Integration of Rational Fractions:

Method of breaking up into partial fractions:

P1: Integrate $\int \frac{x^2+x-1}{x^3+x^2-6x}\,dx$

Soln: Let $I = \int \frac{x^2+x-1}{x^3+x^2-6x}\,dx$

$$= \int \frac{x^2+x-1}{x(x^2+x-6)}\,dx$$

$$= \int \frac{x^2+x-1}{x(x+3)(x-2)}\,dx \quad \text{--- (1)}$$

Let $\frac{x^2+x-1}{x(x+3)(x-2)} = \frac{A}{x} + \frac{B}{x+3} + \frac{C}{x-2}$

$\Rightarrow x^2+x-1 = A(x+3)(x-2) + Bx(x-2) + Cx(x+3)$

Putting $x = 0, -3, 2$ successively on both sides

we get, $A = \frac{1}{6}$, $B = \frac{1}{3}$, $C = \frac{1}{2}$

Then (1) $\Rightarrow$

$I = \frac{1}{6}\int \frac{dx}{x} + \frac{1}{3}\int \frac{dx}{x+3} + \frac{1}{2}\int \frac{dx}{x-2}$

$= \frac{1}{6}\ln|x| + \frac{1}{3}\ln|x+3| + \frac{1}{2}\ln|x-2| + C$

Ans:

P2: Integrate $\int \frac{x^2}{(x+1)^2(x+2)}\,dx$

Sol^n: Let $I = \int \frac{x^2}{(x+1)^2(x+2)}\,dx$ —— (1)

Let $\frac{x^2}{(x+1)^2(x+2)} = \frac{A}{(x+1)^2} + \frac{B}{x+1} + \frac{C}{x+2}$

$\Rightarrow x^2 = A(x+2) + B(x+1)(x+2) + C(x+1)^2$ —— (2)

Putting $x=-1,-2$ successively ~~we~~ on both sides, we get

$A = 1, \; C = 4$

~~Then~~ Also (2) becomes,

$x^2 = A(x+2) + B(x^2+3x+2) + C(x^2+2x+1)$

$= (B+C)x^2 + (A+3B+2C)x + (2A+2B+C)$

Now, equating the coefficients of x^2 on both sides we get

$B+C = 1$

$\Rightarrow B+4 = 1$ (Since $C=4$)

$\therefore B = -3$

Then (1) becomes

$$I = \int \frac{dx}{(x+1)^2} - 3\int \frac{dx}{x+1} + 4\int \frac{dx}{x+2}$$

$$= -\frac{1}{x+1} - 3\ln|x+1| + 4\ln|x+2| + C$$

Ans:

P3: Integrate $\int \frac{x\,dx}{(x-1)(x^2+4)}$

Soln: Let $I = \int \frac{x\,dx}{(x-1)(x^2+4)}$ ——— (1)

Let $\frac{x}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4}$

$\Rightarrow x = A(x^2+4) + (Bx+C)(x-1)$ ——— (2)

Putting $x = 1$ on both sides, we get $A = \frac{1}{5}$

Also (2) becomes

$$x = Ax^2 + A4 + Bx^2 - Bx + Cx - C$$

$$= x^2(A+B) + x(-B+C) + 4A - C$$

Now, equating the coefficients of x^2 and x on both sides, we get

$$A+B=0 \quad \text{and} \quad C-B=1$$

hence $B=-\frac{1}{5}$, $C=\frac{4}{5}$.

Then (1) becomes

$$I=\frac{1}{5}\int\frac{dx}{x-1}-\frac{1}{5}\int\frac{x-4}{x^2+4}\,dx$$

$$=\frac{1}{5}\int\frac{dx}{x-1}-\frac{1}{10}\int\frac{2x}{x^2+4}\,dx+\frac{4}{5}\int\frac{dx}{x^2+4}$$

$$=\frac{1}{5}\ln|x-1|-\frac{1}{10}\ln|x^2+4|+\frac{4}{5}\cdot\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+C$$

$$=\frac{1}{5}\ln|x-1|-\frac{1}{10}\ln|x^2+4|+\frac{2}{5}\tan^{-1}\left(\frac{x}{2}\right)+C$$

Ans:

P4: Integrate $\int \frac{x^2}{x^4+x^2-2}\,dx$

Sol^n: Let $I = \int \frac{x^2}{x^4+x^2-2}\,dx$ ——— (1)

Let $x^2 = z$, we have

$$\frac{x^2}{x^4+x^2-2} = \frac{z}{z^2+z-2} = \frac{z}{(z+2)(z-1)} \text{ ——— (2)}$$

Let, $\frac{z}{(z+2)(z-1)} = \frac{A}{z+2} + \frac{B}{z-1}$

$\Rightarrow z = A(z-1) + B(z+2)$

Putting $z = -2, 1$, we get $A = \frac{2}{3}$, $B = \frac{1}{3}$

Then (2) $\Rightarrow$

$$\frac{x^2}{x^4+x^2-2} = \frac{2}{3}\,\frac{1}{z+2} + \frac{1}{3}\,\frac{1}{z-1}$$

$$= \frac{2}{3}\,\frac{1}{x^2+1} + \frac{1}{3}\,\frac{1}{x^2-1} \quad \left[\text{Since } z = x^2\right]$$

Then (1) be comes

$$I = \frac{2}{3}\int \frac{dx}{x^2+2} + \frac{1}{2}\int \frac{dx}{x^2-1}$$

$$\Rightarrow I = \frac{2}{3}\int \frac{dx}{x^2+(\sqrt{2})^2} + \frac{1}{3}\int \frac{dx}{(x)^2-(1)^2}$$

$$= \frac{2}{3}\cdot\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + \frac{1}{3}\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C$$

$$= \frac{\sqrt{2}}{3}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + \frac{1}{6}\ln\left|\frac{x-1}{x+1}\right| + C$$

Ans:

P5: Integrate $\int \frac{x^3\,dx}{x^4+3x^2+2}$

Soln: Let $I = \int \frac{x^3\,dx}{x^4+3x^2+2}$ ——— (A)

Put $x^2 = z \Rightarrow 2x\,dx = dz \Rightarrow 2x^3\,dx = x^2\,dz$
$= z\,dz$

$\therefore x^3\,dx = \frac{z}{2}\,dz.$

Then (A) becomes,

$$I = \frac{1}{2}\int \frac{z\,dz}{z^2+3z+2} \quad \text{——— (1)}$$

Now, $\frac{z}{z^2+3z+2} = \frac{z}{(z+1)(z+2)} = \frac{A}{(z+1)} + \frac{B}{(z+2)}$

$\Rightarrow z = A(z+2) + B(z+1)$

Putting $z=-1,-2$, then we get $A=-1, B=2$

Then (1) becomes

$$I = \frac{1}{2}\left[2\int\frac{dx}{z+2} - \int\frac{dx}{z+1}\right]$$

$$= \ln|z+2| - \frac{1}{2}\ln|z+1| + C$$

$$= \ln|x^2+2| - \frac{1}{2}\ln|x^2+1| + C \quad \left[\text{Since } z = x^2\right]$$

Ans:

P6: Integrate $\int \frac{e^{-x}\,dx}{e^x+2e^{-x}+3}$

Soln: Let $I = \int \frac{e^{-x}\,dx}{e^x+2e^{-x}+3}$

$$= \int \frac{e^x\,dx}{e^{3x}+2e^x+3e^{2x}}$$

Let $e^x = z \Rightarrow e^x dx = dz$, then

$$I = \int \frac{dz}{z^3+3z^2+2z} = \int \frac{dz}{z(z+1)(z+2)} \quad \text{--- (1)}$$

Let $\dfrac{1}{z(z+1)(z+2)} = \dfrac{A}{z} + \dfrac{B}{z+1} + \dfrac{C}{z+2}$

$$\Rightarrow \quad 1 = A(z+1)(z+2) + B(z+2)z + Cz(z+1)$$

Putting $z = 0, -1, -2$ successively on both sides, we get

$$A = \frac{1}{2}, \; B = -1, \; C = \frac{1}{2}$$

Then (1) becomes,

$$I = \frac{1}{2}\int \frac{dz}{z} - \int \frac{dz}{z+1} + \frac{1}{2}\int \frac{dz}{z+2}$$

$$= \frac{1}{2}\ln|z| - \ln|z+1| + \frac{1}{2}\ln|z+2| + C$$

$$= \frac{1}{2}\ln|e^x| - \ln|e^x+1| + \frac{1}{2}\ln|e^x+2| + C$$

[Since $z = e^x$]

$$= \frac{1}{2}x - \ln|e^x+1| + \frac{1}{2}\ln|e^x+2| + C$$

Ans:

Assignment (Das)

Integrate the followings:

1. $I = \int \frac{x\,dx}{x^4 - x^2 - 2}$

2. $I = \int \frac{x\,dx}{(x+1)(x^2+1)}$

3. $I = \int \frac{x^2\,dx}{x^4 - x^2 - 12}$

4. $I = \int \frac{dx}{x^3 - x^2 - x + 1}$

5. $I = \int \frac{x\,dx}{x^2 - 12x + 35}$

6. $I = \int \frac{x^2\,dx}{(x+1)(x+2)^2}$