Definite Integrals - 7

Fundamental Theorem of Integral Calculus:

If $f(x)$ is integrable in (a,b), $a<b$ and if there exists a function $F(x)$ such that $F'(x) = f(x)$ in (a,b), then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where b = upper limit of integration
a = lower limit of integration.

The definite integral $\int_a^b f(x)\,dx$ represents the area bounded by the curve $y = f(x)$ as x varies from a to b. where $F(x)$ is an integral of $f(x)$.

Relation bet^n definite and indefinite integrals:

If f is continuous on $[a,b]$ and F is any antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)\,dx = \left[\int f(x)\,dx\right]_a^b$$

$$= \left[F(x) + \cancel{c}\right]_a^b$$

Properties of Definites Integrals

P1: $\int_a^b f(x)\,dx = \int_a^b f(z)\,dz \quad , a<b$

P2: $\int_a^b f(z)\,dz = -\int_b^a f(z)\,dz \quad , a<b$

P3: $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \quad , a<b, \; a<c<b$

P4: $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

P5: $\int_0^{na} f(x)\,dx = n\int_0^a f(x)\,dx \text{ , if } f(a+x) = f(x).$

P6: $\int_a^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx \text{ , if } f(2a-x) = f(x).$

P7: $\int_{-a}^{+a} f(x)\,dx = 2\int_0^{+a} f(x)\,dx$

P1: Evaluate the following definite integrals:

a) $\int_0^2 \frac{dx}{4+3\sin x}$ (b) $\int_0^{\pi} \frac{dx}{3+2\sin x+\cos x}$

Soln: Let $I = \int_0^2 \frac{dx}{4+3\sin x}$ ——— (1)

Now $\int \frac{dx}{4+3\sin x} = \int \frac{dx}{4+3\cdot\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}}$

$$= \int \frac{dx}{\frac{4+4\tan^2\frac{x}{2}+6\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}}$$

$$= \int \frac{\left(1+\tan^2\frac{x}{2}\right)dx}{4+4\tan^2\frac{x}{2}+6\tan\frac{x}{2}} = \int \frac{\sec^2\frac{x}{2}\,dx}{4+4\tan^2\frac{x}{2}+6\tan\frac{x}{2}}$$ ——— (2)

Let $\tan\frac{x}{2} = z \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}\,dx = dz$

$\Rightarrow \sec^2\frac{x}{2}\,dx = 2\,dz$

Then (2) $\Rightarrow$

$$\int \frac{dx}{4+3\sin x} = \int \frac{2\,dz}{4z^2+6z+4}$$

$$= \frac{1}{2} \int \frac{dz}{z^2 + 2 \cdot z \cdot \frac{3}{4} + \left(\frac{3}{4}\right)^2 - \left(\frac{3}{4}\right)^2 + 1}$$

$$= \frac{1}{2} \int \frac{dz}{\left(z + \frac{3}{4}\right)^2 - \left(\frac{\sqrt{7}}{4}\right)^2}$$

$$= \frac{1}{2} \cdot \frac{4}{\sqrt{7}} \tan^{-1} \left(\frac{z + \frac{3}{4}}{\frac{\sqrt{7}}{4}} \right) + C$$

$$= \frac{2}{\sqrt{7}} \tan^{-1} \left(\frac{4z + 3}{\sqrt{7}} \right) + C \qquad \left[z = \tan \frac{x}{2} \right]$$

$$= \frac{2}{\sqrt{7}} \tan^{-1} \left(\frac{4 \tan \frac{x}{2} + 3}{\sqrt{7}} \right) + C$$

Now (1) becomes

$$I = \int_0^2 \frac{dx}{4 + 3 \sin x} = \left[\frac{2}{\sqrt{7}} \tan^{-1} \left(\frac{4 \tan \frac{x}{2} + 3}{\sqrt{7}} \right) \right]_0^2$$

$$= \frac{2}{\sqrt{7}} \tan^{-1} \left(\frac{4 \tan 1 + 3}{\sqrt{7}} \right) - \frac{2}{\sqrt{7}} \tan^{-1} \left(\frac{4 \tan 0 + 3}{\sqrt{7}} \right)$$

$$= \frac{2}{\sqrt{7}} \tan^{-1} \left(\frac{4 \tan 1 + 3}{\sqrt{7}} \right) - \frac{2}{\sqrt{7}} \tan^{-1} \frac{3}{\sqrt{7}}$$

Ans.

(b) Let $I = \int_0^{\pi} \frac{dx}{3+2\sin x+\cos x}$ ——— (1)

Now $\int \frac{dx}{3+2\sin x+\cos x}$

$$= \int \frac{dx}{3+2\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} + \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}}$$

$$= \int \frac{dx}{\frac{3+3\tan^2\frac{x}{2}+4\tan\frac{x}{2}+1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}}$$

$$= \int \frac{\left(1+\tan^2\frac{x}{2}\right)dx}{2\tan^2\frac{x}{2}+4\tan\frac{x}{2}+4}$$

$$= \int \frac{\sec^2\frac{x}{2}\,dx}{2\tan^2\frac{x}{2}+4\tan\frac{x}{2}+4}$$

$$= \int \frac{2\,dz}{2z^2+4z+4}$$

Let $\tan\frac{x}{2} = z$

$\Rightarrow \sec^2\frac{x}{2}\cdot\frac{1}{2}\,dx = dz$

$\Rightarrow \sec^2\frac{x}{2}\,dx = 2dz$

$$= \frac{2}{2}\int \frac{dz}{z^2+2z+2}$$

$$= \int \frac{dz}{(z+1)^2+(1)^2}$$

$$= \tan^{-1}(z+1) + C$$

$$= \tan^{-1}\left(1+\tan\frac{x}{2}\right) + c \quad \left[z = \tan\frac{x}{2}\right]$$

Now (1) becomes

$$I = \int_0^{\pi} \frac{dx}{3+2\sin x+\cos x}$$

$$= \left[\tan^{-1}\left(1+\tan\frac{x}{2}\right)+c\right]_0^{\pi}$$

$$= \tan^{-1}\left(1+\tan\frac{\pi}{2}\right) - \tan^{-1}(1+\tan 0) + c - c$$

$$= \tan^{-1}(1+\infty) - \tan^{-1}(1)$$

$$= \tan^{-1}(\infty) - \tan^{-1}(1)$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \frac{2\pi-\pi}{4} = \frac{\pi}{4} \quad \text{Ans:}$$

(7.2)

Evaluate: $\int_2^{\infty}$ Definite Integrals

P = 323 (Golden MS)

P1 $\displaystyle\int_2^8 |x-5|\,dx$

Sol^n: Here,

$x-5=0$ when $x=5$

For $2 \le x < 5$, $(x-5)<0$ $\therefore |x-5| = 5-x$

For $5 < x \le 8$, $(x-5)>0$ $\therefore |x-5| = x-5$

$$\therefore \int_2^8 |x-5|\,dx$$

$$= \int_2^5 (5-x)\,dx + \int_5^8 (x-5)\,dx$$

$$= \left(5x-\frac{x^2}{2}\right)\Big|_2^5 + \left[\frac{x^2}{2}-5x\right]_5^8$$

$$= 9.$$

P2 $\int_0^{\pi} |\cos x|\, dx$

Sol^n: Here $\cos x = 0$ when $x = \frac{\pi}{2}$

For $0 \le x < \frac{\pi}{2}$, $\cos x > 0$, $\therefore |\cos x| = \cos x$

For $\pi/2 < x \le \pi$, $\cos x < 0$, $\therefore |\cos x| = -\cos x$

$$\therefore \int_0^{\pi} |\cos x|\, dx$$

$$= \int_0^{\pi/2} \cos x\, dx - \int_{\pi/2}^{\pi} \cos x\, dx$$

$$= \left[\sin x\right]_0^{\pi/2} - \left[\sin x\right]_{\pi/2}^{\pi}$$

$$= 2.$$

Assignment

Evaluate the following definite integrals:

1. $$\int_0^{\pi/2} \frac{dx}{5 - 13\sin x}$$

2. $$\int_0^{\pi} \frac{dx}{5+3\cos x} = \frac{\pi}{4}$$

3. $$\int_0^{\sqrt{3}} \frac{x^2-1}{x^4-x^2-1}\, dx$$

4. $$\int_0^{1} \frac{dx}{(2x+1)\sqrt{4x+3}}$$

5. $$\int_0^{2} \frac{e^{-x}\, dx}{e^{x}+2e^{-x}+3}$$

6. $$\int_0^{2} (\log\sqrt{x})^2\, dx$$

7. $$\int_0^{1} (\sin^{-1}x)^2\, dx$$