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#1

Solution:

1) $\sum_{n=0}^{\infty} P_n(t) = 1 \longrightarrow \sum_{n=0}^{\infty} \frac{1}{\lambda_n} = \infty$

proof: waiting times in a pure birth process are exponentially distributed and the mean waiting time at state n is $\frac{1}{\lambda_n}$. Therefore $\sum_{n=0}^{\infty} \frac{1}{\lambda_n}$ is the mean time until system reaches infinity.

If $\sum_{n=0}^{\infty} \frac{1}{\lambda_n} < \infty$, then the mean time is finite. Let T_∞ be waiting time until the process goes to infinity or the population size goes to infinity (T_∞ can be defined as a limit of K transitions waiting time, by using monotone convergence theorem)

more precisely, Let $T_K = \sum_{n=0}^{K-1} H_{X(n)}$ be the waiting time of the first K transitions in the system, where $X(n)$ is the embedded markov chain. since in a pure birth process transitions are in one direction : $n \longrightarrow n+1$ so $T_K = \sum_{n=0}^{K-1} H_n$ is the waiting time until process reaches stat k, and by MCT we have $T_\infty = \lim_{K\to\infty} \sum_{n=0}^{K-1} H_n = \sum_{n=0}^{\infty} H_n$ is waiting time for infinitely many jumps or transitions for Yule process.

since $E[H_n] = \frac{1}{\lambda_n} \Rightarrow$ if $\sum \frac{1}{\lambda_n} < \infty \Rightarrow E[T_\infty] < \infty$

which means $P(T_\infty = \infty) = 0$ or $P(T_\infty < \infty) = 1$. so we

have infinitely many jumps in a finite time with prob 1

so for all t, $P(X_t = \infty) > 0$ (If not 1!). But

$$P(X_t = \infty) = 1 - \sum_{n=0}^{\infty} P(X_t = n) \Rightarrow \sum_{n=0}^{\infty} P(X_t = n) < 1$$

so if $\sum_{n=0}^{\infty} P_n(t) = 1 \Rightarrow \sum_{n=0}^{\infty} \frac{1}{\lambda_n} = \infty$

2) $\sum \frac{1}{\lambda_n} = \infty \Rightarrow \sum_{n=0}^{\infty} P_n(t) = 1$

In this case:

$$E[e^{-T_\infty}] = \prod_{n=0}^{\infty} E[e^{-H_n}] \quad \text{; because } H_i\text{'s are independent}$$

$$= \prod_{n=0}^{\infty} \frac{\lambda_n}{1+\lambda_n} = \prod_{n=0}^{\infty} \frac{1}{1+\frac{1}{\lambda_n}}$$

using properties of finite products since $\sum \frac{1}{\lambda_n} = \infty \Rightarrow E[e^{-T_\infty}] = 0$

$\log\left(\prod \frac{1}{1+\frac{1}{\lambda_n}}\right) = -\sum_{n=0}^{\infty} \log\left(1+\frac{1}{\lambda_n}\right)$ and there are two

cases: either $\frac{1}{\lambda_n} \to 0$ or $\frac{1}{\lambda_n} \not\to 0$. In the first case after some N we can approximate $\log(1+\frac{1}{\lambda_n}) \approx \frac{1}{\lambda_n}$; $n > N$

and since $\sum_{n=N}^{\infty} \frac{1}{\lambda_n} = \infty \Rightarrow -\sum_{n=0}^{\infty} \log(1+\frac{1}{\lambda_n}) = -\infty$

in the second case if $\frac{1}{\lambda_n} \not\to 0 \Rightarrow \log(1+\frac{1}{\lambda_n}) \not\to 0$

and obviousely $-\sum_{n=0}^{\infty} \log(1+\frac{1}{\lambda_n}) = -\infty \Rightarrow \prod_{n=0}^{\infty} (\frac{1}{1+\frac{1}{\lambda_n}}) = 0$

so $E[e^{-T_\infty}] = 0 \Rightarrow P(T_\infty = \infty) = 1$ or $P(T_\infty < \infty) = 0$

so $P(X_t = \infty) = 0 \Rightarrow 1 - \sum_{n=0}^{\infty} P(X_t = n) = 0 \Rightarrow \sum_{n=0}^{\infty} P_n(t) = 1$

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Question #2

#2. we can find the mean using Dynkin's formula

$$Af(x) = \lim_{t \to 0} \frac{E^x[f(X_t)] - f(x)}{t}$$

or $$Af(x) = \lim_{\Delta t \to 0} \frac{E[f(X_{t+\Delta t}) \mid X_t = x] - f(X_t)}{\Delta t}$$

where A is the infinitesimal generator of X_t. If we take $f(x) = x$, then we can find the mean from that. Let X_t be the population size at time t. Then we have,

$$E[X_{t+\Delta t} \mid X_t] = (X_t + 1)(\lambda X_t \Delta t + \nu \Delta t) + (X_t - 1)\mu X_t \Delta t + X_t(1 - \lambda X_t \Delta t - \nu \Delta t - \mu X_t \Delta t)$$

Therefore we have

$$E[X_{t+\Delta t} \mid X_t] = \lambda X_t \Delta t + \nu \Delta t - \mu X_t \Delta t + X_t$$

hence,

$$\lim_{\Delta t \to 0} \frac{E[(X_{t+\Delta t}) \mid X_t] - X_t}{\Delta t} = \lambda X_t - \mu X_t + \nu$$

By taking Expectation from both sides and considering that

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$$E\left[E[X_{t+\Delta t} \mid X_t]\right] = E[X_{t+\Delta t}] \quad \text{(law of total expectation)}$$

we have

$$\lim_{\Delta t \to 0} \frac{E[X_{t+\Delta t}] - E[X_t]}{\Delta t} = (\lambda-\mu)E[X_t] + \nu$$

$$\Rightarrow \quad \frac{d}{dt}E[X_t] = (\lambda-\mu)E[X_t] + \nu$$

which is the same as differential equation for deterministic model: $\frac{dn}{dt} = (\lambda-\mu)n + \nu$, so the mean, $E[X_t]$, is the solution of this differential equation.

In general, this is not true. For example assume $\lambda_n = \lambda n^2$ and $\mu_n = \mu n^2$, therefore for deterministic model we have

$$\frac{dn}{dt} = (\lambda-\mu)n^2 + \nu$$

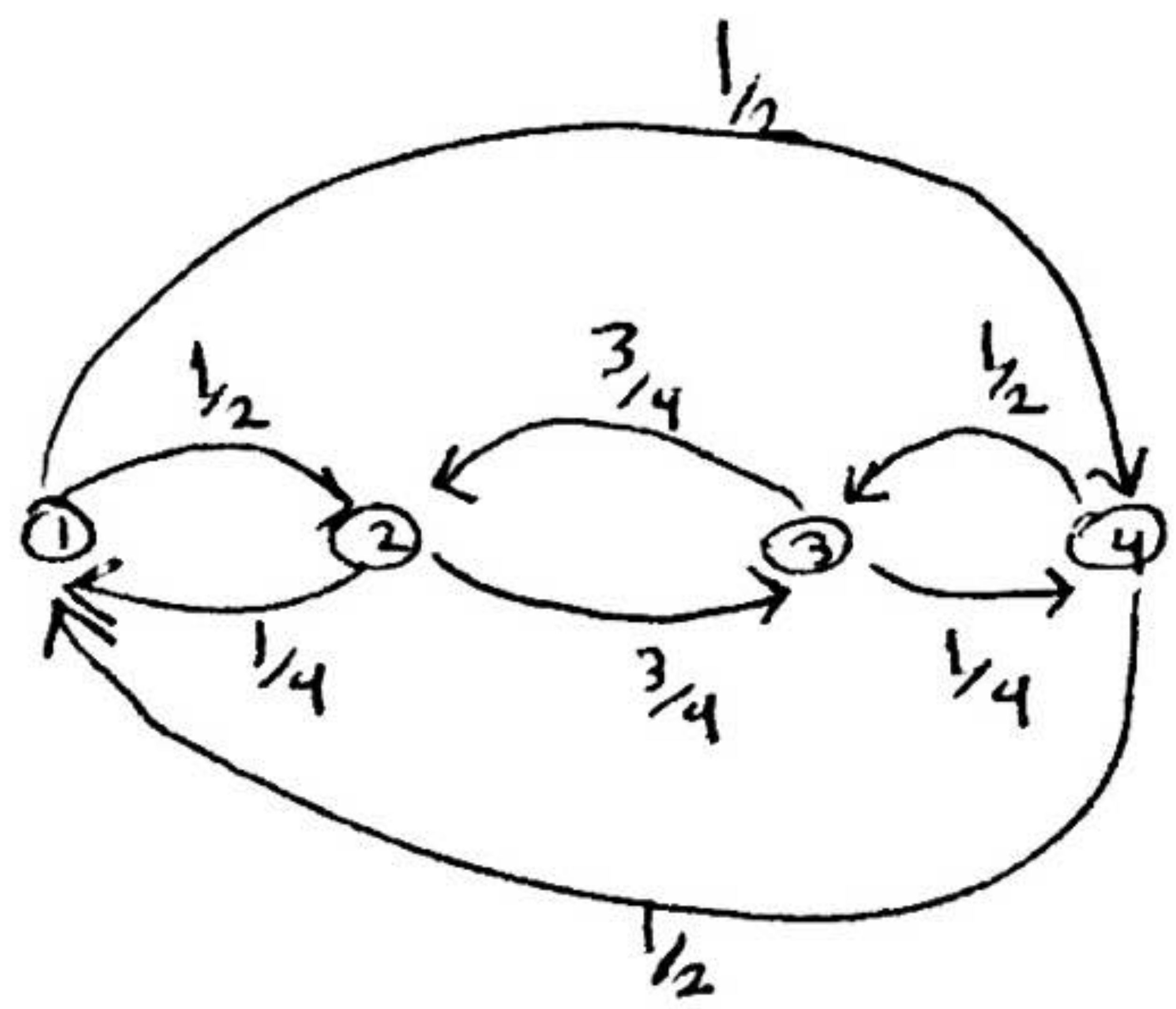
but for stochastic model we have

$$\frac{d\bar{n}}{dt} = (\lambda-\mu)E[\bar{n}^2] + \nu \quad \boxed{\text{or} \quad \frac{d}{dt}E[X_t] = (\lambda-\mu)E[X_t^2] + \nu}$$

which is not the same differential equation unless we assume $E[\bar{n}^2] = E[\bar{n}]^2$! which is not true unless $\bar{n}$ is constant!

#3

Part (a):



Part (b):

Let $P_{ij} = P(X_1 = j \mid X_0 = i)$. Then $P_{12} = \frac{1}{2} > 0$, $P_{14} = \frac{1}{2} > 0$

$P_{21} = \frac{1}{4} > 0$, $P_{23} = \frac{3}{4} > 0$, $P_{32} = \frac{3}{4} > 0$, $P_{34} = \frac{1}{4} > 0$, $P_{41} = \frac{1}{2} > 0$

$P_{43} = \frac{1}{2} > 0$. If we look at the powers of transition

matrix P we have

$$P^2 = \begin{pmatrix} 0.375 & 0 & 0.625 & 0 \\ 0 & 0.687 & 0 & 0.3125 \\ 0.3125 & 0 & 0.687 & 0 \\ 0 & 0.625 & 0 & 0.375 \end{pmatrix}$$

So $P^{(2)}_{11} > 0$, $P^{(2)}_{13} > 0$, $P^{(2)}_{22} > 0$, $P^{(2)}_{24} > 0$, $P^{(2)}_{31} > 0$, $P^{(2)}_{33} > 0$

$P^{(2)}_{42} > 0$, $P^{(2)}_{44} > 0$. As we can see for any i, j there exists

n_{ij} such that $P(X_{n_{ij}} = j \mid X_0 = i) > 0$. In this case

$n_{ij} = 1$ or 2. Therefore the chain is irreducible.

Also we can see $P_{ii}^{(2k-1)} = 0$, $P_{ii}^{(2k)} > 0$, for all k.

So <u>the chain is periodic with period 2</u>.

Since the chain is irreducible so there is only one class of communication and therefore if one state is recurrent, then all other states are also recurrent. Moreover since we have finite states (4 states) so there exists at least one recurrent state.

From part (d) (next part) in this question we can see the chain has a unique stationary distribution and so we have $\pi_i = \dfrac{1}{m_i}$ where $m_i = E[T_{ii}]$

since $\pi_i > 0 \Rightarrow m_i = E[T_{ii}] < \infty$ and hence the chain is <u>positive recurrent</u>.

Part(d)

we solve $\pi P = P \Rightarrow \pi = \left(\frac{1}{4}, \frac{5}{16}, \frac{1}{4}, \frac{3}{16}\right)$.

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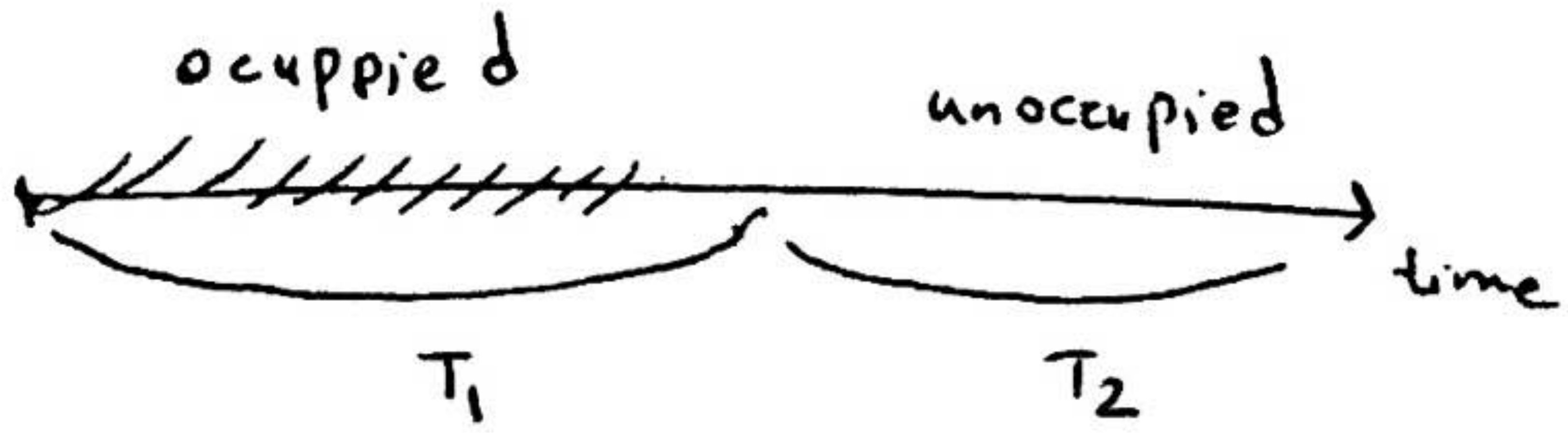
Question #4

#4

Solution: Let T_1 be the time during which a protein is bound and T_2 be the time interral after a protein leaves the prometer until a new protein arrives.

So $E[T_1] = \mu$ and $E[T_2] = \frac{1}{\lambda}$ because T_2 is exponentially distributed with parameter λ.



from above picture it's clear that this process is repeated and the fraction of time that the prometer is unoccupied is

$$\frac{E[T_2]}{E[T_1 + T_2]} = \frac{\frac{1}{\lambda}}{\frac{1}{\lambda} + \mu}$$

$$= \frac{1}{1 + \lambda\mu}$$

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Question #5

#5

Solution: Let λ be parameter of this poisson distribution

$$P(T(t) < u) = P(\text{at least one action occurs in } [t-u, t+u])$$

$$= 1 - P(\text{No action in } [t-u, t+u]) = 1 - e^{-2\lambda u}$$

because $P(\text{n actions in } [0, u]) = \frac{e^{-\lambda u} (\lambda u)^n}{n!}$

so $f_{T(t)}(u) = 2\lambda e^{-2\lambda u} \rightarrow T(t)$ is exponentially distributed

with parameter 2λ and therefore $E[T(t)] = \frac{1}{2\lambda}$

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