Special Trigonometric Function and hyperbolic functions

P1: $I = \int \frac{dx}{4+3\sin x}$

$= \int \frac{dx}{4+3\cdot\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}}$

$= \int \frac{dx}{\frac{4+4\tan^2\frac{x}{2}+6\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}}$

$= \int \frac{\sec^2\frac{x}{2}\,dx}{4+4\tan^2\frac{x}{2}+6\tan\frac{x}{2}}$ ——— (1)

Let $\tan\frac{x}{2} = z \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}\cdot dx = dz$

$\Rightarrow \sec^2\frac{x}{2}\,dx = 2dz$

Then (1) $\Rightarrow$

$I = \int \frac{2dz}{4z^2+6z+4}$

$= \frac{1}{2}\int \frac{dz}{z^2+2\cdot z\cdot\frac{3}{4}+\left(\frac{3}{4}\right)^2-\left(\frac{3}{4}\right)^2+1}$

$= \frac{1}{2}\int \frac{dz}{\left(z+\frac{3}{4}\right)^2+\left(\frac{\sqrt{7}}{4}\right)^2}$

$$\Rightarrow I = \frac{1}{2} \cdot \frac{4}{\sqrt{7}} \tan^{-1} \frac{z + \frac{3}{4}}{\frac{\sqrt{7}}{4}} + C$$

$$= \frac{2}{\sqrt{7}} \tan^{-1} \left(\frac{4z + 3}{\sqrt{7}} \right) + C$$

$$= \frac{2}{\sqrt{7}} \tan^{-1} \left(\frac{4 \tan \frac{x}{2} + 3}{\sqrt{7}} \right) + C$$

Ans:

P2: $I = \int \frac{dx}{3 + 2\sin x + \cos x}$

$$= \int \frac{dx}{3 + 2 \cdot \frac{2\tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} + \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}}$$

$$= \int \frac{dx}{\frac{3 + 3\tan^2 \frac{x}{2} + 4\tan \frac{x}{2} + 1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}}$$

$$= \int \frac{\left(1 + \tan^2 \frac{x}{2}\right) dx}{2\tan^2 \frac{x}{2} + 4\tan \frac{x}{2} + 4}$$

$$\Rightarrow I = \int \frac{\sec^2 \frac{x}{2}\, dx}{2\tan^2 \frac{x}{2} + 4\tan\frac{x}{2} + 4}$$

$$= \int \frac{2\,dz}{2z^2+4z+4}$$

Let

$\tan\frac{x}{2} = z$

$\Rightarrow \sec^2\frac{x}{2} \cdot \frac{1}{2}\,dx = dz$

$\Rightarrow \sec^2\frac{x}{2}\,dx = 2\,dz$

$$= \frac{2}{2}\int \frac{dz}{z^2+2z+2}$$

$$= \int \frac{dz}{(z+1)^2+1}$$

$$= \tan^{-1}(z+1) + C$$

$$= \tan^{-1}\left(1+\tan\frac{x}{2}\right) + C$$

Ans:

P3: $I = \int \frac{2\sin x + 3\cos x}{3\sin x + 4\cos x}\,dx$ —— (A)

Let $2\sin x + 3\cos x$

$= l(\text{denominator}) + m(\text{differential coefficient of denominator})$

numerator

$= l(3\sin x + 4\cos x) + m(3\cos x - 4\sin x)$

$= (3l - 4m)\sin x + (4l + 3m)\cos x$ —— (1)

Now comparing the coefficients of $\sin x$ and $\cos x$ of both sides, we get

$$3l - 4m = 2$$

$$4l + 3m = 3$$

Solving these, we get

$$l = \frac{18}{25}, \quad m = \frac{1}{25}$$

(1) becomes

$$2\sin x + 3\cos x = \frac{18}{25}(3\sin x + 4\cos x) + \frac{1}{25}(3\cos x - 4\sin x)$$

Then (A) becomes,

$$I = \int \frac{18}{25} \frac{3\sin x + 4\cos x}{3\sin x + 4\cos x}\,dx + \int \frac{1}{25} \frac{3\cos x - 4\sin x}{3\sin x + 4\cos x}\,dx$$

$$= \frac{18}{25}\int dx + \frac{1}{25}\int \frac{3\cos x - 4\sin x}{3\sin x + 4\cos x}\,dx$$

$$= \frac{18}{25}x + \frac{1}{25}\ln|3\sin x + 4\cos x| + C$$

Ans:

H.W.

* Note: Generally $\int \frac{a\sin x + b\cos x}{c\sin x + d\cos x}\,dx$ can be treated in the same way.

P4: $I = \int \frac{dx}{5 - 13\sin x}$

$$= \int \frac{dx}{5\left(\sin^2\frac{x}{2} + \cos^2\frac{x}{2}\right) - 13\, 2\sin\frac{x}{2}\cdot\cos\frac{x}{2}}$$

Multiplying the numerator and denominator by $\sec^2\frac{x}{2}$, this

$$I = \int \frac{\sec^2 \frac{x}{2}\, dx}{5\left(\tan^2 \frac{x}{2} + 1\right) - 26 \tan \frac{x}{2}}$$

Let $\tan \frac{x}{2} = z$

$$\Rightarrow \sec^2 \frac{x}{2} \cdot \frac{1}{2}\, dx = dz$$

$$\Rightarrow \sec^2 \frac{x}{2}\, dx = 2\,dz$$

So,

$$I = \int \frac{2\,dz}{5z^2 + 5 - 26z}$$

$$= \frac{2}{5} \int \frac{dz}{z^2 - \frac{26}{5} z + 1}$$

$$= \frac{2}{5} \int \frac{dz}{(z)^2 - 2 \cdot z \cdot \frac{13}{5} + \left(\frac{13}{5}\right)^2 - \left(\frac{13}{5}\right)^2 + 1}$$

$$= \frac{2}{5} \int \frac{dz}{\left(z - \frac{13}{5}\right)^2 - \left(\frac{12}{5}\right)^2}$$

$$= \frac{2}{5} \cdot \frac{1}{2 \cdot \frac{12}{5}} \ln \left| \frac{\left(z - \frac{13}{5}\right) - \frac{12}{5}}{\left(z - \frac{13}{5}\right) + \frac{12}{5}} \right| + C$$

$$= \frac{1}{12} \ln \left| \frac{z - 5}{z - \frac{1}{5}} \right| + C$$

$$I = \frac{1}{12} \ln \left| \frac{5\tan\frac{x}{2} - 25}{5\tan\frac{x}{2} - 1} \right| + C$$

Ans:

PS: $I = \int \frac{dx}{3+4\cosh x}$

$$= \int \frac{dx}{3\left(\cosh^2\frac{x}{2} - \sinh^2\frac{x}{2}\right) + 4\left(\cosh^2\frac{x}{2} + \sinh^2\frac{x}{2}\right)}$$

$$= \int \frac{dx}{7\cosh^2\frac{x}{2} + \sinh^2\frac{x}{2}}$$

Now multiplying the numerator and denominator by $\operatorname{sech}^2\frac{x}{2}$

$$I = \int \frac{\operatorname{sech}^2\frac{x}{2}}{7 + \tanh^2\frac{x}{2}}\, dx$$

put $\tanh\frac{x}{2} = z \Rightarrow \frac{1}{2}\operatorname{sech}^2\frac{x}{2}\, dx = dz$

$$\therefore I = 2\int \frac{dz}{7+z^2} = 2\int \frac{dz}{z^2+(\sqrt{7})^2}$$

$$= 2 \cdot \frac{1}{\sqrt{7}} \tan^{-1}\left(\frac{z}{\sqrt{7}}\right) = \frac{2}{\sqrt{7}} \tan^{-1}\left(\frac{1}{\sqrt{7}} \tanh\frac{x}{2}\right)$$

(Ans)

p6 $I = \int \frac{dx}{a\cos x + b\sin x}$

Let $a = r\sin\alpha$, $b = r\cos\alpha$

$\therefore r = \sqrt{a^2+b^2}$, $\alpha = \tan^{-1}\left(\frac{a}{b}\right)$ —— (1)

$\therefore I = \int \frac{dx}{r(\sin\alpha\cos x + \cos\alpha\sin x)}$

$= \frac{1}{r}\int \frac{dx}{\sin(\alpha+x)}$

$= \frac{1}{r}\int \operatorname{cosec}(\alpha+x)\,dx$

$= \frac{1}{r}\ln\left(\tan\frac{x+\alpha}{2}\right) + C$

$= \frac{1}{\sqrt{a^2+b^2}}\ln\left(\tan\frac{1}{2}\left(x+\tan^{-1}\left(\frac{a}{b}\right)\right)\right) + C$

[by (1)]

A .

Assignment (Das)

$P = 97$ (Das)

SV 1. $I = \int \frac{dx}{5 + 4\sin x}$

1.1. $\int \frac{dx}{a + b\cos x}$

(Case 1 $\rightarrow a > b$)

(Case 2 $\rightarrow a < b$).

SV 2. $I = \int \frac{dx}{5 + 4\cos x}$

SV 3. $I = \int \frac{dx}{2\sin x + 3\cos x + 4}$

4. $I = \int \frac{dx}{1 - \cos x + \sin x}$

SV 5. $I = \int \frac{11\cos x - 16\sin x}{2\cos x + 5\sin x}\, dx$

6. $I = \int \frac{6 + 3\sin x + 14\cos x}{3 + 4\sin x + 5\cos x}\, dx$

Hints: Numerator $= \lambda(\text{denominator}) + m \frac{d}{dx}(\text{denominator}) + n$

7. $I = \int \frac{dx}{4 + 3\sinh x}$

8. $I = \int \frac{dx}{1 + 2\tan x}$, $P = 116$ (Das)