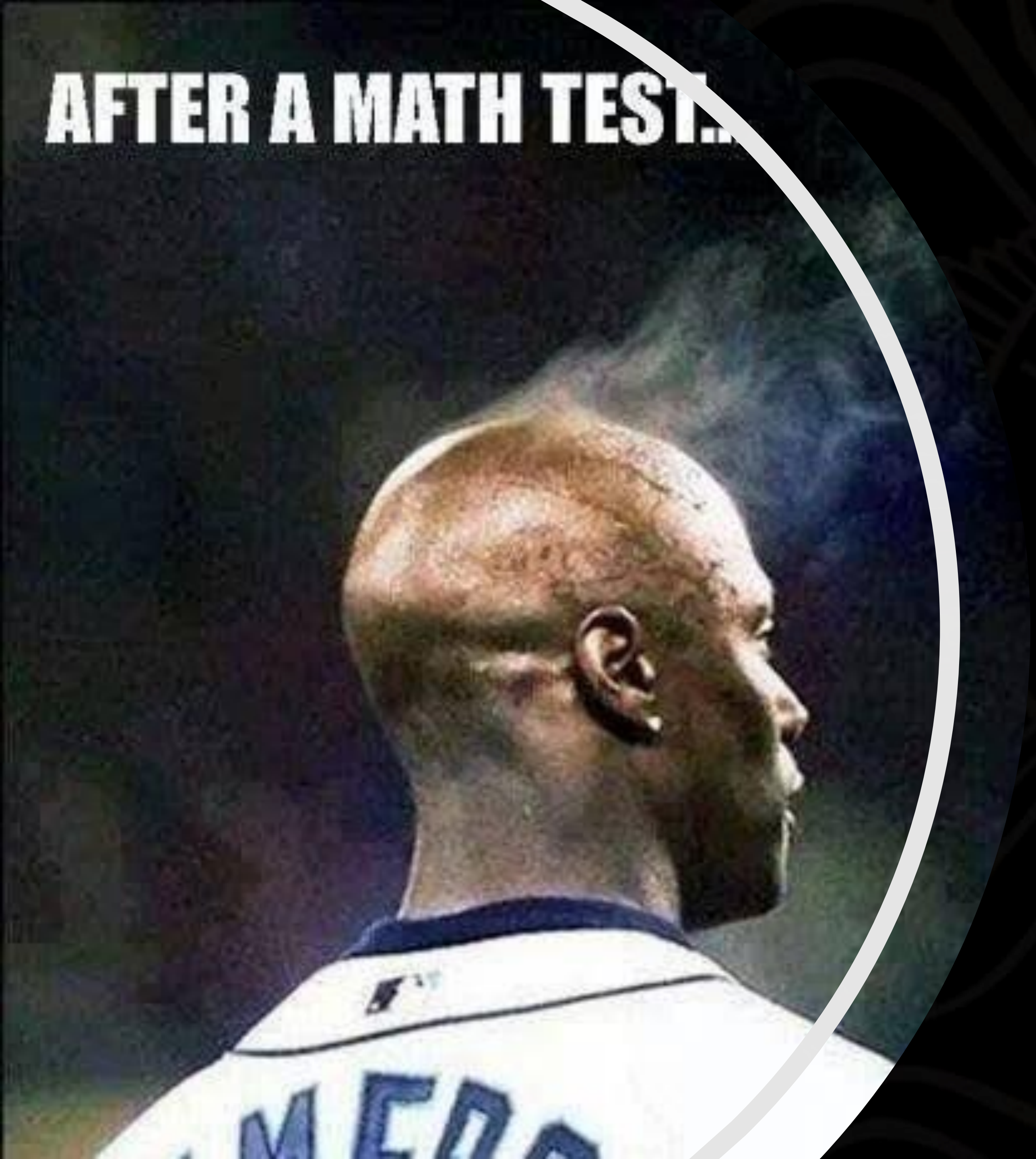

Discrete Mathematics

#5 Induction and Recursion

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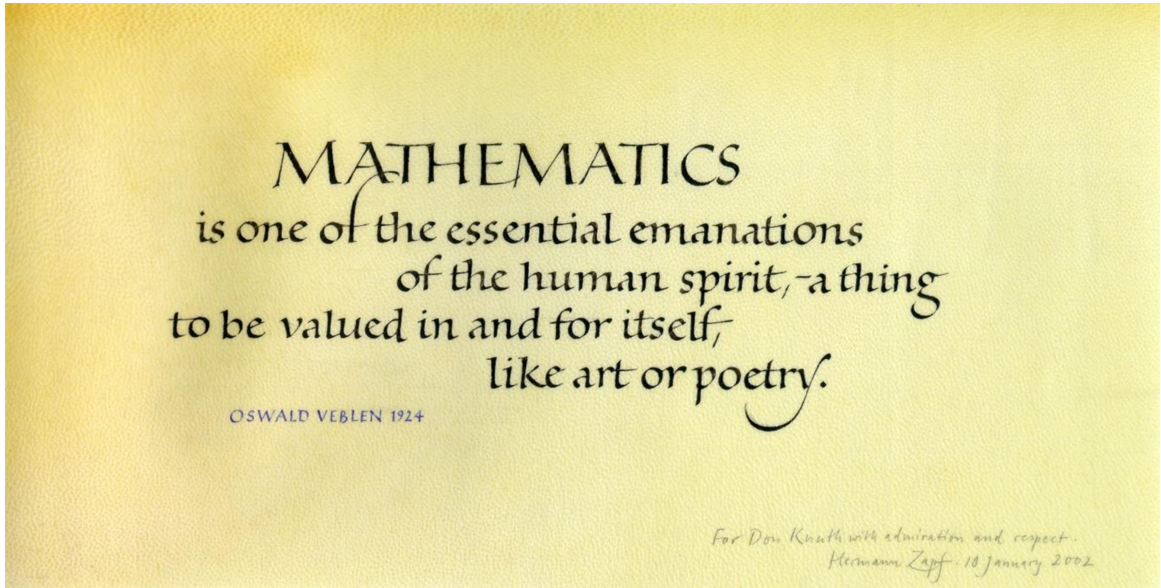
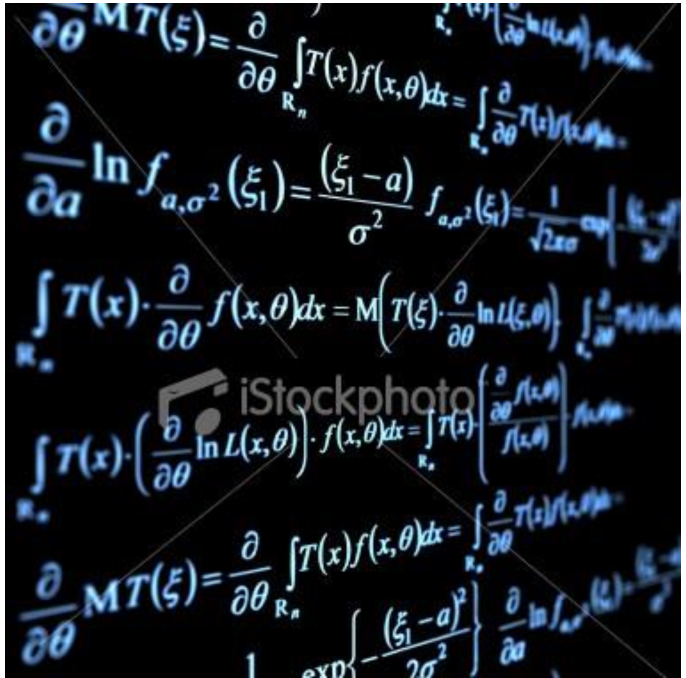
When I was young …

Why study mathematics?

- It is foundation of science
- It has been used in so many areas: economics, engineering, life science, social science, modern physics etc
- Example?





simple drill: read aloud!

- On Thomas Calculus we have this

THEOREM 4—The Sandwich Theorem Suppose that $g(x) \le f(x) \le h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

Then $\lim_{x \to c} f(x) = L$.

Another drillA

- On a technical book there is a passage like this

5.1. Theorem. *Suppose that $f \in \mathcal{F}$ is bounded below by some constant c, and f_α is defined as above with $\alpha > 0$. Then f_α is bounded below by c, and is Lipschitz on each bounded subset of X (and in particular is finite-valued). Furthermore, suppose $x \in X$ is such that $\partial_P f_\alpha(x)$ is nonempty. Then there exists a point $\bar{y} \in X$ satisfying the following:*

(a) *If $\{y_i\} \subset X$ is a minimizing sequence for the infimum in (1), then $\lim_{i\to\infty} y_i = \bar{y}$.*

(b) *The infimum in (1) is attained uniquely at $\bar{y}$.*

(c) *The Fréchet derivative $f'_\alpha(x)$ exists and equals $2\alpha(x - \bar{y})$. Thus the proximal subgradient $\partial_P f_\alpha(x)$ is the singleton $\{2\alpha(x - \bar{y})\}$.*

(d) $2\alpha(x - \bar{y}) \in \partial_P f(\bar{y})$.

Proof. Suppose we are given f and $\alpha > 0$ as above. It is clear from the definition that f_α is bounded below by c. We now show that f_α is Lipschitz on any bounded set $S \subset X$.

For any fixed $x_0 \in \operatorname{dom} f \neq \emptyset$, note that $f_\alpha(x) \leq f(x_0) + \alpha\|x - x_0\|^2$ for all $x \in X$, and thus in particular $m := \sup\{f_\alpha(x) : x \in S\} < \infty$. Since $\alpha > 0$, and f is bounded below, we have that for any $\varepsilon > 0$ the set

$$C := \{z : \exists y \in S \text{ so that } f(z) + \alpha\|y - z\|^2 \leq m + \varepsilon\}$$

is bounded in X.

One more

Theorem 4.1 *Let $x = 0$ be an equilibrium point for (4.1) and $D \subset R^n$ be a domain containing $x = 0$. Let $V : D \to R$ be a continuously differentiable function such that*

$$V(0) = 0 \text{ and } V(x) > 0 \text{ in } D - \{0\} \tag{4.2}$$

$$\dot{V}(x) \leq 0 \text{ in } D \tag{4.3}$$

Then, $x = 0$ is stable. Moreover, if

$$\dot{V}(x) < 0 \text{ in } D - \{0\} \tag{4.4}$$

then $x = 0$ is asymptotically stable. ◇

- A very famous theorem called Lyapunov Theorem

Proof: Given $\varepsilon > 0$, choose $r \in (0, \varepsilon]$ such that

$$B_r = \{x \in R^n \mid \|x\| \leq r\} \subset D$$

Let $\alpha = \min_{\|x\|=r} V(x)$. Then, $\alpha > 0$ by (4.2). Take $\beta \in (0, \alpha)$ and let

$$\Omega_\beta = \{x \in B_r \mid V(x) \leq \beta\}$$

Then, Ω_β is in the interior of B_r.[2] (See Figure 4.1.) The set Ω_β has the property that any trajectory starting in Ω_β at $t = 0$ stays in Ω_β for all $t \geq 0$. This follows from (4.3) since

$$\dot{V}(x(t)) \leq 0 \ \Rightarrow \ V(x(t)) \leq V(x(0)) \leq \beta, \ \forall\, t \geq 0$$

Because Ω_β is a compact set,[3] we conclude from Theorem 3.3 that (4.1) has a unique solution defined for all $t \geq 0$ whenever $x(0) \in \Omega_\beta$. As $V(x)$ is continuous and $V(0) = 0$, there is $\delta > 0$ such that

$$\|x\| \leq \delta \ \Rightarrow \ V(x) < \beta$$

Then,

$$B_\delta \subset \Omega_\beta \subset B_r$$

and

$$x(0) \in B_\delta \Rightarrow x(0) \in \Omega_\beta \Rightarrow x(t) \in \Omega_\beta \Rightarrow x(t) \in B_r$$

Therefore,

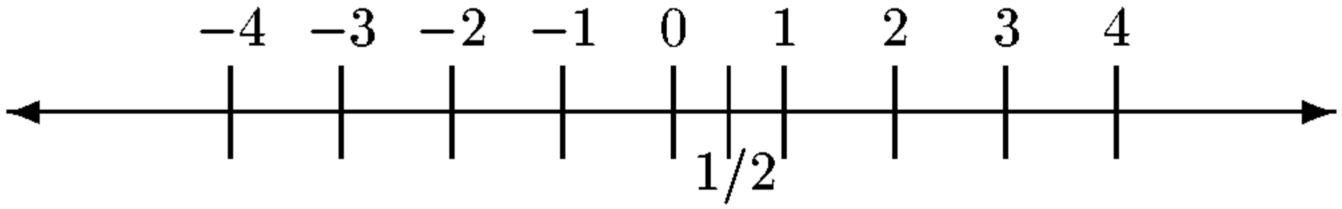
$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < r \leq \varepsilon, \quad \forall\, t \geq 0$$

What can you conclude?

- It is indeed important to open many doors!

Refresher (set and number)

- Number



- Set of number
    - **Z**=the set of integers={0,1,-1,2,-1,3,-3,...}
    - **N**=the set of nonnegative integers or natural numbers
    - $\mathbf{Z}^+$=the set of positive integers
    - **Q**=the set of rational numbers={*a*/*b*| *a*,*b* is integer, *b* not zero}
    - $\mathbf{Q}^+$=the set of positive rational numbers
    - **Q***=the set of nonzero rational numbers
    - **R**=the set of real numbers
    - $\mathbf{R}^+$=the set of positive real numbers
    - **R***=the set of nonzero real numbers
    - **C**=the set of complex numbers
- Infinite sets come in different sizes!

Induction and Recursion

Chapter 4

Take a look at this video, today we are going to talk about this!