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 Mathematical Induction
 Exercise problems

3. $P(n) : 1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$, for all true int n

Basis Step

$$P(1) = \frac{1(1+1)(2 \cdot 1+1)}{6} = 1$$

$$\text{or, } 1 = \frac{2 \times 3}{6} = 1 \quad (1+1) =$$

$$\therefore 1 = 1.$$

Inductive Step:

Let $P(k)$ be true:

$$1^2 + 2^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6} \quad \text{--- (1)}$$

Now, $P(k+1)$ is

$$1^2 + 2^2 + \dots + k^2 + (k+1)^2 = \frac{(k+1)(k+1+1)(2(k+1)+1)}{6} \quad \text{--- (2)}$$

$$\text{or, } 1^2 + 2^2 + \dots + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6} \quad \text{state eqn (1)}$$

Adding $(k+1)^2$ to (1) on both sides:

$$\begin{aligned} 1^2 + 2^2 + \dots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= (k+1) \left[\frac{k(2k+1)}{6} + (k+1) \right] \\ &\text{--- (1) ---} \quad (k+1) \left[\frac{k(2k+1)}{6} + \frac{6(k+1)}{6} \right] \end{aligned}$$

$$= (k+1) \left\{ \frac{2k^2 + k + 6k + 6}{6} \right\}$$

~~common 6~~ common

$$= (k+1) \left\{ \frac{2k^2 + 7k + 6}{6} \right\}$$

$$= (k+1) \left\{ \frac{2k^2 + 4k + 3k + 6}{6} \right\}$$

~~(k+2)(k+1)~~

$$= (k+1) \frac{2k(k+2) + 3(k+2)}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

\therefore If $P(k)$ is true then $P(k+1)$ is true.

$$4. 1^3 + 2^3 + \dots + k^3 = \frac{(n(n+1)/2)^2}{(1+n)(2+n)(3+n)}$$

Basis step: $P(1)$

$$1^3 = (1(1+1)/2)^2 = 1^2 = 1.$$

Inductive step:

$$P(k): 1^3 + 2^3 + \dots + k^3 = \left[\frac{k(k+1)}{2} \right]^2 \quad \textcircled{1}$$

$$P(k+1): 1^3 + 2^3 + \dots + (k+1)^3 = \left[\frac{(k+1)(k+1+1)}{2} \right]^2$$

$$\left[\frac{(k+1)}{2} + \frac{(k+2)}{2} \right]^2 = \frac{(k+1)^2(k+2)^2}{4}$$

→ $\textcircled{2}$

$$\begin{aligned}
 & \text{Adding } (k+1)^3 \text{ to } \textcircled{1} \Rightarrow (1+2+\dots+k+1)^3 = 1^3 + 2^3 + \dots + k^3 + (k+1)^3 \\
 & 1^3 + 2^3 + \dots + k^3 + (k+1)^3 = \left[\frac{k(k+1)}{2} \right]^2 + (k+1)^3 \stackrel{(1+2+\dots+k+1)}{=} \textcircled{2} \stackrel{(1+2+\dots+k+1)}{=} \textcircled{2} \\
 & = \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4} \quad L = R \\
 & = \frac{k^2(k^2+2k+1) + 4(k^3+3k^2+3k+1)}{4} \stackrel{(1+2+\dots+k+1)}{=} \textcircled{3} \\
 & = \frac{k^4 + 2k^3 + k^2 + 4k^3 + 12k^2 + 12k + 4}{4} \\
 & = \frac{k^4 + 6k^3 + 13k^2 + 12k + 4}{4} \quad \textcircled{3} = \textcircled{3}
 \end{aligned}$$

we further expand \textcircled{1}

$$\begin{aligned}
 1^3 + 2^3 + \dots + (k+1)^3 &= \frac{(k^2+2k+1)(k^2+4k+4)}{4} \\
 &= \frac{k^2(k^2+4k+4) + 2k(k^2+4k+4) + k^2+4k+4}{4} \stackrel{\textcircled{2}}{=} \\
 &= \frac{k^4 + 4k^3 + 4k^2 + 2k^3 + 8k^2 + 8k + k^2 + 4k + 4}{4} \\
 &= \frac{k^4 + 6k^3 + 13k^2 + 12k + 4}{4} \stackrel{\textcircled{3}}{=} \textcircled{3}
 \end{aligned}$$

$$\begin{aligned}
 & \frac{(2+1)(2+2)(2+3)}{2} = \\
 & [21 + 22 + 23] (2+3) = \\
 & [21 + 22 + 23] (2+3) =
 \end{aligned}$$

$$\begin{aligned}
 & (21 + 22 + 23) 2 + (21 + 22 + 23) 3 = \\
 & 2(21 + 22 + 23) + 3(21 + 22 + 23) =
 \end{aligned}$$

$$5 \cdot 0^2 + 1^2 + 3^2 + 5^2 + \dots + (2n+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3} \quad \text{Bm66A}$$

$P(0): 0^2 + (2 \cdot 0 + 1)^2 = \frac{(0+1)(2 \cdot 0+1)(2 \cdot 0+3)}{3} = 0^2 + 1^2 + \dots + 1^2 + 0^2$

$$1 = 1.$$

$$\frac{(1+1)P}{3} + \frac{(1+3)S}{3} =$$

$$P(1): 0^2 + 1^2 + (2+1)^2 = \frac{(1+1)(2 \cdot 1+1)(2 \cdot 1+3)}{3} =$$

$$\text{on, } 1 + 3^2 = \frac{2 \times (2+1)(2+3)}{3} =$$

$$\text{or } 10 = \frac{2 \times 3 \times 5}{3} =$$

$$\text{on, } 10 = 10.$$

(ii) by using induction

$$\frac{(P + NP + NS)(1 + N + S)}{3} = \varepsilon_{(1+n)} + \dots + \varepsilon_2 + \varepsilon_1$$

IS:

$$P(k) = 0^2 + 1^2 + 2^2 + \dots + (2k+1)^2 = \frac{(k+1)(2k+1)(2k+3)}{3} \quad \text{--- ①}$$

$$P(k+1): 0^2 + 1^2 + 2^2 + \dots + (2k+1)^2 + (2k+2)^2 = \frac{(k+1+1)(2(k+1)+1)(2(k+1)+3)}{3}$$

$$= \frac{(k+2)(2k+2+1)(2k+2+3)}{3} =$$

$$= \frac{(k+2)(2k+3)(2k+5)}{3}$$

(using)

$$= \frac{(k+2)[(2k^2+8k+15)]}{3}$$

$$= \frac{(k+2)[4k^2+16k+15]}{3}$$

$$= \frac{k(4k^2+16k+15) + 2(4k^2+16k+15)}{3}$$

$$= \frac{4k^3+16k^2+15k+8k^2+32k+30}{3}$$

$$P(k+1) = \frac{4k^3 + 12k^2 + 47k + 30}{3} + \frac{1}{(1+k)(2+k)} + \frac{1}{(1+k)(3+k)} + \dots + \frac{1}{(1+k)(n+k)} \quad (209)$$

Adding $(2(k+1)+1)^2$ to both sides of ①: $E - \frac{1}{(1+k)} = E - (209)$

$$\begin{aligned} 0^2 + 1^2 + \dots + (2k+1)^2 + (2(k+1)+1)^2 &= \frac{(k+1)(2k+1)(2k+3)}{3} + [2(k+1)+1]^2 \\ &= \frac{(k+1)[(2k)^2 + 4 \times 2k + 3]}{3} + [2k+2+1]^2 = E \end{aligned}$$

$$\textcircled{1} \longrightarrow E - \frac{1}{(1+k)} = \frac{1}{(1+k)} + \dots + \frac{1}{(2k+3)} + \frac{1}{(2k+4)} + \dots + \frac{1}{(n+k)} \quad (209)$$

$$E = \frac{4k^3 + 8k^2 + 3k + 4k^2 + 8k + 3}{3} + (4k^2 + 12k + 9) \quad (209)$$

$$E = \frac{4k^3 + 12k^2 + 11k + 3}{3} + \frac{12k^2 + 36k + 27}{3} \quad \text{using method of } \frac{1}{(1+k)} \cdot \frac{1}{(2k+3)} \cdot \text{etc}$$

$$\frac{1}{(1+k)} \cdot \frac{1}{(2k+3)} = \frac{4k^3 + 24k^2 + 47k + 30}{3} \quad (209)$$

$$E - \frac{1}{(1+k)} - \frac{1}{(2k+3)} = \dots$$

$$E - (1+k) \cdot \frac{1}{(1+k)} =$$

$$E - \frac{1}{(1+k)} \cdot \frac{1}{(2k+3)} =$$

$$6. P(n) : 1 \cdot 1! + 2 \cdot 2! + \dots + n \cdot n! = \frac{(n+1)! - 1}{e} = (n+1)!$$

$P(1)$: $1 = (1+1)! - 1$ \rightarrow the value of $(1+1)$ is 2

$$\begin{aligned} 1 &= \frac{2! - 1}{e} \\ 1 &= 2 - 1 \quad \text{So} \\ 1 &= \frac{[1 \cdot 2 + \dots]}{e} + \frac{[(1+1) + (1+2) + \dots]}{e} \end{aligned}$$

$$P(k) : 1 \cdot 1! + 2 \cdot 2! + \dots + k \cdot k! = (k+1)! - 1. \quad \text{--- (i)}$$

$$\begin{aligned} P(k+1) &\stackrel{\text{Def}}{=} 1 \cdot 1! + 2 \cdot 2! + \dots + (k+1) \cdot (k+1)! = (k+1+1)! - 1. \\ &= (k+2)! - 1 \end{aligned}$$

$$\frac{[1 \cdot 2 + \dots] + [(k+1) \cdot (k+1)!]}{e} = (k+2)(k+1)! - 1.$$

Adding: $(k+1) \cdot (k+1)!$ to both sides:

$$\begin{aligned} 1 \cdot 1! + 2 \cdot 2! + \dots + k \cdot k! + (k+1) \cdot (k+1)! &= (k+1)! - 1 + (k+1) \cdot (k+1)! \\ &= (k+1)! + (k+1) \cdot (k+1)! - 1 \end{aligned}$$

$$= (k+1)! (1+k+1) - 1$$

$$= (k+2)(k+1)! - 1$$

$$7. \quad 3 + 3 \cdot 5 + 3 \cdot 5^2 + \dots + 3 \cdot 5^n = 3(5^{n+1} - 1)/4$$

$$\text{Now } 3 \in \mathbb{Z}_{\geq 0}, \text{ so } \left[\frac{3(5^{n+1} - 1)}{4} \right] = 3(F) \underline{\underline{2}} + \dots + 4 \underline{\underline{2}} + F \underline{\underline{2}} + F \underline{\underline{2}} + F \underline{\underline{2}}$$

$\therefore P(0) \equiv$

$$3 \cdot 5^0 = 3(5^{0+1} - 1)/4$$

$$3 = 3 \times \frac{4}{4} = 3.$$

$$\frac{3(5^{0+1} - 1)}{4} = F \underline{\underline{2}}$$

P(k) ≡

$$3 + 3 \cdot 5 + 3 \cdot 5^2 + \dots + 3 \cdot 5^k = \frac{3(5^{k+1} - 1)}{4}$$

P(k+1) ≡

$$3 + 3 \cdot 5^1 + 3 \cdot 5^2 + \dots + 3 \cdot 5^k + 3 \cdot 5^{k+1} = \frac{3(5^{k+2} - 1)}{4}$$

Adding: $3 \cdot 5^{k+1}$ to both side of ①

$$3 + 3 \cdot 5^1 + \dots + 3 \cdot 5^k + 3 \cdot 5^{k+1} = \frac{3(5^{k+1} - 1)}{4} + 3 \cdot 5^{k+1}$$

∴ Now ① $\equiv \frac{3 \cdot 5^{k+1} - 3}{4} + \frac{4 \cdot 3 \cdot 5^{k+1}}{4}$

$$\frac{1+3(F) \underline{\underline{2}}}{4} + \frac{3 \cdot 5^{k+1} - 3}{4} = \frac{3 \cdot 5^{k+1} - 3 + 12 \cdot 5^{k+1}}{4}$$

$$\frac{1+3(F) \underline{\underline{2}}}{4} = \frac{15 \cdot 5^{k+1} - 3 - 1}{4} =$$

$$\frac{1+3(F) \underline{\underline{2}}}{4} + \frac{3[5 \cdot 5^{k+1} - 1]}{4} = \frac{3(5^{k+2} - 1)}{4}$$

(Proved)

8. Prove that $\forall (L-1^m \mathcal{E}) \mathcal{E} = 0 \cdot \mathcal{E} + \dots + 4 \cdot \mathcal{E} + 8 \cdot \mathcal{E} + \dots$

$$2 \cdot 7^0 - 2 \cdot 7^1 + 2 \cdot 7^2 - \dots + 2(-7)^n = \frac{[1 - (-7)^{n+1}]}{4}$$

P(0)

$$2 \cdot 7^0 = \frac{1 - (-7)^{0+1}}{4}$$

$$2 = \frac{1 - (-7)^1}{4}$$

$$2 = \frac{1 - (-7)}{4} = \frac{2}{4} = \frac{2 \cdot \mathcal{E}}{4} = \frac{2 \cdot \mathcal{E}}{4} = 0 \cdot \mathcal{E} + \dots + 2 \cdot \mathcal{E} + 8 \cdot \mathcal{E}$$

P(k)

$$\frac{(L-1^m \mathcal{E}) \mathcal{E}}{4} = \frac{[1 - (-7)^{k+1}]}{4}$$

$$2 \cdot 7^0 - 2 \cdot 7^1 + 2 \cdot 7^2 - \dots + 2(-7)^k = \frac{[1 - (-7)^{k+1}]}{4}$$

P(k+1)

$$2 \cdot 7^0 - 2 \cdot 7^1 + 2 \cdot 7^2 - \dots + 2(-7)^k + 2(-7)^{k+1} = \frac{[1 - (-7)^{k+2}]}{4} = \frac{1 \cdot \mathcal{E} + 4 \cdot \mathcal{E} + \dots + 3 \cdot \mathcal{E} + 8 \cdot \mathcal{E}}{4}$$

Adding: $2(-7)^{k+1}$ to both sides of ① we get:

$$2 \cdot 7^0 - 2 \cdot 7^1 - \dots + 2(-7)^k + 2(-7)^{k+1} = \frac{1 - (-7)^{k+1}}{4} + 2(-7)^{k+1}$$

$$= \frac{1 - (-7)^{k+1}}{4} + \frac{8 \cdot (-7)^{k+1}}{4}$$

$$\frac{(L-1^m \mathcal{E}) \mathcal{E}}{4} = \frac{1 - 1 \cdot (-7)^{k+1} + 8 \cdot (-7)^{k+1}}{4}$$

(b) v o r 9)

$$= \frac{1 - (-7)}{4} = \frac{1 + 7}{4} = \frac{8}{4} = 2$$

$$= \frac{1 + (-7)^{k+1} [8-1]}{4} = \frac{1 + (-7)^{k+1} \cdot 7}{4}$$

$$= \frac{1 + (-7)^{k+1} \cdot 7}{4}$$

$$= \frac{1 + [(-7)^{k+1} \cdot (-1)^{k+1}] \cdot 7}{4} = \frac{1 + (-1)^4 + (-1)^{k+2} \cdot (-1)^{k+1}}{4} = \frac{1 + 1 - 7^{k+2} \cdot (-1)^{k+1}}{4} = \frac{1 - 7^{k+2} \cdot (-1)^{k+1}}{4}$$

$$= \frac{1 + (-1) \cdot (-1)^{k+1} \cdot 7^{k+2} \cdot (-1)^{k+1}}{4} = \frac{1 + (-1)^{k+1} \cdot 7^{k+2}}{4} = \frac{1 + (-1)^{k+1} \cdot 7^{k+2} \cdot (-1)^{k+1}}{4} = \frac{1 - 7^{k+2} \cdot (-1)^{k+1}}{4}$$

~~DE~~ ~~DE~~ ④ To estimate instead of $\frac{1 + (-1)^{k+1} \cdot 7^{k+2}}{4}$ ~~estimate~~

$$= \frac{1 - (-7)^{k+2}}{4}$$

$$= \frac{1 + (-1)}{4} + \frac{(-1) + 1}{4} = \frac{0}{4} + \frac{0}{4} = 0$$

$$= \frac{1 + (-1) \cdot 8}{4} + \frac{(-1) + 1 \cdot 8}{4} = \frac{-7}{4} + \frac{7}{4} = 0$$

$$= \frac{1 + (-1) \cdot 8}{4} + \frac{(-1) + 1 \cdot 8}{4} =$$

$$12. \sum_{j=0}^n \left(-\frac{1}{2}\right)^j = \frac{2^{n+1} + (-1)^n}{3 \cdot 2^n}$$

$$\left[\text{LHS} \right] = \left[-\frac{1}{2}^0 + (-\frac{1}{2})^1 + (-\frac{1}{2})^2 + \dots + (-\frac{1}{2})^n \right] = \frac{2^{n+1} + (-1)^n}{3 \cdot 2^n}$$

P(0)

$$1 = \frac{2^{0+1} + 1}{3} = 1.$$

P(k)

$$\left[\text{LHS} \right] = \left[-\frac{1}{2}^0 + (-\frac{1}{2})^1 + \dots + (-\frac{1}{2})^k \right] = \frac{2^{k+1} + (-1)^k}{3 \cdot 2^k} \quad \text{--- (i)}$$

P(k+1)

$$\left[\text{LHS} \right] = \left[-\frac{1}{2}^0 + (-\frac{1}{2})^1 + \dots + (-\frac{1}{2})^{k+1} \right] = \frac{2^{k+2} + (-1)^{k+1}}{3 \cdot 2^{k+1}}$$

Adding $(-\frac{1}{2})^{k+1}$ to both sides of (i) we get :

$$\left[\text{LHS} \right] = \frac{2^{k+1} + (-1)^k + (-\frac{1}{2})^{k+1}}{3 \cdot 2^k}$$

$$= \frac{2^{k+1} + (-1)^k}{3 \cdot 2^k} + \frac{(-1)^{k+1}}{2^{k+1}}$$

$$= \frac{2^{k+1} + (-1)^k}{3 \cdot 2^k} + \frac{3(-1)^{k+1}}{3 \cdot 2^{k+1}}$$

$$= \frac{2[2^{k+1} + (-1)^k]}{3 \cdot 2^{k+1}} + \frac{3(-1)^{k+1}}{3 \cdot 2^{k+1}}$$

$$= \frac{2 \cdot 2^{k+1} + 2(-1)^k + 3(-1)^{k+1}}{3 \cdot 2^{k+1}}$$

$$\textcircled{1} \quad \frac{2 \cdot 2^{k+1} + 2(-1)^k + 3(-1)^{k+1}}{3 \cdot 2^{k+1}}$$

$$= \frac{(1+\omega)(1+\omega^2) \cdot (-1)^k}{3 \cdot 2^{k+1}} = (1+\omega)^{-1}(-1) + \dots + (-1)^k \quad (1)$$

$$= \frac{2 \cdot 2^{k+1} + 2(-1)^k - 3(-1)^k}{3 \cdot 2^{k+1}}$$

$$= (1+\omega)^{-1}(-1) + \frac{2}{3}(-1)^k + \dots + \frac{4}{3}(-1)^k \quad (2)$$

$$= \frac{2 \cdot 2^{k+1} - (-1)^k}{3 \cdot 2^{k+1}} = (1+\omega)^{-1}(-1) + \frac{2}{3}(-1)^k + \dots + \frac{4}{3}(-1)^k \quad (3)$$

$$= \frac{(1+\omega)^{-1}(-1)}{2^{k+2}} + \frac{(-1)^k}{3 \cdot 2^{k+1}} = (1+\omega)^{-1}(-1) + \frac{2}{3}(-1)^k + \dots + \frac{4}{3}(-1)^k \quad (4)$$

$$= \frac{2^{k+2} - (-1)^{k+1}}{3 \cdot 2^{k+1}} =$$

$$= \frac{2^{k+2} + (-1)^{k+1}}{3 \cdot 2^{k+1}} =$$

(b) ω^3

$$13. \quad 1^2 - 2^2 + 3^2 - \dots + (-1)^{n-1} n^2 = \frac{(-1)^{n-1} \cdot n(n+1)}{2}$$

$P(1)$: can be shown as true.

$$P(k): 1^2 - 2^2 + \dots + (-1)^{k-1} k^2 = \frac{(-1)^{k-1} \cdot k(k+1)}{2} \quad (1)$$

$$P(k+1) \quad 1^2 - 2^2 + \dots + (-1)^{k-1} k^2 + (-1)^k (k+1)^2 = \frac{(-1)^k \cdot (k+1)(k+2)}{2}$$

From (1) we get by adding $(-1)^k (k+1)^2$

$$1^2 - 2^2 + \dots + (-1)^k (k+1)^2 = \frac{(-1)^k \cdot (k+1)(k+2)}{2}$$

$$1^2 - 2^2 + \dots + (-1)^{k-1} k^2 + (-1)^k (k+1)^2 = \frac{(-1)^{k-1} \cdot k(k+1)}{2} + (-1)^k (k+1)^2$$

$$= \frac{(-1)^k \cdot (-1)^{-1} \cdot k(k+1)}{2} + \frac{2(-1)^k (k+1)^2}{2}$$

$$= \frac{(-1)^k (-1)^{-1} k(k+1) + 2(-1)^k (k+1)^2}{2}$$

$$= \frac{(-1)^k (k+1) [k(-1)^{-1} + 2(k+1)]}{2}$$

$$= \frac{(-1)^k (k+1) [\frac{k}{-1} + 2k+2]}{2}$$

$$= \frac{(-1)^k (k+1) (k+2)}{2}$$

(Proved)

$$14. \sum_{k=1}^n k \cdot 2^k = (n-1)2^{n+1} + 2.$$

P(1):

$$1 \cdot 2^1 = (1-1)2^{1+1} + 2$$

$$\text{or}, 2 = 2.$$

POO:

$$1 \cdot 2^1 + 2 \cdot 2^2 + 3 \cdot 2^3 + \dots + k \cdot 2^k = (k-1)2^{k+1} + 2 \quad \text{--- i}$$

P(k+1):

$$1 \cdot 2^1 + 2 \cdot 2^2 + \dots + (k+1)2^{k+1} = (k+1)2^{k+2} + 2$$

From ① we get by adding $(k+1)2^{k+1}$

$$1 \cdot 2^1 + 2 \cdot 2^2 + \dots + k \cdot 2^k + (k+1)2^{k+1} = (k-1)2^{k+1} + 2 + (k+1)2^{k+1}$$

$$= 2^{k+1} (k+1+k-1) + 2 =$$

$$= 2^{k+1} \cdot 2k + 2 =$$

$$= 2^{k+2} \cdot k + 2 \quad (\text{Proved})$$

∴ To prove ① to be true or not

15, 16 it solve similarly.

$$= 2^{k+2} \cdot k + 2 + \dots + 2^2 + 2 + 2$$

$$17 \sum_{j=1}^n j^4 = n(n+1)(2n+1)(3n^2+3n-1)/30 + \frac{(1-3\epsilon+5\epsilon^2)(1+5\epsilon)(1+2\epsilon)}{30}$$

P(1) =

$$1^4 = 1(2)(3)(3+3-1)/30 + (1-3\epsilon+5\epsilon^2)(1+2\epsilon)(1+5\epsilon)/30$$

$$= 1. \quad \boxed{[0.8 + 0.8 + P(1-3\epsilon+5\epsilon^2)(1+2\epsilon)(1+5\epsilon)]/30}$$

P(k)

$$1^4 + 2^4 + \dots + k^4 = \frac{k(k+1)(2k+1)(3k^2+3k-1)}{30} + \frac{(1-3\epsilon+5\epsilon^2)(1+2\epsilon)(1+5\epsilon)}{30}$$

P(k+1)

$$1^4 + 2^4 + \dots + (k+1)^4 = \frac{(k+1)(k+2)(2k+3)(3k^2+6k+3-1)}{30}$$

$$= \frac{(k+1)(k+2)(2k+3)[3(k^2+2k+1)+3k+3-1]}{30}$$

$$= \frac{(k+1)(k+2)(2k+3)[3k^2+6k+3+3k+2]}{30}$$

(i) bracketed
should be equal

Adding $(k+1)^4$ to (i)

$$1^4 + 2^4 + \dots + k^4 + (k+1)^4 = \frac{k(k+1)(2k+1)(3k^2+3k-1)}{30} + (k+1)^4$$

+ 0.8 + 0.8

From (ii):

R.H.S.

$$\begin{aligned} &= \frac{(k+1)(k+2)}{30} \left\{ 2k(3k^2 + 9k + 5) + 3(3k^2 + 9k + 5) \right\} \\ &= \frac{k+1}{30} (k+2) \left\{ 6k^3 + 18k^2 + 10k + 19k^2 + 27k + 15 \right\} \\ &= \frac{k+1}{30} (k+2) \left\{ 6k^3 + 27k^2 + 37k + 15 \right\} \\ &= \frac{k+1}{30} \left\{ k(6k^3 + 27k^2 + 37k + 15) + 2(6k^3 + 27k^2 + 37k + 15) \right\} \\ &= \frac{k+1}{30} \left\{ 6k^4 + 27k^3 + 37k^2 + 15k + 12k^3 + 54k^2 + 74k + 30 \right\} \\ &= \frac{k+1}{30} \left\{ 6k^4 + 39k^3 + 91k^2 + 89k + 30 \right\} \end{aligned}$$

From (iii):

R.H.S.

$$\begin{aligned} &= \frac{k(k+1)(2k+1)(3k^2 + 3k - 1)}{30} + \frac{30(k+1)^4}{30} \\ &= \frac{(k+1)}{30} \left\{ k(2k+1)(3k^2 + 3k - 1) + 30(k+1)^3 \right\} \\ &= \frac{(k+1)}{30} \left\{ (2k^2 + k)(3k^2 + 3k - 1) + 30(k^3 + 3k^2 + 3k + 1) \right\} \\ &= \frac{k+1}{30} \left\{ 2k^2(3k^2 + 3k - 1) + k(3k^2 + 3k - 1) + (30k^3 + 90k^2 + 90k + 30) \right\} \\ &= \frac{k+1}{30} \left\{ 6k^4 + 6k^3 - 2k^2 + 3k^3 + 3k^2 - k + 30k^3 + 90k^2 + 90k + 30 \right\} \\ &= \frac{k+1}{30} \left\{ 6k^4 + 39k^3 + 91k^2 + 89k + 30 \right\} \end{aligned}$$

(Proved),

30)

Harmonic numbers

$$H_1 = \frac{1}{1}$$

$$\left[(\text{E} + \text{F} + \text{G}) + (\text{E} + \text{F} + \text{G}) + \dots \right] \frac{1}{\partial \varepsilon} =$$

$$H_2 = \frac{1}{1} + \frac{1}{2}$$

$$\left[(\text{E} + \text{F} + \text{G}) + (\text{E} + \text{F} + \text{G}) + (\text{E} + \text{F} + \text{G}) + \dots \right] \frac{1}{\partial \varepsilon} =$$

$$H_3 = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} \Rightarrow H_3 = H_2 + \frac{1}{3}$$

$$\left[(\text{E} + \text{F} + \text{G}) + \dots \right] \frac{1}{\partial \varepsilon} =$$

$$H_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \Rightarrow H_4 = H_3 + \frac{1}{4}$$

$$\left[(\text{E} + \text{F} + \text{G}) + \dots \right] \frac{1}{\partial \varepsilon} =$$

$$H_5 = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$$

$$\left[(\text{E} + \text{F} + \text{G}) + \dots \right] \frac{1}{\partial \varepsilon} =$$

$$\left[(\text{E} + \text{F} + \text{G}) + \dots \right] \frac{1}{\partial \varepsilon} =$$

P(1):

$$H_1 = (n+1) H_n - n$$

$$H_1 = (1+1) 1 - 1 =$$

$$1 = 1.$$

P(k):

$$H_1 + H_2 + \dots + H_k = \frac{\frac{(1+k)}{\partial \varepsilon} H_k - k}{\frac{(1-k\varepsilon + \varepsilon)(k+1)}{\partial \varepsilon}} \quad \text{①}$$

P(k+1):

$$H_1 + H_2 + \dots + H_{k+1} = (k+2) H_{k+1} - k - 1$$

Adding H_{k+1} to both sides of ①:

$$H_1 + H_2 + \dots + H_k + H_{k+1} = (k+1) H_k - k + H_{k+1}$$

$$= (k+1) H_k - k + \cancel{H_{k+1}} + \frac{1}{\cancel{k+1}} - \cancel{\frac{1}{k+1}} \quad \text{②}$$

$$= (k+1) H_k + H_k + \frac{1}{k+1} - k \quad \text{③}$$

(break)

$$= H_k (k+1) + \frac{1}{k+1} - k$$

$$= H_k (k+2) + \frac{1}{k+1} - k$$

$$= (k+2) H_k + \frac{1}{k+1} - k$$

Now:

$$H_{k+1} = H_k + \frac{1}{k+1}$$

$$\text{or, } H_k = \textcircled{1} H_{k+1} - \frac{1}{k+1}$$

\therefore In (iii)

$$\begin{aligned}
 H_1 + H_2 + \dots + H_{k+1} &= (k+2) \left(H_{k+1} - \frac{1}{k+1} \right) + \frac{1}{k+1} - k \\
 &= (k+2) H_{k+1} - \frac{k+2}{k+1} + \frac{1}{k+1} - k \\
 &= (k+2) H_{k+1} + \frac{1-k-2}{k+1} - k \\
 &= (k+2) H_{k+1} + \frac{-k-1}{k+1} - k \\
 &\quad + \underline{\frac{1}{k+1} + \dots} \\
 &\stackrel{\text{L.H.S.}}{=} (k+2) H_{k+1} - k - \frac{1}{k+1} + \frac{1}{k+1} + \dots
 \end{aligned}$$

$$\frac{1}{k+1} - \frac{1}{k+2} = \frac{1}{(k+1)k}$$

$$(k+1)(k+2) \geq k(k+1)$$

Inequalities

$$\textcircled{18} \quad n! < n^n$$

P(2) :

$$2! < 2^2$$

$$2 < 4$$

P(k)

$$k! < k^k \quad \textcircled{1}$$

P(k+1)

$$(k+1)! < (k+1)^{k+1} \quad \textcircled{11}$$

Multiplying $\textcircled{1}$ by $k+1$

$$(k+1)k! < (k+1)^k$$

$$\text{or, } (k+1)! < k \cdot k^k + k^k$$

$$\text{or, } (k+1)! < k^{k+1} + k^k$$

$$\text{or, } (k+1)! < k^{k+1} + k^k$$

$$(k+1)^{k+1} = k \cdot 1^{k+1}$$

$$\therefore (k+1)! < (k+1)^{k+1}$$

$$\frac{1}{1+n} + (1+n) - 1 =$$

Binomial theorem:

$$\begin{aligned} (1+n)^n &= \frac{1}{1+n} + (1+n) - 1 \\ (1+n)^n &= nC_0 a^n b^0 + nC_1 a^{n-1} b^1 \\ &\quad + nC_2 a^{n-2} b^2 + \dots + nC_n a^0 b^n \end{aligned}$$

$$\frac{1}{1+n} + nH = nH$$

$$\frac{1}{1+n} + 1 + nH - \textcircled{1} = nH$$

$$(k+1)^{k+1} = (k+1)k^k + \dots + k^k + 1$$

$$= k^{k+1} + \frac{k^k}{k+1} C_{k+1}^{k+1} \cdot 1^0 +$$

$$k+1 C_1 k^{k+1-1} 1^1 +$$

$$k+1 C_2 k^{k+1-2} 1^2 +$$

$$1 + k+1 C_{k+1} k^{k+1-k} 1^k$$

$$= \frac{k^{k+1}}{k+1} + \frac{(k+1)^{k+1}}{k+1} + \dots + 1.$$

$$⑯ 1 + \frac{1}{4} + \frac{1}{9} + \dots + \frac{1}{n^2} < 2 - \frac{1}{n} = \frac{1 + \frac{1}{4} + \frac{1}{9} + \dots + \frac{1}{n^2}}{\frac{1}{(1+n)^2}} =$$

$$\underline{P(2)}$$

$$1 + \frac{1}{4} < 2 - \frac{1}{2} = \frac{1 + \frac{1}{4}}{\frac{1}{(1+2)^2}} =$$

$$\therefore 1 + \frac{1}{4} < 1.5$$

$$\therefore 1.25 < 1.5. =$$

$$\underline{P(k)}$$

$$1 + \frac{1}{4} + \frac{1}{9} + \dots + \frac{1}{k^2} < 2 - \frac{1}{k} = \frac{1 + \frac{1}{4} + \dots + \frac{1}{k^2}}{\frac{1}{(1+k)^2}} =$$

$$\underline{P(k+1)}$$

$$1 + \frac{1}{4} + \frac{1}{9} + \dots + \frac{1}{(k+1)^2} < 2 - \frac{1}{k+1} =$$

$$= \frac{1}{(k+1)} - \frac{1}{(k+1)(k+2)} =$$

$$= \frac{2k+1}{k+1} =$$

$$\therefore 1 + \frac{1}{4} + \frac{1}{9} + \dots + \frac{1}{(k+1)^2} < \frac{2k+1}{k+1} = \frac{1}{k+1}$$

Adding $\frac{1}{(k+1)^2}$ to both sides of ① we gets

$$1 + \frac{1}{4} + \dots + \frac{1}{(k+1)^2} < 2 - \frac{1}{k} + \frac{1}{(k+1)^2} =$$

$$= \frac{(2(k+1)^2)k}{k(k+1)^2} - \frac{2(k+1)^2}{k(k+1)^2} + \frac{k}{k(k+1)^2} + \dots + \frac{1}{k} + 1$$

$$= \frac{2k(k+1)^2 - k^2 - 2k - 1 + k}{k(k+1)^2}$$

$$= \frac{2k(k^2 + 2k + 1) - k^2 - 2k - 1 + k}{k(k+1)^2}$$

$$= \frac{2k^3 + 2k^2 + 2k - k^2 - 2k - 1 + k}{k(k+1)^2}$$

$$= \frac{2k^3 + 3k^2 + k - 1}{k(k+1)^2} \quad \text{...} \quad \frac{1}{k+1} + \dots + \frac{1}{n} + \frac{1}{n+1} + \dots + 1 \quad (1)$$

$$= \frac{2k^3 + 3k^2 + k}{k(k+1)^2} - \frac{1}{k(k+1)^2} \quad \text{...} \quad \frac{1}{k+1} + \dots + \frac{1}{n} + \dots + 1 \quad (2)$$

$$= \frac{k(2k^2 + 3k + 1)}{k(k+1)^2} - \frac{1}{k(k+1)^2} \quad \text{...} \quad \frac{1}{k+1} + \dots + \frac{1}{n} + \dots + 1 \quad (3)$$

$$= \frac{2k^2 + 2k + 1}{(k+1)^2} - \frac{1}{k(k+1)^2} + \dots + \frac{1}{n} + \dots + 1 \quad (4)$$

$$= \frac{2k(k+1) + 1(k+1)}{(k+1)^2} - \frac{1}{k(k+1)^2} + \dots + \frac{1}{n} + \dots + 1 \quad (5)$$

$$= \frac{(2k+1)(k+1)}{(k+1)^2} - \frac{1}{k(k+1)^2}$$

$$= \frac{2k+1}{k+1} - \frac{1}{k(k+1)^2} + \frac{1}{(k+1)} + \dots + \frac{1}{n} + \dots + 1$$

also see Q7B $\frac{2k+1}{k+1} - \frac{1}{k(k+1)^2}$ instead of $\frac{1}{(k+1)}$ (marked)

$$1 + \frac{1}{4} + \dots + \frac{1}{(k+1)^2} < \frac{2k+1}{k+1} - \frac{1}{k(k+1)^2}$$

$$\therefore 1 + \frac{1}{4} + \dots + \frac{1}{(k+1)^2} < \frac{2k+1}{k+1} - \frac{1}{k(k+1)^2} - \text{(Proved)}$$

$$\frac{1 + 1 - \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} - \dots - \cancel{\frac{1}{(k+1)}}}{\cancel{\frac{1}{(k+1)}}} =$$

$$\frac{1 + 1 - \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} - \dots - \cancel{\frac{1}{(k+1)}} - (1 + \cancel{\frac{1}{2}} + \cancel{\frac{1}{3}} + \dots + \cancel{\frac{1}{k}})}{\cancel{\frac{1}{(k+1)}}} =$$

$$\frac{1 + 1 - \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} - \dots - \cancel{\frac{1}{(k+1)}} - (1 + \cancel{\frac{1}{2}} + \cancel{\frac{1}{3}} + \dots + \cancel{\frac{1}{k}})}{\cancel{\frac{1}{(k+1)}}} =$$

(20) Prove that $3^n < n!$ $n \in \{7, 8, 9\}$

<u>P(7)</u>	<u>(100)</u>	<u>(101)</u>
$3^7 < 7! \quad \text{as } 3^7 = 2187 < 5040$	$7! < 4032$	$4032 < 4032$

P(k):

$$3^k < k! \quad \text{--- (1)}$$

P(k+1):

$$3^{k+1} < (k+1)! \quad \text{or, } 3^{k+1} < (k+1)k!.$$

Multiplying (1) by 3, we find $3^{k+1} < 3k!$

Now:

$3k!$ is smaller than $(k+1)k!$ as $(k+1)$ is clearly more than 3 [$\because n \in \{7, 8, 9\}$].

$$\therefore 3^{k+1} < (k+1)!$$

21. $2^n > n^2$ when $n \in \{5, 6, 7, \dots\}$

<u>$P(5)$</u>	<u>$P(k)$</u>	<u>$P(k+1)$</u>
$2^5 > 5^2$	$2^k > k^2 \quad \text{--- (i)}$	$2^{k+1} > (k+1)^2$

or, $32 > 25$.

Now from (i)

$$2^k \cdot 2 > 2 \cdot k^2$$

$$2^{k+1} > 2k^2$$

If we can show that $2k^2 \geq (k+1)^2$, the proof will be complete.

$$\begin{aligned} 2k^2 &\geq k^2 + 2k + 1. & k^2 + k^2 &= k^2 + 2k + 1 \\ \text{or, } 2k^2 - k^2 - 2k - 1 &\geq 0 & k^2 + k^2 &> k^2 + 4k \\ \text{or, } k^2 - 2k - 1 &\geq 0 & k^2 + k^2 &> k^2 + 2k + 2k \\ \text{Since } k \text{ is a natural number, } k^2 - 2k - 1 &\geq 0 & \text{or, } k^2 + k^2 &> k^2 + 2k + 1 \\ \text{or, } k^2 - 2k &\geq 1 & \text{or, } k^2 + k^2 &> k^2 + 2k + 1 \\ \text{or, } k(k-2) &\geq 1. & \text{or, } k^2 + k^2 &> k^2 + 2k + 1 \end{aligned}$$

∴ Proved

$$(k+1)^2 > k^2 + 2k + 1$$

24

$$\frac{1}{2^n} \leq \frac{1 \times 3 \times 5 \times \dots \times (2n-1)}{2 \times 4 \times 6 \times \dots \times 2n}$$

$$\frac{1}{2^n} \leq \frac{1 \times 3 \times 5 \times \dots \times (2n-1)}{2 \times 4 \times 6 \times \dots \times 2n}$$

P(1)

$$\frac{1}{2} \leq \frac{1}{2}$$

P(k)

$$\frac{1}{2^k} \leq \frac{1 \times 3 \times 5 \times \dots \times (2k-1)}{2 \times 4 \times 6 \times \dots \times 2k}$$

P(k)

$$1 \leq \frac{1 \times 3 \times 5 \times \dots \times (2k-1)}{2 \times 4 \times 6 \times \dots \times (2k-2)}$$

P(k+1)

$$\frac{1}{2^{k+1}} \leq \frac{1 \times 3 \times 5 \times \dots \times (2k+1)-1}{2 \times 4 \times 6 \times \dots \times 2(k+1)}$$

$$\frac{1}{2^{k+1}} \leq \frac{1 \times 3 \times 5 \times \dots \times (2k+1)}{2 \times 4 \times 6 \times \dots \times (2k+2)}$$

Mult ① by $\frac{2k}{2k}$ on RHS.

$$1 \leq \frac{1 \times 3 \times 5 \times \dots \times (2k-1) \times 2k}{2 \times 4 \times 6 \times \dots \times (2k-2) \times 2k}$$

$$\therefore 1 \leq \frac{1 \times 3 \times 5 \times \dots \times (2k-1) \times 2(2k+1)}{2 \times 4 \times 6 \times \dots \times (2k-2) \times 2k}$$

P(k+1)

$$1 \leq \frac{1 \times 3 \times 5 \times \dots \times (2k+1)}{2 \times 4 \times 6 \times \dots \times 2k}$$

① most work

②

③

. $\leq d + w_2 - s_2$

. $\leq d + w_2 - s_2$

. $\leq d + w_2 - s_2$

. $\leq d + w_2 - s_2$

. $\leq d + w_2 - s_2$

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. $\leq d + w_2 - s_2$

. $\leq d + w_2 - s_2$

(28) $n^2 - 7n + 12 \geq 0$

for $n \geq 3$, $\left[\text{Index } \times \text{Depth} \right] \sqrt{[(\text{Index}) \times \text{Depth}]} \leq \frac{1}{n}$

$$P(3) = 3^2 - 7 \times 3 + 12 = 0. \quad \checkmark$$

$$P(k): k^2 - 7k + 12 \geq 0 \quad \text{---} \textcircled{1}$$

$$P(k+1) \stackrel{def}{=} (k+1)^2 - 3(k+1) + 12 \geq 0.$$

$$\text{or, } k^2 + 2k + 1 = 7k - 7 + 12 \Rightarrow 0$$

$$\text{or, } k^2 - 5k + 6 \geq 0$$

Now from (i)

$$(1) k^2 - 7k + 12 \geq 0$$

$$\text{or, } k^2 - 5k - 2k + 6 + 6 \geq 0.$$

$$01, \quad k^2 - 5k + 6 \geq 2k - 6.$$

$$\frac{(1-(1+\omega)\Omega) \times \dots \times 2 \times \omega \times 1}{(1+\omega)\Omega \times \dots \times j \times \rho \times \delta} \geq \frac{\frac{1}{1+\omega}\delta}{\delta + \eta \epsilon}$$

$$\frac{(1+\omega\mathbb{E}) \times \dots \times 2 \times \mathbb{E}+1}{(1+\omega\mathbb{E}) \times \dots \times 2 \times \omega\mathbb{E}} \geq \frac{1}{(1+\omega\mathbb{E})^2}$$

We need to show that $2k-6 \geq 0$.

we can see that for

$$\therefore u^2 - 5u + 6 \geq 0.$$

$$\frac{(1-\omega) \otimes x (1-\omega)}{1} \times \dots \times \omega \times \varepsilon \times 1$$

$$N \times (s-N) \times \dots \times p \times s$$

31. n^2+n is divisible by 2 for ~~pos~~ when n is a true int.

→ solution - 28

$P(1)$: ~~$1^2+1=2$~~ = 2 is divisible by 2.

$P(k)$: k^2+k is divisible by 2

$P(k+1)$: $(k+1)^2+(k+1)$ is divisible by 2.

or, $k^2+2k+1+\cancel{k+1}$ is divisible by 2

or, $k^2+k+2k+2$ is divisible by 2

Now based on $P(k)$ we find that for $P(k+1)$,

~~$k^2+k+2k+2$~~ is clearly divisible by 2. Also $2k+2$ is divisible by 2.

∴

(1) \rightarrow no 6 second

32. n^3+2n is divisible by 3.

$P(1)$: $1^3+2 \cdot 1 = 3$ div by 3

$P(k)$: k^3+2k is div by 3.

$P(k+1)$: $(k+1)^3+2(k+1)$ = ~~$k^3+3k^2+3k+1+2k+2$~~

$$= \underbrace{k^3+2k}_{\text{by } P(k)} + \underbrace{3k^2+3k+3}_{\text{div by 3}}$$

by $P(k)$ + ~~terms~~ is div by 3.

by 3

(~~terms~~) mod =

33 - similar

34. $n^3 - n$ is div by 6. when n is non-negative, i.e. $n \geq 0$.

P(0):

$0^3 - 0 = 0$ is divisible by 6

P(k): $k^3 - k$ is divisible by 6

$(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - k - 1 = k^3 + 3k^2 + 2k$ is divisible by 6.

$k^3 + 3k^2 + 2k = k(k^2 + 3k + 2)$ is divisible by 6
based on P(k)

$3k^2 + 3k$

- it is divisible by 3.

It is also div by 2:

If k is even, then $3k^2 + 3k$ is also even.

$$\begin{aligned} & 3k^2 + 3k = 3k(k+1) \\ & k = 2n \quad \text{where } n \in \mathbb{N} \\ & 3(2n)(2n+1) \\ & = 2n(3(2n+1)) \end{aligned}$$

If n is odd then $3n^2 + 3n$ is even

$$n = 2m+1.$$

$$\begin{aligned}3n^2 + 3n &= 3(2m+1)^2 + 3(2m+1) \\&= 3[8m^3 + 3 \cdot (2m^2 + 3 \cdot 2m + 1)] + 6m + 3 \\&= 3[8m^3 + 3 \times 4m^2 + 6m + 1] + 6m + 3 \\&= 24m^3 + 36m^2 + 18m + 3 + 6m + 3 \\&= 24m^3 + 36m^2 + 18m + 6m + 6 \\&= 2[12m^3 + 18m^2 + 9m + 3m + 3].\end{aligned}$$