TKU211121
Classical Mechanics
Chapter #1 : Rotation of Rigid Body

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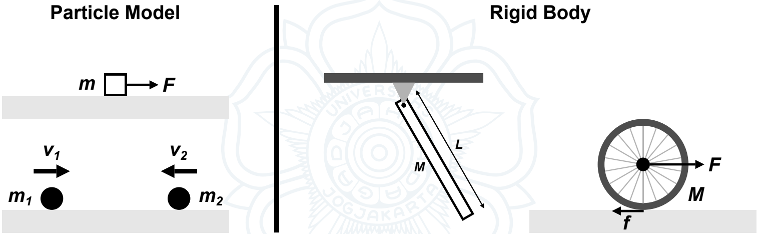
Rigid Body and Center of Mass

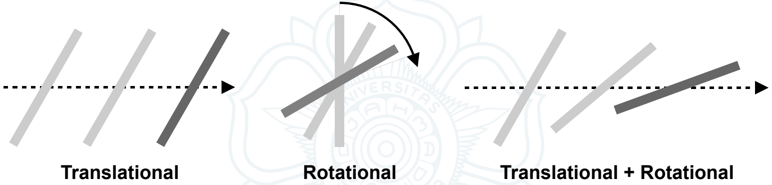
Particle Model vs Rigid Body



Example of A System with Particle Model (left) and Rigid Body (right).

- **Particle Model** : An object is represented as a mass in a single point in space.
- **Rigid Body** : An extended object whose size and shape do not change as it moves.
- Example of Rigid Body : Wheel, Rod, Ball, Axle, etc.

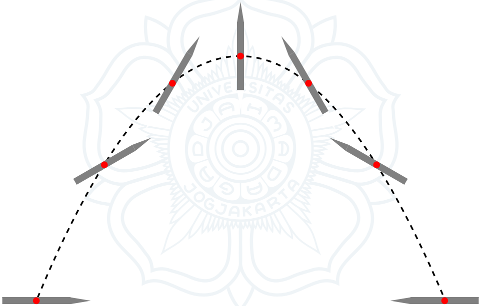
Particle Model vs Rigid Body



Types of Motion of a Rigid Body.

- In a particle model, the motion is very simple.
- Because it is modelled as a single point in space, a rotational motion cannot be defined.
- Only translational motion can be considered in a particle model.
- The types of motion in a rigid body is more complex.
- A rigid body can have a translational motion, a rotational motion, or the combination of both translational and rotational motion.

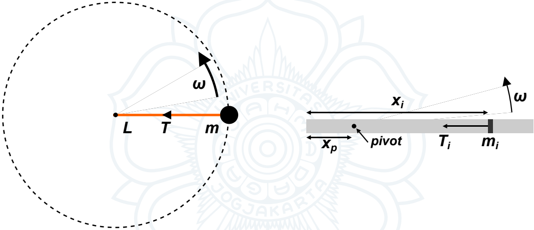
Center of Mass



Motion of a Pencil thrown to the Air.

- A student throws a pencil to the air. He notices that the pencil is moving while rotating about a certain point. **About what point does it rotate?**
- An unconstrained object (i.e., one not on an axle or a pivot) tends to rotates about a point called the **Center of Mass**.

Center of Mass

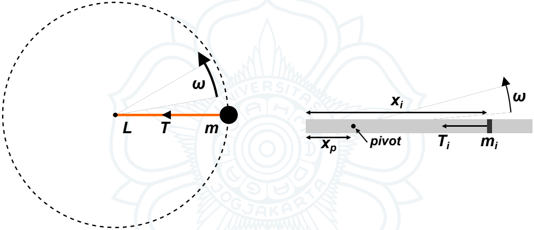


Calculation of Center of Mass Position.

- Where is the center of mass position of an object?
- First, let us consider a ball with mass m tied to a string with length L. The ball is rotating with constant angular velocity ω.
- Because the ball is rotating, the ball is experiencing a centripetal acceleration

$$a_c = \omega^2 L$$

Center of Mass



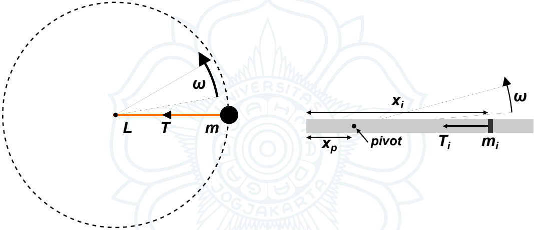
Calculation of Center of Mass Position.

- Acceleration requires a force, which in this case is provided by the tension of the string T and can be calculated as follow

$$T = ma_c = m\omega^2 L$$

- Now let us consider a rigid body (a rod) is rotating with angular velocity ω about a pivot point located at distance x_p from left end of the rod.

Center of Mass

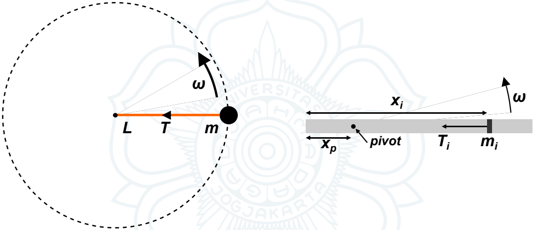


Calculation of Center of Mass Position.

- Consider a small portion of the rod with mass m_i located at distance $\vec{x_i}$ from left end of the rod. It is experiencing a centripetal acceleration given by

$$\vec{a_i} = \omega^2(\vec{x_p} - \vec{x_i})$$

Center of Mass



Calculation of Center of Mass Position.

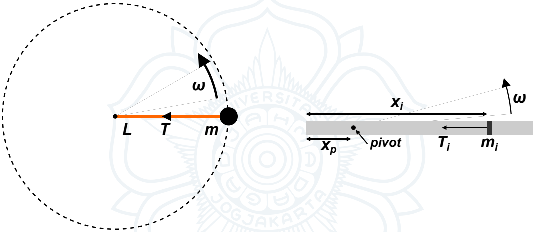
- Therefore the force $\vec{T}_i$ for rotating this mass portion m_i is given by

$$\vec{T}_i = m_i \omega^2 (\vec{x_p} - \vec{x_i})$$

- The sum of all tension forces is given by.

$$\sum_i \vec{T}_i = \sum_i m_i \omega^2 (\vec{x_p} - \vec{x_i}) = \omega^2 \sum_i m_i (\vec{x_p} - \vec{x_i})$$

Center of Mass



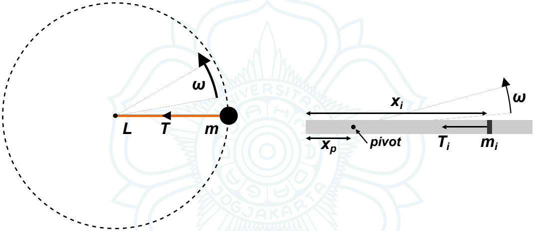
Calculation of Center of Mass Position.

- When the rod is rotating about its center of mass ($\vec{x_p} = \vec{x_{cm}}$), the sum of all tension forces are zero, therefore

$$\sum_i m_i(\vec{x_{cm}} - \vec{x_i}) = \vec{x_{cm}} \sum_i m_i - \sum_i m_i \vec{x_i} = 0$$

- As the result, the center of mass position is given by

$$\vec{x_{cm}} = \frac{\sum_i m_i \vec{x_i}}{\sum_i m_i} = \frac{m_1 \vec{x_1} + m_2 \vec{x_2} + m_3 \vec{x_3} + \cdots}{m_1 + m_2 + m_3 + \cdots} \quad (1)$$



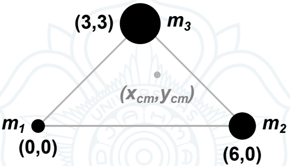
Calculation of Center of Mass Position.

- Using the same method, it can be proved also that the center of mass position in y and z directions are given by

$$\vec{y_{cm}} = \frac{\sum_i m_i \vec{y_i}}{\sum_i m_i} = \frac{m_1 \vec{y_1} + m_2 \vec{y_2} + m_3 \vec{y_3} + \cdots}{m_1 + m_2 + m_3 + \cdots} \tag{2}$$

$$\vec{z_{cm}} = \frac{\sum_i m_i \vec{z_i}}{\sum_i m_i} = \frac{m_1 \vec{z_1} + m_2 \vec{z_2} + m_3 \vec{z_3} + \cdots}{m_1 + m_2 + m_3 + \cdots} \tag{3}$$

Center of Mass (Example)

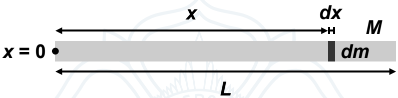


A Three-Bodies System.

- A system consists of three bodies with mass $m_1 = 1kg$, $m_2 = 2kg$ and $m_3 = 3kg$ located at $(0, 0)$, $(6, 0)$ and $(3, 3)$, respectively. Determine the x_{cm} and y_{cm} of this system.
- This problem can be solved easily by applying the x_{cm} and y_{cm} equations

$$\vec{x_{cm}} = \frac{m_1\vec{x_1} + m_2\vec{x_2} + m_3\vec{x_3}}{m_1 + m_2 + m_3} = \frac{(1)(0) + (2)(6) + (3)(3)}{1 + 2 + 3} = 3.5$$

$$\vec{y_{cm}} = \frac{m_1\vec{y_1} + m_2\vec{y_2} + m_3\vec{y_3}}{m_1 + m_2 + m_3} = \frac{(1)(0) + (2)(0) + (3)(3)}{1 + 2 + 3} = 1.5$$



The Center of Mass of a Rigid Object.

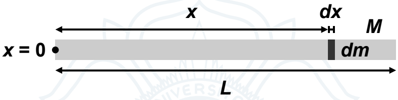
- Now let us consider a rigid object (rod) with mass M and length L. How to determine the center of mass position of this kind of object?
- First, we need to divide this object into many small cell with mass dm, where its position is x. The center of mass position of this object can be calculated as follow

$$\vec{x_{cm}} = \frac{\int \vec{x} \cdot dm}{\int dm} \tag{4}$$

- Similarly, it can be proved also that the center of mass position in y and z directions are given by

$$\vec{y_{cm}} = \frac{\int \vec{y} \cdot dm}{\int dm} \qquad \vec{z_{cm}} = \frac{\int \vec{z} \cdot dm}{\int dm} \tag{5}$$

Center of Mass (Example)



The Center of Mass of a Rigid Object.

- As an example, we will try to determine the center of mass of a rod using the aforementioned equation
- First, we need to determine the *dm* of this rod

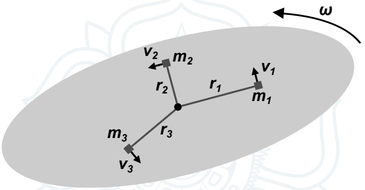
$$dm = \left(\frac{M}{L}\right) dx$$

- Therefore the center of mass position can be calculated as follow

$$x_{\vec{cm}} = \frac{\int \vec{x} \cdot dm}{\int dm} = \frac{(M/L)(\int_0^L xdx)}{(M/L)(\int_0^L dx)} = \frac{L^2/2}{L} = \frac{L}{2}$$

Rotational Energy and Moment of Inertia

Rotational Kinetic Energy

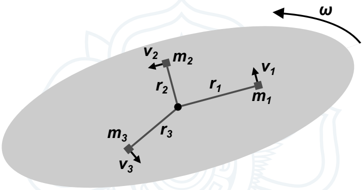


Rotational Kinetic Energy of a Rigid Body.

- A rotating rigid body has kinetic energy because all of the particles are in motion. The kinetic energy due to rotation is called a **Rotational Kinetic Energy** .
- All of the particles in a rigid body are rotating with the same angular velocity ω. However, because the distance of each particles from the rotation axis are different, each has different velocity v_i which is given by

$$v_i = \omega r_i$$

Rotational Kinetic Energy



Rotational Kinetic Energy of a Rigid Body.

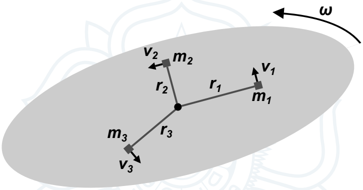
- The rotational kinetic energy of an object is the sum of all the particles' kinetic energy

$$K_{rot} = \frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2} + \frac{m_3 v_3^2}{2} + \cdots$$
$$= \frac{m_1 r_1^2 \omega^2}{2} + \frac{m_2 r_2^2 \omega^2}{2} + \frac{m_3 r_3^2 \omega^2}{2} + \cdots$$

Therefore

$$K_{rot} = \left(\sum_i m_i r_i^2 \right) \frac{\omega^2}{2} = \frac{I\omega^2}{2} \tag{6}$$

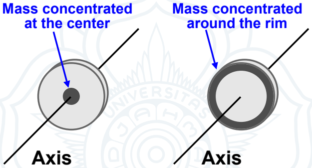
Rotational Kinetic Energy



Rotational Kinetic Energy of a Rigid Body.

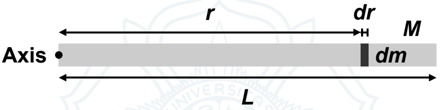
- Parameter $I = \sum_i m_i r_i^2$ in the previous equation is defined as the object's **moment of inertia**.
- Notice that $K_{rot} = I\omega^2/2$, which indicates that for the same angular velocity ω, the object with higher moment of inertia I carries higher rotational kinetic energy. Therefore, it requires more work to accelerate an object with higher moment of inertia.
- Therefore, this fact indicates that it is more difficult to accelerate an object with larger moment of inertia.

Moment of Inertia



Effect of Mass Distribution on the Moment of Inertia.

- The *moment of inertia* in rotational motion indicates how difficult to *rotate* an object. It is analogous with *mass* (or *inertial mass*) in translational motion, which indicates how difficult to *move* an object.
- The main different between *moment of inertia* and *inertial mass* is that the *moment of inertia* of an object also depends on how the mass is distributed.
- Two wheels with the same mass M and radius R, it is easier to rotate the wheel whose mass is concentrated at the center rather than concentrated around the rim.

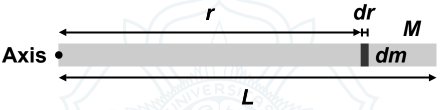


Moment of Inertia Calculation of a Rigid Object.

- Let us once again consider a rigid object (rod) with mass M and length L. How to determine the moment of inertia of this kind of object?
- First, we need to divide this object into many small cell with mass dm, where its position is r from the rotation axis.
- The moment of inertia of this object can be calculated as follow

$$I = \int r^2 \cdot dm \tag{7}$$

Moment of Inertia (Example)



Moment of Inertia Calculation of a Rigid Object.

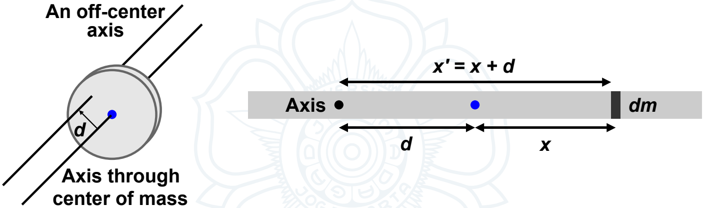
- As an example, we will try to determine the moment of inertia of a rod using the aforementioned equation
- First, we need to determine the *dm* of this rod

$$dm = \left(\frac{M}{L}\right) dr$$

- Therefore the moment of inertia can be calculated as follow

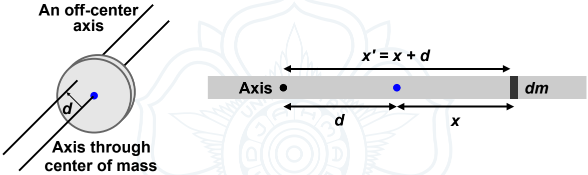
$$I = \int r^2 \cdot dm = \left(\frac{M}{L}\right)\left(\int_0^L r^2 dr\right) = \left(\frac{M}{L}\right)\left(\frac{L^3}{3}\right) = \frac{ML^2}{3}$$

Parallel Axis Theorem



Rotation about an Off-center Axis.

- A rigid body with mass M has moment of inertia I_{cm} when rotating about its center of mass. When the rotation axis is moved toward a new axis with distance d from its center of mass, what is the new moment of inertia of this object?
- Let us take an example of a rod, whose rotation axis is moved toward a new axis with distance d from its center of mass.

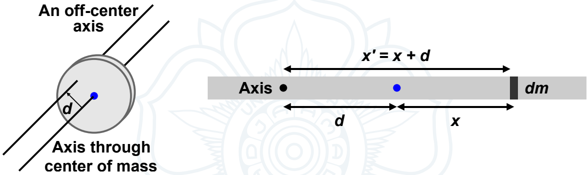


Rotation about an Off-center Axis.

- The moment of inertia of this case can be calculated as follow

$$
\begin{aligned}
I = \int (x')^2 \cdot dm &= \int (x^2 + 2xd + d^2) \cdot dm \\
&= \int (x^2 + 2d(x' - d) + d^2) \cdot dm \\
&= \left(\int x^2 \cdot dm \right) + 2d \left(\int x' \cdot dm - \int d \cdot dm \right) + d^2 \left(\int dm \right) \\
&= I_{cm} + 2d(x'_{cm} - d)M + d^2(M)
\end{aligned}
$$

Parallel Axis Theorem



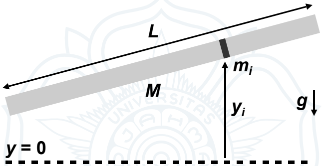
Rotation about an Off-center Axis.

- Note that x'_{cm} is the object's center of mass position assuming that the rotation axis is the center of coordinate (0,0,0). Therefore $x'_{cm} = d$ and the moment of inertia about the new axis is given by

$$I = I_{cm} + Md^2 \tag{8}$$

- This equation is known as the **Parallel Axis Theorem**.

Gravitational Potential Energy of a Rigid Body



Gravitational Potential Energy of a Rigid Body.

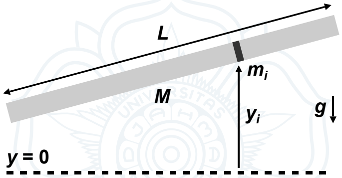
- How to calculate the gravitational potential energy of a rigid body?
- Let us once again consider e rod with mass M and length L. Assume that a particle with mass m_i is located at height y_i, therefore its gravitational potential energy is

$$U_i = m_i g y_i$$

- The total gravitational potential energy of the rod is the sum of the potential energy of all particles, therefore

$$U = \sum_i U_i = g \sum_i m_i y_i$$

Gravitational Potential Energy of a Rigid Body



Gravitational Potential Energy of a Rigid Body.

- From the definition of center of mass in y direction, we know that

$$y_{cm} = \frac{\sum_i m_i y_i}{\sum_i m_i} = \frac{\sum_i m_i y_i}{M} \quad \rightarrow \quad \sum_i m_i y_i = M y_{cm}$$

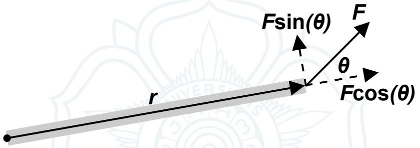
- Therefore, the gravitational potential energy of a rigid body is given by

$$U = M g y_{cm} \tag{9}$$

which only depends on its total mass and the height of its center of mass.

Torque and Static Equilibrium

Torque

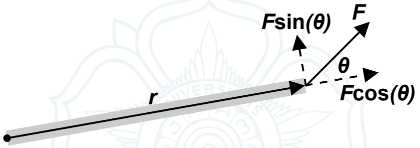


A Force $\vec{F}$ acting on a Rigid Body.

- **Torque** is a quantity that measures the "effectiveness" of a force at causing an object to rotate about a pivot.
- Torque is the rotational equivalent of force.
- Now let us consider a force $\vec{F}$ is working on an object at a distance $\vec{r}$ from its axis/pivot. The *torque* working on this object is defined as

$$\vec{\tau} = \vec{r} \times \vec{F} \tag{10}$$

Torque



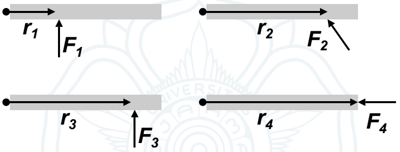
A Force $\vec{F}$ acting on a Rigid Body.

- Remember that what really matter in a cross product between two vectors is the vector's component which is perpendicular to each other.
- Therefore, if there exists an angle θ between $\vec{r}$ and $\vec{F}$, the $|\vec{\tau}|$ is defined as follow

$$|\vec{\tau}| = |\vec{r}||\vec{F}|\sin\theta \tag{11}$$

which indicates that the produced torque is a multiplication between $|\vec{r}|$ and the component of $\vec{F}$ which is perpendicular to $\vec{r}$.

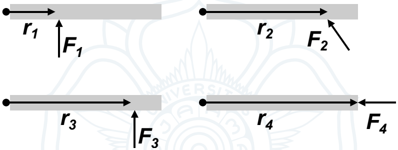
Torque (Example)



Four Different Forces act on a Door.

- Let us consider the case when someone is pushing a door. There are 4 different forces $\vec{F}_1$, $\vec{F}_2$, $\vec{F}_3$ and $\vec{F}_4$ working on a door at 4 different positions and angles. If the all forces has the same magnitude, which force produces the strongest torque? Which one produces the weakest torque?
- For $\vec{F}_1$ and $\vec{F}_3$, the angle between $\vec{r}$ and $\vec{F}$ are $\theta = 90^0$ (perpendicular). However, because $|\vec{r}_1| < |\vec{r}_3|$, therefore $0 < |\vec{\tau}_1| < |\vec{\tau}_3|$.

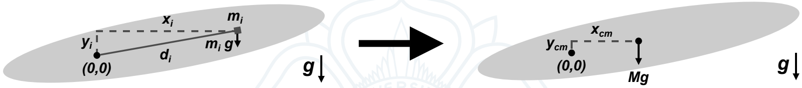
Torque (Example)



Four Different Forces act on a Door.

- $\vec{F}_2$ is working at the same location as $\vec{F}_3$ ($\vec{r}_2 = \vec{r}_3$). However, because the angle θ between $\vec{r}_2$ and $\vec{F}_2$ is between 90^0 and 180^0, therefore $0 < |\vec{\tau}_2| < |\vec{\tau}_3|$.
- Finally, for $\vec{F}_4$, because the angle between $\vec{r}_4$ and $\vec{F}_4$ is 180^0, therefore $|\vec{\tau}_4| = 0$.
- Therefore we can conclude that the strongest torque is produced by $\vec{F}_3$, and $\vec{F}_4$ produces the weakest torque.

Gravitational Torque



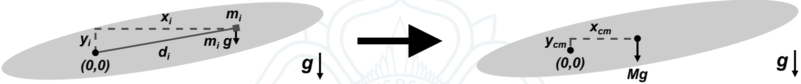
Gravitational Torque acting on a Rigid Body.

- Gravity exerts torque on many objects. Let us consider an object shown in the figure above. A torque due to gravity will cause it to rotate around the axle.
- Every particle inside an object contributes to the gravitational torque. Consider a particle with mass m_i which is located at the position (x_i, y_i) (assuming that the rotation axis is at coordinate $(0, 0)$). The torque produces by this particle is

$$\tau_i = -m_i g x_i$$

Note that the *negative sign* appears because the torque causes the object to rotate in *clockwise direction*.

Gravitational Torque



Gravitational Torque acting on a Rigid Body.

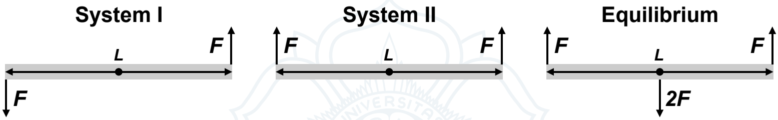
- The total gravitational torque is the sum of all the contribution of each particles

$$\tau_{grav} = -\sum_i m_i g x_i = -g\left(\sum_i m_i x_i\right)$$

- From the definition of center of mass, $\sum_i m_i x_i = M x_{cm}$, therefore

$$\tau_{grav} = -Mgx_{cm} \tag{12}$$

- This equation indicates that the gravitational torque of an object is calculated by **treating the object as if all the mass is concentrated at its center of mass**.



Static Equilibrium of a Rigid Body.

- From Newton's 1st Law we know that a rest object will remain at rest when

$$\sum \vec{F} = 0 \quad \rightarrow \quad \sum \vec{F}_x = 0 \quad \sum \vec{F}_y = 0 \quad \sum \vec{F}_z = 0$$

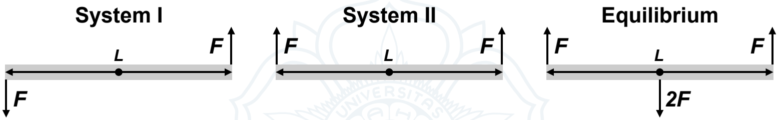
- As torque is a rotational equivalent of force, therefore a rest object (no rotation) will remain at rest (no rotation) when

$$\sum \vec{\tau} = 0 \quad \rightarrow \quad \sum \vec{\tau}_x = 0 \quad \sum \vec{\tau}_y = 0 \quad \sum \vec{\tau}_z = 0$$

- Therefore we can conclude that for an object to be in *static equilibrium* (no movement and no rotation) when

$$\sum \vec{F} = 0 \quad \text{and} \quad \sum \vec{\tau} = 0 \tag{13}$$

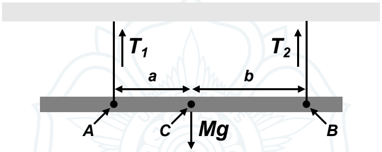
Static Equilibrium



Static Equilibrium of a Rigid Body.

- Now let us consider the systems shown in the figure above.
- For the first system, we can see that $\sum \vec{F} = 0$ and $\sum \vec{\tau} = FL \neq 0$. As the result, the rod is still able to rotate about its center of mass.
- On the other hand, for the second system, we can see that $\sum \vec{F} = 2F \neq 0$ and $\sum \vec{\tau} = 0$. As the result, the rod has no rotation, but it is still able to have a translational motion.
- This example emphasizes that for achieving a *static equilibrium*, both $\sum \vec{F} = 0$ and $\sum \vec{\tau} = 0$ conditions have to be satisfied.

Static Equilibrium (Example)



System under a Static Equilibrium Condition.

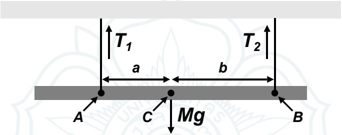
- Consider a system shown in the figure above. How much the tension feels by string 1 and string 2?
- From the equilibrium conditions $\sum \vec{F} = 0$ and $\sum \vec{\tau} = 0$

$$\sum F = T_1 + T_2 - Mg = 0 \quad \sum \tau = T_2 b - T_1 a = 0$$

  where the torque is calculated assuming that the pivot point is at point C (center of mass).
- Solving those two equations we will get

$$T_1 = Mg\left(\frac{b}{a+b}\right) \quad T_2 = Mg\left(\frac{a}{a+b}\right)$$

Static Equilibrium (Example)



System under a Static Equilibrium Condition.

- We can also solve this problem by analyzing the torque at point A and B.
- Let us analyze the $\sum \vec{\tau} = 0$ condition by assuming that the pivot point is at point B, therefore

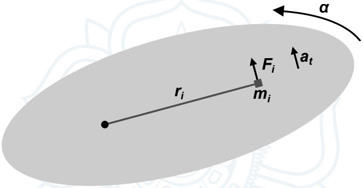
$$\sum \tau = Mgb - T_1(a+b) = 0 \quad \rightarrow \quad T_1 = Mg\left(\frac{b}{a+b}\right)$$

- Next, we analyze the $\sum \vec{\tau} = 0$ condition by assuming that the pivot point is at point A, therefore

$$\sum \tau = Mga - T_2(a+b) = 0 \quad \rightarrow \quad T_2 = Mg\left(\frac{a}{a+b}\right)$$

Rotational Dynamics and Angular Momentum

Rotational Dynamics



Effect of a Force on a Rigid Body.

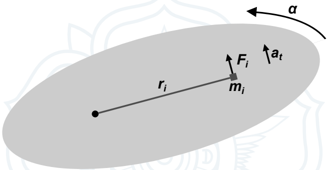
- Consider an object is rotating freely about a pivot point as shown in the figure above. The forces F_i acting on a particle with mass m_i in the object. This force causes the particle to have an acceleration a_t, therefore

$$F_i = m_i a_t$$

- The relation between tangential acceleration a_t and angular acceleration α is

$$a_t = \alpha r_i$$

Rotational Dynamics



Effect of a Force on a Rigid Body.

- Therefore the F_i equation can be re-written as follow

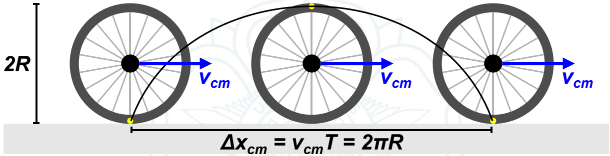
$$F_i = m_i \alpha r_i \quad \rightarrow \quad F_i r_i = m_i r_i^2 \alpha \quad \rightarrow \quad \tau_i = (m_i r_i^2)\alpha$$

- By summing all of the contribution of each individual particles, we get

$$\sum_i \tau_i = \left(\sum_i m_i r_i^2 \right) \alpha \quad \rightarrow \quad \sum \tau = I\alpha \tag{14}$$

which is known as *Newton's second law for rotational motion*

Rolling Motion



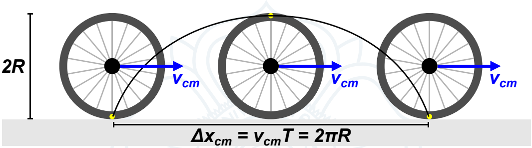
Rolling Motion without Slipping.

- **Rolling Motion** is a combination motion on which an object rotates about an axis that is moving along a straight-line trajectory.
- Consider a ball shown in the figure above is rolling without slipping. After the ball underwent a full rotation, the center of mass position had moved as far as

$$\Delta x_{cm} = v_{cm} T = 2\pi R \quad \rightarrow \quad v_{cm} = \left(\frac{2\pi}{T}\right) R$$

where T is the rotation period.

Rolling Motion



Rolling Motion without Slipping.

- When the ball is rotating with constant angular velocity ω, then $\omega T = 2\pi$. Therefore

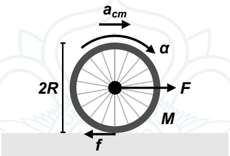
$$v_{cm} = \omega R \tag{15}$$

  which is known as rolling constraint, the basic link between translation and rotation for an objects that roll without slipping.
- By differentiating both sides of the previous equation, we can derive the relation between translational acceleration a_{cm} and angular acceleration α as follow

$$a_{cm} = \alpha R \tag{16}$$

- These 2 equations are the most important equations that are usually used when analyzing an object that rolls without slipping.

Rolling Motion (Example)



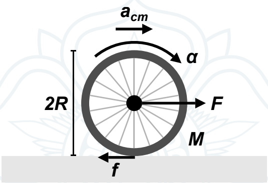
A Wheel is Rolling without Slipping.

- A force F is working at the center of mass of a wheel with mass M, radius R and moment of inertia I. The wheel is rolling without slipping on a rough surface with coefficient of friction μ. Determine the wheel's translational acceleration (a_{cm}).
- To solve this problem, first, we need to draw all the forces acting on the wheel. Then we need apply the Newton's second law for both translational and rotational motion

$$\sum F = Ma_{cm} \quad \rightarrow \quad F - f = Ma_{cm}$$

$$\sum \tau = I\alpha \quad \rightarrow \quad fR = I\alpha$$

Rolling Motion (Example)



A Wheel is Rolling without Slipping.

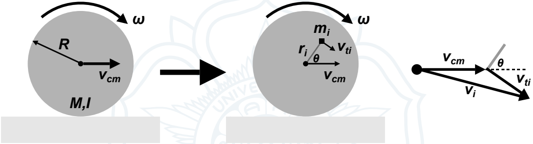
- It is important to note that because the wheel is rotating about its center of mass, therefore the torque is calculated by assuming the pivot point is at its center of mass.
- Finally, because the wheel is rolling without slipping, therefore

$$a_{cm} = \alpha R$$

- By solving those equations, we can get the a_{cm} equation as follow

$$a_{cm} = \frac{F}{M + (I/R^2)}$$

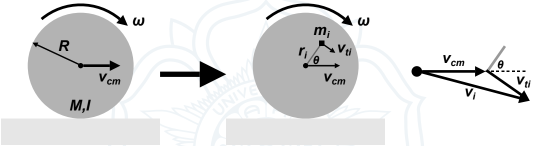
Kinetic energy of a Rolling Object



Kinetic energy of a Rolling Object.

- Consider an object with mass M and moment of inertia I. Its center of mass is moving with speed v_{cm} while rotating about its center of mass with angular velocity ω. How much is the kinetic energy of this object?
- Let us consider a particle in the object with mass m_i. In order to calculate the kinetic energy contribution of this particle, we need to calculate the speed of this particle relative to the ground (v_i).

Kinetic energy of a Rolling Object



Kinetic energy of a Rolling Object.

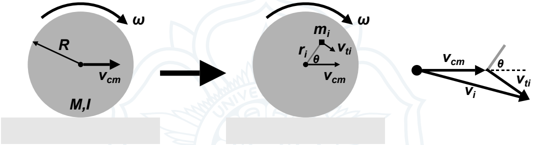
- The particle is moving with speed $v_{ti} = \omega r_i$ relative to the center of mass, where the center of mass is moving with speed v_{cm} relative to the ground. Therefore

$$\vec{v_i} = \vec{v_{cm}} + \vec{v_{ti}}$$

- It can be shown that

$$\begin{aligned} v_i^2 &= (v_{cm} + \omega r_i \sin\theta)^2 + (\omega r_i \cos\theta)^2 \\ &= v_{cm}^2 + 2 v_{cm}\omega (r_i \sin\theta) + \omega^2 r_i^2 \\ &= v_{cm}^2 + 2 v_{cm}\omega y_i + \omega^2 r_i^2 \end{aligned}$$

Kinetic energy of a Rolling Object



Kinetic energy of a Rolling Object.

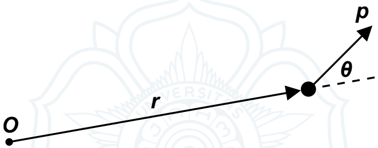
- The total kinetic energy of this object can be calculated as follow

$$K = \sum_i \frac{m_i v_i^2}{2} = \frac{v_{cm}^2}{2}\left(\sum_i m_i\right) + v_{cm}\omega\left(\sum_i m_i y_i\right) + \frac{\omega^2}{2}\left(\sum_i m_i r_i^2\right)$$
$$= \frac{Mv_{cm}^2}{2} + Mv_{cm}\omega y_{cm} + \frac{I\omega^2}{2}$$

- Because $y_{cm} = 0$ (why?), therefore the total kinetic energy is given by

$$K = \frac{Mv_{cm}^2}{2} + \frac{I\omega^2}{2} = K_{translation} + K_{rotation} \tag{17}$$

Angular Momentum

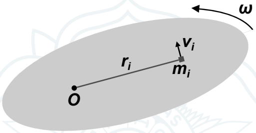


Angular Momentum of a Moving Particle.

- We have studied that a particle with mass m moving with velocity $\vec{v}$ carries a momentum $\vec{p} = m\vec{v}$.
- Similar also for a particle rotating about a certain axis, it also carries an **Angular Momentum**.
- Consider a particle with linear momentum $\vec{p}$ is moving at distance $\vec{r}$ relative to a point O. The *angular momentum* of this particle relative to point O is defined as

$$\vec{L} = \vec{r} \times \vec{p} \tag{18}$$

Angular Momentum



Angular Momentum of a Rigid Body.

- How about the angular momentum of a rigid body? To calculate the angular momentum of a rigid body, let us consider a rigid body shown in the figure above.
- A particle with mass m_i is located at a distance r_i from the rotation axis. Its angular momentum can be calculated as follow

$$L_i = m_i v_i r_i = m_i r_i^2 \omega$$

- The total angular momentum of this object can be calculated as follow

$$L = \sum L_i = \left(\sum_i m_i r_i^2 \right) \omega \quad \rightarrow \quad L = I\omega \tag{19}$$

Angular Momentum and Torque

- From the definition of of angular momentum $\vec{L} = \vec{r} \times \vec{p}$, let us differentiate both sides of this equation

$$\frac{d\vec{L}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt}$$

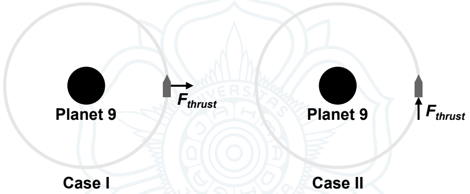
assuming that $\vec{r}$ is constant. It is known that $\frac{d\vec{p}}{dt} = \vec{F}$. Therefore

$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} \quad \rightarrow \quad \frac{d\vec{L}}{dt} = \vec{\tau} \tag{20}$$

- This equation shows that when a torque $\vec{\tau}$ is acting on an object, it will change its angular momentum $\vec{L}$.
- On the other hand, when there is no torque acting on an object, its angular momentum is constant (conserved), i.e.

$$\vec{L}_{initial} = \vec{L}_{final} \quad \text{when} \quad \sum \tau = 0 \tag{21}$$

Conservation of Angular Momentum (Example)



A Space Shuttle Orbits a Planet.

- Consider the case of a space shuttle orbiting a planet. In case I, a thrust F_{thrust} is working on the space shuttle in a radially outward direction. In case II, the thrust is working in the tangential direction. In which case is the angular momentum conserved?
- For case A, because F_{thrust} is working in radially outward direction, therefore the net torque is zero. Therefore its angular momentum is conserved.
- For case B, the force F_{thrust} causes a torque working on the space shuttle. Therefore, its angular momentum is *not* conserved.