Integration by Parts -5

Integration of a product by Parts:

If u and v are two functions of x, then

$$\int uv\,dx = u\int v\,dx - \int\left(\frac{du}{dx}\int v\,dx\right)dx$$

Note:

LIATE ←

- E → Exponential → e^x, e^{5x} etc
- T → Trigonometric → $\sin x$, $\cos 5x$, etc
- A → Algebric → $5x, x^3, x^2$ etc
- I → Inverse → $\sin^{-1}x$, $\tan^{-1}x$ etc
- L → Logarithm → $\ln x$, $\log 5x$ etc.

Integration এর সময় v কোনটি হবে?
উপরের LIATE এর E থেকে শুরু করে প্রথমে যেটি আসবে সেটি v হবে।

Ex: $\int xe^x\,dx$

$= x\int e^x\,dx - \int\left(\frac{dx}{dx}\int e^x\,dx\right)dx$

$= xe^x - \int 1\cdot e^x\,dx$

$= xe^x - e^x + C$ **Ans:**

Here $u = x$ and $v = e^x$

P1: $I = \int x^3 e^x \, dx$

$$= x^3 \int e^x dx - \int \left(\frac{d}{dx} x^3 \int e^x dx\right) dx$$

$$= x^3 e^x - 3\int x^2 e^x dx$$

$$= x^3 e^x - 3\left(x^2 \int e^x dx - \int \left(\frac{d}{dx} x^2 \int e^x dx\right) dx\right)$$

$$= x^3 e^x - 3\left(x^2 e^x - 2\int x e^x dx\right)$$

$$= x^3 e^x - 3\left(x^2 e^x - 2\left(x \int e^x dx - \int \left(\frac{dx}{dx} \int e^x dx\right) dx\right)\right)$$

$$= x^3 e^x - 3x^2 e^x - 6\left(x e^x - \int e^x dx\right)$$

$$= x^3 e^x - 3x^2 e^x - 6x e^x + 6e^x + C$$

$$= (x^3 - 3x^2 - 6x + 6) e^x + C \quad \text{Ans:}$$

P2: Integrate $I = \int e^{ax} \cos bx \, dx$

Sol^n: Let $I = \int e^{ax} \cos bx \, dx$ ——— (1)

Now integrating by parts, we have

$$I = \cos bx \int e^{ax} dx - \int \left(\frac{d}{dx} \cos bx \int e^{x} dx \right) dx$$

$$= \cos bx \cdot \frac{1}{a} e^{ax} - \int (-\sin bx) \cdot b \cdot \frac{1}{a} e^{ax} dx$$

$$= \frac{e^{ax}}{a} \cos bx + \frac{b}{a} \int e^{ax} \sin bx \, dx$$

$$= \frac{e^{ax}}{a} \cos bx + \frac{b}{a} \left[\sin bx \int e^{ax} dx - \int \left(\frac{d}{dx} \sin bx \int e^{ax} dx \right) dx \right]$$

$$= \frac{e^{ax}}{a} \cos bx + \frac{b}{a} \left[\sin bx \cdot \frac{1}{a} e^{ax} - \int b \cos bx \cdot \frac{1}{a} e^{ax} dx \right]$$

$$= \frac{e^{ax}}{a} \cos bx + \frac{b}{a} \frac{e^{ax}}{a} \sin bx - \frac{b^2}{a^2} \int e^{ax} \cos bx \, dx$$

$$\Rightarrow I = \frac{e^{ax}}{a} \cos bx + \frac{b}{a^2} e^{ax} \sin bx - \frac{b^2}{a^2} \cdot I \quad [\text{by } (1)]$$

$$\Rightarrow I \left(1 + \frac{b^2}{a^2} \right) = \frac{(a \cos bx + b \sin bx) e^{ax}}{a^2}$$

$$\Rightarrow I\left(\frac{a^2+b^2}{a^2}\right) = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2}$$

$$\Rightarrow I = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$$

$$\therefore \int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$$

<u>Ans:</u>

Similarly. Find

$$* \int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}$$

P3: Integrate $\int (\log\sqrt{x})^2\,dx$

Soln: Let,

$$I = \int (\log(\sqrt{x}))^2\,dx$$

$$= \int \left(\frac{1}{2}\log x\right)^2 dx$$

$$= \frac{1}{4}\int (\log x)^2\,dx$$

$$= \frac{1}{4}(\log x)^2 \int dx - \frac{1}{4}\int \left(\frac{d}{dx}(\log x)^2 \int dx\right) dx$$

$$= \frac{1}{4}(\log x)^2 x - \frac{1}{4}\int 2\log x \cdot \frac{1}{x} \cdot x\,dx$$

$$= \frac{x}{4}(\log x)^2 - \frac{1}{2}\int \log x\,dx$$

$$= \frac{x}{4}(\log x)^2 - \frac{1}{2}\left[\log x \int dx - \int \left(\frac{d}{dx}\log x \int dx\right) dx\right]$$

$$= \frac{x}{4}(\log x)^2 - \frac{1}{2}\left[\log x \cdot x - \int \frac{1}{x}\cdot x\,dx\right]$$

$$= \frac{x}{4}(\log x)^2 - \frac{x}{2}\log x + \frac{1}{2}\int dx$$

$$= \frac{x}{4}(\log x)^2 - \frac{x}{2}\log x + \frac{1}{2}x + C$$

Ans:

P4: Integrate $\int (\sin^{-1}x)^2 dx$

Soln: Let, $I = \int (\sin^{-1}x)^2 dx$

$$\Rightarrow I = (\sin^{-1}x)^2 \int dx - \int \left(\frac{d}{dx}(\sin^{-1}x)^2 \int dx \right) dx$$

$$= x(\sin^{-1}x)^2 - \int 2\sin^{-1}x \cdot \frac{1}{\sqrt{1-x^2}} \cdot x\, dx$$

$$= x(\sin^{-1}x)^2 - 2\int \frac{x \sin^{-1}x}{\sqrt{1-x^2}} dx$$

$$= x(\sin^{-1}x)^2 - 2I_1 \quad , \text{ where } I_1 = \int \frac{x\sin^{-1}x}{\sqrt{1-x^2}} dx$$

Let $\sin^{-1}x = z$

$$\Rightarrow \frac{1}{\sqrt{1-x^2}} dx = dz$$

Now,

$$I = x(\sin^{-1}x)^2 - 2\int z \sin z\, dz$$

$$= x(\sin^{-1}x)^2 - 2\left[z\int \sin z\, dx - \int \left(\frac{dz}{dz} \int \sin z\, dz \right) dz \right]$$

$$= x(\sin^{-1}x)^2 - 2\left[(-z\cos z) + \int \cos z\, dz \right]$$

$$= x(\sin^{-1}x)^2 + 2z\cos z - 2\sin z + C$$

$$= x(\sin^{-1}x)^2 + 2\cdot \sin^{-1}x \sqrt{1-x^2} - 2x + C$$

Ans.

Assignment

Integrate the followings: (Integration by Parts):

1. $\int x e^{ax} \, dx$
2. $\int \sin^{-1}\sqrt{x} \, dx$
3. $\int e^{ax} \cosh bx \, dx$ and $\int e^{ax} \sinh bx \, dx$
4. $\int (\sin^{-1}x)^3 \, dx$
5. $\int x e^{x} \cos x \, dx$
6. $\int x^3 (\log x)^2 \, dx$