Vector and Matrix Theory

Chapter 3 : Multiplication and Inverse Matrices

Husni Rois Ali & Dzuhri Radityo Utomo & Syukron Abu Ishaq Alfarozi

Department of Electrical and Information Engineering
Universitas Gadjah Mada

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Section 1 :
Matrix Multiplication

Matrix Multiplication

- There are 4 (four) different ways :
  - Element-wise : Dot product of two vectors (rows dot columns)
  - Combination of columns
  - Combination of rows
  - Columns multiply rows: outer product
- Suppose that we have 2 matrices.
  - Matrix A : m by n (m rows and n columns) or $R^{m \times n}$
  - Matrix B : n by p (n rows and p columns) or $R^{n \times p}$
- Matrix A can be multiplied by matrix B **if and only if** the number of columns of matrix A is equal to the number of rows of matrix B.
- Suppose that the multiplication result between A and B is given by $C = AB$.
  - Matrix C : m by p (m rows and p columns)

$$\begin{bmatrix} m \textbf{ rows} \\ n \text{ columns} \end{bmatrix} \times \begin{bmatrix} n \text{ rows} \\ p \textbf{ columns} \end{bmatrix} = \begin{bmatrix} m \textbf{ rows} \\ p \textbf{ columns} \end{bmatrix}$$

Element-wise : Dot Product of Two Vectors

- Suppose that the multiplication result between matrix A by B is C.

$$\begin{bmatrix} & & * & & & \\ * & * & c_{ij} & * & * & * \\ & & * & & & \\ & & * & & & \end{bmatrix} =$$

$$\begin{bmatrix} * & & & \\ a_{i1} & a_{i2} & \cdots & a_{i5} \\ * & & & \\ * & & & \end{bmatrix} \times \begin{bmatrix} * & * & b_{1j} & * & * & * \\ & & b_{2j} & & & \\ & & b_{3j} & & & \\ & & b_{4j} & & & \\ & & b_{5j} & & & \end{bmatrix}$$

- From the example above, the resulting c_{ij} is given by :

$$\begin{aligned} c_{ij} &= A_i^T B_j = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{i5} b_{5j} \\ &= \sum_{k=1}^{5} a_{ik} b_{kj} \end{aligned}$$

Element-wise : Dot Product of Two Vectors

- For a more general case, the resulting c_{ij} is given by :

$$\begin{aligned} c_{ij} = A_i^T B_j &= a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} \\ &= \sum_{k=1}^{n} a_{ik}b_{kj} \end{aligned}$$

- This formula is the same formula for calculating the dot product between vector $A_i^T = (a_{i1}, a_{i2}, \cdots, a_{in})$ and $B_j = (b_{1j}, b_{2j}, \cdots, b_{nj})^T$.
- Therefore, it can be concluded that the i-th row and j-th column of the resulting matrix C is the dot product between the i-th row vector of matrix A and the j-th column vector of matrix B.
- This fact also explains why matrix A can be multiplied by matrix B **if and only if** the number of columns of matrix A is equal to the number of rows of matrix B.
- Dot product operation requires the two vectors to have the same dimension.

Combination of Columns

- We have learn the idea about the multiplication between matrix and vector in the previous chapter when we discuss about the column picture of a matrix.
- Suppose that matrix A is multiplied by a column vector B_1 as follow :

$$c = AB_1 = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{bmatrix}$$

$$= b_{11} \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + b_{21} \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + b_{n1} \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} c_{11} \\ c_{21} \\ \vdots \\ c_{m1} \end{bmatrix}$$

- The resulting column vector c is a linear combination of column vectors of A.

Geometry of $c = AB_1$

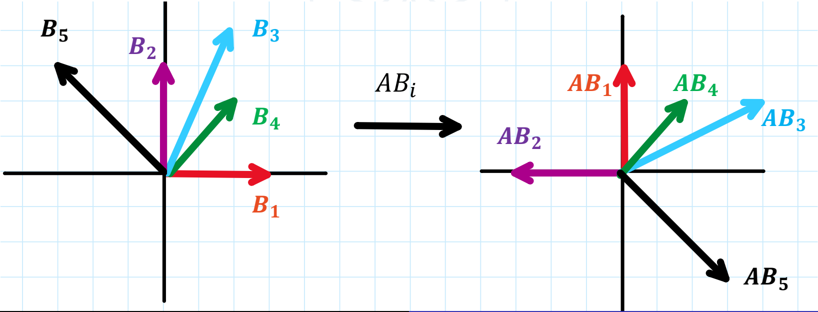
- Suppose that we have :

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- What is the outcome of $c_i = AB_i$ for the following B_i

$$B_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix},\ B_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix},\ B_3 = \begin{bmatrix} 2 \\ 4 \end{bmatrix},\ B_4 = \begin{bmatrix} 2 \\ 2 \end{bmatrix},\ \ B_5 = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$$

- This result:



Combination of Columns

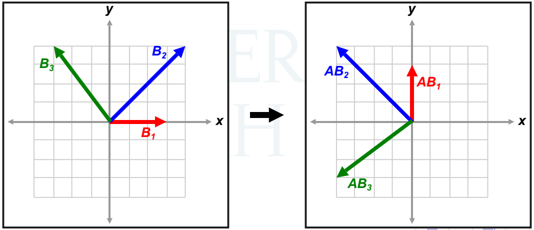
- Suppose that we have several column vectors $B_1, B_2, \cdots B_p$, arranged into matrix B :

$$B = \begin{bmatrix} B_1 & B_2 & \cdots & B_p \end{bmatrix}$$

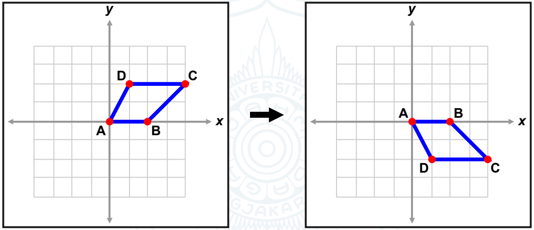
- Therefore, the multiplication between matrix A and B is given by :

$$C = AB = A\begin{bmatrix} B_1 & B_2 & \cdots & B_p \end{bmatrix} = \begin{bmatrix} AB_1 & AB_2 & \cdots & AB_p \end{bmatrix}$$

  i.e. **the columns of C is a linear combination of columns of A**.
- This result indicates the multiplication between matrix A and B can be seen as if we are transforming each column vector of B into new column vectors $AB_1, AB_2, \cdots, AB_p$.



Combination of Columns



- This concept can be seen easily in graphic manipulation as in the figure shown above.
- Using a single matrix multiplication operation, we can move each vertex points of an object into new vertex points, so that the overall object is also moved to a new position/orientation.

Combination of Rows

- Suppose that row vector A_1^T is multiplied by a matrix B as follow :

$$A_1^T B = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix}$$
$$= a_{11} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \end{bmatrix} + \cdots + a_{1n} \begin{bmatrix} b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix}$$
$$= \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1p} \end{bmatrix}$$

- The resulting row vector is a linear combination of row vectors of B.
- Suppose that we have several row vectors $A_1^T, A_2^T, \cdots A_m^T$, arranged into matrix A :

$$A = \begin{bmatrix} A_1^T \\ A_2^T \\ \vdots \\ A_m^T \end{bmatrix}$$

Combination of Columns

- Therefore, the multiplication between matrix A and B is given by :

$$C = AB = \begin{bmatrix} A_1^T \\ A_2^T \\ \vdots \\ A_m^T \end{bmatrix} B = \begin{bmatrix} A_1^T B \\ A_2^T B \\ \vdots \\ A_m^T B \end{bmatrix} = \begin{bmatrix} C_1^T \\ C_2^T \\ \vdots \\ C_m^T \end{bmatrix}$$

- This result indicates in a multiplication between matrix A and B, each rows of the resulting matrix can be seen as the different sets of linear combination of row vectors in matrix B.
- **It can also be seen that the rows of C is a linear combinations of rows of B**
- We will see soon the importance of this concept in the next Section.

Columns Multiply Rows

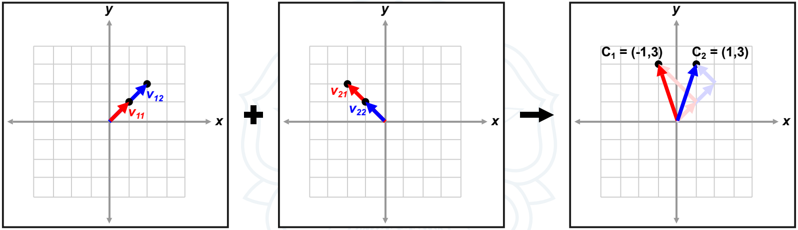
- Suppose that the multiplication between matrix A and B is given as follow :

$$
\begin{aligned}
AB &= \begin{bmatrix} \mathbf{col}_1 & \mathbf{col}_2 & \cdots & \mathbf{col}_n \end{bmatrix} \begin{bmatrix} \mathbf{row}_1^T \\ \mathbf{row}_2^T \\ \vdots \\ \mathbf{row}_n^T \end{bmatrix} \\
&= \mathbf{col}_1\mathbf{row}_1^T + \mathbf{col}_2\mathbf{row}_2^T + \cdots + \mathbf{col}_n\mathbf{row}_n^T
\end{aligned}
$$

- To understand the meaning of this operation, let us take a look in an example :

$$
\begin{aligned}
AB &= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 3 & 3 \end{bmatrix}
\end{aligned}
$$

Columns Multiply Rows



- Suppose that we re-write the aforementioned equation as follow :

$$AB = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 3 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} \boldsymbol{v}_{11} & \boldsymbol{v}_{12} \end{bmatrix} + \begin{bmatrix} \boldsymbol{v}_{21} & \boldsymbol{v}_{22} \end{bmatrix} = \begin{bmatrix} \boldsymbol{C}_1 & \boldsymbol{C}_2 \end{bmatrix}$$

- The illustration above shows the movement of each column vectors as each additional terms is added to the calculation.

Section 2 :
Elimination Using Matrices

Matrix Multiplication : Combination of Rows

- Let us take a look at some examples below.

$$C_1 = A_1 B = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

$$C_2 = A_2 B = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 7 & 8 & 9 \end{bmatrix}$$

$$C_3 = A_3 B = \begin{bmatrix} -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 \end{bmatrix}$$

- The resulting vectors :
  - C_1 : 1st row of B
  - C_2 : 3rd row of B
  - C_3 : (2nd row of B) - (1st row of B)
- Gauss Elimination can be represented using Matrix Multiplication

Augmented Matrix

- Now, let us take a look at a set of equations with 3 unknowns as follow :

$$\begin{array}{ccccccc} 1x_1 & + & 2x_2 & + & 1x_3 & = & 2 \\ 3x_1 & + & 8x_2 & + & 1x_3 & = & 12 \\ 0x_1 & + & 4x_2 & + & 1x_3 & = & 2 \end{array}$$

- This set of equations can be represented as matrix multiplication as follow :

$$\underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 2 \\ 12 \\ 2 \end{bmatrix}}_{b}$$

- This set of equations is usually written as :

$$Ax = b$$

- **Augmented Matrix** is a matrix where b is included as an extra column of A :

$$\textbf{Augmented Matrix} = \left[\begin{array}{c|c} A & b \end{array} \right]$$

Elimination with Matrices

- The augmented matrix for the aforementioned set of linear equations is given by :

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 3 & 8 & 1 & 12 \\ 0 & 4 & 1 & 2 \end{array}\right]$$

- The first step is to make the element $(2,1)$ equal to zero. It is performed by subtracting the 2nd row, by 3 times the 1st row (Row 2=Row 2 - $3 \times$ Row 1):

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{21}} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 3 & 8 & 1 & 12 \\ 0 & 4 & 1 & 2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 4 & 1 & 2 \end{array}\right]$$

- And then we can continue to the second step to make the elements (3,1) and (3,2) equal to zero.
- Can you see what matrix makes the element (3,1) equal to zero?

Elimination with Matrices

- For (3,1), we do nothing. Or to be "exact" subtracting the 3rd row by 0 times the first row (Row 3=Row 3 - 0$\times$ Row 1):

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{31}} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 4 & 1 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 4 & 1 & 2 \end{array}\right]$$

- For (3,2), it is performed by subtracting the 3rd row, by 2 times the 2nd row ($R_3 = R_3 - 2 \times R_2$) :

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}}_{E_{32}} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 4 & 1 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 0 & 5 & -10 \end{array}\right]$$

Elimination with Matrices

- Therefore, we can conclude that for this case, the Gauss Elimination process can be represented as :

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}}_{E_{32}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{31}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{21}} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 3 & 8 & 1 & 12 \\ 0 & 4 & 1 & 2 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 0 & 5 & -10 \end{array}\right]$$

Elimination with Matrices

- It can be also represented using a single elimination matrix E as follow :

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 6 & -2 & 1 \end{bmatrix}}_{E=E_{32}E_{31}E_{21}} \underbrace{\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 3 & 8 & 1 & 12 \\ 0 & 4 & 1 & 2 \end{array}\right]}_{[A \mid b]} \rightarrow \underbrace{\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 0 & 5 & -10 \end{array}\right]}_{[U \mid Eb]}$$

- The elimination matrix E_{ij} indicates the operation to make the element (i, j) equal to zero.
- Finally, we can re-write the aforementioned matrix operation using a single equation :

$$Ax = b \quad \rightarrow \quad EAx = Eb \quad \rightarrow \quad Ux = Eb$$

where

$$U = EA$$

is the resulting upper triangular matrix.

Section 3 :
Inverse Matrices

Inverse Matrix

- The matrix A is *invertible* if there exists a matrix A^{-1} that "inverts" A so that :

$$A^{-1}A = AA^{-1} = I$$

where I is the identity matrix :

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

- Suppose that matrix A transforms a vector x_1 into another vector x_2, so that :

$$Ax_1 = x_2$$

- In the case of vector x_2 and matrix A are known, the original vector x_1 can be calculated using inverse matrix of A as follow :

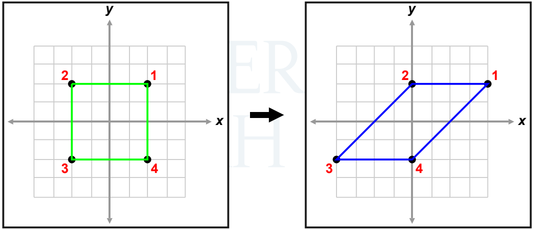
$$x_1 = A^{-1}x_2$$

Invertible Matrix

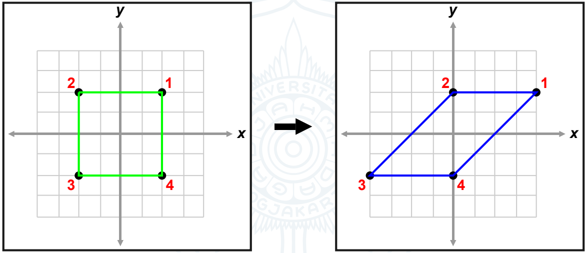
- The matrix A is invertible **if and only if** matrix A is a non-singular matrix.
- On the other hand, if A is a singular matrix, then A is non-invertible.
- Let us take an example of the following matrix :

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} x_{new} \\ y_{new} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{old} \\ y_{old} \end{bmatrix} = \begin{bmatrix} x_{old} + y_{old} \\ y_{old} \end{bmatrix}$$

- The transformation caused by this matrix can be illustrated using the figure below.



Invertible Matrix



- This figure shows that the matrix A transforms the square into a new 2D object.
- This object can be transformed back to the original shape using a the following equation :

$$\begin{bmatrix} x_{old} \\ y_{old} \end{bmatrix} = \begin{bmatrix} x_{new} - y_{new} \\ y_{new} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{new} \\ y_{new} \end{bmatrix} \rightarrow A^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

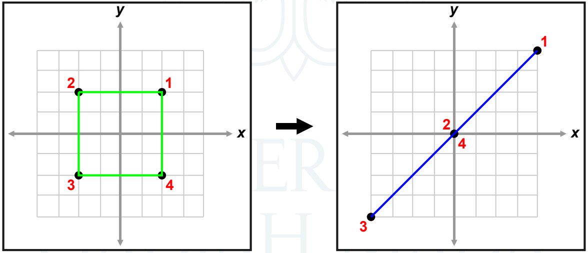
- Therefore, it can be concluded that matrix A is an Invertible Matrix.

Non-Invertible Matrix

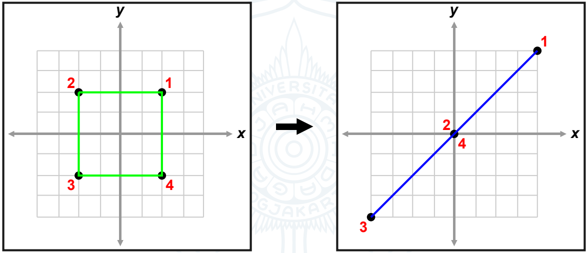
- Now, let us take an example of the following matrix :

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} x_{new} \\ y_{new} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_{old} \\ y_{old} \end{bmatrix} = \begin{bmatrix} x_{old} + y_{old} \\ x_{old} + y_{old} \end{bmatrix}$$

- The transformation caused by this matrix can be illustrated using the figure below.



Non-Invertible Matrix



- This figure shows that the matrix A transforms the 2D object (square) into a new 1D object (line).
- Moreover, it can be seen that point 2 and 4 are transformed into the same new positions.
- Therefore, it is impossible to transformed it back to the original shape.
- Therefore, it can be concluded that matrix A is a Non-Invertible Matrix.

Tests for Invertibility

- Matrix A is non-invertible if :
  - The determinant of A is equal to **zero**.
  - During Gauss Elimination process, it is found a **zero row vector** $\begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix}$.
  - Vector $x = (0, 0, \cdots, 0)$ is not the only solution for $Ax = 0$.
- Now, let us take an example of the following matrix :

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

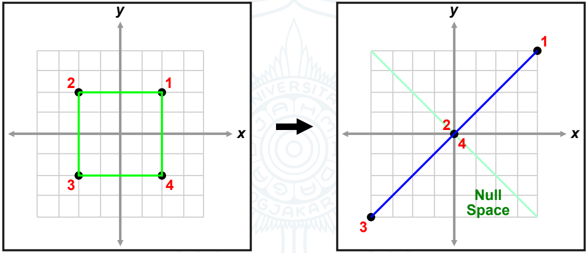
- The $Ax = 0$ form can be written as :

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- This equation indicates that :

$$x + y = 0 \quad \rightarrow \quad y = -x$$

Tests for Invertibility



- This equation indicates that :

$$x + y = 0 \quad \rightarrow \quad y = -x$$

- As the result, all the points crossed by line $y = -x$ is the Null Space of matrix A.
- There are an infinite number of vectors other than $(0, 0)$ that satisfy the $Ax = 0$ equation. Therefore, this matrix is <u>Non-Invertible</u>.

Gauss-Jordan Elimination

- Gauss-Jordan Elimination is an algorithm/method to compute the inverse of a Matrix.
- As an example, let us compute the inverse of the following matrix :

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$$

- Suppose that the inverse matrix is defined as :

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

- Therefore :

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A^{-1}} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I}$$

Gauss-Jordan Elimination

- In other words, we need to simultaneously solve the following equations :

$$\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- We can construct an Augmented Matrix as follow :

$$\left[\ A \mid I\ \right]$$

- Suppose that we multiply this augmented matrix with an elimination matrix E so that :

$$E \left[\ A \mid I\ \right] = \left[\ EA \mid E\ \right]$$

- If $E = A^{-1}$, therefore the resulting augmented matrix is given by :

$$\left[\ A^{-1}A \mid A^{-1}I\ \right] = \left[\ I \mid A^{-1}\ \right]$$

Gauss-Jordan Elimination

- Therefore, to compute the inverse of matrix A, we need to transform an augmented matrix $[\ A \mid I\ ]$ into $[\ I \mid A^{-1}\ ]$.
- Let us take a look at the previous example matrix. The augmented matrix for this case is given by :

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 7 & 0 & 1 \end{array}\right]$$

- Row 2 = (Row 2) - 3×(Row 1) :

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & -3 & 1 \end{array}\right]$$

- After this step, we get an upper triangular matrix as in Gauss Elimination. But we need to go further.
- Row 1 = (Row 1) - 2×(Row 2) :

$$\left[\begin{array}{cc|cc} 1 & 0 & 7 & -2 \\ 0 & 1 & -3 & 1 \end{array}\right] \quad \rightarrow \quad A^{-1} = \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}$$

Gauss-Jordan Elimination (Non-Singular Matrix)

- Now let us take a look at the following matrix :

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 1 \\ 0 & 4 & 1 \end{bmatrix}$$

- And then we construct the following augmented matrix :

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 3 & 6 & 1 & 0 & 1 & 0 \\ 0 & 4 & 1 & 0 & 0 & 1 \end{array}\right]$$

- Row 2 = (Row 2) - 3×(Row 1) : $\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & -3 & 1 & 0 \\ 0 & 4 & 1 & 0 & 0 & 1 \end{array}\right]$

- Exchanging Row 2 and Row 3 : $\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 & 1 \\ 0 & 0 & -2 & -3 & 1 & 0 \end{array}\right]$

Gauss-Jordan Elimination (Non-Singular Matrix)

- Row 1 = (Row 1) - 0.5×(Row 2) : $\left[\begin{array}{ccc|ccc} 1 & 0 & 0.5 & 1 & 0 & -0.5 \\ 0 & 4 & 1 & 0 & 0 & 1 \\ 0 & 0 & -2 & -3 & 1 & 0 \end{array}\right]$
- Row 2 = (Row 2) +0.5×(Row 3):
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0.5 & 1 & 0 & -0.5 \\ 0 & 4 & 0 & -1.5 & 0.5 & 1 \\ 0 & 0 & -2 & -3 & 1 & 0 \end{array}\right]$$
- Row 1 = (Row 1) + 0.25×(Row 3) :
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.25 & 0.25 & -0.5 \\ 0 & 4 & 0 & -1.5 & 0.5 & 1 \\ 0 & 0 & -2 & -3 & 1 & 0 \end{array}\right]$$
- Finally, we scale each rows so that the diagonal components are equal to <u>one</u> :

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.25 & 0.25 & -0.5 \\ 0 & 1 & 0 & -0.375 & 0.125 & 0.25 \\ 0 & 0 & 1 & 1.5 & -0.5 & 0 \end{array}\right]$$

Gauss-Jordan Elimination (Singular Matrix)

- Therefore we can conclude that : $A^{-1} = \begin{bmatrix} 0.25 & 0.25 & -0.5 \\ -0.375 & 0.125 & 0.25 \\ 1.5 & -0.5 & 0 \end{bmatrix}$
- You must be able to see the difference between Gauss and Gauss-Jordan eliminations:
  - Gauss elimination:

$$E_g \left[\begin{array}{c|c} A & b \end{array}\right] = \left[\begin{array}{c|c} E_g A & E_g b \end{array}\right] = \left[\begin{array}{c|c} U & E_g b \end{array}\right]$$

  - Gauss-Jordan elimination:

$$E_{gj} \left[\begin{array}{c|c} A & I \end{array}\right] = \left[\begin{array}{c|c} E_{gj} A & E_{gj} I \end{array}\right] = \left[\begin{array}{c|c} I & A^{-1} \end{array}\right], \quad \text{if } E_{gj} = A^{-1}$$

  So you go few steps further in Gauss-Jordan

Gauss-Jordan Elimination (Singular Matrix)

- Now let us take a look at the following matrix :

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

- And then we construct the following augmented matrix :

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 3 & 6 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right]$$

- Row 2 $=$ (Row 2) - 3$\times$(Row 1) :

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & -3 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right]$$

Gauss-Jordan Elimination (Singular Matrix)

- Row 3 = (Row 2) + (Row 3) :

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & -3 & 1 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -3 & 1 & 1 \end{array}\right]$$

- A zero row vector is found during the elimination process. Therefore, we can conclude that this matrix is <u>Non-Invertible</u>.