设 $f(x)\in[0,1]$ 上有连续导数，$f(0)=0$，$f(1)=1$

求证：$\lim\limits_{n\to\infty} n\left[\int_0^1 f(x)\,dx-\frac{1}{n}\sum\limits_{k=1}^{n} f\left(\frac{k}{n}\right)\right]=-\frac{1}{2}$

证明：由拉氏中值定理：$\exists\, x_0\in(0,1)$，$f'(x_0)=1$

对该点展开：$f(x)=f(x_0)+(x-x_0)+o\left((x-x_0)^2\right)$

$$\int_0^1 f(x)\,dx-\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)=\sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left(f(x)-f\left(\frac{k}{n}\right)\right)dx$$

$$=\sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left[x-\frac{k}{n}+o\left((x-x_0)^2\right)\right]dx=\sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left(x-\frac{k}{n}\right)dx+\sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}}o\left((x-x_0)^2\right)$$

$$=-\frac{1}{2n}+o\left(\frac{1}{n}\right)$$

$$\lim_{n\to\infty} n\left[\int_0^1 f(x)\,dx-\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\right]=-\frac{1}{2}$$