to solve Linear recurrences (matrix exponentiation)

Q : $f(i) = f(i-1) + f(i-2)$ { Note :- $f(i) = f(i-1) * f(i-2)$ Not a Linear recurrance }

find $f(n)$

→ with recursion $O(2^n)$

with dp $O(n)$

with matrix exponentiation $O(K^3 \log N)$

Step1 : Find the value of K in this recurrance relation $K=2$

because $f(i)$ is dependent on previous 2 terms

if $f(i) = f(i-1) + f(i-2) + f(i-4)$

then $K=4$ because this expression can be written as $f(i-1) + f(i-2) + 0*f(i-3) + f(i-4)$

Step2 : Find first K terms, in this case (fibonaci seq) $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} f(i-2) \\ f(i-1) \end{bmatrix}$

Step3 : find transformation matrix in this case it is $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ { there is a method to find this }

Step4 :- Iteratively do the multiplication until you get the result

① → $\underset{T}{\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}} \underset{f_i}{\begin{bmatrix} f(i-2) \\ f(i-1) \end{bmatrix}} = \underset{f_{i+1}}{\begin{bmatrix} f(i-1) \\ f(i) \end{bmatrix}}$

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② → $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} f(i-1) \\ f(i) \end{bmatrix} = \begin{bmatrix} f(i) \\ f(i+1) \end{bmatrix}$

$$f(i-2),\ f(i-1),\ f(i),\ f(i+1),\ f(i+2)\ \ldots$$

general formula :→

$$F_n = T\,F_{n-1}$$

$$= T\,(T \cdot F_{n-2})$$

$$= T\,(T \cdot (T \cdot F_{n-3}))$$

$$\boxed{F_n = T^{n-1} \cdot F_1}$$

$$T \cdot F_{n-1} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} f(n-1) \\ f(n) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} f(n-2) \\ f(n-1) \end{bmatrix}$$

$$\vdots$$

If you want to dry run this formula then dry run it on fibonaci sequence

0, 1, 1, 2, 3, 5, 8, 13 ....

* Calculate Transformation matrix

example →

$$f(i) = f(i-1) + 2f(i-2) + 0^* f(i-3) + 4f(i-4)$$

the Logic is

$$\underset{4\times4}{\begin{bmatrix} & & & \\ & T & & \\ & & & \end{bmatrix}} \begin{bmatrix} f(i-4) \\ f(i-3) \\ f(i-2) \\ f(i-1) \end{bmatrix} = \begin{bmatrix} f(i-3) \\ f(i-2) \\ f(i-1) \\ f(i) \end{bmatrix} \quad \text{O/P}$$

to get the o/p what must be the value of T

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 4 & 0 & 2 & 1 \end{bmatrix}$$

this matrix will give the req result

General trend

$$f(i) = \sum_{j=1}^{K} C_j^* (f(i-j))$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ c_K & \cdots & c_2 & c_1 \end{bmatrix}$$

general trend

$c_1 = 1$ $\quad c_2 = 2$

$c_3 = 0$ $\quad c_4 = 4$

* if you have a constant in the recurrance relation

$$f(i) = f(i-1) + 2f(i-2) + 0^{*}f(i-3) + 4f(i-4) + d$$

$$T_{[5\times5]} \times \begin{bmatrix} f(i-4) \\ f(i-3) \\ f(i-2) \\ f(i-1) \\ d \end{bmatrix} = \begin{bmatrix} f(i-3) \\ f(i-2) \\ f(i-1) \\ f(i) \\ d \end{bmatrix}$$

$$T \rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 4 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{5\times5}$$

Identity matrix.

for 'd' the coefficient is 1