Vector and Matrix Theory

Chapter 2 :

Linear Equations and Gauss Elimination

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Reminder

2.12 Pengulangan Kegiatan Pendidikan

Untuk memperbaiki nilai, mahasiswa diperkenankan mengambil kembali kegiatan pendidikan yang pernah diikuti dalam batas waktu studi yang diizinkan. Nilai yang dipergunakan untuk menghitung Indeks Prestasi Kumulatif (IPK) adalah **nilai terakhir** yang diambil. Untuk itu mahasiswa yang hendak mengulang mata kuliah harus bersungguh-sungguh dalam belajar untuk memperbaiki nilai yang sudah pernah ditempuh sebelumnya. Mahasiswa bisa berkonsultasi dengan dosen pembimbing akademik (DPA) untuk memutuskan mengulang/memperbaiki mata kuliah atau tidak.

I. The Geometry of Linear Equations

II. Non-Singular and Singular Case

III. Gaussian Elimination

IV. Breakdown of Elimination

Section 1 :
The Geometry of Linear Equations

Solving a Set of Linear Equations

- One of central problems in linear algebra is *solving a set of linear equations*.
- Examples (with 2 unknowns) :

$$\begin{array}{rcrcl} x & & & = & 2 \\ x & + & 2y & = & 4 \end{array} \rightarrow \begin{array}{rcrcl} 1x & + & 0y & = & 2 \\ 1x & + & 2y & = & 4 \end{array} \qquad (1)$$

$$\begin{array}{rcrcl} 2x & - & 3y & = & 1 \\ & & y & = & 1 \end{array} \rightarrow \begin{array}{rcrcl} 2x & + & -3y & = & 1 \\ 0x & + & 1y & = & 1 \end{array} \qquad (2)$$

$$\begin{array}{rcrcl} 2x & - & y & = & 3 \\ -x & + & 2y & = & 0 \end{array} \rightarrow \begin{array}{rcrcl} 2x & + & -1y & = & 3 \\ -1x & + & 2y & = & 0 \end{array} \qquad (3)$$

- The number of unknowns can be more than 2, for some cases it can be 1000 or even more.
- **Question :** *What is the solution for x and y for each set of linear equations above?*

Solving a Set of Linear Equations

- For the 1st set of linear equations :

$$\begin{array}{ccccc} 1x & & & = & 2 \\ 1x & + & 2y & = & 4 \end{array} \rightarrow \begin{array}{ccccc} x & + & 0y & = & 2 \\ x & + & 2y & = & 4 \end{array} \rightarrow \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

- From the 1st equation, we directly know that $x = 2$ is the solution for x.
- By substituting $x = 2$ into the 2nd equation, we know that $y = 1$ is the solution for y.
- Similarly, using the same method, it can be proved that the solutions for the 2nd and 3rd set of linear equations are also the same, which is :

$$\begin{array}{ccc} x & = & 2 \\ y & = & 1 \end{array} \rightarrow \begin{array}{ccccc} 1x & + & 0y & = & 2 \\ 0x & + & 1y & = & 1 \end{array} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad (4)$$

The meaning of Ax

- Suppose we have $\boldsymbol{x} = [x \quad y]^T$
- Row picture

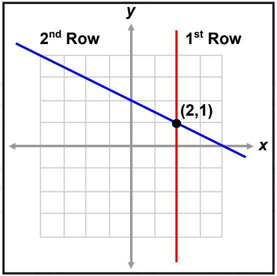
$$A\boldsymbol{x} = \begin{bmatrix} (row\ 1) \cdot \boldsymbol{x} \\ (row\ 2) \cdot \boldsymbol{x} \end{bmatrix}$$

- Column picture

$$A\boldsymbol{x} = \left[x(column\ 1) + y(column\ 2)\right]$$

- **Column picture is a linear combination of the columns of A**
- Column picture will be very important for our future discussion

Row Picture (Two Unknowns)



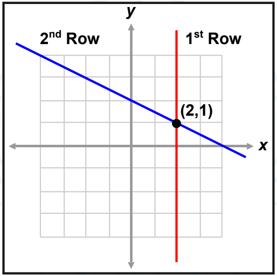
Row Picture of a Set of Linear Equations with 2 Unknowns.

- Let us once again take a look at the following equations :

$$\begin{array}{rcrcr} 1x & + & 0y & = & 2 \\ 1x & + & 2y & = & 4 \end{array}$$

- We can look at that system by *rows* or by *columns*.
- The first approach concentrates on the separate equations (the *rows*).

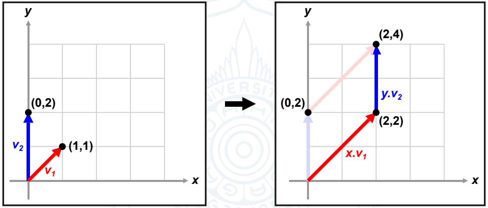
Row Picture (Two Unknowns)



Row Picture of a Set of Linear Equations with 2 Unknowns.

- When we take a look at each rows of this set of equation, these equations can be represented by 2 straight lines in $x - y$ plane.
- The intersection point between these lines represents the solution for this set of equations.

Column Picture (Two Unknowns)



Column Picture of a Set of Linear Equations with 2 Unknowns.

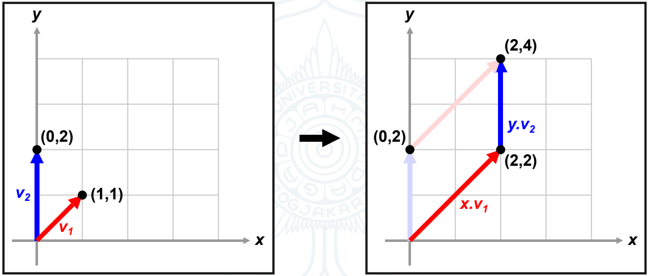
- The second approach concentrates on the *columns* of the following equations :

$$\begin{aligned} 1x &+ 0y &= 2 \\ 1x &+ 2y &= 4 \end{aligned}$$

- As you have been studied in the high school, these equations can be written as :

$$xv_1 + yv_2 = x\begin{bmatrix}1\\1\end{bmatrix} + y\begin{bmatrix}0\\2\end{bmatrix} = \begin{bmatrix}2\\4\end{bmatrix} \tag{5}$$

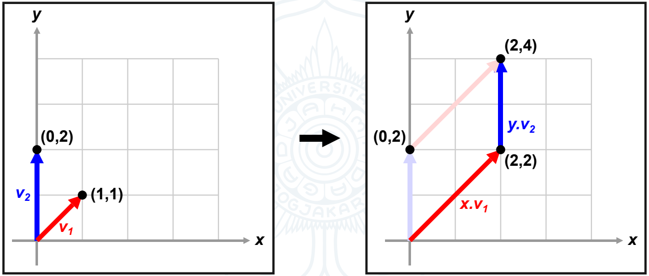
Column Picture (Two Unknowns)



Column Picture of a Set of Linear Equations with 2 Unknowns.

- This equation shows that this set of equations can be seen as a *linear combination* of column vectors $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$.
- This problem now can be seen as *finding the combination of the column vectors on the left side that produces the vector on the right side*.

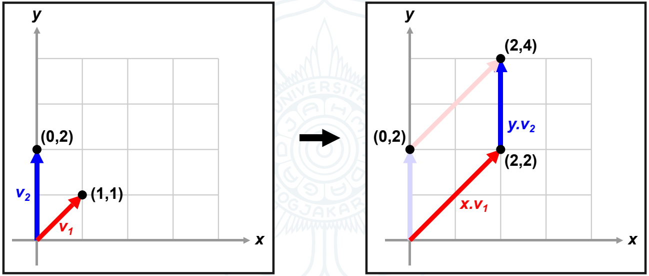
Column Picture (Two Unknowns)



Column Picture of a Set of Linear Equations with 2 Unknowns.

- Or in another words, how much the vectors $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ should be scaled (by x and y, respectively) so that the resultant of those vectors is equal to vector $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

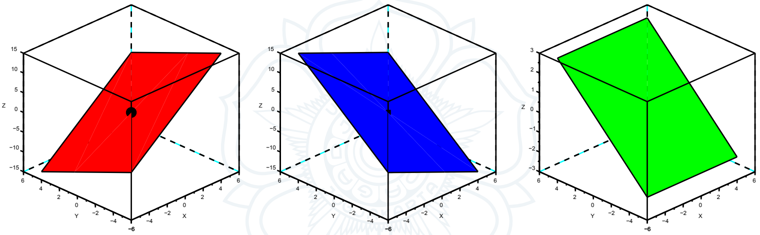
Column Picture (Two Unknowns)



Column Picture of a Set of Linear Equations with 2 Unknowns.

- As shown in the figure above, vector $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ should be scaled by a factor $x = 2$, and vector $v_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ should be scaled by a factor $y = 1$, so that the resultant of those vectors is equal to vector $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

Row Picture (Three Unknowns)



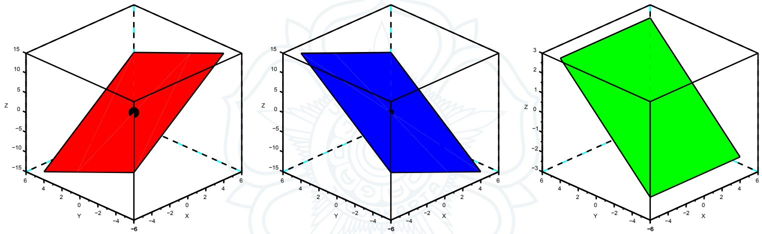
Row (Equation) 1 **Row (Equation) 2** **Row (Equation) 3**

Row Picture of a Set of Linear Equations with 3 Unknowns.

- Now, let us move to a three unknowns case, represented by the following equations :

$$\begin{array}{ccccccc} 2x & + & -1y & + & -1z & = & 0 \\ -1x & + & 2y & + & -1z & = & 0 \\ 0x & + & -3y & + & 4z & = & 1 \end{array}$$

Row Picture (Three Unknowns)

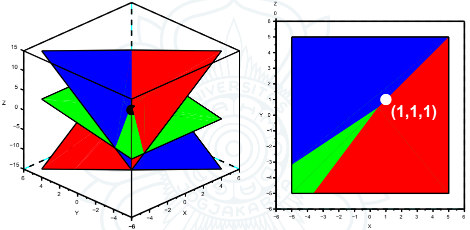


Row (Equation) 1 **Row (Equation) 2** **Row (Equation) 3**

Row Picture of a Set of Linear Equations with 3 Unknowns.

- When we take a look at each rows of this set of equations, these equations can be represented by 3 planes in a three dimensional space.
- The solution (x, y, z) of those equations is the intersection point between those 3 planes.

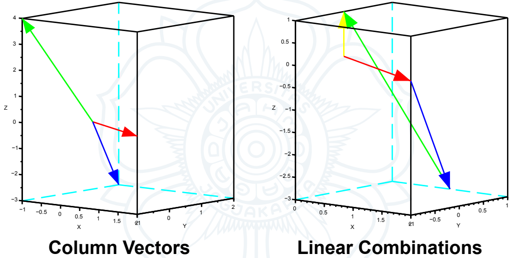
Row Picture (Three Unknowns)



Row Picture of a Set of Linear Equations with 3 Unknowns.

- When we take a look at each rows of this set of equations, these equations can be represented by 3 planes in a three dimensional space.
- The solution (x, y, z) of those equations is the intersection point between those 3 planes.

Column Picture (Three Unknowns)

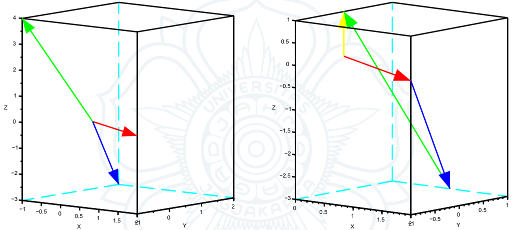


Column Picture of a Set of Linear Equations with 3 Unknowns.

- Now let us try to re-write those equations as follow :

$$x \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} + z \begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{6}$$

Column Picture (Three Unknowns)



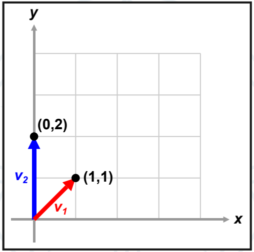
Column Vectors **Linear Combinations**

Column Picture of a Set of Linear Equations with 3 Unknowns.

- This equation indicates a *linear combination* of column vectors $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix}$ to produce vector $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.
- From this view of linear combination, we can directly see that the solutions for x, y and z are $x = 1$, $y = 1$ and $z = 1$.

Section 2 : Non-Singular and Singular Case

Non-Singular Case



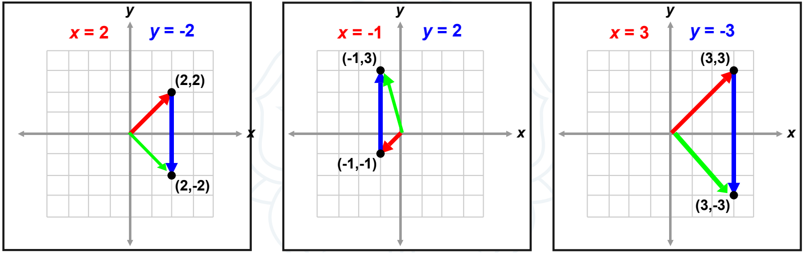
Column Picture of Non-Singular Case in 2-D Space.

- Let us once again consider the following linear combination :

$$x \begin{bmatrix} 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

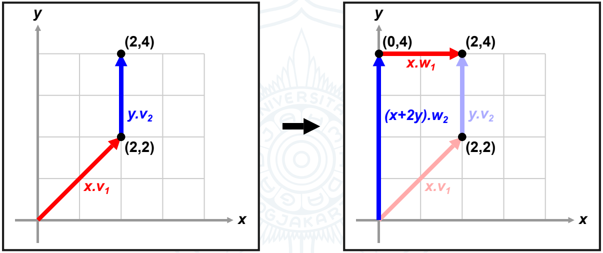
- **Question :** *Does the solution for* (x, y) *exist for every vector* $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$?

Non-Singular Case



Column Picture of Non-Singular Case in 2-D Space.

- The figure above shows the resulting vector $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ for several values of (x, y) pair.
- As can be seen from the figure, it seems like the solution exists for every vector $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$.



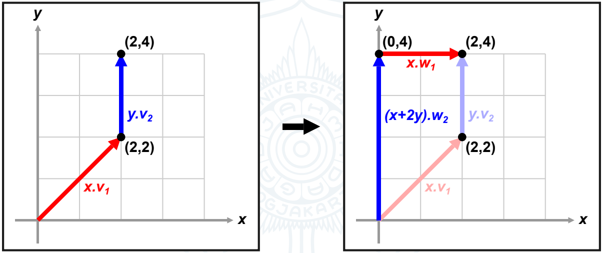
Column Picture of Non-Singular Case in 2-D Space.

- To see that, we can re-write the linear combination as follow :

$$x\begin{bmatrix}1\\1\end{bmatrix}+y\begin{bmatrix}0\\2\end{bmatrix}=x\begin{bmatrix}1\\0\end{bmatrix}+(x+2y)\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}b_1\\b_2\end{bmatrix}$$

- The linear combination of vectors $\boldsymbol{v}_1=\begin{bmatrix}1\\1\end{bmatrix}$ and $\boldsymbol{v}_2=\begin{bmatrix}0\\2\end{bmatrix}$ can be seen also as a linear combination of vectors $\boldsymbol{w}_1=\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\end{bmatrix}$

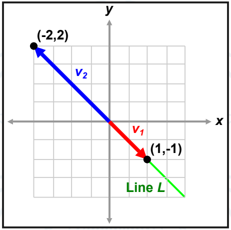
Non-Singular Case



Column Picture of Non-Singular Case in 2-D Space.

- Using these 2 vectors, it can be seen from the figure that it is possible to achieve every vector $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = b_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ in 2-D plane.
- This case, when linear combination of two vectors is able to achieve every points/vectors in 2-D plane, is called as a *Non-Singular Case*.

Singular Case



Column Picture of Singular Case in 2-D Space.

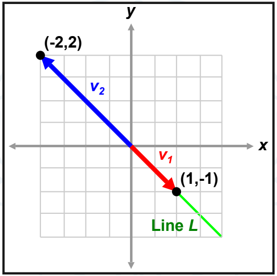
- Now let us consider the 2nd case represented by the following equations :

$$\begin{array}{rcrcl} 1x & + & -2y & = & b_1 \\ -1x & + & 2y & = & b_2 \end{array} \tag{7}$$

- These equations can be represented by the following linear combination :

$$x \begin{bmatrix} 1 \\ -1 \end{bmatrix} + y \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

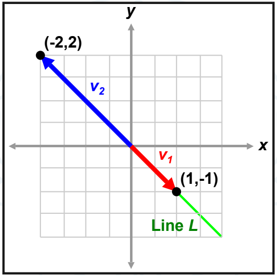
Singular Case



Column Picture of Singular Case in 2-D Space.

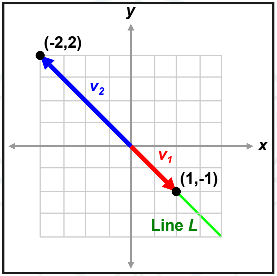
- This equation shows a linear combination of vectors $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\boldsymbol{v}_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$.

Singular Case



Column Picture of Singular Case in 2-D Space.

- As can be seen from the figure, these 2 vectors lies in the same line L. As the result, the combination of these vectors can achieve every points in line L.
- When a particular vector $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ lies in line L, There are infinite number of (x, y) pairs for achieving this vector.



Column Picture of Singular Case in 2-D Space.

- On the other hand, there is no possibility of the linear combination of these vectors to achieve any point outside line L (no solution).
- This kind of case, when a set of linear equations has *infinitely many solutions* or *no solution*, is called as a *Singular Case*.

Section 3 : Gaussian Elimination

Triangular Matrix

- Now let us consider the following sets of linear equations with 3 unknowns :

$$\begin{array}{ccccccc} 1x & + & -2y & + & 3z & = & 2 \\ 0x & + & 1y & + & 2z & = & 4 \\ 0x & + & 0y & + & 1z & = & 1 \end{array} \tag{8}$$

$$\begin{array}{ccccccc} 1x & + & -2y & + & 3z & = & 2 \\ 1x & + & 1y & + & 2z & = & 7 \\ 1x & + & 2y & + & 1z & = & 4 \end{array} \tag{9}$$

- **Question :** *Which set of equation is easier to solve?*
- Both sets of equations have the same (x, y, z) solution. However, solving the 1st set of equations is easier than the 2nd one.
- The 1st set of equation can be represented by the following matrix multiplication :

$$\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} \tag{10}$$

Triangular Matrix

- This equation can be represented by a matrix multiplication, where the non-zero components of the matrix form a triangular shape. Because of that, this kind of matrix is called as *Triangular Matrix*.
- *Upper* and *Lower Triangular Matrix* :

$$\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{Upper Triangular Matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \qquad \text{Lower Triangular Matrix}$$

- As we will see later, forming a triangular matrix (especially the *upper* one) is the main purpose of performing Gauss Elimination.

Elimination Method

- Let us once again consider the following set of equations :

$$\begin{array}{rcrcl} 2x & - & y & = & 3 \\ -x & + & 2y & = & 0 \end{array}$$

- We have learnt in middle and/or high school, we can solve this set of equation using a method called *elimination*, by adding 1/2 the first equation to the second equation as follow :

$$\begin{array}{rcrcc} x & - & 1/2y & = & 3/2 \\ -x & + & 2y & = & 0 \end{array}$$

  so that it becomes

$$3/2y = 3/2 \quad \rightarrow \quad y = 1.$$

- After that, we can substitute this solution of y into the first equation so that we can obtain the solution of x as follow :

$$x = 2.$$

- This result shows a simple illustration about the use of *elimination* method to find a solution of linear equations.

Gauss Elimination

- In reality, the *elimination* method that we have learnt in school is what is called as *Gaussian Elimination*.
- Now let us take a look at the "real" Gaussian elimination method.
- The first step is by representing the set of equations in matrix form :

$$\left[\begin{array}{cc|c} 2 & -1 & 3 \\ -1 & 2 & 0 \end{array}\right] \tag{11}$$

- As we have discussed before, we add the 1/2 1st row of equation/matrix to the 2nd row of equation/matrix. Then we put the result in the 2nd row of matrix as follow :

$$\left[\begin{array}{cc|c} 2 & -1 & 3 \\ -1 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 2 & -1 & 3 \\ 0 & 3/2 & 3/2 \end{array}\right]$$

- As we have seen in this equation, we finally obtain a triangular matrix.

Gauss Elimination

- Now, let us take a look at the second example, a set of equations with 3 unknowns as follow :

$$\begin{array}{ccccccc} 1x & + & 2y & + & 1z & = & 2 \\ 3x & + & 8y & + & 1z & = & 12 \\ 0x & + & 4y & + & 1z & = & 2 \end{array}$$

- The representation in matrix form is given by :

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 3 & 8 & 1 & 12 \\ 0 & 4 & 1 & 2 \end{array}\right] \tag{12}$$

- The first step is by subtracting the 2nd row, by 3 times the 1st row, so that :

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 3 & 8 & 1 & 12 \\ 0 & 4 & 1 & 2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 4 & 1 & 2 \end{array}\right]$$

Gauss Elimination

- Finally, we can subtract the 3rd row, by 2 times the 2nd row :

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 4 & 1 & 2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 0 & 5 & -10 \end{array}\right]$$

- Once again, as we can see in this equation, we obtain a triangular matrix. From this Gaussian elimination method, we can obtain the following solution :

$$x = 2 \quad ; \quad y = 1 \quad ; \quad z = -2$$

Section 4 :
Breakdown of Elimination

Exchanging the Row

- Now let us consider the following sets of linear equations with 3 unknowns :

$$\begin{array}{ccccccc} 1x & + & 2y & + & 1z & = & 2 \\ 3x & + & 6y & + & 1z & = & 10 \\ 0x & + & 4y & + & 1z & = & 2 \end{array} \tag{13}$$

- The representation in matrix form is given by :

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 3 & 6 & 1 & 10 \\ 0 & 4 & 1 & 2 \end{array}\right]$$

- By subtracting the 2nd row, by 3 times the 1st row, we get :

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 3 & 6 & 1 & 10 \\ 0 & 4 & 1 & 2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 0 & -2 & 4 \\ 0 & 4 & 1 & 2 \end{array}\right]$$

- **Question :** *Are we in a trouble?*

Exchanging the Row

- No, we are still in a good position. We can simply exchange the position of row 2 and 3, so that :

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 0 & -2 & 4 \\ 0 & 4 & 1 & 2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 4 & 1 & 2 \\ 0 & 0 & -2 & 4 \end{array}\right]$$

- Exchanging the row of the matrix does not give any significant change in the matrix. We know that the following matrix :

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 0 & -2 & 4 \\ 0 & 4 & 1 & 2 \end{array}\right]$$

represents the following set of linear equations :

$$\begin{array}{ccccccc} 1x & + & 2y & + & 1z & = & 2 \\ 0x & + & 0y & + & -2z & = & 4 \\ 0x & + & 4y & + & 1z & = & 2 \end{array}$$

Exchanging the Row

- Exchanging the position of row 2 and 3, we can get :

$$\begin{array}{ccccccc} 1x & + & 2y & + & 1z & = & 2 \\ 0x & + & 4y & + & 1z & = & 2 \\ 0x & + & 0y & + & -2z & = & 4 \end{array}$$

so that this set of equations can be represented by :

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 4 & 1 & 2 \\ 0 & 0 & -2 & 4 \end{array}\right]$$

- We can see from this process, exchanging the row of matrix, is simply exchanging the position of the equations, without changing the equation itself.
- As the result, we can once again obtain a triangular matrix, and we can get the same solution as before :

$$x = 2 \quad ; \quad y = 1 \quad ; \quad z = -2$$

Singular Case

- Now let us consider the following set of equations :

$$\begin{array}{ccccccc} 1x & + & 2y & + & 1z & = & 2 \\ 3x & + & 6y & + & 1z & = & 10 \\ 0x & + & 0y & + & 2z & = & -4 \end{array} \tag{14}$$

so that this set of equations can be represented by :

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 3 & 6 & 1 & 10 \\ 0 & 0 & 2 & -4 \end{array}\right]$$

- By subtracting the 2nd row, by 3 times the 1st row, we get :

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 3 & 6 & 1 & 10 \\ 0 & 0 & 2 & -4 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 2 & -4 \end{array}\right]$$

- **Question :** *Are we in a trouble?*

Singular Case

- Unfortunately, the answer is Yes, we are in a big trouble. There is no way we can get a row in the form of $\left[\begin{array}{ccc|c} 0 & X & X & X \end{array}\right]$
- As the result, there will be too many (x, y, z) solutions that satisfy this set of equations. Therefore we can conclude that this case is a *Singular Case*.