SINE WAVE GENERATOR USING CORDIC ALGORITHM

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1. Abstract

In this report, the authors have done the hardware implementation of sine wave generator using ***CORDIC*** *Algorithm on "Ico-Board" FPGA and demonstrated the output on the "Analog Discovery" digital Oscilloscope. Basically, CORDIC Algorithm is a 'shift and add' algorithm used for implementing countless transcendental functions like trigonometric,hyperbolic, expotential and logarithmic.*

Keywords: **CORDIC**, **RTL**, **Rotation Mode**

2. INTRODUCTION

CORDIC stand for **CO**ordinate **R**otation **D**igital **C**omputer. In 1959, Jack Volder [1] first proposed this algorithm. In his thesis, he proposed an efficient way of calculating trigonometric function. John Walther [2] and others extended the CORDIC theory to provide solutions to a wider range of functions like transcedental functions.

This paper has been divided into five parts.

1. Unified CORDIC algorithm
2. Implementation of algorithm in MATLAB & C
3. Implementation of algorithm in RTL
4. Sine wave generator implementation in RTL
5. Hardware implementation of Sine Wave Generator

3. CORDIC Algorithm

3.1. Observation

If a unit vector with co-ordinates $(x_1, y_1) = (1, 0)$ is rotated by an angle θ, its new co-ordinate will be $(x_2, y_2) = (\cos\theta, \sin\theta)$. Thus, by finding the (x_2, y_2), $\cos\theta, \sin\theta$ can easily be computed.

3.2. Pseudorotations

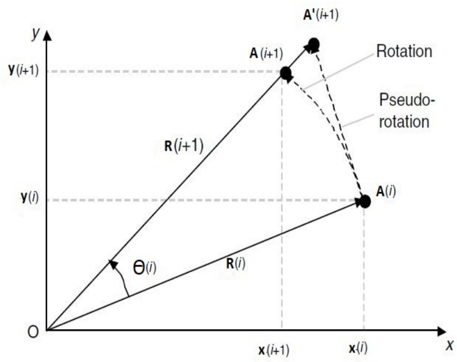


Figure 1: pseudorotation step in CORDIC [3]

Pseudorotation step increases the length of the vector $R(i)$ to

$$R(i+1) = R(i)(1 + \tan^2\theta(i))^{1/2}$$

The coordinates of the new point $A'(i+1)$ after pseudorotation are given by the set of equations:

$$x(i+1) = x(i) - y(i)\tan\theta(i) \tag{1}$$

$$y(i+1) = y(i) + x(i)\tan\theta(i) \tag{2}$$

$$\alpha(i+1) = \alpha(i) - \theta(i) \tag{3}$$

After n pseudorotations by the angle $\theta(1), \theta(2), \cdots, \theta(n)$ with $x(0) = x, y(0) = y$ and $\alpha(0) = \alpha$ we will get,

$$x(n) = \mathbf{K}\,(x\cos(\sum\theta(i)) - y\sin(\sum\theta(i)) \tag{4}$$

$$y(n) = \mathbf{K}\,(y\cos(\sum\theta(i)) + x\sin(\sum\theta(i)) \tag{5}$$

$$\alpha(n) = \alpha - \sum\theta(i) \tag{6}$$

where $\mathbf{K} = \prod(1+\tan^2\theta(i))^{1/2}$

3.3. CORDIC angle

Each pseudorotations should be choosen in such a way that,the **tan** values of these are just bit shifts (i.e divided by the power of two). **A bit shift is a much easier instruction for a CPU to deal with than full integer division.**

$$\theta(i) = \tan^{-1}(d_i\,2^{-i}),\ \ d_i \in \{+1,-1\} \tag{7}$$

RULE: Choose $d_i \in \{+1,-1\}$ such that $\alpha(n) \to 0$

3.4. CORDIC iteration

Thus eqn. 1,2,3 can be written as:

$$x(i+1) = x(i) - d_i\,y(i)\,2^{-i} \tag{8}$$

$$y(i+1) = y(i) + d_i\,x(i)\,2^{-i} \tag{9}$$

$$\alpha(i+1) = \alpha(i) - d_i\,\tan^{-1}2^{-i} \tag{10}$$

Each CORDIC iteration associates three addition,two shifts and a table lookup (it contains a list of precomputed cordic angles). If we always pseudorotate the vector by the same set of CORDIC angles either with positive or negative signs, then the value of scaling factor **K** can be pre-determine and approaches 1.646760258121 after sufficiently large number of iterations. After n pseudorotation steps, when $\alpha(n)$ is well enough close to zero, we will get $\sum\theta(i) = \alpha$.

Finally the CORDIC iterations in ROTATION MODE become:

$$x(n) = \mathbf{K}\,(x\cos\alpha - y\sin\alpha) \tag{11}$$

$$y(n) = \mathbf{K}\,(y\cos\alpha + x\sin\alpha) \tag{12}$$

$$\alpha(n) = 0 \tag{13}$$

3.5. Computation of trigonometric functions

From the eqn. 11,12,13 we observe that if we start with $x(0) = 1/K$ and $y(0) = 0$, then after 'n' pseudorotation steps as $\alpha(n) \to 0$, the $x(n) \to \cos\alpha$ and $y(n) \to \sin\alpha$ with CORDIC iterations in rotation mode.

The domain of convergence is $-99.7° \leq \alpha \leq 99.7°$.

3.6. Implementation in MATLAB & C

Circular Rotation (Basic **CORDIC**)

Algorithm 1 cordic (angle in degree α)

Input: (angle = α)
Output: $(x(n), y(n), \alpha(n)) = (\cos\alpha, \sin\alpha, 0)$
Initialisation: (x(0), y(0), α(0)) = (1/K, 0,α)
$\mu = 1$

%%% Range -pi to pi %%%
if ($\alpha < -90^o$) || ($\alpha > 90^o$) **then**
  if ($\alpha < 0^o$) **then**
    cordic (α + 180^o)
  else
    cordic (α - 180^o)
  end if
  Result $\leftarrow \begin{pmatrix} x(n) \\ y(n) \end{pmatrix}$ = - $\begin{pmatrix} x(n) \\ y(n) \end{pmatrix}$
  return Result
end if
%%% CORDIC iteration %%%
for i = [0:N-1] **do**
  $\theta(i) \leftarrow \tan^{-1}(2^{-i})$ % from lookup table
  if ($\alpha(i) < 0$) **then**
    $d_i \leftarrow (-1)$
  else
    $d_i \leftarrow (+1)$
  end if
  R $\leftarrow \begin{pmatrix} 1 & -\mu\,d_i\,2^{-i} \\ d_i\,2^{-i} & 1 \end{pmatrix}$
  $\begin{pmatrix} x(i+1) \\ y(i+1) \end{pmatrix} \leftarrow R * \begin{pmatrix} x(i) \\ y(i) \end{pmatrix}$
  $\alpha(i+1) \leftarrow \alpha(i) - d_i\,\theta(i)$
end for
Result $\leftarrow \begin{pmatrix} x(n) \\ y(n) \end{pmatrix}$
return *Result*

C-code were executed for 32 bits of precision in the resulting values, thus 32 CORDIC iterations. Trig- functions graph was simulated using MATLAB.

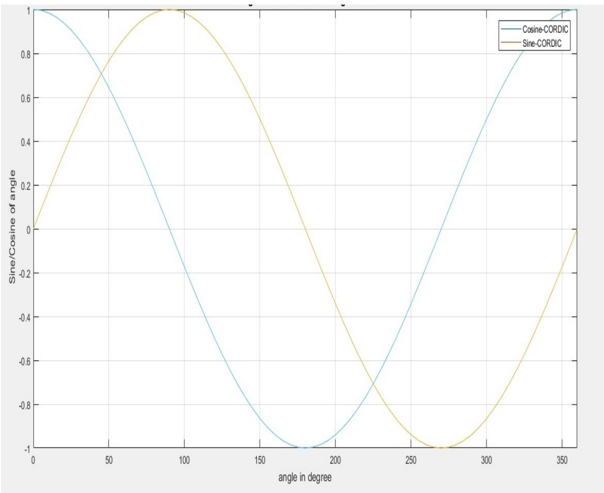


Figure 2: Trigonometric Functions using CORDIC

4. Unified CORDIC Algorithm

J.S Walther [2] improved the CORDIC iteration given by Volder by introducing a system parameter 'μ'. He proposed "Generalised CORDIC iteration" which can used to compute function belongs to three different co-ordinate system i.e Circular, Hyperbolic and Linear.

Generalised CORDIC iteration

$$x(i+1) = x(i) - \mu\, d_i\, y(i)\, 2^{-i} \tag{14}$$

$$y(i+1) = y(i) + d_i\, x(i)\, 2^{-i} \tag{15}$$

$$\alpha(i+1) = \alpha(i) - d_i\, \theta(i) \tag{16}$$

Coordinate system	μ	θ**(i)**
Circular	+1	$\tan^{-1}(2^{-i})$
Hyperbolic	-1	$\tanh^{-1}(2^{-i})$
Linear	0	2^{-i}

RULE:

$$d_i = \begin{cases} +1 & \alpha(i) \geq 0 \\ -1 & \alpha(i) < 0 \end{cases}$$

4.1. Computation of hyperbolic functions

The following equations defines the CORDIC Algorithm for hyperbolic functions:

$$x(n) = \mathbf{K}'\,(x\cosh\alpha + y\sinh\alpha) \tag{17}$$

$$y(n) = \mathbf{K}'\,(y\cosh\alpha + x\sinh\alpha) \tag{18}$$

$$\alpha(n) = 0 \tag{19}$$

Thus, if we start with x(1) = 1/$\mathbf{K}'$ and y(1) = 0 (iteration can not be start from '0', since $\tanh^{-1}(2^{-i})$ does not exist), then after 'n-1' hyperbolic CORDIC rotation steps as $\alpha(n) \to 0$, the $x(n) \to \cosh\alpha$ and $y(n) \to \sinh\alpha$ with hyperbolic CORDIC iterations in rotation mode. Thus, $\tanh\alpha$ can be computed.

4.2. Convergence of hyperbolic CORDIC iteration

Hyperbolic function does not converge with the sequence of CORDIC angles $\tanh^{-1}(2^{-i})$, since

$$\tanh^{-1}(2^{-(i+1)}) \geq 0.5\, \tanh^{-1}(2^{-i}) \tag{20}$$

does not hold in general [4]. To ensure convergence the iterations $i = 4, 13, 40, \cdots, j, 3j+1, \cdots$ must be executed twice. Thus, we get domain of convergence $|\alpha| < 1.13$. where $\mathbf{K}'$ = 0.828159361 after considering the repeated iterations.

4.3. Implementation in MATLAB & C

Hyperbolic CORDIC rotations

Algorithm 2 cordic_hyper (α)

Input: (angle = α)
Output: $(x(n), y(n), \alpha(n)) = (\cosh\alpha, \sinh\alpha, 0)$
Initialisation : (x(1), y(1), $\alpha(1)$) = (1/K$'$, 0, α)
$\mu = -1$

```
%%%     CORDIC iteration     %%%
itr =[4, 13, ··· , j, 3j + 1, ···]
for i = [1:N , itr] do
    θ(i) ← tanh^{-1}(2^{-i})     % from lookup table
    if (α(i) < 0) then
        d_i ← (−1)
    else
        d_i ← (+1)
    end if
    R ← ( 1          −μ d_i 2^{-i} )
        ( d_i 2^{-i}   1           )
    ( x(i+1) ) ← R * ( x(i) )
    ( y(i+1) )       ( y(i) )
    α(i+1) ← α(i) − d_i θ(i)
end for
Result ← y(n) / x(n)
return Result
```

MATLAB code for hyperbolic CORDIC rotation algorithm was executed for various values of iteration count and the error between the simulated and theoritical was observed.

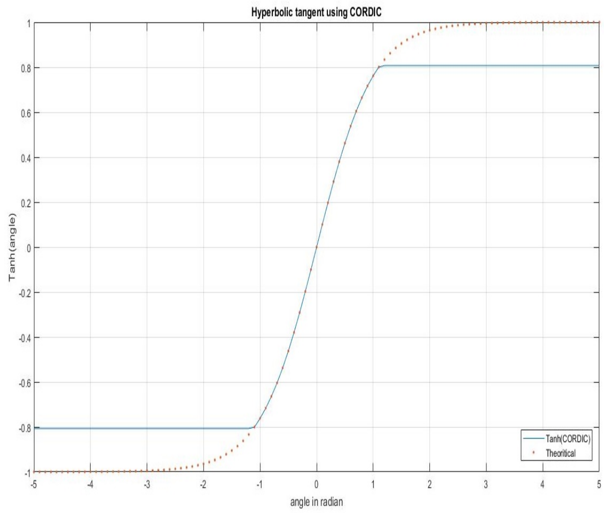


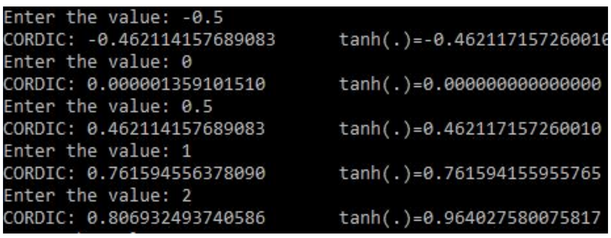
Figure 3: Simulated result i = {8,16,32,64}

From the fig. 3, after the improvement in iteration we get the domain of convergence $|\alpha| < 1.13$ (same as we see in section 4.2).

4.4. Error analysis

Iteration count	Error		
	$\alpha = 0$	$\alpha = 0.5$	$\alpha = 1$
8	9.3×10^{-5}	1.4×10^{-3}	-1.3×10^{-3}
16	-1.4×10^{-5}	7.4×10^{-6}	4.0×10^{-7}
32	-6.3×10^{-10}	5.0×10^{-10}	8.8×10^{-10}
64	-8.6×10^{-10}	6.8×10^{-10}	9.8×10^{-10}

From the above table, it is clear that error almost saturates after the iteration count 32. Hence, in the **C-** code, the iteration count is taken 32. C-code for hyperbolic tangent function using CORDIC was written and executed for different values of angle and also compared with theoritical value using **math.h** library.



```
Enter the value: -0.5
CORDIC: -0.462114157689083      tanh(.)=-0.46211715726001
Enter the value: 0
CORDIC: 0.000001359101510       tanh(.)=0.000000000000000
Enter the value: 0.5
CORDIC: 0.462114157689083       tanh(.)=0.462117157260010
Enter the value: 1
CORDIC: 0.761594556378090       tanh(.)=0.761594155955765
Enter the value: 2
CORDIC: 0.806932493740586       tanh(.)=0.964027580075817
```

Figure 4: Simulated result i = 32

4.5. Improving domain of convergence

Each iterations are repeated with a same repetition factor (**R**) of an even multiple to increase the domain of convergence. MATLAB code for the improved iterations is executed and the following graph has been observed.

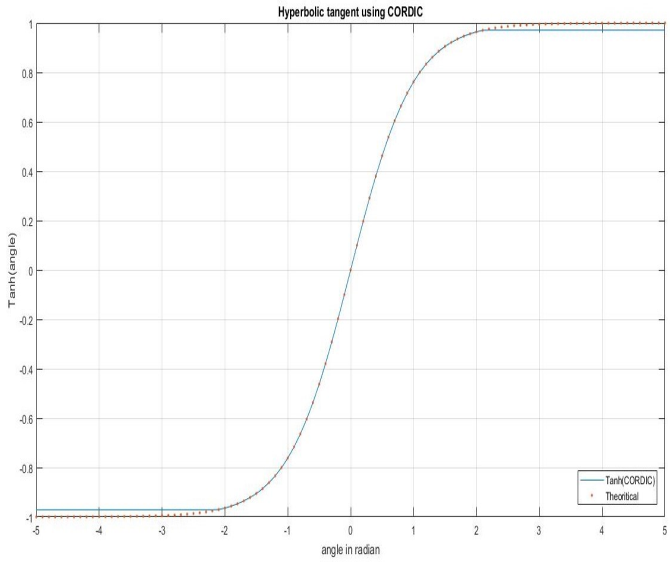


Figure 5: Simulated result i = 32 and R = 2

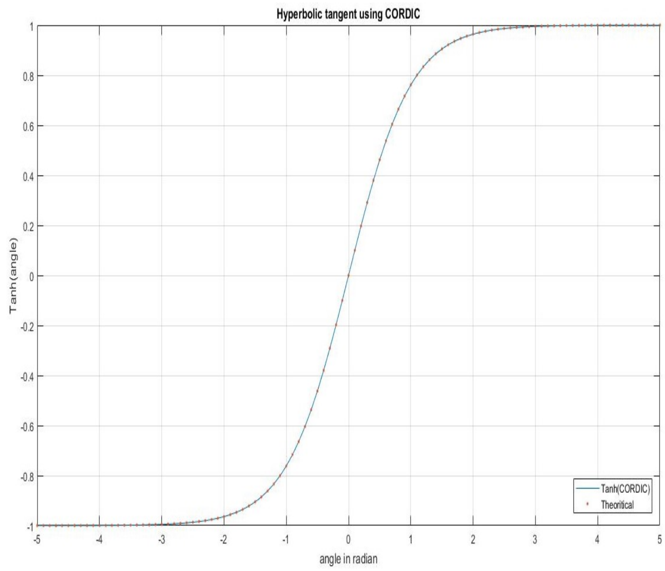


Figure 6: Simulated result i = 32 and R = 4

4.5.1. Conclusion: Error analysis.

		Error		
R	i	$\alpha = 0$	$\alpha = 0.5$	$\alpha = 1$
2	16	0.0	-6.6×10^{-6}	8.2×10^{-7}
	32	0.0	-6.7×10^{-10}	1.7×10^{-10}
	64	0.0	-4.4×10^{-10}	2.8×10^{-10}
4	16	0.0	-6.6×10^{-6}	8.2×10^{-7}
	32	0.0	-6.7×10^{-10}	1.7×10^{-10}
	64	0.0	-4.4×10^{-10}	2.8×10^{-10}

From the above table, it is clear that the error remains almost same as compared to the **Table 1.0** except for α = 0. But the hyperbolic domain of convergence for 32 bits of precision in the resulting values, increases from $|\alpha| < 1.13$ (basic iterations) to $|\alpha| < 2.1$ (R = 2) and $\alpha \in \Re$ (R = 4).

5. Implementation in RTL

The Verilog platform has been used for the implementation of CORDIC Algorithm in RTL.

5.1. Circular CORDIC: Trig-functions

The digital design should be able to compute Sine and Cosine of the input angles $\in [0, 2\pi]$ including the floating point angles like 89.45^o, 29.7^o etc.
A 16-bit binary scaling system has been used to represent angles [5]. The resolution of the design is $360^o/2^{16}$ = $5.493\,164\,063 \times 10^{-3}$. The input angle is scaled to fit in a 16-bit register and user must convert the input angle to $[0, (2^{16} - 1)]$.

To convert the angle from degrees to 16-bit value, multiply the angle^o by 2^{16}, then divide it 360^o. FInally, convert the decimal value to binary. The angle is represented by 16-bit format. The upper two bits represent the quadrant [6].

1. 2b'00 = represents I quadrant i.e (0 - π/2) range
2. 2b'01 = represents II quadrant i.e (π/2 - π) range
3. 2b'10 = represents III quadrant i.e (π - 3π/2) range
4. 2b'11 = represents IV quadrant i.e (3π/2 - 2π) range

Total 8 STAGES have been used (i.e 8-bits of accuracy in the resulting values), which consist of a Pre-rotation STAGE (STAGE 1) to make sure that the rotation angle must lie within $-\pi/2$ to $\pi/2$ range (see section 3.5). The size of the Output data is 8-bits. Clock Frequency of 100MHz was used.

Testbench was written and tested for different values of input angle using Xilinx ISE Design Suite 14.7 (ISim) software. After 7-clock cycles, Output Data was displayed on the simulator screen. Ouput value is scaled by G = 75*GainFactor$_{CORDIC}$ times. Simulation Result: angle = 30^o -> 16 - bit binary = 0001010101010101, Output = 00111110 i.e 62/G = 0.5020141293).

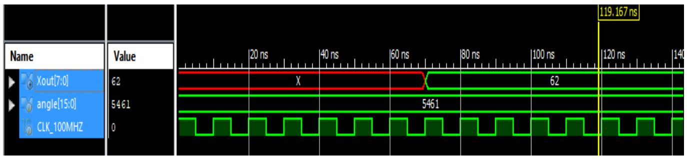


Figure 7: $\alpha = 30^o$

5.2. SINE Wave Generator

Xilinx ISE Design Suite 14.7 and ModelSim PE Student Edition 10.4a software were used to test the Verilog Code and simulate the SINE WAVE GENRATOR module from 0 to 2π respectively. The simulation result is shown below:

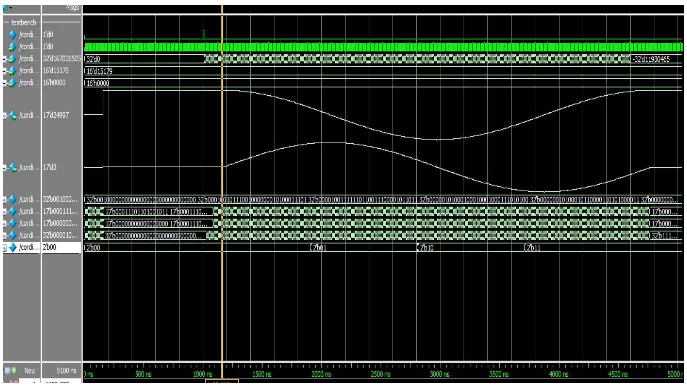


Figure 8: Simulation Result (ModelSim)

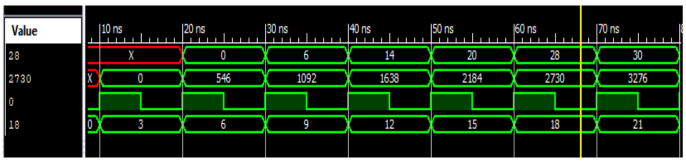


Figure 9: $\alpha \in [0, 2\pi]$

6. Hardware Implementation

The major components used for hardware implementation are as follows:

1. Ico-Board FPGA
2. Arduino MEGA 2560
3. R2R DAC
4. 'Analog Discovery'- Digital Oscilloscope

6.1. R2R DAC circuit

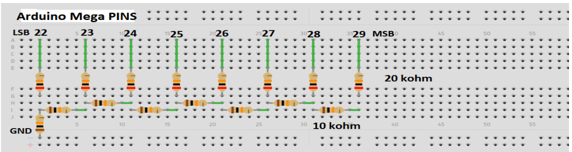


Figure 10: 8-bit 'R2R' DAC Circuit

6.2. Circuit and System setup

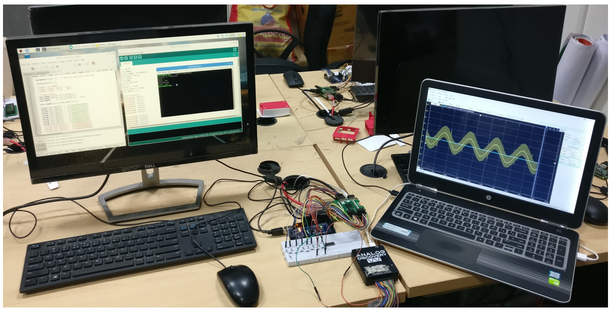


Figure 11: System setup

6.3. Digital Oscilloscope

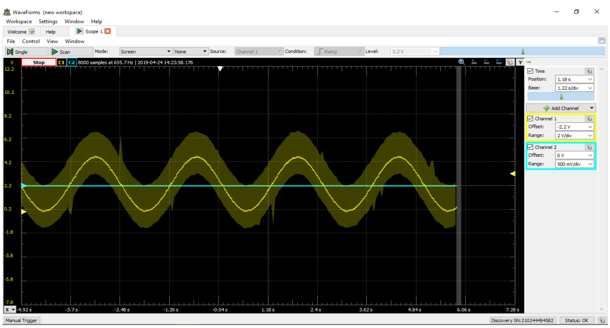


Figure 12: Output Waveform

7. Future Work

Errors in the output like presence of steps in the sine waveform can be reduced by increasing the bit size. Adding a filter in the DAC circuit will help obtain a smoother waveform. Apart from rectifying the quantisation error, we can also extend the project to generate sine waves of desired frequencies.

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