Multiscale Analysis Methods in Astronomy and Engineering

J.L. Starck

CEA/DSM/DAPNIA, CEA-Saclay, and
Dept. Statistics, Stanford University

and

F. Murtagh

Computer Science, Queen's University of Belfast, and
Astronomical Observatory, Strasbourg

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3. The Discrete Wavelet Transform
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    (c) Compression
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Examples and Demonstrations

- Relativizing background
- Faint features
- Linear and clustered patterns
- Noise removal
- Signal processing – shocks, features
- Image decomposition
- Process control

Many other examples are used to illustrate the theory.

Status of the Wavelet Transform in Astronomy

The wavelet transform is used in many astronomical domains:

- galaxy counting: large-scale structure analysis, void detection, ...
- Gamma-ray astronomy: Gamma-ray burst detection
- X-ray images (ROSAT, XMM, AXAF): extended sources and filament detection
- Optical astronomy: asteroid and planetary ring detection, deconvolution
- Infrared (ISO): calibration, source detection, deconvolution, ...
- Cosmic Microwave Background (Planck): fluctuation analysis
- Radio astronomy: aperture synthesis image deconvolution

and for all kinds of astrophysics, from solar to CMB

ADS Abstract Service

keyword in title: wavelet - multiscale - multiresolution

$\rightarrow$ more than 500 papers!

The Continuous Wavelet Transform

$$W(a,b) = K \int_{-\infty}^{+\infty} \psi^*(\frac{x-b}{a}) f(x) dx$$

where:

- $W(a,b)$ is the wavelet coefficient of the function $f(x)$
- $\psi(x)$ is the analyzing wavelet
- a (> 0) is the scale parameter
- b is the position parameter

In Fourier space, we have: $\hat{W}(a,\nu) = \sqrt{a}\hat{f}(\nu)\hat{\psi}^*(a\nu)$
When the scale a varies, the filter $\hat{\psi}^*(a\nu)$ is only reduced or dilated while keeping the same pattern.

Properties

- CWT is a linear transformation:
  - if $f(x) = f_1(x) + f_2(x)$ then $W_f(a,b) = W_{f_1}(a,b) + W_{f_2}(a,b)$
  - if $f(x) = kf_1(x))$ then $W_f(a,b) = kW_{f_1}(a,b)$
- CWT is covariant under translation:

$$\text{if } f_0(x) = f(x - x_0) \text{ then } W_{f_0}(a,b) = W_f(a, b - x_0)$$

- CWT is covariant under dilation:

$$\text{if } f_s(x) = f(sx) \text{ then } W_{f_s}(a,b) = \frac{1}{s^{\frac{1}{2}}} W_f(sa, sb)$$

Whatever the scale and the position, the signal analysis is done using the same function.

The Inverse Transform

The inverse transform is:

$$f(x) = \frac{1}{C_\psi} \int_{-\infty}^{+\infty} \int_{0}^{+\infty} \frac{1}{\sqrt{a}} W(a,b) \psi(\frac{x-b}{a}) \frac{da db}{a^2}$$

where

$$C_\psi = \int_{-\infty}^{+\infty} \mid \hat{\psi}(t) \mid^2 \frac{dt}{t} < +\infty$$

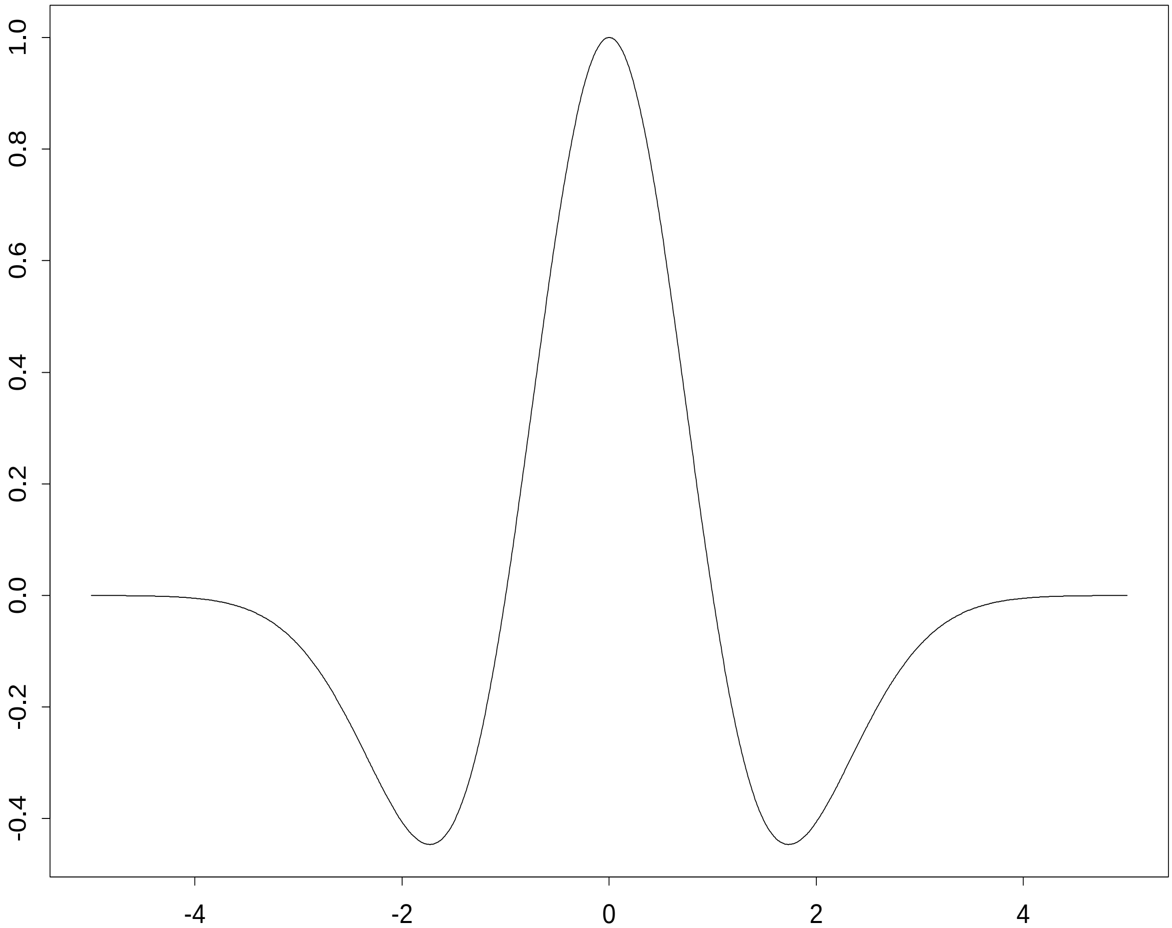
Reconstruction is only possible if C_ψ is defined (admissibility condition). This condition implies $\hat{\psi}(0) = 0$, i.e. the mean of the wavelet function is 0.

Mexican hat

The Mexican hat function is in one dimension:

$$g(x) = (1 - x^2)e^{-\frac{1}{2}x^2}$$

This is the second derivative of a Gaussian.



Discrete Wavelet Transform

Several approaches:

1. Convolution
2. Mallat transform
3. Feauveau transform
4. À trous algorithm
5. Pyramidal algorithm
6. Pyramidal algorithm using the FFT

Multiresolution Analysis

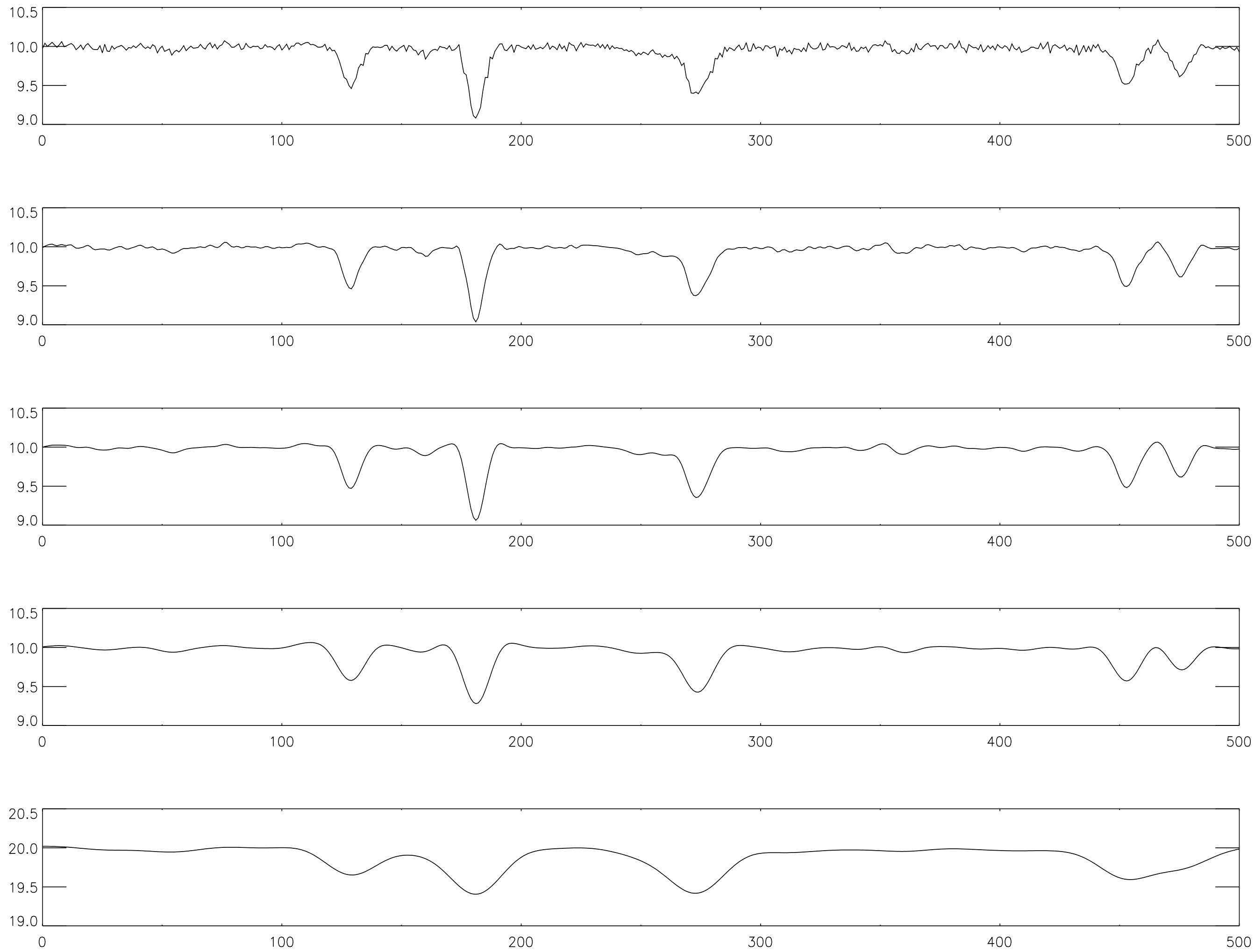
Multiresolution analysis (Mallat, 1989) results from the embedded subsets generated by the interpolations at different scales.

A function $f(x)$ is projected at each step j onto the subset V_j $(\ldots \subset V_{2^{j+1}} \subset V_{2^j} \subset V_{2^{j-1}} \subset V_{2^0} \ldots)$. This projection is defined by the scalar product $c_j(k)$ of $f(x)$ with the scaling function $\phi(x)$ which is dilated and translated:

$$c_j(x) \quad = \quad < f(x), \phi_j(x - 2^j k) >$$

$$\phi_j(x) = 2^j \phi(2^j x)$$

where $\phi(x)$ is the scaling function. ϕ is a low-pass filter.


10.5
10.0
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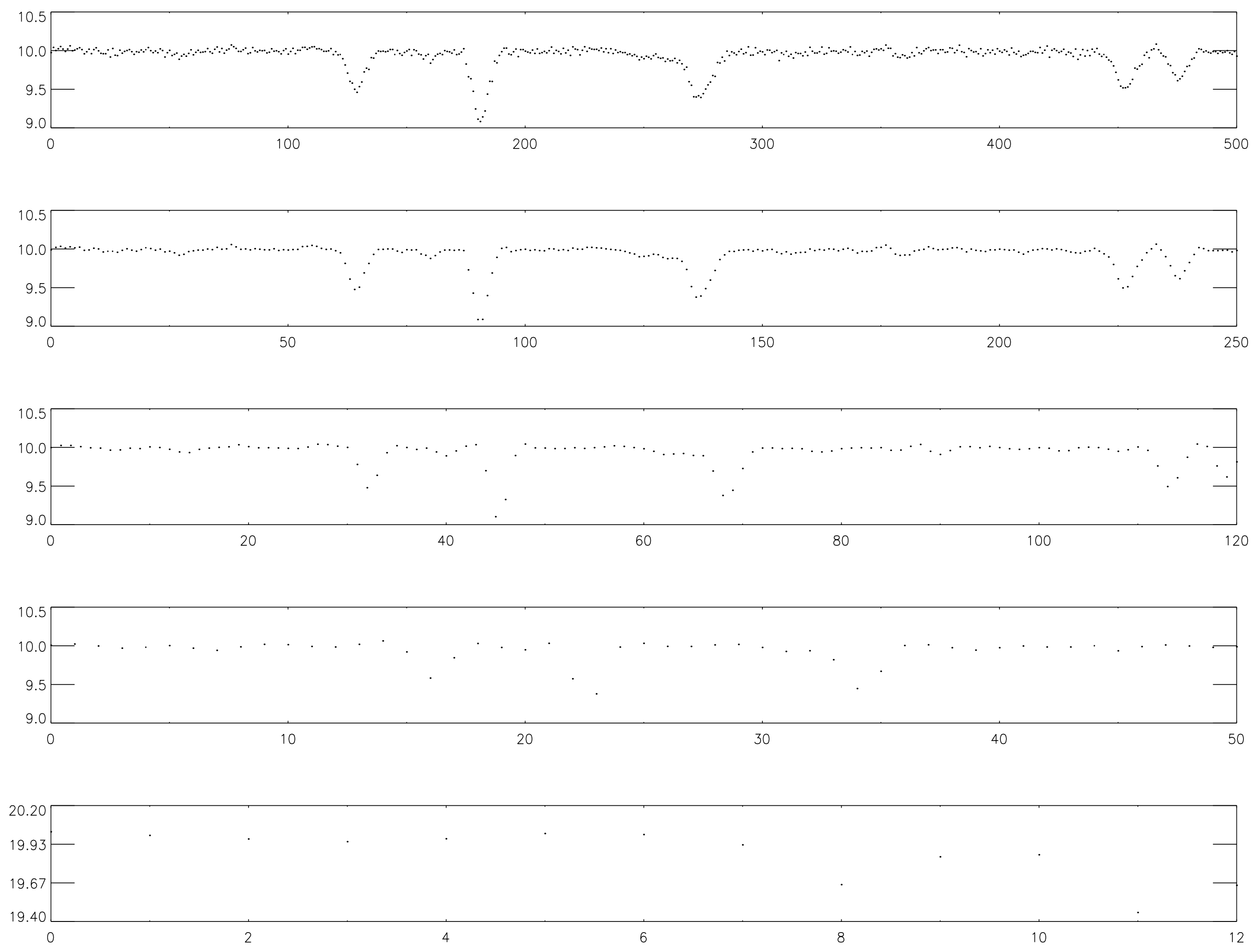
200
300
400
500
20.5
20.0
19.5
19.0


10.5
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9.5
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50
100
150
200
250
10.5
10.0
9.5
9.0
0
20
40
60
80
100
120
10.5
10.0
9.5
9.0
0
10
20
30
40
50
20.20
19.93
19.67
19.40
0
2
4
6
8
10
12

Wavelets and Multiresolution Analysis

The difference between c_{j-1} and c_j is contained in the detail signal belonging to the space O_j orthogonal to V_j.

$$O_j \oplus V_j = V_{j-1}$$

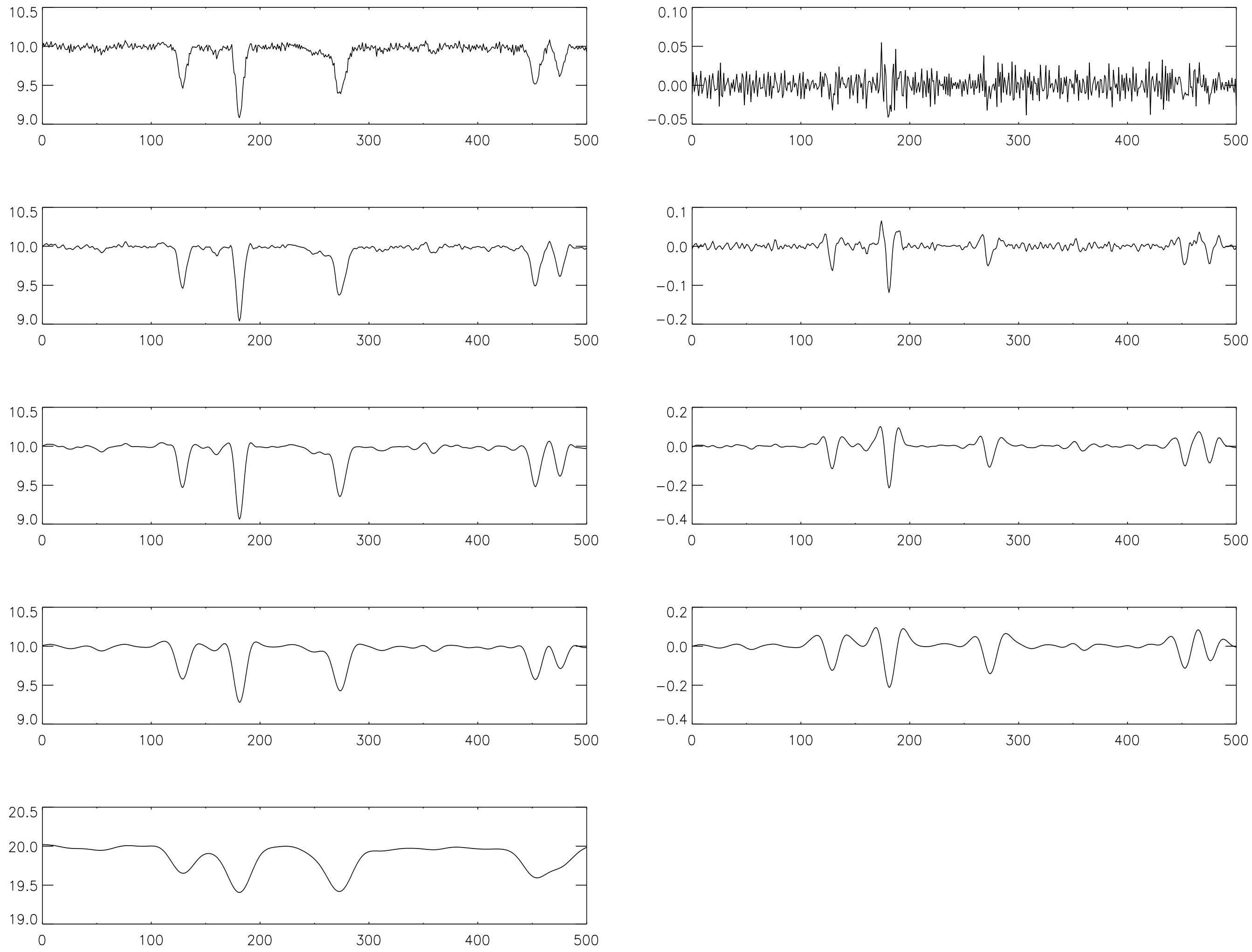
The set $\sqrt{2^{-j}}\psi_j(x - 2^{-j}k)_{k \in \mathcal{Z}}$ forms a basis of O_j.

$$\psi_j(x) = 2^j \psi(2^j x)$$

where $\psi(x)$ is the wavelet function.

The wavelet coefficients are obtained by:

$$w_j(x) \quad = \quad < f(x), \psi_j(x - 2^j k) >$$



The Fast Wavelet Transform

Since $\phi(x)$ is a scaling function which has the property: $\frac{1}{2}\phi(\frac{x}{2}) = \sum_n h(n)\phi(x-n)$
$c_{j+1}(k)$ can be obtained by direct computation from $c_j(k)$

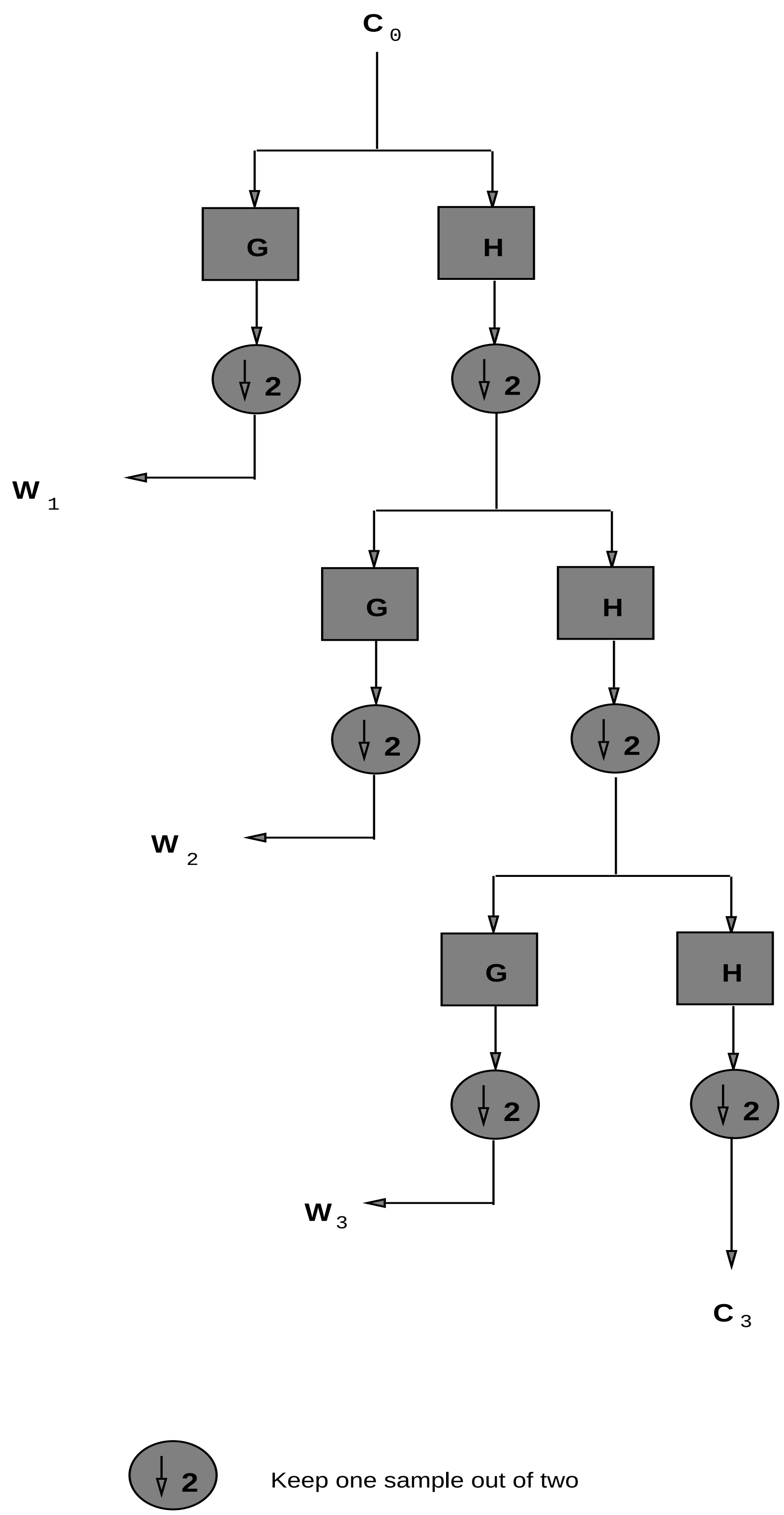
$$c_{j+1}(k) = \sum_n h(n-2k)c_j(n)$$

and $\frac{1}{2}\psi(\frac{x}{2}) = \sum_n g(n)\phi(x-n)$. The scalar products $< f(x), 2^{-(j+1)}\psi(2^{-(j+1)}x-k) >$ are computed with:

$$w_{j+1}(k) = \sum_n g(n-2k)c_j(n)$$

Reconstuction from:

$$c_j(k) = 2\sum_n h(k-2n)c_{j+1}(n) + g(k-2n)w_{j+1}(n)$$


C 0
G
H
2
2
W 1
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W 2
G
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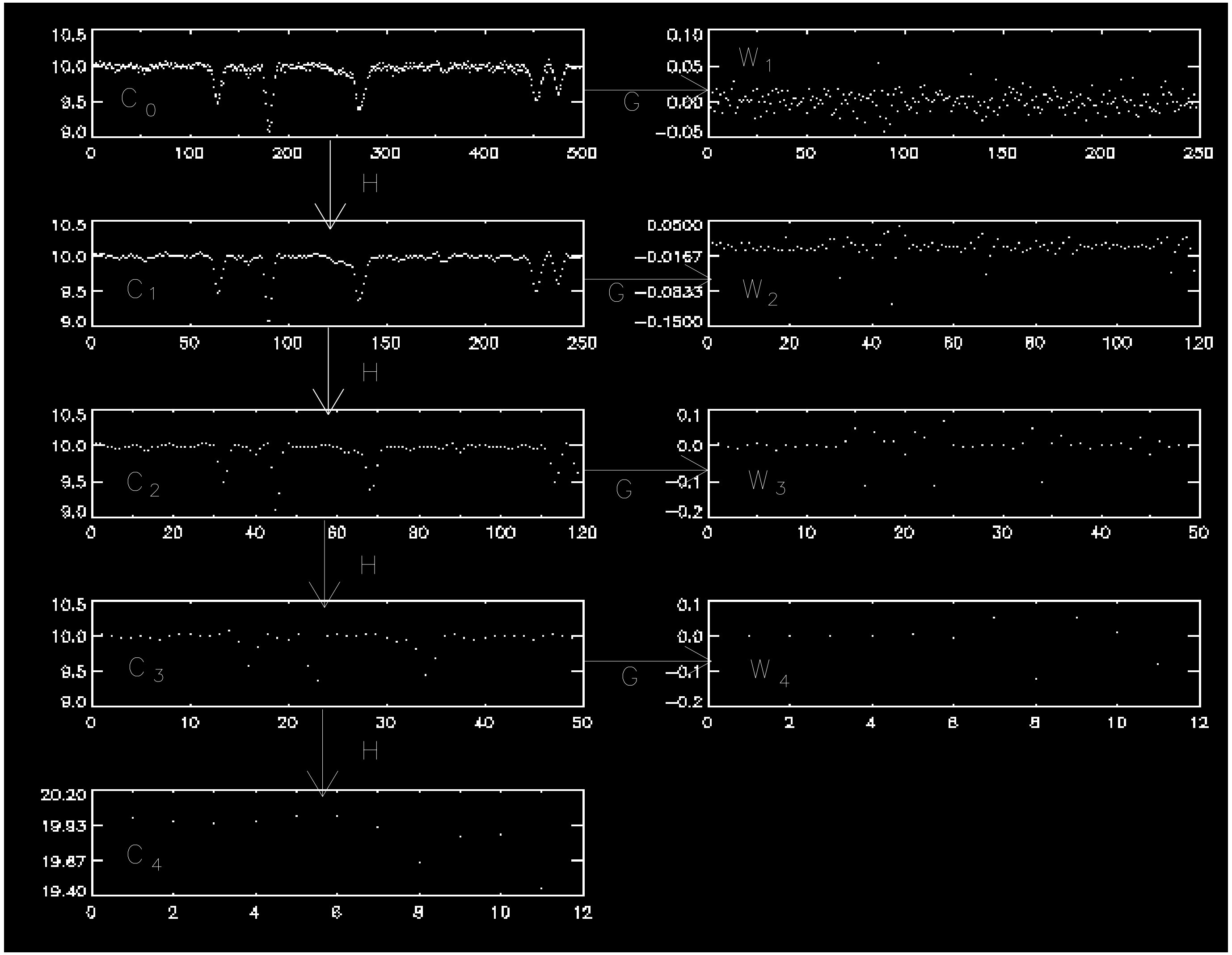
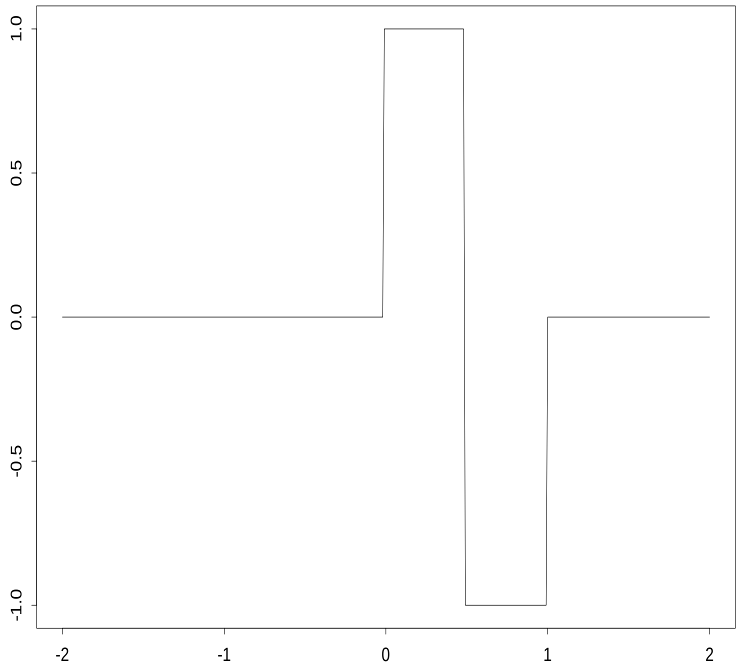
2
2
W 3
C 3
2
Keep one sample out of two


C 0
C 1
C 2
C 3
C 4
W 1
W 2
W 3
W 4
H
G

Example: the Haar Wavelet Transform

$$\begin{aligned} \psi(x) &= 1 && \text{if } 0 \le x < \tfrac{1}{2} \\ \psi(x) &= -1 && \text{if } \tfrac{1}{2} \le x < 1 \\ \psi(x) &= 0 && \text{otherwise} \end{aligned}$$



64	48	16	32	56	56	48	24

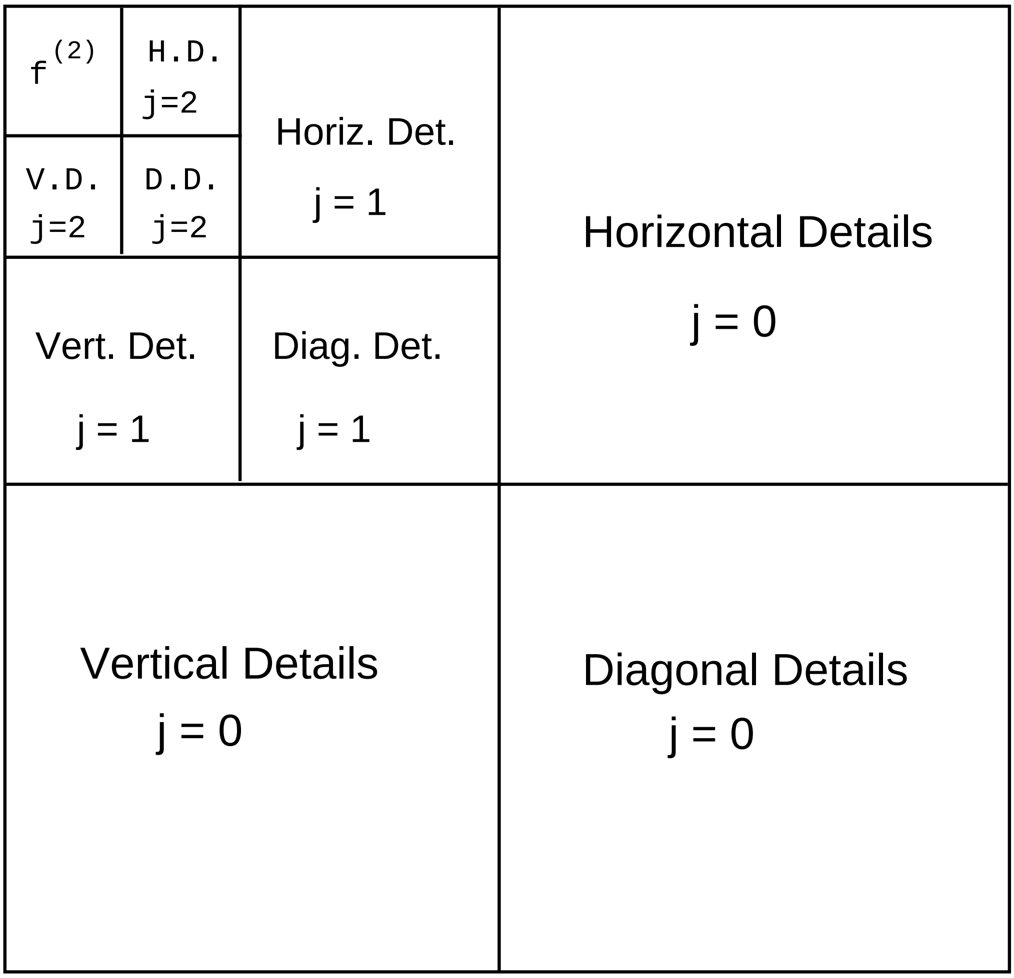
56	24	56	36	**8**	**-8**	**0**	**12**
40	46	**16**	**10**	**8**	**-8**	**0**	**12**
43	**-3**	**16**	**10**	**8**	**-8**	**0**	**12**

In two dimensions, we separate the variables x, y:

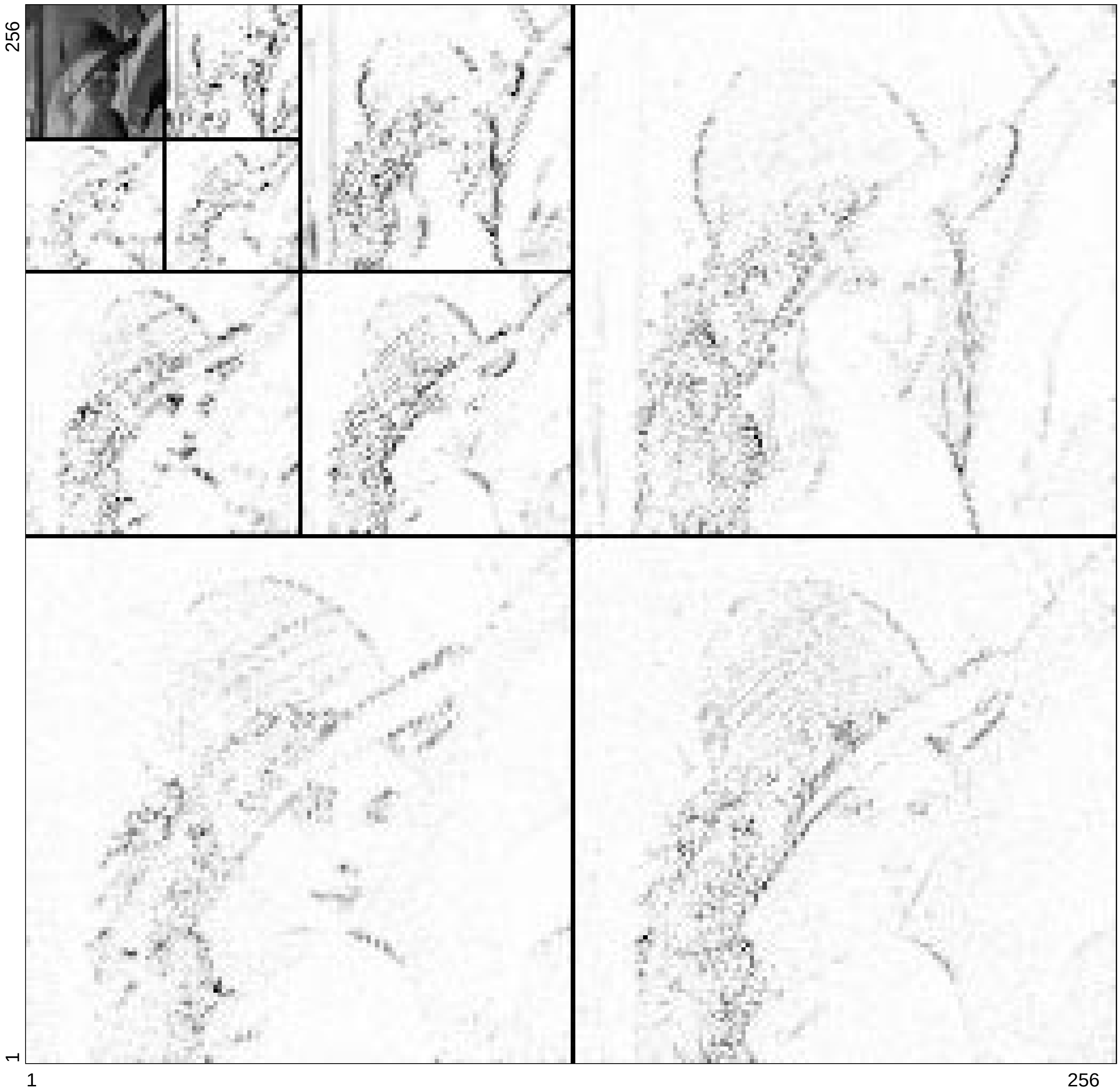
- vertical wavelet: $\psi^1(x,y) = \phi(x)\psi(y)$
- horizontal wavelet: $\psi^2(x,y) = \psi(x)\phi(y)$
- diagonal wavelet: $\psi^3(x,y) = \psi(x)\psi(y)$

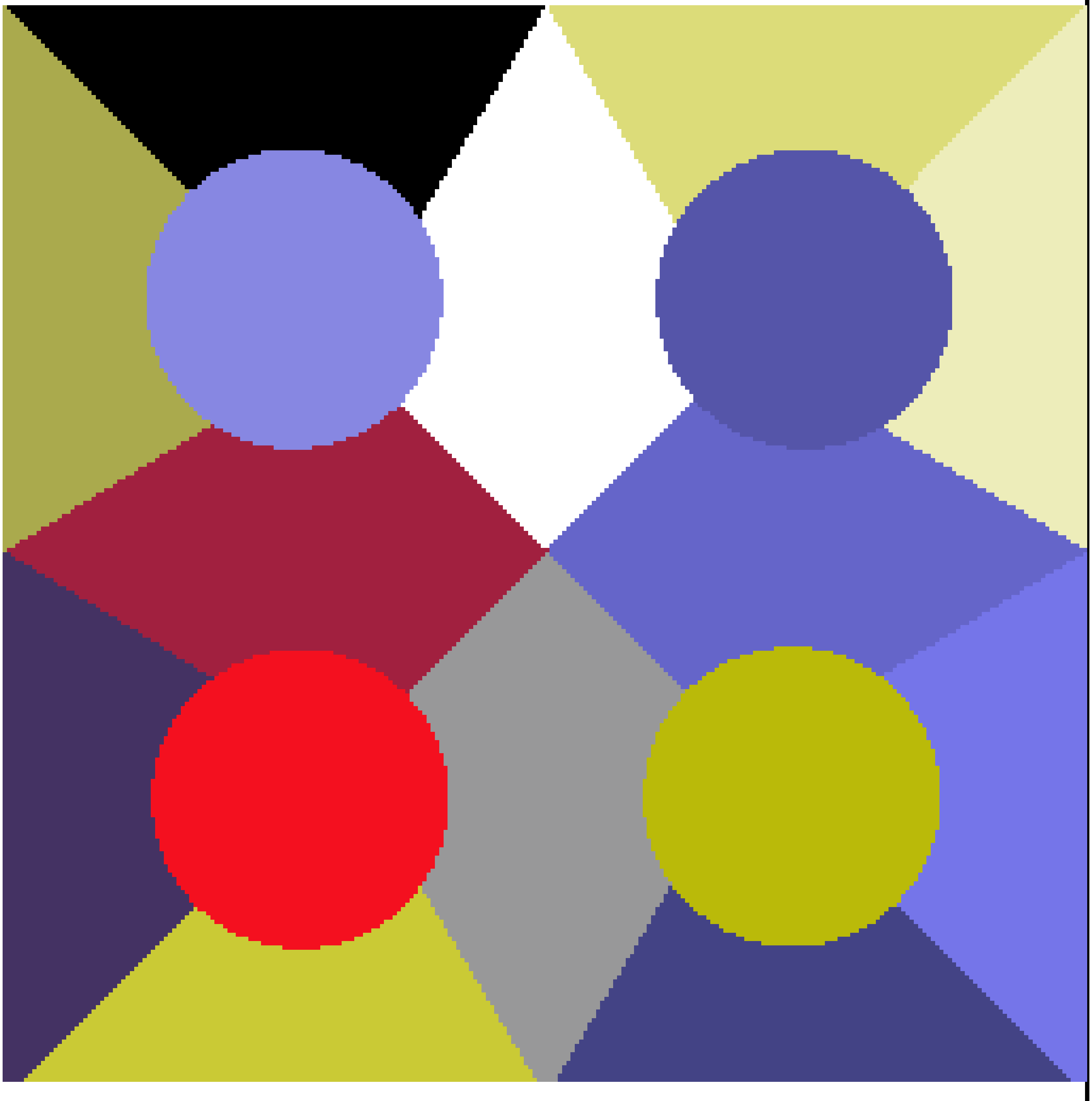
The detail signal is contained in three sub-images:

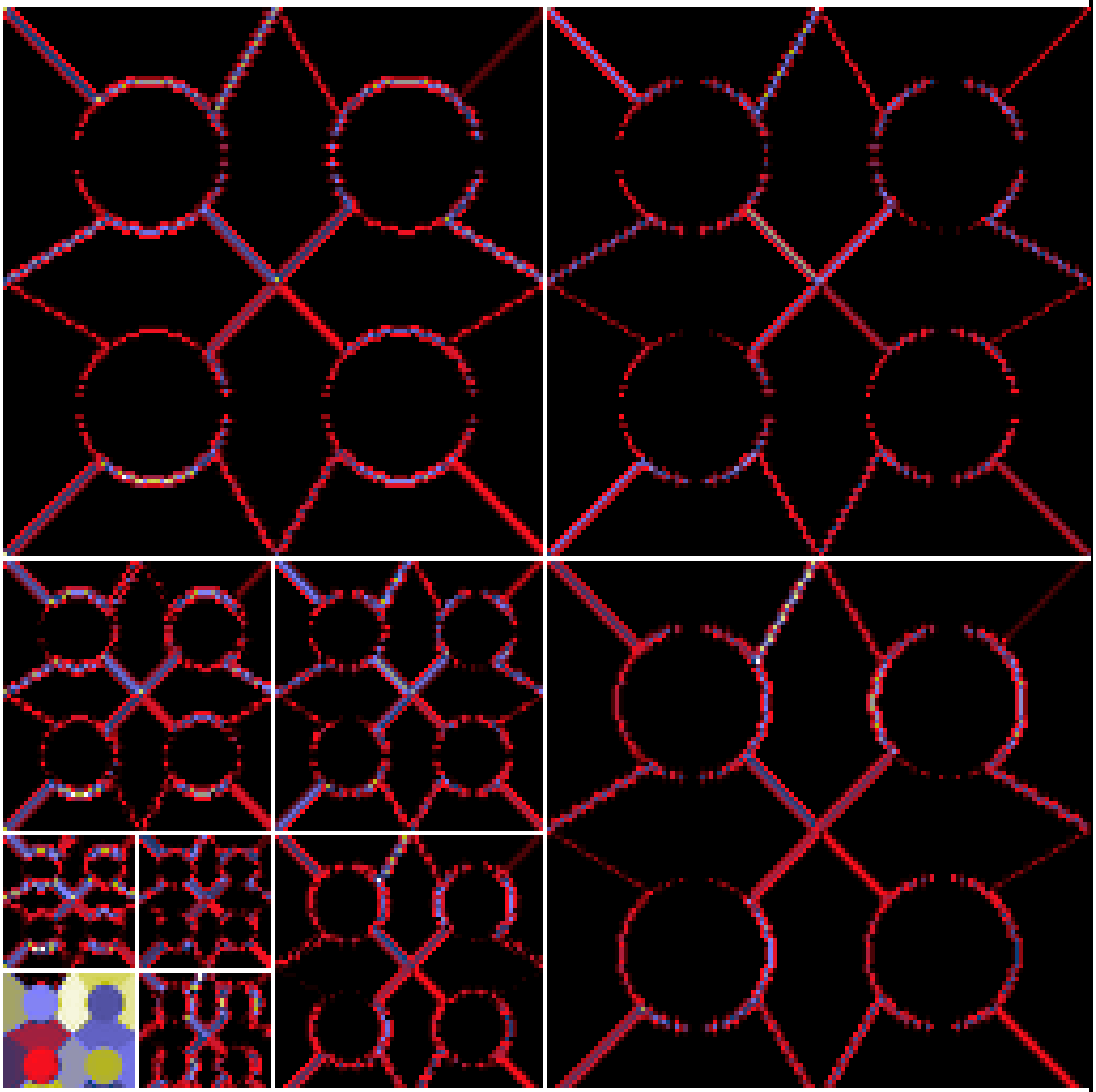
$$
\begin{aligned}
w_j^1(k_x,k_y) &= \sum_{l_x=-\infty}^{+\infty}\sum_{l_y=-\infty}^{+\infty} g(l_x-2k_x)h(l_y-2k_y)c_{j+1}(l_x,l_y) \\
w_j^2(k_x,k_y) &= \sum_{l_x=-\infty}^{+\infty}\sum_{l_y=-\infty}^{+\infty} h(l_x-2k_x)g(l_y-2k_y)c_{j+1}(l_x,l_y) \\
w_j^3(k_x,k_y) &= \sum_{l_x=-\infty}^{+\infty}\sum_{l_y=-\infty}^{+\infty} g(l_x-2k_x)g(l_y-2k_y)c_{j+1}(l_x,l_y)
\end{aligned}
$$


$f^{(2)}$
H.D. j=2
V.D. j=2
D.D. j=2
Horiz. Det. j = 1
Vert. Det. j = 1
Diag. Det. j = 1
Horizontal Details j = 0
Vertical Details j = 0
Diagonal Details j = 0


1
512
1
512


256
1
1
256





Bi-Orthogonal Wavelets

These are a generalization of orthogonal wavelets. Two other spaces $\tilde{O}_j$ and $\tilde{V}_j$ are introduced for the reconstruction:

- $V_{j-1} = V_j \oplus O_j$, and $V_j \not\perp O_j$
- $\tilde{V}_{j-1} = \tilde{V}_j \oplus \tilde{O}_j$, and $\tilde{V}_j \not\perp \tilde{O}_j$
- $\tilde{O}_j \perp V_j$ and $O_j \perp \tilde{V}_j$

Reconstruction with bi-orthogonal wavelets

Two other filters $\tilde{h}$ and $\tilde{g}$ are used, defined to be conjugate to h and g. The reconstruction of the signal is performed with:

$$c_j(k) = 2\sum_l [c_{j+1}(l)\tilde{h}(k-2l) + w_{j+1}(l)\tilde{g}(k-2l)]$$

In order to get an exact reconstruction, two conditions are required for the conjugate filters:

- *Dealiasing condition*: $\hat{h}(\nu+\frac{1}{2})\hat{\tilde{h}}(\nu) + \hat{g}(\nu+\frac{1}{2})\hat{\tilde{g}}(\nu) = 0$
- *Exact restoration:* $\hat{h}(\nu)\hat{\tilde{h}}(\nu) + \hat{g}(\nu)\hat{\tilde{g}}(\nu) = 1$

Orthogonal wavelets

Orthogonal wavelets correspond to the restricted case where:

$$
\begin{aligned}
\hat{g}(\nu) &= e^{-2\pi i\nu}\hat{h}^*(\nu + \frac{1}{2}) \\
\hat{\tilde{h}}(\nu) &= \hat{h}^*(\nu) \\
\hat{\tilde{g}}(\nu) &= \hat{g}^*(\nu)
\end{aligned}
$$

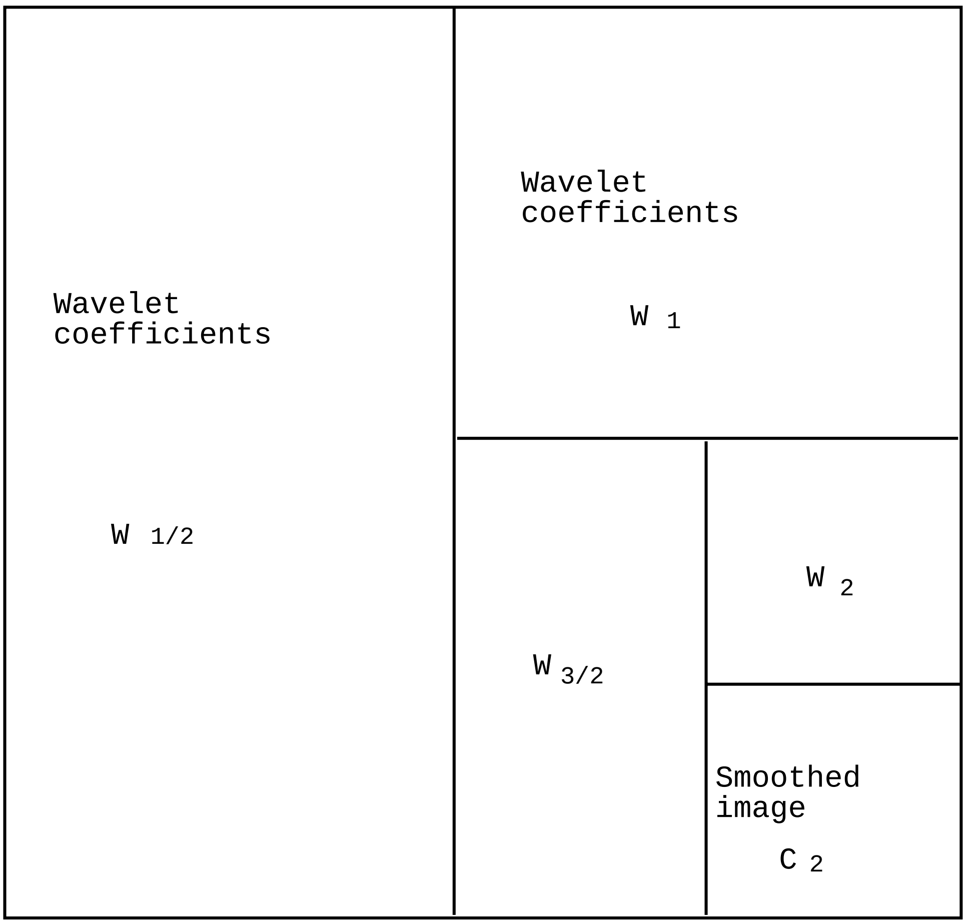
and $\mid \hat{h}(\nu) \mid^2 + \mid \hat{h}(\nu + \frac{1}{2}) \mid^2 = 1$

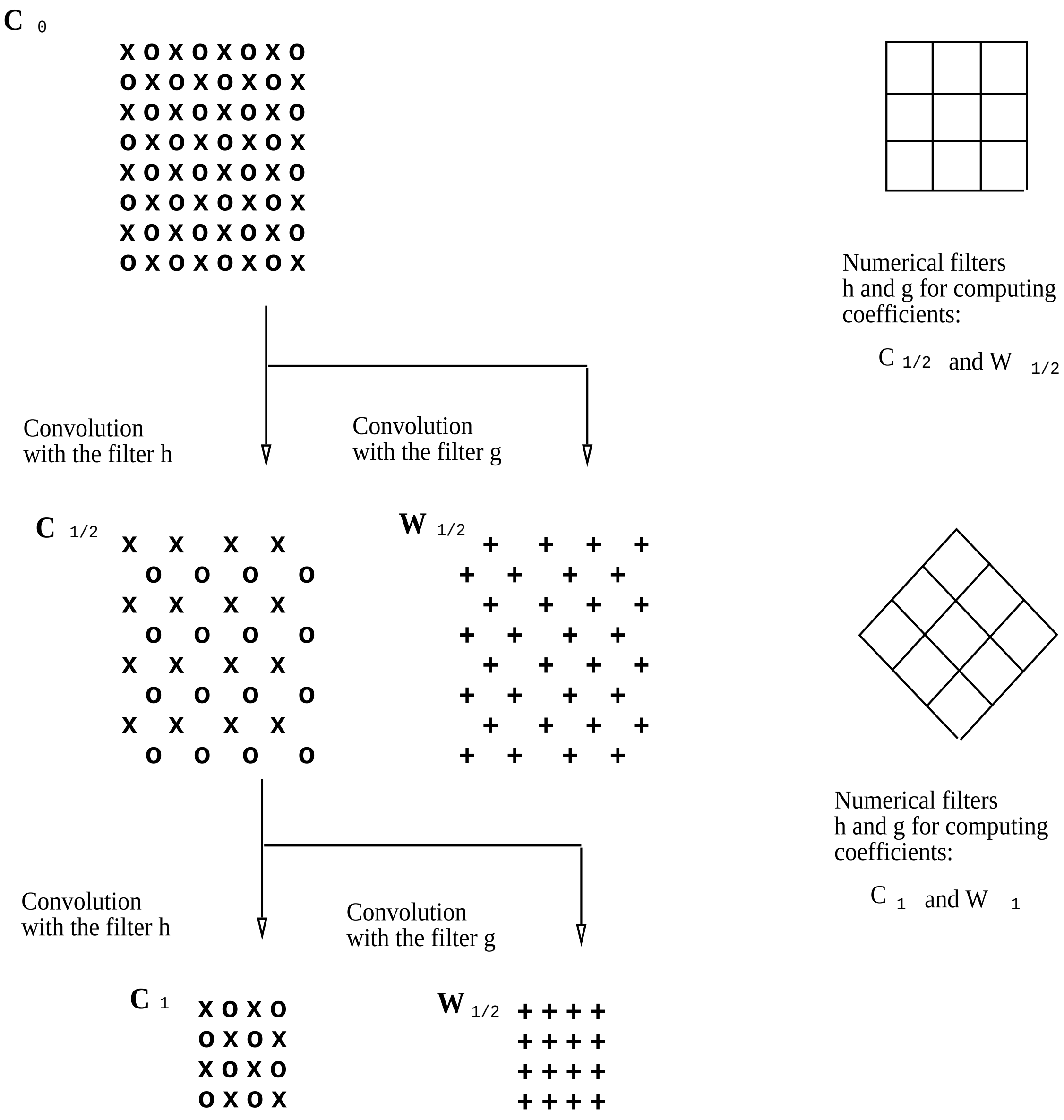
Then we have $g(n) = (-1)^{1-n} h(1-n)$.

$g(n)$ and $h(n)$ are called Conjugate Mirror Filters.

Non-Dyadic Resolution Factor

Feauveau introduced *quincunx* analysis. This analysis is not dyadic and allows an image decomposition with a resolution factor equal to $\sqrt{2}$.





At each step, the image is undersampled by two in one direction (x and y, alternatively). This undersampling is made by keeping one pixel out of two, alternatively even and odd.

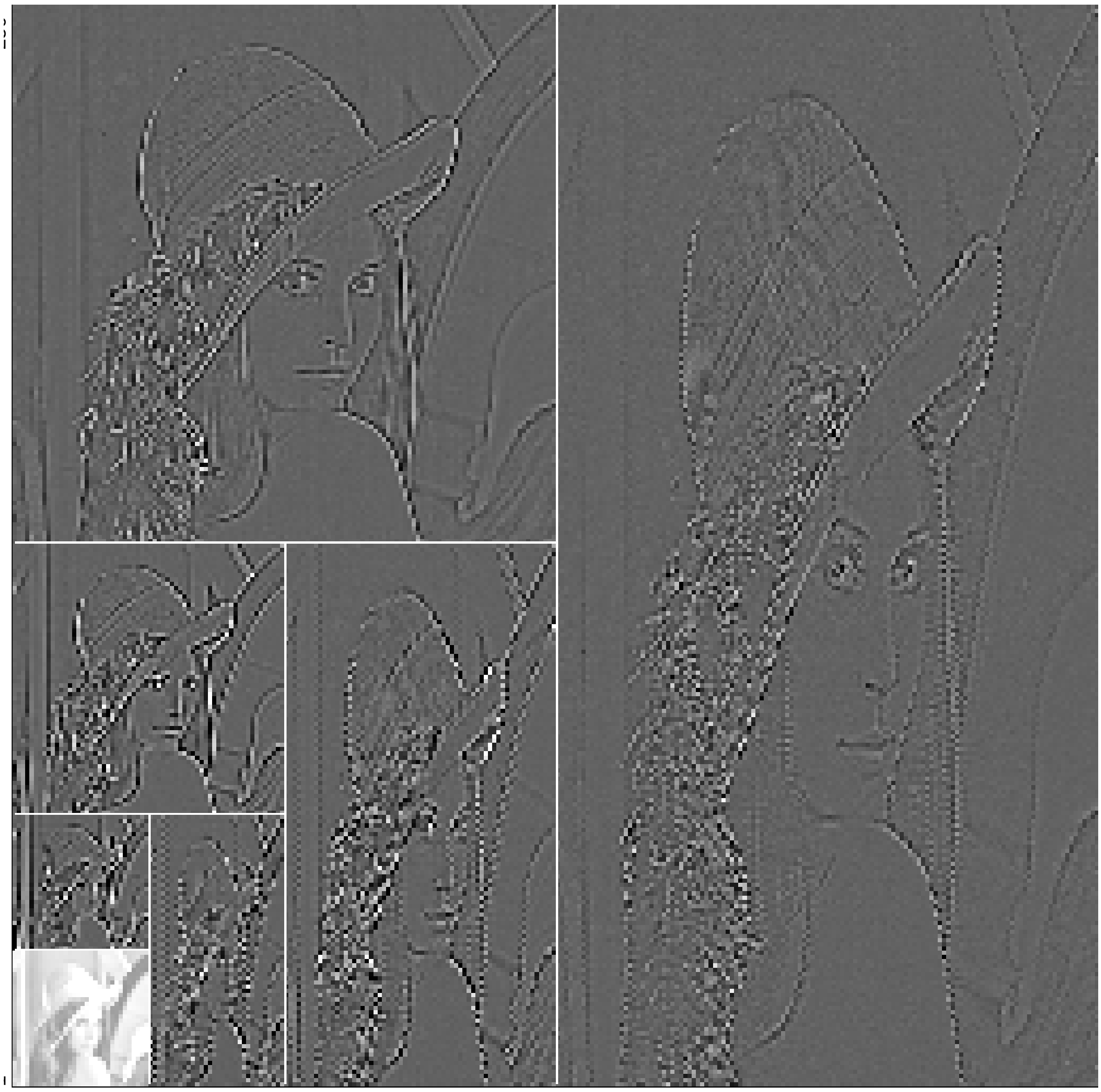
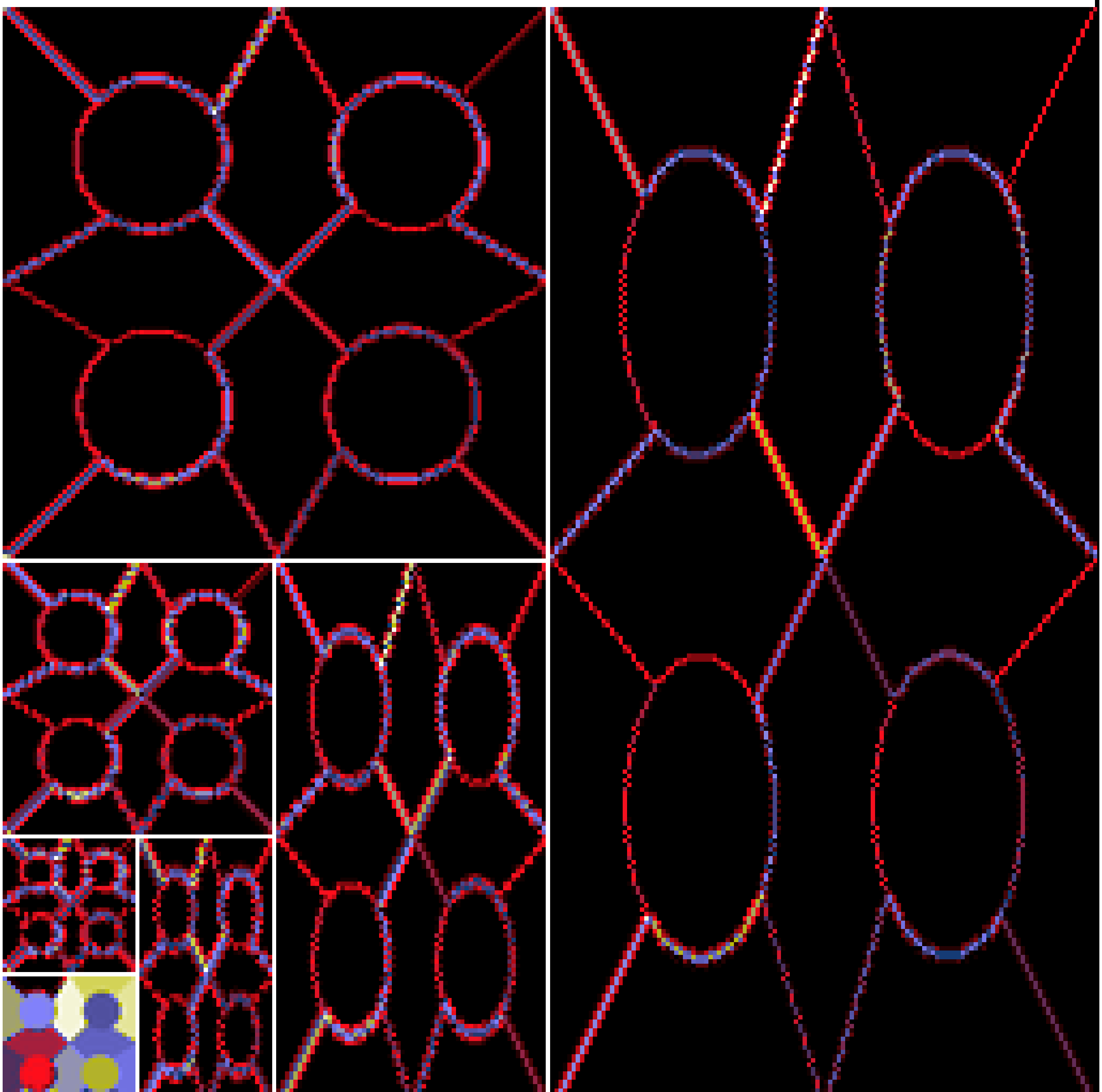
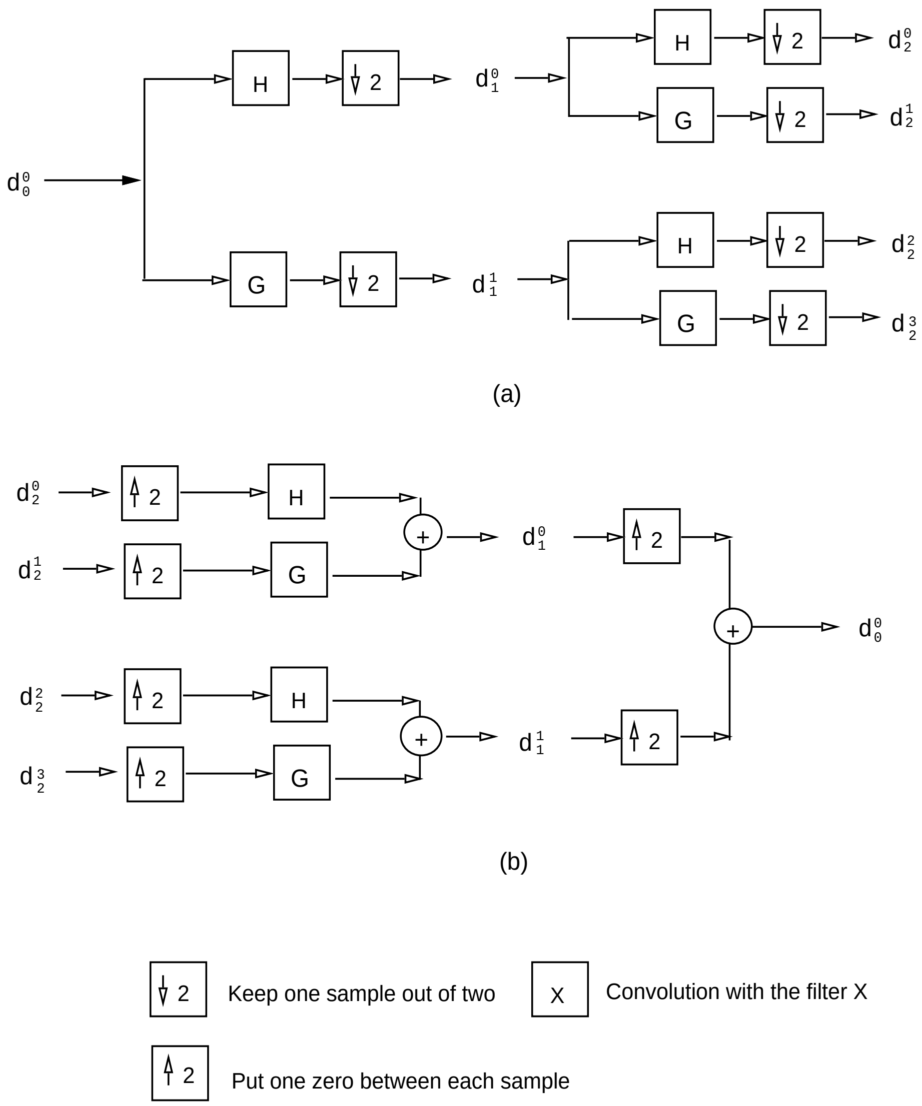


Figure 1: Wavelet transform of Lena by Feauveau's algorithm.



Wavelet Packet

Wavelet packets were introduced by Coifman, Meyer and Wickerhauser (1992). Instead of dividing only the approximation space, as in the standard orthogonal wavelet transform, detail spaces are also divided.



Wavelet packet Scheme: (a) forward direction, (b) reconstruction.

The Lifting Scheme

The lifting scheme (Sweldens, 1996) is a flexible technique that has been used in several different setting, for easy construction and implementation of traditional wavelets, and of second generation wavelets, such as spherical wavelets. The basic principle is to compute the difference between a true coefficient and its prediction:

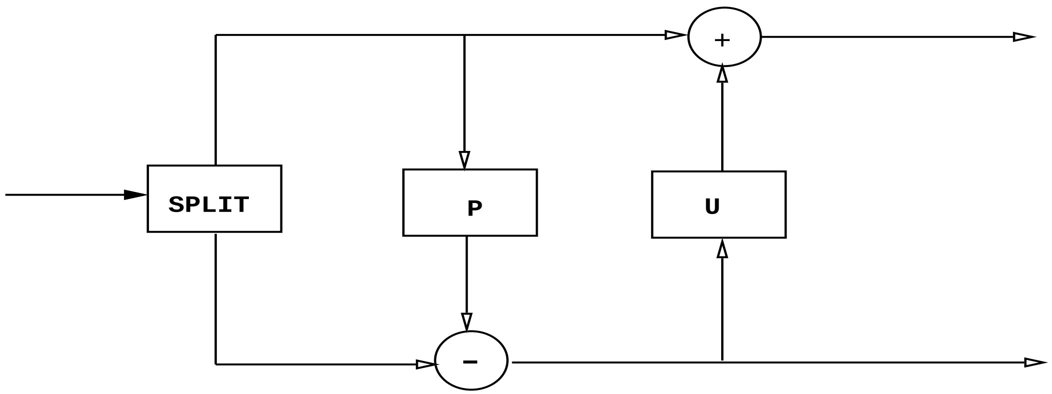
$$w_{j+1,l} \quad = \quad c_{j,2l+1} - \mathcal{P}(c_{j,2l-2L}, ..., c_{j,2l-2}, c_{j,2l}, c_{j,2l+2}, ..., c_{j,2l+2L})$$

A pixel at an odd location 2l+1 is then predicted using pixels at even locations.

Lifting Scheme Transformation in Three Steps:

1. Split the signal into even- and odd-numbered samples: $c_{j+1,l} = c_{j,2l}$ and $w_{j+1,l} = c_{j,2l+1}$.
2. Set $w_{j+1,l} = w_{j+1,l} - \mathcal{P}(c_{j+1,l})$
3. Set $c_{j+1,l} = c_{j+1,l} + \mathcal{U}(w_{j+1,l})$

where $\mathcal{U}$ is the update operator.



The Lifting Scheme - forward direction.

Reconstruction

Reconstruction is obtained by:

$$
\begin{aligned}
c_{j,2l} &= c_{j+1,l} - \mathcal{U}(w_{j+1,l}) \\
c_{j,2l+1} &= w_{j+1,l} + \mathcal{P}(c_{j+1,l})
\end{aligned}
$$

Haar wavelet via lifting

The Haar transform can be performed via the lifting scheme by taking the predict operator equal to the identity, and an update operator which halves the difference. The transform becomes:

$$
\begin{aligned}
w_{j+1,l} &= w_{j+1,l} - c_{j+1,l} \\
c_{j+1,l} &= c_{j+1,l} + \frac{w_{j+1,l}}{2}
\end{aligned}
$$

All computation can be done in place. Every wavelet transform can be written via lifting.

Linear wavelets via lifting

The identity predictor used before is correct when the signal is constant. In the same way, we can use a linear predictor which is correct when the the signal is linear. The predictor and update operators are now:

$$\begin{aligned} \mathcal{P}(c_{j-1,l} &= \frac{1}{2}(c_{j-1,l} + c_{j-1,l+1}) \\ \mathcal{U}(w_{j-1,l}) &= \frac{1}{4}(w_{j-1,l-1} + w_{j-1,l}) \end{aligned}$$

It can be verified that:

$$c_{j-1,l} = -\frac{1}{8}c_{j,2l-2} + \frac{1}{4}c_{j,2l-1} + \frac{3}{4}c_{j,2l} + \frac{1}{4}c_{j,2l+1} - \frac{1}{8}c_{j,2l+2}$$

which is the biorthogonal Cohen-Daubechies-Feauveau wavelet transform.

Integer Wavelet Transform

When the input data are integer values, the wavelet transform no longer consists of integers. For lossless coding, however, we need a wavelet transform which produces integer values. We can built an integer version of every wavelet transform. For instance, the integer Haar transform can be calculated by:

$$
\begin{aligned}
w_{j+1,l} &= c_{j,2l+1} - c_{j,2l} \\
c_{j+1,l} &= c_{j,2l} + \lfloor \frac{w_{j+1,l}}{2} \rfloor = c_{j+1,l} + \lfloor \frac{w_{j+1,l}}{2} \rfloor
\end{aligned}
$$

and the reconstruction is

$$
\begin{aligned}
c_{j,2l} &= c_{j+1,l} - \lfloor \frac{w_{j+1,l}}{2} \rfloor \\
c_{j,2l+1} &= w_{j+1,l} + c_{j,2l}
\end{aligned}
$$

More generally, the lifting operators for an integer version of the wavelet transform are:

$$\begin{aligned}
\mathcal{P}(c_{j+1,l}) &= \lfloor \sum_k p_k c_{j+1,l-k} + \frac{1}{2} \rfloor \\
\mathcal{U}(w_{j+1,l}) &= \lfloor \sum_k u_k w_{j+1,l-k} + \frac{1}{2} \rfloor
\end{aligned}$$

The linear integer wavelet transform is

$$\begin{aligned}
w_{j+1,l} &= w_{j+1,l} - \lfloor \frac{1}{2}(c_{j+1,l} + c_{j+1,l+1}) + \frac{1}{2} \rfloor \\
c_{j+1,l} &= c_{j+1,l} + \lfloor \frac{1}{4}(w_{j+1,l-1} + w_{j+1,l}) + \frac{1}{2} \rfloor
\end{aligned}$$

The à trous Algorithm

This is a "stationary" or redundant transform, i.e. decimation is not carried out. The distance between samples increasing by a factor 2 from scale $(j-1)$ ($j > 0$) to the next, $c_j(k)$, is given by:

$$c_j(k) = \sum_l h(l) c_{j-1}(k + 2^{j-1} l)$$

and the discrete wavelet coefficients by:

$$w_j(k) = \sum_l g(l) c_{j-1}(k + 2^{j-1} l)$$

Generally, the wavelet resulting from the difference between two successive approximations is applied:

$$w_j(k) = c_{j-1}(k) - c_j(k)$$

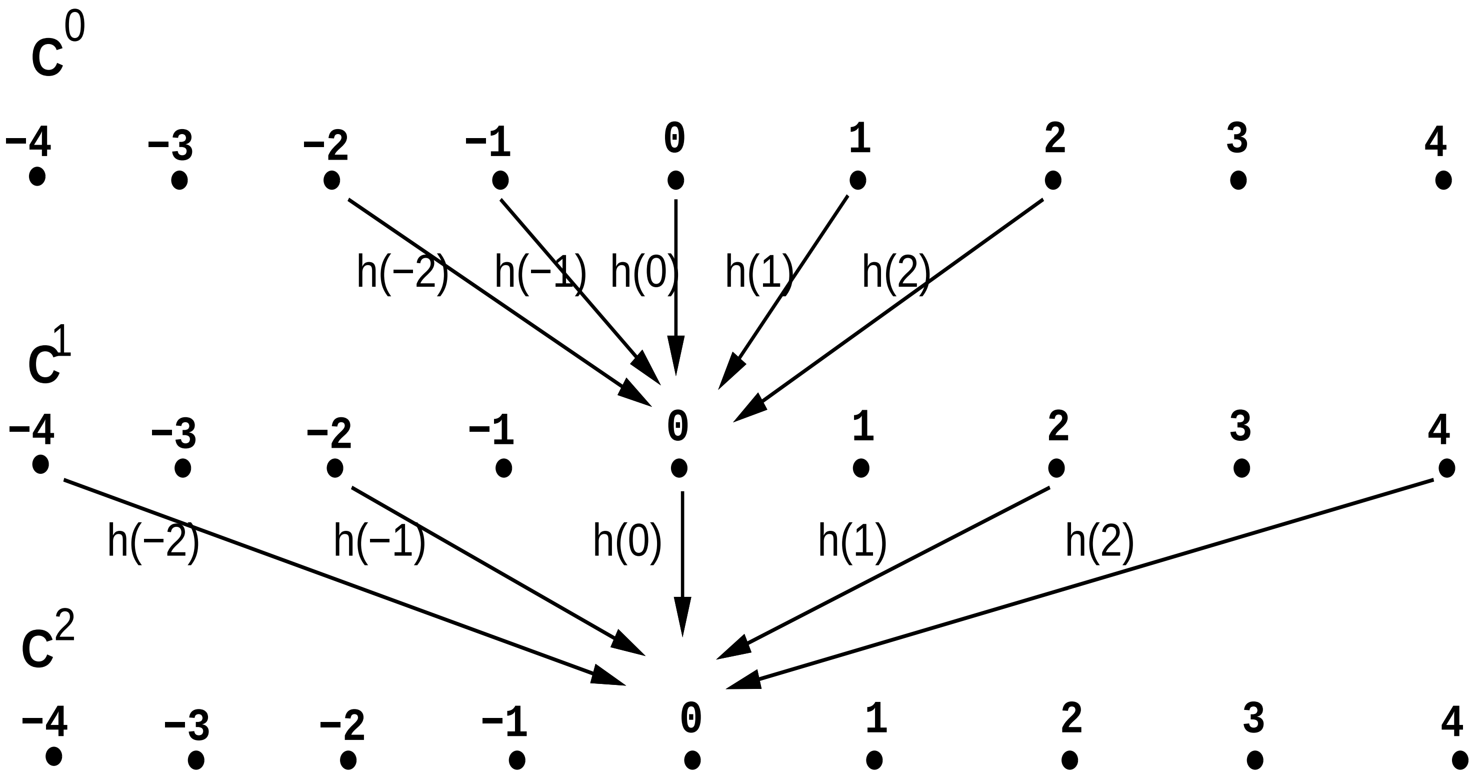
The associated wavelet is $\psi(x)$.

$$\frac{1}{2}\psi(\frac{x}{2}) = \phi(x) - \frac{1}{2}\phi(\frac{x}{2})$$

The reconstruction algorithm is immediate:

$$c_0(k) = c_{n_p}(k) + \sum_{j=1}^{n_p} w_j(k)$$

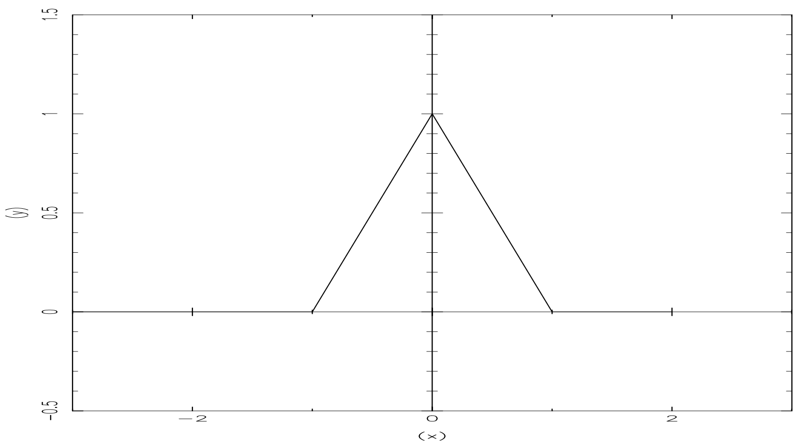
Passage from c_0 to c_1, and from c_1 to c_2

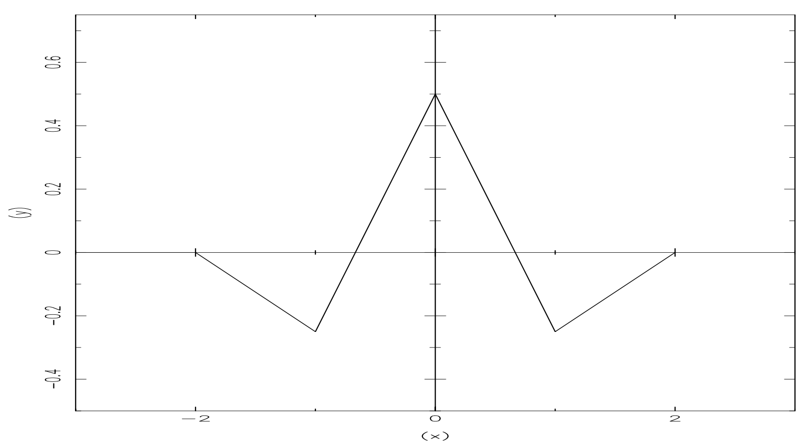


The triangle function

Choosing the triangle function as the scaling function ϕ leads to piecewise linear interpolation:

$$\phi(x) = 1 - |x| \quad \text{if } x \in [-1, 1]$$
$$\phi(x) = 0 \quad \text{if } x \notin [-1, 1]$$





We have

$$\frac{1}{2}\phi(\frac{x}{2}) = \frac{1}{4}\phi(x+1) + \frac{1}{2}\phi(x) + \frac{1}{4}\phi(x-1)$$

c_{j+1} is obtained from c_j by:

$$c_{j+1}(k) = \frac{1}{4}c_j(k-2^j) + \frac{1}{2}c_j(k) + \frac{1}{4}c_j(k+2^j)$$

The wavelet coefficients at scale j are:

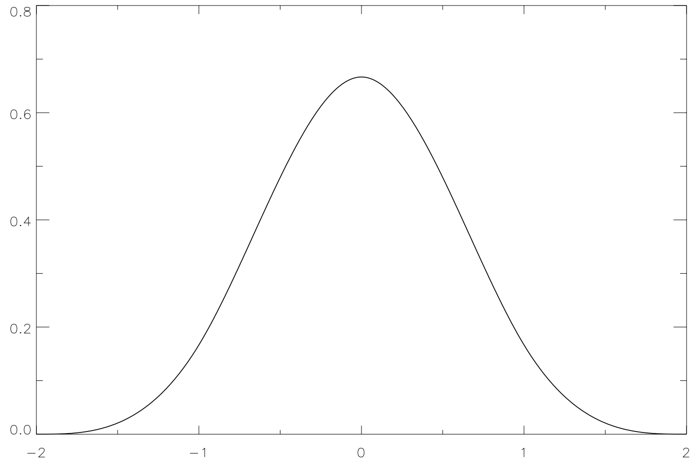
$$w_{j+1}(k) = -\frac{1}{4}c_j(k-2^j) + \frac{1}{2}c_j(k) - \frac{1}{4}c_j(k+2^j)$$

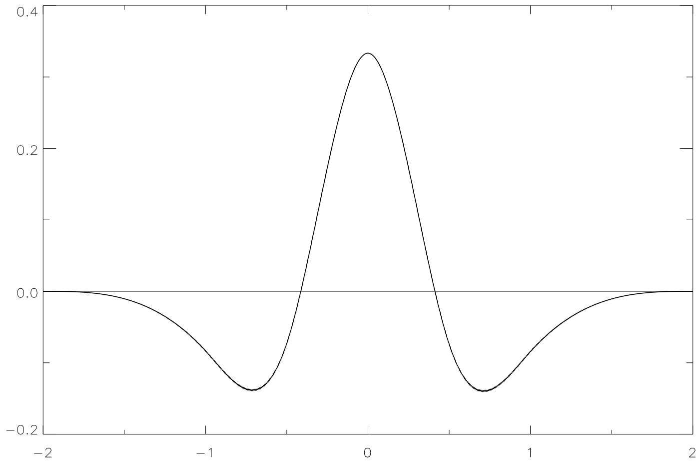
The cubic spline function

$$B_3(x) = \frac{1}{12}(\mid x-2\mid^3 -4\mid x-1\mid^3 +6\mid x\mid^3 -4\mid x+1\mid^3 +\mid x+2\mid^3)$$

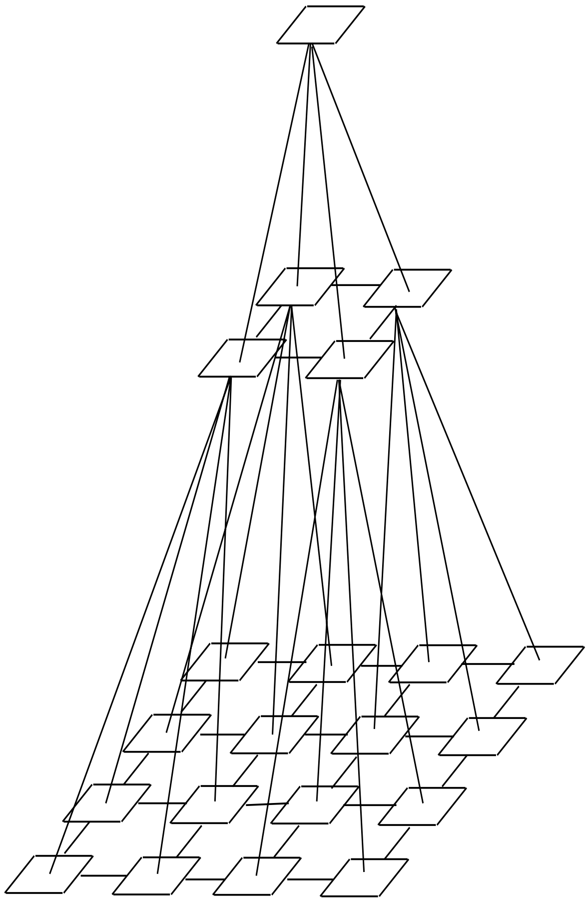
and

$$h(-2) = \frac{1}{16}; h(-1) = \frac{1}{4}; h(0) = \frac{3}{8}; h(1) = \frac{1}{4}; h(2) = \frac{1}{16}$$





Pyramidal Algorithm



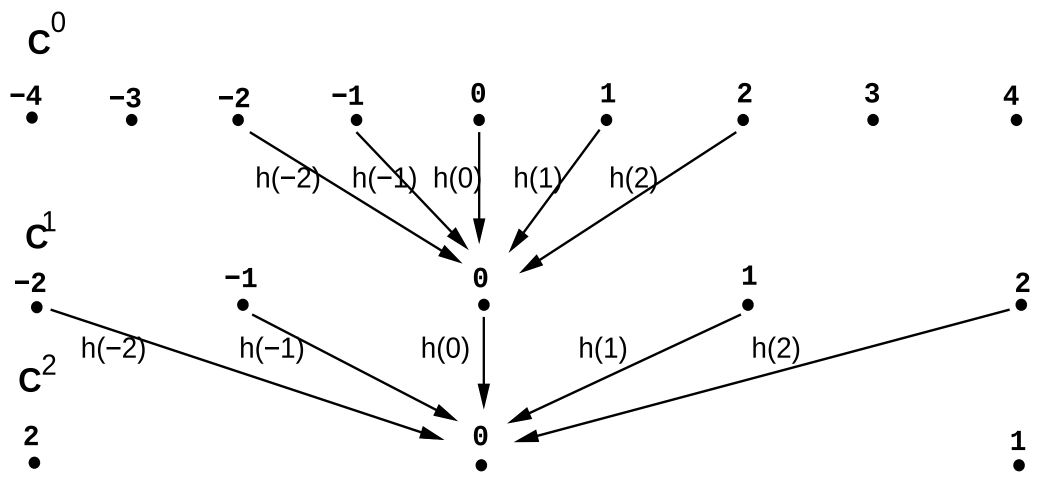
Several algorithms exist:

- The Laplacian Pyramid
- Pyramid with one wavelet
- Pyramidal wavelet transform using the FFT

The Laplacian Pyramid

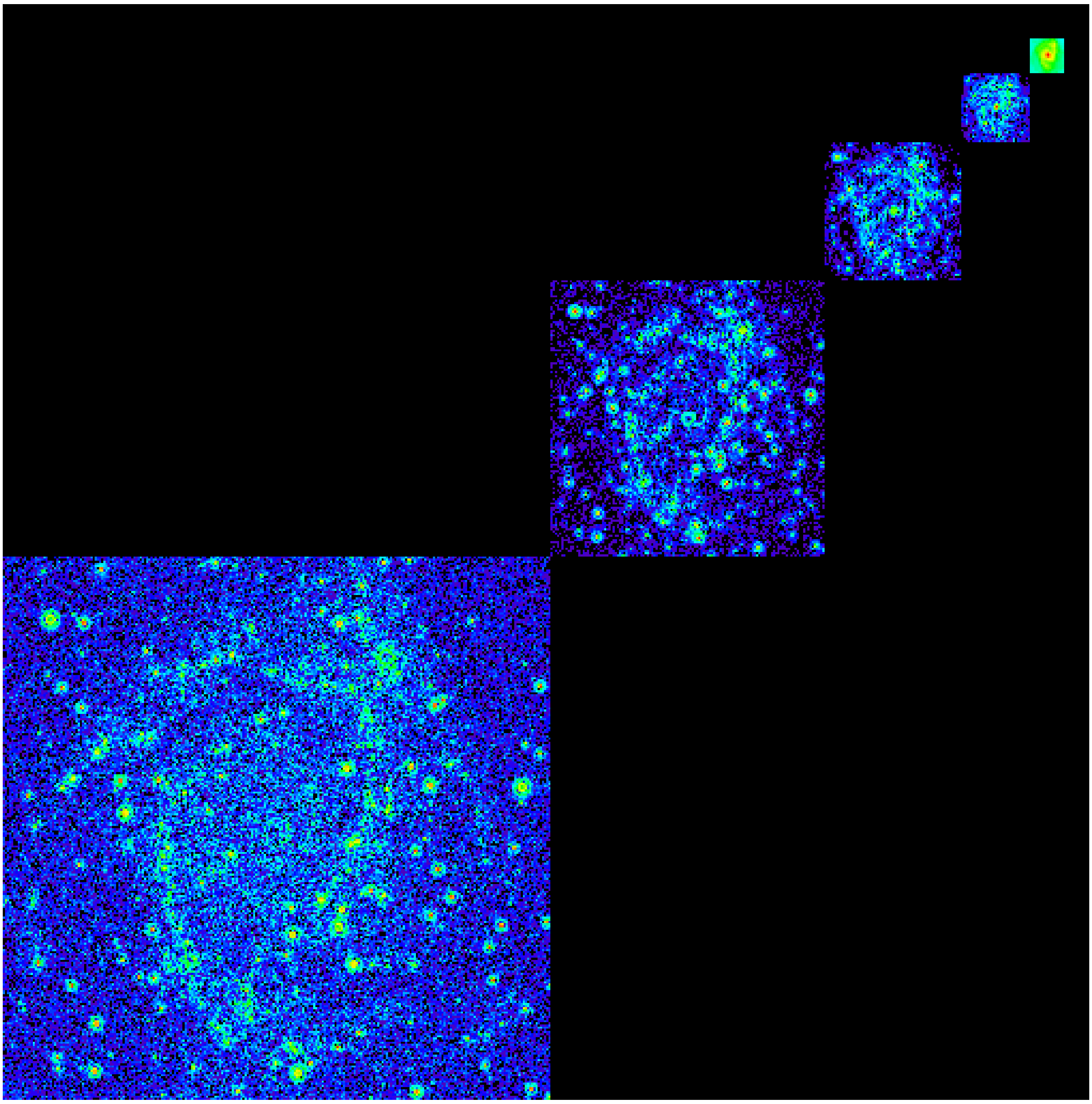
The Laplacian pyramid was developed by Burt and Adelson in 1981 in order to compress images. The convolution is carried out with the filter h by keeping one sample out of two:

$$
\begin{aligned}
c_{j+1}(k) &= \sum_l h(l-2k)c_j(l) \\
w_{j+1}(k) &= c_j(k) - \tilde{c}_j(k) \\
\tilde{c}_j(k) &= 2\sum_l h(k-2l)c_{j+1}(k)
\end{aligned}
$$



There is no invariance to translation.

NGC 2997 pyramidal coefficients



Half Pyramidal Wavelet Transform

At each iteration of the pyramidal transform, there is a smoothing and a decimation of the image. But the smoothing is not strong enough to reduce the cut-off frequency by a factor of two. Then the decimation violates the Shannon theorem. To avoid this, because it may create artifacts, Bijaoui (1997) proposed:

- Not to decimate the image at the first iteration of the algorithm. This means that the two first scales have the same size as the original image.
- To decimate at the following iterations.

Discussion of the Wavelet Transform

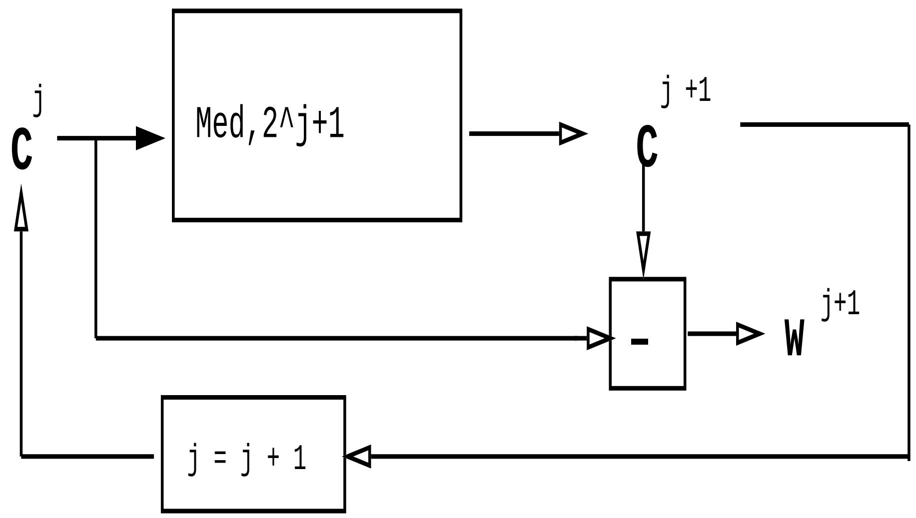
1. Anisotropic Wavelet (Mallat)
2. Invariance by Translation (Mallat, Feauveau)
3. Scale Separation (Mallat, Feauveau, Pyramidal Algorithm)

$$\begin{aligned}\hat{W}^{(I)}(a,u,v) &= \sqrt{a}\hat{\psi}^*(au,av)\hat{P}(u,v)\hat{O}(u,v) \\ &= \hat{O}(u,v)\hat{W}^{(P)}(a,u,v)\end{aligned}$$

4. Negative Values (Bumps)
5. Point Objects

Multiresolution Median Transform

1. Let $c_j = I$ with $j = 1$
2. Determine $c_{j+1} = med(f, 2s + 1)$.
3. The multiresolution coefficients w_{j+1} are defined as: $w_{j+1} = c_j - c_{j+1}$.
4. Let $j \longleftarrow j + 1$; $s \longleftarrow 2s$. Return to step 2 if $j < S$.



Reconstruction by: $I = c_p + \sum_j w_j$

Pyramidal Median Transform

1. Let $c_j = I$ with $j = 1$.
2. Determine $c^*_{j+1} = med(c_j, 2s+1)$ with $s = 1$.
3. The pyramidal multiresolution coefficients w_{j+1} are defined as:

$$w_{j+1} = c_j - c^*_{j+1}$$

4. Let $c_{j+1} = dec(c^*_{j+1})$ (where the decimation operation, dec, entails 1 pixel replacing each 2×2 subimage).
5. Let $j \longleftarrow j + 1$. Return to step 2 so long as $j < S$.

WT-PMT transform

Wavelet transform $\rightarrow$ robust noise estimation in the different scales
PMT $\rightarrow$ better separation of the structures in the scales.

In fact, when there is **no signal in a given region**, a **wavelet transform** would be better, and if a **strong signal** appears, it is the **PMT** that we would like to use. So a novel – and feasible – idea is to **try to merge both transforms**, and to adapt the analysis at each position and at each scale, depending on the measured amplitude of the coefficient.

Noise Modeling

We can introduce a statistical significance test for wavelet coefficients. Let $\mathcal{H}_0$ be the hypothesis that the image is locally constant at scale j. Rejection of hypothesis $\mathcal{H}_0$ depends (for a positive coefficient value) on:

$$P = Prob(W_N > w_j(x, y))$$

and if the coefficient value is negative

$$P = Prob(W_N < w_j(x, y))$$

Given a threshold, ϵ, if $P > \epsilon$ the null hypothesis is not excluded. Although non-null, the value of the coefficient could be due to noise. On the other hand, if $P < \epsilon$, the coefficient value cannot be due only to the noise alone, and so the null hypothesis is rejected. In this case, a significant coefficient has been detected.

Multiresolution Support

The *multiresolution support* M of an image is computed as follows:

- Step 1 is to compute the wavelet transform of the image.
- Booleanization of each scale leads to the multiresolution support.
- A priori knowledge can be introduced by modifying the support.

$M^{(I)}(j, x, y) = 1$ (or $=\ true$) $\Longrightarrow I$ contains information at scale j and at the position (x, y).

Remark:

Incorporating a priori knowledge: e.g. no interesting object smaller or larger than a given size in our image. Suppress, in the support, anything which is due to that kind of object. Can use mathematical morphology to do this.

Noise Modeling in Wavelet Space

Our noise modeling in wavelet space is based on the assumption that the noise in the data follows a statistical distribution, which can be:

- a Gaussian distribution
- a Poisson distribution
- a Poisson + Gaussian distribution (CCD noise)
- Poisson noise with few events (galaxy counting, X-ray images, ...)
- Speckle noise
- Root Mean Square map: we have a noise standard deviation of each data value.

Noise modeling with less constraint on the noise behavior

If the noise doesn't follows any of these distributions, we can derive a noise model from any of the following assumptions:

- it is additive, and non-stationary.
- it is multiplicative and stationary.
- it is multiplicative, but non-stationary.
- it is undefined but stationary.

If none of these assumptions can be applied, the only way to derive a noise model in wavelet space is to consider that the local variation of the wavelet coefficient follows a Gaussian distribution.

Gaussian Noise

$$p(w_j(x,y)) = \frac{1}{\sqrt{2\pi}\sigma_j} e^{-w_j(x,y)^2/2\sigma_j^2}$$

Rejection of hypothesis $\mathcal{H}_0$ depends (for a positive coefficient value) on:

$$P = Prob(w_j(x,y) > W) = \frac{1}{\sqrt{2\pi}\sigma_j} \int_{w_j(x,y)}^{+\infty} e^{-W^2/2\sigma_j^2} dW$$

and if the coefficient value is negative, it depends on

$$P = Prob(w_j(x,y) < W) = \frac{1}{\sqrt{2\pi}\sigma_j} \int_{-\infty}^{w_j(x,y)} e^{-W^2/2\sigma_j^2} dW$$

Given stationary Gaussian noise, it suffices to compare $w_j(x,y)$ to $k\sigma_j$.

if $\mid w_j \mid \geq k\sigma_j$ then w_j is significant

if $\mid w_j \mid < k\sigma_j$ then w_j is not significant

How to find σ_j ?

The appropriate value of σ_j in the succession of wavelet planes is assessed from the standard deviation of the noise σ_I in the original image and from study of the noise in wavelet space. This study consists of simulating an image containing Gaussian noise with a standard deviation equal to 1, and taking the wavelet transform of this image. Then we compute the standard deviation σ_j^e at each scale. We get a curve σ_j^e as a function of j, giving the behavior of the noise in wavelet space. (Note that if we had used an orthogonal wavelet transform, this curve would be linear.) Due to the properties of the wavelet transform, we have $\sigma_j = \sigma_I \sigma_j^e$. The standard deviation of the noise at a scale j of the image is equal to the standard deviation of the noise of the image multiplied by the standard deviation of the noise of the scale j of the wavelet transform.

Poisson Noise

If the noise in the data I is Poisson, the transform

$$t(I(x,y)) = 2\sqrt{I(x,y) + \frac{3}{8}}$$

acts as if the data arose from a Gaussian white noise model (Anscombe, 1948), with $\sigma = 1$, under the assumption that the mean value of I is large.

Poisson Noise + Gaussian

The generalization of the variance stabilizing is:

$$t(I(x,y)) = \frac{2}{\alpha}\sqrt{\alpha I(x,y) + \frac{3}{8}\alpha^2 + \sigma^2 - \alpha g}$$

where α is the gain of the detector, and g and σ are the mean and the standard deviation of the read-out noise.

Poisson Noise with Few Photons

A wavelet coefficient at a given position and at a given scale j is

$$w_j(x,y) = \sum_{k \in K} n_k \psi(\frac{x_k - x}{2^j} \frac{y_k - y}{2^j})$$

where K is the support of the wavelet function ψ and n_k is the number of events which contribute to the calculation of $w_j(x,y)$.

If a wavelet coefficient $w_j(x,y)$ is due to the noise, it can be considered as a realization of the sum $\sum_{k \in K} n_k$ of independent random variables with the same statistical distribution as is given by the wavelet function. Then we compare the wavelet coefficient of the data to the values which can be taken by the sum of n independent variables.

The distribution of one event in wavelet space is then directly given by the histogram H_1 of the wavelet ψ. Assuming independent events, the distribution of a coefficient W_n related to n events is given by n autoconvolutions of H_1

$$H_n = H_1 \otimes H_1 \otimes ... \otimes H_1$$

Speckle Noise

Speckle occurs in all types of coherent imagery such as synthetic aperture radar (SAR) imagery, acoustic imagery and laser illuminated imagery. The probability density function (pdf) of the modulus of a homogeneous scene is a Rayleigh distribution :

$$p(\rho) = \frac{\rho}{\sigma^2} e^{-\frac{\rho^2}{2\sigma^2}} \quad M_\rho = \sqrt{\frac{\pi}{2}}\sigma \quad \sigma_\rho = \sqrt{\frac{4-\pi}{2}}\sigma$$

The ratio σ_ρ / M_ρ is a constant of value $\sqrt{\frac{4-\pi}{\pi}}$. This means that the speckle is multiplicative noise. The pdf of the modulus of log-transformed speckle noise is :

$$p(\ell) = \frac{e^{2\ell}}{\sigma^2} e^{-\frac{e^{2\ell}}{2\sigma^2}} \quad M_\ell = 0.058 + \log(\sigma) \quad \sigma_\ell = \sqrt{\frac{\pi^2}{24}} = 0.641$$

A better estimator of the Rayleigh distribution is the energy ($I = \rho^2$) which is known to have an exponential (*i.e.* Laplace) distribution of parameter $a = 2\sigma^2$

$$p(I) = \frac{1}{a} e^{-\frac{I}{a}}$$

With this transformation the noise is still multiplicative, but it has been shown that the ratio R between the energy of a perfect filtered image and a spatially Laplace-distributed image of local mean value $a(k_x, k_y)$ has a Laplace pdf of mean 1.

RMS map

If, associated with the data $I(x, y)$, we have the root mean square map $R_\sigma(x, y)$, the noise in I is non-homogeneous. For each wavelet coefficient $w_j(x, y)$ of I, the exact standard deviation $\sigma_j(x, y)$ has to be calculated from R_σ. A wavelet coefficient $w_j(x, y)$ is obtained by the correlation product between the image I and a function g_j:
$w_j(x, y) = \sum_k \sum_l I(x, y) g_j(x + k, y + l)$.

Then we have: $\sigma_j^2(x, y) = \sum_k \sum_l R_\sigma^2(x, y) g_j^2(x + k, y + l)$.
In the case of the à trous algorithm, the coefficients $g_j(x, y)$ are not known exactly, but they can easily be computed by taking the wavelet transform of a Dirac w^δ. The map σ_j^2 is calculated by correlating the square of the wavelet scale j of w^δ with $R_\sigma^2(x, y)$.

Other Types of Noise

1. Additive non-stationary noise: $R_\sigma(x, y)$ is calculated from the local variation in the image.
2. Multiplicative noise: the image is transformed by taking its logarithm. In the resulting image, the noise is additive, and a hypothesis of Gaussian noise can be used.
3. Multiplicative non-stationary noise: we take the logarithm of the image, and the resulting image is treated as for additive non-uniform noise above.
4. Undefined uniform noise: A k-sigma clipping is applied at each scale in order to find σ_j.
5. Undefined noise: The standard deviation is estimated for each wavelet coefficient, by considering a box around it, and the calculation of σ is done in the same way as for non-uniform additive noise.

Filtering

- Hard thresholding
- Soft thresholding
- Donoho universal approach
- Multiresolution Wiener filtering
- Hierarchical Wiener filtering
- Hierarchical hard thresholding
- Iterative filtering from the multiresolution support
- Multiscale entropy filtering

Hard and soft thresholding

- Hard thresholding:

$$\begin{aligned} \tilde{w}_j &= w_j \text{ if } \mid w_j \mid \geq T_j \\ &= 0 \text{ otherwise} \end{aligned}$$

  with $T_j = k\sigma_j$. For an energy-normalized wavelet transform algorithm, we have $\sigma_j = \sigma$ for all j.

- Soft thresholding:

$$\begin{aligned} \tilde{w}_j &= sgn(w_j)(\mid w_j \mid - T_j) \text{ if } \mid w_j \mid \geq T_j \\ &= 0 \text{ otherwise} \end{aligned}$$

- Donoho approach:
  $T_j = \sqrt{2\log(n)}\sigma_j$ (where n is the number of pixels) instead of the standard $k\sigma$ value. This leads to a new soft and hard thresholding approach.

Filtering from the multiresolution support

The filtering can be seen as an inverse problem. Indeed, we want to reconstruct an image from the detected wavelet coefficient. The problem of reconstruction consists of searching for a signal $\tilde{I}$ such that its wavelet coefficients are the same as those of the detected structure. Denoting $\mathcal{T}$ the wavelet transform operator, and P the projection operator in the subspace of the detected coefficients (i.e. setting to zero all coefficients at scales and positions where nothing where detected), the solution is found by minimization of

$$J(\tilde{I}) = \parallel W - (P \circ \mathcal{T})\mathcal{S} \parallel$$

where W represents the detected wavelet coefficients of the image I.

Algorithm

The residual at iteration n is:

$$R^{(n)}(x,y) = I(x,y) - \tilde{I}^{(n)}(x,y)$$

By using the *à trous* wavelet transform algorithm:

$$R^{(n)}(x,y) = c_p(x,y) + \sum_{j=1}^{p} w_j(x,y)$$

Significant residual pixels are determined:

$$\bar{R}^{(n)}(x,y) = c_p(x,y) + \sum_{j=1}^{p} M(j,x,y) w_j(x,y)$$

and

$$\tilde{I}^{(n+1)} = \tilde{I}^{(n)} + \bar{R}^{(n)}$$

Multiscale Entropy Filtering method

The multiscale entropy method consists of measuring the information h relative to wavelet coefficients, and of separating this into two parts h_s, and h_n. The expression h_s is called the signal information and represents the part of h which is certainly not contaminated by the noise. The expression h_n is called the “noise information” and represents the part of h which may be contaminated by the noise. We have $h = h_s + h_n$. Following this notation, the corrected coefficient $\tilde{w}$ should minimize:

$$J(\tilde{w}_j) = h_s(w_j - \tilde{w}_j) + \alpha h_n(\tilde{w}_j)$$

i.e. there is a minimum of information in the residual $(w - \tilde{w})$ which can be due to the significant signal, and a minimum of information which could be due to the noise in the solution $\tilde{w}_j$.

Deconvolution

Formation of the image is expressed by the convolution integral

$$\begin{aligned} I(x,y) &= \int_{x_1=-\infty}^{+\infty} \int_{y_1=-\infty}^{+\infty} P(x-x_1, y-y_1) O(x_1,y_1) dx_1 dy_1 + N(x,y) \\ &= (O * P)(x,y) + N(x,y) \end{aligned}$$

where I is the data, P the point spread function (PSF), and O is the object.
In Fourier space we have:

$$\hat{I}(u,v) = \hat{O}(u,v)\hat{P}(u,v) + \hat{N}(u,v)$$

We want to determine O knowing I and P.
The main difficulties are the existence of:

- a cut-off frequency of the point spread function,
- the noise.

This is in fact an **ill-posed problem**, and there is no unique solution.

Other topics related to deconvolution

- blind deconvolution –
  the PSF P is unknown.
- myopic deconvolution –
  the PSF P is partially known.
- superresolution –
  object spatial frequency information outside the spatial bandwith of the image formation system is recovered.
- image reconstruction –
  an image is formed from a series of projections (Computed Tomography, Positron Emission Tomography, ...), and the computation of the object from the projection image data embodies a computer synthesized PSF.

Regularization

To resolve these problems, some constraints must be added to the solution. The addition of such constraints is called regularization.

$$J(O) = \parallel I - P * O \parallel + \alpha C(O)$$

Several regularization methods exist:

- Tikhonov: Tikhonov's regularization consists of minimizing the term:

  $$\parallel I(x,y) - (P * O)(x,y) \parallel + \lambda \parallel H * O \parallel$$

  where H corresponds to a high-pass filter

- MEM: $C(O) = Entropy(O)$

- CLEAN: $Data = Stars +$ const. Background

- Markov Field: $O(x,y)$ is a function of its neighborhood.

Deconvolution Methods

In astronomy:

- without regularization: Lucy $O^{(n+1)} = O^{(n)}[\frac{I}{I^{(n)}} * P^*] = O^{(n)}[\frac{I^{(n)}+R^{(n)}}{I^{(n)}} * P^*]$ where $R^{(n)} = I - I^{(n)}$ and $I^{(n)} = P * O^{(n)}$
- regularized method: MEM and CLEAN.

In the signal processing domain:

- without regularization
  - Van Cittert: $O^{(n+1)} = O^{(n)} + \beta(I - P * O^{(n)})$
  - Fixed step gradient: $O^{(n+1)} = O^{(n)} + \beta P^* * (I - P * O^{(n)})$
- regularized method: Tikhonov and Markov field models.

Deconvolution + Wavelet

CLEAN $\rightarrow$ Multiresolution CLEAN

$$\left.\begin{array}{l}\text{Fixed step gradient}\\ \text{Lucy}\\ \text{Van Cittert}\end{array}\right\} \rightarrow \begin{array}{c}\text{Regularized by}\\ \text{the multiresolution}\\ \text{support}\end{array}$$

MEM $\rightarrow$ Multiscale Entropy Method

CLEAN method

CLEAN decomposes an image into a set of Diracs. We get

- a set

$$\delta_c = \{A_1\delta(x - x_1, y - y_1), \ldots, A_n\delta(x - x_n, y - y_n)\}$$

- a residual R.

The deconvolved image is:

$$O(x, y) = \delta_c * B(x, y) + R(x, y)$$

where B is the clean beam.

Multiresolution CLEAN

- $w_j^{(I)}(x,y)$ = wavelet coefficients of the image at the scale j
- $w_j^{(P)}(x,y)$ = wavelet coefficients of the PSF at the scale j

$$\delta_j = \{A_{j,1}\delta(x - x_{j,1}, y - y_{j,1}), \ldots, A_{j,n_j}\delta(x - x_{j,n_j}, y - y_{j,n_j})\}$$

$$\mathcal{W}_\delta = \{\delta_1, \delta_2, \ldots\}$$

$$\begin{aligned} w_j^{(O)}(x,y) &= \delta_j * w_j^{(B)}(x,y) + w_j^{(R)}(x,y) \\ &= \sum_k A_{j,k} w_j^{(B)}(x - x_{j,k}, y - y_{j,k}) + w_j^{(R)}(x,y) \end{aligned}$$

Iterative Reconstruction

$$\mid \hat{\tilde{O}}(u,v) - V_m(u,v) \mid \leq \Delta_m(u,v)$$

where $\Delta_m(u,v)$ is the error on the measure V_m.
we minimize by an iterative algorithm:

$$\parallel p(\mathcal{K}\mathcal{W}_\delta - V) \parallel^2$$

where $\mathcal{K}$:

- convolves δ_j at each scale j by $w_j^{(B)}$
- reconstructs the object at full resolution
- computes the Fourier transform

and p is a weight function which depends on the quality of the measurements (error bars).

Regularization using the Multiresolution Support

$$R^{(n)} = I - I^{(n)} = c_p + \sum_j w_j \quad \text{and} \quad I^{(n)} = P * O^{(n)}$$

- Van Cittert: $O^{(n+1)} = O^{(n)} + R^{(n)}$
- Fixed Step Gradient: $O^{(n+1)} = O^{(n)} + P^* * R^{(n)}$
- Lucy: $O^{(n+1)} = O^{(n)}[\frac{I}{I^{(n)}} * P^*] = O^{(n)}[\frac{I^{(n)}+R^{(n)}}{I^{(n)}} * P^*]$

The regularization is done by removing the non-significant wavelet coefficients:

$$\bar{R}^{(n)}(x,y) = c_p(x,y) + \sum_j M(j,x,y) w_j(x,y)$$

Bayesian methodology

The Bayesian approach consists of constructing the conditional probability density relationship:

$$p(O \mid I) = \frac{p(I \mid O)p(O)}{p(I)}$$

The Bayes solution is found by maximizing the right hand part of the equation. The maximum likelihood solution (ML) maximizes only the density $p(I \mid O)$ over O:

$$ML(O) = \max_O p(I \mid O)$$

The maximum-a-posteriori solution (MAP) maximizes over O the product $p(I \mid O)p(O)$ of the ML and a prior:

$$MAP(O) = \max_O p(I \mid O)p(O)$$

$p(I)$ is considered to be a constant value which has no effect on the maximization process, and is neglected. The ML solution is equivalent to the MAP solution assuming a uniform density probability for $p(O)$.

Maximum Likehood with Gaussian Noise

The probability $p(I \mid O)$ is

$$p(I \mid O) = \frac{1}{\sqrt{2}\pi\sigma_n} \exp - \frac{(I - P * O)^2}{2\sigma_N^2}$$

and maximizing $p(O \mid I)$ is equivalent to minimizing

$$J(O) = \frac{\| I - P * O \|}{2\sigma_n^2}$$

Using the steepest descent minimization method, a typical iteration is

$$O^{n+1} = O^n + \gamma(I - P * O^n)$$

The solution can also be found directly using the FFT by

$$\hat{O}(u,v) = \frac{\hat{P}^*(u,v)\hat{I}(u,v)}{\hat{P}^*(u,v)\hat{P}(u,v)}$$

Image Compression

Why data compression?

- Transfer of data between satellites and ground-based stations.
- Images archiving.
- Fast access to large pictorial databases.
- Web-based data transmission.

Following the kind of images and the application needs, different strategies can be used:

1. Lossy compression: in this case the compression ratio is relatively low (< 5).
2. Compression without visual loss. This means that one cannot see the difference between the original image and the decompressed one. Generally, compression ratios between 10 and 20 can be obtained.
3. Good quality compression: the decompressed image does not contain any artifact, but some information is lost. Compression ratios up to 40 can be obtained in this case.
4. Fixed compression ratio: for some technical reasons, one may decide to compress all images with a compression ratio higher than a given value, whatever the effect on the decompressed image quality.
5. Signal/noise separation: if noise is present in the data, noise modeling can allow very high compression ratios just by including filtering in wavelet space during the compression.

Following the image type, and the selected strategy, the optimal compression method may differ. The main interest in using a multiresolution framework is to get, naturally, the possibility for progressive information transfer.

Redundancy

Compression methods try to use the redundancy contained in the raw data in order to reduce the number of bits. The main efficient methods belong to the transform coding family, where the image is first transformed into another set of data where the information is more compact (i.e. the entropy of the new set is lower than the original image entropy).

Compression steps

The typical steps are:

1. transform the image (for example using a discrete cosine transform, or a wavelet transform),
2. quantize the obtained coefficients, and
3. code the values by a Huffman or an arithmentic coder.

The first and third points are reversible, while the second is not. The distortion will depend on the way the coefficients are quantized. We may want to minimize the distortion with the minimum of bits, and a trade-off will then be necessary in order to also have “acceptable” quality. “Acceptable” is evidently subjective, and will depend on the application. Sometimes, any loss is unacceptable, and the price to pay is a very low compression ratio (often between one and two).

According to Shannon's theorem, the number of bits we need to code an image I without distorsion is given by its entropy H. If the image (with N pixels) is coded with L intensity levels, each level having a probability p_i to appear, the entropy H is

$$H(I) = \sum_{i=1}^{L} -p_i \log_2 p_i$$

The probabilities p_i can be easily derived from the image histogram. The compression ratio is given by:

$$\mathcal{C}(I) = \frac{\text{number of bits per pixel in the raw data}}{H(I)}$$

A Huffman, or an arithmetic, coder is generally used to transform the set of interger values into the new set of values, in a reversible way.

For lossy compression methods, the distortion is measured by

$$R = \| I - \tilde{I} \| = \sum_{k=1}^{N} (I_k - \tilde{I}_k)^2$$

where $\tilde{I}$ is the decompressed image.
The SNR is:

$$SNR_{dB} = 10 \log_{10} \frac{\sum_{k=1}^{N} I_k^2}{R}$$

and the Peak SNR (PSNR) is

$$PSNR_{dB} = 10 \log_{10} \frac{255^2}{\frac{R}{N}}$$

Quality criteria for lossy methods in Astronomy

1. Visual aspect
2. Signal to noise ratio
3. Photometry
4. Astrometry
5. Detection of real and faint objects
6. Object morphology

Methods in Astronomy

Standard methods in astronomy

- JPEG: is the standard video compression software for single frame images. It decorrelates pixel coefficients within 8 by 8 pixel blocks using the Discrete Cosine Transform and uniform quantization.
- FITSPRESS (Press, 1992): is based on a wavelet transform, using Daubechies-4 filters. It was developed at the Center for Astrophysics, Harvard.
- HCOMPRESS (White et al, 1992): was developed at Space Telescope Science Institute (STScI), Baltimore, and is commonly used to distribute archive images from the Digital Sky Survey. It is based on the Haar wavelet transform.

Other methods in astronomy

- Mathematical Morphology + Quadtree (Huang and Bijaoui, 1991)
- Pyramidal Median Transform (Starck et al, 1996)
- Iterated Hcompress method (Bijaoui et al, 1996)

Wavelet Transform and Compression

- Choice of the filters
  Antonini 7/9 filters are the most often used, with an L^2 normalization.
- Quantization
- Coding

Quantization + Coding Algorithm

- Uniform quantization + Huffman encoding
- Vector quantization
- Uniform quantization + Quadtree + Huffman encoding
- Embedded Zerotree Wavelet

Large Images

With new technology developments, images furnished by detectors are larger and larger. For example, current astronomical projects plan to deal with images of sizes larger than 8000 by 8000 pixels (VLT: 8k $\times$ 8k, MEGACAM 16k $\times$ 16k, ...). Analysis of such images is obviously not easy, but the main problem is clearly archiving and network access to the data by users.

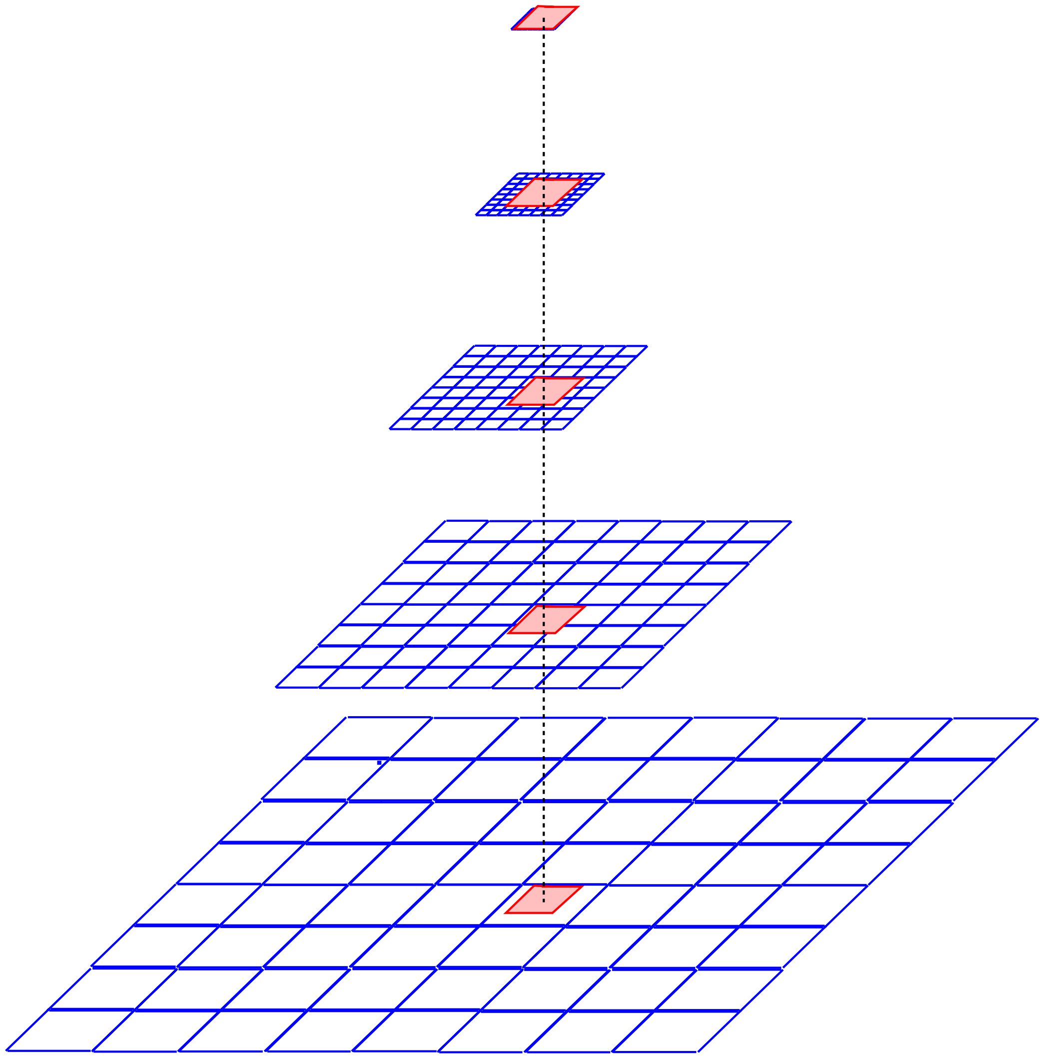
In order to visualize an image in a reasonable amount of time, transmission is based on two concepts:

- data compression
- progressive decompression

Large Image Visualization Environment: LIVE

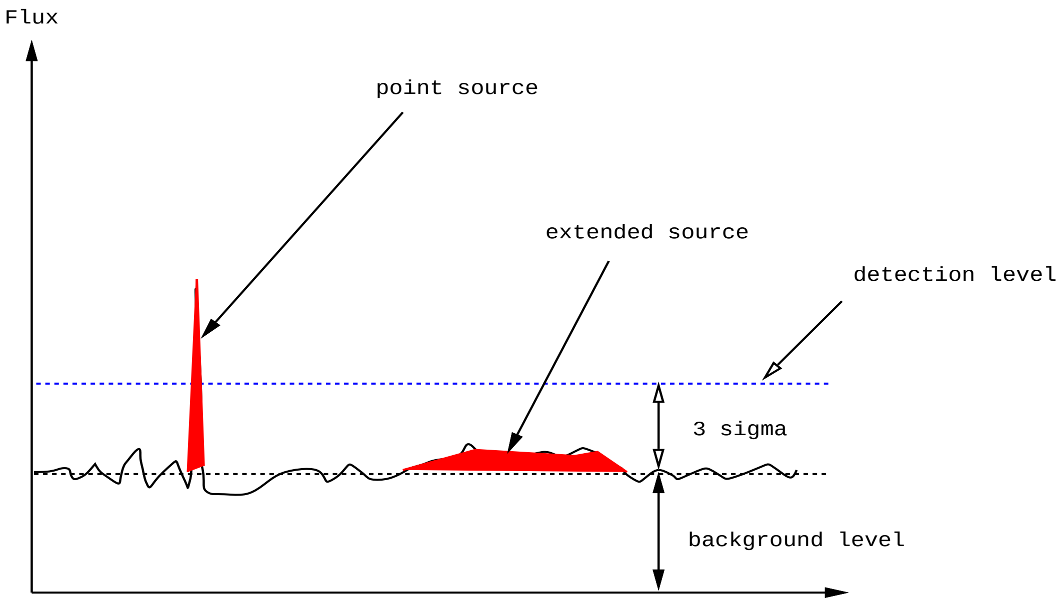
A third concept is necessary, which is the region of interest. Indeed, images become so large it is impossible to display them in a normal window (typically of size 512x512), and we need to have to ability to focus on a given area of the image at a given resolution. To move from one area to another, or to increase the resolution of a part of the area is a user task, and is a new active element of the decompression. The goal of LIVE is to furnish this third concept, by adding the following functionality:

- Full image display at a very low resolution.
- Image navigation: the user can go up (the quality of an area of the image is improved) or down (return to the previous image). Then the new image represents only a quarter of the previous one.



Large image, compressed by block, and represented at five resolution levels. At each resolution level, the visualization window is superimposed at a given position.

Standard source detection



- Background estimation
- Noise etimation
- Detection where flux $>$ Background + k * Noise

Faint extended objects, easely detectable in the wavelet space, are lost.

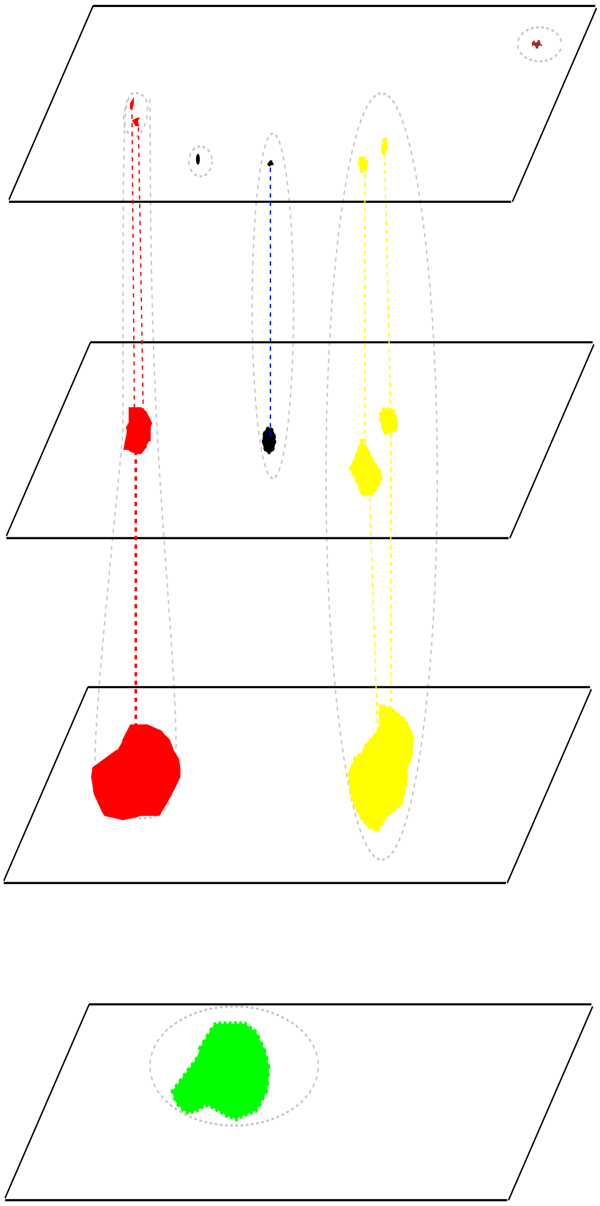
Object extraction

Definition

Structure: a structure $\mathcal{S}_j$ is a set of significant connected wavelet coefficients at the same scale j.
object: an object is a set of structures.
object scale: the scale of an object is given by the scale of the maximum of its wavelet coefficients.
interscale-relation: two structures $\mathcal{S}_j^1, \mathcal{S}_{j+1}^2$ at two successive scales j, $j+1$ are connected if the wavelet coefficients $w_{j+1}(x_m, y_m)$ belongs to the structure $\mathcal{S}_{j+1}^2$, (x_m, y_m) being the position of the maximum wavelet coefficient of the structure $\mathcal{S}_j^1$.

$$\mathcal{S}_j^1 \leftrightarrow \mathcal{S}_{j+1}^2 \text{ if } w_j(x_m, y_m) = max(\mathcal{S}_j^1) \text{ and } w_{j+1}(x_m, y_m) \in \mathcal{S}_{j+1}^2$$

Object extraction



Reconstruction

The problem of reconstruction (Bijaoui and Rué, 1995) consists of searching for a signal O such that its wavelet coefficients are the same than those of the detected structures. If $\mathcal{T}$ describes the wavelet transform operator, and P_w the projection operator in the subspace of the detected coefficients (i.e. set to zero all coefficients at scales and positions where nothing where detected), the solution is found by minimization of

$$J(O) = \parallel W - (P_w \circ \mathcal{T})O \parallel$$

where W represents the detected wavelet coefficients of the data.

The MVM presents many advantages compared to the standard approach

- Faint extended objects can be detected as well as point sources.
- The analysis does not require background estimation.

  If the background varies spatially, its estimation becomes a non-trivial task, and may produce large errors in the object photometry.
- The Point Spread Function (PSF) is not needed.

  It is an advantage when the PSF is unknown, or difficult to estimate, which happens relatively often when it is space-variant.

However, when the PSF is well-determined, it becomes a drawback because known information is not used for the object reconstruction. This leads to systematic errors in the photometry, which depend on PSF and on the source signal to noise ratio.

Image Registration

Image registration is a procedure that determines the best spatial fit between two or more images that overlap the same scene, acquired at the same or at a different time, by identical or different sensors. The geometrical correction is usually performed by three operations

- The measure of a set of well-defined ground control points (GCPs), which are features well located both in the input image and in the reference image.
- The determination of the warping or deformation model, by specifying a mathematical deformation model defining the relation between the coordinates (x, y) and (X, Y) in the reference and input image respectively.
- The construction of the corrected image by output-to-input mapping.

Deformation Model

Geometric correction requires a spatial transformation to invert an unknown distortion function. A general model for characterizing misregistration between two sets of remotely sensed data is a pair of bivariate polynomials of the form:

$$x_i = \sum_{p=0}^{N} \sum_{q=0}^{N-p} a_{pq} X_i^p Y_i^q = Q(X_i, Y_j)$$

$$y_i = \sum_{p=0}^{N} \sum_{q=0}^{N-p} b_{pq} X_i^p Y_i^q = R(X_i, Y_j)$$

where (X_i, Y_i) are the coordinates of the i^{th} GCP in the reference image, (x_i, y_i) the corresponding GCP in the input image and N is the degree of the polynomial.

Some common deformations

Shift	$x = a_0 + X$ $y = b_0 + Y$
Scale	$x = a_1 X$ $y = b_2 Y$
Skew	$x = X + a_2 Y$ $y = Y$
Perspective	$x = a_3 XY$ $y = Y$
Rotation	$x = \cos\theta X + \sin\theta Y$ $x = -\sin\theta X + \cos\theta Y$

Problems

The main difficulty lies in the automated localization of the corresponding GCPs, since the accuracy of their determination will affect the overall quality of the registration. In fact, there are always ambiguities in matching two sets of points, as a given point corresponds to a small region D, which takes into account the prior geometric uncertainty between the two images and many objects could be contained in this region.

One property of the wavelet transform is having a sampling step proportional to the scale. When we compare the images in the wavelet transform space, we can choose a scale corresponding to the size of the region D, so that no more than one object can be detected in this area, and the matching is done automatically.