CSE 2213: Discrete Mathematics

Section - O Room No - 325

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REVIEW :
BICONDITIONAL STATEMENT

Expresses that 2 propositions have the same truth value.

Definition 6: **Let *p* and *q* be propositions.**
The *biconditional statement p* ↔ *q* is the proposition "***p* if and only if *q*.**"
The biconditional statement *p* ↔ *q* is
- **True** when *p* and *q* have the same truth values, and
- **False**, otherwise.

Biconditional statements are also called *bi-implications*.

- You can take the train if and only if you buy ticket.
- $p \leftrightarrow q$
- p and q are necessary and sufficient conditions for each other

p	*q*	*p* ↔ *q*
T	T	T
T	F	F
F	T	F
F	F	T

Examples

1. **You can take the flight if and only if you buy a ticket**

p: You take the flight

q: You buy a ticket

$p \leftrightarrow q$

Build the Truth Table

1. **Below freezing is necessary and sufficient for it to be snowing.**

p: It is below freezing

q: It is snowing

English Words for Biconditional:

- **p is necessary and sufficient for q**
- **p iff q**
- **if p then q and conversely**
  - $p \rightarrow q$
  - $q \rightarrow p$
  - So, combinedly we can write $p \leftrightarrow q$

Determine whether these biconditionals are true or false

1) 2+2=4 if and only if 1+1=2
    a) 2+2=4 is True
    b) 1+1=2 is True
    c) T - T ↔ T
2) 1+1 = 2 if and only if 2+3=4
    a) T - F ↔ F
3) 1+1 = 3 if and only if horse can fly
    a) F - F ↔ T
4) 0 > 1 if and only if 2 > 1
    a) F - T ↔ F

REVIEW :

CONVERSE, CONTRAPOSITIVE AND INVERSE

- We can form some **new conditional statements** from a **conditional statement $p \rightarrow q$.**
  - **Converse**
  - **Contrapositive**
  - **Inverse**

$p \rightarrow q$ If it rains, I will stay home.

$q \rightarrow p$ **Converse :** If I stay home, it is raining.

$\neg q \rightarrow \neg p$ **Contrapositive:** If I do not stay home, it is not raining.

$\neg p \rightarrow \neg q$ **Inverse :** If it does not rain, I will not stay home

Exercise

p: I am in Dhaka

q: I am in Bangladesh

Conditional Statement:

$p \rightarrow q$

if I am in Dhaka then I am in Bangladesh

Converse:

$q \rightarrow p$

if I am in Bangladesh then I am in Dhaka

Inverse:

$\neg p \rightarrow \neg q$

if I am not in Dhaka then I am not in Bangladesh

Contrapositive:

$\neg q \rightarrow \neg p$

if I am not in Bangladesh then I am not in Dhaka

Build the Truth Table.

- **Show that the Conditional Statement and the Contrapositive have the Exact Match.**

Exercise: English Sentence to Poposition

Let p and q be the Propositions

p: It is below Freezing

q: It is snowing

Write the propositions using p, q and logical connectives.

1) It is below freezing and snowing
   a) p □ q
2) It is below freezing but not snowing
   a) p □ ¬q
3) It is either snowing or freezing but not both
   a) $p \oplus q$

Logic Laws

1. Double Negation Law
    1. **¬**(**¬p**) ≡ p
2. Idempotent Law
    1. P

PREDICATES AND QUANTIFIERS

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PREDICATES

Propositional logic, studied earlier, cannot adequately express the meaning of all statements in mathematics and in natural language.

Example 1 : "Every computer connected to the university network is functioning properly."

Example 2 : "There is a computer on the university network that is under attack by an intruder."

Example 3 : "Computer x is under attack by an intruder,"

Example 4 : "Computer x is functioning properly"

Example 5 : Statements with variables **e.g., "$x>3$," "$x=y+3$," "$x+y=z$"**

PREDICATES

$x + y = 10$ – true or false?

- A proposition with gap(s)
- When all the gaps are filled up (directly or indirectly), we get a proposition

For $x = 5$ and $y = 5$, $x + y = 10$ (or, $5 + 5 = 10$) – a proposition

- The gaps are filled up

For $x = 2$ and $y = 6$, $x + y = 10$ (or, $2 + 6 = 10$) – a proposition?

PREDICATES

Example 5 : "x>3,"

- The statement **"*x* is greater than 3"** has two parts.
  - **The first** part, **the variable *x*, is the subject** of the statement.
  - **The second** part—**the predicate, "is greater than 3"**.
- We can denote the statement **"*x* is greater than 3" by *P*(*x*)**,
  - ***P* denotes the predicate "is greater than 3"** and
  - ***x* is the variable**.
- The statement $P(x)$ is also said to be
  - **the value of the propositional function *P* at *x*.**
- Once a value has been assigned to the variable x,
  - the statement $P(x)$ becomes a proposition and has a truth value.

PREDICATES

We use function-like symbols to represent a predicate

Let $P(x, y): x + y = 10$

(Number of gaps = number of parameters)

Propositions	Representation by predicate	Truth value
"$x > 3$."	$P(x)$; *Truth values of* $P(4)$ and $P(2)$?	?
"Computer x is under attack by an intruder."	$A(x)$; Truth values of A(CS1), A(CS2), and A(MATH1)?	If, CS2 and MATH1 Under attack ?
"$x = y + 3$."	$Q(x, y)$; $Q(1, 2)$ and $Q(3, 0)$	?
"$x+y=z$"	?	?

PRECONDITIONS AND POSTCONDITIONS

- The statements that **describe valid input are known as preconditions**

- **The conditions that the output should satisfy** when the program has run are known as **postconditions.**

- **Example 6:** Consider the following program:
  - temp := x
  - x := y
  - y := temp

- The Program is designed to interchange the values of var x and y.
- For the precondition,
  - it is needed that x and y have values before we run the program.
- So, for this **precondition** we can use the predicate P(x, y),
  - where **P(x, y)** is the statement
    - **“x = a and y = b,”**
    - i.e., a and b are the values before we run the program.
- Because we want to verify that the program swaps the values of x and y for all input values,
  - for the **postcondition** we can use **Q(x, y)**,
    - where Q(x, y) is the statement
    - **“x = b and y = a.”**

Classification of Predicates

- **P(x) – Unary Predicates**
  - “x is greater than 3”
  - For x=2, P(x) = 2 is greater than 3 : FALSE
  - For x=4, P(x) = 4 is greater than 3 : TRUE
- **Q(x, y) – Binary Predicates**
  - “x is greater than y”
  - For (2,1), P(2,1) = 2 is greater than 1 : TRUE
  - For (3,4), P(3,4) = 3 is greater than 4 : FALSE
- **R(x, y, z) – Ternary Predicates**
  - “x is greater than y and z”
- **“x can speak English”**

Examples

1) Let P(x) denote the statement “x > 3”, what are the truth value of P(4) and P(2)?
   - P(x) is equal to x > 3
   - Thus, P(4) = 4 > 3 TRUE
   - Thus, P(2) = 2 > 3 FALSE
2) Let Q(x, y) denote the statement “x = y + 3”, what are the truth values of the propositions of Q(1, 2) and Q(3, 0)?
   - Q(x, y) is equal to x = y + 3
   - Thus, Q(1, 2), 1 = 2 + 3 FALSE
   - Thus, Q(3, 0), 3 = 3 + 0 TRUE
3) Let P(x) be the statement “the word x contains the letter E”, what are the truth value of P(UIU), P(Engineering) and P(CSE)?

QUANTIFIERS

- **A Quantification** Creates a proposition from a propositional function.
- **A Quantification** expresses the extent to which
  - a predicate is true over a range of elements.
- The words **all, some, many, none, and few** are used.
- Two types of quantification Assessment here:
  - **Universal quantification (∀)**
    - A predicate is true for every element under consideration, and
  - **Existential quantification (∃)**
    - There is one or more element under consideration for which the predicate is true.
- The area of logic that deals with predicates and quantifiers is called the predicate calculus

QUANTIFIERS

Indirect way to fill up a gap

Let $P(x): x + 5 = 10$

Are the following true or false?

For every real number x, $x + 5 = 10$ False

There exists at least one real number x such that, $x + 5 = 10$ True

We never specified x!!!

We Need to Specify the Domain

QUANTIFIERS

Other than putting concrete values/objects in a predicate, we can also declare that the predicate is true for every or some value/object in the domain

Example: $\underbrace{\text{For every real number } x}_{\forall x}, \underbrace{x + 1 > x}_{P(x)}$ ➡ Universal quantifier

Example: $\underbrace{\text{For some real number value of } x}_{\exists x}, \underbrace{x < 2}_{Q(x)}$ ➡ Existential quantifier

Definition 1

The universal quantification of $P(x)$ is the statement

“**$P(x)$ for all values of x in the domain.**”

The notation $\forall x P(x)$ denotes the universal quantification of $P(x)$.

Here $\forall$ is called the universal quantifier.

We read $\forall x P(x)$ as “for all $x P(x)$” or “for every $x P(x)$.”

An element for which $P(x)$ is false is called a counterexample to $\forall x P(x)$.

Definition 2

The existential quantification of P(x) is the proposition

“**There exists an element x in the domain such that P(x).**”

We use the notation $\exists x P(x)$ for the existential quantification of P(x).

Here $\exists$ is called the existential quantifier.

Examples

"x can speak English"

Domain Student

"**P(x) = x can speak English.**"

Note: Quantifier to assign value for the Predicates i.e., assign value of x.

Here ∀ is called the universal quantifier.

$\forall x P(x)$ i.e. "all students can speak English"

Here ∃ is called the existential quantifier.

$\exists x P(x)$ i.e. "there exists a student who can speak Japanese"

Examples

“Let P(x) be the statement “$x < 2$”. What is the truth value of the quantification $\forall x P(x)$, where the domain consists of all real numbers?

Domain is all real number

$P(x) = x < 2$.

$P(1) = 1 < 2$ TRUE

$P(2) = 2 < 2$ FALSE

$P(3) = 3 < 2$ FALSE

“What is the truth value of $\forall x P(x)$, where P(x) is the statement “$x^2 < 10$” and the domain consists of the positive interegers not exceeding 4 i.e. $x = \{0, 1, 2, 3, 4\}$?

$P(x) = x^2 < 10$.

$P(1) = 1 < 10$ TRUE

$P(2) = 4 < 10$ TRUE

$P(3) = 9 < 10$ TRUE

$P(4) = 16 < 10$ FALSE

QUANTIFIERS AS CONJUNCTIONS AND DISJUNCTIONS

In general, universal quantifier is a shorthand for conjunctions of the predicate over all values of the domain

Domain, $S = \{1, 2, 3, 4, 5\}$

$P(x)$ is true for every value of $x \in S$

So we can say, $\forall x P(x)$

We can also say
–

$P(1) \land P(2) \land P(3) \land P(4) \land P(5)$

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QUANTIFIERS AS CONJUNCTIONS AND DISJUNCTIONS

In general, universal quantifier is a shorthand for conjunctions of the predicate over all values of the domain

Similarly, existential quantifier is a shorthand for disjunctions of the predicate over all values of the domain

Domain, $S = \{1, 2, 3, 4, 5\}$

$P(x)$ is true for some value of $x \in S$

So we can say, $\exists x P(x)$

We can also say –

$$P(1) \vee P(2) \vee P(3) \vee P(4) \vee P(5)$$

EXERCISE

$C(x) \equiv x$ has a cat

$D(x) \equiv x$ has a dog

$F(x) \equiv x$ has a ferret

Express the sentences using these predicates, appropriate quantifiers and logical connectives.

The domain of the variables is the set of all students in your class.

A student in your class has a cat, a dog and a ferret

All students in your class has a cat, a dog or a ferret

No student in your class has a cat, a dog and a ferret

Exercises

1. Let $P(x)$ be the predicate "$x^2 - 1 = 0$" where x is a real number. Determine the truth value of $\exists x\, P(x)$.
2. Let $Q(x,y)$ be the predicate "$x < y$" where x and y are integers. What does $\forall x\, \exists y\, Q(x,y)$ mean in words?
3. Let $R(x)$ be the predicate "x is even" where x is an integer. Write the statement "All integers are even" using predicate logic.
4. Let $S(x)$ be the predicate "x is a mammal" and $T(x)$ be "x can fly" where x is an animal. How would you express "Some mammals can fly" using predicate logic?
5. Let $P(x,y)$ be the predicate "x is the parent of y" where x and y are people. Express "Everyone has a parent" using predicate logic.
6. Let $Q(x)$ be the predicate "x is prime" where x is a positive integer. What is the negation of $\forall x\, Q(x)$?
7. Let $R(x,y)$ be the predicate "$x + y = 10$" where x and y are integers. Is the statement $\forall x\, \forall y\, R(x,y)$ true or false? Explain why.
8. Let $S(x)$ be the predicate "x is a multiple of 5" where x is an integer. Express the statement "There is no integer that is a multiple of 5" using predicate logic.
9. Let $T(x,y)$ be the predicate "x is taller than y" where x and y are people. Express the statement "Nobody is taller than themselves" using predicate logic.

LIMITED LOGICAL INFERENCE

Limitation of propositional logic

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LIMITED LOGICAL INFERENCE

p: Every knight speaks the truth

q: Bahubali is a knight

Can we conclude that Bahubali speaks the truth?

Can we mathematically conclude that Bahubali speaks the truth?

Surprisingly, NO!!!

LIMITED LOGICAL INFERENCE

p: Every knight speaks the truth

q: Bahubali is a knight

r: Bahubali speaks the truth

There is no way we can relate p, q and r

Can we use predicates to relate p, q and r?

LOGICAL INFERENCE BY PREDICATE

$K(x)$: x is a knight

$T(x)$: x speaks the truth

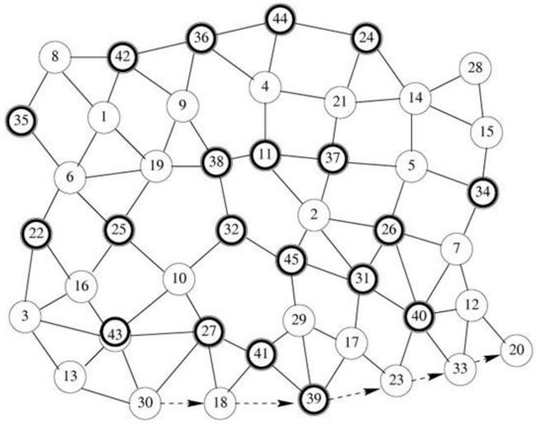
Every knight speaks the truth	$\forall x\big(K(x) \to T(x)\big)$
Bahubali is a knight	$K(Bahubali)$
∴ Bahubali speaks the truth	$T(Bahubali)$

SOME USEFUL EQUIVALENCES FOR PREDICATES, AND THEIR MEANINGS

LOGICAL EQUIVALENCES INVOLVING QUANTIFIERS

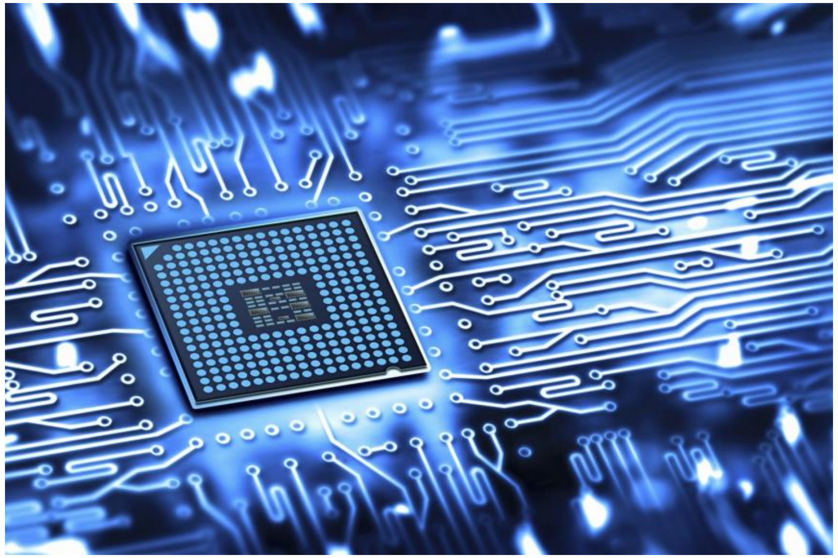
- $p \equiv$ "A knight always speaks the truth“
- $q \equiv$ "Bahubali is a knight“
- Can we conclude that Bahubali speaks the truth?

Surprisingly, NO!!!



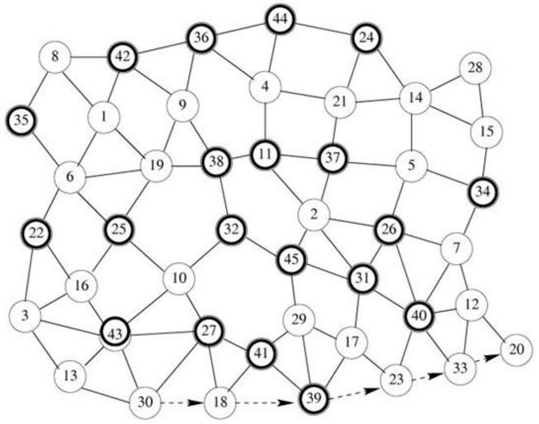
Every CS student learns programming AND discrete mathematics.

≡ Every CS student learns programming
AND every CS student learns discrete mathematics.



Some CS student took VLSI course OR Web Programming course

≡ Some CS student took VLSI course
OR some CS student took Web Programming course.



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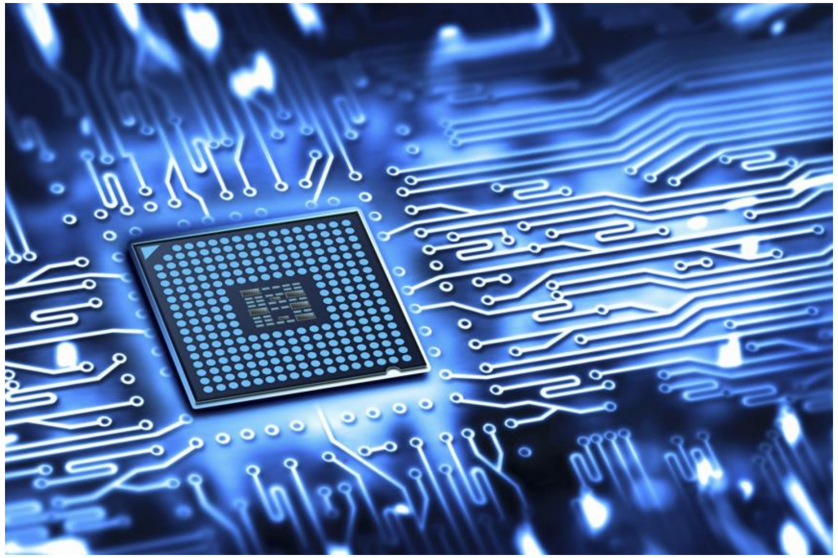

Some CS student took VLSI course AND Web Programming course

≡ Some CS student took VLSI course
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NEGATING QUANTIFIED EXPRESSIONS

De Morgan's Laws for Quantifiers

$$\neg\left(\forall x\, P(x)\right) \equiv \exists x\, \neg P(x)$$

$$\neg\left(\exists x\, P(x)\right) \equiv \forall x\, \neg P(x)$$

What do these mean?

DE MORGAN'S LAW FOR QUANTIFIERS

	$\forall x\, E(x)$	Everyone is evil
Negation	$\neg(\forall x\, E(x))$	Not everyone is evil
	$\equiv \exists x\, \neg E(x)$	Someone is good
	$\exists x\, R(x)$	Rich people exist => Someone is rich
Negation	$\neg(\exists x\, R(x))$	Rich people do not exist => No one is rich
	$\equiv \forall x\, \neg R(x)$	Everyone is poor

PROBLEM

- What are the negations of these statements?
  - $\forall x(x^2 > x)$
  - $\exists x(x^2 = 2)$

TRANSLATING FROM ENGLISH TO LOGICAL EXPRESSIONS

Every CS student learns programming.

EXAMPLE

Every CS student learns programming.

For every CS student, the student learns programming.

For every CS student x, x learns programming.

For every student x, if x is a CS student, x learns programming.

$\forall x\big(C(x) \rightarrow P(x)\big)$

- Here, $C(x) \equiv x$ is a CS student
  and $P(x) \equiv x$ learns programming

EXAMPLE

Every CS student learns programming.

- Why not $\forall x\big(C(x) \land P(x)\big)$??

PROBLEM

Some student in this class has visited Mexico

There is a student in this class who visited Mexico

There is a student x in this class such that x visited Mexico

There is a person x such that x is a student in this class and x visited Mexico

$\exists x\big(S(x) \land M(x)\big)$

- Here, $S(x) \equiv x$ is a student in this class, and $M(x) \equiv x$ visited Mexico

Why not $\exists x\big(S(x) \to M(x)\big)$???

PROBLEM

Express the following statements using the given predicates and quantifiers:

- All lions are fierce.

- Some lions do not drink coffee.

- Some fierce creatures do not drink coffee.

Given predicates:

- $L(x) \equiv x$ is a lion
- $F(x) \equiv x$ is a fierce creature
- $C(x) \equiv x$ drinks coffee

The domain of the variables is all creatures.

PROBLEM

Express the following statements using the given predicates and quantifiers:

- All hummingbirds are richly colored.
- No large birds live on honey.
- Birds that do not live on honey are dull in color.
- Hummingbirds are small.

Given predicates:

- $P(x) \equiv x$ is a hummingbird
- $Q(x) \equiv x$ is large
- $R(x) \equiv x$ lives on honey
- $S(x) \equiv x$ is richly colored

The domain of the variables is all birds.

WHAT HAPPENS FOR MULTIPLE VARIABLES?

- Write down an equivalent conjunction or disjunction for the following propositions, where x belongs to the domain $\{1,2,3,4\}$.
  - $\forall x\ P(x)$
  - $\exists x\ P(x)$

NESTED QUANTIFIERS

Quantifier used within the scope of another quantifier

The example from previous slide: $\forall x \forall y (x + y = y + x)$

We can put any quantifier within the scope of any quantifier

Quantifiers are read left-to-right

ANOTHER EXAMPLE: ADDITIVE INVERSE

For every real number x, there is a corresponding real number y such that $x + y = 0$.

How do we represent it using quantifiers?

$\forall x \exists y (x + y = 0)$

- $Knight(x) \equiv x$ is a knight
- $SpeaksTruth(x) \equiv x$ speaks the truth
- $Knight(x) \rightarrow SpeaksTruth(x)$
- $Knight(\text{Bahubali})$
- Therefore, $SpeaksTruth(\text{Bahubali})$

QUICK EXERCISE

ORDER OF QUANTIFIERS

Commutative law: $\forall x \forall y (x + y = y + x)$

Can we rewrite this as $\forall y \forall x (x + y = y + x)$??

What do they mean?

Do they mean the same?

Yes ☺

ORDER OF QUANTIFIERS

Some random observation: $\exists x \exists y (xy = 6)$

Can we rewrite this as $\exists y \exists x (xy = 6)$??

What do they mean?

Do they mean the same?

ORDER OF QUANTIFIERS

Inverse law: $\forall x \exists y (x + y = 0)$

Can we write it as $\exists y \forall x (x + y = 0)$??

What do they mean?

Do they mean the same?

No ☹

ORDER OF QUANTIFIERS

If only universal or only existential quantifiers are nested, their order can be changed

If both quantifiers are nested, their order cannot be changed, otherwise the meaning will be changed

QUICK EXERCISE

- Let $P(x) \equiv x + 1 > x$
- For every value of x, $x + 1 > x$
- We can write $\forall x\, P(x)$
- $\forall$ is called the universal quantifier

QUICK EXERCISE

Find out the truth values of the following:

$$\forall x \exists y (xy = 0)$$

$$\exists y \forall x (xy = 0)$$

EXERCISE

There is a student at your school who has been a contestant on a television quiz show

No student at your school has ever been a contestant on a television quiz show.

There is a student at your school who has been a contestant on Jeopardy! and on Wheel of Fortune

Every television quiz show has had a student from your school as a contestant.

Given predicate:

$Q(x, y)$: x has been a contestant on the game show y.

Domains:

x: Set of students in your school

y: Set of all television quiz shows

EXERCISE

At least two students from your school have been contestants on Jeopardy!

Given predicate:

$Q(x, y)$: x has been a contestant on the game show y.

Domains:

x: Set of students in your school

y: Set of all television quiz shows

EXERCISE

Jerry does not have an internet connection.

No one in the class has chatted with Bob.

Sanjay has chatted with everyone except Joseph.

Not everyone in your class has an internet connection.

Everyone in your class with an Internet connection has chatted over the Internet with at least one other student in your class.

<u>Given predicate:</u>

$I(x)$: x has an internet connection

$C(x, y)$: x and y have chatted over the internet

<u>Domain:</u> Set of all students in this class

EXERCISE

Exactly one student in your class has an Internet connection.

Someone in your class has an Internet connection but has not chatted with anyone else in your class.

There are two students in your class who have not chatted with each other over the Internet.

There are at least two students in your class who have not chatted with the same person in your class.

<u>Given predicate:</u>

$I(x)$: x has an internet connection

$C(x, y)$: x and y have chatted over the internet

<u>Domain:</u> Set of all students in this class

EXERCISE

There are two students in the class who between them have chatted with everyone else in the class.

<u>Given predicate:</u>

$I(x)$: x has an internet connection

$C(x, y)$: x and y have chatted over the internet

<u>Domain:</u> Set of all students in this class