Proving Classical Theorems with Numerical Methods: The Intermediate Value Theorem Revisited

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13 May 2025

ABSTRACT
The **Intermediate Value Theorem** (IVT) guarantees the existence of roots for continuous functions on closed intervals where the endpoints have opposite signs. While the IVT is traditionally proven using real analysis (by leveraging the idea of completeness in the real numbers), the core idea of the IVT can be demonstrated using practical root-finding algorithms. This paper will examine how numerical methods, specifically the **bisection method**, provides an algorithmic proof to the IVT. By analyzing the **convergence** properties of the bisection method and illustrating its behavior through visualizations, we explore how classical theorems can be reinterpreted through **computational techniques**. This approach highlights the relationship between theoretical mathematics and numerical approximation, offering insight into the correlation of abstract theory and computational geometry.

Key words: Intermediate Value Theorem – Bisection Method – Convergence – Computational Techniques

1 INTRODUCTION

The Intermediate Value Theorem (IVT) is an important result in real analysis. It says that a continuous function f on a closed interval $[a, b]$ that changes signs must contain a root within the interval. The IVT is commonly introduced in early calculus courses, as it is a major consequence that is highlighted throughout the subject. However, a rigorous proof was not formalized until the early 19th century. The general concept of the IVT was widely accepted, so much so that its proof was thought of as necessary as the result seemed obvious and trivial. However, in 1817 Bernard Bolzano provided the first formal proof, emphasizing continuity and inequalities, predating the formal development of the modern real number system. In 1821, Augustin-Louis Cauchy offered an insightful and widely adopted version of the theorem in his *Cours d'Analyse*, helping to cement the IVT's role in mathematical analysis Barany (2013).

In this paper, we revisit the IVT through a numerical lens by implementing the bisection method, a root-finding algorithm, to directly apply the logic of Cauchy's proof. The method algorithmically halves an interval $[a, b]$ where $f(a) \cdot f(b) < 0$, eventually converging to a root to any desired degree of accuracy. Through this computational approach, we explore how the IVT can be not only applied but also constructively demonstrated using numerical methods.

1.1 Classical Theory

Classical analysis, built on the foundation of set theory and logic, treats the real numbers as a complete ordered field. Within this framework, a function is continuous if small changes in the input cause small changes in output. It can be thought of as a function with no "holes". In this context, the proof of the IVT typically depends on the completeness of the real numbers, a foreign concept at the time of the IVT's first appearance. The IVT in classical mathematics is non-constructive, that is, it claims the existence of some c without necessarily providing a method to find it.

It is important to note that the converse of the IVT does not hold, i.e. a function can satisfy the intermediate value property without being continuous. These are called *Darboux functions*, they satisfy the intermediate value property without being continuous Halperin (1959).

At the heart of the IVT lies the concept of continuity. The theorem's validity hinges on the function's continuous nature over the interval, ensuring that it doesn't "jump" over any values.

1.2 Constructive Theory

Constructive mathematics requires explicit examples. Here, mathematical existence is equated with the ability to distinctly construct or compute an object. Errett Bishop, a pioneer in constructive analysis, redefined real numbers in terms of computable sequences of rational numbers, emphasizing the need for explicit constructions in mathematical proofs Bishop & Bridges (1987). So in this context, continuity is defined in terms of effective approximations, and the real numbers themselves are seen as sequences of rational numbers that converge under rules that allow effective (though not necessarily algorithmic) approximation. This interpretation demands that proofs not only assert existence but demonstrate a method for finding what is claimed to exist.

The advancement of computer science has significantly influenced constructive mathematics. The emphasis on algorithms and computability has led to a more nuanced understanding of mathematical existence, focusing on what can be explicitly constructed or computed as opposed to some supposition of truth or falsehood.

In this setting, the Intermediate Value Theorem does not hold in

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its full classical form. In computable analysis, even if a computable function changes sign on an interval, it does not follow that a root can be computed. If we could always compute such a root, it could imply a method for solving certain undecidable problems. For example, the Halting Problem asks whether a given computer program will ever finish running. If there were a computational way to definitively find a precise root of a function (as the classical IVT suggests), it would lead to a method for solving the Halting Problem, contradicting Alan Turing's proof of its undecidability in computational theory. Instead, constructive mathematics accepts a weakened version of the IVT: if f is a continuous function on $[a, b]$ with $f(a) < 0 < f(b)$, then for every $\varepsilon > 0$, there exists $x \in [a, b]$ such that $|f(x)| < \varepsilon$ Constable (2019).

2 IDEAS IN SPACE

Cauchy emphasized the geometric character of the Intermediate Value Theorem. His work helped develop a visual and spatial interpretation of analysis. This intuition laid a foundation not only for the formal epsilon-delta proofs of the 19th century but also for later topological generalizations.

Though the IVT is often first encountered in an analytic context, it can be reinterpreted through a geometric lens when viewed in the framework of topology. Topology abstracts and generalizes concepts of continuity and nearness, providing a natural setting for understanding the IVT.

At the heart of the IVT lies the notion of *connectedness*. In topology, a space is connected if it cannot be partitioned into two nonempty, disjoint open subsets. The real interval $[a, b]$, being connected, cannot allow a continuous function to jump from a negative to a positive value without passing through zero. The classical IVT is a special case of a more general result that the continuous image of a connected space is connected.

This topological perspective allows the theorem to be extended beyond real analysis. One example is the Brouwer Fixed-Point Theorem, which states that any continuous function from a closed disk to itself has at least one fixed point. While not a direct generalization of the IVT, it shares the underlying principle: continuity constrains the behavior of functions on connected and compact domains in nontrivial ways. The fixed-point theorem and the IVT both illustrate how continuity, when paired with the structure of the space, ensures the existence of solutions without necessarily constructing them.

In this way, the Intermediate Value Theorem becomes more than a statement about real-valued functions, but a gateway into the geometry of continuous behavior across abstract spaces.

3 NUMERICAL METHODS

The Intermediate Value Theorem has far-reaching implications beyond pure analysis and topology; it highlights many foundational ideas in numerical computation. This section explores how the theorem manifests in familiar numerical methods.

3.1 Relationship to the Cauchy Sequence

A pillar of analysis, the *Cauchy sequence*, provides a definition of convergence that is independent of knowing the actual limit. This property makes it indispensable in numerical methods, where exact solutions are often unknown or inexpressible. The Intermediate Value Theorem guarantees that a root exists within a certain interval; numerical methods can then generate a sequence (often Cauchy) to approximate that root.

3.2 Fixed-Point Iterative Methods

A variety of fixed-point iterative algorithms reflect the same foundational insight of the IVT which we can use to construct explicit examples of the weakened IVT. These include:

- **Bisection Method:** This is a bracketing method that repeatedly halves the interval to hone in on a root. At each step, the sign of f at the midpoint determines which subinterval to iterate on, ensuring converges to a value arbitrarily close to a root where $f(x) = 0$, provided f is continuous and changes sign over the interval.
- **Newton's Method:** A powerful technique that uses tangent lines to approximate roots. It requires knowledge of the derivative $f'(x)$, and its convergence depends heavily on the choice of initial guess and the behavior of f.

This paper will employ the bisection method to construct an example.

3.3 Why the Bisection Method?

While several numerical methods can be employed to find roots, the Bisection Method is uniquely suited to serve as a *constructive proof* of the Intermediate Value Theorem. Unlike Newton's method, it does not rely on differentiability or an accurate initial guess. Its reliance solely on the effective continuity (meaning its values are computable to arbitrary precision) of f and a sign change on the interval $[a, b]$ ties it directly to the hypothesis of the IVT.

Also, the method's deterministic structure mirrors the constructive conditions discussed earlier. Rather than claiming a root *exists* abstractly, it constructs a converging sequence of intervals in which the root must lie. Each step shrinks that range, revealing arbitrarily accurate approximations of the root, concretely instead of abstractly.

For this reason, the Bisection Method is not only a practical numerical tool but also an elegant computational embodiment of a classical theorem.

4 PROVING THE INTERMEDIATE VALUE THEOREM

Intermediate Value Theorem: Let $f\colon [a, b] \to \mathbb{R}$ be a continuous function such that $f(a)\cdot f(b) < 0$. Then there exists a point $c \in (a, b)$ such that $f(c) = 0$.

Weakened Intermediate Value Theorem: If f is a continuous function on $[a, b]$ with $f(a) < 0 < f(b)$, then for every $\varepsilon > 0$, there exists $x \in [a, b]$ such that $|f(x)| < \varepsilon$

4.1 Classical Proof

Proof. Let $f\colon [a, b] \to \mathbb{R}$ be continuous on the closed interval $[a, b]$, and without loss of generality, suppose that $f(a) < 0$ and $f(b) > 0$.
Goal: show that there exists $c \in (a, b)$ such that $f(c) = 0$.
Define the set

$$S = \{x \in [a, b] \mid f(x) < 0\}.$$

Note that $a \in S$ since $f(a) < 0$, and $b \notin S$ because $f(b) > 0$. Thus, S is nonempty and bounded above by b. By the completeness property of $\mathbb{R}$, there exists the supremum of S $c \in [a, b]$.

We now show that $f(c) = 0$. Suppose, for sake of contradiction, that $f(c) \neq 0$.

Case 1: $f(c) > 0$.

Let $\varepsilon = f(c)/2 > 0$. Since f is continuous at c, there exists $\delta > 0$ such that for all $x \in [a, b]$, if $|x - c| < \delta$, then

$$|f(x) - f(c)| < \frac{f(c)}{2} \implies$$

$$f(x) > f(c) - \frac{f(c)}{2} = \frac{f(c)}{2} > 0$$

for all $x \in (c - \delta, c + \delta) \cap [a, b]$.

In particular, $f(x) > 0$ for all $x \in (c - \delta, c) \cap [a, b]$, contradicting the definition of $c = \sup S$, since no $x < c$ in this neighborhood can belong to S (i.e., $f(x) < 0$). But then c would not be the least upper bound of S, a contradiction.

Case 2: $f(c) < 0$.

A similar argument shows that there exists $\delta > 0$ such that $f(x) < 0$ for all $x \in (c, c + \delta) \cap [a, b]$, implying that values just to the right of c also belong to S, again contradicting that c is an upper bound of S.

Therefore, $f(c) = 0$.

So, there exists $c \in [a, b]$ such that $f(c) = 0$. Since $f(a) < 0 < f(b)$, c cannot equal a or b, so $c \in (a, b)$.

Q.E.D.

4.2 Constructive Proof via the Bisection Method

The bisection algorithm begins with the assumption that $f(a)$ and $f(b)$ have opposite signs. By the continuity of f, the sign of f cannot change without crossing zero. At each iteration, the method computes the midpoint $c_n = \frac{a_n + b_n}{2}$ of the current interval $[a_n, b_n]$. It then evaluates $f(c_n)$ and determines whether the sign change occurs in $[a_n, c_n]$ or $[c_n, b_n]$, replacing the current interval accordingly.

This process creates a nested sequence of closed intervals $[a_n, b_n]$ with the following properties:

(i) The length of the intervals: $b_n - a_n = \frac{b-a}{2^n} \to 0$ as $n \to \infty$.

(ii) Each interval contains a sign change, so a point where $f = 0$ must lie within it.

(iii) The nested intervals converge to a unique point $x^* \in [a, b]$, by the completeness of the real numbers.

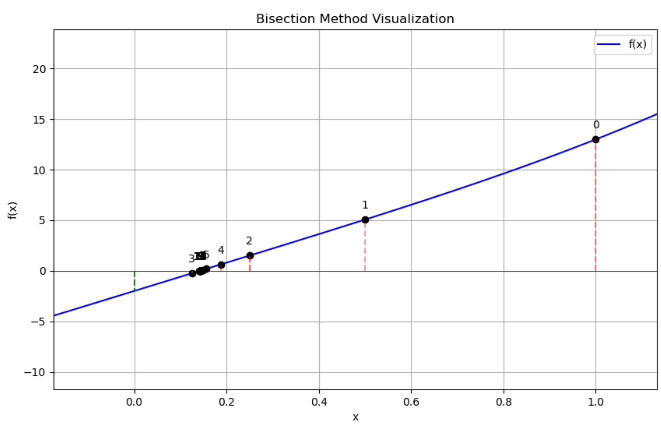
Because f is continuous and each subinterval contains a sign change, the limit point x^* must satisfy $f(x^*) = 0$. Because $(c_n)$is bounded and the interval lengths shrink to zero, the Bisection Method forms a Cauchy sequence converging to $x*$, providing a construction of the Intermediate Value Theorem.

In this way, the algorithm does not randomly assert the existence of a root, but approximates one to arbitrary precision. This approach ties the existence of the root to the computation of its approximation, as dictated by constructive mathematics.

4.3 Code(Python)

```
import numpy as np
import matplotlib.pyplot as plt

def f(x):
    return x**3 - x - 2

def bisection(f, a, b, tol=1e-5, max_iter=25):
    steps = []
    if f(a) * f(b) >= 0:
        raise ValueError("f(a) x f(b) > 0")

    for i in range(max_iter):
        c = (a + b) / 2
        steps.append((a, b, c, f(c)))

        if abs(f(c)) < tol or abs(b - a) < tol:
            break
        elif f(a) * f(c) < 0:
            b = c
        else:
            a = c
    return steps

def visualize_bisection(f, a, b, steps):
    x_vals = np.linspace(a - 1, b + 1, 1000)
    y_vals = f(x_vals)

    plt.figure(figsize=(10, 6))
    plt.plot(x_vals, y_vals, label='f(x)', color='blue')
    plt.axhline(0, color='black', linewidth=0.5)

    for i, (a_i, b_i, c_i, f_c) in enumerate(steps):
        plt.plot([a_i, a_i], [0, f(a_i)], 'g--', alpha=0.4)
        plt.plot([b_i, b_i], [0, f(b_i)], 'r--', alpha=0.4)
        plt.plot(c_i, f_c, 'ko')  # Midpoint
        plt.annotate(f'{i}', (c_i, f_c),
        textcoords="offset points",
        xytext=(0,10), ha='center')

    plt.title("Bisection Method Visualization")
    plt.xlabel("x")
    plt.ylabel("f(x)")
    plt.grid(True)
    plt.legend()
    plt.show()

a, b = 1, 2
steps = bisection(f, a, b)
visualize_bisection(f, a, b, steps)
```

4.4 Output



5 CONCLUSION

The Intermediate Value Theorem, though deceptively simple in its statement, lies at the intersection of classical analysis, topology, constructive mathematics, and numerical computation. By revisiting the theorem through the lens of numerical methods we obtain a constructive and computationally verifiable version of the result. The Bisection Method does more than approximate a root! It embodies the logical structure of the IVT in algorithmic form, offering a sequence of decreasing intervals that converge to a solution. In doing so, it bridges the gap between abstract existence theorems and practical computation.

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