Proving Classical Theorems with Numerical Methods: **The Intermediate Value Theorem Revisited**

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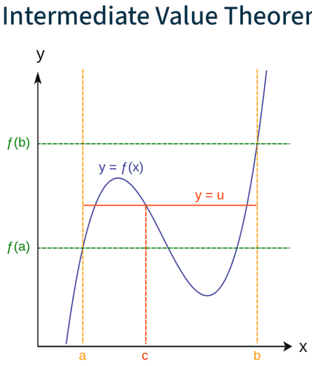
Overview

What is the Intermediate Value Theorem (IVT)?

- If f is continuous on $[a, b]$ and $f(a) \cdot f(b) < 0$, then there exists $c \in (a, b)$ such that $f(c) = 0$.
- Foundational result in calculus and real analysis.
- Typically proven using the completeness of $\mathbb{R}$.



Historical Context

- Intuitively accepted before rigorous proof.
- First formal proof: Bernard Bolzano (1817)
- Refined by Cauchy in 1821 in *Cours d'Analyse*.

Classical vs. Constructive

Classical:

- Based on completeness of real numbers.
- Existence proofs without construction.

Constructive:

- Emphasizes computable constructions.
- No object exists unless it can be explicitly constructed.

Constructive Mathematics

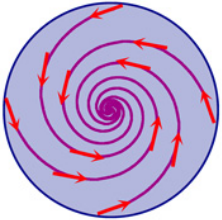
- Errett Bishop: real numbers as converging rational sequences.
- In computable analysis, IVT does not imply a computable root.
- Weakened IVT:

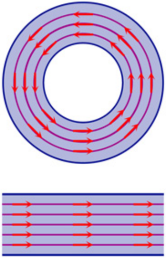
$$\forall \varepsilon > 0, \exists x \in [a, b] \text{ such that } |f(x)| < \varepsilon$$

- Avoids contradiction with Turing's undecidability results.

Ideas in Space

- Cauchy introduced geometric intuition into analysis.
- IVT reflects the **connectedness** of $[a, b]$.
- In topology: a continuous image of a connected space is connected.
- Related idea: Brouwer Fixed-Point Theorem.





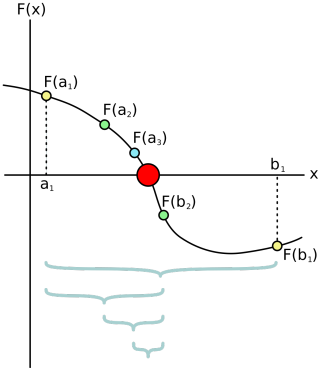
Root-Finding Algorithms

Two major techniques:

- **Bisection Method**
  - Guaranteed convergence.
  - Only requires continuity and sign change.
- **Newton's Method**
  - Fast but requires derivative and good initial guess.
  - May diverge.

- IVT ensures the existence of a root.
- Numerical methods constructively approximate it.
- A bridge between abstract theory and computation.

Why Bisection?



- Constructive and simple.
- Relies only on IVT's hypotheses.
- Does not require derivative or precise guess.
- Each step reduces error: converges to solution.

Classical Proof Outline

Theorem (IVT)

If f is continuous on $[a, b]$ and $f(a) \cdot f(b) < 0$,
then $\exists c \in (a, b)$ such that $f(c) = 0$.

- Define $S = \{x \in [a, b] \mid f(x) < 0\}$.
- Let $c = \sup S$.
- Use continuity to show $f(c) = 0$.

Constructive Proof via Bisection

Given: f continuous, $f(a) < 0$, $f(b) > 0$, and $\varepsilon > 0$.

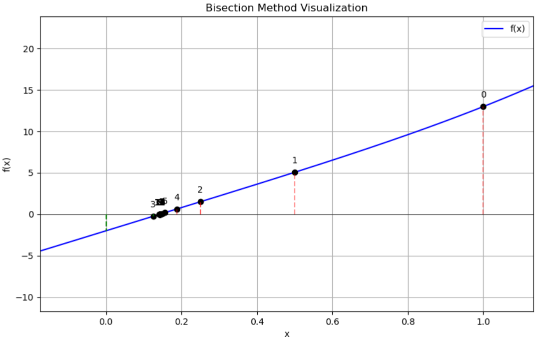
Algorithm:

1. Compute midpoint $m = \frac{a+b}{2}$.
2. Check sign of $f(m)$.
3. Narrow interval to $[a, m]$ or $[m, b]$.
4. Repeat until $|b - a| < \varepsilon$.

Result: Find x such that $|f(x)| < \varepsilon$.

Conclusion

- Classical IVT assumes existence.
- Numerical methods provide algorithmic insight.
- Bisection method embodies constructive logic.
- Bridging abstract theory and computation.



References

- Cauchy, A.-L. *Cours d'Analyse* (1821)
- Bishop, E. *Foundations of Constructive Analysis* (1967)
- Halperin, I. *Discontinuous Functions with the Darboux Property* (1958)

Questions?