# Integration by Successive Reduction

In the general sense of the term, any formula expressing a given integral in terms of another which is simpler then it called a reduction formula for the given integral.

2nd// In practice, however, the reduction formula for a given integral means that the integral belongs to a class of integrals such that it can be expressed in terms of one or more integrals or lower orders belonging to the same class. By successive application of the formula, we arrive at integrals which can be easily integrated and hence the given integral can be evaluated.

—x—

1st/ Many times we see that the value of complicated integral is not determined by the operation of integration by parts only one time. In this case, the power of integrand can be reduced step by step by the repeated application of integration by parts. This method is known as reduction formula.

eg: $I_n = \int \sin^n x \, dx$ , $\int \sin^{14} x \, dx$.

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P1: Establish a reduction formula for $\int x^n e^{ax} dx$ and find $\int x^2 e^{ax} dx$.

Sol$^n$: Let $I_n = \int x^n e^{ax} dx$, (where n is a positive number)

Now, integration by parts, we have

$$I_n = x^n \int e^{ax} dx - \int \left( \frac{d}{dx} x^n \int e^{ax} dx \right) dx$$

$$= x^n e^{ax} \cdot \frac{1}{a} - \int n x^{n-1} \cdot e^{ax} \cdot \frac{1}{a} dx$$

$$= \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax} \cdot dx$$

$$= \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1} \text{; ——(A)}$$ which is the required reduction formula.

where $I_{n-1} = \int x^{n-1} e^{ax} dx$.

Now, put $n = 2$ in above (A)

$$I_2 = \frac{x^2 e^{ax}}{a} - \frac{2}{a} I_{2-1} = \int x^2 e^{ax} dx$$

$$= \frac{x^2 e^{ax}}{a} - \frac{2}{a} I_1 \quad \text{———— (1)}$$

Now, $$I_1 = \int x e^{ax} dx = x \int e^{ax} dx - \int \left( \frac{dx}{dx} \int e^{ax} dx \right) dx$$

$$= \frac{x}{a} e^{ax} - \int \frac{1}{a} e^{ax} dx$$

$$= \frac{x e^{ax}}{a} - \frac{1}{a^2} e^{ax} \quad \text{———— (2)}$$

Then (1) becomes

$$I_2 = \int x^2 e^{ax}\,dx$$

$$= \frac{x^2 e^{ax}}{a} - \frac{2}{a}\left(\frac{x e^{ax}}{a} - \frac{1}{a^2} e^{ax}\right) + c \quad [\text{by } (2)]$$

$$= \frac{x^2 e^{ax}}{a} - \frac{2}{a^2} x e^{ax} + \frac{2}{a^3} e^{ax} + C$$

$$= \frac{e^{ax}}{a^3}\left(a^2x^2 - 2ax + 2\right) + C$$

Ans:

P2: Establish a reduction formula for

$$\int x^m \sin nx\,dx$$

Sol$^n$: Let $I_m = \int x^m \sin nx\,dx$

We have integration by parts,

$$I_m = x^m \int \sin nx\,dx - \int \left(\frac{d}{dx} x^m \int \sin nx\,dx\right) dx$$

$$= \frac{-x^m}{n} \cos nx - \int m x^{m-1} \left(-\frac{1}{n} \cos nx\right) dx$$

$$= -\frac{x^m}{n} \cos nx + \frac{m}{n} \int x^{m-1} \cos nx\,dx$$

$$= -\frac{x^m}{n} \cos nx + \frac{m}{n}\left[x^{m-1} \int \cos nx\,dx - \int \left(\frac{d}{dx} x^{m-1} \int \cos nx\,dx\right) dx\right]$$

$$\Rightarrow I_m = -\frac{x^m}{n}\cos nx + \frac{m}{n^2}x^{m-1}\sin nx - \frac{m(m-1)}{n^2}\int x^{m-2}\sin nx\,dx$$

$$= -\frac{x^m}{n}\cos nx + \frac{m}{n^2}x^{m-1}\sin nx - \frac{m(m-1)}{n^2}\cdot I_{m-2}$$

which is the required reduction formula and where $I_{m-2} = \int x^{m-2}\sin nx\,dx$.

Ans:

P3: Establish a formula of reduction for $\int \cos^n x\,dx$.

Soln: Let $I_n = \int \cos^n x\,dx = \int \cos^{n-1}x\cdot\cos x\,dx$

We have integration by parts

$$I_n = \cos^{n-1}x\int \cos x\,dx - \int\left(\frac{d}{dx}\cos^{n-1}x\int\cos x\,dx\right)dx$$

$$= \sin x\cos^{n-1}x - \int (n-1)\cos^{n-2}x\,(-\sin x)\sin x\,dx$$

$$= \sin x\cos^{n-1}x + (n-1)\int \cos^{n-2}x\,\sin^2 x\,dx$$

$$\Rightarrow I_n = \sin x \cdot \cos^{n-1} x + (n-1) \int \cos^{n-2} x \, (1 - \cos^2 x) \, dx$$

$$= \sin x \cos^{n-1} x + (n-1) \int \cos^{n-2} x \, dx - (n-1) \int \cos^n x \, dx$$

$$= \sin x \cos^{n-1} x + (n-1) I_{n-2} - (n-1) I_n$$

$$\Rightarrow I_n + (n-1) I_n = \sin x \cos^{n-1} x + (n-1) I_{n-2}$$

$$\Rightarrow I_n (1 + n - 1) = \sin x \cos^{n-1} x + (n-1) I_{n-2}$$

$$\Rightarrow I_n \cdot n = \sin x \cos^{n-1} x + (n-1) I_{n-2}$$

$$\therefore I_n = \frac{1}{n} \sin x \cos^{n-1} x + \frac{n-1}{n} I_{n-2}$$

which is the required reduction formula and where $I_{n-2} = \int \cos^{n-2} x \, dx$.

Ans:

P4: Establish a reduction formula for

$$\int \tan^n x \, dx$$

Soln: Let,

$$I_n = \int \tan^n x \, dx = \int \tan^{n-2}x \cdot \tan^2 x \, dx$$

$$= \int \tan^{n-2}x \, (\sec^2 x - 1) \, dx$$

$$= \int \tan^{n-2}x \sec^2 x \, dx - \int \tan^{n-2}x \, dx$$

$$= \int \tan^{n-2}x \sec^2 x \, dx - I_{n-2}$$

we have integration by parts,

$$I_n = \tan^{n-2}x \int \sec^2 x \, dx - \int \left( \frac{d}{dx} \tan^{n-2} \int \sec^2 x \, dx \right) dx - I_{n-2}$$

$$= \tan^{n-2}x \cdot \tan x - \int (n-2) \tan^{n-3}x \sec^2 x \tan x \, dx - I_{n-2}$$

$$= \tan^{n-1}x - (n-2) \int \tan^{n-2}x \, (1 + \tan^2 x) \, dx - I_{n-2}$$

$$= \tan^{n-1}x - (n-2) \int \tan^{n-2}x \, dx - (n-2) \int \tan^n x \, dx - I_{n-2}$$

$$= \tan^{n-1}x - (n-2) I_{n-2} - (n-2) I_n - I_{n-2}$$

$$\Rightarrow I_n + (n-2) I_n = \tan^{n-1}x - I_{n-2}(n-2+1)$$

$\Rightarrow I_n(1+n-2) = \tan^{n-1}x - (n-1)I_{n-2}$

$\Rightarrow I_n(n-1) = \tan^{n-1}x - (n-1)I_{n-2}$

$$\therefore I_n = \frac{\tan^{n-1}x}{n-1} - I_{n-2}$$

which is the required reduction formula and where $I_{n-2} = \int \tan^{n-2}x \, dx$.

Ans:-

# Assignment ([illegible]).

Establish a reduction formula for the followings:

1. $\int \sin^n x \, dx$ and $\int \sin^7 x \, dx$ (i.e $n=7$)

2. $\int \cot^n x \, dx$

3. $\int \sec^n x \, dx$

Sol$^n$:

1) $I_n = \int \sin^n x\,dx = -\frac{1}{n} \sin^{n-1}x \cos x + \frac{n-1}{n} I_{n-2}$

where $I_{n-2} = \int \sin^{n-2} x\, dx$.

and

$I_7 = \int \sin^7 x\, dx = -\cos x + \cos^3 x - \frac{3}{5} \cos^5 x +$

$\frac{1}{7} \cos^7 x + C$.

2) $I_n = \int \cot^n x\, dx = \frac{\cot^{n-1} x}{n-1} - I_{n-2}$

where $I_{n-2} = \int \cot^{n-2} x\, dx$.

3) $I_n = \int \sec^n x\, dx = \frac{\tan x \sec^{n-2} x}{n-1} + \frac{(n-2)}{(n-1)} I_{n-2}$

where $I_{n-2} = \int \sec^{n-2} x\, dx$.

# Reduction Formula

## 1. Integration by Part:

LIATE

L = Logarithm $\to$ log / In ....

I = Inverse $\to \sin^{-1}x, \cos^{-1}x, \ldots$

A = Algebraic $\to x^2, x^3, 5x, \ldots$

T = Trigonometeric $\to \sin x, \cos x, \ldots$

E = Exponential $\to e^x, e^y, e^{-2x} \ldots$

N.B:

$$\int uv\,dx = u\int v\,dx - \int\left(\frac{du}{dx}\int v\,dx\right)dx$$

$\xrightarrow{\hspace{2em}}$ LIATE

কোনটি u হবে তার জন্য

2. $$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}\underbrace{\int \cos^{n-2}x\,dx}_{I_{n-2}}$$

3. $$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int \sin^{n-2}x\,dx$$

4. $$\int x^n e^x\,dx = x^n e^x - n\int x^{n-1}e^x\,dx$$

5. $$\int \sin^m x\cos^n x\,dx = \frac{\sin^{m+1}x\cos^{n-1}x}{m+n} + \frac{n-1}{m+n}I_{m,n-2}$$

6. $\int \tan^n x\,dx = \frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x\,dx$

7. $\int \sec^n x\,dx = \frac{\sec^{n-2}x\tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}x\,dx$

8. $\int_0^{\pi/2} \cos^n x\,dx = \frac{n-1}{n} I_{n-2}$.

where $I_{n-2} = \int_0^{\pi/2} \cos^{n-2}x\,dx$.

9. $\int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n} I_{n-2}$

$I_{n-2} = \int_0^{\pi/2} \sin^{n-2}x\,dx$.

10. $I_{p,q} = \int_0^{\pi/2} \sin^p x \cos^q x\,dx$

$= \frac{p-1}{p+q} I_{p-2,q}$.

11. $\int \sin^m x \cos^n x\,dx = \int \sin^m x\, d(\sin^n x) = \frac{\sin^{m+n}x}{m+n}$

12. Same to $\tan x \sec^n x$

Q: $I_n = \int_0^{\pi/4} \tan^n\theta \, d\theta$ Show that.

$$n\left(I_{n+1} + I_{n-1}\right) = 1.$$

Soln: Given that.

$$I_n = \int_0^{\pi/4} \tan^n\theta \, d\theta.$$

n replaced by n+1.

$$\therefore I_{n+1} = \int_0^{\pi/4} \tan^{n+1}\theta \, d\theta$$

$$= \int_0^{\pi/4} \tan^2\theta \, \tan^{n-1}\theta \, d\theta$$

$$= \int_0^{\pi/4} \tan^{n-1}\theta \left(\sec^2\theta - 1\right) d\theta$$

$$= \int_0^{\pi/4} \tan^{n-1}\theta \sec^2\theta \, d\theta - \int_0^{\pi/4} \tan^{n-1}\theta \, d\theta$$

$$= \int_0^{\pi/4} \tan^{n-1}\theta \, d(\tan\theta) - \int_0^{\pi/4} \tan^{n-1}\theta \, d\theta$$

$$I_{n+1} = \left[\frac{\tan^n\theta}{n}\right]_0^{\pi/4} - I_{n-1}$$

$$\Rightarrow I_{n+1} + I_{n-1} = \frac{1}{n}\left[\left(\tan\frac{\pi}{4}\right)^n - 0\right]$$

$$= \frac{1}{n}\left[1^n - 0\right]$$

$$= \frac{1}{n}$$

$$\therefore n\left(I_{n+1} + I_{n-1}\right) = 1.$$

Proved

# If $U_n = \int_0^1 x^n \tan^{-1}x\,dx$, then prove that

$$(n+1)U_n + (n-1)U_{n-2} = \frac{\pi}{2} - \frac{1}{n}.$$

**Sol$^n$:** Here given, $U_n = \int_0^1 x^n \tan^{-1}x\,dx$

let, $x = \tan\theta \Rightarrow dx = \sec^2\theta\,d\theta$.

| $x$ | $0$ | $1$ |
|---|---|---|
| $\theta$ | $0$ | $\pi/4$ |

So, $U_n = \int_0^{\pi/4} \underline{\theta\tan^n\theta}\cdot\underline{\sec^2\theta}\,d\theta$ $\quad$ (uv) $\quad$ $(*)$

$$\Rightarrow U_n = \left[\theta\tan^n\theta\cdot\tan\theta\right]_0^{\pi/4} - \int_0^{\pi/4}\tan\theta\left\{1\cdot\tan^n\theta + \theta n\tan^{n-1}\theta\cdot\sec^2\theta\right\}d\theta$$

$$= \frac{\pi}{4} - \int_0^{\pi/4}\tan^{n+1}\theta\,d\theta - \int_0^{\pi/4} n\,\theta\tan^n\theta\sec^2\theta\,d\theta$$

$$= \frac{\pi}{4} - \int_0^{\pi/4}\tan^{n+1}\theta\,d\theta - nU_n \qquad \left(\text{Again, let } \boxed{x = \tan\theta} \text{ on by } (*) \Rightarrow \theta = \tan^{-1}x\right)$$

$$\Rightarrow U_n + nU_n = \frac{\pi}{4} - \int_0^{\pi/4}\tan^{n+1}\theta\,d\theta$$

$$\Rightarrow (n+1)U_n = \frac{\pi}{4} - \int_0^{\pi/4}\tan^{n+1}\theta\,d\theta \quad \text{--- (1)}$$

$\therefore \cancel{U_{n-2} \quad \frac{\pi}{1}}$

$$\therefore (n-1)\,U_{n-2} = \frac{\pi}{4} - \int_0^{\pi/4} \tan^{n-1}\theta \, d\theta \quad \text{—— (2)}$$

(1) + (2) $\Rightarrow$

$$(n+1)\,U_n + (n-1)\,U_{n-2}$$

$$= \frac{\pi}{2} - \int_0^{\pi/4} \tan^{n-1}\theta \left(\tan^2\theta + 1\right) d\theta$$

$$= \frac{\pi}{2} - \int_0^{\pi/4} \tan^{n-1}\theta \cdot \sec^2\theta \, d\theta$$

$$= \frac{\pi}{2} - \int_0^{\pi/4} (\tan\theta)^{n-1} \, d(\tan\theta)$$

$$= \frac{\pi}{2} - \left(\frac{\tan^n\theta}{n}\right)_0^{\pi/4}$$

$$= \frac{\pi}{2} - \frac{1}{n}.$$

$$\therefore (n+1)\,U_n + (n-1)\,U_{n-2} = \frac{\pi}{2} - \frac{1}{n}.$$

Proved.