累加：

$$\lim_{n\to\infty}\frac{\sum_{i=1}^{n} i(i+1)(i+2)}{n^4}$$

法一：Stolz定理

$$\lim_{n\to\infty}\frac{\sum_{i=1}^{n} i(i+1)(i+2)}{n^4}=\lim_{n\to\infty}\frac{n(n+1)(n+2)}{n^4-(n-1)^4}$$

$$=\lim_{n\to\infty}\frac{n^3+3n^2+2n}{4n^3-6n^2+4n-1}=\frac{1}{4}$$

法二：幂级数求和

$$\sum_{i=1}^{n} i=\frac{n(n+1)}{2},\ \sum_{i=1}^{n} i^2=\frac{n(n+1)(2n+1)}{6},\ \sum_{i=1}^{n} i^3=\left(\frac{n(n+1)}{2}\right)^2$$

$$\sum_{i=1}^{n} i(i+1)(i+2)=\sum_{i=1}^{n}[i^3+3i^2+2i]=\left(\frac{n(n+1)}{2}\right)^2+3\cdot\frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}$$

n^4的系数为$\frac{1}{4}$，即 $\lim_{n\to\infty}\frac{\sum i(i+1)(i+2)}{n^4}=\frac{1}{4}$

法三：裂项

$$n(n+1)(n+2)=n(n+1)(n+2)\left((n+3)-(n-1)\right)\cdot\frac{1}{4}$$

$$=\frac{1}{4}[n(n+1)(n+2)(n+3)-(n-1)n(n+1)(n+2)]$$

$$\lim_{n\to\infty}\frac{\sum_{i=1}^{n} i(i+1)(i+2)}{n^4}=\lim_{n\to\infty}\frac{n(n+1)(n+2)(n+3)}{4n^4}=\frac{1}{4}$$

下面我们对问题加强，

证明：$\lim_{n\to\infty}\frac{\sum i(i+1)\cdots(i+k)}{n^{k+2}}=\frac{1}{k+2}$ 留给读者证明

累乘： $\lim_{n\to\infty}\prod_{k=2}^{n}\frac{k^3-1}{k^3+1}$

$$=\lim_{n\to\infty}\prod_{k=2}^{n}\frac{(k-1)(k^2+k+1)}{(k+1)(k^2-k+1)}$$

$$=\lim_{n\to\infty}\prod_{k=2}^{n}\frac{k-1}{k+1}\cdot\prod_{k=2}^{n}\frac{k^2+k+1}{k^2-k+1}$$

$$\prod_{k=2}^{n}\frac{k-1}{k+1}=\frac{2}{n(n+1)}\qquad \prod_{k=2}^{n}\frac{k^2+k+1}{k^2-k+1}=\frac{n^2+n+1}{3}$$

故 $\prod_{i=1}^{n}\frac{k^3-1}{k^3+1}=\frac{2}{3}$