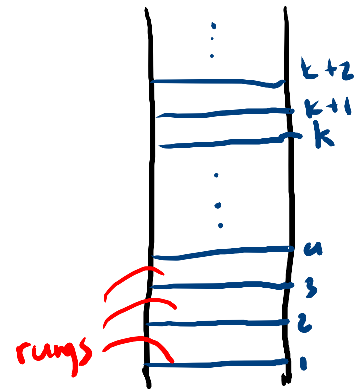
ch.5 Induction & Recursion

infinite ladder



$n \in \mathbb{Z}^+, \mathbb{N}$

How to prove

1) We step 1-st rung

2) We step particular rung
   k-th rung

Show: we also step (k+1)-th rung

then conclude we also step n-th rung $\forall n \in \mathbb{Z}^+$

we are going to use the concept to prove statement that assert $P(n)$ is true $\forall n \in \mathbb{Z}^+$.

$P(n)$ is propositional function.

Challenge Prove that $P(n)$ is true., $\forall n \in \mathbb{Z}^+$

How using 2 steps

Basis step we need to verify that $P(1)$ is true.

Inductive step Prove $P(k) \implies P(k+1)$ is true.

(IH) Induction Hypothesis : $P(k)$ is true.

From IH Prove that $P(k+1)$ is true $\implies P(n)$ is true $\forall n \in \mathbb{Z}^+$

On what we've done

$$\boxed{P(1) \wedge P(k) \Rightarrow P(k+1) \Rightarrow P(n) \quad \forall n \in \mathbb{Z}^+}$$

#1. Prove the summation formula

$$1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

$$P(n): 1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

Basis step

$P(1)$: $\underline{1} = \frac{1(1+1)}{2} = \frac{2}{2}$

$= \underline{1}$ ✓

inductive step $P(k) \Rightarrow P(k+1)$

IH: $P(k)$: $1+2+3+\cdots+k = \frac{k(k+1)}{2}$

Show: $P(k+1)$: $\underbrace{1+2+\cdots+k}_{P(k)=} + \underline{(k+1)} = \frac{(k+1)(k+2)}{2}$ is true.

$\frac{k(k+1)}{2} + k+$ $\quad = \frac{(k+1)(k+2)}{2}$

Show: $P(k+1): \underbrace{1+2+\cdots+k}_{P(k)=} + (k+1) = \frac{(k+1)(k+2)}{2}$ is true.

$$\frac{k(k+1)}{2} + k+1 \qquad = \frac{(k+1)(k+2)}{2}$$

$$5\,☺ + 3\,☺ = ☺(5+3)$$

$$\frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{k(k+1)+2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

therefore, then $P(k+1)$ is true.

$\Rightarrow \therefore P(n)$ is true $\forall n \in \mathbb{Z}^+$

tomb stone

$\blacksquare$

q.e.d

quot erat demonstrandum

#2 Show that the sum of

$$1 + 2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 1, \quad \forall n \in \mathbb{N}$$

$$P(n): 1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1, \quad \forall n \in \mathbb{N}$$

Basis step

$$P(0): 1 = 2^{0+1} - 1$$
$$= 1 \checkmark$$

inductive step $P(k) \Rightarrow P(k+1)$

IH: $P(k): 1 + 2 + 2^2 \cdots + 2^k = 2^{k+1} - 1$

Show: $P(k+1): \underbrace{1 + 2 \quad + 2^k}_{P(k)} + \underbrace{2^{k+1}}_{} = 2^{k+2} - 1$

$$P(k) =$$
$$2^{k+1} - 1 + 2^{k+1} = 2^{k+2} - 1$$
$$2 \cdot 2^{k+1} - 1 = 2^{k+2} - 1$$
$$2^{k+2} - 1 = 2^{k+2} - 1 \checkmark$$

then $P(k+1)$ is true.

$\therefore P(n)$ is true $\forall n \in \mathbb{N}$ $\blacksquare$

To prove a Conjecture

Example of a Conjecture

Sum of first odd n. positive integer

$n=1$ $\quad 1 = 1$

$n=2$ $\quad 1+3 = 4$

$n=3$ $\quad 1+3+5 = 9$

$n=4$ $\quad 1+3+5+7 = 16$

$\vdots$

$n=n$ $\quad \boxed{1+3+5+7+\dots+(2n-1) = n^2} := P(n)$

Let $\boxed{1+3+5+7+\cdots+(2n-1)=n^2} := P(n)$

Basis step

$P(1): \quad 1 = 1^2$ ✓

Inductive step $\quad P(k) \Rightarrow P(k+1)$

IH: $P(k): \quad 1+3+5+\cdots+(2k-1) = k^2$

Show: $P(k+1): \quad \underbrace{1+3+5+\quad+(2k-1)}_{P(k)} + (2k+1) = (k+1)^2$

$+2$

$$k^2 \qquad\qquad + 2k+1 = (k+1)^2$$

$$(k+1)(k+1) = (k+1)^2$$

then $P(k+1)$ is true

$\therefore P(n)$ is true, $\forall n \in \mathbb{Z}^+$. ∎

To prove inequality

#4 Prove $n < 2^n$, $\forall n \in \mathbb{Z}^+$

$P(n)$: $n < 2^n$, $\forall n \in \mathbb{Z}^+$

Basis step

$P(1)$: $1 < 2^1$ ✓

Inductive step $P(k) \Rightarrow P(k+1)$

IH: $P(k)$: $k < 2^k$, $\forall k \in \mathbb{Z}^+$

Show: $P(k+1)$: $\boxed{\underbrace{k}_{P(k)}+1 < 2^{k+1}}$

$$k+1 < 2^k+1 < 2^k+2 < 2^k + 2^k$$

$k+1 < 2^k + 2^k$!

$\boxed{k+1 < 2^{k+1}}$

$P(k+1)$ is true.

$\therefore P(n)$ is true $\forall n \in \mathbb{Z}^+$

$\blacksquare$

#5 Prove $2^n < n!$, $\forall n \in \mathbb{Z}^+$ & $n \geq 4$

$P(n)$: $2^n < n!$, $\forall n \in \mathbb{Z}^+$ & $n \geq 4$

Basis step

$P(4)$: $2^4 < 4!$

$16 < 4 \cdot 3 \cdot 2 \cdot 1$

$16 < 24$ ✓

Inductive step $P(k) \Rightarrow P(k+1)$

IH: $P(k)$: $2^k < k!$

Show: $P(k+1)$: $2^{k+1} < (k+1)!$

$2 \cdot 2^k < k! \cdot 2$

$2^{k+1} < 2k! < (k+1) \cdot k! = (k+1)!$

$2^{k+1} < (k+1)!$ ✓

$P(k+1)$ is true

$\therefore$ $P(n)$ is true, $\forall n \in \mathbb{Z}^+$

$n \geq 4$ ∎

Homework

Exercise pp. 350 –
4, 11, 27

Rossen's
8-th-ed.

- Divisibility Proof
- Strong Induction & Well-ordering principle
- Recursion