Euler's three-body problem

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The simplest three-body problem that attracts physical interest is the one first studied by Euler. In Euler's problem, a primary and secondary mass are fixed in space, a given distance apart, and a test mass is allowed to move unrestricted in their superimposed gravitational fields. The equations of motion are derived and solved via a simple numerical procedure. The algorithm is adaptable to a small programmable calculator and should help to stimulate interest in the classroom. Several examples of motion are discussed and graphically presented.

I. INTRODUCTION

In the classical mechanics course offered in most undergraduate physics curricula, the problem of a fixed center of gravitation and a test particle is usually solved analytically in the standard Newtonian formulation. This analytic solution is characterized by the constants of the motion and the orbit is determined from a knowledge of the conic section and its associated parameters such as eccentricity, semimajor axis, and semiminor axis. In polar coordinates, the orbit is calculated by evaluating

$$r = r_0 (1 + e)/(1 + e \cos\theta)$$

for a range of values for θ. In the equation e is the eccentricity and r_0 is a constant that is dependent on the various constants: angular momentum per unit mass, mass of the primary, eccentricity, and the Newtonian gravitational constant.

In his famous lecture series, Feynmann[1] discusses a refreshing approach to the problem by solving the equations of motion directly using a simple numerical procedure. Such a technique allows one to observe the evolution of the orbit from a set of initial conditions by using a small programmable pocket calculator. Eisberg[2] gives several examples or orbits generated from programs written for the HP-25 and SR-56 calculators.

However, there is usually very little mention of the generalizations of the two-body problem to cases where there are three or more masses involved. Except for special cases where the problem possesses a high degree of symmetry, the n-body problem cannot be analytically solved for $n > 2$. For instance, the famous three-body problem of Lagrange is exactly soluble because of the inherent symmetry in a rotating coordinate system. (The solution of Lagrange's problem indicates that there exists points where a test particle will remain stable, known as the Lagrangian points, or L_4 and L_5 points.)

Owing to the complexities in even the simplest three-body situation where one particle is assumed to have negligable mass, discussion of the three-body problem is often left to specialized courses in celestial mechanics where a more thorough treatment can be afforded through the use of perturbation theory. Approximations may still be necessary to get meaningful results. Examples of approximations which have been extensively studied are the lunar theories and the planetary theories. The former concerns itself with a massive planet orbiting a primary with the test mass in close orbit about the planet. The latter assumes a planet orbiting the primary with a test particle also orbiting the primary though never getting very close to either mass.[3-5]

The simplest three-body problem which retains physical significance is known as Euler's problem.[6] Here there are two massive bodies and a test particle, but the massive bodies are fixed in space for all time and do not influence one another. The motion of the test particle is uninhibited, and it is influenced by the gravitational forces from the two masses. This problem is analytically soluble. However, in order to plot an actual orbit one must evaluate several elliptic integrals—a tedious task.[7,8] For the student, the mathematical difficulties involved with the numerical or tabular evaluation of these integrals would detract from the interesting physics of the problem. One would like to watch the orbit evolve without getting tangled up with mathematical difficulties.

It is our desire to introduce material which may be quite stimulating to the curious student. We intend to discuss Euler's problem in greater depth by first deriving the equations of motion, then showing how to implement these equations on a small programmable calculator. The motion will be restricted to the plane, and several examples will be illustrated.

The numerical method is the simplest one known, also developed by Euler, where for every step forward in time the velocity is assumed constant. Using this algorithm, we have coded the planar equations of motion into a program for the SR-56 calculator (also implementable without change onto TI-58 and TI-59 calculators). The program is given in the appendix. Although many more sophisticated techniques exist, a computer must be used if they are to be used for our equations. In general Euler's method gives good results if the test particle does not approach too close to a primary and the time increment is small.

II. EQUATIONS OF MOTION

For two fixed centers of gravitation, the total gravitational force acting upon a test mess is, by Newton's second law, the sum of the individual forces. In vectorial notation

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2, \tag{1}$$

the subscripts referring to the force due to the primary and secondary masses. This is, of course, a relation that is true in any inertial reference frame.

We shall derive the equations of motion relative to a

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convenient choice of coordinates so that the equations take on the simplest form. Since the primary and secondary masses are fixed for all time, let the primary be situated at the origin and let the secondary be situated on the positive x axis a distance d from the origin. Referring to Fig. 1, let $\mathbf{R}$ represent the distance of the test mass, in the plane $z = 0$, from the primary. Similarly, let $\boldsymbol{\Delta}$ be the distance between the test mass and the secondary.

Let G be Newton's constant of gravitation, M_1 the primary mass, M_2 the secondary mass, and m the mass of the test mass (m will divide out of the final equations, so it can be regarded as being very small). Equation (1) becomes

$$\mathbf{F} = (GM_1m/R^3)\mathbf{R} - (GM_2m/\Delta^3)\boldsymbol{\Delta}, \tag{2a}$$

where for Cartesian (x,y) coordinates,

$$\mathbf{R} = \hat{i}x + \hat{j}y, \quad R^3 = (x^2 + y^2)^{3/2}, \tag{2b}$$

and

$$\boldsymbol{\Delta} = \hat{i}(x - d) + \hat{j}y, \quad \Delta^3 = [(x - d)^2 + y^2]^{3/2}, \tag{2c}$$

for unit vectors $\hat{i}$ and $\hat{j}$. Note that if $M_1 = M_2$, there exists a point at $x = d/2$ where, if placed there with zero velocity, the test particle will remain fixed for all time. For unequal masses, such points also exist and are determined by $F_x = 0$.

For planar motion there are two components of the force vector, namely,

$$F_x = m\ddot{x} = -(GM_1m/R^3)x - (GM_2m/\Delta^3)(x - d) \tag{3a}$$

and

$$F_y = m\ddot{y} = -(GM_1m/R^3)y - (GM_2m/\Delta^3)y. \tag{3b}$$

The mass of the test particle cancels out. It is assumed that m is very small or it will have a perturbing influence causing the problem to become less realistic. Here the double dot indicates a second time derivative, so $\ddot{x} = a_x$, the x component of the acceleration.

The question of units arises. The most intuitive dimensions if we were to explore the dynamics of a solar system would be those that we are most accustomed to, namely, astronomical units (AU—the average earth–sun distance), years, and solar masses. For instance, the earth is one AU from the sun (the primary) which is in turn one solar mass. The earth's velocity around the sun is 2π AU/y.

For our hypothetical solar system, let the mass of the primary be defined as one solar mass, or symbolically, $M_1 = M_\odot$. Let the secondary be expressed as some fraction α of the primary so that $M_2 = \alpha M_1 = \alpha M_\odot$. If $\alpha = 1$, both masses are identical. Other appropriate choices, for α will allow us to simulate other bodies of the solar system.

To adopt these units, set

$$GM_1 = GM_\odot = 4\pi^2,$$

which by Kepler's third law gives us the dimensions of AU/y/solar mass.[9]

Equations (3a) and (3b) take on the form

$$\ddot{x} = -4\pi^2[x/R^3 + \alpha(x - d)/\Delta^3], \tag{4a}$$

$$\ddot{y} = -4\pi^2 y(1/R^3 + \alpha/\Delta^3). \tag{4b}$$

These are the equations that determine the path for the test mass once the initial position and velocity are specified. In Sec. III we shall discuss how the system is coded for use on

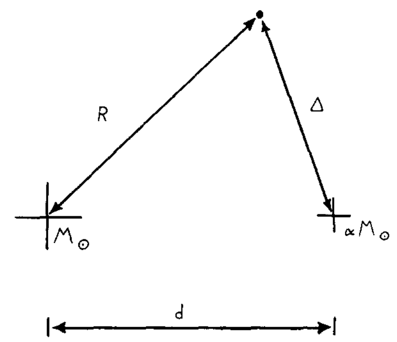

Fig. 1. Geometry of the planar Euler problem.

our calculator. Notice that although the forces add linearly, the actual evolution equations are nonlinear and coupled. These equations can be solved, but only by using coordinate transformations, and then we still have to evaluate several very complicated integrals. Further discussion of these solutions can be found in Wintner[10] and Whittaker.[7]

III. NUMERICAL SOLUTION

The most rapid, and for our purposes, efficient way to solve the equations of motion (4a) and (4b) would be via a numerical technique. Many powerful but very complex numerical codes exist, some of which were developed as a direct result of the space program. However, these software packages are not readily available, although many computers have software for systems of differential equations and can be adapted to our problem. It is our intent to express Eqs. (4) as a simple set of finite difference or iteration formulas, and to code these formulas onto a small calculator.

First we shall rewrite Eqs. (4) as a system of first-order differential equations. Define

$$\dot{x} = v, \tag{5a}$$

$$\dot{y} = u, \tag{5b}$$

where v is the x component of velocity and u the y component of the velocity. Consequently, Eqs. (4) become

$$\dot{v} = -4\pi^2[x/R^3 + \alpha(x - d)/\Delta^3], \tag{5c}$$

$$\dot{u} = -4\pi^2 y[1/R^3 + \alpha/\Delta^3]. \tag{5d}$$

Our numerical method, known as Euler's method, approximates the derivative with a simple finite difference scheme. For example, if we know the solution at $x = x(t)$, then at $x = x(t + \delta t)$, for a small time increment δt, we have by Taylor's theorem

$$x(t + \delta t) = x(t) + \delta t\dot{x}(t) + O(\delta t^2),$$

where $O(\delta t^2)$ represents terms of higher order in δt. Since $\dot{x} = v$, we have for Euler's approximation

$$x(t + \delta t) = x(t) + v\delta t,$$

and higher-order terms are neglected. For small values of δt, this approximation is reasonable. The numerical error attributed to this method is of order δt.[11]

Defining the following notation, for n a non-negative integer,

$$x_n(t) = x(n\delta t), \tag{6}$$

our finite difference formulas take on the general form

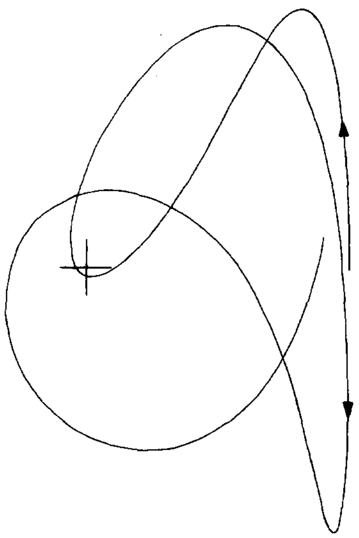

Fig. 2. Path of a test mass released at $x_0 = 1, y_0 = 0, v_0 = 0$, and $u_0 = 2\pi$. The primary and secondary (not shown) are both one solar mass, the secondary being situated at $d = 2.1$ AU from the origin.

$$x_{n+1} = x_n + v_n \delta t, \tag{7a}$$

$$y_{n+1} = y_n + u_n \delta t. \tag{7b}$$

Applying the Taylor series expansions to the velocities, Eqs. (5c) and (5d), yield

$$v(t+\delta t) = v(t) + \dot{v}(t)\delta t + 0(v/dt^2)$$
$$\simeq v(t) - 4\pi^2 \delta t[x/R^3 + \alpha(x-d)/\Delta^3],$$

and therefore

$$v_{n+1} = v_n - 4\pi^2 \delta t[x_{n+1}/R^3_{n+1} + \alpha(x_{n+1} - d)/\Delta^3_{n+1}], \tag{7c}$$

$$u_{n+1} = u_n - 4\pi^2 \delta t y_{n+1}[1/R^3_{n+1} + \alpha/\Delta^3_{n+1}]. \tag{7d}$$

The reason why we use x_{n+1} and y_{n+1} to determine v_{n+1} and u_{n+1}, rather than x_n and y_n, is that it is convenient from the programming point of view (less memory is required and fewer program steps), and further, we have observed that the accuracy increases when we integrate numerically in this manner.

For the calculator program, several variables must be initialized. The distance d (in AU) between the primary and secondary must be specified, as well as the mass of the secondary as some fraction α of a solar mass. Further, the initial coordinates of the test particle, as well as the initial values for the velocity components must be set. The initial position is (x_0,y_0), and the initial velocity is (v_0,u_0). The latter are expressed in units of AU/y.

Once the program is initialized, it may be wise to check that the program has been keyed in without error. This can be readily done by checking the asymptotic limits of the problem. For instance, by choosing d to be very large and setting the initial conditions for an earth orbit, which is nearly circular with radius 1 AU, where

$$x_0 = 1.0, \quad y_0 = 0.0, \quad v_0 = 0.0, \quad u_0 = 2\pi,$$

we should get a circle since the perturbations due to the secondary are very small (if α is also small or zero). For convenience we choose the motion to begin on the x axis so that v_0 can be set to zero. If we were to begin at some other point, we must set x_0, y_0, v_0, and u_0 such that $(x_0^2 + y_0^2)^{1/2} = 1$ AU, and $(v_0^2 + u_0^2)^{1/2} = 2\pi$ AU/y. A suitable choice for δt is 0.01 or 0.005, so 100 or 200 iterations are required for one revolution about the primary, respectively. For the SR-56, with $\delta t = 0.01$ y, the calculator will require

$$(100 \text{ iterations})(5 \text{ sec/iteration}) = 500 \text{ sec}$$

for 1 y of evolution, or about 8 min. After this time, the test particle should be near its starting point.

Another test has the secondary at $d = 2$, and $\alpha = 1$. Then by setting $x_0 = 1$, $y_0 = 0$, $v_0 = 0$, $u_0 = 2\pi$, and $\delta t = 0.01$, we can observe the test particle bobbing up and down midway between the masses. We need not use the above choice for u_0, though did so to simulate the earth's motion under these circumstances. This is an unstable situation since if we set $d = 2 + \epsilon$ for some small ϵ, the test particle will quickly move toward the primary. We shall have more to say about this later on.

Euler's method works quite well if the test mass does not get too close to the masses and evolution times are not excessive. Experimentation of the method with varying δt will enable the student to get an intuitive feeling for the method and its limitations. It should be mentioned that for close approaches one of the outstanding problems in numerical celestial mechanics is to find ways of reducing numerical error to a minimum. One way to do this is by invoking a procedure known as regularization, which entails performing a coordinate transformation on the equations of motion. This procedure can become very complicated for cases where there are more than two bodies involved and is the subject of intense research. Further discussion of this theory is presented in Szebehely,[5] and a computer program is given in Leuhrmann[11] for the case of a test particle subjected to a solar wind.

IV. EXAMPLES

It is now possible, with the aid of calculator, and graph paper, to observe interesting phenomena. For instance, one can ask how the earth's motion would be affected by the presence of another star. Although this problem is artificial in that the two masses are fixed, Szebehely[5] suggests several physical analogies. Nevertheless, the variety of orbits that can be observed in our "universe" makes it interesting in its own right. By placing $\alpha = 1$ and $d = 100$ we can plot the subsequent motion of the earth (using the above initial conditions) and we obtain a circle. For this case the perturbations from the secondary are very small. By placing $d = 10$ the motion deviates slightly from a circle. For successively smaller values of d, these deviations grow significantly. In Fig. 2 we set $d = 2.1$, and the earth crosses the x axis on the near side of the sun. One gets the feeling that if a star approached the earth, that the earth would not be a safe place to live.

According to Wintner,[10] most orbits will be space filling curves, and are not periodic in the true sense. If the evolution were observed for a bound orbit for an extended period of time, the particle may return to an arbitrarily close neighborhood of its initial conditions, but never exactly to its origin.

The next example has the secondary at $d = 0.2$ and $\alpha = 1.0$. The initial conditions are $x_0 = 1$, $y_0 = 0$, $v_0 = 0$, and $u_0 = 2\pi$. Notice how (in Fig. 3) the test mass precesses around the double mass, due to the changing gradient of the fields. This motion is similar to the case when a solar wind is present.[11] A similar effect occurs if the secondary is placed far from the primary, for instance, at $d = 10$. For this case

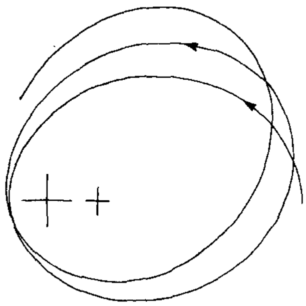

Fig. 3. Motion of a test mass with $x_0 = 1, y_0 = 0, v_0 = 0$, and $u_0 = 2\pi$. Here $\alpha = 1$ and $d = 0.2$.

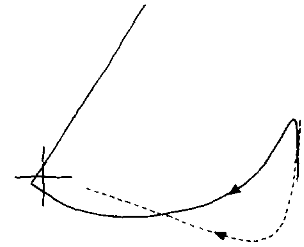

Fig. 4. Motion for which $x_0 = 1, y_0 = 1, v_0 = 0, u_0 = 2$, and $\alpha = 1$. The solid line has $d = 2.01$, whereas the dashed line has $d = 2.001$. The upswing in the motion after the origin is passed in the former case is a result of computational error.

we set $\delta t = 0.001$ and the subsequent evolution requires about

$$(1000 \text{ iterations/y})(2 \text{ y of evolution})(5 \text{ sec/iteration}) \simeq 2.7 \text{ h}.$$

In Fig. 4 we show what happens when the particle is released a very small distance from the equilibrium point, mentioned earlier. Here $x_0 = 1, y_0 = 0, v_0 = 0, u_0 = 2, \alpha = 1$, and $d = 2 + \epsilon$. For $\epsilon = 0$, the particle endlessly bobs up and down. The solid line in Fig. 4 has $\epsilon = 0.01$, and the dashed line depicts the motion for $\epsilon = 0.001$. Notice how the particle moves toward the primary almost immediately.

It can be speculated that the number of axis crossings, denoted by $n(\epsilon)$, before the perihelion point is reached, satifies the condition

$$\lim_{\epsilon \to 0} n(\epsilon) = \infty,$$

though it is seen from our example, that even for small ϵ, $n(\epsilon) \simeq 1$. This statement seems to be intuitively obvious, however, we have not attempted a rigourous proof.

For our last example, we have attempted to illustrate the famous figure "8" orbit. The interested reader may desire to experiment with different initial conditions in order to find a truly periodic figure "8" (see Fig. 5).

In order to check the accuracy of the method as the evolution is observed, especially if a larger calculator such as the TI-58 or TI-59 is being used, one can calculate the energy of the test mass continuously within the program. The total energy is a constant of the motion and is the sum of the kinetic energy and the potential energy components:

$$E = (v^2 + u^2)/2 - 4\pi^2(R^{-1} + \Delta^{-1}). \tag{8}$$

If this quantity begins to deviate significantly from the value calculated at $t = 0$, then the numerical accuracy is decaying so the motion may not represent reality. It may be necessary to experiment with varying time increments δt, and perhaps changing δt suitably, depending on how close the particle is to either of the masses. It is our experience that if δt is changed during the course of a program, numerical error increases. Further, if E is negative the particle is bound to the system, whereas if $E \geq 0$, the particle will escape to infinity.

It should be noted that Euler's numerical integration scheme is one of the simplest methods, and therefore also prone to greater numerical errors than more sophisticated techniques. This method is ideal for problems where the following two conditions are satisfied: (a) there are no very close encounters between the test mass and primary or secondary, and, (b) the time evolution is not excessive. That is, short paths are best unless more sophisticated procedures are used. For long paths the error becomes increasingly greater with time until the solution is no longer representative of real events.

For our examples we have used a fairly small time increment so that we could proceed for fairly long time histories. The accuracy of our examples was checked in two ways. First, we performed the example by using a given time increment; then we repeated the example by using a substantially smaller increment. When the two paths converged (to within a reasonable error) we considered the final result to be accurate. Second, we wrote a Runge–Kutta computer program and repeated the examples. The Runge–Kutta method is very powerful and can generally be trusted to yield good results. The key to understanding the Euler method is to experiment with it by exploring different scenarios with a different choice for time increment each time. In this way the student can build up an intuition for what's going on and can distinguish between accurate and erroneous results.

V. CONCLUSION

It is our hope that this article will be of use to students in the undergraduate mathematics and physics curricula who desire to explore beyond the simple two-body problem in Newtonian dynamics. We have attempted to show how the complex equations for a specific three-body problem can be solved on a small calculator, and that interesting problems can be solved. Our technique can be adapted to the general three-body problem if one is willing to use a larger calculator such as the TI-58 or TI-59, or even a small computer. Further, after experimenting with the procedure given here, one may wish to code the equations using a more powerful method such as the Runge–Kutta method. If one has access to a plotter, it becomes possible to watch the body as it moves. With the availability of home computers, one application would be to write a program whereby the motion of the bodies in, say, a solar system can be observed directly on the cathode ray tube.

We have treated the two-dimensional problem in this article without mentioning the more intriguing three-dimensional case. It is our intention to do this in a future paper; however, it is not difficult to imagine some of the phenomena which occurs. For instance, in the plane perpendicular to the line joining the masses and equidistant

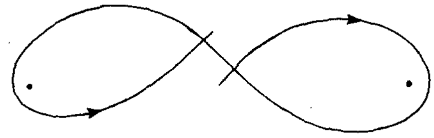

Fig. 5. Figure "8" type motion with $\alpha = 1, d = 3$, and initial conditions $x_0 = 1.5, y_0 = 0, v_0 = u_0 = 3$, and $\delta t = 0.001$.

between them, a test particle will execute a closed path if given an initial velocity within this plane. What is the shape of this path? Further, what are the orbits like when the particle moves around the entire system, analogous to the figure "8" case. Indeed, this is a simple yet interesting universe to ponder.

ACKNOWLEDGMENT

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APPENDIX: CALCULATOR PROGRAM

Below is a copy of the calculator program. It was originally written for the SR-56 100-step programmable, however, it can be entered into the TI-58 and TI-59 calculators without modification. The program has the following features:

(i) The mass of the secondary can be adjusted by specifying α, which represents the fraction of a solar mass desired (α can be greater than one).

(ii) The secondary mass can be placed anywhere along the x axis by specifying d, the distance in AU (astronomical units) from the origin.

(iii) The units are $M_\odot$, AU, years.

(iv) The calculator program can be conveniently halted at a specified time (in years of evolution) by placing that time in the test (t) register.

The following program performs the iterations shown in Eqs. (7a)–(7d).

Register assignments

0—δt	5—u_0
1—x_0	6—used
2—v_0	7—α
3—time (t)	8—used
4—y_0	9—d

Notice that only registers 0, 1, 2, 4, 5, 7, and 9 need to be filled prior to starting the program. Register 3 keeps tract of the time elapsed. The t register contains the time when the program stops running.

Program

00—RCL	20—0	40—RCL	60—1	80—x^2
5	=	8	×	+
×	STO	=	RCL	RCL
RCL	8	×	8	4
0	RCL	RCL	=	x^2
=	1	4	SUM	=
SUM	–	=	2	×
4	RCL	SUM	RCL	√
RCL	9	5	0	=
2	=	RCL	SUM	1/x
10—×	30—SUBR	50—1	70—3	90—+/–
RCL	8	–	RCL	×
0	0	RCL	3	4
=	×	9	PAUSE	×
SUM	RCL	=	$x = t$	π
1	7	×	7	x^2
RCL	=	RCL	8	×
1	STO	6	RST	RCL
SUBR	6	+	R/S	0
8	+	RCL	RST	RTN

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