# Hypothesis Tests II

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# Hypothesis testing

# Hypothesis testing

**Goal:** make decisions about a population parameter based on a sample of data.

**Statistical hypothesis** - a statement made about the value of a population parameter (e.g. $\mu > 80$)

**Hypothesis test** - statistical method for evaluating the degree to which data favors (or does not favor) the “alternative” hypothesis over the null hypothesis.

# Example

**Research question:** Can I read your minds?

# The data

$n$ = Number of people in room

$x$ = Number that I got correct

Is $x$ large enough for us to believe that I'm psychic?

# Guiding mindset

> We want to find a simplest explanation of the observed phenomenon

- Unless there is strong enough evidence to the contrary, we should assume that I am random guessing.
- **Thinking like a skeptic:** If I were random guessing, would getting $x$ correct out of $n$ be surprising?

# Statistics to the rescue

- $x$ is a realization of what sort of random variable?

$$X \sim \text{Binomial}(n, p)$$

**Null hypothesis** - expresses skeptical perspective, i.e., "nothing interesting here" (status quo) - in this example - "He's random guessing." - $H_0 : p = 1/4$

**Alternative hypothesis** - something new, not previously accepted - He's psychic! $H_A : p > 1/4$.

# Is this surprising "under the null?"

$$H_0 : p = 1/4$$

Under null, we think $x$ is a realization of random variable

$$X \sim \text{Binomial}(n, 1/4).$$

$x$ is higher than $n/4$. But is it unlikely under random guessing?

If $P(X \geq x)$ is very small under null hypothesis, perhaps we should favor alternative that $p > 1/4$.

# In R

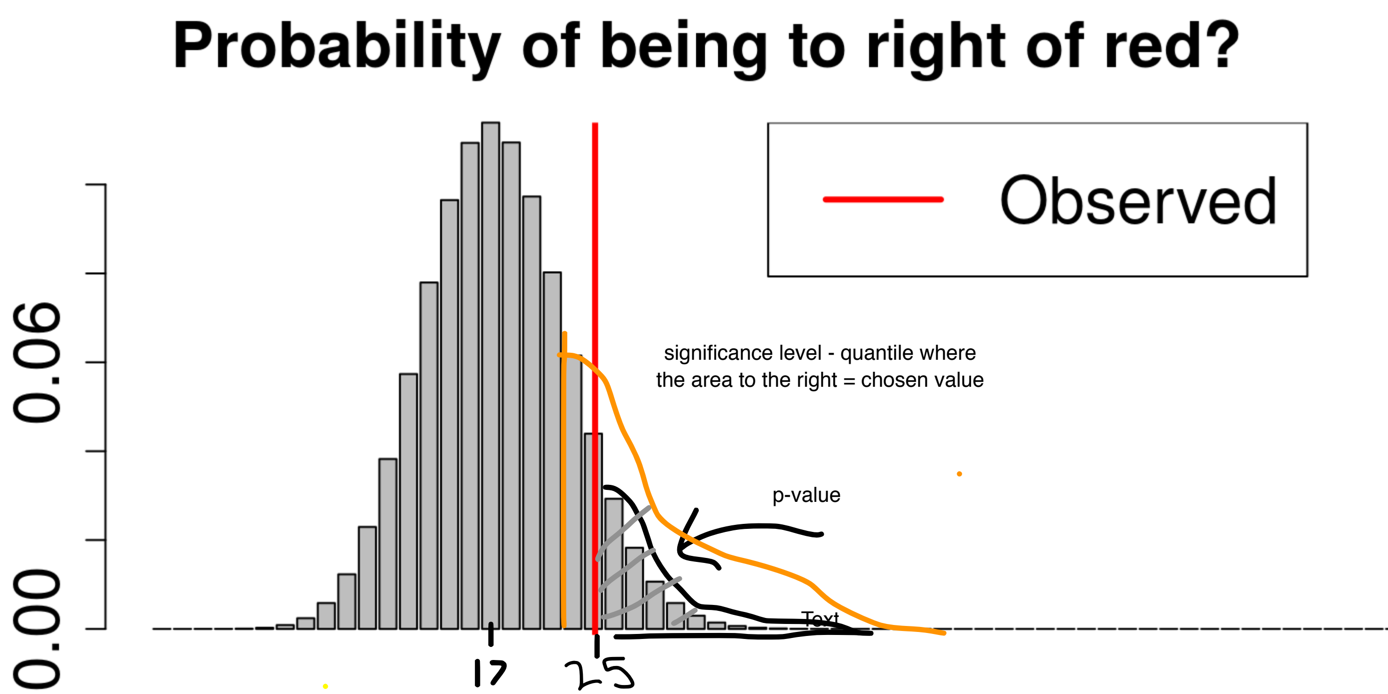
```r
n=65 # number of trials
x=25 # number of successes
p = 1/4 # calculate under null hypothesis
1 - pbinom(x-1, size = n, prob = p)
```

```
## [1] 0.011344
```

If Null were true, what are the odds that I'd see something as or more extreme than what was observed

- This is called a **p-value** - *the probability, calculated under the null, of seeing something as extreme or more extreme than what was observed.*
- Note: direction of "extreme" is defined by alternative hypothesis

# In a picture



Probability of being to right of red?
Observed
0.06
0.00
significance level - quantile where
the area to the right = chosen value
p-value
Text
17
25

# How small is small enough?

Upper Bound on Making Type I error

- the **significance level** $\alpha$ of a hypothesis test is our chosen threshold for the p-value, below which we reject the null.
- most common choice: $0.05$… i.e., 1/20.
- Are you surprised if something happens to you that should only happen 1 out of every 20 times?

> *"A claimed result that overturns all ideas of causality might well require something stricter than .05."* - Brad Efron (NY Times 2011)

# Connecting back to science

**Your goal:** Convince skeptical reader of your “research finding” - Is observed data necessarily inconsistent with a more simple explanation? - **Structure of argument:** - There are two possibilities:

```
1) simple ("null") explanation

2) new ("alternative") science here
```

- Our data would be very unusual if (1) were true.
- We and reader are forced to reject (1) in favor of (2)

# Two kinds of errors

| | $H_0$ true | $H_A$ true |
|---|---|---|
| Reject $H_0$ | Type I error | Good Sensitivity |
| Fail to reject $H_0$ | Good Specificity | Type II error |

**Type I error** - false positive ("gullible")

**Type II error** - false negative ("missed out on an opportunity")

**Goal:** design a procedure that can ensure that

$$P(\text{Type I error}) \leq \alpha$$

Chance of getting it wrong. e.g. if \alpha = 0.05 we have a 5% chance of being wrong. Upper bound on making a type I error

yet still has small $P(\text{Type II error})$.

# Significance level

By $P(\text{Type I error})$ we mean

$$P(\text{Reject } H_0 \mid H_0 \text{ is true})$$

Think of **significance level** as our *level of gullibility*

- thinking I'm psychic when I'm random guessing
- declaring drug works when it actually makes no difference
- jury deciding "guilty" when person is innocent

# Power

•1 - P(Type II Error)
•Enough data to capture the signal
•Related to Sensitivity

The **power** of a test is

$$P(\text{Reject } H_0 \mid H_A \text{ is true})$$

Sensitivity

*Power* is test's **ability to detect** that alternative applies.

- detecting that drug works when it in fact does
- jury deciding "guilty" when person was guilty of crime

Note:

$$P(\text{Type II error}) = P(\text{Fail to reject } H_0 \mid H_A \text{ is true}) = 1 - \text{Power}$$

# Back to example

Test

$$H_0 : p = 1/4 \text{ versus } H_A : p > 1/4$$

Suppose I want a test with significance level $\alpha = 0.05$. How high would observed $x$ need to be for me to reject $H_0$?

Consider decision rule in which I reject $H_0$ if observed $x$ is $\geq c$.

Want to find a cutoff $c$ such that

Type I error

$$P(\text{Reject } H_0 \mid H_0 \text{ true}) = P(X \geq c \mid H_0 \text{ true}) = 0.05$$

# Finding cutoff

$$
\begin{aligned}
P(X \geq c \mid H_0 \text{ true}) &= P(\text{Binomial}(n, 1/4) \geq c) \\
&= 1 - P(\text{Binomial}(n, 1/4) \leq c - 1)
\end{aligned}
$$

```r
alpha = 0.05; p = 1/4 # under null
quantile = qbinom(1-alpha, n, p) # this is c - 1
cutoff = quantile + 1
cutoff
```

```
## [1] 23
```

```r
1 - pbinom(cutoff - 1, n, p)
```

```
## [1] 0.04024569
```

# Rejection region

Our level $\alpha = 0.05$ test rejects if observed number of successes is greater than or equal to 23.

We have designed the **rejection region** of the test (the set of values for which we will reject $H_0$) so that

$$P(\text{Reject } H_0 \mid H_0 \text{ is true}) \leq 0.05$$

# Making the case

Suppose we found that I got $x = 21$ correct out of $n = 65$ guesses.

We fail to reject $H_0$ because 21 is less than 23.

Have we shown that I am not psychic?

> "Absence of evidence is not evidence of absence."

- Difference between "not guilty" and "innocent"
- Similarly we say "fail to reject $H_0$" rather than "accept $H_0$"
- If we fail to reject null, it could just mean we didn't get enough data ("under-powered study")

# Power

Suppose I were *slightly* psychic:

$$p = 1/4 + 0.01 = 0.26$$

**What's the probability under this alternative that our test would have rejected the null at the $\alpha = 0.05$ level?**

Recall: our test rejects $H_0$ if we observe $\geq 23$ successes out of $n = 65$.

$$P(\text{Reject } H_0 \mid p = 0.26) = P(X \geq 23 \mid p = 0.26)$$

where $X \sim \text{Binomial}(n, p)$.

Significance level calculates type 1 error with no power

# Power

$$\text{Power} = P\left(\text{Binomial}(65, 0.26) \geq 23\right) = ?$$

```
cutoff
```

```
## [1] 23
```

```
1 - pbinom(cutoff - 1, n, prob = 0.26)
```

```
## [1] 0.05988829
```

This is very low power, meaning that if $p = 0.26$, my experiment had very little shot at establishing this.

# Power

Power is a function of sample size

Suppose I am *very* psychic, so that $p = 1/2$.

$$\text{Power} = P\left(\text{Binomial}(65, 0.5) \geq 23\right) = ?$$

```
cutoff
```

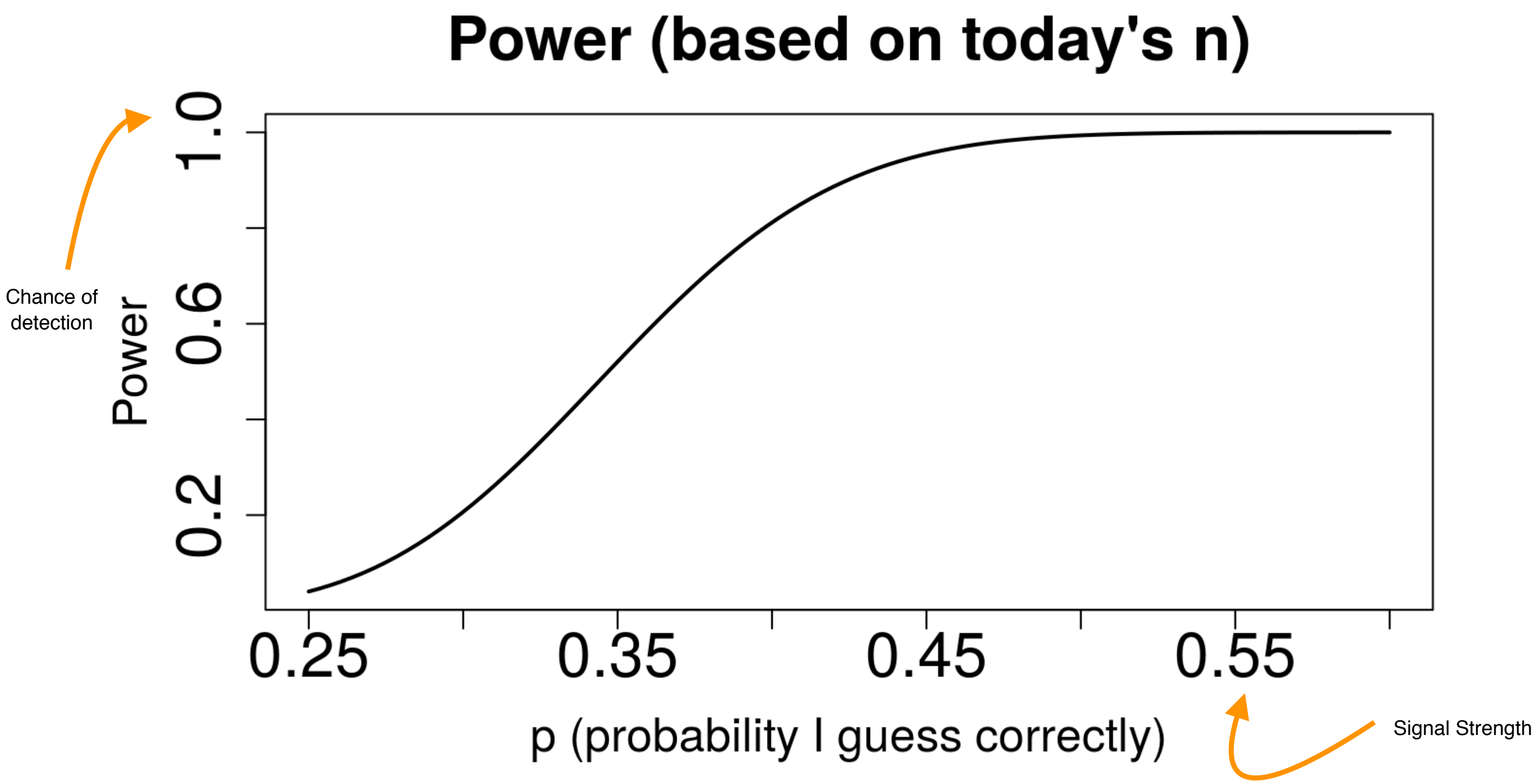
```
## [1] 23
```

```
1 - pbinom(cutoff - 1, n, prob = 0.5)
```

```
## [1] 0.9937487
```

This is very high power, meaning that if $p = 0.5$, my experiment had a very good chance of detecting my abilities.

# Power function



# A power calculation

**Idea:** Before doing an experiment, I should figure out what size sample is needed to have a target power.

Requires that I have a guess of the size of $p$.

# A power calculation

Suppose I think I'm slightly psychic: $p = 1/4 + 0.01 = 0.26$. **What $n$ do I need to have $85\%$ power?**

Is $n = 1000$ enough?

```
n = 1000 # initial guess
alpha = 0.05; pnull = 1/4 # under null
quantile = qbinom(1-alpha, n, pnull) # this is c - 1
cutoff = quantile + 1 # this is the cutoff to ensure a level alpha test
palt = 0.26 # suppose I think I'm slightly psychic: 1/4 + 0.01
power = 1 - pbinom(cutoff - 1, n, prob = palt)
power
```

```
## [1] 0.165125
```

# A power calculation

Is $n = 2000$ enough?

```r
n = 2000 # initial guess
alpha = 0.05; pnull = 1/4 # under null
quantile = qbinom(1-alpha, n, pnull) # this is c - 1
cutoff = quantile + 1 # this is the cutoff to ensure a level alpha test
palt = 0.26 # suppose I think I'm slightly psychic: 1/4 + 0.01
power = 1 - pbinom(cutoff - 1, n, prob = palt)
power
```

```
## [1] 0.2612058
```

# A power calculation

Is $n = 10000$ enough?

```
n = 10000 # initial guess
alpha = 0.05; pnull = 1/4 # under null
quantile = qbinom(1-alpha, n, pnull) # this is c - 1
cutoff = quantile + 1 # this is the cutoff to ensure a level alpha test
palt = 0.26 # suppose I think I'm slightly psychic: 1/4 + 0.01
power = 1 - pbinom(cutoff - 1, n, prob = palt)
power
```

```
## [1] 0.7417297
```

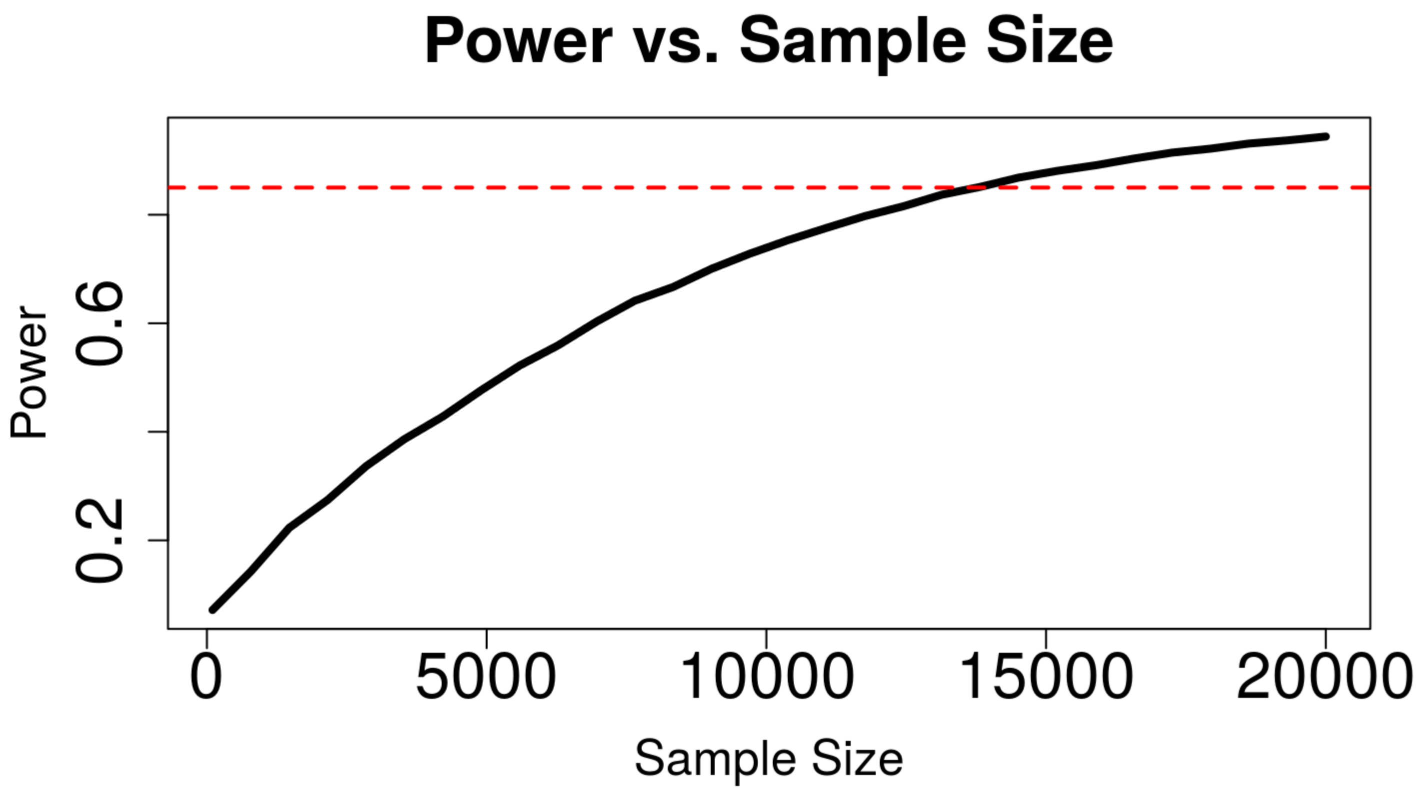
# A power calculation

Is $n = 20000$ enough?

```
n = 20000 # initial guess
alpha = 0.05; pnull = 1/4 # under null
quantile = qbinom(1-alpha, n, pnull) # this is c - 1
cutoff = quantile + 1 # this is the cutoff to ensure a level alpha test
palt = 0.26 # suppose I think I'm slightly psychic: 1/4 + 0.01
power = 1 - pbinom(cutoff - 1, n, prob = palt)
power
```

```
## [1] 0.944068
```

# Power versus sample size



# Power versus sample size

```r
alpha = 0.05; pnull = 1/4 # under null
palt = 0.26 # suppose I think I'm slightly psychic: 1/4 + 0.01
nlist = round(seq(100, 20000, length=50))
power = rep(NA, length(nlist))
for (i in 1:length(nlist)) {
  quantile = qbinom(1-alpha, nlist[i], pnull) # this is c - 1
  cutoff = quantile + 1 # this is the cutoff to ensure a level alpha test
  power[i] = 1 - pbinom(cutoff - 1, nlist[i], prob = palt)
}
```

```r
plot(nlist, power, type="l", xlab="n", ylab="Power", main="Power vs. Sample Size")
abline(h=0.85, col=2, lwd=2, lty=2)
```

To sum up …

# Two kinds of errors

| | $H_0$ true | $H_A$ true |
|---|---|---|
| Reject $H_0$ Specificity | Type I error | Good |
| Fail to reject $H_0$ Sensitivity | Good | Type II error |

**Type I error** - false positive ("gullible")

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**Goal:** design a procedure that can ensure that

$$P(\text{Type I error}) \leq \alpha$$

yet still has small $P(\text{Type II error})$.

# Significance level

By $P(\text{Type I error})$ we mean

$$P(\text{Reject } H_0 \mid H_0 \text{ is true})$$

Think of **significance level** as our *level of gullibility*

- thinking I’m psychic when I’m random guessing
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# Power

The **power** of a test is

$$P(\text{Reject } H_0 \mid H_A \text{ is true})$$

*Power* is test's **ability to detect** that alternative applies.

- detecting that drug works when it in fact does
- jury deciding "guilty" when person was guilty of crime

Note:

$$P(\text{Type II error}) = P(\text{Fail to reject } H_0 \mid H_A \text{ is true}) = 1 - \text{Power}$$

# Rejection region approach

# Rejection region approach

Classic Approach. Later comes the p-value approach

1. Specify $H_0$ and $H_A$
2. Determine **test statistic**

   Distribution to do calculations e.g. X~Binom(n, p)

   - figure out its sampling distribution under $H_0$
3. Determine **rejection region**

   This changes with p-value approach

   - specify for which observed values we will reject $H_0$
   - choose size of it to ensure significance level is $\alpha$
4. **Decision**: Did observed value of test statistic fall in rejection region?
5. Check assumptions

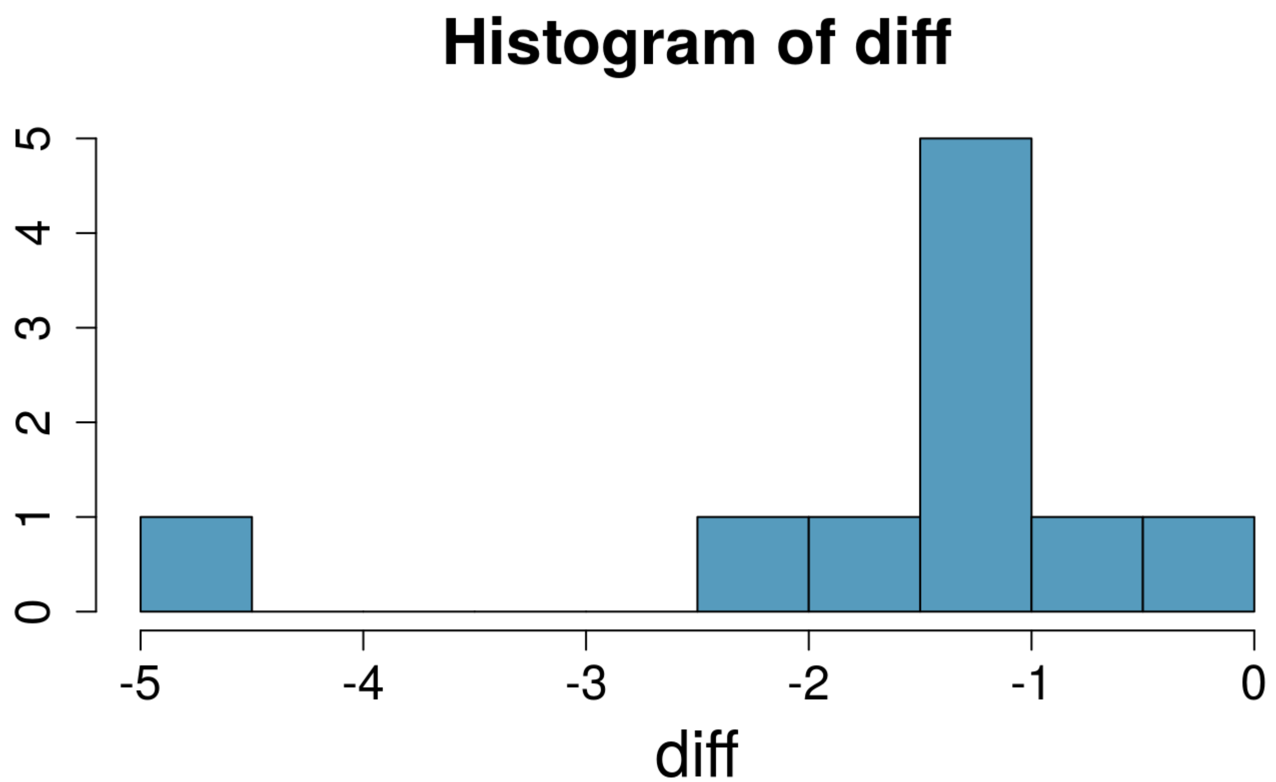
# Example (from Gosset himself!)

Measure effect of sleeping drugs A and B on each of 10 people

| | A | B | diff |
|---|---:|---:|---:|
| person1 | 0.7 | 1.9 | -1.2 |
| person2 | -1.6 | 0.8 | -2.4 |
| person3 | -0.2 | 1.1 | -1.3 |
| person4 | -1.2 | 0.1 | -1.3 |
| person5 | -0.1 | -0.1 | 0.0 |
| person6 | 3.4 | 4.4 | -1.0 |
| person7 | 3.7 | 5.5 | -1.8 |
| person8 | 0.8 | 1.6 | -0.8 |
| person9 | 0.0 | 4.6 | -4.6 |
| person10 | 2.0 | 3.4 | -1.4 |

# Example (from Gosset himself!)

```r
hist(diff, breaks=10)
```



# Step 1: Identify hypotheses

**Null hypothesis** - “no difference between drugs”

- a hypothesis is a statement about the population parameter
- let $\mu$ be the (population) mean difference between drug A and drug B.
- $H_0 : \mu = 0$

**Alternative hypothesis** - “there is a difference between the drugs”

- $H_A : \mu \neq 0$
- this is called a “two-sided” hypothesis
- two-sided hypothesis should be your default choice

# Step 2: Test statistic

$X_i =$ the difference in effect between the drugs for person $i$.

**Test statistic** - let's make decision based on $\bar{X}_n = \frac{X_1 + \cdots + X_n}{n}$

Sample Statistic

Assuming $X_i$ is approximately $N(\mu, \sigma)$, then

$$\bar{X}_n \approx N(\mu, \sigma/\sqrt{n}).$$

Standard Error

Under $H_0 : \mu = 0$, we have $\bar{X}_n \approx N(0, \sigma/\sqrt{n})$ or

$$\frac{\bar{X}_n - 0}{\sigma/\sqrt{n}} \approx N(0, 1)$$ Test Statistic

If $\sigma$ were known we'd have a test statistic with known sampling distribution under the null!

# Step 3: Rejection region

**Rejection region** - range of values of test statistic for which we will reject $H_0$ in favor of $H_A$.

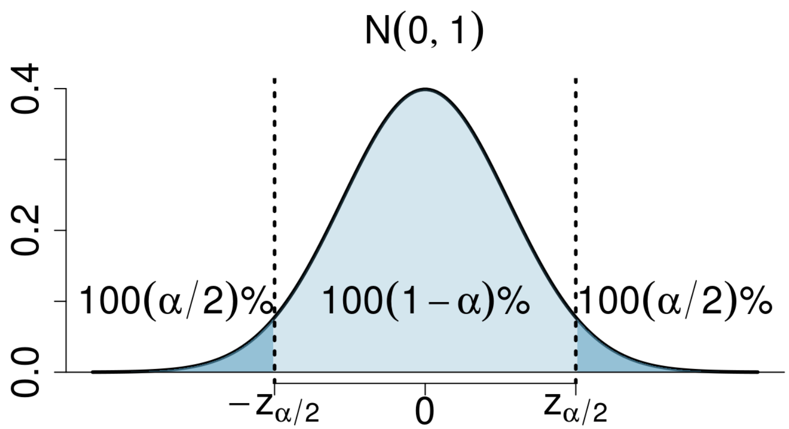
- look at $H_A$ to decide whether low values, high values, or both would be considered evidence against $H_0$ in favor of $H_A$.

**In example:**

$H_A : \mu \neq 0$. So will reject if computed value of our test statistic $\frac{\bar{x}_n - 0}{\sigma/\sqrt{n}}$ is too high or too low.

What does “too high” or “too low” mean?

# What does “too high” or “too low” mean?




$$\frac{\bar{X}_n - 0}{\sigma/\sqrt{n}} \approx N(0, 1)$$

so if we calculate $\frac{\bar{x}_n - 0}{\sigma/\sqrt{n}}$ and it falls in tails, we'd reject $H_0$ in favor of $H_A$.

# Step 3: Rejection region

Reject $H_0$ in favor of $H_A$ if

$$\left|\frac{\bar{x}_n - 0}{\sigma/\sqrt{n}}\right| > z_{\alpha/2}.$$

doing so ensures Type I error rate is $\alpha$:

$$P\left(\text{Reject } H_0 \,\middle|\, H_0 \text{ true}\right) = P\left(\left|\frac{\bar{X}_n - 0}{\sigma/\sqrt{n}}\right| > z_{\alpha/2} \,\middle|\, \mu = 0\right)$$

$$= P\left(|N(0,1)| > z_{\alpha/2}\right) = \alpha$$

# Step 4: Decision

Compute our test statistic (assume we know $\sigma = 1$):

```
alpha = 0.05; mu0 = 0; sigma = 1
xbar = mean(diff); n = length(diff)
(xbar - mu0) / (sigma / sqrt(n))
```

```
## [1] -4.996399
```

```
zvalue = -qnorm(alpha / 2)
zvalue
```

```
## [1] 1.959964
```

We reject $H_0$ in favor of $H_A$ since $4.996 \geq 1.96$.

# Step 5: Check assumptions

*Assumptions made:*

- independence of $X_i$'s
- approximate normality of $X_i$'s

In reality, we don't know $\sigma = 1$. How does the above change?

# Step 1: Identify hypotheses

*(Unchanged)*

- $H_0 : \mu = 0$
- $H_A : \mu \neq 0$

# Step 2: Test statistic

$X_i =$ the difference in effect between the drugs for person $i$.

**Test statistic** - let's make decision based on $\bar{X}_n = \frac{X_1 + \cdots + X_n}{n}$

Assuming $X_i$ is approximately $N(\mu, \sigma)$, then

$$\bar{X}_n \approx N(\mu, \sigma/\sqrt{n}).$$

Under $H_0 : \mu = 0$, we have $\bar{X}_n \approx N(0, \sigma/\sqrt{n})$ or

$$\frac{\bar{X}_n - 0}{\sigma/\sqrt{n}} \approx N(0, 1)$$

However, since $\sigma$ is unknown, we can't calculate its value and so it is **not a usable test statistic.**

# Step 2: Test statistic

**Problem:** we don't know $\sigma$, so we can't compute

$$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}$$

What did we do for confidence intervals?

$$\frac{\bar{X}_n - \mu}{S_n/\sqrt{n}}$$

where

$$S_n = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2}$$

# *Flashback* Why is it not normal?

Intuitively,

$$\frac{\bar{X}_n - \mu}{S_n/\sqrt{n}} \qquad \text{(let's call this } T_n\text{)}$$

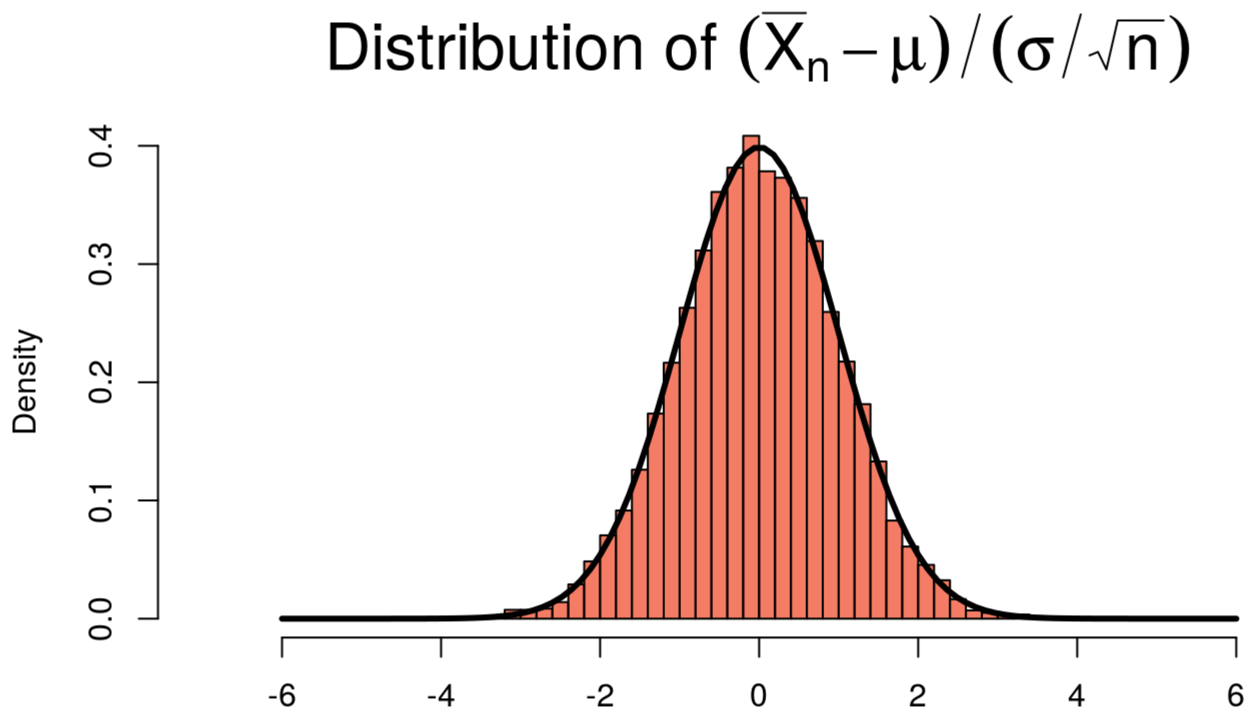
has more variability “in it” than

$$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$$

since $S_n$ is also random.

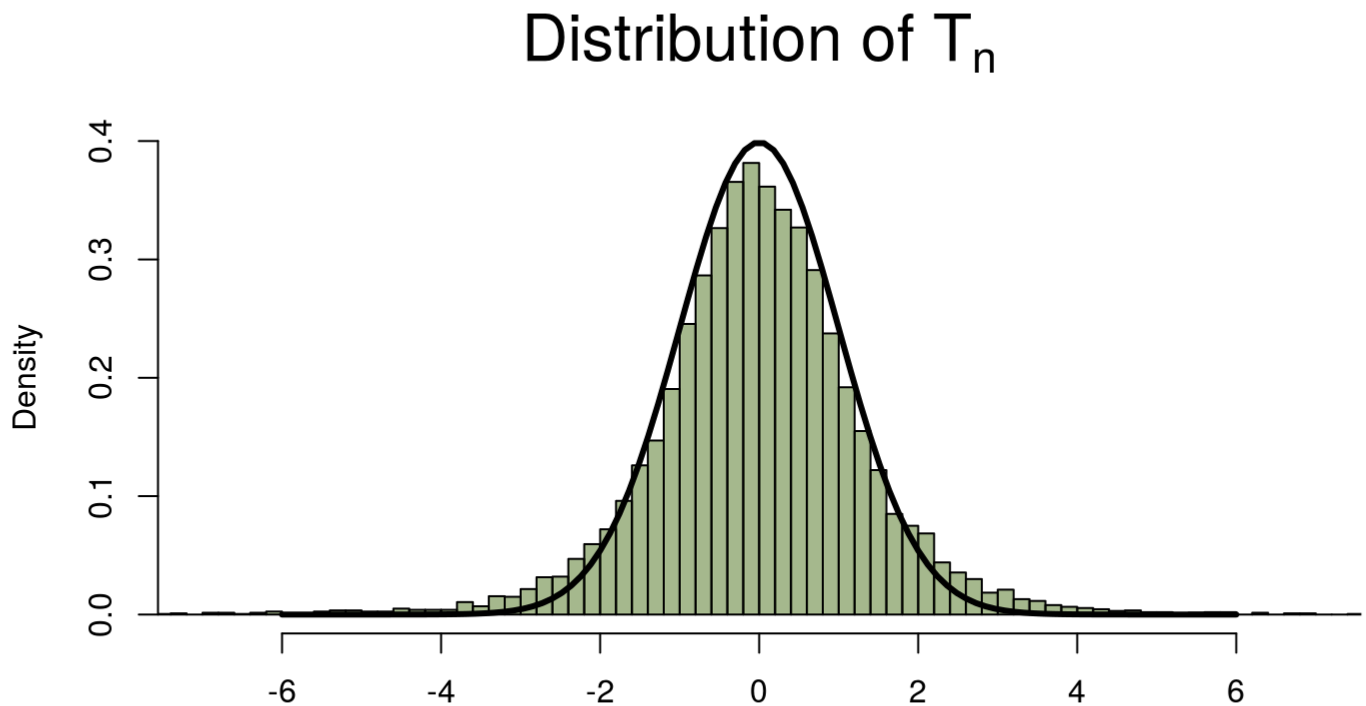
# Monte Carlo simulation (normal data, n=5)

Draw $X_1, \ldots, X_5 \sim N(\mu, \sigma)$: See $\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \approx N(0, 1)$



# Monte Carlo simulation (normal data, n=5)

Same as before. See $T_n = \frac{\bar{X}_n - \mu}{S_n/\sqrt{n}}$ has heavier tails than $N(0,1)$



# Student's t-distribution

If $X_1, \ldots, X_n \sim N(\mu, \sigma)$ are independent, then

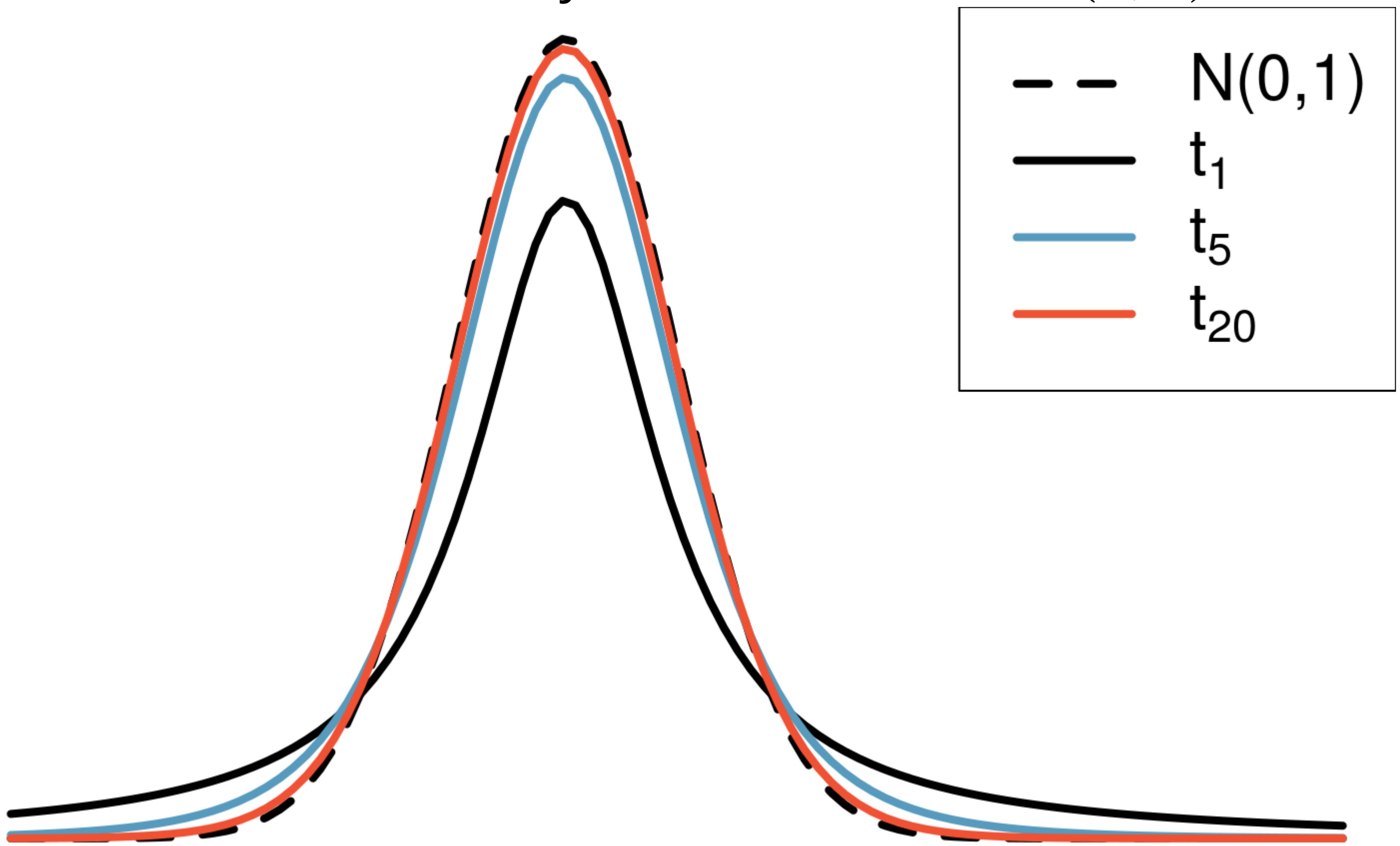
$$T_n = \frac{\bar{X}_n - \mu}{S_n/\sqrt{n}} \sim t_{n-1}$$

In words, we say that $T_n$ has a **t-distribution with $n-1$ degrees of freedom**.

$t_{n-1}$ denotes this distribution.

# Student's t-distribution

For small $n$ has noticeably heavier tails than $N(0, 1)$.



# *Flashforward* Step 2: Test statistic

Let's use

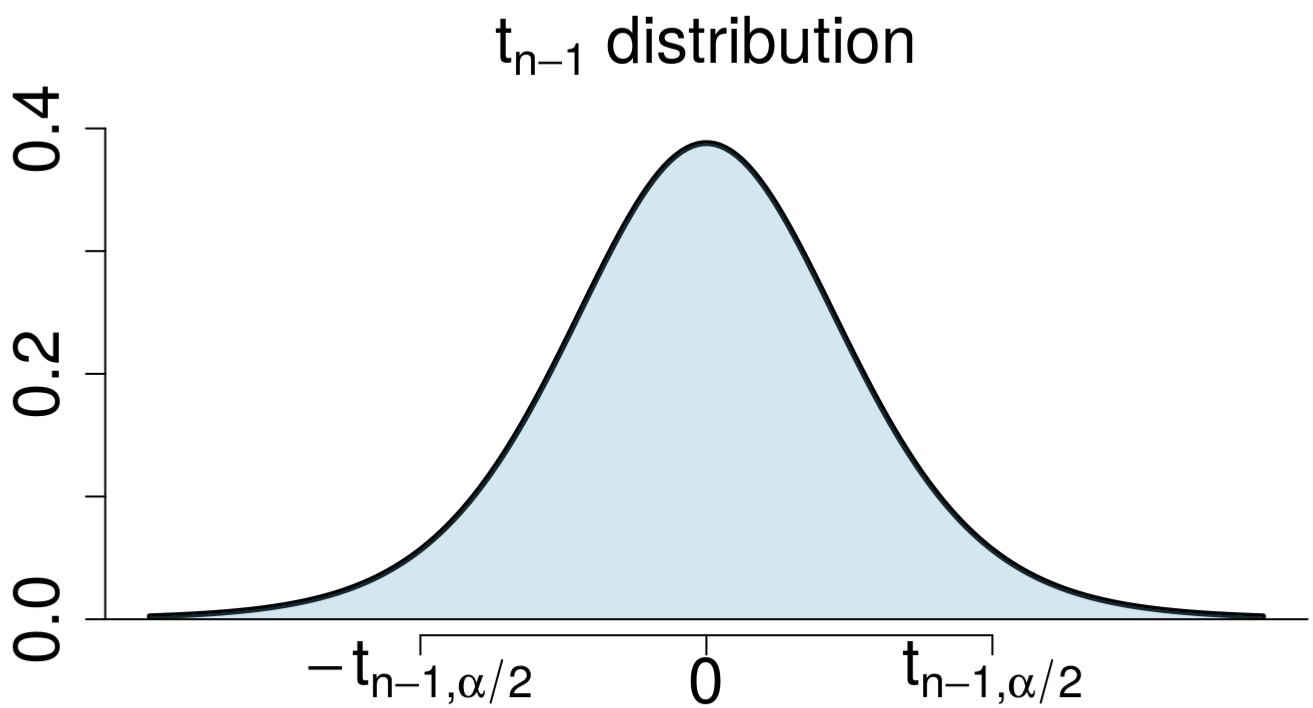
$$\frac{\bar{X}_n - \mu_0}{S_n/\sqrt{n}}$$

as our test statistic.

Under $H_0 : \mu = \mu_0$, its sampling distribution is known: $t_{n-1}$

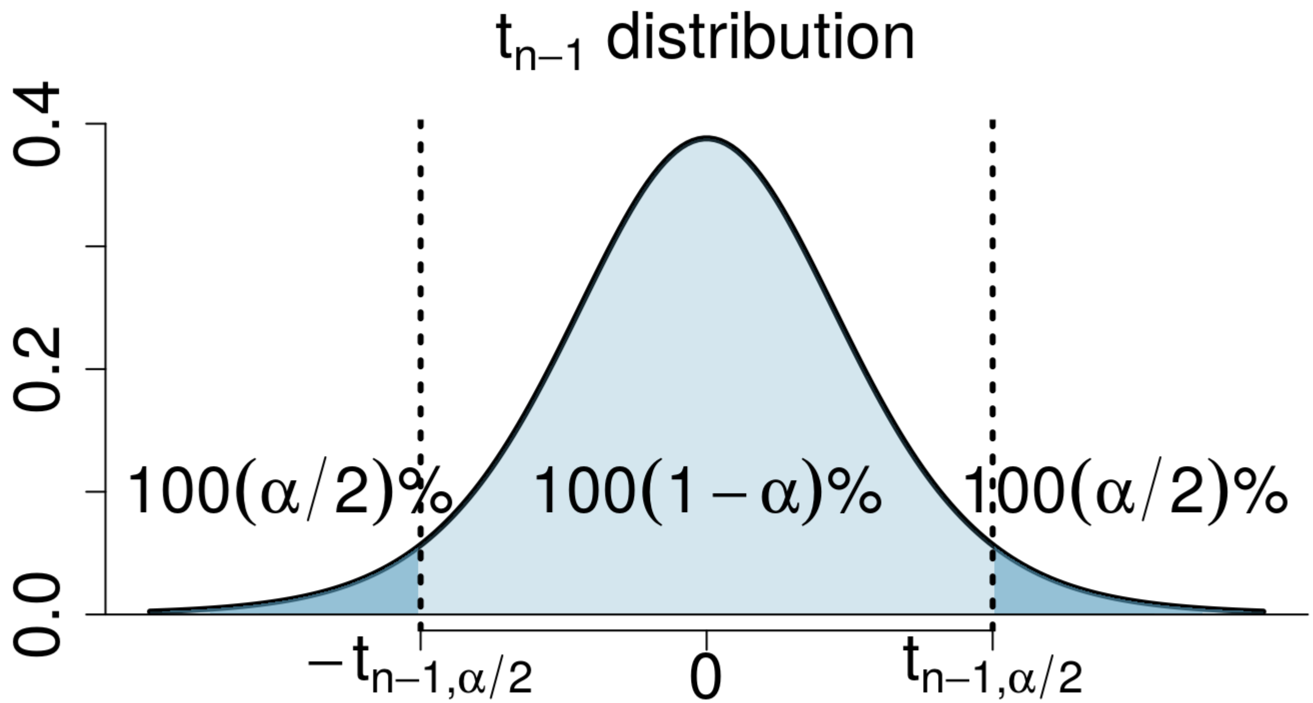
# Step 2: Test statistic

Distribution of $\frac{\bar{X}_n - \mu_0}{S_n/\sqrt{n}}$ under $H_0$



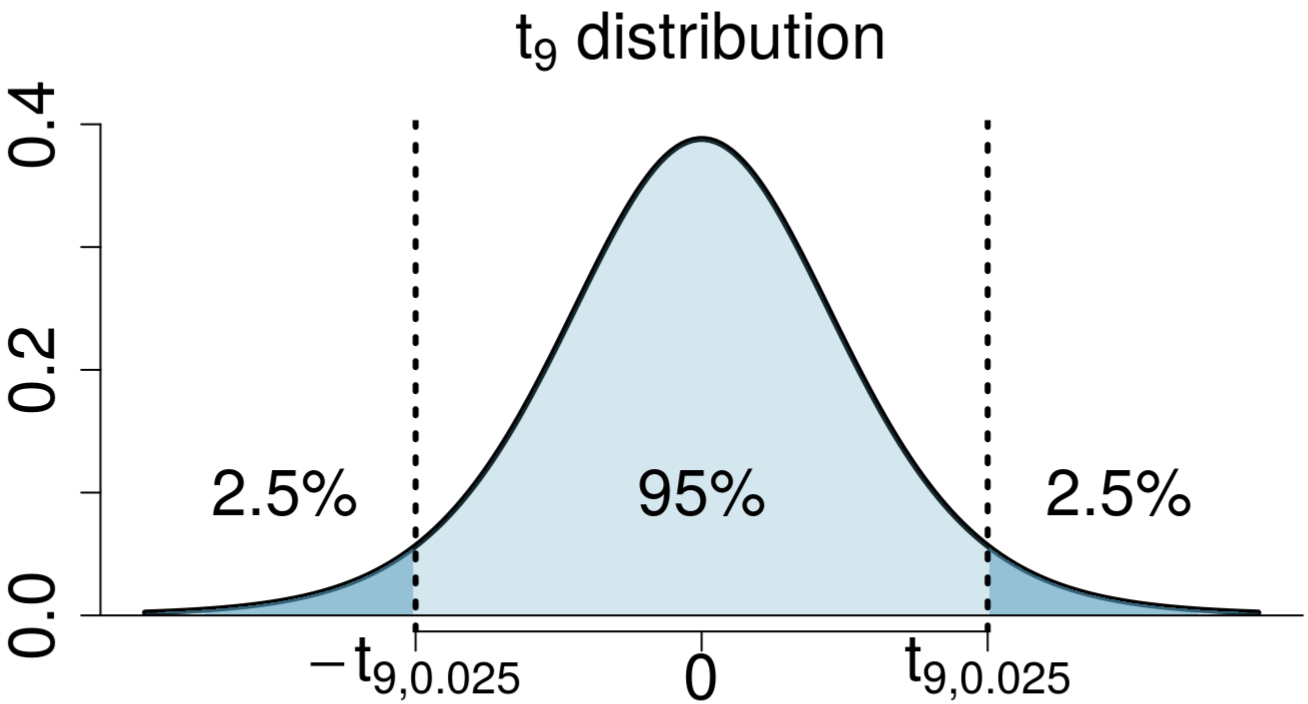
# Step 3: Rejection region

Distribution of $\frac{\bar{X}_n - \mu_0}{S_n/\sqrt{n}}$ under $H_0$

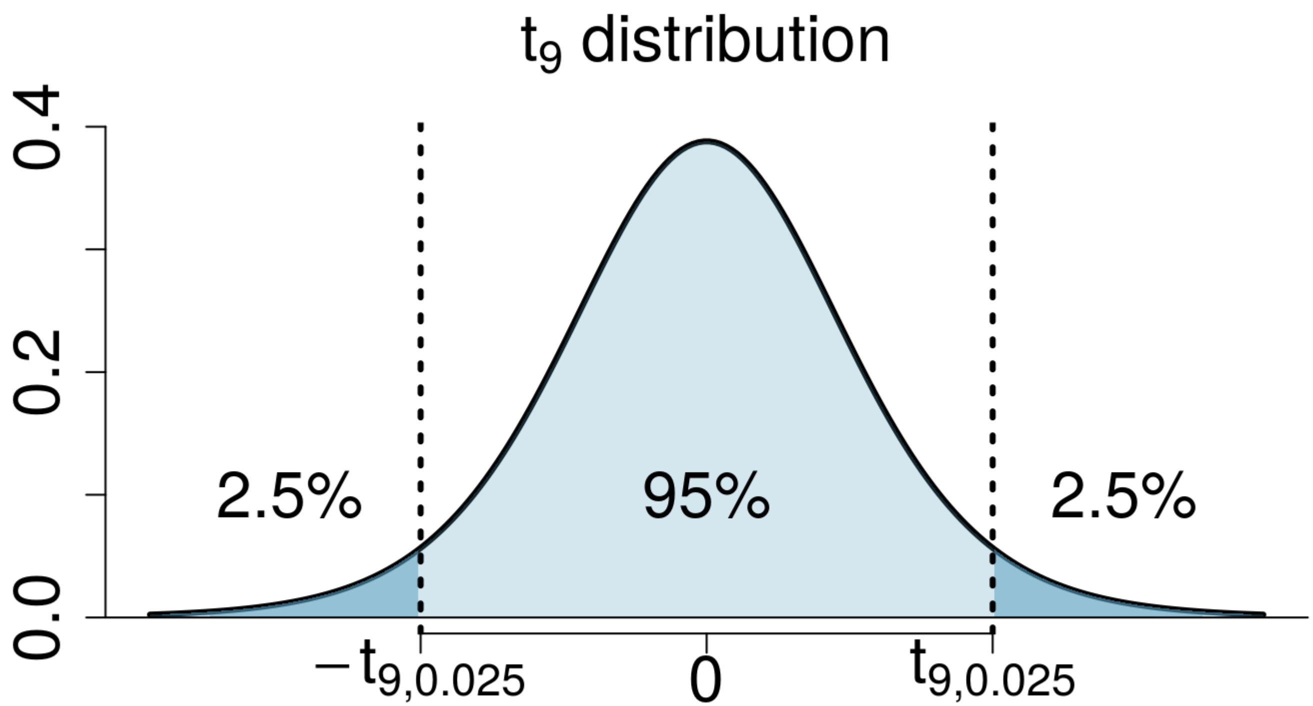


# Step 3: Rejection region

Sleep example: $\frac{\bar{X}_n - 0}{S_n/\sqrt{n}}$ under $H_0$ has distribution $t_9$



# Step 3: Rejection region




Reject $H_0$ in favor of $H_A$ if computed $\left|\frac{\bar{x}_n - 0}{s/\sqrt{n}}\right| > t_{9,0.025}$.

# Step 4: Decision

Compute our test statistic:

```
alpha = 0.05; mu0 = 0
xbar = mean(diff); n = length(diff); s = sd(diff)
(xbar - mu0) / (s / sqrt(n))
```

```
## [1] -4.062128
```

```
tvalue = -qt(alpha / 2, df = n - 1)
tvalue
```

```
## [1] 2.262157
```

We reject $H_0$ in favor of $H_A$ since $4.062 \geq 2.262$.

# Note

For t-test, our cutoff is $t_{n-1,\alpha/2}$ which is bigger than $z_{\alpha/2}$.

$$z_{0.025} = 1.960$$

$$t_{9,0.025} = 2.262$$

**Intuition for higher cutoff:** We're less surprised by a large value of test statistic when if we were using $S_n$.

# Step 5: Check assumptions

t-test assumes $X_i$'s are normal. How to evaulate?

- matters most when $n$ is small
- hardest to verify when $n$ is small (unfortunate!)

# p-value approach

# p-value approach

1. Specify $H_0$ and $H_A$
2. Determine **test statistic**
   - figure out its sampling distribution under $H_0$
3. Compute the **p-value** based on the particular sample of data you collected
4. **Decision**: Did p-value fall below $\alpha$? If so, reject $H_0$ in favor of $H_A$.
5. Check assumptions

# Steps 1 and 2

Identical to what was done in rejection region approach

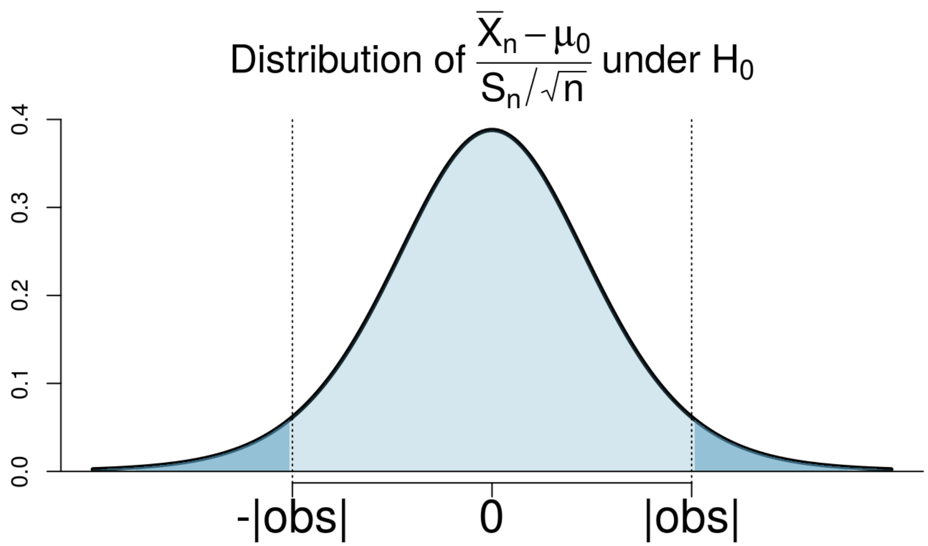
# Step 3: Compute the p-value

> The **p-value** is the probability under the null hypothesis of seeing something as extreme or more extreme than what was actually observed.

- "**extreme**" is defined by $H_A$.
- in example, $H_A : \mu \neq 0$, so "extremeness" means how far $\bar{X}_n$ is from 0, that is $|\bar{X}_n - 0|$.
- "**probability under the null**" - we know $\frac{\bar{X}_n - 0}{S_n/\sqrt{n}} \sim t_{n-1}$
- "**actually observed**" - we observed $\frac{\bar{x}_n - 0}{s/\sqrt{n}} = -4.06$

**Putting it all together:** what is the probability that a $t_{n-1}$ random variable would be larger than 4.06 or smaller than -4.06?

# Always draw a picture!




- sampling distribution is a $t_{n-1}$ distribution
- obs is the observed value (realization of test statistic):

$$\frac{\bar{x}_n - \mu_0}{s/\sqrt{n}}$$

# Calculating p-value

Want

$$P(t_{n-1} < -|obs|) + P(t_{n-1} > |obs|) = 2P(t_{n-1} < -|obs|)$$

In example:

```
mu0 = 0 # null value
xbar = mean(diff); s = sd(diff); n = 10
obs = (xbar - mu0) / (s / sqrt(n))
pvalue = 2*pt(-abs(obs), df = n - 1)
pvalue
```

```
## [1] 0.00283289
```

# Step 4: Decide

- If p-value is less than $\alpha$, we reject $H_0$ in favor of $H_A$ .
- this step is optional if you've calculated the p-value. You can leave it up to the reader whether this is "statistically significant"

# Step 5: Check assumptions

Same as before.

# Other remarks

# What makes a p-value small?

Recall for two-sided t-test, our p-value was

$$2P\left(t_{n-1} > \left|\frac{\bar{x}_n - \mu_0}{s/\sqrt{n}}\right|\right)$$

where $\bar{x}_n$ and $s$ are the computed values from your data.

- p-value is small when $\left|\frac{\bar{x}_n - \mu_0}{s/\sqrt{n}}\right|$ is large.
- this occurs when:
  - $|\bar{x}_n - \mu_0|$ is large (happens when true $\mu$ is far from $\mu_0$)
  - $s$ is small (happens when $\sigma$ is small)
  - $n$ is large (in your control!)

# What makes a p-value small?

**Consequence:**

- As long as true $\mu$ is not exactly equal to $\mu_0$, we can get small p-values by increasing $n$.

- *cynical view:* p-values just reflect your amount of effort (sample size) relative to what's there,

$$\frac{|\mu - \mu_0|}{\sigma}.$$

# Practical significance

## Important distinction

> Statistical significance $\neq$ Practical significance

In example: If drug A increases sleep by 1.2 minutes over drug B, do we care?

- relevant question since we could still get a p-value $< 0.0001$ with a large enough $n$.
- essential to consider the **effect size** **This will be important on the test**
  - such as $|\mu - \mu_0|$
  - or standardized $|\mu - \mu_0|/\sigma$

# Duality between testing and confidence intervals

# There's a connection!

1. Recall **level $\alpha$ test** for $H_0 : \mu = \mu_0$ vs. $H_A : \mu \neq \mu_0$:

> Reject $H_0$ when
>
> $$\frac{|\bar{x}_n - \mu_0|}{s_n/\sqrt{n}} > t_{n-1,\alpha/2}$$

1. **$100(1-\alpha)\%$ confidence interval** for $\mu$:

> $$\left[\bar{x}_n - t_{n-1,\alpha/2} s_n/\sqrt{n}, \bar{x}_n + t_{n-1,\alpha/2} s_n/\sqrt{n}\right]$$

**Observe:** $\mu_0$ in confidence interval is equivalent to failing to reject $H_0$!

# There's a connection!

**Justification:** $\mu_0$ being in CI means

$$\bar{x}_n - t_{n-1,\alpha/2} s_n/\sqrt{n} \le \mu_0 \le \bar{x}_n + t_{n-1,\alpha/2} s_n/\sqrt{n}$$

or

$$|\bar{x}_n - \mu_0| \le t_{n-1,\alpha/2} s_n/\sqrt{n}$$

or

$$\frac{|\bar{x}_n - \mu_0|}{s_n/\sqrt{n}} \le t_{n-1,\alpha/2}$$

Compare to

> Reject $H_0$ when
>
> $$\frac{|\bar{x}_n - \mu_0|}{s_n/\sqrt{n}} > t_{n-1,\alpha/2}$$

# New interpretation of CIs

A $100(1-\alpha)\%$ confidence interval consists of all the values of $\mu_0$ for which a test of the form

$$\begin{aligned} H_0 &: \mu = \mu_0 \\ H_A &: \mu \neq \mu_0 \end{aligned}$$

would fail to reject $H_0$ at the $\alpha$ significance level.

# Why this is useful

Suppose we are interested in testing

$$H_0 : \mu = 0$$
$$H_A : \mu \neq 0.$$

Suppose we get a $95\%$ confidence interval: [-0.6, 1.1]

Test would fail to reject at $0.05$ significance level because the interval [-0.6, 1.1] includes 0.

**Sanity check:** what happens to both if you increase $\alpha$?