Vector and Matrix Theories

Chapter 4 : Factorization into $A = LU$

Husni Rois Ali and Dzuhri Radityo Utomo and Syukron Abu Ishaq Alfarozi

Department of Electrical and Information Engineering
Universitas Gadjah Mada

November 10, 2021

I. Properties of Inverse Matrix

II. Factorization $A = LU$

III. Factorization $PA = LU$

Section 1 :
Properties of Inverse Matrix

Inverting a Product of Matrices

- The matrix A is *invertible* if there exists a matrix A^{-1} that "inverts" A so that :

$$A^{-1}A = AA^{-1} = I$$

where I is the identity matrix :

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

- Suppose we have 2 square matrices A and B which are invertible and both have the same size.
- **Question :** What is the relation between $(AB)^{-1}$ with both $(A)^{-1}$ and $(B)^{-1}$?

Inverting a Product of Matrices

- From the definition of inverse matrices, we know that :

$$(AB)^{-1}AB = I$$

- Multiplying both sides with B^{-1} we get :

$$(AB)^{-1}A(BB^{-1}) = IB^{-1} \quad \rightarrow \quad (AB)^{-1}A = B^{-1}$$

- Finally, multiplying both sides with A^{-1} we get :

$$(AB)^{-1}(AA^{-1}) = B^{-1}A^{-1} \quad \rightarrow \quad (AB)^{-1} = B^{-1}A^{-1}$$

- **Question :** What about the inverting the product of 3 matrices A, B and C?
- For this case, it can be proved that the result is given by :

$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$$

- Note that the inverse matrices multiplication is performed in **reverse order**.

Inverse of Elimination Matrices

- Now, let us take a look at a set of equations with 3 unknowns as follow :

$$\begin{array}{ccccccc} 1x_1 & + & 2x_2 & + & 1x_3 & = & 2 \\ 2x_1 & + & 5x_2 & + & 1x_3 & = & 1 \\ 3x_1 & + & 10x_2 & + & 1x_3 & = & -2 \end{array}$$

- The augmented matrix for the aforementioned set of linear equations is given by :

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 2 & 5 & 1 & 1 \\ 3 & 10 & 1 & -2 \end{array}\right]$$

- **First Step :** Make the the element (2,1) to zero. It is performed by (Row 2)=(Row 2)-2×(Row 1) :

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{21}} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 2 & 5 & 1 & 1 \\ 3 & 10 & 1 & -2 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 1 & -1 & -3 \\ 3 & 10 & 1 & -2 \end{array}\right]$$

Inverse of Elimination Matrices

- **Question :** What should be the matrix multiplier to transform the resulting augmented matrix back to the original one?

$$\underbrace{\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}}_{E_{21}^{-1}} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 1 & -1 & -3 \\ 3 & 10 & 1 & -2 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 2 & 5 & 1 & 1 \\ 3 & 10 & 1 & -2 \end{array}\right]$$

- The original transformation is that :

$$(\text{Row } 2)_{\text{new}} = (\text{Row } 2)_{\text{old}} \text{ - } 2 \times (\text{Row } 1)_{\text{old}}$$

- As $(\text{Row } 1)_{\text{new}} = (\text{Row } 1)_{\text{old}}$, therefore we can conclude that :

$$(\text{Row } 2)_{\text{old}} = (\text{Row } 2)_{\text{new}} + 2 \times (\text{Row } 1)_{\text{new}}$$

- The resulting matrix multiplier is given by :

$$E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Inverse of Elimination Matrices

- **Second Step :** Make the <u>1st column of row 3 is equal</u> to zero. The operation is :

$$(\text{Row } 3)_{\text{new}} = (\text{Row } 3)_{\text{old}} - 3 \times (\text{Row } 1)_{\text{old}}$$

- Therefore :

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}}_{E_{31}} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 1 & -1 & -3 \\ 3 & 10 & 1 & -2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 4 & -2 & -8 \end{array}\right]$$

- Using similar logic, it can be proved that :

$$E_{31}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

Inverse of Elimination Matrices

- **Third Step :** Make the 2nd column of row 3 is equal to zero. The operation is :

$$(\text{Row } 3)_{\text{new}} = (\text{Row } 3)_{\text{old}} - 4 \times (\text{Row } 2)_{\text{old}}$$

- Therefore :

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix}}_{E_{32}} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 4 & -2 & -8 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 2 & 4 \end{array}\right]$$

- Using similar logic, it can be proved that :

$$E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

Inverse of Elimination Matrices

- For this particular case, the overall elimination matrix is given by :

$$E = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix}}_{E_{32}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}}_{E_{31}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{21}} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -4 & 1 \end{bmatrix}$$

- Therefore, it can be shown that :

$$E^{-1} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{21}^{-1}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}}_{E_{31}^{-1}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}}_{E_{32}^{-1}} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix}$$

Inverse of Elimination Matrices (Conclusion)

- Suppose that the overall elimination matrix is given by :

$$E = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\mathcal{L}_{32} & 1 \end{bmatrix}}_{E_{32}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\mathcal{L}_{31} & 0 & 1 \end{bmatrix}}_{E_{31}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -\mathcal{L}_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{21}}$$

- It can be shown that :

$$E^{-1} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ \mathcal{L}_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{21}^{-1}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \mathcal{L}_{31} & 0 & 1 \end{bmatrix}}_{E_{31}^{-1}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \mathcal{L}_{32} & 1 \end{bmatrix}}_{E_{32}^{-1}} = \begin{bmatrix} 1 & 0 & 0 \\ \mathcal{L}_{21} & 1 & 0 \\ \mathcal{L}_{31} & \mathcal{L}_{32} & 1 \end{bmatrix}$$

- The multiplier ($\mathcal{L}$) of each elimination matrices can be clearly identified from the E^{-1} matrix.

Section 2 :

Factorization $A = LU$

Factorization $A = LU$

- We have learnt that a square matrix A can be transformed into an upper triangular matrix U using an elimination matrix E, so that :

$$EA = U \quad \rightarrow \quad A = E^{-1}U$$

- Using the example shown in the previous section, we can see that :

$$\underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 1 \\ 3 & 10 & 1 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix}}_{E^{-1}} \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}}_{U}$$

- It can be shown that the E^{-1} is a Lower Triangular Matrix, which is usually denoted by matrix L. Therefore, the A matrix can be re-written as :

$$A = LU \quad \text{(LU Factorization)}$$

Solving System of Linear Equations

- A system of linear equations is usually written as :

$$Ax = b$$

- Making use of the LU Factorization, the equation can be re-written as :

$$LUx = b$$

- Suppose that we define :

$$Ux = c$$

  Therefore, the $LUx = B$ equation can be written as :

$$Lc = b$$

- This result indicates that we can use the LU factorization to solve a system of linear equations in 2 steps :
  1. Solve $Lc = b$ equation to compute c.
  2. Solve $Ux = c$ equation to compute x.

- As an example, let us take a look at the following set of linear equations :

$$\begin{array}{ccccccc} 1x_1 & + & 2x_2 & + & 1x_3 & = & 2 \\ 2x_1 & + & 5x_2 & + & 1x_3 & = & 1 \\ 3x_1 & + & 10x_2 & + & 1x_3 & = & -2 \end{array}$$

- For this case, we have determined that :

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}}_{U} \quad b = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

- The first step is by solving the $Lc = b$ equation :

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix}}_{L} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \rightarrow \begin{array}{ccccccc} c_1 & & & & & = & 2 \\ 2c_1 & + & c_2 & & & = & 1 \\ 3c_1 & + & 4c_2 & + & c_3 & = & -2 \end{array}$$

Solving System of Linear Equations

- Therefore, we get the solution for c as follow :

$$c_1 = 2 \quad c_2 = -3 \quad c_3 = 4$$

- Finally, we solve the $Ux = c$ equation :

$$\underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}}_{U} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} \rightarrow \begin{array}{ccccccc} x_1 & + & 2x_2 & + & x_3 & = & 2 \\ & & x_2 & - & x_3 & = & -3 \\ & & & & 2x_3 & = & 4 \end{array}$$

- Therefore, we get the solution for x as follow :

$$x_1 = 2 \quad x_2 = -1 \quad x_3 = 2$$

Solving System of Linear Equations

- Suppose that there is a change in the right hand side of the following set of linear equations :

$$
\begin{array}{rcrcrcr}
1x_1 & + & 2x_2 & + & 1x_3 & = & 6 \\
2x_1 & + & 5x_2 & + & 1x_3 & = & 13 \\
3x_1 & + & 10x_2 & + & 1x_3 & = & 24
\end{array}
$$

- **Question :** What do you need to do when solving this set of linear equation using Gauss Elimination method?
- In that case, you need to perform the Gauss Elimination process once again from the beginning.
- This can be avoided when the *LU* Factorization method is adopted. Once the *LU* factorization is obtained, we can easily compute the solution x for any b vector.

Solving System of Linear Equations

- Solving the $Lc = b$ equation :

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix}}_{L} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 13 \\ 24 \end{bmatrix} \rightarrow \begin{array}{ccccccc} c_1 & & & & & = & 6 \\ 2c_1 & + & c_2 & & & = & 13 \\ 3c_1 & + & 4c_2 & + & c_3 & = & 24 \end{array}$$

- Therefore, we get the solution for c as follow :

$$c_1 = 6 \quad c_2 = 1 \quad c_3 = 2$$

- Finally, we solve the $Ux = c$ equation :

$$\underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}}_{U} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 2 \end{bmatrix} \rightarrow \begin{array}{ccccccc} x_1 & + & 2x_2 & + & x_3 & = & 6 \\ & & x_2 & - & x_3 & = & 1 \\ & & & & 2x_3 & = & 2 \end{array}$$

- Therefore, we get the solution for x as follow :

$$x_1 = 1 \quad x_2 = 2 \quad x_3 = 1$$

Computing the Determinant of a Matrix

- Another advantage of performing LU factorization is we can easily compute the determinant of matrix A :

$$det(A) = det(LU) = det(L)det(U)$$

- The determinant of a lower triangular matrix L is given by :

$$L = \begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix} \quad \rightarrow \quad det(L) = l_{11} l_{22} \cdots l_{nn}$$

- Similarly, the determinant of a upper triangular matrix U is given by :

$$U = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{bmatrix} \quad \rightarrow \quad det(U) = u_{11} u_{22} \cdots u_{nn}$$

Computing the Determinant of a Matrix

- Using the example shown in the previous section, we can see that :

$$\underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 1 \\ 3 & 10 & 1 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}}_{U}$$

- The determinant of A is given by :

$$det(A) = det(L)det(U) = (1)(2) \quad \rightarrow \quad det(A) = 2$$

- Using the determinant formula for 3×3 matrix :

$$\begin{aligned} det(A) =& a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + \\ & a_{13}(a_{21}a_{32} - a_{22}a_{31}) \end{aligned}$$

We get the same result as follow :

$$det(A) = 2$$

Section 3 :
Factorization $PA = LU$

Another Example of LU Factorization

- Let us take a look at the following set of linear equations :

$$\begin{array}{ccccccc} 1x_1 & + & 2x_2 & + & 1x_3 & = & 6 \\ 2x_1 & + & 4x_2 & + & 3x_3 & = & 13 \\ 0x_1 & + & 3x_2 & + & 1x_3 & = & 7 \end{array}$$

- For this case, the A matrix is given by :

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 0 & 3 & 1 \end{bmatrix}$$

- The first elimination process is :

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{21}} \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 0 & 3 & 1 \end{bmatrix}}_{A} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 3 & 1 \end{bmatrix}$$

Another Example of LU Factorization

- The second process is row exchange as follows:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{P_{32}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

- Therefore, the A matrix can be written as :

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{21}^{-1}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{P_{32}^{-1}} \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{U} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{E^{-1}} \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{U}$$

- We can see in this problem, E^{-1} is <u>not</u> a lower triangular matrix. Therefore, the $A = LU$ factorization is not possible.
- **Question :** How do we fix this problem?

Factorization into $PA = LU$

- We can multiply both size with a matrix P so that the E^{-1} matrix is transformed into a lower triangular matrix L as follow :

$$\underbrace{\begin{bmatrix}1 & 0 & 0\\0 & 0 & 1\\0 & 1 & 0\end{bmatrix}}_{P}\underbrace{\begin{bmatrix}1 & 2 & 1\\2 & 4 & 3\\0 & 3 & 1\end{bmatrix}}_{A}=\underbrace{\begin{bmatrix}1 & 0 & 0\\0 & 0 & 1\\0 & 1 & 0\end{bmatrix}}_{P}\underbrace{\begin{bmatrix}1 & 0 & 0\\2 & 0 & 1\\0 & 1 & 0\end{bmatrix}}_{E^{-1}}\underbrace{\begin{bmatrix}1 & 2 & 1\\0 & 3 & 1\\0 & 0 & 1\end{bmatrix}}_{U}$$

$$\underbrace{\begin{bmatrix}1 & 2 & 1\\0 & 3 & 1\\2 & 4 & 3\end{bmatrix}}_{PA}=\underbrace{\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\2 & 0 & 1\end{bmatrix}}_{L=PE^{-1}}\underbrace{\begin{bmatrix}1 & 2 & 1\\0 & 3 & 1\\0 & 0 & 1\end{bmatrix}}_{U}$$

- This is what is called as factorization into $PA = LU$.
- Matrix P here is called as a **Permutation Matrix**, whose function is to represent row exchanges operation.

Solving System of Linear Equations

- Because of the addition of P, the way to solve a system of linear equations is a bit different :
  1. Solve $\underline{Lc = Pb}$ equation to compute c.
  2. Solve $Ux = c$ equation to compute x.
- Solving the $\underline{Lc = Pb}$ equation :

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}}_{L} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 6 \\ 13 \\ 7 \end{bmatrix}}_{b} = \begin{bmatrix} 6 \\ 7 \\ 13 \end{bmatrix}$$

- The resulting system of linear equations is given by :

$$\begin{array}{lcccccl} c_1 & & & & & = & 6 \\ & & c_2 & & & = & 7 \\ 2c_1 & + & & & c_3 & = & 13 \end{array}$$

- Therefore, we get the solution for c as follow :

$$c_1 = 6 \quad c_2 = 7 \quad c_3 = 1$$

Solving System of Linear Equations

- Finally, we solve the $Ux = c$ equation :

$$\underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{U} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 1 \end{bmatrix} \rightarrow \begin{array}{ccccccc} x_1 & + & 2x_2 & + & x_3 & = & 6 \\ & & 3x_2 & + & x_3 & = & 7 \\ & & & & x_3 & = & 1 \end{array}$$

- Therefore, we get the solution for x as follow :

$$x_1 = 1 \quad x_2 = 2 \quad x_3 = 1$$

Reading Assignment

- Please read the textbook "Introduction to Linear Algebra" Section 2.7.
- Topics :
  - Transpose of a Matrix
  - Meaning of Inner Product
  - Symmetric Matrices
  - Permutation Matrices