1. 网红极限题: $\lim\limits_{n\to\infty}\dfrac{n+n^{\frac{1}{2}}+\cdots+n^{\frac{1}{n}}}{n}$

法一: 不等式放缩:

由均值不等式得: $\dfrac{n+\sum_{k=2}^{n}1}{n}\le\dfrac{\sum_{k=1}^{n}n^{\frac{1}{k}}}{n}\le\dfrac{n+\sum_{k=2}^{n}\frac{2\sqrt{n}+k-2}{k}}{n}=1+\dfrac{2\sqrt{n}\sum_{k=2}^{n}\frac{1}{k}+(n-1)-2\sum_{k=2}^{n}\frac{1}{k}}{n}$

两端极限为2. 夹逼得: $\lim\limits_{n\to\infty}\dfrac{\sum n^{\frac{1}{k}}}{n}=2$

法二: 分段估值:

令 $m=[\sqrt[3]{n}]$

$$I=\frac{\sum_{k=1}^{n}n^{\frac{1}{k}}}{n}=\frac{\sum_{k=1}^{m}n^{\frac{1}{k}}}{n}+\frac{\sum_{k=m+1}^{n}n^{\frac{1}{k}}}{n}=1+\frac{\sum_{k=2}^{m}n^{\frac{1}{k}}}{n}+\frac{\sum_{k=m+1}^{n}n^{\frac{1}{k}}}{n}$$

$$1+\frac{m-1}{n}n^{\frac{1}{m}}+\frac{n-m}{n}n^{\frac{1}{n}}<I<1+\frac{m-1}{n}\sqrt{n}+\frac{n-m}{n}n^{\frac{1}{m+1}}$$

两端极限都为2. 由夹逼准则: $\lim\limits_{n\to\infty}I=2$

法三: stolz定理:

$$I=\lim_{n\to\infty}\frac{n+n^{\frac{1}{2}}+\cdots+n^{\frac{1}{n}}}{n}=\lim_{n\to\infty}\frac{n-(n-1)+n^{\frac{1}{2}}-(n-1)^{\frac{1}{2}}+\cdots+n^{\frac{1}{n-1}}-(n-1)^{\frac{1}{n-1}}+n^{\frac{1}{n}}}{n-(n-1)}$$

$$2=\lim_{n\to\infty}(1+\sqrt[n]{n})<I=\lim_{n\to\infty}\left(1+\sqrt[n]{n}+\frac{1}{n^{\frac{1}{2}}+(n-1)^{\frac{1}{2}}}+\frac{1}{n^{\frac{2}{3}}+n^{\frac{1}{3}}(n-1)^{\frac{1}{3}}+(n-1)^{\frac{2}{3}}}+\cdots+\frac{1}{\sum_{i=0}^{n-2}(n^{\frac{1}{n-1}})^{n-2-i}[(n-1)^{\frac{1}{n-1}}]^{i}}\right)$$

$$<\lim_{n\to\infty}\left(1+\sqrt[n]{n}+\frac{1}{2(n-1)^{\frac{1}{2}}}+\cdots+\frac{1}{(n-1)(n-1)^{\frac{n-2}{n-1}}}\right)$$

$$<\lim_{n\to\infty}\left(1+\sqrt[n]{n}+\frac{1}{2(n-1)^{\frac{1}{2}}}+\frac{1}{3(n-1)^{\frac{1}{2}}}+\cdots+\frac{1}{(n-1)(n-1)^{\frac{1}{2}}}\right)$$

$$=\lim_{n\to\infty}\left(1+\sqrt[n]{n}+\frac{\sum_{k=2}^{n-1}\frac{1}{k}}{(n-1)^{\frac{1}{2}}}\right)$$

$$=2$$

$\therefore I=2$