Vector and Matrix Theory

Chapter 1 : Introduction to Vectors

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I. Scalars, Vectors and Matrices

II. Vector addition and linear combinations

III. Lengths and Dot Products

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Section 1 :
Scalars, Vectors and Matrices

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Scalars and Vectors

Name	Score		
	HW	Quiz	Exam
Leibniz	60	80	90
Lagrange	70	70	70
Newton	80	50	80

Scalar

S = 70

Name	Score		
	HW	Quiz	Exam
Leibniz	60	80	90
Lagrange	70	70	70
Newton	80	50	80

Row Vector

V = [60 80 90]

Name	Score		
	HW	Quiz	Exam
Leibniz	60	80	90
Lagrange	70	70	70
Newton	80	50	80

Column Vector

$$V = \begin{bmatrix} 80 \\ 70 \\ 50 \end{bmatrix}$$

- **Scalar** is a quantity (variable), described by a single number.
- **Vector** is a single row or column in a data.
- *Row Vector* is when the vector is in the form of a single row, whereas *Column Vector* is when the vector is in the form of a single column.
- The above vectors are normally denoted as $\mathbb{R}^3$ since they have three real elements.

Zero, ones, and unit vectors

- $\mathbb{R}^n$ with all entries 0 is denoted as 0_n or $\mathbf{0}$
- $\mathbb{R}^n$ with all entries 1 is denoted as 1_n or $\mathbf{1}$
- A unit vector is a vector whose length/magnitude is one, $\|\boldsymbol{v}\| = 1$.
- A standard unit vector has one entry 1 and all others 0. Normally denoted as $\boldsymbol{e}_i$ whose the i^{th} entry is 1
- A standard unit vectors in $\mathbb{R}^3$:

$$\boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \boldsymbol{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Convention: We will use the standard notation for a column vector as $\boldsymbol{v}$ and a row vector as $\boldsymbol{v}^T$

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Vectors: Word count and histogram

Word count vectors are used in computer based document analysis. Each entry of the word count vector is the number of times the associated dictionary word appears in the document.

$$\begin{array}{l} \text{word} \\ \text{in} \\ \text{number} \\ \text{horse} \\ \text{the} \\ \text{document} \end{array} \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \\ 4 \\ 2 \end{bmatrix} = \boldsymbol{v}$$

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Matrices

Name	Score		
	HW	Quiz	Exam
Leibniz	60	80	90
Lagrange	70	70	70
Newton	80	50	80

Matrix

$$M = \begin{bmatrix} 60 & 80 & 90 \\ 70 & 70 & 70 \\ 80 & 50 & 80 \end{bmatrix}$$

Name	Score		
	HW	Quiz	Exam
Leibniz	60	80	90
Lagrange	70	70	70
Newton	80	50	80

Matrix

$$M = \begin{bmatrix} 80 & 90 \\ 70 & 70 \\ 50 & 80 \end{bmatrix}$$

- A compilation of several row vectors/column vectors form a **Matrix**.
- **Matrix** is a rectangular array or table of numbers, symbols, or expressions, arranged in rows and columns.
- The above vectors are normally denoted as $\mathbb{R}^{3\times 3}$ and $\mathbb{R}^{3\times 2}$ since their sizes are 3×3 and 3×2 with real elements.

Three Dimensional Matrices

Linear Algebra

Name	Score		
	HW	Quiz	Exam
Leibniz	60	80	90
Lagrange	70	70	70
Newton	80	50	80

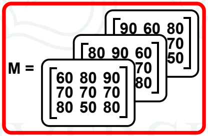
Basic Electronics

Name	Score		
	HW	Quiz	Exam
Leibniz	80	90	60
Lagrange	70	70	70
Newton	50	80	80

Numerical Methods

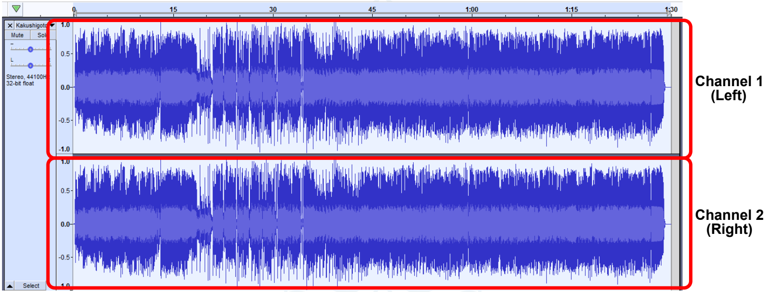
Name	Score		
	HW	Quiz	Exam
Leibniz	90	60	80
Lagrange	70	70	70
Newton	80	80	50

3D Matrix



- Several two dimensional (2D) matrices can be combined into a single three-dimensional (3D) Matrix (in other places, you might call it as a tensor).
- Often used in MATLAB, SCILAB, etc.

Audio Signal



- Data from a single Channel of an Audio signal can be represented as a *Vector*.

$$\mathbf{v}_l = \begin{bmatrix} l_1 & l_2 & l_3 & \cdots l_N \end{bmatrix}^T$$

$$\mathbf{v}_r = \begin{bmatrix} r_1 & r_2 & r_3 & \cdots r_N \end{bmatrix}^T$$

- Data of an Audio signal can be represented as a *Matrix*.

$$M = \begin{bmatrix} l_1 & l_2 & l_3 & \cdots l_N \\ r_1 & r_2 & r_3 & \cdots r_N \end{bmatrix} = \begin{bmatrix} \mathbf{v}_l^T \\ \mathbf{v}_r^T \end{bmatrix}$$

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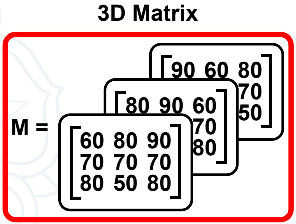
Matrix

$$M = \begin{bmatrix} 60 & 80 & 90 \\ 70 & 70 & 70 \\ 80 & 50 & 80 \end{bmatrix}$$

- Data of an Image Data in Grayscale can be represented as a *Matrix*.
- Each components represents the brightness of each pixels (0 - 1).
- Zero brightness indicate a pitch black pixel, whereas full brightness (1) indicates a white pixel.
- Generally, each pixels is represented by an 8-bit unsigned integer data (0 - 255).

Image Data (Colour)



- Data of an Image Data in full colour can be represented as a *3D Matrix*.
- The matrix represents 3 different basic colour components, Red, Green and Blue (RGB).
- Each matrix components represents the colour components of each pixels (0 - 255).

Section 2 : Vector addition and linear combinations

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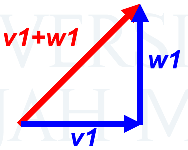
Vector addition

- Two vectors of the same size can be added together by adding the corresponding elements
- Vector addition is denoted by the symbol $+$
- Adding $\boldsymbol{v}_1, \boldsymbol{w}_1 \in \mathbb{R}^n$ is denoted as:

$$\boldsymbol{v}_1 + \boldsymbol{w}_1 = \begin{bmatrix} v_{11} \\ v_{21} \\ \vdots \\ v_{n1} \end{bmatrix} + \begin{bmatrix} w_{11} \\ w_{21} \\ \vdots \\ w_{n1} \end{bmatrix} = \begin{bmatrix} v_{11} + w_{11} \\ v_{21} + w_{21} \\ \vdots \\ v_{n1} + w_{n1} \end{bmatrix}$$



Vector addition: examples

- *Word counts*. If $\boldsymbol{v}$ and $\boldsymbol{w}$ are word count vectors (using the same dictionary) for two documents. what does $\boldsymbol{v} + \boldsymbol{w}$ represent?
- How about $\boldsymbol{v} + \boldsymbol{w}$?
- *Audio addition*. When $\boldsymbol{a}$ and $\boldsymbol{b}$ are vectors representing audio signals over the same period of time, the sum $\boldsymbol{a} + \boldsymbol{b}$ is an audio signal that is perceived as containing both audio signals combined into one. If $\boldsymbol{a}$ represents a recording of a voice, and $\boldsymbol{b}$ is a recording of music (of the same length), the audio signal $\boldsymbol{a} + \boldsymbol{b}$ will be perceived as containing both the voice recording and, simultaneously, the music.

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Scalar multiplication

- If α is a scalar and $\boldsymbol{u} \in \mathbb{R}^n$, then scalar multiplication $\alpha\boldsymbol{u}$ is defined as

$$\alpha\boldsymbol{u} = \begin{bmatrix} \alpha u_{11} \\ \alpha u_{21} \\ \vdots \\ \alpha u_{n1} \end{bmatrix}$$

- Properties:
  - associative $(\alpha_1\alpha_2)\boldsymbol{u} = \alpha_1(\alpha_2\boldsymbol{u})$
  - Left distributive $(\alpha_1 + \alpha_2)\boldsymbol{u} = \alpha_1\boldsymbol{u} + \alpha_2\boldsymbol{u}$
  - Right distributive $\alpha(\boldsymbol{u}_1 + \boldsymbol{u}_2) = \alpha\boldsymbol{u}_1 + \alpha\boldsymbol{u}_2$
- Example: *audio scaling*. If a is $\boldsymbol{u}$ vector representing an audio signal, the scalar-vector product $\alpha\boldsymbol{u}$ is perceived as the same audio signal, but changed in volume (loudness) by the factor $|\alpha|$

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Linear combinations

- Linear combination of vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, u_m$ with all $\boldsymbol{u}_i$ in $\mathbb{R}^n$ and scalars $\alpha_1,\ \alpha_2, \ldots, \alpha_m$ is denoted as

$$\alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \cdots + \alpha_m \boldsymbol{u}_m$$

- $\alpha_1,\ \alpha_2, ..., \alpha_m$ are the coefficients
- It is a very important concept
- A simple example

$$\boldsymbol{u}_1 = u_{11} \boldsymbol{e}_1 + u_{21} \boldsymbol{e}_2 + \cdots + u_{m1} \boldsymbol{e}_m$$

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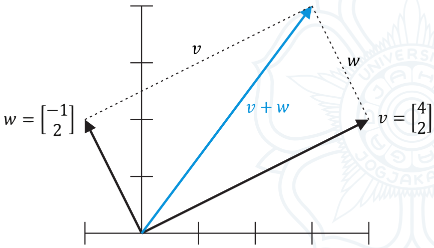
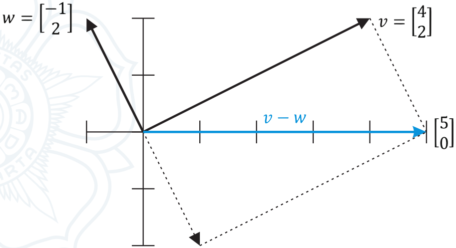
Linear combinations



$$v + w = \begin{bmatrix}4\\2\end{bmatrix} + \begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}3\\4\end{bmatrix}$$



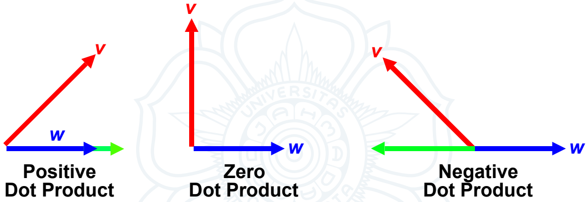
$$v - w = \begin{bmatrix}4\\2\end{bmatrix} - \begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}5\\0\end{bmatrix}$$

Important questions?

- What is the picture of all combinations $\alpha_1 \boldsymbol{u}_1$?
- What is the picture of all combinations $\alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2$?
- What is the picture of all combinations $\alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \alpha_3 \boldsymbol{u}_3$?

Section 2 :
Lengths and Dot Products

Dot Product or Inner Product



- The **Dot Product** or **Inner Product** of two vectors $\boldsymbol{v} = (v_1, v_2, \cdots, v_n)^T$ and $\boldsymbol{w} = (w_1, w_2, \cdots, w_n)^T$ is defined as :

$$\langle \boldsymbol{v}, \boldsymbol{w} \rangle = \boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{v}^T \boldsymbol{w} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n$$

- A dot product of two vectors indicates the alignment of the direction of those vectors.
- When two non-zero vectors v and w form an angle θ between them, the dot product of the two vectors can be defined as :

$$\boldsymbol{v} \cdot \boldsymbol{w} = \|\boldsymbol{v}\| \|\boldsymbol{w}\| \cos \theta$$

Dot Product or Inner Product

Properties of inner product

- $\boldsymbol{u}^T \boldsymbol{v} = \boldsymbol{v}^T \boldsymbol{u}$
- $(\alpha \boldsymbol{u})^T \boldsymbol{v} = \alpha(\boldsymbol{v}^T \boldsymbol{u})$
- $(\boldsymbol{u} + \boldsymbol{v})^T \boldsymbol{w} = \boldsymbol{u}^T \boldsymbol{w} + \boldsymbol{v}^T \boldsymbol{w}$

General examples

- $\boldsymbol{e}_i^T \boldsymbol{u}_1 = u_{i1}$ (picks out ith entry)
- $\mathbf{1}^T \boldsymbol{u}_1 = u_{11} + ... + u_{n1}$ (sum of entries)
- $\boldsymbol{u}_1^T \boldsymbol{u}_1 = u_{11}^2 + ... + u_{n1}^2$ (sum of squares of entries)

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Dot Product or Inner Product : Example

Comp.	Score (S)	Weight (W)	S *x* W
HW	90	0.2	18
Quiz	60	0.3	18
Exam	72	0.5	36
Total			$\boxed{72}$

→ Dot Product or Inner Product

Comp.	Score (S)	Weight (W)	S *x* W
HW	90	0.2	18
Quiz	60	0.3	18
Exam	-72	0.5	-36
Total			$\boxed{0}$

→ Zero Dot Product

- Examples of dot product operation are shown in the figure above.
- Dot product of two vectors is equal to **zero** when two vectors are **perpendicular** to each other.
- There are a lot of score combinations which result on zero dot product.

Dot Product or Inner Product : Example

Comp.	Score (S)	Weight (W)	S x W
HW	90	0.2	18
Quiz	60	0.3	18
Exam	-72	0.5	-36
Total			0

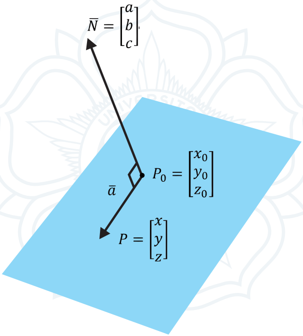
→ Zero Dot Product

Comp.	Score (S)	Weight (W)	S x W
HW	-90	0.2	-18
Quiz	-60	0.3	-18
Exam	72	0.5	36
Total			0

→ Zero Dot Product

- Examples of dot product operation are shown in the figure above.
- Dot product of two vectors is equal to **zero** when two vectors are **perpendicular** to each other.
- There are a lot of score combinations which result on zero dot product.

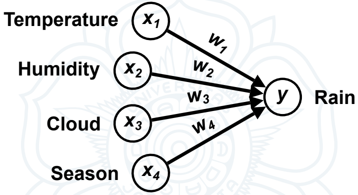
Dot Product or Inner Product : Equation of planes



- The equation of a plane with the normal vector $\boldsymbol{N} = [a\ b\ c]^T$ and passing the point $\boldsymbol{p_0}$ can be found using dot product
- The vector $\boldsymbol{a}$ from $\boldsymbol{P}_0$ to any point $\boldsymbol{P}$ in the plane is $\boldsymbol{a} = \boldsymbol{P} - \boldsymbol{P}_0$
- As $\boldsymbol{N}$ is normal to all vectors $\boldsymbol{a}$, the dot product is zero:

$$\boldsymbol{N}^T\boldsymbol{a} = \begin{bmatrix} a & b & c \end{bmatrix}^T \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} = a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Dot Product or Inner Product : Example

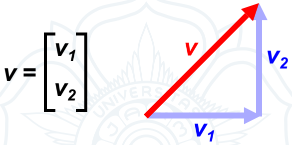


- Another example is in Machine Learning.
- The output value (y) depends on the value of several input parameters (x_1, x_2, x_3 and x_4), so that :

$$y = w_1x_1 + w_2x_2 + w_3x_3 + w_4x_4$$

- The weights w_1, w_2, w_3 and w_4 indicate the importance of each input parameters to the output value.

Length of Vector and Unit Vector



Length of a Vector

- The **Length** of a vector :

$$\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

is defined as :

$$\|\boldsymbol{v}\| = \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

- **Unit Vector** is a vector whose length is equal to <u>one</u>.

Some important formulas

- Cosine formula: $\cos\theta = \dfrac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}$
- Cauchy–Schwarz inequality: $|\mathbf{v}\cdot\mathbf{w}| \leq \|\mathbf{v}\|\|\mathbf{w}\|$
- Triangle inequality: $\|\mathbf{v}+\mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$

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Acknowledgement

The material in this lecture is adopted from:

1. *Introduction to Linear Algebra* by Gilbert Strang.
2. *Introduction to Applied Linear Algebra Vectors, Matrices, and Least Squares* by S. Boyd and L. Vandenberghe

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