Mathematical Induction

Exercise problems

3. $P(n): 1^2 + 2^2 + \dots + n^2 = n(n+1)(2n+1)/6$. for all +ve int n

Basis step:

$P(1)$:

$$1^2 = \frac{1(1+1)(2.1+1)}{6}$$

or, $1 = \frac{2\times 3}{6}$

$\therefore 1 = 1$.

Inductive step:

Let $P(k)$ be true:

$$1^2 + 2^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6} \quad \text{(i)}$$

Now: $P(k+1)$ is

$$1^2 + 2^2 + \dots + k^2 + (k+1)^2 = \frac{(k+1)(k+1+1)(2(k+1)+1)}{6}$$

or, $1^2 + 2^2 + \dots + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$

Adding $(k+1)^2$ to (i) on both sides:

$$1^2 + 2^2 + \dots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= (k+1)\left[\frac{k(2k+1)}{6} + (k+1)\right]$$

$$\text{(ii)} \quad = (k+1)\left\{\frac{k(2k+1)}{6} + \frac{6(k+1)}{6}\right\}$$

$$= (k+1)\left\{\frac{2k^2 + k + 6k + 6}{6}\right\}$$

$$= (k+1)\left\{\frac{2k^2 + 7k + 6}{6}\right\}$$

$2 | 12$
$2 | 6$
3

$$= (k+1)\left\{\frac{2k^2 + 4k + 3k + 6}{6}\right\}$$

$$= (k+1)\,\frac{2k(k+2) + 3(k+2)}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

$\therefore$ If $P(k)$ is true then $P(k+1)$ is true.

4. $1^3 + 2^3 + \dots + n^3 = (n(n+1)/2)^2$

Basis step: $P(1)$

$1^3 = (1(1+1)/2)^2 = 1^2 = 1.$

Inductive step:

$P(k)$: $1^3 + 2^3 + \dots + k^3 = \left[\frac{k(k+1)}{2}\right]^2$ —— (i)

$P(k+1)$ $1^3 + 2^3 + \dots + (k+1)^3 = \left[\frac{(k+1)(k+1+1)}{2}\right]^2$

$$= \frac{(k+1)^2(k+2)^2}{4}$$ —— (ii)

Adding $(k+1)^3$ to ①

$$1^3 + 2^3 + \dots + k^3 + (k+1)^3 = \left[\frac{k(k+1)}{2}\right]^2 + (k+1)^3$$

$$= \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4}$$

$$= \frac{k^2(k^2+2k+1) + 4(k^3+3k^2+3k+1)}{4}$$

$$= \frac{k^4 + 2k^3 + k^2 + 4k^3 + 12k^2 + 12k + 4}{4}$$

$$= \frac{k^4 + 6k^3 + 13k^2 + 12k + 4}{4} \quad \text{— (iii)}$$

We farther expand (ii)

$$1^3 + 2^3 + \dots + (k+1)^3 = \frac{(k^2+2k+1)(k^2+4k+4)}{4}$$

$$= \frac{k^2(k^2+4k+4) + 2k(k^2+4k+4) + k^2+4k+4}{4}$$

$$= \frac{k^4 + 4k^3 + 4k^2 + 2k^3 + 8k^2 + 8k + k^2 + 4k + 4}{4}$$

$$= \frac{k^4 + 6k^3 + 13k^2 + 12k + 4}{4}.$$

(Proved)

5. $0^2 + 1^2 + 3^2 + 5^2 + \dots + (2n+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}$

P(0): $0^2 + (2\times0+1)^2 = \frac{(0+1)(2\times0+1)(2\times0+3)}{3}$

$1 = 1$.

P(1): $0^2 + 1^2 + (2+1)^2 = \frac{(1+1)(2\times1+1)(2\times1+3)}{3}$

or, $1 + 3^2 = \frac{2\times(2+1)(2+3)}{3}$

or, $10 = \frac{2\times\cancel{3}\times5}{\cancel{3}}$

or, $10 = 10$.

IS:

$P(k) = 0^2 + 1^2 + 2^2 + \dots + (2k+1)^2 = \frac{(k+1)(2k+1)(2k+3)}{3}$ —— (1)

$P(k+1): 0^2 + 1^2 + 2^2 + \dots + (2(k+1)+1)^2 = \frac{(k+1+1)(2(k+1)+1)(2(k+1)+3)}{3}$

$= \frac{(k+2)(2k+2+1)(2k+2+3)}{3}$

$= \frac{(k+2)(2k+3)(2k+5)}{3}$

$= \frac{(k+2)[(2k)^2 + 8.2k + 15]}{3}$

$= \frac{(k+2)[4k^2 + 16k + 15]}{3}$

$= \frac{k(4k^2+16k+15) + 2(4k^2+16k+15)}{3}$

$= \frac{4k^3 + 16k^2 + 15k + 8k^2 + 32k + 30}{3}$

$$P(k+1) = \frac{4k^3 + 24k^2 + 47k + 30}{3}.$$

Adding $(2(k+1)+1)^2$ to both sides of (i):

$$0^2 + 1^2 + \dots + (2k+1)^2 + (2(k+1)+1)^2 = \frac{(k+1)(2k+1)(2k+3)}{3} + [2(k+1)+1]^2$$

$$= \frac{(k+1)[(2k)^2 + 4\times 2k + 3]}{3} + [2k+2+1]^2$$

$$= \frac{(k+1)(4k^2+8k+3)}{3} + (2k+3)^2$$

$$= \frac{4k^3+8k^2+3k+4k^2+8k+3}{3} + (4k^2+12k+9)$$

$$= \frac{4k^3+12k^2+11k+3}{3} + \frac{12k^2+36k+27}{3}$$

$$= \frac{4k^3+24k^2+47k+30}{3}$$

6. $P(n): 1.1! + 2.2! + \dots + n.n! = (n+1)! - 1$

$P(1): 1 = (1+1)! - 1$

$1 = 2! - 1$

$1 = 2 - 1$

$1 = 1.$

$P(k): 1.1! + 2.2! + \dots + k.k! = (k+1)! - 1. \quad \text{(i)}$

$P(k+1): 1.1! + 2.2! + \dots + (k+1).(k+1)! = (k+1+1)! - 1.$

$= (k+2)! - 1$

$= (k+2)(k+1)! - 1.$

Adding: $(k+1).(k+1)!$ to both sides:

$1.1! + 2.2! + \dots + k.k! + (k+1)(k+1)! = (k+1)! - 1 + (k+1)(k+1)!$

$= (k+1)! + (k+1)(k+1)! - 1$

$= (k+1)!(1+k+1) - 1$

$= (k+2)(k+1)! - 1$

7. $3 + 3\cdot 5 + 3\cdot 5^2 + \dots + 3\cdot 5^n = 3(5^{n+1}-1)/4$

$n \in \{0, 1, \dots\}$

$P(0)$:

$$3 \cdot 5^0 = 3(5^{0+1}-1)/4$$

$$3 = 3 \times \frac{4}{4} = 3.$$

P(k):

$$3 + 3\cdot 5 + 3\cdot 5^2 + \dots + 3\cdot 5^k = \frac{3(5^{k+1}-1)}{4} \quad \text{(i)}$$

P(k+1):

$$3 + 3\cdot 5^1 + 3\cdot 5^2 + \dots + 3\cdot 5^{k+1} = \frac{3(5^{k+2}-1)}{4}$$

$$\cancel{= 3 \cdot 5^k}$$

Adding $3\cdot 5^{k+1}$ to both side of (i)

$$3 + 3\cdot 5^1 + \dots + 3\cdot 5^k + 3\cdot 5^{k+1} = \frac{3(5^{k+1}-1)}{4} + 3\cdot 5^{k+1}$$

$$= \frac{3\cdot 5^{k+1} - 3}{4} + \frac{4\cdot 3\cdot 5^{k+1}}{4}$$

$$= \frac{3\cdot 5^{k+1} - 3 + 12\cdot 5^{k+1}}{4}$$

$$= \frac{15\cdot 5^{k+1} - 3}{4}$$

$$= \frac{3[5\cdot 5^{k+1} - 1]}{4} = \frac{3(5^{k+2}-1)}{4}.$$

(Proved)

8. Prove that

$$2.7^0 - 2.7^1 + 2.7^2 - \dots + 2(-7)^n = \frac{[1 - (-7)^{n+1}]}{4}$$

$\underline{\underline{P(0)}}$

$$2.7^0 = \frac{1-(-7)^{0+1}}{4}$$

$$2 = \frac{1-(-7)^1}{4}$$

$$2 = \frac{1-(-7)}{4} = 2$$

$\underline{\underline{P(k)}}$

$$2.7^0 - 2.7^1 + 2.7^2 - \dots + 2(-7)^k = \frac{[1-(-7)^{k+1}]}{4} \quad \text{(i)}$$

$\underline{\underline{P(k+1)}}$

$$2.7^0 - 2.7^1 + 2.7^2 - \dots + 2(-7)^{k+1} = \frac{[1-(-7)^{k+2}]}{4}$$

Adding: $2(-7)^{k+1}$ to both sides of (i) we get:

$$2.7^0 - 2.7^1 \dots + 2(-7)^k + 2(-7)^{k+1} = \frac{1-(-7)^{k+1}}{4} + 2(-7)^{k+1}$$

$$= \frac{1-(-7)^{k+1}}{4} + \frac{8(-7)^{k+1}}{4}$$

$$= \frac{[1 - 1.(-7)^{k+1} + 8.(-7)^{k+1}}{4}$$

$$\cancel{= \frac{1 - 1\cdot(-7)}{4}}$$

$$= \frac{1 + (-7)^{k+1}\,[8-1]}{4}$$

$$= \frac{1 + (-7)^{k+1}\cdot 7}{4}$$

$$= \frac{1 + [(7)^{k+1}\cdot(-1)^{k+1}]\cdot 7}{4}$$

$$= \frac{1 + 7^{k+2}\,(-1)^{k+1}}{4}$$

$$= \frac{1 + (-1)(-1)\,7^{k+2}\,(-1)^{k+1}}{4}$$

$$= \frac{1 - 7^{k+2}(-1)^{k+2}}{4}$$

$$= \frac{1 - (-7)^{k+2}}{4}.$$

12. $$\sum_{j=0}^{n}\left(-\frac{1}{2}\right)^{j} = \frac{2^{n+1}+(-1)^{n}}{3.2^{n}}$$

$$\left(-\frac{1}{2}\right)^{0}+\left(-\frac{1}{2}\right)^{1}+\left(-\frac{1}{2}\right)^{2}+\dots+\left(-\frac{1}{2}\right)^{n} = \frac{2^{n+1}+(-1)^{n}}{3.2^{n}}$$

P(0)

$$1 = \frac{2^{0+1}+1}{3} = 1.$$

P(k).

$$\left(-\frac{1}{2}\right)^{0}+\left(-\frac{1}{2}\right)^{1}+\dots+\left(-\frac{1}{2}\right)^{k} = \frac{2^{k+1}+(-1)^{k}}{3.2^{k}} \quad \text{—(i)}$$

P(k+1)

$$\left(-\frac{1}{2}\right)^{0}+\left(-\frac{1}{2}\right)^{1}+\dots+\left(-\frac{1}{2}\right)^{k+1} = \frac{2^{k+2}+(-1)^{k+1}}{3.2^{k+1}}$$

Adding $\left(-\frac{1}{2}\right)^{k+1}$ to both sides of (i) we get:

$$\left(-\frac{1}{2}\right)^{0}+\dots+\left(-\frac{1}{2}\right)^{k}+\left(-\frac{1}{2}\right)^{k+1} = \frac{2^{k+1}+(-1)^{k}}{3.2^{k}}+\left(-\frac{1}{2}\right)^{k+1}$$

$$= \frac{2^{k+1}+(-1)^{k}}{3.2^{k}}+\frac{(-1)^{k+1}}{2^{k+1}}$$

$$= \frac{2^{k+1}+(-1)^{k}}{3.2^{k}}+\frac{3.(-1)^{k+1}}{3.2^{k+1}}$$

$$= \frac{2[2^{k+1}+(-1)^{k}]}{3.2^{k+1}}+\frac{3(-1)^{k+1}}{3.2^{k+1}}$$

$$= \frac{2 \cdot 2^{k+1} + 2(-1)^k + 3(-1)^{k+1}}{3 \cdot 2^{k+1}}$$

$$= \frac{2 \cdot 2^{k+1} + 2(-1)^k + 3(-1)^k \cdot (-1)^1}{3 \cdot 2^{k+1}}$$

$$= \frac{2 \cdot 2^{k+1} + 2(-1)^k - 3(-1)^k}{3 \cdot 2^{k+1}}$$

$$= \frac{2 \cdot 2^{k+1} - (-1)^k}{3 \cdot 2^{k+1}}$$

$$= \frac{2^{k+2} - (-1)^k}{3 \cdot 2^{k+1}}$$

$$= \frac{2^{k+2} + (-1)(-1)^k}{3 \cdot 2^{k+1}}$$

$$= \frac{2^{k+2} + (-1)^{k+1}}{3 \cdot 2^{k+1}} \quad \text{(Proved)}$$

13. $1^2 - 2^2 + 3^2 - \dots + (-1)^{n-1} n^2 = \dfrac{(-1)^{n-1} \cdot n(n+1)}{2}$

$P(1)$: can be shown as true.

$P(k)$: $1^2 - 2^2 + \dots + (-1)^{k-1} k^2 = \dfrac{(-1)^{k-1} \cdot k(k+1)}{2}$ —— (i)

$P(k+1)$ $1^2 - 2^2 + \dots + (-1)^{k-1} k^2 + (-1)^k (k+1)^2 = \dfrac{(-1)^k \cdot (k+1)(k+2)}{2}$

From (i) we get by adding $(-1)^k (k+1)^2$

$\cancel{1^2 - 2^2 + \dots + (-1)^k (k+1)^2 = (-1)^k (k+1)(k+2)}$

$$1^2 - 2^2 + \dots + (-1)^{k-1} k^2 + (-1)^k (k+1)^2 = \frac{(-1)^{k-1} \cdot k(k+1)}{2} + (-1)^k (k+1)^2$$

$$= \frac{(-1)^k \cdot (-1)^{-1} \cdot k(k+1)}{2} + \frac{2(-1)^k (k+1)^2}{2}$$

$$= \frac{(-1)^k (-1)^{-1} k(k+1) + 2(-1)^k (k+1)^2}{2}$$

$$= \frac{(-1)^k (k+1)\left[k(-1)^{-1} + 2(k+1)\right]}{2}$$

$$= \frac{(-1)^k (k+1)\left[\frac{k}{-1} + 2k + 2\right]}{2}$$

$$= \frac{(-1)^k (k+1)(k+2)}{2}$$

(Proved)

14. $\sum_{k=1}^{n} k.2^k = (n-1)2^{n+1} + 2.$

P(1):

$1.2^1 = (1-1)2^{1+1} + 2$

or, $2 = 2.$

P(k):

$1.2^1 + 2.2^2 + 3.2^3 + \dots + k.2^k = (k-1)2^{k+1} + 2$ —— (i)

P(k+1):

$1.2^1 + 2.2^2 + \dots + (k+1)2^{k+1} = k.2^{k+2} + 2$

From (i) we get by adding $(k+1)2^{k+1}$

$1.2^1 + 2.2^2 + \dots + k.2^k + (k+1)2^{k+1} = (k-1)2^{k+1} + 2 + (k+1)2^{k+1}$

$= 2^{k+1}(k+1+k-1) + 2$

$= 2^{k+1}.2k + 2$

$= 2^{k+1}.2^1.k + 2$

$= 2^{k+2}.k + 2$ (Proved)

15, 16 solve similarly.

17 $\sum_{j=1}^{n} j^4 = n(n+1)(2n+1)(3n^2+3n-1)/30$

P(1) =

$1^4 = 1(2)(3)*(3+3-1)/30$

$= 1.$

P(k)

$1^4+2^4+\dots+k^4 = \dfrac{k(k+1)(2k+1)(3k^2+3k-1)}{30}$ — (i)

P(k+1)

$1^4+2^4+\dots+(k+1)^4 = \dfrac{(k+1)(k+2)(2k+3)(3(k+1)^2+3(k+1)-1)}{30}$

$= \dfrac{(k+1)(k+2)(2k+3)[3[k^2+2k+1]+3k+3-1]}{30}$

$= \boxed{\dfrac{(k+1)(k+2)(2k+3)[3k^2+6k+3+3k+2]}{30}}$ — (ii)

should be equal

Adding $(k+1)^4$ to (i)

$1^4+2^4+\dots+k^4+(k+1)^4 = \boxed{\dfrac{k(k+1)(2k+1)(3k^2+3k-1)}{30}+(k+1)^4}$ — (iii)

From (ii):

R.H.S.

$$= \frac{(k+1)(k+2)}{30}\{2k(3k^2+9k+5)+3(3k^2+9k+5)\}$$

$$= \frac{k+1}{30}(k+2)\{6k^3+18k^2+10k+9k^2+27k+15\}$$

$$= \frac{k+1}{30}(k+2)\{6k^3+27k^2+37k+15\}.$$

$$= \frac{k+1}{30}\{k(6k^3+27k^2+37k+15)+2(6k^3+27k^2+37k+15)\}$$

$$= \frac{k+1}{30}\{6k^4+27k^3+37k^2+15k+12k^3+54k^2+74k+30\}$$

$$= \frac{k+1}{30}\{6k^4+39k^3+91k^2+89k+30\}$$

From (iii)

R.H.S.

$$= \frac{k(k+1)(2k+1)(3k^2+3k-1)}{30}+\frac{30(k+1)^4}{30}$$

$$= \frac{(k+1)\{k(2k+1)(3k^2+3k-1)+30(k+1)^3\}}{30}$$

$$= \frac{(k+1)\{(2k^2+k)(3k^2+3k-1)+30(k^3+3k^2+3k+1)\}}{30}$$

$$= \frac{k+1}{30}\{2k^2(3k^2+3k-1)+k(3k^2+3k-1)+30k^3+90k^2+90k+30\}$$

$$= \frac{k+1}{30}\{6k^4+6k^3-2k^2+3k^3+3k^2-k+30k^3+90k^2+90k+30\}$$

$$= \frac{k+1}{30}\{6k^4+39k^3+91k^2+89k+30\}$$

(Proved)

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Harmonic numbers:

$H_1 = \frac{1}{1}$

$H_2 = \frac{1}{1} + \frac{1}{2}$

$H_3 = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} \Rightarrow H_3 = H_2 + \frac{1}{3}$

$H_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \Rightarrow H_4 = H_3 + \frac{1}{4}$

$H_j = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{j}$

P(1):

$H_1 = (n+1) H_n - n$

$H_1 = (1+1)1 - 1$

$1 = 1.$

P(k):

$H_1 + H_2 + \dots + H_k = (k+1) H_k - k$ (i)

P(k+1):

$H_1 + H_2 + \dots + H_{k+1} = (k+2) H_{k+1} - k - 1$ (ii)

Adding H_{k+1} to both sides of (i):

$H_1 + H_2 + \dots + H_k + H_{k+1} = (k+1) H_k - k + H_{k+1}$

$= (k+1) H_k - k + H_k + \frac{1}{k+1}$

$= (k+1) H_k + H_k + \frac{1}{k+1} - k$

$$= H_k(k+1+1) + \frac{1}{k+1} - k$$

$$= H_k(k+2) + \frac{1}{k+1} - k$$

$$= (k+2) H_k + \frac{1}{k+1} - k \quad \text{(iii)}$$

Now:

$$H_{k+1} = H_k + \frac{1}{k+1}$$

$$\text{or, } H_k = H_{k+1} - \frac{1}{k+1}$$

$\therefore$ In (iii)

$$H_1 + H_2 + \dots + H_{k+1} = (k+2)\left(H_{k+1} - \frac{1}{k+1}\right) + \frac{1}{k+1} - k$$

$$= (k+2)H_{k+1} - \frac{k+2}{k+1} + \frac{1}{k+1} - k$$

$$= (k+2) H_{k+1} + \frac{1-k-2}{k+1} - k$$

$$= (k+2) H_{k+1} + \frac{-k-1}{k+1} - k$$

$$= (k+2) H_{k+1} - k - 1 \text{. (Proved)}$$

Inequalities

(18) $n! < n^n$

$P(2)$:

$2! < 2^2$

$2 < 4$

$P(k)$

$k! < k^k$ —— (i)

$P(k+1)$

$(k+1)! < (k+1)^{k+1}$ —— (ii)

Multiplying (i) by $k+1$

$(k+1)k! < (k+1)k^k$

$(k+1)! < (k+1)k^k$

or, $(k+1)! < k.k^k + k^k$

or, $(k+1)! < k^{k+1} + k$

or, $(k+1)! < k^{k+1} + k$

$\cancel{(k+1)^{k+1}} = k.1^{k+1}$

$\therefore (k+1)! < (k+1)^{k+1}$

Binomial Theorem:

$(a+b)^n = {}^nC_0 a^n b^0 + {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 + \dots + {}^nC_n a^0 b^n$

$(k+1)^{k+1}$

$= {}^{k+1}C_0 k^{k+1}.1^0 + {}^{k+1}C_1 k^{k+1-1}1^1 + {}^{k+1}C_2 k^{k+1-2}.1^2 + \dots + {}^{k+1}C_{k+1} k^0 1^{k+1}$

$= k^{k+1} + \cancel{{}^{k+1}C_1}(k+1)k + \dots + 1.$

(19) $1+\frac{1}{4}+\frac{1}{9}+\dots+\frac{1}{n^2} < 2-\frac{1}{n}$

P(2)

$1+\frac{1}{2^2} < 2-\frac{1}{2}$

$\therefore \quad 1+\frac{1}{4} < 1.5$

$\therefore \quad 1.25 < 1.5.$

P(k)

$1+\frac{1}{4}+\frac{1}{9}+\dots+\frac{1}{k^2} < 2-\frac{1}{k}$ —— (1)

P(k+1)

$1+\frac{1}{4}+\frac{1}{9}+\dots+\frac{1}{(k+1)^2} < 2-\frac{1}{k+1}$

$= \frac{2(k+1)}{(k+1)} - \frac{1}{k+1}$

$= \frac{2k+2-1}{k+1} = \frac{2k+1}{k+1}$

$\therefore \quad 1+\frac{1}{4}+\frac{1}{9}+\dots+\frac{1}{(k+1)^2} < \frac{2k+1}{k+1}$

Adding $\frac{1}{(k+1)^2}$ to both sides of (1) we get:

$1+\frac{1}{4}+\dots+\frac{1}{(k+1)^2} < 2-\frac{1}{k}+\frac{1}{(k+1)^2}$

$= \frac{2(k+1)^2 k}{k(k+1)^2} - \frac{(k+1)^2}{k(k+1)^2} + \frac{k}{k(k+1)^2}$

$= \frac{2k(k+1)^2 - k^2-2k-1 + k}{k(k+1)^2}$

$= \frac{2k(k^2+2k+1) - k^2-2k-1+k}{k(k+1)^2}$

$= \frac{2k^3+4k^2+2k-k^2-2k-1+k}{k(k+1)^2}$

$$= \frac{2k^3+3k^2+k-1}{k(k+1)^2}$$

$$= \frac{2k^3+3k^2+k}{k(k+1)^2} - \frac{1}{k(k+1)^2}$$

$$= \frac{k(2k^2+3k+1)}{k(k+1)^2} - \frac{1}{k(k+1)^2}$$

$$= \frac{2k^2+2k+k+1}{(k+1)^2} - \frac{1}{k(k+1)^2}$$

$$= \frac{2k(k+1)+1(k+1)}{(k+1)^2} - \frac{1}{k(k+1)^2}$$

$$= \frac{(2k+1)(k+1)}{(k+1)^2} - \frac{1}{k(k+1)^2}$$

$$= \frac{2k+1}{k+1} - \frac{1}{k(k+1)^2}$$

$$\therefore \quad 1+\frac{1}{4}+\dots+\frac{1}{(k+1)^2} < \frac{2k+1}{k+1} - \frac{1}{k(k+1)^2}$$

$$\therefore \quad 1+\frac{1}{4}+\dots+\frac{1}{(k+1)^2} < \frac{2k+1}{k+1}$$ (Proved)

(20) Prove that $3^n < n!$ $n \in \{7, 8, 9, ...\}$

P(7)

$3^7 < 7!$

$2187 < 5040$.

P(k):

$3^k < k!$ —— (i)

P(k+1):

$3^{k+1} < (k+1)!$ or, $3^{k+1} < (k+1)k!$.

Multiplying (i) by 3.

$3^k \cdot 3 < 3 \cdot k!$

$3^{k+1} < 3 \cdot k!$

Now:

$3k!$ is smaller than $(k+1)k!$ as $(k+1)$ is clearly more than 3 [$\because n \in \{7, 8, 9, ...\}$].

$\therefore 3^{k+1} < (k+1)!$

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21. $2^n > n^2$ when $n \in \{5, 6, 7, \ldots\}$.

P(5)	P(k)	P(k+1)
$2^5 > 5^2$	$2^k > k^2$ —— (i)	$2^{k+1} > (k+1)^2$.
or, $32 > 25$.		

Now. from (i)

$2^k \cdot 2 > 2 \cdot k^2$

$2^{k+1} > 2k^2$

If we can show that $2k^2 \geq (k+1)^2$, the proof will be complete.

$2k^2 \geq k^2 + 2k + 1$.

or, $2k^2 - k^2 - 2k - 1 \geq 0$

or, $k^2 - 2k - 1 \geq 0$

or, $k^2 - 2k \geq 1$

or, $k(k-2) \geq 1$.

$\therefore$ Proved

$k^2 + k^2$

$k^2 + k^2 > k^2 + 4k$

$k^2 + k^2 > k^2 + 2k + 2k$

$k^2 + k^2 > k^2 + 2k + 1$

$k^2 + 2k + 1$

24

$\frac{1}{2n} \leq [1 \times 3 \times 5 \times \dots \times (2n-1)] / [2 \times 4 \times \dots \times 2n]$

$$\frac{1}{2n} \leq \frac{1 \times 3 \times 5 \times \dots \times (2n-1)}{2 \times 4 \times 6 \times \dots \times 2n}$$

P(1)

$$\frac{1}{2} = \frac{1}{2}$$

P(k)

$$\frac{1}{2k} \leq \frac{1 \times 3 \times 5 \times \dots \times (2k-1)}{2 \times 4 \times 6 \times \dots \times 2k}$$

P(k)

$$1 \leq \frac{1 \times 3 \times 5 \times \dots \times (2k-1)}{2 \times 4 \times 6 \times \dots \times (2k-2)} \quad \text{—(i)}$$

P(k+1)

$$\frac{1}{2k+2} \leq \frac{1 \times 3 \times 5 \times \dots \times ((2k+1)-1)}{2 \times 4 \times 6 \times \dots \times 2(k+1)}$$

$$\frac{1}{2(k+1)} \leq \frac{1 \times 3 \times 5 \times \dots \times (2k+1)}{2 \times 4 \times 6 \times \dots \times (2k+2)}$$

P(k+1)

$$1 \leq \frac{1 \times 3 \times 5 \times \dots \times (2k+1)}{2 \times 4 \times 6 \times \dots \times 2k} \quad \text{—(ii)}$$

Mult (i) by $\frac{2k}{2k}$ on RHS.

$$1 \leq \frac{1 \times 3 \times 5 \times \dots \times (2k-1) \times 2k}{2 \times 4 \times 6 \times \dots \times (2k-2) \times 2k}$$

$$\therefore 1 \leq \frac{1 \times 3 \times 5 \times \dots \times (2k-1) \times (2k+1)}{2 \times 4 \times 6 \times \dots \times (2k-2) \times 2k}$$

(28) $n^2 - 7n + 12 \geq 0$

for $n \geq 3$.

$P(3)$: $3^2 - 7 \times 3 + 12 = 0$.

$P(k)$: $k^2 - 7k + 12 \geq 0$ —— (i)

$P(k+1)$: $(k+1)^2 - 7(k+1) + 12 \geq 0$.

or, $k^2 + 2k + 1 - 7k - 7 + 12 \geq 0$

(i) or, $k^2 - 5k + 6 \geq 0$

Now from (i)

(ii) $k^2 - 7k + 12 \geq 0$

or, $k^2 - 5k - 2k + 6 + 6 \geq 0$.

or, $k^2 - 5k + 6 \geq 2k - 6$.

We need to show that $2k - 6$ is ≥ 0.

we can see that for $k \geq 3$, this is the case.

$\therefore$ $k^2 - 5k + 6 \geq 0$.

31. n^2+n is divisible by 2 ~~for pos~~ when n is a +ve int.

$P(1)$: $1^2+1 = 2$ is divisible by 2.

$P(k)$: k^2+k is divisible by 2

$P(k+1)$: $(k+1)^2+(k+1)$ is divisible by 2.

or, $k^2+2k+1+k+1$ is divisiby by 2

or, $k^2+k+2k+2$ is divisible by 2.

Now based on $P(k)$ we find that for $P(k+1)$,

k^2+k is divisible by 2. Also $2k+2$ is clearly divisible by 2.

32. n^3+2n is divisible by 3.

$P(1)$ $1^3+2\cdot1 = 3$ div by 3

$P(k)$ k^3+2k is div by 3.

$P(k+1)$: $(k+1)^3+2(k+1) = k^3+3k^2+3k+1+2k+2$

$$= \underbrace{k^3+2k}_{\text{by } P(k) \text{ is div by } 3} + \underbrace{3k^2+3k+3}_{\text{div by } 3.}$$

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33 - similar

(34). $n^3 - n$ is div by 6. when n is non-negative, i.e. $n \geq 0$.

~~P(1):~~

P(0): $0^3 - 0 = 0$ is divisible by 6

P(k): $k^3 - k$ is div by 6

P(k+1): $(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - k - 1$ is div by 6.

$$= \underbrace{k^3 - k}_{\text{div by 6 (based on } P(k))} + 3k^2 + 3k$$

$3k^2 + 3k$

- it is divisible by 3.

It is also div by 2:

If k is even, then $3k^2 + 3k$ is also even.

~~3(2n~~

$k = 2n$

$3 \times (2n)^2 + 3 \times 2n$

$3 \times 2n \times 2n + 3 \times 2n$

$= 2n(3 \times 2n + 3)$

If k is odd then $3k^2+3k$ is even

$k = 2n+1.$

$$3k^2+3k = 3(2n+1)^2 + 3(2n+1)$$

$$= 3\left[(2n)^3 + 3.(2n)^2 + 3.2n.+1\right] + 6n+3$$

$$= 3\left[8n^3 + 3\times 4n^2 + 6n+1\right] + 6n+3$$

$$= 24n^3 + 36n^2 + 18n + 3 + 6n+3$$

$$= 24n^3 + 36n^2 + 18n + 6n+6$$

$$= 2\left[12n^3 + 18n^2 + 9n + 3n + 3\right].$$