9.2、 二重积分的计算

一、 利用直角坐标系计算二重积分

二、 利用极坐标计算二重积分

9.2、二重积分的计算法

二、 利用极坐标计算二重积分

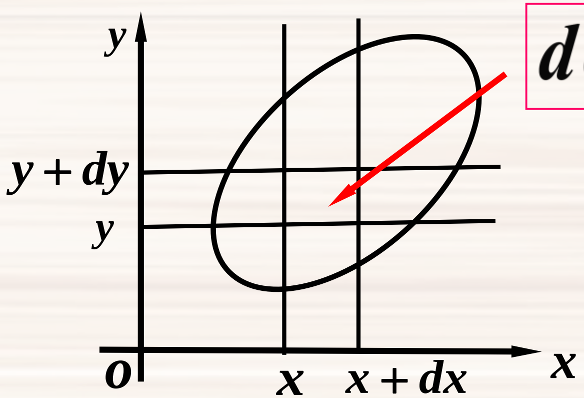
有些二重积分，积分区域D的边界用极坐标方程来表示比较方便，且被积函数用极坐标变量ρ (或r)、θ表达比较简单。这时，我们就可以利用极坐标计算二重积分。

按二重积分的定义

$$\iint_D f(x,y)d\sigma = \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i,\eta_i)\Delta\sigma_i$$

下面我们来研究这个和的极限在极坐标系中的形式。

注： 在直角坐标下，用平行于坐标轴的直线网划分区域D，



$$\boxed{d\sigma = dxdy}$$

------直角坐标系中的面积元素

1、极坐标系下的二重积分的形式

假定从极点O出发且穿过闭区域D内部的射线与D的边界曲线相交不多于两点。

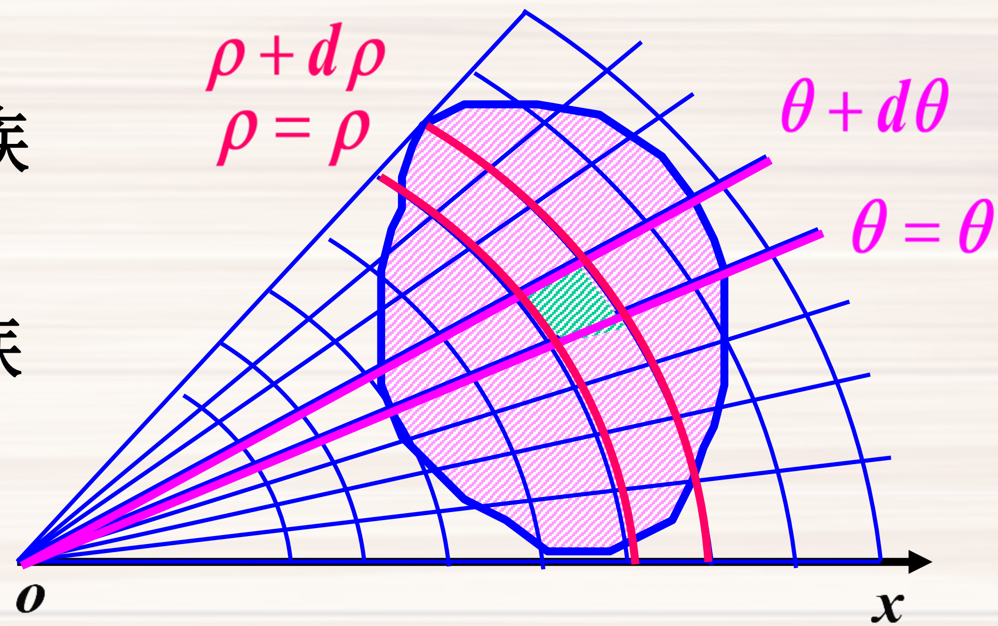
$d\sigma = \rho d\rho d\theta$

我们用下面方法分割

(1)以极点为中心的一族同心圆：ρ =常数，

(2)从极点出发的一族射线：θ=常数，

$\boxed{d\sigma = \rho d\rho d\theta}$



------极坐标系中的面积元素

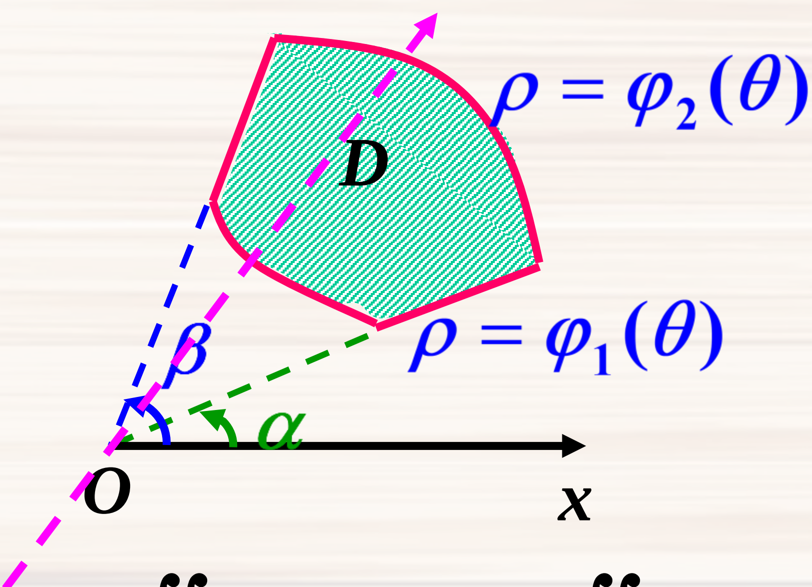
$$\iint_D f(x,y)d\sigma = \iint_D f(\rho\cos\theta, \rho\sin\theta)\rho d\rho d\theta.$$

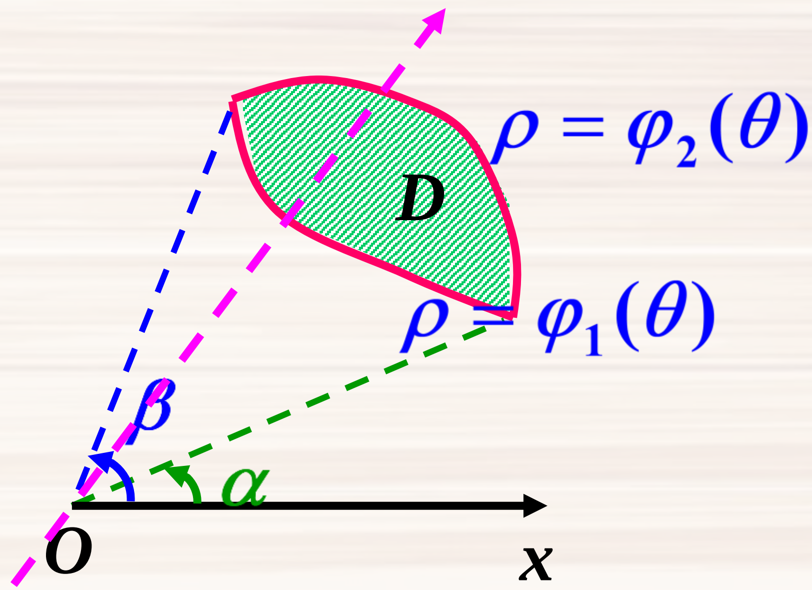
2、如何化为两次单积分

积分顺序：一般是先ρ后θ

(1) 极点在D外　　$D: \varphi_1(\theta) \le \rho \le \varphi_2(\theta), \alpha \le \theta \le \beta$

其中函数$\varphi_1(\theta), \varphi_2(\theta)$在区间$[\alpha, \beta]$上连续



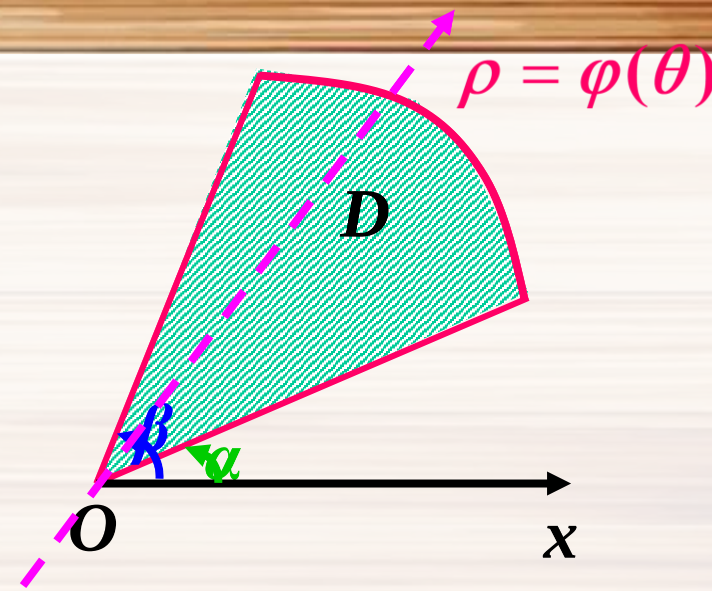


$$\iint_D f(x, y) d\sigma = \iint_D f(\rho\cos\theta, \rho\sin\theta)\rho d\rho d\theta$$

$$= \int_\alpha^\beta d\theta \int_{\varphi_1(\theta)}^{\varphi_2(\theta)} f(\rho\cos\theta, \rho\sin\theta)\rho d\rho$$

(2) 极点在D的边界上时

闭区域D用不等式表示

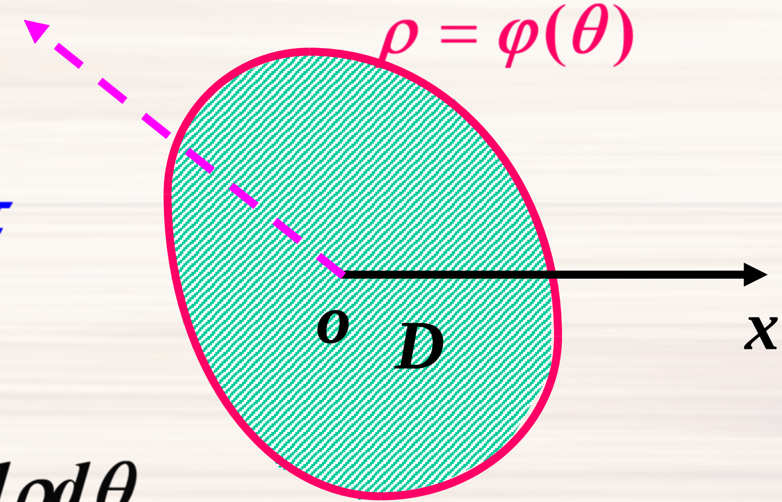


$$0\le\rho\le\varphi(\theta),\ \alpha\le\theta\le\beta$$

$$\iint_D f(x,y)d\sigma=\iint_D f(\rho\cos\theta,\rho\sin\theta)\rho d\rho d\theta$$

$$=\int_\alpha^\beta d\theta\int_0^{\varphi(\theta)} f(\rho\cos\theta,\rho\sin\theta)\rho d\rho。$$

(3) 极点在 D 的内部时



闭区域D用不等式表示

$$D: 0 \le \rho \le \varphi(\theta), 0 \le \theta \le 2\pi$$

$$\iint_D f(x,y)d\sigma$$

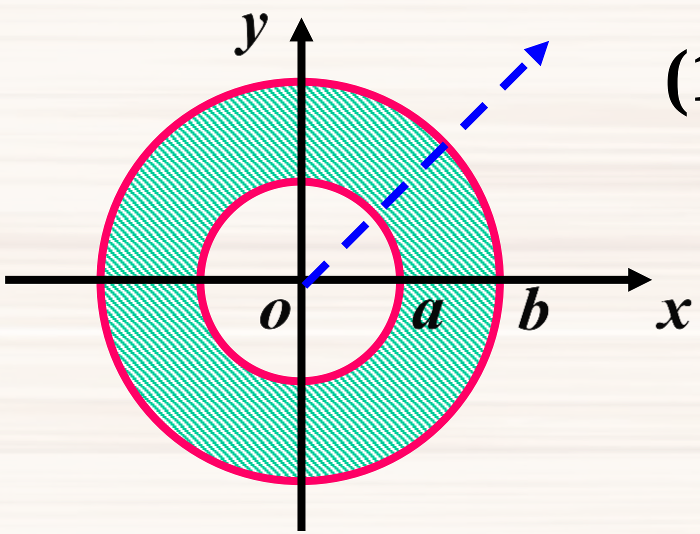
$$= \iint_D f(\rho\cos\theta, \rho\sin\theta)\rho d\rho d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\varphi(\theta)} f(\rho\cos\theta, \rho\sin\theta)\rho d\rho。$$

由二重积分的性质，闭区域D的面积 σ 可以表示为

$$\sigma = \iint_D d\sigma = \iint_D \rho d\rho d\theta$$

例1 将二重积分 $\iint\limits_D f(x,y)d\sigma$ 化为极坐标系下的二次积分,其积分区域D如下图所示。

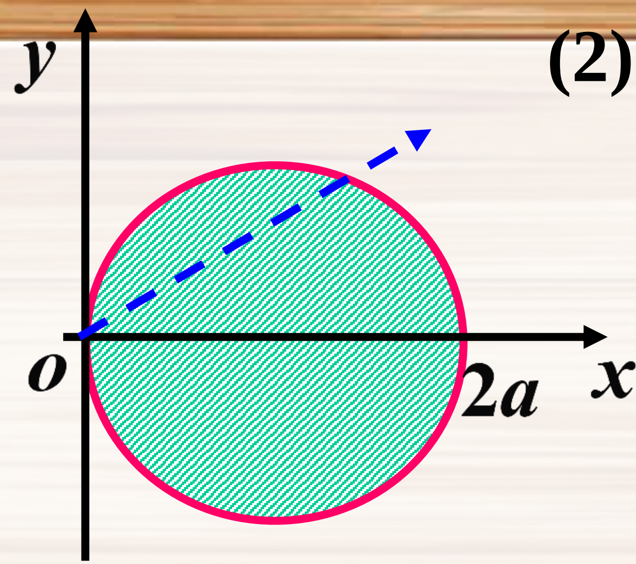


(1) 闭区域D用不等式表示

$$D: a^2 \le x^2 + y^2 \le b^2$$

$$D: a \le \rho \le b,\ 0 \le \theta \le 2\pi$$

$$\iint\limits_D f(x,y)d\sigma = \int_0^{2\pi} d\theta \int_a^b f(\rho\cos\theta, \rho\sin\theta)\rho d\rho$$



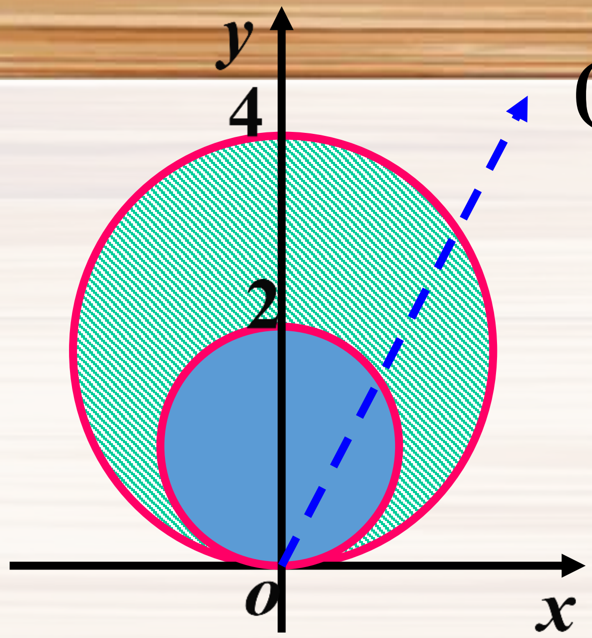
(2) 闭区域D用不等式表示

$$D: x^2 + y^2 \le 2ax$$

$$D:\begin{cases} 0 \le \rho \le 2a\cos\theta \\ -\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2} \end{cases}$$

$$\iint_D f(x,y)d\sigma = \iint_D f(\rho\cos\theta, \rho\sin\theta)\rho d\rho d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{2a\cos\theta} f(\rho\cos\theta, \rho\sin\theta)\rho d\rho$$



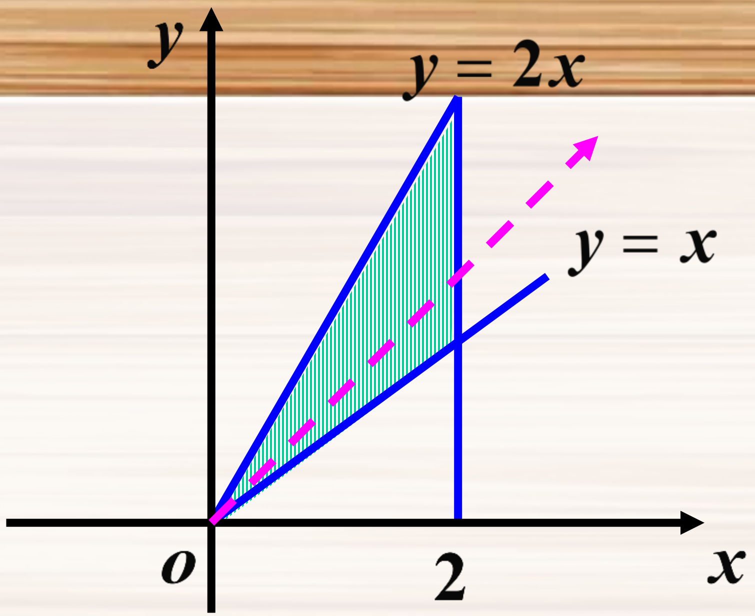
(3) 闭区域D用不等式表示

$$D:\quad 2y \le x^2 + y^2 \le 4y$$

$$D:\quad 2\sin\theta \le \rho \le 4\sin\theta,\ 0 \le \theta \le \pi$$

$$\iint\limits_D f(x,y)d\sigma = \iint\limits_D f(\rho\cos\theta,\rho\sin\theta)\rho d\rho d\theta$$

$$= \int_0^{\pi} d\theta \int_{2\sin\theta}^{4\sin\theta} f(\rho\cos\theta,\rho\sin\theta)\rho d\rho$$



(4) 闭区域D用不等式表示

$$D:\quad x \le y \le 2x,\ 0 \le x \le 2$$

$$D:\quad 0 \le \rho \le \frac{2}{\cos\theta},$$

$$\frac{\pi}{4} \le \theta \le \arctan 2$$

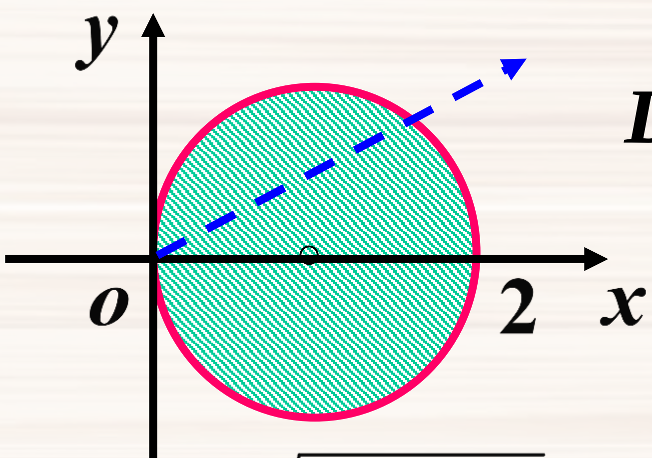
$$\iint_D f(x,y)\,d\sigma = \iint_D f(\rho\cos\theta, \rho\sin\theta)\,\rho\,d\rho\,d\theta$$

$$= \int_{\frac{\pi}{4}}^{\arctan 2} d\theta \int_0^{\frac{2}{\cos\theta}} f(\rho\cos\theta, \rho\sin\theta)\,\rho\,d\rho$$

例2：将下列积分化为极坐标形式，并计算积分值.

(1) $\iint\limits_D \sqrt{x^2+y^2}\,dxdy$,其中$D: x^2+y^2\le 2x$。

解 积分区域 $D: x^2+y^2\le 2x$



$$D:\quad 0\le\rho\le 2\cos\theta, -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$$

$$\therefore \iint\limits_D \sqrt{x^2+y^2}\,dxdy = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{2\cos\theta} \rho\cdot\rho d\rho$$

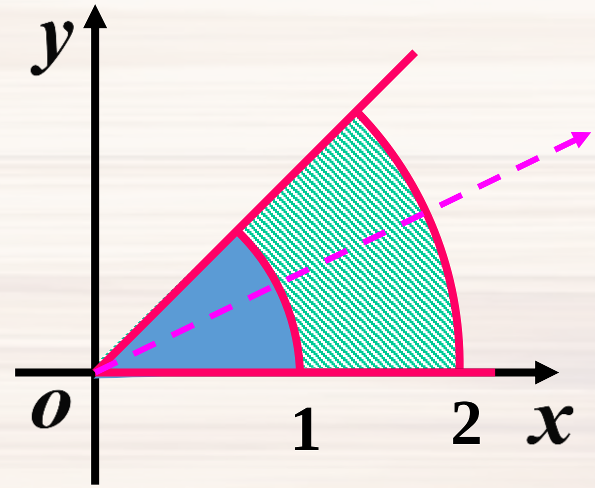
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{8}{3}\cos^3\theta d\theta \quad = \frac{8}{3}\cdot 2\cdot I_3 = \frac{8}{3}\cdot 2\cdot\frac{2}{3}\cdot 1 = \frac{32}{9}.$$

(2) $\iint\limits_D (x^2+y^2)\arctan\frac{y}{x}dxdy$，积分区域$D$的图形为：

$D: 1 \le x^2+y^2 \le 4, y=x,$

$y=0$所围成的位于第 I

象限的部分.

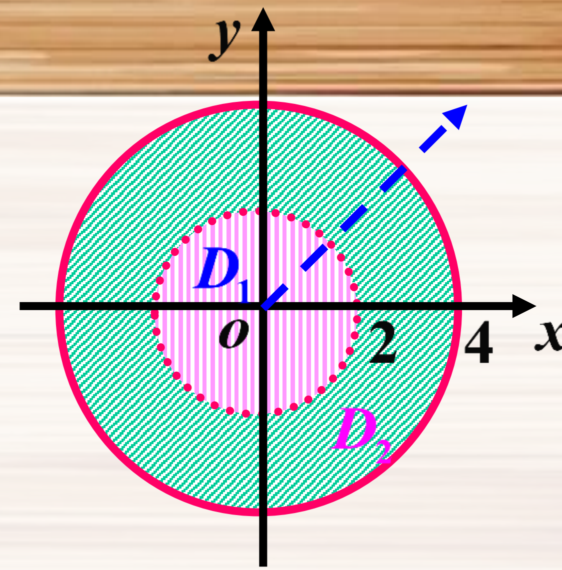


解 $D: 1 \le \rho \le 2, 0 \le \theta \le \frac{\pi}{4}$

$$\therefore \iint\limits_D (x^2+y^2)\arctan\frac{y}{x}dxdy = \int_0^{\frac{\pi}{4}} d\theta \int_1^2 \rho^2 \cdot \theta \cdot \rho d\rho$$

$$= \int_0^{\frac{\pi}{4}} \theta d\theta \int_1^2 \rho^3 d\rho = \frac{15}{128}\pi^2.$$

(3) $\iint\limits_D \left|4 - x^2 - y^2\right| dxdy$,



$D: x^2 + y^2 \le 16 = D_1 \cup D_2$

$D_1: x^2 + y^2 \le 4$,

$D_2: 4 \le x^2 + y^2 \le 16$。

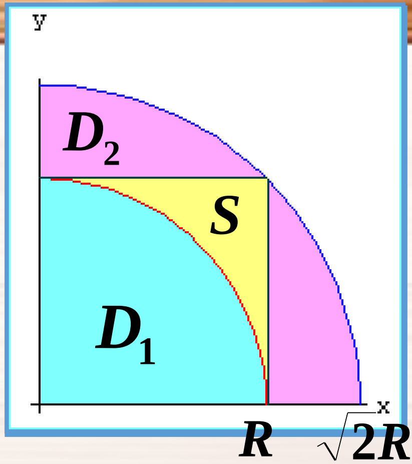
$D_1: 0 \le \rho \le 2, 0 \le \theta \le 2\pi \quad D_2: 2 \le \rho \le 4, 0 \le \theta \le 2\pi$

$$\therefore \iint\limits_D \left|4 - x^2 - y^2\right| dxdy = \iint\limits_{D_1 \cup D_2} \left|4 - x^2 - y^2\right| dxdy$$

$$= \int_0^{2\pi} d\theta \int_0^2 (4 - \rho^2) \cdot \rho d\rho + \int_0^{2\pi} d\theta \int_2^4 (\rho^2 - 4) \cdot \rho d\rho$$

$$= 8\pi + 72\pi = 80\pi$$

例 3 求广义积分 $\int_0^{\infty} e^{-x^2} dx$.



解 $\int_0^{+\infty} e^{-x^2} dx = \lim\limits_{R\to+\infty} \int_0^R e^{-x^2} dx$

$$(\int_0^R e^{-x^2} dx)^2 = \int_0^R e^{-x^2} dx \cdot \int_0^R e^{-x^2} dx$$

$$= \int_0^R e^{-x^2} dx \cdot \int_0^R e^{-y^2} dy = \iint\limits_S e^{-x^2-y^2} dxdy$$

$$S = \{(x, y) \mid 0 \le x \le R, 0 \le y \le R\}$$

$$D_1 = \{(x, y) \mid x^2 + y^2 \le R^2\}$$

$$D_2 = \{(x, y) \mid x^2 + y^2 \le 2R^2\}$$

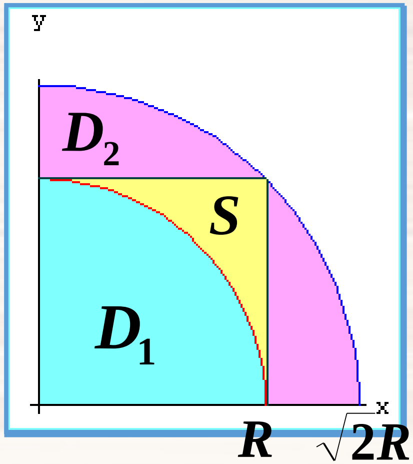
$\{x \geq 0, y \geq 0\}$ 显然有 $D_1 \subset S \subset D_2$

$$\because \quad e^{-x^2-y^2} > 0,$$

$$\therefore \quad \iint_{D_1} e^{-x^2-y^2}dxdy \leq \iint_{S} e^{-x^2-y^2}dxdy \leq \iint_{D_2} e^{-x^2-y^2}dxdy.$$

$$I_1 = \iint_{D_1} e^{-x^2-y^2}dxdy$$

$$= \int_0^{\frac{\pi}{2}} d\theta \int_0^R e^{-\rho^2}\rho d\rho = \frac{\pi}{4}(1-e^{-R^2});$$



同理$I_2 = \iint_{D_2} e^{-x^2-y^2}dxdy = \frac{\pi}{4}(1-e^{-2R^2});$

$\because I_1 < I < I_2,$

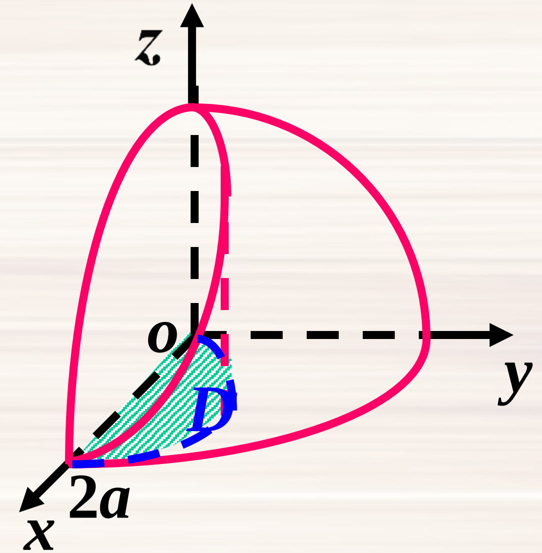
$$\therefore \frac{\pi}{4}(1-e^{-R^2}) < \left(\int_0^R e^{-x^2}dx\right)^2 < \frac{\pi}{4}(1-e^{-2R^2});$$

当$R \to \infty$时, $I_1 \to \dfrac{\pi}{4}$, $I_2 \to \dfrac{\pi}{4}$,

故当$R \to \infty$时, $I \to \dfrac{\pi}{4}$, 即$\left(\int_0^\infty e^{-x^2}dx\right)^2 = \dfrac{\pi}{4}$,

所求广义积分$\int_0^\infty e^{-x^2}dx = \dfrac{\sqrt{\pi}}{2}$.

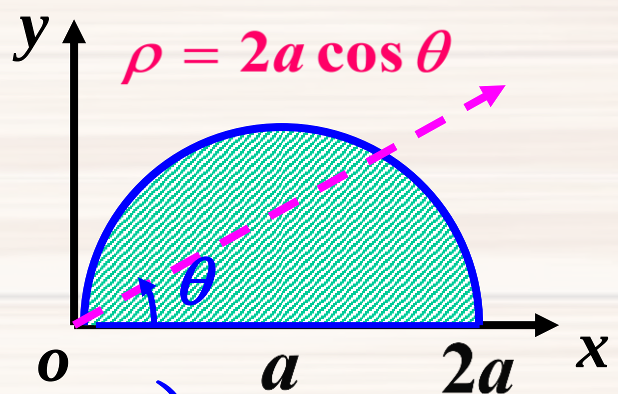
例4 求球面 $x^2+y^2+z^2=4a^2$ 被圆柱 $x^2+y^2=2ax(a>0)$ 所截得的（含在圆柱面内的部分）立体的体积。



解 由对称性知

$$V=4\iint_D \sqrt{4a^2-x^2-y^2}\,dxdy,$$

$$D: 0\le\rho\le 2a\cos\theta, 0\le\theta\le\frac{\pi}{2}$$



$$V=4\iint_D \sqrt{4a^2-x^2-y^2}\,dxdy,$$

$$=4\iint_D \sqrt{4a^2-\rho^2}\,\rho d\rho d\theta,$$

$$=4\int_0^{\frac{\pi}{2}} d\theta\int_0^{2a\cos\theta}\sqrt{4a^2-\rho^2}\,\rho d\rho$$

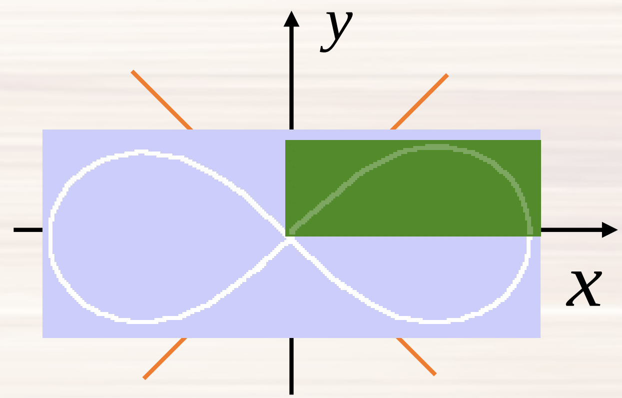
$$=\frac{32}{3}a^3\int_0^{\frac{\pi}{2}}(1-\sin^3\theta)d\theta=\frac{32}{3}a^3\left(\frac{\pi}{2}-\frac{2}{3}\right).$$

例5. 求双纽线 $(x^2+y^2)^2=2a^2(x^2-y^2)$所围区域的面积:

解:记双纽线在第一象限的弧与 X轴围成的区域为 D_1

用对称性,整个双纽线所围区域

的面积为 $s=4\iint\limits_{D_1}dxdy$



$$D_1=\left\{(\rho,\theta)0\le\rho\le a\sqrt{2\cos 2\theta},0\le\theta\le\frac{\pi}{4}\right\}$$

$$s=4\int_0^{\frac{\pi}{4}}d\theta\int_0^{a\sqrt{2\cos 2\theta}}\rho d\rho=4a^2\int_0^{\frac{\pi}{4}}\cos 2\theta d\theta$$

$$=2a^2$$

内容小结

1、会把二重积分化成极坐标下的二次积分.

2、 会适当选取坐标系来计算二重积分。

注：极坐标系的选取方法:

1：如果积分区域D为圆形域，扇型域等

或边界曲线含有: $(1) x^2+y^2=a^2 \quad \Rightarrow \rho=a$

$(2) x^2+y^2=2ax \quad \Rightarrow \rho=2a\cos\theta$

$(3) x^2+y^2=2ay \quad \Rightarrow \rho=2a\sin\theta$

2：被积函数用极坐标表示比较简单，如

$f(x,y)=\varphi(x^2+y^2)=\varphi(\rho^2)$ $\quad \psi(\arctan\frac{y}{x})=\psi(\theta)$

作业 同步练习册 习题 8.2.2

讨论

1、交换极坐标下的积分次序:

$$I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_{0}^{a\cos\theta}f(\rho,\theta)d\rho \quad (a\geq 0).$$

并举出先θ后ρ的计算二重积分的例子.

2、设$f(u)$在$u=0$可导，$f(0)=0, D: x^2+y^2\leq 2t\,x$,

$y\geq 0$,求$I=\lim\limits_{t\to 0^+}\dfrac{1}{t^4}\iint\limits_{D} yf(\sqrt{x^2+y^2})dxdy$.