Integrating Boundary Work for $PV^n = \text{constant}$

Remember calculus? There's a reason it's required. Let's take a look at polytropic boundary work, including two special cases.

$n = 0:$

$$PV^n = \Phi = P \tag{1}$$

$$\int_1^2 P \, \mathrm{d}V = P(V_2 - V_1) \tag{2}$$

$n = 1:$

$$PV^n = \Psi = PV \tag{3}$$

$$\begin{aligned} \int_1^2 P \, \mathrm{d}V &= \int_1^2 \frac{\Psi}{V} \, \mathrm{d}V \\ &= \Psi \ln \frac{V_2}{V_1} \end{aligned}$$

Exploit the constant to express the final form in favorable terms. From (3), $\Psi = P_1V_1 = P_2V_2$, so

$$\int_1^2 P \, \mathrm{d}V = P_1V_1 \ln \frac{V_2}{V_1} = P_2V_2 \ln \frac{V_2}{V_1} = P_1V_1 \ln \frac{P_1}{P_2} = P_2V_2 \ln \frac{P_1}{P_2} \tag{4}$$

$n \notin \{0, 1\}:$

$$PV^n = \Omega \tag{5}$$

$$\begin{aligned} \int_1^2 P \, \mathrm{d}V &= \int_1^2 \frac{\Omega}{V^n} \, \mathrm{d}V \\ &= \Omega \int_1^2 V^{-n} \, \mathrm{d}V \\ &= \Omega \left. \frac{V^{1-n}}{1-n} \right|_1^2 \\ &= \frac{\Omega V_2^{1-n} - \Omega V_1^{1-n}}{1-n} \end{aligned}$$

Exploit the constant (5): $\Omega = P_2V_2^n = P_1V_1^n$, and

$$\int_1^2 P \, \mathrm{d}V = \frac{P_2V_2^n - P_1V_1^n}{1-n} \tag{6}$$