#1. Solution:

We can find r_k using the formula $P = 100\, e^{-kr_k}$ and so

$$r_k = \frac{52}{M} \ln\left(\frac{100}{P}\right)$$

* For 13-week T-bill : $M = 13$ and $P = 99.7$. Therefore

$$r_k = 4 \ln\left(\frac{100}{99.7}\right) = 0.012 \Rightarrow r_k = 1.2\,\%$$

** For 26-week T-bill : $M = 26$ and $P = 99.4$, and so

$$r_k = 2 \ln\left(\frac{100}{99.4}\right) = 0.012 \text{ or } r_k = 1.2\,\%$$

*** For 52-week T-bill we have $M = 52$ and $P = 99$ and so

$$r_k = \ln\left(\frac{100}{99}\right) = 0.10 \Rightarrow r_k = 1\,\%$$

New spot rates for the next week can be calculated in the Same way and we get

* For 13-week T-bill, $r_k = 0.0128 = 1.28\,\%$

** For 26-week T-bill, $r_k = 0.0122 = 1.22\,\%$

*** For 52-week T-bill, $r_k = 0.103 = 1.03\,\%$

and the percentage change is

For 13-week T-bill : $\frac{1.28-1.2}{1.2} = 0.067 = 6.7\%$

For 26-week T-bill : $\frac{1.22-1.2}{1.2} = 1.66\%$

For 52-week T-bill : $\frac{1.03-1.00}{1.00} = 3\%$

All values are positive, so the yield curve is an increasing function of time to maturity.

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Question #2 : Solution

The price - yield formula is

$$\frac{P(y)\, e^{yT}}{100} = e^{-My/2} + \frac{C}{1000\,(1-e^{-y/2})}\left[1-e^{-My/2}\right]$$

If we assume she pays the first payment immidiately $(T=0)$

and time to maturity is 5, then

$$\frac{P(y)}{100} = e^{-5y} + \frac{C}{1000\,(1-e^{-y/2})}\left[1-e^{-5y}\right]$$

we are given $C = 18$ and $y = 0.012$ and so

$$P(y) \simeq \$111.69$$

*** we could also use the approximation $e^{y} \simeq 1+y$ and

so $$\frac{P(y)}{100} = (1+y)^{-M/2} + \frac{C}{500\,y}\left[1-(1+y)^{-M/2}\right]$$

Total ~~tim~~ yield to maturity is $y_T = \frac{C}{5P} + \frac{1}{T}\ln\left(\frac{100}{P}\right)$

$$\Rightarrow y_T = \frac{18}{(5)\,111.69} + \frac{1}{5}\ln\left(\frac{100}{111.69}\right) \simeq 0.01 > 0$$

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Question #3: solution

For the first bond with $T=5$, $c=16$ we have

$$\frac{P_1(y)\, e^{yT}}{100} = e^{-5y} + \frac{16}{1000\,(1-e^{-y/2})}\left(1-e^{-5y}\right)$$

and for the second bond with $T=4$, $c=20$ we have

$$\frac{P_2(y)\, e^{yT}}{100} = e^{-4y} + \frac{20}{1000\,(1-e^{-y/2})}\left(1-e^{-4y}\right)$$

we can show that for $y \geq 0$, we have always

$P_2 \geq P_1 \rightarrow P_2$ is likely to cost more

and the reason is as follows:

First, we can see for $y \geq 0$: $e^{-4y} \geq e^{-5y}$. Moreover

we can show $20\,(1-e^{-4y}) \geq 16\,(1-e^{-5y})$ for all $y \geq 0$.

because let $f(y) = 20\,(1-e^{-4y})$ and $g(y) = 16\,(1-e^{-5y})$

so $f'(y) = \frac{\partial f}{\partial y} = 80\, e^{-4y}$ and $g'(y) = 80\, e^{-5y}$ and since

$e^{-4y} \geq e^{-5y} \Rightarrow f'(y) \geq g'(y)$. This means $(f(y)-g(y))' \geq 0$

or $f(y)-g(y)$ is an increasing function of y. Therefore for $y \geq 0 \Rightarrow \quad f(y)-g(y) \geq f(0)-g(0) = 0 \Rightarrow f(y) \geq g(y)$

so $\quad 20(1-e^{-4y}) \geq 16(1-e^{-5y})$, for all $y \geq 0$.

This proves that $P_2(y) \geq P_1(y)$, for $y > 0$.

If you don't want to use a mathematical proof, you can show this by any Programming language. For example in the next page you can see the plot for $P_1(y)$ and $P_2(y)$ and as you can see: $P_2(y) \geq P_1(y)$ for $y \in (0, \infty)$ and also it's clear that $\lim_{y \to 0} P_2(y) = \lim_{y \to 0} P_1(y)$.

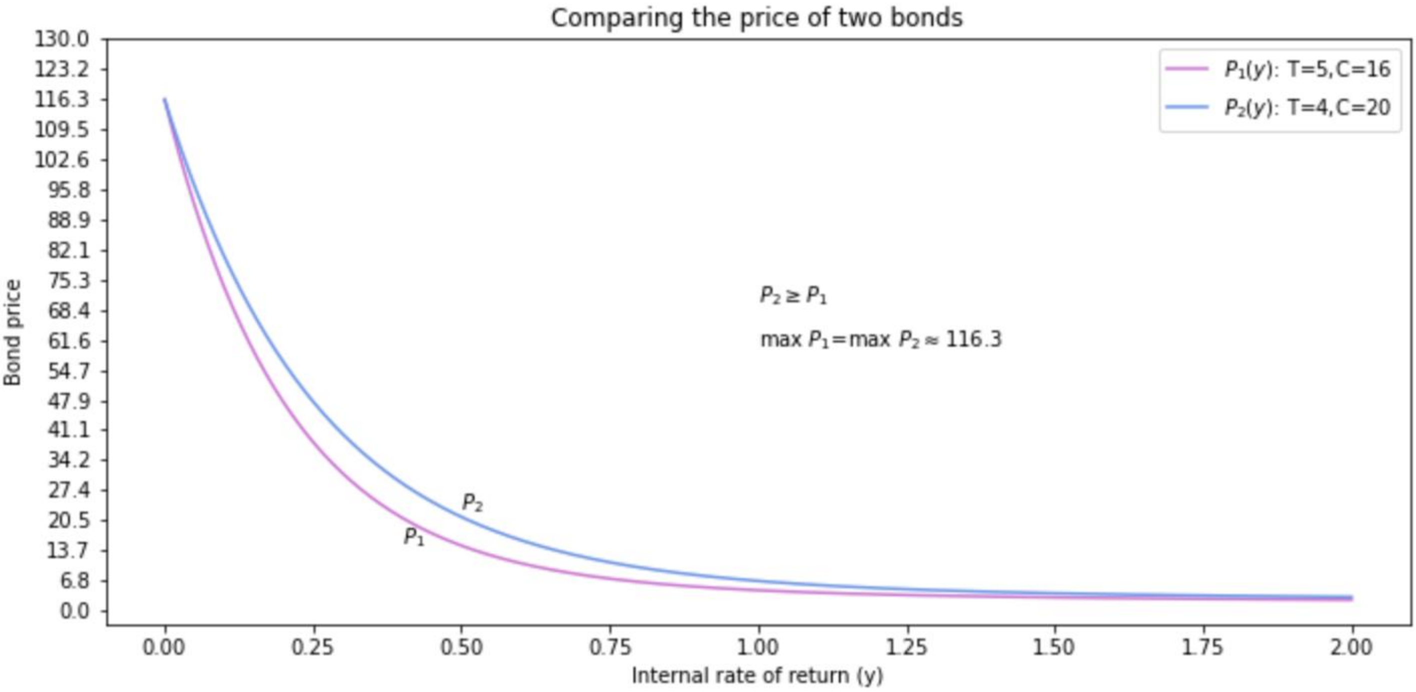
This is where the maximum of two functions occurs and so you can conclude that both P_1 and P_2 have the same maximum which also answers the second part of the question, but we answer the second part separately.

(ii) For part (2), we must have $\frac{c}{5P} + \frac{1}{T}\ln\left(\frac{100}{P}\right) \geq 0$

and so the value $P_{max} = \frac{100}{X}$ is the maximum

where $\ln x = -\frac{cTx}{500}$. For both bonds we have $cT = 80$

and so they must have the same maximum (It's also clear from the plot)


Comparing the price of two bonds
$P_1(y)$: T=5,C=16
$P_2(y)$: T=4,C=20
$P_2 \geq P_1$
max P_1=max $P_2 \approx$ 116.3
P_1
P_2
Bond price
Internal rate of return (y)
130.0
123.2
116.3
109.5
102.6
95.8
88.9
82.1
75.3
68.4
61.6
54.7
47.9
41.1
34.2
27.4
20.5
13.7
6.8
0.0
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00

and the maximum is obtained by solving $\ln x = -\frac{80x}{500}$ for x and then $P_{max} = \frac{100}{x}$. To slove $\ln x = -\frac{80x}{500}$ you can use Excel or any programming language and the solution is $\hat{x} \approx 0.87$ and hence $P_{max} = \frac{100}{0.87} = 114.94$ which is close to the exact P_{max} from the plot (≈ 116).

The results are reasonable, because the relation $\ln x = -\frac{CTx}{500}$ only depends on CT and CT is the same for both bonds although $P_2 \geqslant P_1$ for all $y \geqslant 0$, they have the same maximum and this can be seen from the plot.

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Question # 4: Solution

$c = 12$, $T = 4$, $P = 106$

1) current yield $= y_c = \dfrac{c}{5P} = \dfrac{12}{5(106)} = 0.02264 = 2.264\%$

2) capital gains $= y_{cg} = \dfrac{1}{T} \ln\left(\dfrac{100}{P}\right) = \dfrac{1}{4} \ln\left(\dfrac{100}{106}\right) = -0.0145$

3) $y_T = y_c + y_{cg} \approx 0.008 = 0.8\%$

4) To find IRR, we solve the equation

$$\frac{106}{100} = e^{-4y} + \frac{12}{1000(1-e^{y/2})}\left[1-e^{-4y}\right]$$

for y and again this can be done by Excel, Mathematica

MATLAB, Python or any other PL and the solution is

$$\hat{y} \approx 0.009 = 0.9\%$$

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Question 5: solution:

half-year $\Rightarrow$ $t = \frac{1}{2}$

$r = 0.06$

$E_0 = 10000 \Rightarrow B_0 = 160(100) - 10000 = 6000$

(a) $V(\frac{1}{2}) = 164(100) = 16400 \Rightarrow E(\frac{1}{2}) = V(\frac{1}{2}) - B_0 e^{\frac{1}{2}r}$

$$= V(\tfrac{1}{2}) - 6000\, e^{0.03}$$

$$= 16400 - 6000\, e^{0.03}$$

$$= 10217.27$$

and return of equity after 6 months ($t = \frac{1}{2}$) is

$$r_E(t) = r_E(\tfrac{1}{2}) = \frac{1}{1/2} \ln\left(\frac{E(\frac{1}{2})}{E_0}\right) = 2 \ln\left(\frac{10217.27}{10000}\right)$$

$$= 0.043$$

(b) $V(\frac{1}{2}) = 17000 \Rightarrow E(\frac{1}{2}) = 17000 - 6000\, e^{0.03} = 10817.27$

and $r_E(\frac{1}{2}) = 2 \ln\left(\frac{10817.27}{10000}\right) = 0.157$

(c) $V(\frac{1}{2}) = 18000 \Rightarrow E(\frac{1}{2}) = 18000 - 6000\, e^{0.03} = 11817.27$

and $r_E(\frac{1}{2}) = 2 \ln\left(\frac{11817.27}{10000}\right) = 0.333$

Part (ii)

$E_0 = 12000 \Rightarrow B_0 = 16000 - 12000 = 4000$ and all other

parameters are the same. The same calculations give us

(a) $r_E(t) = r_E(\frac{1}{2}) \simeq 0.0458$ ✓

$\boxed{\text{better return on equity}}$

(b) $r_E(\frac{1}{2}) \approx 0.141$ X

(c) $r_E(t) \simeq 0.29$ X

Comparing with the previous results, we see in part (a) we have a better return on equity. (For parts (b) and (c) $r_E(t)$ have decreased)

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