Gamma and Beta Functions - (9)

Gamma Function:

The integral $\int_0^\infty x^{n-1} e^{-x} dx$ (1) is called the Eulerian integral. The function (1) defined for positive value of n (i.e. $n>0$) is known as Gamma Function and denoted by $\Gamma(n)$. i.e.

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx \quad , n>0$$

Note: $\dfrac{\Gamma(n)}{z^n} = \int_0^\infty y^{n-1} e^{-zy} dy$

$$\Rightarrow \Gamma(n) = \int_0^\infty z^n y^{n-1} e^{-zy} dy .$$

Beta Function:

The integral $\int_0^1 x^{m-1} (1-x)^{n-1} dx$ (2) is know as First Eulerian integral. The function (2) is defined for positive value of m and n (i.e. $m>0, n>0$) is known as Beta Function and denoted by $\beta(m,n)$.

$$\therefore \beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx \quad , m>0, n>0$$

Note: $\beta(m,n) = \int_0^\infty \dfrac{y^{n-1}}{(1+y)^{m+n}} dy$

Relation between Gamma and Beta Functions:

Prove that,

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Sol^n: we know that

$$\Gamma(n) = \int_0^\infty z^n y^{n-1} e^{-zy}\, dy \qquad \text{—— (1)}$$

Multiply both sides by $z^{m-1} e^{-z}$

$$\Gamma(n)\, z^{m-1} e^{-z} = \int_0^\infty z^n y^{n-1} e^{-zy} \cdot z^{m-1} e^{-z}\, dy$$

$$= \int_0^\infty y^{n-1} z^{m+n-1} e^{-z(1+y)}\, dy$$

Integrate both sides w.r.t z for $z=0$ to $z=\infty$

$$\Gamma(n) \int_0^\infty z^{m-1} e^{-z}\, dz = \int_0^\infty \left[\int_0^\infty z^{m+n-1} e^{-z(1+y)} dz \right] y^{n-1}\, dy$$

$$\Rightarrow \Gamma(n)\cdot\Gamma(m) = \int_0^\infty \frac{\Gamma(m+n)}{(y+1)^{m+n}}\, y^{n-1}\, dy$$

[~~$\Gamma(m+n)=$~~ by (1) and defn of Gamma]

$$= \Gamma(m+n) \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\, dy$$

$$= \Gamma(m+n)\, B(m,n) \qquad \left[B(m,n) = \int_0^\infty \frac{y^{n-1}}{y^{m+n}}\, dy \right]$$

$$\Rightarrow B(m,n) = \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)} \quad \text{(Proved)}$$

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Properties of Beta and Gamma Functions:

P1: Prove that $\Gamma(n+1) = n\Gamma(n)$ if $n>0$

Soln: By definition of Gamma Function,

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx \quad \text{———} (1) \quad , \text{if } n>0$$

$$\therefore \Gamma(n+1) = \int_0^\infty x^{n+1-1} e^{-x}\, dx$$

$$= \int_0^\infty x^n e^{-x} dx$$

$$= x^n \int_0^\infty e^{-x} dx - \int_0^\infty \left(\frac{d}{dx} x^n \int_0^\infty e^{-x} dx\right) dx$$

$$= \left[-x^n e^{-x}\right]_0^\infty - \int_0^\infty n x^{n-1} (-1) e^{-x} dx$$

$$= 0 + n \int_0^\infty x^{n-1} e^{-x} dx$$

$$= n\Gamma(n) \quad [\text{by } (1)]$$

$$\therefore \Gamma(n+1) = n\Gamma(n)$$

Proved

P2: $\Gamma(n+1) = n!$ if $n>0$

P3: $\Gamma(1) = 1$

P4: $\frac{\Gamma(n)}{z^n} = \int_0^\infty y^{n-1} e^{-zy}\, dy$

P5: $\beta(m,n) = \beta(n,m)$ if $n>0, m>0$

P6: $$\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx.$$

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Q. Prove that, $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$.

(von)

Then prove that,

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$$

Sol^n: By definition of beta Function,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad \text{——— (1)}$$

let $x = \sin^2\theta \Rightarrow dx = 2\sin\theta\cos\theta\,d\theta$

x	0	1
θ	0	$\pi/2$

Then (1) becomes,

$$\beta(m,n) = \int_0^{\pi/2} \sin^{2m-2}\theta\,(1-\sin^2\theta)^{n-1}\,2\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta \quad \text{——— (2)}$$

(Proved)

we know that

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Now we put $2m-1 = p$ and $2n-1 = q$ in (2)

Then we have $m = \frac{p+1}{2}$ and $n = \frac{q+1}{2}$. Then (2)

becomes

$$\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = 2\int_0^{\infty} \sin^p\theta \cos^q\theta \, d\theta$$

$$\Rightarrow \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)} = 2\int_0^{\infty} \sin^p\theta \cos^q\theta \, d\theta$$

$$\Rightarrow \int_0^{\infty} \sin^p\theta \cos^q\theta \, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$$

Proved

P7: $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

P8: $\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$, $0 < n < 1$

$$\int_0^{\pi/6} \cos^6 3x (\sin 6x)^4 \, dx = ?$$

Soln:

$$\int_0^{\pi/6} \cos^6 3x (\sin 6x)^4 \, dx$$

$$= \int_0^{\pi/6} \cos^6 3x \, (2\sin 3x \cos 3x)^4 \, dx$$

$$= 2^4 \int_0^{\pi/6} \cos^6 3x \, \cos^4 3x \, \sin^4 3x \, dx$$

$$= 2^4 \int_0^{\pi/6} \cos^{10} 3x \, \sin^4 3x \, dx$$

Let $3x = z$, $d\cancel{x} = \frac{1}{3} dz$

x	0	$\pi/6$
z	0	$\pi/2$

$$= 2^4 \int_0^{\pi/2} \cos^{10} z \, \sin^4 z \, \frac{1}{3} \, dz$$

$$= \frac{2^4}{3} \int_0^{\pi/2} \cos^{10} z \, \sin^4 z \, dz$$

$$= \frac{16}{3} \frac{\Gamma\left(\frac{10+1}{2}\right) \Gamma\left(\frac{4+1}{2}\right)}{2\Gamma\left(\frac{10+4+2}{2}\right)} = \frac{16}{3\cancel{6}} \frac{\Gamma\left(\frac{11}{2}\right)\Gamma\left(\frac{5}{2}\right)}{2\cancel{6}\,\Gamma(8)}$$

$$= \frac{16}{3} \frac{\frac{9}{2}\frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma(\frac{1}{2})\ \frac{3}{2}\frac{1}{2}\Gamma(\frac{1}{2})}{2\,(8-1)!}$$

$$= \frac{3\pi}{256} \quad \underline{\underline{\text{Ans.}}}$$

H.W.

Q. $\int_0^{2\pi} \sin^4 x \cos^6 x \, dx$

<u>Soln:</u> $\int_0^{2\pi} \sin^4 x \cos^6 x \, dx$

$$= \int_0^{4\pi/2} \sin^4 x \cos^6 x \, dx$$

[we know that,

$\int_0^{na} f(x)\, dx = n \int_0^{a} f(x)\, dx$]

$$= 4 \int_0^{\pi/2} \sin^4 x \cos^6 x \, dx$$

$$= 4 \frac{\Gamma\left(\frac{4+1}{2}\right)\Gamma\left(\frac{6+1}{2}\right)}{2\,\Gamma\left(\frac{4+6+2}{2}\right)}$$

$$= 2 \frac{\Gamma(\frac{5}{2})\ \Gamma(\frac{7}{2})}{\Gamma(6)} = \frac{2\ \frac{3}{2}\frac{1}{2}\Gamma(\frac{1}{2})\frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma(\frac{1}{2})}{(6-1)!}$$

$$= \frac{3\pi}{128} \quad \text{A.}$$

Q. $\int_0^{\pi} x \sin^6 x \cos^4 x \, dx$

Soln:

$$I = \int_0^{\pi} x \sin^6 x \cos^4 x \, dx \quad \text{(1)}$$

[we know that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$]

$$I = \int_0^{\pi} (\pi - x) \sin^6 (\pi - x) \cos^4 (\pi - x)\, dx$$

$$= \int_0^{\pi} (\pi - x) \sin^6 x \cos^4 x \, dx \quad \text{(11)}$$

(1) and (11) adding,

$$2I = \int_0^{\pi} (\pi - x + x) \sin^6 x \cos^4 x \, dx$$

$$I = \frac{1}{2} \int_0^{\pi} \pi \sin^6 x \cos^4 x \, dx.$$

$$= \frac{\pi}{2} \int_0^{2\pi/2} \sin^6 x \cos^4 x \, dx$$

$$= \frac{2\pi}{2} \int_0^{\pi/2} \sin^6 x \cos^4 x \, dx$$

$$I = \pi \int_0^{\pi/2} \sin^6 x \cos^4 x \, dx$$

$$= \pi \frac{\Gamma\left(\frac{6+1}{2}\right) \Gamma\left(\frac{4+1}{2}\right)}{2\Gamma\left(\frac{6+4+2}{2}\right)}$$

$$= \pi \frac{\Gamma\left(\frac{7}{2}\right) \Gamma\left(\frac{5}{2}\right)}{2\Gamma(6)}$$

$$= \pi \frac{\frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right) \frac{3}{2} \cdot \frac{1}{2} \cdot \Gamma\left(\frac{1}{2}\right)}{2 \, (6-1)!}$$

$$= \frac{\pi \, \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi} \cdot \frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi}}{2 \cdot \cancel{(6)} \; 5!}$$

$$= \frac{3\pi^2}{512} \cdot \text{Pr.}$$