Quantum Approximate Optimization Algorithm (QAOA)

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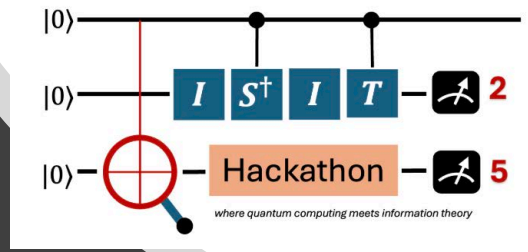


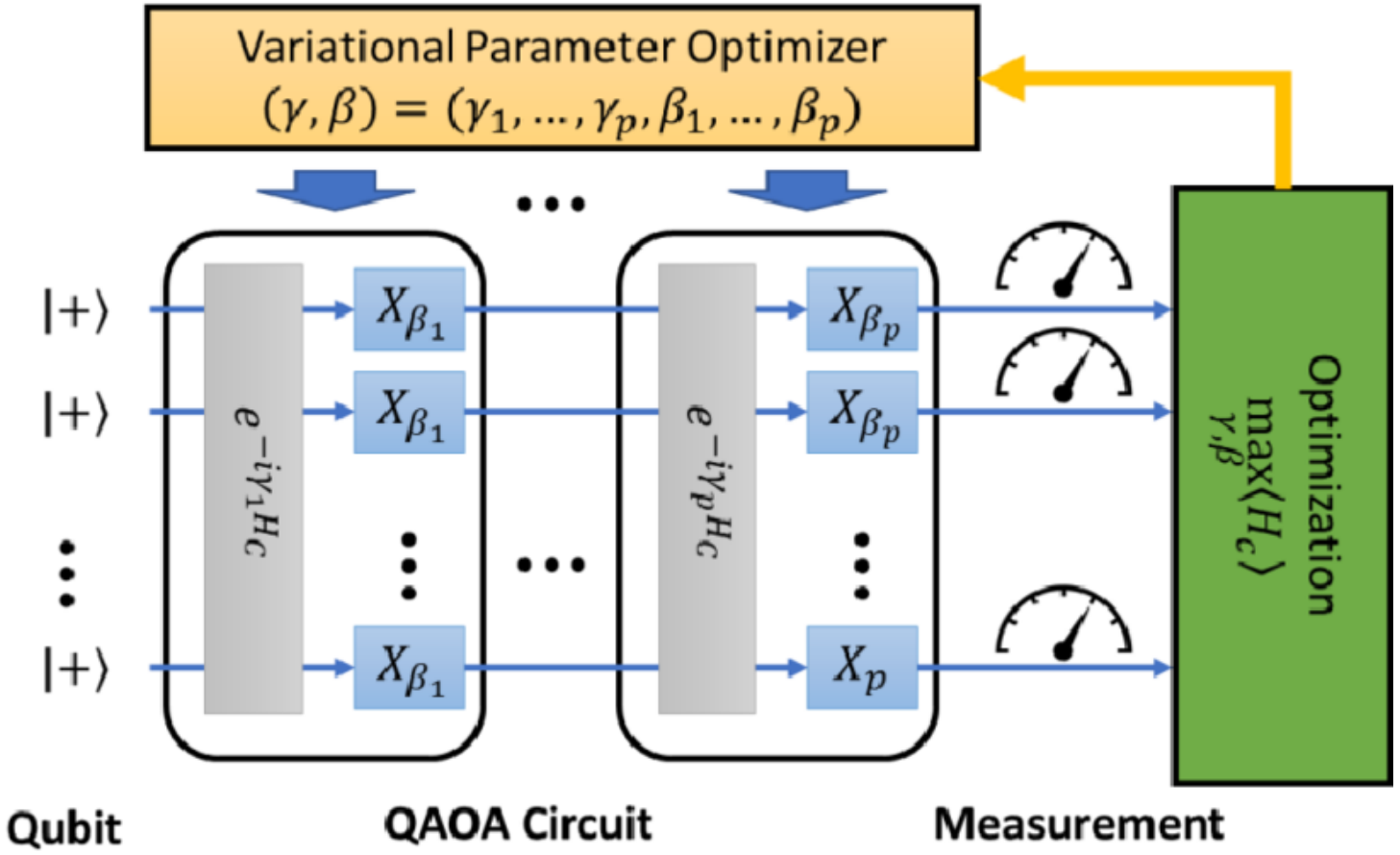
Slides Courtesy: Ruslan Shaydulin

Noisy, error-prone, small devices

What can we do now?

Variational Models

QAOA outline



Algorithms with proven performance

Classical

- Matrix multiplication
- Dijkstra's algorithm for shortest path

Quantum

- Shor's algorithm for integer factoring
- Grover's algorithm for unstructured search

Heuristic methods

Classical

- Gradient descent (for non-convex problems)
- Simulated annealing
- Genetic algorithm

Quantum

- Quantum annealing
- QAOA
- VQE
- More to be discovered...

PART 1: MAPPING COMBINATORIAL OPTIMIZATION PROBLEMS ONTO QUANTUM COMPUTERS

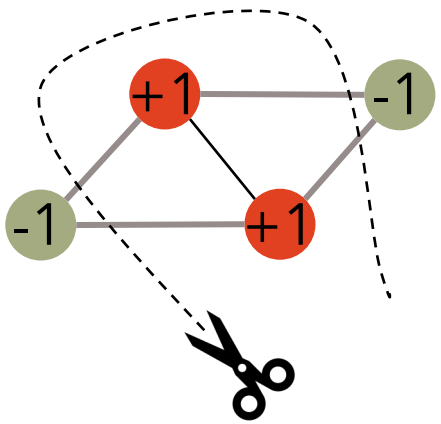
Outline

1. Maximum cut as a paradigmatic example

2. (Optional) General rules for constructing Hamiltonians representing Boolean Functions

Maximum Cut Problem (MAXCUT)

- The goal of maximum cut is to split the set of vertices V of a graph into two disjoint parts such that the number of edges spanning two parts is maximized.
- For example, if color denotes part, in the graph on the right 4 edges are cut:



- Maximum cut can be formulated as an optimization problem:

$$\max_{\mathbf{s}} \frac{1}{2} \sum_{ij \in E} (1 - s_i s_j) \qquad s_i \in \{-1, +1\}$$

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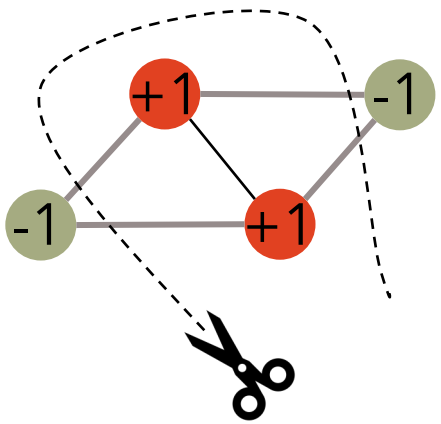

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$$\max_{\mathbf{s}} \frac{1}{2} \sum_{ij \in E} (1 - s_i s_j) \qquad s_i \in \{-1, +1\}$$

Same sign – no edge is cut (no contribution to the objective): $\frac{1}{2}(1 - s_i s_j) = 0$

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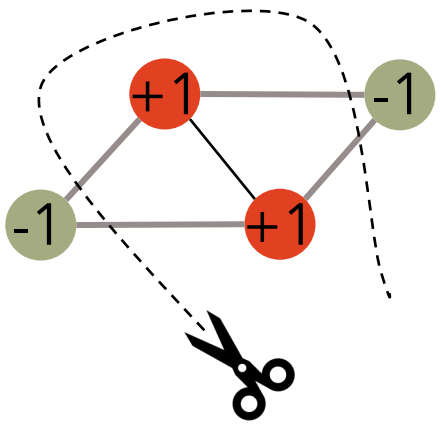
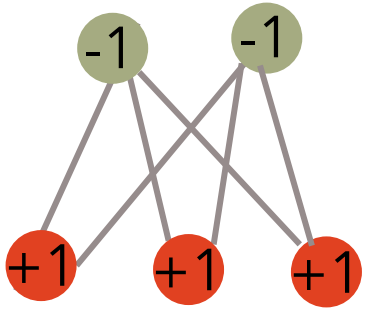


- Maximum cut can be formulated as an optimization problem:

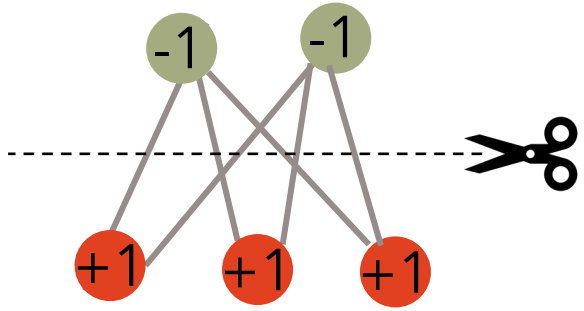
$$\max_{\mathbf{s}} \frac{1}{2} \sum_{ij \in E} (1 - s_i s_j) \qquad s_i \in \{-1, +1\}$$

Different sign – an edge is cut (contribution to the objective = 1): $\frac{1}{2}(1 - s_i s_j) = 1$

Q: What is the cut in the graph below?



Q: What is the cut in the graph below?



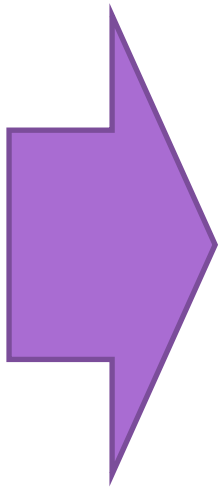
- A: six edges are cut

Solving Combinatorial Optimization Problems on a Quantum Computer

Classical

Maximizing objective

$$\max_{\mathbf{s}} C(\mathbf{s}) = \frac{1}{2} \sum_{ij \in E} (1 - s_i s_j)$$



Quantum

Characterizing a Hamiltonian (Hermitian operator) C

To solve an optimization problem on a quantum computer, we need to convert it into a problem of characterizing a quantum Hamiltonian

Solving Combinatorial Optimization Problems on a Quantum Computer

Classical		Quantum
Objective	$\max_{\mathbf{s}} C(\mathbf{s}) = \frac{1}{2} \sum_{ij \in E} (1 - s_i s_j)$	Hamiltonian (Hermitian operator) C
Solution	$\mathbf{s} \in \{-1, +1\}^n$	Highest energy eigenstate (largest eigenvalue eigenvector) $\lvert s \rangle$

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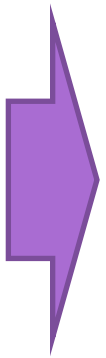
Objective $\max_{\mathbf{s}} C(\mathbf{s}) = \frac{1}{2}\sum_{ij\in E}(1 - s_i s_j)$	Hamiltonian (Hermitian operator) C
Solution $\mathbf{s} \in \{-1, +1\}^n$	Highest energy eigenstate (largest eigenvalue eigenvector) $\lvert s\rangle$

Since the Hamiltonian is classical, this eigenstate is a computational basis state*, we can measure it and **get the solution with certainty**

What does this Hamiltonian look like?

- Hamiltonian is diagonal, with values on the diagonal corresponding to the values of the objective function

$$\max_{x \in \{0,1\}^n} f(x)$$



$$C = \begin{pmatrix} f(0\ldots00) & 0 & 0 & 0 & 0 \\ 0 & f(0\ldots01) & 0 & 0 & 0 \\ \vdots & & \ddots & & \\ 0 & 0 & 0 & f(1\ldots10) & 0 \\ 0 & 0 & 0 & 0 & f(1\ldots11) \end{pmatrix}$$

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What does this Hamiltonian look like?

- Hamiltonian is diagonal, with values on the diagonal corresponding to the values of the objective function
- In general:

$$C\left|x\right\rangle = \begin{pmatrix} f(0\ldots00) & 0 & 0 & 0 & 0 \\ 0 & f(0\ldots01) & 0 & 0 & 0 \\ \vdots & & \ddots & & \\ 0 & 0 & 0 & f(1\ldots10) & 0 \\ 0 & 0 & 0 & 0 & f(1\ldots11) \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = f(x)\left|x\right\rangle \qquad \forall x \in \{0,1\}^n$$

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- This Hamiltonian is too large to construct explicitly!

Solving Combinatorial Optimization Problems on a Quantum Computer

	Classical	Quantum
Objective	$\max_{\mathbf{s}} C(\mathbf{s}) = \frac{1}{2} \sum_{ij \in E} (1 - s_i s_j)$	Hamiltonian (Hermitian operator) C
Evaluation	c	state s
Solution	$\mathbf{s} \in \{-1, +1\}^n$	Highest energy eigenstate $\lvert s \rangle$

How do we construct this Hamiltonian?

Notation Reminder

$$|x\rangle = \mathbf{x} = \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{column vector} \qquad \langle y| = |y\rangle^\dagger = \begin{bmatrix} y_1^* & y_2^* & \cdots & y_n^* \end{bmatrix} \quad \text{conjugate transpose}$$

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$$\langle y|x\rangle = y_1^* x_1 + \cdots + y_n^* x_n = \begin{bmatrix} y_1^* & y_2^* & \cdots & y_n^* \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{inner product}$$

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$$|x\rangle\langle y| = \begin{bmatrix} x_1 y_1^* & \cdots & x_1 y_n^* \\ \vdots & \ddots & \vdots \\ x_n y_1^* & \cdots & x_n y_n^* \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \begin{bmatrix} y_1^* & y_2^* & \cdots & y_n^* \end{bmatrix} \quad \text{outer product}$$

Constructing MAXCUT Hamiltonian

- MAXCUT objective:

$$\max_{\mathbf{s}} \frac{1}{2} \sum_{ij \in E} (1 - s_i s_j) \qquad s_i \in \{-1, +1\}$$

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- MAXCUT Hamiltonian is constructed by **mapping binary variables s_i onto the eigenvalues of Z**

$$C = \frac{1}{2} \sum_{ij \in E} (I - Z_i Z_j)$$

- Want to show:

$$C\,|x\rangle = C(\mathbf{x})\,|x\rangle$$

Pauli Z operator

- Consider Pauli Z operator

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

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- Note that it has eigenvalues -1, +1 with eigenvectors being computational basis states

$$Z|0\rangle = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle$$

$$Z|1\rangle = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = (-1)|1\rangle$$

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- Therefore,

$$Z|x\rangle = (-1)^x |x\rangle \quad x \in \{0, 1\}$$

Pauli Z operator

- Acting on the i-th qubit:

$$
\begin{aligned}
Z_i \left|x_0 \ldots x_n\right\rangle &= I \otimes \ldots \otimes Z_i \otimes \ldots \otimes I \left|x_0 \ldots x_n\right\rangle \\
&= (-1)^{x_i} \left|x_0 \ldots x_n\right\rangle \qquad x_i \in \{0, 1\}, \quad i = 1, \ldots n
\end{aligned}
$$

Pauli Z operator

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$$= (-1)^{x_i} \left|x_0 \dots x_n\right\rangle \qquad x_i \in \{0, 1\}, \quad i = 1, \dots n$$

- Acting on the i-th and j-th qubit:

$$Z_i Z_j \left|x_0 \dots x_n\right\rangle = I \otimes \dots \otimes Z_i \otimes Z_j \otimes \dots \otimes I \left|x_0 \dots x_n\right\rangle$$
$$= (-1)^{x_i} (-1)^{x_j} \left|x_0 \dots x_n\right\rangle \qquad x_i \in \{0, 1\}, \quad i = 1, \dots n$$

- (note that here we reorder qubits such that i-th and j-th qubit are adjacent)

Constructing MAXCUT Hamiltonian

- MAXCUT objective:

$$\max_{\mathbf{s}} \frac{1}{2} \sum_{ij \in E} (1 - s_i s_j) \qquad s_i \in \{-1, +1\}$$

- MAXCUT Hamiltonian is constructed by **mapping binary variables s_i onto the eigenvalues of Z**

$$C = \frac{1}{2} \sum_{ij \in E} (I - Z_i Z_j)$$

Verifying MAXCUT Hamiltonian

- MAXCUT objective:

$$\max_{\mathbf{s}} \frac{1}{2} \sum_{ij \in E} (1 - s_i s_j) \qquad s_i \in \{-1, +1\}$$

- Let's reformulate it in the following way:

$$x_i = \frac{1}{2}(1 - s_i)$$

$$s_i = 1 \rightarrow x_i = 0, \qquad (-1)^{x_i} = 1 = s_i$$

$$s_i = -1 \rightarrow x_i = 1, \qquad (-1)^{x_i} = -1 = s_i$$

Verifying MAXCUT Hamiltonian

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- New objective:

$$\max_{\mathbf{x}} \frac{1}{2} \sum_{ij \in E} (1 - (-1)^{x_i}(-1)^{x_j}) \qquad x_i \in \{0, 1\}$$

Verifying MAXCUT Hamiltonian

- MAXCUT objective:

$$C(\mathbf{x}) = \frac{1}{2} \sum_{ij \in E} (1 - (-1)^{x_i} (-1)^{x_j}) \qquad x_i \in \{0, 1\}$$

- MAXCUT Hamiltonian:

$$C = \frac{1}{2} \sum_{ij \in E} (I - Z_i Z_j)$$

- Want to show:

$$C \left| x \right\rangle = C(\mathbf{x}) \left| x \right\rangle$$

Verifying MAXCUT Hamiltonian

$$
\begin{aligned}
C\left|x\right\rangle &= \frac{1}{2}\sum_{ij\in E}\left(I - Z_i Z_j\right)\left|x_0 \ldots x_n\right\rangle \\
&= \frac{1}{2}\sum_{ij\in E}\left(I - Z_i Z_j\right)\left|x_0 \ldots x_n\right\rangle \\
&= \frac{1}{2}\sum_{ij\in E}\left(\left|x_0 \ldots x_n\right\rangle - Z_i Z_j\left|x_0 \ldots x_n\right\rangle\right) \\
&= \frac{1}{2}\sum_{ij\in E}\left(1 - (-1)^{x_i}(-1)^{x_j}\right)\right)\left|x_0 \ldots x_n\right\rangle = C(\mathbf{x})\left|x\right\rangle \\
x_i &\in \{0,1\}, \quad i = 1, \ldots n
\end{aligned}
$$

Constructing MAXCUT Hamiltonian

- MAXCUT objective:

$$\max_{\mathbf{s}} \frac{1}{2} \sum_{ij \in E} (1 - s_i s_j) \qquad s_i \in \{-1, +1\}$$

- MAXCUT Hamiltonian is constructed by **mapping binary variables s_i onto the eigenvalues of Z**

$$C = \frac{1}{2} \sum_{ij \in E} (I - Z_i Z_j)$$

- Note that the same procedure would work for any (unconstrained) binary objective!

Q: what is the size of Hamiltonian C?

$$C = \frac{1}{2} \sum_{ij \in E} (I - Z_i Z_j)$$

- Recall that: $Z_i Z_j = I \otimes \ldots \otimes Z_i \otimes Z_j \otimes \ldots \otimes I$

- Therefore each term in the sum is a 2^nx2^n matrix if written explicitly!
- Luckily, we do not have to write them out explicitly

Constructing a Hamiltonian for a general problem

- Recall that we want to construct a Hamiltonian *C* such that:

$$C\left|x\right\rangle = f(x)\left|x\right\rangle \qquad \forall x \in \{0,1\}^n$$

- How can we do this efficiently for an arbitrary function *f*?
- In other words, how do we map Boolean and real functions to diagonal Hamiltonians acting on qubits?

This method is due to Hadfield, Stuart. "On the representation of Boolean and real functions as Hamiltonians for quantum computing." arXiv preprint arXiv:1804.09130 (2018).

Constructing a Hamiltonian for a general problem: a *simpler* recipe

- Start with the table of building blocks:

$f(x)$	H_f	$f(x)$	H_f
x	$\frac{1}{2}I - \frac{1}{2}Z$	$\bar{x}$	$\frac{1}{2}I + \frac{1}{2}Z$
$x_1 \oplus x_2$	$\frac{1}{2}I - \frac{1}{2}Z_1Z_2$	$\bigoplus_{j=1}^{k} x_j$	$\frac{1}{2}I - \frac{1}{2}Z_1Z_2 \dots Z_k$
$x_1 \wedge x_2$	$\frac{1}{4}I - \frac{1}{4}(Z_1 + Z_2 - Z_1Z_2)$	$\Lambda_{j=1}^{k} x_j$	$\frac{1}{2^k}\prod_j (I - Z_j)$
$x_1 \vee x_2$	$\frac{3}{4}I - \frac{1}{4}(Z_1 + Z_2 + Z_1Z_2)$	$V_{j=1}^{k} x_j$	$I - \frac{1}{2^k}\prod_j (I + Z_j)$
$\overline{x_1 x_2}$	$\frac{3}{4}I + \frac{1}{4}(Z_1 + Z_2 - Z_1Z_2)$	$x_1 \Rightarrow x_2$	$\frac{3}{4}I + \frac{1}{4}(Z_1 - Z_2 + Z_1Z_2)$

- Combine them as follows:

$$H_{\neg f} = H_{\bar{f}} = I - H_f \qquad H_{f \Rightarrow g} = I - H_f + H_f H_g$$
$$H_{f \wedge g} = H_{fg} = H_f H_g \qquad H_{f \vee g} = H_f + H_g - H_f H_g$$
$$H_{f \oplus g} = H_f + H_g - 2H_f H_g \qquad H_{af+bg} = aH_f + bH_g \quad a, b \in \mathbb{R}.$$

Hadfield, Stuart. "On the representation of Boolean and real functions as Hamiltonians for quantum computing." arXiv preprint arXiv:1804.09130 (2018).

PART 2:
QUANTUM APPROXIMATE OPTIMIZATION ALGORITHM (QAOA)

Quantum Approximate Optimization Algorithm (QAOA)

- QAOA prepares a parameterized "trial" (ansatz) state of the form:

$$
\begin{aligned}
|\psi(\boldsymbol{\theta})\rangle &= |\psi(\boldsymbol{\beta}, \boldsymbol{\gamma})\rangle \\
&= e^{-i\beta_p B} e^{-i\gamma_p C} \cdots e^{-i\beta_1 B} e^{-i\gamma_1 C} H^{\otimes n} |0\rangle .
\end{aligned}
$$

- Here *C* is the problem Hamiltonian, e.g. for MAXCUT: $C = \frac{1}{2} \sum_{ij \in E} (I - Z_i Z_j)$
- *B* is the mixer Hamiltonian: $B = \sum_i X_i$

Quantum Approximate Optimization Algorithm (QAOA)

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\end{aligned}
$$

- Then a classical optimizer is used to vary the parameters $\boldsymbol{\beta}, \boldsymbol{\gamma}$ to maximize:

$$
f(\boldsymbol{\beta}, \boldsymbol{\gamma}) = \langle \psi(\boldsymbol{\beta}, \boldsymbol{\gamma}) | \, C \, | \psi(\boldsymbol{\beta}, \boldsymbol{\gamma}) \rangle \,.
$$

Quantum Approximate Optimization Algorithm (QAOA)

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&= e^{-i\beta_p B} e^{-i\gamma_p C} \cdots e^{-i\beta_1 B} e^{-i\gamma_1 C} H^{\otimes n} |0\rangle .
\end{aligned}
$$

- Note that for p → ∞ QAOA can *at least* exactly approximate adiabatic quantum evolution and can therefore find the exact optimal solution
- For small p, picture is more mixed, but there is some indication of the potential for quantum advantage

Quantum Approximate Optimization Algorithm (QAOA)

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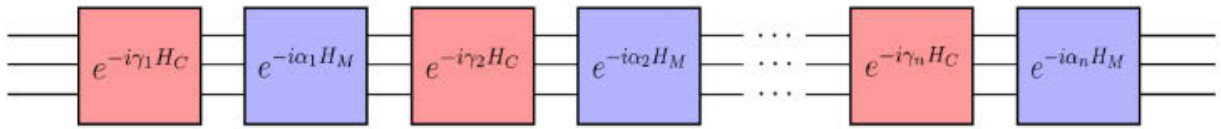
$$|\psi(\boldsymbol{\theta})\rangle = |\psi(\boldsymbol{\beta}, \boldsymbol{\gamma})\rangle$$
$$= e^{-i\beta_p B} e^{-i\gamma_p C} \cdots e^{-i\beta_1 B} e^{-i\gamma_1 C} H^{\otimes n} |0\rangle .$$



How do we implement this circuit in gates?

Implementing QAOA

- Let's assume the following gate set

$$X \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad Y \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; \quad Z \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}; \quad H \equiv \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix};$$

$$S \equiv \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}; \quad R_z(\theta) \equiv e^{-i\theta Z/2} = \cos\frac{\theta}{2} I - i \sin\frac{\theta}{2} Z = \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix}$$

Implementing QAOA

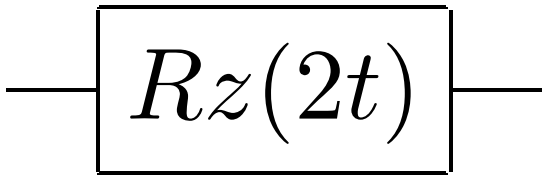
- Let's start with simple operator $e^{-iZt} = R_z(2t)$

$$Z \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad R_z(\theta) \equiv e^{-i\theta Z/2} = \cos\frac{\theta}{2} I - i \sin\frac{\theta}{2} Z = \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix}$$

Implementing QAOA

- Let's start with simple operator $e^{-iZt} = R_z(2t)$

- Circuit:



$$Z \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad R_z(\theta) \equiv e^{-i\theta Z/2} = \cos\frac{\theta}{2} I - i \sin\frac{\theta}{2} Z = \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix}$$

Implementing QAOA

- Slightly more complicated operator: $e^{-iZZt} = e^{-iZ \otimes Zt}$
- Remember that Z has eigenvectors $|0\rangle, |1\rangle$ with eigenvalues 1,-1 and $e^A |v\rangle = e^\lambda |v\rangle$ if $A|v\rangle = \lambda |v\rangle$
- Then:

$$
\begin{aligned}
e^{-iZ\otimes Zt}|00\rangle &= e^{-i(1\times 1)t}|00\rangle = e^{-it}|00\rangle \\
e^{-iZ\otimes Zt}|01\rangle &= e^{-i(1\times -1)t}|01\rangle = e^{it}|01\rangle \\
e^{-iZ\otimes Zt}|10\rangle &= e^{-i(-1\times 1)t}|10\rangle = e^{it}|10\rangle \\
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- Adds a phase factor with the sign depending on parity! In general:

$$e^{-iZ\otimes Zt}|ab\rangle = e^{-i(-1)^{a\oplus b}t}|ab\rangle$$

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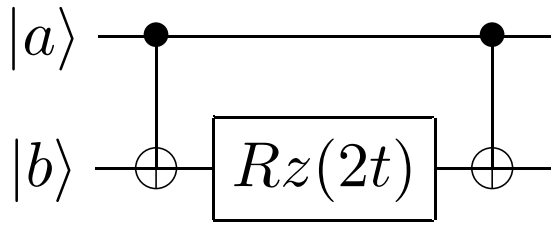

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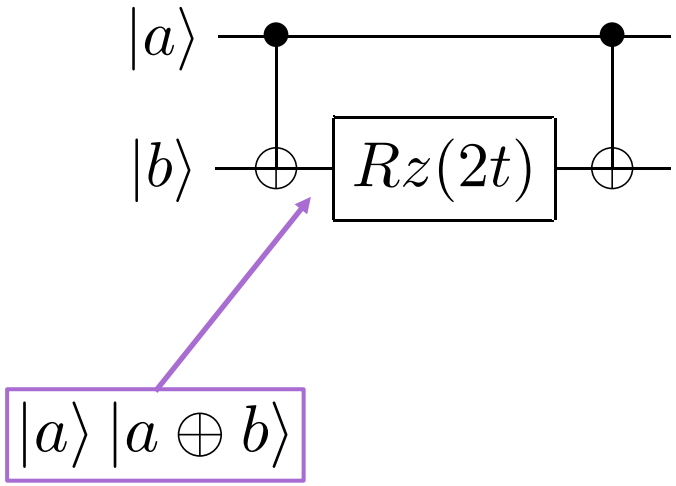

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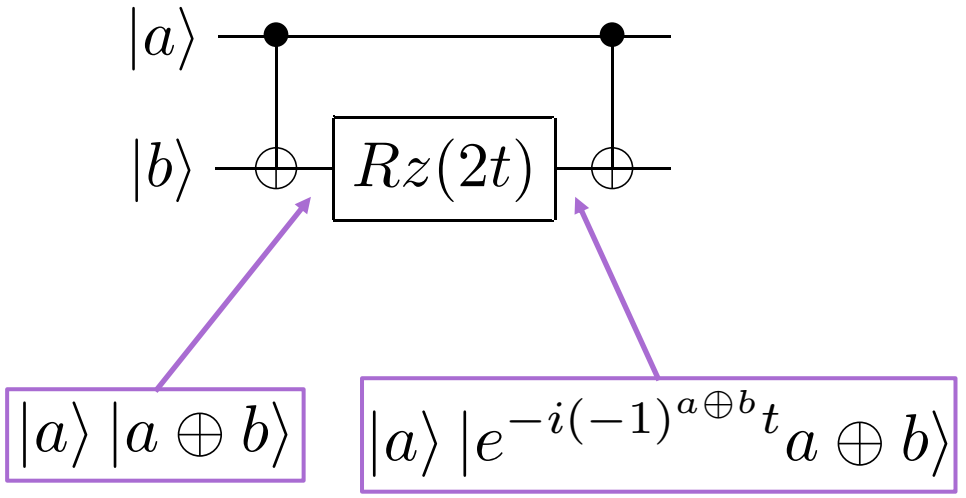

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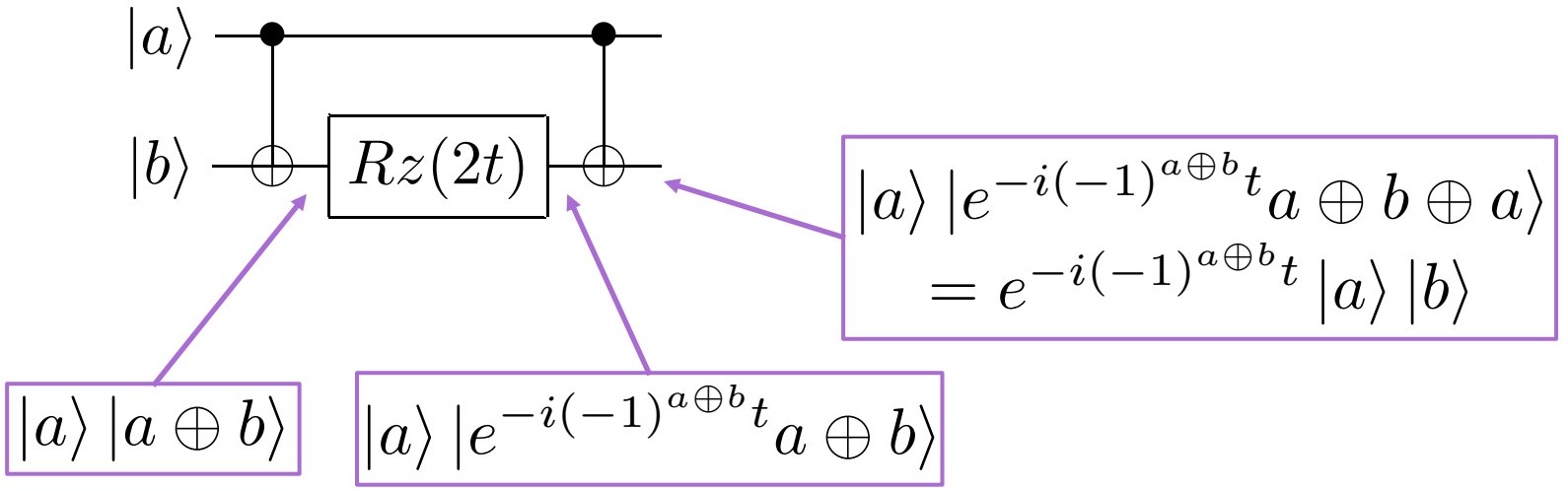

Implementing QAOA

- If terms of our Hamiltonian commute, we can just concatenate corresponding circuits:

$$e^{-iaZ_iZ_jt-ibZ_jZ_kt} = e^{-iaZ_iZ_jt}e^{-ibZ_jZ_kt}$$

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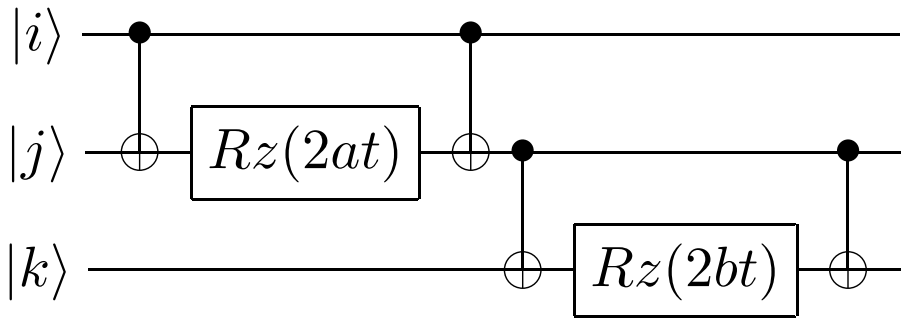

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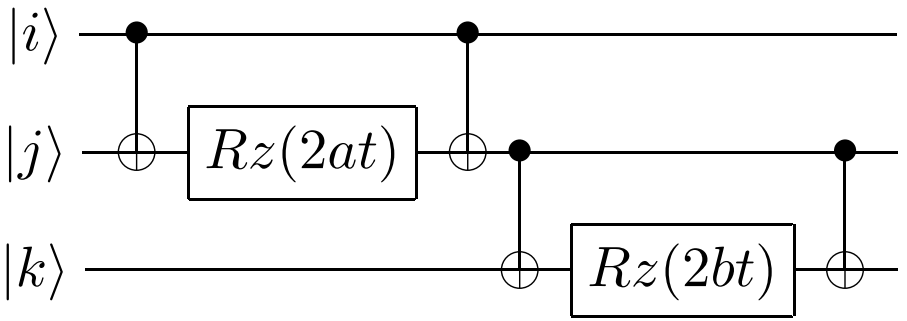

- Now we can simulate the operator corresponding to the problem Hamiltonian:

$$e^{-i\gamma C} = e^{-i\gamma\frac{1}{2}\sum_{ij\in E}(I-Z_iZ_j)}$$

Implementing QAOA

- QAOA prepares a parameterized "trial" (ansatz) state of the form:

$$
\begin{aligned}
|\psi(\boldsymbol{\theta})\rangle &= |\psi(\boldsymbol{\beta}, \boldsymbol{\gamma})\rangle \\
&= e^{-i\beta_p B} e^{-i\gamma_p C} \cdots e^{-i\beta_1 B} e^{-i\gamma_1 C} H^{\otimes n} |0\rangle .
\end{aligned}
$$

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- Mixer operator: $e^{-i\beta B} = e^{-i\beta \sum_j X_j}$
- One term: e^{-iXt}

- Note that

$$e^{-iZt} = \sum_{j=0}^{\infty} \frac{(-iZt)^j}{j!} = I - iZt - \frac{Z^2t^2}{2!} + \dots$$

$$\begin{aligned} He^{-iZt}H &= H\left(I - iZt - \tfrac{Z^2t^2}{2!} + \dots\right)H = I - iHZHt - \tfrac{HZHHZHt^2}{2!} + \dots \\ &= I - iXt - \tfrac{X^2t^2}{2!} + \dots = e^{-iXt} \end{aligned}$$

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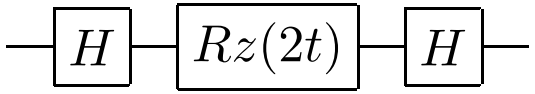
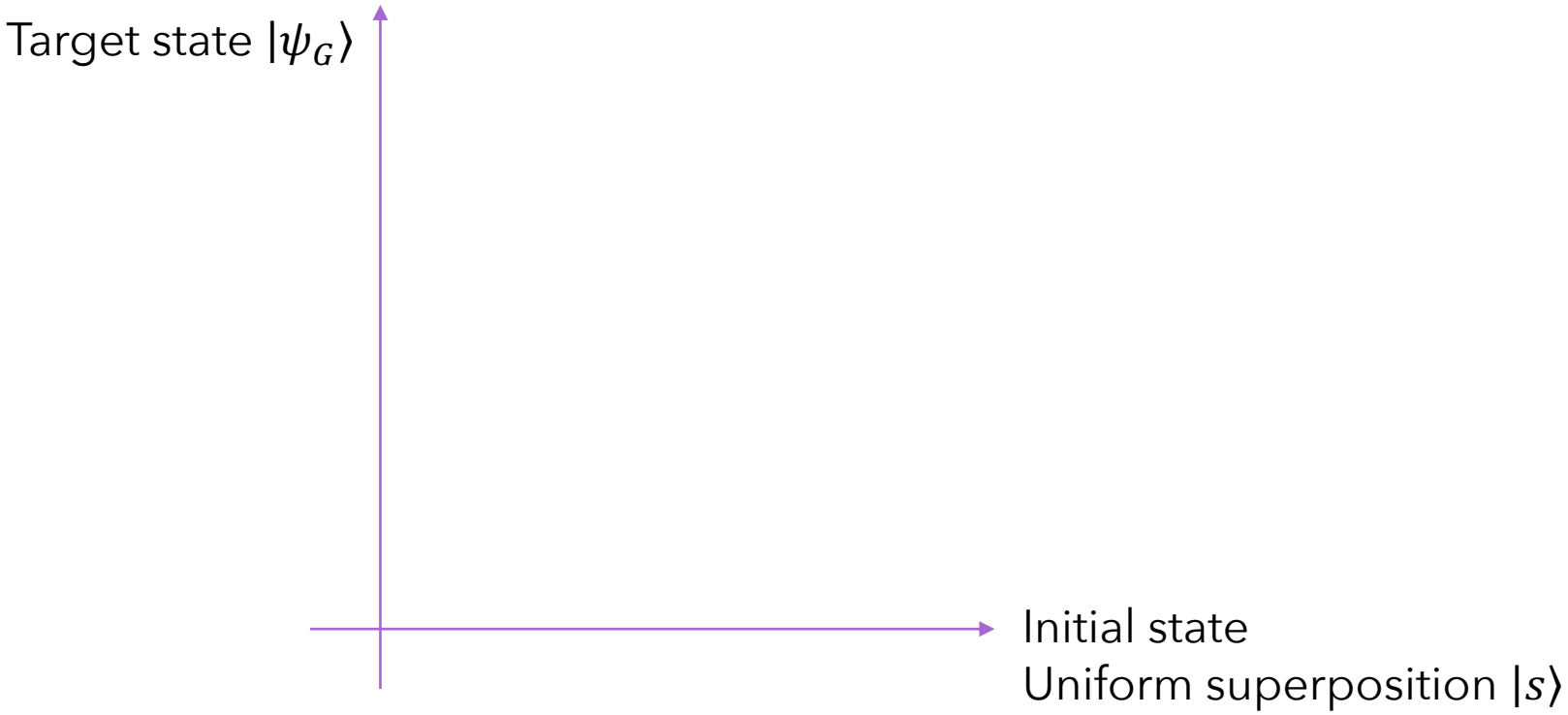
- Circuit: 

Quantum Approximate Optimization Algorithm (QAOA): Geometric Interpretation

- QAOA prepares a parameterized "trial" (ansatz) state $|\psi(\boldsymbol{\beta}, \boldsymbol{\gamma})\rangle$



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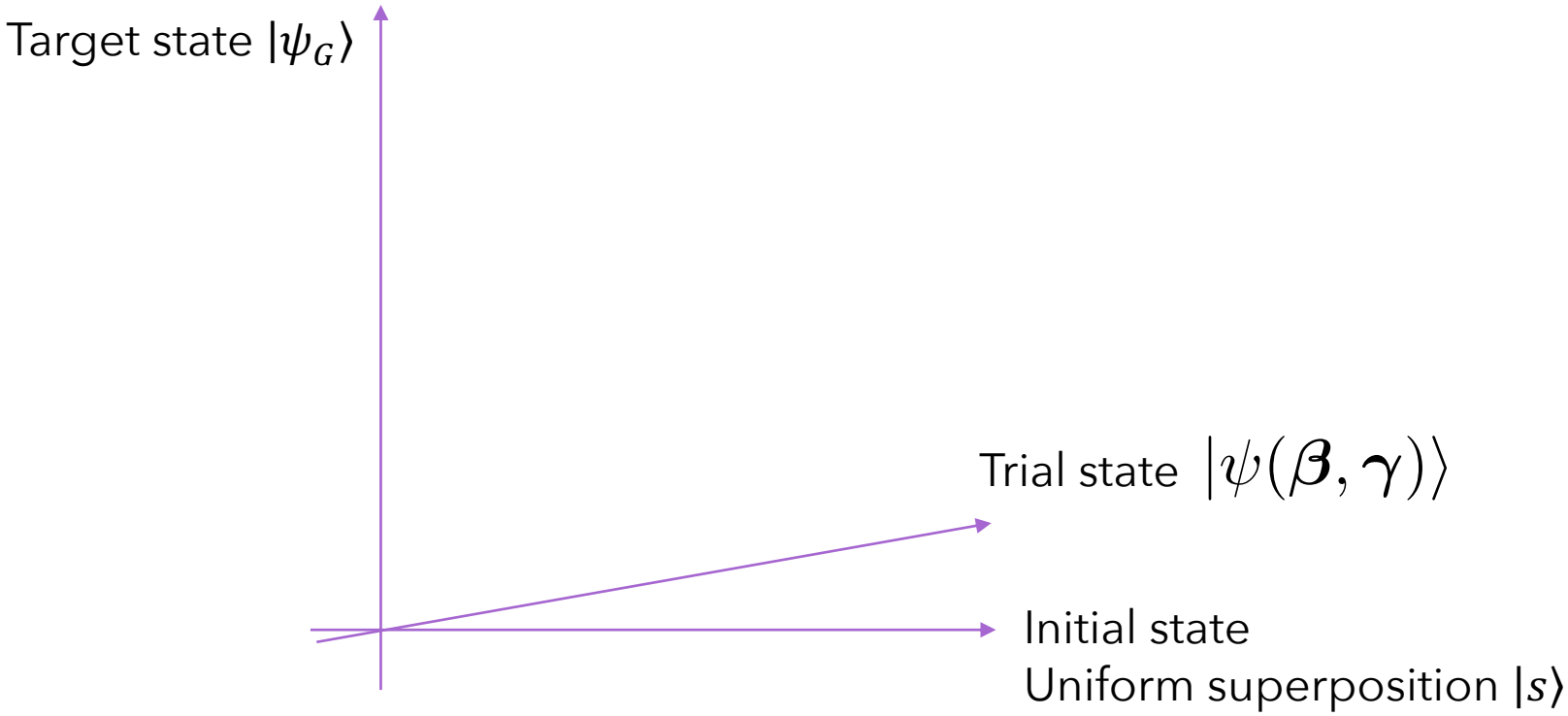

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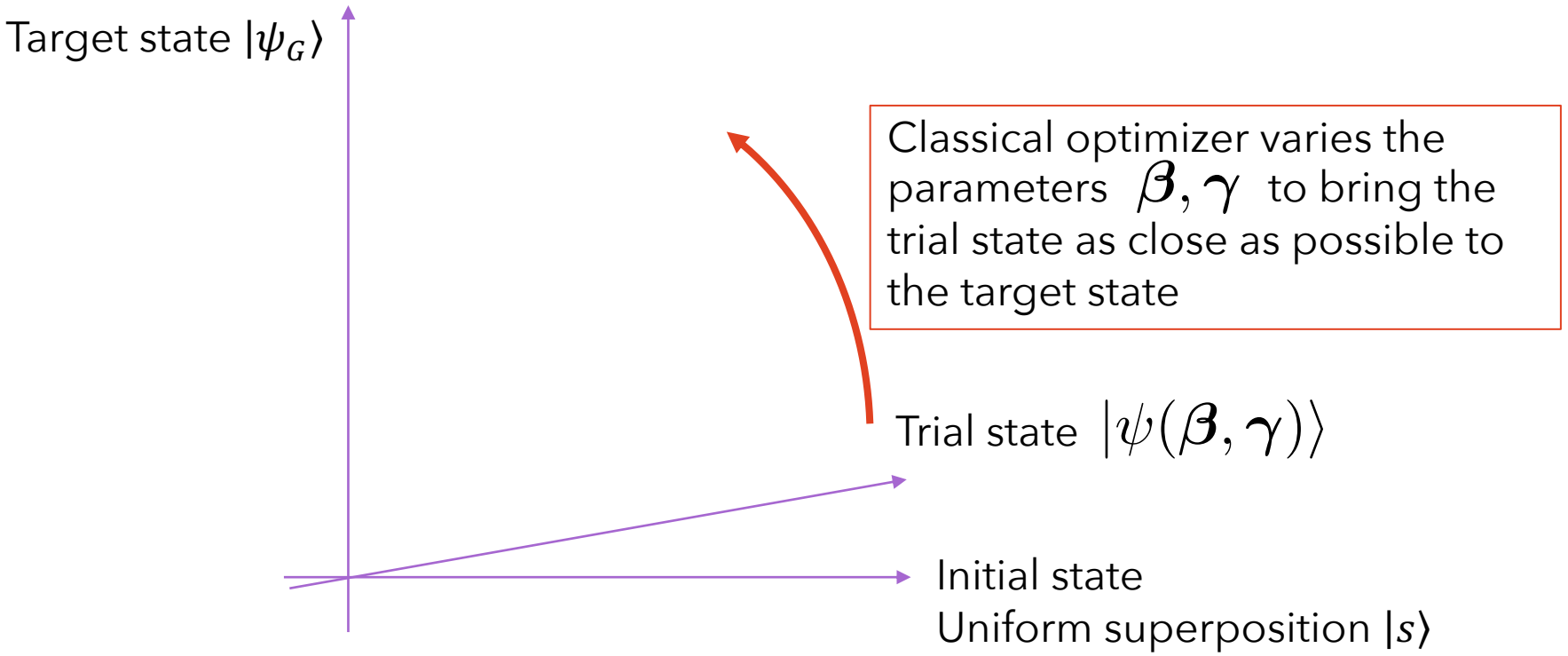

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\end{aligned}
$$

- Crucially, the quality of QAOA solution heavily depends on the quality of the parameters found by the classical optimizer

Adiabatic Quantum Computation (AQC)

- A way to find the ground / highest energy state of the problem Hamiltonian, inspired by adiabatic approximation theorems
- Adiabatic approximation states, roughly, that a system prepared in an eigenstate (e.g. a ground state) of some time-dependent Hamiltonian H(t) will remain in the corresponding eigenstate provided that H(t) is varied ``slowly enough''.
- **Idea:** prepare a system in an eigenstate of a simple Hamiltonian and adiabatically interpolate it into the corresponding eigenstate of problem Hamiltonian
- Interpolation is given by a time-dependent Hamiltonian:

$$H(t) = (1 - s(t))H_D + s(t)H_P$$

- s(t) is a smooth function s.t. s(t = 0) = 0 and s(t = T) = 1

Adiabatic Quantum Computation for MAXCUT

- Note minus because by convention AQC goes from ground state to ground state

$$
\begin{aligned}
H(t) &= (1 - s(t))H_D + s(t)H_P \\
H_P &= -C = -\tfrac{1}{2}\textstyle\sum_{ij \in E}(I - Z_i Z_j) \\
H_D &= -\textstyle\sum_j X_j \\
s &= |+\rangle^{\otimes n} = \tfrac{1}{2^n}\textstyle\sum_{x \in \{0,1\}^n} |x\rangle
\end{aligned}
$$

- s is the ground state of driver Hamiltonian H_D and is the initial state

Simulation of AQC

- How can we simulate this time-dependent Hamiltonian on a quantum computer?

$$
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H(t) &= (1 - s(t))H_D + s(t)H_P \\
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- To answer this, let's go to the basics

Simulation of AQC

- Consider a general quantum evolution, defined by a unitary U:

$$|\psi(t)\rangle = U(t)\,|\psi(0)\rangle$$

- Evolution is related to system Hamiltonian by:

$$i\frac{dU(t)}{dt} = H(t)U(t)$$

- For a time-dependent Hamiltonian such that $[H(t_1), H(t_2)] = 0, \qquad \forall t_1, t_2$ the solution of this equation is:

$$U(t) = e^{-i\int_0^t H(t)dt}$$

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- For a time-independent Hamiltonian solution this simplifies to:

$$U(t) = e^{-iHt}$$

Approximating Non-Commuting Hamiltonians

- (**Trotter formula**) For Hermitian A and B and any real t:

$$\lim_{n\to\infty}(e^{iAt/n}e^{iBt/n})^n = e^{i(A+B)t}$$

Approximating Non-Commuting Hamiltonians

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$$e^{i(A+B)t} = \lim_{n\to\infty}(e^{iAt/n}e^{iBt/n})^n$$

- Corollary (**Lie-Trotter-Suzuki** decompositions):

$$e^{i(A+B)t} = e^{iAt}e^{iBt} + O(t^2)$$

$$e^{i(A+B)t} = e^{iBt/2}e^{iAt}e^{iBt/2} + O(t^3)$$

Discretizing AQC

- Let's now consider our time-dependent Hamiltonian H(t) and evolution operator:

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- Now, choose time step Δt to be small enough so that H(t) is approximately constant over the interval:

$$U(j\Delta t,(j-1)\Delta t) = e^{-i\int_{(j-1)\Delta t}^{j\Delta t} H(t)dt} \approx e^{-iH(j\Delta t)\Delta t}$$

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- Combining both, we get:

$$U(T,0) \approx \prod_{j=1}^{p} e^{-iH(j\Delta t)\Delta t}$$

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U(T,0) \quad &\approx \prod_{j=1}^{p} e^{-iH(j\Delta t)\Delta t} = \prod_{j=1}^{p} e^{i((1-s(j\Delta t))H_D + s(j\Delta t)H_P)\Delta t} \\
&= \left[\text{applying Lie-Trotter-Suzuki decomposition: } e^{i(A+B)t} = e^{iAt}e^{iBt} + O(t^2) \right] \\
&\approx \prod_{j=1}^{p} e^{i(1-s(j\Delta t))\Delta t H_D} e^{i\, s(j\Delta t)\Delta t H_P}
\end{aligned}
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Understanding QAOA

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\begin{aligned}
|\gamma,\beta\rangle &\equiv U_{\mathrm{QAOA}_p}(\gamma,\beta)\,|s\rangle \\
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Simulated AQC

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Simulated AQC

- QAOA is equivalent to simulated adiabatic computation if we set

$$\gamma_j = -s(j\Delta t)\Delta t, \qquad \beta_j = -1[1 - s(j\Delta t)]\Delta t$$

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Simulated AQC

- However, the non-adiabatic mechanism of QAOA is quite different!

  See e.g. Jiang, Zhang, Eleanor G. Rieffel, and Zhihui Wang. "Near-optimal quantum circuit for Grover's unstructured search using a transverse field." Physical Review A 95.6 (2017): 062317.

Summary

- QAOA will not solve NP-complete problems in polynomial time
- QAOA is a promising *heuristic* for the NISQ era
- For any Boolean function on n bits we can construct a unique n-qubit Hamiltonian representing it
- QAOA is deeply connected to Adiabatic Quantum Optimization Algorithm
- QAOA energy can often be evaluated purely classically

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PART 3:
HANDS-ON