

**INTRODUCTION  
TO  
LINEAR  
ALGEBRA  
Fourth Edition**

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**MANUAL FOR INSTRUCTORS**

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## Problem Set 1.1, page 8

- 1 The combinations give (a) a line in  $\mathbf{R}^3$  (b) a plane in  $\mathbf{R}^3$  (c) all of  $\mathbf{R}^3$ .
- 2  $\mathbf{v} + \mathbf{w} = (2, 3)$  and  $\mathbf{v} - \mathbf{w} = (6, -1)$  will be the diagonals of the parallelogram with  $\mathbf{v}$  and  $\mathbf{w}$  as two sides going out from  $(0, 0)$ .
- 3 This problem gives the diagonals  $\mathbf{v} + \mathbf{w}$  and  $\mathbf{v} - \mathbf{w}$  of the parallelogram and asks for the sides: The opposite of Problem 2. In this example  $\mathbf{v} = (3, 3)$  and  $\mathbf{w} = (2, -2)$ .
- 4  $3\mathbf{v} + \mathbf{w} = (7, 5)$  and  $c\mathbf{v} + d\mathbf{w} = (2c + d, c + 2d)$ .
- 5  $\mathbf{u} + \mathbf{v} = (-2, 3, 1)$  and  $\mathbf{u} + \mathbf{v} + \mathbf{w} = (0, 0, 0)$  and  $2\mathbf{u} + 2\mathbf{v} + \mathbf{w} = (-2, 3, 1)$ . The vectors  $\mathbf{u}, \mathbf{v}, \mathbf{w}$  are in the same plane because a combination gives  $(0, 0, 0)$ . Stated another way:  $\mathbf{u} = -\mathbf{v} - \mathbf{w}$  is in the plane of  $\mathbf{v}$  and  $\mathbf{w}$ .
- 6 The components of every  $c\mathbf{v} + d\mathbf{w}$  add to zero.  $c = 3$  and  $d = 9$  give  $(3, 3, -6)$ .
- 7 The nine combinations  $c(2, 1) + d(0, 1)$  with  $c = 0, 1, 2$  and  $d = 0, 1, 2$  will lie on a lattice. If we took all whole numbers  $c$  and  $d$ , the lattice would lie over the whole plane.
- 8 The other diagonal is  $\mathbf{v} - \mathbf{w}$  (or else  $\mathbf{w} - \mathbf{v}$ ). Adding diagonals gives  $2\mathbf{v}$  (or  $2\mathbf{w}$ ).
- 9 The fourth corner can be  $(4, 4)$  or  $(4, 0)$  or  $(-2, 2)$ . Three possible parallelograms!
- 10  $\mathbf{i} - \mathbf{j} = (1, 1, 0)$  is in the base ( $x$ - $y$  plane).  $\mathbf{i} + \mathbf{j} + \mathbf{k} = (1, 1, 1)$  is the opposite corner from  $(0, 0, 0)$ . Points in the cube have  $0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1$ .
- 11 Four more corners  $(1, 1, 0), (1, 0, 1), (0, 1, 1), (1, 1, 1)$ . The center point is  $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ . Centers of faces are  $(\frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{2}, \frac{1}{2}, 1)$  and  $(0, \frac{1}{2}, \frac{1}{2}), (1, \frac{1}{2}, \frac{1}{2})$  and  $(\frac{1}{2}, 0, \frac{1}{2}), (\frac{1}{2}, 1, \frac{1}{2})$ .
- 12 A four-dimensional cube has  $2^4 = 16$  corners and  $2 \cdot 4 = 8$  three-dimensional faces and 24 two-dimensional faces and 32 edges in Worked Example 2.4 A.
- 13 Sum = zero vector. Sum = -2:00 vector = 8:00 vector. 2:00 is  $30^\circ$  from horizontal =  $(\cos \frac{\pi}{6}, \sin \frac{\pi}{6}) = (\sqrt{3}/2, 1/2)$ .
- 14 Moving the origin to 6:00 adds  $\mathbf{j} = (0, 1)$  to every vector. So the sum of twelve vectors changes from  $\mathbf{0}$  to  $12\mathbf{j} = (0, 12)$ .
- 15 The point  $\frac{3}{4}\mathbf{v} + \frac{1}{4}\mathbf{w}$  is three-fourths of the way to  $\mathbf{v}$  starting from  $\mathbf{w}$ . The vector  $\frac{1}{4}\mathbf{v} + \frac{1}{4}\mathbf{w}$  is halfway to  $\mathbf{u} = \frac{1}{2}\mathbf{v} + \frac{1}{2}\mathbf{w}$ . The vector  $\mathbf{v} + \mathbf{w}$  is  $2\mathbf{u}$  (the far corner of the parallelogram).
- 16 All combinations with  $c + d = 1$  are on the line that passes through  $\mathbf{v}$  and  $\mathbf{w}$ . The point  $\mathbf{V} = -\mathbf{v} + 2\mathbf{w}$  is on that line but it is beyond  $\mathbf{w}$ .
- 17 All vectors  $c\mathbf{v} + c\mathbf{w}$  are on the line passing through  $(0, 0)$  and  $\mathbf{u} = \frac{1}{2}\mathbf{v} + \frac{1}{2}\mathbf{w}$ . That line continues out beyond  $\mathbf{v} + \mathbf{w}$  and back beyond  $(0, 0)$ . With  $c \geq 0$ , half of this line is removed, leaving a ray that starts at  $(0, 0)$ .
- 18 The combinations  $c\mathbf{v} + d\mathbf{w}$  with  $0 \leq c \leq 1$  and  $0 \leq d \leq 1$  fill the parallelogram with sides  $\mathbf{v}$  and  $\mathbf{w}$ . For example, if  $\mathbf{v} = (1, 0)$  and  $\mathbf{w} = (0, 1)$  then  $c\mathbf{v} + d\mathbf{w}$  fills the unit square.
- 19 With  $c \geq 0$  and  $d \geq 0$  we get the infinite “cone” or “wedge” between  $\mathbf{v}$  and  $\mathbf{w}$ . For example, if  $\mathbf{v} = (1, 0)$  and  $\mathbf{w} = (0, 1)$ , then the cone is the whole quadrant  $x \geq 0, y \geq 0$ . Question: What if  $\mathbf{w} = -\mathbf{v}$ ? The cone opens to a half-space.

- 20 (a)  $\frac{1}{3}\mathbf{u} + \frac{1}{3}\mathbf{v} + \frac{1}{3}\mathbf{w}$  is the center of the triangle between  $\mathbf{u}$ ,  $\mathbf{v}$  and  $\mathbf{w}$ ;  $\frac{1}{2}\mathbf{u} + \frac{1}{2}\mathbf{w}$  lies between  $\mathbf{u}$  and  $\mathbf{w}$  (b) To fill the triangle keep  $c \geq 0$ ,  $d \geq 0$ ,  $e \geq 0$ , and  $c + d + e = 1$ .
- 21 The sum is  $(\mathbf{v} - \mathbf{u}) + (\mathbf{w} - \mathbf{v}) + (\mathbf{u} - \mathbf{w}) = \mathbf{zero\ vector}$ . Those three sides of a triangle are in the same plane!
- 22 The vector  $\frac{1}{2}(\mathbf{u} + \mathbf{v} + \mathbf{w})$  is *outside* the pyramid because  $c + d + e = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} > 1$ .
- 23 All vectors are combinations of  $\mathbf{u}$ ,  $\mathbf{v}$ ,  $\mathbf{w}$  as drawn (not in the same plane). Start by seeing that  $c\mathbf{u} + d\mathbf{v}$  fills a plane, then adding  $e\mathbf{w}$  fills all of  $\mathbf{R}^3$ .
- 24 The combinations of  $\mathbf{u}$  and  $\mathbf{v}$  fill one plane. The combinations of  $\mathbf{v}$  and  $\mathbf{w}$  fill another plane. Those planes meet in a *line*: *only the vectors*  $c\mathbf{v}$  are in both planes.
- 25 (a) For a line, choose  $\mathbf{u} = \mathbf{v} = \mathbf{w} =$  any nonzero vector (b) For a plane, choose  $\mathbf{u}$  and  $\mathbf{v}$  in different directions. A combination like  $\mathbf{w} = \mathbf{u} + \mathbf{v}$  is in the same plane.
- 26 Two equations come from the two components:  $c + 3d = 14$  and  $2c + d = 8$ . The solution is  $c = 2$  and  $d = 4$ . Then  $2(1, 2) + 4(3, 1) = (14, 8)$ .
- 27 The combinations of  $\mathbf{i} = (1, 0, 0)$  and  $\mathbf{i} + \mathbf{j} = (1, 1, 0)$  fill the  $xy$  plane in  $xyz$  space.
- 28 There are 6 unknown numbers  $v_1, v_2, v_3, w_1, w_2, w_3$ . The six equations come from the components of  $\mathbf{v} + \mathbf{w} = (4, 5, 6)$  and  $\mathbf{v} - \mathbf{w} = (2, 5, 8)$ . Add to find  $2\mathbf{v} = (6, 10, 14)$  so  $\mathbf{v} = (3, 5, 7)$  and  $\mathbf{w} = (1, 0, -1)$ .
- 29 Two combinations out of infinitely many that produce  $\mathbf{b} = (0, 1)$  are  $-2\mathbf{u} + \mathbf{v}$  and  $\frac{1}{2}\mathbf{w} - \frac{1}{2}\mathbf{v}$ . **No**, three vectors  $\mathbf{u}, \mathbf{v}, \mathbf{w}$  in the  $x$ - $y$  plane could fail to produce  $\mathbf{b}$  if all three lie on a line that does not contain  $\mathbf{b}$ . *Yes*, if one combination produces  $\mathbf{b}$  then two (and infinitely many) combinations will produce  $\mathbf{b}$ . This is true even if  $\mathbf{u} = \mathbf{0}$ ; the combinations can have different  $c\mathbf{u}$ .
- 30 The combinations of  $\mathbf{v}$  and  $\mathbf{w}$  fill the plane *unless*  $\mathbf{v}$  and  $\mathbf{w}$  lie on the same line through  $(0, 0)$ . Four vectors whose combinations fill 4-dimensional space: one example is the “standard basis”  $(1, 0, 0, 0)$ ,  $(0, 1, 0, 0)$ ,  $(0, 0, 1, 0)$ , and  $(0, 0, 0, 1)$ .
- 31 The equations  $c\mathbf{u} + d\mathbf{v} + e\mathbf{w} = \mathbf{b}$  are

$$\begin{array}{rcl} 2c - d & = & 1 \\ -c + 2d - e & = & 0 \\ -d + 2e & = & 0 \end{array} \quad \begin{array}{l} \text{So } d = 2e \\ \text{then } c = 3e \\ \text{then } 4e = 1 \end{array} \quad \begin{array}{l} c = 3/4 \\ d = 2/4 \\ e = 1/4 \end{array}$$

## Problem Set 1.2, page 19

- 1  $\mathbf{u} \cdot \mathbf{v} = -1.8 + 3.2 = 1.4$ ,  $\mathbf{u} \cdot \mathbf{w} = -4.8 + 4.8 = 0$ ,  $\mathbf{v} \cdot \mathbf{w} = 24 + 24 = 48 = \mathbf{w} \cdot \mathbf{v}$ .
- 2  $\|\mathbf{u}\| = 1$  and  $\|\mathbf{v}\| = 5$  and  $\|\mathbf{w}\| = 10$ . Then  $1.4 < (1)(5)$  and  $48 < (5)(10)$ , confirming the Schwarz inequality.
- 3 Unit vectors  $\mathbf{v}/\|\mathbf{v}\| = (\frac{3}{5}, \frac{4}{5}) = (.6, .8)$  and  $\mathbf{w}/\|\mathbf{w}\| = (\frac{4}{5}, \frac{3}{5}) = (.8, .6)$ . The cosine of  $\theta$  is  $\frac{\mathbf{v}}{\|\mathbf{v}\|} \cdot \frac{\mathbf{w}}{\|\mathbf{w}\|} = \frac{24}{25}$ . The vectors  $\mathbf{w}, \mathbf{u}, -\mathbf{w}$  make  $0^\circ, 90^\circ, 180^\circ$  angles with  $\mathbf{w}$ .
- 4 (a)  $\mathbf{v} \cdot (-\mathbf{v}) = -1$  (b)  $(\mathbf{v} + \mathbf{w}) \cdot (\mathbf{v} - \mathbf{w}) = \mathbf{v} \cdot \mathbf{v} + \mathbf{w} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{w} - \mathbf{w} \cdot \mathbf{w} = 1 + ( ) - ( ) - 1 = 0$  so  $\theta = 90^\circ$  (notice  $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$ ) (c)  $(\mathbf{v} - 2\mathbf{w}) \cdot (\mathbf{v} + 2\mathbf{w}) = \mathbf{v} \cdot \mathbf{v} - 4\mathbf{w} \cdot \mathbf{w} = 1 - 4 = -3$ .

- 5  $\mathbf{u}_1 = \mathbf{v}/\|\mathbf{v}\| = (3, 1)/\sqrt{10}$  and  $\mathbf{u}_2 = \mathbf{w}/\|\mathbf{w}\| = (2, 1, 2)/3$ .  $\mathbf{U}_1 = (1, -3)/\sqrt{10}$  is perpendicular to  $\mathbf{u}_1$  (and so is  $(-1, 3)/\sqrt{10}$ ).  $\mathbf{U}_2$  could be  $(1, -2, 0)/\sqrt{5}$ : There is a whole plane of vectors perpendicular to  $\mathbf{u}_2$ , and a whole circle of unit vectors in that plane.
- 6 All vectors  $\mathbf{w} = (c, 2c)$  are perpendicular to  $\mathbf{v}$ . All vectors  $(x, y, z)$  with  $x + y + z = 0$  lie on a *plane*. All vectors perpendicular to  $(1, 1, 1)$  and  $(1, 2, 3)$  lie on a *line*.
- 7 (a)  $\cos \theta = \mathbf{v} \cdot \mathbf{w}/\|\mathbf{v}\|\|\mathbf{w}\| = 1/(2)(1)$  so  $\theta = 60^\circ$  or  $\pi/3$  radians (b)  $\cos \theta = 0$  so  $\theta = 90^\circ$  or  $\pi/2$  radians (c)  $\cos \theta = 2/(2)(2) = 1/2$  so  $\theta = 60^\circ$  or  $\pi/3$  (d)  $\cos \theta = -1/\sqrt{2}$  so  $\theta = 135^\circ$  or  $3\pi/4$ .
- 8 (a) False:  $\mathbf{v}$  and  $\mathbf{w}$  are any vectors in the plane perpendicular to  $\mathbf{u}$  (b) True:  $\mathbf{u} \cdot (\mathbf{v} + 2\mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + 2\mathbf{u} \cdot \mathbf{w} = 0$  (c) True,  $\|\mathbf{u} - \mathbf{v}\|^2 = (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})$  splits into  $\mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} = 2$  when  $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} = 0$ .
- 9 If  $v_2 w_2 / v_1 w_1 = -1$  then  $v_2 w_2 = -v_1 w_1$  or  $v_1 w_1 + v_2 w_2 = \mathbf{v} \cdot \mathbf{w} = 0$ : perpendicular!
- 10 Slopes  $2/1$  and  $-1/2$  multiply to give  $-1$ : then  $\mathbf{v} \cdot \mathbf{w} = 0$  and the vectors (the directions) are perpendicular.
- 11  $\mathbf{v} \cdot \mathbf{w} < 0$  means angle  $> 90^\circ$ ; these  $\mathbf{w}$ 's fill half of 3-dimensional space.
- 12  $(1, 1)$  perpendicular to  $(1, 5) - c(1, 1)$  if  $6 - 2c = 0$  or  $c = 3$ ;  $\mathbf{v} \cdot (\mathbf{w} - c\mathbf{v}) = 0$  if  $c = \mathbf{v} \cdot \mathbf{w} / \mathbf{v} \cdot \mathbf{v}$ . Subtracting  $c\mathbf{v}$  is the key to perpendicular vectors.
- 13 The plane perpendicular to  $(1, 0, 1)$  contains all vectors  $(c, d, -c)$ . In that plane,  $\mathbf{v} = (1, 0, -1)$  and  $\mathbf{w} = (0, 1, 0)$  are perpendicular.
- 14 One possibility among many:  $\mathbf{u} = (1, -1, 0, 0)$ ,  $\mathbf{v} = (0, 0, 1, -1)$ ,  $\mathbf{w} = (1, 1, -1, -1)$  and  $(1, 1, 1, 1)$  are perpendicular to each other. "We can rotate those  $\mathbf{u}, \mathbf{v}, \mathbf{w}$  in their 3D hyperplane."
- 15  $\frac{1}{2}(x + y) = (2 + 8)/2 = 5$ ;  $\cos \theta = 2\sqrt{16}/\sqrt{10}\sqrt{10} = 8/10$ .
- 16  $\|\mathbf{v}\|^2 = 1 + 1 + \dots + 1 = 9$  so  $\|\mathbf{v}\| = 3$ ;  $\mathbf{u} = \mathbf{v}/3 = (\frac{1}{3}, \dots, \frac{1}{3})$  is a unit vector in 9D;  $\mathbf{w} = (1, -1, 0, \dots, 0)/\sqrt{2}$  is a unit vector in the 8D hyperplane perpendicular to  $\mathbf{v}$ .
- 17  $\cos \alpha = 1/\sqrt{2}$ ,  $\cos \beta = 0$ ,  $\cos \gamma = -1/\sqrt{2}$ . For any vector  $\mathbf{v}$ ,  $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = (v_1^2 + v_2^2 + v_3^2)/\|\mathbf{v}\|^2 = 1$ .
- 18  $\|\mathbf{v}\|^2 = 4^2 + 2^2 = 20$  and  $\|\mathbf{w}\|^2 = (-1)^2 + 2^2 = 5$ . Pythagoras is  $\|(3, 4)\|^2 = 25 = 20 + 5$ .
- 19 Start from the rules (1), (2), (3) for  $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$  and  $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w})$  and  $(c\mathbf{v}) \cdot \mathbf{w}$ . Use rule (2) for  $(\mathbf{v} + \mathbf{w}) \cdot (\mathbf{v} + \mathbf{w}) = (\mathbf{v} + \mathbf{w}) \cdot \mathbf{v} + (\mathbf{v} + \mathbf{w}) \cdot \mathbf{w}$ . By rule (1) this is  $\mathbf{v} \cdot (\mathbf{v} + \mathbf{w}) + \mathbf{w} \cdot (\mathbf{v} + \mathbf{w})$ . Rule (2) again gives  $\mathbf{v} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{v} + \mathbf{w} \cdot \mathbf{w} = \mathbf{v} \cdot \mathbf{v} + 2\mathbf{v} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{w}$ . Notice  $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$ ! The main point is to be free to open up parentheses.
- 20 We know that  $(\mathbf{v} - \mathbf{w}) \cdot (\mathbf{v} - \mathbf{w}) = \mathbf{v} \cdot \mathbf{v} - 2\mathbf{v} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{w}$ . The Law of Cosines writes  $\|\mathbf{v}\|\|\mathbf{w}\|\cos \theta$  for  $\mathbf{v} \cdot \mathbf{w}$ . When  $\theta < 90^\circ$  this  $\mathbf{v} \cdot \mathbf{w}$  is positive, so in this case  $\mathbf{v} \cdot \mathbf{v} + \mathbf{w} \cdot \mathbf{w}$  is larger than  $\|\mathbf{v} - \mathbf{w}\|^2$ .
- 21  $2\mathbf{v} \cdot \mathbf{w} \leq 2\|\mathbf{v}\|\|\mathbf{w}\|$  leads to  $\|\mathbf{v} + \mathbf{w}\|^2 = \mathbf{v} \cdot \mathbf{v} + 2\mathbf{v} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{w} \leq \|\mathbf{v}\|^2 + 2\|\mathbf{v}\|\|\mathbf{w}\| + \|\mathbf{w}\|^2$ . This is  $(\|\mathbf{v}\| + \|\mathbf{w}\|)^2$ . Taking square roots gives  $\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$ .
- 22  $v_1^2 w_1^2 + 2v_1 w_1 v_2 w_2 + v_2^2 w_2^2 \leq v_1^2 w_1^2 + v_1^2 w_2^2 + v_2^2 w_1^2 + v_2^2 w_2^2$  is true (cancel 4 terms) because the difference is  $v_1^2 w_2^2 + v_2^2 w_1^2 - 2v_1 w_1 v_2 w_2$  which is  $(v_1 w_2 - v_2 w_1)^2 \geq 0$ .

- 23**  $\cos \beta = w_1/\|\mathbf{w}\|$  and  $\sin \beta = w_2/\|\mathbf{w}\|$ . Then  $\cos(\beta - \alpha) = \cos \beta \cos \alpha + \sin \beta \sin \alpha = v_1 w_1/\|\mathbf{v}\|\|\mathbf{w}\| + v_2 w_2/\|\mathbf{v}\|\|\mathbf{w}\| = \mathbf{v} \cdot \mathbf{w}/\|\mathbf{v}\|\|\mathbf{w}\|$ . This is  $\cos \theta$  because  $\beta - \alpha = \theta$ .
- 24** Example 6 gives  $|u_1|U_1| \leq \frac{1}{2}(u_1^2 + U_1^2)$  and  $|u_2|U_2| \leq \frac{1}{2}(u_2^2 + U_2^2)$ . The whole line becomes  $.96 \leq (.6)(.8) + (.8)(.6) \leq \frac{1}{2}(.6^2 + .8^2) + \frac{1}{2}(.8^2 + .6^2) = 1$ . True:  $.96 < 1$ .
- 25** The cosine of  $\theta$  is  $x/\sqrt{x^2 + y^2}$ , near side over hypotenuse. Then  $|\cos \theta|^2$  is not greater than 1:  $x^2/(x^2 + y^2) \leq 1$ .
- 26** The vectors  $\mathbf{w} = (x, y)$  with  $(1, 2) \cdot \mathbf{w} = x + 2y = 5$  lie on a line in the  $xy$  plane. The shortest  $\mathbf{w}$  on that line is  $(1, 2)$ . (The Schwarz inequality  $\|\mathbf{w}\| \geq \mathbf{v} \cdot \mathbf{w}/\|\mathbf{v}\| = \sqrt{5}$  is an equality when  $\cos \theta = 0$  and  $\mathbf{w} = (1, 2)$  and  $\|\mathbf{w}\| = \sqrt{5}$ .)
- 27** The length  $\|\mathbf{v} - \mathbf{w}\|$  is between 2 and 8 (triangle inequality when  $\|\mathbf{v}\| = 5$  and  $\|\mathbf{w}\| = 3$ ). The dot product  $\mathbf{v} \cdot \mathbf{w}$  is between  $-15$  and  $15$  by the Schwarz inequality.
- 28** Three vectors in the plane could make angles greater than  $90^\circ$  with each other: for example  $(1, 0), (-1, 4), (-1, -4)$ . Four vectors could *not* do this ( $360^\circ$  total angle). How many can do this in  $\mathbf{R}^3$  or  $\mathbf{R}^n$ ? Ben Harris and Greg Marks showed me that the answer is  $n + 1$ . The vectors from the center of a regular simplex in  $\mathbf{R}^n$  to its  $n + 1$  vertices all have negative dot products. If  $n + 2$  vectors in  $\mathbf{R}^n$  had negative dot products, project them onto the plane orthogonal to the last one. Now you have  $n + 1$  vectors in  $\mathbf{R}^{n-1}$  with negative dot products. Keep going to 4 vectors in  $\mathbf{R}^2$ : no way!
- 29** For a specific example, pick  $\mathbf{v} = (1, 2, -3)$  and then  $\mathbf{w} = (-3, 1, 2)$ . In this example  $\cos \theta = \mathbf{v} \cdot \mathbf{w}/\|\mathbf{v}\|\|\mathbf{w}\| = -7/\sqrt{14}\sqrt{14} = -1/2$  and  $\theta = 120^\circ$ . This always happens when  $x + y + z = 0$ :

$$\mathbf{v} \cdot \mathbf{w} = xz + xy + yz = \frac{1}{2}(x + y + z)^2 - \frac{1}{2}(x^2 + y^2 + z^2)$$

This is the same as  $\mathbf{v} \cdot \mathbf{w} = 0 - \frac{1}{2}\|\mathbf{v}\|\|\mathbf{w}\|$ . Then  $\cos \theta = \frac{1}{2}$ .

- 30** Wikipedia gives this proof of geometric mean  $G = \sqrt[3]{xyz} \leq$  arithmetic mean  $A = (x + y + z)/3$ . First there is equality in case  $x = y = z$ . Otherwise  $A$  is somewhere between the three positive numbers, say for example  $z < A < y$ .

Use the known inequality  $g \leq a$  for the *two* positive numbers  $x$  and  $y + z - A$ . Their mean  $a = \frac{1}{2}(x + y + z - A)$  is  $\frac{1}{2}(3A - A) =$  same as  $A$ ! So  $a \geq g$  says that  $A^3 \geq g^2 A = x(y + z - A)A$ . But  $(y + z - A)A = (y - A)(A - z) + yz > yz$ . Substitute to find  $A^3 > xyz = G^3$  as we wanted to prove. Not easy!

There are many proofs of  $G = (x_1 x_2 \cdots x_n)^{1/n} \leq A = (x_1 + x_2 + \cdots + x_n)/n$ . In calculus you are maximizing  $G$  on the plane  $x_1 + x_2 + \cdots + x_n = n$ . The maximum occurs when all  $x$ 's are equal.

- 31** The columns of the 4 by 4 "Hadamard matrix" (times  $\frac{1}{2}$ ) are perpendicular unit vectors:

$$\frac{1}{2}H = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

- 32** The commands  $V = \text{randn}(3, 30)$ ;  $D = \text{sqrt}(\text{diag}(V' * V))$ ;  $U = V \setminus D$ ; will give 30 random unit vectors in the columns of  $U$ . Then  $u' * U$  is a row matrix of 30 dot products whose average absolute value may be close to  $2/\pi$ .

### Problem Set 1.3, page 29

- 1  $2s_1 + 3s_2 + 4s_3 = (2, 5, 9)$ . The same vector  $\mathbf{b}$  comes from  $S$  times  $\mathbf{x} = (2, 3, 4)$ :

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} (\text{row 1}) \cdot \mathbf{x} \\ (\text{row 2}) \cdot \mathbf{x} \\ (\text{row 3}) \cdot \mathbf{x} \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix}.$$

- 2 The solutions are  $y_1 = 1, y_2 = 0, y_3 = 0$  (right side = column 1) and  $y_1 = 1, y_2 = 3, y_3 = 5$ . That second example illustrates that the first  $n$  odd numbers add to  $n^2$ .

$$\begin{array}{rcl} y_1 & = & B_1 \\ y_1 + y_2 & = & B_2 \\ y_1 + y_2 + y_3 & = & B_3 \end{array} \quad \text{gives} \quad \begin{array}{rcl} y_1 & = & B_1 \\ y_2 & = & -B_1 + B_2 \\ y_3 & = & -B_2 + B_3 \end{array} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}$$

The inverse of  $S = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$  is  $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ : independent columns in  $A$  and  $S$ !

- 4 The combination  $0\mathbf{w}_1 + 0\mathbf{w}_2 + 0\mathbf{w}_3$  always gives the zero vector, but this problem looks for other *zero* combinations (then the vectors are *dependent*, they lie in a plane):  $\mathbf{w}_2 = (\mathbf{w}_1 + \mathbf{w}_3)/2$  so one combination that gives zero is  $\frac{1}{2}\mathbf{w}_1 - \mathbf{w}_2 + \frac{1}{2}\mathbf{w}_3$ .

- 5 The rows of the 3 by 3 matrix in Problem 4 must also be *dependent*:  $\mathbf{r}_2 = \frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_3)$ . The column and row combinations that produce  $\mathbf{0}$  are the same: this is unusual.

$$\begin{array}{lcl} 6 \ c = 3 & \begin{bmatrix} 1 & 3 & 5 \\ 1 & 2 & 4 \\ 1 & 1 & 3 \end{bmatrix} & \text{has column 3} = 2(\text{column 1}) + \text{column 2} \\ c = -1 & \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} & \text{has column 3} = -\text{column 1} + \text{column 2} \\ c = 0 & \begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & 5 \\ 3 & 3 & 6 \end{bmatrix} & \text{has column 3} = 3(\text{column 1}) - \text{column 2} \end{array}$$

- 7 All three rows are perpendicular to the solution  $\mathbf{x}$  (the three equations  $\mathbf{r}_1 \cdot \mathbf{x} = 0$  and  $\mathbf{r}_2 \cdot \mathbf{x} = 0$  and  $\mathbf{r}_3 \cdot \mathbf{x} = 0$  tell us this). Then the whole plane of the rows is perpendicular to  $\mathbf{x}$  (the plane is also perpendicular to all multiples  $c\mathbf{x}$ ).

$$\begin{array}{rcl} x_1 - 0 & = & b_1 \\ x_2 - x_1 & = & b_2 \\ x_3 - x_2 & = & b_3 \\ x_4 - x_3 & = & b_4 \end{array} \quad \begin{array}{rcl} x_1 & = & b_1 \\ x_2 & = & b_1 + b_2 \\ x_3 & = & b_1 + b_2 + b_3 \\ x_4 & = & b_1 + b_2 + b_3 + b_4 \end{array} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = A^{-1}\mathbf{b}$$

- 9 The cyclic difference matrix  $C$  has a line of solutions (in 4 dimensions) to  $C\mathbf{x} = \mathbf{0}$ :

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{when } \mathbf{x} = \begin{bmatrix} c \\ c \\ c \\ c \end{bmatrix} = \text{any constant vector.}$$

$$\begin{array}{lcl}
 z_2 - z_1 = b_1 & z_1 = -b_1 - b_2 - b_3 & \\
 10 \quad z_3 - z_2 = b_2 & z_2 = -b_2 - b_3 & \\
 0 - z_3 = b_3 & z_3 = -b_3 & 
 \end{array}
 = \begin{bmatrix} -1 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \Delta^{-1} \mathbf{b}$$

11 The forward differences of the squares are  $(t+1)^2 - t^2 = t^2 + 2t + 1 - t^2 = 2t + 1$ . Differences of the  $n$ th power are  $(t+1)^n - t^n = t^n - t^n + nt^{n-1} + \dots$ . The leading term is the derivative  $nt^{n-1}$ . The binomial theorem gives all the terms of  $(t+1)^n$ .

12 Centered difference matrices of *even* size seem to be invertible. Look at eqns. 1 and 4:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \quad \begin{array}{l} \text{First} \\ \text{solve} \\ x_2 = b_1 \\ -x_3 = b_4 \end{array} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -b_2 - b_4 \\ b_1 \\ -b_4 \\ b_1 + b_3 \end{bmatrix}$$

13 *Odd size*: The five centered difference equations lead to  $b_1 + b_3 + b_5 = 0$ .

$$\begin{array}{ll}
 x_2 = b_1 & \text{Add equations 1, 3, 5} \\
 x_3 - x_1 = b_2 & \text{The left side of the sum is zero} \\
 x_4 - x_2 = b_3 & \text{The right side is } b_1 + b_3 + b_5 \\
 x_5 - x_3 = b_4 & \text{There cannot be a solution unless } b_1 + b_3 + b_5 = 0. \\
 -x_4 = b_5 & 
 \end{array}$$

14 An example is  $(a, b) = (3, 6)$  and  $(c, d) = (1, 2)$ . The ratios  $a/c$  and  $b/d$  are equal. Then  $ad = bc$ . Then (when you divide by  $bd$ ) the ratios  $a/b$  and  $c/d$  are equal!

## Problem Set 2.1, page 40

- The columns are  $\mathbf{i} = (1, 0, 0)$  and  $\mathbf{j} = (0, 1, 0)$  and  $\mathbf{k} = (0, 0, 1)$  and  $\mathbf{b} = (2, 3, 4) = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ .
- The planes are the same:  $2x = 4$  is  $x = 2$ ,  $3y = 9$  is  $y = 3$ , and  $4z = 16$  is  $z = 4$ . The solution is the same point  $\mathbf{X} = \mathbf{x}$ . The columns are changed; but same combination.
- The solution is not changed! The second plane and row 2 of the matrix and all columns of the matrix (vectors in the column picture) are changed.
- If  $z = 2$  then  $x + y = 0$  and  $x - y = z$  give the point  $(1, -1, 2)$ . If  $z = 0$  then  $x + y = 6$  and  $x - y = 4$  produce  $(5, 1, 0)$ . Halfway between those is  $(3, 0, 1)$ .
- If  $x, y, z$  satisfy the first two equations they also satisfy the third equation. The line  $\mathbf{L}$  of solutions contains  $\mathbf{v} = (1, 1, 0)$  and  $\mathbf{w} = (\frac{1}{2}, 1, \frac{1}{2})$  and  $\mathbf{u} = \frac{1}{2}\mathbf{v} + \frac{1}{2}\mathbf{w}$  and all combinations  $c\mathbf{v} + d\mathbf{w}$  with  $c + d = 1$ .
- Equation 1 + equation 2 - equation 3 is now  $0 = -4$ . Line misses plane; *no solution*.
- Column 3 = Column 1 makes the matrix singular. Solutions  $(x, y, z) = (1, 1, 0)$  or  $(0, 1, 1)$  and you can add any multiple of  $(-1, 0, 1)$ ;  $\mathbf{b} = (4, 6, c)$  needs  $c = 10$  for solvability (then  $\mathbf{b}$  lies in the plane of the columns).
- Four planes in 4-dimensional space normally meet at a *point*. The solution to  $A\mathbf{x} = (3, 3, 3, 2)$  is  $\mathbf{x} = (0, 0, 1, 2)$  if  $A$  has columns  $(1, 0, 0, 0)$ ,  $(1, 1, 0, 0)$ ,  $(1, 1, 1, 0)$ ,  $(1, 1, 1, 1)$ . The equations are  $x + y + z + t = 3$ ,  $y + z + t = 3$ ,  $z + t = 3$ ,  $t = 2$ .
- (a)  $A\mathbf{x} = (18, 5, 0)$  and (b)  $A\mathbf{x} = (3, 4, 5, 5)$ .

- 10** Multiplying as linear combinations of the columns gives the same  $A\mathbf{x}$ . By rows or by columns: **9** separate multiplications for 3 by 3.
- 11**  $A\mathbf{x}$  equals (14, 22) and (0, 0) and (9, 7).
- 12**  $A\mathbf{x}$  equals (z, y, x) and (0, 0, 0) and (3, 3, 6).
- 13** (a)  $\mathbf{x}$  has  $n$  components and  $A\mathbf{x}$  has  $m$  components (b) Planes from each equation in  $A\mathbf{x} = \mathbf{b}$  are in  $n$ -dimensional space, but the columns are in  $m$ -dimensional space.
- 14**  $2x + 3y + z + 5t = 8$  is  $A\mathbf{x} = \mathbf{b}$  with the 1 by 4 matrix  $A = [2 \ 3 \ 1 \ 5]$ . The solutions  $\mathbf{x}$  fill a 3D “plane” in 4 dimensions. It could be called a *hyperplane*.
- 15** (a)  $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  (b)  $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
- 16**  $90^\circ$  rotation from  $R = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ ,  $180^\circ$  rotation from  $R^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$ .
- 17**  $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$  produces (y, z, x) and  $Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$  recovers (x, y, z).  $Q$  is the inverse of  $P$ .
- 18**  $E = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$  and  $E = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$  subtract the first component from the second.
- 19**  $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$  and  $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ ,  $E\mathbf{v} = (3, 4, 8)$  and  $E^{-1}E\mathbf{v}$  recovers (3, 4, 5).
- 20**  $P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$  projects onto the  $x$ -axis and  $P_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$  projects onto the  $y$ -axis.  $\mathbf{v} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$  has  $P_1\mathbf{v} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$  and  $P_2P_1\mathbf{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ .
- 21**  $R = \frac{1}{2} \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix}$  rotates all vectors by  $45^\circ$ . The columns of  $R$  are the results from rotating (1, 0) and (0, 1)!
- 22** The dot product  $A\mathbf{x} = [1 \ 4 \ 5] \begin{bmatrix} x \\ y \\ z \end{bmatrix} = (1 \text{ by } 3)(3 \text{ by } 1)$  is zero for points (x, y, z) on a plane in three dimensions. The columns of  $A$  are one-dimensional vectors.
- 23**  $A = [1 \ 2 \ ; \ 3 \ 4]$  and  $\mathbf{x} = [5 \ -2]'$  and  $\mathbf{b} = [1 \ 7]'$ .  $\mathbf{r} = \mathbf{b} - A*\mathbf{x}$  prints as zero.
- 24**  $A*\mathbf{v} = [3 \ 4 \ 5]'$  and  $\mathbf{v}'*\mathbf{v} = 50$ . But  $\mathbf{v}*A$  gives an error message from 3 by 1 times 3 by 3.
- 25**  $\mathbf{ones}(4, 4) * \mathbf{ones}(4, 1) = [4 \ 4 \ 4 \ 4]'$ ;  $B*\mathbf{w} = [10 \ 10 \ 10 \ 10]'$ .
- 26** The row picture has two lines meeting at the solution (4, 2). The column picture will have  $4(1, 1) + 2(-2, 1) = 4(\text{column } 1) + 2(\text{column } 2) = \text{right side } (0, 6)$ .
- 27** The row picture shows **2 planes in 3-dimensional space**. The column picture is in **2-dimensional space**. The solutions normally lie on a *line*.



- 28** The row picture shows four *lines* in the 2D plane. The column picture is in *four*-dimensional space. No solution unless the right side is a combination of *the two columns*.
- 29**  $u_2 = \begin{bmatrix} .7 \\ .3 \end{bmatrix}$  and  $u_3 = \begin{bmatrix} .65 \\ .35 \end{bmatrix}$ . The components add to 1. They are always positive.  $u_7, v_7, w_7$  are all close to  $(.6, .4)$ . Their components still add to 1.
- 30**  $\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} .6 \\ .4 \end{bmatrix} = \begin{bmatrix} .6 \\ .4 \end{bmatrix} = \text{steady state } s$ . No change when multiplied by  $\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}$ .
- 31**  $M = \begin{bmatrix} 8 & 3 & 4 \\ 1 & 5 & 9 \\ 6 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 5+u & 5-u+v & 5-v \\ 5-u-v & 5 & 5+u+v \\ 5+v & 5+u-v & 5-u \end{bmatrix}$ ;  $M_3(1, 1, 1) = (15, 15, 15)$ ;  $M_4(1, 1, 1, 1) = (34, 34, 34, 34)$  because  $1 + 2 + \cdots + 16 = 136$  which is  $4(34)$ .
- 32**  $A$  is singular when its third column  $w$  is a combination  $cu + dv$  of the first columns. A typical column picture has  $b$  outside the plane of  $u, v, w$ . A typical row picture has the intersection line of two planes parallel to the third plane. *Then no solution.*
- 33**  $w = (5, 7)$  is  $5u + 7v$ . Then  $Aw$  equals 5 times  $Au$  plus 7 times  $Av$ .
- 34**  $\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$  has the solution  $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 8 \\ 6 \end{bmatrix}$ .
- 35**  $x = (1, \dots, 1)$  gives  $Sx = \text{sum of each row} = 1 + \cdots + 9 = 45$  for Sudoku matrices. 6 row orders  $(1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1)$  are in Section 2.7. The same 6 permutations of *blocks* of rows produce Sudoku matrices, so  $6^4 = 1296$  orders of the 9 rows all stay Sudoku. (And also 1296 permutations of the 9 columns.)

## Problem Set 2.2, page 51

- Multiply by  $\ell_{21} = \frac{10}{2} = 5$  and subtract to find  $2x + 3y = 14$  and  $-6y = 6$ . The pivots to circle are 2 and  $-6$ .
- $-6y = 6$  gives  $y = -1$ . Then  $2x + 3y = 1$  gives  $x = 2$ . Multiplying the right side  $(1, 11)$  by 4 will multiply the solution by 4 to give the new solution  $(x, y) = (8, -4)$ .
- Subtract  $-\frac{1}{2}$  (or add  $\frac{1}{2}$ ) times equation 1. The new second equation is  $3y = 3$ . Then  $y = 1$  and  $x = 5$ . If the right side changes sign, so does the solution:  $(x, y) = (-5, -1)$ .
- Subtract  $\ell = \frac{c}{a}$  times equation 1. The new second pivot multiplying  $y$  is  $d - (cb/a)$  or  $(ad - bc)/a$ . Then  $y = (ag - cf)/(ad - bc)$ .
- $6x + 4y$  is 2 times  $3x + 2y$ . There is no solution unless the right side is  $2 \cdot 10 = 20$ . Then all the points on the line  $3x + 2y = 10$  are solutions, including  $(0, 5)$  and  $(4, -1)$ . (The two lines in the row picture are the same line, containing all solutions).
- Singular system if  $b = 4$ , because  $4x + 8y$  is 2 times  $2x + 4y$ . Then  $g = 32$  makes the lines become the *same*: infinitely many solutions like  $(8, 0)$  and  $(0, 4)$ .
- If  $a = 2$  elimination must fail (two parallel lines in the row picture). The equations have no solution. With  $a = 0$ , elimination will stop for a row exchange. Then  $3y = -3$  gives  $y = -1$  and  $4x + 6y = 6$  gives  $x = 3$ .

- 8 If  $k = 3$  elimination must fail: no solution. If  $k = -3$ , elimination gives  $0 = 0$  in equation 2: infinitely many solutions. If  $k = 0$  a row exchange is needed: one solution.
- 9 On the left side,  $6x - 4y$  is 2 times  $(3x - 2y)$ . Therefore we need  $b_2 = 2b_1$  on the right side. Then there will be infinitely many solutions (two parallel lines become one single line).
- 10 The equation  $y = 1$  comes from elimination (subtract  $x + y = 5$  from  $x + 2y = 6$ ). Then  $x = 4$  and  $5x - 4y = c = 16$ .
- 11 (a) Another solution is  $\frac{1}{2}(x + X, y + Y, z + Z)$ . (b) If 25 planes meet at two points, they meet along the whole line through those two points.
- 12 Elimination leads to an upper triangular system; then comes back substitution.
- $$\begin{array}{rcl} 2x + 3y + z = 8 & & x = 2 \\ y + 3z = 4 & \text{gives} & y = 1 \\ 8z = 8 & & z = 1 \end{array}$$
- If a zero is at the start of row 2 or 3, that avoids a row operation.
- 13 
$$\begin{array}{rcl} 2x - 3y & = & 3 \\ 4x - 5y + z = 7 & \text{gives} & y + z = 1 \\ 2x - y - 3z = 5 & & 2y + 3z = 2 \end{array}$$

$$\begin{array}{rcl} 2x - 3y & = & 3 \\ y + z & = & 1 \\ -5z & = & 0 \end{array}$$

Subtract  $2 \times$  row 1 from row 2, subtract  $1 \times$  row 1 from row 3, subtract  $2 \times$  row 2 from row 3

14 Subtract 2 times row 1 from row 2 to reach  $(d - 10)y - z = 2$ . Equation (3) is  $y - z = 3$ . If  $d = 10$  exchange rows 2 and 3. If  $d = 11$  the system becomes singular.

15 The second pivot position will contain  $-2 - b$ . If  $b = -2$  we exchange with row 3. If  $b = -1$  (singular case) the second equation is  $-y - z = 0$ . A solution is  $(1, 1, -1)$ .

**Example of**  $0x + 0y + 2z = 4$   
 $x + 2y + 2z = 5$   
 $0x + 3y + 4z = 6$   
 (exchange 1 and 2, then 2 and 3)

**Exchange**  $0x + 3y + 4z = 4$   
**but then**  $x + 2y + 2z = 5$   
**break down**  $0x + 3y + 4z = 6$   
 (rows 1 and 3 are not consistent)

16 (a) **2 exchanges** (b) **break down**

17 If row 1 = row 2, then row 2 is zero after the first step; exchange the zero row with row 3 and there is no *third* pivot. If column 2 = column 1, then column 2 has no pivot.

18 *Example*  $x + 2y + 3z = 0, 4x + 8y + 12z = 0, 5x + 10y + 15z = 0$  has 9 different coefficients but rows 2 and 3 become  $0 = 0$ : infinitely many solutions.

19 Row 2 becomes  $3y - 4z = 5$ , then row 3 becomes  $(q + 4)z = t - 5$ . If  $q = -4$  the system is singular—no third pivot. Then if  $t = 5$  the third equation is  $0 = 0$ . Choosing  $z = 1$  the equation  $3y - 4z = 5$  gives  $y = 3$  and equation 1 gives  $x = -9$ .

20 Singular if row 3 is a combination of rows 1 and 2. From the end view, the three planes form a triangle. This happens if rows  $1 + 2 =$  row 3 on the left side but not the right side:  $x + y + z = 0, x - 2y - z = 1, 2x - y = 4$ . No parallel planes but still no solution.

21 (a) Pivots  $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}$  in the equations  $2x + y = 0, \frac{3}{2}y + z = 0, \frac{4}{3}z + t = 0, \frac{5}{4}t = 5$  after elimination. Back substitution gives  $t = 4, z = -3, y = 2, x = -1$ . (b) If the off-diagonal entries change from  $+1$  to  $-1$ , the pivots are the same. The solution is  $(1, 2, 3, 4)$  instead of  $(-1, 2, -3, 4)$ .

22 The fifth pivot is  $\frac{6}{5}$  for both matrices ( $1$ 's or  $-1$ 's off the diagonal). The  $n$ th pivot is  $\frac{n+1}{n}$ .

- 23** If ordinary elimination leads to  $x + y = 1$  and  $2y = 3$ , the original second equation could be  $2y + \ell(x + y) = 3 + \ell$  for any  $\ell$ . Then  $\ell$  will be the multiplier to reach  $2y = 3$ .
- 24** Elimination fails on  $\begin{bmatrix} a & 2 \\ a & a \end{bmatrix}$  if  $a = 2$  or  $a = 0$ .
- 25**  $a = 2$  (equal columns),  $a = 4$  (equal rows),  $a = 0$  (zero column).
- 26** Solvable for  $s = 10$  (add the two pairs of equations to get  $a + b + c + d$  on the left sides, 12 and  $2 + s$  on the right sides). The four equations for  $a, b, c, d$  are **singular**! Two solutions are  $\begin{bmatrix} 1 & 3 \\ 1 & 7 \end{bmatrix}$  and  $\begin{bmatrix} 0 & 4 \\ 2 & 6 \end{bmatrix}$ ,  $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$  and  $U = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ .
- 27** Elimination leaves the diagonal matrix  $\text{diag}(3, 2, 1)$  in  $3x = 3, 2y = 2, z = 4$ . Then  $x = 1, y = 1, z = 4$ .
- 28**  $A(2, :) = A(2, :) - 3 * A(1, :)$  subtracts 3 times row 1 from row 2.
- 29** The average pivots for  $\text{rand}(3)$  *without* row exchanges were  $\frac{1}{2}, 5, 10$  in one experiment—but pivots 2 and 3 can be arbitrarily large. Their averages are actually infinite! *With row exchanges* in MATLAB's  $\text{lu}$  code, the averages .75 and .50 and .365 are much more stable (and should be predictable, also for  $\text{randn}$  with normal instead of uniform probability distribution).
- 30** If  $A(5, 5)$  is 7 not 11, then the last pivot will be 0 not 4.
- 31** Row  $j$  of  $U$  is a combination of rows  $1, \dots, j$  of  $A$ . If  $A\mathbf{x} = \mathbf{0}$  then  $U\mathbf{x} = \mathbf{0}$  (not true if  $\mathbf{b}$  replaces  $\mathbf{0}$ ).  $U$  is the diagonal of  $A$  when  $A$  is *lower triangular*.
- 32** The question deals with 100 equations  $A\mathbf{x} = \mathbf{0}$  when  $A$  is singular.
- (a) Some linear combination of the 100 rows is **the row of 100 zeros**.
  - (b) Some linear combination of the 100 **columns** is **the column of zeros**.
  - (c) A very singular matrix has all ones:  $A = \text{eye}(100)$ . A better example has 99 random rows (or the numbers  $1^i, \dots, 100^i$  in those rows). The 100th row could be the sum of the first 99 rows (or any other combination of those rows with no zeros).
  - (d) The row picture has 100 planes **meeting along a common line through 0**. The column picture has 100 vectors all in the same 99-dimensional hyperplane.

### Problem Set 2.3, page 63

- 1**  $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ,  $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 7 & 1 \end{bmatrix}$ ,  $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ .
- 2**  $E_{32}E_{21}\mathbf{b} = (1, -5, -35)$  but  $E_{21}E_{32}\mathbf{b} = (1, -5, 0)$ . When  $E_{32}$  comes first, row 3 feels no effect from row 1.
- 3**  $\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$   $M = E_{32}E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 10 & -2 & 1 \end{bmatrix}$ .

4 Elimination on column 4:  $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{E_{21}} \begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix} \xrightarrow{E_{31}} \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix} \xrightarrow{E_{32}} \begin{bmatrix} 1 \\ -4 \\ 10 \end{bmatrix}$ . The original  $A\mathbf{x} = \mathbf{b}$  has become  $U\mathbf{x} = \mathbf{c} = (1, -4, 10)$ . Then back substitution gives  $z = -5, y = \frac{1}{2}, x = \frac{1}{2}$ . This solves  $A\mathbf{x} = (1, 0, 0)$ .

5 Changing  $a_{33}$  from 7 to 11 will change the third pivot from 5 to 9. Changing  $a_{33}$  from 7 to 2 will change the pivot from 5 to *no pivot*.

6 Example:  $\begin{bmatrix} 2 & 3 & 7 \\ 2 & 3 & 7 \\ 2 & 3 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$ . If all columns are multiples of column 1, there is no second pivot.

7 To reverse  $E_{31}$ , **add** 7 times row 1 to row 3. The inverse of the elimination matrix

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -7 & 0 & 1 \end{bmatrix} \text{ is } E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 7 & 0 & 1 \end{bmatrix}.$$

8  $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  and  $M^* = \begin{bmatrix} a & b \\ c - \ell a & d - \ell b \end{bmatrix}$ .  $\det M^* = a(d - \ell b) - b(c - \ell a)$  reduces to  $ad - bc$ !

9  $M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$ . After the exchange, we need  $E_{31}$  (not  $E_{21}$ ) to act on the new row 3.

10  $E_{13} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}; E_{31}E_{13} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ . Test on the identity matrix!

11 An example with two negative pivots is  $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$ . The diagonal entries can change sign during elimination.

12 The first product is  $\begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$  rows and also columns reversed. The second product is  $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 2 & -3 \end{bmatrix}$ .

13 (a)  $E$  times the third column of  $B$  is the third column of  $EB$ . A column that starts at zero will stay at zero. (b)  $E$  could add row 2 to row 3 to change a zero row to a nonzero row.

14  $E_{21}$  has  $-\ell_{21} = \frac{1}{2}$ ,  $E_{32}$  has  $-\ell_{32} = \frac{2}{3}$ ,  $E_{43}$  has  $-\ell_{43} = \frac{3}{4}$ . Otherwise the  $E$ 's match  $I$ .

15  $a_{ij} = 2i - 3j$ :  $A = \begin{bmatrix} -1 & -4 & -7 \\ 1 & -2 & -5 \\ 3 & 0 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & -4 & -7 \\ 0 & -6 & -12 \\ 0 & -12 & -24 \end{bmatrix}$ . The zero became  $-12$ ,

an example of *fill-in*. To remove that  $-12$ , choose  $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ .

- 16** (a) The ages of  $X$  and  $Y$  are  $x$  and  $y$ :  $x - 2y = 0$  and  $x + y = 33$ ;  $x = 22$  and  $y = 11$  (b) The line  $y = mx + c$  contains  $x = 2, y = 5$  and  $x = 3, y = 7$  when  $2m + c = 5$  and  $3m + c = 7$ . Then  $m = 2$  is the slope.

- 17** The parabola  $y = a + bx + cx^2$  goes through the 3 given points when 
$$\begin{aligned} a + b + c &= 4 \\ a + 2b + 4c &= 8 \\ a + 3b + 9c &= 14 \end{aligned}$$
 Then  $a = 2, b = 1$ , and  $c = 1$ . This matrix with columns  $(1, 1, 1), (1, 2, 3), (1, 4, 9)$  is a "Vandermonde matrix."

**18**  $EF = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}, FE = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b+ac & c & 1 \end{bmatrix}, E^2 = \begin{bmatrix} 1 & 0 & 0 \\ 2a & 1 & 0 \\ 2b & 0 & 1 \end{bmatrix}, F^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3c & 1 \end{bmatrix}.$

**19**  $PQ = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ . In the opposite order, two row exchanges give  $QP = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$

If  $M$  exchanges rows 2 and 3 then  $M^2 = I$  (also  $(-M)^2 = I$ ). There are many square roots of  $I$ : Any matrix  $M = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$  has  $M^2 = I$  if  $a^2 + bc = 1$ .

- 20** (a) Each column of  $EB$  is  $E$  times a column of  $B$  (b)  $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix}$ . All rows of  $EB$  are multiples of  $[1 \ 2 \ 4]$ .

**21** No.  $E = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$  and  $F = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$  give  $EF = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$  but  $FE = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ .

**22** (a)  $\sum a_{3j}x_j$  (b)  $a_{21} - a_{11}$  (c)  $a_{21} - 2a_{11}$  (d)  $(EA\mathbf{x})_1 = (A\mathbf{x})_1 = \sum a_{1j}x_j$ .

- 23**  $E(EA)$  subtracts 4 times row 1 from row 2 ( $EEA$  does the row operation twice).  $AE$  subtracts 2 times column 2 of  $A$  from column 1 (multiplication by  $E$  on the right side acts on **columns** instead of rows).

**24**  $\begin{bmatrix} A & b \end{bmatrix} = \begin{bmatrix} 2 & 3 & \mathbf{1} \\ 4 & 1 & \mathbf{17} \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 3 & \mathbf{1} \\ 0 & -5 & \mathbf{15} \end{bmatrix}$ . The triangular system is  $\begin{aligned} 2x_1 + 3x_2 &= 1 \\ -5x_2 &= 15 \end{aligned}$   
Back substitution gives  $x_1 = 5$  and  $x_2 = -3$ .

- 25** The last equation becomes  $0 = 3$ . If the original 6 is 3, then row 1 + row 2 = row 3.

**26** (a) Add two columns  $b$  and  $b^*$   $\begin{bmatrix} 1 & 4 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{bmatrix} \rightarrow \mathbf{x} = \begin{bmatrix} -7 \\ 2 \end{bmatrix}$   
and  $\mathbf{x}^* = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ .

- 27** (a) No solution if  $d = 0$  and  $c \neq 0$  (b) Many solutions if  $d = 0 = c$ . No effect from  $a, b$ .

**28**  $A = AI = A(BC) = (AB)C = IC = C$ . That middle equation is crucial.

**29**  $E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$  subtracts each row from the next row. The result  $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix}$  still has multipliers = 1 in a 3 by 3 Pascal matrix. The product  $M$  of all elimination matrices is  $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$ . This “alternating sign Pascal matrix” is on page 88.

**30** Given positive integers with  $ad - bc = 1$ . Certainly  $c < a$  and  $b < d$  would be impossible. Also  $c > a$  and  $b > d$  would be impossible with integers. This leaves row 1 < row 2 OR row 2 < row 1. An example is  $M = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$ . Multiply by  $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$  to get  $\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ , then multiply twice by  $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$  to get  $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ . This shows that  $M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ .

$$\mathbf{31} \quad E_{21} = \begin{bmatrix} 1 & & & \\ 1/2 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E_{32} = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 2/3 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E_{43} = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 3/4 & 1 \end{bmatrix},$$

$$E_{43} E_{32} E_{21} = \begin{bmatrix} 1 & & & \\ 1/2 & 1 & & \\ 1/3 & 2/3 & 1 & \\ 1/4 & 2/4 & 3/4 & 1 \end{bmatrix}$$

### Problem Set 2.4, page 75

- 1** If all entries of  $A, B, C, D$  are 1, then  $BA = 3 \text{ ones}(5)$  is 5 by 5;  $AB = 5 \text{ ones}(3)$  is 3 by 3;  $ABD = 15 \text{ ones}(3, 1)$  is 3 by 1.  $DBA$  and  $A(B + C)$  are not defined.
- 2** (a)  $A$  (column 3 of  $B$ )      (b) (Row 1 of  $A$ )  $B$       (c) (Row 3 of  $A$ )(column 4 of  $B$ )  
(d) (Row 1 of  $C$ ) $D$ (column 1 of  $E$ ).
- 3**  $AB + AC$  is the same as  $A(B + C) = \begin{bmatrix} 3 & 8 \\ 6 & 9 \end{bmatrix}$ . (*Distributive law*).
- 4**  $A(BC) = (AB)C$  by the *associative law*. In this example both answers are  $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$  from column 1 of  $AB$  and row 2 of  $C$  (multiply columns times rows).
- 5** (a)  $A^2 = \begin{bmatrix} 1 & 2b \\ 0 & 1 \end{bmatrix}$  and  $A^n = \begin{bmatrix} 1 & nb \\ 0 & 1 \end{bmatrix}$ .      (b)  $A^2 = \begin{bmatrix} 4 & 4 \\ 0 & 0 \end{bmatrix}$  and  $A^n = \begin{bmatrix} 2^n & 2^n \\ 0 & 0 \end{bmatrix}$ .
- 6**  $(A + B)^2 = \begin{bmatrix} 10 & 4 \\ 6 & 6 \end{bmatrix} = A^2 + AB + BA + B^2$ . But  $A^2 + 2AB + B^2 = \begin{bmatrix} 16 & 2 \\ 3 & 0 \end{bmatrix}$ .
- 7** (a) True      (b) False      (c) True      (d) False: usually  $(AB)^2 \neq A^2B^2$ .

- 8** The rows of  $DA$  are 3 (row 1 of  $A$ ) and 5 (row 2 of  $A$ ). Both rows of  $EA$  are row 2 of  $A$ . The columns of  $AD$  are 3 (column 1 of  $A$ ) and 5 (column 2 of  $A$ ). The first column of  $AE$  is zero, the second is column 1 of  $A$  + column 2 of  $A$ .
- 9**  $AF = \begin{bmatrix} a & a+b \\ c & c+d \end{bmatrix}$  and  $E(AF)$  equals  $(EA)F$  because matrix multiplication is associative.
- 10**  $FA = \begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix}$  and then  $E(FA) = \begin{bmatrix} a+c & b+d \\ a+2c & b+2d \end{bmatrix}$ .  $E(FA)$  is not the same as  $F(EA)$  because multiplication is not commutative.
- 11** (a)  $B = 4I$  (b)  $B = 0$  (c)  $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$  (d) Every row of  $B$  is 1, 0, 0.
- 12**  $AB = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} = BA = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$  gives  $b = c = 0$ . Then  $AC = CA$  gives  $a = d$ . The only matrices that commute with  $B$  and  $C$  (and all other matrices) are multiples of  $I$ :  $A = aI$ .
- 13**  $(A - B)^2 = (B - A)^2 = A(A - B) - B(A - B) = A^2 - AB - BA + B^2$ . In a typical case (when  $AB \neq BA$ ) the matrix  $A^2 - 2AB + B^2$  is different from  $(A - B)^2$ .
- 14** (a) True ( $A^2$  is only defined when  $A$  is square) (b) False (if  $A$  is  $m$  by  $n$  and  $B$  is  $n$  by  $m$ , then  $AB$  is  $m$  by  $m$  and  $BA$  is  $n$  by  $n$ ). (c) True (d) False (take  $B = 0$ ).
- 15** (a)  $mn$  (use every entry of  $A$ ) (b)  $mnp = p \times \text{part (a)}$  (c)  $n^3$  ( $n^2$  dot products).
- 16** (a) Use only column 2 of  $B$  (b) Use only row 2 of  $A$  (c)–(d) Use row 2 of first  $A$ .
- 17**  $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$  has  $a_{ij} = \min(i, j)$ .  $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$  has  $a_{ij} = (-1)^{i+j} =$   
 “alternating sign matrix”.  $A = \begin{bmatrix} 1/1 & 1/2 & 1/3 \\ 2/1 & 2/2 & 2/3 \\ 3/1 & 3/2 & 3/3 \end{bmatrix}$  has  $a_{ij} = i/j$  (this will be an example of a *rank one matrix*).
- 18** Diagonal matrix, lower triangular, symmetric, all rows equal. Zero matrix fits all four.
- 19** (a)  $a_{11}$  (b)  $\ell_{31} = a_{31}/a_{11}$  (c)  $a_{32} - (\frac{a_{31}}{a_{11}})a_{12}$  (d)  $a_{22} - (\frac{a_{21}}{a_{11}})a_{12}$ .
- 20**  $A^2 = \begin{bmatrix} 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ ,  $A^3 = \begin{bmatrix} 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ ,  $A^4 =$  zero matrix for *strictly triangular*  $A$ .  
 Then  $A\mathbf{v} = A \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 2y \\ 2z \\ 2t \\ 0 \end{bmatrix}$ ,  $A^2\mathbf{v} = \begin{bmatrix} 4z \\ 4t \\ 0 \\ 0 \end{bmatrix}$ ,  $A^3\mathbf{v} = \begin{bmatrix} 8t \\ 0 \\ 0 \\ 0 \end{bmatrix}$ ,  $A^4\mathbf{v} = 0$ .

**21**  $A = A^2 = A^3 = \dots = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}$  but  $AB = \begin{bmatrix} .5 & -.5 \\ .5 & -.5 \end{bmatrix}$  and  $(AB)^2 = \text{zero matrix!}$

**22**  $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$  has  $A^2 = -I$ ;  $BC = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ ;  
 $DE = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = -ED$ . You can find more examples.

**23**  $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$  has  $A^2 = 0$ . Note: Any matrix  $A = \text{column times row} = \mathbf{uv}^T$  will

have  $A^2 = \mathbf{uv}^T \mathbf{uv}^T = 0$  if  $\mathbf{v}^T \mathbf{u} = 0$ .  $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$  has  $A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

but  $A^3 = 0$ ; strictly triangular as in Problem 20.

**24**  $(A_1)^n = \begin{bmatrix} 2^n & 2^n - 1 \\ 0 & 1 \end{bmatrix}$ ,  $(A_2)^n = 2^{n-1} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ ,  $(A_3)^n = \begin{bmatrix} a^n & a^{n-1}b \\ 0 & 0 \end{bmatrix}$ .

**25**  $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a \\ d \\ g \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} b \\ e \\ h \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} c \\ f \\ i \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$ .

**26** Columns of  $A$  times rows of  $B$   $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 0 \\ 6 & 6 & 0 \\ 6 & 6 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 4 & 8 & 4 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 0 \\ 10 & 14 & 4 \\ 7 & 8 & 1 \end{bmatrix} = AB$ .

**27** (a) (row 3 of  $A$ )  $\cdot$  (column 1 of  $B$ ) and (row 3 of  $A$ )  $\cdot$  (column 2 of  $B$ ) are both zero.

(b)  $\begin{bmatrix} x \\ x \\ 0 \end{bmatrix} \begin{bmatrix} 0 & x & x \end{bmatrix} = \begin{bmatrix} 0 & x & x \\ 0 & x & x \\ 0 & 0 & 0 \end{bmatrix}$  and  $\begin{bmatrix} x \\ x \\ x \end{bmatrix} \begin{bmatrix} 0 & 0 & x \end{bmatrix} = \begin{bmatrix} 0 & 0 & x \\ 0 & 0 & x \\ 0 & 0 & x \end{bmatrix}$ : **both upper**.

**28**  $A$  times  $B$  with cuts  $A \begin{bmatrix} | & | & | & | \end{bmatrix}, \begin{bmatrix} \text{---} \end{bmatrix} B, \begin{bmatrix} \text{---} \end{bmatrix} \begin{bmatrix} | & | & | & | \end{bmatrix}, \begin{bmatrix} | & | & | \end{bmatrix} \begin{bmatrix} \text{=} \end{bmatrix}$

**29**  $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$  and  $E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$  produce zeros in the 2, 1 and 3, 1 entries.

Multiply  $E$ 's to get  $E = E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$ . Then  $EA = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix}$  is the result of both  $E$ 's since  $(E_{31}E_{21})A = E_{31}(E_{21}A)$ .

**30** In **29**,  $\mathbf{c} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$ ,  $D = \begin{bmatrix} 0 & 1 \\ 5 & 3 \end{bmatrix}$ ,  $D - \mathbf{c}\mathbf{b}/a = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$  in the lower corner of  $EA$ .

**31**  $\begin{bmatrix} A & -B \\ B & A \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} A\mathbf{x} - B\mathbf{y} \\ B\mathbf{x} + A\mathbf{y} \end{bmatrix}$  real part imaginary part. Complex matrix times complex vector needs **4** real times real multiplications.



**32**  $A$  times  $X = [\mathbf{x}_1 \ \mathbf{x}_2 \ \mathbf{x}_3]$  will be the identity matrix  $I = [A\mathbf{x}_1 \ A\mathbf{x}_2 \ A\mathbf{x}_3]$ .

**33**  $\mathbf{b} = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix}$  gives  $\mathbf{x} = 3\mathbf{x}_1 + 5\mathbf{x}_2 + 8\mathbf{x}_3 = \begin{bmatrix} 3 \\ 8 \\ 16 \end{bmatrix}$ ;  $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$  will have those  $\mathbf{x}_1 = (1, 1, 1)$ ,  $\mathbf{x}_2 = (0, 1, 1)$ ,  $\mathbf{x}_3 = (0, 0, 1)$  as columns of its “inverse”  $A^{-1}$ .

**34**  $A * \mathbf{ones} = \begin{bmatrix} a+b & a+b \\ c+d & c+d \end{bmatrix}$  agrees with  $\mathbf{ones} * A = \begin{bmatrix} a+c & b+b \\ a+c & b+d \end{bmatrix}$  when  $b = c$  and  $a = d$ .  
Then  $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ .

**35**  $A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ ,  $A^2 = \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix}$ , **aba, ada cba, cda** These show  
**bab, bcb dab, dcb** 16 2-step  
**abc, adc cbc, cdc** paths in  
**bad, bcd dad, dcd** the graph

**36** Multiplying  $AB = (m \text{ by } n)(n \text{ by } p)$  needs  $mnp$  multiplications. Then  $(AB)C$  needs  $mpq$  more. Multiply  $BC = (n \text{ by } p)(p \text{ by } q)$  needs  $npq$  and then  $A(BC)$  needs  $mnq$ .

(a) If  $m, n, p, q$  are 2, 4, 7, 10 we compare  $(2)(4)(7) + (2)(7)(10) = \mathbf{196}$  with the larger number  $(2)(4)(10) + (4)(7)(10) = \mathbf{360}$ . So  $AB$  first is better, so that we multiply that 7 by 10 matrix by as few rows as possible.

(b) If  $\mathbf{u}, \mathbf{v}, \mathbf{w}$  are  $N$  by 1, then  $(\mathbf{u}^T \mathbf{v})\mathbf{w}^T$  needs  $2N$  multiplications but  $\mathbf{u}^T(\mathbf{v}\mathbf{w}^T)$  needs  $N^2$  to find  $\mathbf{v}\mathbf{w}^T$  and  $N^2$  more to multiply by the row vector  $\mathbf{u}^T$ . Apologies to use the transpose symbol so early.

(c) We are comparing  $mnp + mpq$  with  $mnq + npq$ . Divide all terms by  $mnpq$ : Now we are comparing  $q^{-1} + n^{-1}$  with  $p^{-1} + m^{-1}$ . This yields a simple important rule. If matrices  $A$  and  $B$  are multiplying  $\mathbf{v}$  for  $AB\mathbf{v}$ , **don't multiply the matrices first**.

**37** The proof of  $(AB)\mathbf{c} = A(B\mathbf{c})$  used the column rule for matrix multiplication—this rule is clearly linear, column by column.

Even for nonlinear transformations,  $A(B(\mathbf{c}))$  would be the “composition” of  $A$  with  $B$  (applying  $B$  then  $A$ ). This composition  $A \circ B$  is just  $AB$  for matrices.

One of many uses for the associative law: The left-inverse  $B = \text{right-inverse } C$  from  $B = B(AC) = (BA)C = C$ .

## Problem Set 2.5, page 89

**1**  $A^{-1} = \begin{bmatrix} 0 & \frac{1}{4} \\ \frac{1}{3} & 0 \end{bmatrix}$  and  $B^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ -1 & \frac{1}{2} \end{bmatrix}$  and  $C^{-1} = \begin{bmatrix} 7 & -4 \\ -5 & 3 \end{bmatrix}$ .

**2** A simple row exchange has  $P^2 = I$  so  $P^{-1} = P$ . Here  $P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ . Always  $P^{-1} = \text{“transpose” of } P$ , coming in Section 2.7.

- 3  $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} .5 \\ -.2 \end{bmatrix}$  and  $\begin{bmatrix} t \\ z \end{bmatrix} = \begin{bmatrix} -.2 \\ .1 \end{bmatrix}$  so  $A^{-1} = \frac{1}{10} \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$ . This question solved  $AA^{-1} = I$  column by column, the main idea of Gauss-Jordan elimination.
- 4 The equations are  $x + 2y = 1$  and  $3x + 6y = 0$ . No solution because 3 times equation 1 gives  $3x + 6y = 3$ .
- 5 An upper triangular  $U$  with  $U^2 = I$  is  $U = \begin{bmatrix} 1 & a \\ 0 & -1 \end{bmatrix}$  for any  $a$ . And also  $-U$ .
- 6 (a) Multiply  $AB = AC$  by  $A^{-1}$  to find  $B = C$  (since  $A$  is invertible) (b) As long as  $B - C$  has the form  $\begin{bmatrix} x & y \\ -x & -y \end{bmatrix}$ , we have  $AB = AC$  for  $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ .
- 7 (a) In  $A\mathbf{x} = (1, 0, 0)$ , equation 1 + equation 2 - equation 3 is  $0 = 1$  (b) Right sides must satisfy  $b_1 + b_2 = b_3$  (c) Row 3 becomes a row of zeros—no third pivot.
- 8 (a) The vector  $\mathbf{x} = (1, 1, -1)$  solves  $A\mathbf{x} = \mathbf{0}$  (b) After elimination, columns 1 and 2 end in zeros. Then so does column 3 = column 1 + 2: no third pivot.
- 9 If you exchange rows 1 and 2 of  $A$  to reach  $B$ , you exchange **columns** 1 and 2 of  $A^{-1}$  to reach  $B^{-1}$ . In matrix notation,  $B = PA$  has  $B^{-1} = A^{-1}P^{-1} = A^{-1}P$  for this  $P$ .
- 10  $A^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1/5 \\ 0 & 0 & 1/4 & 0 \\ 0 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \end{bmatrix}$  and  $B^{-1} = \begin{bmatrix} 3 & -2 & 0 & 0 \\ -4 & 3 & 0 & 0 \\ 0 & 0 & 6 & -5 \\ 0 & 0 & -7 & 6 \end{bmatrix}$  (invert each block of  $B$ ).
- 11 (a) If  $B = -A$  then certainly  $A + B =$  zero matrix is not invertible. (b)  $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$  and  $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$  are both singular but  $A + B = I$  is invertible.
- 12 Multiply  $C = AB$  on the left by  $A^{-1}$  and on the right by  $C^{-1}$ . Then  $A^{-1} = BC^{-1}$ .
- 13  $M^{-1} = C^{-1}B^{-1}A^{-1}$  so multiply on the left by  $C$  and the right by  $A$  :  $B^{-1} = CM^{-1}A$ .
- 14  $B^{-1} = A^{-1} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} = A^{-1} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ : subtract column 2 of  $A^{-1}$  from column 1.
- 15 If  $A$  has a column of zeros, so does  $BA$ . Then  $BA = I$  is impossible. There is no  $A^{-1}$ .
- 16  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix}$ . The inverse of each matrix is the other divided by  $ad - bc$ .
- 17  $E_{32}E_{31}E_{21} = \begin{bmatrix} 1 & & \\ & 1 & \\ & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ 1 & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ -1 & 1 & \\ & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ -1 & 1 & \\ 0 & -1 & 1 \end{bmatrix} = E$ .  
Reverse the order and change  $-1$  to  $+1$  to get inverses  $E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} = \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 1 & 1 & 1 \end{bmatrix} = L = E^{-1}$ . Notice the 1's unchanged by multiplying in this order.
- 18  $A^2B = I$  can also be written as  $A(AB) = I$ . Therefore  $A^{-1}$  is  $AB$ .

**19** The  $(1, 1)$  entry requires  $4a - 3b = 1$ ; the  $(1, 2)$  entry requires  $2b - a = 0$ . Then  $b = \frac{1}{5}$  and  $a = \frac{2}{5}$ . For the 5 by 5 case  $5a - 4b = 1$  and  $2b = a$  give  $b = \frac{1}{6}$  and  $a = \frac{2}{6}$ .

**20**  $A * \text{ones}(4, 1)$  is the zero vector so  $A$  cannot be invertible.

**21** Six of the sixteen  $0 - 1$  matrices are invertible, including all four with three 1's.

$$\begin{aligned} \mathbf{22} \quad \begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{bmatrix} &\rightarrow \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{bmatrix} = [I \ A^{-1}]; \\ \begin{bmatrix} 1 & 4 & 1 & 0 \\ 3 & 9 & 0 & 1 \end{bmatrix} &\rightarrow \begin{bmatrix} 1 & 4 & 1 & 0 \\ 0 & -3 & -3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -3 & 4/3 \\ 0 & 1 & 1 & -1/3 \end{bmatrix} = [I \ A^{-1}]. \end{aligned}$$

$$\begin{aligned} \mathbf{23} \quad [A \ I] &= \left[ \begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array} \right] \rightarrow \left[ \begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & 1 & -1/2 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array} \right] \rightarrow \\ &\left[ \begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & 1 & -1/2 & 1 & 0 \\ 0 & 0 & 4/3 & 1/3 & -2/3 & 1 \end{array} \right] \rightarrow \left[ \begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & 0 & -3/4 & 3/2 & -3/4 \\ 0 & 0 & 4/3 & 1/3 & -2/3 & 1 \end{array} \right] \rightarrow \\ &\left[ \begin{array}{ccc|ccc} 2 & 0 & 0 & 3/2 & -1 & 1/2 \\ 0 & 3/2 & 0 & -3/4 & 3/2 & -3/4 \\ 0 & 0 & 4/3 & 1/3 & -2/3 & 1 \end{array} \right] \rightarrow \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 3/4 & -1/2 & 1/4 \\ 0 & 1 & 0 & -1/2 & 1 & -1/2 \\ 0 & 0 & 1 & 1/4 & -1/2 & 3/4 \end{array} \right] = \\ &[I \ A^{-1}]. \end{aligned}$$

$$\mathbf{24} \quad \begin{bmatrix} 1 & a & b & 1 & 0 & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & a & 0 & 1 & 0 & -b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & -a & ac - b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

$$\mathbf{25} \quad \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}^{-1} = \frac{1}{4} \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}; \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ so } B^{-1} \text{ does not exist.}$$

$$\begin{aligned} \mathbf{26} \quad E_{21}A &= \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}. E_{12}E_{21}A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}. \\ \text{Multiply by } D &= \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \text{ to reach } DE_{12}E_{21}A = I. \text{ Then } A^{-1} = DE_{12}E_{21} = \\ &\frac{1}{2} \begin{bmatrix} 6 & -2 \\ -2 & 1 \end{bmatrix}. \end{aligned}$$

$$\mathbf{27} \quad A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \text{ (notice the pattern); } A^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

$$\mathbf{28} \quad \begin{bmatrix} 0 & 2 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 2 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & -1 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1/2 & 1/2 \\ 0 & 1 & 1/2 & 0 \end{bmatrix}.$$

This is  $[I \ A^{-1}]$ : row exchanges are certainly allowed in Gauss-Jordan.

**29** (a) True (If  $A$  has a row of zeros, then every  $AB$  has too, and  $AB = I$  is impossible)  
 (b) False (the matrix of all ones is singular even with diagonal 1's: *ones* (3) has 3 equal rows)  
 (c) True (the inverse of  $A^{-1}$  is  $A$  and the inverse of  $A^2$  is  $(A^{-1})^2$ ).

30 This  $A$  is not invertible for  $c = 7$  (equal columns),  $c = 2$  (equal rows),  $c = 0$  (zero column).

31 Elimination produces the pivots  $a$  and  $a-b$  and  $a-b$ .  $A^{-1} = \frac{1}{a(a-b)} \begin{bmatrix} a & 0 & -b \\ -a & a & 0 \\ 0 & -a & a \end{bmatrix}$ .

32  $A^{-1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ . When the triangular  $A$  alternates 1 and  $-1$  on its diagonal,

$A^{-1}$  is *bidagonal* with 1's on the diagonal and first superdiagonal.

33  $\mathbf{x} = (1, 1, \dots, 1)$  has  $P\mathbf{x} = Q\mathbf{x}$  so  $(P - Q)\mathbf{x} = \mathbf{0}$ .

34  $\begin{bmatrix} I & 0 \\ -C & I \end{bmatrix}$  and  $\begin{bmatrix} A^{-1} & 0 \\ -D^{-1}CA^{-1} & D^{-1} \end{bmatrix}$  and  $\begin{bmatrix} -D & I \\ I & 0 \end{bmatrix}$ .

35  $A$  can be invertible with diagonal zeros.  $B$  is singular because each row adds to zero.

36 The equation  $LDLD = I$  says that  $LD = \text{pascal}(4, 1)$  is its own inverse.

37  $\text{hilb}(6)$  is not the exact Hilbert matrix because fractions are rounded off. So  $\text{inv}(\text{hilb}(6))$  is not the exact either.

38 The three Pascal matrices have  $P = LU = LL^T$  and then  $\text{inv}(P) = \text{inv}(L^T)\text{inv}(L)$ .

39  $A\mathbf{x} = \mathbf{b}$  has many solutions when  $A = \text{ones}(4, 4) = \text{singular matrix}$  and  $\mathbf{b} = \text{ones}(4, 1)$ .  $A \backslash \mathbf{b}$  in MATLAB will pick the shortest solution  $\mathbf{x} = (1, 1, 1, 1)/4$ . This is the only solution that is combination of the rows of  $A$  (later it comes from the "pseudoinverse"  $A^+ = \text{pinv}(A)$  which replaces  $A^{-1}$  when  $A$  is singular). Any vector that solves  $A\mathbf{x} = \mathbf{0}$  could be added to this particular solution  $\mathbf{x}$ .

40 The inverse of  $A = \begin{bmatrix} 1 & -a & 0 & 0 \\ 0 & 1 & -b & 0 \\ 0 & 0 & 1 & -c \\ 0 & 0 & 0 & 1 \end{bmatrix}$  is  $A^{-1} = \begin{bmatrix} 1 & a & ab & abc \\ 0 & 1 & b & bc \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}$ . (This

would be a good example for the cofactor formula  $A^{-1} = C^T / \det A$  in Section 5.3)

41 The product  $\begin{bmatrix} 1 & & & \\ a & 1 & & \\ b & 0 & 1 & \\ c & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & d & 1 & \\ 0 & e & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & f & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ a & 1 & & \\ b & d & 1 & \\ c & e & f & 1 \end{bmatrix}$

that in this order the multipliers shows  $a, b, c, d, e, f$  are unchanged in the product (**important for  $A = LU$  in Section 2.6**).

42  $MM^{-1} = (I_n - UV)(I_n + U(I_m - VU)^{-1}V)$  (this is testing formula 3)  
 $= I_n - UV + U(I_m - VU)^{-1}V - UVU(I_m - VU)^{-1}V$  (keep simplifying)  
 $= I_n - UV + U(I_m - VU)(I_m - VU)^{-1}V = I_n$  (formulas 1, 2, 4 are similar)

43 4 by 4 still with  $T_{11} = 1$  has pivots 1, 1, 1, 1; reversing to  $T^* = UL$  makes  $T_{44}^* = 1$ .

44 Add the equations  $C\mathbf{x} = \mathbf{b}$  to find  $0 = b_1 + b_2 + b_3 + b_4$ . Same for  $F\mathbf{x} = \mathbf{b}$ .

45 The block pivots are  $A$  and  $S = D - CA^{-1}B$  (and  $d - cb/a$  is the correct second pivot of an ordinary 2 by 2 matrix). The example problem has

$$S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} -5 & -6 \\ -6 & -5 \end{bmatrix}.$$

- 46** Inverting the identity  $A(I + BA) = (I + AB)A$  gives  $(I + BA)^{-1}A^{-1} = A^{-1}(I + AB)^{-1}$ . So  $I + BA$  and  $I + AB$  are both invertible or both singular when  $A$  is invertible. (This remains true also when  $A$  is singular: Problem 6.6.19 will show that  $AB$  and  $BA$  have the same nonzero eigenvalues, and we are looking here at  $\lambda = -1$ .)

## Problem Set 2.6, page 102

- 1**  $\ell_{21} = 1$  multiplied row 1;  $L = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$  times  $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \mathbf{c}$  is  $A\mathbf{x} = \mathbf{b}$ :  
 $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$ .
- 2**  $L\mathbf{c} = \mathbf{b}$  is  $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$ , solved by  $\mathbf{c} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$  as elimination goes forward.  
 $U\mathbf{x} = \mathbf{c}$  is  $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ , solved by  $\mathbf{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$  in back substitution.
- 3**  $\ell_{31} = 1$  and  $\ell_{32} = 2$  (and  $\ell_{33} = 1$ ): reverse steps to get  $A\mathbf{u} = \mathbf{b}$  from  $U\mathbf{x} = \mathbf{c}$ :  
 1 times  $(x + y + z = 5) + 2$  times  $(y + 2z = 2) + 1$  times  $(z = 2)$  gives  $x + 3y + 6z = 11$ .
- 4**  $L\mathbf{c} = \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 11 \end{bmatrix}$ ;  $U\mathbf{x} = \begin{bmatrix} 1 & 1 & 1 \\ & 1 & 2 \\ & & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix}$ ;  $\mathbf{x} = \begin{bmatrix} 5 \\ -2 \\ 2 \end{bmatrix}$ .
- 5**  $EA = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix} = U$ . With  $E^{-1}$  as  $L$ ,  $A = LU =$   
 $\begin{bmatrix} 1 & & \\ 0 & 1 & \\ 3 & 0 & 1 \end{bmatrix} U$ .
- 6**  $\begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ -2 & 1 & \\ 0 & 0 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -6 \end{bmatrix} = U$ . Then  $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} U$  is  
 the same as  $E_{21}^{-1}E_{32}^{-1}U = LU$ . The multipliers  $\ell_{21}, \ell_{32} = 2$  fall into place in  $L$ .
- 7**  $E_{32}E_{31}E_{21} A = \begin{bmatrix} 1 & & \\ & 1 & \\ & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ -3 & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ -2 & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix}$ . This is  
 $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = U$ . Put those multipliers 2, 3, 2 into  $L$ . Then  $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} U = LU$ .
- 8**  $E = E_{32}E_{31}E_{21} = \begin{bmatrix} 1 & & \\ & 1 & \\ & -c & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ -b & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ -a & 1 & \\ & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ -a & 1 & \\ ac - b & -c & 1 \end{bmatrix}$ .  
 The multipliers are just  $a, b, c$  and the upper triangular  $U$  is  $I$ . In this case  $A = L$  and its inverse is that matrix  $E = L^{-1}$ .
- 9** 2 by 2:  $d = 0$  not allowed;  $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ l & 1 & \\ m & n & 1 \end{bmatrix} \begin{bmatrix} d & e & g \\ f & h & \\ i & & \end{bmatrix}$   $d = 1, e = 1$ , then  $l = 1$   
 $f = 0$  is not allowed  
**no pivot in row 2**

- 10**  $c = 2$  leads to zero in the second pivot position: exchange rows and not singular.  
 $c = 1$  leads to zero in the third pivot position. In this case the matrix is *singular*.

**11**  $A = \begin{bmatrix} 2 & 4 & 8 \\ 0 & 3 & 9 \\ 0 & 0 & 7 \end{bmatrix}$  has  $L = I$  ( $A$  is already upper triangular) and  $D = \begin{bmatrix} 2 & & \\ & 3 & \\ & & 7 \end{bmatrix}$ ;

$A = LU$  has  $U = A$ ;  $A = LDU$  has  $U = D^{-1}A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$  with 1's on the diagonal.

**12**  $A = \begin{bmatrix} 2 & 4 \\ 4 & 11 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = LDU$ ;  $U$  is  $L^T$   
 $\begin{bmatrix} 1 & & \\ 4 & 1 & \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 0 \\ 0 & -4 & 4 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & & \\ 4 & 1 & \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & -4 & \\ & & 4 \end{bmatrix} \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = LDL^T$ .

**13**  $\begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & a & a & a \\ b-a & b-a & b-a & b-a \\ c-b & c-b & c-b & c-b \\ d-c & d-c & d-c & d-c \end{bmatrix}$ . Need  $\begin{matrix} a \neq 0 \\ b \neq a \\ c \neq b \\ d \neq c \end{matrix}$  All of the multipliers are  $\ell_{ij} = 1$  for this  $A$

**14**  $\begin{bmatrix} a & r & r & r \\ a & b & s & s \\ a & b & c & t \\ a & b & c & d \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & r & r & r \\ b-r & s-r & s-r & s-r \\ c-s & t-s & t-s & t-s \\ d-t & d-t & d-t & d-t \end{bmatrix}$ . Need  $\begin{matrix} a \neq 0 \\ b \neq r \\ c \neq s \\ d \neq t \end{matrix}$

**15**  $\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \mathbf{c} = \begin{bmatrix} 2 \\ 11 \end{bmatrix}$  gives  $\mathbf{c} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ . Then  $\begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$  gives  $\mathbf{x} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$ .  
 $A\mathbf{x} = \mathbf{b}$  is  $LU\mathbf{x} = \begin{bmatrix} 2 & 4 \\ 8 & 17 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 2 \\ 11 \end{bmatrix}$ . Forward to  $\begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \mathbf{c}$ .

**16**  $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \mathbf{c} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$  gives  $\mathbf{c} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$ . Then  $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$  gives  $\mathbf{x} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ .  
Those are the forward elimination and back substitution steps for  
 $A\mathbf{x} = \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ .

- 17** (a)  $L$  goes to  $I$  (b)  $I$  goes to  $L^{-1}$  (c)  $LU$  goes to  $U$ . Elimination multiply by  $L^{-1}$ !

- 18** (a) Multiply  $LDU = L_1 D_1 U_1$  by inverses to get  $L_1^{-1} L D = D_1 U_1 U^{-1}$ . The left side is lower triangular, the right side is upper triangular  $\Rightarrow$  both sides are diagonal.  
(b)  $L, U, L_1, U_1$  have diagonal 1's so  $D = D_1$ . Then  $L_1^{-1} L$  and  $U_1 U^{-1}$  are both  $I$ .

**19**  $\begin{bmatrix} 1 & & \\ 1 & 1 & \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ & 1 & 1 \\ & & 1 \end{bmatrix} = LIU$ ;  $\begin{bmatrix} a & a & 0 \\ a & a+b & b \\ 0 & b & b+c \end{bmatrix} = (\text{same } L) \begin{bmatrix} a & & \\ & b & \\ & & c \end{bmatrix}$   
(same  $U$ ). A tridiagonal matrix  $A$  has **bidiagonal factors**  $L$  and  $U$ .

- 20** A tridiagonal  $T$  has 2 nonzeros in the pivot row and only one nonzero below the pivot (one operation to find  $\ell$  and then one for the new pivot!).  $T = \text{bidiagonal } L \text{ times bidiagonal } U$ .

- 21** For the first matrix  $A$ ,  $L$  keeps the 3 lower zeros at the start of rows. But  $U$  may not have the upper zero where  $A_{24} = 0$ . For the second matrix  $B$ ,  $L$  keeps the bottom left zero at the start of row 4.  $U$  keeps the upper right zero at the start of column 4. One zero in  $A$  and two zeros in  $B$  are filled in.
- 22** Eliminating upwards,  $\begin{bmatrix} 5 & 3 & 1 \\ 3 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 4 & 2 & 0 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} = L$ . We reach a lower triangular  $L$ , and the multipliers are in an upper triangular  $U$ .  $A = UL$  with  $U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ .
- 23** The 2 by 2 upper submatrix  $A_2$  has the first two pivots 5, 9. Reason: Elimination on  $A$  starts in the upper left corner with elimination on  $A_2$ .
- 24** The upper left blocks all factor at the same time as  $A$ :  $A_k$  is  $L_k U_k$ .
- 25** The  $i, j$  entry of  $L^{-1}$  is  $j/i$  for  $i \geq j$ . And  $L_{i, i-1}$  is  $(1-i)/i$  below the diagonal
- 26**  $(K^{-1})_{ij} = j(n-i+1)/(n+1)$  for  $i \geq j$  (and symmetric):  $(n+1)K^{-1}$  looks good.

### Problem Set 2.7, page 115

- 1**  $A = \begin{bmatrix} 1 & 0 \\ 9 & 3 \end{bmatrix}$  has  $A^T = \begin{bmatrix} 1 & 9 \\ 0 & 3 \end{bmatrix}$ ,  $A^{-1} = \begin{bmatrix} 1 & 0 \\ -3 & 1/3 \end{bmatrix}$ ,  $(A^{-1})^T = (A^T)^{-1} = \begin{bmatrix} 1 & -3 \\ 0 & 1/3 \end{bmatrix}$ ;  
 $A = \begin{bmatrix} 1 & c \\ c & 0 \end{bmatrix}$  has  $A^T = A$  and  $A^{-1} = \frac{1}{c^2} \begin{bmatrix} 0 & c \\ c & -1 \end{bmatrix} = (A^{-1})^T$ .
- 2**  $(AB)^T$  is not  $A^T B^T$  except when  $AB = BA$ . Transpose that to find:  $B^T A^T = A^T B^T$ .
- 3** (a)  $((AB)^{-1})^T = (B^{-1} A^{-1})^T = (A^{-1})^T (B^{-1})^T$ . This is also  $(A^T)^{-1} (B^T)^{-1}$ .  
 (b) If  $U$  is upper triangular, so is  $U^{-1}$ : then  $(U^{-1})^T$  is lower triangular.
- 4**  $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$  has  $A^2 = 0$ . The diagonal of  $A^T A$  has dot products of columns of  $A$  with themselves. If  $A^T A = 0$ , zero dot products  $\Rightarrow$  zero columns  $\Rightarrow A =$  zero matrix.
- 5** (a)  $\mathbf{x}^T A \mathbf{y} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 5$  (b)  $\mathbf{x}^T A = \begin{bmatrix} 4 & 5 & 6 \end{bmatrix}$  (c)  $A \mathbf{y} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ .
- 6**  $M^T = \begin{bmatrix} A^T & C^T \\ B^T & D^T \end{bmatrix}$ ;  $M^T = M$  needs  $A^T = A$  and  $B^T = C$  and  $D^T = D$ .
- 7** (a) False:  $\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$  is symmetric only if  $A = A^T$ . (b) False: The transpose of  $AB$  is  $B^T A^T = BA$  when  $A$  and  $B$  are symmetric  $\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$  transposes to  $\begin{bmatrix} 0 & A^T \\ A^T & 0 \end{bmatrix}$ . So  $(AB)^T = AB$  needs  $BA = AB$ . (c) True: Invertible symmetric matrices have symmetric inverses! Easiest proof is to transpose  $AA^{-1} = I$ . (d) True:  $(ABC)^T$  is  $C^T B^T A^T (= CBA$  for symmetric matrices  $A, B$ , and  $C$ ).
- 8** The 1 in row 1 has  $n$  choices; then the 1 in row 2 has  $n-1$  choices  $\dots (n!$  overall).

$$\mathbf{9} \quad P_1 P_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ but } P_2 P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If  $P_3$  and  $P_4$  exchange *different* pairs of rows,  $P_3 P_4 = P_4 P_3$  does both exchanges.

- 10** (3, 1, 2, 4) and (2, 3, 1, 4) keep 4 in place; 6 more even  $P$ 's keep 1 or 2 or 3 in place; (2, 1, 4, 3) and (3, 4, 1, 2) exchange 2 pairs. (1, 2, 3, 4), (4, 3, 2, 1) make 12 even  $P$ 's.

$$\mathbf{11} \quad PA = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \text{ is upper triangular. Multiplying on the right by a permutation matrix } P_2 \text{ exchanges the columns. To make this } A \text{ lower triangular, we also need } P_1 \text{ to exchange rows 2 and 3: } P_1 A P_2 = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

$$A \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 2 & 1 \end{bmatrix}.$$

- 12**  $(Px)^T(Py) = x^T P^T P y = x^T y$  since  $P^T P = I$ . In general  $Px \cdot y = x \cdot P^T y \neq x \cdot Py$ :

$$\text{Non-equality where } P \neq P^T: \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

- 13** A cyclic  $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$  or its transpose will have  $P^3 = I : (1, 2, 3) \rightarrow (2, 3, 1) \rightarrow (3, 1, 2) \rightarrow (1, 2, 3)$ .  $\hat{P} = \begin{bmatrix} 1 & 0 \\ 0 & P \end{bmatrix}$  for the same  $P$  has  $\hat{P}^4 = \hat{P} \neq I$ .

- 14** The “reverse identity”  $P$  takes  $(1, \dots, n)$  into  $(n, \dots, 1)$ . When rows and also columns are reversed,  $(PAP)_{ij}$  is  $(A)_{n-i+1, n-j+1}$ . In particular  $(PAP)_{11}$  is  $A_{nn}$ .

- 15** (a) If  $P$  sends row 1 to row 4, then  $P^T$  sends row 4 to row 1 (b)  $P = \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix} = P^T$  with  $E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$  moves all rows: 1 and 2 are exchanged, 3 and 4 are exchanged.

- 16**  $A^2 - B^2$  (but not  $(A + B)(A - B)$ , this is different) and also  $ABA$  are symmetric if  $A$  and  $B$  are symmetric.

- 17** (a)  $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = A^T$  is not invertible (b)  $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$  needs row exchange (c)  $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$  has  $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ .

- 18** (a)  $5 + 4 + 3 + 2 + 1 = 15$  independent entries if  $A = A^T$  (b)  $L$  has 10 and  $D$  has 5; total 15 in  $LDL^T$  (c) Zero diagonal if  $A^T = -A$ , leaving  $4 + 3 + 2 + 1 = 10$  choices.

- 19** (a) The transpose of  $R^T A R$  is  $R^T A^T R^T = R^T A R = n$  by  $n$  when  $A^T = A$  (any  $m$  by  $n$  matrix  $R$ ) (b)  $(R^T R)_{jj} = (\text{column } j \text{ of } R) \cdot (\text{column } j \text{ of } R) = (\text{length squared of column } j) \geq 0$ .



$$20 \quad \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -7 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}; \quad \begin{bmatrix} 1 & b \\ b & c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & c - b^2 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & & \\ -\frac{1}{2} & 1 & \\ 0 & -\frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 2 & & \\ \frac{3}{2} & & \\ \frac{4}{3} & & \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ & 1 & -\frac{2}{3} \\ & & 1 \end{bmatrix} = LDL^T.$$

21 Elimination on a symmetric 3 by 3 matrix leaves a symmetric lower right 2 by 2 matrix.

The examples  $\begin{bmatrix} 2 & 4 & 8 \\ 4 & 3 & 9 \\ 8 & 9 & 0 \end{bmatrix}$  and  $\begin{bmatrix} 1 & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$  lead to  $\begin{bmatrix} -5 & -7 \\ -7 & -32 \end{bmatrix}$  and  $\begin{bmatrix} d - b^2 & e - bc \\ e - bc & f - c^2 \end{bmatrix}$ .

$$22 \quad \begin{bmatrix} & 1 \\ 1 & \\ & 1 \end{bmatrix} A = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ & 1 & 1 \\ & & -1 \end{bmatrix}; \quad \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} A = \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ & -1 & 1 \\ & & 1 \end{bmatrix}$$

$$23 \quad A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = P \text{ and } L = U = I. \quad \text{This cyclic } P \text{ exchanges rows 1-2 then rows 2-3 then rows 3-4.}$$

$$24 \quad PA = LU \text{ is } \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 0 & 3 & 8 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & 1/3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ & 3 & 8 \\ & & -2/3 \end{bmatrix}. \text{ If we wait}$$

to exchange and  $a_{12}$  is the pivot,  $A = L_1 P_1 U_1 = \begin{bmatrix} 1 & & \\ 3 & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ 1 & & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}.$

25 The **splu** code will not end when  $\mathbf{abs}(A(k, k)) < \text{tol}$  line 4 of the **slu** code on page 100. Instead **splu** looks for a nonzero entry below the diagonal in the current column  $k$ , and executes a row exchange. The 4 lines to exchange row  $k$  with row  $r$  are at the end of Section 2.7 (page 113). To find that nonzero entry  $A(r, k)$ , follow  $\mathbf{abs}(A(k, k)) < \text{tol}$  by locating the first nonzero (or the largest  $A(r, k)$  out of  $r = k + 1, \dots, n$ ).

26 One way to decide even vs. odd is to count all pairs that  $P$  has in the wrong order. Then  $P$  is even or odd when that count is even or odd. Hard step: Show that an exchange always switches that count! Then 3 or 5 exchanges will leave that count odd.

$$27 \quad (a) \quad E_{21} = \begin{bmatrix} 1 & & \\ -3 & 1 & \\ & & 1 \end{bmatrix} \text{ puts 0 in the 2, 1 entry of } E_{21}A. \text{ Then } E_{21}AE_{21}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 9 \end{bmatrix}$$

is still symmetric, with zero also in its 1, 2 entry. (b) Now use  $E_{32} = \begin{bmatrix} 1 & & \\ & 1 & \\ & -4 & 1 \end{bmatrix}$

to make the 3, 2 entry zero and  $E_{32}E_{21}AE_{21}^TE_{32}^T = D$  also has zero in its 2, 3 entry. Key point: Elimination from both sides gives the symmetric  $LDL^T$  directly.

$$28 \quad A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{bmatrix} = A^T \text{ has 0, 1, 2, 3 in every row. (I don't know any rules for a symmetric construction like this)}$$

**29** Reordering the rows and/or the columns of  $\begin{bmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{bmatrix}$  will move the entry  $\mathbf{a}$ . So the result cannot be the transpose (which doesn't move  $\mathbf{a}$ ).

**30** (a) Total currents are  $A^T \mathbf{y} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} y_{BC} \\ y_{CS} \\ y_{BS} \end{bmatrix} = \begin{bmatrix} y_{BC} + y_{BS} \\ -y_{BC} + y_{CS} \\ -y_{CS} - y_{BS} \end{bmatrix}.$

(b) Either way  $(A\mathbf{x})^T \mathbf{y} = \mathbf{x}^T (A^T \mathbf{y}) = x_B y_{BC} + x_B y_{BS} - x_C y_{BC} + x_C y_{CS} - x_S y_{CS} - x_S y_{BS}.$

**31**  $\begin{bmatrix} 1 & 50 \\ 40 & 1000 \\ 2 & 50 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\mathbf{x}; A^T \mathbf{y} = \begin{bmatrix} 1 & 40 & 2 \\ 50 & 1000 & 50 \end{bmatrix} \begin{bmatrix} 700 \\ 3 \\ 3000 \end{bmatrix} = \begin{bmatrix} 6820 \\ 188000 \end{bmatrix} \begin{matrix} 1 \text{ truck} \\ 1 \text{ plane} \end{matrix}$

**32**  $A\mathbf{x} \cdot \mathbf{y}$  is the *cost* of inputs while  $\mathbf{x} \cdot A^T \mathbf{y}$  is the *value* of outputs.

**33**  $P^3 = I$  so three rotations for  $360^\circ$ ;  $P$  rotates around  $(1, 1, 1)$  by  $120^\circ$ .

**34**  $\begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = EH = (\text{elementary matrix}) \text{ times } (\text{symmetric matrix}).$

**35**  $L(U^T)^{-1}$  is lower triangular times lower triangular, so lower triangular. The transpose of  $U^T D U$  is  $U^T D^T U^{TT} = U^T D U$  again, so  $U^T D U$  is symmetric. The factorization multiplies lower triangular by symmetric to get  $L D U$  which is  $A$ .

**36** These are groups: Lower triangular with diagonal 1's, diagonal invertible  $D$ , permutations  $P$ , orthogonal matrices with  $Q^T = Q^{-1}$ .

**37** Certainly  $B^T$  is northwest.  $B^2$  is a full matrix!  $B^{-1}$  is southeast:  $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$ . The rows of  $B$  are in reverse order from a lower triangular  $L$ , so  $B = PL$ . Then  $B^{-1} = L^{-1}P^{-1}$  has the *columns* in reverse order from  $L^{-1}$ . So  $B^{-1}$  is *southeast*. Northwest  $B = PL$  times southeast  $PU$  is  $(PLP)U =$  upper triangular.

**38** There are  $n!$  permutation matrices of order  $n$ . Eventually *two powers of  $P$  must be the same*: If  $P^r = P^s$  then  $P^{r-s} = I$ . Certainly  $r - s \leq n!$

$P = \begin{bmatrix} P_2 & \\ & P_3 \end{bmatrix}$  is 5 by 5 with  $P_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$  and  $P_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$  and  $P^6 = I$ .

**39** To split  $A$  into (symmetric  $B$ ) + (anti-symmetric  $C$ ), the only choice is  $B = \frac{1}{2}(A + A^T)$  and  $C = \frac{1}{2}(A - A^T)$ .

**40** Start from  $Q^T Q = I$ , as in  $\begin{bmatrix} \mathbf{q}_1^T \\ \mathbf{q}_2^T \end{bmatrix} \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(a) The diagonal entries give  $\mathbf{q}_1^T \mathbf{q}_1 = 1$  and  $\mathbf{q}_2^T \mathbf{q}_2 = 1$ : *unit vectors*

(b) The off-diagonal entry is  $\mathbf{q}_1^T \mathbf{q}_2 = 0$  (and in general  $\mathbf{q}_i^T \mathbf{q}_j = 0$ )

(c) The leading example for  $Q$  is the rotation matrix  $\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}.$

### Problem Set 3.1, page 127

- 1  $x + y \neq y + x$  and  $x + (y + z) \neq (x + y) + z$  and  $(c_1 + c_2)x \neq c_1x + c_2x$ .
- 2 When  $c(x_1, x_2) = (cx_1, 0)$ , the only broken rule is 1 times  $x$  equals  $x$ . Rules (1)-(4) for addition  $x + y$  still hold since addition is not changed.
- 3 (a)  $cx$  may not be in our set: not closed under multiplication. Also no  $0$  and no  $-x$   
 (b)  $c(x + y)$  is the usual  $(xy)^c$ , while  $cx + cy$  is the usual  $(x^c)(y^c)$ . Those are equal. With  $c = 3$ ,  $x = 2$ ,  $y = 1$  this is  $3(2 + 1) = 8$ . The zero vector is the number 1.
- 4 The zero vector in matrix space  $\mathbf{M}$  is  $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ ;  $\frac{1}{2}A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$  and  $-A = \begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix}$ .  
 The smallest subspace of  $\mathbf{M}$  containing the matrix  $A$  consists of all matrices  $cA$ .
- 5 (a) One possibility: The matrices  $cA$  form a subspace not containing  $B$  (b) Yes: the subspace must contain  $A - B = I$  (c) Matrices whose main diagonal is all zero.
- 6 When  $f(x) = x^2$  and  $g(x) = 5x$ , the combination  $3f - 4g$  in function space is  $h(x) = 3f(x) - 4g(x) = 3x^2 - 20x$ .
- 7 Rule 8 is broken: If  $cf(x)$  is defined to be the usual  $f(cx)$  then  $(c_1 + c_2)f = f((c_1 + c_2)x)$  is not generally the same as  $c_1f + c_2f = f(c_1x) + f(c_2x)$ .
- 8 If  $(f + g)(x)$  is the usual  $f(g(x))$  then  $(g + f)x$  is  $g(f(x))$  which is different. In Rule 2 both sides are  $f(g(h(x)))$ . Rule 4 is broken there might be no inverse function  $f^{-1}(x)$  such that  $f(f^{-1}(x)) = x$ . If the inverse function exists it will be the vector  $-f$ .
- 9 (a) The vectors with integer components allow addition, but not multiplication by  $\frac{1}{2}$   
 (b) Remove the  $x$  axis from the  $xy$  plane (but leave the origin). Multiplication by any  $c$  is allowed but not all vector additions.
- 10 The only subspaces are (a) the plane with  $b_1 = b_2$  (d) the linear combinations of  $v$  and  $w$  (e) the plane with  $b_1 + b_2 + b_3 = 0$ .
- 11 (a) All matrices  $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$  (b) All matrices  $\begin{bmatrix} a & a \\ 0 & 0 \end{bmatrix}$  (c) All diagonal matrices.
- 12 For the plane  $x + y - 2z = 4$ , the sum of  $(4, 0, 0)$  and  $(0, 4, 0)$  is not on the plane. (The key is that this plane does not go through  $(0, 0, 0)$ .)
- 13 The parallel plane  $\mathbf{P}_0$  has the equation  $x + y - 2z = 0$ . Pick two points, for example  $(2, 0, 1)$  and  $(0, 2, 1)$ , and their sum  $(2, 2, 2)$  is in  $\mathbf{P}_0$ .
- 14 (a) The subspaces of  $\mathbf{R}^2$  are  $\mathbf{R}^2$  itself, lines through  $(0, 0)$ , and  $(0, 0)$  by itself (b) The subspaces of  $\mathbf{R}^4$  are  $\mathbf{R}^4$  itself, three-dimensional planes  $n \cdot v = 0$ , two-dimensional subspaces ( $n_1 \cdot v = 0$  and  $n_2 \cdot v = 0$ ), one-dimensional lines through  $(0, 0, 0, 0)$ , and  $(0, 0, 0, 0)$  by itself.
- 15 (a) Two planes through  $(0, 0, 0)$  probably intersect in a line through  $(0, 0, 0)$   
 (b) The plane and line probably intersect in the point  $(0, 0, 0)$   
 (c) If  $x$  and  $y$  are in both  $S$  and  $T$ ,  $x + y$  and  $cx$  are in both subspaces.
- 16 The smallest subspace containing a plane  $\mathbf{P}$  and a line  $\mathbf{L}$  is *either*  $\mathbf{P}$  (when the line  $\mathbf{L}$  is in the plane  $\mathbf{P}$ ) *or*  $\mathbf{R}^3$  (when  $\mathbf{L}$  is not in  $\mathbf{P}$ ).
- 17 (a) The invertible matrices do not include the zero matrix, so they are not a subspace  
 (b) The sum of singular matrices  $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$  is not singular: not a subspace.

- 18** (a) *True*: The symmetric matrices do form a subspace (b) *True*: The matrices with  $A^T = -A$  do form a subspace (c) *False*: The sum of two unsymmetric matrices could be symmetric.
- 19** The column space of  $A$  is the  $x$ -axis = all vectors  $(x, 0, 0)$ . The column space of  $B$  is the  $xy$  plane = all vectors  $(x, y, 0)$ . The column space of  $C$  is the line of vectors  $(x, 2x, 0)$ .
- 20** (a) Elimination leads to  $0 = b_2 - 2b_1$  and  $0 = b_1 + b_3$  in equations 2 and 3: Solution only if  $b_2 = 2b_1$  and  $b_3 = -b_1$  (b) Elimination leads to  $0 = b_1 + 2b_3$  in equation 3: Solution only if  $b_3 = -b_1$ .
- 21** A combination of the columns of  $C$  is also a combination of the columns of  $A$ . Then  $C = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$  and  $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$  have the same column space.  $B = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$  has a different column space.
- 22** (a) Solution for every  $\mathbf{b}$  (b) Solvable only if  $b_3 = 0$  (c) Solvable only if  $b_3 = b_2$ .
- 23** The extra column  $\mathbf{b}$  enlarges the column space unless  $\mathbf{b}$  is *already in* the column space.  
 $[A \ \mathbf{b}] = \begin{bmatrix} 1 & 0 & \mathbf{1} \\ 0 & 0 & \mathbf{1} \end{bmatrix}$  (larger column space)  $\begin{bmatrix} 1 & 0 & \mathbf{1} \\ 0 & 1 & \mathbf{1} \end{bmatrix}$  ( $\mathbf{b}$  is in column space)  
 (no solution to  $A\mathbf{x} = \mathbf{b}$ ) ( $A\mathbf{x} = \mathbf{b}$  has a solution)
- 24** The column space of  $AB$  is *contained in* (possibly equal to) the column space of  $A$ . The example  $B = 0$  and  $A \neq 0$  is a case when  $AB = 0$  has a smaller column space than  $A$ .
- 25** The solution to  $A\mathbf{z} = \mathbf{b} + \mathbf{b}^*$  is  $\mathbf{z} = \mathbf{x} + \mathbf{y}$ . If  $\mathbf{b}$  and  $\mathbf{b}^*$  are in  $C(A)$  so is  $\mathbf{b} + \mathbf{b}^*$ .
- 26** The column space of any invertible 5 by 5 matrix is  $\mathbf{R}^5$ . The equation  $A\mathbf{x} = \mathbf{b}$  is always solvable (by  $\mathbf{x} = A^{-1}\mathbf{b}$ ) so every  $\mathbf{b}$  is in the column space of that invertible matrix.
- 27** (a) *False*: Vectors that are *not* in a column space don't form a subspace. (b) *True*: Only the zero matrix has  $C(A) = \{\mathbf{0}\}$ . (c) *True*:  $C(A) = C(2A)$ .  
 (d) *False*:  $C(A - I) \neq C(A)$  when  $A = I$  or  $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$  (or other examples).
- 28**  $A = \begin{bmatrix} 1 & 1 & \mathbf{0} \\ 1 & 0 & \mathbf{0} \\ 0 & 1 & \mathbf{0} \end{bmatrix}$  and  $\begin{bmatrix} 1 & 1 & \mathbf{2} \\ 1 & 0 & \mathbf{1} \\ 0 & 1 & \mathbf{1} \end{bmatrix}$  do not have  $(1, 1, 1)$  in  $C(A)$ .  $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 3 & 6 & 0 \end{bmatrix}$  has  $C(A) = \text{line}$ .
- 29** When  $A\mathbf{x} = \mathbf{b}$  is solvable for all  $\mathbf{b}$ , every  $\mathbf{b}$  is in the column space of  $A$ . So that space is  $\mathbf{R}^9$ .
- 30** (a) If  $\mathbf{u}$  and  $\mathbf{v}$  are both in  $S + T$ , then  $\mathbf{u} = \mathbf{s}_1 + \mathbf{t}_1$  and  $\mathbf{v} = \mathbf{s}_2 + \mathbf{t}_2$ . So  $\mathbf{u} + \mathbf{v} = (\mathbf{s}_1 + \mathbf{s}_2) + (\mathbf{t}_1 + \mathbf{t}_2)$  is also in  $S + T$ . And so is  $c\mathbf{u} = c\mathbf{s}_1 + c\mathbf{t}_1$ : a subspace.  
 (b) If  $S$  and  $T$  are different lines, then  $S \cup T$  is just the two lines (*not a subspace*) but  $S + T$  is the whole plane that they span.
- 31** If  $S = C(A)$  and  $T = C(B)$  then  $S + T$  is the column space of  $M = [A \ B]$ .
- 32** The columns of  $AB$  are combinations of the columns of  $A$ . So all columns of  $[A \ AB]$  are already in  $C(A)$ . But  $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$  has a larger column space than  $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ .  
 For square matrices, the column space is  $\mathbf{R}^n$  when  $A$  is *invertible*.

### Problem Set 3.2, page 140

- 1 (a)  $U = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$  Free variables  $x_2, x_4, x_5$   
Pivot variables  $x_1, x_3$  (b)  $U = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix}$  Free  $x_3$   
Pivot  $x_1, x_2$
- 2 (a) Free variables  $x_2, x_4, x_5$  and solutions  $(-2, 1, 0, 0, 0), (0, 0, -2, 1, 0), (0, 0, -3, 0, 1)$   
(b) Free variable  $x_3$ : solution  $(1, -1, 1)$ . Special solution for each free variable.
- 3 The complete solution to  $A\mathbf{x} = \mathbf{0}$  is  $(-2x_2, x_2, -2x_4 - 3x_5, x_4, x_5)$  with  $x_2, x_4, x_5$  free. The complete solution to  $B\mathbf{x} = \mathbf{0}$  is  $(2x_3, -x_3, x_3)$ . The nullspace contains only  $\mathbf{x} = \mathbf{0}$  when there are no free variables.
- 4  $R = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ ,  $R = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ ,  $R$  has the same nullspace as  $U$  and  $A$ .
- 5  $A = \begin{bmatrix} -1 & 3 & 5 \\ -2 & 6 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & 0 \end{bmatrix}$ ;  $B = \begin{bmatrix} -1 & 3 & 5 \\ -2 & 6 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & -3 \end{bmatrix} = LU$ .
- 6 (a) Special solutions  $(3, 1, 0)$  and  $(5, 0, 1)$  (b)  $(3, 1, 0)$ . Total of pivot and free is  $n$ .
- 7 (a) The nullspace of  $A$  in Problem 5 is the plane  $-x + 3y + 5z = 0$ ; it contains all the vectors  $(3y + 5z, y, z) = y(3, 1, 0) + z(5, 0, 1) =$  combination of special solutions.  
(b) The line through  $(3, 1, 0)$  has equations  $-x + 3y + 5z = 0$  and  $-2x + 6y + 7z = 0$ . The special solution for the free variable  $x_2$  is  $(3, 1, 0)$ .
- 8  $R = \begin{bmatrix} 1 & -3 & -5 \\ 0 & 0 & 0 \end{bmatrix}$  with  $I = [1]$ ;  $R = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$  with  $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ .
- 9 (a) *False*: Any singular square matrix would have free variables (b) *True*: An invertible square matrix has *no* free variables. (c) *True* (only  $n$  columns to hold pivots)  
(d) *True* (only  $m$  rows to hold pivots)
- 10 (a) Impossible row 1 (b)  $A$  is invertible (c)  $A =$  all ones (d)  $A = 2I, R = I$ .
- 11  $\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$
- 12  $\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ . Notice the identity matrix in the pivot columns of these *reduced* row echelon forms  $R$ .
- 13 If column 4 of a 3 by 5 matrix is all zero then  $x_4$  is a *free* variable. Its special solution is  $\mathbf{x} = (0, 0, 0, 1, 0)$ , because 1 will multiply that zero column to give  $A\mathbf{x} = \mathbf{0}$ .
- 14 If column 1 = column 5 then  $x_5$  is a free variable. Its special solution is  $(-1, 0, 0, 0, 1)$ .
- 15 If a matrix has  $n$  columns and  $r$  pivots, there are  $n - r$  special solutions. The nullspace contains only  $\mathbf{x} = \mathbf{0}$  when  $r = n$ . The column space is all of  $\mathbf{R}^m$  when  $r = m$ . All important!

- 16** The nullspace contains only  $\mathbf{x} = \mathbf{0}$  when  $A$  has 5 pivots. Also the column space is  $\mathbf{R}^5$ , because we can solve  $A\mathbf{x} = \mathbf{b}$  and every  $\mathbf{b}$  is in the column space.
- 17**  $A = \begin{bmatrix} 1 & -3 & -1 \end{bmatrix}$  gives the plane  $x - 3y - z = 0$ ;  $y$  and  $z$  are free variables. The special solutions are  $(3, 1, 0)$  and  $(1, 0, 1)$ .
- 18** Fill in **12** then **4** then **1** to get the complete solution to  $x - 3y - z = 12$ :  $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \mathbf{x}_{\text{particular}} + \mathbf{x}_{\text{nullspace}}.$
- 19** If  $LU\mathbf{x} = \mathbf{0}$ , multiply by  $L^{-1}$  to find  $U\mathbf{x} = \mathbf{0}$ . Then  $U$  and  $LU$  have the same nullspace.
- 20** Column 5 is sure to have no pivot since it is a combination of earlier columns. With 4 pivots in the other columns, the special solution is  $\mathbf{s} = (1, 0, 1, 0, 1)$ . The nullspace contains all multiples of this vector  $\mathbf{s}$  (a line in  $\mathbf{R}^5$ ).
- 21** For special solutions  $(2, 2, 1, 0)$  and  $(3, 1, 0, 1)$  with free variables  $x_3, x_4$ :  $R = \begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & -2 & -1 \end{bmatrix}$  and  $A$  can be any invertible 2 by 2 matrix times this  $R$ .
- 22** The nullspace of  $A = \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$  is the line through  $(4, 3, 2, 1)$ .
- 23**  $A = \begin{bmatrix} 1 & 0 & -1/2 \\ 1 & 3 & -2 \\ 5 & 1 & -3 \end{bmatrix}$  has  $(1, 1, 5)$  and  $(0, 3, 1)$  in  $C(A)$  and  $(1, 1, 2)$  in  $N(A)$ . Which other  $A$ 's?
- 24** This construction is impossible: 2 pivot columns and 2 free variables, only 3 columns.
- 25**  $A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$  has  $(1, 1, 1)$  in  $C(A)$  and only the line  $(c, c, c, c)$  in  $N(A)$ .
- 26**  $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$  has  $N(A) = C(A)$  and also (a)(b)(c) are all false. Notice  $\text{rref}(A^T) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ .
- 30**
- 27** If nullspace = column space (with  $r$  pivots) then  $n - r = r$ . If  $n = 3$  then  $3 = 2r$  is impossible.
- 28** If  $A$  times every column of  $B$  is zero, the column space of  $B$  is contained in the nullspace of  $A$ . An example is  $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$  and  $B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ . Here  $C(B)$  equals  $N(A)$ . (For  $B = 0$ ,  $C(B)$  is smaller.)
- 29** For  $A =$  random 3 by 3 matrix,  $R$  is almost sure to be  $I$ . For 4 by 3,  $R$  is most likely to be  $I$  with fourth row of zeros. What about a random 3 by 4 matrix?
- 31** If  $N(A) =$  line through  $\mathbf{x} = (2, 1, 0, 1)$ ,  $A$  has *three pivots* (4 columns and 1 special solution). Its reduced echelon form can be  $R = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$  (add any zero rows).

- 32** Any zero rows come after these rows:  $R = [1 \ -2 \ -3]$ ,  $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ ,  $R = I$ .
- 33** (a)  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ,  $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ ,  $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ ,  $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ ,  $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$  (b) All 8 matrices are  $R$ 's!
- 34** One reason that  $R$  is the same for  $A$  and  $-A$ : They have the same nullspace. They also have the same column space, but that is not required for two matrices to share the same  $R$ . ( $R$  tells us the nullspace and row space.)
- 35** The nullspace of  $B = [A \ A]$  contains all vectors  $x = \begin{bmatrix} y \\ -y \end{bmatrix}$  for  $y$  in  $\mathbf{R}^4$ .
- 36** If  $Cx = 0$  then  $Ax = 0$  and  $Bx = 0$ . So  $N(C) = N(A) \cap N(B) = \text{intersection}$ .
- 37** Currents:  $y_1 - y_3 + y_4 = -y_1 + y_2 + y_5 = -y_2 + y_4 + y_6 = -y_4 - y_5 - y_6 = 0$ . These equations add to  $0 = 0$ . Free variables  $y_3, y_5, y_6$ : watch for flows around loops.

### Problem Set 3.3, page 151

- 1** (a) and (c) are correct; (b) is completely false; (d) is false because  $R$  might have 1's in nonpivot columns.

**2**  $A = \begin{bmatrix} 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \end{bmatrix}$  has  $R = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ . The rank is  $r = 1$ ;

$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{bmatrix}$  has  $R = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ . The rank is  $r = 2$ ;

$A = \begin{bmatrix} -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 \end{bmatrix}$  has  $R = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ . The rank is  $r = 1$

**3**  $R_A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$   $R_B = [R_A \ R_A]$   $R_C \longrightarrow \begin{bmatrix} R_A & 0 \\ 0 & R_A \end{bmatrix} \longrightarrow$  Zero rows go to the bottom

**4** If all pivot variables come last then  $R = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}$ . The nullspace matrix is  $N = \begin{bmatrix} I \\ 0 \end{bmatrix}$ .

- 5** I think  $R_1 = A_1, R_2 = A_2$  is true. But  $R_1 - R_2$  may have  $-1$ 's in some pivots.

- 6**  $A$  and  $A^T$  have the same rank  $r = \text{number of pivots}$ . But *pivcol* (the column number)

is 2 for this matrix  $A$  and 1 for  $A^T$ :  $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ .

- 7** Special solutions in  $N = [-2 \ -4 \ 1 \ 0; -3 \ -5 \ 0 \ 1]$  and  $[1 \ 0 \ 0; 0 \ -2 \ 1]$ .

**8** The new entries keep rank 1:  $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 4 & 8 & 16 \end{bmatrix}$ ,  $B = \begin{bmatrix} 2 & 6 & -3 \\ 1 & 3 & -3/2 \\ 2 & 6 & -3 \end{bmatrix}$ ,

$M = \begin{bmatrix} a & b \\ c & bc/a \end{bmatrix}$ .

- 9 If  $A$  has rank 1, the column space is a *line* in  $\mathbf{R}^m$ . The nullspace is a *plane* in  $\mathbf{R}^n$  (given by one equation). The nullspace matrix  $N$  is  $n$  by  $n - 1$  (with  $n - 1$  special solutions in its columns). The column space of  $A^T$  is a *line* in  $\mathbf{R}^n$ .

$$10 \begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ 4 & 8 & 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 2 & 6 & 4 \\ -1 & -1 & -3 & -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 & 2 \end{bmatrix}$$

- 11 A rank one matrix has one pivot. (That pivot is in row 1 after possible row exchange; it could come in any column.) The second row of  $U$  is zero.

$$12 \text{ Invertible } r \text{ by } r \text{ submatrices} \quad S = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} \text{ and } S = [1] \text{ and } S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Use pivot rows and columns

- 13  $P$  has rank  $r$  (the same as  $A$ ) because elimination produces the same pivot columns.

- 14 The rank of  $R^T$  is also  $r$ . The example matrix  $A$  has rank 2 with invertible  $S$ :

$$P = \begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 2 & 7 \end{bmatrix} \quad P^T = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 6 & 7 \end{bmatrix} \quad S^T = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} \quad S = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}.$$

- 15 The product of rank one matrices has rank one or zero. These particular matrices have  $\text{rank}(AB) = 1$ ;  $\text{rank}(AM) = 1$  except  $AM = 0$  if  $c = -1/2$ .

- 16  $(uv^T)(wz^T) = u(v^T w)z^T$  has rank one unless the inner product is  $v^T w = 0$ .

- 17 (a) By matrix multiplication, each column of  $AB$  is  $A$  times the corresponding column of  $B$ . So if column  $j$  of  $B$  is a combination of earlier columns, then column  $j$  of  $AB$  is the same combination of earlier columns of  $AB$ . Then  $\text{rank}(AB) \leq \text{rank}(B)$ . No new pivot columns! (b) The rank of  $B$  is  $r = 1$ . Multiplying by  $A$  cannot increase this rank. The rank of  $AB$  stays the same for  $A_1 = I$  and  $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ . It drops to zero for  $A_2 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ .

- 18 If we know that  $\text{rank}(B^T A^T) \leq \text{rank}(A^T)$ , then since rank stays the same for transposes, (apologies that this fact is not yet proved), we have  $\text{rank}(AB) \leq \text{rank}(A)$ .

- 19 We are given  $AB = I$  which has rank  $n$ . Then  $\text{rank}(AB) \leq \text{rank}(A)$  forces  $\text{rank}(A) = n$ . This means that  $A$  is invertible. The right-inverse  $B$  is also a left-inverse:  $BA = I$  and  $B = A^{-1}$ .

- 20 Certainly  $A$  and  $B$  have at most rank 2. Then their product  $AB$  has at most rank 2. Since  $BA$  is 3 by 3, it cannot be  $I$  even if  $AB = I$ .

- 21 (a)  $A$  and  $B$  will both have the same nullspace and row space as the  $R$  they share.  
(b)  $A$  equals an *invertible* matrix times  $B$ , when they share the same  $R$ . A key fact!

$$22 A = (\text{pivot columns})(\text{nonzero rows of } R) = \begin{bmatrix} 1 & 0 \\ 1 & 4 \\ 1 & 8 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} +$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 8 \end{bmatrix}. \quad B = \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{matrix} \text{columns} \\ \text{times rows} \end{matrix} = \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix}$$



- 23** If  $c = 1$ ,  $R = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$  has  $x_2, x_3, x_4$  free. If  $c \neq 1$ ,  $R = \begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$  has  $x_3, x_4$  free. Special solutions in  $N = \begin{bmatrix} -1 & -2 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$  (for  $c = 1$ ) and  $N = \begin{bmatrix} -2 & -2 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$  (for  $c \neq 1$ ). If  $c = 1$ ,  $R = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$  and  $x_1$  free; if  $c = 2$ ,  $R = \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$  and  $x_2$  free;  $R = I$  if  $c \neq 1, 2$ . Special solutions in  $N = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$  ( $c = 1$ ) or  $N = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$  ( $c = 2$ ) or  $N = 2$  by 0 empty matrix.
- 24**  $A = \begin{bmatrix} I & I \end{bmatrix}$  has  $N = \begin{bmatrix} I \\ -I \end{bmatrix}$ ;  $B = \begin{bmatrix} I & I \\ 0 & 0 \end{bmatrix}$  has the same  $N$ ;  $C = \begin{bmatrix} I & I & I \end{bmatrix}$  has  $N = \begin{bmatrix} -I & -I \\ I & 0 \\ 0 & I \end{bmatrix}$ .
- 25**  $A = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 2 & 5 \\ 1 & 3 & 2 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & 1 \end{bmatrix}$  (pivot columns) times  $R$ .
- 26** The  $m$  by  $n$  matrix  $Z$  has  $r$  ones to start its main diagonal. Otherwise  $Z$  is all zeros.
- 27**  $R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} r \text{ by } r & r \text{ by } n-r \\ m-r \text{ by } r & m-r \text{ by } n-r \end{bmatrix}$ ;  $\text{rref}(R^T) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ ;  $\text{rref}(R^T R) = \text{same } R$
- 28** The row-column reduced echelon form is always  $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ ;  $I$  is  $r$  by  $r$ .

### Problem Set 3.4, page 163

- 1**  $\begin{bmatrix} 2 & 4 & 6 & 4 & \mathbf{b}_1 \\ 2 & 5 & 7 & 6 & \mathbf{b}_2 \\ 2 & 3 & 5 & 2 & \mathbf{b}_3 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 4 & 6 & 4 & \mathbf{b}_1 \\ 0 & 1 & 1 & 2 & \mathbf{b}_2 - \mathbf{b}_1 \\ 0 & -1 & -1 & -2 & \mathbf{b}_3 - \mathbf{b}_1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 4 & 6 & 4 & \mathbf{b}_1 \\ 0 & 1 & 1 & 2 & \mathbf{b}_2 - \mathbf{b}_1 \\ 0 & 0 & 0 & 0 & \mathbf{b}_3 + \mathbf{b}_2 - 2\mathbf{b}_1 \end{bmatrix}$   
 $A\mathbf{x} = \mathbf{b}$  has a solution when  $b_3 + b_2 - 2b_1 = 0$ ; the column space contains all combinations of  $(2, 2, 2)$  and  $(4, 5, 3)$ . **This is the plane**  $b_3 + b_2 - 2b_1 = 0$  (!). The nullspace contains all combinations of  $s_1 = (-1, -1, 1, 0)$  and  $s_2 = (2, -2, 0, 1)$ ;  $x_{\text{complete}} = x_p + c_1 s_1 + c_2 s_2$ ;

$$\begin{bmatrix} R & d \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & -2 & 4 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ gives the particular solution } x_p = (4, -1, 0, 0).$$

$$2 \quad \begin{bmatrix} 2 & 1 & 3 & \mathbf{b}_1 \\ 6 & 3 & 9 & \mathbf{b}_2 \\ 4 & 2 & 6 & \mathbf{b}_3 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 3 & \mathbf{b}_1 \\ 0 & 0 & 0 & \mathbf{b}_2 - 3\mathbf{b}_1 \\ 0 & 0 & 0 & \mathbf{b}_3 - 2\mathbf{b}_1 \end{bmatrix} \quad \text{Then } [R \quad \mathbf{d}] = \begin{bmatrix} 1 & 1/2 & 3/2 & \mathbf{5} \\ 0 & 0 & 0 & \mathbf{0} \\ 0 & 0 & 0 & \mathbf{0} \end{bmatrix}$$

$A\mathbf{x} = \mathbf{b}$  has a solution when  $b_2 - 3b_1 = 0$  and  $b_3 - 2b_1 = 0$ ;  $C(A)$  = line through  $(2, 6, 4)$  which is the intersection of the planes  $b_2 - 3b_1 = 0$  and  $b_3 - 2b_1 = 0$ ; the nullspace contains all combinations of  $\mathbf{s}_1 = (-1/2, 1, 0)$  and  $\mathbf{s}_2 = (-3/2, 0, 1)$ ; particular solution  $\mathbf{x}_p = \mathbf{d} = (5, 0, 0)$  and complete solution  $\mathbf{x}_p + c_1\mathbf{s}_1 + c_2\mathbf{s}_2$ .

$$3 \quad \mathbf{x}_{\text{complete}} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}. \quad \text{The matrix is singular but the equations are still solvable; } \mathbf{b} \text{ is in the column space. Our particular solution has free variable } y = 0.$$

$$4 \quad \mathbf{x}_{\text{complete}} = \mathbf{x}_p + \mathbf{x}_n = \left(\frac{1}{2}, 0, \frac{1}{2}, 0\right) + x_2(-3, 1, 0, 0) + x_4(0, 0, -2, 1).$$

$$5 \quad \begin{bmatrix} 1 & 2 & -2 & b_1 \\ 2 & 5 & -4 & b_2 \\ 4 & 9 & -8 & b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -2 & b_1 \\ 0 & 1 & 0 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 - 2b_1 - b_2 \end{bmatrix} \quad \text{solvable if } b_3 - 2b_1 - b_2 = 0.$$

Back-substitution gives the particular solution to  $A\mathbf{x} = \mathbf{b}$  and the special solution to

$$A\mathbf{x} = \mathbf{0}: \mathbf{x} = \begin{bmatrix} 5b_1 - 2b_2 \\ b_2 - 2b_1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

$$6 \quad (a) \text{ Solvable if } b_2 = 2b_1 \text{ and } 3b_1 - 3b_3 + b_4 = 0. \text{ Then } \mathbf{x} = \begin{bmatrix} 5b_1 - 2b_3 \\ b_3 - 2b_1 \end{bmatrix} = \mathbf{x}_p$$

$$(b) \text{ Solvable if } b_2 = 2b_1 \text{ and } 3b_1 - 3b_3 + b_4 = 0. \mathbf{x} = \begin{bmatrix} 5b_1 - 2b_3 \\ b_3 - 2b_1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.$$

$$7 \quad \begin{bmatrix} 1 & 3 & 1 & b_1 \\ 3 & 8 & 2 & b_2 \\ 2 & 4 & 0 & b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 1 & b_2 \\ 0 & -1 & -1 & b_2 - 3b_1 \\ 0 & -2 & -2 & b_3 - 2b_1 \end{bmatrix} \quad \text{One more step gives } [0 \ 0 \ 0 \ 0] = \text{row } 3 - 2(\text{row } 2) + 4(\text{row } 1) \text{ provided } \mathbf{b}_3 - 2\mathbf{b}_2 + 4\mathbf{b}_1 = \mathbf{0}.$$

8 (a) Every  $\mathbf{b}$  is in  $C(A)$ : independent rows, only the zero combination gives  $\mathbf{0}$ .

(b) We need  $b_3 = 2b_2$ , because  $(\text{row } 3) - 2(\text{row } 2) = \mathbf{0}$ .

$$9 \quad L[U \quad \mathbf{c}] = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 5 & b_1 \\ 0 & 0 & 2 & 2 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 + b_2 - 5b_1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 5 & b_1 \\ 2 & 4 & 8 & 12 & b_2 \\ 3 & 6 & 7 & 13 & b_3 \end{bmatrix} \\ = [A \quad \mathbf{b}]; \text{ particular } \mathbf{x}_p = (-9, 0, 3, 0) \text{ means } -9(1, 2, 3) + 3(3, 8, 7) = (0, 6, -6). \\ \text{This is } A\mathbf{x}_p = \mathbf{b}.$$

$$10 \quad \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \text{ has } \mathbf{x}_p = (2, 4, 0) \text{ and } \mathbf{x}_{\text{null}} = (c, c, c).$$

11 A 1 by 3 system has at least **two** free variables. But  $\mathbf{x}_{\text{null}}$  in Problem 10 only has **one**.

12 (a)  $\mathbf{x}_1 - \mathbf{x}_2$  and  $\mathbf{0}$  solve  $A\mathbf{x} = \mathbf{0}$  (b)  $A(2\mathbf{x}_1 - 2\mathbf{x}_2) = \mathbf{0}$ ,  $A(2\mathbf{x}_1 - \mathbf{x}_2) = \mathbf{b}$

13 (a) The particular solution  $\mathbf{x}_p$  is always multiplied by 1 (b) Any solution can be  $\mathbf{x}_p$

$$(c) \quad \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}. \text{ Then } \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ is shorter (length } \sqrt{2}) \text{ than } \begin{bmatrix} 2 \\ 0 \end{bmatrix} \text{ (length } 2)$$

(d) The only "homogeneous" solution in the nullspace is  $\mathbf{x}_n = \mathbf{0}$  when  $A$  is invertible.

- 14 If column 5 has no pivot,  $x_5$  is a *free* variable. The zero vector is *not* the only solution to  $A\mathbf{x} = \mathbf{0}$ . If this system  $A\mathbf{x} = \mathbf{b}$  has a solution, it has *infinitely many* solutions.
- 15 If row 3 of  $U$  has no pivot, that is a *zero row*.  $U\mathbf{x} = \mathbf{c}$  is only solvable provided  $c_3 = 0$ .  $A\mathbf{x} = \mathbf{b}$  *might not be solvable*, because  $U$  may have other zero rows needing more  $c_i = 0$ .
- 16 The largest rank is 3. Then there is a pivot in every *row*. The solution *always exists*. The column space is  $\mathbf{R}^3$ . An example is  $A = [I \ F]$  for any 3 by 2 matrix  $F$ .
- 17 The largest rank of a 6 by 4 matrix is 4. Then there is a pivot in every *column*. The solution is *unique*. The nullspace contains only the zero *vector*. An example is  $A = R = [I \ F]$  for any 4 by 2 matrix  $F$ .
- 18 Rank = 2; rank = 3 unless  $q = 2$  (then rank = 2). Transpose has the same rank!
- 19 Both matrices  $A$  have rank 2. Always  $A^T A$  and  $A A^T$  have **the same rank** as  $A$ .
- 20  $A = LU = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{bmatrix}; A = LU \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & -2 & 3 \\ 0 & 0 & 11 & -5 \end{bmatrix}$ .
- 21 (a)  $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$  (b)  $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ . The second equation in part (b) removed one special solution.
- 22 If  $A\mathbf{x}_1 = \mathbf{b}$  and also  $A\mathbf{x}_2 = \mathbf{b}$  then we can add  $\mathbf{x}_1 - \mathbf{x}_2$  to any solution of  $A\mathbf{x} = \mathbf{B}$ : the solution  $\mathbf{x}$  is not unique. But there will be **no solution** to  $A\mathbf{x} = \mathbf{B}$  if  $\mathbf{B}$  is not in the column space.
- 23 For  $A$ ,  $q = 3$  gives rank 1, every other  $q$  gives rank 2. For  $B$ ,  $q = 6$  gives rank 1, every other  $q$  gives rank 2. These matrices cannot have rank 3.
- 24 (a)  $\begin{bmatrix} 1 \\ 1 \end{bmatrix} [x] = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$  has 0 or 1 solutions, depending on  $\mathbf{b}$  (b)  $\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [b]$  has infinitely many solutions for every  $b$  (c) There are 0 or  $\infty$  solutions when  $A$  has rank  $r < m$  and  $r < n$ : the simplest example is a zero matrix. (d) *one* solution for all  $\mathbf{b}$  when  $A$  is square and invertible (like  $A = I$ ).
- 25 (a)  $r < m$ , always  $r \leq n$  (b)  $r = m$ ,  $r < n$  (c)  $r < m$ ,  $r = n$  (d)  $r = m = n$ .
- 26  $\begin{bmatrix} 2 & 4 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow R = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$  and  $\begin{bmatrix} 2 & 4 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{bmatrix} \rightarrow R = I$ .
- 27 If  $U$  has  $n$  pivots, then  $R$  has  $n$  pivots **equal to 1**. Zeros above and below those pivots make  $R = I$ .
- 28  $\begin{bmatrix} 1 & 2 & 3 & \mathbf{0} \\ 0 & 0 & 4 & \mathbf{0} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & \mathbf{0} \\ 0 & 0 & 1 & \mathbf{0} \end{bmatrix}; \mathbf{x}_n = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}; \begin{bmatrix} 1 & 2 & 3 & \mathbf{5} \\ 0 & 0 & 4 & \mathbf{8} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & -\mathbf{1} \\ 0 & 0 & 1 & \mathbf{2} \end{bmatrix}$ .  
Free  $x_2 = 0$  gives  $\mathbf{x}_p = (-1, 0, 2)$  because the pivot columns contain  $I$ .
- 29  $[R \ \mathbf{d}] = \begin{bmatrix} 1 & 0 & 0 & \mathbf{0} \\ 0 & 0 & 1 & \mathbf{0} \\ 0 & 0 & 0 & \mathbf{0} \end{bmatrix}$  leads to  $\mathbf{x}_n = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; [R \ \mathbf{d}] = \begin{bmatrix} 1 & 0 & 0 & -\mathbf{1} \\ 0 & 0 & 1 & \mathbf{2} \\ 0 & 0 & 0 & \mathbf{5} \end{bmatrix}$ :  
no solution because of the 3rd equation

$$30 \quad \begin{bmatrix} 1 & 0 & 2 & 3 & 2 \\ 1 & 3 & 2 & 0 & 5 \\ 2 & 0 & 4 & 9 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & 3 & 2 \\ 0 & 3 & 0 & -3 & 3 \\ 0 & 0 & 0 & 3 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & 0 & -4 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}; \begin{bmatrix} -4 \\ 3 \\ 0 \\ 2 \end{bmatrix}; x_n = x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

31 For  $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 3 \end{bmatrix}$ , the only solution to  $A\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$  is  $\mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ .  $B$  cannot exist since 2 equations in 3 unknowns cannot have a unique solution.

32  $A = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 5 \end{bmatrix}$  factors into  $LU = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 2 & 2 & 1 & \\ 1 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$  and the rank is  $r = 2$ . The special solution to  $A\mathbf{x} = \mathbf{0}$  and  $U\mathbf{x} = \mathbf{0}$  is  $\mathbf{s} = (-7, 2, 1)$ . Since  $\mathbf{b} = (1, 3, 6, 5)$  is also the last column of  $A$ , a particular solution to  $A\mathbf{x} = \mathbf{b}$  is  $(0, 0, 1)$  and the complete solution is  $\mathbf{x} = (0, 0, 1) + c\mathbf{s}$ . (Or use the particular solution  $\mathbf{x}_p = (7, -2, 0)$  with free variable  $x_3 = 0$ .)

For  $\mathbf{b} = (1, 0, 0, 0)$  elimination leads to  $U\mathbf{x} = (1, -1, 0, 1)$  and the fourth equation is  $0 = 1$ . No solution for this  $\mathbf{b}$ .

33 If the complete solution to  $A\mathbf{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$  is  $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ c \end{bmatrix}$  then  $A = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$ .

34 (a) If  $\mathbf{s} = (2, 3, 1, 0)$  is the only special solution to  $A\mathbf{x} = \mathbf{0}$ , the complete solution is  $\mathbf{x} = c\mathbf{s}$  (line of solution!). The rank of  $A$  must be  $4 - 1 = 3$ .

(b) The fourth variable  $x_4$  is *not free* in  $\mathbf{s}$ , and  $R$  must be  $\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ .

(c)  $A\mathbf{x} = \mathbf{b}$  can be solve for all  $\mathbf{b}$ , because  $A$  and  $R$  have *full row rank*  $r = 3$ .

35 For the  $-1, 2, -1$  matrix  $K$  (9 by 9) and constant right side  $\mathbf{b} = (10, \dots, 10)$ , the solution  $\mathbf{x} = K^{-1}\mathbf{b} = (45, 80, 105, 120, 125, 120, 105, 80, 45)$  rises and falls along the parabola  $x_i = 50i - 5i^2$ . (A formula for  $K^{-1}$  is later in the text.)

36 If  $A\mathbf{x} = \mathbf{b}$  and  $C\mathbf{x} = \mathbf{b}$  have the same solutions,  $A$  and  $C$  have the same shape and the same nullspace (take  $\mathbf{b} = \mathbf{0}$ ). If  $\mathbf{b} =$  column 1 of  $A$ ,  $\mathbf{x} = (1, 0, \dots, 0)$  solves  $A\mathbf{x} = \mathbf{b}$  so it solves  $C\mathbf{x} = \mathbf{b}$ . Then  $A$  and  $C$  share column 1. Other columns too:  $A = C$ !

## Problem Set 3.5, page 178

1  $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \mathbf{0}$  gives  $c_3 = c_2 = c_1 = 0$ . So those 3 column vectors are

independent. But  $\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix} [\mathbf{c}] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$  is solved by  $\mathbf{c} = (1, 1, -4, 1)$ . Then

$\mathbf{v}_1 + \mathbf{v}_2 - 4\mathbf{v}_3 + \mathbf{v}_4 = \mathbf{0}$  (dependent).

2  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  are independent (the  $-1$ 's are in different positions). All six vectors are on the plane  $(1, 1, 1, 1) \cdot \mathbf{v} = 0$  so no four of these six vectors can be independent.

- 3 If  $a = 0$  then column 1 =  $\mathbf{0}$ ; if  $d = 0$  then  $b(\text{column 1}) - a(\text{column 2}) = \mathbf{0}$ ; if  $f = 0$  then all columns end in zero (they are all in the  $xy$  plane, they must be dependent).
- 4  $U\mathbf{x} = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$  gives  $z = 0$  then  $y = 0$  then  $x = 0$ . A square triangular matrix has independent columns (invertible matrix) when its diagonal has no zeros.
- 5 (a)  $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & -1 & -5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & 0 & -18/5 \end{bmatrix}$ : invertible  $\Rightarrow$  independent columns.
- (b)  $\begin{bmatrix} 1 & 2 & -3 \\ -3 & 1 & 2 \\ 2 & -3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -3 \\ 0 & 7 & -7 \\ 0 & -7 & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -3 \\ 0 & 7 & -7 \\ 0 & 0 & 0 \end{bmatrix}$ ;  $A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ , columns add to  $\mathbf{0}$ .
- 6 Columns 1, 2, 4 are independent. Also 1, 3, 4 and 2, 3, 4 and others (but not 1, 2, 3). Same column numbers (not same columns!) for  $A$ .
- 7 The sum  $\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$  because  $(\mathbf{w}_2 - \mathbf{w}_3) - (\mathbf{w}_1 - \mathbf{w}_3) + (\mathbf{w}_1 - \mathbf{w}_2) = \mathbf{0}$ . So the difference are *dependent* and the difference matrix is singular:  $A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$ .
- 8 If  $c_1(\mathbf{w}_2 + \mathbf{w}_3) + c_2(\mathbf{w}_1 + \mathbf{w}_3) + c_3(\mathbf{w}_1 + \mathbf{w}_2) = \mathbf{0}$  then  $(c_2 + c_3)\mathbf{w}_1 + (c_1 + c_3)\mathbf{w}_2 + (c_1 + c_2)\mathbf{w}_3 = \mathbf{0}$ . Since the  $\mathbf{w}$ 's are independent,  $c_2 + c_3 = c_1 + c_3 = c_1 + c_2 = 0$ . The only solution is  $c_1 = c_2 = c_3 = 0$ . Only this combination of  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  gives  $\mathbf{0}$ .
- 9 (a) The four vectors in  $\mathbf{R}^3$  are the columns of a 3 by 4 matrix  $A$ . There is a nonzero solution to  $A\mathbf{x} = \mathbf{0}$  because there is at least one free variable (b) Two vectors are dependent if  $[\mathbf{v}_1 \ \mathbf{v}_2]$  has rank 0 or 1. (OK to say "they are on the same line" or "one is a multiple of the other" but *not* " $\mathbf{v}_2$  is a multiple of  $\mathbf{v}_1$ "—since  $\mathbf{v}_1$  might be  $\mathbf{0}$ .) (c) A nontrivial combination of  $\mathbf{v}_1$  and  $\mathbf{0}$  gives  $\mathbf{0}$ :  $0\mathbf{v}_1 + 3(0, 0, 0) = \mathbf{0}$ .
- 10 The plane is the nullspace of  $A = [1 \ 2 \ -3 \ -1]$ . Three free variables give three solutions  $(x, y, z, t) = (2, -1, 0, 0)$  and  $(3, 0, 1, 0)$  and  $(1, 0, 0, 1)$ . Combinations of those special solutions give more solutions (all solutions).
- 11 (a) Line in  $\mathbf{R}^3$  (b) Plane in  $\mathbf{R}^3$  (c) All of  $\mathbf{R}^3$  (d) All of  $\mathbf{R}^3$ .
- 12  $\mathbf{b}$  is in the column space when  $A\mathbf{x} = \mathbf{b}$  has a solution;  $\mathbf{c}$  is in the row space when  $A^T\mathbf{y} = \mathbf{c}$  has a solution. *False*. The zero vector is always in the row space.
- 13 The column space and row space of  $A$  and  $U$  all have the same dimension = 2. *The row spaces of  $A$  and  $U$  are the same*, because the rows of  $U$  are combinations of the rows of  $A$  (and vice versa!).
- 14  $\mathbf{v} = \frac{1}{2}(\mathbf{v} + \mathbf{w}) + \frac{1}{2}(\mathbf{v} - \mathbf{w})$  and  $\mathbf{w} = \frac{1}{2}(\mathbf{v} + \mathbf{w}) - \frac{1}{2}(\mathbf{v} - \mathbf{w})$ . The two pairs *span* the same space. They are a basis when  $\mathbf{v}$  and  $\mathbf{w}$  are *independent*.
- 15 The  $n$  independent vectors span a space of dimension  $n$ . They are a *basis* for that space. If they are the columns of  $A$  then  $m$  is *not less* than  $n$  ( $m \geq n$ ).

- 16** These bases are not unique! (a)  $(1, 1, 1, 1)$  for the space of all constant vectors  $(c, c, c, c)$  (b)  $(1, -1, 0, 0), (1, 0, -1, 0), (1, 0, 0, -1)$  for the space of vectors with sum of components = 0 (c)  $(1, -1, -1, 0), (1, -1, 0, -1)$  for the space perpendicular to  $(1, 1, 0, 0)$  and  $(1, 0, 1, 1)$  (d) The columns of  $I$  are a basis for its column space, the empty set is a basis (by convention) for  $N(I) = \{\text{zero vector}\}$ .
- 17** The column space of  $U = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$  is  $\mathbf{R}^2$  so take any bases for  $\mathbf{R}^2$ ; (row 1 and row 2) or (row 1 and row 1 + row 2) and (row 1 and - row 2) are bases for the row spaces of  $U$ .
- 18** (a) The 6 vectors *might not* span  $\mathbf{R}^4$  (b) The 6 vectors *are not* independent (c) Any four *might be* a basis.
- 19**  $n$ -independent columns  $\Rightarrow$  rank  $n$ . Columns span  $\mathbf{R}^m \Rightarrow$  rank  $m$ . Columns are basis for  $\mathbf{R}^m \Rightarrow \text{rank} = m = n$ . The rank counts the number of *independent* columns.
- 20** One basis is  $(2, 1, 0), (-3, 0, 1)$ . A basis for the intersection with the  $xy$  plane is  $(2, 1, 0)$ . The normal vector  $(1, -2, 3)$  is a basis for the line perpendicular to the plane.
- 21** (a) The only solution to  $A\mathbf{x} = \mathbf{0}$  is  $\mathbf{x} = \mathbf{0}$  because *the columns are independent* (b)  $A\mathbf{x} = \mathbf{b}$  is solvable because *the columns span  $\mathbf{R}^5$* . Key point:  $A$  basis gives exactly one solution for every  $\mathbf{b}$ .
- 22** (a) True (b) False because the basis vectors for  $\mathbf{R}^6$  might not be in  $\mathbf{S}$ .
- 23** Columns 1 and 2 are bases for the (**different**) column spaces of  $A$  and  $U$ ; rows 1 and 2 are bases for the (**equal**) row spaces of  $A$  and  $U$ ;  $(1, -1, 1)$  is a basis for the (**equal**) nullspaces.
- 24** (a) *False*  $A = \begin{bmatrix} 1 & 1 \end{bmatrix}$  has dependent columns, independent row (b) *False* column space  $\neq$  row space for  $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$  (c) *True*: Both dimensions = 2 if  $A$  is invertible, dimensions = 0 if  $A = 0$ , otherwise dimensions = 1 (d) *False*, columns may be dependent, in that case not a basis for  $\mathcal{C}(A)$ .
- 25**  $A$  has rank 2 if  $c = 0$  and  $d = 2$ ;  $B = \begin{bmatrix} c & d \\ d & c \end{bmatrix}$  has rank 2 except when  $c = d$  or  $c = -d$ .
- 26** (a)  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$   
 (b) Add  $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$   
 (c)  $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$
- These are simple bases (among many others) for (a) diagonal matrices (b) symmetric matrices (c) skew-symmetric matrices. The dimensions are 3, 6, 3.

- 27  $I, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ ; echelon matrices do *not* form a subspace; they *span* the upper triangular matrices (not every  $U$  is echelon).
- 28  $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}; \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$  and  $\begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}$ .
- 29 (a) The invertible matrices span the space of all 3 by 3 matrices (b) The rank one matrices also span the space of all 3 by 3 matrices (c)  $I$  by itself spans the space of all multiples  $cI$ .
- 30  $\begin{bmatrix} -1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ -1 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 2 \end{bmatrix}$ .
- 31 (a)  $y(x) = \text{constant } C$  (b)  $y(x) = 3x$  this is one basis for the 2 by 3 matrices with  $(2, 1, 1)$  in their nullspace (4-dim subspace). (c)  $y(x) = 3x + C = y_p + y_n$  solves  $dy/dx = 3$ .
- 32  $y(0) = 0$  requires  $A + B + C = 0$ . One basis is  $\cos x - \cos 2x$  and  $\cos x - \cos 3x$ .
- 33 (a)  $y(x) = e^{2x}$  is a basis for, all solutions to  $y' = 2y$  (b)  $y = x$  is a basis for all solutions to  $dy/dx = y/x$  (First-order linear equation  $\Rightarrow$  1 basis function in solution space).
- 34  $y_1(x), y_2(x), y_3(x)$  can be  $x, 2x, 3x$  (dim 1) or  $x, 2x, x^2$  (dim 2) or  $x, x^2, x^3$  (dim 3).
- 35 Basis  $1, x, x^2, x^3$ , for cubic polynomials; basis  $x - 1, x^2 - 1, x^3 - 1$  for the subspace with  $p(1) = 0$ .
- 36 Basis for  $\mathbf{S}$ :  $(1, 0, -1, 0), (0, 1, 0, 0), (1, 0, 0, -1)$ ; basis for  $\mathbf{T}$ :  $(1, -1, 0, 0)$  and  $(0, 0, 2, 1)$ ;  $\mathbf{S} \cap \mathbf{T} =$  multiples of  $(3, -3, 2, 1) =$  nullspace for 3 equation in  $\mathbf{R}^4$  has dimension 1.
- 37 The subspace of matrices that have  $AS = SA$  has dimension *three*.
- 38 (a) No, 2 vectors don't span  $\mathbf{R}^3$  (b) No, 4 vectors in  $\mathbf{R}^3$  are dependent (c) Yes, a basis (d) No, these three vectors are dependent
- 39 If the 5 by 5 matrix  $[A \ b]$  is invertible,  $b$  is not a combination of the columns of  $A$ . If  $[A \ b]$  is singular, and the 4 columns of  $A$  are independent,  $b$  is a combination of those columns. In this case  $Ax = b$  has a solution.
- 40 (a) The functions  $y = \sin x, y = \cos x, y = e^x, y = e^{-x}$  are a basis for solutions to  $d^4y/dx^4 = y(x)$ .  
(b) A particular solution to  $d^4y/dx^4 = y(x) + 1$  is  $y(x) = -1$ . The complete solution is  $y(x) = -1 + c_1 \sin x + c_2 \cos x + c_3 e^x + c_4 e^{-x}$  (or use another basis for the nullspace of the 4th derivative).
- 41  $I = \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix} - \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix} + \begin{bmatrix} & & 1 \\ 1 & 1 & \\ & & 1 \end{bmatrix} + \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} - \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$ . The six  $P$ 's are dependent.  
Those five are independent: The 4th has  $P_{11} = 1$  and cannot be a combination of the others. Then the 2nd cannot be (from  $P_{32} = 1$ ) and also 5th ( $P_{32} = 1$ ). Continuing, a nonzero combination of all five could not be zero. Further challenge: How many independent 4 by 4 permutation matrices?

- 42 The dimension of  $\mathcal{S}$  spanned by all rearrangements of  $\mathbf{x}$  is (a) zero when  $\mathbf{x} = \mathbf{0}$  (b) one when  $\mathbf{x} = (1, 1, 1, 1)$  (c) three when  $\mathbf{x} = (1, 1, -1, -1)$  because all rearrangements of this  $\mathbf{x}$  are perpendicular to  $(1, 1, 1, 1)$  (d) four when the  $\mathbf{x}$ 's are not equal and don't add to zero. **No  $\mathbf{x}$  gives  $\dim \mathcal{S} = 2$ .** I owe this nice problem to Mike Artin—the answers are the same in higher dimensions:  $0, 1, n-1, n$ .
- 43 The problem is to show that the  $\mathbf{u}$ 's,  $\mathbf{v}$ 's,  $\mathbf{w}$ 's together are independent. We know the  $\mathbf{u}$ 's and  $\mathbf{v}$ 's together are a basis for  $V$ , and the  $\mathbf{u}$ 's and  $\mathbf{w}$ 's together are a basis for  $W$ . Suppose a combination of  $\mathbf{u}$ 's,  $\mathbf{v}$ 's,  $\mathbf{w}$ 's gives  $\mathbf{0}$ . **To be proved:** All coefficients = zero.  
*Key idea:* In that combination giving  $\mathbf{0}$ , the part  $\mathbf{x}$  from the  $\mathbf{u}$ 's and  $\mathbf{v}$ 's is in  $V$ . So the part from the  $\mathbf{w}$ 's is  $-\mathbf{x}$ . This part is now in  $V$  and also in  $W$ . But if  $-\mathbf{x}$  is in  $V \cap W$  it is a combination of  $\mathbf{u}$ 's only. Now the combination uses only  $\mathbf{u}$ 's and  $\mathbf{v}$ 's (independent in  $V$ !) so all coefficients of  $\mathbf{u}$ 's and  $\mathbf{v}$ 's must be zero. Then  $\mathbf{x} = \mathbf{0}$  and the coefficients of the  $\mathbf{w}$ 's are also zero.
- 44 The inputs to an  $m$  by  $n$  matrix fill  $\mathbf{R}^n$ . The outputs (column space!) have dimension  $r$ . The nullspace has  $n - r$  special solutions. The formula becomes  $r + (n - r) = n$ .
- 45 If the left side of  $\dim(V) + \dim(W) = \dim(V \cap W) + \dim(V + W)$  is greater than  $n$ , then  $\dim(V \cap W)$  must be greater than zero. So  $V \cap W$  contains nonzero vectors.
- 46 If  $A^2 = \text{zero matrix}$ , this says that each column of  $A$  is in the nullspace of  $A$ . If the column space has dimension  $r$ , the nullspace has dimension  $10 - r$ , and we must have  $r \leq 10 - r$  and  $r \leq 5$ .

### Problem Set 3.6, page 190

- 1 (a) Row and column space dimensions = 5, nullspace dimension = 4,  $\dim(N(A^T)) = 2$  sum =  $16 = m + n$  (b) Column space is  $\mathbf{R}^3$ ; left nullspace contains only  $\mathbf{0}$ .
- 2  $A$ : Row space basis = row 1 =  $(1, 2, 4)$ ; nullspace  $(-2, 1, 0)$  and  $(-4, 0, 1)$ ; column space basis = column 1 =  $(1, 2)$ ; left nullspace  $(-2, 1)$ .  $B$ : Row space basis = both rows =  $(1, 2, 4)$  and  $(2, 5, 8)$ ; column space basis = two columns =  $(1, 2)$  and  $(2, 5)$ ; nullspace  $(-4, 0, 1)$ ; left nullspace basis is empty because the space contains only  $\mathbf{y} = \mathbf{0}$ .
- 3 Row space basis = rows of  $U = (0, 1, 2, 3, 4)$  and  $(0, 0, 0, 1, 2)$ ; column space basis = pivot columns (of  $A$  not  $U$ ) =  $(1, 1, 0)$  and  $(3, 4, 1)$ ; nullspace basis  $(1, 0, 0, 0, 0)$ ,  $(0, 2, -1, 0, 0)$ ,  $(0, 2, 0, -2, 1)$ ; left nullspace  $(1, -1, 1) = \text{last row of } E^{-1}$ !
- 4 (a)  $\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$  (b) Impossible:  $r + (n - r)$  must be 3 (c)  $\begin{bmatrix} 1 & 1 \end{bmatrix}$  (d)  $\begin{bmatrix} -9 & -3 \\ 3 & 1 \end{bmatrix}$   
 (e) *Impossible* Row space = column space requires  $m = n$ . Then  $m - r = n - r$ ; nullspaces have the same dimension. Section 4.1 will prove  $N(A)$  and  $N(A^T)$  orthogonal to the row and column spaces respectively—here those are the same space.
- 5  $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$  has those rows spanning its row space  $B = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix}$  has the same rows spanning its nullspace and  $BA^T = 0$ .
- 6  $A$ : dim **2, 2, 2, 1**: Rows  $(0, 3, 3, 3)$  and  $(0, 1, 0, 1)$ ; columns  $(3, 0, 1)$  and  $(3, 0, 0)$ ; nullspace  $(1, 0, 0, 0)$  and  $(0, -1, 0, 1)$ ;  $N(A^T) (0, 1, 0)$ .  $B$ : dim **1, 1, 0, 2** Row space  $(1)$ , column space  $(1, 4, 5)$ , nullspace: empty basis,  $N(A^T) (-4, 1, 0)$  and  $(-5, 0, 1)$ .



- 7 Invertible 3 by 3 matrix  $A$ : row space basis = column space basis =  $(1, 0, 0)$ ,  $(0, 1, 0)$ ,  $(0, 0, 1)$ ; nullspace basis and left nullspace basis are *empty*. Matrix  $B = \begin{bmatrix} A & A \end{bmatrix}$ : row space basis  $(1, 0, 0, 1, 0, 0)$ ,  $(0, 1, 0, 0, 1, 0)$  and  $(0, 0, 1, 0, 0, 1)$ ; column space basis  $(1, 0, 0)$ ,  $(0, 1, 0)$ ,  $(0, 0, 1)$ ; nullspace basis  $(-1, 0, 0, 1, 0, 0)$  and  $(0, -1, 0, 0, 1, 0)$  and  $(0, 0, -1, 0, 0, 1)$ ; left nullspace basis is empty.
- 8  $\begin{bmatrix} I & 0 \end{bmatrix}$  and  $\begin{bmatrix} I & I & 0 & 0 \end{bmatrix}$  and  $\begin{bmatrix} 0 \end{bmatrix}$  = 3 by 2 have row space dimensions = 3, 3, 0 = column space dimensions; nullspace dimensions 2, 3, 2; left nullspace dimensions 0, 2, 3.
- 9 (a) Same row space and nullspace. So rank (dimension of row space) is the same  
(b) Same column space and left nullspace. Same rank (dimension of column space).
- 10 For **rand** (3), almost surely rank = 3, nullspace and left nullspace contain only  $(0, 0, 0)$ .  
For **rand** (3, 5) the rank is almost surely 3 and the dimension of the nullspace is 2.
- 11 (a) No solution means that  $r < m$ . Always  $r \leq n$ . Can't compare  $m$  and  $n$  here.  
(b) Since  $m - r > 0$ , the left nullspace must contain a nonzero vector.
- 12 A neat choice is  $\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 4 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ ;  $r + (n - r) = n = 3$  does not match  $2 + 2 = 4$ . Only  $\mathbf{v} = \mathbf{0}$  is in both  $N(A)$  and  $C(A^T)$ .
- 13 (a) *False*: Usually row space  $\neq$  column space (same dimension!) (b) *True*:  $A$  and  $-A$  have the same four subspaces (c) *False* (choose  $A$  and  $B$  same size and invertible: then they have the same four subspaces)
- 14 Row space basis can be the nonzero rows of  $U$ :  $(1, 2, 3, 4)$ ,  $(0, 1, 2, 3)$ ,  $(0, 0, 1, 2)$ ; nullspace basis  $(0, 1, -2, 1)$  as for  $U$ ; column space basis  $(1, 0, 0)$ ,  $(0, 1, 0)$ ,  $(0, 0, 1)$  (happen to have  $C(A) = C(U) = \mathbf{R}^3$ ); left nullspace has empty basis.
- 15 After a row exchange, the row space and nullspace stay the same;  $(2, 1, 3, 4)$  is in the new left nullspace after the row exchange.
- 16 If  $A\mathbf{v} = \mathbf{0}$  and  $\mathbf{v}$  is a row of  $A$  then  $\mathbf{v} \cdot \mathbf{v} = 0$ .
- 17 Row space =  $yz$  plane; column space =  $xy$  plane; nullspace =  $x$  axis; left nullspace =  $z$  axis. For  $I + A$ : Row space = column space =  $\mathbf{R}^3$ , both nullspaces contain only the zero vector.
- 18 Row  $3 - 2$  row  $2 +$  row  $1 =$  zero row so the vectors  $c(1, -2, 1)$  are in the left nullspace. The same vectors happen to be in the nullspace (an accident for this matrix).
- 19 (a) Elimination on  $A\mathbf{x} = \mathbf{0}$  leads to  $0 = b_3 - b_2 - b_1$  so  $(-1, -1, 1)$  is in the left nullspace. (b) 4 by 3: Elimination leads to  $b_3 - 2b_1 = 0$  and  $b_4 + b_2 - 4b_1 = 0$ , so  $(-2, 0, 1, 0)$  and  $(-4, 1, 0, 1)$  are in the left nullspace. *Why?* Those vectors multiply the matrix to give *zero rows*. Section 4.1 will show another approach:  $A\mathbf{x} = \mathbf{b}$  is solvable ( $\mathbf{b}$  is in  $C(A)$ ) when  $\mathbf{b}$  is orthogonal to the left nullspace.
- 20 (a) Special solutions  $(-1, 2, 0, 0)$  and  $(-\frac{1}{4}, 0, -3, 1)$  are perpendicular to the rows of  $R$  (and then  $ER$ ). (b)  $A^T\mathbf{y} = \mathbf{0}$  has 1 independent solution = last row of  $E^{-1}$ . ( $E^{-1}A = R$  has a zero row, which is just the transpose of  $A^T\mathbf{y} = \mathbf{0}$ ).
- 21 (a)  $\mathbf{u}$  and  $\mathbf{w}$  (b)  $\mathbf{v}$  and  $\mathbf{z}$  (c) rank  $< 2$  if  $\mathbf{u}$  and  $\mathbf{w}$  are dependent or if  $\mathbf{v}$  and  $\mathbf{z}$  are dependent (d) The rank of  $\mathbf{u}\mathbf{v}^T + \mathbf{w}\mathbf{z}^T$  is 2.
- 22  $A = \begin{bmatrix} \mathbf{u} & \mathbf{w} \end{bmatrix} \begin{bmatrix} \mathbf{v}^T & \mathbf{z}^T \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 2 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & 2 \\ 5 & 1 \end{bmatrix}$  has column space spanned by  $\mathbf{u}$  and  $\mathbf{w}$ , row space spanned by  $\mathbf{v}$  and  $\mathbf{z}$ .

- 23 As in Problem 22: Row space basis  $(3, 0, 3), (1, 1, 2)$ ; column space basis  $(1, 4, 2), (2, 5, 7)$ ; the rank of  $(3 \text{ by } 2) \text{ times } (2 \text{ by } 3)$  cannot be larger than the rank of either factor, so  $\text{rank} \leq 2$  and the  $3 \text{ by } 3$  product is not invertible.
- 24  $A^T y = d$  puts  $d$  in the row space of  $A$ ; unique solution if the left nullspace (nullspace of  $A^T$ ) contains only  $y = 0$ .
- 25 (a) True ( $A$  and  $A^T$  have the same rank) (b) False  $A = \begin{bmatrix} 1 & 0 \end{bmatrix}$  and  $A^T$  have very different left nullspaces (c) False ( $A$  can be invertible and unsymmetric even if  $C(A) = C(A^T)$ ) (d) True (The subspaces for  $A$  and  $-A$  are always the same. If  $A^T = A$  or  $A^T = -A$  they are also the same for  $A^T$ )
- 26 The rows of  $C = AB$  are combinations of the rows of  $B$ . So  $\text{rank } C \leq \text{rank } B$ . Also  $\text{rank } C \leq \text{rank } A$ , because the columns of  $C$  are combinations of the columns of  $A$ .
- 27 Choose  $d = bc/a$  to make  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  a rank-1 matrix. Then the row space has basis  $(a, b)$  and the nullspace has basis  $(-b, a)$ . Those two vectors are perpendicular!
- 28  $B$  and  $C$  (checkers and chess) both have rank 2 if  $p \neq 0$ . Row 1 and 2 are a basis for the row space of  $C$ ,  $B^T y = 0$  has 6 special solutions with  $-1$  and  $1$  separated by a zero;  $N(C^T)$  has  $(-1, 0, 0, 0, 0, 0, 1)$  and  $(0, -1, 0, 0, 0, 0, 1, 0)$  and columns 3, 4, 5, 6 of  $I$ ;  $N(C)$  is a challenge.
- 29  $a_{11} = 1, a_{12} = 0, a_{13} = 1, a_{22} = 0, a_{32} = 1, a_{31} = 0, a_{23} = 1, a_{33} = 0, a_{21} = 1$ .
- 30 The subspaces for  $A = uv^T$  are pairs of orthogonal lines ( $v$  and  $v^\perp$ ,  $u$  and  $u^\perp$ ). If  $B$  has those same four subspaces then  $B = cA$  with  $c \neq 0$ .
- 31 (a)  $AX = 0$  if each column of  $X$  is a multiple of  $(1, 1, 1)$ ;  $\dim(\text{nullspace}) = 3$ .  
 (b) If  $AX = B$  then all columns of  $B$  add to zero; dimension of the  $B$ 's = 6.  
 (c)  $3 + 6 = \dim(M^{3 \times 3}) = 9$  entries in a  $3 \text{ by } 3$  matrix.
- 32 The key is equal row spaces. First row of  $A =$  combination of the rows of  $B$ : only possible combination (notice  $I$ ) is 1 (row 1 of  $B$ ). Same for each row so  $F = G$ .

## Problem Set 4.1, page 202

- 1 Both nullspace vectors are orthogonal to the row space vector in  $\mathbf{R}^3$ . The column space is perpendicular to the nullspace of  $A^T$  (two lines in  $\mathbf{R}^2$  because  $\text{rank} = 1$ ).
- 2 The nullspace of a  $3 \text{ by } 2$  matrix with rank 2 is  $\mathbf{Z}$  (only zero vector) so  $x_n = 0$ , and row space =  $\mathbf{R}^2$ . Column space = plane perpendicular to left nullspace = line in  $\mathbf{R}^3$ .
- 3 (a)  $\begin{bmatrix} 1 & 2 & -3 \\ 2 & -3 & 1 \\ -3 & 5 & -2 \end{bmatrix}$  (b) Impossible,  $\begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$  not orthogonal to  $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$  (c)  $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$  in  $C(A)$  and  $N(A^T)$  is impossible: not perpendicular (d) Need  $A^2 = 0$ ; take  $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$  (e)  $(1, 1, 1)$  in the nullspace (columns add to 0) and also row space; no such matrix.
- 4 If  $AB = 0$ , the columns of  $B$  are in the nullspace of  $A$ . The rows of  $A$  are in the left nullspace of  $B$ . If  $\text{rank} = 2$ , those four subspaces would have dimension 2 which is impossible for  $3 \text{ by } 3$ .
- 5 (a) If  $Ax = b$  has a solution and  $A^T y = 0$ , then  $y$  is perpendicular to  $b$ .  $b^T y = (Ax)^T y = x^T (A^T y) = 0$ . (b) If  $A^T y = (1, 1, 1)$  has a solution,  $(1, 1, 1)$  is in the row space and is orthogonal to every  $x$  in the nullspace.

- 6 Multiply the equations by  $y_1, y_2, y_3 = 1, 1, -1$ . Equations add to  $0 = 1$  so no solution:  $\mathbf{y} = (1, 1, -1)$  is in the left nullspace.  $A\mathbf{x} = \mathbf{b}$  would need  $0 = (\mathbf{y}^T A)\mathbf{x} = \mathbf{y}^T \mathbf{b} = 1$ .
- 7 Multiply the 3 equations by  $\mathbf{y} = (1, 1, -1)$ . Then  $x_1 - x_2 = 1$  plus  $x_2 - x_3 = 1$  minus  $x_1 - x_3 = 1$  is  $0 = 1$ . Key point: This  $\mathbf{y}$  in  $N(A^T)$  is not orthogonal to  $\mathbf{b} = (1, 1, 1)$  so  $\mathbf{b}$  is not in the column space and  $A\mathbf{x} = \mathbf{b}$  has *no solution*.
- 8  $\mathbf{x} = \mathbf{x}_r + \mathbf{x}_n$ , where  $\mathbf{x}_r$  is in the row space and  $\mathbf{x}_n$  is in the nullspace. Then  $A\mathbf{x}_n = \mathbf{0}$  and  $A\mathbf{x} = A\mathbf{x}_r + A\mathbf{x}_n = A\mathbf{x}_r$ . All  $A\mathbf{x}$  are in  $C(A)$ .
- 9  $A\mathbf{x}$  is always in the *column space* of  $A$ . If  $A^T A\mathbf{x} = \mathbf{0}$  then  $A\mathbf{x}$  is also in the nullspace of  $A^T$ . So  $A\mathbf{x}$  is perpendicular to itself. Conclusion:  $A\mathbf{x} = \mathbf{0}$  if  $A^T A\mathbf{x} = \mathbf{0}$ .
- 10 (a) With  $A^T = A$ , the column and row spaces are the same (b)  $\mathbf{x}$  is in the nullspace and  $\mathbf{z}$  is in the column space = row space: so these “eigenvectors” have  $\mathbf{x}^T \mathbf{z} = 0$ .
- 11 **For A:** The nullspace is spanned by  $(-2, 1)$ , the row space is spanned by  $(1, 2)$ . The column space is the line through  $(1, 3)$  and  $N(A^T)$  is the perpendicular line through  $(3, -1)$ . **For B:** The nullspace of  $B$  is spanned by  $(0, 1)$ , the row space is spanned by  $(1, 0)$ . The column space and left nullspace are the same as for  $A$ .
- 12  $\mathbf{x}$  splits into  $\mathbf{x}_r + \mathbf{x}_n = (1, -1) + (1, 1) = (2, 0)$ . Notice  $N(A^T)$  is a plane  $(1, 0) = (1, 1)/2 + (1, -1)/2 = \mathbf{x}_r + \mathbf{x}_n$ .
- 13  $V^T W = \text{zero}$  makes each basis vector for  $V$  orthogonal to each basis vector for  $W$ . Then every  $\mathbf{v}$  in  $V$  is orthogonal to every  $\mathbf{w}$  in  $W$  (combinations of the basis vectors).
- 14  $A\mathbf{x} = B\hat{\mathbf{x}}$  means that  $\begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ -\hat{\mathbf{x}} \end{bmatrix} = \mathbf{0}$ . Three homogeneous equations in four unknowns always have a nonzero solution. Here  $\mathbf{x} = (3, 1)$  and  $\hat{\mathbf{x}} = (1, 0)$  and  $A\mathbf{x} = B\hat{\mathbf{x}} = (5, 6, 5)$  is in both column spaces. Two planes in  $\mathbf{R}^3$  must share a line.
- 15 A  $p$ -dimensional and a  $q$ -dimensional subspace of  $\mathbf{R}^n$  share at least a line if  $p + q > n$ . (The  $p + q$  basis vectors of  $V$  and  $W$  cannot be independent.)
- 16  $A^T \mathbf{y} = \mathbf{0}$  leads to  $(A\mathbf{x})^T \mathbf{y} = \mathbf{x}^T A^T \mathbf{y} = 0$ . Then  $\mathbf{y} \perp A\mathbf{x}$  and  $N(A^T) \perp C(A)$ .
- 17 If  $S$  is the subspace of  $\mathbf{R}^3$  containing only the zero vector, then  $S^\perp$  is  $\mathbf{R}^3$ . If  $S$  is spanned by  $(1, 1, 1)$ , then  $S^\perp$  is the plane spanned by  $(1, -1, 0)$  and  $(1, 0, -1)$ . If  $S$  is spanned by  $(2, 0, 0)$  and  $(0, 0, 3)$ , then  $S^\perp$  is the line spanned by  $(0, 1, 0)$ .
- 18  $S^\perp$  is the nullspace of  $A = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 2 & 2 \end{bmatrix}$ . Therefore  $S^\perp$  is a *subspace* even if  $S$  is not.
- 19  $L^\perp$  is the 2-dimensional subspace (a plane) in  $\mathbf{R}^3$  perpendicular to  $L$ . Then  $(L^\perp)^\perp$  is a 1-dimensional subspace (a line) perpendicular to  $L^\perp$ . In fact  $(L^\perp)^\perp$  is  $L$ .
- 20 If  $V$  is the whole space  $\mathbf{R}^4$ , then  $V^\perp$  contains only the *zero vector*. Then  $(V^\perp)^\perp = \mathbf{R}^4 = V$ .
- 21 For example  $(-5, 0, 1, 1)$  and  $(0, 1, -1, 0)$  span  $S^\perp = \text{nullspace of } A = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 3 & 3 & 2 \end{bmatrix}$ .
- 22  $(1, 1, 1, 1)$  is a basis for  $P^\perp$ .  $A = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$  has  $P$  as its nullspace and  $P^\perp$  as row space.
- 23  $\mathbf{x}$  in  $V^\perp$  is perpendicular to any vector in  $V$ . Since  $V$  contains all the vectors in  $S$ ,  $\mathbf{x}$  is also perpendicular to any vector in  $S$ . So every  $\mathbf{x}$  in  $V^\perp$  is also in  $S^\perp$ .

- 24  $AA^{-1} = I$ : Column 1 of  $A^{-1}$  is orthogonal to the space spanned by the 2nd, 3rd, ...,  $n$ th rows of  $A$ .
- 25 If the columns of  $A$  are unit vectors, all mutually perpendicular, then  $A^T A = I$ .
- 26  $A = \begin{bmatrix} 2 & 2 & -1 \\ -1 & 2 & 2 \\ 2 & -1 & 2 \end{bmatrix}$ , This example shows a matrix with perpendicular columns.  $A^T A = 9I$  is *diagonal*:  $(A^T A)_{ij} = (\text{column } i \text{ of } A) \cdot (\text{column } j \text{ of } A)$ . When the columns are *unit vectors*, then  $A^T A = I$ .
- 27 The lines  $3x + y = b_1$  and  $6x + 2y = b_2$  are **parallel**. They are the same line if  $b_2 = 2b_1$ . In that case  $(b_1, b_2)$  is perpendicular to  $(-2, 1)$ . The nullspace of the 2 by 2 matrix is the line  $3x + y = 0$ . One particular vector in the nullspace is  $(-1, 3)$ .
- 28 (a)  $(1, -1, 0)$  is in both planes. Normal vectors are perpendicular, but planes still intersect! (b) Need *three* orthogonal vectors to span the whole orthogonal complement. (c) Lines can meet at the zero vector without being orthogonal.
- 29  $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ ,  $B = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & 0 \\ 3 & 0 & -1 \end{bmatrix}$ ;  $A$  has  $\mathbf{v} = (1, 2, 3)$  in row space and column space.  $B$  has  $\mathbf{v}$  in its column space and nullspace.  $\mathbf{v}$  **can not** be in the nullspace and row space, or in the left nullspace and column space. These spaces are orthogonal and  $\mathbf{v}^T \mathbf{v} \neq 0$ .
- 30 When  $AB = 0$ , the column space of  $B$  is contained in the nullspace of  $A$ . Therefore the dimension of  $C(B) \leq \text{dimension of } N(A)$ . This means  $\text{rank}(B) \leq 4 - \text{rank}(A)$ .
- 31  $\text{null}(N')$  produces a basis for the *row space* of  $A$  (perpendicular to  $N(A)$ ).
- 32 We need  $\mathbf{r}^T \mathbf{n} = 0$  and  $\mathbf{c}^T \boldsymbol{\ell} = 0$ . All possible examples have the form  $a\mathbf{c}\mathbf{r}^T$  with  $a \neq 0$ .
- 33 Both  $\mathbf{r}$ 's orthogonal to both  $\mathbf{n}$ 's, both  $\mathbf{c}$ 's orthogonal to both  $\boldsymbol{\ell}$ 's, each pair independent. All  $A$ 's with these subspaces have the form  $[\mathbf{c}_1 \ \mathbf{c}_2]M[\mathbf{r}_1 \ \mathbf{r}_2]^T$  for a 2 by 2 invertible  $M$ .

## Problem Set 4.2, page 214

- 1 (a)  $\mathbf{a}^T \mathbf{b} / \mathbf{a}^T \mathbf{a} = 5/3$ ;  $\mathbf{p} = 5\mathbf{a}/3$ ;  $\mathbf{e} = (-2, 1, 1)/3$  (b)  $\mathbf{a}^T \mathbf{b} / \mathbf{a}^T \mathbf{a} = -1$ ;  $\mathbf{p} = \mathbf{a}$ ;  $\mathbf{e} = \mathbf{0}$ .
- 2 (a) The projection of  $\mathbf{b} = (\cos \theta, \sin \theta)$  onto  $\mathbf{a} = (1, 0)$  is  $\mathbf{p} = (\cos \theta, 0)$   
 (b) The projection of  $\mathbf{b} = (1, 1)$  onto  $\mathbf{a} = (1, -1)$  is  $\mathbf{p} = (0, 0)$  since  $\mathbf{a}^T \mathbf{b} = 0$ .
- 3  $P_1 = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$  and  $P_1 \mathbf{b} = \frac{1}{3} \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}$ .  $P_2 = \frac{1}{11} \begin{bmatrix} 1 & 3 & 1 \\ 3 & 9 & 3 \\ 1 & 3 & 1 \end{bmatrix}$  and  $P_2 \mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ .
- 4  $P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ ,  $P_2 = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ .  $P_1$  projects onto  $(1, 0)$ ,  $P_2$  projects onto  $(1, -1)$ .  $P_1 P_2 \neq 0$  and  $P_1 + P_2$  is not a projection matrix.
- 5  $P_1 = \frac{1}{9} \begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix}$ ,  $P_2 = \frac{1}{9} \begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix}$ .  $P_1$  and  $P_2$  are the projection matrices onto the lines through  $\mathbf{a}_1 = (-1, 2, 2)$  and  $\mathbf{a}_2 = (2, 2, -1)$ .  $P_1 P_2 = \text{zero matrix}$  because  $\mathbf{a}_1 \perp \mathbf{a}_2$ .
- XXX Above solution does not fit in 3 lines.
- 6  $\mathbf{p}_1 = (\frac{1}{9}, -\frac{2}{9}, -\frac{2}{9})$  and  $\mathbf{p}_2 = (\frac{4}{9}, \frac{4}{9}, -\frac{2}{9})$  and  $\mathbf{p}_3 = (\frac{4}{9}, -\frac{2}{9}, \frac{4}{9})$ . So  $\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 = \mathbf{b}$ .

$$7 \quad P_1 + P_2 + P_3 = \frac{1}{9} \begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix} + \frac{1}{9} \begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix} + \frac{1}{9} \begin{bmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{bmatrix} = I.$$

We can add projections onto *orthogonal vectors*. This is important.

8 The projections of  $(1, 1)$  onto  $(1, 0)$  and  $(1, 2)$  are  $\mathbf{p}_1 = (1, 0)$  and  $\mathbf{p}_2 = (0.6, 1.2)$ . Then  $\mathbf{p}_1 + \mathbf{p}_2 \neq \mathbf{b}$ .

9 Since  $A$  is invertible,  $P = A(A^T A)^{-1} A^T = A A^{-1} (A^T)^{-1} A^T = I$ : project on all of  $\mathbf{R}^2$ .

10  $P_2 = \begin{bmatrix} 0.2 & 0.4 \\ 0.4 & 0.8 \end{bmatrix}$ ,  $P_2 \mathbf{a}_1 = \begin{bmatrix} 0.2 \\ 0.4 \end{bmatrix}$ ,  $P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ ,  $P_1 P_2 \mathbf{a}_1 = \begin{bmatrix} 0.2 \\ 0 \end{bmatrix}$ . This is not  $\mathbf{a}_1 = (1, 0)$ . No,  $P_1 P_2 \neq (P_1 P_2)^2$ .

11 (a)  $\mathbf{p} = A(A^T A)^{-1} A^T \mathbf{b} = (2, 3, 0)$ ,  $\mathbf{e} = (0, 0, 4)$ ,  $A^T \mathbf{e} = \mathbf{0}$  (b)  $\mathbf{p} = (4, 4, 6)$ ,  $\mathbf{e} = \mathbf{0}$ .

12  $P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$  = projection matrix onto the column space of  $A$  (the  $xy$  plane)

$P_2 = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$  = Projection matrix onto the second column space. Certainly  $(P_2)^2 = P_2$ .

13  $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ ,  $P$  = square matrix =  $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ ,  $\mathbf{p} = P \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$ .

14 The projection of this  $\mathbf{b}$  onto the column space of  $A$  is  $\mathbf{b}$  itself when  $\mathbf{b}$  is in that space.

But  $P$  is not necessarily  $I$ .  $P = \frac{1}{21} \begin{bmatrix} 5 & 8 & -4 \\ 8 & 17 & 2 \\ -4 & 2 & 20 \end{bmatrix}$  and  $\mathbf{b} = P\mathbf{b} = \mathbf{p} = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}$ .

15  $2A$  has the same column space as  $A$ .  $\hat{\mathbf{x}}$  for  $2A$  is *half* of  $\hat{\mathbf{x}}$  for  $A$ .

16  $\frac{1}{2}(1, 2, -1) + \frac{3}{2}(1, 0, 1) = (2, 1, 1)$ . So  $\mathbf{b}$  is in the plane. Projection shows  $P\mathbf{b} = \mathbf{b}$ .

17 If  $P^2 = P$  then  $(I - P)^2 = (I - P)(I - P) = I - PI - IP + P^2 = I - P$ . When  $P$  projects onto the column space,  $I - P$  projects onto the *left nullspace*.

18 (a)  $I - P$  is the projection matrix onto  $(1, -1)$  in the perpendicular direction to  $(1, 1)$   
(b)  $I - P$  projects onto the plane  $x + y + z = 0$  perpendicular to  $(1, 1, 1)$ .

19 For any basis vectors in the plane  $x - y - 2z = 0$ , say  $(1, 1, 0)$  and  $(2, 0, 1)$ , the matrix  $P$  is  $\begin{bmatrix} 5/6 & 1/6 & 1/3 \\ 1/6 & 5/6 & -1/3 \\ 1/3 & -1/3 & 1/3 \end{bmatrix}$ .

20  $\mathbf{e} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$ ,  $Q = \frac{\mathbf{e}\mathbf{e}^T}{\mathbf{e}^T \mathbf{e}} = \begin{bmatrix} 1/6 & -1/6 & -1/3 \\ -1/6 & 1/6 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$ ,  $I - Q = \begin{bmatrix} 5/6 & 1/6 & 1/3 \\ 1/6 & 5/6 & -1/3 \\ 1/3 & -1/3 & 1/3 \end{bmatrix}$ .

21  $(A(A^T A)^{-1} A^T)^2 = A(A^T A)^{-1} (A^T A) (A^T A)^{-1} A^T = A(A^T A)^{-1} A^T$ . So  $P^2 = P$ .  $P\mathbf{b}$  is in the column space (where  $P$  projects). Then its projection  $P(P\mathbf{b})$  is  $P\mathbf{b}$ .

22  $P^T = (A(A^T A)^{-1} A^T)^T = A((A^T A)^{-1})^T A^T = A(A^T A)^{-1} A^T = P$ . ( $A^T A$  is symmetric!)

23 If  $A$  is invertible then its column space is all of  $\mathbf{R}^n$ . So  $P = I$  and  $\mathbf{e} = \mathbf{0}$ .

24 The nullspace of  $A^T$  is *orthogonal* to the column space  $C(A)$ . So if  $A^T \mathbf{b} = \mathbf{0}$ , the projection of  $\mathbf{b}$  onto  $C(A)$  should be  $\mathbf{p} = \mathbf{0}$ . Check  $P\mathbf{b} = A(A^T A)^{-1} A^T \mathbf{b} = A(A^T A)^{-1} \mathbf{0}$ .

- 25** The column space of  $P$  will be  $S$ . Then  $r = \text{dimension of } S = n$ .
- 26**  $A^{-1}$  exists since the rank is  $r = m$ . Multiply  $A^2 = A$  by  $A^{-1}$  to get  $A = I$ .
- 27** If  $A^T A \mathbf{x} = \mathbf{0}$  then  $A \mathbf{x}$  is in the nullspace of  $A^T$ . But  $A \mathbf{x}$  is always in the column space of  $A$ . To be in both of those perpendicular spaces,  $A \mathbf{x}$  must be zero. So  $A$  and  $A^T A$  have the same nullspace.
- 28**  $P^2 = P = P^T$  give  $P^T P = P$ . Then the  $(2, 2)$  entry of  $P$  equals the  $(2, 2)$  entry of  $P^T P$  which is the length squared of column 2.
- 29**  $A = B^T$  has independent columns, so  $A^T A$  (which is  $BB^T$ ) must be invertible.
- 30** (a) The column space is the line through  $\mathbf{a} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$  so  $P_C = \frac{\mathbf{a}\mathbf{a}^T}{\mathbf{a}^T\mathbf{a}} = \frac{1}{25} \begin{bmatrix} 9 & 12 \\ 12 & 25 \end{bmatrix}$ .  
 (b) The row space is the line through  $\mathbf{v} = (1, 2, 2)$  and  $P_R = \frac{\mathbf{v}\mathbf{v}^T}{\mathbf{v}^T\mathbf{v}}$ . Always  $P_C A = A$  (columns of  $A$  project to themselves) and  $A P_R = A$ . Then  $P_C A P_R = A$ !
- 31** The error  $\mathbf{e} = \mathbf{b} - \mathbf{p}$  must be perpendicular to all the  $\mathbf{a}$ 's.
- 32** Since  $P_1 \mathbf{b}$  is in  $C(A)$ ,  $P_2(P_1 \mathbf{b})$  equals  $P_1 \mathbf{b}$ . So  $P_2 P_1 = P_1 = \mathbf{a}\mathbf{a}^T / \mathbf{a}^T \mathbf{a}$  where  $\mathbf{a} = (1, 2, 0)$ .
- 33** If  $P_1 P_2 = P_2 P_1$  then  $S$  is contained in  $T$  or  $T$  is contained in  $S$ .
- 34**  $BB^T$  is invertible as in Problem 29. Then  $(A^T A)(BB^T) = \text{product of } r \text{ by } r \text{ invertible matrices, so rank } r$ .  $AB$  can't have rank  $< r$ , since  $A^T$  and  $B^T$  cannot increase the rank.  
*Conclusion:*  $A$  ( $m$  by  $r$  of rank  $r$ ) times  $B$  ( $r$  by  $n$  of rank  $r$ ) produces  $AB$  of rank  $r$ .

### Problem Set 4.3, page 226

**1**  $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$  and  $\mathbf{b} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$  give  $A^T A = \begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix}$  and  $A^T \mathbf{b} = \begin{bmatrix} 36 \\ 112 \end{bmatrix}$ .

$$A^T A \hat{\mathbf{x}} = A^T \mathbf{b} \text{ gives } \hat{\mathbf{x}} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \text{ and } \mathbf{p} = A \hat{\mathbf{x}} = \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix} \text{ and } \mathbf{e} = \mathbf{b} - \mathbf{p} = \begin{bmatrix} -1 \\ 3 \\ -5 \\ 3 \end{bmatrix} \\ E = \|\mathbf{e}\|^2 = 44$$

**2**  $\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$ . This  $A\mathbf{x} = \mathbf{b}$  is unsolvable. Change  $\mathbf{b}$  to  $\mathbf{p} = P\mathbf{b} = \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix}$ ;  $\hat{\mathbf{x}} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$  exactly solves  $A\hat{\mathbf{x}} = \mathbf{p}$ .

**3** In Problem 2,  $\mathbf{p} = A(A^T A)^{-1} A^T \mathbf{b} = (1, 5, 13, 17)$  and  $\mathbf{e} = \mathbf{b} - \mathbf{p} = (-1, 3, -5, 3)$ .  $\mathbf{e}$  is perpendicular to both columns of  $A$ . This shortest distance  $\|\mathbf{e}\|$  is  $\sqrt{44}$ .

**4**  $E = (C + 0D)^2 + (C + 1D - 8)^2 + (C + 3D - 8)^2 + (C + 4D - 20)^2$ . Then  $\partial E / \partial C = 2C + 2(C + D - 8) + 2(C + 3D - 8) + 2(C + 4D - 20) = 0$  and  $\partial E / \partial D = 1 \cdot 2(C + D - 8) + 3 \cdot 2(C + 3D - 8) + 4 \cdot 2(C + 4D - 20) = 0$ . These normal equations are again  $\begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \end{bmatrix}$ .

- 5  $E = (C-0)^2 + (C-8)^2 + (C-8)^2 + (C-20)^2$ .  $A^T = [1 \ 1 \ 1 \ 1]$  and  $A^T A = [4]$ .  $A^T \mathbf{b} = [36]$  and  $(A^T A)^{-1} A^T \mathbf{b} = 9 = \text{best height } C$ . Errors  $\mathbf{e} = (-9, -1, -1, 11)$ .
- 6  $\mathbf{a} = (1, 1, 1, 1)$  and  $\mathbf{b} = (0, 8, 8, 20)$  give  $\hat{x} = \mathbf{a}^T \mathbf{b} / \mathbf{a}^T \mathbf{a} = 9$  and the projection is  $\hat{x} \mathbf{a} = \mathbf{p} = (9, 9, 9, 9)$ . Then  $\mathbf{e}^T \mathbf{a} = (-9, -1, -1, 11)^T (1, 1, 1, 1) = 0$  and  $\|\mathbf{e}\| = \sqrt{204}$ .
- 7  $A = [0 \ 1 \ 3 \ 4]^T$ ,  $A^T A = [26]$  and  $A^T \mathbf{b} = [112]$ . Best  $D = 112/26 = 56/13$ .
- 8  $\hat{x} = 56/13$ ,  $\mathbf{p} = (56/13)(0, 1, 3, 4)$ .  $(C, D) = (9, 56/13)$  don't match  $(C, D) = (1, 4)$ . Columns of  $A$  were not perpendicular so we can't project separately to find  $C$  and  $D$ .
- 9 Parabola  
Project  $\mathbf{b}$   
4D to 3D  $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$ .  $A^T A \hat{\mathbf{x}} = \begin{bmatrix} 4 & 8 & 26 \\ 8 & 26 & 92 \\ 26 & 92 & 338 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \\ 400 \end{bmatrix}$ .
- 10  $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \\ F \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$ . Then  $\begin{bmatrix} C \\ D \\ E \\ F \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 0 \\ 47 \\ -28 \\ 5 \end{bmatrix}$ . Exact cubic so  $\mathbf{p} = \mathbf{b}$ ,  $\mathbf{e} = \mathbf{0}$ . This Vandermonde matrix gives exact interpolation by a cubic at 0, 1, 3, 4.
- 11 (a) The best line  $x = 1 + 4t$  gives the center point  $\hat{\mathbf{b}} = 9$  when  $\hat{t} = 2$ .  
(b) The first equation  $Cm + D \sum t_i = \sum b_i$  divided by  $m$  gives  $C + D\hat{t} = \hat{\mathbf{b}}$ .
- 12 (a)  $\mathbf{a} = (1, \dots, 1)$  has  $\mathbf{a}^T \mathbf{a} = m$ ,  $\mathbf{a}^T \mathbf{b} = b_1 + \dots + b_m$ . Therefore  $\hat{x} = \mathbf{a}^T \mathbf{b} / m$  is the **mean** of the  $b$ 's. (b)  $\mathbf{e} = \mathbf{b} - \hat{x} \mathbf{a}$ ,  $\mathbf{b} = (1, 2, b)$ ,  $\|\mathbf{e}\|^2 = \sum_{i=1}^m (b_i - \hat{x})^2 = \text{variance}$ .
- (c)  $\mathbf{p} = (3, 3, 3)$   
 $\mathbf{e} = (-2, -1, 3)$   $\mathbf{p}^T \mathbf{e} = 0$ .  $P = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ .
- 13  $(A^T A)^{-1} A^T (\mathbf{b} - A\mathbf{x}) = \hat{\mathbf{x}} - \mathbf{x}$ . When  $\mathbf{e} = \mathbf{b} - A\mathbf{x}$  averages to  $\mathbf{0}$ , so does  $\hat{\mathbf{x}} - \mathbf{x}$ .
- 14 The matrix  $(\hat{\mathbf{x}} - \mathbf{x})(\hat{\mathbf{x}} - \mathbf{x})^T$  is  $(A^T A)^{-1} A^T (\mathbf{b} - A\mathbf{x})(\mathbf{b} - A\mathbf{x})^T A (A^T A)^{-1}$ . When the average of  $(\mathbf{b} - A\mathbf{x})(\mathbf{b} - A\mathbf{x})^T$  is  $\sigma^2 I$ , the average of  $(\hat{\mathbf{x}} - \mathbf{x})(\hat{\mathbf{x}} - \mathbf{x})^T$  will be the *output covariance matrix*  $(A^T A)^{-1} A^T \sigma^2 A (A^T A)^{-1}$  which simplifies to  $\sigma^2 (A^T A)^{-1}$ .
- 15 When  $A$  has 1 column of ones, Problem 14 gives the expected error  $(\hat{\mathbf{x}} - \mathbf{x})^2$  as  $\sigma^2 (A^T A)^{-1} = \sigma^2 / m$ . By taking  $m$  measurements, the variance drops from  $\sigma^2$  to  $\sigma^2 / m$ .
- 16  $\frac{1}{10} b_{10} + \frac{9}{10} \hat{x}_9 = \frac{1}{10} (b_1 + \dots + b_{10})$ . Knowing  $\hat{x}_9$  avoids adding all  $b$ 's.
- 17  $\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix}$ . The solution  $\hat{\mathbf{x}} = \begin{bmatrix} 9 \\ 4 \end{bmatrix}$  comes from  $\begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 35 \\ 42 \end{bmatrix}$ .
- 18  $\mathbf{p} = A\hat{\mathbf{x}} = (5, 13, 17)$  gives the heights of the closest line. The error is  $\mathbf{b} - \mathbf{p} = (2, -6, 4)$ . This error  $\mathbf{e}$  has  $P\mathbf{e} = P\mathbf{b} - P\mathbf{p} = \mathbf{p} - \mathbf{p} = \mathbf{0}$ .
- 19 If  $\mathbf{b} = \text{error } \mathbf{e}$  then  $\mathbf{b}$  is perpendicular to the column space of  $A$ . Projection  $\mathbf{p} = \mathbf{0}$ .
- 20 If  $\mathbf{b} = A\hat{\mathbf{x}} = (5, 13, 17)$  then  $\hat{\mathbf{x}} = (9, 4)$  and  $\mathbf{e} = \mathbf{0}$  since  $\mathbf{b}$  is in the column space of  $A$ .
- 21  $\mathbf{e}$  is in  $N(A^T)$ ;  $\mathbf{p}$  is in  $C(A)$ ;  $\hat{\mathbf{x}}$  is in  $C(A^T)$ ;  $N(A) = \{\mathbf{0}\} = \text{zero vector only}$ .

- 22 The least squares equation is  $\begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 5 \\ -10 \end{bmatrix}$ . Solution:  $C = 1, D = -1$ .  
Line  $1 - t$ . Symmetric  $t$ 's  $\Rightarrow$  diagonal  $A^T A$
- 23  $e$  is orthogonal to  $p$ ; then  $\|e\|^2 = e^T(b - p) = e^T b = b^T b - b^T p$ .
- 24 The derivatives of  $\|Ax - b\|^2 = x^T A^T A x - 2b^T A x + b^T b$  (this term is constant) are zero when  $2A^T A x = 2A^T b$ , or  $x = (A^T A)^{-1} A^T b$ .
- 25 3 points on a line: *Equal slopes*  $(b_2 - b_1)/(t_2 - t_1) = (b_3 - b_2)/(t_3 - t_2)$ . Linear algebra: Orthogonal to  $(1, 1, 1)$  and  $(t_1, t_2, t_3)$  is  $y = (t_2 - t_3, t_3 - t_1, t_1 - t_2)$  in the left nullspace.  $b$  is in the column space. Then  $y^T b = 0$  is the same equal slopes condition written as  $(b_2 - b_1)(t_3 - t_2) = (b_3 - b_2)(t_2 - t_1)$ .
- 26  $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 4 \end{bmatrix}$  has  $A^T A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ ,  $A^T b = \begin{bmatrix} 8 \\ -2 \\ -3 \end{bmatrix}$ ,  $\begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -3/2 \end{bmatrix}$ . At  $x, y = 0, 0$  the best plane  $2 - x - \frac{3}{2}y$  has height  $C = 2 =$  average of  $0, 1, 3, 4$ .
- 27 The shortest link connecting two lines in space is *perpendicular to those lines*.
- 28 Only 1 plane contains  $0, a_1, a_2$  unless  $a_1, a_2$  are *dependent*. Same test for  $a_1, \dots, a_n$ .
- 29 There is exactly one hyperplane containing the  $n$  points  $0, a_1, \dots, a_{n-1}$  *When the  $n - 1$  vectors  $a_1, \dots, a_{n-1}$  are linearly independent*. (For  $n = 3$ , the vectors  $a_1$  and  $a_2$  must be independent. Then the three points  $0, a_1, a_2$  determine a plane.) The equation of the plane in  $\mathbf{R}^n$  will be  $a_n^T x = 0$ . Here  $a_n$  is any nonzero vector on the line (it is only a line!) perpendicular to  $a_1, \dots, a_{n-1}$ .

### Problem Set 4.4, page 239

- 1 (a) *Independent* (b) *Independent and orthogonal* (c) *Independent and orthonormal*.  
For orthonormal vectors, (a) becomes  $(1, 0), (0, 1)$  and (b) is  $(.6, .8), (.8, -.6)$ .
- 2 Divide by length 3 to get  
 $q_1 = (\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}), q_2 = (-\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$ .  $Q^T Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  but  $Q Q^T = \begin{bmatrix} 5/9 & 2/9 & -4/9 \\ 2/9 & 8/9 & 2/9 \\ -4/9 & 2/9 & 5/9 \end{bmatrix}$ .
- 3 (a)  $A^T A$  will be  $16I$  (b)  $A^T A$  will be diagonal with entries 1, 4, 9.
- 4 (a)  $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ ,  $Q Q^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq I$ . Any  $Q$  with  $n < m$  has  $Q Q^T \neq I$ . (b)  $(1, 0)$  and  $(0, 0)$  are *orthogonal*, not *independent*. Nonzero orthogonal vectors are independent. (c) Starting from  $q_1 = (1, 1, 1)/\sqrt{3}$  my favorite is  $q_2 = (1, -1, 0)/\sqrt{2}$  and  $q_3 = (1, 1, -2)/\sqrt{6}$ .
- 5 *Orthogonal* vectors are  $(1, -1, 0)$  and  $(1, 1, -1)$ . *Orthonormal* are  $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0), (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$ .



- 6  $Q_1 Q_2$  is orthogonal because  $(Q_1 Q_2)^T Q_1 Q_2 = Q_2^T Q_1^T Q_1 Q_2 = Q_2^T Q_2 = I$ .
- 7 When Gram-Schmidt gives  $Q$  with orthonormal columns,  $Q^T Q \hat{x} = Q^T b$  becomes  $\hat{x} = Q^T b$ .
- 8 If  $q_1$  and  $q_2$  are orthonormal vectors in  $\mathbf{R}^5$  then  $(q_1^T b)q_1 + (q_2^T b)q_2$  is closest to  $b$ .
- 9 (a)  $Q = \begin{bmatrix} .8 & -.6 \\ .6 & .8 \\ 0 & 0 \end{bmatrix}$  has  $P = Q Q^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$  (b)  $(Q Q^T)(Q Q^T) = Q(Q^T Q)Q^T = Q Q^T$ .
- 10 (a) If  $q_1, q_2, q_3$  are orthonormal then the dot product of  $q_1$  with  $c_1 q_1 + c_2 q_2 + c_3 q_3 = \mathbf{0}$  gives  $c_1 = 0$ . Similarly  $c_2 = c_3 = 0$ . Independent  $q$ 's (b)  $Qx = \mathbf{0} \Rightarrow Q^T Qx = \mathbf{0} \Rightarrow x = \mathbf{0}$ .
- 11 (a) Two orthonormal vectors are  $q_1 = \frac{1}{10}(1, 3, 4, 5, 7)$  and  $q_2 = \frac{1}{10}(-7, 3, 4, -5, 1)$   
(b) Closest in the plane:  $\text{project } Q Q^T(1, 0, 0, 0, 0) = (0.5, -0.18, -0.24, 0.4, 0)$ .
- 12 (a) Orthonormal  $a$ 's:  $a_1^T b = a_1^T(x_1 a_1 + x_2 a_2 + x_3 a_3) = x_1(a_1^T a_1) = x_1$   
(b) Orthogonal  $a$ 's:  $a_1^T b = a_1^T(x_1 a_1 + x_2 a_2 + x_3 a_3) = x_1(a_1^T a_1)$ . Therefore  $x_1 = a_1^T b / a_1^T a_1$   
(c)  $x_1$  is the first component of  $A^{-1}$  times  $b$ .
- 13 The multiple to subtract is  $\frac{a^T b}{a^T a}$ . Then  $B = b - \frac{a^T b}{a^T a} a = (4, 0) - 2 \cdot (1, 1) = (2, -2)$ .
- 14  $\begin{bmatrix} 1 & 4 \\ 1 & 0 \end{bmatrix} = [q_1 \ q_2] \begin{bmatrix} \|a\| & q_1^T b \\ 0 & \|B\| \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 2\sqrt{2} \\ 0 & 2\sqrt{2} \end{bmatrix} = QR$ .
- 15 (a)  $q_1 = \frac{1}{3}(1, 2, -2)$ ,  $q_2 = \frac{1}{3}(2, 1, 2)$ ,  $q_3 = \frac{1}{3}(2, -2, -1)$  (b) The nullspace of  $A^T$  contains  $q_3$  (c)  $\hat{x} = (A^T A)^{-1} A^T(1, 2, 7) = (1, 2)$ .
- 16 The projection  $p = (a^T b / a^T a)a = 14a/49 = 2a/7$  is closest to  $b$ ;  $q_1 = a/\|a\| = a/7$  is  $(4, 5, 2, 2)/7$ .  $B = b - p = (-1, 4, -4, -4)/7$  has  $\|B\| = 1$  so  $q_2 = B$ .
- 17  $p = (a^T b / a^T a)a = (3, 3, 3)$  and  $e = (-2, 0, 2)$ .  $q_1 = (1, 1, 1)/\sqrt{3}$  and  $q_2 = (-1, 0, 1)/\sqrt{2}$ .
- 18  $A = a = (1, -1, 0, 0)$ ;  $B = b - p = (\frac{1}{2}, \frac{1}{2}, -1, 0)$ ;  $C = c - p_A - p_B = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -1)$ . Notice the pattern in those orthogonal  $A, B, C$ . In  $\mathbf{R}^5$ ,  $D$  would be  $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -1)$ .
- 19 If  $A = QR$  then  $A^T A = R^T Q^T Q R = R^T R = \text{lower triangular times upper triangular}$  (this Cholesky factorization of  $A^T A$  uses the same  $R$  as Gram-Schmidt!). The example has  $A = \begin{bmatrix} -1 & 1 \\ 2 & 1 \\ 2 & 4 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -1 & 2 \\ 2 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 3 & 3 \\ 0 & 3 \end{bmatrix} = QR$  and the same  $R$  appears in  $A^T A = \begin{bmatrix} 9 & 9 \\ 9 & 18 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 3 & 3 \\ 0 & 3 \end{bmatrix} = R^T R$ .
- 20 (a) True (b) True.  $Qx = x_1 q_1 + x_2 q_2$ .  $\|Qx\|^2 = x_1^2 + x_2^2$  because  $q_1 \cdot q_2 = 0$ .
- 21 The orthonormal vectors are  $q_1 = (1, 1, 1, 1)/2$  and  $q_2 = (-5, -1, 1, 5)/\sqrt{52}$ . Then  $b = (-4, -3, 3, 0)$  projects to  $p = (-7, -3, -1, 3)/2$ . And  $b - p = (-1, -3, 7, -3)/2$  is orthogonal to both  $q_1$  and  $q_2$ .
- 22  $A = (1, 1, 2)$ ,  $B = (1, -1, 0)$ ,  $C = (-1, -1, 1)$ . These are not yet unit vectors.

- 23 You can see why  $\mathbf{q}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ ,  $\mathbf{q}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ ,  $\mathbf{q}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ .  $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{bmatrix} = QR$ .
- 24 (a) One basis for the subspace  $\mathcal{S}$  of solutions to  $x_1 + x_2 + x_3 - x_4 = 0$  is  $\mathbf{v}_1 = (1, -1, 0, 0)$ ,  $\mathbf{v}_2 = (1, 0, -1, 0)$ ,  $\mathbf{v}_3 = (1, 0, 0, 1)$  (b) Since  $\mathcal{S}$  contains solutions to  $(1, 1, 1, -1)^T \mathbf{x} = 0$ , a basis for  $\mathcal{S}^\perp$  is  $(1, 1, 1, -1)$  (c) Split  $(1, 1, 1, 1) = \mathbf{b}_1 + \mathbf{b}_2$  by projection on  $\mathcal{S}^\perp$  and  $\mathcal{S}$ :  $\mathbf{b}_2 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$  and  $\mathbf{b}_1 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2})$ .
- 25 This question shows 2 by 2 formulas for  $QR$ ; breakdown  $R_{22} = 0$  when  $A$  is singular.  $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \cdot \frac{1}{\sqrt{5}} \begin{bmatrix} 5 & 3 \\ 0 & 1 \end{bmatrix}$ . Singular  $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}$ . The Gram-Schmidt process breaks down when  $ad - bc = 0$ .
- 26  $(\mathbf{q}_2^T C^*) \mathbf{q}_2 = \frac{\mathbf{B}^T \mathbf{c}}{\mathbf{B}^T \mathbf{B}} \mathbf{B}$  because  $\mathbf{q}_2 = \frac{\mathbf{B}}{\|\mathbf{B}\|}$  and the extra  $\mathbf{q}_1$  in  $C^*$  is orthogonal to  $\mathbf{q}_2$ .
- 27 When  $a$  and  $b$  are not orthogonal, the projections onto these lines *do not add* to the projection onto the plane of  $a$  and  $b$ . We must use the orthogonal  $A$  and  $B$  (or orthonormal  $\mathbf{q}_1$  and  $\mathbf{q}_2$ ) to be allowed to add 1D projections.
- 28 There are  $mn$  multiplications in (11) and  $\frac{1}{2}m^2n$  multiplications in each part of (12).
- 29  $\mathbf{q}_1 = \frac{1}{3}(2, 2, -1)$ ,  $\mathbf{q}_2 = \frac{1}{3}(2, -1, 2)$ ,  $\mathbf{q}_3 = \frac{1}{3}(1, -2, -2)$ .
- 30 The columns of the wavelet matrix  $W$  are *orthonormal*. Then  $W^{-1} = W^T$ . See Section 7.2 for more about wavelets: a useful orthonormal basis with many zeros.
- 31 (a)  $c = \frac{1}{2}$  normalizes all the orthogonal columns to have unit length (b) The projection  $(\mathbf{a}^T \mathbf{b} / \mathbf{a}^T \mathbf{a}) \mathbf{a}$  of  $\mathbf{b} = (1, 1, 1, 1)$  onto the first column is  $\mathbf{p}_1 = \frac{1}{2}(-1, 1, 1, 1)$ . (Check  $\mathbf{e} = \mathbf{0}$ .) To project onto the plane, add  $\mathbf{p}_2 = \frac{1}{2}(1, -1, 1, 1)$  to get  $(0, 0, 1, 1)$ .
- 32  $Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$  reflects across  $x$  axis,  $Q_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$  across plane  $y + z = 0$ .
- 33 Orthogonal and lower triangular  $\Rightarrow \pm 1$  on the main diagonal and zeros elsewhere.
- 34 (a)  $Q\mathbf{u} = (I - 2\mathbf{u}\mathbf{u}^T)\mathbf{u} = \mathbf{u} - 2\mathbf{u}\mathbf{u}^T\mathbf{u}$ . This is  $-\mathbf{u}$ , provided that  $\mathbf{u}^T\mathbf{u}$  equals 1 (b)  $Q\mathbf{v} = (I - 2\mathbf{u}\mathbf{u}^T)\mathbf{v} = \mathbf{v} - 2\mathbf{u}\mathbf{u}^T\mathbf{v} = \mathbf{v}$ , provided that  $\mathbf{u}^T\mathbf{v} = 0$ .
- 35 Starting from  $\mathbf{A} = (1, -1, 0, 0)$ , the orthogonal (not orthonormal) vectors  $\mathbf{B} = (1, 1, -2, 0)$  and  $\mathbf{C} = (1, 1, 1, -3)$  and  $\mathbf{D} = (1, 1, 1, 1)$  are in the directions of  $\mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4$ . The 4 by 4 and 5 by 5 matrices with *integer orthogonal columns* (not orthogonal rows, since not orthonormal  $Q$ !) are  $\begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 0 & -2 & 1 & 1 \\ 0 & 0 & -3 & 1 \end{bmatrix}$  and  $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & 1 & 1 \\ 0 & 0 & -3 & 1 & 1 \\ 0 & 0 & 0 & -4 & 1 \end{bmatrix}$

- 36**  $[Q, R] = \mathbf{q}_r(A)$  produces from  $A$  ( $m$  by  $n$  of rank  $n$ ) a “full-size” square  $Q = [Q_1 \ Q_2]$  and  $\begin{bmatrix} R \\ 0 \end{bmatrix}$ . The columns of  $Q_1$  are the orthonormal basis from Gram-Schmidt of the column space of  $A$ . The  $m - n$  columns of  $Q_2$  are an orthonormal basis for the left nullspace of  $A$ . Together the columns of  $Q = [Q_1 \ Q_2]$  are an orthonormal basis for  $\mathbf{R}^m$ .
- 37** This question describes the next  $\mathbf{q}_{n+1}$  in Gram-Schmidt using the matrix  $Q$  with the columns  $\mathbf{q}_1, \dots, \mathbf{q}_n$  (instead of using those  $\mathbf{q}$ 's separately). Start from  $\mathbf{a}$ , subtract its projection  $\mathbf{p} = Q^T \mathbf{a}$  onto the earlier  $\mathbf{q}$ 's, divide by the length of  $\mathbf{e} = \mathbf{a} - Q^T \mathbf{a}$  to get  $\mathbf{q}_{n+1} = \mathbf{e} / \|\mathbf{e}\|$ .

### Problem Set 5.1, page 251

- 1**  $\det(2A) = 8$ ;  $\det(-A) = (-1)^4 \det A = \frac{1}{2}$ ;  $\det(A^2) = \frac{1}{4}$ ;  $\det(A^{-1}) = 2 = \det(A^T)^{-1}$ .
- 2**  $\det(\frac{1}{2}A) = (\frac{1}{2})^3 \det A = -\frac{1}{8}$  and  $\det(-A) = (-1)^3 \det A = 1$ ;  $\det(A^2) = 1$ ;  $\det(A^{-1}) = -1$ .
- 3** (a) *False*:  $\det(I + I)$  is not  $1 + 1$  (b) *True*: The product rule extends to  $ABC$  (use it twice) (c) *False*:  $\det(4A)$  is  $4^n \det A$  (d) *False*:  $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ ,  $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ ,  $AB - BA = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$  is invertible.
- 4** Exchange rows 1 and 3 to show  $|J_3| = -1$ . Exchange rows 1 and 4, then 2 and 3 to show  $|J_4| = 1$ .
- 5**  $|J_5| = 1$ ,  $|J_6| = -1$ ,  $|J_7| = -1$ . Determinants 1, 1, -1, -1 repeat so  $|J_{101}| = 1$ .
- 6** To prove Rule 6, multiply the zero row by  $t = 2$ . The determinant is multiplied by 2 (Rule 3) but the matrix is the same. So  $2 \det(A) = \det(A)$  and  $\det(A) = 0$ .
- 7**  $\det(Q) = 1$  for rotation and  $\det(Q) = -1$  for reflection ( $1 - 2 \sin^2 \theta - 2 \cos^2 \theta = -1$ ).
- 8**  $Q^T Q = I \Rightarrow |Q|^2 = 1 \Rightarrow |Q| = \pm 1$ ;  $Q^n$  stays orthogonal so  $\det$  can't blow up.
- 9**  $\det A = 1$  from two row exchanges.  $\det B = 2$  (subtract rows 1 and 2 from row 3, then columns 1 and 2 from column 3).  $\det C = 0$  (equal rows) even though  $C = A + B$ !
- 10** If the entries in every row add to zero, then  $(1, 1, \dots, 1)$  is in the nullspace: singular  $A$  has  $\det = 0$ . (The columns add to the zero column so they are linearly dependent.) If every row adds to one, then rows of  $A - I$  add to zero (not necessarily  $\det A = 1$ ).
- 11**  $CD = -DC \Rightarrow \det CD = (-1)^n \det DC$  and *not*  $-\det DC$ . If  $n$  is even we can have an invertible  $CD$ .
- 12**  $\det(A^{-1})$  divides twice by  $ad - bc$  (once for each row). This gives  $\frac{ad-bc}{(ad-bc)^2} = \frac{1}{ad-bc}$ .
- 13** Pivots 1, 1, 1 give determinant = 1; pivots 1, -2, -3/2 give determinant = 3.
- 14**  $\det(A) = 36$  and the 4 by 4 second difference matrix has  $\det = 5$ .
- 15** The first determinant is 0, the second is  $1 - 2t^2 + t^4 = (1 - t^2)^2$ .

- 16** A singular rank one matrix has determinant = 0. The skew-symmetric  $K$  also  $\det K = 0$  (see #17).
- 17** Any 3 by 3 skew-symmetric  $K$  has  $\det(K^T) = \det(-K) = (-1)^3 \det(K)$ . This is  $-\det(K)$ . But always  $\det(K^T) = \det(K)$ . So we must have  $\det(K) = 0$  for 3 by 3.
- 18** 
$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} = \begin{vmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{vmatrix} \quad (\text{to reach 2 by 2, eliminate } a \text{ and } a^2 \text{ in row 1 by column operations}).$$
 Factor out  $b-a$  and  $c-a$  from the 2 by 2:  $(b-a)(c-a) \begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix} = (b-a)(c-a)(c-b)$ .
- 19** For triangular matrices, just multiply the diagonal entries:  $\det(U) = 6$ ,  $\det(U^{-1}) = \frac{1}{6}$ , and  $\det(U^2) = 36$ . 2 by 2 matrix:  $\det(U) = ad$ ,  $\det(U^2) = a^2 d^2$ . If  $ad \neq 0$  then  $\det(U^{-1}) = 1/ad$ .
- 20**  $\det \begin{bmatrix} a-Lc & b-Ld \\ c-\ell a & d-\ell b \end{bmatrix}$  reduces to  $(ad-bc)(1-L\ell)$ . The determinant changes if you do two row operations at once.
- 21** Rules 5 and 3 give Rule 2. (Since Rules 4 and 3 give 5, they also give Rule 2.)
- 22**  $\det(A) = 3$ ,  $\det(A^{-1}) = \frac{1}{3}$ ,  $\det(A - \lambda I) = \lambda^2 - 4\lambda + 3$ . The numbers  $\lambda = 1$  and  $\lambda = 3$  give  $\det(A - \lambda I) = 0$ . *Note to instructor:* If you discuss this exercise, you can explain that this is the reason determinants come before eigenvalues. Identify  $\lambda = 1$  and  $\lambda = 3$  as the eigenvalues of  $A$ .
- 23**  $\det(A) = 10$ ,  $A^2 = \begin{bmatrix} 18 & 7 \\ 14 & 11 \end{bmatrix}$ ,  $\det(A^2) = 100$ ,  $A^{-1} = \frac{1}{10} \begin{bmatrix} 3 & -1 \\ -2 & 4 \end{bmatrix}$  has  $\det \frac{1}{10}$ .  $\det(A - \lambda I) = \lambda^2 - 7\lambda + 10 = 0$  when  $\lambda = 2$  or  $\lambda = 5$ ; those are eigenvalues.
- 24** Here  $A = LU$  with  $\det(L) = 1$  and  $\det(U) = -6$  product of pivots, so also  $\det(A) = -6$ .  $\det(U^{-1}L^{-1}) = -\frac{1}{6} = 1/\det(A)$  and  $\det(U^{-1}L^{-1}A)$  is  $\det I = 1$ .
- 25** When the  $ij$  entry is  $ij$ , row 2 = 2 times row 1 so  $\det A = 0$ .
- 26** When the  $ij$  entry is  $i + j$ , row 3 - row 2 = row 2 - row 1 so  $A$  is singular:  $\det A = 0$ .
- 27**  $\det A = abc$ ,  $\det B = -abcd$ ,  $\det C = a(b-a)(c-b)$  by doing elimination.
- 28** (a) *True*:  $\det(AB) = \det(A)\det(B) = 0$  (b) *False*: A row exchange gives  $-\det =$  product of pivots. (c) *False*:  $A = 2I$  and  $B = I$  have  $A - B = I$  but the determinants have  $2^n - 1 \neq 1$  (d) *True*:  $\det(AB) = \det(A)\det(B) = \det(BA)$ .
- 29**  $A$  is rectangular so  $\det(A^T A) \neq (\det A^T)(\det A)$ : these determinants are not defined.
- 30** Derivatives of  $f = \ln(ad - bc)$ : 
$$\begin{bmatrix} \partial f / \partial a & \partial f / \partial c \\ \partial f / \partial b & \partial f / \partial d \end{bmatrix} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = A^{-1}.$$
- 31** The Hilbert determinants are  $1, 8 \times 10^{-2}, 4.6 \times 10^{-4}, 1.6 \times 10^{-7}, 3.7 \times 10^{-12}, 5.4 \times 10^{-18}, 4.8 \times 10^{-25}, 2.7 \times 10^{-33}, 9.7 \times 10^{-43}, 2.2 \times 10^{-53}$ . Pivots are ratios of determinants so the 10th pivot is near  $10^{-10}$ . The Hilbert matrix is numerically difficult (*ill-conditioned*).

- 32** Typical determinants of  $\text{rand}(n)$  are  $10^6, 10^{25}, 10^{79}, 10^{218}$  for  $n = 50, 100, 200, 400$ .  $\text{randn}(n)$  with normal distribution gives  $10^{31}, 10^{78}, 10^{186}, \text{Inf}$  which means  $\geq 2^{1024}$ . MATLAB allows  $1.999999999999999 \times 2^{1023} \approx 1.8 \times 10^{308}$  but one more 9 gives Inf!
- 33** I now know that maximizing the determinant for  $1, -1$  matrices is **Hadamard's problem** (1893): see Brenner in American Math. Monthly volume 79 (1972) 626-630. Neil Sloane's wonderful On-Line Encyclopedia of Integer Sequences ([research.att.com/~njas](http://research.att.com/~njas)) includes the solution for small  $n$  (and more references) when the problem is changed to  $0, 1$  matrices. That sequence A003432 starts from  $n = 0$  with 1, 1, 1, 2, 3, 5, 9. Then the  $1, -1$  maximum for size  $n$  is  $2^{n-1}$  times the  $0, 1$  maximum for size  $n - 1$  (so  $(32)(5) = 160$  for  $n = 6$  in sequence **A003433**).

To reduce the  $1, -1$  problem from 6 by 6 to the  $0, 1$  problem for 5 by 5, multiply the six rows by  $\pm 1$  to put +1 in column 1. Then subtract row 1 from rows 2 to 6 to get a 5 by 5 submatrix  $S$  of  $-2, 0$  and divide  $S$  by  $-2$ .

Here is an advanced MATLAB code and a  $1, -1$  matrix with largest  $\det A = 48$  for  $n = 5$ :

```
n = 5; p = (n - 1)^2; A0 = ones(n); maxdet = 0;
for k = 0 : 2^p - 1
    Asub = rem(floor(k ./ 2.^(-p + 1 : 0)), 2); A = A0; A(2 : n, 2 : n) = 1 - 2*
    reshape(Asub, n - 1, n - 1);
    if abs(det(A)) > maxdet, maxdet = abs(det(A)); maxA = A;
end
end
```

```
Output: maxA =   1   1   1   1   1   maxdet = 48.
                1   1   1  -1  -1
                1   1  -1   1  -1
                1  -1   1   1  -1
                1  -1  -1  -1   1
```

- 34** Reduce  $B$  by row operations to [row 3; row 2; row 1]. Then  $\det B = -6$  (odd permutation).

## Problem Set 5.2, page 263

- $\det A = 1 + 18 + 12 - 9 - 4 - 6 = 12$ , rows are independent;  $\det B = 0$ , row 1 + row 2 = row 3;  $\det C = -1$ , independent rows ( $\det C$  has one term, odd permutation)
- $\det A = -2$ , independent;  $\det B = 0$ , dependent;  $\det C = -1$ , independent.
- All cofactors of row 1 are zero.  $A$  has rank  $\leq 2$ . Each of the 6 terms in  $\det A$  is zero. Column 2 has no pivot.
- $a_{11}a_{23}a_{32}a_{44}$  gives  $-1$ , because  $2 \leftrightarrow 3$ ,  $a_{14}a_{23}a_{32}a_{41}$  gives  $+1$ ,  $\det A = 1 - 1 = 0$ ;  $\det B = 2 \cdot 4 \cdot 4 \cdot 2 - 1 \cdot 4 \cdot 4 \cdot 1 = 64 - 16 = 48$ .
- Four zeros in the same row guarantee  $\det = 0$ .  $A = I$  has 12 zeros (maximum with  $\det \neq 0$ ).
- (a) If  $a_{11} = a_{22} = a_{33} = 0$  then 4 terms are sure zeros (b) 15 terms must be zero.

- 7  $5!/2 = 60$  permutation matrices have  $\det = +1$ . Move row 5 of  $I$  to the top; starting from  $(5, 1, 2, 3, 4)$  elimination will do four row exchanges.
- 8 Some term  $a_{1\alpha}a_{2\beta}\cdots a_{n\omega}$  in the big formula is not zero! Move rows  $1, 2, \dots, n$  into rows  $\alpha, \beta, \dots, \omega$ . Then these nonzero  $a$ 's will be on the main diagonal.
- 9 To get  $+1$  for the even permutations, the matrix needs an *even* number of  $-1$ 's. To get  $+1$  for the odd  $P$ 's, the matrix needs an *odd* number of  $-1$ 's. So all six terms  $= +1$  in the big formula and  $\det = 6$  are impossible:  $\max(\det) = 4$ .
- 10 The  $4!/2 = 12$  even permutations are  $(1, 2, 3, 4), (2, 1, 4, 3), (3, 1, 4, 2), (4, 3, 2, 1)$ , and 8  $P$ 's with one number in place and even permutation of the other three numbers.  $\det(I + P_{\text{even}}) = 16$  or  $4$  or  $0$  ( $16$  comes from  $I + I$ ).
- 11  $C = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ .  $D = \begin{bmatrix} 0 & 42 & -35 \\ 0 & -21 & 14 \\ -3 & 6 & -3 \end{bmatrix}$ .  $\det B = 1(0) + 2(42) + 3(-35) = -21$ . Puzzle:  $\det D = 441 = (-21)^2$ . Why?
- 12  $C = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$  and  $AC^T = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ . Therefore  $A^{-1} = \frac{1}{4}C^T = C^T/\det A$ .
- 13 (a)  $C_1 = 0, C_2 = -1, C_3 = 0, C_4 = 1$  (b)  $C_n = -C_{n-2}$  by cofactors of row 1 then cofactors of column 1. Therefore  $C_{10} = -C_8 = C_6 = -C_4 = C_2 = -1$ .
- 14 We must choose 1's from column 2 then column 1, column 4 then column 3, and so on. Therefore  $n$  must be even to have  $\det A_n \neq 0$ . The number of row exchanges is  $n/2$  so  $C_n = (-1)^{n/2}$ .
- 15 The 1, 1 cofactor of the  $n$  by  $n$  matrix is  $E_{n-1}$ . The 1, 2 cofactor has a single 1 in its first column, with cofactor  $E_{n-2}$ : sign gives  $-E_{n-2}$ . So  $E_n = E_{n-1} - E_{n-2}$ . Then  $E_1$  to  $E_6$  is  $1, 0, -1, -1, 0, 1$  and this cycle of six will repeat:  $E_{100} = E_4 = -1$ .
- 16 The 1, 1 cofactor of the  $n$  by  $n$  matrix is  $F_{n-1}$ . The 1, 2 cofactor has a 1 in column 1, with cofactor  $F_{n-2}$ . Multiply by  $(-1)^{1+2}$  and also  $(-1)$  from the 1, 2 entry to find  $F_n = F_{n-1} + F_{n-2}$  (so these determinants are Fibonacci numbers).
- 17  $|B_4| = 2 \det \begin{bmatrix} 1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} + \det \begin{bmatrix} 1 & -1 \\ -1 & 2 \\ -1 & -1 \end{bmatrix} = 2|B_3| - \det \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = 2|B_3| - |B_2|$ .  $|B_3|$  and  $-|B_2|$  are cofactors of row 4 of  $B_4$ .
- 18 Rule 3 (linearity in row 1) gives  $|B_n| = |A_n| - |A_{n-1}| = (n+1) - n = 1$ .
- 19 Since  $x, x^2, x^3$  are all in the same row, they are never multiplied in  $\det V_4$ . The determinant is zero at  $x = a$  or  $b$  or  $c$ , so  $\det V$  has factors  $(x-a)(x-b)(x-c)$ . Multiply by the cofactor  $V_3$ . The Vandermonde matrix  $V_{ij} = (x_i)^{j-1}$  is for fitting a polynomial  $p(x) = b$  at the points  $x_i$ . It has  $\det V = \text{product of all } x_k - x_m \text{ for } k > m$ .
- 20  $G_2 = -1, G_3 = 2, G_4 = -3$ , and  $G_n = (-1)^{n-1}(n-1) = (\text{product of the } \lambda\text{'s})$ .
- 21  $S_1 = 3, S_2 = 8, S_3 = 21$ . The rule looks like every second number in Fibonacci's sequence  $\dots 3, 5, 8, 13, 21, 34, 55, \dots$  so the guess is  $S_4 = 55$ . Following the solution to Problem 30 with 3's instead of 2's confirms  $S_4 = 81 + 1 - 9 - 9 - 9 = 55$ . Problem 33 directly proves  $S_n = F_{2n+2}$ .
- 22 Changing 3 to 2 in the corner reduces the determinant  $F_{2n+2}$  by 1 times the cofactor of that corner entry. This cofactor is the determinant of  $S_{n-1}$  (one size smaller) which is  $F_{2n}$ . Therefore changing 3 to 2 changes the determinant to  $F_{2n+2} - F_{2n}$  which is  $F_{2n+1}$ .

- 23** (a) If we choose an entry from  $B$  we must choose an entry from the zero block; result zero. This leaves entries from  $A$  times entries from  $D$  leading to  $(\det A)(\det D)$   
 (b) and (c) Take  $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ ,  $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ ,  $C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ ,  $D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ . See #25.
- 24** (a) All  $L$ 's have  $\det = 1$ ;  $\det U_k = \det A_k = 2, 6, -6$  for  $k = 1, 2, 3$  (b) Pivots  $2, \frac{3}{2}, \frac{-1}{3}$ .
- 25** Problem 23 gives  $\det \begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} = 1$  and  $\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = |A|$  times  $|D - CA^{-1}B|$  which is  $|AD - ACA^{-1}B|$ . If  $AC = CA$  this is  $|AD - CAA^{-1}B| = \det(AD - CB)$ .
- 26** If  $A$  is a row and  $B$  is a column then  $\det M = \det AB = \text{dot product of } A \text{ and } B$ . If  $A$  is a column and  $B$  is a row then  $AB$  has rank 1 and  $\det M = \det AB = 0$  (unless  $m = n = 1$ ). This block matrix is invertible when  $AB$  is invertible which certainly requires  $m \leq n$ .
- 27** (a)  $\det A = a_{11}C_{11} + \cdots + a_{1n}C_{1n}$ . Derivative with respect to  $a_{11} = \text{cofactor } C_{11}$ .
- 28** Row 1  $- 2$  row 2  $+ \text{row } 3 = 0$  so this matrix is singular.
- 29** There are five nonzero products, all 1's with a plus or minus sign. Here are the (row, column) numbers and the signs:  $+(1, 1)(2, 2)(3, 3)(4, 4) + (1, 2)(2, 1)(3, 4)(4, 3) - (1, 2)(2, 1)(3, 3)(4, 4) - (1, 1)(2, 2)(3, 4)(4, 3) - (1, 1)(2, 3)(3, 2)(4, 4)$ . Total  $-1$ .
- 30** The 5 products in solution 29 change to  $16 + 1 - 4 - 4 - 4$  since  $A$  has 2's and  $-1$ 's:  
 $(2)(2)(2)(2) + (-1)(-1)(-1)(-1) - (-1)(-1)(2)(2) - (2)(2)(-1)(-1) - (2)(-1)(-1)(2)$ .
- 31**  $\det P = -1$  because the cofactor of  $P_{14} = 1$  in row one has sign  $(-1)^{1+4}$ . The big formula for  $\det P$  has only one term  $(1 \cdot 1 \cdot 1 \cdot 1)$  with minus sign because three exchanges take 4, 1, 2, 3 into 1, 2, 3, 4;  $\det(P^2) = (\det P)(\det P) = +1$  so  $\det \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} = \det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$  is *not right*.
- 32** The problem is to show that  $F_{2n+2} = 3F_{2n} - F_{2n-2}$ . Keep using Fibonacci's rule:  
 $F_{2n+2} = F_{2n+1} + F_{2n} = F_{2n} + F_{2n-1} + F_{2n} = 2F_{2n} + (F_{2n} - F_{2n-2}) = 3F_{2n} - F_{2n-2}$ .
- 33** The difference from 20 to 19 multiplies its 3 by 3 cofactor  $= 1$ : then  $\det$  drops by 1.
- 34** (a) The last three rows must be dependent (b) In each of the 120 terms: Choices from the last 3 rows must use 3 columns; at least one of those choices will be zero.
- 35** Subtracting 1 from the  $n, n$  entry subtracts its cofactor  $C_{nn}$  from the determinant. That cofactor is  $C_{nn} = 1$  (smaller Pascal matrix). Subtracting 1 from 1 leaves 0.

### Problem Set 5.3, page 279

- 1** (a)  $\begin{vmatrix} 2 & 5 \\ 1 & 4 \end{vmatrix} = 3$ ,  $\begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix} = 6$ ,  $\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$  so  $x_1 = -6/3 = -2$  and  $x_2 = 3/3 = 1$  (b)  $|A| = 4$ ,  $|B_1| = 3$ ,  $|B_2| = 2$ ,  $|B_3| = 1$ . Therefore  $x_1 = 3/4$  and  $x_2 = -1/2$  and  $x_3 = 1/4$ .

- 2 (a)  $y = \begin{vmatrix} a & 1 \\ c & 0 \end{vmatrix} / \begin{vmatrix} a & b \\ c & d \end{vmatrix} = c/(ad - bc)$  (b)  $y = \det B_2 / \det A = (fg - id)/D$ .
- 3 (a)  $x_1 = 3/0$  and  $x_2 = -2/0$ : no solution (b)  $x_1 = x_2 = 0/0$ : undetermined.
- 4 (a)  $x_1 = \det([b \ a_2 \ a_3]) / \det A$ , if  $\det A \neq 0$  (b) The determinant is linear in its first column so  $x_1|a_1 \ a_2 \ a_3| + x_2|a_2 \ a_2 \ a_3| + x_3|a_3 \ a_2 \ a_3|$ . The last two determinants are zero because of repeated columns, leaving  $x_1|a_1 \ a_2 \ a_3|$  which is  $x_1 \det A$ .
- 5 If the first column in  $A$  is also the right side  $b$  then  $\det A = \det B_1$ . Both  $B_2$  and  $B_3$  are singular since a column is repeated. Therefore  $x_1 = |B_1|/|A| = 1$  and  $x_2 = x_3 = 0$ .
- 6 (a)  $\begin{bmatrix} 1 & -\frac{2}{3} & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & -\frac{7}{3} & 1 \end{bmatrix}$  (b)  $\frac{1}{4} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ . An invertible symmetric matrix has a symmetric inverse.
- 7 If all cofactors = 0 then  $A^{-1}$  would be the zero matrix if it existed; cannot exist. (And the cofactor formula gives  $\det A = 0$ .)  $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$  has no zero cofactors but it is not invertible.
- 8  $C = \begin{bmatrix} 6 & -3 & 0 \\ 3 & 1 & -1 \\ -6 & 2 & 1 \end{bmatrix}$  and  $AC^T = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ . This is  $(\det A)I$  and  $\det A = 3$ . The 1, 3 cofactor of  $A$  is 0. Multiplying by 4 or 100: no change.
- 9 If we know the cofactors and  $\det A = 1$ , then  $C^T = A^{-1}$  and also  $\det A^{-1} = 1$ . Now  $A$  is the inverse of  $C^T$ , so  $A$  can be found from the cofactor matrix for  $C$ .
- 10 Take the determinant of  $AC^T = (\det A)I$ . The left side gives  $\det AC^T = (\det A)(\det C)$  while the right side gives  $(\det A)^n$ . Divide by  $\det A$  to reach  $\det C = (\det A)^{n-1}$ .
- 11 The cofactors of  $A$  are integers. Division by  $\det A = \pm 1$  gives integer entries in  $A^{-1}$ .
- 12 Both  $\det A$  and  $\det A^{-1}$  are integers since the matrices contain only integers. But  $\det A^{-1} = 1/\det A$  so  $\det A$  must be 1 or  $-1$ .
- 13  $A = \begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$  has cofactor matrix  $C = \begin{bmatrix} -1 & 2 & 1 \\ 3 & -6 & 2 \\ 1 & 3 & -1 \end{bmatrix}$  and  $A^{-1} = \frac{1}{5}C^T$ .
- 14 (a) Lower triangular  $L$  has cofactors  $C_{21} = C_{31} = C_{32} = 0$  (b)  $C_{12} = C_{21}$ ,  $C_{31} = C_{13}$ ,  $C_{32} = C_{23}$  make  $S^{-1}$  symmetric. (c) Orthogonal  $Q$  has cofactor matrix  $C = (\det Q)(Q^{-1})^T = \pm Q$  also orthogonal. Note  $\det Q = 1$  or  $-1$ .
- 15 For  $n = 5$ ,  $C$  contains 25 cofactors and each 4 by 4 cofactor has 24 terms. Each term needs 3 multiplications: total 1800 multiplications vs. 125 for Gauss-Jordan.
- 16 (a) Area  $|\begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix}| = 10$  (b) and (c) Area  $10/2 = 5$ , these triangles are half of the parallelogram in (a).
- 17 Volume =  $|\begin{vmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{vmatrix}| = 20$ . Area of faces = length of cross product =  $|\begin{vmatrix} i & j & k \\ 3 & 1 & 1 \\ 1 & 3 & 1 \end{vmatrix}| = \frac{-2i - 2j + 8k}{\text{length} = 6\sqrt{2}}$
- 18 (a) Area  $\frac{1}{2}|\begin{vmatrix} 2 & 1 & 1 \\ 3 & 4 & 1 \\ 0 & 5 & 1 \end{vmatrix}| = 5$  (b)  $5 +$  new triangle area  $\frac{1}{2}|\begin{vmatrix} 2 & 1 & 1 \\ 0 & 5 & 1 \\ -1 & 0 & 1 \end{vmatrix}| = 5 + 7 = 12$ .
- 19  $|\begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix}| = 4 = |\begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix}|$  because the transpose has the same determinant. See #22.



**20** The edges of the hypercube have length  $\sqrt{1+1+1+1} = 2$ . The volume  $\det H$  is  $2^4 = 16$ . ( $H/2$  has orthonormal columns. Then  $\det(H/2) = 1$  leads again to  $\det H = 16$ .)

**21** The maximum volume  $L_1 L_2 L_3 L_4$  is reached when the edges are orthogonal in  $\mathbf{R}^4$ . With entries 1 and  $-1$  all lengths are  $\sqrt{4} = 2$ . The maximum determinant is  $2^4 = 16$ , achieved in Problem 20. For a 3 by 3 matrix,  $\det A = (\sqrt{3})^3$  can't be achieved by  $\pm 1$ .

**22** This question is still waiting for a solution! An 18.06 student showed me how to transform the parallelogram for  $A$  to the parallelogram for  $A^T$ , without changing its area. (Edges slide along themselves, so no change in baselength or height or area.)

$$\mathbf{23} \quad A^T A = \begin{bmatrix} \mathbf{a}^T \\ \mathbf{b}^T \\ \mathbf{c}^T \end{bmatrix} \begin{bmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \end{bmatrix} = \begin{bmatrix} \mathbf{a}^T \mathbf{a} & 0 & 0 \\ 0 & \mathbf{b}^T \mathbf{b} & 0 \\ 0 & 0 & \mathbf{c}^T \mathbf{c} \end{bmatrix} \text{ has } \begin{array}{l} \det A^T A = (\|\mathbf{a}\| \|\mathbf{b}\| \|\mathbf{c}\|)^2 \\ \det A = \pm \|\mathbf{a}\| \|\mathbf{b}\| \|\mathbf{c}\| \end{array}$$

**24** The box has height 4 and volume  $= \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 4 \end{bmatrix} = 4$ .  $\mathbf{i} \times \mathbf{j} = \mathbf{k}$  and  $(\mathbf{k} \cdot \mathbf{w}) = 4$ .

**25** The  $n$ -dimensional cube has  $2^n$  corners,  $n2^{n-1}$  edges and  $2n(n-1)$ -dimensional faces. Coefficients from  $(2+x)^n$  in Worked Example 2.4A. Cube from  $2I$  has volume  $2^n$ .

**26** The pyramid has volume  $\frac{1}{6}$ . The 4-dimensional pyramid has volume  $\frac{1}{24}$  (and  $\frac{1}{n!}$  in  $\mathbf{R}^n$ )

**27**  $x = r \cos \theta$ ,  $y = r \sin \theta$  give  $J = r$ . The columns are orthogonal and their lengths are 1 and  $r$ .

**28**  $J = \begin{vmatrix} \sin \varphi \cos \theta & \rho \cos \varphi \sin \theta & -\rho \sin \varphi \sin \theta \\ \sin \varphi \sin \theta & \rho \cos \varphi \cos \theta & \rho \sin \varphi \cos \theta \\ \cos \varphi & -\rho \sin \varphi & \theta \end{vmatrix} = \rho^2 \sin \varphi$ . This Jacobian is needed for triple integrals inside spheres.

**29** From  $x, y$  to  $r, \theta$ :  $\begin{vmatrix} \partial r / \partial x & \partial r / \partial y \\ \partial \theta / \partial x & \partial \theta / \partial y \end{vmatrix} = \begin{vmatrix} x/r & y/r \\ -y/r^2 & x/r^2 \end{vmatrix} = \begin{vmatrix} \cos \theta & \sin \theta \\ (-\sin \theta)/r & (\cos \theta)/r \end{vmatrix} = \frac{1}{r} = \frac{1}{\text{Jacobian in 27}}$ .

**30** The triangle with corners  $(0, 0)$ ,  $(6, 0)$ ,  $(1, 4)$  has area 24. Rotated by  $\theta = 60^\circ$  the area is *unchanged*. The determinant of the rotation matrix is  $J = \begin{vmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{vmatrix} = \begin{vmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{vmatrix} = 1$ .

**31** Base area 10, height 2, volume 20.

**32** The volume of the box is  $\det \begin{bmatrix} 2 & 4 & 0 \\ -1 & 3 & 0 \\ 1 & 2 & 2 \end{bmatrix} = 20$ .

**33**  $\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = u_1 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$ . This is  $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ .

**34**  $(\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v} = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ : Even permutation of  $(\mathbf{u}, \mathbf{v}, \mathbf{w})$  keeps the same determinant. Odd permutations reverse the sign.

- 35**  $S = (2, 1, -1)$ , area  $\|PQ \times PS\| = \|(-2, -2, -1)\| = 3$ . The other four corners can be  $(0, 0, 0)$ ,  $(0, 0, 2)$ ,  $(1, 2, 2)$ ,  $(1, 1, 0)$ . The volume of the tilted box is  $|\det| = 1$ .
- 36** If  $(1, 1, 0)$ ,  $(1, 2, 1)$ ,  $(x, y, z)$  are in a plane the volume is  $\det \begin{bmatrix} x & y & z \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} = x - y + z = 0$ . The “box” with those edges is flattened to zero height.
- 37**  $\det \begin{bmatrix} x & y & z \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{bmatrix} = 7x - 5y + z$  will be *zero* when  $(x, y, z)$  is a combination of  $(2, 3, 1)$  and  $(1, 2, 3)$ . The plane containing those two vectors has equation  $7x - 5y + z = 0$ .
- 38** Doubling each row multiplies the volume by  $2^n$ . Then  $2 \det A = \det(2A)$  only if  $n = 1$ .
- 39**  $AC^T = (\det A)I$  gives  $(\det A)(\det C) = (\det A)^n$ . Then  $\det A = (\det C)^{1/3}$  with  $n = 4$ . With  $\det A^{-1} = 1/\det A$ , construct  $A^{-1}$  using the cofactors. *Invert to find  $A$ .*
- 40** The cofactor formula adds 1 by 1 determinants (which are just entries) *times* their cofactors of size  $n - 1$ . Jacobi discovered that this formula can be generalized. For  $n = 5$ , Jacobi multiplied each 2 by 2 determinant from rows 1-2 (with columns  $a < b$ ) times a 3 by 3 determinant from rows 3-5 (using the remaining columns  $c < d < e$ ).  
The key question is  $+$  or  $-$  sign (as for cofactors). The product is given a  $+$  sign when  $a, b, c, d, e$  is an even permutation of 1, 2, 3, 4, 5. This gives the correct determinant  $+1$  for that permutation matrix. More than that, all other  $P$  that permute  $a, b$  and separately  $c, d, e$  will come out with the correct sign when the 2 by 2 determinant for columns  $a, b$  multiplies the 3 by 3 determinant for columns  $c, d, e$ .
- 41** The Cauchy-Binet formula gives the determinant of a square matrix  $AB$  (and  $AA^T$  in particular) when the factors  $A, B$  are rectangular. For (2 by 3) times (3 by 2) there are 3 products of 2 by 2 determinants from  $A$  and  $B$  (printed in boldface):

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} g & j \\ h & k \\ i & \ell \end{bmatrix} \quad \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} g & j \\ h & k \\ i & \ell \end{bmatrix} \quad \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} g & j \\ h & k \\ i & \ell \end{bmatrix}$$

$$\text{Check } A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 7 \end{bmatrix} \quad AB = \begin{bmatrix} 14 & 30 \\ 30 & 66 \end{bmatrix}$$

$$\text{Cauchy-Binet: } (4-2)(4-2) + (7-3)(7-3) + (14-12)(14-12) = \mathbf{24}$$

$$(14)(66) - (30)(30) = \mathbf{24}$$

## Problem Set 6.1, page 293

- 1** The eigenvalues are 1 and 0.5 for  $A$ , 1 and 0.25 for  $A^2$ , 1 and 0 for  $A^\infty$ . Exchanging the rows of  $A$  changes the eigenvalues to 1 and  $-0.5$  (the trace is now  $0.2 + 0.3$ ). Singular matrices stay singular during elimination, so  $\lambda = 0$  does not change.
- 2**  $A$  has  $\lambda_1 = -1$  and  $\lambda_2 = 5$  with eigenvectors  $x_1 = (-2, 1)$  and  $x_2 = (1, 1)$ . The matrix  $A + I$  has the same eigenvectors, with eigenvalues increased by 1 to **0** and **6**. That zero eigenvalue correctly indicates that  $A + I$  is singular.
- 3**  $A$  has  $\lambda_1 = 2$  and  $\lambda_2 = -1$  (check trace and determinant) with  $x_1 = (1, 1)$  and  $x_2 = (2, -1)$ .  $A^{-1}$  has the same eigenvectors, with eigenvalues  $1/\lambda = \frac{1}{2}$  and  $-1$ .

- 4  $A$  has  $\lambda_1 = -3$  and  $\lambda_2 = 2$  (check trace  $= -1$  and determinant  $= -6$ ) with  $x_1 = (3, -2)$  and  $x_2 = (1, 1)$ .  $A^2$  has the *same eigenvectors* as  $A$ , with eigenvalues  $\lambda_1^2 = 9$  and  $\lambda_2^2 = 4$ .
- 5  $A$  and  $B$  have eigenvalues 1 and 3.  $A + B$  has  $\lambda_1 = 3, \lambda_2 = 5$ . Eigenvalues of  $A + B$  are *not equal* to eigenvalues of  $A$  plus eigenvalues of  $B$ .
- 6  $A$  and  $B$  have  $\lambda_1 = 1$  and  $\lambda_2 = 1$ .  $AB$  and  $BA$  have  $\lambda = 2 \pm \sqrt{3}$ . Eigenvalues of  $AB$  are *not equal* to eigenvalues of  $A$  times eigenvalues of  $B$ . Eigenvalues of  $AB$  and  $BA$  are *equal* (this is proved in section 6.6, Problems 18-19).
- 7 The eigenvalues of  $U$  (on its diagonal) are the *pivots* of  $A$ . The eigenvalues of  $L$  (on its diagonal) are all 1's. The eigenvalues of  $A$  are *not* the same as the pivots.
- 8 (a) Multiply  $Ax$  to see  $\lambda x$  which reveals  $\lambda$  (b) Solve  $(A - \lambda I)x = 0$  to find  $x$ .
- 9 (a) Multiply by  $A$ :  $A(Ax) = A(\lambda x) = \lambda Ax$  gives  $A^2x = \lambda^2x$  (b) Multiply by  $A^{-1}$ :  $x = A^{-1}Ax = A^{-1}\lambda x = \lambda A^{-1}x$  gives  $A^{-1}x = \frac{1}{\lambda}x$  (c) Add  $Ix = x$ :  $(A + I)x = (\lambda + 1)x$ .
- 10  $A$  has  $\lambda_1 = 1$  and  $\lambda_2 = .4$  with  $x_1 = (1, 2)$  and  $x_2 = (1, -1)$ .  $A^\infty$  has  $\lambda_1 = 1$  and  $\lambda_2 = 0$  (same eigenvectors).  $A^{100}$  has  $\lambda_1 = 1$  and  $\lambda_2 = (.4)^{100}$  which is near zero. So  $A^{100}$  is very near  $A^\infty$ : same eigenvectors and close eigenvalues.
- 11 Columns of  $A - \lambda_1 I$  are in the nullspace of  $A - \lambda_2 I$  because  $M = (A - \lambda_2 I)(A - \lambda_1 I) = \text{zero matrix}$  [this is the *Cayley-Hamilton Theorem* in Problem 6.2.32]. Notice that  $M$  has *zero eigenvalues*  $(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_1) = 0$  and  $(\lambda_2 - \lambda_2)(\lambda_2 - \lambda_1) = 0$ .
- 12 The projection matrix  $P$  has  $\lambda = 1, 0, 1$  with eigenvectors  $(1, 2, 0), (2, -1, 0), (0, 0, 1)$ . Add the first and last vectors:  $(1, 2, 1)$  also has  $\lambda = 1$ . Note  $P^2 = P$  leads to  $\lambda^2 = \lambda$  so  $\lambda = 0$  or  $1$ .
- 13 (a)  $Pu = (uu^T)u = u(u^Tu) = u$  so  $\lambda = 1$  (b)  $Pv = (uu^T)v = u(u^Tv) = 0$   
(c)  $x_1 = (-1, 1, 0, 0), x_2 = (-3, 0, 1, 0), x_3 = (-5, 0, 0, 1)$  all have  $Px = 0x = 0$ .
- 14 Two eigenvectors of this rotation matrix are  $x_1 = (1, i)$  and  $x_2 = (1, -i)$  (more generally  $cx_1$ , and  $dx_2$  with  $cd \neq 0$ ).
- 15 The other two eigenvalues are  $\lambda = \frac{1}{2}(-1 \pm i\sqrt{3})$ ; the three eigenvalues are  $1, 1, -1$ .
- 16 Set  $\lambda = 0$  in  $\det(A - \lambda I) = (\lambda_1 - \lambda) \dots (\lambda_n - \lambda)$  to find  $\det A = (\lambda_1)(\lambda_2) \dots (\lambda_n)$ .
- 17  $\lambda_1 = \frac{1}{2}(a + d + \sqrt{(a-d)^2 + 4bc})$  and  $\lambda_2 = \frac{1}{2}(a + d - \sqrt{(a-d)^2 + 4bc})$  add to  $a + d$ . If  $A$  has  $\lambda_1 = 3$  and  $\lambda_2 = 4$  then  $\det(A - \lambda I) = (\lambda - 3)(\lambda - 4) = \lambda^2 - 7\lambda + 12$ .
- 18 These 3 matrices have  $\lambda = 4$  and 5, trace 9, det 20:  $\begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ -1 & 6 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ -3 & 7 \end{bmatrix}$ .
- 19 (a) rank  $= 2$  (b)  $\det(B^T B) = 0$  (d) eigenvalues of  $(B^2 + I)^{-1}$  are  $1, \frac{1}{2}, \frac{1}{5}$ .
- 20  $A = \begin{bmatrix} 0 & 1 \\ -28 & 11 \end{bmatrix}$  has trace 11 and determinant 28, so  $\lambda = 4$  and 7. Moving to a 3 by 3 companion matrix,  $C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix}$  has  $\det(C - \lambda I) = -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = (1 - \lambda)(2 - \lambda)(3 - \lambda)$ . Notice the trace  $6 = 1 + 2 + 3$ , determinant  $6 = (1)(2)(3)$ , and also  $11 = (1)(2) + (1)(3) + (2)(3)$ .

- 21  $(A - \lambda I)$  has the same determinant as  $(A - \lambda I)^T$ .  $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$  and  $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$  have different eigenvectors. because every square matrix has  $\det M = \det M^T$ .
- 22  $\lambda = 1$  (for Markov), 0 (for singular),  $-\frac{1}{2}$  (so sum of eigenvalues = trace =  $\frac{1}{2}$ ).
- 23  $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ ,  $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ ,  $\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$ . Always  $A^2$  is the zero matrix if  $\lambda = 0$  and 0, by the Cayley-Hamilton Theorem in Problem 6.2.32.
- 24  $\lambda = 0, 0, 6$  (notice rank 1 and trace 6) with  $\mathbf{x}_1 = (0, -2, 1)$ ,  $\mathbf{x}_2 = (1, -2, 0)$ ,  $\mathbf{x}_3 = (1, 2, 1)$ .
- 25 With the same  $n$   $\lambda$ 's and  $\mathbf{x}$ 's,  $A\mathbf{x} = c_1\lambda_1\mathbf{x}_1 + \cdots + c_n\lambda_n\mathbf{x}_n$  equals  $B\mathbf{x} = c_1\lambda_1\mathbf{x}_1 + \cdots + c_n\lambda_n\mathbf{x}_n$  for all vectors  $\mathbf{x}$ . So  $A = B$ .
- 26 The block matrix has  $\lambda = 1, 2$  from  $B$  and 5, 7 from  $D$ . All entries of  $C$  are multiplied by zeros in  $\det(A - \lambda I)$ , so  $C$  has no effect on the eigenvalues.
- 27  $A$  has rank 1 with eigenvalues 0, 0, 0, 4 (the 4 comes from the trace of  $A$ ).  $C$  has rank 2 (ensuring two zero eigenvalues) and  $(1, 1, 1, 1)$  is an eigenvector with  $\lambda = 2$ . With trace 4, the other eigenvalue is also  $\lambda = 2$ , and its eigenvector is  $(1, -1, 1, -1)$ .
- 28  $B$  has  $\lambda = -1, -1, -1, 3$  and  $C$  has  $\lambda = 1, 1, 1, -3$ . Both have  $\det = -3$ .
- 29 Triangular matrix:  $\lambda(A) = 1, 4, 6$ ;  $\lambda(B) = 2, \sqrt{3}, -\sqrt{3}$ ; Rank-1 matrix:  $\lambda(C) = 0, 0, 6$ .
- 30  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a+b \\ c+d \end{bmatrix} = (a+b) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ;  $\lambda_2 = d-b$  to produce the correct trace  $(a+b) + (d-b) = a+d$ .
- 31 Eigenvector  $(1, 3, 4)$  for  $A$  with  $\lambda = 11$  and eigenvector  $(3, 1, 4)$  for  $PAP^T$ . Eigenvectors with  $\lambda \neq 0$  must be in the column space since  $A\mathbf{x}$  is always in the column space, and  $\mathbf{x} = A\mathbf{x}/\lambda$ .
- 32 (a)  $\mathbf{u}$  is a basis for the nullspace,  $\mathbf{v}$  and  $\mathbf{w}$  give a basis for the column space  
 (b)  $\mathbf{x} = (0, \frac{1}{3}, \frac{1}{5})$  is a particular solution. Add any  $c\mathbf{u}$  from the nullspace  
 (c) If  $A\mathbf{x} = \mathbf{u}$  had a solution,  $\mathbf{u}$  would be in the column space: wrong dimension 3.
- 33 If  $\mathbf{v}^T\mathbf{u} = 0$  then  $A^2 = \mathbf{u}(\mathbf{v}^T\mathbf{u})\mathbf{v}^T$  is the zero matrix and  $\lambda^2 = 0, 0$  and  $\lambda = 0, 0$  and trace  $(A) = 0$ . This zero trace also comes from adding the diagonal entries of  $A = \mathbf{u}\mathbf{v}^T$ :
- $$A = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} u_1v_1 & u_1v_2 \\ u_2v_1 & u_2v_2 \end{bmatrix} \quad \text{has trace } u_1v_1 + u_2v_2 = \mathbf{v}^T\mathbf{u} = 0$$
- 34  $\det(P - \lambda I) = 0$  gives the equation  $\lambda^4 = 1$ . This reflects the fact that  $P^4 = I$ . The solutions of  $\lambda^4 = 1$  are  $\lambda = 1, i, -1, -i$ . The real eigenvector  $\mathbf{x}_1 = (1, 1, 1, 1)$  is not changed by the permutation  $P$ . Three more eigenvectors are  $(i, i^2, i^3, i^4)$  and  $(1, -1, 1, -1)$  and  $(-i, (-i)^2, (-i)^3, (-i)^4)$ .
- 35 3 by 3 permutation matrices: Since  $P^T P = I$  gives  $(\det P)^2 = 1$ , the determinant is 1 or  $-1$ . The pivots are always 1 (but there may be row exchanges). The trace of  $P$  can be 3 (for  $P = I$ ) or 1 (for row exchange) or 0 (for double exchange). The possible eigenvalues are 1 and  $-1$  and  $e^{2\pi i/3}$  and  $e^{-2\pi i/3}$ .

- 36**  $\lambda_1 = e^{2\pi i/3}$  and  $\lambda_2 = e^{-2\pi i/3}$  give  $\det \lambda_1 \lambda_2 = 1$  and trace  $\lambda_1 + \lambda_2 = -1$ .  
 $A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$  with  $\theta = \frac{2\pi}{3}$  has this trace and det. So does every  $M^{-1}AM$ !
- 37** (a) Since the columns of  $A$  add to 1, one eigenvalue is  $\lambda = 1$  and the other is  $c - .6$  (to give the correct trace  $c + .4$ ).  
 (b) If  $c = 1.6$  then both eigenvalues are 1, and all solutions to  $(A - I)\mathbf{x} = \mathbf{0}$  are multiples of  $\mathbf{x} = (1, -1)$ .  
 (c) If  $c = .8$ , the eigenvectors for  $\lambda = 1$  are multiples of  $(1, 3)$ . Since all powers  $A^n$  also have column sums = 1,  $A^n$  will approach  $\frac{1}{4} \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} = \text{rank-1 matrix } A^\infty$  with eigenvalues 1, 0 and correct eigenvectors.  $(1, 3)$  and  $(1, -1)$ .

## Problem Set 6.2, page 307

- 1**  $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ ;  $\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$ .
- 2** Put the eigenvectors in  $S$  and eigenvalues in  $\Lambda$ .  $A = S\Lambda S^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$ .
- 3** If  $A = S\Lambda S^{-1}$  then the eigenvalue matrix for  $A + 2I$  is  $\Lambda + 2I$  and the eigenvector matrix is still  $S$ .  $A + 2I = S(\Lambda + 2I)S^{-1} = S\Lambda S^{-1} + S(2I)S^{-1} = A + 2I$ .
- 4** (a) False: don't know  $\lambda$ 's (b) True (c) True (d) False: need eigenvectors of  $S$
- 5** With  $S = I$ ,  $A = S\Lambda S^{-1} = \Lambda$  is a diagonal matrix. If  $S$  is triangular, then  $S^{-1}$  is triangular, so  $S\Lambda S^{-1}$  is also triangular.
- 6** The columns of  $S$  are nonzero multiples of  $(2, 1)$  and  $(0, 1)$ : either order. Same for  $A^{-1}$ .
- 7**  $A = S\Lambda S^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} / 2 = \begin{bmatrix} \lambda_1 + \lambda_2 & \lambda_1 - \lambda_2 \\ \lambda_1 - \lambda_2 & \lambda_1 + \lambda_2 \end{bmatrix} / 2 = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$  for any  $a$  and  $b$ .
- 8**  $A = S\Lambda S^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}$ .  $S\Lambda^k S^{-1} = \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2\text{nd component is } F_k \\ (\lambda_1^k - \lambda_2^k)/(\lambda_1 - \lambda_2) \end{bmatrix}$ .
- 9** (a)  $A = \begin{bmatrix} .5 & .5 \\ 1 & 0 \end{bmatrix}$  has  $\lambda_1 = 1$ ,  $\lambda_2 = -\frac{1}{2}$  with  $\mathbf{x}_1 = (1, 1)$ ,  $\mathbf{x}_2 = (1, -2)$   
 (b)  $A^n = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1^n & 0 \\ 0 & (-.5)^n \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix} \rightarrow A^\infty = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$
- 10** The rule  $F_{k+2} = F_{k+1} + F_k$  produces the pattern: even, odd, odd, even, odd, odd, ...
- 11** (a) True (no zero eigenvalues) (b) False (repeated  $\lambda = 2$  may have only one line of eigenvectors) (c) False (repeated  $\lambda$  may have a full set of eigenvectors)

- 12 (a) False: don't know  $\lambda$  (b) True: an eigenvector is missing (c) True.
- 13  $A = \begin{bmatrix} 8 & 3 \\ -3 & 2 \end{bmatrix}$  (or other),  $A = \begin{bmatrix} 9 & 4 \\ -4 & 1 \end{bmatrix}$ ,  $A = \begin{bmatrix} 10 & 5 \\ -5 & 0 \end{bmatrix}$ ; only eigenvectors are  $\mathbf{x} = (c, -c)$ .
- 14 The rank of  $A - 3I$  is  $r = 1$ . Changing any entry except  $a_{12} = 1$  makes  $A$  diagonalizable ( $A$  will have two different eigenvalues).
- 15  $A^k = S\Lambda^k S^{-1}$  approaches zero **if and only if every**  $|\lambda| < 1$ ;  $A_1^k \rightarrow A_1^\infty$ ,  $A_2^k \rightarrow 0$ .
- 16  $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & .2 \end{bmatrix}$  and  $S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ ;  $\Lambda^k \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$  and  $S\Lambda^k S^{-1} \rightarrow \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ : steady state.
- 17  $\Lambda = \begin{bmatrix} .9 & 0 \\ 0 & .3 \end{bmatrix}$ ,  $S = \begin{bmatrix} 3 & -3 \\ 1 & 1 \end{bmatrix}$ ;  $A_2^{10} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = (.9)^{10} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ ,  $A_2^{10} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = (.3)^{10} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ ,  $A_2^{10} \begin{bmatrix} 6 \\ 0 \end{bmatrix} = (.9)^{10} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + (.3)^{10} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$  because  $\begin{bmatrix} 6 \\ 0 \end{bmatrix}$  is the sum of  $\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ .
- 18  $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$  and  $A^k = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ . Multiply those last three matrices to get  $A^k = \frac{1}{2} \begin{bmatrix} 1+3^k & 1-3^k \\ 1-3^k & 1+3^k \end{bmatrix}$ .
- 19  $B^k = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}^k \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 5^k & 5^k - 4^k \\ 0 & 4^k \end{bmatrix}$ .
- 20  $\det A = (\det S)(\det \Lambda)(\det S^{-1}) = \det \Lambda = \lambda_1 \cdots \lambda_n$ . This proof works when  $A$  is diagonalizable.
- 21 trace  $ST = (aq + bs) + (cr + dt)$  is equal to  $(qa + rc) + (sb + td) = \text{trace } TS$ . Diagonalizable case: the trace of  $S\Lambda S^{-1} = \text{trace of } (\Lambda S^{-1})S = \Lambda$ : sum of the  $\lambda$ 's.
- 22  $AB - BA = I$  is impossible since trace  $AB - \text{trace } BA = \text{zero} \neq \text{trace } I$ .  $AB - BA = C$  is possible when trace  $(C) = 0$ , and  $E = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$  has  $EE^T - E^T E = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ .
- 23 If  $A = S\Lambda S^{-1}$  then  $B = \begin{bmatrix} A & 0 \\ 0 & 2A \end{bmatrix} = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} \Lambda & 0 \\ 0 & 2\Lambda \end{bmatrix} \begin{bmatrix} S^{-1} & 0 \\ 0 & S^{-1} \end{bmatrix}$ . So  $B$  has the additional eigenvalues  $2\lambda_1, \dots, 2\lambda_n$ .
- 24 The  $A$ 's form a subspace since  $cA$  and  $A_1 + A_2$  all have the same  $S$ . When  $S = I$  the  $A$ 's with those eigenvectors give the subspace of diagonal matrices. Dimension 4.
- 25 If  $A$  has columns  $\mathbf{x}_1, \dots, \mathbf{x}_n$  then column by column,  $A^2 = A$  means every  $A\mathbf{x}_i = \mathbf{x}_i$ . All vectors in the column space (combinations of those columns  $\mathbf{x}_i$ ) are eigenvectors with  $\lambda = 1$ . Always the nullspace has  $\lambda = 0$  ( $A$  might have dependent columns, so there could be less than  $n$  eigenvectors with  $\lambda = 1$ ). Dimensions of those spaces add to  $n$  by the Fundamental Theorem, so  $A$  is diagonalizable ( $n$  independent eigenvectors altogether).
- 26 Two problems: The nullspace and column space can overlap, so  $\mathbf{x}$  could be in both. There may not be  $r$  independent eigenvectors in the column space.

- 27**  $R = S\sqrt{\Lambda}S^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$  has  $R^2 = A$ .  $\sqrt{B}$  needs  $\lambda = \sqrt{9}$  and  $\sqrt{-1}$ , trace is not real. Note that  $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$  can have  $\sqrt{-1} = i$  and  $-i$ , trace 0, real square root  $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ .
- 28**  $A^T = A$  gives  $\mathbf{x}^T A B \mathbf{x} = (A\mathbf{x})^T (B\mathbf{x}) \leq \|A\mathbf{x}\| \|B\mathbf{x}\|$  by the Schwarz inequality.  $B^T = -B$  gives  $-\mathbf{x}^T B A \mathbf{x} = (B\mathbf{x})^T (A\mathbf{x}) \leq \|A\mathbf{x}\| \|B\mathbf{x}\|$ . Add to get Heisenberg's Uncertainty Principle when  $AB - BA = I$ . Position-momentum, also time-energy.
- 29** The factorizations of  $A$  and  $B$  into  $S\Lambda S^{-1}$  are the same. So  $A = B$ . (This is the same as Problem 6.1.25, expressed in matrix form.)
- 30**  $A = S\Lambda_1 S^{-1}$  and  $B = S\Lambda_2 S^{-1}$ . Diagonal matrices always give  $\Lambda_1 \Lambda_2 = \Lambda_2 \Lambda_1$ . Then  $AB = BA$  from  $S\Lambda_1 S^{-1} S\Lambda_2 S^{-1} = S\Lambda_1 \Lambda_2 S^{-1} = S\Lambda_2 \Lambda_1 S^{-1} = S\Lambda_2 S^{-1} S\Lambda_1 S^{-1} = BA$ .
- 31** (a)  $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$  has  $\lambda = a$  and  $\lambda = d$ :  $(A - aI)(A - dI) = \begin{bmatrix} 0 & b \\ 0 & d - a \end{bmatrix} \begin{bmatrix} a - d & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ . (b)  $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$  has  $A^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$  and  $A^2 - A - I = 0$  is true, matching  $\lambda^2 - \lambda - 1 = 0$  as the Cayley-Hamilton Theorem predicts.
- 32** When  $A = S\Lambda S^{-1}$  is diagonalizable, the matrix  $A - \lambda_j I = S(\Lambda - \lambda_j I)S^{-1}$  will have 0 in the  $j, j$  diagonal entry of  $\Lambda - \lambda_j I$ . In the product  $p(A) = (A - \lambda_1 I) \cdots (A - \lambda_n I)$ , each inside  $S^{-1}$  cancels  $S$ . This leaves  $S$  times (product of diagonal matrices  $\Lambda - \lambda_j I$ ) times  $S^{-1}$ . That product is the zero matrix because the factors produce a zero in each diagonal position. Then  $p(A) =$  zero matrix, which is the Cayley-Hamilton Theorem. (If  $A$  is not diagonalizable, one proof is to take a sequence of diagonalizable matrices approaching  $A$ .)

**Comment** I have also seen this reasoning but I am not convinced:

Apply the formula  $AC^T = (\det A)I$  from Section 5.3 to  $A - \lambda I$  with variable  $\lambda$ . Its cofactor matrix  $C$  will be a polynomial in  $\lambda$ , since cofactors are determinants:

$$(A - \lambda I) \operatorname{cof}(A - \lambda I)^T = \det(A - \lambda I)I = p(\lambda)I.$$

"For fixed  $A$ , this is an identity between two matrix polynomials." Set  $\lambda = A$  to find the zero matrix on the left, so  $p(A) =$  zero matrix on the right—which is the Cayley-Hamilton Theorem.

I am not certain about the key step of substituting a matrix for  $\lambda$ . If other matrices  $B$  are substituted, does the identity remain true? If  $AB \neq BA$ , even the order of multiplication seems unclear...

- 33**  $\lambda = 2, -1, 0$  are in  $\Lambda$  and the eigenvectors are in  $S$  (below).  $A^k = S\Lambda^k S^{-1}$  is

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix} \Lambda^k \frac{1}{6} \begin{bmatrix} 2 & 1 & 1 \\ 2 & -2 & -2 \\ 0 & 3 & -3 \end{bmatrix} = \frac{2^k}{6} \begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} + \frac{(-1)^k}{3} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

Check  $k = 4$ . The  $(2, 2)$  entry of  $A^4$  is  $2^4/6 + (-1)^4/3 = 18/6 = 3$ . The 4-step paths that begin and end at node 2 are 2 to 1 to 1 to 1 to 2, 2 to 1 to 2 to 1 to 2, and 2 to 1 to 3 to 1 to 2. Much harder to find the eleven 4-step paths that start and end at node 1.

- 34** If  $AB = BA$ , then  $B$  has the same eigenvectors  $(1, 0)$  and  $(0, 1)$  as  $A$ . So  $B$  is also diagonal  $b = c = 0$ . The nullspace for the following equation is 2-dimensional:  
 $AB - BA = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -b \\ c & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ . The coefficient matrix has rank  $4 - 2 = 2$ .
- 35**  $B$  has  $\lambda = i$  and  $-i$ , so  $B^4$  has  $\lambda^4 = 1$  and  $1$  and  $B^4 = I$ .  $C$  has  $\lambda = (1 \pm \sqrt{3}i)/2$ . This is  $\exp(\pm\pi i/3)$  so  $\lambda^3 = -1$  and  $-1$ . Then  $C^3 = -I$  and  $C^{1024} = -C$ .
- 36** The eigenvalues of  $A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$  are  $\lambda = e^{i\theta}$  and  $e^{-i\theta}$  (trace  $2 \cos \theta$  and  $\det = 1$ ). Their eigenvectors are  $(1, -i)$  and  $(1, i)$ :

$$\begin{aligned} A^n &= S \Lambda^n S^{-1} = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} e^{in\theta} & \\ & e^{-in\theta} \end{bmatrix} \begin{bmatrix} i & -1 \\ i & 1 \end{bmatrix} / 2i \\ &= \begin{bmatrix} (e^{in\theta} + e^{-in\theta})/2 & \dots \\ (e^{in\theta} - e^{-in\theta})/2i & \dots \end{bmatrix} = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}. \end{aligned}$$

Geometrically,  $n$  rotations by  $\theta$  give one rotation by  $n\theta$ .

- 37** Columns of  $S$  times rows of  $\Lambda S^{-1}$  will give  $r$  rank-1 matrices ( $r = \text{rank of } A$ ).
- 38** Note that  $\text{ones}(n) * \text{ones}(n) = n * \text{ones}(n)$ . This leads to  $C = 1/(n+1)$ .

$$\begin{aligned} AA^{-1} &= (\text{eye}(n) + \text{ones}(n)) * (\text{eye}(n) + C * \text{ones}(n)) \\ &= \text{eye}(n) + (1 + C + Cn) * \text{ones}(n) = \text{eye}(n). \end{aligned}$$

### Problem Set 6.3, page 325

- 1**  $\mathbf{u}_1 = e^{4t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ,  $\mathbf{u}_2 = e^t \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ . If  $\mathbf{u}(0) = (5, -2)$ , then  $\mathbf{u}(t) = 3e^{4t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2e^t \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ .
- 2**  $z(t) = 2e^t$ ; then  $dy/dt = 4y - 6e^t$  with  $y(0) = 5$  gives  $y(t) = 3e^{4t} + 2e^t$  as in Problem 1.
- 3** (a) If every column of  $A$  adds to zero, this means that the rows add to the zero row. So the rows are dependent, and  $A$  is singular, and  $\lambda = 0$  is an eigenvalue.
- (b) The eigenvalues of  $A = \begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix}$  are  $\lambda_1 = 0$  with eigenvector  $\mathbf{x}_1 = (3, 2)$  and  $\lambda_2 = -5$  (to give trace  $= -5$ ) with  $\mathbf{x}_2 = (1, -1)$ . Then the usual 3 steps:
1. Write  $\mathbf{u}(0) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$  as  $\begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \mathbf{x}_1 + \mathbf{x}_2$
  2. Follow those eigenvectors by  $e^{0t} \mathbf{x}_1$  and  $e^{-5t} \mathbf{x}_2$
  3. The solution  $\mathbf{u}(t) = \mathbf{x}_1 + e^{-5t} \mathbf{x}_2$  has steady state  $\mathbf{x}_1 = (3, 2)$ .
- 4**  $d(v+w)/dt = (w-v) + (v-w) = 0$ , so the total  $v+w$  is constant.  $A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$   
 has  $\lambda_1 = 0$  with  $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ,  $\mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ ;  $v(1) = 20 + 10e^{-2}$   $v(\infty) = 20$   
 $w(1) = 20 - 10e^{-2}$   $w(\infty) = 20$



- 5  $\frac{d}{dt} \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$  has  $\lambda = 0$  and  $+2$ :  $v(t) = 20 + 10e^{2t} - \infty$  as  $t \rightarrow \infty$ .
- 6  $A = \begin{bmatrix} a & 1 \\ 1 & a \end{bmatrix}$  has real eigenvalues  $a + 1$  and  $a - 1$ . These are both negative if  $a < -1$ , and the solutions of  $\mathbf{u}' = A\mathbf{u}$  approach zero.  $B = \begin{bmatrix} b & -1 \\ 1 & b \end{bmatrix}$  has complex eigenvalues  $b + i$  and  $b - i$ . These have negative real parts if  $b < 0$ , and all solutions of  $\mathbf{v}' = B\mathbf{v}$  approach zero.
- 7 A projection matrix has eigenvalues  $\lambda = 1$  and  $\lambda = 0$ . Eigenvectors  $P\mathbf{x} = \mathbf{x}$  fill the subspace that  $P$  projects onto: here  $\mathbf{x} = (1, 1)$ . Eigenvectors  $P\mathbf{x} = \mathbf{0}$  fill the perpendicular subspace: here  $\mathbf{x} = (1, -1)$ . For the solution to  $\mathbf{u}' = -P\mathbf{u}$ ,
- $$\mathbf{u}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \mathbf{u}(t) = e^{-t} \begin{bmatrix} 2 \\ 2 \end{bmatrix} + e^{0t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ approaches } \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$
- 8  $\begin{bmatrix} 6 & -2 \\ 2 & 1 \end{bmatrix}$  has  $\lambda_1 = 5$ ,  $\mathbf{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ ,  $\lambda_2 = 2$ ,  $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ ; rabbits  $r(t) = 20e^{5t} + 10e^{2t}$ , and  $w(t) = 10e^{5t} + 20e^{2t}$ . The ratio of rabbits to wolves approaches  $20/10$ ;  $e^{5t}$  dominates.
- 9 (a)  $\begin{bmatrix} 4 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ i \end{bmatrix} + 2 \begin{bmatrix} 1 \\ -i \end{bmatrix}$ . (b) Then  $\mathbf{u}(t) = 2e^{it} \begin{bmatrix} 1 \\ i \end{bmatrix} + 2e^{-it} \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} 4 \cos t \\ 4 \sin t \end{bmatrix}$ .
- 10  $\frac{d}{dt} \begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix}$ .  $A = \begin{bmatrix} 0 & 1 \\ 4 & 5 \end{bmatrix}$  has  $\det(A - \lambda I) = \lambda^2 - 5\lambda - 4 = 0$ . Directly substituting  $y = e^{\lambda t}$  into  $y'' = 5y' + 4y$  also gives  $\lambda^2 = 5\lambda + 4$  and the same two values of  $\lambda$ . Those values are  $\frac{1}{2}(5 \pm \sqrt{41})$  by the quadratic formula.
- 11  $e^{At} = I + t \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \text{zeros} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$ . Then  $\begin{bmatrix} y(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix} = \begin{bmatrix} y(0) + y'(0)t \\ y'(0) \end{bmatrix}$ . This  $y(t) = y(0) + y'(0)t$  solves the equation.
- 12  $A = \begin{bmatrix} 0 & 1 \\ -9 & 6 \end{bmatrix}$  has trace 6, det 9,  $\lambda = 3$  and 3 with *one* independent eigenvector  $(1, 3)$ .
- 13 (a)  $y(t) = \cos 3t$  and  $\sin 3t$  solve  $y'' = -9y$ . It is  $3 \cos 3t$  that starts with  $y(0) = 3$  and  $y'(0) = 0$ . (b)  $A = \begin{bmatrix} 0 & 1 \\ -9 & 0 \end{bmatrix}$  has  $\det = 9$ :  $\lambda = 3i$  and  $-3i$  with  $\mathbf{x} = (1, 3i)$  and  $(1, -3i)$ . Then  $\mathbf{u}(t) = \frac{3}{2}e^{3it} \begin{bmatrix} 1 \\ 3i \end{bmatrix} + \frac{3}{2}e^{-3it} \begin{bmatrix} 1 \\ -3i \end{bmatrix} = \begin{bmatrix} 3 \cos 3t \\ -9 \sin 3t \end{bmatrix}$ .
- 14 When  $A$  is skew-symmetric,  $\|\mathbf{u}(t)\| = \|e^{At}\mathbf{u}(0)\|$  is  $\|\mathbf{u}(0)\|$ . So  $e^{At}$  is *orthogonal*.
- 15  $\mathbf{u}_p = 4$  and  $\mathbf{u}(t) = ce^t + 4$ ;  $\mathbf{u}_p = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$  and  $\mathbf{u}(t) = c_1 e^t \begin{bmatrix} 1 \\ t \end{bmatrix} + c_2 e^t \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ .
- 16 Substituting  $\mathbf{u} = e^{ct}\mathbf{v}$  gives  $ce^{ct}\mathbf{v} = Ae^{ct}\mathbf{v} - e^{ct}\mathbf{b}$  or  $(A - cI)\mathbf{v} = \mathbf{b}$  or  $\mathbf{v} = (A - cI)^{-1}\mathbf{b}$  = particular solution. If  $c$  is an eigenvalue then  $A - cI$  is not invertible.

- 17** (a)  $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$  (b)  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  (c)  $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ . These show the unstable cases  
 (a)  $\lambda_1 < 0$  and  $\lambda_2 > 0$  (b)  $\lambda_1 > 0$  and  $\lambda_2 > 0$  (c)  $\lambda = a \pm ib$  with  $a > 0$
- 18**  $d/dt(e^{At}) = A + A^2t + \frac{1}{2}A^3t^2 + \frac{1}{6}A^4t^3 + \dots = A(I + At + \frac{1}{2}A^2t^2 + \frac{1}{6}A^3t^3 + \dots)$ .  
 This is exactly  $Ae^{At}$ , the derivative we expect.
- 19**  $e^{Bt} = I + Bt$  (short series with  $B^2 = 0$ )  $= \begin{bmatrix} 1 & -4t \\ 0 & 1 \end{bmatrix}$ . Derivative  $= \begin{bmatrix} 0 & -4 \\ 0 & 0 \end{bmatrix} = B$ .
- 20** The solution at time  $t + T$  is also  $e^{A(t+T)}\mathbf{u}(0)$ . Thus  $e^{At}$  times  $e^{AT}$  equals  $e^{A(t+T)}$ .
- 21**  $\begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 0 & -1 \end{bmatrix}; \begin{bmatrix} 1 & 4 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} e^t & 4e^t - 4 \\ 0 & 1 \end{bmatrix}$ .
- 22**  $A^2 = A$  gives  $e^{At} = I + At + \frac{1}{2}At^2 + \frac{1}{6}At^3 + \dots = I + (e^t - 1)A = \begin{bmatrix} e^t & e^t - 1 \\ 0 & 1 \end{bmatrix}$ .
- 23**  $e^A = \begin{bmatrix} e & 4(e-1) \\ 0 & 1 \end{bmatrix}$  from **21** and  $e^B = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}$  from **19**. By direct multiplication  
 $e^A e^B \neq e^B e^A \neq e^{A+B} = \begin{bmatrix} e & 0 \\ 0 & 1 \end{bmatrix}$ .
- 24**  $A = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}$ . Then  $e^{At} = \begin{bmatrix} e^t & \frac{1}{2}(e^{3t} - e^t) \\ 0 & e^{3t} \end{bmatrix}$ .
- 25** The matrix has  $A^2 = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} = A$ . Then all  $A^n = A$ . So  $e^{At} = I + (t + t^2/2! + \dots)A = I + (e^t - 1)A = \begin{bmatrix} e^t & 3(e^t - 1) \\ 0 & 0 \end{bmatrix}$  as in Problem 22.
- 26** (a) The inverse of  $e^{At}$  is  $e^{-At}$  (b) If  $A\mathbf{x} = \lambda\mathbf{x}$  then  $e^{At}\mathbf{x} = e^{\lambda t}\mathbf{x}$  and  $e^{\lambda t} \neq 0$ .  
 To see  $e^{At}\mathbf{x}$ , write  $(I + At + \frac{1}{2}A^2t^2 + \dots)\mathbf{x} = (1 + \lambda t + \frac{1}{2}\lambda^2t^2 + \dots)\mathbf{x} = e^{\lambda t}\mathbf{x}$ .
- 27**  $(x, y) = (e^{4t}, e^{-4t})$  is a growing solution. The correct matrix for the exchanged  $\mathbf{u} = (y, x)$  is  $\begin{bmatrix} 2 & -2 \\ -4 & 0 \end{bmatrix}$ . It *does* have the same eigenvalues as the original matrix.
- 28** Centering produces  $U_{n+1} = \begin{bmatrix} 1 & \Delta t \\ -\Delta t & 1 - (\Delta t)^2 \end{bmatrix} U_n$ . At  $\Delta t = 1$ ,  $\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$  has  $\lambda = e^{i\pi/3}$  and  $e^{-i\pi/3}$ . Both eigenvalues have  $\lambda^6 = 1$  so  $A^6 = I$ . Therefore  $U_6 = A^6 U_0$  comes exactly back to  $U_0$ .
- 29** First  $A$  has  $\lambda = \pm i$  and  $A^4 = I$ .  $A^n = (-1)^n \begin{bmatrix} 1 - 2n & -2n \\ 2n & 2n + 1 \end{bmatrix}$  Linear growth.  
 Second  $A$  has  $\lambda = -1, -1$  and
- 30** With  $a = \Delta t/2$  the trapezoidal step is  $U_{n+1} = \frac{1}{1+a^2} \begin{bmatrix} 1-a^2 & 2a \\ -2a & 1-a^2 \end{bmatrix} U_n$ .
- That matrix has orthonormal columns  $\Rightarrow$  orthogonal matrix  $\Rightarrow \|U_{n+1}\| = \|U_n\|$
- 31** (a)  $(\cos A)\mathbf{x} = (\cos \lambda)\mathbf{x}$  (b)  $\lambda(A) = 2\pi$  and  $0$  so  $\cos \lambda = 1, 1$  and  $\cos A = I$   
 (c)  $\mathbf{u}(t) = 3(\cos 2\pi t)(1, 1) + 1(\cos 0t)(1, -1)$  [ $\mathbf{u}' = A\mathbf{u}$  has **exp**,  $\mathbf{u}'' = A\mathbf{u}$  has **cos**]

## Problem Set 6.4, page 337

**Note** A way to complete the proof at the end of page 334, (perturbing the matrix to produce distinct eigenvalues) is now on the course website: “*Proofs of the Spectral Theorem.*” [math.mit.edu/linearalgebra](http://math.mit.edu/linearalgebra).

- 1  $A = \begin{bmatrix} 1 & 3 & 6 \\ 3 & 3 & 3 \\ 6 & 3 & 5 \end{bmatrix} + \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -3 \\ 2 & 3 & 0 \end{bmatrix} = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$   
 $= \text{symmetric} + \text{skew-symmetric}.$
- 2  $(A^T C A)^T = A^T C^T (A^T)^T = A^T C A.$  When  $A$  is 6 by 3,  $C$  will be 6 by 6 and the triple product  $A^T C A$  is 3 by 3.
- 3  $\lambda = 0, 4, -2$ ; unit vectors  $\pm(0, 1, -1)/\sqrt{2}$  and  $\pm(2, 1, 1)/\sqrt{6}$  and  $\pm(1, -1, -1)/\sqrt{3}.$
- 4  $\lambda = 10$  and  $-5$  in  $\Lambda = \begin{bmatrix} 10 & 0 \\ 0 & -5 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$  and  $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$  have to be normalized to unit vectors in  $Q = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}.$
- 5  $Q = \frac{1}{3} \begin{bmatrix} 2 & 1 & 2 \\ 2 & -2 & -1 \\ -1 & -2 & 2 \end{bmatrix}.$  The columns of  $Q$  are unit eigenvectors of  $A$ . Each unit eigenvector could be multiplied by  $-1$ .
- 6  $A = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$  has  $\lambda = 0$  and  $25$  so the columns of  $Q$  are the two eigenvectors:  
 $Q = \begin{bmatrix} .8 & .6 \\ -.6 & .8 \end{bmatrix}$  or we can exchange columns or reverse the signs of any column.
- 7 (a)  $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$  has  $\lambda = -1$  and  $3$  (b) The pivots have the same signs as the  $\lambda$ 's (c) trace  $= \lambda_1 + \lambda_2 = 2$ , so  $A$  can't have two negative eigenvalues.
- 8 If  $A^3 = 0$  then all  $\lambda^3 = 0$  so all  $\lambda = 0$  as in  $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$  If  $A$  is symmetric then  $A^3 = Q \Lambda^3 Q^T = 0$  requires  $\Lambda = 0.$  The only symmetric  $A$  is  $Q 0 Q^T =$  zero matrix.
- 9 If  $\lambda$  is complex then  $\bar{\lambda}$  is also an eigenvalue ( $A\bar{\mathbf{x}} = \bar{\lambda}\bar{\mathbf{x}}).$  Always  $\lambda + \bar{\lambda}$  is real. The trace is real so the third eigenvalue of a 3 by 3 real matrix must be real.
- 10 If  $\mathbf{x}$  is not real then  $\lambda = \mathbf{x}^T A \mathbf{x} / \mathbf{x}^T \mathbf{x}$  is *not* always real. Can't assume real eigenvectors!
- 11  $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = 2 \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} + 4 \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}; \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} = 0 \begin{bmatrix} .64 & -.48 \\ -.48 & .36 \end{bmatrix} + 25 \begin{bmatrix} .36 & .48 \\ .48 & .64 \end{bmatrix}$
- 12  $[\mathbf{x}_1 \ \mathbf{x}_2]$  is an orthogonal matrix so  $P_1 + P_2 = \mathbf{x}_1 \mathbf{x}_1^T + \mathbf{x}_2 \mathbf{x}_2^T = [\mathbf{x}_1 \ \mathbf{x}_2] \begin{bmatrix} \mathbf{x}_1^T \\ \mathbf{x}_2^T \end{bmatrix} = I;$   
 $P_1 P_2 = \mathbf{x}_1 (\mathbf{x}_1^T \mathbf{x}_2) \mathbf{x}_2^T = 0.$  Second proof:  $P_1 P_2 = P_1 (I - P_1) = P_1 - P_1 = 0$  since  $P_1^2 = P_1.$
- 13  $A = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$  has  $\lambda = ib$  and  $-ib.$  The block matrices  $\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$  and  $\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$  are also skew-symmetric with  $\lambda = ib$  (twice) and  $\lambda = -ib$  (twice).

14  $M$  is skew-symmetric and orthogonal;  $\lambda$ 's must be  $i, i, -i, -i$  to have trace zero.

15  $A = \begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix}$  has  $\lambda = 0, 0$  and only one independent eigenvector  $\mathbf{x} = (i, 1)$ . The good property for complex matrices is not  $A^T = A$  (symmetric) but  $\bar{A}^T = A$  (Hermitian with real eigenvalues and orthogonal eigenvectors: see Problem 20 and Section 10.2).

16 (a) If  $Az = \lambda \mathbf{y}$  and  $A^T \mathbf{y} = \lambda \mathbf{z}$  then  $B[\mathbf{y}; -\mathbf{z}] = [-Az; A^T \mathbf{y}] = -\lambda[\mathbf{y}; -\mathbf{z}]$ . So  $-\lambda$  is also an eigenvalue of  $B$ . (b)  $A^T A \mathbf{z} = A^T(\lambda \mathbf{y}) = \lambda^2 \mathbf{z}$ . (c)  $\lambda = -1, -1, 1, 1$ ;  $\mathbf{x}_1 = (1, 0, -1, 0)$ ,  $\mathbf{x}_2 = (0, 1, 0, -1)$ ,  $\mathbf{x}_3 = (1, 0, 1, 0)$ ,  $\mathbf{x}_4 = (0, 1, 0, 1)$ .

17 The eigenvalues of  $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$  are  $0, \sqrt{2}, -\sqrt{2}$  by Problem 16 with  $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ ,  
 $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ \sqrt{2} \end{bmatrix}$ ,  $\mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \\ -\sqrt{2} \end{bmatrix}$ .

18 1.  $\mathbf{y}$  is in the nullspace of  $A$  and  $\mathbf{x}$  is in the column space = row space because  $A = A^T$ . Those spaces are perpendicular so  $\mathbf{y}^T \mathbf{x} = 0$ .

2. If  $A\mathbf{x} = \lambda \mathbf{x}$  and  $A\mathbf{y} = \beta \mathbf{y}$  then shift by  $\beta$ :  $(A - \beta I)\mathbf{x} = (\lambda - \beta)\mathbf{x}$  and  $(A - \beta I)\mathbf{y} = \mathbf{0}$  and again  $\mathbf{x} \perp \mathbf{y}$ .

19  $A$  has  $S = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ;  $B$  has  $S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2d \end{bmatrix}$ . Perpendicular for  $A$   
 Not perpendicular for  $B$   
 since  $B^T \neq B$

20  $A = \begin{bmatrix} 1 & 3 + 4i \\ 3 - 4i & 1 \end{bmatrix}$  is a Hermitian matrix ( $\bar{A}^T = A$ ). Its eigenvalues 6 and  $-4$  are real. Adjust equations (1)–(2) in the text to prove that  $\lambda$  is always real when  $\bar{A}^T = A$ :

$A\mathbf{x} = \lambda \mathbf{x}$  leads to  $\bar{A}\bar{\mathbf{x}} = \bar{\lambda}\bar{\mathbf{x}}$ . Transpose to  $\bar{\mathbf{x}}^T A = \bar{\mathbf{x}}^T \bar{\lambda}$  using  $\bar{A}^T = A$ .

Then  $\bar{\mathbf{x}}^T A\mathbf{x} = \bar{\mathbf{x}}^T \lambda \mathbf{x}$  and also  $\bar{\mathbf{x}}^T A\mathbf{x} = \bar{\mathbf{x}}^T \bar{\lambda} \mathbf{x}$ . So  $\lambda = \bar{\lambda}$  is real.

21 (a) False.  $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$  (b) True from  $A^T = Q\Lambda Q^T$  (c) True from  $A^{-1} = Q\Lambda^{-1}Q^T$  (d) False!

22  $A$  and  $A^T$  have the same  $\lambda$ 's but the order of the  $\mathbf{x}$ 's can change.  $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$  has  $\lambda_1 = i$  and  $\lambda_2 = -i$  with  $\mathbf{x}_1 = (1, i)$  first for  $A$  but  $\mathbf{x}_1 = (1, -i)$  first for  $A^T$ .

23  $A$  is invertible, orthogonal, permutation, diagonalizable, Markov;  $B$  is projection, diagonalizable, Markov.  $A$  allows  $QR, S\Lambda S^{-1}, Q\Lambda Q^T$ ;  $B$  allows  $S\Lambda S^{-1}$  and  $Q\Lambda Q^T$ .

24 Symmetry gives  $Q\Lambda Q^T$  if  $b = 1$ ; repeated  $\lambda$  and no  $S$  if  $b = -1$ ; singular if  $b = 0$ .

25 Orthogonal and symmetric requires  $|\lambda| = 1$  and  $\lambda$  real, so  $\lambda = \pm 1$ . Then  $A = \pm I$  or  
 $A = Q\Lambda Q^T = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$ .

26 Eigenvectors  $(1, 0)$  and  $(1, 1)$  give a  $45^\circ$  angle even with  $A^T$  very close to  $A$ .

- 27** The roots of  $\lambda^2 + b\lambda + c = 0$  are  $\frac{1}{2}(-b \pm \sqrt{b^2 - 4ac})$ . Then  $\lambda_1 - \lambda_2$  is  $\sqrt{b^2 - 4c}$ . For  $\det(A + tB - \lambda I)$  we have  $b = -3 - 8t$  and  $c = 2 + 16t - t^2$ . The minimum of  $b^2 - 4c$  is  $1/17$  at  $t = 2/17$ . Then  $\lambda_2 - \lambda_1 = 1/\sqrt{17}$ .
- 28**  $A = \begin{bmatrix} 4 & 2+i \\ 2-i & 0 \end{bmatrix} = \overline{A}^T$  has real eigenvalues  $\lambda = 5$  and  $-1$  with trace  $= 4$  and  $\det = -5$ . The solution to **20** proves that  $\lambda$  is real when  $\overline{A}^T = A$  is Hermitian; I did not intend to repeat this part.
- 29** (a)  $A = Q\Lambda\overline{Q}^T$  times  $\overline{A}^T = Q\overline{\Lambda}^T\overline{Q}^T$  equals  $\overline{A}^T$  times  $A$  because  $\Lambda\overline{\Lambda}^T = \overline{\Lambda}^T\Lambda$  (diagonal!) (b) step 2: The 1, 1 entries of  $\overline{T}^T T$  and  $T\overline{T}^T$  are  $|a|^2$  and  $|a|^2 + |b|^2$ . This makes  $b = 0$  and  $T = \Lambda$ .
- 30**  $a_{11}$  is  $[q_{11} \dots q_{1n}] [\lambda_1 \overline{q}_{11} \dots \lambda_n \overline{q}_{1n}]^T \leq \lambda_{\max} (|q_{11}|^2 + \dots + |q_{1n}|^2) = \lambda_{\max}$ .
- 31** (a)  $\mathbf{x}^T(A\mathbf{x}) = (A\mathbf{x})^T\mathbf{x} = \mathbf{x}^T A^T \mathbf{x} = -\mathbf{x}^T A \mathbf{x}$ . (b)  $\overline{\mathbf{z}}^T A \mathbf{z}$  is pure imaginary, its real part is  $\mathbf{x}^T A \mathbf{x} + \mathbf{y}^T A \mathbf{y} = 0 + 0$  (c)  $\det A = \lambda_1 \dots \lambda_n \geq 0$ : pairs of  $\lambda$ 's  $= ib, -ib$ .
- 32** Since  $A$  is diagonalizable with eigenvalue matrix  $\Lambda = 2I$ , the matrix  $A$  itself has to be  $S\Lambda S^{-1} = S(2I)S^{-1} = 2I$ . (The unsymmetric matrix  $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$  also has  $\lambda = 2, 2$ .)

## Problem Set 6.5, page 350

- 1** Suppose  $a > 0$  and  $ac > b^2$  so that also  $c > b^2/a > 0$ . (i) The eigenvalues have the *same sign* because  $\lambda_1 \lambda_2 = \det = ac - b^2 > 0$ . (ii) That sign is *positive* because  $\lambda_1 + \lambda_2 > 0$  (it equals the trace  $a + c > 0$ ).
- 2** Only  $A_4 = \begin{bmatrix} 1 & 10 \\ 10 & 101 \end{bmatrix}$  has two positive eigenvalues.  $\mathbf{x}^T A_1 \mathbf{x} = 5x_1^2 + 12x_1x_2 + 7x_2^2$  is negative for example when  $x_1 = 4$  and  $x_2 = -3$ :  $A_1$  is not positive definite as its determinant confirms.
- 3** Positive definite for  $-3 < b < 3$   $\begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 9-b^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 9-b^2 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = LDL^T$   
Positive definite for  $c > 8$   $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 0 & c-8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & c-8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = LDL^T$ .
- 4**  $f(x, y) = x^2 + 4xy + 9y^2 = (x + 2y)^2 + 5y^2$ ;  $x^2 + 6xy + 9y^2 = (x + 3y)^2$ .
- 5**  $x^2 + 4xy + 3y^2 = (x + 2y)^2 - y^2 = \text{difference of squares}$  is negative at  $x = 2$ ,  $y = -1$ , where the first square is zero.
- 6**  $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$  produces  $f(x, y) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2xy$ .  $A$  has  $\lambda = 1$  and  $-1$ . Then  $A$  is an *indefinite matrix* and  $f(x, y) = 2xy$  has a *saddle point*.
- 7**  $R^T R = \begin{bmatrix} 1 & 2 \\ 2 & 13 \end{bmatrix}$  and  $R^T R = \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix}$  are positive definite;  $R^T R = \begin{bmatrix} 2 & 3 & 3 \\ 3 & 5 & 4 \\ 3 & 4 & 5 \end{bmatrix}$  is singular (and positive semidefinite). The first two  $R$ 's have independent columns. The 2 by 3  $R$  cannot have full column rank 3, with only 2 rows.
- 8**  $A = \begin{bmatrix} 3 & 6 \\ 6 & 16 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ . Pivots 3, 4 outside squares,  $\ell_{ij}$  inside.  $\mathbf{x}^T A \mathbf{x} = 3(x + 2y)^2 + 4y^2$ .

- 9  $A = \begin{bmatrix} 4 & -4 & 8 \\ -4 & 4 & -8 \\ 8 & -8 & 16 \end{bmatrix}$  has only one pivot = 4, rank  $A = 1$ , eigenvalues are 24, 0, 0,  $\det A = 0$ .
- 10  $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$  has pivots  $2, \frac{3}{2}, \frac{4}{3}$ ;  $B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$  is singular;  $B \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ .
- 11 Corner determinants  $|A_1| = 2$ ,  $|A_2| = 6$ ,  $|A_3| = 30$ . The pivots are  $2/1, 6/2, 30/6$ .
- 12  $A$  is positive definite for  $c > 1$ ; determinants  $c, c^2 - 1$ , and  $(c - 1)^2(c + 2) > 0$ .  $B$  is *never* positive definite (determinants  $d - 4$  and  $-4d + 12$  are never both positive).
- 13  $A = \begin{bmatrix} 1 & 5 \\ 5 & 10 \end{bmatrix}$  is an example with  $a + c > 2b$  but  $ac < b^2$ , so not positive definite.
- 14 The eigenvalues of  $A^{-1}$  are positive because they are  $1/\lambda(A)$ . And the entries of  $A^{-1}$  pass the determinant tests. And  $\mathbf{x}^T A^{-1} \mathbf{x} = (A^{-1} \mathbf{x})^T A (A^{-1} \mathbf{x}) > 0$  for all  $\mathbf{x} \neq \mathbf{0}$ .
- 15 Since  $\mathbf{x}^T A \mathbf{x} > 0$  and  $\mathbf{x}^T B \mathbf{x} > 0$  we have  $\mathbf{x}^T (A + B) \mathbf{x} = \mathbf{x}^T A \mathbf{x} + \mathbf{x}^T B \mathbf{x} > 0$  for all  $\mathbf{x} \neq \mathbf{0}$ . Then  $A + B$  is a positive definite matrix. The second proof uses the test  $A = R^T R$  (independent columns in  $R$ ): If  $A = R^T R$  and  $B = S^T S$  pass this test, then  $A + B = \begin{bmatrix} R & S \end{bmatrix}^T \begin{bmatrix} R \\ S \end{bmatrix}$  also passes, and must be positive definite.
- 16  $\mathbf{x}^T A \mathbf{x}$  is zero when  $(x_1, x_2, x_3) = (0, 1, 0)$  because of the zero on the diagonal. Actually  $\mathbf{x}^T A \mathbf{x}$  goes *negative* for  $\mathbf{x} = (1, -10, 0)$  because the second pivot is *negative*.
- 17 If  $a_{jj}$  were smaller than all  $\lambda$ 's,  $A - a_{jj}I$  would have all eigenvalues  $> 0$  (positive definite). But  $A - a_{jj}I$  has a *zero* in the  $(j, j)$  position; impossible by Problem 16.
- 18 If  $A\mathbf{x} = \lambda\mathbf{x}$  then  $\mathbf{x}^T A \mathbf{x} = \lambda \mathbf{x}^T \mathbf{x}$ . If  $A$  is positive definite this leads to  $\lambda = \mathbf{x}^T A \mathbf{x} / \mathbf{x}^T \mathbf{x} > 0$  (ratio of positive numbers). So positive energy  $\Rightarrow$  positive eigenvalues.
- 19 All cross terms are  $\mathbf{x}_i^T \mathbf{x}_j = 0$  because symmetric matrices have orthogonal eigenvectors. So positive eigenvalues  $\Rightarrow$  positive energy.
- 20 (a) The determinant is positive; all  $\lambda > 0$  (b) All projection matrices except  $I$  are singular (c) The diagonal entries of  $D$  are its eigenvalues (d)  $A = -I$  has  $\det = +1$  when  $n$  is even.
- 21  $A$  is positive definite when  $s > 8$ ;  $B$  is positive definite when  $t > 5$  by determinants.
- 22  $R = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{9} & \\ & \sqrt{1} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ ;  $R = Q \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} Q^T = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ .
- 23  $x^2/a^2 + y^2/b^2$  is  $\mathbf{x}^T A \mathbf{x}$  when  $A = \text{diag}(1/a^2, 1/b^2)$ . Then  $\lambda_1 = 1/a^2$  and  $\lambda_2 = 1/b^2$  so  $a = 1/\sqrt{\lambda_1}$  and  $b = 1/\sqrt{\lambda_2}$ . The ellipse  $9x^2 + 16y^2 = 1$  has axes with half-lengths  $a = \frac{1}{3}$  and  $b = \frac{1}{4}$ . The points  $(\frac{1}{3}, 0)$  and  $(0, \frac{1}{4})$  are at the ends of the axes.
- 24 The ellipse  $x^2 + xy + y^2 = 1$  has axes with half-lengths  $1/\sqrt{\lambda} = \sqrt{2}$  and  $\sqrt{2/3}$ .
- 25  $A = C^T C = \begin{bmatrix} 9 & 3 \\ 3 & 5 \end{bmatrix}$ ;  $\begin{bmatrix} 4 & 8 \\ 8 & 25 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$  and  $C = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}$

- 26 The Cholesky factors  $C = (L\sqrt{D})^T = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$  and  $C = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & \sqrt{5} \end{bmatrix}$  have square roots of the pivots from  $D$ . Note again  $C^T C = L D L^T = A$ .
- 27 Writing out  $\mathbf{x}^T A \mathbf{x} = \mathbf{x}^T L D L^T \mathbf{x}$  gives  $ax^2 + 2bxy + cy^2 = a(x + \frac{b}{a}y)^2 + \frac{ac-b^2}{a}y^2$ . So the  $LDL^T$  from elimination is exactly the same as *completing the square*. The example  $2x^2 + 8xy + 10y^2 = 2(x + 2y)^2 + 2y^2$  with pivots 2, 2 outside the squares and multiplier 2 inside.
- 28  $\det A = (1)(10)(1) = 10$ ;  $\lambda = 2$  and  $5$ ;  $\mathbf{x}_1 = (\cos \theta, \sin \theta)$ ,  $\mathbf{x}_2 = (-\sin \theta, \cos \theta)$ ; the  $\lambda$ 's are positive. So  $A$  is positive definite.
- 29  $H_1 = \begin{bmatrix} 6x^2 & 2x \\ 2x & 2 \end{bmatrix}$  is semidefinite;  $f_1 = (\frac{1}{2}x^2 + y)^2 = 0$  on the curve  $\frac{1}{2}x^2 + y = 0$ ;  
 $H_2 = \begin{bmatrix} 6x & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$  is indefinite at  $(0, 1)$  where 1st derivatives = 0. This is a saddle point of the function  $f_2(x, y)$ .
- 30  $ax^2 + 2bxy + cy^2$  has a saddle point if  $ac < b^2$ . The matrix is *indefinite* ( $\lambda < 0$  and  $\lambda > 0$ ) because the determinant  $ac - b^2$  is *negative*.
- 31 If  $c > 9$  the graph of  $z$  is a bowl, if  $c < 9$  the graph has a saddle point. When  $c = 9$  the graph of  $z = (2x + 3y)^2$  is a “trough” staying at zero along the line  $2x + 3y = 0$ .
- 32 Orthogonal matrices, exponentials  $e^{At}$ , matrices with  $\det = 1$  are groups. Examples of subgroups are orthogonal matrices with  $\det = 1$ , exponentials  $e^{An}$  for integer  $n$ . Another subgroup: lower triangular elimination matrices  $E$  with diagonal 1's.
- 33 A product  $AB$  of symmetric positive definite matrices comes into many applications. The “generalized” eigenvalue problem  $K\mathbf{x} = \lambda M\mathbf{x}$  has  $AB = M^{-1}K$ . (often we use  $\text{eig}(K, M)$  without actually inverting  $M$ .) All eigenvalues  $\lambda$  are positive:  
 $AB\mathbf{x} = \lambda\mathbf{x}$  gives  $(B\mathbf{x})^T AB\mathbf{x} = (B\mathbf{x})^T \lambda\mathbf{x}$ . Then  $\lambda = \mathbf{x}^T B^T AB\mathbf{x} / \mathbf{x}^T B\mathbf{x} > 0$ .
- 34 The five eigenvalues of  $K$  are  $2 - 2 \cos \frac{k\pi}{6} = 2 - \sqrt{3}, 2 - 1, 2, 2 + 1, 2 + \sqrt{3}$ . The product of those eigenvalues is  $6 = \det K$ .
- 35 Put parentheses in  $\mathbf{x}^T A^T C A \mathbf{x} = (A\mathbf{x})^T C (A\mathbf{x})$ . Since  $C$  is assumed positive definite, this energy can drop to zero only when  $A\mathbf{x} = \mathbf{0}$ . Since  $A$  is assumed to have independent columns,  $A\mathbf{x} = \mathbf{0}$  only happens when  $\mathbf{x} = \mathbf{0}$ . Thus  $A^T C A$  has positive energy and is positive definite.

My textbooks *Computational Science and Engineering* and *Introduction to Applied Mathematics* start with many examples of  $A^T C A$  in a wide range of applications. I believe this is a unifying concept from linear algebra.

## Problem Set 6.6, page 360

- 1  $B = G C G^{-1} = G F^{-1} A F G^{-1}$  so  $M = F G^{-1}$ .  $C$  similar to  $A$  and  $B \Rightarrow A$  similar to  $B$ .
- 2  $A = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$  is similar to  $B = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} = M^{-1} A M$  with  $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ .

$$3 \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = M^{-1}AM;$$

$$B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix};$$

$$B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

4  $A$  has no repeated  $\lambda$  so it can be diagonalized:  $S^{-1}AS = \Lambda$  makes  $A$  similar to  $\Lambda$ .

5  $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ ,  $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ ,  $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ ,  $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$  are similar (they all have eigenvalues 1 and 0).  
 $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  is by itself and also  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$  is by itself with eigenvalues 1 and  $-1$ .

6 Eight families of similar matrices: six matrices have  $\lambda = 0, 1$  (one family); three matrices have  $\lambda = 1, 1$  and three have  $\lambda = 0, 0$  (two families each!); one has  $\lambda = 1, -1$ ; one has  $\lambda = 2, 0$ ; two matrices have  $\lambda = \frac{1}{2}(1 \pm \sqrt{5})$  (they are in one family).

7 (a)  $(M^{-1}AM)(M^{-1}\mathbf{x}) = M^{-1}(A\mathbf{x}) = M^{-1}\mathbf{0} = \mathbf{0}$  (b) The nullspaces of  $A$  and of  $M^{-1}AM$  have the same dimension. Different vectors and different bases.

8 Same  $\Lambda$  But  $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$  and  $B = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$  have the same line of eigenvectors and the same eigenvalues  $\lambda = 0, 0$ .  
 Same  $S$

$$9 \quad A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, A^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \text{ every } A^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}. A^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } A^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

$$10 \quad J^2 = \begin{bmatrix} c^2 & 2c \\ 0 & c^2 \end{bmatrix} \text{ and } J^k = \begin{bmatrix} c^k & kc^{k-1} \\ 0 & c^k \end{bmatrix}; J^0 = I \text{ and } J^{-1} = \begin{bmatrix} c^{-1} & -c^{-2} \\ 0 & c^{-1} \end{bmatrix}.$$

11  $\mathbf{u}(0) = \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} v(0) \\ w(0) \end{bmatrix}$ . The equation  $\frac{d\mathbf{u}}{dt} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \mathbf{u}$  has  $\frac{dv}{dt} = \lambda v + w$  and  $\frac{dw}{dt} = \lambda w$ . Then  $w(t) = 2e^{\lambda t}$  and  $v(t)$  must include  $2te^{\lambda t}$  (this comes from the repeated  $\lambda$ ). To match  $v(0) = 5$ , the solution is  $v(t) = 2te^{\lambda t} + 5e^{\lambda t}$ .

$$12 \quad \text{If } M^{-1}JM = K \text{ then } JM = \begin{bmatrix} m_{21} & m_{22} & m_{23} & m_{24} \\ 0 & 0 & 0 & 0 \\ m_{41} & m_{42} & m_{43} & m_{44} \\ 0 & 0 & 0 & 0 \end{bmatrix} = MK = \begin{bmatrix} 0 & m_{12} & m_{13} & 0 \\ 0 & m_{22} & m_{23} & 0 \\ 0 & m_{32} & m_{33} & 0 \\ 0 & m_{42} & m_{43} & 0 \end{bmatrix}.$$

That means  $m_{21} = m_{22} = m_{23} = m_{24} = 0$ .  $M$  is not invertible,  $J$  not similar to  $K$ .

13 The five 4 by 4 Jordan forms with  $\lambda = 0, 0, 0, 0$  are  $J_1 =$  zero matrix and

$$J_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad J_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$J_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad J_5 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



Problem 12 showed that  $J_3$  and  $J_4$  are *not similar*, even with the same rank. Every matrix with all  $\lambda = 0$  is “*nilpotent*” (its  $n$ th power is  $A^n = \text{zero matrix}$ ). You see  $J^4 = 0$  for these matrices. How many possible Jordan forms for  $n = 5$  and all  $\lambda = 0$ ?

- 14** (1) Choose  $M_i =$  reverse diagonal matrix to get  $M_i^{-1}J_iM_i = M_i^T$  in each block  
 (2)  $M_0$  has those diagonal blocks  $M_i$  to get  $M_0^{-1}JM_0 = J^T$ . (3)  $A^T = (M^{-1})^T J^T M^T$  equals  $(M^{-1})^T M_0^{-1}JM_0M^T = (MM_0M^T)^{-1}A(MM_0M^T)$ , and  $A^T$  is similar to  $A$ .
- 15**  $\det(M^{-1}AM - \lambda I) = \det(M^{-1}AM - M^{-1}\lambda IM)$ . This is  $\det(M^{-1}(A - \lambda I)M)$ . By the product rule, the determinants of  $M$  and  $M^{-1}$  cancel to leave  $\det(A - \lambda I)$ .
- 16**  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  is similar to  $\begin{bmatrix} d & c \\ b & a \end{bmatrix}$ ;  $\begin{bmatrix} b & a \\ d & c \end{bmatrix}$  is similar to  $\begin{bmatrix} c & d \\ a & b \end{bmatrix}$ . So two pairs of similar matrices but  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  is not similar to  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ : different eigenvalues!
- 17** (a) *False*: Diagonalize a nonsymmetric  $A = S\Lambda S^{-1}$ . Then  $\Lambda$  is symmetric and similar  
 (b) *True*: A singular matrix has  $\lambda = 0$ . (c) *False*:  $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$  and  $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$  are similar (they have  $\lambda = \pm i$ ) (d) *True*: Adding  $I$  increases all eigenvalues by 1
- 18**  $AB = B^{-1}(BA)B$  so  $AB$  is similar to  $BA$ . If  $ABx = \lambda x$  then  $BA(Bx) = \lambda(Bx)$ .
- 19** Diagonal blocks 6 by 6, 4 by 4;  $AB$  has the same eigenvalues as  $BA$  plus 6 – 4 zeros.
- 20** (a)  $A = M^{-1}BM \Rightarrow A^2 = (M^{-1}BM)(M^{-1}BM) = M^{-1}B^2M$ . So  $A^2$  is similar to  $B^2$ . (b)  $A^2$  equals  $(-A)^2$  but  $A$  may not be similar to  $B = -A$  (it could be!).  
 (c)  $\begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix}$  is diagonalizable to  $\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$  because  $\lambda_1 \neq \lambda_2$ , so these matrices are similar.  
 (d)  $\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$  has only one eigenvector, so not diagonalizable (e)  $PAP^T$  is similar to  $A$ .
- 21**  $J^2$  has three 1's down the *second* superdiagonal, and *two* independent eigenvectors for  $\lambda = 0$ . Its 5 by 5 Jordan form is  $\begin{bmatrix} J_3 & \\ & J_2 \end{bmatrix}$  with  $J_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$  and  $J_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ .

**Note to professors:** An interesting question: Which matrices  $A$  have (complex) square roots  $R^2 = A$ ? If  $A$  is invertible, no problem. But any Jordan blocks for  $\lambda = 0$  must have sizes  $n_1 \geq n_2 \geq \dots \geq n_k \geq n_{k+1} = 0$  that come in pairs like 3 and 2 in this example:  $n_1 = (n_2 \text{ or } n_2 + 1)$  and  $n_3 = (n_4 \text{ or } n_4 + 1)$  and so on.

A list of all 3 by 3 and 4 by 4 Jordan forms could be  $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}, \begin{bmatrix} a & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix},$

$\begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{bmatrix}$  (for any numbers  $a, b, c$ )  
 with 3, 2, 1 eigenvectors;  $\text{diag}(a, b, c, d)$  and  $\begin{bmatrix} a & 1 & & \\ & a & & \\ & & b & \\ & & & c \end{bmatrix},$

$\begin{bmatrix} a & 1 & & \\ & a & & \\ & & b & 1 \\ & & & b \end{bmatrix}, \begin{bmatrix} a & 1 & & \\ & a & 1 & \\ & & a & 1 \\ & & & b \end{bmatrix}, \begin{bmatrix} a & 1 & & \\ & a & 1 & \\ & & a & 1 \\ & & & a \end{bmatrix}$  with 4, 3, 2, 1 eigenvectors.

- 22 If all roots are  $\lambda = 0$ , this means that  $\det(A - \lambda I)$  must be just  $\lambda^n$ . The Cayley-Hamilton Theorem in Problem 6.2.32 immediately says that  $A^n = \text{zero matrix}$ . The key example is a single  $n$  by  $n$  Jordan block (with  $n - 1$  ones above the diagonal): Check directly that  $J^n = \text{zero matrix}$ .
- 23 Certainly  $Q_1 R_1$  is similar to  $R_1 Q_1 = Q_1^{-1}(Q_1 R_1)Q_1$ . Then  $A_1 = Q_1 R_1 - cs^2 I$  is similar to  $A_2 = R_1 Q_1 - cs^2 I$ .
- 24  $A$  could have eigenvalues  $\lambda = 2$  and  $\lambda = \frac{1}{2}$  ( $A$  could be diagonal). Then  $A^{-1}$  has the same two eigenvalues (and is similar to  $A$ ).

### Problem Set 6.7, page 371

$$1 \quad A = U \Sigma V^T = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} \sqrt{50} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{matrix} \sqrt{10} \\ \sqrt{5} \end{matrix}$$

- 2 This  $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$  is a 2 by 2 matrix of rank 1. Its row space has basis  $\mathbf{v}_1$ , its nullspace has basis  $\mathbf{v}_2$ , its column space has basis  $\mathbf{u}_1$ , its left nullspace has basis  $\mathbf{u}_2$ :

$$\begin{aligned} \text{Row space} & \quad \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix} & \text{Nullspace} & \quad \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \end{bmatrix} \\ \text{Column space} & \quad \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 3 \end{bmatrix}, & N(A^T) & \quad \frac{1}{\sqrt{10}} \begin{bmatrix} 3 \\ -1 \end{bmatrix}. \end{aligned}$$

- 3 If  $A$  has rank 1 then so does  $A^T A$ . The only nonzero eigenvalue of  $A^T A$  is its trace, which is the sum of all  $a_{ij}^2$ . (Each diagonal entry of  $A^T A$  is the sum of  $a_{ij}^2$  down one column, so the trace is the sum down all columns.) Then  $\sigma_1 = \text{square root of this sum}$ , and  $\sigma_1^2 = \text{this sum of all } a_{ij}^2$ .
- 4  $A^T A = A A^T = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$  has eigenvalues  $\sigma_1^2 = \frac{3 + \sqrt{5}}{2}$ ,  $\sigma_2^2 = \frac{3 - \sqrt{5}}{2}$ . But  $A$  is indefinite  $\sigma_1 = (1 + \sqrt{5})/2 = \lambda_1(A)$ ,  $\sigma_2 = (\sqrt{5} - 1)/2 = -\lambda_2(A)$ ;  $\mathbf{u}_1 = \mathbf{v}_1$  but  $\mathbf{u}_2 = -\mathbf{v}_2$ .
- 5 A proof that **eigshow** finds the SVD. When  $\mathbf{V}_1 = (1, 0)$ ,  $\mathbf{V}_2 = (0, 1)$  the demo finds  $A\mathbf{V}_1$  and  $A\mathbf{V}_2$  at some angle  $\theta$ . A  $90^\circ$  turn by the mouse to  $\mathbf{V}_2, -\mathbf{V}_1$  finds  $A\mathbf{V}_2$  and  $-A\mathbf{V}_1$  at the angle  $\pi - \theta$ . Somewhere between, the constantly orthogonal  $\mathbf{v}_1$  and  $\mathbf{v}_2$  must produce  $A\mathbf{v}_1$  and  $A\mathbf{v}_2$  at angle  $\pi/2$ . Those orthogonal directions give  $\mathbf{u}_1$  and  $\mathbf{u}_2$ .
- 6  $A A^T = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$  has  $\sigma_1^2 = 3$  with  $\mathbf{u}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$  and  $\sigma_2^2 = 1$  with  $\mathbf{u}_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$ .  
 $A^T A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$  has  $\sigma_1^2 = 3$  with  $\mathbf{v}_1 = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$ ,  $\sigma_2^2 = 1$  with  $\mathbf{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$ ;  
and  $\mathbf{v}_3 = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$ . Then  $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = [\mathbf{u}_1 \quad \mathbf{u}_2] \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3]^T$ .

- 7 The matrix  $A$  in Problem 6 had  $\sigma_1 = \sqrt{3}$  and  $\sigma_2 = 1$  in  $\Sigma$ . The smallest change to rank 1 is **to make  $\sigma_2 = 0$** . In the factorization

$$A = U\Sigma V^T = \mathbf{u}_1\sigma_1\mathbf{v}_1^T + \mathbf{u}_2\sigma_2\mathbf{v}_2^T$$

this change  $\sigma_2 \rightarrow 0$  will leave the closest rank-1 matrix as  $\mathbf{u}_1\sigma_1\mathbf{v}_1^T$ . See Problem 14 for the general case of this problem.

- 8 The number  $\sigma_{\max}(A^{-1})\sigma_{\max}(A)$  is the same as  $\sigma_{\max}(A)/\sigma_{\min}(A)$ . This is certainly  $\geq 1$ . It equals 1 if all  $\sigma$ 's are equal, and  $A = U\Sigma V^T$  is a multiple of an orthogonal matrix. The ratio  $\sigma_{\max}/\sigma_{\min}$  is the important **condition number** of  $A$  studied in Section 9.2.
- 9  $A = UV^T$  since all  $\sigma_j = 1$ , which means that  $\Sigma = I$ .
- 10 A rank-1 matrix with  $A\mathbf{v} = 12\mathbf{u}$  would have  $\mathbf{u}$  in its column space, so  $A = \mathbf{u}\mathbf{w}^T$  for some vector  $\mathbf{w}$ . I intended (but didn't say) that  $\mathbf{w}$  is a multiple of the unit vector  $\mathbf{v} = \frac{1}{2}(1, 1, 1, 1)$  in the problem. Then  $A = 12\mathbf{u}\mathbf{v}^T$  to get  $A\mathbf{v} = 12\mathbf{u}$  when  $\mathbf{v}^T\mathbf{v} = 1$ .
- 11 If  $A$  has orthogonal columns  $\mathbf{w}_1, \dots, \mathbf{w}_n$  of lengths  $\sigma_1, \dots, \sigma_n$ , then  $A^T A$  will be diagonal with entries  $\sigma_1^2, \dots, \sigma_n^2$ . So the  $\sigma$ 's are definitely the singular values of  $A$  (as expected). The eigenvalues of that diagonal matrix  $A^T A$  are the columns of  $I$ , so  $V = I$  in the SVD. Then the  $\mathbf{u}_i$  are  $A\mathbf{v}_i/\sigma_i$  which is the unit vector  $\mathbf{w}_i/\sigma_i$ .

The SVD of this  $A$  with orthogonal columns is  $A = U\Sigma V^T = (A\Sigma^{-1})(\Sigma)(I)$ .

- 12 Since  $A^T = A$  we have  $\sigma_1^2 = \lambda_1^2$  and  $\sigma_2^2 = \lambda_2^2$ . But  $\lambda_2$  is negative, so  $\sigma_1 = 3$  and  $\sigma_2 = 2$ . The unit eigenvectors of  $A$  are the same  $\mathbf{u}_1 = \mathbf{v}_1$  as for  $A^T A = AA^T$  and  $\mathbf{u}_2 = -\mathbf{v}_2$  (notice the sign change because  $\sigma_2 = -\lambda_2$ , as in Problem 4).
- 13 Suppose the SVD of  $R$  is  $R = U\Sigma V^T$ . Then multiply by  $Q$  to get  $A = QR$ . So the SVD of this  $A$  is  $(QU)\Sigma V^T$ . (Orthogonal  $Q$  times orthogonal  $U =$  orthogonal  $QU$ .)
- 14 The smallest change in  $A$  is to set its smallest singular value  $\sigma_2$  to zero. See #7.
- 15 The singular values of  $A + I$  are not  $\sigma_j + 1$ . They come from eigenvalues of  $(A + I)^T(A + I)$ .
- 16 This simulates the random walk used by *Google* on billions of sites to solve  $A\mathbf{p} = \mathbf{p}$ . It is like the power method of Section 9.3 except that it follows the links in one "walk" where the vector  $\mathbf{p}_k = A^k \mathbf{p}_0$  averages over all walks.
- 17  $A = U\Sigma V^T = [\text{cosines including } \mathbf{u}_4] \text{diag}(\text{sqrt}(2 - \sqrt{2}, 2, 2 + \sqrt{2})) [\text{sine matrix}]^T$ .  $AV = U\Sigma$  says that differences of sines in  $V$  are cosines in  $U$  times  $\sigma$ 's.

The SVD of the *derivative* on  $[0, \pi]$  with  $f(0) = 0$  has  $\mathbf{u} = \sin nx$ ,  $\sigma = n$ ,  $\mathbf{v} = \cos nx$ !

## Problem Set 7.1, page 380

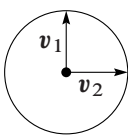
- 1 With  $\mathbf{w} = \mathbf{0}$  linearity gives  $T(\mathbf{v} + \mathbf{0}) = T(\mathbf{v}) + T(\mathbf{0})$ . Thus  $T(\mathbf{0}) = \mathbf{0}$ . With  $c = -1$  linearity gives  $T(-\mathbf{0}) = -T(\mathbf{0})$ . This is a second proof that  $T(\mathbf{0}) = \mathbf{0}$ .
- 2 Combining  $T(c\mathbf{v}) = cT(\mathbf{v})$  and  $T(d\mathbf{w}) = dT(\mathbf{w})$  with addition gives  $T(c\mathbf{v} + d\mathbf{w}) = cT(\mathbf{v}) + dT(\mathbf{w})$ . Then one more addition gives  $cT(\mathbf{v}) + dT(\mathbf{w}) + eT(\mathbf{u})$ .
- 3 (d) is not linear.

- 4 (a)  $S(T(\mathbf{v})) = \mathbf{v}$  (b)  $S(T(\mathbf{v}_1) + T(\mathbf{v}_2)) = S(T(\mathbf{v}_1)) + S(T(\mathbf{v}_2))$ .
- 5 Choose  $\mathbf{v} = (1, 1)$  and  $\mathbf{w} = (-1, 0)$ . Then  $T(\mathbf{v}) + T(\mathbf{w}) = (\mathbf{v} + \mathbf{w})$  but  $T(\mathbf{v} + \mathbf{w}) = (0, 0)$ .
- 6 (a)  $T(\mathbf{v}) = \mathbf{v}/\|\mathbf{v}\|$  does not satisfy  $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$  or  $T(c\mathbf{v}) = cT(\mathbf{v})$   
 (b) and (c) are linear (d) satisfies  $T(c\mathbf{v}) = cT(\mathbf{v})$ .
- 7 (a)  $T(T(\mathbf{v})) = \mathbf{v}$  (b)  $T(T(\mathbf{v})) = \mathbf{v} + (2, 2)$  (c)  $T(T(\mathbf{v})) = -\mathbf{v}$  (d)  $T(T(\mathbf{v})) = T(\mathbf{v})$ .
- 8 (a) The range of  $T(v_1, v_2) = (v_1 - v_2, 0)$  is the line of vectors  $(c, 0)$ . The nullspace is the line of vectors  $(c, c)$ . (b)  $T(v_1, v_2, v_3) = (v_1, v_2)$  has Range  $\mathbf{R}^2$ , kernel  $\{(0, 0, v_3)\}$  (c)  $T(\mathbf{v}) = \mathbf{0}$  has Range  $\{\mathbf{0}\}$ , kernel  $\mathbf{R}^2$  (d)  $T(v_1, v_2) = (v_1, v_1)$  has Range = multiples of  $(1, 1)$ , kernel = multiples of  $(1, -1)$ .
- 9 If  $T(v_1, v_2, v_3) = (v_2, v_3, v_1)$  then  $T(T(\mathbf{v})) = (v_3, v_1, v_2)$ ;  $T^3(\mathbf{v}) = \mathbf{v}$ ;  $T^{100}(\mathbf{v}) = T(\mathbf{v})$ .
- 10 (a)  $T(1, 0) = \mathbf{0}$  (b)  $(0, 0, 1)$  is not in the range (c)  $T(0, 1) = \mathbf{0}$ .
- 11 For multiplication  $T(\mathbf{v}) = A\mathbf{v}$ :  $\mathbf{V} = \mathbf{R}^n$ ,  $\mathbf{W} = \mathbf{R}^m$ ; the outputs fill the column space;  $\mathbf{v}$  is in the kernel if  $A\mathbf{v} = \mathbf{0}$ .
- 12  $T(\mathbf{v}) = (4, 4); (2, 2); (2, 2)$ ; if  $\mathbf{v} = (a, b) = b(1, 1) + \frac{a-b}{2}(2, 0)$  then  $T(\mathbf{v}) = b(2, 2) + (0, 0)$ .
- 13 The distributive law (page 69) gives  $A(M_1 + M_2) = AM_1 + AM_2$ . The distributive law over  $c$ 's gives  $A(cM) = c(AM)$ .
- 14 This  $A$  is invertible. Multiply  $AM = 0$  and  $AM = B$  by  $A^{-1}$  to get  $M = 0$  and  $M = A^{-1}B$ . The kernel contains only the zero matrix  $M = 0$ .
- 15 This  $A$  is not invertible.  $AM = I$  is impossible.  $A \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ . The range contains only matrices  $AM$  whose columns are multiples of  $(1, 3)$ .
- 16 No matrix  $A$  gives  $A \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ . To professors: Linear transformations on matrix space come from  $4$  by  $4$  matrices. Those in Problems 13–15 were special.
- 17 For  $T(M) = MT$  (a)  $T^2 = I$  is True (b) True (c) True (d) False.
- 18  $T(I) = 0$  but  $M = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} = T(M)$ ; these  $M$ 's fill the range. Every  $M = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}$  is in the kernel. Notice that  $\dim(\text{range}) + \dim(\text{kernel}) = 3 + 1 = \dim(\text{input space of } 2 \text{ by } 2 \text{ } M\text{'s})$ .
- 19  $T(T^{-1}(M)) = M$  so  $T^{-1}(M) = A^{-1}MB^{-1}$ .
- 20 (a) Horizontal lines stay horizontal, vertical lines stay vertical (b) House squashes onto a line (c) Vertical lines stay vertical because  $T(1, 0) = (a_{11}, 0)$ .
- 21  $D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$  doubles the width of the house.  $A = \begin{bmatrix} .7 & .7 \\ .3 & .3 \end{bmatrix}$  projects the house (since  $A^2 = A$  from trace = 1 and  $\lambda = 0, 1$ ). The projection is onto the column space of  $A =$  line through  $(.7, .3)$ .  $U = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$  will shear the house horizontally: The point at  $(x, y)$  moves over to  $(x + y, y)$ .

- 22 (a)  $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$  with  $d > 0$  leaves the house  $AH$  sitting straight up (b)  $A = 3I$  expands the house by 3 (c)  $A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$  rotates the house.
- 23  $T(\mathbf{v}) = -\mathbf{v}$  rotates the house by  $180^\circ$  around the origin. Then the affine transformation  $T(\mathbf{v}) = -\mathbf{v} + (1, 0)$  shifts the rotated house one unit to the right.
- 24 A code to add a chimney will be gratefully received!
- 25 This code needs a correction: add spaces between `-10 10 -10 10`
- 26  $\begin{bmatrix} 1 & 0 \\ 0 & .1 \end{bmatrix}$  compresses vertical distances by 10 to 1.  $\begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}$  projects onto the  $45^\circ$  line.  $\begin{bmatrix} .5 & .5 \\ -.5 & .5 \end{bmatrix}$  rotates by  $45^\circ$  clockwise and contracts by a factor of  $\sqrt{2}$  (the columns have length  $1/\sqrt{2}$ ).  $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$  has determinant  $-1$  so the house is “flipped and sheared.” One way to see this is to factor the matrix as  $LDL^T$ :

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = (\text{shear}) (\text{flip left-right}) (\text{shear}).$$

- 27 Also 30 emphasizes that circles are transformed to ellipses (see figure in Section 6.7).
- 28 A code that adds two eyes and a smile will be included here with public credit given!
- 29 (a)  $ad - bc = 0$  (b)  $ad - bc > 0$  (c)  $|ad - bc| = 1$ . If vectors to two corners transform to themselves then by linearity  $T = I$ . (Fails if one corner is  $(0, 0)$ .)

- 30 The circle  transforms to the ellipse by rotating  $30^\circ$  and stretching the first axis by 2.

- 31 Linear transformations keep straight lines straight! And two parallel edges of a square (edges differing by a fixed  $\mathbf{v}$ ) go to two parallel edges (edges differing by  $T(\mathbf{v})$ ). So the output is a parallelogram.

## Problem Set 7.2, page 395

- For  $S\mathbf{v} = d^2\mathbf{v}/dx^2$
- 1  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 = 1, x, x^2, x^3$  The matrix for  $S$  is  $B = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ .  
 $S\mathbf{v}_1 = S\mathbf{v}_2 = \mathbf{0}, S\mathbf{v}_3 = 2\mathbf{v}_1, S\mathbf{v}_4 = 6\mathbf{v}_2$ ;
- 2  $S\mathbf{v} = d^2\mathbf{v}/dx^2 = 0$  for linear functions  $\mathbf{v}(x) = a + bx$ . All  $(a, b, 0, 0)$  are in the nullspace of the second derivative matrix  $B$ .
- 3 (Matrix  $A$ ) $^2 = B$  when (transformation  $T$ ) $^2 = S$  and output basis = input basis.

- 4** The third derivative matrix has **6** in the  $(1, 4)$  position; since the third derivative of  $x^3$  is 6. This matrix also comes from  $AB$ . The fourth derivative of a cubic is zero, and  $B^2$  is the zero matrix.
- 5**  $T(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) = 2\mathbf{w}_1 + \mathbf{w}_2 + 2\mathbf{w}_3$ ;  $A$  times  $(1, 1, 1)$  gives  $(2, 1, 2)$ .
- 6**  $\mathbf{v} = c(\mathbf{v}_2 - \mathbf{v}_3)$  gives  $T(\mathbf{v}) = \mathbf{0}$ ; nullspace is  $(0, c, -c)$ ; solutions  $(1, 0, 0) + (0, c, -c)$ .
- 7**  $(1, 0, 0)$  is not in the column space of the matrix  $A$ , and  $\mathbf{w}_1$  is not in the range of the linear transformation  $T$ . Key point: *Column space* of matrix matches *range* of transformation.
- 8** We don't know  $T(\mathbf{w})$  unless the  $\mathbf{w}$ 's are the same as the  $\mathbf{v}$ 's. In that case the matrix is  $A^2$ .
- 9** Rank of  $A = 2 = \text{dimension of the range of } T$ . The outputs  $A\mathbf{v}$  (column space) match the outputs  $T(\mathbf{v})$  (the range of  $T$ ). The "output space"  $W$  is like  $\mathbf{R}^m$ : it contains all outputs but may not be filled up.
- 10** The matrix for  $T$  is  $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ . For the output  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$  choose input  $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = A^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ . This means: For the output  $\mathbf{w}_1$  choose the input  $\mathbf{v}_1 - \mathbf{v}_2$ .
- 11**  $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$  so  $T^{-1}(\mathbf{w}_1) = \mathbf{v}_1 - \mathbf{v}_2$ ,  $T^{-1}(\mathbf{w}_2) = \mathbf{v}_2 - \mathbf{v}_3$ ,  $T^{-1}(\mathbf{w}_3) = \mathbf{v}_3$ .  
The columns of  $A^{-1}$  describe  $T^{-1}$  from  $W$  back to  $V$ . The only solution to  $T(\mathbf{v}) = \mathbf{0}$  is  $\mathbf{v} = \mathbf{0}$ .
- 12** (c)  $T^{-1}(T(\mathbf{w}_1)) = \mathbf{w}_1$  is wrong because  $\mathbf{w}_1$  is not generally in the input space.
- 13** (a)  $T(\mathbf{v}_1) = \mathbf{v}_2$ ,  $T(\mathbf{v}_2) = \mathbf{v}_1$  is its own inverse (b)  $T(\mathbf{v}_1) = \mathbf{v}_1$ ,  $T(\mathbf{v}_2) = \mathbf{0}$  has  $T^2 = T$  (c) If  $T^2 = I$  for part (a) and  $T^2 = T$  for part (b), then  $T$  must be  $I$ .
- 14** (a)  $\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$  (b)  $\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$  = inverse of (a) (c)  $A \begin{bmatrix} 2 \\ 6 \end{bmatrix}$  must be  $2A \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ .
- 15** (a)  $M = \begin{bmatrix} r & s \\ t & u \end{bmatrix}$  transforms  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$  to  $\begin{bmatrix} r \\ t \end{bmatrix}$  and  $\begin{bmatrix} s \\ u \end{bmatrix}$ ; this is the "easy" direction. (b)  $N = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$  transforms in the inverse direction, back to the standard basis vectors. (c)  $ad = bc$  will make the forward matrix singular and the inverse impossible.
- 16**  $MW = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -1 \\ -7 & 3 \end{bmatrix}$ .
- 17** Recording basis vectors is done by a *Permutation matrix*. Changing lengths is done by a *positive diagonal matrix*.
- 18**  $(a, b) = (\cos \theta, -\sin \theta)$ . Minus sign from  $Q^{-1} = Q^T$ .

- 19  $M = \begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix}; \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \end{bmatrix} = \text{first column of } M^{-1} = \text{coordinates of } \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ in basis } \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$
- 20  $w_2(x) = 1 - x^2; w_3(x) = \frac{1}{2}(x^2 - x); y = 4w_1 + 5w_2 + 6w_3.$
- 21  $w$ 's to  $v$ 's:  $\begin{bmatrix} 0 & 1 & 0 \\ .5 & 0 & -.5 \\ .5 & -1 & .5 \end{bmatrix}$ .  $v$ 's to  $w$ 's: inverse matrix  $= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix}$ . The key idea: The matrix multiplies the coordinates in the  $v$  basis to give the coordinates in the  $w$  basis.
- 22 The 3 equations to match 4, 5, 6 at  $x = a, b, c$  are  $\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ . This Vandermonde determinant equals  $(b - a)(c - a)(c - b)$ . So  $a, b, c$  must be distinct to have  $\det \neq 0$  and one solution  $A, B, C$ .
- 23 The matrix  $M$  with these nine entries must be invertible.
- 24 Start from  $A = QR$ . Column 2 is  $a_2 = r_{12}q_1 + r_{22}q_2$ . This gives  $a_2$  as a combination of the  $q$ 's. So the change of basis matrix is  $R$ .
- 25 Start from  $A = LU$ . Row 2 of  $A$  is  $\ell_{21}(\text{row 1 of } U) + \ell_{22}(\text{row 2 of } U)$ . The change of basis matrix is always *invertible*, because basis goes to basis.
- 26 The matrix for  $T(v_i) = \lambda_i v_i$  is  $\Lambda = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$ .
- 27 If  $T$  is not invertible,  $T(v_1), \dots, T(v_n)$  is not a basis. We couldn't choose  $w_i = T(v_i)$ .
- 28 (a)  $\begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$  gives  $T(v_1) = \mathbf{0}$  and  $T(v_2) = 3v_1$ . (b)  $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$  gives  $T(v_1) = v_1$  and  $T(v_1 + v_2) = v_1$  (which combine into  $T(v_2) = \mathbf{0}$  by *linearity*).
- 29  $T(x, y) = (x, -y)$  is reflection across the  $x$ -axis. Then reflect across the  $y$ -axis to get  $S(x, -y) = (-x, -y)$ . Thus  $ST = -I$ .
- 30  $S$  takes  $(x, y)$  to  $(-x, y)$ .  $S(T(v)) = (-1, 2)$ .  $S(v) = (-2, 1)$  and  $T(S(v)) = (1, -2)$ .
- 31 Multiply the two reflections to get  $\begin{bmatrix} \cos 2(\theta - \alpha) & -\sin 2(\theta - \alpha) \\ \sin 2(\theta - \alpha) & \cos 2(\theta - \alpha) \end{bmatrix}$  which is *rotation* by  $2(\theta - \alpha)$ . In words:  $(1, 0)$  is reflected to have angle  $2\alpha$ , and that is reflected again to angle  $2\theta - 2\alpha$ .
- 32 False: We will not know  $T(v)$  for *energy*  $v$  unless the  $n$   $v$ 's are linearly independent.
- 33 To find coordinates in the wavelet basis, multiply by  $W^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$ .
- Then  $e = \frac{1}{4}w_1 + \frac{1}{4}w_2 + \frac{1}{2}w_3$  and  $v = w_3 + w_4$ . Notice again:  $W$  tells us how the bases change,  $W^{-1}$  tells us how the coordinates change.
- 34 The last step writes 6, 6, 2, 2 as the overall average 4, 4, 4, 4 plus the difference 2, 2, -2, -2. Therefore  $c_1 = 4$  and  $c_2 = 2$  and  $c_3 = 1$  and  $c_4 = 1$ .

- 35** The wavelet basis is  $(1, 1, 1, 1, 1, 1, 1, 1)$  and the long wavelet and two medium wavelets  $(1, 1, -1, -1, 0, 0, 0, 0)$ ,  $(0, 0, 0, 0, 1, 1, -1, -1)$  and 4 wavelets with a single pair  $1, -1$ .
- 36** If  $V\mathbf{b} = W\mathbf{c}$  then  $\mathbf{b} = V^{-1}W\mathbf{c}$ . The change of basis matrix is  $V^{-1}W$ .
- 37** Multiplying by  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  gives  $T(\mathbf{v}_1) = A \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} = a\mathbf{v}_1 + c\mathbf{v}_3$ . Similarly  $T(\mathbf{v}_2) = a\mathbf{v}_2 + c\mathbf{v}_4$  and  $T(\mathbf{v}_3) = b\mathbf{v}_1 + d\mathbf{v}_3$  and  $T(\mathbf{v}_4) = b\mathbf{v}_2 + d\mathbf{v}_4$ . The matrix for  $T$  in this basis is  $\begin{bmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{bmatrix}$ .
- 38** The matrix for  $T$  in this basis is  $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ .

### Problem Set 7.3, page 406

- 1**  $A^T A = \begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix}$  has  $\lambda = 50$  and  $0$ ,  $\mathbf{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ ,  $\mathbf{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ ;  $\sigma_1 = \sqrt{50}$ .
- 2** Orthonormal bases:  $\mathbf{v}_1$  for row space,  $\mathbf{v}_2$  for nullspace,  $\mathbf{u}_1$  for column space,  $\mathbf{u}_2$  for  $N(A^T)$ . All matrices with those four subspaces are multiples  $cA$ , since the subspaces are just lines. Normally many more matrices share the same 4 subspaces. (For example, all  $n$  by  $n$  invertible matrices share  $\mathbf{R}^n$ .)
- 3**  $A = QH = \frac{1}{\sqrt{50}} \begin{bmatrix} 7 & -1 \\ 1 & 7 \end{bmatrix} \frac{1}{\sqrt{50}} \begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix}$ .  $H$  is semidefinite because  $A$  is singular.
- 4**  $A^+ = V \begin{bmatrix} 1/\sqrt{50} & 0 \\ 0 & 0 \end{bmatrix} U^T = \frac{1}{50} \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ ;  $A^+ A = \begin{bmatrix} .2 & .4 \\ .4 & .8 \end{bmatrix}$ ,  $AA^+ = \begin{bmatrix} .1 & .3 \\ .3 & .9 \end{bmatrix}$ .
- 5**  $A^T A = \begin{bmatrix} 10 & 8 \\ 8 & 10 \end{bmatrix}$  has  $\lambda = 18$  and  $2$ ,  $\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ,  $\mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ ,  $\sigma_1 = \sqrt{18}$  and  $\sigma_2 = \sqrt{2}$ .
- 6**  $AA^T = \begin{bmatrix} 18 & 0 \\ 0 & 2 \end{bmatrix}$  has  $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ,  $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ . The same  $\sqrt{18}$  and  $\sqrt{2}$  go into  $\Sigma$ .
- 7**  $[\sigma_1 \mathbf{u}_1 \quad \sigma_2 \mathbf{u}_2] \begin{bmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \end{bmatrix} = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T$ . In general this is  $\sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \cdots + \sigma_r \mathbf{u}_r \mathbf{v}_r^T$ .
- 8**  $A = U\Sigma V^T$  splits into  $QK$  (polar):  $Q = UV^T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$  and  $K = V\Sigma V^T = \begin{bmatrix} \sqrt{18} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$ .
- 9**  $A^+$  is  $A^{-1}$  because  $A$  is invertible. Pseudoinverse equals inverse when  $A^{-1}$  exists!
- 10**  $A^T A = \begin{bmatrix} 9 & 12 & 0 \\ 12 & 16 & 0 \\ 0 & 0 & 0 \end{bmatrix}$  has  $\lambda = 25, 0, 0$  and  $\mathbf{v}_1 = \begin{bmatrix} .6 \\ .8 \\ 0 \end{bmatrix}$ ,  $\mathbf{v}_2 = \begin{bmatrix} .8 \\ -.6 \\ 0 \end{bmatrix}$ ,  $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ . Here  $A = \begin{bmatrix} 3 & 4 & 0 \end{bmatrix}$  has rank 1 and  $AA^T = \begin{bmatrix} 25 \end{bmatrix}$  and  $\sigma_1 = 5$  is the only singular value in  $\Sigma = \begin{bmatrix} 5 & 0 & 0 \end{bmatrix}$ .



- 11  $A = [1] [5 \ 0 \ 0] V^T$  and  $A^+ = V \begin{bmatrix} .2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} .12 \\ .16 \\ 0 \end{bmatrix}$ ;  $A^+ A = \begin{bmatrix} .36 & .48 & 0 \\ .48 & .64 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ ;  $AA^+ = [1]$
- 12 The zero matrix has no pivots or singular values. Then  $\Sigma =$  same 2 by 3 zero matrix and the pseudoinverse is the 3 by 2 zero matrix.
- 13 If  $\det A = 0$  then  $\text{rank}(A) < n$ ; thus  $\text{rank}(A^+) < n$  and  $\det A^+ = 0$ .
- 14  $A$  must be *symmetric and positive definite*, if  $\Sigma = \Lambda$  and  $U = V =$  eigenvector matrix  $Q$  is orthogonal.
- 15 (a)  $A^T A$  is singular (b) This  $\mathbf{x}^+$  in the row space does give  $A^T A \mathbf{x}^+ = A^T \mathbf{b}$  (c) If  $(1, -1)$  in the nullspace of  $A$  is added to  $\mathbf{x}^+$ , we get another solution to  $A^T A \hat{\mathbf{x}} = A^T \mathbf{b}$ . But this  $\hat{\mathbf{x}}$  is longer than  $\mathbf{x}^+$  because the added part is orthogonal to  $\mathbf{x}^+$  in the row space.
- 16  $\mathbf{x}^+$  in the row space of  $A$  is perpendicular to  $\hat{\mathbf{x}} - \mathbf{x}^+$  in the nullspace of  $A^T A =$  nullspace of  $A$ . The right triangle has  $c^2 = a^2 + b^2$ .
- 17  $AA^+ \mathbf{p} = \mathbf{p}$ ,  $AA^+ \mathbf{e} = \mathbf{0}$ ,  $A^+ A \mathbf{x}_r = \mathbf{x}_r$ ,  $A^+ A \mathbf{x}_n = \mathbf{0}$ .
- 18  $A^+ = V \Sigma^+ U^T$  is  $\frac{1}{5} [.6 \ .8] = [.12 \ .16]$  and  $A^+ A = [1]$  and  $AA^+ = \begin{bmatrix} .36 & .48 \\ .48 & .64 \end{bmatrix} =$  projection.
- 19  $L$  is determined by  $\ell_{21}$ . Each eigenvector in  $S$  is determined by one number. The counts are 1 + 3 for  $LU$ , 1 + 2 + 1 for  $LDU$ , 1 + 3 for  $QR$ , 1 + 2 + 1 for  $U \Sigma V^T$ , 2 + 2 + 0 for  $S \Lambda S^{-1}$ .
- 20  $LDL^T$  and  $Q \Lambda Q^T$  are determined by  $1 + 2 + 0$  numbers because  $A$  is *symmetric*.
- 21 Column times row multiplication gives  $A = U \Sigma V^T = \sum \sigma_i \mathbf{u}_i \mathbf{v}_i^T$  and also  $A^+ = V \Sigma^+ U^T = \sum \sigma_i^{-1} \mathbf{v}_i \mathbf{u}_i^T$ . Multiplying  $A^+ A$  and using orthogonality of each  $\mathbf{u}_i$  to all other  $\mathbf{u}_j$  leaves the projection matrix  $A^+ A = \sum 1 \mathbf{v}_i \mathbf{v}_i^T$ . Similarly  $AA^+ = \sum 1 \mathbf{u}_i \mathbf{u}_i^T$  from  $V V^T = I$ .
- 22 Keep only the  $r$  by  $r$  corner  $\Sigma_r$  of  $\Sigma$  (the rest is all zero). Then  $A = U \Sigma V^T$  has the required form  $A = \hat{U} M_1 \Sigma_r M_2^T \hat{V}^T$  with an invertible  $M = M_1 \Sigma_r M_2^T$  in the middle.
- 23  $\begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} A\mathbf{v} \\ A^T \mathbf{u} \end{bmatrix} = \sigma \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix}$ . The singular values of  $A$  are *eigenvalues* of this block matrix.

## Problem Set 8.1, page 418

- 1  $\det A_0^T C_0 A_0 = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 + c_4 \end{bmatrix}$  is by direct calculation. Set  $c_4 = 0$  to find  $\det A_1^T C_1 A_1 = c_1 c_2 c_3$ .
- 2  $(A_1^T C_1 A_1)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1^{-1} & & \\ & c_2^{-1} & \\ & & c_3^{-1} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} =$
- $$\begin{bmatrix} c_1^{-1} & c_1^{-1} & c_1^{-1} \\ c_1^{-1} & c_1^{-1} + c_2^{-1} & c_1^{-1} + c_2^{-1} \\ c_1^{-1} & c_1^{-1} + c_2^{-1} & c_1^{-1} + c_2^{-1} + c_3^{-1} \end{bmatrix}.$$

- 3 The rows of the free-free matrix in equation (9) add to  $[0 \ 0 \ 0]$  so the right side needs  $f_1 + f_2 + f_3 = 0$ .  $\mathbf{f} = (-1, 0, 1)$  gives  $c_2 u_1 - c_2 u_2 = -1$ ,  $c_3 u_2 - c_3 u_3 = -1$ ,  $0 = 0$ . Then  $\mathbf{u}_{\text{particular}} = (-c_2^{-1} - c_3^{-1}, -c_3^{-1}, 0)$ . Add any multiple of  $\mathbf{u}_{\text{nullspace}} = (1, 1, 1)$ .
- 4  $\int -\frac{d}{dx} \left( c(x) \frac{du}{dx} \right) dx = - \left[ c(x) \frac{du}{dx} \right]_0^1 = 0$  (bdry cond) so we need  $\int f(x) dx = 0$ .
- 5  $-\frac{dy}{dx} = f(x)$  gives  $y(x) = C - \int_0^x f(t) dt$ . Then  $y(1) = 0$  gives  $C = \int_0^1 f(t) dt$  and  $y(x) = \int_x^1 f(t) dt$ . If the load is  $f(x) = 1$  then the displacement is  $y(x) = 1 - x$ .
- 6 Multiply  $A_1^T C_1 A_1$  as columns of  $A_1^T$  times  $c$ 's times rows of  $A_1$ . The first 3 by 3 "element matrix"  $c_1 E_1 = [1 \ 0 \ 0]^T c_1 [1 \ 0 \ 0]$  has  $c_1$  in the top left corner.
- 7 For 5 springs and 4 masses, the 5 by 4  $A$  has two nonzero diagonals: all  $a_{ii} = 1$  and  $a_{i+1,i} = -1$ . With  $C = \text{diag}(c_1, c_2, c_3, c_4, c_5)$  we get  $K = A^T C A$ , symmetric tridiagonal with diagonal entries  $K_{ii} = c_i + c_{i+1}$  and off-diagonals  $K_{i+1,i} = -c_{i+1}$ . With  $C = I$  this  $K$  is the  $-1, 2, -1$  matrix and  $K(2, 3, 3, 2) = (1, 1, 1, 1)$  solves  $K\mathbf{u} = \text{ones}(4, 1)$ . ( $K^{-1}$  will solve  $K\mathbf{u} = \text{ones}(4)$ .)
- 8 The solution to  $-u'' = 1$  with  $u(0) = u(1) = 0$  is  $u(x) = \frac{1}{2}(x - x^2)$ . At  $x = \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$  this gives  $\mathbf{u} = 2, 3, 3, 2$  (discrete solution in Problem 7) times  $(\Delta x)^2 = 1/25$ .
- 9  $-u'' = mg$  has complete solution  $u(x) = A + Bx - \frac{1}{2}mgx^2$ . From  $u(0) = 0$  we get  $A = 0$ . From  $u'(1) = 0$  we get  $B = mg$ . Then  $u(x) = \frac{1}{2}mg(2x - x^2)$  at  $x = \frac{1}{3}, \frac{2}{3}, \frac{3}{3}$  equals  $mg/6, 4mg/9, mg/2$ . This  $u(x)$  is *not* proportional to the discrete  $\mathbf{u} = (3mg, 5mg, 6mg)$  at the meshpoints. This imperfection is because the discrete problem uses a 1-sided difference, less accurate at the free end. Perfect accuracy is recovered by a centered difference (discussed on page 21 of my CSE textbook).
- 10 (added in later printing, changing 10-11 below into 11-12). The solution in this fixed-fixed case is (2.25, 2.50, 1.75) so the second mass moves furthest.
- 11 The two graphs of 100 points are "discrete parabolas" starting at (0, 0): symmetric around 50 in the fixed-fixed case, ending with slope zero in the fixed-free case.
- 12 Forward/backward/centered for  $du/dx$  has a big effect because that term has the large coefficient. MATLAB:  $E = \text{diag}(\text{ones}(6, 1), 1)$ ;  $K = 64 * (2 * \text{eye}(7) - E - E')$ ;  $D = 80 * (E - \text{eye}(7))$ ;  $(K + D) \setminus \text{ones}(7, 1)$ ; % forward;  $(K - D') \setminus \text{ones}(7, 1)$ ; % backward;  $(K + D/2 - D'/2) \setminus \text{ones}(7, 1)$ ; % centered is usually the best: more accurate

## Problem Set 8.2, page 428

- 1  $A = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$ ; nullspace contains  $\begin{bmatrix} c \\ c \\ c \end{bmatrix}$ ;  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$  is not orthogonal to that nullspace.
- 2  $A^T \mathbf{y} = \mathbf{0}$  for  $\mathbf{y} = (1, -1, 1)$ ; current along edge 1, edge 3, back on edge 2 (full loop).

3 Elimination on  $b_1[A \ \mathbf{b}] = \begin{bmatrix} -1 & 1 & 0 & b_1 \\ -1 & 0 & 1 & b_2 \\ 0 & -1 & 1 & b_3 \end{bmatrix}$  leads to  $[U \ \mathbf{c}] = \begin{bmatrix} -1 & 1 & 0 & b_1 \\ 0 & -1 & 1 & b_2 - b_1 \\ 0 & 0 & 0 & b_3 - b_2 + b_1 \end{bmatrix}$ . The nonzero rows of  $U$  come from edges 1 and 3 in a tree. The zero row comes from the loop (all 3 edges).

4 For the matrix in Problem 3,  $A\mathbf{x} = \mathbf{b}$  is solvable for  $\mathbf{b} = (1, 1, 0)$  and not solvable for  $\mathbf{b} = (1, 0, 0)$ . For solvable  $\mathbf{b}$  (in the column space),  $\mathbf{b}$  must be orthogonal to  $\mathbf{y} = (1, -1, 1)$ ; that combination of rows is the zero row, and  $b_1 - b_2 + b_3 = 0$  is the third equation after elimination.

5 Kirchhoff's Current Law  $A^T\mathbf{y} = \mathbf{f}$  is solvable for  $\mathbf{f} = (1, -1, 0)$  and not solvable for  $\mathbf{f} = (1, 0, 0)$ ;  $\mathbf{f}$  must be orthogonal to  $(1, 1, 1)$  in the nullspace:  $f_1 + f_2 + f_3 = 0$ .

6  $A^T A\mathbf{x} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} = \mathbf{f}$  produces  $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} c \\ c \\ c \end{bmatrix}$ ; potentials  $\mathbf{x} = 1, -1, 0$  and currents  $-A\mathbf{x} = 2, 1, -1$ ;  $\mathbf{f}$  sends 3 units from node 2 into node 1.

7  $A^T \begin{bmatrix} 1 & & \\ & 2 & \\ & & 2 \end{bmatrix} A = \begin{bmatrix} 3 & -1 & -2 \\ -1 & 3 & -2 \\ -2 & -2 & 4 \end{bmatrix}$ ;  $\mathbf{f} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$  yields  $\mathbf{x} = \begin{bmatrix} 5/4 \\ 1 \\ 7/8 \end{bmatrix} + \text{any } \begin{bmatrix} c \\ c \\ c \end{bmatrix}$ ; potentials  $\mathbf{x} = \frac{5}{4}, 1, \frac{7}{8}$  and currents  $-CA\mathbf{x} = \frac{1}{4}, \frac{3}{4}, \frac{1}{4}$ .

8  $A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$  leads to  $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$  and  $\mathbf{y} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$  solving  $A^T\mathbf{y} = \mathbf{0}$ .

9 Elimination on  $A\mathbf{x} = \mathbf{b}$  always leads to  $\mathbf{y}^T\mathbf{b} = 0$  in the zero rows of  $U$  and  $R$ :  $-b_1 + b_2 - b_3 = 0$  and  $b_3 - b_4 + b_5 = 0$  (those  $\mathbf{y}$ 's are from Problem 8 in the left nullspace). This is Kirchhoff's *Voltage Law* around the two *loops*.

10 The echelon form of  $A$  is  $U = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$  The nonzero rows of  $U$  keep edges 1, 2, 4. Other spanning trees from edges, 1, 2, 5; 1, 3, 4; 1, 3, 5; 1, 4, 5; 2, 3, 4; 2, 3, 5; 2, 4, 5.

11  $A^T A = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$  diagonal entry = number of edges into the node  
the trace is 2 times the number of nodes  
off-diagonal entry = -1 if nodes are connected  
 $A^T A$  is the **graph Laplacian**,  $A^T C A$  is **weighted** by  $C$

12 (a) The nullspace and rank of  $A^T A$  and  $A$  are always the same (b)  $A^T A$  is always positive semidefinite because  $\mathbf{x}^T A^T A \mathbf{x} = \|A\mathbf{x}\|^2 \geq 0$ . Not positive definite because rank is only 3 and  $(1, 1, 1, 1)$  is in the nullspace (c) Real eigenvalues all  $\geq 0$  because positive semidefinite.

- 13  $A^T C A \mathbf{x} = \begin{bmatrix} 4 & -2 & -2 & 0 \\ -2 & 8 & -3 & -3 \\ -2 & -3 & 8 & -3 \\ 0 & -3 & -3 & 6 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$  gives four potentials  $\mathbf{x} = (\frac{5}{12}, \frac{1}{6}, \frac{1}{6}, 0)$   
I grounded  $x_4 = 0$  and solved for  $\mathbf{x}$   
currents  $\mathbf{y} = -C A \mathbf{x} = (\frac{2}{3}, \frac{2}{3}, 0, \frac{1}{2}, \frac{1}{2})$
- 14  $A^T C A \mathbf{x} = \mathbf{0}$  for  $\mathbf{x} = c(1, 1, 1, 1) = (c, c, c, c)$ . If  $A^T C A \mathbf{x} = \mathbf{f}$  is solvable, then  $\mathbf{f}$  in the column space (= row space by symmetry) must be orthogonal to  $\mathbf{x}$  in the nullspace:  $f_1 + f_2 + f_3 + f_4 = 0$ .
- 15 The number of loops in this connected graph is  $n - m + 1 = 7 - 7 + 1 = 1$ . What answer if the graph has two separate components (no edges between)?
- 16 Start from (4 nodes) - (6 edges) + (3 loops) = 1. If a new node connects to 1 old node,  $5 - 7 + 3 = 1$ . If the new node connects to 2 old nodes, a new loop is formed:  $5 - 8 + 4 = 1$ .
- 17 (a) 8 independent columns (b)  $\mathbf{f}$  must be orthogonal to the nullspace so  $\mathbf{f}$ 's add to zero (c) Each edge goes into 2 nodes, 12 edges make diagonal entries sum to 24.
- 18 A complete graph has  $5 + 4 + 3 + 2 + 1 = 15$  edges. With  $n$  nodes that count is  $1 + \cdots + (n - 1) = n(n - 1)/2$ . Tree has 5 edges.

### Problem Set 8.3, page 437

- 1 Eigenvalues  $\lambda = 1$  and  $.75$ ;  $(A - I)\mathbf{x} = \mathbf{0}$  gives the steady state  $\mathbf{x} = (.6, .4)$  with  $A\mathbf{x} = \mathbf{x}$ .
- 2  $A = \begin{bmatrix} .6 & -1 \\ .4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ .75 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -.4 & .6 \end{bmatrix}$ ;  $A^\infty = \begin{bmatrix} .6 & -1 \\ .4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -.4 & .6 \end{bmatrix} = \begin{bmatrix} .6 & .6 \\ .4 & .4 \end{bmatrix}$ .
- 3  $\lambda = 1$  and  $.8$ ,  $\mathbf{x} = (1, 0)$ ;  $1$  and  $-.8$ ,  $\mathbf{x} = (\frac{5}{9}, \frac{4}{9})$ ;  $1, \frac{1}{4}$ , and  $\frac{1}{4}$ ,  $\mathbf{x} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ .
- 4  $A^T$  always has the eigenvector  $(1, 1, \dots, 1)$  for  $\lambda = 1$ , because each row of  $A^T$  adds to 1. (Note again that many authors use row vectors multiplying Markov matrices. So they transpose our form of  $A$ .)
- 5 The steady state eigenvector for  $\lambda = 1$  is  $(0, 0, 1) = \text{everyone is dead}$ .
- 6 Add the components of  $A\mathbf{x} = \lambda\mathbf{x}$  to find sum  $s = \lambda s$ . If  $\lambda \neq 1$  the sum must be  $s = 0$ .
- 7  $(.5)^k \rightarrow 0$  gives  $A^k \rightarrow A^\infty$ ; any  $A = \begin{bmatrix} .6 + .4a & .6 - .6a \\ .4 - .4a & .4 + .6a \end{bmatrix}$  with  $\begin{matrix} a \leq 1 \\ .4 + .6a \geq 0 \end{matrix}$
- 8 If  $P = \text{cyclic permutation}$  and  $\mathbf{u}_0 = (1, 0, 0, 0)$  then  $\mathbf{u}_1 = (0, 0, 1, 0)$ ;  $\mathbf{u}_2 = (0, 1, 0, 0)$ ;  $\mathbf{u}_3 = (1, 0, 0, 0)$ ;  $\mathbf{u}_4 = \mathbf{u}_0$ . The eigenvalues  $1, i, -1, -i$  are all on the unit circle. This Markov matrix contains zeros; a positive matrix has one largest eigenvalue  $\lambda = 1$ .
- 9  $M^2$  is still nonnegative;  $[1 \ \cdots \ 1]M = [1 \ \cdots \ 1]$  so multiply on the right by  $M$  to find  $[1 \ \cdots \ 1]M^2 = [1 \ \cdots \ 1] \Rightarrow \text{columns of } M^2 \text{ add to } 1$ .
- 10  $\lambda = 1$  and  $a + d - 1$  from the trace; steady state is a multiple of  $\mathbf{x}_1 = (b, 1 - a)$ .
- 11 Last row  $.2, .3, .5$  makes  $A = A^T$ ; rows also add to 1 so  $(1, \dots, 1)$  is also an eigenvector of  $A$ .
- 12  $B$  has  $\lambda = 0$  and  $-.5$  with  $\mathbf{x}_1 = (.3, .2)$  and  $\mathbf{x}_2 = (-1, 1)$ ;  $A$  has  $\lambda = 1$  so  $A - I$  has  $\lambda = 0$ .  $e^{-.5t}$  approaches zero and the solution approaches  $c_1 e^{0t} \mathbf{x}_1 = c_1 \mathbf{x}_1$ .
- 13  $\mathbf{x} = (1, 1, 1)$  is an eigenvector when the row sums are equal;  $A\mathbf{x} = (.9, .9, .9)$

- 14**  $(I - A)(I + A + A^2 + \cdots) = (I + A + A^2 + \cdots) - (A + A^2 + A^3 + \cdots) = I$ . This says that  $I + A + A^2 + \cdots$  is  $(I - A)^{-1}$ . When  $A = \begin{bmatrix} 0 & .5 \\ 1 & 0 \end{bmatrix}$ ,  $A^2 = \frac{1}{2}I$ ,  $A^3 = \frac{1}{2}A$ ,  $A^4 = \frac{1}{4}I$  and the series adds to  $\begin{bmatrix} 1 + \frac{1}{2} + \cdots & \frac{1}{2} + \frac{1}{4} + \cdots \\ 1 + \frac{1}{2} + \cdots & 1 + \frac{1}{2} + \cdots \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix} = (I - A)^{-1}$ .
- 15** The first two  $A$ 's have  $\lambda_{\max} < 1$ ;  $p = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$  and  $\begin{bmatrix} 130 \\ 32 \end{bmatrix}$ ;  $I - \begin{bmatrix} .5 & 1 \\ .5 & 0 \end{bmatrix}$  has no inverse.
- 16**  $\lambda = 1$  (Markov), 0 (singular), .2 (from trace). Steady state (.3, .3, .4) and (30, 30, 40).
- 17** No,  $A$  has an eigenvalue  $\lambda = 1$  and  $(I - A)^{-1}$  does not exist.
- 18** The Leslie matrix on page 435 has  $\det(A - \lambda I) = \det \begin{bmatrix} F_1 - \lambda & F_2 & F_3 \\ P_1 & -\lambda & 0 \\ 0 & P_2 & -\lambda \end{bmatrix} = -\lambda^3 + F_1\lambda^2 + F_2P_1\lambda + F_3P_1P_2$ . This is negative for large  $\lambda$ . It is positive at  $\lambda = 1$  provided that  $F_1 + F_2P_1 + F_3P_1P_2 > 1$ . Under this key condition,  $\det(A - \lambda I)$  must be zero at some  $\lambda$  between 1 and  $\infty$ . That eigenvalue means that the population grows (under this condition connecting  $F$ 's and  $P$ 's reproduction and survival rates).
- 19**  $\Lambda$  times  $S^{-1}\Delta S$  has the same diagonal as  $S^{-1}\Delta S$  times  $\Lambda$  because  $\Lambda$  is diagonal.
- 20** If  $B > A > 0$  and  $Ax = \lambda_{\max}(A)x > 0$  then  $Bx > \lambda_{\max}(A)x$  and  $\lambda_{\max}(B) > \lambda_{\max}(A)$ .

## Problem Set 8.4, page 446

- Feasible set = line segment (6, 0) to (0, 3); minimum cost at (6, 0), maximum at (0, 3).
- Feasible set has corners (0, 0), (6, 0), (2, 2), (0, 6). Minimum cost  $2x - y$  at (6, 0).
- Only two corners (4, 0, 0) and (0, 2, 0); let  $x_i \rightarrow -\infty$ ,  $x_2 = 0$ , and  $x_3 = x_1 - 4$ .
- From (0, 0, 2) move to  $x = (0, 1, 1.5)$  with the constraint  $x_1 + x_2 + 2x_3 = 4$ . The new cost is  $3(1) + 8(1.5) = \$15$  so  $r = -1$  is the reduced cost. The simplex method also checks  $x = (1, 0, 1.5)$  with cost  $5(1) + 8(1.5) = \$17$ ;  $r = 1$  means more expensive.
- Cost = 20 at start (4, 0, 0); keeping  $x_1 + x_2 + 2x_3 = 4$  move to (3, 1, 0) with cost 18 and  $r = -2$ ; or move to (2, 0, 1) with cost 17 and  $r = -3$ . Choose  $x_3$  as entering variable and move to (0, 0, 2) with cost 14. Another step will reach (0, 4, 0) with minimum cost 12.
- If we reduce the Ph.D. cost to \$1 or \$2 (below the student cost of \$3), the job will go to the Ph.D. with cost vector  $c = (2, 3, 8)$  the Ph.D. takes 4 hours ( $x_1 + x_2 + 2x_3 = 4$ ) and charges \$8.  
The teacher in the dual problem now has  $y \leq 2$ ,  $y \leq 3$ ,  $2y \leq 8$  as constraints  $A^T y \leq c$  on the charge of  $y$  per problem. So the dual has maximum at  $y = 2$ . The dual cost is also \$8 for 4 problems and maximum = minimum.
- $x = (2, 2, 0)$  is a corner of the feasible set with  $x_1 + x_2 + 2x_3 = 4$  and the new constraint  $2x_1 + x_2 + x_3 = 6$ . The cost of this corner is  $c^T x = (5, 3, 8) \cdot (2, 2, 0) = 16$ . Is this the minimum cost?

Compute the reduced cost  $r$  if  $x_3 = 1$  enters ( $x_3$  was previously zero). The two constraint equations now require  $x_1 = 3$  and  $x_2 = -1$ . With  $x = (3, -1, 1)$  the new

cost is  $3.5 - 1.3 + 1.8 = 20$ . This is higher than 16, so the original  $\mathbf{x} = (2, 2, 0)$  was optimal.

Note that  $x_3 = 1$  led to  $x_2 = -1$  and a negative  $x_2$  is not allowed. If  $x_3$  reduced the cost (it didn't) we would not have used as much as  $x_3 = 1$ .

8  $\mathbf{y}^T \mathbf{b} \leq \mathbf{y}^T \mathbf{A} \mathbf{x} = (\mathbf{A}^T \mathbf{y})^T \mathbf{x} \leq \mathbf{c}^T \mathbf{x}$ . The first inequality needed  $\mathbf{y} \geq 0$  and  $\mathbf{A} \mathbf{x} - \mathbf{b} \geq 0$ .

## Problem Set 8.5, page 451

1  $\int_0^{2\pi} \cos((j+k)x) dx = \left[ \frac{\sin((j+k)x)}{j+k} \right]_0^{2\pi} = 0$  and similarly  $\int_0^{2\pi} \cos((j-k)x) dx = 0$ . Notice  $j-k \neq 0$  in the denominator. If  $j=k$  then  $\int_0^{2\pi} \cos^2 jx dx = \pi$ .

2 Three integral tests show that  $1, x, x^2 - \frac{1}{3}$  are orthogonal on the interval  $[-1, 1]$ :  $\int_{-1}^1 (1)(x) dx = 0$ ,  $\int_{-1}^1 (1)(x^2 - \frac{1}{3}) dx = 0$ ,  $\int_{-1}^1 (x)(x^2 - \frac{1}{3}) dx = 0$ . Then  $2x^2 = 2(x^2 - \frac{1}{3}) + 0(x) + \frac{2}{3}(1)$ . Those coefficients  $2, 0, \frac{2}{3}$  can come from integrating  $f(x) = 2x^2$  times the 3 basis functions and dividing by their lengths squared—in other words using  $\mathbf{a}^T \mathbf{b} / \mathbf{a}^T \mathbf{a}$  for functions (where  $\mathbf{b}$  is  $f(x)$  and  $\mathbf{a}$  is  $1$  or  $x$  or  $x^2 - \frac{1}{3}$ ) exactly as for vectors.

3 One example orthogonal to  $\mathbf{v} = (1, \frac{1}{2}, \dots)$  is  $\mathbf{w} = (2, -1, 0, 0, \dots)$  with  $\|\mathbf{w}\| = \sqrt{5}$ .

4  $\int_{-1}^1 (1)(x^3 - cx) dx = 0$  and  $\int_{-1}^1 (x^2 - \frac{1}{3})(x^3 - cx) dx = 0$  for all  $c$  (odd functions). Choose  $c$  so that  $\int_{-1}^1 x(x^3 - cx) dx = [\frac{1}{5}x^5 - \frac{c}{3}x^3]_{-1}^1 = \frac{2}{5} - c\frac{2}{3} = 0$ . Then  $c = \frac{3}{5}$ .

5 The integrals lead to the Fourier coefficients  $a_1 = 0$ ,  $b_1 = 4/\pi$ ,  $b_2 = 0$ .

6 From eqn. (3)  $a_k = 0$  and  $b_k = 4/\pi k$  (odd  $k$ ). The square wave has  $\|f\|^2 = 2\pi$ . Then eqn. (6) is  $2\pi = \pi(16/\pi^2)(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots)$ . That infinite series equals  $\pi^2/8$ .

7 The  $-1, 1$  odd square wave is  $f(x) = x/|x|$  for  $0 < |x| < \pi$ . Its Fourier series in equation (8) is  $4/\pi$  times  $[\sin x + (\sin 3x)/3 + (\sin 5x)/5 + \dots]$ . The sum of the first  $N$  terms has an interesting shape, close to the square wave except where the wave jumps between  $-1$  and  $1$ . At those jumps, the Fourier sum spikes the wrong way to  $\pm 1.09$  (the *Gibbs phenomenon*) before it takes the jump with the true  $f(x)$ .

This happens for the Fourier sums of all functions with jumps. It makes shock waves hard to compute. You can see it clearly in a graph of the sum of 10 terms.

8  $\|\mathbf{v}\|^2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = 2$  so  $\|\mathbf{v}\| = \sqrt{2}$ ;  $\|\mathbf{v}\|^2 = 1 + a^2 + a^4 + \dots = 1/(1-a^2)$  so  $\|\mathbf{v}\| = 1/\sqrt{1-a^2}$ ;  $\int_0^{2\pi} (1 + 2\sin x + \sin^2 x) dx = 2\pi + 0 + \pi$  so  $\|f\| = \sqrt{3\pi}$ .

9 (a)  $f(x) = (1 + \text{square wave})/2$  so the  $a$ 's are  $\frac{1}{2}, 0, 0, \dots$  and the  $b$ 's are  $2/\pi, 0, -2/3\pi, 0, 2/5\pi, \dots$  (b)  $a_0 = \int_0^{2\pi} x dx / 2\pi = \pi$ , all other  $a_k = 0$ ,  $b_k = -2/k$ .

10 The integral from  $-\pi$  to  $\pi$  or from  $0$  to  $2\pi$  (or from any  $a$  to  $a + 2\pi$ ) is over one complete period of the function. If  $f(x)$  is periodic this changes  $\int_0^{2\pi} f(x) dx$  to  $\int_0^\pi f(x) dx + \int_{-\pi}^0 f(x) dx$ . If  $f(x)$  is **odd**, those integrals cancel to give  $\int f(x) dx = 0$  over one period.

11  $\cos^2 x = \frac{1}{2} + \frac{1}{2} \cos 2x$ ;  $\cos(x + \frac{\pi}{3}) = \cos x \cos \frac{\pi}{3} - \sin x \sin \frac{\pi}{3} = \frac{1}{2} \cos x - \frac{\sqrt{3}}{2} \sin x$ .

$$12 \quad \frac{d}{dx} \begin{bmatrix} 1 \\ \cos x \\ \sin x \\ \cos 2x \\ \sin 2x \end{bmatrix} = \begin{bmatrix} 0 \\ -\sin x \\ \cos x \\ -2 \sin 2x \\ 2 \cos 2x \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \cos x \\ \sin x \\ \cos 2x \\ \sin 2x \end{bmatrix}. \quad \text{This shows the differentiation matrix.}$$

- 13 The square pulse with  $F(x) = 1/h$  for  $-x \leq h/2 \leq x$  is an even function, so all sine coefficients  $b_k$  are zero. The average  $a_0$  and the cosine coefficients  $a_k$  are

$$a_0 = \frac{1}{2\pi} \int_{-h/2}^{h/2} (1/h) dx = \frac{1}{2\pi}$$

$$a_k = \frac{1}{\pi} \int_{-h/2}^{h/2} (1/h) \cos kx dx = \frac{2}{\pi kh} \left( \sin \frac{kh}{2} \right) \text{ which is } \frac{1}{\pi} \operatorname{sinc} \left( \frac{kh}{2} \right)$$

(introducing the sinc function  $(\sin x)/x$ ). As  $h$  approaches zero, the number  $x = kh/2$  approaches zero, and  $(\sin x)/x$  approaches 1. So all those  $a_k$  approach  $1/\pi$ .

The limiting “delta function” contains an equal amount of all cosines: a very irregular function.

## Problem Set 8.6, page 458

- 1 The diagonal matrix  $C = W^T W$  is  $\Sigma^{-1} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1/2 \end{bmatrix}$  with no covariances (independent trials). Then solve  $A^T C A \hat{x} = A^T C b$  for this weighted least squares problem (notice  $Ct + D$  instead of  $C + Dt$ ):

$$Ax = \hat{b} \quad \text{is} \quad \begin{matrix} 0C + D = 1 \\ 1C + D = 2 \\ 2C + D = 4 \end{matrix} \quad \text{or} \quad \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}.$$

$$A^T C A = \begin{bmatrix} 3 & 2 \\ 2 & 2.5 \end{bmatrix} \quad A^T C b = \begin{bmatrix} 6 \\ 5 \end{bmatrix} \quad \hat{x} = \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 10/7 \\ 6/7 \end{bmatrix}.$$

- 2 If the measurement  $b_3$  is totally unreliable and  $\sigma_3^2 = \infty$ , then the best line will not use  $b_3$ . In this example, the system  $Ax = b$  becomes square (first two equations from Problem 1):

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{gives} \quad \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \quad \text{The line } b = t + 1 \text{ fits exactly.}$$

- 3 If  $\sigma_3 = 0$  the third equation is exact. Then the best line has  $Ct + D = b_3$  which is  $2C + D = 4$ . The errors  $Ct + D - b$  in the measurements at  $t = 0$  and  $1$  are  $D - 1$  and  $C + D - 2$ . Since  $D = 4 - 2C$  from the exact  $b_3 = 4$ , those two errors are  $D - 1 = 3 - 2C$  and  $C + D - 2 = 2 - C$ . The sum of squares  $(3 - 2C)^2 + (2 - C)^2$  is a minimum at  $8 = 5C$  (calculus or linear algebra in 1D). Then  $C = 8/5$  and  $D = 4 - 2C = 4/5$ .

- 4 0, 1, 2 have probabilities  $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$  and  $\sigma^2 = (0-1)^2\frac{1}{4} + (1-1)^2\frac{1}{2} + (2-1)^2\frac{1}{4} = \frac{1}{2}$ .
- 5 Mean  $(\frac{1}{2}, \frac{1}{2})$ . Independent flips lead to  $\Sigma = \text{diag}(\frac{1}{4}, \frac{1}{4})$ . Trace  $= \sigma_{\text{total}}^2 = \frac{1}{2}$ .
- 6 Mean  $m = p_0$  and variance  $\sigma^2 = (1-p_0)^2 p_0 + (0-p_0)^2(1-p_0) = p_0(1-p_0)$ .
- 7 Minimize  $P = a^2\sigma_1^2 + (1-a)^2\sigma_2^2$  at  $P' = 2a\sigma_1^2 - 2(1-a)\sigma_2^2 = 0$ ;  $a = \sigma_2^2/(\sigma_1^2 + \sigma_2^2)$  recovers equation (2) for the statistically correct choice with minimum variance.
- 8 Multiply  $L\Sigma L^T = (A^T\Sigma^{-1}A)^{-1}A^T\Sigma^{-1}\Sigma\Sigma^{-1}A(A^T\Sigma^{-1}A)^{-1} = P = (A^T\Sigma^{-1}A)^{-1}$ .
- 9 The new grade matrix  $A$  has row 3 = - row 1 and row 4 = - row 2, so the rank is 7. The nullspace of  $A$  now includes  $(1, -1, -1, 1)$  as well as  $(1, 1, 1, 1)$ . Compare to the grade matrix in Example 6 (not Example 5). The other two singular vectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$  for Example 6 are still correct for this new  $A$  ( $A\mathbf{v}_1$  is still orthogonal to  $A\mathbf{v}_2$ ):

$$A[2\mathbf{v}_1 \quad 2\mathbf{v}_2] = \begin{bmatrix} 3 & -1 & 1 & -3 \\ -1 & 3 & -3 & 1 \\ -3 & 1 & -1 & -3 \\ 1 & -3 & 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & -1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 8 & -4 \\ -8 & -4 \\ -8 & 4 \\ 8 & 4 \end{bmatrix}.$$

Those last orthogonal columns are multiples of the orthonormal  $\mathbf{u}_1$  and  $\mathbf{u}_2$ . This matrix  $A$  has  $\sigma_1 = 8$  and  $\sigma_2 = 4$  (only two singular values since the rank is 2). If you compute  $A^T A$  to find those singular vectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$  from scratch, notice that its trace is  $\sigma_1^2 + \sigma_2^2 = 64 + 16 = 80$ :

$$A^T A = \begin{bmatrix} 20 & -12 & -20 & 12 \\ -12 & 20 & 12 & -20 \\ -20 & 12 & 20 & -12 \\ 12 & -20 & -12 & 20 \end{bmatrix}.$$

## Problem Set 8.7, page 463

- 1  $(x, y, z)$  has homogeneous coordinates  $(cx, cy, cz, c)$  for  $c = 1$  and all  $c \neq 0$ .
- 2 For an affine transformation we also need  $T$  (origin), because  $T(\mathbf{0})$  need not be  $\mathbf{0}$  for affine  $T$ . Including this translation by  $T(\mathbf{0})$ ,  $(x, y, z, 1)$  is transformed to  $xT(\mathbf{i}) + yT(\mathbf{j}) + zT(\mathbf{k}) + T(\mathbf{0})$ .
- 3  $T T_1 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 1 & 4 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 0 & 2 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 1 & 6 & 8 & 1 \end{bmatrix}$  is translation along  $(1, 6, 8)$ .
- 4  $S = \text{diag}(c, c, c, 1)$ ; row 4 of  $ST$  and  $TS$  is  $1, 4, 3, 1$  and  $c, 4c, 3c, 1$ ; use  $\mathbf{v}TS$ !
- 5  $S = \begin{bmatrix} 1/8.5 & & \\ & 1/11 & \\ & & 1 \end{bmatrix}$  for a 1 by 1 square, starting from an 8.5 by 11 page.
- 6  $[x \ y \ z \ 1] \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ -1 & -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & & & \\ & 2 & & \\ & & 2 & \\ & & & 1 \end{bmatrix} = [x \ y \ z \ 1] \begin{bmatrix} 2 & & & \\ & 2 & & \\ & & 2 & \\ -2 & -2 & -4 & 1 \end{bmatrix}.$
- The first matrix translates by  $(-1, -1, -2)$ . The second matrix rescales by 2.



- 7 The three parts of  $Q$  in equation (1) are  $(\cos \theta)I$  and  $(1 - \cos \theta)\mathbf{a}\mathbf{a}^T$  and  $-\sin \theta(\mathbf{a}\mathbf{x})$ . Then  $Q\mathbf{a} = \mathbf{a}$  because  $\mathbf{a}\mathbf{a}^T\mathbf{a} = \mathbf{a}$  (unit vector) and  $\mathbf{a}\mathbf{x}\mathbf{a} = \mathbf{0}$ .
- 8 If  $\mathbf{a}^T\mathbf{b} = 0$  and those three parts of  $Q$  (Problem 7) multiply  $\mathbf{b}$ , the results in  $Q\mathbf{b}$  are  $(\cos \theta)\mathbf{b}$  and  $\mathbf{a}\mathbf{a}^T\mathbf{b} = \mathbf{0}$  and  $(-\sin \theta)\mathbf{a}\mathbf{x}\mathbf{b}$ . The component along  $\mathbf{b}$  is  $(\cos \theta)\mathbf{b}$ .
- 9  $\mathbf{n} = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$  has  $P = I - \mathbf{n}\mathbf{n}^T = \frac{1}{9} \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix}$ . Notice  $\|\mathbf{n}\| = 1$ .
- 10 We can choose  $(0, 0, 3)$  on the plane and multiply  $T_-PT_+ = \frac{1}{9} \begin{bmatrix} 5 & -4 & -2 & 0 \\ -4 & 5 & -2 & 0 \\ -2 & -2 & 8 & 0 \\ 6 & 6 & 3 & 9 \end{bmatrix}$ .
- 11  $(3, 3, 3)$  projects to  $\frac{1}{3}(-1, -1, 4)$  and  $(3, 3, 3, 1)$  projects to  $(\frac{1}{3}, \frac{1}{3}, \frac{5}{3}, 1)$ . Row vectors!
- 12 The projection of a square onto a plane is a parallelogram (or a line segment). The sides of the square are perpendicular, but their projections may not be ( $\mathbf{x}^T\mathbf{y} = 0$  but  $(P\mathbf{x})^T(P\mathbf{y}) = \mathbf{x}^TP^TP\mathbf{y} = \mathbf{x}^TP\mathbf{y}$  may be nonzero).
- 13 That projection of a cube onto a plane produces a hexagon.
- 14  $(3, 3, 3)(I - 2\mathbf{n}\mathbf{n}^T) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \begin{bmatrix} 1 & -8 & -4 \\ -8 & 1 & -4 \\ -4 & -4 & 7 \end{bmatrix} = \left(-\frac{11}{3}, -\frac{11}{3}, -\frac{1}{3}\right)$ .
- 15  $(3, 3, 3, 1) \rightarrow (3, 3, 0, 1) \rightarrow \left(-\frac{7}{3}, -\frac{7}{3}, -\frac{8}{3}, 1\right) \rightarrow \left(-\frac{7}{3}, -\frac{7}{3}, \frac{1}{3}, 1\right)$ .
- 16 Just subtracting vectors would give  $\mathbf{v} = (x, y, z, 0)$  ending in 0 (not 1). In homogeneous coordinates, add a **vector** to a point.
- 17 Space is rescaled by  $1/c$  because  $(x, y, z, c)$  is the same point as  $(x/c, y/c, z/c, 1)$ .

## Problem Set 9.1, page 472

- 1 Without exchange, pivots .001 and 1000; with exchange, 1 and  $-1$ . When the pivot is larger than the entries below it, all  $|\ell_{ij}| = |\text{entry}/\text{pivot}| \leq 1$ .  $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$ .
- 2 The exact inverse of  $\text{hilb}(3)$  is  $A^{-1} = \begin{bmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{bmatrix}$ .
- 3  $A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 11/6 \\ 13/12 \\ 47/60 \end{bmatrix} = \begin{bmatrix} 1.833 \\ 1.083 \\ 0.783 \end{bmatrix}$  compares with  $A \begin{bmatrix} 0 \\ 6 \\ -3.6 \end{bmatrix} = \begin{bmatrix} 1.80 \\ 1.10 \\ 0.78 \end{bmatrix}$ .  $\|\Delta\mathbf{b}\| < .04$  but  $\|\Delta\mathbf{x}\| > 6$ .  
The difference  $(1, 1, 1) - (0, 6, -3.6)$  is in a direction  $\Delta\mathbf{x}$  that has  $A\Delta\mathbf{x}$  near zero.
- 4 The largest  $\|\mathbf{x}\| = \|A^{-1}\mathbf{b}\|$  is  $\|A^{-1}\| = 1/\lambda_{\min}$  since  $A^T = A$ ; largest error  $10^{-16}/\lambda_{\min}$ .
- 5 Each row of  $U$  has at most  $w$  entries. Then  $w$  multiplications to substitute components of  $\mathbf{x}$  (already known from below) and divide by the pivot. Total for  $n$  rows  $< wn$ .
- 6 The triangular  $L^{-1}$ ,  $U^{-1}$ ,  $R^{-1}$  need  $\frac{1}{2}n^2$  multiplications.  $Q$  needs  $n^2$  to multiply the right side by  $Q^{-1} = Q^T$ . So  $QR\mathbf{x} = \mathbf{b}$  takes 1.5 times longer than  $LU\mathbf{x} = \mathbf{b}$ .

- 7**  $UU^{-1} = I$ : Back substitution needs  $\frac{1}{2}j^2$  multiplications on column  $j$ , using the  $j$  by  $j$  upper left block. Then  $\frac{1}{2}(1^2 + 2^2 + \cdots + n^2) \approx \frac{1}{2}(\frac{1}{3}n^3) = \text{total to find } U^{-1}$ .
- 8**  $\begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 2 \\ 0 & -1 \end{bmatrix} = U$  with  $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$  and  $L = \begin{bmatrix} 1 & 0 \\ .5 & 1 \end{bmatrix}$ ;  
 $A \rightarrow \begin{bmatrix} 2 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = U$  with  
 $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$  and  $L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ .5 & -.5 & 1 \end{bmatrix}$ .
- 9**  $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$  has cofactors  $C_{13} = C_{31} = C_{24} = C_{42} = 1$  and  $C_{14} = C_{41} = -1$ .  $A^{-1}$  is a full matrix!
- 10** With 16-digit floating point arithmetic the errors  $\|\mathbf{x} - \mathbf{x}_{\text{computed}}\|$  for  $\varepsilon = 10^{-3}, 10^{-6}, 10^{-9}, 10^{-12}, 10^{-15}$  are of order  $10^{-16}, 10^{-11}, 10^{-7}, 10^{-4}, 10^{-3}$ .
- 11** (a)  $\cos \theta = 1/\sqrt{10}$ ,  $\sin \theta = -3/\sqrt{10}$ ,  $R = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 3 & 5 \end{bmatrix} = \frac{1}{\sqrt{10}} \begin{bmatrix} 10 & 14 \\ 0 & 8 \end{bmatrix}$ .  
 (b)  $A$  has eigenvalues 4 and 2. Put one of the unit eigenvectors in row 1 of  $Q$ : either  $Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$  and  $Q A Q^{-1} = \begin{bmatrix} 2 & -4 \\ 0 & 4 \end{bmatrix}$  or  $Q = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}$  and  $Q A Q^{-1} = \begin{bmatrix} 4 & -4 \\ 0 & 2 \end{bmatrix}$ .
- 12** When  $A$  is multiplied by a plane rotation  $Q_{ij}$ , this changes the  $2n$  (not  $n^2$ ) entries in rows  $i$  and  $j$ . Then multiplying on the right by  $(Q_{ij})^{-1} = (Q_{ij})^T$  changes the  $2n$  entries in columns  $i$  and  $j$ .
- 13**  $Q_{ij} A$  uses  $4n$  multiplications (2 for each entry in rows  $i$  and  $j$ ). By factoring out  $\cos \theta$ , the entries 1 and  $\pm \tan \theta$  need only  $2n$  multiplications, which leads to  $\frac{2}{3}n^3$  for  $QR$ .
- 14** The  $(2, 1)$  entry of  $Q_{21} A$  is  $\frac{1}{3}(-\sin \theta + 2 \cos \theta)$ . This is zero if  $\sin \theta = 2 \cos \theta$  or  $\tan \theta = 2$ . Then the 2, 1,  $\sqrt{5}$  right triangle has  $\sin \theta = 2/\sqrt{5}$  and  $\cos \theta = 1/\sqrt{5}$ .  
 Every 3 by 3 rotation with  $\det Q = +1$  is the product of 3 plane rotations.
- 15** This problem shows how elimination is more expensive (the nonzero multipliers are counted by  $\mathbf{nnz}(L)$  and  $\mathbf{nnz}(LL)$ ) when we spoil the tridiagonal  $K$  by a random permutation.  
 If on the other hand we start with a poorly ordered matrix  $K$ , an improved ordering is found by the code **symamd** discussed in this section.
- 16** The “red-black ordering” puts rows and columns 1 to 10 in the odd-even order 1, 3, 5, 7, 9, 2, 4, 6, 8, 10. When  $K$  is the  $-1, 2, -1$  tridiagonal matrix, odd points are connected

only to even points (and 2 stays on the diagonal, connecting every point to itself):

$$K = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 2 \end{bmatrix} \text{ and } PKP^T = \begin{bmatrix} 2I & D \\ D^T & 2I \end{bmatrix} \text{ with}$$

$$D = \begin{bmatrix} -1 & & & & \\ -1 & -1 & & & \\ 0 & -1 & -1 & & \\ & & -1 & -1 & \\ & & & -1 & -1 \end{bmatrix} \begin{array}{l} 1 \text{ to } 2 \\ 3 \text{ to } 2, 4 \\ 5 \text{ to } 4, 6 \\ 7 \text{ to } 6, 8 \\ 9 \text{ to } 8, 10 \end{array}$$

- 17 Jeff Stuart's **Shake a Stick** activity has long sticks representing the graphs of two linear equations in the  $x$ - $y$  plane. The matrix is nearly singular and Section 9.2 shows how to compute its condition number  $c = \|A\|\|A^{-1}\| = \sigma_{\max}/\sigma_{\min} \approx 80,000$ :

$$A = \begin{bmatrix} 1 & 1.0001 \\ 1 & 1.0000 \end{bmatrix} \quad \|A\| \approx 2 \quad A^{-1} = 10000 \begin{bmatrix} -1 & 1.0001 \\ 1 & -1 \end{bmatrix} \quad \begin{array}{l} \|A^{-1}\| \approx 20000 \\ c \approx 40000. \end{array}$$

## Problem Set 9.2, page 478

- 1  $\|A\| = 2$ ,  $\|A^{-1}\| = 2$ ,  $c = 4$ ;  $\|A\| = 3$ ,  $\|A^{-1}\| = 1$ ,  $c = 3$ ;  $\|A\| = 2 + \sqrt{2} = \lambda_{\max}$  for positive definite  $A$ ,  $\|A^{-1}\| = 1/\lambda_{\min}$ ,  $c = (2 + \sqrt{2})/(2 - \sqrt{2}) = 5.83$ .
- 2  $\|A\| = 2$ ,  $c = 1$ ;  $\|A\| = \sqrt{2}$ ,  $c = \text{infinite}$  (singular matrix);  $A^T A = 2I$ ,  $\|A\| = \sqrt{2}$ ,  $c = 1$ .
- 3 For the first inequality replace  $\mathbf{x}$  by  $B\mathbf{x}$  in  $\|A\mathbf{x}\| \leq \|A\|\|\mathbf{x}\|$ ; the second inequality is just  $\|B\mathbf{x}\| \leq \|B\|\|\mathbf{x}\|$ . Then  $\|AB\| = \max(\|AB\mathbf{x}\|/\|\mathbf{x}\|) \leq \|A\|\|B\|$ .
- 4  $1 = \|I\| = \|AA^{-1}\| \leq \|A\|\|A^{-1}\| = c(A)$ .
- 5 If  $\Lambda_{\max} = \Lambda_{\min} = 1$  then all  $\Lambda_i = 1$  and  $A = SIS^{-1} = I$ . The only matrices with  $\|A\| = \|A^{-1}\| = 1$  are *orthogonal matrices*.
- 6 All orthogonal matrices have norm 1, so  $\|A\| \leq \|Q\|\|R\| = \|R\|$  and in reverse  $\|R\| \leq \|Q^{-1}\|\|A\| = \|A\|$ , then  $\|A\| = \|R\|$ . Inequality is usual in  $\|A\| < \|L\|\|U\|$  when  $A^T A \neq AA^T$ . Use **norm** on a random  $A$ .
- 7 The triangle inequality gives  $\|A\mathbf{x} + B\mathbf{x}\| \leq \|A\mathbf{x}\| + \|B\mathbf{x}\|$ . Divide by  $\|\mathbf{x}\|$  and take the maximum over all nonzero vectors to find  $\|A + B\| \leq \|A\| + \|B\|$ .
- 8 If  $A\mathbf{x} = \lambda\mathbf{x}$  then  $\|A\mathbf{x}\|/\|\mathbf{x}\| = |\lambda|$  for that particular vector  $\mathbf{x}$ . When we maximize the ratio over all vectors we get  $\|A\| \geq |\lambda|$ .
- 9  $A + B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$  has  $\rho(A) = 0$  and  $\rho(B) = 0$  but  $\rho(A + B) = 1$ .

The triangle inequality  $\|A + B\| \leq \|A\| + \|B\|$  fails for  $\rho(A)$ .  $AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$  also has  $\rho(AB) = 1$ ; thus  $\rho(A) = \max |\lambda(A)| = \text{spectral radius}$  is not a norm.

- 10** (a) The condition number of  $A^{-1}$  is  $\|A^{-1}\| \|(A^{-1})^{-1}\|$  which is  $\|A^{-1}\| \|A\| = c(A)$ .  
 (b) Since  $A^T A$  and  $A A^T$  have the same nonzero eigenvalues,  $A$  and  $A^T$  have the same norm.
- 11** Use the quadratic formula for  $\lambda_{\max}/\lambda_{\min}$ , which is  $c = \sigma_{\max}/\sigma_{\min}$  since this  $A = A^T$  is positive definite:

$$c(A) = \left(1.00005 + \sqrt{(1.00005)^2 - .0001}\right) / \left(1.00005 - \sqrt{\quad}\right) \approx 40,000.$$

- 12**  $\det(2A)$  is not  $2 \det A$ ;  $\det(A + B)$  is not always less than  $\det A + \det B$ ; taking  $|\det A|$  does not help. The only reasonable property is  $\det AB = (\det A)(\det B)$ . The condition number should not change when  $A$  is multiplied by 10.
- 13** The residual  $\mathbf{b} - A\mathbf{y} = (10^{-7}, 0)$  is much smaller than  $\mathbf{b} - A\mathbf{z} = (.0013, .0016)$ . But  $\mathbf{z}$  is much closer to the solution than  $\mathbf{y}$ .
- 14**  $\det A = 10^{-6}$  so  $A^{-1} = 10^3 \begin{bmatrix} 659 & -563 \\ -913 & 780 \end{bmatrix}$ :  $\|A\| > 1$ ,  $\|A^{-1}\| > 10^6$ , then  $c > 10^6$ .
- 15**  $\mathbf{x} = (1, 1, 1, 1, 1)$  has  $\|\mathbf{x}\| = \sqrt{5}$ ,  $\|\mathbf{x}\|_1 = 5$ ,  $\|\mathbf{x}\|_\infty = 1$ .  $\mathbf{x} = (.1, .7, .3, .4, .5)$  has  $\|\mathbf{x}\| = 1$ ,  $\|\mathbf{x}\|_1 = 2$  (sum)  $\|\mathbf{x}\|_\infty = .7$  (largest).
- 16**  $x_1^2 + \cdots + x_n^2$  is not smaller than  $\max(x_i^2)$  and not larger than  $(|x_1| + \cdots + |x_n|)^2 = \|\mathbf{x}\|_1^2$ .  $x_1^2 + \cdots + x_n^2 \leq n \max(x_i^2)$  so  $\|\mathbf{x}\| \leq \sqrt{n} \|\mathbf{x}\|_\infty$ . Choose  $y_i = \text{sign } x_i = \pm 1$  to get  $\|\mathbf{x}\|_1 = \mathbf{x} \cdot \mathbf{y} \leq \|\mathbf{x}\| \|\mathbf{y}\| = \sqrt{n} \|\mathbf{x}\|$ .  $\mathbf{x} = (1, \dots, 1)$  has  $\|\mathbf{x}\|_1 = \sqrt{n} \|\mathbf{x}\|$ .
- 17** For the  $\ell^\infty$  norm, the largest component of  $\mathbf{x}$  plus the largest component of  $\mathbf{y}$  is not less than  $\|\mathbf{x} + \mathbf{y}\|_\infty = \text{largest component of } \mathbf{x} + \mathbf{y}$ .
- For the  $\ell^1$  norm, each component has  $|x_i + y_i| \leq |x_i| + |y_i|$ . Sum on  $i = 1$  to  $n$ :  $\|\mathbf{x} + \mathbf{y}\|_1 \leq \|\mathbf{x}\|_1 + \|\mathbf{y}\|_1$ .
- 18**  $|x_1| + 2|x_2|$  is a norm but  $\min(|x_1|, |x_2|)$  is not a norm.  $\|\mathbf{x}\| + \|\mathbf{x}\|_\infty$  is a norm;  $\|A\mathbf{x}\|$  is a norm provided  $A$  is invertible (otherwise a nonzero vector has norm zero; for rectangular  $A$  we require independent columns to avoid  $\|A\mathbf{x}\| = 0$ ).
- 19**  $\mathbf{x}^T \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots \leq (\max |y_i|)(|x_1| + |x_2| + \cdots) = \|\mathbf{x}\|_1 \|\mathbf{y}\|_\infty$ .
- 20** With  $\lambda_j = 2 - 2 \cos(j\pi/(n+1))$ , the largest eigenvalue is  $\lambda_n \approx 2 + 2 = 4$ . The smallest is  $\lambda_1 = 2 - 2 \cos(\pi/(n+1)) \approx \left(\frac{\pi}{n+1}\right)^2$ , using  $2 \cos \theta \approx 2 - \theta^2$ . So the condition number is  $c = \lambda_{\max}/\lambda_{\min} \approx (4/\pi^2) n^2$ , growing with  $n$ .
- 21**  $A = \begin{bmatrix} 1 & 1 \\ 0 & 1.1 \end{bmatrix}$  has  $A^n = \begin{bmatrix} 1 & q \\ 0 & (1.1)^n \end{bmatrix}$  with  $q = 1 + 1.1 + \cdots + (1.1)^{n-1} = (1.1^n - 1)/(1.1 - 1) \approx 1.1^n/.1$ . So the growing part of  $A^n$  is  $1.1^n \begin{bmatrix} 0 & 10 \\ 0 & 1 \end{bmatrix}$  with  $\|A^n\| \approx \sqrt{101}$  times  $1.1^n$  for larger  $n$ .

### Problem Set 9.3, page 489

- 1** The iteration  $\mathbf{x}_{k+1} = (I - A)\mathbf{x}_k + \mathbf{b}$  has  $S = I$  and  $T = I - A$  and  $S^{-1}T = I - A$ .

- 2 If  $A\mathbf{x} = \lambda\mathbf{x}$  then  $(I - A)\mathbf{x} = (1 - \lambda)\mathbf{x}$ . Real eigenvalues of  $B = I - A$  have  $|1 - \lambda| < 1$  provided  $\lambda$  is between 0 and 2.
- 3 This matrix  $A$  has  $I - A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$  which has  $|\lambda| = 2$ . The iteration diverges.
- 4 Always  $\|AB\| \leq \|A\|\|B\|$ . Choose  $A = B$  to find  $\|B^2\| \leq \|B\|^2$ . Then choose  $A = B^2$  to find  $\|B^3\| \leq \|B^2\|\|B\| \leq \|B\|^3$ . Continue (or use induction) to find  $\|B^k\| \leq \|B\|^k$ . Since  $\|B\| \geq \max |\lambda(B)|$  it is no surprise that  $\|B\| < 1$  gives convergence.
- 5  $A\mathbf{x} = \mathbf{0}$  gives  $(S - T)\mathbf{x} = \mathbf{0}$ . Then  $S\mathbf{x} = T\mathbf{x}$  and  $S^{-1}T\mathbf{x} = \mathbf{x}$ . Then  $\lambda = 1$  means that the errors do not approach zero. We can't expect convergence when  $A$  is singular and  $A\mathbf{x} = \mathbf{b}$  is unsolvable!
- 6 Jacobi has  $S^{-1}T = \frac{1}{3} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$  with  $|\lambda|_{\max} = \frac{1}{3}$ . Small problem, fast convergence.
- 7 Gauss-Seidel has  $S^{-1}T = \begin{bmatrix} 0 & \frac{1}{3} \\ 0 & \frac{1}{9} \end{bmatrix}$  with  $|\lambda|_{\max} = \frac{1}{9}$  which is  $(|\lambda|_{\max} \text{ for Jacobi})^2$ .
- 8 Jacobi has  $S^{-1}T = \begin{bmatrix} a & \\ & d \end{bmatrix}^{-1} \begin{bmatrix} 0 & -b \\ -c & 0 \end{bmatrix} = \begin{bmatrix} 0 & -b/a \\ -c/d & 0 \end{bmatrix}$  with  $|\lambda| = |bc/ad|^{1/2}$ .  
 Gauss-Seidel has  $S^{-1}T = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} 0 & -b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -b/a \\ 0 & -bc/ad \end{bmatrix}$  with  $|\lambda| = |bc/ad|$ .  
 So Gauss-Seidel is twice as fast to converge (or to explode if  $|bc| > |ad|$ ).
- 9 Set the trace  $2 - 2\omega + \frac{1}{4}\omega^2$  equal to  $(\omega - 1) + (\omega - 1)$  to find  $\omega_{\text{opt}} = 4(2 - \sqrt{3}) \approx 1.07$ . The eigenvalues  $\omega - 1$  are about .07, a big improvement.
- 10 Gauss-Seidel will converge for the  $-1, 2, -1$  matrix.  $|\lambda|_{\max} = \cos^2(\pi/n + 1)$  is given on page 485, with the improvement from successive over relaxation.
- 11 If the iteration gives all  $x_i^{\text{new}} = x_i^{\text{old}}$  then the quantity in parentheses is zero, which means  $A\mathbf{x} = \mathbf{b}$ . For Jacobi change  $\mathbf{x}^{\text{new}}$  on the right side to  $\mathbf{x}^{\text{old}}$ .
- 12 A lot of energy went into SOR in the 1950's! Now incomplete  $LU$  is simpler and preferred.
- 13  $\mathbf{u}_k/\lambda_1^k = c_1\mathbf{x}_1 + c_2\mathbf{x}_2(\lambda_2/\lambda_1)^k + \cdots + c_n\mathbf{x}_n(\lambda_n/\lambda_1)^k \rightarrow c_1\mathbf{x}_1$  if all ratios  $|\lambda_i/\lambda_1| < 1$ . The largest ratio controls the rate of convergence (when  $k$  is large).  $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$  has  $|\lambda_2| = |\lambda_1|$  and no convergence.
- 14 The eigenvectors of  $A$  and also  $A^{-1}$  are  $\mathbf{x}_1 = (.75, .25)$  and  $\mathbf{x}_2 = (1, -1)$ . The inverse power method converges to a multiple of  $\mathbf{x}_2$ , since  $|1/\lambda_2| > |1/\lambda_1|$ .
- 15 In the  $j$ th component of  $A\mathbf{x}_1$ ,  $\lambda_1 \sin \frac{j\pi}{n+1} = 2 \sin \frac{j\pi}{n+1} - \sin \frac{(j-1)\pi}{n+1} - \sin \frac{(j+1)\pi}{n+1}$ . The last two terms combine into  $-2 \sin \frac{j\pi}{n+1} \cos \frac{\pi}{n+1}$ . Then  $\lambda_1 = 2 - 2 \cos \frac{\pi}{n+1}$ .
- 16  $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$  produces  $\mathbf{u}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ,  $\mathbf{u}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ ,  $\mathbf{u}_2 = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$ ,  $\mathbf{u}_3 = \begin{bmatrix} 14 \\ -13 \end{bmatrix}$ . This is converging to the eigenvector direction  $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$  with largest eigenvalue  $\lambda = 3$ . Divide  $\mathbf{u}_k$  by  $\|\mathbf{u}_k\|$ .

- 17  $A^{-1} = \frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$  gives  $\mathbf{u}_1 = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ ,  $\mathbf{u}_2 = \frac{1}{9} \begin{bmatrix} 5 \\ 4 \end{bmatrix}$ ,  $\mathbf{u}_3 = \frac{1}{27} \begin{bmatrix} 14 \\ 13 \end{bmatrix} \rightarrow \mathbf{u}_\infty = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$ .
- 18  $R = Q^T A = \begin{bmatrix} 1 & \cos \theta \sin \theta \\ 0 & -\sin^2 \theta \end{bmatrix}$  and  $A_1 = RQ = \begin{bmatrix} \cos \theta (1 + \sin^2 \theta) & -\sin^3 \theta \\ -\sin^3 \theta & -\cos \theta \sin^2 \theta \end{bmatrix}$ .
- 19 If  $A$  is orthogonal then  $Q = A$  and  $R = I$ . Therefore  $A_1 = RQ = A$  again, and the “QR method” doesn’t move from  $A$ . But shift  $A$  slightly and the method goes quickly to  $\Lambda$ .
- 20 If  $A - cI = QR$  then  $A_1 = RQ + cI = Q^{-1}(QR + cI)Q = Q^{-1}AQ$ . No change in eigenvalues because  $A_1$  is similar to  $A$ .
- 21 Multiply  $A\mathbf{q}_j = b_{j-1}\mathbf{q}_{j-1} + a_j\mathbf{q}_j + b_j\mathbf{q}_{j+1}$  by  $\mathbf{q}_j^T$  to find  $\mathbf{q}_j^T A\mathbf{q}_j = a_j$  (because the  $\mathbf{q}$ ’s are orthonormal). The matrix form (multiplying by columns) is  $AQ = QT$  where  $T$  is *tridiagonal*. The entries down the diagonals of  $T$  are the  $a$ ’s and  $b$ ’s.
- 22 Theoretically the  $\mathbf{q}$ ’s are orthonormal. In reality this important algorithm is not very stable. We must stop every few steps to reorthogonalize—or find another more stable way to orthogonalize  $\mathbf{q}, A\mathbf{q}, A^2\mathbf{q}, \dots$ .
- 23 If  $A$  is symmetric then  $A_1 = Q^{-1}AQ = Q^T AQ$  is also symmetric.  $A_1 = RQ = R(QR)R^{-1} = RAR^{-1}$  has  $R$  and  $R^{-1}$  upper triangular, so  $A_1$  cannot have nonzeros on a lower diagonal than  $A$ . If  $A$  is tridiagonal and symmetric then (by using symmetry for the upper part of  $A_1$ ) the matrix  $A_1 = RAR^{-1}$  is also tridiagonal.
- 24 The proof of  $|\lambda| < 1$  when every absolute row sum  $< 1$  uses  $|\sum a_{ij}x_j| \leq \sum |a_{ij}||x_i| < |x_i|$ . (Here  $x_i$  is the largest component.) The application to the Gershgorin circle theorem (very useful) is printed after its statement in this problem.
- 25 For  $A$  and  $K$ , the maximum row sums give all  $|\lambda| \leq 1$  and all  $|\lambda| \leq 4$ . The circles  $|\lambda - .5| \leq .5$  and  $|\lambda - .4| \leq .6$  around diagonal entries of  $A$  give tighter bounds. The circle  $|\lambda - 2| \leq 2$  for  $K$  contains the circle  $|\lambda - 2| \leq 1$  and all three eigenvalues  $2 + \sqrt{2}$ ,  $2$ , and  $2 - \sqrt{2}$ .
- 26 With diagonal dominance  $a_{ii} > r_i$ , the circles  $|\lambda - a_{ii}| \leq r_i$  don’t include  $\lambda = 0$  (so  $A$  is invertible!). Notice that the  $-1, 2, -1$  matrix is also invertible even though its diagonals are only weakly dominant. They *equal* the off-diagonal row sums,  $2 = 2$  except in the first and last rows, and more care is needed to prove invertibility.
- 27 From the last line of code,  $\mathbf{q}_2$  is in the direction of  $\mathbf{v} = A\mathbf{q}_1 - h_{11}\mathbf{q}_1 = A\mathbf{q}_1 - (\mathbf{q}_1^T A\mathbf{q}_1)\mathbf{q}_1$ . The dot product with  $\mathbf{q}_1$  is zero. This is Gram-Schmidt with  $A\mathbf{q}_1$  as the second input vector.
- 28 *Note* The five lines in Solutions to Selected Exercises prove two key properties of conjugate gradients—the residuals  $\mathbf{r}_k = \mathbf{b} - A\mathbf{x}_k$  are orthogonal and the search directions are  $A$ -orthogonal ( $\mathbf{p}_i^T A\mathbf{p}_i = 0$ ). Then each new guess  $\mathbf{x}_{k+1}$  is the **closest vector to  $\mathbf{x}$**  among all combinations of  $\mathbf{b}, A\mathbf{b}, A^k\mathbf{b}$ . Ordinary iteration  $S\mathbf{x}_{k+1} = T\mathbf{x}_k + \mathbf{b}$  does not find this best possible combination  $\mathbf{x}_{k+1}$ .

The solution to Problem 28 in this Fourth Edition is straightforward and important. Since  $H = Q^{-1}AQ = Q^T AQ$  is symmetric if  $A = A^T$ , and since  $H$  has only one lower diagonal by construction, then  $H$  has only one upper diagonal:  $H$  is tridiagonal and all the recursions in Arnoldi’s method have only 3 terms (Problem 29).

- 29**  $H = Q^{-1}AQ$  is similar to  $A$ , so  $H$  has the same eigenvalues as  $A$  (at the end of Arnoldi). When Arnoldi stops sooner because the matrix size is large, the eigenvalues of  $H_k$  (called *Ritz values*) are close to eigenvalues of  $A$ . This is an important way to compute approximations to  $\lambda$  for large matrices.
- 30** In principle the conjugate gradient method converges in 100 (or 99) steps to the exact solution  $\mathbf{x}$ . But it is slower than elimination and its all-important property is to give good approximations to  $\mathbf{x}$  much sooner. (Stopping elimination part way leaves you nothing.) The problem asks how close  $\mathbf{x}_{10}$  and  $\mathbf{x}_{20}$  are to  $\mathbf{x}_{100}$ , which equals  $\mathbf{x}$  except for roundoff errors.

### Problem Set 10.1, page 498

- 1** (a)(b)(c) have sums 4,  $-2 + 2i$ ,  $2 \cos \theta$  and products 5,  $-2i$ , 1. Note  $(e^{i\theta})(e^{-i\theta}) = 1$ .
- 2** In polar form these are  $\sqrt{5}e^{i\theta}$ ,  $5e^{2i\theta}$ ,  $\frac{1}{\sqrt{5}}e^{-i\theta}$ ,  $\sqrt{5}$ .
- 3** The absolute values are  $r = 10, 100, \frac{1}{10}$ , and 100. The angles are  $\theta, 2\theta, -\theta$  and  $-2\theta$ .
- 4**  $|z \times w| = 6$ ,  $|z + w| \leq 5$ ,  $|z/w| = \frac{2}{3}$ ,  $|z - w| \leq 5$ .
- 5**  $a + ib = \frac{\sqrt{3}}{2} + \frac{1}{2}i$ ,  $\frac{1}{2} + \frac{\sqrt{3}}{2}i$ ,  $i$ ,  $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$ ;  $w^{12} = 1$ .
- 6**  $1/z$  has absolute value  $1/r$  and angle  $-\theta$ ;  $(1/r)e^{-i\theta}$  times  $re^{i\theta}$  equals 1.
- 7**  $\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} \begin{bmatrix} ac - bd \\ bc + ad \end{bmatrix}$  **real part**  $\begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$  is the matrix form of  $(1 + 3i)(1 - 3i) = 10$ .
- 8**  $\begin{bmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$  gives complex matrix = vector multiplication  $(A_1 + iA_2)(x_1 + ix_2) = b_1 + ib_2$ .
- 9**  $2 + i$ ;  $(2 + i)(1 + i) = 1 + 3i$ ;  $e^{-i\pi/2} = -i$ ;  $e^{-i\pi} = -1$ ;  $\frac{1-i}{1+i} = -i$ ;  $(-i)^{103} = i$ .
- 10**  $z + \bar{z}$  is real;  $z - \bar{z}$  is pure imaginary;  $z\bar{z}$  is positive;  $z/\bar{z}$  has absolute value 1.
- 11**  $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$  includes  $aI$  (which just adds  $a$  to the eigenvalues and  $b \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ ). So the eigenvectors are  $\mathbf{x}_1 = (1, i)$  and  $\mathbf{x}_2 = (1, -i)$ . The eigenvalues are  $\lambda_1 = a + bi$  and  $\lambda_2 = a - bi$ . We see  $\bar{\mathbf{x}}_1 = \mathbf{x}_2$  and  $\bar{\lambda}_1 = \lambda_2$  as expected for real matrices with complex eigenvalues.
- 12** (a) When  $a = b = d = 1$  the square root becomes  $\sqrt{4c}$ ;  $\lambda$  is complex if  $c < 0$   
 (b)  $\lambda = 0$  and  $\lambda = a + d$  when  $ad = bc$  (c) the  $\lambda$ 's can be real and different.
- 13** Complex  $\lambda$ 's when  $(a+d)^2 < 4(ad-bc)$ ; write  $(a+d)^2 - 4(ad-bc)$  as  $(a-d)^2 + 4bc$  which is positive when  $bc > 0$ .
- 14**  $\det(P - \lambda I) = \lambda^4 - 1 = 0$  has  $\lambda = 1, -1, i, -i$  with eigenvectors  $(1, 1, 1, 1)$  and  $(1, -1, 1, -1)$  and  $(1, i, -1, -i)$  and  $(1, -i, -1, i)$  = columns of Fourier matrix.
- 15** The 6 by 6 cyclic shift  $P$  has  $\det(P_6 - \lambda I) = \lambda^6 - 1 = 0$ . Then  $\lambda = 1, w, w^2, w^3, w^4, w^5$  with  $w = e^{2\pi i/6}$ . These are the six solutions to  $\lambda^6 = 1$  as in Figure 10.3 (The sixth roots of 1).

- 16 The symmetric block matrix has real eigenvalues; so  $i\lambda$  is real and  $\lambda$  is pure imaginary.
- 17 (a)  $2e^{i\pi/3}, 4e^{2i\pi/3}$  (b)  $e^{2i\theta}, e^{4i\theta}$  (c)  $7e^{3\pi i/2}, 49e^{3\pi i} (= -49)$  (d)  $\sqrt{50}e^{-\pi i/4}, 50e^{-\pi i/2}$ .
- 18  $r = 1$ , angle  $\frac{\pi}{2} - \theta$ ; multiply by  $e^{i\theta}$  to get  $e^{i\pi/2} = i$ .
- 19  $a + ib = 1, i, -1, -i, \pm \frac{1}{\sqrt{2}} \pm \frac{i}{\sqrt{2}}$ . The root  $\bar{w} = w^{-1} = e^{-2\pi i/8}$  is  $1/\sqrt{2} - i/\sqrt{2}$ .
- 20  $1, e^{2\pi i/3}, e^{4\pi i/3}$  are cube roots of 1. The cube roots of  $-1$  are  $-1, e^{\pi i/3}, e^{-\pi i/3}$ . Altogether six roots of  $z^6 = 1$ .
- 21  $\cos 3\theta = \operatorname{Re}[(\cos \theta + i \sin \theta)^3] = \cos^3 \theta - 3 \cos \theta \sin^2 \theta$ ;  $\sin 3\theta = 3 \cos^2 \theta \sin \theta - \sin^3 \theta$ .
- 22 If the conjugate  $\bar{z} = 1/z$  then  $|z|^2 = 1$  and  $z$  is any point  $e^{i\theta}$  on the unit circle.
- 23  $e^i$  is at angle  $\theta = 1$  on the unit circle;  $|e^i| = 1^e$ ; Infinitely many  $i^e = e^{i(\pi/2 + 2\pi n)e}$ .
- 24 (a) Unit circle (b) Spiral in to  $e^{-2\pi}$  (c) Circle continuing around to angle  $\theta = 2\pi^2$ .

### Problem Set 10.2, page 506

- 1  $\|u\| = \sqrt{9} = 3, \|v\| = \sqrt{3}, u^H v = 3i + 2, v^H u = -3i + 2$  (this is the conjugate of  $u^H v$ ).
- 2  $A^H A = \begin{bmatrix} 2 & 0 & 1+i \\ 0 & 2 & 1+i \\ 1-i & 1-i & 2 \end{bmatrix}$  and  $AA^H = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$  are Hermitian matrices. They share the eigenvalues 4 and 2.
- 3  $z =$  multiple of  $(1+i, 1+i, -2)$ ;  $Az = \mathbf{0}$  gives  $z^H A^H = \mathbf{0}^H$  so  $z$  (not  $\bar{z}$ !) is orthogonal to all columns of  $A^H$  (using complex inner product  $z^H$  times columns of  $A^H$ ).
- 4 The four fundamental subspaces are now  $C(A), N(A), C(A^H), N(A^H)$ .  $A^H$  **and not**  $A^T$ .
- 5 (a)  $(A^H A)^H = A^H A^{HH} = A^H A$  again (b) If  $A^H A z = \mathbf{0}$  then  $(z^H A^H)(Az) = 0$ . This is  $\|Az\|^2 = 0$  so  $Az = \mathbf{0}$ . The nullspaces of  $A$  and  $A^H A$  are always the **same**.
- 6 (a) False (c) False  $A = U = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$  (b) True:  $-i$  is not an eigenvalue when  $A = A^H$ .
- 7  $cA$  is still Hermitian for real  $c$ ;  $(iA)^H = -iA^H = -iA$  is skew-Hermitian.
- 8 This  $P$  is invertible and unitary.  $P^2 = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, P^3 = \begin{bmatrix} -i & & \\ & -i & \\ & & -i \end{bmatrix} = -iI$ . Then  $P^{100} = (-i)^{33} P = -iP$ . The eigenvalues of  $P$  are the roots of  $\lambda^3 = -i$ , which are  $i$  and  $ie^{2\pi i/3}$  and  $ie^{4\pi i/3}$ .
- 9 One unit eigenvector is certainly  $x_1 = (1, 1, 1)$  with  $\lambda_1 = i$ . The other eigenvectors are  $x_2 = (1, w, w^2)$  and  $x_3 = (1, w^2, w^4)$  with  $w = e^{2\pi i/3}$ . The eigenvector matrix is the Fourier matrix  $F_3$ . The eigenvectors of any unitary matrix like  $P$  are orthogonal (using the correct complex form  $x^H y$  of the inner product).
- 10  $(1, 1, 1), (1, e^{2\pi i/3}, e^{4\pi i/3}), (1, e^{4\pi i/3}, e^{2\pi i/3})$  are orthogonal (complex inner product!) because  $P$  is an orthogonal matrix—and therefore its eigenvector matrix is unitary.



- 11 Not included in 4<sup>th</sup> edition  $C = \begin{bmatrix} 2 & 5 & 4 \\ 4 & 2 & 5 \\ 5 & 4 & 2 \end{bmatrix} = 2 + 5P + 4P^2$  has  $\lambda = 2 + 5 + 4 = 11$ ,  
 $2 + 5e^{2\pi i/3} + 4e^{4\pi i/3}$ ,  
 $2 + 5e^{4\pi i/3} + 4e^{8\pi i/3}$ .
- 11 If  $U^H U = I$  then  $U^{-1}(U^H)^{-1} = U^{-1}(U^{-1})^H = I$  so  $U^{-1}$  is also unitary. Also  $(UV)^H(UV) = V^H U^H U V = V^H V = I$  so  $UV$  is unitary.
- 12 Determinant = product of the eigenvalues (*all real*). And  $A = A^H$  gives  $\det A = \overline{\det A}$ .
- 13  $(z^H A^H)(Az) = \|Az\|^2$  is positive unless  $Az = \mathbf{0}$ . When  $A$  has independent columns this means  $z = \mathbf{0}$ ; so  $A^H A$  is positive definite.
- 14  $A = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & -1+i \\ 1+i & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1-i \\ -1-i & 1 \end{bmatrix}$ .
- 15  $K = (iA^T$  in Problem 14)  $= \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & -1-i \\ 1-i & 1 \end{bmatrix} \begin{bmatrix} 2i & 0 \\ 0 & -i \end{bmatrix} \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ -1+i & 1 \end{bmatrix}$ ;  
 $\lambda$ 's are imaginary.
- 16  $Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} \begin{bmatrix} \cos \theta + i \sin \theta & 0 \\ 0 & \cos \theta - i \sin \theta \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$  has  $|\lambda| = 1$ .
- 17  $V = \frac{1}{L} \begin{bmatrix} 1+\sqrt{3} & -1+i \\ 1+i & 1+\sqrt{3} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \frac{1}{L} \begin{bmatrix} 1+\sqrt{3} & 1-i \\ -1-i & 1+\sqrt{3} \end{bmatrix}$  with  $L^2 = 6+2\sqrt{3}$ .  
Unitary means  $|\lambda| = 1$ .  $V = V^H$  gives real  $\lambda$ . Then trace zero gives  $\lambda = 1$  and  $-1$ .
- 18 The  $\mathbf{v}$ 's are columns of a unitary matrix  $U$ , so  $U^H$  is  $U^{-1}$ . Then  $\mathbf{z} = U U^H \mathbf{z} =$  (multiply by columns)  $= \mathbf{v}_1(\mathbf{v}_1^H \mathbf{z}) + \cdots + \mathbf{v}_n(\mathbf{v}_n^H \mathbf{z})$ : a typical orthonormal expansion.
- 19 Don't multiply  $(e^{-ix})(e^{ix})$ . Conjugate the first, then  $\int_0^{2\pi} e^{2ix} dx = [e^{2ix}/2i]_0^{2\pi} = 0$ .
- 20  $\mathbf{z} = (1, i, -2)$  completes an orthogonal basis for  $\mathbb{C}^3$ . So does any  $e^{i\theta} \mathbf{z}$ .
- 21  $R + iS = (R + iS)^H = R^T - iS^T$ ;  $R$  is symmetric but  $S$  is skew-symmetric.
- 22  $\mathbb{C}^n$  has dimension  $n$ ; the columns of any unitary matrix are a basis. For example use the columns of  $iI$ :  $(i, 0, \dots, 0), \dots, (0, \dots, 0, i)$
- 23  $[1]$  and  $[-1]$ ; any  $[e^{i\theta}]$ ;  $\begin{bmatrix} a & b+ic \\ b-ic & d \end{bmatrix}$ ;  $\begin{bmatrix} w & e^{i\phi}\bar{z} \\ -z & e^{i\phi}\bar{w} \end{bmatrix}$  with  $|w|^2 + |z|^2 = 1$  and any angle  $\phi$
- 24 The eigenvalues of  $A^H$  are *complex conjugates* of the eigenvalues of  $A$ :  $\det(A - \lambda I) = 0$  gives  $\det(A^H - \bar{\lambda} I) = 0$ .
- 25  $(I - 2\mathbf{u}\mathbf{u}^H)^H = I - 2\mathbf{u}\mathbf{u}^H$  and also  $(I - 2\mathbf{u}\mathbf{u}^H)^2 = I - 4\mathbf{u}\mathbf{u}^H + 4\mathbf{u}(\mathbf{u}^H \mathbf{u})\mathbf{u}^H = I$ . The rank-1 matrix  $\mathbf{u}\mathbf{u}^H$  projects onto the line through  $\mathbf{u}$ .
- 26 Unitary  $U^H U = I$  means  $(A^T - iB^T)(A + iB) = (A^T A + B^T B) + i(A^T B - B^T A) = I$ .  
 $A^T A + B^T B = I$  and  $A^T B - B^T A = 0$  which makes the block matrix orthogonal.
- 27 We are given  $A + iB = (A + iB)^H = A^T - iB^T$ . Then  $A = A^T$  and  $B = -B^T$ . So that  $\begin{bmatrix} A & -B \\ B & A \end{bmatrix}$  is symmetric.
- 28  $AA^{-1} = I$  gives  $(A^{-1})^H A^H = I$ . Therefore  $(A^{-1})^H$  is  $(A^H)^{-1} = A^{-1}$  and  $A^{-1}$  is Hermitian.
- 29  $A = \begin{bmatrix} 1-i & 1-i \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \frac{1}{6} \begin{bmatrix} 2+2i & -2 \\ 1+i & 2 \end{bmatrix} = S \Lambda S^{-1}$ . Note real  $\lambda = 1$  and  $4$ .

- 30** If  $U$  has (complex) orthonormal columns, then  $U^H U = I$  and  $U$  is *unitary*. If those columns are eigenvectors of  $A$ , then  $A = U \Lambda U^{-1} = U \Lambda U^H$  is *normal*. The direct test for a normal matrix (which is  $AA^H = A^H A$  because diagonals could be real!) and  $\Lambda^H$  surely commute:

$$AA^H = (U \Lambda U^H)(U \Lambda^H U^H) = U(\Lambda \Lambda^H)U^H = U(\Lambda^H \Lambda)U^H = (U \Lambda^H U^H)(U \Lambda U^H) = A^H A.$$

An easy way to construct a normal matrix is  $1 + i$  times a symmetric matrix. Or take  $A = S + iT$  where the real symmetric  $S$  and  $T$  commute (Then  $A^H = S - iT$  and  $AA^H = A^H A$ ).

### Problem Set 10.3, page 514

- 1** Equation (3) (the FFT) is correct using  $i^2 = -1$  in the last two rows and three columns.

$$\mathbf{2} \quad F^{-1} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 & & \\ & 1 & i^2 & \\ & & 1 & 1 \\ & & & 1 & i^2 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & & 1 & \\ & 1 & & 1 \\ & & -1 & \\ & & & i \end{bmatrix} = \frac{1}{4} F^H.$$

$$\mathbf{3} \quad F = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & & \\ & 1 & i^2 & \\ & & 1 & 1 \\ & & & 1 & i^2 \end{bmatrix} \begin{bmatrix} 1 & & 1 & \\ & 1 & & 1 \\ & & -1 & \\ & & & i \end{bmatrix} \text{ permutation last.}$$

$$\mathbf{4} \quad D = \begin{bmatrix} 1 & & \\ & e^{2\pi i/6} & \\ & & e^{4\pi i/6} \end{bmatrix} \text{ (note 6 not 3) and } F_3 \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i/3} & e^{4\pi i/3} \\ 1 & e^{4\pi i/3} & e^{2\pi i/3} \end{bmatrix}.$$

- 5**  $F^{-1} \mathbf{w} = \mathbf{v}$  and  $F^{-1} \mathbf{v} = \mathbf{w}/4$ . Delta vector  $\leftrightarrow$  all-ones vector.

$$\mathbf{6} \quad (F_4)^2 = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \\ 0 & 4 & 0 & 0 \end{bmatrix} \text{ and } (F_4)^4 = 16I. \text{ Four transforms recover the signal!}$$

$$\mathbf{7} \quad \mathbf{c} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix} = F \mathbf{c}. \text{ Also } \mathbf{C} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix} = F \mathbf{C}.$$

Adding  $\mathbf{c} + \mathbf{C}$  gives  $(1, 1, 1, 1)$  to  $(4, 0, 0, 0) = 4$  (delta vector).

- 8**  $\mathbf{c} \rightarrow (1, 1, 1, 1, 0, 0, 0, 0) \rightarrow (4, 0, 0, 0, 0, 0, 0, 0) \rightarrow (4, 0, 0, 0, 4, 0, 0, 0) = F_8 \mathbf{c}$ .  
 $\mathbf{C} \rightarrow (0, 0, 0, 0, 1, 1, 1, 1) \rightarrow (0, 0, 0, 0, 4, 0, 0, 0) \rightarrow (4, 0, 0, 0, -4, 0, 0, 0) = F_8 \mathbf{C}$ .

- 9** If  $w^{64} = 1$  then  $w^2$  is a 32nd root of 1 and  $\sqrt{w}$  is a 128th root of 1: Key to FFT.

- 10** For every integer  $n$ , the  $n$ th roots of 1 add to zero. For even  $n$ , they cancel in pairs. For any  $n$ , use the geometric series formula  $1 + w + \dots + w^{n-1} = (w^n - 1)/(w - 1) = 0$ . In particular for  $n = 3$ ,  $1 + (-1 + i\sqrt{3})/2 + (-1 - i\sqrt{3})/2 = 0$ .

- 11** The eigenvalues of  $P$  are  $1, i, i^2 = -1$ , and  $i^3 = -i$ . Problem 11 displays the eigenvectors. And also  $\det(P - \lambda I) = \lambda^4 - 1$ .

- 12**  $\Lambda = \text{diag}(1, i, i^2, i^3)$ ;  $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$  and  $P^T$  lead to  $\lambda^3 - 1 = 0$ .
- 13**  $e_1 = c_0 + c_1 + c_2 + c_3$  and  $e_2 = c_0 + c_1i + c_2i^2 + c_3i^3$ ;  $E$  contains the four eigenvalues of  $C = FEF^{-1}$  because  $F$  contains the eigenvectors.
- 14** Eigenvalues  $e_1 = 2 - 1 - 1 = 0$ ,  $e_2 = 2 - i - i^3 = 2$ ,  $e_3 = 2 - (-1) - (-1) = 4$ ,  $e_4 = 2 - i^3 - i^9 = 2$ . Just transform column 0 of  $C$ . Check trace  $0 + 2 + 4 + 2 = 8$ .
- 15** Diagonal  $E$  needs  $n$  multiplications, Fourier matrix  $F$  and  $F^{-1}$  need  $\frac{1}{2}n \log_2 n$  multiplications each by the **FFT**. The total is much less than the ordinary  $n^2$  for  $C$  times  $\mathbf{x}$ .
- 16** The row 1,  $\bar{w}^k, \bar{w}^{2k}, \dots$  in  $\bar{F}$  is the same as the row 1,  $w^{N-k}, w^{N-2k}, \dots$  in  $F$  because  $w^{N-k} = e^{(2\pi i/N)(N-k)}$  is  $e^{2\pi i} e^{-(2\pi i/N)k} = 1$  times  $\bar{w}^k$ . So  $F$  and  $\bar{F}$  have the **same rows in reversed order** (except for row 0 which is all ones).