(n+\lambda)-\nu}}. \tag{0.5}$$

2. Numerically solve the set of equations Eq. (0.1) assuming the boundary conditions: $p(0) = t(0) = 1, x(0) = l(0) = 0$ and the following power-law indices:

   - *Ideal gas equation of state*: $a = b = 1, \rho_0 = \mu/\mathcal{R}$ where $\mu \simeq 0.5X^{-0.57}$ is the mean molecular weight, $\mathcal{R} = 8.3 \times 10^7$ ergs/mol/K is the gas constant and X is the mass fraction of hydrogen (assume $X \simeq 0.75$).
   - *Electron scattering opacity*: $n = s = 0, \kappa_0 = 0.2(1 + X)\,\mathrm{cm}^2/\mathrm{g}$.
   - *Energy generation by $p - p$ chain*: $\lambda = 1, \nu = 4, \epsilon_0 \sim 10^{-30} X^2\,\mathrm{ergs\,cm^3\,g^{-2}\,s^{-1}\,K^{-4}}$.

   You can start by solving the dimensionless equations Eq.(0.1) by setting $A = A_0$, some constant. (Numerical experimentation shows that $p(m)$ goes to zero only for the choice $A_0 \lesssim 0.55$). Plot $p(m), x(m), l(m)$ and $t(m)$. Identify the value of m at which the $p(m)$ goes to zero, denoted by $m_\star$; i.e. $p(m_\star) = 0$. Physically we require that $t(m)$ should go to zero at the same value of m; i.e., $t(m_\star) = 0$. This is satisfied only for a specific value of A, say $A_\star$. Using $A = A_\star$, re-plot $p(m), x(m), l(m)$ and $t(m)$. These relations are independent of the mass of the star.

3. Setting $A = A_\star$ in Eq.(0.5), we can express P_c fully in terms of T_c. This allows us to express R, T_c and L in terms of M using Eqs.(0.4). Now, the only free parameter is $M := M_\star/m_\star$, where $M_\star$ is the actual mass of 4the star. Now plot the stellar structure quantities $M(r), \rho(r), P(r)$ and $L(r)$ for a star of mass $M_\star = 1M_\odot$ in terms of the physical radial coordinate $r/R_\odot$.

4. Repeat the calculation using stars of different masses. Plot the radius $r_\star$, central temperature T_c, and surface luminosity $L_\star$ of the stars as a function of their masses (hint: It would be useful to plot them in log-log scale so that the power-law relationships become apparent).

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Assignment 2: Stellar Evolution

1. Derive the homology relations for the pressure P, temperature T and luminosity L (express them in terms of the mass M and radius R, as well as the mean molecular weight μ of the gas particles). Compare these with the numerical results that you obtained in the previous section (Problem 4).

2. Using the homologous relations derived above, show that, for a star of mass M in quasi-equilibrium,

$$Pc = CGM^{2/3}\rho_c^{4/3}. \tag{0.6}$$

3. Consider a slowly contracting star in quasi-hydrostatic equilibrium for which the pressure is given by a combination of ideal gas and electron degeneracy : $P = (\mathcal{R}/\mu)\rho T + K(\rho/\mu_e)^\gamma$, where γ varies between 5/3 (non-relativistic) and 4/3 (extreme relativistic). Plot the variation of T_c with ρ_c.

4. Derive an expression for the maximum central temperature reached by a star of mass M.

Assignment 3: Stellar Collapse

1. Neutron stars have a radius of $\simeq$ 10 km. Use this to estimate the energy generated during a core collapse supernova (Hint: assume that before the collapse the core is like a white dwarf with Chandrasekhar mass, and that it suffers no significant mass loss after the collapse).

2. Compute the kinetic energy of the ejecta as well as the energy lost into photons and neutrinos. Assume that the progenitor star has an original mass of 10 km, the measured velocity of the ejecta is $\sim 10^4$ km/s, and the supernova shines with an (electromagnetic) luminosity of $2 \times 10^8 L_\odot$ for $\sim$ two months.

3. Assuming that these neutrinos have an average energy of $\simeq$ 5 MeV, how many neutrinos are produced in the collapse? Estimate the expected flux of neutrinos from a galactic supernova ($d_L \sim 10$ kpc) here on earth. Compare it with he neutrino flux from the Sun.